COMPLEX ANALYSIS AND PROBABILITY DISTRIBUTIONS

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BY

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CO's	Course outcomes
CO1	Discuss about continuity/differentiability/analyticity of a Complex function using Cauchy-Riemann Equations. Estimate complex conjugate using Milne Thomson method and understand the concept of Bilinear transformation.
CO2	Recognize and apply the Cauchy's integral formula and the generalized Cauchy's integral formula. Evaluate complex functions as power series and radius of convergence of power series
CO3	Establish the contour integral with an integrand which has singularities lying inside or outside the simple closed contour. Expand complex function as power series using Taylor's and Laurent series.
CO4	Enrich the knowledge of Probability to discrete and continuous random variables.
CO5	Analyze probability distributions for Binomial, Poisson and normal distributions and study its properties.

MODULE– I
COMPLEX FUNCTION AND DIFFERENTIATION

CLOs	Course Learning Outcome
CLO 1	Recall continuity, differentiability, analyticity of a function using limits.
CLO 2	Interpret the conditions for a complex variable to be analytic and/or entire function.
CLO 3	Interpret the concepts of Cauchy-Riemann relations and harmonic functions.
CLO 4	Analyze the Bilinear transformation by cross ratio property.
CLO 5	Identify the conditions of fixed and critical point of Bilinear Transformation.

Complex number:

For a complex number $z = x + iy$, the number $\operatorname{Re} z = x$ is called the real part of z and the number $\operatorname{Im} z = y$ is said to be the its imaginary part. If $x = 0$, z is said to be a purely imaginary number.

Definition : Let $z = x + iy \in C$. The complex number $z = x - iy$ is called the complex conjugate of z and $|z| = \sqrt{x^2+y^2}$ is said to be the absolute value or the modulus of the complex number z.

COMPLEX FUNCTIONS AND DIFFERENTIATION

Let D be a nonempty set in C. A single-valued complex function or, simply, a complex function $f : D \to C$ is a map that assigns to each complex argument $z = x + iy$ in D a unique complex number $w = u + iv$. We write $w = f(z)$.

The set D is called the domain of the function f and the set f(D) is the range or the image of f. So, a complex-valued function f of a complex variable z is a rule that assigns to each complex number z in a set D one and only one complex number w. We call w the image of z under f.

COMPLEX FUNCTIONS AND DIFFERENTIATION

If $z = x + iy \in D$, we shall write $f(z) = u(x, y) + iv(x, y)$ or $f(z) = u(z) + iv(z)$. The real functions u and v are called the real and, respectively, the imaginary part of the complex function f. Therefore, we can describe a complex function with the aid of two real functions depending on two real variables.

COMPLEX FUNCTIONS AND DIFFERENTIATION

Example 1.The function $f : C \to C$, defined by $f(z) = z^3$, can be written as $f(z) = u(x, y) + iv(x, y)$, with $u, v : R^2 \to R$ given by $u(x, y) = x^3 - 3xy^2$, $v(x, y) = 3x^2 y - y^3$.

Example 2.For the function $f : C \to C$, defined by $f(z) = e^z$, we have

$u(x, y) = e^x \cos y$, $v(x, y) = e^x \sin y$, for any $(x, y) \in R^2$.

COMPLEX FUNCTIONS AND DIFFERENTIATION

Limits of Functions :

Let $D \subseteq C$, $a \in D'$ and $f : D \to C$. A number $l \in C$ is called a limit of the function f at the point a if for any $V \in V(l)$, there exists $U \in V(a)$ such that, for any $z \in U \cap D \setminus \{a\}$, it follows that $f(z) \in V$. We shall use the notation

$$l = \lim_{z \to z_0} f(z).$$

Remark :

If a complex function $f : D \to C$ possesses a limit l at a given point a, then this limit is unique

Exercise 1: Prove that $\lim_{z \to z_0} \frac{\bar{z}}{z}$ does not exist.

Solution : To prove that the above limit does not exist, we compute this limit as z → 0 on the real and on the imaginary axis, respectively. In the first situation, i.e. for z = x ∈ R, the value of the limit is 1. In the second situation,

i.e. for z = i y, with y ∈ R, the limit is −1. Thus, the limit depends on the direction from which we approach 0, which implies that the limit does not exist.

COMPLEX FUNCTIONS AND DIFFERENTIATION

Let w = f(z) be a given function defined for all z in a neighbourhood of z_0.If $\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ exists,the function f(z) is said to be derivable at z_0 and the limit is denoted by $f'(z_0)$. $f'(z_0)$ if exists is called the derivative of f(z) at z_0

Exercise : f(z)= $|z^2|$ is a function which is continuous at all z but not derivable at any $z \neq 0$

Solution: Let f(z)= $|z^2| = z\bar{z}$

Then f(z)= $z_0 \bar{z}_0$

We have to prove that $\lim_{z \to z_0} z = z_0$ and $\lim_{z \to z_0} \bar{z} = \bar{z}_0$ Thus $\lim_{z \to z_0} z\bar{z} = z_0 \bar{z}_0$

$\therefore \lim_{z \to z_0} f(z) = f(z_0)$

$\therefore$ The function is continuous at all z

COMPLEX FUNCTIONS AND DIFFERENTIATION

The Cauchy–Riemann equations on a pair of real-valued functions of two real variables $u(x,y)$ and $v(x,y)$ are the two equations:

1. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$
2. $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Typically u and v are taken to be the real and imaginary parts respectively of a complex-valued function of a single complex variable $z = x + iy$, $f(x + iy) = u(x,y) + iv(x,y)$

COMPLEX FUNCTIONS AND DIFFERENTIATION

- We found

$$\boxed{\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}} \qquad \boxed{\frac{df}{dz} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}}$$

- For a unique derivative, these expressions must be equal. That is, a *necessary* condition for the existence of a derivative of function of a complex variable is that

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{Cauchy-Riemann conditions}}$$

- We've proved that if $\frac{df}{dz}$ exists,

(implies)
$\Rightarrow$ Cauchy-Riemann conditions (necessity).

polar form of Cauchy-Riemann equation

polar form of Cauchy-Riemann equation:

Theorem:

If $f(z)=f(re^{i\theta})=u(r,\theta)+iv(r,\theta)$ and f(z) is derivable at $z_0=r_0e^{i\theta_0}$ then

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Proof: Let $z=re^{i\theta}$ Then $f(z)=f(re^{i\theta})=u(r,\theta)+iv(r,\theta)$

Differentiating it with respect to r partially,

$$\frac{\partial}{\partial r}f(z)=f'(z)\frac{\partial z}{\partial r}=f'(z)e^{i\theta}$$

$$\therefore f'(z)=\frac{1}{e^{i\theta}}\frac{\partial f}{\partial r}=\frac{1}{e^{i\theta}}(u_r+iv_r) \qquad \rightarrow (1)$$

COMPLEX FUNCTIONS AND DIFFERENTIATION

Similarly differentiating partially with respectto θ

$$\frac{\partial f}{\partial \theta} = f'(z)\frac{\partial z}{\partial \theta} = f'(z).rie^{i\theta}$$

$$\therefore f'(z) = \frac{1}{rie^{i\theta}}(u_\theta + iv_\theta) \qquad \rightarrow (2)$$

From (1) and (2) we have

$$\frac{1}{e^{i\theta}}(u_r + iv_r) = \frac{1}{rie^{i\theta}}(u_\theta + iv_\theta)$$

$$\therefore u_r + iv_r = \frac{1}{r}\frac{\partial v}{\partial \theta} - i\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Equating real and imaginary parts ,we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} and \ \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Analytic function

A complex function is said to be analytic on a region R if it is complex differentiable at every point in R. The terms holomorphic function, differentiable function, and complex differentiable function are sometimes used interchangeably with "analytic function"

If a complex function is analytic on a region R , it is infinitely differentiable in R.

Singularities:

A complex function may fail to be analytic at one or more points through the presence of singularities, or along lines or line segments through the presence of branch cuts.

Eg. f(z)= $\frac{1}{z}$ is analytic every where except at z=0.

At z=0 $f'(z)$ does not exist.

So z=0 is an isolated singular point.

Entire function

A complex function that is analytic at all finite points of the complex plane is said to be entire. A single-valued function that is analytic in all but possibly a discrete subset of its domain, and at those singularities goes to infinity like a polynomial (i.e., these exceptional points must be poles and not essential singularities), is called a meromorphic function.

Cauchy–Riemann equations

The Cauchy–Riemann equations on a pair of real-valued functions of two real variables $u(x,y)$ and $v(x,y)$ are the two equations:

1. $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
2. $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Typically u and v are taken to be the real and imaginary parts respectively of a complex-valued function of a single complex variable $z = x + iy$, $f(x + iy) = u(x,y) + iv(x,y)$

Conjugate harmonic function:

If two harmonic functions u and v satisfy the Cauchy-Reimann equations in a domain D and they are real and imaginary parts of an analytic function f in D then v is said to be a conjugate harmonic function of u in D.If f(z)=u+iv is an analytic function and if u and v satisfy Laplace's equation ,then u and v are called conjugate harmonic functions.

C-R equations in polar form

The Cauchy-Riemann equations can be written in other coordinate systems. For instance, it is not difficult to see that in the system of coordinates given by the polar representation $z = re^{i\theta}$ these equations take the following form:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

COMPLEX FUNCTIONS AND DIFFERENTIATION

Problem: Show that the function $f(z) = e^z$ ***satisfies the Cauchy-Riemann equations.***

Solution:

since $e^z = e^x(\cos y + i \sin y),$

Indeed it follows that

$u(x, y) = e^x \cos y, \quad v(x, y) = e^x \sin y \quad , u(x, y) = e^x \cos y, \quad v(x, y) = e^x s$

and $\dfrac{\partial u}{\partial x} = e^x \cos y = \dfrac{\partial v}{\partial y}$; $\dfrac{\partial u}{\partial y} = e^x \sin y = -\dfrac{\partial v}{\partial x}$;

Moreover, e^z is complex derivable and it follows immediately that its complex derivative is

e^z.

Construction of analytic function whose real or imaginary part is known:

Suppose $f(z)=u+iv$ is an analytic function ,whose real part u is known .We can find v, the imaginary part and also the function $f(z)$.

COMPLEX FUNCTIONS AND DIFFERENTIATION

Problem: Showthat $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)|=0$ where f(z) is an analytic function.

Solution: Taking $x=\frac{z+\bar{z}}{2}, y=\frac{z-\bar{z}}{2}=\frac{-i}{2}(z-\bar{z})$

We have $\frac{\partial}{\partial z}=\frac{\partial}{\partial x}\left(\frac{\partial x}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial y}{\partial z}\right)=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)$

And $$\frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$$

COMPLEX FUNCTIONS AND DIFFERENTIATION

$$\therefore \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \cdot \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

Hence $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\log|f'(z)| = 4 \quad \frac{\partial^2}{\partial z \partial \bar{z}}\left(\frac{1}{2}\log|f'(z)|^2\right)$

$$= 2\frac{\partial^2}{\partial z \partial \bar{z}} \; [(\log f'(z) \; f'(\bar{z}))] \qquad (\because |z|^2 = zz)$$

$$= 2\frac{\partial^2}{\partial z \partial \bar{z}} [(\log f'(z) + f'(\bar{z}))]$$

$$= 2\left[\frac{\partial}{\partial \bar{z}} \frac{f''(z)}{f'(z)} + \frac{\partial}{\partial z} \frac{f''(z)}{f'(z_)}\right]$$

$$= 2(0+0)=0$$

Since f(z) is analytic , f(z) is analytic, $f'(\bar{z})$ is also analytic

and $\dfrac{\partial f'(z)}{\partial z} = 0, \dfrac{\partial f'(\bar{z})}{\partial z} = 0$

Problem: Show that $f(z) = \begin{cases} \dfrac{xy(x+iy)}{x^2+y^4} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$ is not analytic

at z=0 although C-R equations satisified at origin.

Solution: $$\frac{f(z)-f(0)}{z-0}=\frac{f(z)-0}{z}=\frac{f(z)}{z}$$

$$=\frac{xy^2(x+iy)}{(x^2+y^4).z}=\frac{xy^2(z)}{(x^2+y^4).z}=\frac{xy^2}{(x^2+y^4)}$$

Clearly $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{xy^2}{(x^2+y^4)} = \lim\limits_{\substack{y\to 0\\ x\to 0}} \frac{xy^2}{(x^2+y^4)} = 0$

Along path y=mx

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{x\to 0}\frac{x(m^2.x^2)}{x^2+m^4.x^4}=\lim_{x\to 0}\frac{m^2.x^2}{1+m^4.x^2}=0$$

Along path $x=my^2$

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{y\to 0}\frac{y^2(m.y^2)}{y^4+m^2.y^4}=\lim_{y\to 0}\frac{m}{1+m^2}\neq 0$$

Limit value depends on m i.e on the path of approach and its different for the different paths

Followed and therefore limit does not exists.

Hence f(z) is not differentiable at z=0.Thus f(z) is not analytic at z=0

To prove that C-R conditions are satisified at origin

Let $f(z) = u + iv = \dfrac{xy^2(x+iy)}{(x^2+y^4)}$

Then u(x,y)= $\dfrac{x^2y^2}{(x^2+y^4)}$ and v(x,y)= $\dfrac{xy^3}{(x^2+y^4)}$

for $z \neq 0$

Also u(0,0)=0 and v(0,0)= 0

[$\because$ f(z)=0 at z=0]

COMPLEX FUNCTIONS AND DIFFERENTIATION

Now $\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \frac{0}{x} = 0$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{x \to 0} \frac{0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x \to 0} \frac{0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{x \to 0} \frac{0}{y} = 0$$

Thus C-R equations are satisified are satisified at the origin

Hence f(z) is not analytic at z=0 even C-R equations are satisified at origin.

Milne Thomson method:

Problem : Find the regular function whose imaginary part is $\log(x^2 + y^2) + x - 2y$.

Solution: Given $v = \log(x^2 + y^2) + x - 2y$

$\therefore \frac{\partial v}{\partial x} = \frac{2x}{x^2 + y^2} + 1$ -→ (1) and $\frac{\partial v}{\partial y} = \frac{2y}{x^2 + y^2} - 2$ -→ (2)

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$ (Using C-R equation)

$$= \frac{2y}{x^2 + y^2} - 2 + \left(\frac{2x}{x^2 + y^2} + 1\right)$$ (using (1),(2))

COMPLEX FUNCTIONS AND DIFFERENTIATION

By Milne Thomson method ,f'(z) is expressed in terms of z by replacing x z and y by 0.

Hence $f'(z) = -2 + i\left(\frac{2z}{z^2} + 1\right) = -2 + i\left(\frac{2}{z} + 1\right)$

On integrating, $f(z) = \int\left[-2 + i\left(\frac{2}{z} + 1\right)\right]dz + c$

$$= -2z + i(2\log z + z) + c = 2i\log z - (2 - i)z + c.$$

Problem: Show that the function $u = 4xy - 3x + 2$ is harmonic .construct the corresponding analytic function f(z)=u+iv in terms of z.

Solution: Given $u = 4xy - 3x + 2 \rightarrow (1)$

Differentiating (1) partially w.r.t .x, $\frac{\partial u}{\partial x} = 4y - 3$

Again differentiating $\frac{\partial^2 u}{\partial x^2} = 0$

Again differentiating (1) partially w.r.t .y, $\frac{\partial u}{\partial x} = 4x$

Again differentiating $\frac{\partial^2 u}{\partial y^2} = 0$

Bilinear Transformations - Mobius Transformations

Let a,b,c,d denote four complex constants with the restriction

$ad \neq bc$ that . Then the function

$$w = s\ z = \frac{az + b}{cz + d}$$

is called a bilinear transformation, a Möbius transformation, or a linear fractional transformation.

There exists a unique bilinear transformation that maps three distinct points z_1, z_2 and z_3 onto three distinct points $w_1, w_2\ and w_3$ respectively. An implicit formula for the mapping is given by the equation

$$\frac{z - z_1}{z - z_3}\frac{(z_2 - z_3)}{(z_2 - z_1)} = \frac{w - w_1}{w - w_3}\frac{(w_2 - w_3)}{(w_2 - w_1)}$$

Problem: Construct the bilinear transformation $w = S(z)$ that maps the points $z_1 = -i$, $z_2 = 1$, $z_3 = i$ onto the points $w_1 = -1$, $w_2 = 0$, $w_3 = 1$, respectively.

Solution: We use the implicit formula and write

$$\frac{(z-(-i))(1-i)}{(z-i)(1-(-i))} = \frac{(w-(-1))(0-1)}{(w-1)(0-(-1))}$$

$$\frac{(z+i)(1-i)}{(z-i)(1+i)} = \frac{(w+1)(0-1)}{(w-1)(0+1)}$$

$$\frac{(z+i)(1-i)}{(z-i)(1+i)} = \frac{w+1}{-w+1}.$$

COMPLEX FUNCTIONS AND DIFFERENTIATION

Expanding this equation, collecting terms involving w and zw on the left and then simplify.

$$(z - i)(1 + i)(w + 1) = (z + i)(1 - i)(-w + 1)$$

$$(1 + i)zw + (1 - i)w + (1 + i)z + (1 - i)$$

$$=$$

$$(-1 + i)zw + (-1 - i)w + (1 - i)z + (1 + i)$$

$$zw + izw + w - iw + z + iz + 1 - i$$

$$=$$

$$-zw + izw - w - iw + z - iz + 1 + i$$

$$2zw + 2w = -2iz + 2i$$

$$zw + w = -iz + i$$

$$w(1+z) = i(1-z)$$

Therefore the desired bilinear transformation is

$$w = S(z) = \frac{i(1-z)}{1+z}.$$

Problem: Find the bilinear transformation $w = S(z)$ that maps the points $z_1 = -2,\ z_2 = -1 - i,\ z_3 = 0$ onto the points $w_1 = -1,\ w_2 = 0,\ w_3 = 1$, respectively.

Solution. Again, we use the implicit formula and write

$$\frac{(z-(-2))\,((-1-i)-0)}{(z-0)\,((-1-i)-(-2))} = \frac{(w-(-1))\,(0-1)}{(w-1)\,(0-(-1))}$$

$$\frac{(z+2)\,(-1-i)}{(z)\,(-1-i+2)} = \frac{(w+1)\,(-1)}{(w-1)\,(1)}$$

$$\frac{z+2}{z}\;\frac{-1-i}{1-i} = \frac{1+w}{1-w}$$

Using the fact that $\frac{-1-i}{1-i}=\frac{1}{i}$, we rewrite this equation as

$$\frac{z+2}{iz}=\frac{1+w}{1-w}.$$

We now expand the equation and obtain

$$(z+2)(1-w)=iz(1+w)$$

$$z+2-zw-2w=iz+izw$$

$$z-iz+2=zw+2w+izw$$

$$z(1-i)+2=w(z+iz+2)$$

$$(1-i)z+2=w((1+i)z+2)$$

which can be solved for w in terms of z, giving the desired solution

$$w = s(z) = \frac{(1-i)z+2}{(1+i)z+2}$$

COMPLEX FUNCTIONS AND DIFFERENTIATION

EXCERCISE PROBLEMS:

1) Show that the real part of an analytic function f (z) where u = e^{-2xy} sin $x^2 - y^2$ is a harmonic function. Hence find its harmonic conjugate.

2) Prove that the real part of analytic function f (z) where u = log z^2 is harmonic function. If so find the analytic function by Milne Thompson method.

3) Obtain the regular function f (z) whose imaginary part of an analytic function is $\frac{x}{x^2+y^2}$

4) Find an analytic function f (z) whose real part of an analytic function is u = $\frac{\sin 2x}{\cosh 2y - \cos 2x}$ by Milne-Thompson method.

5) Find an analytic function f (z) = u +iv if the real part of an analytic function is u = a (1+cosθ) using Cauchy-Riemann equations in polar form.

COMPLEX FUNCTIONS AND DIFFERENTIATION

6) Prove that if $u = x^2 - y^2, v = -\frac{y}{x^2+y^2}$ both u and v satisfy Laplace's equation, but u + iv is not a regular (analytic) function of z.

7) Show that the function f(z) = $x\overline{y}$ is not analytic at the origin although Cauchy –Riemann equations are satisfied at origin.

8) If $w = \emptyset + i\varphi$ represents the complex potential for an electric field where $\varphi = x^2 - y^2 + \frac{x}{x^2+y^2}$ then determine the function φ.

9) State and Prove the necessary condition for f(z) to be an analytic function in Cartesian form.

10) If u and v are conjugate harmonic functions then show that uv is also a harmonic function.

11) Find the orthogonal trajectories of the family of curves $r^2 cos 2\theta$ = c

12) Find an analytic function whose real part is $u = \frac{sin\, 2x}{cosh\, 2y - cos\, 2x}$

COMPLEX FUNCTIONS AND DIFFERENTIATION

13) Find an analytic function whose imaginary part is v = $e^{x}\, x\sin y + y\cos y$

14) Find an analytic function whose real part is (i) u = $\frac{x}{x^2+y^2}$ (ii) u = $\frac{y}{x^2+y^2}$

15) Find an analytic function whose imaginary part is v = $\frac{2\sin x\sin y}{\cosh 2x+\cosh 2y}$

16) Find an analytic function f(z) = u +iv if u = a(1+cos θ)

17) Find the conjugate harmonic of u = $e^{x^2-y^2}\cos 2xy$ and find f(z) in terms of z.

18) If f(z) is an analytic function of z and if u - v = $e^{x}\cos y - \sin y$ find f(z) in terms of z.

19) If f(z) is an analytic function of z and if u - v = (x-y) $x^2 + 4xy + y^2$) find f(z) in terms of z.

20) Find the orthogonal trajectories of the family of curves $x^3y - xy^3$ = C = constant

MODULE– II
COMPLEX INTEGRATION

CLOs	Course Learning Outcome
CLO 6	Demonstrate the area under a curve using the concepts of indefinite integration
CLO 7	Interpret the concepts of the Cauchy's integral formula and the generalized Cauchy's integral formula.
CLO 8	Demonstrate complex functions as power series and radius of convergence of power series.
CLO 9	Interpret the concept of complex integration to the real-world problems of flow with circulation around a cylinder.

COMPLEX INTEGRATION

Defination: In mathematics, a line integral is an integral where the function to be integrated is evaluated along a curve. The terms path integral, curve integral, and curvilinear integral are also used; contour integral as well, although that is typically reserved for line integrals in the complex plane.

The function to be integrated may be a scalar field or a vector field. The value of the line integral is the sum of values of the field at all points on the curve, weighted by some scalar function on the curve (commonly arc length or, for a vector field, the scalar product of the vector field with a differential vector in the curve). This weighting distinguishes the line integral from simpler integrals defined on intervals. Many simple formulae in physics (for example, $W = F \cdot s$) have natural continuous analogs in terms of line integrals ($W = \int_C F \cdot ds$). The line integral finds the work done on an object moving through an atomic or gravitational field.

In complex analysis, the line integral is defined in terms of multiplication and addition of complex numbers.

Let us consider F(t)= u(t)+i v(t) , $a \le t \le b$. Where u and v are real valued continuous functions of t in [a,b].

we define $\int_a^b F(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$

Thus, $\int_a^b F(t)\,dt$ is a complex number such that real part of $\int_a^b F(t)\,dt$ is $\int_a^b u(t)\,dt$ and imaginary part of $\int_a^b F(t)\,dt$ is $\int_a^b v(t)\,dt$.

COMPLEX INTEGRATION

Problem: Evaluate $\int_0^{1+i}(x^2 - iy)\,dz$ along the paths

1) y=x 2) $y=x^2$

Solution: 1) along the line y=x, dy= dx so that dz = dx+idx=(1+i) dx

$$\therefore \int_0^{1+i}(x^2 - iy)\,dz = \int_0^1 (x^2 - ix)(1+i)\,dx,$$ since y=x

$$=(1+i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1$$

$$=(1+i)\left[\frac{1}{3} - \frac{1}{2}i\right]$$

$$= \frac{5}{6} - \frac{1}{6}i$$

2) along the parabola $y = x^2$, $dy = 2x\,dx$

so that $dz = dx + 2ix\,dx$

$dz = (1 + 2ix)\,dx$ and x varies from 0 to 1

$$\therefore \int_0^{1+i} (x^2 - iy)\,dz = \int_0^1 (x^2 - ix^2)(1 + 2ix)\,dx$$

$$= (1 - i)\int_0^1 x^2(1 + 2ix)\,dx$$

$$= (1 - i)\left(\frac{1}{3} + \frac{1}{2}i\right)$$

$$= \frac{(1 - i)(2 + 3i)}{6}$$

$$= \frac{5}{6} + \frac{1}{6}i$$

Problem: Evaluate $\int_{z=0}^{z=1+i} (x^2 + 2xy + i(y^2 - x))dz$ along $y = x^2$

Solution: Given $f(z) = x^2 + 2xy + i(y^2 - x))dz$

$z = x + iy, dz = dx + idy$

$\therefore$ the curve, $y = x^2, dy = 2xdx$

$\therefore \; dz = dx + 2xidx = (1 + 2ix)dx$

$F(z) = x^2 + 2xx^2 + i(x^4 - x)$

$= x^2 + 2x^3 + i(x^4 - x)$

$F(Z)dz = (x^2 + 2x^3) + i((x^4 - x)(1 + 2ix))dx$

$= x^2 + 2x^3 + i(x^4 - x) + 2ix^3 + 4ix^4 - 2x^5 + 2x^2$

$$\therefore \int_c f(z)\,dz = \int_{z=0}^{1+i} x^2 + 2xy + i(y^2 - x)\,dz$$

$$= \int_0^1 (-2x^5 + 3x^2 + 2x^3 + i(5x^4 - x + 2x^3))\,dx)$$

$$= \left[-\frac{x^6}{3} + x^3 + \frac{x^4}{2} + i\left(\frac{5x^5}{5} - \frac{x^2}{2} + \frac{x^4}{2}\right)\right]_0^1$$

$$= \left[\left(\frac{-1}{3} + 1 + \frac{1}{2}\right) + \left(\frac{5}{5} - \frac{1}{2} + \frac{1}{2}\right)\right] - 0$$

$$= \frac{7}{6} + \frac{5}{5}i = \frac{7}{6} + i$$

$$\int_c f(z)\,dz = \frac{7}{6} + i$$

CAUCHY INTEGRAL THEOREM

STATEMENT : let F(z)=u(x,y)+iv(x,y) be analytic on and within a simple closed contour (or curve) 'c' and let f ′(z) be continuous there,then

$$f\ z\ dz = 0$$

Proof: f (z)=u(x,y)+iv(x,y)

And dz=dx+idy

$\Rightarrow$f(z).dz = (u(x,y)+iv(x,y))dx+idy

f(z).dz = u(x,y)dx+i u(x,y)dy+iv(x,y)dx+i^2 v(x,y)dy

f(z).dz= u(x,y)dx- v(x,y)dy+i(u(x,y)dy+ v(x,y)dx

Integrate both sides, we get

$$f\ \ z\ \ dz\ =\ \ (\mathrm{udx} -\ \mathrm{vdy}) + i(\ \mathrm{udy} +\ \ \mathrm{vdx})$$

By greens theorem ,we have

$$M\,dx + N\,dy = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial Y}\,dx\,dy$$

Now $f\ z\ dz = \left(-\frac{\partial v}{\partial x} \quad \frac{\partial u}{\partial Y}\right) dx\,dy + i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial Y}\right) dx\,dy$

Since $f'(z)$ is continuous &four partial derivatives i.e $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial Y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial Y}$ are also continuous in the region R enclosed by C, Hence we can apply Green's Theorem.

Using Green's Theorem in plane ,assuming that R is the region bounded by C.

It is given that f (z)=u(x,y)+iv(x,y) is analytic on and within c.

Hence $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Using this we have

$$\int_c f(z)dz = \iint_R 0 \quad dxdy + i\iint_R 0 \quad dxdy \quad = 0$$

Hence the theorem.

Cauchy's integral formula

Cauchy's integral formula states that

$$f(z_0) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)}dz$$

where the integral is a contour integral along the contour c enclosing the point z_0.

Generalization of Cauchy's integral formula:

Generalization of Cauchy's integral formula states that

$$f^{n}(z_0) = \frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-a)^{n+1}}dz$$

where the integral is a contour integral along the contour c enclosing the point z_0

Problem: Evaluate using cauchy's integral formula $\int_c \frac{e^{2z}}{(z-1)(z-2)}dz$ where c is the circle $|z| = 3$

Solution: Given $\int_c \frac{e^{2z}}{(z-1)(z-2)}dz$ …… (1)

Both the points z=1,z=2 line inside $|z|=3$

Resolving into partial fractions

$$\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

A=-1, B=1

From(1)

$$\int_c \frac{e^{2z}}{(z-1)(z-2)}dz = \int_c \frac{-e^{2z}}{(z-1)}dz + \int_c \frac{e^{2z}}{(z-2)}dz$$ (by cauchy's integral formula)

$$=-2\pi i f(1)+2\pi i f(2)$$

$$=-2\pi i e^{2.1}+2\pi i e^{2.2}$$

$$=-2\pi i e^{2}+2\pi i e^{4}=2\pi i(e^{4}-e^{2})$$

Problem: Using cauchy'sintegralformula to evaluate $\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)z-2)} dz$, where c is the circle $|z| = 3$

Solution: $\int_c \frac{f(z)}{(z-1)z-2)} dz = (\int_c \frac{1}{(z-2)} dz + \int_c \frac{1}{(z-1)} dz)$f(z)dz

$$= \int_c \frac{f(z)}{(z-2)} dz + \int_c \frac{f(z)}{(z-1)} dz$$

=2Πif(2)- 2∏if(1)

=2Πi(sin4Π+cos4Π)-(sinΠ+cosΠ))

=2Πi(1-(-1))=4Πi

$$\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)z-2)} dz = 4\Pi i$$

Problem: Evaluate $\int_c \frac{(z-1)}{(z+1)^2(z-2)}dz$ whrere c is $|Z - i| = 2$

Solution: the singularities of $\frac{(z-1)}{(z+1)^2(z-2)}$ are given by

$(z+1)^2(z-2)=0$

$\Rightarrow$ Z=-1 and z=2

Z=-1 lies inside the circle since $|-1-i| - 2 < 0$

Z=2 lies outside the circle sinceI2-iI-2>0 $|2-i| - 2 > 0$

The given line integral can be written as

$$\int_c \frac{(z-1)}{(z+1)^2(z-2)}dz = \int_c \frac{\frac{(z-1)}{(z-2)}}{(z+1)^2} \text{----------}(1)$$

The derivative of analytic function is given by

$$\int_c \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i f^n(a)}{n!} \text{----------} (2)$$

From (1) and (2) $f(z)=\frac{(z-1)}{(z-2)}$, $a=-1, n=1$

$$f^1(z) = \frac{1(z-2)-1(z-1)}{(z-2)^2} = \frac{1}{(z-2)^2}$$

$$f^1(-1) = \frac{1}{-9}$$

Substituting in (2), we get

$$\int_c \frac{(z-1)}{(z+1)^2(z-2)}dz = \frac{2\pi i}{1}\left(-\frac{1}{9}\right)$$

$$= \frac{-2}{9}\Pi i$$

Problem: Evaluate $\int_c \frac{e^{2z}}{(z+1)^4} dz$ where c: $|z-1|=1$

Solution: the singular points of $\frac{e^{2z}}{(z+1)^4}$ dz are given by7

$(z+1)^4=0 \Rightarrow z=-1$

The singular point z=-1 lies insidethecirclec: $|z-1|=3$

Applying cauchy's integral formula for derivatives

$$\int_c \frac{f(z)}{(z-a)^{n+1}} dz = \int_c \frac{2\pi i f^n(-1)}{n!} dz \text{----------(1)}$$

$f(z) = e^{2z}$, $n = 3$, $a = -1$

$f(z) = 2e^{2z}$

$f^{1}(z) = 4e^{2z}$

$f^{11}(z) = 8e^{2z}$

$f^{111})z) = 16e^{2z}$

$f^{111}(-1) = 16e^{-2}$

substituting in (1)

$$\int_c \frac{e^{2z}}{(z+1)^4} dz = \int_c \frac{2\pi i f^{111}(-1)}{n!}$$

$$= \frac{2\pi i 16 e^{-2}}{2!}$$

$$= 16 \Pi i e^{-2}$$

Problem: Use cauchy's integral formula to evaluate $\int_c \frac{e^{-2z}}{(z+1)^3} dz$ with $c: |z| = 2$

Solution: Given $\int_c \frac{e^{-2z}}{(z+1)^3} dz$

$f(z)=e^{-2z}$

the singular poin $z=-1$ lies inside the given circle $|z| = 2$

apply Cauchy's integral formula for derivatives

$$\int_c \frac{e^{-2z}}{(z+1)^3}dz = \frac{2\pi i f^1(-1)}{2!} \qquad \left[\because \int_c \frac{f(z)}{(z-a)^3} = \frac{2\pi i f^1(a)}{2!}\right]$$

Where $f(z)=e^{-2z}$

$$f^1(z)=-2\ e^{-2z}$$

$$f^{11}(z)=4\ e^{-2z}$$

$$f^{11}(-1)= 4\ e^2$$

$$\therefore \int_c \frac{e^{-2z}}{(z+1)^3}dz = \frac{2\pi i 4e^2}{2} = 4\pi i e^2$$

Problem: Evaluate $\int_c \frac{dz}{z^8(z+4)} dz$ with $c: |z| = 2$

Solution:

The singularities of $\int_c \frac{dz}{z^8(z+4)} dz$ are given by

$$Z^8(z+4) = 0 \Rightarrow z = 0, z = -4$$

The point z=0 lie inside $|z| = 2$

and the z=-4 lies outside the circle

By the derivative of analytic function.

Problem: Evaluate using integral formula $\oint_c \frac{e^{2z}dz}{(z-1)(z-2)}$ where c is the circle $|z| = 3$

Solution: Let (z)= e^z which is analytic within the circle c: $|z| = 3$ and the two singular points a=1,a=2 lie inside c.

$$\oint_c \frac{e^{2z}dz}{(z-1)(z-2)} = \oint e^{2z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right)dz$$

$$\oint_c \frac{e^{2z}}{z-2}dz - \oint_c \frac{e^{2z}}{z-1}dz$$

Now using cauchy's integral formula ,we obtain

$$\oint_c \frac{e^{2z}dz}{(z-1)(z-2)} = 2\pi i e^4 - 2\pi i e^2$$

$$= 2\pi i(e^4 - e^2)$$

$$\oint_c \frac{e^{2z}dz}{(z-1)(z-2)} = 2\pi i(e^4 - e^2)$$

Problem : Evaluate $\oint_c \frac{3z^2+z}{z^2-1}dz$ where cisthe circle

$|z-1|=1$

Solution: Given f(z)= $3z^2+z$

Z=a=+1or -1

The circle $|z-1| = 1$ has centre at z=1 and radius 1 and includes the point z=1,f(z)=$3z^2+z$ is an analytic function

Also $\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$

$$\int_c \frac{3z_2 + z}{z^2-1} = \frac{1}{2}\left[\int_c \frac{3z^2+z}{z-1}dz\right] - \frac{1}{2}\left[\int_c \frac{3z^2+z}{z+1}dz\right] \text{----------(1)}$$

Since z=1 lies inside c,we have by cauchy's integralformula

$$\oint_c \frac{3z^2+z}{z^2-1}dz = 2\pi i f(i)$$

$$= 2\pi i * 4$$

Bycauchy'sintegral theorem ,since z=-1 lies out side c,we have

$$\oint_c \frac{3z^2+z}{z-1}dz = 0$$

From equation(1) we have

$$\oint_c \frac{3z^2+z}{z^2-1}dz = \frac{1}{2}(8\pi i) - 0$$

$$= 4\pi i$$

EXCERCISE PROBLEMS:

1) Evaluate $\int \frac{dz}{z - z_0}$ where c: $|z - z_0| = r$

2) Evaluate $\int_{(1,1)}^{(2,2)} (x + y)dx + (y - x)dy$ along the parabola $y^2 = x$

3)Evaluate $\int_c \frac{z^2 + 4}{z^2 - 1} dz$ where C: $|z| = 2$ using Cauchy's Integral formula

4)Evaluate $\int_c \frac{e^{2z}}{(z-1)(z-2)} dz$ where C: $|z| = 4$ using Cauchy's integral formula

5) Evaluate $\int_c \frac{z^3 - z}{(z-2)^3}$ where $C : |z| = 3$ using Cauchy's integral formula

6) Expand f(z) = $\int_c \frac{e^{2z}}{(z-1)^3}$ at a point z=1

7) Expand f(z)= $\int_c \frac{1}{z^2 - 4z + 3}$ for $1 < |z| < 3$

8)Evaluate $\int (y^2+z^2)dx+(z^2+x^2)dy+(x^2+y^2)dz$ from (0,0,0) to (1,1,1) , where C is the curve $x=t, x=t^2, x=t^3$

9) Evaluate $\int_{(0,0)}^{(1,1)} (3x^2+4xy+ix^2)dz$ along $y=x^2$

10) Evaluate $\int_{0}^{1+i} (x - y + ix^2)\, dz$

(i) along the straight from z = 0 to z = 1+i .

(ii) along the real axis from z = 0 to z = 1 and then along a line parallel to real axis from z = 1 to z = 1+i

(iii) along the imaginary axis from z = 0 to z = I and then along a line parallel to real axis z = i to z = 1+ i .

11) Evaluate $\int_{1-i}^{2+i} (2x + 1 + iy)\, dz$ along (1-i) to (2+i)

12) Evaluate $\int_{c} (y^2 + 2xy)dx + (x^2 - 2xy)dy$ where c is boundary of the region $y=x^2$ and $x=y^2$

MODULE– III
POWER SERIES EXPANSION OF COMPLEX FUNCTION

CLOs	Course Learning Outcome
CLO 10	Asses the Taylor's and Laurent series expansion of complex functions.
CLO 11	Interpret the concept of different types of singularities for analytic function.
CLO 12	Identify the poles, residues and solve integrals using Cauchy's residue theorem.
CLO 13	Interpret the concept of Cauchy's residue theorem to the real-world problems of Quantum Mechanical scattering and Quantum theory of atomic collisions.

POWER SERIES EXPANSION OF COMPLEX FUNCTION

Power series:

A series expansion is a representation of a particular function as a sum of powers in one of its variables, or by a sum of powers of another (usually elementary) function f(z).

A power series in a variable z is an infinite <u>sum</u> <u>of the form</u>

$$\sum a_i z^i$$

A series of the form $\sum a_n z^n$ is called as power series.

That is $\sum a_n z^n = a_1 z + a_2 z^2 + \dots\dots\dots + a_n z^n + \dots\dots$

Taylor's series:

Taylor's theorem states that any function satisfying certain conditions may be represented by a Taylor series.

The Taylor series is an infinite series, whereas a Taylor polynomial is a polynomial of degree n and has a finite number of terms. The form of a Taylor polynomial of degree n for a function

f (z) at x = a is

$$f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{(z-a)^2}{2!} + f'''(a)\frac{(z-a)^3}{3!} + \dots\dots\dots + f^{n}(a)\frac{(z-a)^n}{n!} + \dots\dots, |z-a| < r$$

POWER SERIES EXPANSION OF COMPLEX FUNCTION

A Maclaurin series is a <u>Taylor series</u> expansion of a function about x=0,

$$f(z) = f(0) + f'(0)(z) + f''(0)\frac{(z)^2}{2!} + f'''(0)\frac{(z)^3}{3!} + \dots\dots\dots + f^{n}(0)\frac{(z)^n}{n!} + \dots\dots$$

This series is called as maclurins series expansion of f(z).

Some important result:

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \dots \quad \text{for } -1 < x \le 1$$

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \dots \quad \text{for } -\infty < x < \infty$$

$$\cos x = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \tfrac{1}{720}x^6 + \dots \quad \text{for } -\infty < x < \infty$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots \quad \text{for } -1 < x < 1$$

POWER SERIES EXPANSION OF COMPLEX FUNCTION

Problem: Determine the first four terms of the power series for sin 2x using Maclaurin's series.

Solution:

Let

$f(x) = \sin 2x \qquad f(0) = \sin 0 = 0$

$f'(x) = 2\cos 2x \qquad f'(0) = 2\cos 0 = 2$

$f''(x) = -4\sin 2x \qquad f''(0) = -4\sin 0 = 0$

$f'''(x) = -8\cos 2x \qquad f'''(0) = -8\cos 0 = -8$

$f^{iv}(x) = 16\sin 2x \qquad f^{iv}(0) = 16\sin 0 = 0$

$f^{v}(x) = 32\cos 2x(0) \qquad f^{v}(0) = 32\cos 0 = 32$

$f^{vi}(x) = -64\sin 2x \qquad f^{vi}(0) = -64\sin 0 = 0$

$f^{vii}(x) = -128\cos 2x \qquad f^{vii}(0) = -128\cos 0 = -128$

$$f(x) = \sin 2x = 0 + 2x + 0\,x^2 + (-8)\frac{x^3}{3!} + 0.x^4 + 32\frac{x^5}{5!}$$

$$= 2x - \frac{4x^3}{3} + \frac{4x^5}{15}$$

Problem : Find the Maclaurin series for $f(z) = z e^{z}$ Express your answer in sigma notation.

Solution:

Let $f(z) = z e^{z}$ $\qquad f(0) = 0$

$f' = e^{z} + z e^{z}$ $\qquad f'(0) = 1 + 0 = 1$

$f'' = e^{z} + e^{z} + z e^{z}$ $\qquad f''(0) = 1 + 1 + 0 = 2$

$f''' = e^{z} + e^{z} + e^{z} + z e^{z}$ $\qquad f'''(0) = 1 + 1 + 1 + 0 = 3$

$f'''' = e^{z} + e^{z} + e^{z} + e^{z} + z e^{z}$ $\qquad f''''(0) = 1 + 1 + 1 + 1 + 0 = 4$

$$f(z) = 0 + 1z + \frac{2}{2!}z^2 + \frac{3}{3!}z^3 + \frac{4}{4!}z^4 + \ldots\ldots\ldots\ldots$$

$$= z + z^2 + \frac{1}{2}z^3 + \frac{1}{6}z^4 + \ldots\ldots\ldots\ldots$$

$$= \sum_{n=1}^{\infty} \frac{z^n}{(n-1)!}$$

POWER SERIES EXPANSION OF COMPLEX FUNCTION

Laurent series:

In mathematics, the **Laurent series** of a complex function $f(z)$ is a representation of that function as a power series which includes terms of negative degree. It may be used to express complex functions in cases where a Taylor series expansion cannot be applied.

The Laurent series for a complex function $f(z)$ about a point c is given by:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z-a)^n}$$

where the a_n and a are constants.

Zero's of an analytic function:

A zero of an analytic function f(z) is a value of z such that f(z)=0 .Particularly a point a is called a zero of an analytic function f (z) if f(a) = 0.

Eg: $f(z) = \dfrac{(z+1)^2}{(z^2+1)^2}$

Now, $(z+1)^2 = 0$

Z = -1, z = -1 are zero's of an analytic function.

Zero's of mth order:

If an analytic function f(z) can be expressed in the form $f(z) = (z-a)^m \Phi(z)$ where $\Phi(z)$ is analytic function and $\Phi(a) \neq 0$ then z=a is called zero of mth order of the function f(z).

- A simple zero is a zero of order 1. Eg:

1. $f(z) = (z-1)^3$

$\Rightarrow (z-1)^3 = 0$

z=1 is a zero of order 3 of the function f(z).

2. $f(z) = \dfrac{1}{1-z}$

i.e $z = \infty$ is a simple zero of f(z).

3. $f(z) = \sin z$

i.e $z = n\pi \quad \forall n = 0,1,2,3,.....$ are simple zero's of f(z).

Problem: Find the first four terms of the Taylor's series expansion of the complex function $f(z) = \frac{z+1}{(z-3)(z-4)}$ about $z = 2$. Find the region of convergence.

Solution:

The singularities of the function $f(z) = \frac{z+1}{(z-3)(z-4)}$ are $z = 3$ and $z = 4$

Draw a circle with centre at $z = 2$ and radius 1. Then the distance of singularities from the centre are 1 and 2.

Hence within the circle $|z-2| = 1$, the given function is analytic. Hence, it can be extended in Taylor's series within the circle $|z-2| = 1$.

Hence $|z-2| = 1$ is the circle of convergence.

POWER SERIES EXPANSION OF COMPLEX FUNCTION

Now $f(z) = \frac{5}{z-4} - \frac{4}{z-3}$ (partial fraction) , f(2)= 3/2

$$f'(z) = -\frac{5}{(z-4)^2} + \frac{4}{(z-3)^2} \quad , \quad f'(2) = \frac{11}{4}$$

$$f''(z) = -\frac{8}{(z-3)^3} + \frac{10}{(z-4)^3} \quad , \quad f''(2) = \frac{27}{4}$$

$$f'''(z) = \frac{24}{(z-3)^4} - \frac{30}{(z-4)^4} \quad , \quad f'''(2) = \frac{177}{8}$$

Taylor's series expansion for f(z) at z=a is

$$f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{(z-a)^2}{2!} + f'''(a)\frac{(z-a)^3}{3!} + \dots\dots\dots + f^{n}(a)\frac{(z-a)^n}{n!} + \dots\dots$$

$$\frac{z+1}{(z-3)(z-4)} = \frac{3}{2} + (z-2)\frac{11}{4} + \frac{(z-2)^2}{2!}\left(\frac{27}{4}\right) + \frac{(z-2)^3}{3!}\left(\frac{177}{8}\right)$$

$$f(z) = \frac{3}{2} + (z-2)\frac{11}{4} + (z-2)^2\left(\frac{27}{8}\right) + (z-2)^3\left(\frac{59}{16}\right).$$

POWER SERIES EXPANSION OF COMPLEX FUNCTION

Problem: Obtain Laurent series for $f(z) = \dfrac{e^{2z}}{(z-1)^3}$ about $z = 1$.

Solution:

Given $f(z) = \dfrac{e^{2z}}{(z-1)^3}$

Put $z-1 = w$ so that $z = w+1$

$$f(z) = \frac{e^{2(1+w)}}{w^3}$$

$$f(z) = \frac{e^2 e^{2w}}{w^3}$$

$$= \frac{e^2}{w^3}\left[1 + 2w + \frac{(2w)^2}{2!} + \frac{(2w)^3}{3!} + \ldots\ldots\right] \text{ if } w \neq 0$$

$$= e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!} w^{n-3}$$

$$= e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!} (z-1)^{n-3} \quad , \text{ if } z-1 \neq 0$$

$$= e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!} (z-1)^{n-3} \quad , \text{ if } |z-1| \neq 0$$

$$f(z) = e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!} (z-1)^{n-3} \quad , \text{ if } |z-1| > 0$$

Since points $|z-1| \leq 0$ will be singular points.

Singular point of an analytic function: A point at which an analytic function f(z) is not analytic, i.e. at which f '(z) fails to exist, is called a **singular point** or **singularity** of the function.

There are different types of singular points:

Isolated and non-isolated singular points: A singular point z_0 is called an **isolated singular point** of an analytic function f(z) if there exists a deleted ε-spherical neighborhood of z_0 that contains no singularity. If no such neighborhood can be found, z_0 is called a non-isolated singular point. Thus an isolated singular point is a singular point that stands completely by itself, embedded in regular points. In fig 1a where z_1, z_2 and z_3 are isolated singular points. Most singular points are isolated singular points. A non-isolated singular point is a singular point such that every deleted ε-spherical neighborhood of it contains singular points. See Fig. 1b where z_0 is the limit point of a set of singular points. Isolated singular points include poles, removable singularities, essential singularities and branch points.

POWER SERIES EXPANSION OF COMPLEX FUNCTION

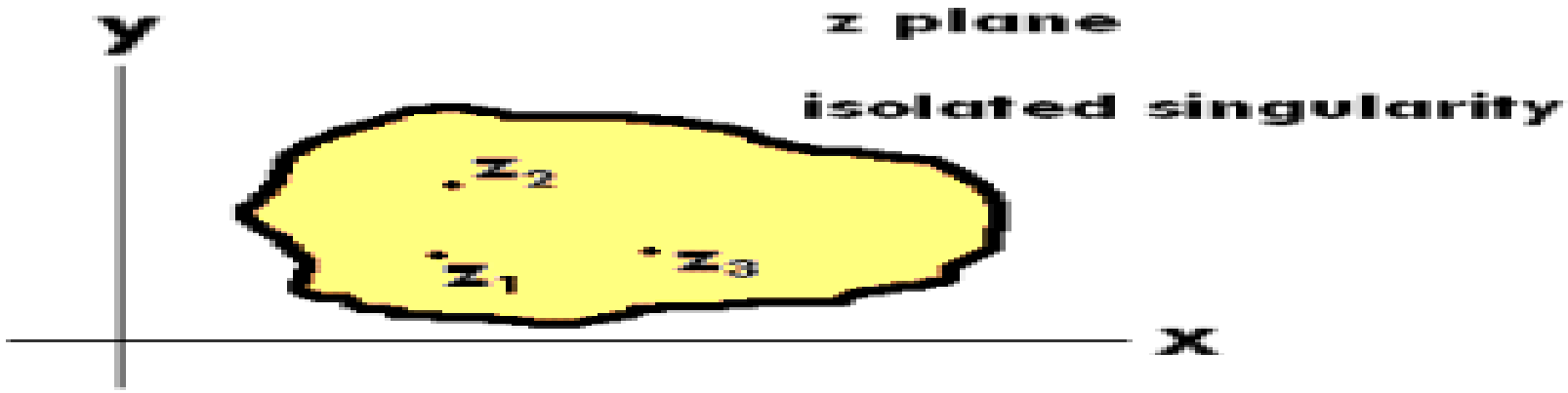


Fig. 1a

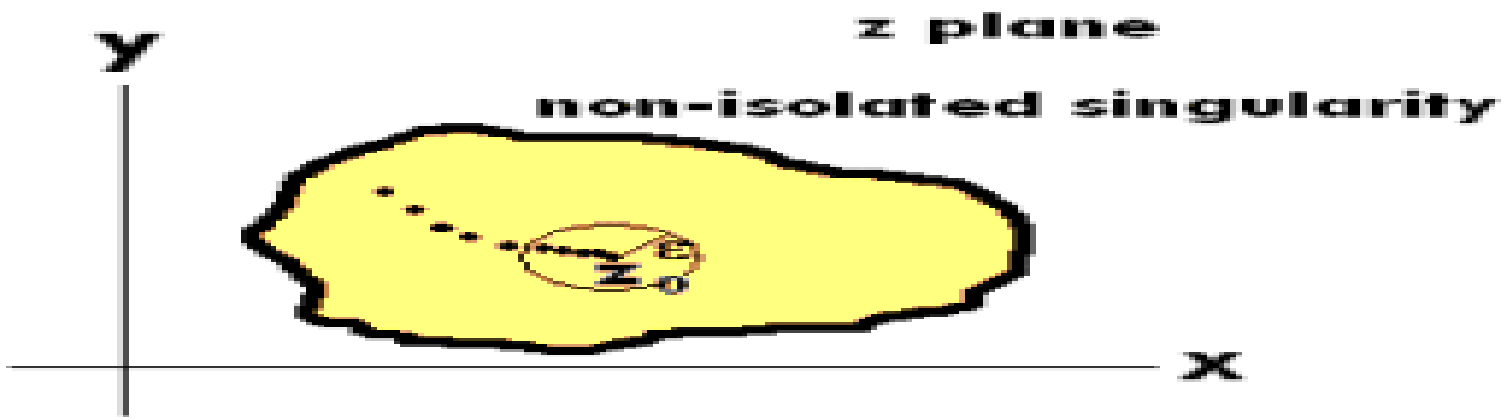


Fig. 1b

The Taylors Series

Taylor's theorem is named after the mathematician Brook Taylor, who stated a version of it in 1712. Yet an explicit expression of the error was not provided until much later on by Joseph-Louis Lagrange. An earlier version of the result was already mentioned in 1671 by James Gregory.

Brook Taylor (1685-1731)

The Taylors Series

This expansion assumes we have a function that is analytic in a disk.

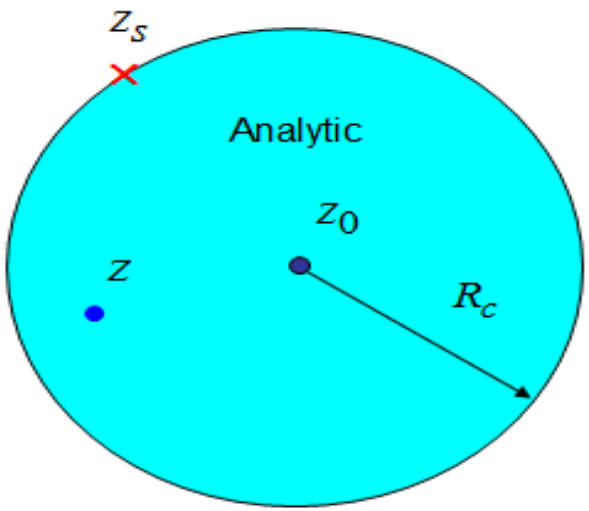
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz'$$

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

"derivative formula"

Note: Both forms are useful.



Here z_s is the closest singularity.

The path C is any counterclockwise closed path that encircles the point z_0.

$R_c =$ radius of convergence of the Taylor series

The Taylor series will converge within the radius of convergence, and diverge outside.

Taylors Series Expansion

- **Write the Cauchy integral formula in the form**

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')}{z'-z}dz'$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)-(z-z_0)}dz'$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)\left[1-\frac{z-z_0}{z'-z_0}\right]}dz' \quad (|z'-z_0| > |z-z_0|)$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n dz'$$

$$\overset{\text{uniform convergence}}{=} \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \oint_C \frac{f(z')}{(z'-z_0)^{n+1}}dz'$$

$$\overset{\text{derivative formulas}}{=} \sum_{n=0}^{\infty}(z-z_0)^n \frac{f^{(n)}(z_0)}{n!}$$

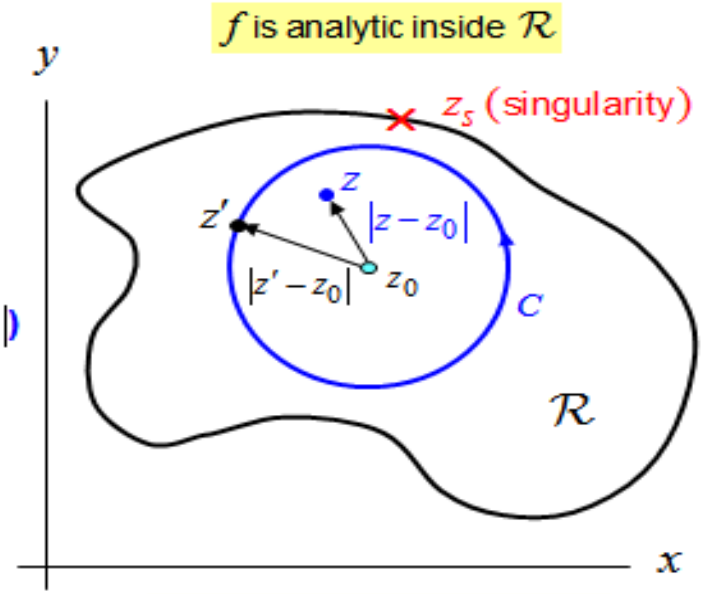
(**recall** $f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}}dz'$)

$$\Rightarrow \boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n} \Leftrightarrow \text{Taylor series expansion of } f(z) \text{ about } z_0$$

where

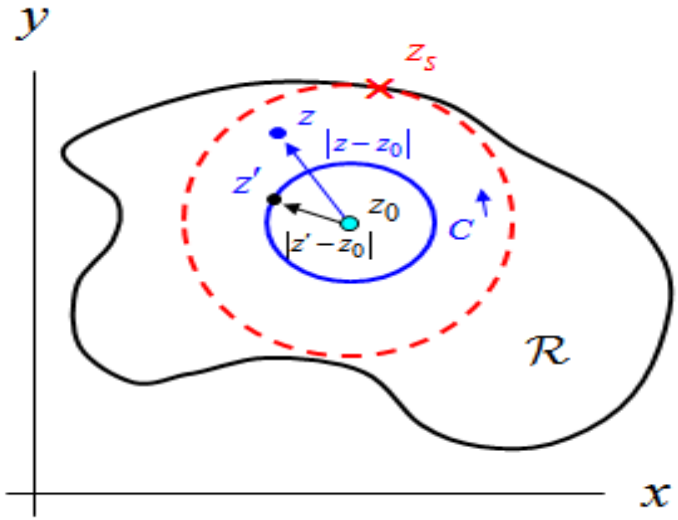
$$\boxed{a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}}dz' = \frac{f^{(n)}(z_0)}{n!}}$$

(both forms are useful!)



$|z-z_0| < |z'-z_0| < |z_s-z_0|$

POWER SERIES EXPANSION OF COMPLEX FUNCTION



Note that in the result for a_n, the integrand is analytic away from z_0, and hence the path is now arbitrary, as long as it encircles z_0.

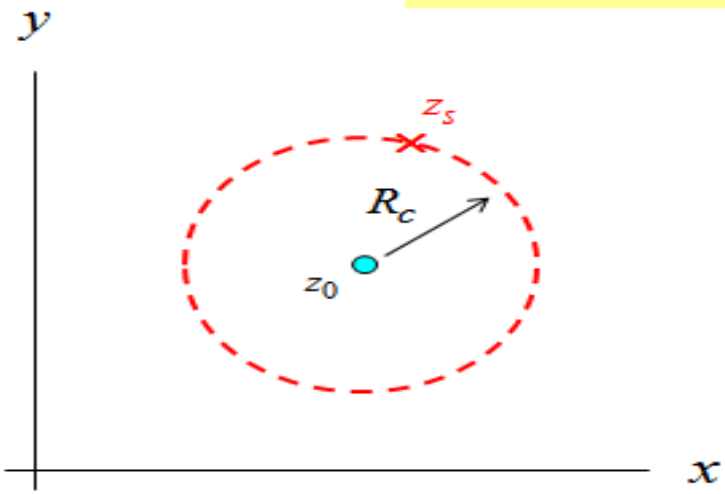
$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}}\,dz'$$

- Note that the construction is valid for any $|z-z_0|<|z_s-z_0|$ where z_s is the singularity nearest z_0 ; hence the series will converge if

$$|z-z_0|<|z_s-z_0|$$

Note: It can also be shown that the series will diverge for $|z-z_0|>|z_s-z_0|$

The radius of convergence of a Taylor series is the distance out to the <u>closest</u> singularity.



Key point:
The point z_0 about which the expansion is made is *arbitrary*, but It determines the region of convergence of the Taylor series.

Converges for : $|z - z_0| < R_c \equiv |z_s - z_0|$

Properties of Taylor Series

R_c = radius of convergence = distance to closest singularity

- A Taylor series will converge for $|z-z_0| < R_c$ (i.e., inside the radius of convergence).
- A Taylor series will diverge for $|z-z_0| > R_c$ (i.e., outside the radius of convergence).
- A Taylor series may be differentiated or integrated term-by-term within the radius of convergence. This does not change the radius of convergence.
- A Taylor series *converges absolutely* inside the radius of convergence (i.e., the series of absolute values converges).
- A Taylor series *converges uniformly* for $|z-z_0| \leq R < R_c$.
- When a Taylor series converges, the resulting function is an analytic function.
- Within the common region of convergence, we can add and multiply Taylor series, collecting terms to find the resulting Taylor series.

Maclaurine's Series Expansion

Maclaurin series:

A Maclaurin series is a Taylor series expansion of a function about x=0,

$$f(z) = f(0) + f'(0)(z) + f''(0)\frac{(z)^2}{2!} + f'''(0)\frac{(z)^3}{3!} + \ldots\ldots\ldots + f^n(0)\frac{(z)^n}{n!} + \ldots\ldots$$

This series is called as maclurins series expansion of f(z).

Some important result:

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots \quad \text{for } -1 < x \le 1$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \ldots \quad \text{for } -\infty < x < \infty$$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \ldots \quad \text{for } -\infty < x < \infty$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots \quad \text{for } -1 < x < 1$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Problem on Maclaurine's Series

Problem: Determine the first four terms of the power series for sin 2x using Maclaurin's series.

Solution:

Let

$f(x) = \sin 2x \qquad f(0) = \sin 0 = 0$

$f'(x) = 2\cos 2x \qquad f'(0) = 2\cos 0 = 2$

$f''(x) = -4\sin 2x \qquad f''(0) = -4\sin 0 = 0$

$f'''(x) = -8\cos 2x \qquad f'''(0) = -8\cos 0 = -8$

$f^{iv}(x) = 16\sin 2x \qquad f^{iv}(0) = 16\sin 0 = 0$

$f^{v}(x) = 32\cos 2x(0) \qquad f^{v}(0) = 32\cos 0 = 32$

$f^{vi}(x) = -64\sin 2x \qquad f^{vi}(0) = -64\sin 0 = 0$

$f^{vii}(x) = -128\cos 2x \qquad f^{vii}(0) = -128\cos 0 = -128$

$$f(x) = \sin 2x = 0 + 2x + 0x^2 + (-8)\frac{x^3}{3!} + 0.x^4 + 32\frac{x^5}{5!}$$

$$= 2x - \frac{4x^3}{3} + \frac{4x^5}{15}$$

The Laurent's Series

The Laurent series was named after and first published by Pierre Alphonse Laurent in 1843. Karl Weierstrass may have discovered it first in a paper written in 1841, but it was not published until after his death.

Pierre Alphonse Laurent (1813 -1854)

The Laurent's Series

This generalizes the concept of a Taylor series to include cases where the function is analytic in an annulus.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

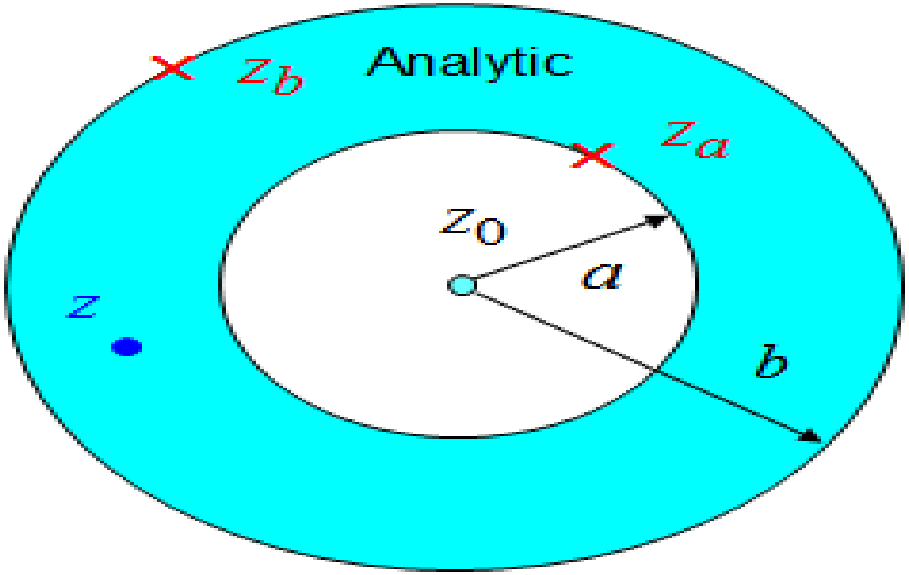
or

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z - z_0)^n}$$

where $b_n = a_{-n}$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz'$$ (derived later)

POWER SERIES EXPANSION OF COMPLEX FUNCTION



Here z_a and z_b are two singularities.

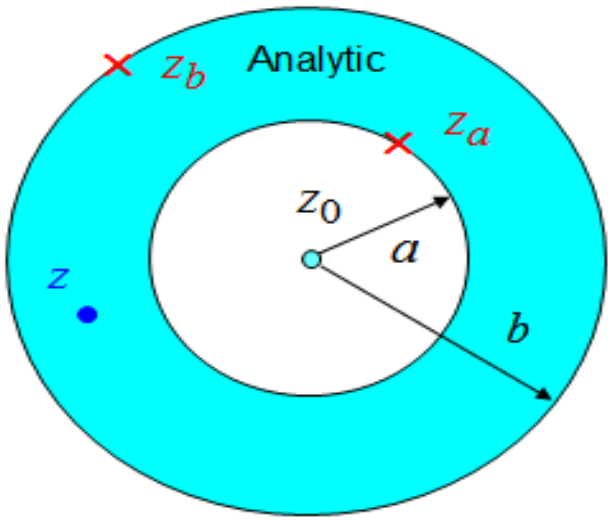
Note:
The point z_b may be at infinity.

The path C is any counterclockwise closed path that stays inside the annulus an encircles the point z_0.

Note: We no longer have the "derivative formula" as we do for a Taylor series.

Laurent series:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$



The Laurent series converges <u>inside</u> the region

$$a < |z - z_0| < b$$

The Laurent series diverges <u>outside</u> this region if there are singularities at

$$|z - z_0| = a, b$$

The Laurent's Series

This is particularly useful for functions that have poles.

Examples of functions with poles, and how we can choose a Laurent series:

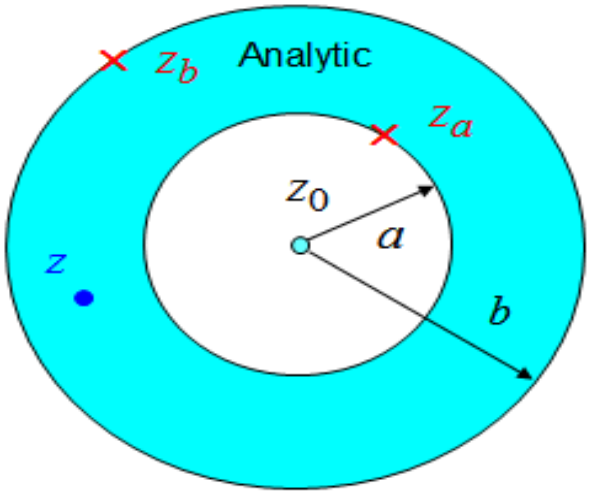
$$f(z)=\frac{1}{z} \qquad (\text{Choose } z_0=0:\ a=0,\ b<\infty)$$

$$f(z)=\frac{z}{z-1} \qquad (\text{Choose } z_0=0:\ a=1,\ b<\infty)$$

$$f(z)=\frac{z}{z-1} \qquad (\text{Choose } z_0=1:\ a=0,\ b<\infty)$$

$$f(z)=\frac{z}{(z-1)(z-2)} \qquad (\text{Choose } z_0=0:\ a=1,\ b=2)$$

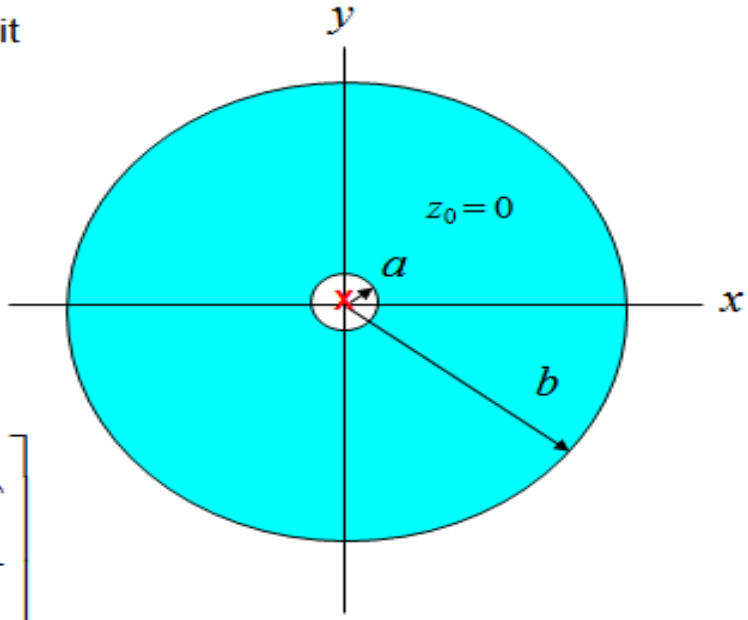
$$f(z)=\frac{z}{(z-1)(z-2)} \qquad (\text{Choose } z_0=0:\ a=2,\ b<\infty)$$



Theorem:
The Laurent series expansion in the annulus region is unique.

(So it doesn't matter how we get it; once we obtain it by any series of valid steps, it is correct!)

This is justified by our Laurent series expansion formula, derived later.



Example:

$$f(z) = \frac{\cos(z)}{z} \qquad (z_0 = 0,\ a = 0,\ b < \infty)$$

$$\Rightarrow f(z) = \overbrace{\frac{1}{z}}^{\text{analytic for } |z|>0} \left[\overbrace{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots}^{\text{valid for } |z|<\infty} \right]$$

Hence $$f(z) = \frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} - \frac{z^5}{6!} \ldots, \qquad 0 < |z| < \infty$$

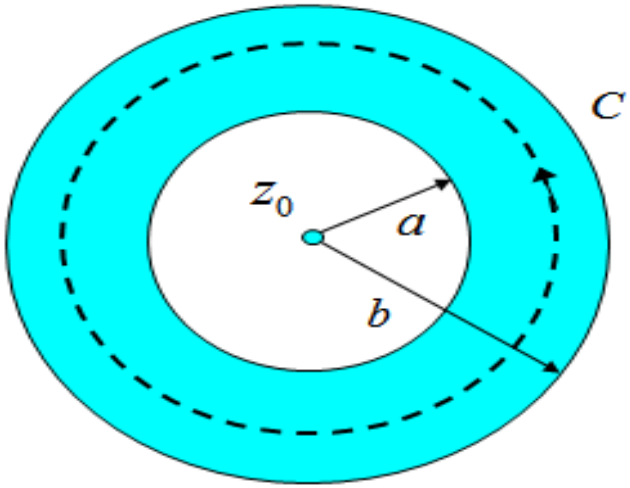
POWER SERIES EXPANSION OF COMPLEX FUNCTION

A Taylor series is a special case of a Laurent series.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz'$$

$$a_n = 0, \quad n = -1, -2, -3 \ldots$$



Here f is assumed to be analytic within C.

If $f(z)$ is analytic within C, the integrand is analytic for negative values of n. Hence, all coefficients a_n for negative n become zero (by Cauchy's theorem).

Derivation of the Laurent's Series

Derivation of Laurent Series

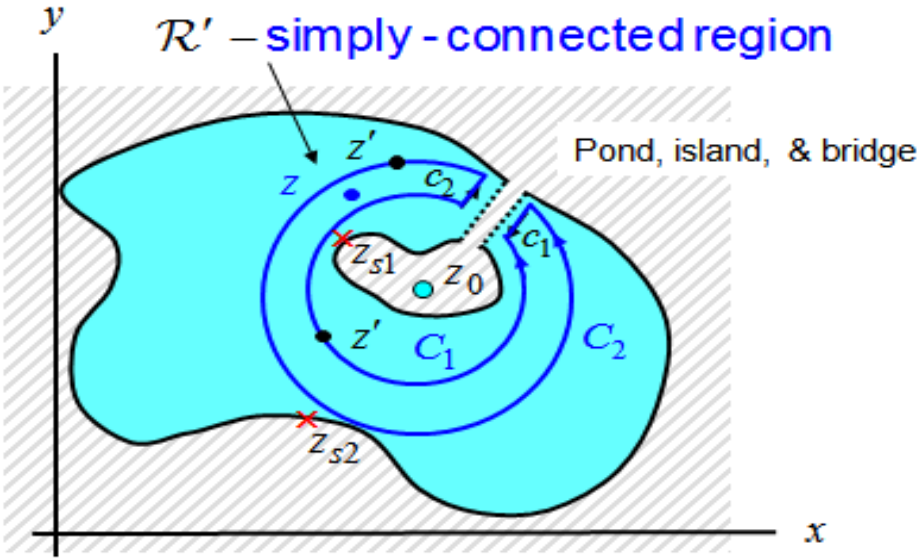
We use the "bridge" principle again

Pond, island, & bridge

Pond: Domain of analyticity
Island: Region containing singularities
Bridge: Region connecting island and boundary of pond

POWER SERIES EXPANSION OF COMPLEX FUNCTION

Contributions from the paths c_1 and c_2 cancel!



- **By Cauchy's Integral Formula,**

$$f(z)=\frac{1}{2\pi i}\oint_{C_2+\cancel{c_1}-\cancel{c_2}-C_1}\frac{f(z')}{z'-z}dz'$$

$$=\frac{1}{2\pi i}\oint_{C_2}\frac{f(z')}{z'-z}dz'-\frac{1}{2\pi i}\oint_{C_1}\frac{f(z')}{z'-z}dz'$$

where on C_2, $|z'-z_0|>|z-z_0|$,

$$\Rightarrow \frac{1}{z'-z}=\frac{1}{(z'-z_0)-(z-z_0)}=\frac{1}{(z'-z_0)\left(1-\dfrac{z-z_0}{z'-z_0}\right)}=\frac{1}{z'-z_0}\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(z'-z_0)^n}=\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(z'-z_0)^{n+1}}$$

and on C_1, $|z'-z_0|<|z-z_0|$ (note the convergence regions for C_1, C_2 overlap!)

$$\Rightarrow \frac{1}{z'-z}=\frac{1}{(z'-z_0)-(z-z_0)}=\frac{-1}{(z-z_0)\left(1-\dfrac{z'-z_0}{z-z_0}\right)}=-\sum_{n=0}^{\infty}\frac{(z'-z_0)^n}{(z-z_0)^{n+1}}\overset{\substack{n\to -n'-1,\\ n'\to n}}{=}-\sum_{n=-1}^{-\infty}\frac{(z-z_0)^n}{(z'-z_0)^{n+1}}$$

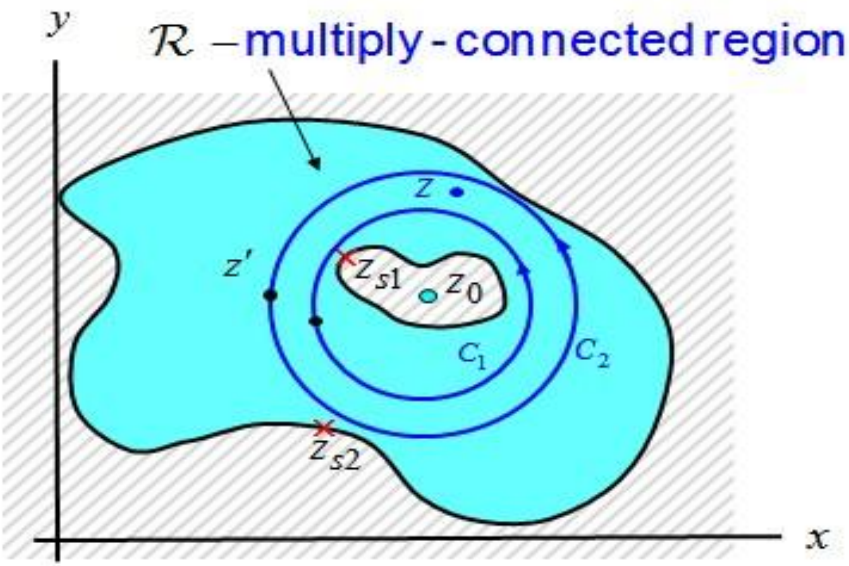
Hence,

$$f(z)=\frac{1}{2\pi i}\oint_{C_2+\cancel{\alpha_1}-\cancel{\alpha_2}-C_1}\frac{f(z')}{z'-z}dz'$$

uniform convergence

$$=\frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\oint_{C_2}\frac{f(z')}{(z'-z_0)^{n+1}}dz'$$

$$+\frac{1}{2\pi i}\sum_{n=-1}^{-\infty}(z-z_0)^n\oint_{C_1}\frac{f(z')}{(z'-z_0)^{n+1}}dz'$$



Let $C_1 \to C_2$ (call them path C).

We thus have

$$f(z)=\frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\oint_{C}\frac{f(z')}{(z'-z_0)^{n+1}}dz'$$

$$+\frac{1}{2\pi i}\sum_{n=-1}^{-\infty}(z-z_0)^n\oint_{C}\frac{f(z')}{(z'-z_0)^{n+1}}dz'$$

POWER SERIES EXPANSION OF COMPLEX FUNCTION

Because the integrand for the coefficient is analytic with $\mathcal{R}$, the path C is arbitrary as long as it stays within $\mathcal{R}$.

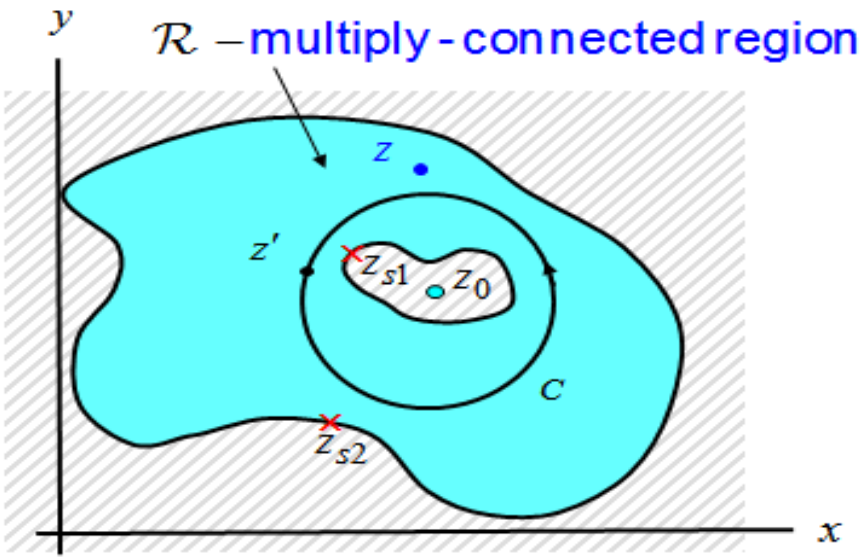
$$f(z)=\frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\oint_C\frac{f(z')}{(z'-z_0)^{n+1}}dz'$$
$$+\frac{1}{2\pi i}\sum_{n=-1}^{-\infty}(z-z_0)^n\oint_C\frac{f(z')}{(z'-z_0)^{n+1}}dz'$$

We thus have

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n$$

where

$$a_n=\frac{1}{2\pi i}\oint_C\frac{f(z')}{(z'-z_0)^{n+1}}dz'$$



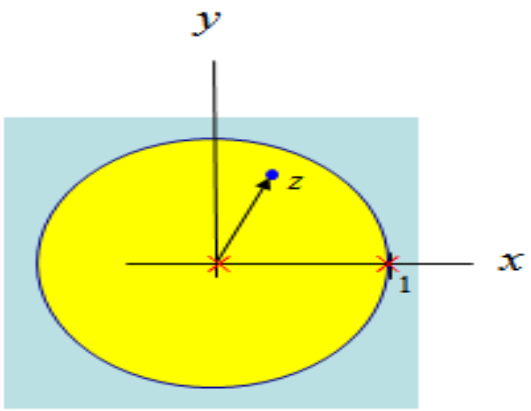
The path C is now arbitrary, as long as it stays in the analytic (blue) region.

Problem on Taylor's and Laurent's Series

Example 1: Obtain all expansions of $f(z) = \dfrac{1}{z(z-1)}$ about the origin.

Use the integral formula for the a_n coefficients.

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz'$$



The path C can be inside the yellow region or outside of it (parts (a) and (b)).

POWER SERIES EXPANSION OF COMPLEX FUNCTION

a) Laurent series with $a=0,\ b=1$

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')}{z'^{\,n+1}}dz' = \frac{1}{2\pi i}\oint_C \frac{1}{(z'-1)z'^{\,n+2}}dz' = \frac{1}{2\pi i}\oint_C \frac{-1}{z'^{\,n+2}}\sum_{m=0}^{\infty} z'^{m}dz', \quad (|z'|<1)$$

$$= \frac{1}{2\pi i}\oint_C \sum_{m=0}^{\infty}\frac{-1}{z'^{\,n-m+2}}dz' = \frac{-1}{2\pi i}\sum_{m=0}^{\infty}\oint_C \frac{1}{z'^{\,n-m+2}}dz' = \frac{-1}{2\pi i}\cdot\begin{cases} 2\pi i, & m=n+1 \\ 0, & m\neq n+1 \end{cases}$$

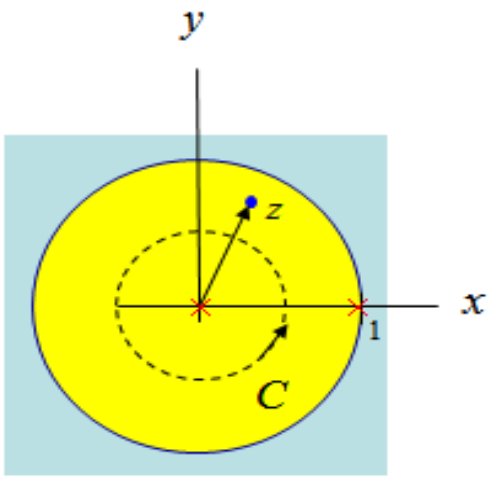
$= -1$ (for $m=n+1\geq 0$)

From uniform convergence

From previous example in Notes 3

$$a_n = -1, \quad n\geq -1$$

$$f(z) = \frac{1}{z(z-1)}$$



Hence

$$f(z) = -\frac{1}{z} - 1 - z - z^2 - z^3 - \cdots, \quad 0<|z|<1$$

The path C is <u>inside</u> the yellow region.

b) Laurent series with $a=1,\ b=\infty$

$$a_n=\frac{1}{2\pi i}\oint_C \frac{f(z')}{z'^{\,n+1}}dz'=\frac{1}{2\pi i}\oint_C \frac{1}{(z'-1)z'^{\,n+2}}dz'=\frac{1}{2\pi i}\oint_C \frac{1}{\left(1-\frac{1}{z'}\right)z'^{\,n+3}}dz'$$

$$=\frac{1}{2\pi i}\oint_C \frac{1}{z'^{\,n+3}}\sum_{m=0}^{\infty}\frac{1}{z'^{\,m}}dz',\quad (|z'|>1)$$

From previous example in Notes 3

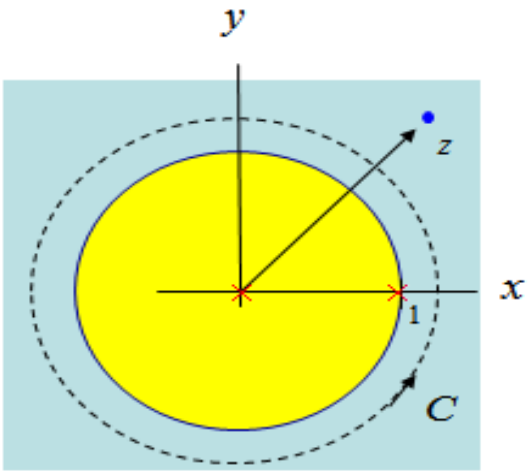
$$=\frac{1}{2\pi i}\oint_C \sum_{m=0}^{\infty}\frac{1}{z'^{\,n+m+3}}\,dz'=\frac{1}{2\pi i}\sum_{m=0}^{\infty}\oint_C \frac{1}{z'^{\,n+m+3}}\,dz'=\frac{1}{2\pi i}\cdot\begin{cases}2\pi i, & m=-n-2\\ 0, & m\neq -n-2\end{cases}$$

From uniform convergence

$$=1\quad (\text{for } m=-n-2\geq 0)$$

$$\Rightarrow\quad a_n=1,\quad n\leq -2$$

$$f(z)=\frac{1}{z(z-1)}$$



Hence

$$f(z)=\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots,\quad |z|>1$$

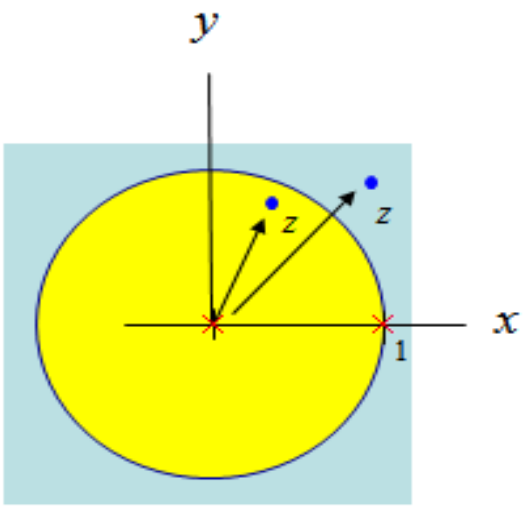
The path C is <u>outside</u> the yellow region.

POWER SERIES EXPANSION OF COMPLEX FUNCTION

Summary of results for the example:

$$f(z)=\frac{1}{z(z-1)}$$

$$f(z)=-\frac{1}{z}-1-z-z^2-z^3-\cdots,\ \ 0<|z|<1$$



$$f(z)=\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots,\ \ |z|>1$$

Note:

Often it is <u>easier</u> to directly use the geometric series (GS) formula together with some algebra, instead of the contour integral approach, to determine the coefficients of the Laurent expanson.

This is illustrated next (using the same example as in Example 1).

Problem on the Laurent's series

◇ **Example 1**

Expand $f(z)=\dfrac{1}{z(z-1)}$ about the origin (we use partial fractions and GS):

$$f(z)=\frac{1}{z(z-1)} = \frac{A}{z}+\frac{B}{z-1}$$

$$A=\lim_{z\to 0} z\,f(z)=\lim_{z\to 0}\frac{\cancel{z}}{\cancel{z}(z-1)}=-1$$

$$B=\lim_{z\to 1}(z-1)\,f(z)=\lim_{z\to 1}\frac{\cancel{z-1}}{z\,\cancel{(z-1)}}=1$$

$$\Rightarrow f(z)=\frac{1}{z(z-1)} = \frac{-1}{z}+\frac{1}{z-1}=\frac{-1}{z}-\frac{1}{1-z}$$

$$= \frac{-1}{z}-\left(1+z+z^2+\ldots\right)$$

Hence

$$f(z) = -\frac{1}{z}-1-z-z^2-z^3-\cdots,\quad 0<|z|<1$$

Alternative expansion:

$$f(z)=\frac{1}{z(z-1)} = \frac{-1}{z} + \frac{1}{z-1} = \frac{-1}{z} + \frac{1}{z}\left(\frac{1}{1-1/z}\right)$$

$$\Rightarrow f(z)=\cancel{\frac{-1}{z}} + \frac{1}{z}\left(\cancel{1}+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$$

Hence

$$f(z)=\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots, \quad |z|>1$$

Problem on the Laurent's series

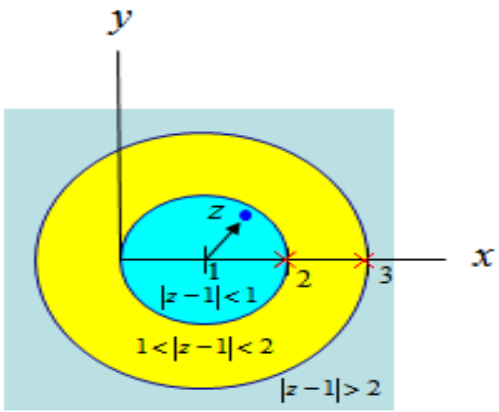
◊ **Example 2**

Expand $f(z)=\dfrac{1}{(z-2)(z-3)}$ in a Taylor / Laurent series about $z_0=1$, valid following in the annular regions :

(a) $0\le|z-1|<1$,

(b) $1<|z-1|<2$,

(c) $|z-1|>2$.



a) For $0\le|z-1|<1$:

Using partial fraction expansion and GS,

$$f(z)=\frac{1}{(z-2)(z-3)}=\frac{1}{z-3}-\frac{1}{z-2}$$

$$=\frac{1}{(z-1)-2}-\frac{1}{(z-1)-1}=\frac{-1}{2[1-(z-1)/2]}+\frac{1}{1-(z-1)}$$

$$=-\frac{1}{2}\left[1+\frac{(z-1)}{2}+\frac{(z-1)^2}{2^2}+\cdots\right]+\left[1+(z-1)+(z-1)^2+\cdots\right]$$

Hence

$$f(z)=\frac{1}{2}+\frac{3}{4}(z-1)+\frac{7}{8}(z-1)^2+\frac{15}{16}(z-1)^3+\cdots,\quad 0\le|z-1|<1$$ (Taylor series)

POWER SERIES EXPANSION OF COMPLEX FUNCTION

$$f(z)=\frac{1}{(z-2)(z-3)}$$

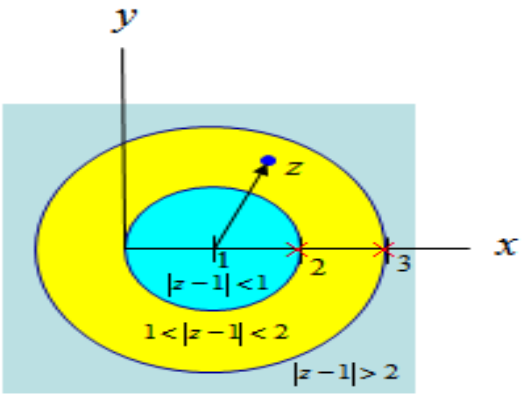
b) For $1<|z-1|<2$:

$$f(z)=\frac{1}{(z-1)-2}-\frac{1}{(z-1)-1}=\frac{-1}{2[1-(z-1)/2]}-\frac{1}{(z-1)[1-1/(z-1)]}$$

so

$$f(z)=-\frac{1}{2}\left[1+\frac{(z-1)}{2}+\frac{(z-1)^2}{2^2}+\cdots\right]-\frac{1}{(z-1)}\left[1+\frac{1}{(z-1)}+\frac{1}{(z-1)^2}+\cdots\right]$$

(Laurent series)



$$f(z)=\frac{1}{(z-2)(z-3)}$$

c) For $|z-1|>2$:

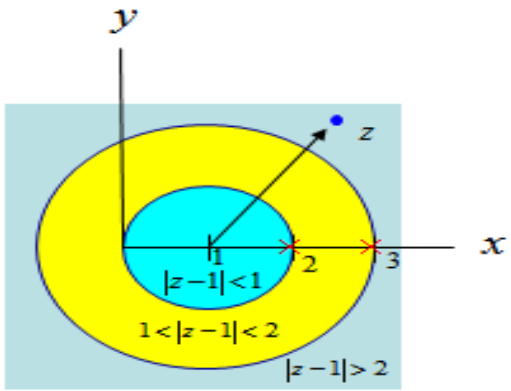
$$f(z)=\frac{1}{(z-1)-2}-\frac{1}{(z-1)-1}=\frac{1}{(z-1)[1-2/(z-1)]}-\frac{1}{(z-1)[1-1/(z-1)]}$$

$$=\frac{1}{(z-1)}\left[\cancel{1}+\frac{2}{(z-1)}+\frac{2^2}{(z-1)^2}+\cdots\right]-\frac{1}{(z-1)}\left[\cancel{1}+\frac{1}{(z-1)}+\frac{1}{(z-1)^2}+\cdots\right]$$

so

$$f(z)=\frac{1}{(z-1)^2}+\frac{3}{(z-1)^3}+\frac{7}{(z-1)^4}+\cdots$$

(Laurent series)



POWER SERIES EXPANSION OF COMPLEX FUNCTION

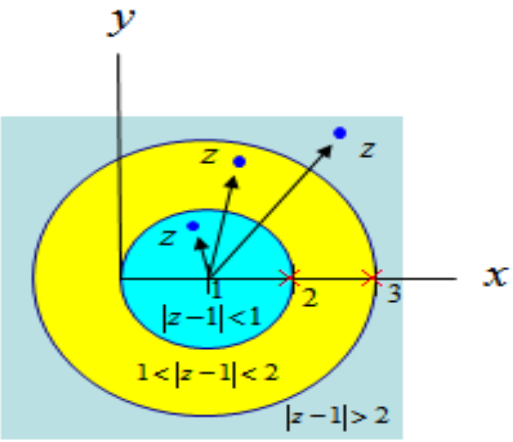
Summary of results for example

$$f(z)=\frac{1}{(z-2)(z-3)}$$

$$f(z)=\frac{1}{2}+\frac{3}{4}(z-1)+\frac{7}{8}(z-1)^2+\frac{15}{16}(z-1)^3+\cdots,\quad 0\le|z-1|<1$$

$$f(z)=-\frac{1}{2}\left[1+\frac{(z-1)}{2}+\frac{(z-1)^2}{2^2}+\cdots\right]-\frac{1}{(z-1)}\left[1+\frac{1}{(z-1)}+\frac{1}{(z-1)^2}+\cdots\right],\quad 1<|z-1|<2$$

$$f(z)=\frac{1}{(z-1)^2}+\frac{3}{(z-1)^3}+\frac{7}{(z-1)^4}+\cdots,\quad |z-1|>2$$



Problem on the Laurent's series

◊ **Example 3**

Find the series expansion about $z = 0$:

$$f(z) = \begin{cases} \dfrac{1-\cos z}{z^2}, & z \neq 0 \\ 1/2, & z = 0 \end{cases}$$

($z = 0$ is a "removable" singularity)

$$1-\cos z = \cancel{1} - \left(\cancel{1} - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) = \frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots$$

Hence

$$f(z) = \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots, \quad |z| < \infty$$

Similarly, we have $f(z) = \dfrac{\sin z}{z} = 1 - \dfrac{z^2}{3!} + \dfrac{z^4}{5!} - \cdots \quad |z| < \infty$

Note : $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots \quad |z| < \infty$

Problem on the Laurent's series

◊ **Example 4**

Note : $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \quad |z| < \infty$

Find the series for $\sin z$ about $z = \pi$:

$$f(z) = \sin z = \sin\left[(z-\pi)+\pi\right]$$
$$= \sin(z-\pi)\cos\pi + \cos(z-\pi)\sin\pi = -\sin(z-\pi)$$

$$f(z) = -(z-\pi) + \frac{(z-\pi)^3}{3!} - \frac{(z-\pi)^5}{5!} + \cdots, \ |z| < \infty$$

Alternatively, directly use the derivative formula for Taylor series :

$$f(\pi) = \sin\pi = 0$$
$$f'(\pi) = \cos\pi = -1$$
$$f''(\pi) = -\sin\pi = 0$$
$$f'''(\pi) = -\cos\pi = +1$$
$$f^{(iv)}(\pi) = \sin\pi = 0$$
$$f^{(v)}(\pi) = \cos\pi = -1$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

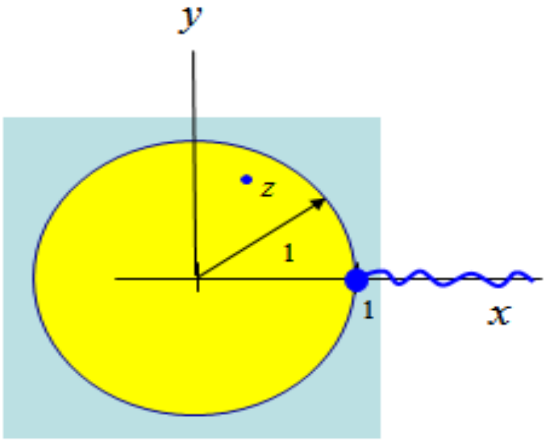
$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}}\,dz' = \frac{f^{(n)}(z_0)}{n!}$$

$$f(z) = -(z-\pi) + \frac{(z-\pi)^3}{3!} - \frac{(z-\pi)^5}{5!} + \cdots, \ |z| < \infty$$

Problem on the Laurent's series

◊ **Example 5**

Find the first three terms of the series for $\sin^2 z \ln(1-z)$ about $z=0$.



The branch cut is chosen away from the yellow region.

Since $\dfrac{1}{1-z} = 1+z+z^2+\cdots,\ |z|<1$ then

$$\int_0^z \frac{1}{1-z}dz = -\ln(1-z) = z+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\cdots,\ |z|<1$$

$$\Rightarrow \ln(1-z) = -z-\frac{z^2}{2}-\frac{z^3}{3}-\frac{z^4}{4}-\cdots,\ |z|<1$$

Also $\sin^2 z = \left[z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-\cdots\right]\left[z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-\cdots\right]$

$$= z^2-\frac{z^4}{3}+\frac{2}{45}z^6+\cdots$$

Hence

$$\sin^2 z \ln(1-z) = -\left(z^2-\frac{z^4}{3}+\frac{2}{45}z^6+\cdots\right)\left(z+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\cdots\right)$$

$$= -z^3-\frac{z^4}{2}+0\cdot z^5+\cdots,\quad |z|<1$$

Summary of the methods for generating Taylors and Laurent's Series

Summary of Methods

- Taylor (*not* Laurent) series, $f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n$, can be generated using $a_n = \dfrac{f^{(n)}(z_0)}{n!}$
- Taylor *and* Laurent series, $f(z)=\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$, can be generated using $a_n = \dfrac{1}{2\pi i}\oint_C \dfrac{f(z')}{(z'-z_0)^{n+1}}dz'$
- To expand about $z=z_0$, first write $f(z)$ in the form $f\left[(z-z_0)+z_0\right]$, rearrange and expand using geometric series or other methods.
- Use partial fraction expansion and geometric series to generate series for rational functions (ratios of polynomials, degree of numerator less than degree of denominator).
- Laurent / Taylor series can be integrated or differentiated term - by - term within their region of convergence.

Summary of Methods

- Note that for two Taylor or Laurent series,

$$f(z)=\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n, \quad g(z)=\sum_{n=-\infty}^{\infty} b_n (z-z_0)^n$$

then $f(z)+g(z)=\sum_{n=-\infty}^{\infty} (a_n+b_n)(z-z_0)^n$ *in their common region of convergence.*

- Two Taylor series can be multiplied term - by - term *within their common region of convergence* :

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n, \qquad g(z)=\sum_{m=0}^{\infty} b_m (z-z_0)^m$$

$$\Rightarrow f(z)g(z)=\left(\sum_{n=0}^{\infty} a_n (z-z_0)^n\right)\left(\sum_{m=0}^{\infty} b_m (z-z_0)^m\right)=\sum_{p=0}^{\infty} c_p (z-z_0)^p \text{ where } c_p=\sum_{n=0}^{p} a_n b_{p-n}$$

Some more Examples on Taylor's and Laurent's series

1. Expand $\frac{1}{z-2}$ at x = 1 in Taylor series

Solution: Given f (z) = $\frac{1}{z-2}$ **f(1) = -1**

$$f'(z) = \frac{-1}{(z-2)^2}, f'(1) = -1$$

$$f''(z) = \frac{2}{(z-2)^3}, f''(1) = -2$$

$$f'''(z) = \frac{-6}{(z-2)^4}, f'''(1) = -6$$

∴ The Taylor series at z = 1 is given by

$$f(z) = f(a) + (z-a)\frac{f'(a)}{1!} + (z-a)^2\frac{f''(a)}{2!} + \ldots\ldots\ldots\ldots$$

$$= -1 + (z-1)(-1) + \frac{(z-1)^2}{2!}(-2) + \frac{(z-1)^3}{3!}(-6) + \ldots\ldots$$

$$= -\left[1 + (z-1) + (z-1)^2 + (z-1)^3 + \ldots\ldots\ldots\ldots\ldots\right]$$

2. Expand $f(z)=\dfrac{z^2-1}{(z+2)(z+3)}$ **as a Laurent's series if (i)** $|z|<2$ **(ii)** $|z|>3$

(iii) $2<|z|<3$

Solution:

$$\text{Let } f(z)=\frac{z^2-1}{(z+2)(z+3)}$$

$$=1+\frac{-5z-7}{(z+2)(z+3)}$$

$$\text{Now } \frac{-5z-7}{(z+2)(z+3)}=\frac{A}{z+2}+\frac{B}{z+3}$$

$$-5z-7=A(z+3)+B(z+2)$$

Put z = -3 $\Rightarrow 15-7=-B \Rightarrow -B=8 \Rightarrow B=-8$

Put $z=-2 \Rightarrow 10-7=A \Rightarrow A=3$

$$\therefore f(z)=1+\frac{3}{z+2}-\frac{8}{z+3}$$

POWER SERIES EXPANSION OF COMPLEX FUNCTION

(i) $|z| < 2 \Rightarrow \frac{|z|}{2} < 1$ also $\frac{|z|}{3} < 1$

$$\therefore f(z) = 1 + \frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left(1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \ldots\ldots\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \ldots\ldots\right)$$

(ii) $|z| > 3 \Rightarrow 3 < |z| \Rightarrow \frac{3}{|z|} < 1, 2 < 3 < |z|, \frac{2}{|z|} < 1$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \ldots\ldots\right) - \frac{8}{z}\left(1 - \frac{3}{z} + \left(\frac{3}{z}\right)^2 - \ldots\ldots\right)$$

POWER SERIES EXPANSION OF COMPLEX FUNCTION

(iii) $2 < |z| < 3$

$$\Rightarrow 2 < |z|, |z| < 3$$

$$\Rightarrow \frac{2}{|z|} < 1, \frac{|z|}{3} < 1$$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \ldots\ldots\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \ldots\ldots\right)$$

3. Find Laurent's series expansion of f (z) = $\dfrac{7z-2}{z(z-2)(z+1)}$ **in** $1<|z+1|<3$

Solution: Let f (z) = $\dfrac{7z-2}{z(z-2)(z+1)}$

$$\frac{7z-2}{z(z-2)(z+1)}=\frac{A}{z}+\frac{B}{z-2}+\frac{C}{z+1}$$

$$7z-2=A(z-2)(z+1)+Bz(z+1)+Cz(z-2)$$

Put $z=0, A=1$

$z=2, B=2$

$z=-1, C=-3$

$$F(z)=\frac{7z-2}{z(z-2)(z+1)}$$

$$f(z)=\frac{1}{z}+\frac{2}{z-2}-\frac{3}{z+1}$$

Given $1<|z+1|<3$

Now put t = z + 1, then $1<|t|<3$

$$\Rightarrow 1<|t|, |t|<3$$

$$\Rightarrow \frac{1}{|t|}<1, \frac{|t|}{3}<1$$

POWER SERIES EXPANSION OF COMPLEX FUNCTION

$$\therefore f(z) = \frac{1}{t-1} + \frac{2}{t-3} - \frac{3}{t}$$

$$= -\frac{3}{t} + \frac{1}{t\left(1-\frac{1}{t}\right)} - \frac{2}{3\left(1-\frac{t}{3}\right)}$$

$$= -\frac{3}{t} + \frac{1}{t}\left(1-\frac{1}{t}\right)^{-1} - \frac{2}{3}\left(1-\frac{t}{3}\right)^{-1}$$

$$= -\frac{3}{t} + \frac{1}{t}\left(1+\frac{1}{t}+\left(\frac{1}{t}\right)^2+\ldots\ldots\right) - \frac{2}{3}\left(1+\frac{t}{3}+\left(\frac{t}{3}\right)^2+\ldots\ldots\right)$$

$$= -\frac{3}{z+1} + \frac{1}{z+1}\left(1+\frac{1}{z+1}+\left(\frac{1}{z+1}\right)^2+\ldots\ldots\right) - \frac{2}{3}\left(1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\ldots\ldots\right)$$

Definition

Definition: Zero of an analytic function

A point z = a is said to be a zero of an analytic function f (z) if f(z) is zero at z = a.

If f (a) = 0 and f' (a) $\neq 0$ **then z = a is called a simple zero of f (z) (or) a zero of the first order.**

If $f(a) = f'(a) = ... = f^{n-1}(a) = 0 \,\&\, f^{n}(a) \neq 0$ **then z = a is a zero of order n.**

Example; Let f (z) = z^2

Then $f'(z) = 2z, f''(z) = 2$

$f(0), f'(0) = 0, f''(0) = 2 \neq 0$

$\therefore z = 0$ **is a zero of order 2.**

Definition

Singular point:

A point z = a is said to be a singular point (or) singularity of f (z) is not analytic at z =a.

Types of Singular Point:

Isolated Singular Point:

A point z = a is said to be an isolated singular point of f (z) if

(i) f (z) is not analytic at z = a

(ii) f (z) is analytic at all points for some neighbourhood of z = a

Example: $f(z) = \dfrac{z}{(z-1)(z-2)}$

Then z = 1, 2 are isolated points.

Definition

Pole:

A point z = a is said to be a pole of f(z) of order n if we can find a positive integer such that $\lim_{z \to a}(z-a)^n f(z) \neq 0$

Essential singular point:

A singular point z = a is said to be an essential point f (z) if the Laurent's series of f (z) about z = a possesses the infinite number of terms in the principal part (terms containing negative powers).

Example

Let $f(z) = e^{1/z}$

Clearly z = 0 is a singular point

Also $f(z) = e^{1/z} = 1 + \frac{1/z}{1!} + \frac{\left(1/z\right)^2}{2!} + \ldots\ldots$

$$= 1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z^2}\right) + \ldots\ldots$$

∴ z = 0 is an essential singular point

Definition

Removable singular point:

A singular point z = a is said to be a removable singular point of f(z) if the Laurent's series of f(z) about z = a does not contain the principal part.

Example

Let $f(z) = \dfrac{\sin z}{z}$

Clearly z = 0 is a singular point

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\ldots\right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots\ldots$$

Z = 0 is a removable singular point.

Problem on Definitions

1. Classify the nature of the singular point of $f(z)=\frac{\tan z}{z}$

Solution : $f(z) = \frac{\tan z}{z}$

$$=\frac{1}{z}\left(z+\frac{z^3}{3}+\ldots\ldots\right)$$

$$= 1+\frac{z^2}{3}+\ldots..$$

This is the Laurent's series of $f(z)$ **about z=0 and there is no principal part.**

$\therefore$ **Z=0 is a removable singular point.**

Also f $(z) = \frac{\tan z}{z}=\frac{\sin z}{z\cos z}$

Poles of f(z) are z cos z = 0

$\Rightarrow z=0, z=n\pi, n=0,\pm1,\ldots.$

$\Rightarrow z=0, n\pi$ **are simple poles (pole of order 1)**

Problem on Definitions

2. Consider the function f $(z) = \dfrac{\sin z}{z^4}$ **. Find the pole and its order.**

Solution: $f(z) = \dfrac{\sin z}{z^4}$

$$= \frac{1}{z^4}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\ldots\right]$$

$$= \frac{1}{z^3} - \frac{1}{6z} + \frac{z}{120} - \ldots\ldots$$

$\therefore z = 0$ **is a pole of order 3.**

Definition

Definition

The residue of a function $f(z)$ **at a singular point** $z = a$ **is the coefficien**

b_1 **of** $\frac{1}{z-a}$ **in the Laurent's series of** $f(z)$ **about the point** $z = a$

EVALUATION OF RESIDUES

1. Suppose z = a is a pole of order 1

Then $\{\text{Re } sf(z)\}_{z=a} = \lim_{z \to a} (z-a) f(z)$

2. Suppose z = a is a pole of order n

Then $\{\text{Re } sf(Z)\}_{z=a} = \lim_{z=a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z)$

3. Suppose z =a is a pole of order 1 and $f(z) = \frac{P(z)}{Q(z)}$

Then $\{\text{Res } f(z)\}_{z=a} = \frac{P(a)}{Q'(a)}$

Problems on Residues of a function by definition, Laurent's Series and by formulae method

Ex : Find the Laurent series of $f(z)=\dfrac{1}{z(z-2)^3}$ about the singularities $z=0$ and $z=2$. Hence verify that $z=0$ is a pole of order 1 and $z=2$ is a pole of order 3, and find the residue of $f(z)$ at each pole.

(1) point $z=0$

$$f(z)=\frac{-1}{8z(1-z/2)^3}=\frac{-1}{8z}[1+(-3)(\frac{-z}{2})+\frac{(-3)(-4)}{2!}(\frac{-z}{2})^2+\frac{(-3)(-4)(-5)}{3!}(\frac{-z}{2})^3+\ldots]$$

$$=-\frac{1}{8z}-\frac{3}{16}-\frac{3}{16}z-\frac{5z^2}{32}-\ldots \qquad z=0 \text{ is a pole of order 1}$$

(2) point $z=2 \Rightarrow$ set $z-2=\xi \Rightarrow z(z-2)^3=(2+\xi)\xi^3=2\xi^3(1+\xi/2)$

$$f(z)=\frac{1}{2\xi^3(1+\xi/2)}=\frac{1}{2\xi^3}[1-(\frac{\xi}{2})+(\frac{\xi}{2})^2-(\frac{\xi}{2})^3+(\frac{\xi}{2})^4-\ldots]$$

$$=\frac{1}{2\xi^3}-\frac{1}{4\xi^2}+\frac{1}{8\xi}-\frac{1}{16}+\frac{\xi}{32}-..=\frac{1}{2(z-2)^3}-\frac{1}{4(z-2)^2}+\frac{1}{8(z-2)}-\frac{1}{16}+\frac{z-2}{32}-$$

$z=2$ is a pole of order 3, the residue of $f(z)$ at $z=2$ is $1/8$.

POWER SERIES EXPANSION OF COMPLEX FUNCTION

Ex : Suppose that $f(z)$ has a pole of order m at the point $z = z_0$. By considering the Laurent series of $f(z)$ about z_0, deriving a general expression for the residue $R(z_0)$ of $f(z)$ at $z = z_0$. Hence evaluate the residue of the function $f(z) = \dfrac{\exp iz}{(z^2+1)^2}$ at the point $z = i$.

$$f(z) = \frac{\exp iz}{(z^2+1)^2} = \frac{\exp iz}{(z+i)^2(z-i)^2}$$ poles of order 2 at $z = i$ and $z = -i$

for pole at $z = i$:

$$\frac{d}{dz}[(z-i)^2 f(z)] = \frac{d}{dz}[\frac{\exp iz}{(z+i)^2}] = \frac{i}{(z+i)^2}\exp iz - \frac{2}{(z+i)^3}\exp iz$$

$$R(i) = \frac{1}{1!}[\frac{i}{(2i)^2}e^{-1} - \frac{2}{(2i)^3}e^{-1}] = \frac{-i}{2e}$$

Problem

Example 1 Find the residue of f(z) $= \dfrac{z+2}{(z-2)(z+1)^2}$ **about each singularity.**

Solution: The poles of f (z) are given by

$$(z-2)=0, z+1=0$$

$$\Rightarrow z=2, z=-1$$

∴ The poles of f (z) are z = 2 is a simple poles and z = -1 is a pole of order 2.

$$\therefore [\operatorname{Re} sf(z)]_{z=2} = \lim_{z\to 2}(z-2)f(z)$$

$$= \lim_{z\to 2}(z-2)\frac{z+2}{(z-2)(z+1)^2} = \frac{4}{9}$$

$$\therefore [\operatorname{Re} sf(z)]_{z=-1} = \lim_{z\to -1}\frac{d}{dz}[(z-2)^2 f(z)]$$

$$= \lim_{z\to -1}\frac{d}{dz}[(z-2)^2 \frac{z+2}{(z-2)(z+1)^2}]$$

$$= \lim_{z\to -1}\frac{d}{dz}\left[\frac{z+2}{z-2}\right]$$

$$= \lim_{z\to -1}\left[\frac{(z-2)(1)-(z+2)(1)}{(z-2)^2}\right] = -\frac{4}{9}$$

Problem

Problem: Determine the poles of the function $f(z)=\dfrac{z+2}{(z+1)^2(z-2)}$ and the residue at each pole.

Solution: The poles of f(z) are given by $(z+1)^2(z-2)=0$

Here z=2 is a simple pole and z= -1 is a pole of order 2 .

Residue at z=2 is

$$\lim_{z\to 2}(z-2)f(z)=\lim_{z\to 2}(z-2)\frac{z+2}{(z+1)^2(z-2)}=\frac{4}{9}$$

Residue at z=-1 is

$$\lim_{z\to -1}\frac{d}{dz}(z+1)^2 f(z)=\lim_{z\to -1}\frac{d}{dz}(z+1)^2\frac{z+2}{(z+1)^2(z-2)}$$

$$\lim_{z\to -1}\frac{d}{dz}\frac{(z+2)}{(z-2)}=\lim_{z\to -1}\frac{-4}{(z-2)^2}=\frac{-4}{9}$$

Problem

Problem: Find the residue of the function $f(z)=\dfrac{1-e^{2z}}{z^4}$ at the poles.

Solution: Let $f(z)=\dfrac{1-e^{2z}}{z^4}$

z =0 is a pole of order 4

Residue of f(z) at z=0 is

$$=\frac{1}{3!}\lim_{z\to 0}\frac{d^3}{dz^3}(z-0)^4\frac{(1-e^{2z})}{z^4}$$

$$=\frac{1}{3!}\lim_{z\to 0}\frac{d^3}{dz^3}(1-e^{2z})$$

$$=\frac{1}{3!}\lim_{z\to 0}\frac{d^2}{dz^2}(-2e^{2z})$$

$$=\frac{1}{3!}\lim_{z\to 0}\frac{d}{dz}(-4e^{2z})$$

$$=\frac{1}{3!}\lim_{z\to 0}(-8e^{2z})$$

$$=\frac{-8}{3!}=\frac{-4}{3}.$$

Problem

Example:

$f(z)=\dfrac{1}{\sin z}$ has simple poles at $z=n\pi,\ \ n=0,\ \pm 1,\ \pm 2,\ \cdots$

$$\operatorname{Res} f(n\pi) = \lim_{z\to n\pi}(z-n\pi)\times\frac{1}{\sin z} \overset{\text{L'Hospital's rule}}{=} \lim_{z\to n\pi}\frac{1}{\cos z} = (-1)^n$$

Alternatively,

$$\operatorname{Res} f(n\pi) = \lim_{z\to n\pi}(z-n\pi)\frac{1}{\sin z} = \lim_{z\to n\pi}(z-n\pi)\frac{1}{\sin(z-n\pi+n\pi)}$$

$$= \lim_{z\to n\pi}(z-n\pi)\frac{1}{\sin(z-n\pi)\cos n\pi+\cos(z-n\pi)\sin n\pi}$$

$$= \lim_{z\to n\pi}\frac{(z-n\pi)}{\sin(z-n\pi)\cos n\pi} = (-1)^n$$ since we already know $\lim_{z\to 0}\dfrac{z}{\sin z}=1$

$$\boxed{\operatorname{Res} f(n\pi) = (-1)^n}$$

Problem

Example: $f(z) = \tan z$

$$\tan z = \frac{\sin z}{\cos z} \text{ has simple poles at } z = \frac{(2n+1)\pi}{2},$$

$$\text{for } n = 0, \pm 1, \pm 2, \cdots$$

$$\Rightarrow \operatorname{Res}(\tan z) = \frac{\cancel{\sin \frac{(2n+1)\pi}{2}}}{-\cancel{\sin \frac{(2n+1)\pi}{2}}} = -1$$

$$\boxed{\operatorname{Res}(\tan z) = -1, \qquad z = \frac{(2n+1)\pi}{2}}$$

Problem

Example:

$f(z)=\dfrac{1}{\sin^2 z}$ has poles of order 2 at $z=n\pi,\ n=0,\pm1,\pm2,\cdots$

$$\Rightarrow \operatorname{Res} f(n\pi) = \frac{d}{dz}\left[\frac{(z-n\pi)^2}{\sin^2 z}\right]\Bigg|_{z=n\pi} = \left[\frac{2(z-n\pi)\sin^2 z-2(z-n\pi)^2\sin z\cos z}{\sin^4 z}\right]_{z=n\pi}$$

$$= \left[\frac{2(z-n\pi)\sin z-2(z-n\pi)^2\cos z}{\sin^3 z}\right]_{z=n\pi}$$

After <u>three</u> applications of L'Hospital's rule:

$$\operatorname{Res} f(n\pi)=0$$

Alternative calculation:

$$f(z) = \frac{1}{\sin^2 z} = \frac{1}{\sin^2(z - n\pi + n\pi)} = \frac{1}{\sin^2(z - n\pi)} = \frac{1}{\sin^2 u} \quad (u = z - n\pi)$$

$$= \frac{1}{\left(u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots\right)^2} = \frac{1}{u^2\left(1 - \frac{u^2}{3!} + \frac{u^4}{5!} - \cdots\right)^2}$$

Geometric Series

$$= \frac{1}{u^2}\left[1 + \left(\frac{u^2}{3!} - \frac{u^4}{5!} + \frac{u^6}{7!}\cdots\right) + \left(\frac{u^2}{3!} - \frac{u^4}{5!} + \frac{u^6}{7!}\cdots\right)^2 + \cdots\right]^2$$

$$= \frac{1}{u^2}\left[1 + \frac{u^2}{3!} + \left(-\frac{1}{5!} + \frac{1}{(3!)^2}\right)u^4 + \cdots\right]^2 = \frac{1}{u^2} + \frac{2}{3!} + \cdots$$

missing $\frac{a_{-1}}{u}$ term $\Rightarrow$ $\operatorname{Res} f(n\pi) = 0$

Residue at Infinity

Residue at infinity:

The residue at infinity is given by:

$$\operatorname{Res}[f(z)]_{Z=\infty} = -\frac{1}{2\pi i}\int_C f(z)dz$$

Where f is an analytic function except at finite number of singular points and C is a closed countour so all singular points lie inside it.

Problem

Problem: Find the residue of the function $f(z) = z^3 \cos(\frac{1}{z})$ at $z = \infty$.

Solution: Let $f(z) = z^3 \cos(\frac{1}{z})$

$$g(t) = f(\frac{1}{t}) = \frac{1}{t^3}\cos t$$

$$= \frac{1}{t^3}\left[1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots\dots\dots\right]$$

$$= \left[\frac{1}{t^3} - \frac{1}{2t} + \frac{t}{24} - \dots\dots\dots\right]$$

There fore $\operatorname{Res}_{z\to\infty} f(z)$ = - coefficient ot t in the eapansion of g(t) about t=0

= -1/24.

Cauchy's Residue Theorem

CAUCHY'S RESIDUE THEOREM

If $f(z)$ is analytic at all point inside and on a simple closed curve C Except at a finite number of point $z_1, z_2, z_3 \ldots\ldots z_n$ inside C

Then

$$\int_c f(z)dz = 2\pi i \text{ [Sum of residues of f(z) at } z_1, z_2, z_3, \ldots z_n \text{]}$$

Proof

Given that $f(z)$ is not analytic Only at $z_1, z_2, z_3, \ldots z_n$

Draw the non intersecting small Circles $c_1, c_2, c_3 \ldots c_n$ with centre at $z_1, z_2, z_3, \ldots z_n$ and radii $\rho_1, \rho_2, \rho_3 \ldots \rho_n$

Then $f(z)$ is analytic in the region Between c and $c_1, c_2, c_3 \ldots c_n$

$$\int_c f(z)dz = \int_{c_1} f(z)dz + \int_{c_2} f(z)dz + \ldots + \int_{c_n} f(z)dz \qquad \ldots\ldots(1)$$

Now $z_1, z_2, z_3 \ldots z_n$ **are the singular points of** $f(z)$.

$\therefore$ **Res** $f(z)_{z=z_i}$ = **the coefficient of** $\dfrac{1}{z-z_i}$ **in the Laurent's series of** $f(z)$ **about** $z = z_i$ **(by definition of residues)**

$$= b_1 = \frac{1}{2\pi i}\int_{c_1} \frac{f(z)}{(z-z_1)^{1-n}} dz$$

Since $\left(b_n = \dfrac{1}{2\pi i}\displaystyle\int_{c_1} \frac{f(z)}{(z-z_1)^{z-n}} dz \right)$

POWER SERIES EXPANSION OF COMPLEX FUNCTION

$$= \frac{1}{2\pi i}\int_{c_1} \frac{f(z)}{(z-z_1)^0}\,dz$$

$$= \frac{1}{2\pi i}\int_{c_1} f(z)dz$$

$$\Rightarrow \int_{c_1} f(z)dz = 2\pi i\left|\text{Re}\, sf(z)\right|_{z=z_i} \qquad \text{.....(2)}$$

From (1) and (2)

$$\int_{c_1} f(z)dz = 2\pi\left|\text{Re}\, sf(z)\right|_{z=z_1} + 2\pi\left|\text{Re}\, sf(z)\right|_{z=z_2} + 2\pi\left|\text{Re}\, sf(z)\right|_{z=z_n}$$

$$= 2\pi\left|\text{Re}\, sf(z)\right|_{z=z_1} + \left|\text{Re}\, sf(z)\right|_{z=z_2} + \ldots + \left|\text{Re}\, sf(z)\right|_{z=z_n}$$

$= 2\pi i$ **{Sum of residues of** $f(z)$ **at** $z = z_1, z_2, z_3 \ldots z_n$**}**

Problem

Example 1 Find the residue of f(z) $= \dfrac{z+2}{(z-2)(z+1)^2}$ **about each singularity.**

Solution: The poles of f (z) are given by

$$(z-2)=0, z+1=0$$

$$\Rightarrow z=2, z=-1$$

∴ The poles of f (z) are z = 2 is a simple poles and z = -1 is a pole of order 2.

$$\therefore [\operatorname{Re} sf(z)]_{z=2} = \lim_{z\to 2}(z-2)f(z)$$

$$= \lim_{z\to 2}(z-2)\frac{z+2}{(z-2)(z+1)^2} = \frac{4}{9}$$

$$\therefore [\operatorname{Re} sf(z)]_{z=-1} = \lim_{z\to -1}\frac{d}{dz}[(z-2)^2 f(z)]$$

$$= \lim_{z\to -1}\frac{d}{dz}[(z-2)^2 \frac{z+2}{(z-2)(z+1)^2}]$$

$$= \lim_{z\to -1}\frac{d}{dz}\left[\frac{z+2}{z-2}\right]$$

$$= \lim_{z\to -1}\left[\frac{(z-2)(1)-(z+2)(1)}{(z-2)^2}\right] = -\frac{4}{9}$$

Problem

Example 2. Evaluate $\int_c \frac{dz}{(z^2+4)^2}$ **where c is the circle** $|z-i|=2$

Solution: Given $f(z)=\frac{1}{(z^2+4)^2}$

$z=\pm 2i$ **is a pole of order 2**

Here $z=2i$ **lies inside the circle** $|z-i|=2$

And $z=-2i$ **lies outside the circle and is of order 2.**

$$\therefore [\operatorname{Re} sf(z)]_{z=zi} = \lim_{z\to 2i} \frac{d}{dz}(z-2i)^2 f(z)$$

$$= \lim_{z\to 2i} \frac{d}{dz}(z-2i)^2 \frac{1}{(z-2i)^2(z+2i)^2}$$

$$= \lim_{z \to 2i} \frac{d}{dz}(z-2i)^2$$

$$= \lim_{z \to 2i} \frac{(z+2i)^2(0) - 2(z+2i)}{(z+2i)^4}$$

$$= -\frac{8i}{256}$$

$\therefore$ **By Residue theorem**

$$\int_c \frac{dz}{(z^2+4)^4} = 2\pi i\left(\frac{-8i}{256}\right) = \frac{\pi}{6}$$

Problem

Problem: Evaluate $\int_c \frac{4-3z}{z(z-1)(z-2)}dz$ where c is the circle $|z| = \frac{3}{2}$. Using Residue theorem.

Solution: Let $\int_c \frac{4-3z}{z(z-1)(z-2)}dz$

The poles of f(z) are z (z-1)(z-2)=0

z=0, z= 1, z=2

These poles are simple poles.

The poles z=0 and z=1 lie within the circle c: $|z| = \frac{3}{2}$

Residue of f(z) at z=0 is $R_1 = \lim_{z\to 0}(z-0)f(z) = \lim_{z\to 0} z\frac{4-3z}{z(z-1)(z-2)} = \frac{4}{2} = 2$

Residue of f(z) at z=1 is $R_2 = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}(z-1)\frac{4-3z}{z(z-1)(z-2)} = \frac{4-3}{1-2} = -1$

By Residue theorem , $\int_c \frac{4-3z}{z(z-1)(z-2)}dz = 2\pi i(R_1+R_2) = 2\pi i(2-1) = 2\pi i$.

Problem

Evaluate $\int_C \frac{z \sec z}{(1-z^2)} dz$ where C is the ellipse $4x^2+9y^2=9$, using Cauchy's residue theorem.

Solution:

Equation of ellipse is

$$4x^2+9y^2=9$$

$$\frac{x^2}{9/4}+\frac{y^2}{1}=1$$

i.e., $$\frac{x^2}{\left(3/2\right)^2}+\frac{y^2}{1}=1$$

$\therefore$ Major axis is $\frac{3}{2}$, Minor axis is 1.

The ellipse meets the x axis at $\pm\frac{3}{2}$ and the y axis at ± 1

Given $f(z) = \dfrac{z \sec z}{1 - z^2}$

$$= \frac{z}{(1+z)(1-z)\cos z}$$

The poles are the solutions of $(1+z)(1-z)\cos z = 0$

i.e., $z = -1$, $z = 1$ are simple poles and $z = (2n+1)\dfrac{\pi}{2}$

Out of these poles $z \pm 1$ lies inside the ellipse

$z = \pm \dfrac{\pi}{4}, \ \pm 3\dfrac{\pi}{4}$ lies outside the ellipse

$$\left[\operatorname{Res} f(z)\right]_{z=1} = \underset{z \to 1}{Lt} (z-1) \frac{z}{(1+z)(1-z)\cos z}$$

$$= \underset{z \to 1}{Lt} \frac{-z}{(1+z)\cos z} = \frac{-1}{2\cos 1}$$

POWER SERIES EXPANSION OF COMPLEX FUNCTION

$$\left[\operatorname{Res} f(z)\right]_{z=-1} = \underset{z \to -1}{Lt} (z+1)\frac{z}{(1+z)(1-z)\cos z}$$

$$= \underset{z \to -1}{Lt} \frac{z}{(1-z)\cos z}$$

$$= \frac{-1}{2\cos 1} = \frac{-1}{2\cos 1}$$

$$\therefore \int_C \frac{z\sec z}{1-z^2}\,dz = 2\pi i\left[\text{sum of the residues }\right]$$

$$= 2\pi i\left[\frac{-1}{2\cos 1} - \frac{1}{2\cos 1}\right]$$

$$= -2\pi i\left[\sec 1\right].$$

Contour Integration

TYPE: I

Integrals of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$ **where** $f(\cos\theta, \sin\theta)$ **is A rational function of** $\sin\theta$ **&** $\cos\theta$**. In this case we take unit circle** $|z| = 1$ **as the contour. On** $|z| = 1$ **.**

$$z = e^{i\theta}, \cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$dz = ie^{i\theta}d\theta, \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\Rightarrow d\theta = \frac{dz}{iz}$$

Also θ **various from 0 to 2** π

$$\therefore \int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta = \int_c f\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right)\frac{dz}{iz}$$

Now applying Cauchy's residue theorem, we can evaluate the Integral on the right side.

Problem on Type-1

Example 1. Evaluate $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$ **by using contour Integration.**

Solution: $I = \int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$ **limit : 0 to 2** π

Contour: $|z| = 1$

Put $z = e^{i\theta}$

$$dz = ie^{i\theta}d\theta, \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\Rightarrow d\theta = \frac{dz}{iz}$$

$$\mathbf{I} = \int_c \frac{dz}{iz\left(13 + 5\left(\frac{z^2-1}{2iz}\right)\right)}$$

$$= 2\int_c \frac{dz}{5z^2 + 26iz - 5}$$

Where $f(z) = \frac{1}{5z^2 + 26iz - 5}$

$$\therefore I = 2\int_c f(z)dz \quad \text{.....(1)}$$

POWER SERIES EXPANSION OF COMPLEX FUNCTION

To find Residue

The poles of f (z) are $5z^2 + 26iz - 5$

$$Z = \frac{-26 \pm \sqrt{(26i)^2 + 100}}{10}$$

$$= \frac{-26 \pm 24}{10} = \frac{-i}{5}, -5i$$

The pole z = $\frac{-i}{5}$ **lies inside the circle** $|z| = 1$ **and z = -5i lies outside the circle** $|z| = 1$

Now $\therefore [\text{Res} f(z)]_{z=-i/5} = \lim_{z \to -i/5} \left(z + \frac{i}{5} \right) f(z)$

$$= \lim_{z \to -i/5} \left(z + \frac{i}{5} \right) \frac{1}{5\left(z + \frac{i}{5} \right)(z + 5i)}$$

$$= \frac{1}{5\left(\frac{-i}{5} + 5i \right)} = \frac{1}{24i}$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i \sum R$$

$$= 2\pi i \frac{1}{24i} = \frac{\pi}{12}$$

(1) becomes

$$I = \frac{2\pi}{12} = \frac{\pi}{6}$$

Problem on Type-1

Example 2: Evaluate $\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta} d\theta, a>b>0$ **using contour integration.**

Solution: Let $I = \int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta} d\theta,$

$$= \int_0^{2\pi} \frac{1-\cos 2\theta}{2a+2b\cos\theta} d\theta$$

We can write $\cos 2\theta =$ **Real part of** $e^{2i\theta}, \because e^{2i\theta} = \cos 2\theta + i\sin 2\theta$

$$\therefore I = R.P \int_0^{2\pi} \frac{1-e^{2i\theta}}{2a+2b\cos\theta} d\theta$$

Put $z = e^{i\theta}$

$$d\theta = \frac{dz}{iz}, \cos\theta = \frac{z^2+1}{z}$$

$$\therefore I = R.P \int_0^{2\pi} \frac{1-z^2}{2a+b\left(\frac{z^2+1}{z}\right)} \frac{dz}{iz}$$

$$= R.P \frac{1}{i} \int_0^{2\pi} \frac{1-z^2}{bz^2+2az+b} dz$$

$$= R.P \frac{1}{i} \int_0^{2\pi} f(z) dz \text{ .....(1)}$$

POWER SERIES EXPANSION OF COMPLEX FUNCTION

Where $f(z) = \dfrac{1 - z^2}{bz^2 + 2az + b}$

To find Residues:

Poles of f(z) are given by $bz^2 + 2az + b = 0$

$$\mathbf{Z} = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b}$$

$$= \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

Let $\alpha = \dfrac{-a + \sqrt{a^2 - b^2}}{b}$ **and** $\beta = \dfrac{-a - \sqrt{a^2 - b^2}}{b}$**, since a>b,**

$$|\alpha| < 1 \;\&\; |\beta| > 1$$

$\therefore$ **The simple pole z =** α **lies inside C,**

$$\therefore [\operatorname{Re} s f(z)]_{z=\alpha} = \lim_{z\to\alpha}(z-\alpha) f(z)$$

$$= \lim_{z\to\alpha}(z-\alpha)\frac{1-z^2}{b(z-\alpha)(z-\beta)}$$

$$= \frac{1-\alpha^2}{b(\alpha-\beta)} = \frac{a-\sqrt{a^2-b^2}}{b^2}$$

$$\therefore \sum R = \frac{a-\sqrt{a^2-b^2}}{b^2}$$

$\therefore$ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i \sum R$$

$$= 2\pi i\left(\frac{a-\sqrt{a^2-b^2}}{b^2}\right)$$

(1) becomes

$$\mathbf{I} = R.P\frac{1}{i}\int_0^{2\pi} f(z)dz$$

$$= R.P\,2\pi\left(\frac{a-\sqrt{a^2-b^2}}{b^2}\right)$$

$$= \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right)$$

Problem on Type-1

Prove that $\int_0^{2\pi} \frac{d\theta}{a^2 - 2a\cos\theta + 1} = \frac{2\pi}{1-a^2}$, given $a^2 < 1$.

Solution: Let $I = \int_0^{2\pi} \frac{d\theta}{a^2 - 2a\cos\theta + 1}$

Put $z = e^{i\theta}$

Then $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$$\therefore\ I = \int_C \frac{dz/iz}{a^2 - a\left(z + \frac{1}{z}\right) + 1} \text{ where C is } |z| = 1.$$

$$= \frac{1}{ai}\int_C \frac{dz}{\left(a + \frac{1}{a}\right)z - z^2 - 1}$$

$$= \frac{i}{a}\int_C \frac{dz}{z^2 - \left(a + \frac{1}{a}\right)z + 1}$$

$$= \int_C f(z)dz \text{ where } f(z) = \left(\frac{i}{a}\right)\frac{1}{z^2 - \left(a + \frac{1}{a}\right)z + 1}$$

$$= \left(\frac{i}{a}\right)\frac{1}{(z-a)\left(z - \frac{1}{a}\right)}$$

The singularities of f(z) are simple poles at a and $\frac{1}{a}$. $a^2 < 1$ implies $|a| < 1$ and $\frac{1}{|a|} > 1$

$\therefore$ The pole that lies inside C is $z = a$.

$$\text{Res}[f(z); a] = \lim_{z \to a}(z-a).\left(\frac{i}{a}\right)\frac{1}{(z-a)\left(z - \frac{1}{a}\right)}$$

$$= \left(\frac{i}{a}\right)\frac{1}{\left(a - \frac{1}{a}\right)}$$

$$= \frac{i}{a^2 - 1}$$

$$\text{Hence } I = 2\pi i.\frac{i}{a^2 - 1} = \frac{2\pi}{1 - a^2}$$

Contour Integration and Problem on Type-2

Type II Integration around semi-circular contour

Consider the integral

Improper integrals of the form $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$ **,where P(x) and Q(x)**

Are polynomials in x such that the degree of Q exceeds that of P atleast by two and Q(x) does not vanish for any x.

Example 1: Prove that $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx = \frac{5\pi}{12}$ **using contour integration.**

Solution: Let $f(z) = \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}$

Consider $\int_C f(z)dz$ **where C is the closed contour consisting of** Γ**, semi- large circle of radius R and the real axis from –R to R.**

Then $\int f(z)dz = \int_{\Gamma} f(z)dz + \int_{-R}^{R} f(x)dx$ **.....(1)**

Now $f(z)=\dfrac{z^2-z+2}{z^4+10z^2+9}\to 0$ **as** $z\to\infty$

$$\therefore \lim_{z\to\infty} zf(z)=0$$

Hence from (1) $\int_{-\infty}^{\infty} f(x)dx=\int_C f(z)dz$

By using residue theorem, $\int_C f(z)dz=2\pi i\sum R$

$$\therefore \int_{-\infty}^{\infty} f(x)dx=2\pi i\,\mathrm{Re}\,sf(z)$$

The poles of f(z) are given by

$$z^4+10z^2+9=0$$

$$\left(z^2+9\right)\left(z^2+1\right)=0$$

$$z=\pm i,\ z=\pm 3i$$

The poles z = 3i, z = i lies in the upper half of the z – plane

POWER SERIES EXPANSION OF COMPLEX FUNCTION

$$[\operatorname{Re} sf(z)]_{z=i} = \lim_{z\to i}(z-i)f(z)$$

$$= \lim_{z\to i}(z-i)\frac{z^2-z+2}{(z+i)(z-i)(z^2+9)}$$

$$= \frac{1-i}{8(2i)} = \frac{1-i}{16i}$$

$$[\operatorname{Re} sf(z)]_{z=3i} = \lim_{z\to 3i}(z-3i)f(z)$$

$$= \lim_{z\to 3i}(z-3i)\frac{z^2-z+2}{(z-3i)(z+3i)(z^2+9)}$$

$$= \lim_{z\to 3i}\frac{z^2-z+2}{(z+3i)(z^2+9)}$$

$$= \frac{-7-3i}{(6i)(-8)} = \frac{7+3i}{48i}$$

$$\therefore \sum \operatorname{Re} sf(z) = \frac{1-i}{16i} + \frac{7+3i}{48i}$$

$$= \frac{10}{48i}$$

$$= \frac{5}{24i}$$

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i\left(\frac{5}{24i}\right) = \frac{5\pi}{12}$$

Problem on Type-2

Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$ **by using contour**

Solution: Consider the integral $\int_C f(z)dz$ **where**

$$f(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)}$$

And C is the closed contour consisting of Γ**, the upper semi large Circle** $|z| = R$ **and the real axis from –R to R**

$$\int_c f(z)dz = \int_\Gamma f(z)dz + \int_{-R}^{R} f(x)dx$$

When R $\to \infty$, $\int_\Gamma f(z)dz$

Hence $\int_{-\infty}^{\infty} f(x)dx = \int_C f(z)dz$

POWER SERIES EXPANSION OF COMPLEX FUNCTION

By using residue theorem, $\int_C f(z)dz = 2\pi i \sum R$

Poles of f(z) are given by $z^2 + a^2 = 0, z^2 + b^2 = 0$

$$z = \pm ai, z = \pm bi$$

The pole z = ai and z = bi lies in the upper half plane

$$[\text{Re}\, sf(z)]_{z=ai} = \lim_{z \to ai} (z - ai) f(z)$$

$$= \lim_{z \to ai} (z - ai) \frac{z^2}{(z + ai)(z - ai)(z^2 + b^2)}$$

$$= \lim_{z \to ai} \frac{z^2}{(z + ai)(z^2 + b^2)}$$

$$= \frac{(ai)^2}{2ai(-a^2 + b^2)}$$

$$= \frac{a}{2i(a^2 - b^2)}$$

$$\left[\operatorname{Re} sf(z)\right]_{z=bi} = \lim_{z\to bi}(z-bi)f(z)$$

$$= \lim_{z\to bi}(z-bi)\frac{z^2}{(z+bi)(z-bi)(z^2+a^2)}$$

$$= \frac{-b^2}{2bi(-b^2+a^2)}$$

$$= \frac{-b}{2i(a^2-b^2)}$$

$$\therefore \int_{-\infty}^{\infty} f(z)dz = \frac{2\pi i}{2i}\left[\frac{a}{a^2-b^2} - \frac{b}{a^2-b^2}\right]$$

$$= \pi\left[\frac{a-b}{(a+b)(a-b)}\right]$$

$$= \pi\left[\frac{1}{(a+b)}\right]$$

Problem on Type-2

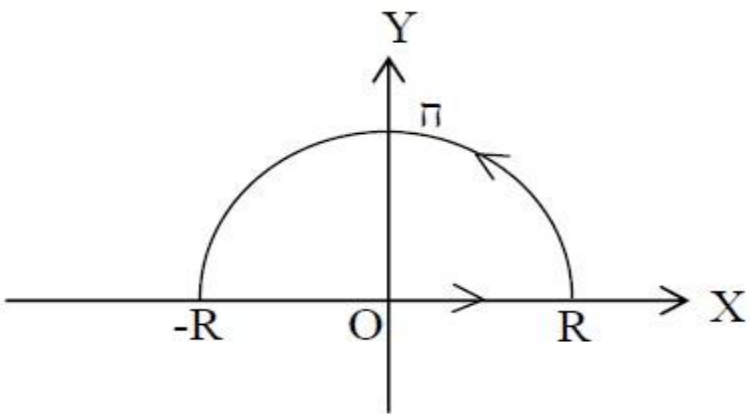
Problem Prove that $\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$

Solution:

Let $\int_C \phi(z)\,dz = \int_C \frac{dz}{(z^2+1)^2}$

Where $\phi(z) = \frac{1}{(z^2+1)^2}$

Here C is the semicircle Γ bounded by the diameter $[-R, R]$



By Cauchy residue theorem,

$$\int_C \phi(z)\,dz = \int_{-R}^{R} \phi(x)\,dx + \int_{\Gamma} \phi(z)\,dz \ldots\ldots(1)$$

To evaluate of $\int_C \phi(z)\,dz$

The poles of $\phi(z) = \dfrac{1}{\left(z^2+1\right)^2}$ is the solution of $\left(z^2+1\right)^2 = 0$

i.e., $(z+i)^2(z-i)^2 = 0$

i.e., the poles are $z = i,\ z = -i$

$z = i$ lies with inside the semi circle

$z = -i$ lies outside the semi circle

Now $\left[Res\,\phi(z)\right]_{z=i} = \underset{z \to i}{Lt} \dfrac{1}{1!}\dfrac{d}{dz}(z-i)^2\,\phi(z)$

POWER SERIES EXPANSION OF COMPLEX FUNCTION

$$= \underset{z \to i}{Lt} \frac{1}{1!}\left[(z-i)^2 \frac{1}{(z^2+1)^2}\right]$$

$$= \underset{z \to i}{Lt} \frac{1}{1!} \frac{d}{dz}\left[\frac{1}{(z+i)^2}\right] \qquad \because z^2+1=(z+i)(z-i)$$

$$= \underset{z \to i}{Lt} \frac{-2}{(z+i)^3}$$

$$= \frac{-2}{i+i} = \frac{-2}{(2i)^3} = \frac{1}{4i}$$

$$\therefore \int_C \phi(z)\, dz = 2\pi i\left[\text{Sum of residues of } \phi(z) \text{ at its poles which lies in } C\right]$$

$$= 2\pi i\left[\frac{1}{4i}\right] = \frac{\pi}{2} \text{.....(2)}$$

Let $R \to \infty$, then $|z| \to \infty$ so that $\phi(z) = 0$

$$\therefore \underset{|z| \to \infty}{Lt} \int_{\Gamma} \phi(z) dz = 0 \ldots\ldots\ldots(3)$$

Sub (2) and (3) in (1)

$$\int_{C} \phi(z)\, dz = \int_{-\infty}^{\infty} \phi(x)\, dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{\left(x^2+1\right)^2} = \frac{\pi}{2}$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{dx}{\left(x^2+1\right)^2} = \frac{\pi}{2}$$

$$\Rightarrow \int_{0}^{\infty} \frac{dx}{\left(x^2+1\right)^2} = \frac{\pi}{4}.$$

MODULE– IV
SINGLE RANDOM VARIABLES

CLOs	Course Learning Outcome
CLO 14	Demonstrate an understanding of the basic concepts of probability and random variables.
CLO 15	Classify the types of random variables and calculate mean, variance.
CLO 16	Estimate moment about origin, central moments, moment generating function of probability distribution.

Exhaustive Events:

The total no. of possible outcomes in any trial is called exhaustive event.

E.g.: (1) In tossing of a coin experiment there are two exhaustive events.

(2) In throwing an n-dice experiment, there are 6^n exhaustive events.

Favorable event:

The no of cases favorable to an event in a trial is the no of outcomes which entities the happening of the event.

E.g. (1) In tossing a coin, there is one and only one favorable case to get either head or tail.

Mutually exclusive Event:

If two or more of them cannot happen simultaneously in the same trial then the event are called mutually exclusive event.

E.g. In throwing a dice experiment, the events 1,2,3,------6 are M.E. events

Equally likely Events:

Outcomes of events are said to be equally likely if there is no reason for one to be preferred over other. E.g. tossing a coin. Chance of getting 1,2,3,4,5,6 is equally likely.

Independent Event:

Several events are said to be independent if the happening or the non-happening of the event is not affected by the concerning of the occurrence of any one of the remaining events.

An event that always happen is called **Certain event,** it is denoted by 'S'.

An event that never happens is called **Impossible event**, it is denoted by 'ϕ'.

Eg: In tossing a coin and throwing a die, getting head or tail is independent of getting no's 1 or 2 or 3 or 4 or 5 or 6.

Probability (Mathematical Definition)

Definition: If a trial results in n-exhaustive mutually exclusive, and equally likely cases and m of them are favorable to the happening of an event E then the probability of an event E is denoted by P(E) and is defined as

$$P(E) = \frac{noof\ \ favourablecasestoevent}{Total\ noof\ exaustivecases} = \frac{m}{n}$$

Sample Space:

The set of all possible outcomes of a random experiment is called Sample Space .The elements of this set are called sample points. Sample Space is denoted by S.

Eg. (1) In throwing two dies experiment, Sample S contains 36 Sample points.

S = {(1,1) ,(1,2) ,----------(1,6), --------(6,1),(6,2),--------(6,6)}

Eg. (2) In tossing two coins experiment , S = {HH ,HT,TH,TT}

A sample space is called **discrete** if it contains only finitely or infinitely many points which can be arranged into a simple sequence $w_1, w_2, \ldots\ldots$ .while a sample space containing non denumerable no. of points is called a continuous sample space.

Statistical or Empirical Probability:

If a trial is repeated a no. of times under essential homogenous and identical conditions, then the limiting value of the ratio of the no. of times the event happens to the total no. of trials, as the number of trials become indefinitely large, is called the probability of happening of the event.(It is assumed the limit is finite and unique)

Conditional Probability

The ***conditional probability*** of an event *B* is the probability that the event will occur given the knowledge that an event *A* has already occurred. This probability is written *P(B|A)*, notation for the *probability of B given A*. In the case where events *A* and *B* are *independent* (where event *A* has no effect on the probability of event *B*), the conditional probability of event *B* given event *A* is simply the probability of event *B*, that is *P(B)*.

If events *A* and *B* are not independent, then the probability of the *intersection of A and B* (the probability that both events occur) is defined by *P(A and B) = P(A)P(B|A).*

From this definition, the conditional probability *P(B|A)* is easily obtained by dividing by *P(A)*:

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

Random Variables

- A random variable X on a sample space S is a function $X : S \rightarrow R$ from S onto the set of real numbers R, which assigns a real number X (s) to each sample point 's' of S.
- Random variables (r.v.) bare denoted by the capital letters X,Y,Z,etc..
- Random variable is a single valued function.
- Sum, difference, product of two random variables is also a random variable .Finite linear combination of r.v is also a r.v .Scalar multiple of a random variable is also random variable.

- A random variable, which takes at most a countable number of values, it is called a discrete r.v. In other words, a real valued function defined on a discrete sample space is called discrete r.v.
- A random variable X is said to be continuous if it can take all possible values between certain limits .In other words, a r.v is said to be continuous when it’s different values cannot be put in 1-1 correspondence with a set of positive integers.

- A continuous r.v is a r.v that can be measured to any desired degree of accuracy. Ex : age , height, weight etc..
- Discrete Probability distribution: Each event in a sample has a certain probability of occurrence . A formula representing all these probabilities which a discrete r.v. assumes is known as the discrete probability distribution.

- The probability function or probability mass function (p.m.f) of a discrete random variable X is the function f(x) satisfying the following conditions.

  i) $f(x) \geq 0$

  ii) $\sum_{x} f(x) = 1$

  iii) $P(X = x) = f(x)$

- Cumulative distribution or simply distribution of a discrete r.v. X is F(x) defined by $F(x) = P(X \leq x) = \sum_{t \leq x} f(t)$ for $-\infty < x < \infty$

- For a continuous r.v. X, the function f(x) satisfying the following is known as the probability density function(p.d.f.) or simply density function:

  i) $f(x) \geq 0$,$-\infty < x < \infty$

  ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

  iii) $P(a<X<b) = \int_{a}^{b} f(x)dx$ = Area under f(x) between ordinates x=a and x=b

- Cumulative distribution for a continuous r.v. X with p.d.f. f(x), the cumulative distribution F(x) is defined as

  $F(x)= P(X\leq x)= \int_{-\infty}^{\infty} f(t)dt \quad -\infty<x<\infty$

  It follows that $F(-\infty) = 0$, $F(\infty)=1$, $0\leq F(x)\leq 1$ for $-\infty<x<\infty$

  $f(x)= d/dx(F(x))= F^{1}(x)\geq 0$ and $P(a < x < b)= F(b)-F(a)$

- In case of discrete r.v. the probability at a point i.e., P(x=c) is not zero for some fixed c however in case of continuous random variables the probability at appoint is always zero. I.e., P(x=c) = 0 for all possible values of c.

  P(E) = 0 does not imply that the event E is null or impossible event.

1:A random variable x has the following probability function:

x	0	1	3	4	5	6	7
P(x)	0	k	2k	2k	3k	k^2	$7k^2+k$

Find (i) k (ii) P(x<6) (iii) P(x>6)

Solution:since the total probability is unity, we have $\sum_{x=0}^{n} p(x)=1$

i.e., $0 + k + 2k + 2k + 3k + k^2 + 7k^2 + k = 1$

i.e., $8k^2 + 9k - 1 = 0$

$k = 1, -1/8$

$P(x<6) = 0 + k + 2k + 2k + 3k$

$= 1+2+2+3 = 8$

iii) $P(x>6) = k^2 + 7k^2 + k$

$= 9$

2. Let X denotes the minimum of the two numbers that appear when a pair of fair dice is thrown once. Determine (i) Discrete probability distribution (ii) Expectation (iii) Variance

Solution:

When two dice are thrown, total number of outcomes is 6x6=36

In this case, sample space S= $\begin{Bmatrix} (1,1)(1,2)(1,3)(1,4)(1,5)(1,6) \\ (2,1)(2,2)(2,3)(2,4)(2,5)(2,6) \\ (3,1)(3,2)(3,3)(3,4)(3,5)(3,6) \\ (4,1)(4,2)(4,3)(4,4)(4,5)(4,6) \\ (5,1)(5,2)(5,3)(5,4)(5,5)(5,6) \\ (6,1)(6,2)(6,3)(6,4)(6,5)(6,6) \end{Bmatrix}$

If the random variable X assigns the minimum of its number in S, then the sample space S=

$$\begin{Bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{Bmatrix}$$

The minimum number could be 1,2,3,4,5,6

For minimum 1, the favorable cases are 11

Therefore, $P(x=1)=11/36$

$P(x=2)=9/36$, $P(x=3)=7/36$, $P(x=4)=5/36$, $P(x=5)=3/36$, $P(x=6)=1/36$

The probability distribution is

X	1	2	3	4	5	6
P(x)	11/36	9/36	7/36	5/36	3/36	1/36

(i)Expectation mean $=\sum p_i x_i$

$$E(x) = 1\frac{11}{36} + 2\frac{9}{36} + 3\frac{7}{36} + 4\frac{5}{36} + 5\frac{3}{36} + 6\frac{1}{36}$$

Or $\mu = \frac{1}{36}\left[11+8+21+20+15+6\right] = \frac{9}{36} = 2.5278$

ii) variance $=\sum p_i x^2 i - \mu^2$

$$E(x)=\frac{11}{36}1+\frac{9}{36}4+\frac{7}{36}9+\frac{5}{36}16+\frac{3}{36}25+\frac{1}{36}36-(2.52$$

=1.9713

A continuous random variable has the probability density function

$$f(x) = \begin{cases} kxe^{-\lambda x}, \ for\ x \geq 0, \lambda > 0 \\ 0, otherwise \end{cases}$$

Determine (i) k (ii) Mean (iii) Variance

Solution:

(i) since the total probability is unity, we have $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} kxe^{-\lambda x}dx = 1$$

i.e., $\int_{0}^{\infty} kxe^{-\lambda x}dx = 1$

$$k\left[x\left(\frac{e^{-\lambda x}}{-\lambda}\right) - 1\left(\frac{e^{-\lambda x}}{\lambda^2}\right)\right]_0^{\infty} \ or\ k = \lambda^2$$

(i) mean of the distribution $\mu = \int_{-\infty}^{\infty} x f(x) dx$

$$\int_{-\infty}^{0} 0 dx + \int_{0}^{\infty} k x^2 e^{-\lambda x} dx$$

$$\lambda^2 \left[x^2 \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 2x \left(\frac{e^{-\lambda x}}{\lambda^2} \right) + 2 \left(\frac{e^{-\lambda x}}{\lambda^3} \right) \right]_0^{\infty}$$

$$= \frac{2}{\lambda}$$

Variance of the distribution

$$o^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$$

$$o^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - \frac{4}{\lambda^2}$$

$$= \lambda^2 \left[x^3 \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 3x^2 \left(\frac{e^{-\lambda x}}{\lambda^2} \right) + 6x \left(\frac{e^{-\lambda x}}{\lambda^3} \right) - 6 \left(\frac{e^{-\lambda x}}{\lambda^4} \right) \right]_0$$

$$= \frac{2}{\lambda^2}$$

Out of 800 families with 5 children each, how many would you expect to have (i)3 boys (ii)5girls (iii)either 2 or 3 boys ? Assume equal probabilities for boys and girls

Solution(i)

P(3boys)=P(r=3)=P(3)= $\frac{1}{2^5}{}^5C_3 = \frac{5}{16}$ per family

Thus for 800 families the probability of number of families having 3 boys= $\frac{5}{16}(800) = 250$families

P(5 girls)=P(no boys)=P(r=0)=$\frac{1}{2^5}{}^5C_0=\frac{1}{32}$ per

Family Thus for 800 families the probability of

number

of families having 5girls= $\frac{1}{32}(800)=25$families

P(either 2 or 3 boys =P(r=2)+P(r=3)=P(2)+P(3)

$\frac{1}{2^5}{}^5C_2+\frac{1}{2^5}{}^5C_3$=5/8 per family

Expected number of families with 2 or 3 boys = $\frac{5}{8}(800)=500$families.

4. In 1000 sets of trials per an event of small probability the frequencies f of the number of x of successes are

X	0	1	2	3	4	5	6	7	Total
P(X)	305	365	210	80	28	9	2	1	1000

of 10 randomly chosen tape recorders. Find (i) P(X=0) (ii) P(X=1) (iii) P(X=2) (iv) P (1<X<4).

MODULE– V
PROBABILITY DISTRIBUTIONS

CLOs	Course Learning Outcome
CLO 17	Recognize where the Binomial distribution could be appropriate model of the distributions.
CLO 18	Recognize where the Poisson distribution could be appropriate model of the distributions.
CLO 19	Recognize where the Binomial distribution and Poisson distribution could be appropriate to find mean, variance of the distributions.
CLO 20	Apply the inferential methods relating to the means of normal distributions

CLOs	Course Learning Outcome
CLO 21	Interpret Binomial distribution to the phenomena of real-world problem like sick versus healthy.
CLO 22	Identify the mapping of Normal distribution in real-world problem to analyze the stock market.
CLO 23	Use Poisson distribution in real-world problem to predict soccer scores.
CLO 24	Possess the knowledge and skills for employability and to succeed in national and international level competitive examinations.

Bionomial Distribution

A random variable X is said to follow binomial distribution if it assumes only non-negative values and its probability mass function is given by

$$P(X = x) = \begin{cases} P(x) = \binom{n}{x} p^x q^{n-x} & \text{where } x = 0,1,2,3,\ldots.n \quad q = 1-p \\ 0 \text{ other wise} & \end{cases}$$

where n, p are known as parameters, n- number of independent trials p- probability of success in each trial, q- probability of failure.

Mode of the Binomial distribution: Mode of B.D. Depending upon the values of (n+1)p

(i) If (n+1)p is not an integer then there exists a unique modal value for binomial distribution and it is ‘m’= integral part of (n+1)p

(ii) If (n+1)p is an integer say m then the distribution is Bi-Modal and the two modal values are m and m-1

Poisson distribution

Poisson distribution:

Poisson Distribution is a limiting case of the Binomial distribution under the following conditions:

(i) n, the number of trials is infinitely large.

(ii) P, the constant probability of success for each trial is indefinitely small.

(iii) np= λ, is finite where λ is a positive real number.

A random variable X is said to follow a Poisson distribution if it assumes only non-negative values and its p.m.f. is given by

$$P(x,\lambda)=\begin{cases} P(X=x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!} : & x=0,1,2,3,\ldots\ldots \lambda>0 \\ 0 & \text{Other wise} \end{cases}$$

Here λ is known as the parameter of the distribution.

We shall use the notation $X \sim P(\lambda)$ to denote that X is a Poisson variate with parameter λ

Mean and variance of Poisson distribution are equal to λ.

The coefficient of skewness and kurtosis of the poisson distribution are $\gamma_1 = \sqrt{\beta_1} = 1/\sqrt{\lambda}$ and $\gamma_2 = \beta_2 - 3 = 1/\lambda$. Hence the poisson distribution is always a skewed distribution. Proceeding to limit as λ tends to infinity we get $\beta_1 = 0$ and $\beta_2 = 3$

Mode of Poisson Distribution: Mode of P.D. Depending upon the value of λ

(i) when λ is not an integer the distribution is uni- modal and integral part of λ is the unique modal value.

(ii) When $\lambda = k$ is an integer the distribution is bi-modal and the two modals are k-1 and k.

(iii) Sum of independent poisson variates is also poisson variate.

(iv) The difference of two independent poisson variates is not a poisson
(v) variate.

Moment generating function of the P.D.

If X~ P(λ) then $M_X(t) = e^{\lambda(e^t - 1)}$

Recurrence formula for the probabilities of P.D. (Fitting of P.D.)

$$P(x+1) = \frac{\lambda}{x+1} p(x)$$

Recurrence relation for the probabilities of B.D. (Fitting of B.D.)

$$P(x+1) = \left\{ \frac{n-x}{x+1} \cdot \frac{p}{q} \right\} p(x)$$

1. Average number of accidents on any day on a national highway is 1.8. Determine th probability that the number of accidents is (i) at least one (ii) at most one

Solution:

Mean= $\lambda = 1.8$

We have P(X=x)=p(x) $\frac{e^{-\lambda}\lambda^{x}}{x!}$ =$\frac{e^{-1.8}1.8^{x}}{x!}$

i) P (at least one) $=P(x\geq 1)=1-P(x=0)$

$=1-0.1653$

$=0.8347$

ii) P (at most one) $=P(x\leq 1)$

$=P(x=0)+P(x=1)$

$= 0.4628$

Exercise problems:

2.If a Poisson distribution is such that $P(X=1)=\frac{3}{2}P(X=3)$ then find

(i) $P(X \geq 1)$

(ii) $P(X \leq 3)$ (iii) $P(2 \leq X \leq 5)$.

Normal Distribution

A random variable X is said to have a normal distribution with parameters μ called mean and σ^2 called variance if its density function is given by the probability law

$$f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-1}{2}\left\{\frac{x-\mu}{\sigma}\right\}^2\right], \quad -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0$$

A r.v. X with mean μ and variance σ^2 follows the normal distribution is denoted by

$X \sim N(\mu, \sigma^2)$

If $X \sim N(\mu, \sigma^2)$ then $Z = \frac{X-\mu}{\sigma}$ is a standard normal variate with $E(Z) = 0$ and $var(Z)=0$ and we write $Z \sim N(0,1)$

The p.d.f. of standard normal variate Z is given by f(Z) = $\frac{1}{\sqrt{2\pi}} e^{-z^2/2}$,

$-\infty < Z < \infty$

The distribution function F(Z) = P(Z $\leq$ z) = $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$

F(-z) = 1 – F(z)

P(a < z $\leq$ b) = P(a $\leq$ z < b)= P(a <z < b)= P(a $\leq$ z $\leq$b)= F(b) – F(a)

If X~ N(μ, σ^2) then Z = $\frac{X-\mu}{\sigma}$ then P(a $\leq$ X $\leq$ b) = $F\left(\frac{b-\mu}{\sigma}\right) - F\left(\frac{a-\mu}{\sigma}\right)$

N.D. is another limiting form of the B.D. under the following conditions:

i)n , the number of trials is infinitely large.

ii)Neither p nor q is very small

i) The maximum probability occurring at the point $x = \mu$ and is given by

$[P(x)]_{max} = 1/\sigma\sqrt{2\Pi}$

ii) $\beta_1 = 0$ and $\beta_2 = 3$

$\mu_{2r+1} = 0$ (r = 0,1,2……) and $\mu_{2r} = 1.3.5....(2r-1)\sigma^{2r}$

iii) Since f(x) being the probability can never be negative no portion of the curve lies below x- axis.

i) Linear combination of independent normal variate is also a normal variate.

ii) X- axis is an asymptote to the curve.

iii) The points of inflexion of the curve are given by x = $\mu \pm \sigma$, f(x) = $\frac{1}{o\sqrt{2\pi}} e^{-1/2}$

iv) Q.D. : M.D.: S.D. :: $\frac{2}{3}\sigma$: $\frac{4}{5}\sigma$: σ ::: $\frac{2}{3}$ $\frac{4}{5}$: 1 Or Q.D. : M.D.: S.D. **::10:12:15**

Problems

Problems:

1. The mean weight of 800 male students at a certain college is 140kg and the standard deviation is 10kg assuming that the weights are normally distributed find how many students weigh I) Between 130 and 148kg ii) more than 152kg

Solution:

Let μ be the mean and σ be the standard deviation. Then μ=140kg and σ =10pounds

(i) When x= 138, $z = \frac{x-\mu}{\sigma} = \frac{138-140}{10} = -0.2 = z_1$

When x= 138, $z = \frac{x-\mu}{\sigma} = \frac{148-140}{10} = 0.8 = z_2$

ii) When x=152, $\frac{x-\mu}{\sigma}=\frac{152-140}{10}=1.2=z_1$

Therefore $P(x>152)=P(z>z_1)=0.5-A(z_1)$

=0.5-0.3849=0.1151

Therefore number of students whose weights are more than 152kg

=800x0.1151=92.

Exercise Problems:

1. Two coins are tossed simultaneously. Let X denotes the number of heads then find i) E(X) ii) $E(X^2)$ iii)$E(X^3)$ iv) V(X)
2. If f(x)=k $e^{-|x|}$is probability density function in the interval, $-\infty<x<\infty$, then find i) k ii) Mean iii) Variance iv) P(0<x<4)

3. If X is a normal variate with mean 30 and standard deviation 5. Find the probabilities that i) $P(26 \leq X \leq 40)$ ii) $P(X \geq 45)$

4. The marks obtained in Statistics in a certain examination found to be normally distributed. If 15% of the students greater than or equal to 60 marks, 40% less than 30 marks. Find the mean and standard deviation.

Thank you