INSTITUTE OF AERONAUTICAL ENGINEERING

(Autonomous)

Dundigal, Hyderabad -500 043

AERONAUTICAL ENGINEERING

COURSE LECTURE NOTES

Course Name	**FINITE ELEMENT METHODS**
Course Code	**AAE009**
Programme	B.Tech
Year	2019-2020
Semester	V
Course Coordinator	Mr. S Devaraj, Assistant Professor,AE
Course Faculty	Mr. S Devaraj, Assistant,Professor, AE Ms.Ch Ragha Leena,Assistant Professor,AE

COURSE OBJECTIVES:

I. Introduce basic concepts of finite element methods including domain discretization, polynomial interpolation and application of boundary conditions.
II. Understand the theoretical basics of governing equations and convergence criteria of finite element method.
III. Develop of mathematical model for physical problems and concept of discretization of continuum.
IV. Discuss the accurate Finite Element Solutions for the various field problems.
V. Use the commercial Finite Element packages to build Finite Element models and solve a selected range of engineering problems.

COURSE OUTCOMES:

I. Describe the concept of FEM and difference between the FEM with other methods and problems based on 1-D bar elements and shape functions.
II. Derive elemental properties and shape functions for truss and beam elements and related problems.
III. Understand the concept deriving the elemental matrix and solving the basic problems of CST and axi-symmetric solids.
IV. Explore the concept of steady state heat transfer in fin and composite slab.
V. Understand the concept of consistent and lumped mass models and slove the dynamic analysis of all types of elements.

COURSE LEARNING OUTCOMES:

CLO Code	Description
AAE009.01	Describe the basic concepts of FEM and steps involved in it.
AAE009.02	Understand the difference between the FEM and Other methods.
AAE009.03	Understand the stress-strain relation for 2-D and their field problem.

AAE009.04	Understand the concepts of shape functions for one dimensional and quadratic elements, stiffness matrix and boundary conditions
AAE009.05	Apply numerical methods for solving one dimensional bar problems
AAE009.06	Derive the elemental property matrix for beam and bar elements.
AAE009.07	Solve the equations of truss and beam elements
AAE009.08	Understand the concepts of shape functions for beam element.
AAE009.09	Apply the numerical methods for solving truss and beam problems
AAE009.10	Derive the element stiffness matrices for triangular elements and axi- symmetric solids and estimate the load vector and stresses.
AAE009.11	Formulate simple and complex problems into finite elements and solve structural and thermal problems
AAE009.12	Understand the concept of CST and LST and their shape functions.
AAE009.13	Understand the concepts of steady state heat transfer analysis for one dimensional slab, fin and thin plate.
AAE009.14	Derive the stiffness matrix for fin element.
AAE009.15	Solve the steady state heat transfer problems for fin and composite slab.
AAE009.16	Understand the concepts of mass and spring system and derive the equations for various structural problems
AAE009.17	Understand the concept of dynamic analysis for all types of elements.
AAE009.18	Calculate the mass matrices, Eigen values, Eigen vectors, natural frequency and mode shapes for dynamic problems.

SYLLABUS:

UNIT – I **INTRODUCTION** Introduction to Finite Element Method for solving field problems. Stress and Equilibrium. Boundary conditions. Strain - displacement relations. Stress-strain relations for 2-D and3-D elastic problems. One Dimensional Problems: Finite element modeling coordinates and shape functions. Assembly of Global stiffness matrix and load vector. Finite element equations – Treatment of boundary conditions, Quadratic shape functions.
UNIT – II **ANALYSIS OF TRUSSES AND BEAMS** Analysis of Trusses: Stiffness matrix for plane Truss Elements, stress calculations and problems. Analysis of beams: Element stiffness matrix for two noded, two degrees of freedom per node beam element and simple problems.
UNIT – III **CONTINUUM ELEMENTS** Finite element modeling of two dimensional stress analysis with constant strain triangles and treatment of boundary conditions. Estimation of load vector and stresses. Finite element modeling of Axi-symmetric solids subjected to Axi-symmetric loading with triangular elements Two dimensional four noded isoparametric elements and problems.

UNIT – IV **STEADY STATE HEAT TRANSFER ANALYSIS** Steady state Heat Transfer Analysis: one dimensional analysis of slab, fin and two dimensional analysis of thin plate. Analysis of a uniform shaft subjected to torsion.
UNIT – V **DYNAMIC ANALYSIS** Dynamic Analysis: Formulation of finite element model, element –Mass matrices, evaluation of Eigen values and Eigen Vectors for a stepped bar, truss. Finite element-formulation to 3D problems in stress analysis, convergence requirements, mesh generation, techniques such as semi-automatic and fully automatic use of software such as ANSYS,NISA,NASTRAN etc.

TEXT BOOKS:

1	Tirupathi. R. Chandrapatla, Ashok D. Belegundu, "Introduction to Finite Elements in Engineering", Printice Hall India, 3rd Edition, 2003.
2	Rao. S.S., "Finite Element Methods in Engineering," Butterworth and Heinemann, 2001
3	Reddy J.N., "An Introduction to Finite Element Method", McGraw Hill, 2000.

REFERENCES:

1	Krishnamurthy, C.S., "Finite Element Analysis", Tata McGraw Hill, 2000.
2	J. Bathe, E. L. Wilson, "Numerical Methods in Finite Elements Analysis", Prentice Hall of India, 1985.
3	Robert D Cook, David S Malkus, Michael E Plesha, "Concepts and Applications of Finite Element Analysis", 4th edition, John Wiley and Sons, Inc., 2003.
4	Larry J Segerlind, "Applied Finite Element Analysis", 2nd Edition, John Wiley and Sons, Inc. 1984.

UNIT-I

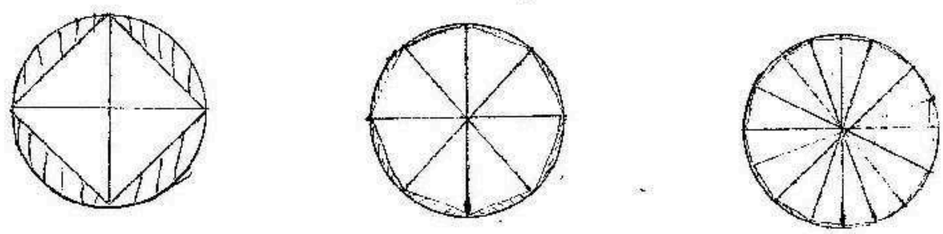
INTRODUCTION

INTRODUCTION TO FEM :

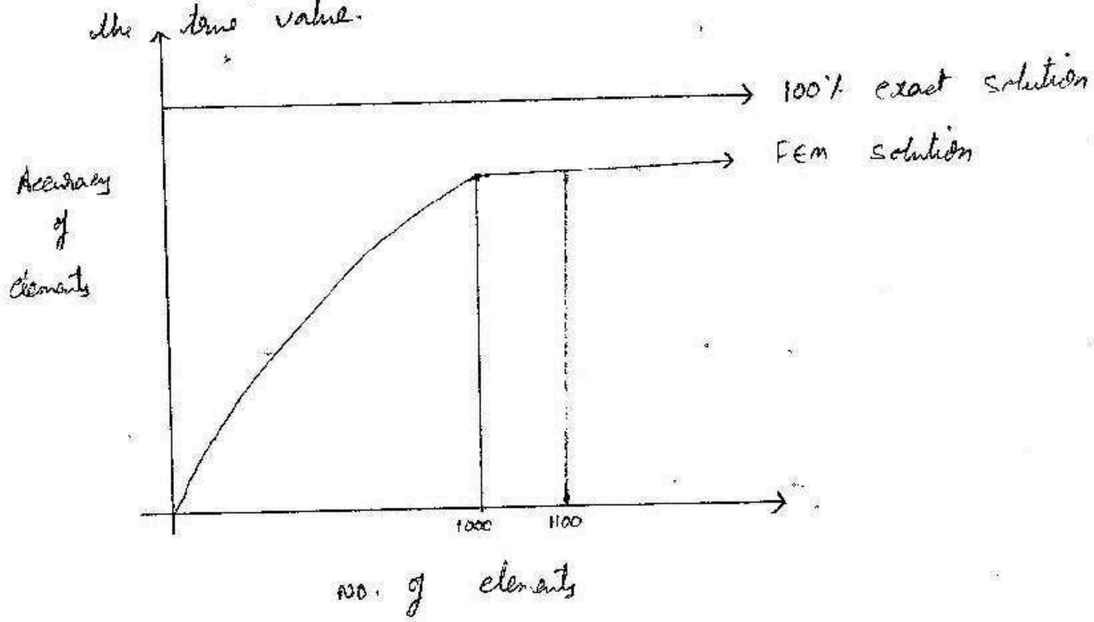
BASIC CONCEPT :

→ The basic idea in the finite element method is to find the solutions of a complicated problem by replacing it by a simpler one.

→ Actual problem is replaced by a simpler one in finding the solutions, we will be able to find only an approximate solution rather than the exact solutions.

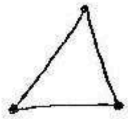


→ As no. of elements increases the approximate values converge to the true value.



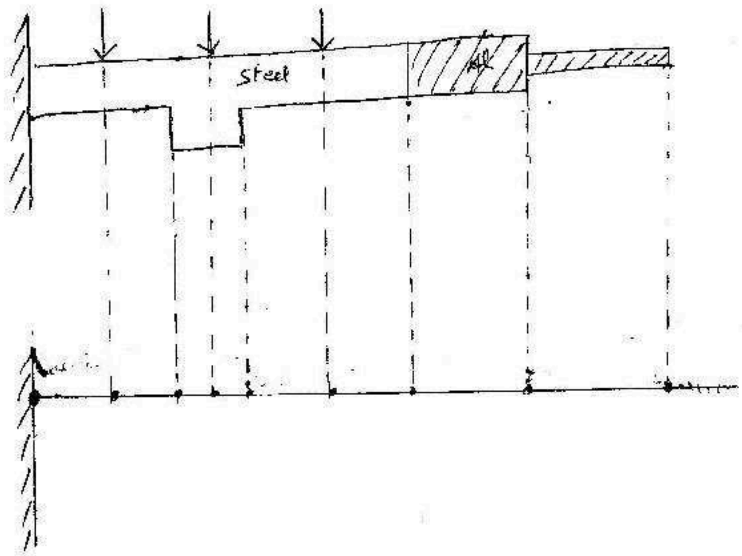
→ A/c structures is considered as one of the key contributions in the development of the FEM.

1 element with 2 nodes



1 elements with 3 nodes

criterian influencing the no. of elements & nodes:



1) where ever the ~~lod~~ load is acting there we have to consider a node.

2) wherever ~~whenever~~ the cross-section is changing we have to consider a node.

3) whenever the material is changing we adopt a node there

4) whenever the boundary conditions is applied we consider another node.

.PLICATIONS OF FEM :

→ FEM method has been extensively used in the field of structural mechanics, it has been successfully applie to solve several other types of engineering problems. such as heat conduction, fluid dynamics, electric & magnet fields.

- → mechanical engg
- → geo mechanics
- → civil Engg
- → A/C structures
- → cell towers / Bridges
- → fluid mechanics

} Regular structural problems.

Heat conduction → for thermal problems.

Field variables :

	variables	problems	software used
1)	Displacement	structural	Ansys
2)	Temperature	thermal	Ansys.

3) pressure & velocity fluid mechanics CFD.

GENERAL DESCRIPTION & steps Involved in FEA:

o→ select the suitable field variable for the given body of structure. (displacement, force, heat conduction etc).

o→ Descretize the structure into the finite elements & The no, type, size & arrangement of the elements are to be decided.

o→ Selection of a proper interpolation or displacement model.

ex: 1-D-element : $u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \cdots + \alpha_m x^n$

where $\alpha_1\ \alpha_2\ \alpha_3$ are the generalized coordinates or co-efficient of polynomials

n = DOF / Degree of polynomial

m = no. of polynomial co-efficients.

for 1-D element $m = n+1$

o→ find the element properties < stiffness matrices (K) / load vector. (P)

o→ Assemble the element properties to get global properties e the structure is composed of several finite elements, the individual element stiffness matrices & load vectors are to be assembled in a suitable manner & the overall equilibrium equations have to be formulated as.

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$$[K]\vec{\phi} = \vec{\underset{\sim}{P}}$$

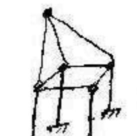
$[\underset{\sim}{K}] \rightarrow$ Assembled stiffness matrices

$\vec{\phi} \rightarrow$ nodal displacement vector

$\vec{\underset{\sim}{P}} \rightarrow$ nodal force vector (Assembled)

o→ Impose boundary conditions (Applying BC'S)

$v=0$ $\quad$ $v=0$
$\theta=0$ $\quad$ $\theta=0$



— displacement = 0

o→ solve the system equations to get nodal unknowns / field unknowns / field variables

δ, ϕ, v — displacement notations.

for solving the system equations we always follows two methods.

1) elimination approch ✓

2) Penality approch. ✗

for linear problems the nodal vector ϕ can be easily solved for non-linear problems it can be obtained by sequential steps by modify K & P.

→ computation of element strains & stresses

stress & strains can be computed by using necessary structural equations.

COMPARISON OF FEM WITH OTHER METHODS:

The common analysis methods available for the solutions of a general field problems can be classified as.

method of analysis

- Analytical method.
  - exact methods
    examples:
    seperation of variables & laplace transformation method.
  - Approximate method.
    examples.
    Rayleigh-Ritz & Galerkins method.
- Numerical methods
  - Numerical solutions of differential equations
    - Numerical integration
    - finite difference method. (FDM)
  - FE

FEM with classical methods

1. In classical methods exact solutions are formed & where as in FEA, exact ~~solutions~~ equations are formed but approximate solutions are obtained.

2. solutions have been obtained for few standard cases by classical methods, where as solutions can be obtained for all problems by FEA.

3. Shape BC's and loading conditions make the classical method solution more complex but FEA can make the solutions very simple & easier.

4 when material property is not isotropic, solution for the problems in classical methods is very difficult but FEM can handle any type of problem without difficulty.

5. FEA can handle two or more different materials in a single problem very easily but it is difficult in classical method.

6. problems with material & geometric non-linearity can not be handled by classical methods but there is no difficult in FEM.

FEM WITH FDM:

1. FDM makes point wise approximation to the governing equations (ie) it ensures continuity only at the nodal points continuity along the sides of gridlines are not ensured.

FEM makes piecewise approximation (ie) it ensures the continuity at node points as well as along the sides of the element.

2. FDM do not gives the values at any point except at node points i.e it does not gives any approximating function to evaluate the basic values using nodal values.

FEM can gives the values at any point by using suitable interpolation formulae.

3. FDM makes use of large no. of nodes to get good results while FEM needs fewer nodes

4. with FDM few complicated problems can be handled where as FEM can handle all types of problems.

ADVATAGES AND DISADVANTAGES OF FEM.

The main advantage of the finite element analysis is that physical problems which were so far intractable & complex for any closed boundary solutions can be analysed by this method

→ The method can efficiently be applied to cater irregular geometry.

→ It can take care of any type of boundary.

→ material anisotropy & inhomogeneity can be treated without much difficulty.

→ Any type of loading can be handled.

Disadvantages:

1. There are many types of problems where some other method of analysis may prove efficient than the FEM.
2. Another disadvantages of this method is cost involved in the solution of the problems.
3. For vibration & stability problems in many cases the cost of analysis by FEM may be prohibitive.
4. stress values may vary by 25% from fine mesh analysis to average mesh analysis.

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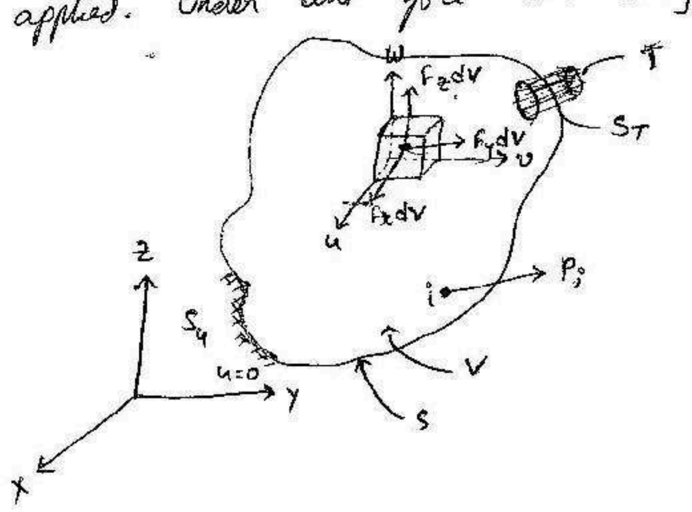
Basic equations:

stresses & equilibrium:

A three-dimensional body occupying a volume V & having a surface S is shown in fig 1.1 points in the body are located by x, y, z co-ordinates. The boundary is constrained on some region, where displacement is specified. On part of the bodary, distributed force per unit area T, also called traction, is applied. Under the force the body deforms.



The deformation of point $x = [x, y, z]^T$ is given by the components of its displacement.

$$u = [u, v, w]^T$$

The distributed force per unit volume, for example, the wt per unit volume, is the vector f given by

$$f = [f_x, f_y, f_z]^T$$

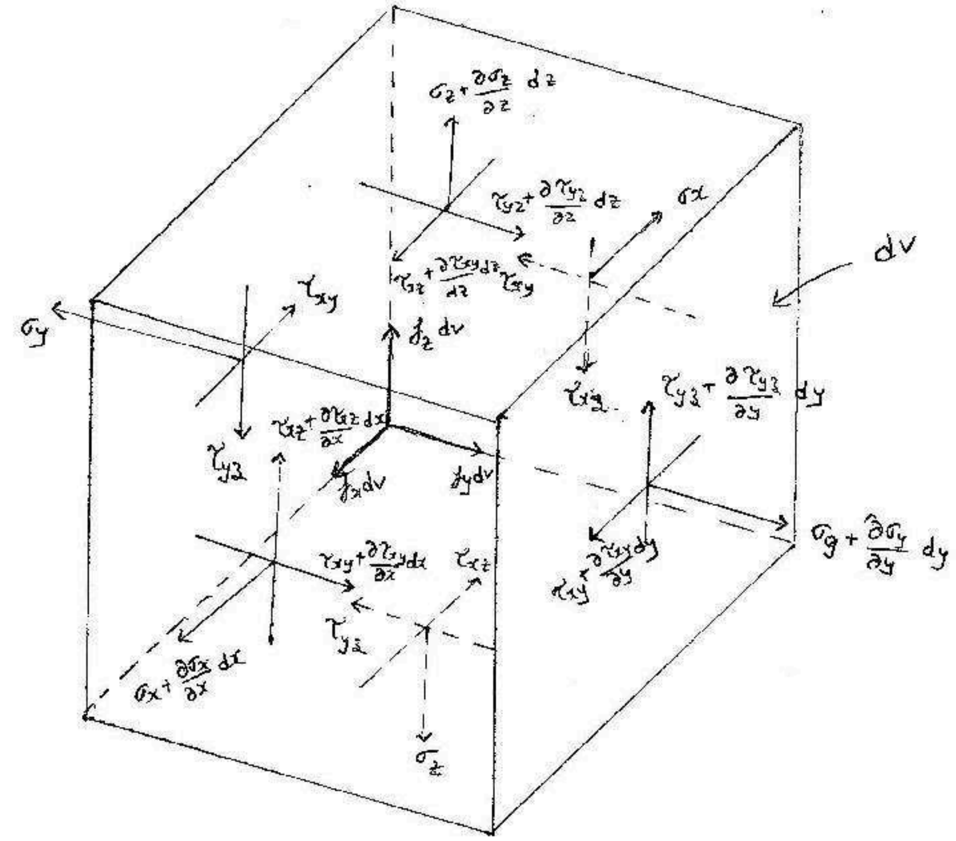
The body force acting on the elemental volume dv is shown in fig. The surface traction T may be given by its component values at points on the surfaces.

$$T = [T_x, T_y, T_z]^T$$

examples of traction are distributed contact force & action of pressure. A load P acting at a point i is represented by the three components.

$$P_i = [P_x, P_y, P_z]_i^T$$

The stresses acting on the elemental volume dv as shown in below figure. when the volume dv shrinks to a point, the stress tensor is represented by placing its components in a (3x3) symmetric matrix. However, we represent stress by the six independent components as in.



$$\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix}$$

3 - Normal stresses &

3 - Shear stresses

for satisfying equilibrium equation $\Sigma F_x = 0$; $\Sigma F_y = 0$ & $\Sigma F_z = 0$ & $dv = dx$

$$\left.\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0. \\
&\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y = 0 \\
&\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z = 0.
\end{aligned}\right\} \text{equilibrium equations}$$

Stress - strain relations :

for 1D problems

$$\boxed{\sigma = E\epsilon}$$

where ϵ - strain $\frac{\delta l}{l}$

σ - stress P/A

E - Young's modulus.

for 2D problems.

$$\epsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E}$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E} \cdot \tau_{xy}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \underbrace{\frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}}_{\text{material matrix } [D]} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\boxed{[\sigma]_{3\times1} = [D]_{3\times3} [\epsilon]_{3\times1}}$$

3D- Problems :

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xz} \\ \tau_{xy} \\ \tau_{yz} \end{Bmatrix} = \underbrace{\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-\nu \end{bmatrix} \frac{E}{(1+\nu)(1-2\nu)}}_{\text{material matrix } [D]} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xz} \\ \gamma_{yz} \\ \gamma_{xy} \end{Bmatrix}$$

$$\boxed{[\sigma]_{6\times1} = [D]_{6\times6}\,[\epsilon]_{6\times1}}$$

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Strain displacement ~~relation~~ matrix : [B]

$\epsilon = [B]\,[q]$ — Nodal displacement.

strain

$$\epsilon = \frac{du}{dx} = \frac{d}{dx}\left[N_1 q_1 + N_2 q_2\right]$$

$$= \left[\frac{dN_1}{dx},\ \frac{dN_2}{dx}\right]\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\epsilon = \left[-\frac{1}{l}\quad \frac{1}{l}\right]\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\boxed{B = \frac{1}{l}\left[-1,\ 1\right]}$$

$$\boxed{\epsilon = \left[\epsilon_x,\ \epsilon_y,\ \epsilon_z,\ \gamma_{xz},\ \gamma_{yz}\ \gamma_{xy}\right]^T}$$

Rayleigh - Ritz Method:

Strain displacement relation:

Normal strains

$$\begin{cases} \epsilon_x = \dfrac{\partial u}{\partial x} \\ \epsilon_y = \dfrac{\partial v}{\partial y} \\ \epsilon_z = \dfrac{\partial w}{\partial z} \end{cases}$$

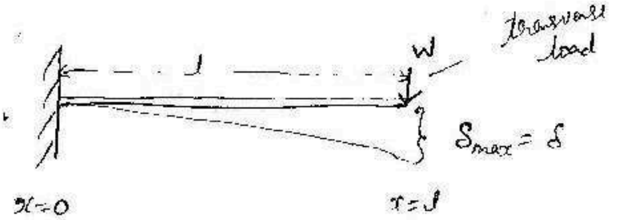
shear strain

$$\begin{cases} \gamma_{xy} = \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} \\ \gamma_{yz} = \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z} \\ \gamma_{zx} = \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} \end{cases}$$

$$\boxed{\epsilon = \left[\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}, \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right]^T}$$

Rayleigh-Ritz method:

Prob: find the deflection of the beam if one end is fixed & the other end is free & carrying a point load W at its free end.



$$\delta = \frac{Wl^3}{3EI}$$

Sol:

Let us assume the deflection function

$$\delta = C_1\left(1 - \cos\frac{\pi x}{2l}\right) \qquad C_1 = \text{Ritz constant}.$$

P.E = strain energy − W.D.

$$\boxed{\pi = U - WD}$$

ONE - DIMENSIONAL PROBLEMS :-

Linear shape function : [N]

The polynominal equations are

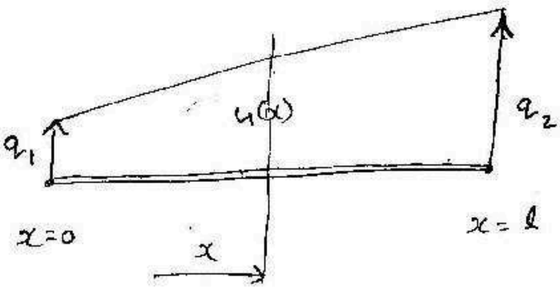
$$u(x) = ax + b \quad \text{— linear}$$

$$u(x) = ax^2 + bx + c \quad \text{— quadratic}$$

$$u(x) = ax^3 + bx^2 + cx + d \quad \text{— cubic.}$$

for finding the linear shape function take linear polynomial equation.

$$u(x) = ax + b$$



at $x = 0$; $\quad u(x) = q_1$

$x = l \quad u(x) = q_2$

substitute above Bc's in linear polynomial eq.

$$q_1 = a(0) + b \qquad \boxed{b = q_1}$$

$$q_2 = a(l) + b$$

$$q_2 = al + q_1$$

$$\boxed{a = \frac{q_2 - q_1}{l}}$$

9/2

substitute a & b in linear polynomial equation

$$u(x) = \left(\frac{q_2 - q_1}{l}\right) x + q_1$$

$$= x\frac{q_2}{l} - x\frac{q_1}{l} + q_1$$

$$= \frac{x}{l} q_2 + \left(1 - \frac{x}{l}\right) q_1$$

$$u(x) = \left[\left(1 - \frac{x}{l}\right) \left(\frac{x}{l}\right)\right] \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$u(x) \rightarrow$ linear displacement

$\left[\left(1 - \frac{x}{l}\right) \left(\frac{x}{l}\right)\right] \longrightarrow$ interpolation functions (or) shape function.

$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \rightarrow$ nodal displacement.

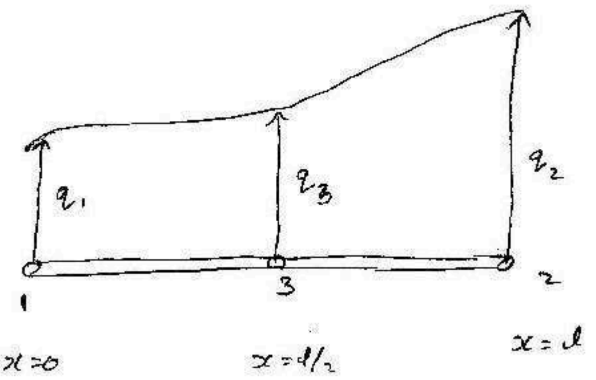
Here

$$\left.\boxed{\begin{aligned} N_1(x) &= \left(1 - \frac{x}{l}\right) \\ N_2(x) &= x/l \end{aligned}}\right\}$$ linear shape function

$$u = N_1 q_1 + N_2 q_2$$

$\boxed{\{u\} = \{N\}\{q\}}$ — vector form.

Quadratic shape functions :-



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$x = 0$ $\qquad u(x) = q_1$

$x = l/2$ $\qquad u(x) = q_3$

$x = l$ $\qquad u(x) = q_2$

$$u(x) = ax^2 + bx + c$$

$x = 0$ $\qquad q_1 = c$

$$\boxed{c = q_1} \quad \text{—— (1)}$$

$x = l$ $\qquad q_2 = al^2 + bl + c \quad \text{—— (2)}$

$x = l/2$ $\qquad q_3 = \frac{al^2}{4} + \frac{bl}{2} + c$

$$4q_3 = al^2 + 2bl + 4c \quad \text{—— (3)}$$

solve the equation (2) and (3)

$$\cancel{al^2} + bl + q_1 = q_2$$

$$-\cancel{al^2} \oplus 2bl \oplus 4q_1 = -4q_3$$

$$-bl - 3q_1 = q_2 - 4q_3$$

$$-bl - 3q_1 = q_2 - 4q_3$$

$$-bl = q_2 - 4q_3 + 3q_1$$

$$bl = 4q_3 - q_2 - 3q_1$$

$$\boxed{b = \frac{4q_3 - q_2 - 3q_1}{l}}$$

$$al^2 + \frac{4q_3 - q_2 - 3q_1}{\cancel{l}}\cancel{l} + q_1 = q_2$$

$$al^2 + 4q_3 - q_2 - 3q_1 = q_2 - q_1$$

$$al^2 = 2q_2 + 2q_1 - 4q_3$$

$$al^2 = 2q_1 + 2q_2 - 4q_3$$

$$\boxed{a = \frac{2q_1 + 2q_2 - 4q_3}{l^2}}$$

$$u(x) = ax^2 + bx + c$$

$$= \frac{2q_1 + 2q_2 - 4q_3}{l^2}x^2 + \frac{4q_3 - q_2 - 3q_1}{l}x + q_1$$

$$= \frac{2q_1x^2}{l^2} + \frac{2q_2x^2}{l^2} - \frac{4q_3x^2}{l^2} + \frac{4q_3x}{l} - \frac{q_2x}{l} - \frac{3q_1}{l}x + q_1$$

$$= q_1\left[\frac{2x^2}{l^2} - \frac{3x}{l} + 1\right] + q_2\left[\frac{2x^2}{l^2} - \frac{x}{l}\right] + q_3\left[\frac{4x}{l} - \frac{4x^2}{l^2}\right]$$

$$q_1\left[\frac{2x^2}{l^2} - \frac{3x}{l} + 1\right] + q_2\left[\frac{2x^2}{l^2} - \frac{x}{l}\right] + q_3\left[\frac{4x}{l} - \frac{4x^2}{l^2}\right]$$

$$q_1 N_1 + q_2 N_2 + q_3 N_3 = u(x)$$

$$\left.\begin{array}{l} \boxed{N_1 = \frac{2x^2}{l^2} - \frac{3x}{l} + 1} \\[2ex] \boxed{N_2 = \frac{2x^2}{l^2} - \frac{x}{l}} \\[2ex] \boxed{N_3 = \frac{4x}{l} - \frac{4x^2}{l^2}} \end{array}\right\} \text{Quadratic shape functions.}$$

$$u(x) = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$\boxed{\begin{array}{l} U = N_1 q_1 + N_2 q_2 + N_3 q_3 \\[2ex] \{U\} = \{N\}\{q\} \end{array}}$$

- To form element matrices we require

1. shape function matrices $[N]$
2. strain displacement matrices $[B]$
3. stiffness matrix $[K]$

→ strain - displacement matrix $[B]$

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$$\epsilon = [B][q] \text{ — nodal displacement}$$

$$\epsilon = \frac{du}{dx} = \frac{d}{dx}\left[N_1 q_1 + N_2 q_2\right]$$

$$= \left[\frac{dN_1}{dx}, \frac{dN_2}{dx}\right]\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$= \left[\frac{d}{dx}\left(1-\frac{x}{l}\right) \quad \frac{d}{dx}\left(\frac{x}{l}\right)\right]\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix}}_{[B]}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\boxed{[B] = \frac{1}{l}\begin{bmatrix} -1 & 1 \end{bmatrix}}$$

→ stiffness matrix $[K]$

$$[K] = \int_V [B]^T [D][B]\, dV$$

$$= \iint_{A\, l} \frac{1}{l}\begin{bmatrix} -1 \\ 1 \end{bmatrix} E \frac{1}{l}\begin{bmatrix} -1 & 1 \end{bmatrix} dA\, dx$$

$$= \iint_{A\,l} \frac{1}{l}\begin{bmatrix}-1\\1\end{bmatrix} E \,\frac{1}{l}\begin{bmatrix}-1 & 1\end{bmatrix} \, dA\, dx$$

$$= \frac{AE}{l^{\cancel{2}}} \times \cancel{l} \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

$$\boxed{K = \frac{AE}{l}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}}$$

Element body force vector matrix.

$$\boxed{F = \frac{A\,l\,f}{2}\begin{bmatrix}1\\1\end{bmatrix}}$$

Formulation of element equilibrium equation: $[K][q] = [F]$

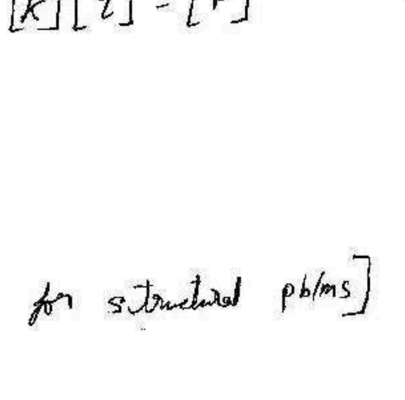
Two methods to get equilibrium equation

1. minimization of functional element.
2. principal of virtual work [only used for structural pblms]

①

Potential energy [P.E] = strain energy − work done by external loads

$$\boxed{\pi = U - W}$$



Minimization method:

$$\boxed{\frac{d\pi}{dq} = 0}$$

$$U = \int_V \frac{1}{2}[\epsilon]^T[\sigma]\, dv$$

Point load (P_i) − N

Traction load (P_t) − N/m²

Body force load (P_b) − N/m³

$$W_1 = -\int_V [u]^T [P_b]\, dv$$

$$W_2 = -\int_S [u]^T [P_t]\, ds$$

$$W_3 = -\sum_{i=1}^{n} P_i\, q_i$$

$\pi = U - W$

$$= \int_V \frac{1}{2} [\epsilon]^T [\sigma] \, dv - \int_V [u]^T [P_b] \, dv - \int_S [u]^T [P_t] \, ds - \sum_{i=1}^{n} P_i q$$

$\because$ $[\epsilon] = [B][q]$

$[\sigma] = [D][B][q]$

$[u] = [N][q]$

$$= \int_V \frac{1}{2} [B]^T [q]^T [D][B][q] \, dv - \int_V [N]^T [q]^T [P_b] \, dv - \int_S [N]^T [q]^T [P_t] \, ds - \sum_{i=1}^{n} P_i q.$$

$$= \frac{1}{2} [q]^T \underbrace{\left(\int_V [B]^T [D][B] \, dv \right)}_{K} [q] - \underbrace{\left(\int_V [N]^T [P_b] \, dv \right)}_{P_b} [q]^T - \underbrace{\left(\int_S [N]^T [P_t] \, ds \right)}_{P_t} [q]^T - \sum_{i=1}^{n} P_i q$$

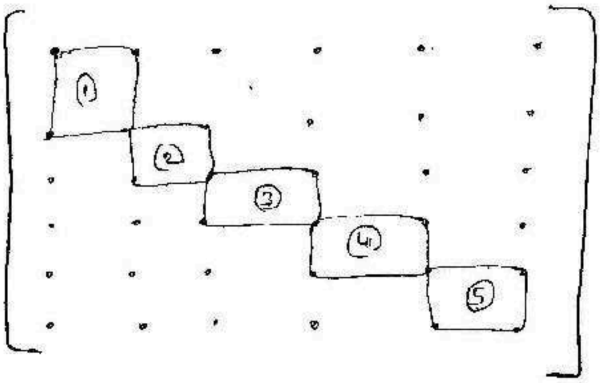
$$\pi = \frac{1}{2} [q]^T [K][q] - [P_b][q]^T - [P_t][q]^T - P_i q$$

$$\boxed{\frac{d\pi}{dq} = 0} \qquad \frac{1}{\cancel{2}} \times \cancel{2} [q][k] - \underbrace{[P_b] - [P_t] - P_i}_{F} = 0$$

$$\boxed{[k][q] = [F]}$$ //

The above equation is the element equilibrium equation.

Assembly of global stiffness matrix :



$$\boxed{\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}} \qquad k_1 = \begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} & \begin{array}{c} 1 \\ 2 \end{array} \end{array} \qquad k_2 = \begin{array}{cc} \begin{array}{cc} 2 & 3 \end{array} & \\ \begin{bmatrix} 2 & 6 \\ 4 & 3 \end{bmatrix}_{2\times 2} & \begin{array}{c} 2 \\ 3 \end{array} \end{array}$$

$$[\underset{\sim}{k}] = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 6 & 6 \\ 0 & 4 & 3 \end{bmatrix}_{3\times 3}$$

Lecture Notes
by
S. Durairaj

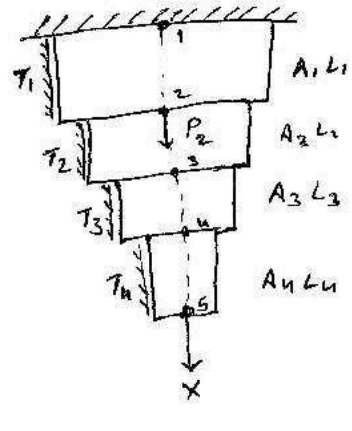
Properties of stiffness matrix :

Several important comments will now be made regarding the global stiffness matrix for the linear one dimensional problems discussed earlier:

1. The dimension of the global stiffness K is $(N \times N)$, where N is the no. of nodes. This follows from the fact that each nodes has only one degree of freedom.

2. K is symmetric

3. K is a banded matrix. That is all element outside of the band are zero. ~~This can be see in example~~



EF = Constant.

In above example K can be compactly represented in banded form as

$$K_{banded} = E \begin{bmatrix} A_1/l_1 & -A_1/l_1 \\ A_1/l_1 + A_2/l_2 & -A_2/l_2 \\ A_2/l_2 + A_3/l_3 & -A_3/l_3 \\ A_3/l_3 + A_4/l_4 & -A_4/l_4 \\ A_4/l_4 & 0 \end{bmatrix}$$

PROBLEMS :

1) Consider the thin (steel) plate as shown in figure. The plate has a uniform thickness $t = 1 in$ & $E = 30\times10^6 psi$ & weight density $\rho = 0.2836 lb$ In addition to it self-wt; the plate is subjected to a point load $P = 100$ at its mid point.

(a) find element stiffness & element force vectors for each element?

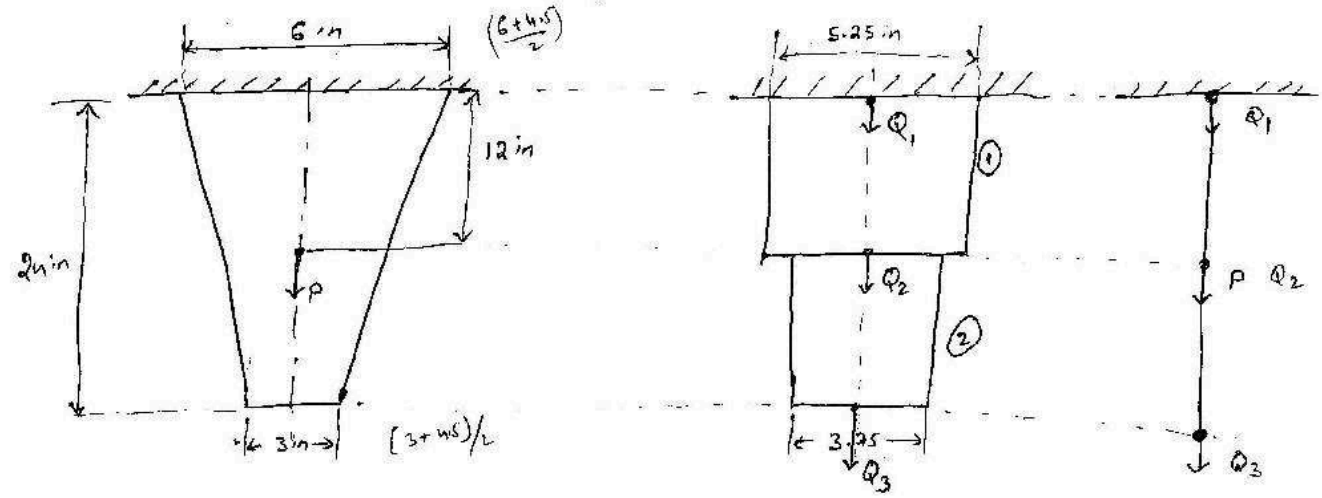
(b) Assemble the force vector & stiffness matrices?

(c) find the global displacement vector Q?

(d) Evaluate the stresses in each element?

(F) Determine the reaction force at the support?

Sol:



(a) stiffness matrix

$$K_1 = \frac{EA}{l} \begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

$$K_1 = \frac{30\times10^6 \times 5.25}{12} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_2 = \frac{30\times10^6 \times 3.75}{12} \begin{array}{cc} \begin{array}{cc} 2 & 3 \end{array} & \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array}$$

force matrix

$$F_1 = \frac{A l f}{2}\begin{bmatrix}1\\1\end{bmatrix}$$

$$F_1 = \frac{5.25 \times 12 \times 0.2836}{2}\begin{bmatrix}1\\1\end{bmatrix}\begin{array}{c}1\\2\end{array}$$

$$f_2 = \frac{3.75 \times 12 \times 0.2836}{2}\begin{bmatrix}1\\1\end{bmatrix}\begin{array}{c}2\\3\end{array}$$

(b)

Assembled stiffness matrix
$$\underset{\sim}{K} = \frac{30\times10^6}{12}\begin{array}{c} \begin{array}{ccc}1 & 2 & 3\end{array} \\ \begin{bmatrix}5.25 & -5.25 & 0\\ -5.25 & \overset{9}{(5.25+3.75)} & -3.75\\ 0 & -3.75 & 3.75\end{bmatrix}\end{array}\begin{array}{c}1\\2\\3\end{array}$$

Assembled force vector matrix
$$\underset{\sim}{F} = \begin{bmatrix}8.9334\\ 8.9334+6.381+100\\ 6.381\end{bmatrix} \Rightarrow \underset{\sim}{F} = \begin{bmatrix}8.9334\\115.3144\\6.3810\end{bmatrix}$$

(c) By elimination Approach.

To get global displacement vector

$$KQ = F$$

$$\frac{30\times10^6}{12}\begin{bmatrix}9 & -3.75\\ -3.75 & 3.75\end{bmatrix}\begin{bmatrix}Q_2\\Q_3\end{bmatrix} = \begin{bmatrix}115.3144\\6.3810\end{bmatrix}$$

By solving the equations

$$Q_2 = 9.272\times10^{-6} \text{ in}$$

$$Q_3 = 9.952\times10^{-6} \text{ in}$$

$$Q = \left[9.272\times10^{-6},\quad 0.9952\times10^{-6}\right]^T$$

(d) stresses in each element

$$\sigma = E B q \qquad \therefore \; B = \frac{1}{x_2 - x_1}\begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$\sigma_1 = 30\times10^6 \times \frac{1}{24-12}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 9.272\times10^{-6} \end{Bmatrix}$$

$$\underline{\sigma_1 = 23.18 \text{ psi}}$$

$$\sigma_2 = 30\times10^6 \times \frac{1}{24-12}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 9.272\times10^{-6} \\ 9.952\times10^{-6} \end{bmatrix}$$

$$\underline{\sigma_2 = 1.70 \text{ psi}}$$

Lecture Notes
by
S. Durasaj

(e) Reaction forces

$$R_1 = K Q - F$$

self wt

$$= \frac{30\times10^6}{12}\begin{bmatrix} 5.25 & -5.25 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 9.272\times10^{-6} \\ 9.952\times10^{-6} \end{bmatrix} - 8.9334$$

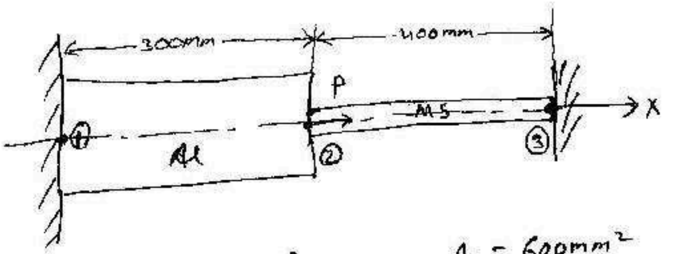
$$\underline{R_1 = -130.6 \text{ lb}}$$

② Consider the bar shown in below fig. An axial load $P = 200 \times 10^3$ N is applied as shown. Using the penalty approach for handling B.C's do the following.

(a) Determine the nodal displacement

(b) Determine the stress in each material

(c) Determine the reaction forces.



$A_1 = 2400 mm^2$ $\quad$ $A_2 = 600 mm^2$

$E_1 = 70 \times 10^9 \ N/m^2$ $\quad$ $E_2 = 200 \times 10^9 \ N/m^2$

$= 70 \times 10^3 \ N/mm^2$ $\quad$ $= 200 \times 10^3 \ N/mm^2$

Sol) (a) element stiffness matrix.

$$K_1 = \frac{70 \times 10^9 \times 2400}{300} \begin{array}{cc} 1 & 2 \\ \end{array}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{array}{c} 1 \\ 2 \end{array}$$

$$K_2 = \frac{200 \times 10^9 \times 600}{400} \begin{array}{cc} 2 & 3 \\ \end{array}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{array}{c} 2 \\ 3 \end{array}$$

global stiffness matrix

$$K = 10^6 \begin{bmatrix} 0.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.30 \\ 0 & -0.30 & 0.30 \end{bmatrix}$$

global load vector is

$$F = \begin{bmatrix} 0. & 200 \times 10^3, & 0 \end{bmatrix}^T$$

- By using penalty approch

$$\boxed{C = \max |k_{ij}| \times 10^4}$$

$$C = [0.86 \times 10^6] \times 10^4$$

Thus modified stiffness matrix is

$$K = 10^6 \begin{bmatrix} 8600.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.30 \\ 0 & -0.36 & 8600.30 \end{bmatrix}$$

<u>finite element equation is</u>

$$\underline{kQ = F}$$

$$10^6 \begin{bmatrix} 8600.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.30 \\ 0 & -0.30 & 8600.30 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 200 \times 10^3 \\ 0 \end{bmatrix}$$

$$\underline{\underline{Q = [15.1432 \times 10^{-6}, \quad 0.23257, \quad 8.1127 \times 10^{-6}]^T \text{ mm}}}$$

(b) <u>elemental stresses</u>

$$\underline{\sigma = EBq}$$

$$\sigma_1 = 70 \times 10^3 \times \frac{1}{300} [-1 \quad 1] \begin{bmatrix} 15.1432 \times 10^{-6} \\ 0.23257 \end{bmatrix}$$

$$\boxed{\sigma_1 = 54.27 \text{ MPa}}$$

where $1 \text{MPa} = 10^6 \text{ N/m}^2 = 1 \text{ N/mm}^2$

$$\sigma_2 = 200 \times 10^3 \times \frac{1}{400} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0.23257 \\ 8.1127 \times 10^{-6} \end{bmatrix}$$

$$\boxed{\sigma_2 = -116.29 \text{ MPa}}$$

(C) <u>Reaction forces</u>

$$R_1 = -C\,Q_1$$

$$= -\left[0.86 \times 10^{10}\right] \times 15.1432 \times 10^{-6}$$

$$R_1 = -130.23 \times 10^3 \text{ N}.$$

&

$$R_3 = -C\,Q_3$$

$$= -\left[0.86 \times 10^{10}\right] \times 8.1127 \times 10^{-6}$$

$$R_3 = -69.77 \times 10^3 \text{ N}$$

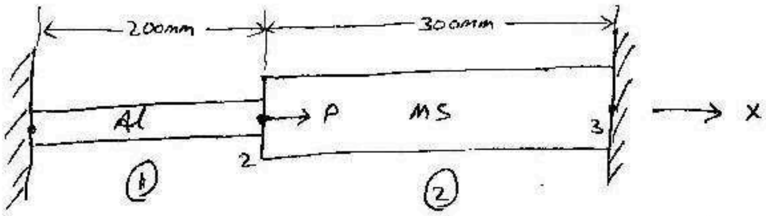
$$\boxed{R_1 = -130.23 \times 10^3 \text{ N} \quad \& \quad R_3 = -69.77 \times 10^3 \text{ N}}$$

//

An axial load $P = 300 \times 10^3 N$ is applied at $20°c$ to the rod as shown in figure. The temperature is then raised to $60°c$

(a) Assemble the K & F matrices

(b) Determine the nodal displacements & elemental stresses



$E_1 = 70 \times 10^9 \ N/m^2$ $\qquad$ $E_2 = 200 \times 10^9 \ N/m^2$

$= 70 \times 10^3 \ N/mm^2$ $\qquad$ $= 200 \times 10^3 \ N/mm^2$

$A_1 = 900 mm^2$ $\qquad$ $A_2 = 1200 mm^2$

$\alpha_1 = 23 \times 10^{-6}$ per°c $\qquad$ $\alpha_2 = 11.7 \times 10^{-6}$ per°c

Lecture Notes by S. Deivaraj

Sol:

(a) element stiffness matrices

$$K_1 = \frac{70 \times 10^3 \times 900}{200} \begin{bmatrix} +1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_2 = \frac{200 \times 10^3 \times 1200}{300} \begin{bmatrix} +1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K = 10^3 \begin{bmatrix} 315 & -315 & 0 \\ -315 & 1115 & -800 \\ 0 & -800 & 800 \end{bmatrix} \ N/mm$$

Now in assembling force matrix both temperature & point loads are considered $\{ \Delta T = 40°c \}$

$$F = \Theta_1 = E_1 A_1 \alpha_1 \Delta T \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$f_1 = \Theta_1 = 70 \times 10^3 \times 900 \times 23 \times 10^{-6} \times 40 \begin{array}{c} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \end{array}$$

$$f_1 = \Theta_1 = \begin{bmatrix} 57.96 & 57.96 \end{bmatrix}^T$$

$$f_2 = \Theta_2 = 200 \times 10^3 \times 1200 \times 11.7 \times 10^{-6} \times 40 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

$$f_2 = \Theta_2 = \begin{bmatrix} -112.32 & 112.32 \end{bmatrix}^T$$

$$F = 10^3 \begin{bmatrix} -57.96 \\ 57.96 \ -112.32 + 300 \\ 112.32 \end{bmatrix}$$

$$F = 10^3 \begin{bmatrix} -57.96 & 245.64 & 112.32 \end{bmatrix}^T \ N.$$

(b) <u>By elimination approach 1 & 3 are zero.</u>

$$10^3 \begin{bmatrix} 1115 \end{bmatrix} Q_2 = 10^3 \times 245.64$$

$$Q_2 = 0.220 \text{mm}$$

$$Q = \begin{bmatrix} 0 & 0.220 & 0 \end{bmatrix}^T \text{ mm}$$

stresses in each element.

$$\sigma_1 = EBq - E\alpha \Delta T$$

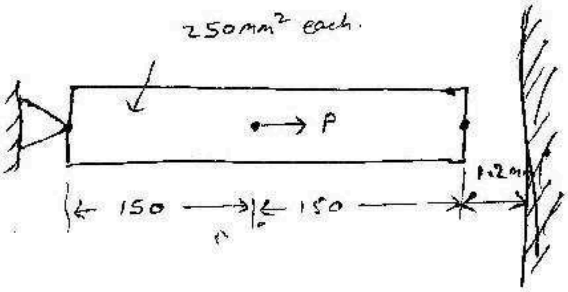
$$= \frac{70 \times 10^3}{200} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.220 \end{bmatrix} - 70 \times 10^3 \times 23 \times 10^{-6} \times 40$$

$$\boxed{\sigma_1 = -12.60 \text{ MPa}}$$

$$\sigma_2 = \frac{200 \times 10^3}{300} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0.220 \\ 0 \end{bmatrix} - 200 \times 10^3 \times 11.7 \times 10^{-6} \times 40$$

$$\boxed{\sigma_2 = -240.27 \text{ MPa}}$$

A load $P = 60 \times 10^3$ N is applied as shown in figure for a bar element. Determine the displacement fields, stresses & support reactions in the body. Take $E = 20 \times 10^3$ N/mm².



Sol:

$$\delta = \frac{PL}{AE} \Rightarrow q_2 = \frac{PL_1}{AE}$$

$$q_2 = \frac{60 \times 10^3 \times 150}{250 \times 20 \times 10^3}$$

$$\boxed{q_2 = 1.8 \text{ mm}}$$

$$K_1 = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{250 \times 20 \times 10^3}{150} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boxed{k_1 = k_2}$$

$$\underset{\sim}{K} = 10^3 \begin{bmatrix} 33.33 & -33.33 & 0 \\ -33.33 & 66.66 & -33.33 \\ 0 & -33.33 & 33.33 \end{bmatrix}$$

$$q = \begin{bmatrix} q_1 = 0 \\ q_2 = ? \\ q_3 = 1.2 \end{bmatrix} \qquad q = \begin{bmatrix} 0 \\ q_2 \\ 1.2 \end{bmatrix}$$

$$F = \begin{bmatrix} 0 \\ 60\times10^3 \\ 0 \end{bmatrix}$$

$K\,q = F$ (By elimination approch)

$$10^3 \begin{bmatrix} \cancel{33.33} & \cancel{-33.33} & \cancel{0} \\ \cancel{-33.33} & 66.66 & -33.33 \\ \cancel{0} & -33.33 & 33.33 \end{bmatrix} \begin{bmatrix} \cancel{0} \\ q_2 \\ 1.2 \end{bmatrix} = \begin{bmatrix} \cancel{0} \\ 60\times10^3 \\ 0 \end{bmatrix}$$

$$10^3 \begin{bmatrix} 66.66 & -33.33 \\ -33.33 & 33.33 \end{bmatrix} \begin{bmatrix} q_2 \\ 1.2 \end{bmatrix} = \begin{bmatrix} 60\times10^3 \\ 0 \end{bmatrix}$$

$$\cancel{10^3}\left[66.66\,q_2 - 33.33\times1.2\right] = 60\times10^3$$

$$66.66\,q_2\times10^3 = 60\times10^3 + (33.33\times1.2\times10^3)$$

$\boxed{q_2 = 1.5\,mm}$ ✓ $\boxed{q_3 = 1.2\,mm}$ $\boxed{q_1 = 0}$

STRESSES:-

$$\sigma_1 = EBq$$

$$= 20\times10^3 \times \frac{1}{150}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1.5 \end{bmatrix}$$

$\boxed{\sigma_1 = 200\ MPa}$

$$\sigma_2 = EBq$$

$$= 20\times10^3 \times \frac{1}{150}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 1.5 \\ 1.2 \end{bmatrix}$$

$\boxed{\sigma_2 = -39.99\ MPa}$

Lecture Notes
by
S. Deivaraj

- Reaction forces :

$$R_1 = KQ - F$$

$$= 10^3 \begin{bmatrix} 33.33 & -33.33 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1.5 \\ 1.2 \end{bmatrix}$$

$$\boxed{R_1 = -49.995 \times 10^3 \, N.}$$

$$R_3 = KQ - F$$

$$\cancel{= 10^3 \begin{bmatrix} -33.33 & 66.66 & -33.33 \end{bmatrix} \begin{bmatrix} 0 \\ 1.5 \\ 1.2 \end{bmatrix}}$$

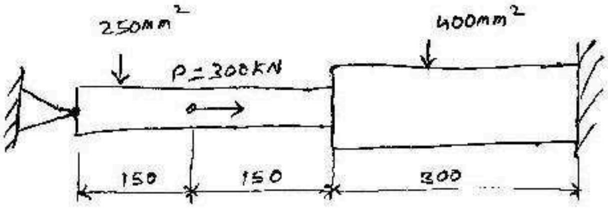
$$= 10^3 \begin{bmatrix} 0 & -33.33 & 33.33 \end{bmatrix} \begin{bmatrix} 0 \\ 1.5 \\ 1.2 \end{bmatrix}$$

$$\boxed{R_3 = -9.999 \times 10^3 \, N}$$

PREVIOUS JNTU QUESTIONS:

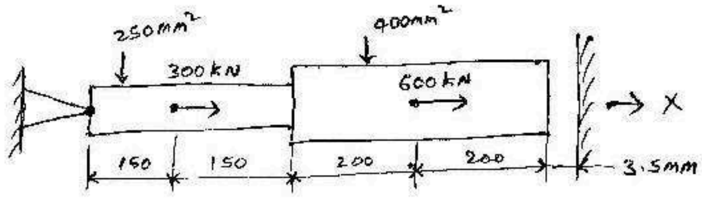
Exercise problems:

(1)



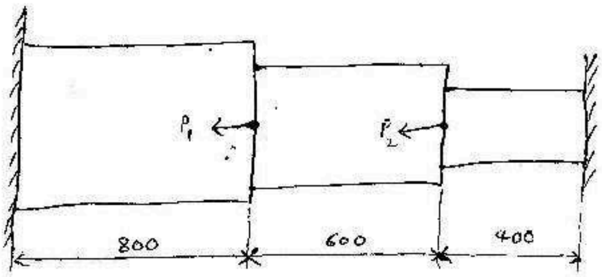
$E = 200 \times 10^9 \, N/mm^2$

(2)



$E = 200 \times 10^9 \, N/mm^2$

(3)



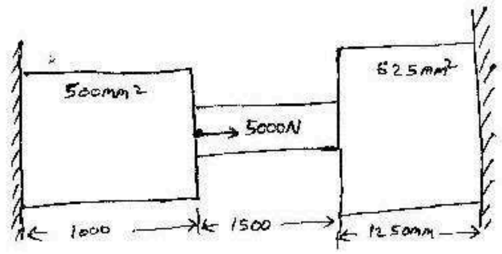
$P_1 = 60kN$ $P_2 = 75kN$

$\Delta T = 80°C$

Bronze	Aluminium	steel
$A_1 = 2400mm^2$	$A_2 = 1200mm^2$	$A_3 = 600mm^2$
$E = 83Gpa$	$E = 70Gpa$	$E = 200Gpa$
$\alpha = 18.9 \times 10^{-6}/°c$	$\alpha = 23 \times 10^{-6}/°c$	$\alpha = 11.7 \times 10^{-6}/°c$

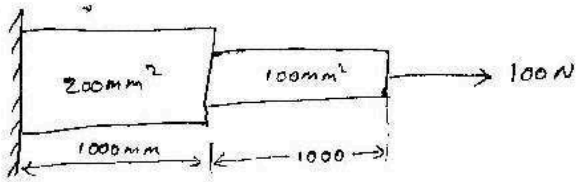
$(1Gpa = 10^9 \, N/m^2)$

(4)

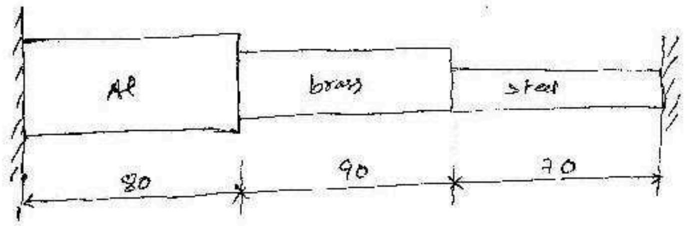


$E = 2 \times 10^5 \, N/mm^2$

(5)



$E = 2 \times 10^5 \, N/mm^2$

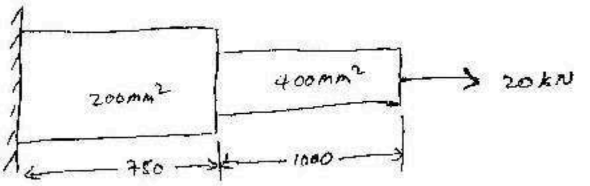


$E = 70Gpa$ | $E = 105Gpa$ | $E = 200Gpa$ | $\Delta T = 40^\circ C$

$A = 900mm^2$ | $A = 400mm^2$ | $A = 200mm^2$

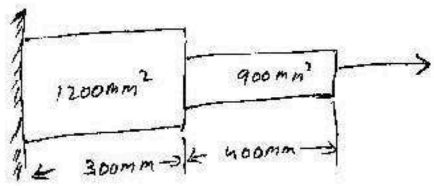
$\alpha = 23\times10^{-6}/^\circ C$ | $\alpha = 19\times10^{-6}/^\circ C$ | $\alpha = 12\times10^{-6}/^\circ C$

(7)



$E_1 = 220Gpa$ $E_2 = 150Gpa$

(8)



$E = 2\times10^5 N/mm^2$

$\rho = 720 kg/mm^3$

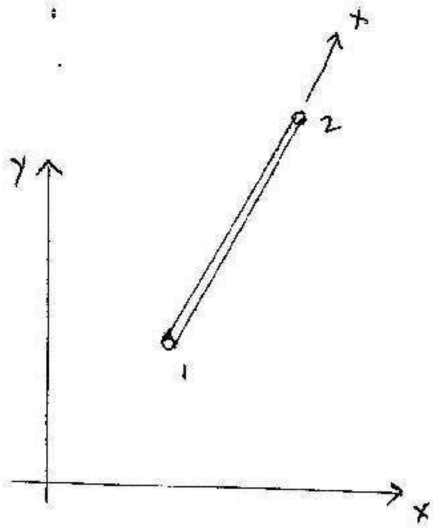
UNIT II

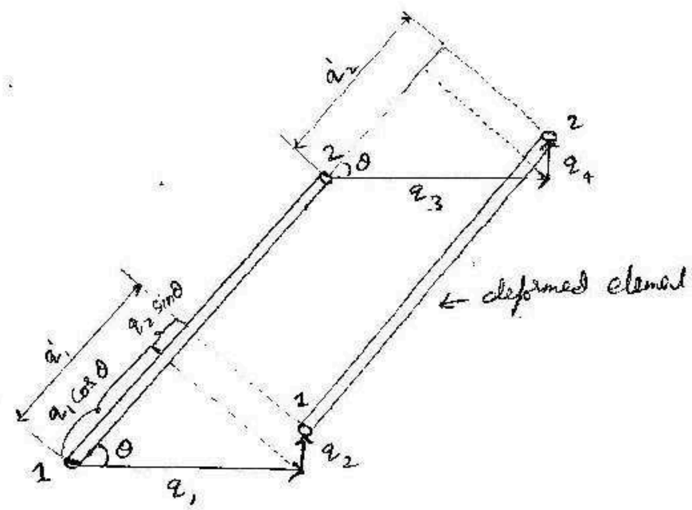
ANALYSIS OF TRUSSES AND BEAMS

TRUSSES

Analysis of trusses :-

Truss is an 2D-element it will displace both in x & y directions, Thus it has 2 DOF at each node, hence totally 1-truss element have 4 DOF as shown in figure below.





$$q_1' = q_1 \cos\theta + q_2 \sin\theta$$

$$q_2' = q_3 \cos\theta + q_4 \sin\theta$$

$$q = [q_1', q_2']^T$$

Introducing l & m as a direction cosines

i.e; $l = \cos\theta$ & $m = \sin\theta$.

$$q = \begin{bmatrix} q_1 \cos\theta + q_2 \sin\theta \\ q_3 \cos\theta + q_4 \sin\theta \end{bmatrix}$$

$$q = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

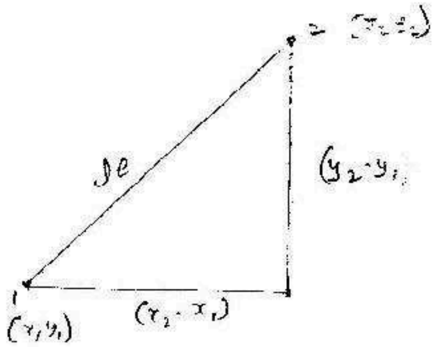
$$q = \underbrace{\begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}}_{q}$$

$$\boxed{q = L q'}$$

$$l = \cos\theta = \frac{x_2 - x_1}{l_e}$$

$$m = \sin\theta = \frac{y_2 - y_1}{l_e}$$

$$l_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Method - 1

Expression for elemental stiffness matrix of a truss element

strain energy.

$$U = \tfrac{1}{2} (q)^T k' q$$

$$= \frac{1}{2} [L q]^T \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} [L q]$$

$$= \frac{1}{2} q^T \underbrace{L^T \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} [L}\ q]$$

$$= \frac{1}{2} q^T k \ q$$

Here k for truss element is

$$k = L^T \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} L$$

$$k = \frac{AE}{l} \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

$$K = \frac{AE}{l}\begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

$$K = \frac{AE}{l}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$ — Stiffness matrix for truss element

The above k matrix is known as element stiffness matrix for a truss element.

Lecture Notes by S. Devaraj

STRESS matrix equation:-

$$\sigma = E\,\epsilon$$

$$= E\, B\, q^{e}$$

$$= E\,\frac{1}{l}\,[-1 \;\; 1]\; L\, q'$$

$$= \frac{E}{l}\,[-1 \;\; 1]\begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} q'$$

$$\boxed{\sigma = \frac{E}{l}\,[-l \;\; -m \;\; l \;\; m]\; q'}$$

method - 2

$$\epsilon = \frac{\partial u}{\partial x}$$

$$= \frac{q_2' - q_1'}{l}$$

$$= \frac{1}{l}\left((q_3 \cos\theta + q_4 \sin\theta) - (q_1 \cos\theta + q_2 \sin\theta) \right)$$

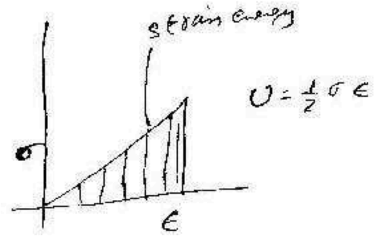
$$= \frac{1}{l}\begin{pmatrix} -\cos\theta & -\sin\theta & \cos\theta & \sin\theta \end{pmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

$$= \frac{1}{l}\begin{bmatrix} -l & -m & l & m \end{bmatrix} q$$

$$\epsilon = B\, q.$$

$$\boxed{B = \frac{1}{l_e}\begin{bmatrix} -l & -m & l & m \end{bmatrix}}$$

$$U = \frac{1}{2}\int \sigma^T \epsilon \, dv$$



$$= \frac{1}{2}\int (E\epsilon)^T \epsilon \, dv$$

$$= \frac{1}{2}\int E^T \epsilon^T \epsilon \, dv$$

$$= \frac{1}{2} E \int (Bq)^T (Bq)\, A\, dx$$

$$= \frac{AE}{2}\, q^T B^T B q\, l$$

$$= \frac{1}{2}\, q^T \underbrace{A E B^T B l}_{k}\, q$$

$$U = \frac{1}{2} q^T k q$$

$$K = A E B^T B l$$

$$K = AEB^TBl$$

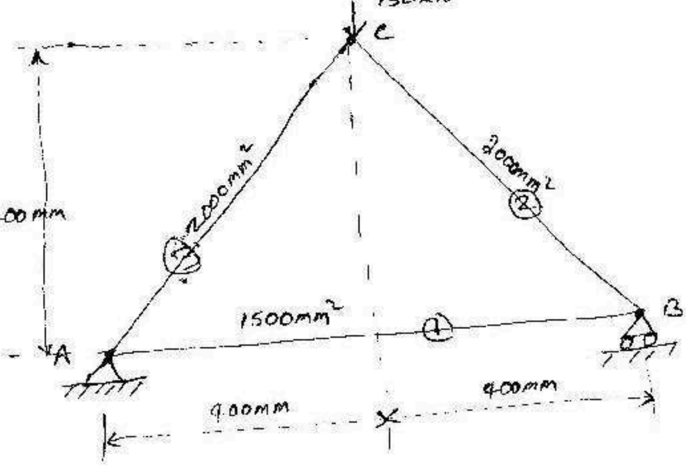
$$K = AEl \begin{bmatrix} -l \\ -m \\ l \\ m \end{bmatrix} \begin{bmatrix} -l & -m & l & m \end{bmatrix}$$

$$K = AEl \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$ — Stiffness matrix for truss (2D) element.

Hence the above K equation is the element stiffness matrix for a truss element.

<u>PLANE TRUSS PROBLEMS:</u>

(1) For the three-bar truss as shown in figure below. Determine the nodal displacements and the stress in each member. find the support reactions. Take modulus of elasticity as 200 GPa.



<u>Some times</u>

1 (0,0) 2 (800, 0) 3 (400, 400)

Sol: first we have to find the elemental lengths i.e;

$$l_{e1} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

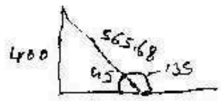
$$l_{e_1} = \sqrt{(800-0)^2+(0-0)^2} = 800mm$$

$$l_{e_2} = \sqrt{(400-800)^2+(400-0)^2} = 400\sqrt{2} \Rightarrow 565.68mm$$

$$l_{e_3} = \sqrt{(400-0)^2+(400-0)^2} = 400\sqrt{2} \Rightarrow 565.68mm$$

find the angles of each elements.

$\theta_1 = 0$ $\qquad$ $\theta_2 = 135$ $\qquad$ $\theta_3 = 45^\circ$



$$\theta_2 = \cos^{-1}\left(\frac{400}{565.68}\right)$$

$$\theta_2 = 135^\circ$$

$\boxed{l = \cos\theta}$ $\qquad$ $\boxed{m = \sin\theta}$

$l_1 = \cos(0) = 1$ $\qquad$ $l_2 = \cos(135) = -0.707$ $\qquad$ $l_3 = \cos(45) = 0.70$

$m_1 = \sin(0) = 0$ $\qquad$ $m_2 = \sin(135) = 0.707$ $\qquad$ $m_3 = \sin(45) = 0.70$

So know we get.

$\theta_1 = 0^\circ$ $\qquad$ $\theta_2 = 135^\circ$ $\qquad$ $\theta_3 = 45^\circ$

$l_1 = 1$ $\qquad$ $l_2 = -0.707$ $\qquad$ $l_3 = 0.707$

$m_1 = 0$ $\qquad$ $m_2 = 0.707$ $\qquad$ $m_3 = 0.707$

$A_1 = 1500mm^2$ $\qquad$ $A_2 = 2000mm^2$ $\qquad$ $A_3 = 2000mm^2$

$$E_1 = E_2 = E_3 = 200Gpa = 200kN/mm^2$$

$$K_1 = \frac{EA}{l}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

$$K_1 = \frac{200 \times 1500}{800}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_1 = \begin{array}{c c} \begin{matrix} 1 & \quad 2 & \quad 3 & \quad 4 \end{matrix} & \\ \begin{bmatrix} 375 & 0 & -375 & 0 \\ 0 & 0 & 0 & 0 \\ -375 & 0 & 375 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \end{array}$$

Lecture Notes by S. Devaraj

$$K_2 = \begin{array}{c c} \begin{matrix} 3 & \qquad 4 & \qquad 5 & \qquad 6 \end{matrix} & \\ \begin{bmatrix} 353.55 & -353.55 & -353.55 & 353.55 \\ -353.55 & 353.55 & 353.55 & -353.55 \\ -353.55 & 353.55 & 353.55 & -353.55 \\ 353.55 & -353.55 & -353.55 & 353.55 \end{bmatrix} & \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix} \end{array}$$

$$K_3 = \begin{array}{c c} \begin{matrix} 1 & \qquad 2 & \qquad 5 & \qquad 6 \end{matrix} & \\ \begin{bmatrix} 353.55 & 353.55 & -353.55 & -353.55 \\ 353.55 & 353.55 & -353.55 & -353.55 \\ -353.55 & -353.55 & 353.55 & 353.55 \\ -353.55 & -353.55 & 353.55 & 353.55 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 5 \\ 6 \end{matrix} \end{array}$$

$$\underset{\sim}{K} = \begin{bmatrix} 728.55 & 353.55 & -375.0 & 0 & -353.55 & -353.55 \\ 353.55 & 353.55 & 0 & 0 & -353.55 & -353.55 \\ -375.0 & 0 & 728.55 & -353.75 & -353.55 & 353.55 \\ 0 & 0 & -353.55 & 353.55 & 353.55 & -353.55 \\ -353.55 & -353.55 & -353.55 & 353.55 & 707.1 & 0 \\ -353.55 & -353.55 & 353.55 & -353.55 & 0 & 707.1 \end{bmatrix}_{6\times 6}$$

$$F_1 = F_2 = F_3 = F_4 = F_5 = 0 \quad \& \quad F_6 = -150$$

$$k\,Q = F$$

$$\begin{bmatrix} 728.55 & 353.55 & -375.0 & 0 & -353.55 & -353.55 \\ 353.55 & 353.55 & 0 & 0 & -353.55 & -353.55 \\ -375.0 & 0 & 728.55 & -353.75 & -353.55 & 353.55 \\ 0 & 0 & -353.55 & -353.55 & 353.55 & -353.55 \\ -353.55 & -353.55 & -353.55 & 353.55 & 707.1 & 0 \\ -353.55 & -353.55 & 353.55 & -353.55 & 0 & 707.1 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -150 \end{bmatrix}$$

Boundary conditions i.e; $Q_1 = Q_2 = Q_4 = 0$.

<u>By elimination approch.</u>

$$\begin{bmatrix} 728.55 & -353.75 & 353.55 \\ -353.55 & 707.1 & 0 \\ 353.55 & 0 & 707.1 \end{bmatrix} \begin{bmatrix} Q_3 \\ Q_5 \\ Q_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -150 \end{bmatrix}$$

By solving the above matrix equation weget

Q_1 Q_5 & Q_6 as below.

$\boxed{Q_3 = 0.2\,mm}$ $\boxed{Q_5 = 0.1\,mm}$ $\boxed{Q_6 = -0.312\,mm}$

St

$$\sigma_1 = \frac{E_1}{l_1}\begin{bmatrix}-l_1 & -m_1 & l_1 & m_1\end{bmatrix}\begin{bmatrix}Q_1\\Q_2\\Q_3\\Q_4\end{bmatrix}$$

$$= \frac{200}{800}\begin{bmatrix}-1 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}0\\0\\0.2\\0\end{bmatrix}$$

$$\sigma_1 = \begin{bmatrix}-0.25 & 0 & 0.25 & 0\end{bmatrix}\begin{bmatrix}0\\0\\0.2\\0\end{bmatrix}$$

$$\boxed{\sigma_1 = 0.05 \ kN/mm^2}$$

$$\sigma_2 = \frac{E_2}{l_2}\begin{bmatrix}-l_2 & -m_2 & l_2 & m_2\end{bmatrix}\begin{bmatrix}Q_3\\Q_4\\Q_5\\Q_6\end{bmatrix}$$

$$\sigma_2 = \frac{200}{565.68}\begin{bmatrix}+0.707 & -0.707 & -0.707 & 0.707\end{bmatrix}\begin{bmatrix}0.2\\0\\0.1\\-0.312\end{bmatrix}$$

$$\boxed{\sigma_2 = -0.053 \ kN/mm^2}$$

$$\sigma_3 = \frac{200}{565.68}\begin{bmatrix}-0.707 & -0.707 & 0.707 & 0.707\end{bmatrix}\begin{bmatrix}0.0\\0.0\\0.1\\-0.312\end{bmatrix}$$

$$\boxed{\sigma_3 = -0.053 \ kN/mm^2}$$

$P_1 = \sigma_1 A_1 \quad = 0.05 \times 1500 = 75 kN \qquad P_1 = 75 kN$

$P_2 = \sigma_2 A_2 \quad = -0.053 \times 2000 = -106 kN \qquad P_2 = -106 kN$

$P_3 = \sigma_3 A_3. \quad = -0.053 \times 2000 = -106 kN \qquad P_3 = -106 kN.$

Reaction support

$R_1 = KQ - F$

$R_1 + F = KQ.$

$R_1 + F = K_1 Q$

$$R_1 + 0 = \begin{bmatrix} 728.55 & 35355 & -375.0 & 0 & -353.55 & -353.55 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.2 \\ 0 \\ 0.1 \\ -0.312 \end{bmatrix}$$

$R_1 = 0 \qquad \boxed{R_1 = -0.0474}$

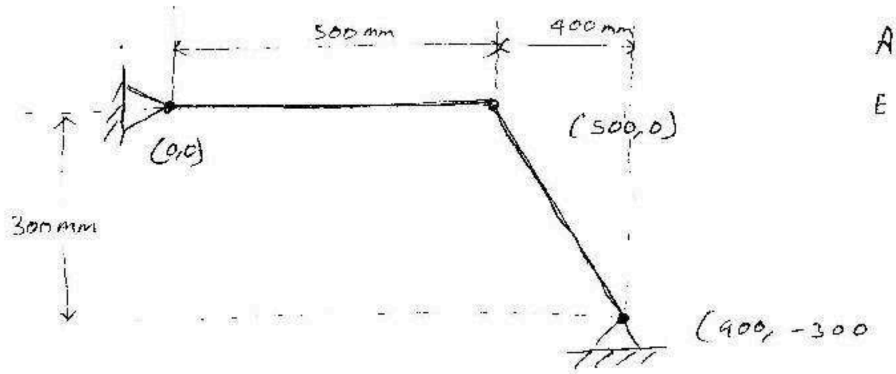
$$R_2 + 0 = \begin{bmatrix} 353.55 & 353.55 & 0 & 0 & -353.55 & -353.55 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.2 \\ 0 \\ 0.1 \\ -0.312 \end{bmatrix}$$

$R_2 = 75 kN. \qquad \boxed{R_1 = 74.95}$

$$R_3 + 0 = \begin{bmatrix} -375.0 & 0 & 728.55 & -353.75 & -353.55 & 353.55 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.2 \\ 0 \\ 0.1 \\ -0.312 \end{bmatrix}$$

$R_3 =$

find the stresses in each element of the truss as shown below.



$A = 200\,mm^2$

$E = 70 \times 10^3\,N/mm^2$

Lecture Notes
by
S. Durairaj

Soln

$$l_{e_1} = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

$$l_{e_1} = \sqrt{(500-0)^2 + (0-0)^2} \qquad l_{e_1} = 500$$

$$l_1 = \frac{x_2-x_1}{l_{e_1}} = \frac{500}{500} = \quad l_1 = 1$$

$$m_1 = \frac{y_2-y_1}{l_{e_1}} = \quad m_1 = 0$$

$$l_{e_2} = \sqrt{(900-500)^2 + (0+300)^2}$$

$$l_{e_2} = 500$$

$$l_2 = \frac{900-500}{500} = \quad l_2 = 4/5$$

$$m_2 = \frac{-300}{500} = \quad m_2 = -3/5$$

$$k_1 = \frac{AE}{l_{e_1}} \begin{bmatrix} l^2 & ml & -l^2 & -ml \\ ml & m^2 & -ml & -m^2 \\ -l^2 & -ml & l^2 & ml \\ -ml & -m^2 & +ml & m^2 \end{bmatrix}$$

$$k_1 = \frac{200 \times 70 \times 10^3}{500} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$k_1 = 28 \times 10^3 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$k_2 = 28 \times 10^3 \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix} \end{array} \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$\underset{\sim}{K} = 28 \times 10^3 \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1.64 & -0.48 & -0.64 & 0.48 \\ 0 & 0 & -0.48 & 0.48 & 0.48 & -0.36 \\ 0 & 0 & -0.64 & 0.48 & 0.64 & -0.48 \\ 0 & 0 & 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

By elimination Approch node 1 & node 3 were fixed.

$$K = 28 \cdot \times 10^3 \begin{bmatrix} 1.64 & -0.48 \\ -0.48 & 0.48 \end{bmatrix} \begin{bmatrix} q_3 \\ q_4 \end{bmatrix} = F \begin{bmatrix} 0 \\ -10 \times 10^3 \end{bmatrix}$$

$$q_3 = -0.307 \text{ mm}$$

$$q_4 = -1.051 \text{ mm}$$

$$\sigma_1 = \frac{70 \times 10^3}{500} \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.307 \\ -1.051 \end{bmatrix}$$

$$\sigma_1 = \frac{70 \times 10^3}{500} \begin{bmatrix} -0.307 \end{bmatrix}$$

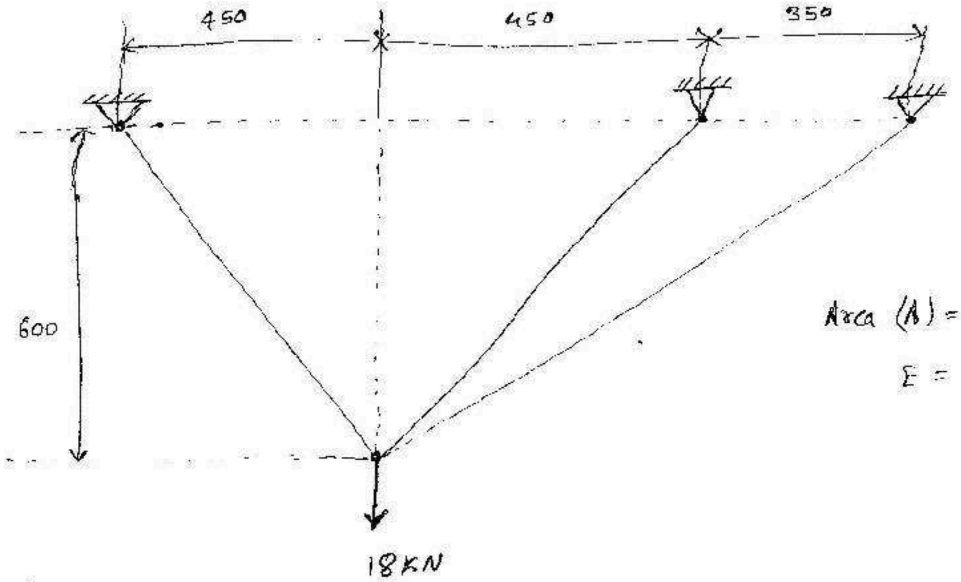
$$\sigma_1 = -42.98 \text{ N/mm}^2$$

$$\sigma_2 = \frac{70 \times 10^3}{500} \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} & \frac{4}{5} & -\frac{3}{5} \end{bmatrix} \begin{bmatrix} -0.307 \\ -1.051 \\ 0 \\ 0 \end{bmatrix}$$

$$\sigma_2 = \begin{bmatrix} \frac{4}{5} \times 0.307 & -\frac{3}{5} \times 1.051 \end{bmatrix} \times \frac{70 \times 10^3}{500}$$

$$\sigma_2 = -53.9 \text{ N/mm}^2$$

→ For the three - bar truss shown in figure. determine the
at node 1 and the stress in element 3.

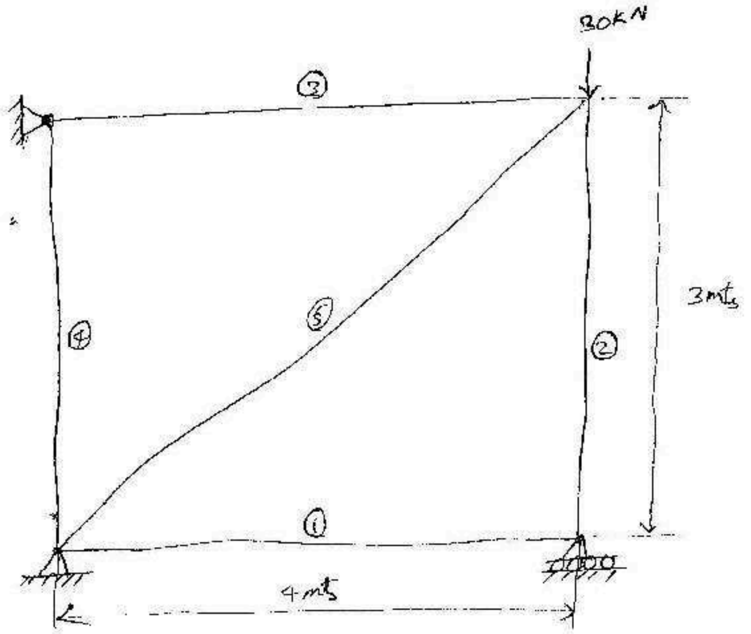


Area (A) = $250 mm^2$

E = 200Gpa

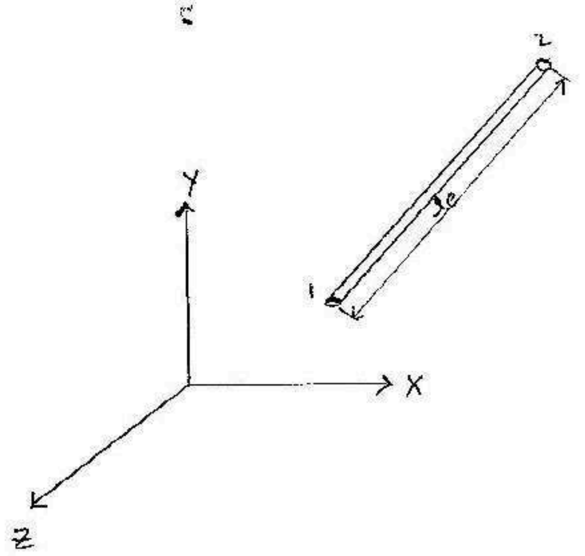
→ Determine the stresses in the members of the truss shown in figure

Take E = 200Gpa A = 2000



Stiffness matrix derivation for a 3D truss element.

Lecture Notes
by
S. Deivaraj



$$q' = \begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix} \qquad q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}$$

$$\boxed{l = \frac{x_2 - x_1}{l_e} \qquad m = \frac{y_2 - y_1}{l_e} \qquad \& \qquad n = \frac{z_2 - z_1}{l_e}}$$

length of a member $[l_e]$

$$\boxed{l_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

$$q_1' = l q_1 + m q_2 + n q_3$$

$$q_2' = l q_4 + m q_5 + n q_6$$

$$q = \begin{bmatrix} q_1' \\ q_2' \end{bmatrix} = \underbrace{\begin{bmatrix} l & m & n & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & n \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}}_{q'}$$

$$q = L q'$$

$$L = \begin{bmatrix} l & m & n & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & n \end{bmatrix}$$

As we that

$$U = \frac{1}{2} q^T k q$$

$$= \frac{1}{2} \left[L^T q^T\right] k \left[q L\right]$$

$$= \frac{1}{2} q^T \underbrace{L^T k L}_{K} q$$

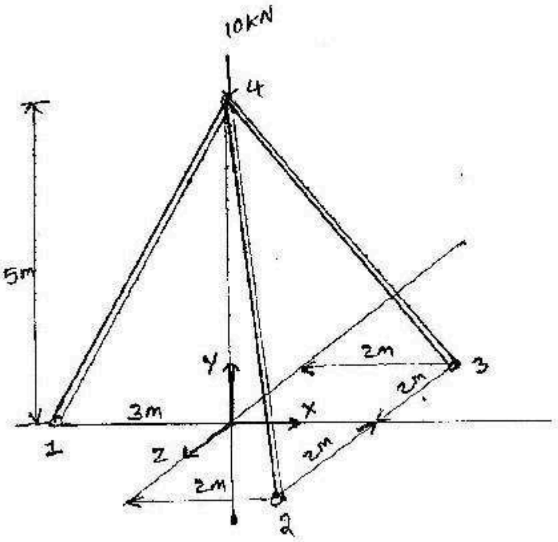
$$K = L^T k L$$

$$= \begin{bmatrix} l & 0 \\ m & 0 \\ n & 0 \\ 0 & l \\ 0 & m \\ 0 & n \end{bmatrix} \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} l & m & n & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & n \end{bmatrix}$$

$$K = \frac{AE}{l} \begin{bmatrix} l & 0 \\ m & 0 \\ n & 0 \\ 0 & l \\ 0 & m \\ 0 & n \end{bmatrix} \begin{bmatrix} l & m & n & -l & -m & -n \\ -l & -m & -n & l & m & n \end{bmatrix}$$

$$K = \frac{AE}{l} \begin{bmatrix} l^2 & lm & ln & -l^2 & -lm & -ln \\ lm & m^2 & mn & -lm & -m^2 & -mn \\ ln & mn & n^2 & -ln & -mn & -n^2 \\ -l^2 & -lm & -ln & l^2 & lm & ln \\ -lm & -m^2 & -mn & lm & m^2 & mn \\ -ln & -mn & -n^2 & ln & mn & n^2 \end{bmatrix}$$ Stiffness matrix for a 3D-truss.

The tripod as shown in figure carries a vertically downward load of 10 kN at joint 4. If young's modulus of the material of tripod stand is 200 kN/mm^2 & the cross sectional area of each leg is 2000 mm^2, determine the forces developed in the legs of the tripod.



Node : 1

$(-3, 0\ 0)$

Node : 2

$(2, 0, 2)$

Node : 3

$(2, 0, -2)$

Node : 4

$(0, 5, 0)$

elemental length

$$l_e = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

Directional cosines are

$$l = \frac{x_2-x_1}{l_e}, \quad m = \frac{y_2-y_1}{l_e} \quad \& \quad n = \frac{z_2-z_1}{l_e}$$

$$l_{e_1} = \sqrt{(0-(-3))^2+(5-0)^2+(0-0)^2} \qquad l_{e_1} = 5831 \text{ mm}$$

$$l_{e_2} = \sqrt{(0-2)^2+(5-0)^2+(0-2)^2} \qquad l_{e_2} = 5744 \text{ mm}$$

$$l_{e_3} = \sqrt{(0-2)^2+(5-0)^2+(0-(-2))^2} \qquad l_{e_3} = 5744 \text{ mm}$$

$$l_1 = \frac{x_2-x_1}{l_{e_1}} = \frac{0-3000}{5831} = l_1 = 0.514 \checkmark$$

$$m_1 = \frac{y_2-y_1}{l_{e_1}} = \frac{5000-0}{5831} = m_1 = -0.348 \checkmark$$

$$n_1 = \frac{z_2-z_1}{l_{e_1}} = \frac{0-0}{5831} = n_1 = -0.348$$

$$l_2 = \frac{x_2-x_1}{l_{e_2}} = \frac{0-2000}{5744} = l_2 = -0.348 \text{ mm}$$

$$m_2 = \frac{y_2-y_1}{l_{e_2}} = \frac{5000-0}{5744} = m_2 = 0.870 \text{ mm}$$

$$n_1 = \frac{z_2-z_1}{l_e} = \frac{0-2000}{5744} = n_2 = -0.348 \text{ mm}$$

$$l_3 = \frac{x_2-x_1}{l_{e_3}} = \frac{0-2000}{5744} = l_3 = -0.348$$

Ans

$$m_3 = \frac{y_2-y_1}{l_{e_3}} = \frac{5000-0}{5744} = m_3 = 0.870$$

$$n_3 = \frac{z_2-z_1}{l_{e_3}} = \frac{0-(-2000)}{5744} = n_3 = +0.348 \checkmark$$

$$K_1 = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 10 & 11 & 12 \end{array} \\ \begin{bmatrix} 18.12 & 30.18 & 0 & -18.12 & -30.18 & 0 \\ 30.18 & 50.35 & 0 & -30.18 & -50.35 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -18.11 & -30.184 & 0 & 18.11 & 30.18 & 0 \\ -30.18 & -50.35 & 0 & 30.18 & 50.35 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 10 \\ 11 \\ 12 \end{array}$$

$$K_2 = \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 10 & 11 & 12 \end{array} \\ \begin{bmatrix} 8.42 & -21.09 & 8.42 & -8.42 & 21.09 & -8.42 \\ -21.09 & 52.70 & -21.03 & 21.09 & -52.70 & 21.09 \\ 8.42 & -21.09 & 8.42 & -8.42 & 21.09 & -8.42 \\ -8.42 & 21.09 & -8.42 & 8.42 & -21.09 & 8.42 \\ 21.09 & -52.70 & 21.09 & -21.09 & 52.70 & -21.09 \\ -8.42 & 21.09 & -8.42 & 8.42 & -21.09 & 8.42 \end{bmatrix} \end{array} \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 10 \\ 11 \\ 12 \end{array}$$

$$K_3 = \begin{array}{c} \begin{array}{cccccc} 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\ \begin{bmatrix} 8.42 & -21.09 & -8.42 & -8.42 & 21.09 & 8.42 \\ -21.09 & 52.70 & 21.09 & 21.09 & -52.70 & -21.09 \\ -8.42 & 21.09 & 8.42 & 8.42 & -21.09 & -8.42 \\ -8.42 & 21.09 & 8.42 & 8.42 & -21.09 & -8.42 \\ 21.09 & -52.70 & -21.09 & -21.09 & 52.70 & 21.09 \\ 8.24 & -21.09 & -8.42 & -8.42 & 21.09 & 8.42 \end{bmatrix} \end{array} \begin{array}{c} \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array}$$

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 0 \end{bmatrix}$$

Here by eliminating Approach

$$q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8 \,\&\, q_9 = 0$$

$$K = \begin{bmatrix} 34.96 & -12.04 & 0 \\ -12.04 & 155.76 & 0 \\ 0 & 0 & 16.95 \end{bmatrix} \begin{bmatrix} q_{10} \\ q_{11} \\ q_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ -10 \\ 0 \end{bmatrix}$$

$$q_{10} = -0.0227$$

$$q_{11} = -0.0659$$

$$q_{12} = 0.$$

By solving the equlibrium equations we get above four unknowns.

~~Answer~~ now we are findg the ~~forces~~ stresses in each element.

$$\sigma_1 = \frac{AE}{l} \begin{bmatrix} -l & -m & -n & l & m & n \end{bmatrix} q'$$

$$= \frac{200 \times 2000}{5831} \begin{bmatrix} -0.514 & -0.857 & 0 & 0.514 & 0.857 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.0227 \\ -0.0659 \\ 0 \end{bmatrix}$$

$$\boxed{\sigma_1 = -4.680 \text{ kN/mm}^2}$$

$$\sigma_2 = \frac{200 \times 2000}{5745} \begin{bmatrix} +0.348 & -0.870 & +0.348 & -0.348 & 0.870 & -0.348 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.0227 \\ -0.0659 \\ 0 \end{bmatrix}$$

$$\boxed{\sigma_2 = -3.445 \text{ kN/mm}^2}$$

$$\sigma_3 = \frac{200 \times 2000}{5745} \begin{bmatrix} +0.348 & -0.870 & +0.348 & -0.348 & 0.870 & -0.348 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.02223 \\ -0.0659 \\ 0 \end{bmatrix}$$

$$\boxed{\sigma_3 = -3.445 \ kN/mm^2}$$

TEMPERATURE EFFECTS:

A truss element is simply a one-dimensional element when viewed in the local coordinate system. The element temperature load in the local coordinate system is given by :- Lecture Notes by S. Devaraj

$$F = E A \epsilon_0 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

where ϵ_0 is the initial strain associated with a temperature change is given by.

$$\epsilon_0 = \alpha \Delta T$$

where α is the coefficient of thermal expansion & ΔT is the change in temperature in the element. then the equation becomes as

$$F = E A \alpha \Delta T \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

then in 2D truss problems then equation changes as below.

$$\boxed{F = E A \alpha \Delta T \begin{bmatrix} -l \\ -m \\ l \\ m \end{bmatrix}}$$

stress in the element equation changes as.

$$\sigma = E \epsilon$$

$$\sigma = E(\epsilon - \epsilon_0)$$

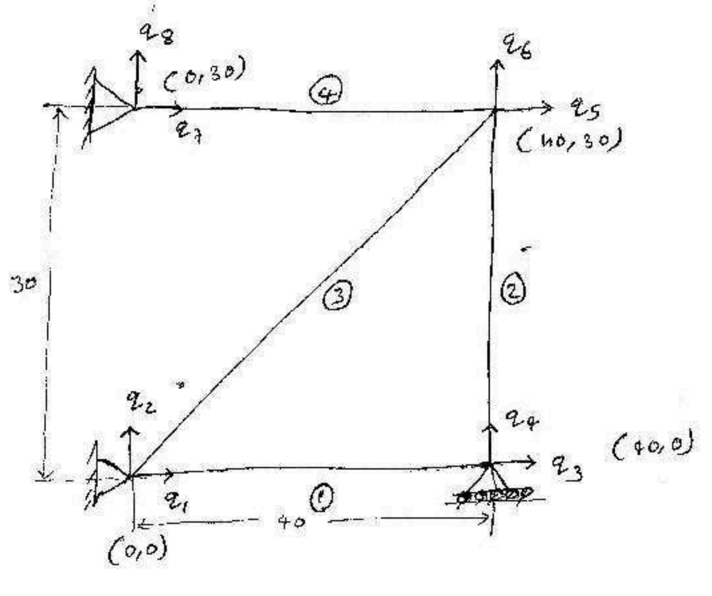
$$\left[\because \epsilon_0 = \alpha \Delta T\right]$$

$$\boxed{\sigma = \frac{E}{l}\begin{bmatrix} -l & -m & l & m \end{bmatrix} q - E\alpha\Delta T}$$

9/2

TEMPERATURE EFFECT PROBLEM :

④ Find the stresses due to temperature effect in the truss given below problem



$E = 29.5 \times 10^6$ psi

$A = 1$ inch^2

$$le_1 = \sqrt{(40-0)^2 + (0-0)^2} = 40$$

$L_1 = 1 \qquad m_1 = 0.$

$$le_2 = \sqrt{(40-40)^2 + (30-0)^2} = 30$$

$L_2 = 0 \qquad m_2 = 1$

$$le_3 = \sqrt{(0-40)^2 + (0-30)^2} = 50$$

$L_3 = -0.8 \qquad m_3 = -0.6$

$$le_4 = \sqrt{(0-40)^2 + (30-30)^2} = 40$$

$L_4 = -1 \qquad m_4 = 0$

$$k_1 = \frac{1 \times 29.5 \times 10^6}{40} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$k_2 = \frac{1 \times 29.5 \times 10^6}{30} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$k_3 = \frac{1 \times 29.5 \times 10^6}{50} \begin{array}{c} \begin{array}{cccc} 5 & 6 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix} \end{array} \begin{array}{c} \\ 5 \\ 6 \\ 1 \\ 2 \end{array}$$

$$K_4 = 10^3 \begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 737.5 & 0 & -737.0 & 0 \\ 0 & 0 & 0 & 0 \\ -737.5 & 0 & 737.5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

$$\underset{\sim}{K} = 10^3 \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1115.1 & 28.32 & -737.5 & 0 & -377.6 & -28.32 & 0 & 0 \\ 283.2 & 212.4 & 0 & 0 & -283.2 & 212.4 & 0 & 0 \\ 737.5 & 0 & 737.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 983.3 & 0 & -983.3 & 0 & 0 \\ -377.6 & -283.2 & 0 & 0 & 1115.1 & 283.2 & -737.5 & 0 \\ -283.2 & -212.4 & 0 & 0 & 283.2 & 1195.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & -737.5 & 0 & 737.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

By elimination Approch.

$$\underset{\sim}{K} = 10^3 \begin{bmatrix} 737.5 & 0 & 0 \\ 0 & 1115.7 & 283.2 \\ 0 & 283.2 & 1195.7 \end{bmatrix} \begin{bmatrix} q_3 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 25 \end{bmatrix} \times 10^3$$

$$\underset{\sim}{K} = \begin{bmatrix} 737.5 & 0 & 0 \\ 0 & 1115.7 & 283.2 \\ 0 & 283.2 & 1195.7 \end{bmatrix} \begin{bmatrix} q_3 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 25 \end{bmatrix}$$

$$737.5\, q_3 = 0$$

$$1115.7\, q_5 + 283.2\, q_6 = 0$$

$$283.2\, q_5 + 1195.7\, q_6 = 25$$

By solving above 3 equations we get 3 unknown's i.e;

$q_3 = 0 \qquad q_5 = -5.64 \times 10^{-3}$ in $\quad$ & $\quad q_6 = 0.0222$ in.

effects. The above problem there is increase in temperature of in ② & 3 elements there are no other loads on structure. Determine nodal displacements due to temperature effects & elemental stresses. Take co-efficient of thermal expansion of elements is $\alpha = \frac{1}{150000} /°F$

Lecture Notes by S. Devaraj

$$F_2 = EA\alpha\Delta T \begin{bmatrix} -l_2 \\ -m_2 \\ l_2 \\ m_2 \end{bmatrix}$$

$$= 29.5\times10^9 \times 1 \times 6.66\times10^{-6} \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$F_2 = \begin{array}{c} \begin{bmatrix} 0 \\ -196.47 \\ 0 \\ 196.47 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$F_3 = 196.47 \begin{bmatrix} 0.8 \\ 0.6 \\ -0.8 \\ -0.6 \end{bmatrix}$$

$$F_3 = \begin{bmatrix} 157.17 \\ 117.8 \\ -157.17 \\ -117.8 \end{bmatrix} \begin{array}{c} 5 \\ 6 \\ 1 \\ 2 \end{array}$$

$$\underset{\sim}{F} = \begin{bmatrix} -157.17 \\ -117.8 \\ 0 \\ -196.47 \\ 157.17+0 \\ 196.47+117.8 \\ 0 \\ 0 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

By elimination Approch.

$$K\,Q = F$$

$$\begin{bmatrix} 737.5 & 0 & 0 \\ 0 & 1115.7 & 283.2 \\ 0 & 283.2 & 1195.7 \end{bmatrix} \begin{bmatrix} q_3 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 157.17 \\ 314.27 \end{bmatrix}$$

$$737.5\, q_3 = 0$$

$$1115.7\, q_5 + 283.2\, q_6 = 157.17$$

$$283.2\, q_5 + 1195.7\, q_6 = 314.27$$

By solving 3 equations we get 3 unknown's

$$q_3 = 0$$

$$q_5 = 0.078\,mm$$

$$q_6 = 0.244\,mm$$

stresses:

$$\sigma_1 = \frac{29.5\times10^6}{40}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sigma_1 = 0\ psi/in^2$$

$$\sigma_2 = \frac{29.5\times10^6}{30}\begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0.078 \\ 0.244 \end{bmatrix}$$

$$= 983.3\times10^3\,[1\times0.244]$$

$$\sigma_2 = 239.925\times10^3\ psi/in^2$$

$$\sigma_3 = \frac{29.5 \times 10^6}{56} \begin{bmatrix} 0.8 & 0.6 & -0.8 & -0.6 \end{bmatrix} \begin{bmatrix} 0.078 \\ 0.24 \\ 0 \\ 0 \end{bmatrix}$$

$$\sigma_3 = 121.77 \times 10^3 \ psi/in^2$$

$$\sigma_4 = \frac{29.5 \times 10^6}{40} \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0.078 \\ 0.24 \\ 0 \\ 0 \end{bmatrix}$$

$$\sigma_4 = 983.3 \times 10^6 \times 0.078$$

$$\sigma_4 = 76.69 \times 10^3 \ psi/in^2$$

$\sigma_1 = 0$ $\quad\quad$ $\sigma_2 = 239.92 \times 10^3 \ psi/in^2$ $\quad\quad$ $\sigma_3 = 121.77 \times 10^3 \ psi/in^2$ $\quad\quad$ $\sigma_4 = 76.69 \times 10^3 \ psi/in^2$

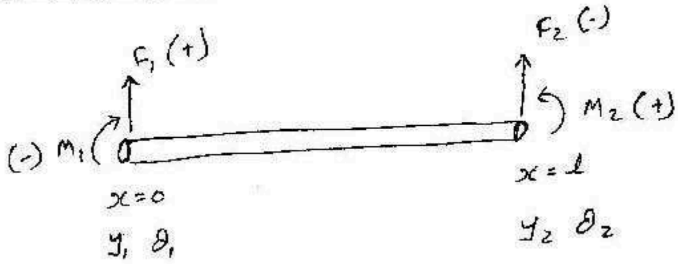
Analysis of beams:

Expression for element stiffness matrix for a beam element:

→ Consider a beam element as shown in figure length L.

The governing differential equation for a beam is

$$\boxed{EI \frac{d^4y}{dx^4} = 0.}$$



$$EI \frac{d^3y}{dx^3} = c_1 \qquad \text{Shear force}$$

$$EI \frac{d^2y}{dx^2} = c_1x + c_2 \qquad \text{Bending moment}$$

$$EI \frac{dy}{dx} = \frac{c_1x^2}{2} + c_2x + c_3 \qquad \text{slop } (\theta) = dy/dx$$

$$EIy = \frac{c_1x^3}{6} + \frac{c_2x^2}{2} + c_3x + c_4 \qquad \text{deflection.}$$

By applying Boundary conditions..

$x = 0 \qquad y = y_1 \qquad \theta = \theta_1$

$x = l \qquad y = y_2 \qquad \theta = \theta_2$

$EIy_1 = c_4$

$EI\theta_1 = c_3$

$$EIy_2 = \frac{c_1 l^3}{6} + \frac{c_2 l^2}{2} + c_3 l + c_4$$

$$EIy_2 = \frac{c_1 l^3}{6} + \frac{c_2 l^2}{2} + EI\theta_1 l + EIy_1 \quad \text{--- (1)}$$

$$EI\theta_2 = \frac{c_1 l^2}{2} + c_2 l + EI\theta_1 \quad \text{--- (2)}$$

By solving the above (1) & (2) equations we get.

$$c_1 = \frac{6EI}{l^2}(\theta_1 + \theta_2) + \frac{12EI}{l^3}(y_1 - y_2)$$

$$c_2 = -\frac{2EI}{l}(2\theta_1 + \theta_2) - \frac{6EI}{l^2}(y_1 - y_2)$$

Hence the shear force F & Bending moment M at node is given by.

$$F_1 = c_1 = \frac{6EI}{l^2}(\theta_1 + \theta_2) + \frac{12EI}{l^3}(y_1 - y_2)$$

$$= \frac{12EI}{l^3}y_1 + \frac{6EI}{l^2}\theta_1 - \frac{12EI}{l^3}y_2 + \frac{6EI}{l^2}\theta_2$$

$$M_1 = -C_2 = \frac{6EI}{l^2}(y_1 - y_2) + \frac{2EI}{l^2}(2\theta_1 + \theta_2)$$

$$= \frac{6EI}{l^2} y_1 + \frac{4EI}{l}\theta_1 - \frac{6EI}{l^2} y_2 + \frac{2EI}{l}\theta_2$$

$$F_2 = -C_1 = -\frac{12EI}{l^3}(y_1 - y_2) - \frac{6EI}{l^2}(\theta_1 + \theta_2)$$

$$= \frac{-12EI}{l^3} y_1 - \frac{6EI}{l^2}\theta_1 + \frac{12EI}{l^3} y_2 - \frac{6EI}{l^2}\theta_2$$

$$M_2 = (C_1 L + C_2) = \frac{6EI}{l^2} y_1 + \frac{2EI}{l}\theta_1 - \frac{6EI}{l^2} y_2 + \frac{4EI}{l}\theta_2$$

Then these all F_1 F_2 M_1 & M_2 equations can be written in matrix form.

$$\begin{bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{bmatrix} y_1 \\ \theta_1 \\ y_2 \\ \theta_2 \end{bmatrix}$$

$$K = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$ — Stiffness matrix for beam element.

Solving the equations ① & ② to get C_1 & C_2 values.

$$EIy_2 = \frac{C_1 l^3}{6} + \frac{C_2 l^2}{2} + EI\theta_1 l + EIy_1 \quad \text{— eq ①}$$

$$EI\theta_2 = \frac{C_1 l^2}{2} + C_2 l + EI\theta_1 \quad \text{— eq ②}$$

$$EIy_2 = \frac{C_1}{6} l^3 + \cancel{\frac{C_2}{2} l^2} + EI\theta_1 l + EIy_1$$

$$-\frac{l}{2} \times EI\theta_2 = -\frac{C_1}{4} l^3 \oplus \cancel{\frac{C_2 l^2}{2}} \oplus \frac{EI\theta_1 l}{2}$$

$$\left(EIy_2 - EI\theta_2 \frac{l}{2}\right) = \left(\frac{C_1}{6} l^3 - \frac{C_1}{4} l^3\right] + EI\theta_1 l - \frac{EI\theta_1}{2} l + EIy_1$$

$$EIy_2 = EI\theta_2 \, l/2 + EI\theta_1 \, l/2 + EIy_1 + \frac{2C_1 l^3 - 3C_1 l^3}{12}$$

$$EIy_2 = EI \, l/2 \, (\theta_1 + \theta_2) + EIy_1 - \frac{C_1 l^3}{12}.$$

$$\frac{C_1 l^3}{12} = EI \frac{l}{2} (\theta_1 + \theta_2) + EI(y_1 - y_2)$$

$$C_1 = \frac{6EI}{l^2} (\theta_1 + \theta_2) + \frac{12EI}{l^3} (y_1 - y_2)$$

Substituting C_1 value in eq ② we get C_2

$$EI\theta_2 = \frac{l}{2} \left[\frac{6EI}{l^2} (\theta_1 + \theta_2) + \frac{12EI}{l^3} (y_1 - y_2)\right] + C_2 l + EI\theta_1$$

$$EI\theta_2 = 3EI(\theta_1 + \theta_2) + \frac{6EI}{l} (y_1 - y_2) + C_2 l + EI\theta_1$$

$$C_2 l_2 = EI(\theta_2 - \theta_1) - 3EI(\theta_1 + \theta_2) - \frac{6EI}{l}(y_1 - y_2)$$

$$= EI\theta_2 - EI\theta_1 - 3EI\theta_1 - 3EI\theta_2 - \frac{6EI}{l}(y_1 + y_2)$$

$$C_2 l = -4EI\theta_1 - 2EI\theta_2 - \frac{6EI}{l}(y_1 - y_2)$$

$$C_2 = -\frac{2EI}{l}(2\theta_1 + \theta_2) - \frac{6EI}{l^2}(y_1 - y_2)$$

$\therefore$ we got the C_1 & C_2 values these values are important for finding the stiffness matrix for beam element.

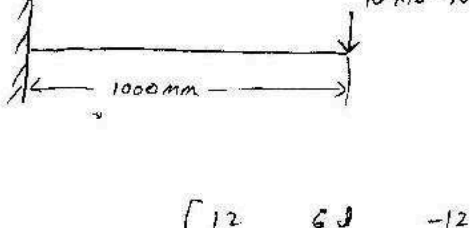
(1). A cantilever beam of length 1m carries a point load 10kN at its free end. Determine the deflection at the end of the beam & also determine S.F & B.M's Take E = 70kPa A = 500mm^2

I = 2500 mm^4 length l = 1000mm

Soln

E = 70kPa A = 500mm^2 L = 1000mm

= 70×10^3 N/mm^2 I = 2500mm^4



$$k = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

$$\frac{EI}{l^3} = \frac{70\times10^3 \times 2500}{1000^3} = 0.175$$

$$K = 0.175 \begin{bmatrix} 12 & 6\times1000 & -12 & 6\times1000 \\ 6\times1000 & 4\times1000^2 & -6\times1000 & 2\times1000^2 \\ -12 & -6\times1000 & 12 & -6\times1000 \\ 6\times1000 & 2\times1000^2 & -6\times1000 & 4\times1000^2 \end{bmatrix}$$

$$K = 0.175 \begin{bmatrix} 12 & 6\times10^3 & -12 & 6\times10^3 \\ 6\times10^3 & 4\times10^6 & -6\times10^3 & 2\times10^6 \\ -12 & -6\times10^3 & 12 & -6\times10^3 \\ 6\times10^3 & 2\times10^6 & -6\times10^3 & 4\times10^6 \end{bmatrix}$$

<u>By elimination Approach.</u>

$$K = 0.175 \begin{bmatrix} 12 & 6\times10^3 & -12 & 6\times10^3 \\ 6\times10^3 & 4\times10^6 & -6\times10^3 & 2\times10^6 \\ -12 & -6\times10^3 & 12 & -6\times10^3 \\ 6\times10^3 & 2\times10^6 & -6\times10^3 & 4\times10^6 \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \\ y_2 \\ \theta_2 \end{bmatrix}$$

$$K = 0.175 \begin{bmatrix} 12 & -6\times10^3 \\ -6\times10^3 & 4\times10^6 \end{bmatrix} \begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} -10\times10^3 \\ 0 \end{bmatrix}$$

By solving the above equations we get y_2 & θ_2

$$y_2 = -19047.61 \text{ mm}$$

$$\theta_2 = -28.571^\circ$$

ermining the S.F & B.M

$$\begin{bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \\ y_2 \\ \theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2.1 & 1.05\times10^3 & -2.1 & 1.05\times10^3 \\ 1.05\times10^3 & 0.7\times10^6 & -1.05\times10^3 & 0.35\times10^6 \\ -2.1 & -1.05\times10^3 & 2.1 & -1.05\times10^3 \\ 1.05\times10^3 & 0.35\times10^6 & -1.05\times10^3 & 0.7\times10^6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -19047.6 \\ -28.571 \end{bmatrix}$$

$$\begin{bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} 10.0\times10^3 \\ 10.0\times10^6 \\ -10.0\times10^3 \\ 0.28\times10^3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -10\times10^3 \\ 0 \end{bmatrix}$$

$F_1 = 10.0\times10^3$

$F_2 = -10.0\times10^3 - (-10\times10^3) = 0$

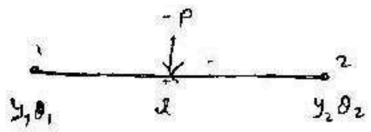
$M_1 = 10.0\times10^6$

$M_2 = 0.28\times10^3$

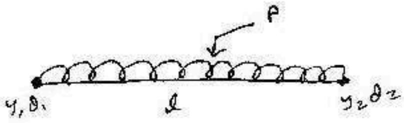
Load vector

① 

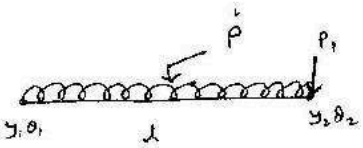
$$= \quad F = \begin{bmatrix} 0 \\ 0 \\ -P \\ 0 \end{bmatrix}$$

② 

$$= \quad F = \begin{bmatrix} -P/2 \\ -Pl/8 \\ -P/2 \\ Pl/8 \end{bmatrix}$$

③ 

$$= \quad F = \begin{bmatrix} -Pl/2 \\ -Pl^2/12 \\ -Pl/2 \\ Pl^2/12 \end{bmatrix}$$

④ 

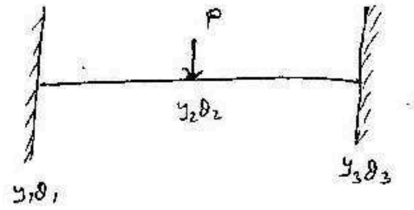
$$= \quad F = \begin{bmatrix} -Pl/2 \\ -Pl^2/12 \\ -Pl/2 - P_1 \\ Pl^2/12 \end{bmatrix}$$

ary Conditions:

Lecture Notes
by
S. Devaraj

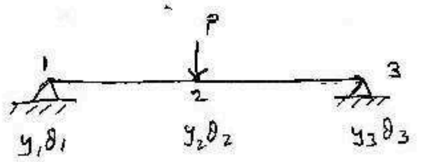
BC's

$y_1 = 0 \quad \theta_1 = 0$



$y_1 = 0 \quad \theta_1 = 0$

$y_3 = 0 \quad \theta_3 = 0$



$y_1 = 0 \quad y_3 = 0$

$\theta_2 = 0$

$\therefore \theta_1 = -\theta_3$

HERMITE SHAPE FUNCTIONS:

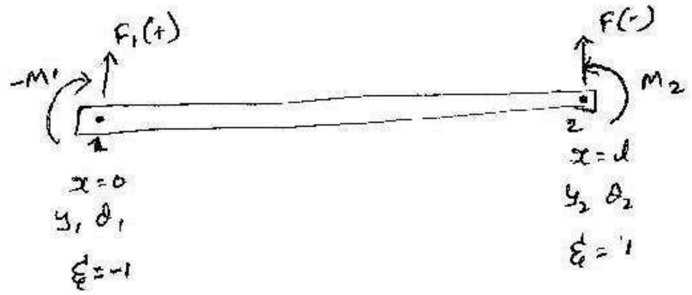
The shape function is used to describe the analysis of beam is known as HERMITE shape function is denoted by (H).

For finding the HERMITE shape function we have to consider the hermitian polynomial which is derived from the lagrangian equation.

hermitian polynomial

$$\boxed{H = a + b\xi + c\xi^2 + d\xi^3}$$

For finding the hermitian shape functions take a beam with 2 nodal points as shown in below figure.



from the properties of shape functions the boundary conditions are given in the following table.

$$\left[\begin{array}{c} \\ \xi=-1 \\ \xi=+1 \end{array}\ \begin{bmatrix} H_1 & \frac{dH_1}{d\xi} \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} H_2 & \frac{dH_2}{d\xi} \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} H_3 & \frac{dH_3}{d\xi} \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} H_4 & \frac{dH_4}{d\xi} \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\right]$$

$$H = a + b\xi + c\xi^2 + d\xi^3$$

$$\frac{dH_1}{d\xi} = b + 2c\xi + 3d\xi^2$$

$H_1 = 1$ at $\xi = -1$

$$1 = a + b(-1) + c(-1)^2 + d(-1)^3$$

$$1 = a - b + c - d \quad \text{———} \quad (1)$$

$H_1 = 0$ at $\xi = +1$

$$0 = a + b + c + d \quad \text{———} \quad (2)$$

$\frac{dH_1}{d\xi} = 0$ at $\xi = -1$

$$0 = b - 2c + 3d \quad \text{———} \quad (3)$$

$\frac{dH_1}{d\xi} = 0$ at $\xi = 1$

$$0 = b + 2c + 3d \quad \text{———} \quad (4)$$

know wegot four equations for I^{st} B.C's

i.e;

$1 = a - b + c - d$ — ①

$0 = a + b + c + d$ — ②

$0 = b - 2c + 3d$ — ③

$0 = b + 2c + 3d$ — ④

from these four equations we can find four unknown's

By solving ① & ②

$1 = a - \cancel{b} + c - \cancel{d}$

$0 = a + \cancel{b} + c + \cancel{d}$

$2a + 2c = 1$

$\boxed{a + c = 1/2}$

By solving ③ & ④

$0 = \cancel{b} - 2c + \cancel{3d}$

$0 = -\cancel{b} \oplus 2c \oplus \cancel{3d}$

$0 = -4c$

$\boxed{c = 0}$

Substituting $c = 0$ in above we get

$a + c = 1/2$

$\boxed{a = 1/2}$

By solving ③ & ④ equations

$0 = b - 2\cancel{c} + 3d$

$0 = b + \cancel{2c} + 3d$

$0 = 2b + 6d$

$\boxed{b + 3d = 0}$

$\boxed{b = -3d}$

$b = -3(1/4$

$\boxed{b = \frac{-3}{4}}$

Substituting All in eq ②

$a + b + c + d = 0$

$\frac{1}{2} + (-3d) + 0 + d = 0$

$-\frac{1}{2} = -3d + d$

$\cancel{-}\frac{1}{2} = \cancel{-}2d$

$\boxed{d = \frac{1}{4}}$

now we got all unknown's a, b, c & d.

$$a = \frac{1}{2} \qquad b = -\frac{3}{4}$$

$$c = 0 \qquad d = \frac{1}{4}$$

substitute All these unknown's in hermitian polynomial equation

$$H = a + b\xi + c\xi^2 + d\xi^3$$

$$H_1 = \frac{1}{2} + \left(-\frac{3}{4}\right)\xi + 0\left(\xi^2\right) + \frac{1}{4}\left(\xi^3\right)$$

$$H_1 = \frac{1}{2} - \frac{3}{4}\xi + \frac{1}{4}\xi^3$$

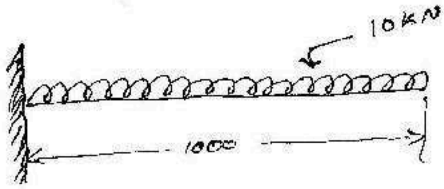
$$H_1 = \frac{1}{4}\left[2 - 3\xi + \xi^3\right]$$

In same way find out the H_2, H_3 & H_4 know these are the Hermite shape functions.

$$\left.\begin{aligned} H_1 &= \frac{1}{4}\left[2 - 3\xi + \xi^3\right] \\ H_2 &= \frac{1}{4}\left[1 - \xi - \xi^2 + \xi^3\right] \\ H_3 &= \frac{1}{4}\left[2 + 3\xi - \xi^3\right] \\ H_4 &= \frac{1}{4}\left[-1 - \xi + \xi^2 + \xi^3\right] \end{aligned}\right\} \text{Hermite shape functions}$$

H_1, H_2, H_3 & H_4 are the HERMITE SHAPE FUNCTIONS.

ed the slope & deflection for a given cantilever beam with a udl load on it?



$E = 70\times10^3\ kN/mm$

$I = 2500\ mm^4$

Soln

$$K = \frac{70\times10^3\times2500}{1000^3}\begin{bmatrix} 12 & -6000 & -12 & 6000 \\ 6000 & 4\times10^6 & -6000 & 2\times10^6 \\ -12 & -6000 & 12 & -6000 \\ 6000 & 2\times10^6 & -6000 & 4\times10^6 \end{bmatrix}$$

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elimination Approch.

$$K = \frac{70\times10^3\times2500}{1000^3}\begin{bmatrix} 12 & -6000 \\ -6000 & 4\times10^6 \end{bmatrix}\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \frac{-10\times10^3\times1000}{2} \\ \frac{10\times10^3\times(1000)^2}{12} \end{bmatrix}$$

$$12y_2 - 6000\,\theta_2 = \frac{-10\times10^3\times1000}{2}\times\frac{1000^3}{70\times10^3\times2500}$$

$$-6000y_2 + 4\times10^6\,\theta_2 = \frac{10\times10^3\times(1000)^2}{12}\times\frac{1000^3}{70\times10^3\times2500}$$

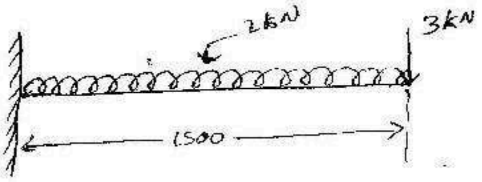
$$12y_2 - 6000\theta_2 = -28.5\times10^6$$

$$-6000y_2 + 4\times10^6\theta_2 = 4.76\times10^9$$

$$y_2 = -712\times10^6\ mm$$

$$\theta_2 = -9490^\circ = -9.49\times10^3\ ^\circ$$

③ solve below pblm?



$E = 2\times10^5\ N/mm^2$

$I = 2500\ mm^4$

Sol

$$F = \begin{bmatrix} -\dfrac{2\times10^3\times1500}{2} \\ -\dfrac{2\times10^3\times1500^2}{12} \\ -\dfrac{2\times10^3\times1500}{2} - 3\times10^3 \\ \dfrac{2\times10^3\times1500^2}{12} \end{bmatrix}$$

$$K = \frac{2\times10^5\times2500}{1500^3}\begin{bmatrix} 12 & 6\times1500 & -12 & 6\times1500 \\ 6\times1500 & 4\times1500^2 & -6\times1500 & 2\times1500^2 \\ -12 & -6\times1500 & 12 & -6\times1500 \\ 6\times1500 & 2\times1500^2 & -6\times1500 & 4\times1500^2 \end{bmatrix}$$

$$\frac{2\times10^5\times2500}{1500^3}\begin{bmatrix} 12 & 6\times1500 & -12 & 6\times1500 \\ 6\times1500 & 4\times1500^2 & -6\times1500 & 2\times1500^2 \\ -12 & -6\times1500 & 12 & -6\times1500 \\ 6\times1500 & 2\times1500^2 & -6\times1500 & 4\times1500^2 \end{bmatrix}\begin{bmatrix} y_1 \\ \theta_1 \\ y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} -2\times10^3\times1500/2 \\ -2\times10^3\times1500^2/12 \\ -\dfrac{2\times10^3\times1500}{2} - 3\times10^3 \\ 2\times10^3\times1500^2/12 \end{bmatrix}$$

By elimination Approach.

$$\frac{2\times10^5\times2500}{1500^3}\begin{bmatrix} 12 & 6\times1500 \\ -6\times1500 & 4\times1500^2 \end{bmatrix}\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{2\times10^3\times1500}{2} - 3\times10^3 \\ \dfrac{2\times10^3\times1500^2}{12} \end{bmatrix}$$

solving the matrix

$$12y_2 - 6\times 1500\,\theta_2 = -10.14\times 10^6$$

$$-(6\times 1500)y_2 + 4\times 1500^2\,\theta_2 = 2.53\times 10^9$$

$$y_2 = -2.53\times 10^6\ \text{mm}$$

$$\theta_2 = -2.25\times 10^3\ {}^\circ$$

Substituting y_2 & θ_2 in the S.F & B.M equation we get SF & B.M

$$\begin{bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{bmatrix} = \frac{2\times 10^5\times 2500}{1500^3}\begin{bmatrix} 12 & 6\times 1500 & -12 & 6\times 1500 \\ 6\times 1500 & 4\times 1500^2 & -6\times 1500 & 2\times 1500^2 \\ -12 & -6\times 1500 & 12 & -6\times 1500 \\ 6\times 1500 & 2\times 1500^2 & -6\times 1500 & 4\times 1500^2 \end{bmatrix}\begin{bmatrix} y_1 \\ \theta_1 \\ y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} -\frac{2\times 10^3\times 15}{2} \\ -\frac{2\times 10^3\times 15}{12} \\ -\frac{2\times 10^3\times 1500}{2}\ 3 \\ \frac{2\times 10^3\times 15}{12} \end{bmatrix}$$

Substitute y_1 θ_1 y_2 & θ_2 in above equation.

$$\begin{bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{bmatrix} = \frac{2\times 10^5\times 2500}{1500^3}\begin{bmatrix} 12 & 6\times 1500 & -12 & 6\times 1500 \\ 6\times 1500 & 4\times 1500^2 & -6\times 1500 & 2\times 1500^2 \\ -12 & -6\times 1500 & 12 & -6\times 1500 \\ 6\times 1500 & 2\times 1500^2 & -6\times 1500 & 4\times 1500^2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ -2.53\times 10^6 \\ -2.25\times 10^3 \end{bmatrix} - \begin{bmatrix} -2\times 10^3\times 1500/2 \\ -\frac{2\times 10^3\times 1500^2}{12} \\ -\frac{2\times 10^3\times 1500}{2} - 3\times 10^3 \\ 2\times 10^3\times 1500^2/2 \end{bmatrix}$$

By solving above matrix equations we get F_1 M_1 F_2 & M_2 these are shearforces at node 1 & 2 [i.e; F_1 F_2] & Bending moments at node 1 & 2 [i.e; M_1 & M_2]

$F_1 =$

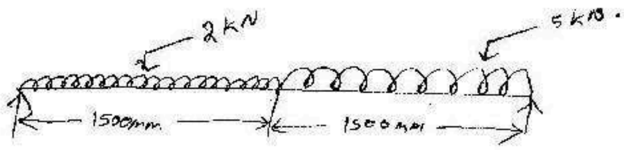
$M_1 =$

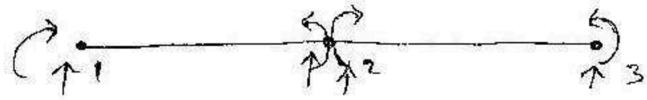
$F_2 =$

$M_2 =$

④ Load vector Assembling problem?

Soln





$$F_1 = \begin{array}{cc} \begin{bmatrix} -\frac{2\times10^3\times1500}{2} \\ -\frac{2\times10^3\times1500^2}{12} \\ -\frac{2\times10^3\times1500}{2} \\ \frac{2\times10^3\times1500^2}{12} \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

$$\therefore F_2 = \begin{array}{cc} \begin{bmatrix} -\frac{5\times10^3\times1500}{2} \\ -\frac{5\times10^3\times1500^2}{12} \\ -\frac{5\times10^3\times1500}{2} \\ \frac{5\times10^3\times1500^2}{12} \end{bmatrix} & \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

$$F_1 + F_2 = \underset{\sim}{F}$$

$$\underset{\sim}{F} = \begin{bmatrix} -\frac{2\times10^3\times1500}{2} \\ -\frac{2\times10^3\times1500^2}{12} \\ -5.25\times10^6 \\ -5.62\times10^8 \\ -\frac{5\times10^3\times1500}{2} \\ +\frac{5\times10^3\times1500^2}{12} \end{bmatrix} = \begin{bmatrix} -1.5\times10^6 \\ -375\times10^6 \\ -5.25\times10^6 \\ -562.5\times10^6 \\ -3.75\times10^6 \\ -937.5\times10^6 \end{bmatrix}$$

$$\underset{\sim}{F} = -10^6 \begin{bmatrix} 1.5 \\ 375 \\ 5.25 \\ 562.5 \\ 3.75 \\ 937.5 \end{bmatrix}$$

: the pblm?

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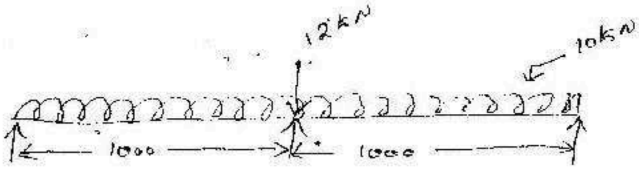

$E = 2\times10^5 \ N/mm^2$

$I = 2500 mm^4$

Sol:

$$k = \frac{2\times10^5 \times 2500}{1000^3} \begin{bmatrix} 12 & 6000 & -12 & 6000 \\ 6000 & 4\times10^6 & -6000 & 2\times10^6 \\ -12 & -6000 & 12 & -6000 \\ 6000 & 2\times10^6 & -6000 & 4\times10^6 \end{bmatrix}$$

$$k_1 = 0.5 \begin{bmatrix} 12 & 6000 & -12 & 6000 \\ 6000 & 4\times10^6 & -6000 & 2\times10^6 \\ -12 & -6000 & 12 & -6000 \\ 6000 & 2\times10^6 & -6000 & 4\times10^6 \end{bmatrix}$$

$k_1 = k_2$

Assembling of Stiffness matrix

$$\underset{\sim}{k} = 0.5 \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 12 & 6\times10^3 & -12 & 6\times10^3 & 0 & 0 \\ 6\times10^3 & 4\times10^6 & -6\times10^3 & 2\times10^6 & 0 & 0 \\ -12 & -6\times10^3 & 24 & 0 & -12 & 6\times10^3 \\ 6\times10^3 & 2\times10^6 & 0 & 8\times10^6 & -6\times10^3 & 2\times10^6 \\ 0 & 0 & -12 & -6\times10^3 & 12 & 6\times10^3 \\ 0 & 0 & 6\times10^3 & 2\times10^6 & -6\times10^3 & 4\times10^6 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

By load vectory

$$F_1 = \begin{bmatrix} \dfrac{-10\times10^3\times10^3}{2} \\ \dfrac{-10\times10^3\times10^6}{12} \\ \dfrac{-10\times10^3\times10^3}{2} - 12\times10^3 \\ \dfrac{10\times10^3\times10^6}{12} \end{bmatrix} = \begin{array}{c} \begin{bmatrix} -5\times10^6 \\ -833.3\times10^6 \\ -5.01\times10^6 \\ 833.3\times10^6 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$F_2 = \begin{bmatrix} \dfrac{-10\times10^3\times10^3}{2} - 12\times10^3 \\ \dfrac{-10\times10^3\times10^6}{12} \\ \dfrac{-10\times10^3\times10^3}{2} \\ \dfrac{-10\times10^3\times10^6}{12} \end{bmatrix} = \begin{bmatrix} -5.01\times10^6 \\ -833.3\times10^6 \\ -5.0\times10^6 \\ 833.3\times10^6 \end{bmatrix} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$F_1 + F_2 = \underset{\sim}{F}$.

$$\underset{\sim}{F} = \begin{bmatrix} -5\times10^6 \\ -833.3\times10^6 \\ -10.0\times10^6 \\ 0 \\ -5\times10^6 \\ 8.33\times10^8 \end{bmatrix}$$

Modified K – By elimination Approach.

$$0.5 * \begin{bmatrix} 4\times10^6 & 2\times10^6 & 0 \\ 2\times10^6 & 8\times10^6 & 2\times10^6 \\ 0 & 2\times10^6 & 4\times10^6 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = 10^6 \begin{bmatrix} -833.33 \\ 0 \\ 833.33 \end{bmatrix}$$

$$0.5\times10^{6}\begin{bmatrix}4 & 2 & 0\\ 2 & 8 & 2\\ 0 & 2 & 4\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix} = 10^{6}\begin{bmatrix}-833.33\\ 0\\ 833.33\end{bmatrix}$$

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$$4\theta_1 + 2\theta_2 = -833.33/0.5$$

$$2\theta_1 + 8\theta_2 + 2\theta_3 = 0$$

$$2\theta_2 + 4\theta_3 = 833.33/0.5$$

By solving above equations we get θ_1, θ_2 & θ_3

$$\theta_1 = -416.6^{\circ}$$

$$\theta_2 = 0$$

$$\theta_3 = 416.6^{\circ}$$

UNIT III

CONTINUUM ELEMENTS

UNIT - V

TWO DIMENSIONAL PROBLEMS

Finite Element Modeling:

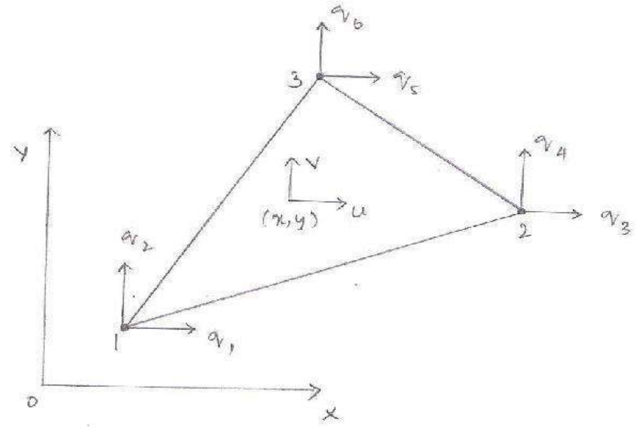
The two dimensional region is devided into straight - sided triangles. the points where the corners of the triangles meet are called nodes. and each triangle formed by three nodes and three side is called an element. For the triangulation, the nodes number are indicated at the corners and element numbers are circled.

In the 2-D problem, each node is permitted to displace in the two directions x and y. Thus, each nodes has two degrees of freedom (dofs)

$$Q = [q_1, q_2, \cdots q_N]^T$$

where N is the number of degrees of freedom.

computationally, the information on the triangulation is to be represented in the form of nodal coordinates and connectivity.



CONSTANT - STRAIN TRIANGLE (CST) :

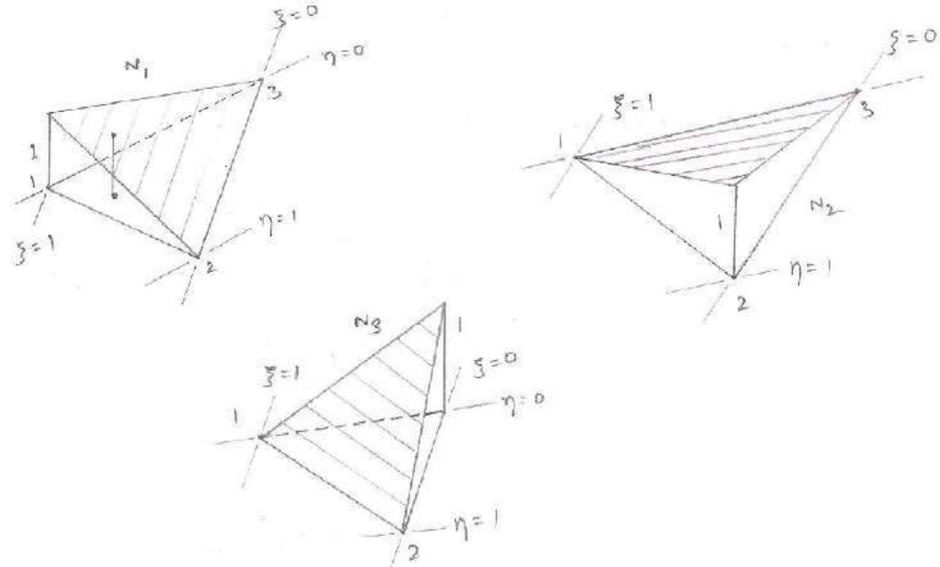
The displacements at points inside an element need to be represented in terms of nodal displacements of the elements. For the constant-strain triangle (CST), the shape functions are linear over the element. The three shape functions N_1, N_2 and N_3 corresponding to nodes 1, 2 and 3 respectively, are shown in fig. Shape function N_1 is 1 at node 1 and linearly reduces to 0 at nodes 2 and 3. The values of shape function N_1 thus defines a plane shown shaded in fig. N_2 and N_3 are represented by similar surfaces having values 1 at nodes 2 and 3, respectively, and dropping to 0 at the opposite edges. Any linear combination of these shape functions also represents a plane surface. In particular, $N_1 + N_2 + N_3$ represents a plane at a height of 1 at nodes 1, 2 and 3, thus it is parallel to the triangle 123. consequently, for every N_1, N_2 and N_3.

$$N_1 + N_2 + N_3 = 1$$

N_1, N_2 and N_3 are therefore not linearly independent, only two of these are independent. The independent shape functions are conveniently represented by the pair ξ, η as

$$N_1 = \xi \qquad N_2 = \eta \qquad N_3 = 1 - \xi - \eta$$

where ξ, η are natural coordinates.

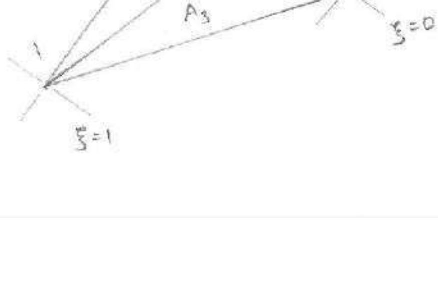


In 1-D, the x-coordinates were mapped onto the ξ coordinate and shape functions were defined ~~of~~ as functions of ξ. but here in the 2D problem, the x-, y-coordinates are mapped onto the ξ-, η-coordinates, and shape functions are defined as fun of ξ & η.

The shape functions can be physically represented by area coordinates. A point (x,y) in a triangle divides it into three areas, A_1, A_2 and A_3 as shown in figure. The shape functions N_1, N_2 and N_3 are precisely represented by

$$N_1 = A_1/A \quad N_2 = A_2/A \quad N_3 = A_3/A$$

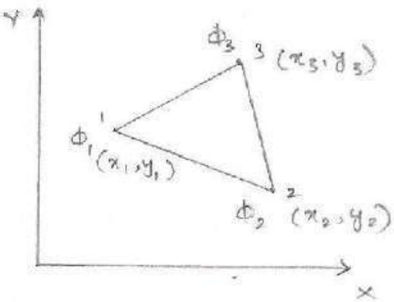
where A is the area of the element. Clearly, $N_1 + N_2 + N_3 = 1$ at every point inside the triangle.



Shape function for two Dimensional linear Element:

Consider 2D, Δ^{le} element of 3 side node of linear or simple element. A general simple variable element is assumed vary linearly inside the element including boundary.

Let a general form of 2-D polynomial function



$$\phi = a_1 + a_2 x + a_3 y \quad \text{——— (1)}$$

where a_1, a_2, a_3 are polynomial constant.

$$\phi_1 = a_1 + a_2 x_1 + a_3 y_1 \quad \text{——— (2a)}$$

$$\phi_2 = a_1 + a_2 x_2 + a_3 y_2 \quad \text{——— (2b)}$$

$$\phi_3 = a_1 + a_2 x_3 + a_3 y_3 \quad \text{——— (2c)}$$

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \text{— (3)} \qquad A = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} \quad \text{— (4)} \qquad [a] = [D]^{-1} [\phi] \quad \text{— (4a)}$$

$[D]$ - coordinate matrix.

$$[D]^{-1} = \frac{1}{|D|} \begin{bmatrix} x_2 y_3 - x_3 y_2 & y_2 - y_3 & x_3 - x_2 \\ y_1 x_3 - x_1 y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1 y_2 - y_1 x_2 & y_1 - y_2 & x_2 - x_1 \end{bmatrix}^{T} \qquad |D| = 2A$$

$$= \frac{1}{2A} \begin{bmatrix} x_2 y_3 - x_3 y_2 & y_1 x_3 - x_1 y_3 & x_1 y_2 - y_1 x_2 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \quad \text{—— (5)}$$

sub (5) in (4d), then

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & y_1x_3 - x_1y_3 & x_1y_2 - y_1x_2 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} \quad \text{--- (6)}$$

from eqn (1) $\Rightarrow$ $\phi = a_1 + a_2 x + a_3 y$.

$$\phi = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \text{--- (7)}$$

sub eqn (6) in (7).

$$\phi = \begin{bmatrix} 1 & x & y \end{bmatrix} \frac{1}{2A} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} \alpha_1 + \beta_1 x + \gamma_1 y & \alpha_2 + \beta_2 x + \beta_2 y & \alpha_3 + \beta_3 x + \gamma_3 y \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\alpha_1 + \beta_1 x + \gamma_1 y}{2A} & \frac{\alpha_2 + \beta_2 x + \gamma_2 y}{2A} & \frac{\alpha_3 + \beta_3 x + \gamma_3 y}{2A} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}$$

$$= \left(\frac{\alpha_1 + \beta_1 x + \gamma_1 y}{2A}\right)\phi_1 + \left(\frac{\alpha_2 + \beta_2 x + \gamma_2 y}{2A}\right)\phi_2 + \left(\frac{\alpha_3 + \beta_3 x + \gamma_3 y}{2A}\right)\phi_3$$

$$\phi = N_1\phi_1 + N_2\phi_2 + N_3\phi_3 \quad \longleftarrow (8)$$

where,
$$N_1 = \frac{\alpha_1 + \beta_1 x + \gamma_1 y}{2A}$$

$$N_2 = \frac{\alpha_2 + \beta_2 x + \gamma_2 y}{2A}$$

$$N_3 = \frac{\alpha_3 + \beta_3 x + \gamma_3 y}{2A}$$

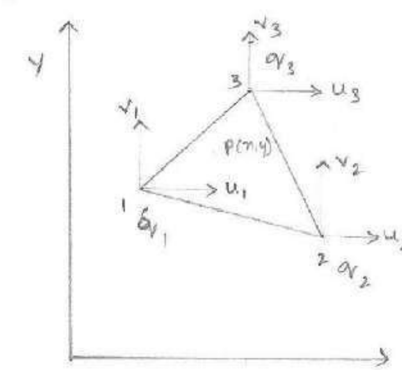
Two Dimensional vector variable problem (Isoparametric element Representation)

The linear Δ^{le} element selected for analysis is specified as constant strain triangular (CST) element because of producing constant strain at the specified triangle.

According to Hooke's law, $\sigma = E e$

for any triangle element, if σ & E are constant, then automatically the strain (e) also constant in that Δ^{le} element, and hence called Constant Strain triangle.

Consider a 3 noded linear triangular (CST) element, where nodes may be specified as 1, 2 & 3.



Let q_1, q_2 and q_3 are displacements at nodes 1, 2 and 3 respectively.

Let u & v are components of q in x & y directions

For the linear element, the displacements u & v are linearly varying inside the element and their values at any point 'P' (x,y) inside the element can be expressed by polynomial

$$u(x,y) = a_1 + a_2 x + a_3 y \quad \text{--- (1)}$$

$$v(x,y) = a_4 + a_5 x + a_6 y. \quad \text{--- (2)}$$

[In CST element, each node has two DOF, totaly we have 6 DOF, so we considered 6 polynomial coefficient]

Now at node 1, 2 & 3, the displacement components are written as

$$\left.\begin{aligned} u_1 &= a_1 + a_2 x_1 + a_3 y_1 \\ u_2 &= a_1 + a_2 x_2 + a_3 y_2 \\ u_3 &= a_1 + a_2 x_3 + a_3 y_3 \end{aligned}\right\} \rightarrow (3a) \qquad \left.\begin{aligned} v_1 &= a_4 + a_5 x_1 + a_6 y_1 \\ v_2 &= a_4 + a_5 x_2 + a_6 y_2 \\ v_3 &= a_4 + a_5 x_3 + a_6 y_3 \end{aligned}\right\} \longrightarrow (3b)$$

From eqn (1) & (3a) [Ref. previous derivation], eqn (2) & (3b)

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 \quad \text{---} (4a) \qquad v = N_1 v_1 + N_2 v_2 + N_3 v_3 \quad \text{---} (4b)$$

The nodal displacement of the point 'P' can be written as

$$[q]_p = \begin{bmatrix} q_x \\ q_y \end{bmatrix}_p = \begin{bmatrix} u \\ v \end{bmatrix}_p$$

$$= \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} \quad \text{———} (5)$$

From eqn (4) & (5).

$$[q]_p = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$$

$$= \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} \quad \text{———} (6).$$

$$\boxed{q_p = [N][q]}$$

NOTE: We can prove the sum of shape function is equal to one.

i.e $N_1 + N_2 + N_3 = 1$. [work out].

STRESS - STRAIN Relationship (matrix formulation)

For 3-D system.

$$e_x = \frac{1}{E}\left[\sigma_x - \mu(\sigma_y + \sigma_z)\right]$$

$$E e_x = \sigma_x - \mu\sigma_y - \mu\sigma_z \qquad \text{ily} \quad E e_y = \sigma_y - \mu\sigma_x - \mu\sigma_z$$

$$E e_z = \sigma_z - \mu\sigma_x - \mu\sigma_y$$

$$E e_x = \sigma_x - \mu\sigma_y - \mu\cancel{\sigma_z}$$

$$\mu E e_z = -\mu^2\sigma_x - \mu^2\sigma_y + \cancel{\mu\sigma_z}$$

$$E(e_x + \mu e_z) = \sigma_x(1-\mu^2) - \mu\sigma_y(1+\mu)$$

$$E e_y = \sigma_y - \mu\sigma_x - \mu\cancel{\sigma_z}$$

$$\mu E e_z = -\mu^2 y - \mu^2\sigma_x + \cancel{\mu\sigma_z}$$

$$E(e_y + \mu e_z) = (1-\mu^2)\sigma_y - \mu(1+\mu)\sigma_x$$

$$(1-\mu)E(e_x + \mu e_z) = (1-\mu)(1-\mu^2)\sigma_x - \mu(1-\cancel{\mu^2})\sigma_y$$

$$\mu E(e_y + \mu e_z) = -\mu^2(1+\mu)\sigma_x + \mu(1-\cancel{\mu^2})\sigma_y$$

$$E\left[(1-\mu)e_x + \mu(1-\mu)e_z + \mu e_y + \mu^2 e_z\right] = \sigma_x\left[(1-\mu)(1-\mu^2) - \mu^2(1+\mu)\right]$$

$$\Rightarrow \sigma_x = \frac{E\left[e_x(1-\mu) + \mu e_y + \mu e_z\right]}{(1+\mu)(1-2\mu)} \quad \text{--- (1a)}$$

$$\text{ily} \quad \sigma_y = \frac{E\left[\mu e_x + (1-\mu)e_y + \mu e_z\right]}{(1+\mu)(1-2\mu)} \quad \text{--- (1b)}$$

$$\sigma_z = \frac{E\left[\mu e_x + \mu e_y + (1-\mu)e_z\right]}{(1+\mu)(1-2\mu)} \quad \text{--- (1c)}$$

Two Dimensional system:

Plane Stress: A state of plane stress is said to be exist, when the elastic body is very thin and there are no loads applied in the co-ordinate direction parallel to thickness.

$$\sigma_z = 0 \quad \tau_{yz} = 0 \quad \tau_{zx} = 0$$

$$e_z \neq 0 \quad \gamma_{yz} = 0 \quad \gamma_{zx} = 0$$

$$E e_x = \sigma_x - \mu\sigma_y \Rightarrow E e_x = \sigma_x - \mu\sigma_y$$

$$E e_y = \sigma_y - \mu\sigma_x \Rightarrow \mu E e_y = -\mu^2\sigma_x + \mu\sigma_y$$

$$E(e_x + \mu e_y) = (1-\mu^2)\sigma_x$$

$$\Rightarrow \sigma_x = \frac{E}{1-\mu^2}(e_x + \mu e_y)$$

$$\mu E e_x = \mu\sigma_x - \mu^2\sigma_y$$

$$E e_y = -\mu\sigma_x + \sigma_y$$

$$E(\mu e_x + e_y) = (1-\mu^2)\sigma_y \Rightarrow \sigma_y = \frac{E}{1-\mu^2}(e_y + \mu e_x)$$

$$\tau_{xy} = \frac{E}{1-\mu^2}\left(\frac{1-\mu}{2}\right)\gamma_{xy}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}\begin{bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{bmatrix} \quad \text{---} (4)$$

$$[\sigma] = [D][e]$$

Plane strain: A state of plane strain occurs in a member that are non free to expand in the direction perpendicular to the plane of applied load.

$$e_z = 0, \ \gamma_{yz} = 0, \ \gamma_{zx} = 0$$

$$\sigma_x = \frac{E}{(1+\mu)(1-2\mu)} \left[(1-\mu)e_x + \mu e_y \right]$$

$$\sigma_y = \frac{E}{(1+\mu)(1-2\mu)} \left[\mu e_x + (1-\mu) e_y \right]$$

$$\tau_{xy} = \frac{E}{(1+\mu)(1-2\mu)} \left(\frac{1-2\mu}{2} \right) \gamma_{xy}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{bmatrix} \quad \text{——} \ (5)$$

$$[\sigma] = [D][e]$$

Strain - displacement relationship matrix:

$$q(x,y)_p = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}_p = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$

$$e_x = \partial u/\partial x = \partial N_1/\partial x \, u_1 + \partial N_2/\partial x \, u_2 + \frac{\partial N_3}{\partial x} u_3$$

$$e_y = \partial v/\partial y = \partial N_1/\partial y \, v_1 + \partial N_2/\partial y \, v_2 + \partial N_3/\partial y \, v_3$$

$$\gamma_{xy} = \partial u/\partial y + \partial v/\partial x$$

$$= \partial N_1/\partial y\, u_1 + \partial N_2/\partial y\, u_2 + \partial N_3/\partial y\, u_3 + \partial N_1/\partial x\, v_1 + \partial N_2/\partial x\, v_2 + \partial N_3/\partial x\, v_3$$

$$\begin{bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \partial N_1/\partial x & 0 & \partial N_2/\partial x & 0 & \partial N_3/\partial x & 0 \\ 0 & \partial N_1/\partial y & 0 & \partial N_2/\partial y & 0 & \partial N_3/\partial y \\ \partial N_1/\partial y & \partial N_1/\partial x & \partial N_2/\partial y & \partial N_2/\partial x & \partial N_3/\partial y & \partial N_3/\partial x \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

$$N_1 = \frac{1}{2A}(\alpha_1 + \beta_1 x + \gamma_1 y)$$

$$N_2 = \frac{1}{2A}(\alpha_2 + \beta_2 x + \gamma_2 y)$$

$$N_3 = \frac{1}{2A}(\alpha_3 + \beta_3 x + \gamma_3 y)$$

$$\begin{bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

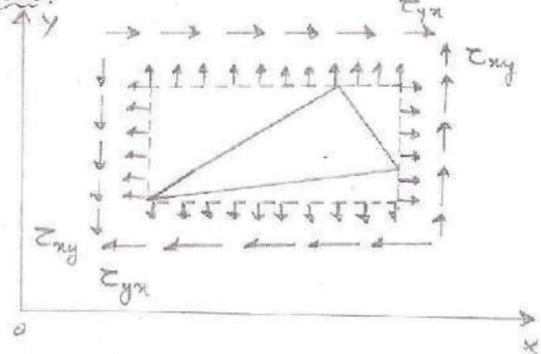
$[e] = [B][q]$ → Strain - displacement matrix.

Stress - displacement Relationship matrix:

$$[\sigma] = [D][e] = [D][B][q]$$

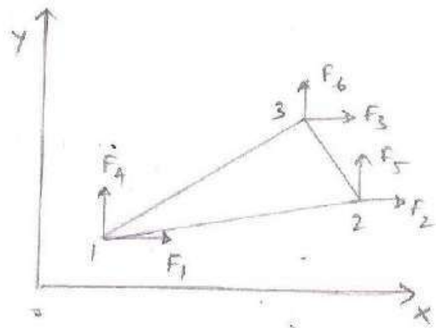
$[\sigma] = [D][B][q]$ → Stress - displacement matrix.

Element stiffness matrix for CST element:



Consider the typical element shown in figure. It is subjected to constant stresses along its all the three edges. Let the constant stresses be $\sigma_x, \sigma_y, \tau_{xy} = \tau_{yx}$.

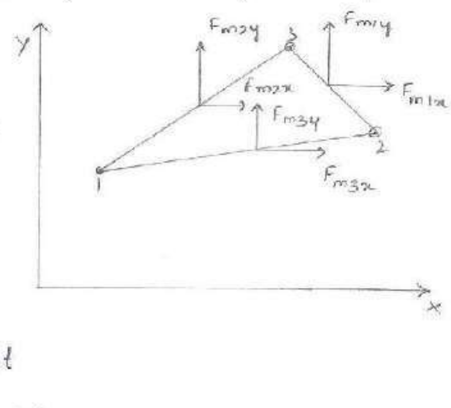
Assembling stiffness matrix means finding nodal equivalent set of forces which are statically equivalent to constant stress field acting at the edges of the elements.



The equivalent nodal forces to be found are $F_1, F_2, F_3 \ldots F_6$ as shown in figure.

We have six unknown nodal forces, but only three eqns of equilibrium. Hence it is not possible to determine $F_1, F_2 \ldots F_6$ in terms of $\sigma_x, \sigma_y, \tau_{xy}$ mathematically.

Turner resolved the uniform stress distribution into an equivalent force system at midsides as shown in figure.



$$F_{m1x} = \sigma_x (y_3 - y_2) t + \tau_{xy} (x_2 - x_3) t$$

where t is the thickness of the element

$$F_{m1y} = \sigma_y (x_2 - x_3) t + \tau_{xy} (y_3 - y_2) t$$

$$F_{m2x} = -\sigma_x (y_3 - y_1) t + \tau_{xy} (x_3 - x_1) t$$

$$F_{m2y} = \sigma_y (x_3 - x_1) t + \tau_{xy} (y_3 - y_1) t$$

$$F_{m3x} = \sigma_x (y_2 - y_1) t - \tau_{xy} (x_2 - x_1) t$$

$$F_{m3y} = -\sigma_y (x_2 - x_1) t + \tau_{xy} (y_2 - y_1) t$$

After this Turner transferred half of mid side forces to nodes at the end of sides to get equivalent nodal force. Thus let

$$F_1 = \frac{1}{2}\,(F_{m2x} + F_{m3x})$$

$$= \frac{t}{2}\left[\sigma_x\,(y_2 - y_3) + \tau_{xy}\,(x_3 - x_2)\right]$$

$$F_2 = \frac{1}{2}\,(F_{m1x} + F_{m3x})$$

$$= \frac{t}{2}\left[\sigma_x\,(y_3 - y_1) + \tau_{xy}\,(x_1 - x_3)\right]$$

$$F_3 = \frac{1}{2}\,(F_{m1x} + F_{m2x})$$

$$= \frac{t}{2}\left[\sigma_x\,(y_1 - y_2) + \tau_{xy}\,(x_2 - x_1)\right]$$

$$F_4 = \frac{1}{2}\,(F_{m2y} + F_{m3y})$$

$$= \frac{t}{2}\left[\sigma_y\,(x_3 - x_2) + \tau_{xy}\,(y_2 - y_3)\right]$$

$$F_5 = \frac{1}{2}\,(F_{m1y} + F_{m3y})$$

$$= \frac{t}{2}\left[\sigma_y\,(x_1 - x_3) + \tau_{xy}\,(y_3 - y_1)\right]$$

$$F_6 = \frac{1}{2}\,(F_{m1y} + F_{m2y})$$

$$= \frac{t}{2}\left[\sigma_y\,(x_2 - x_1) + \tau_{xy}\,(y_1 - y_2)\right]$$

Thus the force vector as derived by Turner is written as

$$[F] = \frac{t}{2}\begin{bmatrix} b_1 & 0 & c_1 \\ b_2 & 0 & c_2 \\ b_3 & 0 & c_3 \\ 0 & c_1 & b_1 \\ 0 & c_2 & b_2 \\ 0 & c_3 & b_3 \end{bmatrix}\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

where, $b_1 = y_2 - y_3 \quad b_2 = y_3 - y_1 \quad b_3 = y_1 - y_2$

$c_1 = x_3 - x_2 \quad c_2 = x_1 - x_3 \quad c_3 = x_2 - x_1$

$$\begin{bmatrix} b_1 & 0 & c_1 \\ b_2 & 0 & c_2 \\ b_3 & 0 & c_3 \\ 0 & c_1 & b_1 \\ 0 & c_2 & b_2 \\ 0 & c_3 & b_3 \end{bmatrix} = 2A[B]^T$$

$$[\sigma] = [D][e] = [D][B][q]_p$$

$$[F] = \frac{t}{2}\ 2A[B]^T[D][B][q]_p$$

$$= A \cdot t\ [B]^T[B][B][q]_p$$

$$= [B]^T[D][B]\, V\, [q]_p \qquad V - \text{volume}$$

$$\therefore [F] = [k][q]$$

$$[k] = [B]^T[D][B]\, V$$

$$\boxed{[k] = \int [B]^T[D][B]\, dV}$$

UNIT IV

STEADY STATE HEAT TRANSFER ANALYSIS

One-Dimensional Heat Conduction:

In 1-D steady state problems, a temperature gradient exists along only one coordinate axis, and the temperature at each point is independent of time.

For 1-D, steady state heat transfer conduction problem, the governing differential equation is given by.

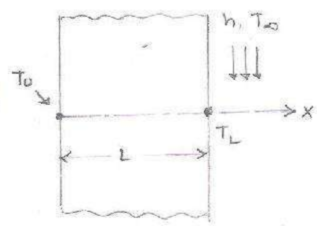
$$\frac{d}{dx}\left(K\frac{dT}{dx}\right) + G = 0$$

where, G = internal heat generated per unit volume (W/m^3)

Boundary conditions:

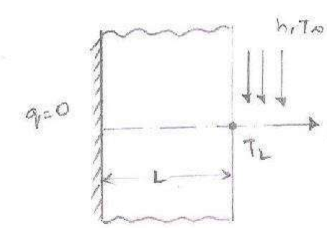
The B.C's are mainly of three kinds

1. Specified temperature
2. Specified heat flux (insulated)
3. Convection.



$$T|_{x=0} = T_0$$

$$q|_{x=L} = h(T_L - T_\infty)$$



$$q|_{x=0} = 0$$

$$q|_{x=L} = h(T_L - T_\infty)$$

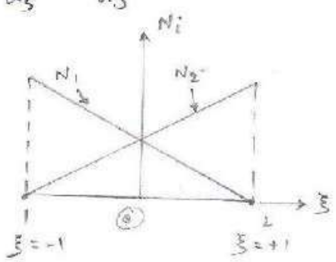
The One-Dimensional Element:

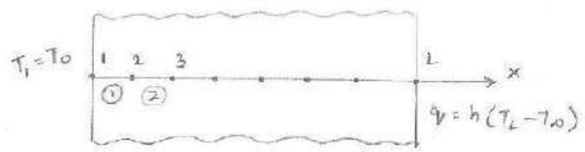
The two-node element with linear shape functions is considered. The temperature at the various nodal points, denoted by T, are the unknowns (except at node 1, where $T_1 = T_0$). Within a typical element e, whose local node numbers are 1 and 2, the temperature field is approximated using shape function N_1 and N_2 as

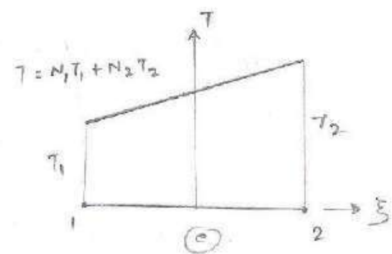
$$T(\xi) = N_1T_1 + N_2T_2$$
$$= \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = NT^e \quad —(1)$$

where, $N_1 = \frac{(1-\xi)}{2}$ $N_2 = \frac{(1+\xi)}{2}$, $\quad \frac{dT}{d\xi} = \frac{dN}{d\xi} \cdot T^e$

ξ various from -1 to $+1$







$$\xi = \frac{2}{x_2 - x_1}(x - x_1) - 1$$

$$d\xi = \frac{2}{x_2 - x_1}\, dx \quad —(2)$$

$$\frac{dT}{dx} = \frac{dT}{d\xi} \cdot \frac{d\xi}{dx} = \frac{2}{x_2 - x_1} \cdot \frac{dN}{d\xi} \cdot T^e = \frac{1}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \quad —(3)$$

$$\frac{dT}{dx} = B_T T^e \quad —(4)$$

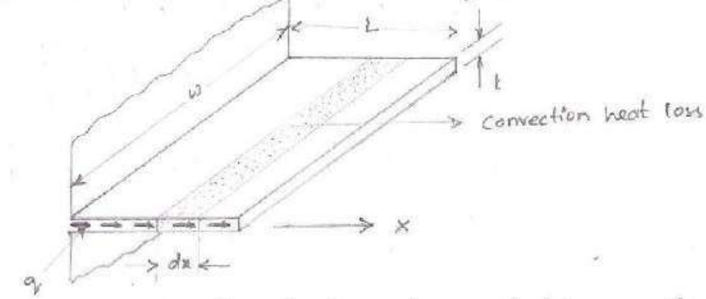
where $B_T = \frac{1}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \quad —(5).$

One-Dimensional Heat Transfer in Thin Fins:

A fin is an extended surface that is added onto a structure to increase the rate of heat removal.

Example: In the motorcycle, where fins extended from the cylinder head to quickly dissipate heat through convection.

Consider a thin rectangular fin as shown in figure.
This problem can be treated as 1-D, because the temperature gradients along the width and across the thickness are negligible.



The governing equation may be derived from the conduction equation with heat source, given by

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + Q = 0 \qquad (1)$$

The convection heat loss in the fin can be considered as a negative heat source

$$Q = -\frac{P\,dx \cdot h\,(T - T_\infty)}{A_c\,dx} = -\frac{Ph}{A_c}(T - T_\infty) \qquad (2)$$

where P = perimeter of fin
A_c - Area of cross section.

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) - \frac{Ph}{A_c}(T - T_\infty) = 0 \qquad (3)$$

Boundary Conditions are

$$T = T_0 \quad @ \quad x = 0$$
$$q = 0 \quad @ \quad x = L$$

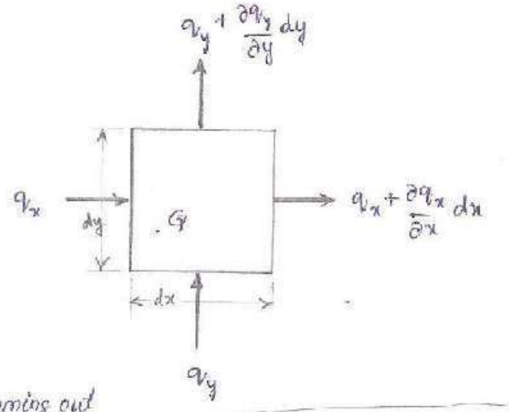
Two-Dimensional Heat conduction :

In two-dimensional conduction, a long prismatic solid is considered to determine the temperature distribution $T(x,y)$.

eg: A chimney of rectangular cross section.

Using Fourier's law, the heat flux can be determained, when the temperature distribution is known.

Consider a differential control volume in the body, as shown in fig. The control volume has thickness 't' in the z-direction. The heat generated is denoted by Q (W/m^3).



Heat rate entering the control volume + Heat rate generated } = Heat rate coming out —— (1)

$\because$ heat rate = (heat flux) × (area)

$$q_x (dy)(t) + q_y (dx)(t) + Q\, dx\, dy (t) = \left(q_x + \frac{\partial q_x}{\partial x} dx \right) dy (t) + \left(q_y + \frac{\partial q_y}{\partial y} dy \right) dx (t) \quad \text{—— (2)}$$

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} - Q = 0 \quad \text{—— (3)}$$

Substitute $q_x = -k\, \partial T/\partial x$ & $q_y = -k\, \partial T/\partial y$

$$\frac{\partial}{\partial x}\left(k\, \partial T/\partial x\right) + \frac{\partial}{\partial y}\left(k\, \partial T/\partial y\right) + Q = 0 \quad \text{—— (4)}$$

Boundary Conditions :

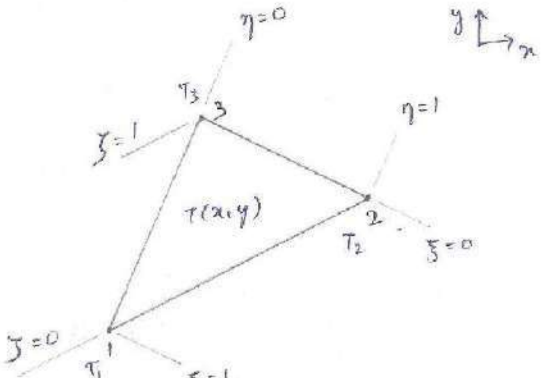
1. Specified temperature (S_T) . $T = T_0$
2. Specified Heat flux (S_q) , $q_n = q_0$
3. Convection (S_c) , $q_n = h(T - T_\infty)$

The triangular element will be used to solve the heat-conduction problem. Consider a constant length of the body perpendicular to the $x\,y$ plane. The temperature field with in an element is given by.

$$T = N_1T_1 + N_2T_2 + N_3T_3$$

$$[T] = \begin{bmatrix} N_1, & N_2, & N_3 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = [N][T]^e \quad — \text{(3)}$$

where, $N_1 = \xi$
$N_2 = \eta$
$N_3 = 1 - \xi - \eta$



$x = N_1x_1 + N_2x_2 + N_3x_3$
$y = N_1y_1 + N_2y_2 + N_3y_3$

Chain rule of differentiation,

$$\frac{\partial T}{\partial \xi} = \frac{\partial T}{\partial x}\cdot\frac{\partial x}{\partial \xi} + \frac{\partial T}{\partial y}\cdot\frac{\partial y}{\partial \xi}$$

$$\frac{\partial T}{\partial \eta} = \frac{\partial T}{\partial x}\cdot\frac{\partial x}{\partial \eta} + \frac{\partial T}{\partial y}\cdot\frac{\partial y}{\partial \eta}$$

$$\begin{bmatrix} \partial T/\partial \xi \\ \partial T/\partial \eta \end{bmatrix} = \underbrace{\begin{bmatrix} \partial x/\partial \xi \;(x_{13}) & \partial y/\partial \xi \;(y_{13}) \\ \partial x/\partial \eta \;(x_{23}) & \partial y/\partial \eta \;(y_{23}) \end{bmatrix}}_{J} \begin{bmatrix} \partial T/\partial x \\ \partial T/\partial y \end{bmatrix}$$

$$\begin{bmatrix} \partial T/\partial x \\ \partial T/\partial y \end{bmatrix} = J^{-1} \begin{bmatrix} \partial T/\partial \xi \\ \partial T/\partial \eta \end{bmatrix} = \frac{1}{\det J} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} [T]$$

$$= \frac{1}{\det J} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} T^e \Rightarrow \begin{bmatrix} \partial T/\partial x \\ \partial T/\partial y \end{bmatrix} = B_T\, T^e$$

$$B_T = \frac{1}{\det J} \begin{bmatrix} y_{23} & -y_{13} & (y_{13} - y_{23}) \\ -x_{23} & x_{13} & (x_{23} - x_{13}) \end{bmatrix} = \frac{1}{\det J} \begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix} \text{ //}$$

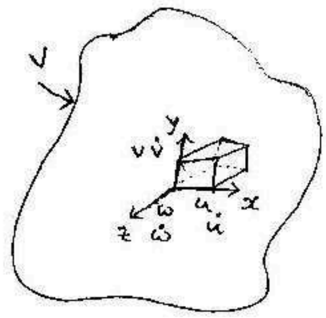
UNIT V

DYNAMIC ANALYSIS

DYNAMIC ANALYSIS :-

General expression for elemental mass :-

Consider a solid body of mass elemental volume dv as shown in figure



Let ρ is the density of element

$\dot{u}$ nodal velocity vector.

$$\dot{u} = [\dot{u}\ \dot{v}\ \dot{w}]^T$$

$$u = Nq$$

$$\dot{u} = N\dot{q}$$

Hence K.E is given as below.

$$K.E = \frac{1}{2} m v^2 \qquad \left[\because V^2 = [V][V]^T \quad m = \rho\right]$$

$$= \frac{1}{2}\rho\, dv$$

$$K.E = \int \frac{1}{2}\rho\, \dot{u}^T \dot{u}\ dv$$

$$= \int \frac{1}{2}\rho\, [N\dot{q}]^T [N\dot{q}]\, dv$$

$$= \frac{1}{2}\int \rho\ \dot{q}^T N^T N \dot{q}\ dv$$

$$= \frac{1}{2}\dot{q}^T \underbrace{\left[\int \rho\, N^T N\, dv\right]}_{\text{Mass}} q$$

In the above expression the term introduced is known as elemental mass matrix & it is given by

$$\boxed{m^e = \int \rho N^T N \, dv}$$

This mass matrix is consistent with the shape functions chosen & it is called the consistent mass matrix.

<u>Element mass matrices for various finite elements:</u>

<u>1. Bar element:</u> let N_1 & N_2 are the linear shape functions for the 1-D bar element in natural coordinate system.

$$\boxed{N_1 = \frac{1-\xi}{2} \quad \Big| \quad N_2 = \frac{1+\xi}{2}}$$

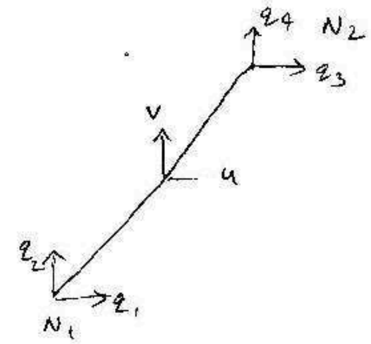
$$m^e = \int \rho N^T N \, dv$$

$$= \int \rho \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \begin{bmatrix} N_1 & N_2 \end{bmatrix} A \, dx$$

$$= \int_{-1}^{1} \rho \begin{bmatrix} \left(\frac{1-\xi}{2}\right)^2 & \frac{1-\xi^2}{4} \\ \frac{1-\xi^2}{4} & \left(\frac{1+\xi}{2}\right)^2 \end{bmatrix} A \frac{l}{2} \, d\xi \qquad \left[\because dx = \frac{l}{2} d\xi\right]$$

$$\boxed{m^e = \frac{\rho A l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}$$

2. Truss element :



$$u = N_1 q_1 + N_2 q_3$$

$$v = N_1 q_2 + N_2 q_4$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

Hence where

$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix}$$

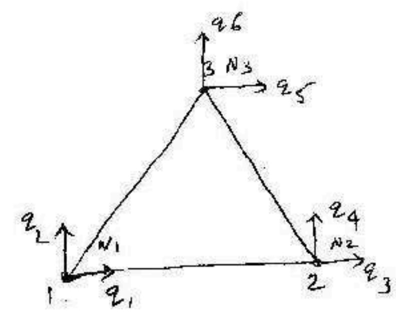
$$\left[\because dx = \frac{l}{2} d\xi\right]$$

$$m^e = \int \rho N^T N \, dv$$

$$= \int \rho \begin{bmatrix} N_1^2 & 0 & N_1 N_2 & 0 \\ 0 & N_1^2 & 0 & N_1 N_2 \\ N_1 N_2 & 0 & N_2^2 & 0 \\ 0 & N_1 N_2 & 0 & N_2^2 \end{bmatrix} A \, dx$$

$$\boxed{m^e = \frac{\rho A l}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}}$$

3. C.S.T element :



$$u = N_1 q_1 + N_2 q_3 + N_3 q_5$$

$$v = N_1 q_2 + N_2 q_4 + N_3 q_6$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_6 \end{bmatrix}$$

$$m^e = \int \rho N^T N \, dV$$

$$= \int \rho \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} t \, dA$$

$$= \rho t \int \begin{bmatrix} \text{multiplication} \\ \text{of matrix} \end{bmatrix} dA$$

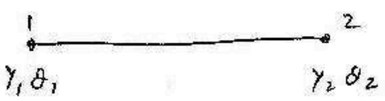
By the principle of lagrangian $\int N^2 dA = A/6$

$$\int N_1 N_2 \, dA = A/12$$

$$\boxed{m^e = \frac{\rho t A}{12} \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix}}$$

4. Beam element :

$$m = \int \rho H^T H \, dv$$



where

$$H = [H_1 \ H_2 \ H_3 \ H_4]$$

$$\left[\because dv = A\,dx \quad dx = \frac{l}{2} d\xi \right]$$

$$m = \int_{-1}^{1} \rho \begin{bmatrix} H_1^2 & H_1H_2 & H_1H_3 & H_1H_4 \\ H_2H_1 & H_2^2 & H_3H_2 & H_2H_4 \\ H_1H_3 & H_2H_3 & H_3^2 & H_3H_4 \\ H_1H_4 & H_2H_4 & H_3H_4 & H_4^2 \end{bmatrix} A \frac{l}{2} d\xi$$

H_1 to H_4 are the hermite shape function for beam element

$H_1 = \frac{1}{4}(2-3\xi+\xi^3)$ $\quad$ $H_2 = \frac{1}{4}(1-\xi-\xi^2+\xi^3)$

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$H_3 = \frac{1}{4}(2+3\xi-\xi^3)$ $\quad$ $H_4 = \frac{1}{4}(-1-\xi+\xi^2+\xi^3)$

On integrating we obtain the mass matrix.

$$\boxed{m^e = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}}$$

The above all elemental mass matrices are known as consistent mass matrices the other method adopted to determine the elemental mass matrices are known as LUMPED parametric mass matrices in which the total mass in each direction is equally distributed to the nodes of the element.

LUMPED mass matrices for elements :-

1) Bar element :-

$$m^e = \frac{\rho A l}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2) Truss element :-

$$m^e = \frac{\rho A l}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

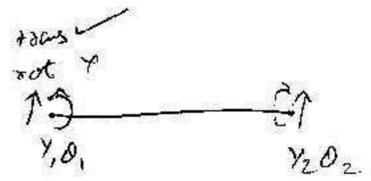
3) C·S·T element :-

$$m^e = \frac{\rho A l}{3}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

NOTE: In lumped parametric model, the total mass is equally distributed among the nodes in each transolational directions hence the total mass is equal to the sum of the nodal masses in each direction.

4) Beam element :-

$$m^e = \frac{\rho A l}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



LUMPED MASS MODEL	CONSISTANT MASS MODEL
→ Masses are Associated with traslation DOF only.	→ The total mass is distributed over both the translation & rotational DOF.
→ masses are equally distributed over nodes	→ not equally distributed
→ only diagonal elements exists other are zero's	→ All elements are exists in matrix
→ moderate results.	→ more Accurate results
→ easy to handle	→ difficult to handle
→ Lower than the exact value	→ exactly equal to the original value.

termination of Eigen values & Eigen vectors:-

The equation of motion for forced vibration is given by

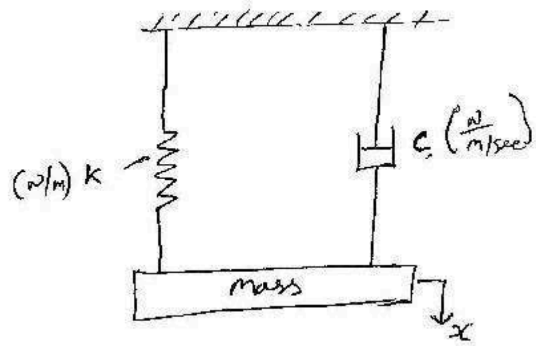
$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F$$

If for free vibration $F = 0$. then

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

for nodal displacement it is given as

$$M\frac{d^2q}{dt^2} + kq = 0$$



Damper [C] doesn't have nodal displacement.

$$\boxed{M\ddot{Q} + KQ = 0}$$

The above equation represents the equation of S.H.M

let the solution of above equation is

let $Q = U\sin\omega t$

$$\dot{Q} = U\omega\cos\omega t$$

$$\ddot{Q} = -U\omega^2\sin\omega t$$

$$\ddot{Q} = -Q\omega^2$$

$$M(-Q\omega^2) + kQ = 0$$

$$kQ = QM\omega^2$$

[where $\omega^2 = \lambda$]

$$kQ = \lambda QM$$

$$\boxed{k - \lambda M = 0}$$

Dynamic equation

where λ – is known as eigenvalues
k – global stiffness matrix
M – global mass matrix.

problem

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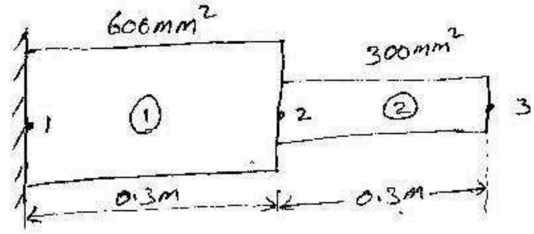
(1) Determine the eigen values, eigen vectors, natural frequencies & mode shapes for the bar shown in the figure below.



Take

$\rho = 7800\ kg/m^3$

$E = 200GPa$

Sol: stiffness:

$$K^1 = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{600\times10^{-6}\times 200\times10^{9}}{0.3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K^1 = 10^6\begin{bmatrix} 400 & -400 \\ -400 & 400 \end{bmatrix}$$

$$k^2 = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{300\times10^{-6}\times 200\times10^{9}}{0.3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$k^2 = 10^6\begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix}$$

$\underset{\sim}{K} = k^1 + k^2$

node 1 is fixed so 1 row & 1 column is eliminated & modified $\underset{\sim}{K}$ becomes as below.

$$\underset{\sim}{K} = 10^6\begin{bmatrix} 600 & -200 \\ -200 & 200 \end{bmatrix}$$

mass matrix:

$$m^e = \frac{\rho A L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$m^1 = \frac{7800 \times 600 \times 10^{-6} \times 0.3}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$m^1 = \begin{bmatrix} 0.468 & 0.234 \\ 0.234 & 0.468 \end{bmatrix}$$

$$m^2 = \frac{7800 \times 300 \times 10^{-6} \times 0.3}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$m^2 = \begin{bmatrix} 0.234 & 0.117 \\ 0.117 & 0.234 \end{bmatrix}$$

Hence node 1 is fixed so modified mass matrix becomes as below

$$\underset{\sim}{M} = \begin{bmatrix} 0.702 & 0.117 \\ 0.117 & 0.234 \end{bmatrix}$$

By using charectaristic polynomial equation $K - \lambda M = 0$

$$\underset{\sim}{K} - \lambda \underset{\sim}{M} = 0$$

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$$\underset{\sim}{k} - \lambda \underset{\sim}{m} = 0$$

$$10^6 \begin{bmatrix} 600 & -200 \\ -200 & 200 \end{bmatrix} - \lambda \begin{bmatrix} 0.702 & 0.117 \\ 0.117 & 0.234 \end{bmatrix} = 0$$

$$\begin{bmatrix} 6 \times 10^8 - 0.702\lambda & -2\times10^8 - 0.117\lambda \\ -2\times10^8 - 0.117\lambda & 2\times10^8 - 0.234\lambda \end{bmatrix} = 0.$$

To find the eigen values for the above equation first find the

det $k - \lambda m$

$$|k - \lambda m| = 0.$$

$$\begin{vmatrix} 6\times10^8 - 0.702\lambda & -2\times10^8 - 0.117\lambda \\ -2\times10^8 - 0.117\lambda & 2\times10^8 - 0.234\lambda \end{vmatrix} = 0.$$

$$(6\times10^8 - 0.702\lambda)(2\times10^8 - 0.234\lambda) - (-2\times10^8 - 0.117\lambda)(-2\times10^8 - 0.117\lambda) = 0.$$

$$\left[1.2\times10^{17} - 140.4\times10^6\lambda - 140.4\times10^6\lambda + 0.164\lambda^2\right] - \left[4\times10^{16} + 23.4\times10^6\lambda + 23.4\times10^6\lambda + 0.013\lambda^2\right]$$

$$\left[0.164\lambda^2 - 280.8\times10^6\lambda + 1.2\times10^{17}\right] - \left[0.013\lambda^2 + 46.8\times10^6\lambda + 4\times10^{16}\right]$$

$$0.151\lambda^2 - 327.6\times10^6\lambda + 8\times10^{16} = 0.$$

By solving the above quadratic equation we get eigen values.

$$\left.\begin{aligned}\lambda_1 &= 18.84\times10^8\\ \lambda_2 &= 2.8115\times10^8\end{aligned}\right\}\text{ eigen values.}$$

So = $\lambda = \omega^2$, where ω is the circular frequency given by

$$\omega = 2\pi F$$

where $F = \dfrac{\omega}{2\pi}$

$\omega_1 = \sqrt{\lambda_1}$ $\qquad$ $\omega_2 = \sqrt{\lambda_2}$

$\omega_1 = 43.4\times10^3$ rad/sec. $\qquad$ $\omega_2 = 16.76\times10^3$ rad/sec

$F_1 = \dfrac{\omega_1}{2\pi}$ $\qquad$ $F_2 = \dfrac{\omega_2}{2\pi}$

$F_1 = 68.17\times10^3$ Hzs $\qquad$ $F_2 = 26.3\times10^3$ Hz

F_1 & F_2 are the frequencies.

igen vectors :

$$\lambda_1 = 18.84 \times 10^8$$

$$[k - \lambda m][u] = 0$$

$$\cancel{10^8}\begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix} - 18.84 \times \cancel{10^8} \begin{bmatrix} 0.702 & 0.117 \\ 0.117 & 0.234 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -7.22 & -4.2 \\ -4.2 & -2.4 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = 0.$$

$$-7.22 u_2 = 4.2 u_3$$

$$u_3 = -1.72 u_2$$

By orthogonal property of eigen vectors

$$U^T M U = 1$$

$$[U_2, -1.72 u_2] \begin{bmatrix} 0.702 & 0.117 \\ 0.117 & 0.234 \end{bmatrix} \begin{bmatrix} u_2 \\ -1.72 u_2 \end{bmatrix} = 1$$

$$[u_2 \quad -1.72 u_2] \begin{bmatrix} 0.702 u_2 & -0.2012 u_2 \\ 0.117 u_2 & -0.4024 u_2 \end{bmatrix} = 1.$$

$$\begin{bmatrix} u_2 & -1.72u_2 \end{bmatrix} \begin{bmatrix} 0.5008u_2 \\ -0.2854u_2 \end{bmatrix} = 1$$

$$0.5008u_2^2 + 0.49u_2^2 = 1$$

$$0.99u_2^2 = 1$$

$$u_2 = \sqrt{\frac{1}{0.99}}$$

$$u_2 = 1.005$$

$$u_3 = -1.72u_2$$

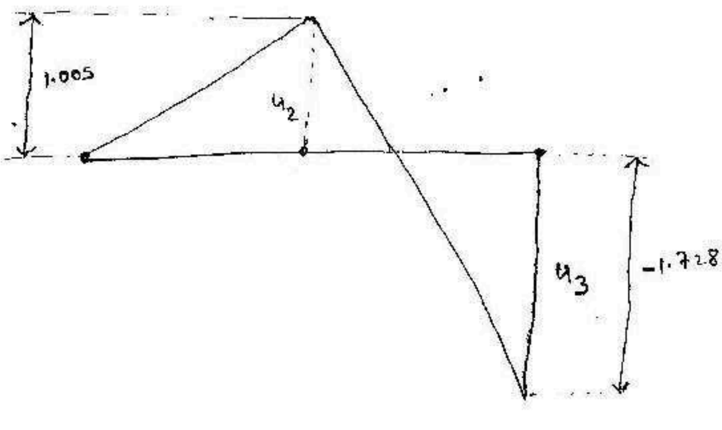
$$= -1.72 \times 1.005$$

$u_2 = 1.005$ $\quad u_3 = -1.728$ $\quad$ & $u_1 = 0$.

∴ eigen vectors for 1st eigen value are given below.

$$\begin{bmatrix} 0 & 1.005 & -1.728 \end{bmatrix}^T$$

The mode shape corresponding to the first eigen vector as shown in below figure.



eigen value for $\lambda_2 = 2.8115 \times 10^8$

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$$[K - \lambda M][u] = 0.$$

$$10^8 \begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix} - 2.8115 \times 10^8 \begin{bmatrix} 0.702 & 0.117 \\ 0.117 & 0.234 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = 0.$$

$$\begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix} - \begin{bmatrix} 1.973 & 0.329 \\ 0.329 & 0.657 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = 0.$$

$$\begin{bmatrix} 4.027 & -2.329 \\ -2.329 & 1.343 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = 0.$$

$$4.027\, u_2 - 2.329\, u_3 = 0.$$

$$4.027\, u_2 = 2.329\, u_3$$

$$u_3 = 1.73\, u_2$$

So get the eigen vectors are previous & draw the corresponding mode shape

Beam Problems

② Determine the natural frequencies of simply supported beam of length 800mm with cross sectional area of 75mm × 20mm as shown in figure. Take $E = 200\,GPa$ $\rho = 7850\ kg/m^3$

Sol:

$$\rho = 7850\ kg/m^3$$

$$l = 0.8\ m$$

$$E = 200 \times 10^9\ N/m^2$$

$$A = 75 \times 25 \times 10^{-6}\ m^2$$

$$I = \frac{bd^3}{12} = \frac{75 \times 25^3 \times 10^{-12}}{12} = 9.76 \times 10^{-8}\ m^4$$

$$K = \frac{EI}{l^3} \begin{array}{c} \begin{array}{cccc} y_1 & \theta_1 & y_2 & \theta_2 \end{array} \\ \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \end{array}$$

$$K = \begin{bmatrix} 97600 & 48800 \\ 48800 & 97600 \end{bmatrix}$$

matrix

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$$M = \frac{\rho A l}{420}\begin{array}{c} \begin{array}{cccc} y_1 & \theta_1 & y_2 & \theta_2 \end{array} \\ \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \end{array}\begin{array}{c} y_1 \\ \theta_1 \\ y_2 \\ \theta_2 \end{array}$$

$$M = \begin{bmatrix} 0.071 & -0.0531 \\ -0.0531 & 0.071 \end{bmatrix}$$

By using characteristic polynomial equation

$$|K - \lambda M| = 0$$

$$\begin{bmatrix} 97600 & 48800 \\ 48800 & 97600 \end{bmatrix} - \lambda \begin{bmatrix} 0.071 & -0.053 \\ -0.053 & 0.071 \end{bmatrix} = 0$$

$$\begin{vmatrix} 97600 - 0.071\lambda & 48800 + 0.053\lambda \\ 48800 + 0.053\lambda & 97600 - 0.071\lambda \end{vmatrix} = 0.$$

$$(97600 - 0.071\lambda)^2 - (48800 + 0.053\lambda)^2 = 0$$

By solving the above equation weget.

$$\lambda = 39.3 \times 10^4$$

$$\omega = \sqrt{\lambda} \quad rad/sec$$

$$f = \frac{\omega}{2\pi}$$

$$F_n = 99.8 \quad Hz \ (cycles/sec)$$

As like previous problem repeat the same steps for finding the eigen vectors & mode shapes.