INSTITUTE OF AERONAUTICAL ENGINEERING
(Autonomous)
Dundigal – 500043, Hyderabad

LINEAR ALGEBRA AND CALCULUS(LAC)
B.TECH ISEM
FRESHMAN ENGINEERING

Ms. L Indira
Associate Professor

CONTENTS

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MODULE -I

THEORY OF MATRICES

Solution for linear systems

Matrix : A system of mn numbers real (or) complex arranged in the form of an ordered set of 'm' rows, each row consisting of an ordered set of 'n' numbers between [ ] (or) ( ) (or) || || is called a matrix of order m xn.

Eg: $$\begin{bmatrix} a_{11} & a_{12} \cdots\cdots & a_{1n} \\ a_{21} & a_{12} \cdots\cdots & a_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots \\ \cdots\cdots & \cdots\cdots & \cdots \\ a_{m1} & a_{m2} \cdots\cdots & a_{mn} \end{bmatrix}_{m \times n} = [a_{ij}]_{m\times n} \text{ where } 1\le i\le m,\ 1\le j\le n.$$

some types of matrics :

1. **square matrix :** A square matrix A of order nxn is sometimes called as a n- rowed matrix A (or) simply a square matrix of order n

   eg : $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is 2^{nd} order matrix

2. Rectangular matrix : A matrix which is not a square matrix is called a rectangular matrix,

$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$ is a 2x3 matrix

3. Row matrix : A matrix of order 1xm is called a row matrix

eg: $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1x3}$

4. Column matrix : A matrix of order nx1 is called a column matrix

Eg: $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}_{3\times1}$

5. **Unit matrix :** if A= $[a_{ij}]_{n\times n}$ such that a_{ij} = 1 for i = j and a_{ij} = 0 for i≠j, then A is called a unit matrx.

Eg: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6. **Zero matrix :** it A = $[a_{ij}]_{m\times n}$ such that a_{ij} = 0 ∀ I and j then A is called a zero matrix (or) null matrix

7. **Diagonal elements in a matrix** A= $[a_{ij}]_{n\times n}$, the elements a_{ij} of A for which i = j. i.e. (a_{11}, a_{22}....a_{nn}) are called the diagonal elements of A

Eg: A= $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ diagonal elements are 1,5,9

Note: the line along which the diagonal elements lie is called the principle diagonal of A

8. **Diagonal matrix:** A square matrix all of whose elements except those in leading diagonal are zero is called diagonal matrix.

If d_1, d_2..... d_n are diagonal elements of a diagonal matrix A, then A is written as A = diag (d_1,d_2....d_n)

E.g. : A = diag (3, 1,-2)= $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

12. The conjugate of a matrix: The matrix obtained from any given matrix A, on replacing its elements by corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\bar{A}$

Eg ; if A= $\begin{bmatrix} 2 & 3i & 2-5i \\ -i & 0 & 4i+3 \end{bmatrix}_{2x3}$ $then\ \bar{A} = \begin{bmatrix} 2 & -3i & 2+5i \\ i & 0 & -4i+3 \end{bmatrix}_{2x3}$

13. <u>The conjugate Transpose of a matrix</u>

The conjugate of the transpose of the matrix A is called the conjugate transpose of A and is denoted by A^{θ} Thus $A^{\theta} = (\bar{A^1})$ where A^1 is the transpose of A. Now A = $[a_{ij}]_{m\times n}$ $\Rightarrow$ $A^{\theta} = [b_{ij}]_{n\times m}$, where bij = $\bar{a}$ij i.e. the $(i,j)^{th}$ element of A^{θ} conjugate complex of the $(j, i)^{th}$ element of A

Eg: if A = $\begin{bmatrix} 5 & 3-i & -2i \\ 0 & 1+i & 4-i \end{bmatrix}_{2X3}$ then A^{θ} = $\begin{bmatrix} 5 & 0 \\ 3+i & 1-i \\ 2i & 4+i \end{bmatrix}_{3x2}$

Note: $A^{\theta} = \bar{A^1} = (\bar{A})^1\ and (A^{\theta})^{\theta}$ =A

14.

(i) Upper Triangular matrix : A square matrix all of whose elements below the leading diagonal are zero is called an Upper triangular matrix. i.e, $a_{ij}=0$ for $i>j$

E.g.; $\begin{bmatrix} 1 & 3 & 8 \\ 0 & 4 & -5 \\ 0 & 0 & 2 \end{bmatrix}$ is an **Upper triangular matrix**

(ii) **Lower triangular matrix:** A square matrix all of whose elements above the leading diagonal are zero is called a lower triangular matrix. i.e., $a_{ij}=0$ for $i<j$

E.g.: $\begin{bmatrix} 4 & 0 & 0 \\ 5 & 2 & 0 \\ 7 & 3 & 6 \end{bmatrix}$ is an **Lower triangular matrix**

(iii) **Triangular matrix:** A matrix is said to be triangular matrix it is either an upper triangular matrix or a lower triangular matrix

15. Symmetric matrix: A square matrix A =[a_{ij}] is said to be symmetric if a_{ij} = a_{ji} for every i and j

Thus A is a symmetric matrix if A^T= A

Eg: $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix

16. Skew – Symmetric: A square matrix A = [a_{ij}] is said to be skew – symmetric if a_{ij} = – a_{ji} for every i and j.

Eg : $\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$ is a skew – symmetric matrix

Thus A is a skew – symmetric iff $A= -A^1$ (or) $-A= A^1$

Note: Every diagonal element of a skew – symmetric matrix is necessarily zero.

Since $a_{ij} = -a_{ij} \Rightarrow a_{ij} = 0$

17. Multiplication of a matrix by a scalar.

Let 'A' be a matrix. The matrix obtain by multiplying every element of A by a scalar K, is called the product of A by K and is denoted by KA (or) AK

Thus : $A + [a_{ij}]_{m\times n}$ then $KA = [ka_{ij}]_{m\times n} = k[a_{ij}]_{m\times n}$

18. Sum of matrices:

Let $A = [a_{ij}]_{m\times n}$,$B = [b_{ij}]_{m\times n}$ be two matrices. The matrix $C = [c_{ij}]_{m\times n}$ where $c_{ij} = a_{ij}+b_{ij}$ is called the sum of the matrices A and B.

The sum of A and B is denoted by A+B. Thus $[a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [a_{ij}+b_{ij}]_{m\times n}$ and

$[a_{ij}+b_{ij}]_{m\times n} = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}$

19. The difference of two matrices: If A, B are two matrices of the same type then A+(-B) is taken as A – B

22. **Trace of A square matrix :** Let A = $[a_{ij}]_{n\times n}$ the trace of the square matrix A is defined as $\sum_{i=1}^{n} a_{ii}$. And is denoted by 'tr A'

Thus trA = $\sum_{i=1}^{n} a_{ii}$ = $a_{11}+a_{22}+ \ldots\ldots a_{nn}$

Eg : A = $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ then trA = a+b+c

Properties : If A and B are square matrices of order n and λ is any scalar, then

(i) tr (λ A) = λ tr A

(ii) tr (A+B) = trA + tr B

(iii) tr(AB) = tr(BA)

23. Idempotent matrix: If A is a square matrix such that $A^2 = A$ then 'A' is called idempotent matrix

24. Nilpotent Matrix: If A is a square matrix such that $A^m=0$ where m is a +ve integer then A is called nilpotent matrix.

Note : If m is least positive integer such that $A^m = 0$ then A is called nilpotent of index m

25. Involutary : If A is a square matrix such that $A^2 = I$ then A is called involuntary matrix.

26. Orthogonal Matrix: A square matrix A is said to be orthogonal if $AA^1 = A^1A = I$

Examples:

1. Show that A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ **is orthogonal.**

Sol: Given A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

A^T = $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Consider $A.A^T$ = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is orthogonal matrix.

27 . Minors and cofactors of a square matrix

Let A =[a_{ij}] $_{n\times n}$ be a square matrix when form A the elements of i^{th} row and j^{th} column are deleted the determinant of (n-1) rowed matrix [Mij] is called the minor of aij of A and is denoted by $|M_{ij}|$

The signed minor $(-1)^{i+j}\ |M_{ij}|$ is called the cofactor of a_{ij} and is denoted by A_{ij}.

If A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$|A| = a_{11}\ |M_{11}| + a_{12}\ |M_{12}| + a_{13}\ |M_{13}|$ (or)

$= a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$

Note 1: If A is a square matrix of order n then $|KA| = K^n |A|$, where k is a scalar.

Note 2: If A is a square matrix of order n, then $|A| = |A^T|$

Note 3: If A and B be two square matrices of the same order, then $|AB| = |A|\ |B|$

28. Inverse of a Matrix: Let A be any square matrix, then a matrix B, if exists such that AB = BA =I then B is called inverse of A and is denoted by A^{-1}.

29. Adjoint of a matrix:

Let A be a square matrix of order n. The transpose of the matrix got from A

By replacing the elements of A by the corresponding co-factors is called the adjoint of A and is denoted by adj A.

Note: For any scalar k, adj(kA) = k^{n-1} adj A

Note : The necessary and sufficient condition for a square matrix to posses inverse is that $|A| \neq 0$

Note: if $|A| \neq 0$ then $A^{-1} = \frac{1}{|A|}(adj\ A)$

30. Singular and Non-singular Matrices:

A square matrix A is said to be singular if $|A| = 0$. If $|A| \neq 0$ then A is said to be non-singular.

Note: 1. Only non-singular matrices posses inverses.

2. The product of non-singular matrices is also non-singular.

Real and complex matrices

Conjugate of a matrix:

If the elements of a matrix A are replaced by their conjugates then the resulting matrix is defined as the conjugate of the given matrix. We denote it with $\overline{A}$

e.g If A= $\begin{bmatrix} 2+3i & 5 \\ 6-7i & -5+i \end{bmatrix}$ then $\overline{A}$= $\begin{bmatrix} 2-3i & 5 \\ 6+7i & -5-i \end{bmatrix}$

The transpose of the conjugate of a square matrix:

If A is a square matrix and its conjugate is $\overline{A}$, then the transpose of $\overline{A}$ is $\left(\overline{A}\right)^T$.

It can be easily seen that $\left(\overline{A}\right)^T = \overline{A^T}$

It is denoted by A^θ

$$A^\theta = \left(\overline{A}\right)^T = \overline{A^T}$$

Note: If A^θ and B^θ be the transposed conjugates of A and B respectively, then

i) $\left(A^\theta\right)^\theta = A$ ii) $\left(A \pm B\right)^\theta = A^\theta \pm B^\theta$ iii) $\left(KA\right)^\theta = \overline{K}A^\theta$ iv) $\left(AB\right)^\theta = B^\theta A^\theta$

Hermitian matrix:

A square matrix A such that $\overline{A}$= A^T (or) $\left(\overline{A}\right)^T$ =A is called a hermitian matrix e.g $A=\begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$ then $\bar{A}=\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ and $A^{\theta}=\begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$

Here $\left(\bar{A}\right)^{T}$=A, Hence A is called Hermitian

Note:

1) The element of the principal diagonal of a Hermitian matrix must be real
2) A hermitian matrix over the field of real numbers is nothing but a real symmetric.

Skew-Hermitian matrix

A square matrix A such that $A^{T}=-\bar{A}$(or) $\left(\bar{A}\right)^{T}$=-A is called a Skew-Hermitian matrix

e.g. Let $A=\begin{bmatrix} -3i & 2+i \\ -2+i & -i \end{bmatrix}$ then $\bar{A}=\begin{bmatrix} 3i & 2-i \\ -2-i & i \end{bmatrix}$ and $\left(\bar{A}\right)^{T}=\begin{bmatrix} 3i & -2-i \\ 2-i & i \end{bmatrix}$

$\therefore \left(\bar{A}\right)^{T}$=-A

A is skew-Hermitian matrix

Note:

1) The elements of the leading diagonal must be zero (or) all are purely imaginary
2) A skew-Hermitian matrix over the field of real numbers is nothing but a real skew-symmetric matrix.

Elementary row and column transformations

The matrix operations of

1. Interchanging two rows or columns,
2. Adding a multiple of one row or column to another,
3. Multiplying any row or column by a nonzero element.

Elementary Matrix Operations

Elementary matrix operations play an important role in many matrix algebra applications, such as finding the inverse of a matrix and solving simultaneous linear equations.

Elementary Operations

There are three kinds of elementary matrix operations.

1. Interchange two rows (or columns).
2. Multiply each element in a row (or column) by a non-zero number.
3. Multiply a row (or column) by a non-zero number and add the result to another row (or column).

When these operations are performed on rows, they are called **elementary row operations**; and when they are performed on columns, they are called **elementary column operations**.

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Operation description	Notation
Row operations	
1. Interchange rows i and j	R_i <--> R_j
2. Multiply row i by s, where $s \neq 0$	sR_i --> R_i
3. Add s times row i to row j	$sR_i + R_j$ --> R_j
Column operations	
1. Interchange columns i and j	C_i <--> C_j
2. Multiply column i by s, where $s \neq 0$	sC_i --> C_i
3. Add s times column i to column j	$sC_i + C_j$ --> C_j

Elementary Operators

Each type of elementary operation may be performed by matrix multiplication, using square matrices called **elementary operators**.

For example, suppose you want to interchange rows 1 and 2 of Matrix **A**. To accomplish this, you could pre-multiply **A** by **E** to produce **B**, as shown below.

$$R_1 <--> R_2 \quad = \quad \underset{\mathbf{E}}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \quad \underset{\mathbf{A}}{\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}}$$

$$R_1 <--> R_2 \quad = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix} = \mathbf{B}$$

How to Perform Elementary Row Operations

To perform an elementary row operation on a **A**, an $r \times c$ matrix, take the following steps.

1. To find **E**, the **elementary row operator**, apply the operation to an $r \times r$ identity matrix
2. To carry out the elementary row operation, premultiply **A** by **E**.

We illustrate this process below for each of the three types of elementary row operations.

- **Interchange two rows**. Suppose we want to interchange the second and third rows of **A**, a 3 x 2 matrix.

  To create the elementary row operator **E**, we interchange the second and third rows of the identity matrix $\mathbf{I}_3$.

$$\underset{\mathbf{I}_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \Rightarrow \underset{\mathbf{E}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}$$

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- Then, to interchange the second and third rows of **A**, we pre-multiply **A** by **E**, as shown below.

$$R_2 <--> R_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}$$

E **A**

- $$R_2 <--> R_3 = \begin{bmatrix} 1*0 + 0*2 + 0*4 & 1*1 + 0*3 + 0*5 \\ 0*0 + 0*2 + 1*4 & 0*1 + 0*3 + 1*5 \\ 0*0 + 1*2 + 0*4 & 0*1 + 1*3 + 0*5 \end{bmatrix}$$

- $$R_2 <--> R_3 = \begin{bmatrix} 0 & 1 \\ 4 & 5 \\ 2 & 3 \end{bmatrix}$$

- **Multiply a row by a number.** Suppose we want to multiply each element in the second row of Matrix **A** by 7. Assume **A** is a 2 x 3 matrix. To create the elementary row operator **E**, we multiply each element in the second row of the identity matrix $\mathbf{I}_2$ by 7.

$$\underset{\mathbf{I}_2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \Rightarrow \underset{\mathbf{E}}{\begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}}$$

- Then, to multiply each element in the second row of **A** by 7, we premultiply **A** by **E**.

$$7R_2 \dashrightarrow R_2 = \underset{\mathbf{E}}{\begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}} \underset{\mathbf{A}}{\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}}$$

$$7R_2 \dashrightarrow R_2 = \begin{bmatrix} 1*0 + 0*3 & 1*1 + 0*4 & 1*2 + 0*5 \\ 0*0 + 7*3 & 0*1 + 7*4 & 0*2 + 7*5 \end{bmatrix}$$

$$7R_2 \dashrightarrow R_2 = \begin{bmatrix} 1*0 + 0*3 & 1*1 + 0*4 & 1*2 + 0*5 \\ 0*0 + 7*3 & 0*1 + 7*4 & 0*2 + 7*5 \end{bmatrix}$$

$$7R_2 \dashrightarrow R_2 = \begin{bmatrix} 0 & 1 & 2 \\ 21 & 28 & 35 \end{bmatrix}$$

- **Multiply a row and add it to another row.** Assume **A** is a 2 x 2 matrix. Suppose we want to multiply each element in the first row of **A** by 3; and we want to add that result to the second row of **A**. For this operation, creating the elementary row operator is a two-step process. First, we multiply each element in the first row of the identity matrix $\mathbf{I}_2$ by 3. Next, we add the result of that multiplication to the second row of $\mathbf{I}_2$ to produce **E**.

$$\underset{\mathbf{I}_2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 + 3*1 & 1 + 3*0 \end{bmatrix} \Rightarrow \underset{\mathbf{E}}{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}$$

- Then, to multiply each element in the first row of **A** by 3 and add that result to the second row, we pre-multiply **A** by **E**.

$$3R_1 + R_2 \dashrightarrow R_2 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

- Then, to multiply each element in the first row of **A** by 3 and add that result to the second row, we pre-multiply **A** by **E**.

$$3R_1 + R_2 \dashrightarrow R_2 = \underset{\mathbf{E}}{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}} \underset{\mathbf{A}}{\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}}$$

$$3R_1 + R_2 \dashrightarrow R_2 = \begin{bmatrix} 1*0 + 0*2 & 1*1 + 0*3 \end{bmatrix}$$

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How to Perform Elementary Column Operations

To perform an elementary column operation on **A**, an *r* x *c* matrix, takes the following steps.

1. To find **E**, the **elementary column operator**, apply the operation to an *c* x *c* identity matrix.
2. To carry out the elementary column operation, post-multiply **A** by **E**.

Let's work through an elementary column operation to illustrate the process. For example, suppose we want to interchange the first and second columns of **A**, a 3 x 2 matrix. To create the elementary column operator **E**, we interchange the first and second columns of the identity matrix $\mathbf{I}_2$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\mathbf{I}_2$ **E**

Then, to interchange the first and second columns of **A**, we postmultiply **A** by **E**, as shown below.

$$C_1 <--> C_2 \quad = \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

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$$C_1 <--> C_2 = \begin{bmatrix} 0*0 + 1*1 & 0*1 + 1*0 \\ 2*0 + 3*1 & 2*1 + 3*0 \\ 4*0 + 5*1 & 4*1 + 5*0 \end{bmatrix}$$

$$C_1 <--> C_2 = \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 5 & 4 \end{bmatrix}$$

Note that the process for performing an elementary column operation on an *r* x *c* matrix is very similar to the process for performing an elementary row operation. The main differences are:

- To operate on the *r* x *c* matrix **A**, the row operator **E** is created from an *r* x *r* identity matrix; whereas the column operator **E** is created from an *c* x *c* identity matrix.

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- To perform a row operation, **A** is *pre-multiplied* by **E**; whereas to perform a column operation, **A** is *post-multiplied* by **E**.
- **Problem 1**
- Assume that **A** is a 4 x 3 matrix. Suppose you want to multiply each element in the second column of matrix **A** by 9. Find the elementary column operator **E**.
- **Solution**
- To find the elementary column operator **E**, we multiply each element in the second column of the identity matrix $\mathbf{I}_3$ by 9.

$$\underset{\mathbf{I}_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \Rightarrow \underset{\mathbf{E}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Rank of a Matrix:

Let A be m x n matrix. If A is a null matrix, we define its rank to be '0'. If A is a non-zero matrix, we say that r is the rank of A if

(i) Every $(r+1)^{th}$ order minor of A is '0' (zero) &

(ii) At least one r^{th} order minor of A which is not zero.

Note: 1. It is denoted by $\rho(A)$

2. Rank of a matrix is unique.

3. Every matrix will have a rank.

4. If A is a matrix of order mxn,

Rank of A $\leq$ min(m,n)

5. If $\rho(A) = r$ then every minor of A of order r+1, or more is zero.

6. Rank of the Identity matrix I_n is n.

7. If A is a matrix of order n and A is non-singular then $\rho(A) = n$

Important Note:

1. The rank of a matrix is $\leq r$ if all minors of $(r+1)^{th}$ order are zero.
2. The rank of a matrix is $\geq r$, if there is at least one minor of order 'r' which is not equal to zero.

PROBLEMS

1. **Find the rank of the given matrix** $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

Sol: Given matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

→ det A = 1(48-40)-2(36-28)+3(30-28)

= 8-16+6 = -2 ≠ 0

We have minor of order 3

$\rho(A) = 3$

2. Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$

Sol: Given the matrix is of order 3x4

Its Rank $\leq$ min(3,4) = 3

Highest order of the minor will be 3.

Let us consider the minor $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{bmatrix}$

Determinant of minor is 1(-49)-2(-56)+3(35-48)

= -49+112-39 = 24 $\neq$ 0.

Hence rank of the given matrix is '3'.

*** Elementary Transformations on a Matrix:**

i). Interchange of i^{th} row and j^{th} row is denoted by $R_i \leftrightarrow R_j$

(ii). If i^{th} row is multiplied with k then it is denoted by $R_i \to K\, R_i$

(iii). If all the elements of i^{th} row are multiplied with k and added to the corresponding elements of j^{th} row then it is denoted by $R_j \to R_j + KR_i$

Note: 1. The corresponding column transformations will be denoted by writing 'c'. i.e

$c_i \leftrightarrow c_j$, $c_i \to k\, c_j$ $c_j \to c_j + kc_i$

2. The elementary operations on a matrix do not change its rank.

Equivalence of Matrices: If B is obtained from A after a finite number of elementary transformations on A, then B is said to be equivalent to A.

It is denoted as B~A.

Note : 1. If A and B are two equivalent matrices, then rank A = rank B.

2. If A and B have the same size and the same rank, then the two matrices are equivalent.

Echelon form of a matrix:

A matrix is said to be in Echelon form, if

(i). Zero rows, if any exists, they should be below the non-zero row.

(ii). The first non-zero entry in each non-zero row is equal to '1'.

(iii). The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

<u>Note:</u> 1. the number of non-zero rows in echelon form of A is the rank of 'A'.

2. The rank of the transpose of a matrix is the same as that of original matrix.
3. The condition (ii) is optional.

E.g.: 1. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a row echelon form.

2. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is a row echelon form.

PROBLEMS

1. Find the rank of the matrix A = $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$ **by reducing it to Echelon form.**

sol: Given A = $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

Applying row transformations on A.

$A \sim \begin{bmatrix} 1 & -3 & -1 \\ 3 & -2 & 4 \\ 2 & 3 & 7 \end{bmatrix} R_1 \leftrightarrow R_3$

$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 7 & 7 \\ 0 & 9 & 9 \end{bmatrix} R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad R_2 \to R_2/7 \; R_3 \to R_3/9$$

$$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

This is the Echelon form of matrix A.

The rank of a matrix A.

= Number of non – zero rows =2

2. For what values of k the matrix $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$ **has rank '3'.**

Sol: The given matrix is of the order 4x4

If its rank is 3 ⇨ det A =0

$$A = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$$

Applying $R_2 \rightarrow 4R_2 - R_1$, $R_3 \rightarrow 4R_3 - kR_1$, $R_4 \rightarrow 4R_4 - 9R_1$

We get $A \sim \begin{bmatrix} 4 & 4 & -3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 8-4k & 8+3k & 8-k \\ 0 & 0 & 4k+27 & 3 \end{bmatrix}$

Since Rank A = 3 ⇨ det A =0

$$\Rightarrow 4 \begin{vmatrix} 0 & -1 & -1 \\ 8-4k & 8+3k & 8-k \\ 0 & 4k+27 & 3 \end{vmatrix} = 0$$

- ⇨ 1[(8-4k)3]-1(8-4k)(4k+27)] = 0
- ⇨ (8-4k) (3-4k-27) = 0
- ⇨ (8-4k)(-24-4k) =0
- ⇨ (2-k)(6+k)=0
- ⇨ k =2 or k = -6

Normal form

Normal Form:

Every mxn matrix of rank r can be reduced to the form $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ (or) (I_r) (or) $\begin{pmatrix} I_r \\ 0 \end{pmatrix}(or)\begin{pmatrix} I_r & 0 \end{pmatrix}$ by a finite number of elementary transformations, where I_r is the r – rowed unit matrix.

Note: 1. If A is an mxn matrix of rank r, there exists non-singular matrices P and Q such that PAQ = $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$

Normal form another name is "canonical form"

Normal form

e.g.: By reducing the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ into normal form, find its rank.

Sol: Given A = $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

A ~ $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & 5 \\ 0 & -6 & -4 & -22 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

A ~ $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 3 & 2 & 11 \end{bmatrix}$ $R_3 \to R_3/-2$

Normal form

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix} R_3 \to R_3+R_2$$

$$A \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix} c_2 \to c_2 - 2c_1,\ c_3 \to c_3-3c_1,\ c_4 \to c_4-4c_1$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 18 \end{bmatrix} c_3 \to 3\,c_3 -2c_2,\ c_4 \to 3c_4-5c_2$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} c_2 \to c_2/-3,\ c_4 \to c_4/18$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} c_4 \leftrightarrow c_3$$

This is in normal form $[I_3\ 0]$

Hence Rank of A is '3'.

Normal form

Inverse of a matrix by Gauss-Jordhan method

Definition

- An $n \cdot n$ matrix A is *nonsingular* or *invertible* if $n \cdot n$ A^{-1} exists with $AA^{-1} = A^{-1}A = I$
- The matrix A^{-1} is called the *inverse* of A
- A matrix without an inverse is called *singular* or *noninvertible*

Inverse of a Matrix:

Gauss – Jordan method

- The inverse of a matrix by elementary Transformations: **(Gauss – Jordan method)**

1. suppose A is a non-singular matrix of order 'n' then we write $A = I_n A$
2. Now we apply elementary row-operations only to the matrix A and the pre-factor I_n of the R.H.S
3. We will do this till we get $I_n = BA$ then obviously B is the inverse of A.

Inverse of a matrix by Gauss-Jordan method

*Find the inverse of the matrix A using elementary operations where A= $\begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

Sol:

Given A = $\begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

We can write A = I_3 A

$$\begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \rightarrow 2R_3 - R_2$, we get

$$\begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} A$$

Inverse of a matrix by Gauss-Jordan method

Applying $R_1 \rightarrow R_1 - 3R_2$, we get

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} A$$

Applying $R_1 \rightarrow R_1 + 5R_3$, $R_2 \rightarrow R_2 - 3R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -8 & 10 \\ 0 & 4 & -6 \\ 0 & -1 & 2 \end{bmatrix} A$$

Applying $R_2 \rightarrow R_2/2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -8 & 10 \\ 0 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix} A \Rightarrow I_3 = BA$$

B is the inverse of A.

Inverse of a matrix by Gauss-Jordan method

Example

Find the inverse of $A=\begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix}$

We augment the matrix to form $\begin{bmatrix} 7 & 3 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{bmatrix}$ And perform row operations to reduce the left-side to the identity.

$$\begin{bmatrix} 7 & 3 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{bmatrix} \quad \underset{\rightarrow}{\frac{1}{7}R_1} \begin{bmatrix} 1 & \frac{3}{7} & \frac{1}{7} & 0 \\ 5 & 2 & 0 & 1 \end{bmatrix} \quad \underset{\rightarrow}{-5R_1} \begin{bmatrix} 1 & 3/7 & 1/7 & 0 \\ 0 & 5 & 0 & 1 \end{bmatrix}$$

$$\underset{\rightarrow}{-7R_2} \begin{bmatrix} 1 & \frac{3}{7} & \frac{1}{7} & 0 \\ 0 & 1 & 5 & -7 \end{bmatrix} \quad \underset{\rightarrow}{\frac{-3}{7}R_2+R_1} \begin{bmatrix} 1-0 & \frac{3}{7}-\frac{3}{7} & \frac{1}{7}-\frac{15}{7} & 0+3 \\ 0 & 1 & 5 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 5 & -7 \end{bmatrix}$$

So $A^{-1}=\begin{vmatrix} -2 & 3 \\ 5 & -7 \end{vmatrix}$

Inverse of a matrix by Gauss-Jordan method

Example 2

Find the inverse of $B=\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

We augment B to form $\left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ which, after Gauss-Jordan elimination, we get

Example 3

Find the inverse of $C=\begin{bmatrix} 1 & 3 & 1 \\ -1 & 2 & 0 \\ 2 & 11 & 3 \end{bmatrix}$

Answer: C has no inverse as we get zeros in last row

Inverse of a matrix by Gauss-Jordan method

Example :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Example : Do row operations to get upper triangular form: (Like Gaussian Elimination)

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 1 & 0 \\ 0 & 2 & 5 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 5 & 2 & 1 \end{array}\right]$$

Inverse of a matrix by Gauss-Jordan method

Continue doing row operations to

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 5 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & 5 & 3 \\ 0 & 0 & 1 & 5 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & 2 & 1 \end{array}\right]$$

Inverse of a matrix by Gauss-Jordan method

At this point the last matrix on the left is the Identity. Thus, the right matrix must be the inverse to A:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & 5 & 3 \\ 0 & 0 & 1 & 5 & 2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Inverse of a matrix by Gauss-Jordan method

ex) Find the inverse of $A=\begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix}$

We augment the matrix to form $\left[\begin{matrix} 7 & 3 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{matrix}\right]$ And perform row operations to reduce the left-side to the identity.

$$\left[\begin{matrix} 7 & 3 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{matrix}\right] \quad \underset{\rightarrow}{\frac{1}{7}R_1}\left[\begin{matrix} 1 & \frac{3}{7} & \frac{1}{7} & 0 \\ 5 & 2 & 0 & 1 \end{matrix}\right] \quad \underset{\rightarrow}{-5R_1}\left[\begin{matrix} 1 & 3/7 & 1/7 & 0 \\ 0 & 5 & 0 & 1 \end{matrix}\right]$$

$$\underset{\rightarrow}{-7R_2}\left[\begin{matrix} 1 & \frac{3}{7} & \frac{1}{7} & 0 \\ 0 & 1 & 5 & -7 \end{matrix}\right] \quad \underset{\rightarrow}{\frac{-3}{7}R_2+R_1}\left[\begin{matrix} 1-0 & \frac{3}{7}-\frac{3}{7} & \frac{1}{7}-\frac{15}{7} & 0+3 \\ 0 & 1 & 5 & -7 \end{matrix}\right]=\left[\begin{matrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 5 & -7 \end{matrix}\right]$$

So $A^{-1}=\left[\begin{matrix} -2 & 3 \\ 5 & -7 \end{matrix}\right]$

ex) Find the inverse of $B=\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-2} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$

We augment B to form $\left[\begin{matrix} 1 & 2 & -2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{matrix}\right]$ which, after Gauss-Jordan elimination, we get

Inverse of a matrix by Gauss-Jordan method

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 2 & -4 \\ 0 & 1 & 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \text{ so } B^{-1} = \begin{bmatrix} -1 & 2 & -4 \\ 1 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

ex) Find the inverse of $C = \begin{bmatrix} 1 & 3 & 1 \\ -1 & 2 & 0 \\ 2 & 11 & 3 \end{bmatrix}$

The augmented matrix $\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 \\ 2 & 11 & 3 & 0 & 0 & 1 \end{array}\right]$ reduces to $\left[\begin{array}{ccc|ccc} 1 & 0 & 2/5 & 0 & -11/15 & 2/15 \\ 0 & 1 & 1/5 & 0 & 2/15 & 1/15 \\ 0 & 0 & 0 & 1 & 1/3 & -1/3 \end{array}\right]$.

Because we have the 3 zeroes in the first 3 columns of the last row, we can say that C has no inverse.

Eigen values and Eigen vectors of a matrix

Eigen Values & Eigen Vectors

Def: Characteristic vector of a matrix:

Let $A = [a_{ij}]$ be an n x n matrix. A non-zero vector X is said to be a Characteristic Vector of A if there exists a scalar such that $AX=\lambda X$.

Note: If $AX=\lambda X$ ($X\neq 0$), then we say 'λ' is the Eigen value (or) characteristic root of 'A'.

Eg: Let $A=\begin{bmatrix}5 & 4\\1 & 2\end{bmatrix}$ $X=\begin{bmatrix}1\\-1\end{bmatrix}$

$AX = \begin{bmatrix}5 & 4\\1 & 2\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix} = 1.\begin{bmatrix}1\\-1\end{bmatrix}$

$= 1.X$

Here Characteristic vector of A is $\begin{bmatrix}1\\-1\end{bmatrix}$ and Characteristic root of A is "1".

Eigen values and Eigen vectors of a matrix

Note: We notice that an Eigen value of a square matrix A can be 0. But a zero vector cannot be an Eigen vector of A.

Method of finding the Eigen vectors of a matrix.

Let $A = [a_{ij}]$ be a nxn matrix. Let X be an Eigen vector of A corresponding to the Eigen value λ.

Then by definition $AX = \lambda X$.

$AX = \lambda IX$

⇨ $AX - \lambda IX = 0$

⇨ $(A-\lambda I)X = 0$ ------- (1)

This is a homogeneous system of n equations in n unknowns.

(1) Will have a non-zero solution X if and only $|A-\lambda I| = 0$

- $A-\lambda I$ is called characteristic matrix of A
- $|A-\lambda I|$ is a polynomial in λ of degree n and is called the characteristic polynomial of A

Eigen values and Eigen vectors of a matrix

- $|A-\lambda I|=0$ is called the characteristic equation

Solving characteristic equation of A, we get the roots, $\lambda_1, \lambda_2, \lambda_3, \ldots \ldots . \lambda_n,$ these are called the characteristic roots or Eigen values of the matrix.

- Corresponding to each one of these n Eigen values, we can find the characteristic vectors.

Procedure to find Eigen values and Eigen vectors

Let A = $\begin{bmatrix} a_{11} & a_{12} \ldots \ldots & a_{1n} \\ a_{21} & a_{22} \ldots .. & a_{2n} \\ \ldots \ldots . & \ldots \ldots & \ldots \ldots . \\ a_{n1} & a_{n2} \ldots \ldots & a_{nn} \end{bmatrix}$ *be a given matrix*

$$Characteristic\ matrix\ of\ A\ is\ A - \lambda I$$

$$i.e., A-\lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$$

Then the characteristic polynomial is $|A-\lambda I|$

Eigen values and Eigen vectors of a matrix

$$say\,\phi(\lambda) = |A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix}$$

The characteristic equation is $|A\text{-}\lambda I| = 0$ we solve the $\emptyset(\lambda) = |A - \lambda I| = 0$, we get n roots, these are called

Eigen values or latent values or proper values.

Let each one of these Eigen values say λ their Eigen vector X corresponding the given value λ is obtained by

solving Homogeneous system

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ 0 \end{bmatrix}$$ and determining the non-trivial solution.

PROBLEMS

1. **Find the Eigen values and the corresponding Eigen vectors of** $\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

$$sol: Let\ A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$$

$$Characteristic\ matrix = A - \lambda I$$

$$= \begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix}$$

$$Characteristic\ equation\ of\ A\ is\ |A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda)(2-\lambda) + 8 = 0$$

$$\Rightarrow 16 + \lambda^2 - 10\lambda + 8 = 0$$

$$\Rightarrow \lambda^2 - 10\,\lambda + 24 = 0$$

$$\Rightarrow (\lambda - 6)(\lambda - 4) = 0$$

$$\Rightarrow \lambda = 6, 4 \text{ are eigen values of A}$$

$$Consider\ system \begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Eigen values and Eigen vectors of a matrix

Eigen vector corresponding to $\lambda = 4$

Put $\lambda = 4$ in the above system, we get

$$\begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow\ 4x_1 - 4x_2 = 0 - - - (1)$$

$$2x_1 - 2x_2 = 0 - - - (2)$$

$$from\ (1) and\ (2) we\ have\ x_1 = x_2$$

Let $x_1 = \alpha$

$$Eigen\ vector\ is \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}\ is\ a\ Eigen\ vector\ of\ matrix\ A, corresponding\ eigen\ value\ \lambda = 4$$

Eigen values and Eigen vectors of a matrix

Eigen Vector corresponding to $\lambda = 6$

$put\ \lambda = 6$ in the above system, we get

$$\begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow\ 2x_1 - 4x_2 = 0 - - - (1)$$

$$2x_1 - 4x_2 = 0 - - - (2)$$

from (1) and (2) we have $x_1 = 2x_2$

Say $x_2 = \alpha \Rightarrow x_1 = 2\alpha$

$$Eigen\ vector = \begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}\ is\ eigen\ vector\ of\ matrix\ A\ corresponding\ eigen\ value\ \lambda = 6$$

Eigen values and Eigen vectors of a matrix

1. **Find the eigen values and the corresponding eigen vectors of matrix** $\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$

ol: Let A = $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

he characteristic equation is |A-λI|=0

e. |A-λI| = $\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$

$$\Rightarrow (2-\lambda)(2-\lambda)^2 - 0 + [-(2-\lambda)] = 0$$

$$\Rightarrow (2-\lambda)^3 — (\lambda - 2) = 0$$

$$\Rightarrow \lambda - 2\,[-(\lambda-2)^2 - 1] = 0$$

$$\Rightarrow \lambda - 2\,\left[-\lambda^2 + 4\lambda - 3\right] = 0$$

$$\Rightarrow (\lambda-2)(\lambda-3)(\lambda-1) = 0$$

> λ=1,2,3

he eigen values of A is 1,2,3.

Eigen values and Eigen vectors of a matrix

$$\Rightarrow \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigen vector corresponding to $\lambda = 1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_3 = 0$

$x_2 = 0$

$x_1 + x_3 = 0$

$x_1 = -x_3, x_2 = 0$

$say\ x_3 = \alpha$

$x_1 = -\alpha \quad x_2 = 0, \quad x_3 = \alpha$

Eigen values and Eigen vectors of a matrix

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\alpha \\ 0 \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ *is Eigen vector*

Eigen vector corresponding to $\lambda = 2$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here $x_1 = 0$ *and* $x_3 = 0$ *and we can take any arbitary value* x_2 *i.e* $x_2 = \alpha$ *(say)*

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Eigen vector is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Eigen vector corresponding to $\lambda = 3$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_3 = 0$$

$$-x_2 = 0$$

Eigen values and Eigen vectors of a matrix

$x_1 - x_3 = 0$

here by solving we get $x_1 = x_3, x_2 = 0$ *say* $x_3 = \propto$

$x_1 = \propto \;,\quad x_2 = 0 \quad, x_3 = \propto$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Eigen vector is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Properties of Eigen values and Eigen vectors of real and complex matrices

Properties of Eigen Values:

Theorem 1: The sum of the eigen values of a square matrix is equal to its trace and product of the eigen values is equal to its determinant.

Example: if A= $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 2 & -1 & 1 \end{bmatrix}$ then trace=1+2+1=4 and determinant=15

Theorem 2: If λ is an Eigen value of A corresponding to the Eigen vector X, then λ^n is Eigen value A^n corresponding to the Eigen vector X.

Example: if A= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then Eigen values of A^3 are 1,8,1

Theorem 3: A Square matrix A and its transpose A^T have the same Eigen values.

Example: if A= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then Eigen values of A^T are 1,2,1.

Theorem 4: If A and B are n-rowed square matrices and If A is invertible show that $A^{-1}B$ and $B A^{-1}$ have same Eigen values.

Theorem 5: If $\lambda_1, \lambda_2, \ldots\ldots\ldots \lambda_n$ are the Eigen values of a matrix A then k λ_1, k λ_2, ... k λ_n are the Eigen value of the matrix KA, where K is a non-zero scalar.

Example:

If 1,2,3 are eigen values of A then eigen values of 3A are 3,3,9

Properties of Eigen values and Eigen vectors of real and complex matrices

Theorem 6: If λ is an Eigen values of the matrix A then λ+K is an Eigen value of the matrix A+KI

Example:

If 1,2,3 are eigen values of A then eigen values of 3+A are 4,5,6

Theorem 7: If λ_1, λ_2 ... λ_n are the Eigen values of A, then $\lambda_1 - K$, $\lambda_2 - K$, ... $\lambda_n - K$,

$$aretheeigenvaluesof\ thematrix(A-KI), whereK\ is\ anon-zero\ scalar$$

Example:

If 1,2,3 are eigen values of A then eigen values of 3-A are 2,1,0

Theorem 8: If $\lambda_1, \lambda_2 \ldots \lambda_n$ are the Eigen values of A, find the Eigen values of the matrix $(A - \lambda I)^2$

Theorem 9: If λ is an Eigen value of a non-singular matrix A corresponding to the Eigen vector X, then λ^{-1} *is an Eigen value of* A^{-1} *and corresponding Eigen vector X itself.*

Theorem 10: If

λ is an eigen value of a non $-$ singular matrix A, then $\frac{|A|}{\lambda}$ is an eigen value of the matrix Adj A

Properties of Eigen values and Eigen vectors of real and complex matrices

Theorem 11: If λ is an eigen value of an orthogonal matrix then $\frac{1}{\lambda}$ is also an eigen value

Theorem 12: If λ is Eigen value of A then prove that the Eigen value of $B = a_0A^2+a_1A+a_2I$ is $a_0\lambda^2+a_1\lambda+a_2$

Theorem 14: Suppose that A and P be square matrices of order n such that P is non singular. Then A and $P^{-1}AP$ have the same Eigen values.

Corollary 1: If A and B are square matrices such that A is non-singular, then $A^{-1}B$ and BA^{-1} have the same Eigen values.

Corollary 2: If A and B are non-singular matrices of the same order, then AB and BA have the same Eigen

Theorem 15: The Eigen values of a triangular matrix are just the diagonal elements of the matrix.

Theorem 16: The Eigen values of a real symmetric matrix are always real.

Theorem 17: For a real symmetric matrix, the Eigen vectors corresponding to two distinct Eigen values are orthogonal.

PROBLEMS

1. Find the Eigen values and Eigen vectors of the matrix A and its inverse, where

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

Sol: Given $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

Sol: Given A = $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

The characteristic equation of A is given by $|A-\lambda I| = 0$

$$\Longrightarrow \begin{vmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Longrightarrow (1-\lambda)[(2-\lambda)(3-\lambda)] = 0$$

$$\Longrightarrow \lambda = 1, 2, 3$$

$Characteristic\ roots\ are\ 1, 2, 3$

$Characteristic\ vector\ for\ \lambda = 1$

$$For\ \lambda = 1, becomes \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow 3x_2 + 4x_3 = 0$$

$$x_2 + 5x_3 = 0$$

$$2x_3 = 0$$

Properties of Eigen values and Eigen vectors of real and complex matrices

$x_2 = 0, x_3 = 0$ and $x_1 = \alpha$

$X = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the solution where α is arbitrary constant

$\therefore X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ Is the Eigen vector corresponding to $\lambda = 1$

Characteristic vector for $\lambda = 2$

For $\lambda = 2$, *becomes* $\begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow -x_1 + 3x_2 + 4x_3 = 0$$

$$5x_3 = 0 \Rightarrow x_3 = 0$$

$$-x_1 + 3x_2 = 0 \Rightarrow x_1 = 3x_2$$

$$Let x_2 = k$$

$$x_1 = 3k$$

$$X = \begin{bmatrix} 3k \\ k \\ 0 \end{bmatrix} = k\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

is the solution $\therefore X = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

Is the Eigen vector corresponding to $\lambda = 2$

Hence the characteristic vector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Characteristic vector for $\lambda = 3$

$$For\ \lambda = 3, becomes \begin{bmatrix} -2 & 3 & 4 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow\ -2x_1 + 3x_2 + 4x_3 = 0$

$-x_2 + 5x_3 = 0$

$Say\ x_3 = K\ \Longrightarrow\ x_2 = 5K$

$x_1 = \frac{19}{2}K$

X = $\begin{bmatrix} \frac{19}{2}K \\ 5K \\ K \end{bmatrix} = \frac{K}{2}\begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix}$

$\therefore\ X = \begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix}$ is the Eigen vector corresponding to $\lambda = 3$

Eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$

$\Rightarrow\ Eigen\ values\ of\ A^{-1}\ are\ 1, \frac{1}{2}, \frac{1}{3}$

We know Eigen vectors of A^{-1} are same as Eigen vectors of A.

Cayley-Hamilton theorem

Cayley - Hamilton Theorem:

Statement

Every square matrix satisfies its own characteristic equation

PROBLEMS

1. Show that the matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ satisfies its characteristic equation Hence find A^{-1}

Sol: Characteristic equation of A is det $(A-\lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 & 2 \\ 1 & -2-\lambda & 3 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0 \quad C2 \rightarrow C2+C3$$

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1-\lambda & 3 \\ 0 & 1-\lambda & 2-\lambda \end{vmatrix} = 0$$

Cayley-Hamilton theorem

$$(1-\lambda)\begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$

By Cayley – Hamilton theorem, we have $A^3 - A^2 + A - I = 0$

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} \quad A^2 = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \quad A^3 = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$A^3 - A^2 + A - I = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Cayley-Hamilton theorem

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Multiplying with A^{-1} we get $A^2 - A + I = A^{-1}$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

2. Using Cayley - Hamilton Theorem find the inverse and A^4 of the matrix $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

The characteristic equation is given by $|A-\lambda I|=0$

$$i.e., \begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{vmatrix} = 0$$

Cayley-Hamilton theorem

$$(1-\lambda)^2 \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 6 & 2 & -(1+\lambda) \end{vmatrix} = 0$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley – Hamilton theorem we have A^3-$5A^2$+$7A$-$3I$=0…..(1)

Multiply with A^{-1} we get

$A^{-1} = \frac{1}{3}\left[A^2 - 5A + 7I\right]$

Cayley-Hamilton theorem

$$A^2 = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} \quad A^3 = \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$$

Multiply (1)with A,we get

$$A^4 - 5A^3 + 7A^2 - 3A = 0$$

$$= \begin{bmatrix} 395 & 130 & -130 \\ -390 & -125 & 130 \\ 390 & 130 & -125 \end{bmatrix} - \begin{bmatrix} 175 & 56 & -56 \\ -168 & -49 & 56 \\ 168 & 56 & -69 \end{bmatrix} + \begin{bmatrix} 21 & 6 & -6 \\ -18 & -3 & 6 \\ 18 & 6 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 241 & 80 & -80 \\ -240 & -79 & 80 \\ 240 & 80 & -79 \end{bmatrix}$$

Cayley-Hamilton theorem

Problems

1. Diagonalize the matrix (i) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

1. Verify Cayley – Hamilton Theorem for A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$. Hence find A^{-1}.

Linear dependence and independence of Vectors:

1. Show that the vectors (1,2,3), (3,-2,1), (1,-6,-5) from a linearly dependent set.

Sol. The Given Vector $X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} X_2 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} X_3 = \begin{bmatrix} 1 \\ -6 \\ -5 \end{bmatrix}$

The Vectors X_1, X_2, X_3 from a square matrix.

Let $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -2 & -6 \\ 3 & 1 & -5 \end{bmatrix}$

Then $|A| = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -2 & -6 \\ 3 & 1 & -5 \end{bmatrix}$

= 1(10+6)-2(15-1)+3(-18+2)

=16+32-48=0

The given vectors are linearly dependent $\because |A|=0$

2. Show that the Vector $X_1=(2,2,1)$, $X_2=(1,4,-1)$ and $X_3=(4,6,-3)$ are linearly independent.

Sol. Given Vectors $X_1=(2,-2,1)$ $X_2=(1,4,-1)$ and $X_3=(4,6,-3)$ The Vectors X_1, X_2, X_3 form a squar matrix.

$$A=\begin{bmatrix} 2 & 1 & 4 \\ -2 & 4 & 6 \\ 1 & -1 & -3 \end{bmatrix}$$

Then $|A|=\begin{bmatrix} 2 & 1 & 4 \\ -2 & 4 & 6 \\ 1 & -1 & -3 \end{bmatrix}$

$=2(-12+6)+2(-3+4)+1(6-16)$

$=-20\neq 0$

$\therefore$ The given vectors are linearly independent

$\therefore$ $|A|\neq 0$

Diagonalization of matrix

Diagonalization of Symmetric Matrices:

NOTE:

a matrix A is diagonalizable if and only if there is an invertible matrix P such that $A = P D P^{-1}$ where D is a diagonal matrix.

A matrix A is orthogonally diagonalizable if and only if there is an orthogonal matrix P such that

$A = P D P^{-1}$ where D is a diagonal matrix.

Remark : Recall that any orthogonal matrix A is invertible and also that $A^{-1} = A^T$. Thus we can say that A matrix A is orthogonally diagonalizable if there is a square matrix P such that $A = P D P^T$ where D is a diagonal matrix.

Remark : The formula for transpose of a product: $(MN)^T = N^T M^T$. Using this we can see that any orthogonally diagonalizable A must be symmetric. This is because $A^T = (P D P^T)^T = ((P^T)^T D^T P^T = P D P^T = A$.

If A is symmetric then any two Eigen values from different Eigen spaces are orthogonal

Proposition : (The Spectral Theorem) An $n \times n$ symmetric matrix has the following properties:

1. A has n real Eigen values if we count multiplicity

2. For each Eigen values the dimension of the corresponding Eigen spaces is equal to the algebraic multiplicity of that Eigen values

3. The Eigen spaces are mutually orthogonal.

4. A is orthogonally diagonalizable.

Diagonalization of matrix

NOTE:

All Eigen values (all roots of the characteristic polynomial) of a symmetric matrix are real.

Eigenvectors of a symmetric matrix corresponding to different Eigen values are orthogonal.

Problems:

1)Find an orthogonal matrix P which diagonalizes $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

Sol: Eigen systems:

Eigen values and Eigenvector are 3 ,3,0 and $(-1, 0, 1)$,$(-1, 1, 0)$, $(1, 1, 1)$

Using the Gram-Schmidt process we find that an orthonormal basis for the eigenspace of A corresponding to $\lambda_1 = 3$ is

$$\begin{aligned}
\mathbf{p}_1 &= \frac{(-1,0,1)}{\|(-1,0,1)\|} = (-1/\sqrt{2}, 0, 1/\sqrt{2}) \\
\mathbf{u}_2 &= (-1,1,0) - \langle(-1,1,0), \mathbf{p}_1\rangle \mathbf{p}_1 = (-1/2, 1, -1/2) \\
\mathbf{p}_2 &= \frac{(-1/2,1,-1/2)}{\|(-1/2,1,-1/2)\|} = (-1/\sqrt{6}, \sqrt{2/3}, -1/\sqrt{6}) \\
\mathbf{p}_3 &= \frac{(1,1,1)}{\|(1,1,1)\|} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})
\end{aligned}$$

Diagonalization of matrix

Let orthogonal matrix $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ then

$$P^T A P$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Diagonalization of matrix

1) Find an orthogonal matrix P which diagonalizes $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$

Sol:

Let

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{bmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{bmatrix}$$

$$= (3-\lambda)[(3-\lambda)(5-\lambda) - 1] - 1[5-\lambda-1] - 1[-1-3-\lambda]$$
$$= [(3-\lambda)(3-\lambda)(5-\lambda) - (3-\lambda)] - (4-\lambda) - (2-\lambda)$$
$$= [(3-\lambda)(3-\lambda)(5-\lambda) - (3-\lambda)] - 2(3-\lambda)$$
$$= (3-\lambda)[(3-\lambda)(5-\lambda) - 1 - 2]$$
$$= (3-\lambda)[15 - 5\lambda - 3\lambda - \lambda^2 - 3]$$
$$= (3-\lambda)(\lambda^2 - 8\lambda - 12)$$
$$= (3-\lambda)(\lambda - 6)(\lambda - 2)$$

Thus, $\lambda = 2, 3, 6$ are the eigenvalues of A. Let us find an eigenvector corresponding to each eigenvalue. For the eigenvalue $\lambda = 2$, since

Diagonalization of matrix

$$A-2I=\begin{bmatrix}1 & 1 & -1\\ 1 & 1 & -1\\ -1 & -1 & 3\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 1 & -1\\ 0 & 0 & 0\\ 0 & 0 & 2\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 1 & -1\\ 0 & 0 & 2\\ 0 & 0 & 0\end{bmatrix}$$

A vector $X^t = (x_1, x_2, x_3)$ will be an eigenvector for eigenvalue $\lambda = 2$ if

$$\begin{bmatrix}1 & 1 & -1\\ 0 & 0 & 2\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = 0,$$

i.e., $x_3 = 0$

$x_1 - x_2 = x_3 = 0.$

If we choose x_2=1, then x_1 = -1. Hence

$$X_1 = \begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}$$

Diagonalization of matrix

is an eigenvector for $\lambda = 2$. For the eigenvalue $\lambda = 3$,

$$A - 3I = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$\boldsymbol{X}_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

will be an eigenvector for the eigenvalue $\lambda = 3$, if

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0},$$

Diagonalization of matrix

i.e., $x_2 = x_3$ and $x_1 - x_3 = 0$. Hence

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ is an eigenvector for } \lambda = 3.$$

Finally, for $\lambda = 6$

$$A - 4I = \begin{bmatrix} -3 & 1 & -1 \\ 1 & -3 & -1 \\ -1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & -1 \\ 0 & -4 & -2 \\ 0 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & 1 \\ 0 & -4 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus ,

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ will be an eigenvector for eigenvalue } \lambda = 6 \text{ if}$$

Diagonalization of matrix

$-4x_2 - 2x_3 = 0, \quad -x_1 - x_2 - x_3 = 0.$

Thus, if we take $x_3 = 2$, then $x_2 = -1$ and $x_1 = -x_2 - x_3 = -1.$
Hence

$$X_3 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \text{ is an eigenvector for the eigenvalue } \lambda = 6.$$

Note that for

$$P := [X_1 \quad X_2 \quad X_3] = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}, \text{ the columns of } P \text{ are orthogonal.}$$

Diagonalization of matrix

To make P orthogonal, we normalize each x_1, x_2 and x_3, and define

$$\tilde{P} := \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix},$$

It is easy to verify that $\tilde{P} A \tilde{P}^{-1} = D$, where

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

In fact, $\tilde{P}^{-1} = \tilde{P}^{-t}$. Thus, we checks $\tilde{P}^{-1} A \tilde{P}^{-t} = D$.

MODULE-II

FUNCTIONS OF SINGLE AND SEVERAL VARIABLES

CONTENTS

- Rolle’s mean value theorem
- Geometric representation of Rolle’s mean value theorem
- Applications of Rolle’s mean value theorem

OBJECTIVE AND OUTCOME

OBJECTIVE:

Rolle’s mean value theorem.

OUTCOME:

Student get to understand the concept of Rolle’s mean value theorem and its applications.

Statement of Rolle 's Mean Value Theorem

Let $f(x)$ be a function defined in $[a,b]$ such that

(i) $f(x)$ It is continuous in closed interval $[a,b]$

(ii) $f(x)$ is differentiable in open interval (a,b) and

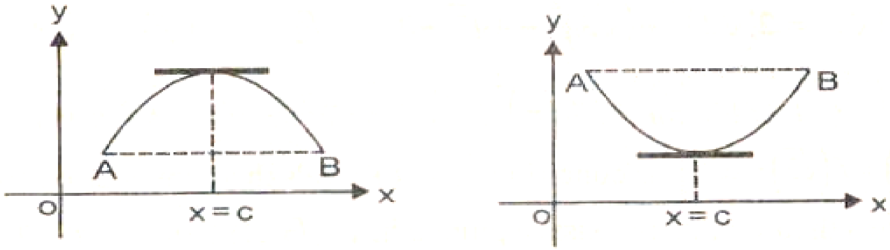
(iii) $f(a) = f(b)$.

Then there exists at least one point 'c' in (a,b) such that $f^1(c) = 0$.

Geometrical Representation of Rolle 's Mean Value Theorem

Let $f : [a, b] \to R$ be a function satisfying the three conditions of Rolle's theorem.

Then the graph drawn is as follows



Geometrically Rolles mean value theorem means the following

1. y=f(x) in a continuous curve in [a,b].
2. There exist a unique tangent line at every point x=c, where a<c<b
3. The ordinates f(a), f(b) at the end points A,B are equal so that the points A and B are equidistant from the X-axis.

   By Rolle's Theorem, There is at least one point x=c between A and B on the curve at which the tangent line is parallel to the x-axis and also it is parallel to chord of the curve.

Applications of Rolle 's Mean Value Theorem

Example 1. Verify Rolle's theorem for the function $f(x) = \sin x/e^x$ or $e^{-x}\sin x$ in $[0,\pi]$

Sol: i) Since sinx and e^x are both continuous functions in $[0, \pi]$.

Therefore, $\sin x/e^x$ is also continuous in $[0,\pi]$.

ii) Since sinx and e^x be derivable in $(0,\pi)$, then f is also derivable in $(0,\pi)$.

iii) $f(0) = \sin 0/e^0 = 0$ and $f(\pi) = \sin \pi/e^{\pi} = 0$

$\therefore\ f(0) = f(\pi)$

Thus all three conditions of Rolle's theorem are satisfied.

$\therefore$ There exists $c \in (0, \pi)$ such that $f^1(c)=0$

Now $f^1(x) = \dfrac{e^x \cos x - \sin x\, e^x}{(e^x)^2} = \dfrac{\cos x - \sin x}{e^x}$

$f^1(c) = 0 \Rightarrow \dfrac{\cos c - \sin c}{e^c} = 0$

$\cos c = \sin c \Rightarrow \tan c = 1$

$c = \pi/4 \in (0,\pi)$

Hence Rolle's theorem is verified.

Applications of Rolle 's Mean Value Theorem Contd.,

Example 2. Verify Rolle's theorem for the functions $\log\left(\frac{x^2+ab}{x(a+b)}\right)$ in[a,b] , a>0, b>0,

Sol: Let f(x)= $\log\left(\frac{x^2+ab}{x(a+b)}\right)$

$= \log(x^2+ab) - \log x - \log(a+b)$

(i). Since f(x) is a composite function of continuous functions in [a,b], it is continuous in [a,b].

(ii). $f^1(x) = \frac{1}{x^2+ab}.2x - \frac{1}{x} = \frac{x^2-ab}{x(x^2+ab)}$

$f^1(x)$ exists for all xϵ (a,b)

(iii). $f(a) = \log\left[\frac{a^2+ab}{a^2+ab}\right] = \log 1 = 0$

$f(b) = \log\left[\frac{b^2+ab}{b^2+ab}\right] = \log 1 = 0$

f(a) = f(b)

Thus f(x) satisfies all the three conditions of Rolle's theorem.

So, $\exists$ c $\in$ (a, b) $\Rightarrow$ $f^1(c) = 0$,

$f^1(c) = 0, \Rightarrow \frac{c^2-ab}{c(c^2+ab)} = 0 \Rightarrow c^2 = ab$

$\Rightarrow c = \sqrt{ab} \in (a,b)$

Hence Rolle's theorem verified.

Example 3. Verify whether Rolle 's Theorem can be applied to the following functions in the intervals.

i) $f(x) = \tan x$ in $[0, \pi]$ and ii) $f(x) = 1/x^2$ in $[-1,1]$

(i) f(x) is discontinuous at $x = \pi/2$ as it is not defined there. Thus condition (i) of Rolle 's Theorem is not satisfied. Hence we cannot apply Rolle 's Theorem here.

$\therefore$ Rolle's theorem cannot be applicable to $f(x) = \tan x$ in $[0,\pi]$.

(ii). $f(x) = 1/x^2$ in $[-1,1]$

f(x) is discontinuous at $x= 0$.

Hence Rolle 's Theorem cannot be applied.

Applications of Rolle 's Mean Value Theorem Contd.,

Example 4. Using Rolle 's Theorem, show that $g(x) = 8x^3-6x^2-2x+1$ has a zero between 0 and 1.

Sol: $g(x) = 8x^3-6x^2-2x+1$ being a polynomial, it is continuous on [0,1] and differentiable on (0,1)

Now $g(0) = 1$ and $g(1)= 8-6-2+1 = 1$

Also $g(0)=g(1)$

Hence, all the conditions of Rolle's theorem are satisfied on [0,1].

Therefore, there exists a number $c\epsilon(0,1)$ such that $g^1(c)=0$.

Now $g^1(x) = 24x^2-12x-2$

$\therefore\ g^1(c)= 0 \Rightarrow 24c^2-12c-2 =0$

$\Rightarrow c= \dfrac{3 \pm \sqrt{21}}{12}$ *ie* $c= 0.63$ or -0.132

only the value $c = 0.63$ lies in (0,1)

Thus there exists at least one root between 0 and 1.

Applications of Rolle's Mean Value Theorem Contd.,

Example 5. Verify Rolle's theorem for $f(x) = x^{2/3} - 2x^{1/3}$ in the interval (0,8).

Sol: Given $f(x) = x^{2/3} - 2x^{1/3}$

f(x) is continuous in [0,8]

$f^1(x) = 2/3 \, . \, 1/x^{1/3} - 2/3 \, . \, 1/x^{2/3} = 2/3(1/x^{1/3} - 1/x^{2/3})$

Which exists for all x in the interval (0,8)

$\therefore$ f is derivable (0,8).

Now $f(0) = 0$ and $f(8) = (8)^{2/3} - 2(8)^{1/3} = 4 - 4 = 0$

i.e., $f(0) = f(8)$

Thus all the three conditions of Rolle's Theorem are satisfied.

$\therefore$ There exists at least one value of c in(0,8) such that $f^1(c)=0$

ie. $\frac{1}{c^{\frac{1}{3}}} - \frac{1}{c^{\frac{2}{3}}} = 0 \Rightarrow c = 1 \in (0,8)$

Hence Rolle's Theorem is verified.

Applications of Rolle 's Mean Value Theorem Contd.,

Example 6. Verify Rolle's theorem for $f(x) = x(x+3)e^{-x/2}$ in [-3,0].

Sol: - (i). Since x(x+3) being a polynomial is continuous for all values of x and $e^{-x/2}$ is also continuous for all x, their product $x(x+3)e^{-x/2} = f(x)$ is also continuous for every value of x and in particular f(x) is continuous in the [-3,0].

(ii). we have $f^1(x) = x(x+3)\left(-1/2\ e^{-x/2}\right)+(2x+3)e^{-x/2}$

$$= e^{-x/2}\left[2x+3-\frac{x^2+3x}{2}\right]$$

$$=e^{-x/2}[6+x-x^2/2]$$

Since $f^1(x)$ does not become infinite or indeterminate at any point of the interval(-3,0).

f(x) is derivable in (-3,0)

Applications of Rolle 's Mean Value Theorem Contd.,

(i) Also we have $f(-3) = 0$ and $f(0) = 0$

$\therefore f(-3)=f(0)$

Thus f(x) satisfies all the three conditions of Rolle's theorem in the interval [-3,0].

Hence there exist at least one value c of x in the interval (-3,0) such that $f^1(c)=0$

i.e., $\frac{1}{2} e^{-c/2}(6+c-c^2)=0 \Rightarrow 6+c-c^2=0$ ($e^{-c/2} \neq 0$ for any c)

$\Rightarrow c^2+c-6 = 0 \Rightarrow (c-3)(c+2)=0$

$c=3,-2$

Clearly, the value c= -2 lies within the (-3,0) which verifies Rolle's theorem.

Example 7.

Verify Rolle's theorem for the function

$f(x) = (x-a)^m (x-b)^n$ *in* $[a, b]$ *where* $m > 1$ *and* $n > 1$.

Solution $f(x) = (x-a)^m (x-b)^n$ is continuous in $[a, b]$

$$f'(x) = (x-a)^m \cdot n(x-b)^{n-1} + m(x-a)^{m-1}(x-b)^n$$
$$= (x-a)^{m-1}(x-b)^{n-1}[n(x-a) + m(x-b)]$$
$$f'(x) = (x-a)^{m-1}(x-b)^{n-1}[nx - na + mx - mb]$$
$$= (x-a)^{m-1}(x-b)^{n-1}[(m+n)x - (na+mb)] \quad \ldots(1)$$

$f'(x)$ exists in (a, b)

Also $f(a) = 0 = f(b)$

Hence all the conditions of the theorem are satisfied.

Now consider $f'(c) = 0$

From (1) $(c-a)^{m-1}(c-b)^{n-1}[(m+n)c - (na+mb)] = 0$

$\Rightarrow$ $c - a = 0, c - b = 0, (m+n)c - (na+mb) = 0$

i.e., $c = a, c = b, c = \dfrac{na+mb}{m+n}$

a, b are the end points.

$c = \dfrac{na+mb}{m+n}$ is the x-coordinate of the point which divides the line joining $[a, f(a)]$, $[b, f(b)]$ internally in the ratio $m : n$.

$\therefore$ $c = \dfrac{na+mb}{m+n} \in (a, b)$

Thus the Rolle's theorem is verified.

CONTENTS

- Lagrange’s mean value theorem
- Geometric Representation of Lagrange’s mean value theorem
- Applications of Lagrange’s mean value theorem

OBJECTIVE AND OUTCOME

OBJECTIVE:

Lagrange's mean value theorem.

OUTCOME:

Student get to understand the concept of Lagrange's mean value theorem and its applications.

Statement of Lagrange 's Mean Value Theorem

Let f(x) be a function defined in [a,b] such that

(i) f(x) is continuous in closed interval [a,b] &

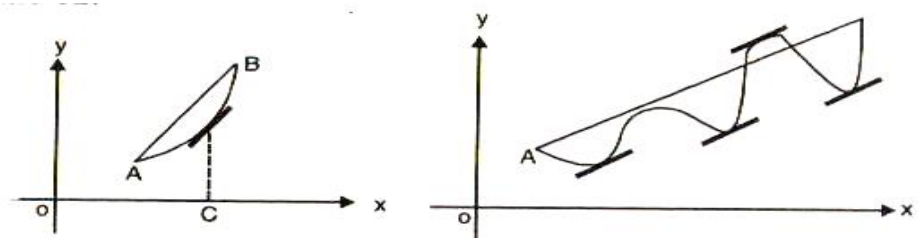
(ii) f(x) is differentiable in (a,b).

Then there exists at least one point c in (a,b) such that $f^{1}(c) = \dfrac{f(b) - f(a)}{b - a}$

Geometrical Representation of Lagrange 's Mean Value Theorem

Let $f : [a, b] \to R$ be a function satisfying the two conditions of Lagrange's theorem.

Then the graph is as follows



Geometrically Lagrange's mean value theorem means the following

1. y=f(x) is continuous curve in [a,b]

2. At every point x=c, when a<c<b, on the curve y=f(x), there is unique tangent to the curve. By Lagrange's theorem there exists at least one point $c \in (a, b) \ni f^{1}(c) = \dfrac{f(b) - f(a)}{b - a}$

Geometrically there exist at least one point c on the curve between A and B such that the tangent line is parallel to the chord $\overleftrightarrow{AB}$

Applications of Lagrange 's Mean Value Theorem

Example 1. Verify Lagrange's Mean value theorem for $f(x) = x^3 - x^2 - 5x + 3$ in [0,4]

Sol: Let $f(x) = x^3 - x^2 - 5x + 3$ is a polynomial in x.

$\therefore$ It is continuous & derivable for every value of x.

In particular, f(x) is continuous [0,4] & derivable in (0,4)

Hence by Lagrange's Mean value theorem $\exists$ $c \in (0,4) \ni$

$$f^1(c) = \frac{f(4) - f(0)}{4 - 0}$$

i.e., $3c^2 - 2c - 5 = \frac{f(4) - f(0)}{4}$ ....(1)

Now $f(4) = 4^3 - 4^2 - 5.4 + 3 = 64 - 16 - 20 - 3 = 67 - 36 = 31$ & $f(0) = 3$

$$\frac{f(4) - f(0)}{4} = \frac{(31 - 3)}{4} = 7$$

From equation (1), we have

$3c^2 - 2c - 5 = 7 \Rightarrow 3c^2 - 2c - 12 = 0$

$$c = \frac{2 \pm \sqrt{4 + 144}}{6} = \frac{2 \pm \sqrt{148}}{6} = \frac{1 \pm \sqrt{37}}{3}$$

We see that $\frac{1 + \sqrt{37}}{3}$ lies in open interval (0,4) & thus Lagrange's Mean value theorem is verified.

Example 2. Verify Lagrange's Mean value theorem for f(x) = $\log_e x$ in [1,e]

Sol: - f(x) = $\log_e x$

This function is continuous in closed interval [1,e] & derivable in (1,e). Hence L.M.V.T is applicable here. By this theorem, ∃ a point c in open interval (1,e) such that

$$f^1(c) = \frac{f(e) - f(1)}{e-1} = \frac{1-0}{e-1} = \frac{1}{e-1}$$

$$\text{But } f^1(c) = \frac{1}{e-1} \implies \frac{1}{c} = \frac{1}{e-1}$$

$\therefore$ c = e - 1

Note that (e-1) is in the interval (1,e).

Hence Lagrange's mean value theorem is verified.

Example 3. Give an example of a function that is continuous on [-1, 1] and for which mean value theorem does not hold with explanations.

Sol:- The function f(x) = $|x|$ is continuous on [-1,1]

But Lagrange Mean value theorem is not applicable for the function f(x) as its derivative does not exist in (-1,1) at x=0.

Applications of Lagrange 's Mean Value Theorem Contd.,

Example 4. If a<b, P.T $\frac{b-a}{1+b^2} < Tan^{-1}b - Tan^{-1}a < \frac{b-a}{1+a^2}$ **using Lagrange's Mean value theorem. Deduce the following.**

i). $\frac{\pi}{4} + \frac{3}{25} < Tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

ii). $\frac{5\pi + 4}{20} < Tan^{-1} 2 < \frac{\pi + 2}{4}$

Sol: consider f(x) = Tan^{-1} x in [a,b] for 0<a<b<1

Since f(x) is continuous in closed interval [a,b] & derivable in open interval (a,b).

We can apply Lagrange's Mean value theorem here.

Applications of Lagrange 's Mean Value Theorem Contd.,

Hence there exists a point c in (a,b) ∍

$$f^1(c) = \frac{f(b) - f(a)}{b - a}$$

Here $f^1(x) = \frac{1}{1+x^2}$ & *hence* $f^1(c) = \frac{1}{1+c^2}$

Thus ∃ c, a<c<b ∍

$$\frac{1}{1+c^2} = \frac{Tan^{-1}b - Tan^{-1}a}{b-a} \quad \text{-------} \ (1)$$

We have $1+a^2<1+c^2<1+b^2$

$$\therefore \frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2} \quad \ldots\ldots\ldots\ldots \ (2)$$

From (1) and (2), we have

$$\frac{1}{1+a^2} > \frac{Tan^{-1}b - Tan^{-1}a}{b-a} > \frac{1}{1+b^2}$$

or

$$\frac{b-a}{1+a^2} < Tan^{-1}b - Tan^{-1}a < \frac{b-a}{1+b^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Hence the result

Deductions: -

(i) We have $\frac{b-a}{1+b^2} < Tan^{-1}b - Tan^{-1}a < \frac{b-a}{1+a^2}$ Take $b = \frac{4}{3}$ & a=1, we get

$$\frac{\frac{4}{3}-1}{1+\frac{16}{9}} < Tan^{-1}(\frac{4}{3}) - Tan^{-1}(1) < \frac{\frac{4}{3}-1}{1+1^2} ==> \frac{\frac{4-3}{3}}{\frac{25}{9}} < Tan^{-1}(\frac{4}{3}) - \frac{\pi}{4} < \frac{\frac{4-3}{3}}{2}$$

$$\Rightarrow \frac{3}{25} + \frac{\pi}{4} < Tan^{-1}(\frac{4}{3}) < \frac{\pi}{4} + \frac{1}{6}$$

(ii) Taking b=2 and a=1, we get

$$\frac{2-1}{1+2^2} < Tan^{-1}2 - Tan^{-1}1 < \frac{2-1}{1+1^2} \Rightarrow \frac{1}{5} < Tan^{-1}2 - \frac{\pi}{4} < \frac{1}{2}$$

$$\Rightarrow \frac{1}{5} + \frac{\pi}{4} < Tan^{-1}2 < \frac{2+\pi}{4}$$

$$\Rightarrow \frac{4+5\pi}{20} + < Tan^{-1}2 < \frac{2+\pi}{4}$$

Applications of Lagrange 's Mean Value Theorem Contd.,

Example 5. Show that for any $x > 0$, $1 + x < e^x < 1 + xe^x$.

Sol: - Let $f(x) = e^x$ defined on $[0,x]$. Then $f(x)$ is continuous on $[0,x]$ & derivable on $(0,x)$.

By Lagrange's Mean value theorem $\exists$ a real number $c \in (0,x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f^1(c)$$

$$\Rightarrow \frac{e^x - e^0}{x - 0} = e^c \Rightarrow \frac{e^x - 1}{x} = e^c \quad \ldots\ldots\ldots\ldots(1)$$

Note that $0<c<x \Rightarrow e^0<e^c<e^x$ (e^x is an increasing function)

$\Rightarrow 1 < \frac{e^x - 1}{x} < e^x$ From (1)

$\Rightarrow x<e^x-1<xe^x$

$\Rightarrow 1+x<e^x<1+xe^x$.

Applications of Lagrange 's Mean Value Theorem Contd.,

Example 6. Calculate approximately $\sqrt[5]{245}$ by using L.M.V.T.

Sol:- Let $f(x) = \sqrt[5]{x} = x^{1/5}$ & a=243 , b=245

Then $f^1(x) = 1/5\, x^{-4/5}$ & $f^1(c) = 1/5 c^{-4/5}$

By L.M.V.T, we have

$$\frac{f(b) - f(a)}{b - a} = f^1(c)$$

$$\Rightarrow \frac{f(245) - f(243)}{245 - 243} = \frac{1}{5} c^{\frac{-4}{5}}$$

$\Rightarrow \quad f(245) = f(243) + 2/5 c^{-4/5}$

$\Rightarrow$ c lies b/w 243 & 245 take c= 243

$\Rightarrow \quad \sqrt[5]{245} = (243)^{1/5} + 2/5(243)^{-4/5} = (3^5)^{\frac{1}{5}} + \frac{2}{5}(3^5)^{\frac{-4}{5}}$

= 3+ (2/5)(1/81) = 3+2/405 = 3.0049

Example 7. Find the region in which $f(x) = 1-4x-x^2$ is increasing & the region in which it is decreasing using M.V.T.

Sol: - Given $f(x) = 1-4x-x^2$

f(x) being a polynomial function is continuous on [a,b] & differentiable on (a,b) $\forall$ a,b $\in$R

$\therefore$ f satisfies the conditions of L.M.V.T on every interval on the real line.

$f^1(x)= - 4-2x= -2(2+x) \forall\ x \in R$

$f^1(x)= 0$ if $x = -2$

for $x<-2$, $f^1(x) >0$ & for $x>-2$, $f^1(x)<0$

Hence f(x) is strictly increasing on $(-\infty, -2)$ & strictly decreasing on $(-2,\infty)$

Example 8. Using Mean value theorem prove that Tan x > x in 0<x<π/2

Sol:- Consider f(x) = Tan x in $[\xi, x]$ where $0<\xi<x<\pi/2$

Apply L.M.V.T to f(x)

∃ a points c such that $0<\xi<c<x<\pi/2$ such that

$$\frac{Tan\ x - Tan\ \xi}{x-\xi} = \sec^2 c \implies$$

$$\text{Tan x} - \text{Tan}\ \xi = (x-\xi)\sec^2 c$$

$$Take\ \ \xi \to 0+0\ then\ \ Tan\ x = x\sec^2 x$$

But $\sec^2 c > 1$.

Hence Tan x > x

Applications of Lagrange 's Mean Value Theorem Contd.,

Example 9. If $f^1(x) = 0$ Through out an interval [a,b], prove using M.V.T f(x) is a constant in that interval.

Sol:- Let f(x) be function defined in [a,b] & let $f^1(x) = 0 \ \forall \ x$ in [a,b].

Then $f^1(t)$ is defined & continuous in [a,x] where $a \leq x \leq b$.

& f(t) exist in open interval (a,x).

By L.M.V.T $\exists$ a point c in open interval (a,x) $\ni$

$$\frac{f(x) - f(a)}{x - a} = f^1(c)$$

But it is given that $f^1(c) = 0$

$\therefore \ f(x) - f(a) = 0$

$\therefore \ f(x) = f(a) \ \forall \ x$

Hence f(x) is constant.

CONTENTS

- Cauchy’ s Mean value theorem
- Applications of Cauchy’ s Mean value theorem

OBJECTIVE AND OUTCOME

OBJECTIVE:

Cauchy' s Mean value theorem.

OUTCOME:

Student get to understand the concept of Cauchy' s Mean value theorem and its applications.

Statement of Cauchy 's Mean Value Theorem

If f: [a,b] $\rightarrow$R, g:[a,b] $\rightarrow$R are any two functions such that

(i) f,g are continuous on [a,b]

(ii) (ii) f,g are differentiable on (a,b)

$(iii)\ g^{1}(x) \neq 0 \,\forall\, x \in (a,b),\quad then$

$$\exists\, a\ \ po\text{int}\ \ c \in (a,b) \ni \frac{f^{1}(c)}{g^{1}(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Applications of Cauchy 's Mean Value Theorem

Example 1.Find c of Cauchy's mean value theorem for

$f(x) = \sqrt{x}$ & $g(x) = \frac{1}{\sqrt{x}}$ **in [a,b] where 0<a<b**

Sol: - Clearly f, g are continuous on $[a,b] \subseteq R^+$

We have $f^1(x) = \frac{1}{2\sqrt{x}}$ and $g^1(x) = \frac{-1}{2x\sqrt{x}}$ which exits on (a,b)

$\therefore$ f, g are differenti able on $(a, b) \subseteq R^+$

Also $g^1(x) \neq 0, \forall x \in (a,b) \subseteq R^+$

Conditions of Cauchy's Mean value theorem are satisfied on (a,b) so $\exists c \in (a,b) \ni$

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f^1(c)}{g^1(c)}$$

$$\frac{\sqrt{b} - \sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}} = \frac{\frac{1}{2\sqrt{c}}}{\frac{-1}{2c\sqrt{c}}} \Longrightarrow \frac{\sqrt{b} - \sqrt{a}}{\frac{\sqrt{a} - \sqrt{b}}{\sqrt{ab}}} = \frac{-2c\sqrt{c}}{2\sqrt{c}} \Rightarrow \sqrt{ab} = c$$

Since a,b >0 , $\sqrt{ab}$ is their geometric mean and we have $a < \sqrt{ab} < b$

$c \in (a,b)$ which verifies Cauchy's mean value theorem.

Applications of Cauchy 's Mean Value Theorem Contd.,

Example 2. Verify Cauchy's Mean value theorem for $f(x) = e^x$ & $g(x) = e^{-x}$ in [3,7] & find the value of c.

Sol: We are given $f(x) = e^x$ & $g(x) = e^{-x}$

f(x) & g(x) are continuous and derivable for all values of x.

=>f & g are continuous in [3,7]

=> f & g are derivable on (3,7)

Also $g^1(x) = e^{-x} \neq 0 \ \forall \ x \in (3,7)$

Thus f & g satisfies the conditions of Cauchy's mean value theorem.

Consequently, $\exists$ a point $c \in (3,7)$ such that

$$\frac{f(7)-f(3)}{g(7)-g(3)} = \frac{f^1(c)}{g^1(c)} \Longrightarrow \frac{e^7-e^3}{e^{-7}-e^{-3}} = \frac{e^c}{-e^{-c}} \Longrightarrow \frac{e^7-e^3}{\frac{1}{e^7}-\frac{1}{e^3}} = -e^{2c}$$

=> $-e^{7+3} = -e^{2c}$

=> 2c = 10

=> $c = 5 \in (3,7)$

Hence C.M.T. is verified

Example 3.

Verify Cauchy's mean value theorem for the following pairs of functions.

(i) $f(x) = x^2 + 3$, $g(x) = x^3 + 1$ in $[1, 3]$.

(ii) $f(x) = \sin x$, $g(x) = \cos x$ in $\left[0, \frac{\pi}{2}\right]$.

(iii) $f(x) = e^x$, $g(x) = e^{-x}$ in $[a, b]$,

Applications of Cauchy 's Mean Value Theorem Contd.,

Solution

(*i*) We have Cauchy's mean value theorem

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Here, $a = 1,\ b = 3$

$$f(x) = x^2 + 3$$

$$g(x) = x^3 + 1$$

$\therefore$ $$f'(x) = 2x$$

$$g'(x) = 3x^2$$

$f(x)$ and $g(x)$ are continuous in [1, 3], differentiable in (1, 3)

$$g'(x) \neq 0 \ \forall\ x \in (1, 3)$$

Hence the theorems becomes

$$\frac{f(3)-f(1)}{g(3)-g(1)} = \frac{2c}{3c^2}$$

$$\frac{12-4}{28-2} = \frac{2}{3c} \Rightarrow \frac{8}{26} = \frac{2}{3c}$$

$$\frac{2}{13} = \frac{1}{3c}$$

or $$c = \frac{13}{6} = 2\frac{1}{6} \in (1, 3)$$

Thus the theorem is verified.

Applications of Cauchy 's Mean Value Theorem Contd.,

(*ii*) We have Cauchy's mean value theorem

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Here,

$$f(x) = \sin x$$
$$g(x) = \cos x$$
$$f'(x) = \cos x$$
$$g'(x) = -\sin x$$

$\therefore$ $g'(x) \neq 0$

Applications of Cauchy 's Mean Value Theorem Contd.,

Clearly both $f(x)$ and $g(x)$ are continuous in $\left[0, \frac{\pi}{2}\right]$ and differentiable in $\left(0, \frac{\pi}{2}\right)$. Therefore from Cauchy's mean value theorem

$$\frac{f\left(\frac{\pi}{2}\right) - f(0)}{g\left(\frac{\pi}{2}\right) - g(0)} = \frac{f'(c)}{g'(c)} \text{ for some } c : 0 < c < \frac{\pi}{2}$$

i.e.,

$$\frac{1-0}{0-1} = \frac{\cos c}{-\sin c}$$

$$-1 = -\cot c \text{ or } \cot c = 1$$

$$\therefore \qquad c = \frac{\pi}{4}$$

Clearly $\qquad c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$

Thus Cauchy's theorem is verified.

Applications of Cauchy ’s Mean Value Theorem Contd.,

(*iii*) We have Cauchy's mean value theorem

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Here $f(x) = e^x$

and $g(x) = e^{-x}$

$f'(x) = e^x$

$g'(x) = -e^{-x}$

$\therefore$ $f(x)$ and $g(x)$ are continuous in $[a, b]$ and differentiable in (a, b)

and also $g'(x) \neq 0$

$\therefore$ From Cauchy's mean value theorem

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

i.e., $$\frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}}$$

i.e., $$\frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}} = -e^{2c}$$

i.e., $$\frac{e^a e^b \left(e^b - e^a\right)}{\left(e^a - e^b\right)} = -e^{2c}$$

$$e^{a+b} = e^{2c}$$

Applications of Cauchy ’s Mean Value Theorem Contd.,

i.e., $a + b = 2c$

or $c = \dfrac{a+b}{2}$

$\therefore$ $c = \dfrac{a+b}{2} \in (a, b)$

Hence Cauchy's theorem holds good for the given functions.

CONTENTS

- Partial Differentiation
- Chain rule of Partial Differentiation

OBJECTIVE AND OUTCOME

OBJECTIVE:

Partial Differentiation and Chain rule of Partial Differentiation.

OUTCOME:

Student get to understand the concept of Partial Differentiation and Chain rule of Partial Differentiation.

Partial Differentiation

Partial differential coefficients : The Partial differential coefficient of f(x,y) with respect to x is the ordinary differential coefficient of f(x,y) when y is regarded as a constant. It is written as

$\frac{\partial f}{\partial x}$ or $\partial f/\partial x$ or $D_x f$

Thus $\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$

Again, the partial differential coefficient $\partial f/\partial y$ of f(x,y) with respect to y is the ordinary differential coefficient of f(x,y) when x is regarded as a constant.

Thus $\frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(x.y+k)-f(x,y)}{k}$

Similarly, if f is a function of the n variables $x_1, x_2, \ldots\ldots\ldots x_n$, the partial differential coefficient of f with respect to x_1 is the ordinary differential coefficient of f when all the variables except x_1 are regarded as constants and is written as $\partial f/\partial x_1$.

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are also denoted by f_x and f_y respectively.

The partial differential coefficients of f_x and f_y are $f_{xx}, f_{xy}, f_{yx}, f_{yy}$

or $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}$, respectively.

It should be specially noted that $\frac{\partial^2 f}{\partial y \partial x}$ means $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ and $\frac{\partial^2 f}{\partial x \partial y}$ means $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$.

The student will be able to convince himself that in all ordinary cases

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Applications on Partial Differentiation

Example 1 : If u = log $(x^3 + y^3 + z^3 - 3xyz)$ show that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+x)^2}$$

Solution : The given relation is

$u = \log(x^3 + y^3 + x^3 - 3xyz)$

Differentiate it w.r.t. x partially, we get

$$\frac{\partial u}{\partial x}=\frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

similarly $\dfrac{\partial u}{\partial y}=\dfrac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$

and $\dfrac{\partial u}{\partial z}=\dfrac{3z^2-3xyz}{x^3+y^3+z^3-3xyz}$

$$\therefore \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^2+y^2+z^2-yz-zx-xy)}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3(x^2+y^2+z^2-yz-xz-xy)}{(x+y+z)(x^2+y^2+z^2-yz-zx-xy)}$$

$$=\frac{3}{x+y+z}$$

Now $\left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}+\dfrac{\partial}{\partial z}\right)^2 u=\left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}+\dfrac{\partial}{\partial z}\right)\left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}+\dfrac{\partial}{\partial z}\right)u$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$=3\left[\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right)\right]$$

$$=3\left[-\frac{1}{(x+y+z)^2}+\frac{-1}{(x+y+z)^2}+\frac{-1}{(x+y+z)^2}\right]$$

$$=3\left[\frac{-3}{(x+y+z)^2}\right]$$

$$=-\frac{9}{(x+y+z)^2}$$ Hence Proved.

Applications on Partial Differentiation Contd.,

Example 2: If $u = e^{xyz}$, show that

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (1+3xyz+x^2y^2z^2)\, e^{xyz}$$

Solution : Given $u = e^{xyz}$

$$\therefore \frac{\partial u}{\partial z} = xy\, e^{xyz}$$

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left(xy\, e^{xyz}\right) = x \frac{\partial}{\partial y} y e^{xyz}$$

$= x[y\ xz\ e^{xyz} + e^{xyz}]$

$= e^{xyz}\ (x^2yz + x)$

Hence $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = \dfrac{\partial}{\partial x}\ [e^{xyz}\ (x^2yz+x)]$

$= e^{xyz}\ (2xyz+1) + yz\ e^{xyz}\ (x^2yz+x)$

$= e^{xyz}\ [2xyz + 1 + x^2y^2z^2 + xyz]$

$= e^{xyz}\ (1+3xyz + x^2y^2z^2)$ Hence Proved.

Applications on Partial Differentiation Contd.,

Example 3 : If $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$, prove that ,
$u_x^2+u_y^2+u_z^2 = 2\,(xu_x + yu_y + zu_z)$

Solution : Given $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$ ....(i)
where u is a function of x,y and z,
Differentiating (i) partially with respect to x, we get

$$\frac{(a^2+u).2x - x^2\frac{\partial u}{\partial x}}{(a^2+u)^2}+\frac{(b^2+u).0 - y^2\frac{\partial u}{\partial x}}{(b^2+u)^2}+\frac{(c^2+u).0 - z^2\frac{\partial u}{\partial x}}{(c^2+u)^2}=0$$

$$\text{or}\quad \frac{2x}{a^2+u}-\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial x}=0$$

$$\text{or}\quad \frac{\partial u}{\partial x}=\frac{2x/(a^2+u)}{\left[x^2/(a^2+u)^2+y^2/(b^2+u)^2+z^2/(c^2+u)^2\right]}$$

$$=\frac{2x/a^2+u}{\sum\left[x^2/(a^2+u)^2\right]}$$

$$\text{Similarly}\quad \frac{\partial u}{\partial y}=\frac{2y/(b^2+u)}{\sum\left[x^2/(a^2+u)^2\right]}$$

$$\text{and}\quad \frac{\partial u}{\partial z}=\frac{2z/(c^2+u)}{\sum\left[x^2/(a^2+u)^2\right]}$$

Applications on Partial Differentiation Contd.,

$$\therefore \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 4\frac{\left[x^2/(a^2+u)^2 + y^2/(b^2+u)^2 + z^2/(c^2+u)^2\right]}{\left[\sum\left\{x^2/(a^2+u)^2\right\}\right]^2}$$

$$\text{or } u_x^2 + u_y^2 + u_z^2 = \frac{4}{\sum\left[\left\{x^2/(a^2+u)^2\right\}\right]} \text{......................(ii)}$$

$$\text{Also } xu_x + yu_y + zu_z = = x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) + z\left(\frac{\partial u}{\partial z}\right)$$

$$= \frac{1}{\sum\left[x^2/(a^2+u)^2\right]}\left[\frac{2x^2}{(a^2+u)} + \frac{2y^2}{(b^2+u)} + \frac{2z^2}{(c^2+u)}\right]$$

$$= \frac{2}{\sum\left[x^2/(a^2+u)^2\right]}[1] \text{..................(iii)}$$

From (i), (ii) (iii) and we have

$u_x^2 + u_y^2 + u_z^2 = 2(xu_x + yu_y + zu_z)$ Hence Proved.

Applications on Partial Differentiation Contd.,

Example 4 : If $u = f(r)$ and $x = r\cos\theta$, $y = r\sin\theta$ i.e. $r^2 = x^2+y^2$, Prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

Solution : Given $u = f(r)$…………….(i)

Differentiating (i) partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = f'(r).\frac{\partial r}{\partial x}$$

$$= f'(r).\ \frac{x}{r} \qquad \because\ r^2 = x^2 + y^2$$

$$\Rightarrow 2r\ \frac{\partial r}{\partial x} = 2x$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$

Differentiating above once again, we get

Applications on Partial Differentiation Contd.,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{xf'(r)}{r}\right]$$

$$= \frac{r\left[f'(r).1 + xf''(r)(\partial r/\partial x)\right] - xf'(r)(\partial r/\partial x)}{r^2}$$

$$\text{or } \frac{\partial^2 u}{\partial x^2} = \frac{1}{r^2}\left[rf'(r) + x^2 f''(r) - \frac{x^2}{r} f'(r)\right] \qquad \text{(ii)}$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left[rf'(r) + y^2 f''(r) - \frac{y^2}{r} f'(r)\right] \qquad \text{(iii)}$$

Adding (ii) and (iii), we get

$$\frac{\partial u^2}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left[2rf'(r) + \left(x^2 + y^2\right)f''(r) - \frac{\left(x^2 + y^2\right)}{r} f'(r)\right]$$

$$= \frac{1}{r^2}\left[2r\, f'(r) + r^2 f''(r) - r\, f'(r)\right]$$

$$= \frac{1}{r} f'(r) + f''(r), \text{ Hence proved.}$$

Applications on Partial Differentiation Contd.,

Example 5 : If $x^x y^y z^z = c$, show that at $x = y = z$,

$$\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$$

Solution : Given $x^x y^y z^z = c$, where z is a function of x and y

Taking logarithms, $x \log x + y \log y + z \log z = \log c$ (i)

Differentiating (i) partially with respect to x, we get

$$\left[x\left(\frac{1}{x}\right) + (\log x)1\right] + \left[z\left(\frac{1}{z}\right) + (\log z)1\right]\frac{\partial z}{\partial x} = 0$$

or $\frac{\partial z}{\partial x} = -\frac{(1 + \log x)}{(1 + \log z)}$ (ii)

Similarly from (i) we have

$\frac{\partial z}{\partial y} = -\frac{(1 + \log y)}{(1 + \log z)}$ (iii)

$$\therefore \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$= \frac{\partial}{\partial x}\left[-\left(\frac{1 + \log y}{1 + \log z}\right)\right]$ From (iii)

Applications on Partial Differentiation Contd.,

or $\dfrac{\partial^2 z}{\partial x \partial y} = -(1+\log y).\dfrac{\partial}{\partial x}\left[(1+\log z)^{-1}\right]$

$= -(1+\log y).\left[-(1+\log z)^{-2}.\dfrac{1}{z}.\dfrac{\partial z}{\partial x}\right]$

or $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{(1+\log y)}{z(1+\log z)^2}.\left\{-\left(\dfrac{1+\log x}{1+\log z}\right)\right\}$, using (ii)

At $x = y = z$, we have $\dfrac{\partial^2 z}{\partial x \partial y} = -\dfrac{(1+\log x)^2}{x(1+\log x)^3}$

Substituting x for y and z

i.e. $\dfrac{\partial^2 z}{\partial x \partial y} = -\dfrac{1}{x(1+\log x)}$

$= -\dfrac{1}{x(\log e + \log x)}$ $\quad\quad \therefore \log e = 1$

$= -\dfrac{1}{x \log(ex)}$

$= -\{x \log(ex)\}^{-1}$ Hence Proved.

Chain rule of Partial Differentiation

Change of Variables : If u is a function of x, y and x, y are functions of t and r, then u is called a composite function of t and r.

Let u =f(x, y) and x = g(t, r), y = h(t, r) then the continuous first order partial derivatives are

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial t}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial r}$$

This is called as Chain rule of Partial Differentiation.

Example 1:

If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$ show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

Solution : Here given $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$

$= u\,(r, s)$

where $r = \frac{y-x}{xy}$ and $s = \frac{z-x}{zx}$

Applications of Chain rule of Partial Differentiation contd.,

$$\Rightarrow r = \frac{1}{x} - \frac{1}{y} \text{ and } s = \frac{1}{x} - \frac{1}{z} \quad \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

we know that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}$$

$$= \frac{\partial u}{\partial r}\left(-\frac{1}{x^2}\right) + \frac{\partial u}{\partial s}\left(-\frac{1}{x^2}\right) \qquad \because r = \frac{1}{x} - \frac{1}{y}$$

$$\Rightarrow \frac{\partial r}{\partial x} = -\frac{1}{x^2}$$

$$= -\frac{1}{x^2}\frac{\partial u}{\partial r} - \frac{1}{x^2}\frac{\partial u}{\partial s} \qquad \because s = \frac{1}{x} - \frac{1}{z}$$

$$\Rightarrow \frac{\partial s}{\partial x} = -\frac{1}{x^2}$$

$$\text{or } x^2\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial r} - \frac{\partial u}{\partial s} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Applications of Chain rule of Partial Differentiation contd.,

Similarly $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial y}$

$= \frac{\partial u}{\partial y}.\frac{1}{y^2} + \frac{\partial u}{\partial s}.0$ from (i)

or $y^2 \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}$ ....(iii)

and $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial z}$

$= \frac{\partial u}{\partial r}.0 + \frac{\partial u}{\partial s}.\frac{1}{z^2}$

$\Rightarrow z^2 \frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}$ ....(iv)

Adding (i) (ii) and (iii) we get

$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$ Hence Proved.

Example 2:

If $u = u(y - z, z - x, x - y)$ Prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Solution : Here given $u = u(y - z, z - x, x - y)$

Let $X = y - z$, $Y = z - x$ and $Z = x - y$................(i)

Then $u = u(X,Y,Z)$, where X, Y, Z are function of x,y and z.

Then

Applications of Chain rule of Partial Differentiation contd.,

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial X}\frac{\partial X}{\partial x}+\frac{\partial u}{\partial Y}\frac{\partial Y}{\partial x}+\frac{\partial u}{\partial Z}\frac{\partial Z}{\partial x}\quad\ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial X}\frac{\partial X}{\partial y}+\frac{\partial u}{\partial Y}\frac{\partial Y}{\partial y}+\frac{\partial u}{\partial Z}\frac{\partial Z}{\partial y}\quad\ldots\ldots\ldots\ldots\ldots\ldots(iii)$$

and $$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial X}\frac{\partial X}{\partial z}+\frac{\partial u}{\partial Y}\frac{\partial Y}{\partial z}+\frac{\partial u}{\partial Z}\frac{\partial Z}{\partial z}\quad\ldots\ldots\ldots\ldots(iv)$$

with the help of (i), equations (ii), (iii) and (iv) gives.

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial X}0+\frac{\partial u}{\partial Y}(-1)+\frac{\partial u}{\partial Z}(1)=-\frac{\partial u}{\partial Y}+\frac{\partial u}{\partial Z}\quad\ldots\ldots\ldots\ldots\ldots\ldots(v)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial X}.1+\frac{\partial u}{\partial Y}.0+\frac{\partial u}{\partial Z}(-1)=\frac{\partial u}{\partial X}-\frac{\partial u}{\partial Z}\quad\ldots\ldots\ldots\ldots\ldots\ldots(vi)$$

and $$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial X}(-1)+\frac{\partial u}{\partial Y}(1)+\frac{\partial u}{\partial Z}(0)=-\frac{\partial u}{\partial X}+\frac{\partial u}{\partial Y}\quad\ldots\ldots\ldots\ldots\ldots(vii)$$

Adding (v), (vi) and (vii) we get $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0.$ Hence Proved.

Applications of Chain rule of Partial Differentiation contd.,

Example 3: If z is a function of x and y and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$,

Prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

Solution : Here z is a function of x and y, where x and y are functions of u and v.

$$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \ldots\ldots\ldots\ldots\ldots\ldots(i)$$

and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v} \quad \ldots\ldots\ldots\ldots(ii)$

Also given that

$x = e^u + e^{-v}$ and $y = e^{-u} - e^v$

$$\therefore \frac{\partial x}{\partial u} = e^u, \ \frac{\partial x}{\partial v} = -e^{-v}, \ \frac{\partial y}{\partial u} = -e^{-u}, \ \frac{\partial y}{\partial v} = -e^v$$

$\therefore$ From (i) we get

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(e^u) + \frac{\partial z}{\partial y}(-e^{-u}) \quad \ldots\ldots\ldots\ldots\ldots\ldots(iii)$$

and from (ii) we get

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^{-v}) \quad \ldots\ldots\ldots\ldots\ldots(iv)$$

Subtracting (iv) from (iii) we get

Applications of Chain rule of Partial Differentiation contd.,

$$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=\left(e^{u}+e^{-v}\right)\frac{\partial z}{\partial x}-\left(e^{-u}-e^{v}\right)\frac{\partial z}{\partial y}$$

$$=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}.$$ Hence Proved.

Example 4:

If V = f(2x -3y, 3y -4z, 4z -2x), compute the value of $6V_x+4V_y+3V_z$.

Applications of Chain rule of Partial Differentiation contd.,

Solution : Here given $V = f(2x - 3y, 3y - 4z, 4z - 2x)$

Let $X = 2x - 3y$, $Y = 3y - 4z$ and $Z = 4z - 2x$.............(i)

Then $u = f(X, Y, Z)$, where X, Y, Z are function of x, y and z.

$$\text{Then } V_x = \frac{\partial V}{\partial x} = \frac{\partial V}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial V}{\partial Y}\frac{\partial Y}{\partial x} + \frac{\partial V}{\partial Z}\frac{\partial Z}{\partial x} \;\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

$$V_y = \frac{\partial V}{\partial y} = \frac{\partial V}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial V}{\partial Y}\frac{\partial Y}{\partial y} + \frac{\partial V}{\partial Z}\frac{\partial Z}{\partial y} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

$$\text{and } V_z = \frac{\partial V}{\partial z} = \frac{\partial V}{\partial X}\frac{\partial X}{\partial z} + \frac{\partial V}{\partial Y}\frac{\partial Y}{\partial z} + \frac{\partial V}{\partial Z}\frac{\partial Z}{\partial z} \;\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}$$

Applications of Chain rule of Partial Differentiation contd.,

with the help of (i), equations (ii), (iii) and (iv) gives

$$V_x = \frac{\partial V}{\partial X}(2) + \frac{\partial V}{\partial Y}(0) + \frac{\partial V}{\partial Z}(-2)$$

or $V_x = 2\left(\frac{\partial V}{\partial X} - \frac{\partial V}{\partial Z}\right)$

$$\Rightarrow 6V_x = 12\left(\frac{\partial V}{\partial X} - \frac{\partial V}{\partial Z}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(v)}$$

Now $V_y = \frac{\partial V}{\partial X}(-3) + \frac{\partial V}{\partial Y}(3) + \frac{\partial V}{\partial Z}(0)$

or $V_y = 3\left(-\frac{\partial V}{\partial X} + \frac{\partial V}{\partial Y}\right)$

$$\Rightarrow 4V_y = 12\left(-\frac{\partial V}{\partial X} + \frac{\partial V}{\partial Y}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(vi)}$$

and $V_z = \frac{\partial V}{\partial X}(0) + \frac{\partial V}{\partial Y}(-4) + \frac{\partial V}{\partial Z}(4)$

or $V_z = 4\left(-\frac{\partial V}{\partial Y} + \frac{\partial V}{\partial Z}\right)$

$$\Rightarrow 3V_z = 12\left(-\frac{\partial V}{\partial Y} + \frac{\partial V}{\partial Z}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(vii)}$$

Adding (v), (vi) and (vii) we get

$6V_x + 4V_y + 3V_z = 0$ Answer.

CONTENTS

- Total derivatives of partial differentiation
- Euler’s homogeneous function
- Euler’s theorem of homogeneous function

OBJECTIVE AND OUTCOME

OBJECTIVE:

Total derivatives of partial differentiation, Euler's theorem of homogeneous function.

OUTCOME:

Student get to understand the concept of Total derivatives of partial differentiation and Euler's theorem of homogeneous function.

Total derivatives of Partial Differentiation

Total Differentiation

Introduction : In partial differentiation of a function of two or more variables, only one variable varies. But in total differentiation, increments are given in all the variables.

Total differential Coefficient : If $u = f(x,y)$

where $x = \phi(t)$, and $y = \Psi(t)$ then we can find the value of u in terms of t by substituting from the last two equations in the first equation. Hence we can regard u as a function of the single variable t, and find the ordinary differential coefficient $\frac{du}{dt}$.

Then $\frac{du}{dt}$ is called the total differential coefficient of u, to distinguish it from the partial differential coefficient $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

Total derivatives of Partial Differentiation

Hence

$$\frac{du}{dt} = \frac{\partial f}{\partial x}.\frac{dx}{dt} + \frac{\partial f}{\partial y}.\frac{dy}{dt}$$

i.e. $$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

Similarly, if $u = f(x_1, x_2 \ldots\ldots\ldots x_n)$ and $x_1, x_2 \ldots\ldots\ldots\ldots\ldots\ldots, x_n$ are all functions of t, we can prove that

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1}.\frac{dx_1}{dt} + \frac{\partial u}{\partial x_2}.\frac{dx_2}{dt} + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{\partial u}{\partial x_n}.\frac{dx_n}{dt}$$

Total derivatives of Partial Differentiation

An important case : By supposing t to be the same, as x in the formula for two variables, we get the following proposition :

When f(x,y) is a function of x and y, and y is a function of x, the total (i.e., the ordinary) differential coefficient of f with respect to x is given by

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

Now, if we have an implicit relation between x and y of the form f(x,y) = C where C is a constant and y is a function of x, the above formula becomes

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

Which gives the important formula

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} / \frac{\partial f}{\partial y}$$

Again, if f is a function of n variables $x_1, x_2, x_3, \ldots\ldots\ldots x_n$, and $x_2, x_3 \ldots\ldots\ldots x_n$ are all functions of x_1, the total (i.e. the ordinary) differential coefficient of f with respect to x_1 is given by

$$\frac{df}{dx_1} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} \cdot \frac{dx_2}{dx_1} + \frac{\partial f}{\partial x_3} \cdot \frac{dx_3}{dx_1} + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{\partial f}{\partial x_n} \cdot \frac{dx_n}{dx_1}$$

Applications of Total derivatives

Example 1:

: If $u = x \log xy$, where $x^3+y^3+3xy = 1$, find $\frac{du}{dx}$.

Solution : Given $u = x \log xy$ ....(i)

we know $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$ ....(ii)

Now from (i) $\frac{\partial u}{\partial x} = x.\frac{1}{xy} y + \log x y$

$= 1 + \log x y$

and $\frac{\partial u}{\partial y} = x \frac{1}{xy} x = \frac{x}{y}$

Again, we are given $x^3+y^3+3xy = 1$, whence differentiating, we get

$3x^2 + 3y^2 \frac{dy}{dx} + 3\left(x\frac{dy}{dx} + y.1\right) = 0$

or $\frac{dy}{dx} = -\frac{(x^2+y)}{(y^2+x)}$

Substituting these values in (ii) we get

$\frac{du}{dx} = (1 + \log xy) + \frac{x}{y}\left[-\frac{(x^2+y)}{(y^2+x)}\right]$ Answer.

Applications of Total derivatives Contd.,

Example 2:

If $f(x, y) = 0$, $\phi(y, z) = 0$ show that $\dfrac{\partial f}{\partial y}\dfrac{\partial \phi}{\partial z}\dfrac{dz}{dx} = \dfrac{\partial f}{\partial x}\dfrac{\partial \phi}{\partial y}$

Solution : If $f(x, y) = 0$ then $\dfrac{dy}{dx} = -\left(\dfrac{\partial f}{\partial x}\right)/\left(\dfrac{\partial f}{\partial y}\right)$ ....(i)

if $\phi(y, z) = 0$, then $\dfrac{dz}{dy} = -\left(\dfrac{\partial \phi}{\partial y}\right)/\left(\dfrac{\partial \phi}{\partial z}\right)$ ....(ii)

Multiplying (i) and (ii), we have

$$\frac{dy}{dx}\cdot\frac{dz}{dy} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial \phi}{\partial y}\right)/\left(\frac{\partial f}{\partial y}\right)\cdot\left(\frac{\partial \phi}{\partial z}\right)$$

or $\left(\dfrac{\partial f}{\partial y}\right)\cdot\left(\dfrac{\partial \phi}{\partial z}\right)\dfrac{dz}{dx} = \left(\dfrac{\partial f}{\partial x}\right)/\left(\dfrac{\partial \phi}{\partial y}\right)$. Hence Proved

Applications of Total derivatives Contd.,

Example 3:

If the curves $f(x,y) = 0$ and $\phi(x, y) = 0$ touch, show that at the point of contact $\frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial \phi}{\partial x} = 0$

Solution : For the curve $f(x, y) = 0$, we have

$\frac{dy}{dx} = -\left(\frac{\partial f}{\partial x}\right) / \left(\frac{\partial f}{\partial y}\right)$ and for the curve $\phi(x, y) = 0$, $\frac{dy}{dx} = -\left(\frac{\partial \phi}{\partial x}\right) / \left(\frac{\partial \phi}{\partial y}\right)$

Also if two curves touch each other at a point then at that point the values of (dy/dx) for the two curves must be the same,

Hence at the point of contact

$$-\left(\frac{\partial f}{\partial x}\right) / \left(\frac{\partial f}{\partial y}\right) = -\left(\frac{\partial \phi}{\partial x}\right) / \left(\frac{\partial \phi}{\partial y}\right)$$

or $\left(\frac{\partial f}{\partial x}\right) \cdot \left(\frac{\partial \phi}{\partial y}\right) - \left(\frac{\partial \phi}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right) = 0.$ Hence Proved

Applications of Total derivatives Contd.,

Example 4:

If $\phi(x,y,z)=0$ show that $\left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z = -1$

Solution : The given relation defines y as a function of x and z. treating x as constant

$$\left(\frac{\partial y}{\partial z}\right)_x = -\frac{\partial \phi / \partial z}{\partial \phi / \partial y} \text{.................(i)}$$

The given relation defines z as a function of x and y. Treating y as constant

$$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\partial \phi / \partial x}{\partial \phi / \partial z} \text{.........(ii)}$$

Similarly, $\left(\frac{\partial x}{\partial z}\right)_z = -\frac{\partial \phi / \partial y}{\partial \phi / \partial x}$ ....(iii)

Multiplying (i), (ii) and (iii) we get

$$\left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z = -1$$

Hence Proved.

Euler's Theorem of Homogeneous functions

Euler's Theorem on Homogeneous Functions :

Statement : If f(x,y) is a homogeneous function of x and y of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

Proof : Since f(x,y) is a homogeneous function of degree n, it can be expressed in the form

$$f(x,y) = x^n F(y/x) \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\{x^n F(y/x)\} = nx^{n-1} F(y/x) + x^n F'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$\text{or } x\frac{\partial f}{\partial x} = n\, x^n F\left(\frac{y}{x}\right) - yx^{n-1} F'\left(\frac{y}{x}\right) \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Again from (i), we have

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\{x^n F(y/x)\}$$

$$= x^n F'(y/x).\ \frac{1}{x}$$

Euler's Theorem of Homogeneous functions Contd.,

or $y\frac{\partial f}{\partial y} = yx^{n-1} F'(y/x)$..............(iii)

Adding (ii) and (iii), we get

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^n F(y/x)$

= nf using(i) Hence Proved.

Note. In general if f $(x_1, x_2 \ldots\ldots x_n)$ be a homogeneous function of degree n, then by Euler's theorem, we have

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \ldots\ldots\ldots\ldots + x_n\frac{\partial f}{\partial x_n} = nf$$

Applications of Euler's Theorem of Homogeneous functions

Example 1: If $u = \log\left(\frac{x^2 + y^2}{x + y}\right)$, Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$

Solution : We are given that

$$u = \log\left(\frac{x^2 + y^2}{x + y}\right)$$

$$\therefore\ e^u = \frac{x^2 + y^2}{x + y} = f(\text{say})$$

Clearly f is a homogeneous function in x and y of degree 2-1 i.e. 1

$\therefore$ By Euler's theorem for f, we should have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f$$

$$x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = e^u \qquad \because\ f = e^u$$

$$\text{or}\ \ xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = e^u$$

$$\text{or}\ \ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$$ Hence Proved.

Applications of Euler's Theorem of Homogeneous functions

Example 2: If $u = \sin^{-1}\left\{\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right\}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\tan u$

Solution : Here $u = \sin^{-1}\left\{\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right\}$

$\Rightarrow \sin u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}} = f\ (\text{say})$

Here f is a homogeneous function in x and y of degree $\left(1-\dfrac{1}{2}\right)$ i.e $\dfrac{1}{2}$

$\therefore$ By Euler's theorem for f, we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{1}{2}f$$

$$\text{or } x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \frac{1}{2}\sin u$$

$$\therefore f = \sin u$$

$$\text{or } x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

$$\text{or } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u.$$

Hence Proved

Applications of Euler's Theorem of Homogeneous functions

Example 3:

If $u = \tan^{-1}\left(\dfrac{x^2 + y^2}{x + y}\right)$, then prove that $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\sin 2u$

Solution : Here $\tan u = \dfrac{x^2 + y^2}{x + y} = f$ (say)

Then for $\dfrac{x^2 + y^2}{x + y}$ is a homogeneous function in x and y of degree 2-1 i.e 1.

$\therefore$ By Euler's theorem for f, we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 1.f$$

or $x\dfrac{\partial}{\partial x}(\tan u) + y\dfrac{\partial}{\partial y}(\tan u) = \tan u$

$\because f = \tan u$

or $x \sec^2 u \dfrac{\partial u}{\partial x} + y \sec^2 u \dfrac{\partial u}{\partial y} = \tan u$

or $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{\tan u}{\sec^2 u} = \sin u \cos u = \dfrac{1}{2}\sin 2u.$ Hence Proved

Example 4:

If u be a homogeneous function of degree n, then prove that

(i) $x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x}$

(ii) $x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y}$ (iii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$

Solution : Since u is a homogenous function of degree n, therefore by Euler's theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \;\ldots\ldots\ldots\ldots(1)$$

Differentiating (i) partially w.r.t. x, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}.1+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x}$$

or $x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x}-\frac{\partial u}{\partial x}$

or $x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x}\;\ldots\ldots\ldots\ldots(2)$

which prove the result (i)

Applications of Euler's Theorem of Homogeneous functions

Now differentiating (i) partially w.r.t. y, we get

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} + 1.\frac{\partial u}{\partial y} = n\frac{\partial u}{\partial y}$$

$$\text{or } x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} = n\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}$$

$$\text{or } x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y} \quad \ldots\ldots\ldots\ldots\ldots(3)$$

Which proves the result (ii)

Multiplying (2) by x and (3) by y and then adding, we get

$$x\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} + xy\frac{\partial^2 u}{\partial y \partial x} + y^2\frac{\partial^2 u}{\partial y^2} = (n-1)\left\{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right\}$$

$$\text{or } x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (n-1)\,nu$$

which proves the result (iii). Hence Proved

Example 5:

If $u(x,y,z) = \log(\tan x + \tan y + \tan z)$ Prove that

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$$

Solution : we have

$u(x,y,z) = \log(\tan x + \tan y + \tan z)$.................(i)

Differentiating (i) w.r.t. 'x' partially, we get

$$\frac{\partial u}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y + \tan z} \text{.....................(ii)}$$

Differentiating (i) w.r.t. 'y' partially we get

$$\frac{\partial u}{\partial y} = \frac{\sec^2 y}{\tan x + \tan y + \tan z} \text{..................(iii)}$$

Again differentiating (i) w.r.t 'z' partially we get

$$\frac{\partial u}{\partial z} = \frac{\sec^2 z}{\tan x + \tan y + \tan z} \text{................(iv)}$$

Multiplying (ii), (iii) and (iv) by sin 2x, sin 2y and sin 2z respectively and adding them, we get

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = \frac{\sin 2x \sec^2 x + \sin 2y \sec^2 y + \sin 2z \sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{2\sin x \cos x.\sec^2 x + 2\sin y \cos y.\sec^2 y + 2\sin z \cos z.\sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{2(\tan x + \tan y + \tan z)}{\tan x + \tan y + \tan z}$$

$= 2$

$$\Rightarrow \sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2. \text{ Hence Proved}$$

CONTENTS

- Jacobian's of two and three variables
- Functional dependence and Independence

OBJECTIVE AND OUTCOME

OBJECTIVE:

Jacobian's of two and three variables , Functional Dependence and Independence.

OUTCOME:

Student get to understand the concept of Jacobian's of two and three variables , Functional Dependence and Independence.

Jacobian (J) of two and three variables

Let u = u (x , y) , v = v(x , y) are two functions of the independent variables x , y.

The jacobian of (u , v) w.r.t (x , y) is given by

$$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Note: $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J^{1} = \frac{\partial(x,y)}{\partial(u,v)}$ then $JJ^{1} = 1$

Similarly of u = u(x, y, z) , v = v (x, y , z) , w = w(x, y , z)

Then the Jacobian of u , v , w w.r.to x , y , z is given by

$$J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Properties:

- $\partial(u,v)/\partial(x,y) \times \partial(x,y)/\partial(u,v) = 1$
- If u, are functions of r,s and r,s are functions of x,y then

  $\partial(u,v)/\partial(x,y) = \partial(u,v)/\partial(r,s) . \partial(r,s)/\partial(x,y)$

Applications of Jacobian's

Example 1.

If $x + y^2 = u$, $y + z^2 = v$, $z + x^2 = w$ find $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$

Sol : Given $x + y^2 = u$, $y + z^2 = v$, $z + x^2 = w$

We have $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 1 & 2y & 0 \\ 0 & 1 & 2z \\ 2x & 0 & 1 \end{vmatrix}$

$= 1(1-0) - 2y(0 - 4xz) + 0$

$= 1 - 2y(-4xz)$

$= 1 + 8xyz$

$\Rightarrow \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \dfrac{1}{\left[\dfrac{\partial(u,v,w)}{\partial(x,y,z)}\right]} = \dfrac{1}{1 + 8xyz}$

Applications of Jacobian's Contd.,

Example 2.

S.T the functions $u = x + y + z$, $v = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$ and $w = x^3 + y^3 + z^3 - 3xyz$ are functionally related.

Sol: Given $u = x + y + z$

$$v = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$$

$$w = x^3 + y^3 + z^3 - 3xyz$$

we have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x-2y-2z & 2y-2x-2z & 2z-2y-2x \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$=6 \begin{vmatrix} 1 & 1 & 1 \\ x-y-z & y-x-z & z-y-x \\ x^2-yz & y^2-xz & z^2-xy \end{vmatrix}$$

$c_1 \to c_1 - c_2$

$c_2 \to c_2 - c_3$

$$= 6 \begin{vmatrix} 0 & 0 & 1 \\ 2x-2y & 2y-2z & z-y-x \\ x^2-yz-y^2+xz & y^2-xz-z^2+xy & z^2-xy \end{vmatrix}$$

$$=6[2(x-y)(y^2+xy-xz-z^2)-2(y-z)(x^2+xz-yz-y^2)]$$

$$=6[2(x-y)(y-z)(x+y+z)-2(y-z)(x-y)(x+y+z)]$$

$$=0$$

Hence there is a relation between u,v,w.

Applications of Jacobian's Contd.,

Example 3.

If x + y + z = u , y + z = uv , z = uvw then evaluate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

Sol: x + y + z = u

y + z = uv

z = uvw

y = uv – uvw = uv (1 – w)

x = u – uv = u (1 – v)

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$R_2 \to R_2 + R_3$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix}$$

= uv [u –uv +uv]

= u^2v

Applications of Jacobian's Contd.,

Example 4.

If $u = x^2 - y^2$, $v = 2xy$ where $x = r\cos\theta$, $y = r\sin\theta$ S.T $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$

Sol: Given $u = x^2 - y^2$, $v = 2xy$

$= r^2\cos^2\theta - r^2\sin^2\theta$ $\quad$ $= 2r\cos\theta \, r\sin\theta$

$= r^2(\cos^2\theta - \sin^2\theta)$ $\quad$ $= r^2\sin 2\theta$

$= r^2\cos 2\theta$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} u_r & u_\theta \\ v_r & v_\theta \end{vmatrix} = \begin{vmatrix} 2r\cos 2\theta & r^2(-\sin 2\theta)2 \\ 2r\sin 2\theta & r^2(\cos 2\theta)2 \end{vmatrix}$$

$$= (2r)(2r)\begin{vmatrix} \cos 2\theta & -r\sin 2\theta \\ \sin 2\theta & r(\cos 2\theta) \end{vmatrix}$$

$$= 4r^2[r\cos^2 2\theta + r\sin^2 2\theta]$$

$$= 4r^2(r)[\cos^2 2\theta + \sin^2 2\theta]$$

$$= 4r^3$$

Applications of Jacobian's Contd.,

Example 5.

If u = $\frac{yz}{x}$ **, v =** $\frac{xz}{y}$ **, w =** $\frac{xy}{z}$ **find** $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Sol: Given u = $\frac{yz}{x}$, v = $\frac{xz}{y}$, w = $\frac{xy}{z}$

We have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$u_x = yz(-1/x^2) = \frac{-yz}{x^2}$, $u_y = \frac{z}{x}$, $u_z = \frac{y}{x}$

$v_x = \frac{z}{y}$, $v_y = xz(-1/y^2) = \frac{-xz}{y^2}$, $v_z = \frac{x}{y}$

$w_x = \frac{y}{z}$, $w_y = \frac{x}{z}$, $w_z = xy\,(-1/z^2) = \frac{-xy}{z^2}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{x^2} \cdot \frac{1}{y^2} \cdot \frac{1}{z^2} \begin{vmatrix} -yz & xz & xy \\ yz & -xz & xy \\ yz & xz & -xy \end{vmatrix}$$

$$= \frac{(yz)(xz)(xy)}{x^2y^2z^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= 1[-1(1-1) -1(-1-1) + (1+1)\,]$

$= 0 -1(-2) + (2)$

$=2 + 2$

$=4$

Example 6.

If $x = e^r \sec\theta$, $y = e^r \tan\theta$ P.T $\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$

Sol: Given $x = e^r \sec\theta$, $y = e^r \tan\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}, \qquad \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix}$$

$x_r = e^r \sec\theta = x$, $\quad x_\theta = e^r \sec\theta \tan\theta$

$y_r = e^r \tan\theta = y$, $\quad y_\theta = e^r \sec^2\theta$

$x^2 - y^2 = e^{2r} (\sec^2\theta - \tan^2\theta)$

⇨ $2r = \log (x^2 - y^2)$

⇨ $r = \frac{1}{2} \log (x^2 - y^2)$

Applications of Jacobian's Contd.,

$$r_x = \frac{1}{2}\frac{1}{x^2 - y^2}(2x) = \frac{x}{(x^2 - y^2)}$$

$$r_y = \frac{1}{2}\frac{1}{x^2 - y^2}(-2y) = \frac{-y}{(x^2 - y^2)}$$

$$\frac{x}{y} = \frac{\sec\theta}{\tan\theta} = \frac{1/\cos\theta}{\sin\theta/\cos\theta} = \frac{1}{\sin\theta}$$

$$\Rightarrow \text{Sin}\theta = \frac{y}{x} \quad , \theta = \sin^{-1}\left(\frac{y}{x}\right)$$

$$\theta_x = \frac{1}{\sqrt{1 - \frac{y^2}{x^2}}}\, y\left(-\frac{1}{x^2}\right) = \frac{-y}{x\sqrt{x^2 - y^2}}$$

$$\theta_y = \frac{1}{\sqrt{1-\frac{y^2}{x^2}}}\,(1/x) \qquad = \frac{1}{\sqrt{x^2 - y^2}}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} e^r\sec\theta & \tan\theta \\ & e^r\sec 2\theta \end{vmatrix} = e^{2r}\sec^2\theta - y\, e^r\sec\theta\tan\theta$$

$$= e^{2r}\sec\theta[\sec^2\theta - \tan^2\theta\,] = e^{2r}\sec\theta$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{x}{(x^2 - y^2)} & \frac{-y}{(x^2 - y^2)} \\ \frac{-y}{x\sqrt{x^2 - y^2}} & \frac{1}{\sqrt{x^2 - y^2}} \end{vmatrix}$$

$$=\left[\frac{x}{(x^2-y^2)\sqrt{x^2-y^2}} - \frac{y^2}{x(x^2-y^2)\sqrt{x^2-y^2}}\right]$$

$$= \frac{x^2 - y^2}{x(x^2-y^2)\sqrt{x^2-y^2}} = \frac{1}{x\sqrt{x^2-y^2}} = \frac{1}{e^{2r}\sec\theta}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$$

Functional Dependence and Independence

Functional Dependence: Two functions $u\ (x,)\ ,v\ (x,y)$ are said to be functional dependent on one another if the Jacobian of u,v $w.r.t$ x,y is zero. If they are functionally dependent on one another, then it is possible to find the relation between these two functions.

Example 1.

Prove that the functions u= xy +yz +zx v = $x^2 + y^2 + z^2$, w = x + y+z are functionally dependent and find the relation between the

Answer: Given u= xy +yz +zx v = $x^2 + y^2 + z^2$, w = x + y+ z -(i)

$$\frac{\partial u}{\partial x} = y + z, \frac{\partial u}{\partial y} = z + x, \frac{\partial u}{\partial z} = x + y$$

and

$$\frac{\partial v}{\partial x} = 2x, \frac{\partial v}{\partial y} = 2y, \frac{\partial v}{\partial z} = 2z \text{ and } \frac{\partial w}{\partial x} = 1, \frac{\partial w}{\partial y} = 1, \frac{\partial w}{\partial z} = 1$$

$$\text{therefore} \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} y+z & z+x & x+y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} = 2 \begin{vmatrix} y+z & z+x & x+y \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2$

$$=2 \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$=2(X+Y+Z) \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 0$$ Since $R_1 and \ R_2$ are same

Hence u,v are functionally dependent. i.e functionalrelationship exists between u,v and w.

Now find that relation

We have w=x+y+z => $w^2 = (x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy+yz+zx) = v+2u$

Therefore $w^2 = 2u+v$ is the functional relationship between u,v and w.

Functional Dependence and Independence Contd.,

Example 2.

If $x = \frac{vw}{u}, y == \frac{uw}{v}, z == \frac{uv}{w}$ then show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = 4$. Are x,y,z functional dependence?

Solution: By the definition of Jacobian,

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} -\frac{vw}{u^2} & \frac{w}{u} & \frac{v}{u} \\ \frac{w}{v} & -\frac{wu}{v^2} & \frac{u}{v} \\ \frac{v}{w} & \frac{u}{w} & -\frac{uv}{w^2} \end{vmatrix}$$

$$= \begin{vmatrix} -vw & wu & uv \\ vw & -wu & uv \\ vw & uw & -uv \end{vmatrix} \cdot \frac{1}{u^2} \cdot \frac{1}{v^2} \cdot \frac{1}{w^2} = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 4$$

Since, Jacobian is non-zero, x, y, z are functionally independent.

CONTENTS

- Maxima and Minima of two variables with constraints
- Working Rule of Maxima and Minima of two variables with constraints
- Applications of Maxima and Minima of two variables with constraints

OBJECTIVE AND OUTCOME

OBJECTIVE:

Maxima and Minima of two variables with constraints.

OUTCOME:

Student get to understand the concept of Maxima and Minima of two variables with constraints.

Maximum & Minimum for function of a single Variable

To find the Maxima & Minima of $f(x)$ we use the following procedure.

(i) Find $f^1(x)$ and equate it to zero

(ii) Solve the above equation we get x_0, x_1 as roots.

(iii) Then find $f^{11}(x)$.

If $f^{11}(x)_{(x = x0)} > 0$, then $f(x)$ is minimum at x_0

If $f^{11}(x)_{(x = x0)}$, < 0, $f(x)$ is maximum at x_0. Similarly we do this for other stationary

points.

Example. Find the max & min of the function $f(x) = x^5 - 3x^4 + 5$

Sol: Given $f(x) = x^5 - 3x^4 + 5$

$f^1(x) = 5x^4 - 12x^3$

for maxima or minima $f^1(x) = 0$

$5x^4 - 12x^3 = 0$

$x = 0, x = 12/5$

$f^{11}(x) = 20x^3 - 36x^2$

At $x = 0 \Rightarrow f^{11}(x) = 0$. So f is neither maximum nor minimum at x = 0

At $x = (12/5) \Rightarrow f^{11}(x) = 20(12/5)^3 - 36(12/5)$

$= 144(48-36)/25 \quad = 1728/25 > 0$

So f(x) is minimum at x = 12/5

The minimum value is $f(12/5) = (12/5)^5 - 3(12/5)^4 + 5$

Definitions

***Extremum** : A function which have a maximum or minimum or both is called ‘extremum’

***Extreme value** :- The maximum value or minimum value or both of a function is Extreme value.

***Stationary points:** - To get stationary points we solve the equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ i.e the pairs (a_1, b_1), (a_2, b_2) ………….. are called Stationary.

Necessary and Sufficient Conditions for Maxima and Minima:

The necessary conditions for a function $f(x, y)$ to have either a maximum or a minimum at a point (a, b) are $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

The points (x, y) where x and y satisfy $f_x(x, y) = 0$ and $f_y(x, y) = 0$ are called the stationary or the critical values of the function.

Suppose (a, b) is a critical value of the function $f(x, y)$. Then $f_x(a, b) = 0$, $f_y(a, b) = 0$.

Now denote

$$f_{xx}(a, b) = A, \quad f_{xy}(a, b) = B, \quad f_{yy}(a, b) = C$$

1. Then, the function $f(x, y)$ has a maximum at (a, b) if $AC - B^2 > 0$ and $A < 0$.
2. The function $f(x, y)$ has a minimum at (a, b) if $AC - B^2 > 0$ and $A > 0$.

Maximum and minimum values of a function are called the **"extreme values of the function".**

Maximum & Minimum for function of two Variables

Working procedure:

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ Equate each to zero. Solve these equations for x & y we get the pair of values $(a_1, b1)$ (a_2,b_2) (a_3 ,b_3) ....

2. Find $l = \frac{\partial^2 f}{\partial x^2}, m = \frac{\partial^2 f}{\partial x \partial y}$, n = $\frac{\partial^2 f}{\partial y^2}$

3. i. If $ln - m^2 > 0$ and $l < 0$ at (a_1,b_1) then f(x ,y) is maximum at (a_1,b_1) and maximum value is $f(a_1,b_1)$

ii. If $ln - m^2 > 0$ and $l > 0$ at (a_1,b_1) then f(x ,y) is minimum at (a_1,b_1) and minimum value is $f(a_1,b_1)$.

iii. If $ln - m^2 < 0$ and at $(a_1, b1)$ then f(x, y) is neither maximum nor minimum at $(a_1, b1)$. In this case $(a_1, b1)$ is saddle point.

iv. If $ln - m^2 = 0$ and at $(a_1, b1)$, no conclusion can be drawn about maximum or minimum and needs further investigation. Similarly we do this for other stationary points.

Example 1. Locate the stationary points & examine their nature of the following functions.

$u = x^4 + y^4 - 2x^2 + 4xy - 2y^2, \quad (x > 0, y > 0)$

Sol: Given $u(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

For maxima & minima $\frac{\partial u}{\partial x} = 0, \ \frac{\partial u}{\partial y} = 0$

$\frac{\partial u}{\partial x} = 4x^3 - 4x + 4y = 0 \Rightarrow x^3 - x + y = 0$ ------------------> (1)

$\frac{\partial u}{\partial y} = 4y^3 + 4x - 4y = 0 \Rightarrow y^3 + x - y = 0$ ------------------> (2)

Adding (1) & (2),

$x^3 + y^3 = 0$

$\Rightarrow x = -y$ ------------------> (3)

(1) $\Rightarrow$ $x^3 - 2x \Rightarrow x = 0, \sqrt{2}, -\sqrt{2}$

Hence (3) $\Rightarrow$ $y = 0, -\sqrt{2}, \sqrt{2}$

$l = \frac{\partial^2 u}{\partial x^2} = 12x^2 - 4$, $m = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 4$ & $n = \frac{\partial^2 u}{\partial y^2} = 12y^2 - 4$

$ln - m^2 = (12x^2 - 4)(12y^2 - 4) - 16$

At $(-\sqrt{2}, \sqrt{2})$, $ln - m^2 = (24 - 4)(24 - 4) - 16 = (20)(20) - 16 > 0$ and $l = 20 > 0$

The function has minimum value at $(-\sqrt{2}, \sqrt{2})$

At (0,0), $ln - m^2 = (0 - 4)(0 - 4) - 16 = 0$

(0,0) is not a extreme value.

Example 2. Investigate the maxima & minima, if any, of the function

$f(x) = x^3y^2 (1-x-y)$.

Sol: Given $f(x) = x^3y^2 (1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$

$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$ $\frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$

For maxima & minima $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

⇨ $3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0 \Rightarrow x^2y^2(3 - 4x - 3y) = 0$ ---------------> (1)

⇨ $2x^3y - 2x^4y - 3x^3y^2 = 0 \Rightarrow x^3y(2 - 2x - 3y) = 0$ ----------------> (2)

From (1) & (2) $4x + 3y - 3 = 0$

$2x + 3y - 2 = 0$

$2x = 1 \Rightarrow x = ½$

$4 (½) + 3y - 3 = 0 \Rightarrow 3y = 3 - 2 , y = (1/3)$

$l = \frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$

$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(1/2,1/3)} = 6(1/2)(1/3)^2 - 12(1/2)^2(1/3)^2 - 6(1/2)(1/3)^3 = 1/3 - 1/3 - 1/9$

$= -1/9$

$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 6x^2y - 8x^3y - 9x^2y^2$

$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(1/2\ ,1/3)} = 6(1/2)^2(1/3) - 8(1/2)^3(1/3) - 9(1/2)^2(1/3)^3 = \frac{6-4-3}{12} = \frac{-1}{12}$

$n = \frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y$

$\left(\frac{\partial^2 f}{\partial y^2}\right)_{(1/2,1/3)} = 2(1/2)^3 - 2(1/2)^4 - 6(1/2)^3(1/3) = \frac{1}{4} - \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}$

$ln - m^2 = (-1/9)(-1/8) - (-1/12)^2 = \frac{1}{72} - \frac{1}{144} = \frac{2-1}{144} = \frac{1}{144} > 0$

and $l = \frac{-1}{9} < 0$

The function has a maximum value at (1/2 , 1/3)

Maximum value is $f\left(\frac{1}{2}, \frac{1}{3}\right) = \left(\frac{1}{8} \times \frac{1}{9}\right)\left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{72}\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{432}$

Example 3. Find the maxima & minima of the function $f(x) = 2(x^2 - y^2) - x^4 + y^4$

Sol: Given $f(x) = 2(x^2 - y^2) - x^4 + y^4 = 2x^2 - 2y^2 - x^4 + y^4$

For maxima & minima $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 4x - 4x^3 = 0 \Rightarrow 4x(1-x^2) = 0 \Rightarrow x = 0$, $x = \pm 1$

$\frac{\partial f}{\partial y} = -4y + 4y^3 = 0 \Rightarrow -4y(1-y^2) = 0 \Rightarrow y = 0, y = \pm 1$

$$l = \left(\frac{\partial^2 f}{\partial x^2}\right) = 4 - 12x^2$$

$$m = \left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 0$$

$$n = \left(\frac{\partial^2 f}{\partial y^2}\right) = -4 + 12y^2$$

we have $ln - m^2 = (4-12x^2)(-4 +12y^2) - 0$

$= -16 +48x^2 +48y^2 -144x^2y^2$

$= 48x^2 +48y^2 -144x^2y^2 -16$

i) At $(0, \pm 1)$

$ln - m^2 = 0 + 48 - 0 -16 = 32 > 0$

$l = 4-0 = 4 > 0$

f has minimum value at $(0, \pm 1)$

$f(x, y) = 2(x^2 - y^2) - x^4 + y^4$

$f(0, \pm 1) = 0 - 2 - 0 + 1 = -1$

The minimum value is '-1 '.

ii) At $(\pm 1, 0)$

$ln - m^2 = 48 + 0 - 0 -16 = 32 > 0$

$l = 4-12 = -8 < 0$

f has maximum value at $(\pm 1, 0)$

$f(x, y) = 2(x^2 - y^2) - x^4 + y^4$

$f(\pm 1, 0) = 2 - 0 - 1 + 0 = 1$

The maximum value is '1 '.

iii) At $(0,0)$, $(\pm 1, \pm 1)$

$ln - m^2 < 0$

$l = 4 - 12x^2$

$(0, 0)$ & $(\pm 1, \pm 1)$ are saddle points.

f has no max & min values at $(0, 0)$, $(\pm 1, \pm 1)$.

Example 4.

Determine the maxima/minima of the function

$$\sin x + \sin y + \sin (x + y).$$

Solution

Let $f(x, y) = \sin x + \sin y + \sin (x + y)$

We have $f_x = \cos x + \cos (x + y)$

$f_y = \cos y + \cos (x + y)$

Now $f_x = 0$ and $f_y = 0$ implies

$\cos (x + y) = -\cos x$ and $\cos (x + y) = -\cos y$

i.e., $-\cos x = -\cos y$ or $\cos x = \cos y$

or $x = y$

Then, $\cos 2x = -\cos x = \cos (\pi - x)$

Applications of Maximum & Minimum for function of two Variables

or $$2x = \pi - x \text{ or } x = \frac{\pi}{3}$$

So that $$y = \frac{\pi}{3}$$

The critical point is $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

Further,
$$A = f_{xx} = -\sin x - \sin (x + y)$$
$$B = f_{xy} = -\sin (x + y)$$
$$C = f_{yy} = -\sin y - \sin (x + y)$$

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$,
$$A = -\sin \frac{\pi}{3} - \sin \frac{2\pi}{3} = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$$
$$B = -\sin \frac{2\pi}{3} = -\frac{\sqrt{3}}{2}$$
$$C = -\sin \frac{\pi}{3} - \sin \frac{2\pi}{3} = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$$

and $$AC - B^2 = \left(-\sqrt{3}\right)\left(-\sqrt{3}\right) - \left(-\frac{\sqrt{3}}{2}\right)^2 = \frac{9}{4} > 0$$

Also $$A = -\sqrt{3} < 0$$

$\therefore f(x, y)$ attains its maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

and $$\text{maximum } f(x, y) = f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \sin \frac{\pi}{3} + \sin \frac{\pi}{3} + \sin\left(\frac{2\pi}{3}\right)$$
$$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}.$$

Example 5.

Find the extreme values of the function $x^4 + 2x^2y - x^2 + 3y^2$.

Solution

Let $f(x, y) = x^4 + 2x^2y - x^2 + 3y^2$

We have $f_x = 4x^3 + 4xy - 2x$

and $f_y = 2x^2 + 6y$

Then $f_x = 0$ and $f_y = 0$ implies

$$2x(2x^2 + 2y - 1) = 0 \text{ and } 2(x^2 + 3y) = 0$$

i.e., $x = 0$ or $2x^2 + 2y - 1 = 0$ and $x^2 + 3y = 0$

which is same as

$$\{x = 0 \text{ and } x^2 + 3y = 0\} \text{ or } \{2x^2 + 2y - 1 = 0 \text{ and } x^2 + 3y = 0\}$$

i.e., $x = 0$ and $y = 0$.

where $x^2 = -3y$

$\therefore$ $2x^2 + 2y - 1 = 0$

$$2(-3y) + 2y - 1 = 0$$

which implies $y = \dfrac{-1}{4}$

Hence, $x^2 = \dfrac{3}{4}$ or $x = \pm\dfrac{\sqrt{3}}{2}$

Hence, the critical values are (0, 0), $\left(\frac{\sqrt{3}}{2}, \frac{-1}{4}\right)$ and $\left(\frac{-\sqrt{3}}{2}, \frac{-1}{4}\right)$

Further

$$A = f_{xx} = 12x^2 + 4y - 2,\ B = f_{xy} = 4x,\ C = f_{yy} = 6$$

(*i*) At (0, 0), $A = -2$, $B = 0$, $C = 6$ and $AC - B^2 = -12 < 0$

Hence, there is neither a maximum nor a minimum at (0, 0)

(*ii*) At $\left(\frac{\sqrt{3}}{2}, \frac{-1}{4}\right)$, $\quad A = 12 \cdot \frac{3}{4} + 4\left(\frac{-1}{4}\right) - 2 = 6$

$$B = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3},\ C = 6$$

Then, $\quad AC - B^2 = 6\ (6) - \left(2\sqrt{3}\right)^2 = 24 > 0$ and

$$A = 6 > 0$$

$\therefore f\ (x, y)$ has a minimum at $\left(\frac{\sqrt{3}}{2}, \frac{-1}{4}\right)$

Hence, $f\ (x, y)$ attains its minimum value at $\left(\frac{\sqrt{3}}{2}, \frac{-1}{4}\right)$

Also, $\quad$ minimum $f\ (x, y) = \left(\frac{-3}{8}\right)$

(iii) Similarly, at $\left(\frac{-\sqrt{3}}{2}, \frac{-1}{4}\right)$

$f(x, y)$ attains its minimum.

Thus, $f(x, y)$ attains minimum $\frac{-3}{8}$ at $\left(\pm\frac{\sqrt{3}}{2}, \frac{-1}{4}\right)$.

Example 6. Find three positive numbers whose sum is 100 and whose product is maximum.

Sol: Let x ,y ,z be three +ve numbers.

Then $x + y + z = 100$

$\Rightarrow \quad z = 100 - x - y$

Let $f(x,y) = xyz = xy(100 - x - y) = 100xy - x^2y - xy^2$

For maxima or minima $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 100y - 2xy - y^2 = 0 \Rightarrow y(100 - 2x - y) = 0$ ----------------> (1)

$\frac{\partial f}{\partial y} = 100x - x^2 - 2xy = 0 \Rightarrow x(100 - x - 2y) = 0$ ------------------> (2)

From (1) & (2)

$100 - 2x - y = 0$

$200 - 2x - 4y = 0$

$-100 + 3y = 0 \Rightarrow 3y = 100 \Rightarrow y = 100/3$

$100 - x - (200/3) = 0 \Rightarrow x = 100/3$

Applications of Maximum & Minimum for function of two Variables

$$l = \frac{\partial^2 f}{\partial x^2} = -2y$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)(100/3\ ,\ 100/3\ ) = -\ 200/3$$

$$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 100 - 2x - 2y$$

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)(100/3\ ,\ 100/3\ ) = 100 - (200/3) - (200/3) = -(100/3)$$

$$n = \frac{\partial^2 f}{\partial y^2} = -2x$$

$$\left(\frac{\partial^2 f}{\partial y^2}\right)(100/3\ ,\ 100/3\ ) = -\ 200/3$$

$$ln - m^2 = (-200/3)\,(-200/3) - (-100/3)^2 = (100)^2/3$$

The function has a maximum value at (100/3 , 100/3)

i.e. at x = 100/3, y = 100/3 $\quad \therefore\ z = 100 - \frac{100}{3} - \frac{100}{3} = \frac{100}{3}$

The required numbers are x = 100/3, y = 100/3, z = 100/3

CONTENTS

OBJECTIVE AND OUTCOME

OBJECTIVE:

Maxima and Minima of two variables without constraints.

OUTCOME:

Student get to understand the concept of Maxima and Minima of two variables without constraints.

Definitions

***Extremum** : A function which have a maximum or minimum or both is called 'extremum'

***Extreme value** :- The maximum value or minimum value or both of a function is Extreme value.

***Stationary points:** - To get stationary points we solve the equations $\frac{\partial f}{\partial x}$ = 0 and $\frac{\partial f}{\partial y}$ = 0 i.e the pairs (a_1, b_1), (a_2, b_2) …………. are called Stationary.

Example 1.

A rectangular box, open at the top, is to have a given capacity. Find the dimensions of the box requiring least material for its construction.

Solution: Let x, y and z be the length, breadth and height respectively, let V be the given capacity and S, the surface

V is given $\Rightarrow$ V is constant

$V = xyz$

$$\text{or } Z = \frac{V}{xy} \qquad \text{(i)}$$

$S = xy + 2xz + 2yz$

Applications of Maximum & Minimum for function of two Variables

$$= xy + \frac{2V}{y} + \frac{2V}{x} = f(x, y) \text{ using (i)}$$

$$\therefore p = \frac{\partial f}{\partial x} = y - \frac{2V}{x^2},\ q = \frac{\partial f}{\partial y} = x - \frac{2V}{y^2}$$

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{4V}{x^3},\quad s = \frac{\partial^2 f}{\partial x \partial y} = 1$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{4V}{y^3}$$

Now $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$$\Rightarrow y - \frac{2V}{x^2} = 0 \text{ ....(ii) and } x - \frac{2V}{y^2} = 0 \text{ ....(iii)}$$

From (i) $y = \frac{2V}{x^2}$

$\therefore$ From (ii), $x - 2V \dfrac{x^4}{4V^2} = 0$

$$\text{or } x\left(1-\frac{x^3}{2V}\right)=0$$

or $x = (2V)^{1/3}$ (As $x \neq 0$)

$$\text{and } y = \frac{2V}{x^2} = \frac{2V}{(2V)^{3/2}} = (2V)^{1/3}$$

$\therefore$ $x = y = (2V)^{1/3}$ is a stationary point. At this point, $r = \dfrac{4V}{2V} = 2 > 0$, $s = 1$,

$$t = \frac{4V}{2V} = 2$$

So that $rt - s^2 = 4 - 1 = 3 > 0$ and $r > 0$

$\Rightarrow$S is minimum when $x = y = (2V)^{1/3}$

$$\text{Also } z = \frac{V}{xy} = \frac{V}{(2V)^{2/3}}$$

$$= \frac{V^{1/3}}{2^{2/3}} = \frac{(2V)^{1/3}}{2}$$

$$= \frac{y}{2}$$

Example 2.

Find the volume of largest parallelopiped that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Solution: Let (x,y,z) denote the co-ordinates of one vertices of the parallelopied which lies in the positive octant and V denote its volume so that

V = 8xyz As 2x, 2y and 2z be the length, breadth and height respectively

$\therefore V^2 = 64x^2 y^2 z^2$

$$= 64\, x^2 y^2 c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$$

$$= 64c^2\left(x^2y^2-\frac{x^4y^2}{a^2}-\frac{x^2y^4}{b^2}\right)= f(x,y) \text{ say}$$

$$\Rightarrow \frac{\partial f}{\partial x}=64c^2\left(2xy^2-\frac{4x^3y^2}{a^2}-\frac{2xy^4}{b^2}\right)$$

$$\text{and } \frac{\partial f}{\partial y}=64c^2\left(2x^2y-\frac{2x^4y}{a^2}-\frac{4x^2y^3}{b^2}\right)$$

Now putting $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ we get

$$\Rightarrow 2xy^2 - \frac{4x^3y^2}{a^2} - \frac{2xy^4}{b^2} = 0$$

$$\Rightarrow 1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2} = 0 \qquad \text{(i)}$$

and $2x^2y - \frac{2x^4y}{a^2} - \frac{4x^2y^3}{b^2} = 0$

$$\Rightarrow 1 - \frac{x^2}{a^2} - \frac{2y^2}{b^2} = 0 \qquad \text{(ii)}$$

Now multiply (i) by 2, we have

$$2 - \frac{4x^2}{a^2} - \frac{2y^2}{b^2} = 0 \qquad \text{(iii)}$$

subtracting (iii) from (ii) we have

$$-1 + \frac{3x^2}{a^2} = 0 \Rightarrow 3x^2 = a^2$$

$\therefore x = \frac{a}{\sqrt{3}}$ and $y = \frac{b}{\sqrt{3}}$

$$\text{Now } r = \frac{\partial^2 f}{\partial x^2} = 64c^2\left[2y^2 - \frac{12x^2y^2}{a^2} - \frac{2y^4}{b^2}\right]$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 64c^2\left[4xy - \frac{8x^3y}{a^2} - \frac{8xy^3}{b^2}\right]$$

$$t = \frac{\partial^2 f}{\partial y^2} = 64c^2\left[2x^2 - \frac{2x^4}{a^2} - \frac{12x^2y^2}{b^2}\right]$$

$$\text{and } rt - s^2 = (64\,c^2)\left(2y^2 - \frac{2y^4}{b^2} - \frac{12x^2y^2}{a^2}\right)(64c^2)\left(2x^2 - \frac{2x^4}{a^2} - \frac{12x^2y^2}{b^2}\right)$$

$$-\left[(64c^2)\left(4xy - \frac{8x^3y}{a^2} - \frac{8xy^3}{b^2}\right)\right]^2$$

$$\text{At } \left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}\right),\ rt - s^2 > 0 \text{ and } r < 0$$

$$\text{Hence } f(x,y) \text{ is max at} \left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}\right)$$

$$\Rightarrow V_{max}^2 = 64c^2\left[\frac{a^2}{3}\cdot\frac{b^2}{3} - \frac{1}{a^2}\cdot\frac{a^4}{9}\cdot\frac{b^2}{3} - \frac{1}{b^2}\times\frac{a^2}{3}\times\frac{b^4}{9}\right]$$

$$= \frac{64a^2b^2c^2}{27}$$

$$\Rightarrow V_{max} = \frac{8abc}{3\sqrt{3}} \qquad \text{Answer.}$$

CONTENT

- Maxima and Minima of two variable function by method of Lagrange multipliers

OBJECTIVE AND OUTCOME

OBJECTIVE:

- Maxima and Minima of two variable function by method of Lagrange multipliers.

OUTCOME:

- Student get to understand the concept of Maxima and Minima of two variable function by method of Lagrange multipliers.

Method of Lagrange Multipliers

In many problems, a function of two or more variables is to be optimized, subjected to a restriction or constraint on the variables, here we will consider a function of three variables to study Lagrange's method of undetermined multipliers.

Let

$u = f(x, y, z)$ (i)

be a function of three variables connected by the relation

$\phi(x, y, z) = 0$ (ii)

The necessary conditions for u to have stationary values are

$$\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0, \frac{\partial u}{\partial z} = 0$$

Differentiating equation (i), we get du =0 i.e.

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \qquad \text{(iii)}$$

Differentiating equation (ii) we get $d\phi = 0$ i.e.

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0 \qquad \text{(iv)}$$

Multiplying equation (iv) by λ and adding to equation (iii) we get

$$\left(\frac{\partial u}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial u}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0$$

Method of Lagrange Multipliers

This equation will be satisfied if

$$\frac{\partial u}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \qquad \text{(v)}$$

$$\frac{\partial u}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \qquad \text{(vi)}$$

$$\frac{\partial u}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \qquad \text{(vii)}$$

where λ is the Lagrange multiplier. On solving equations (ii), (v) (vi) and (vii), we get the values of x, y, z and λ which determine the stationary points and hence the stationary values of f(x, y, z).

Note: (i) Lagrange's method gives only the stationary values of f(x, y, z). The nature of stationary points cannot be determined by this method.

(ii) If there are two constraints $\phi_1(x, y, z) = 0$ & $\phi_2(x, y, z) = 0$, then the auxiliary function is $F(x, y, z) = f(x, y, z) + \lambda_1, \phi, (x, y, z) + \lambda_2 \phi_2 (x, y, z)$ here λ_1 and λ_2 are the two Lagrange multipliers. The stationary values are obtained by solving the five equations $F_x = 0$, $F_y = 0$, $F_z = 0$, $F_{\lambda_1} = 0$ and $F_{\lambda_2} = 0$

Method of Lagrange Multipliers

Summary:

Suppose $f(x , y , z) = 0$ ------------(1)

$\varnothing(x , y , z) = 0$ ------------- (2)

$F(x , y , z) = f(x , y , z) + \gamma \varnothing(x , y , z)$ where γ is called Lagrange's constant.

1. $\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \gamma \frac{\partial \varnothing}{\partial x} = 0$ --------------- (3)

   $\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + \gamma \frac{\partial \varnothing}{\partial y} = 0$ --------------- (4)

   $\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{\partial f}{\partial z} + \gamma \frac{\partial \varnothing}{\partial z} = 0$ --------------- (5)
2. Solving the equations (2) (3) (4) & (5) we get the stationary point (x, y, z).
3. Substitute the value of x , y , z in equation (1) we get the extremum.

Applications of Method of Lagrange Multipliers

Example 1. Find the minimum value of $x^2 + y^2 + z^2$, given $x + y + z = 3a$

Sol: $u = x^2 + y^2 + z^2$

$\varnothing = x + y + z - 3a = 0$

Using Lagrange's function

$F(x, y, z) = u(x, y, z) + \gamma\, \varnothing(x, y, z)$

For maxima or minima

$$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x} + \gamma \frac{\partial \varnothing}{\partial x} = 2x + \gamma = 0 \text{ ------------ (1)}$$

$$\frac{\partial F}{\partial y} = \frac{\partial u}{\partial y} + \gamma \frac{\partial \varnothing}{\partial y} = 2y + \gamma = 0 \text{ ------------ (2)}$$

$$\frac{\partial F}{\partial z} = \frac{\partial u}{\partial z} + \gamma \frac{\partial \varnothing}{\partial z} = 2z + \gamma = 0 \text{ ------------ (3)}$$

From (1), (2) & (3)

$\gamma = -2x = -2y = -2z$

$x = y = z$

$\varnothing = x + x + x - 3a = 0$

$x = a$

$x = y = z = a$

Minimum value of $u = a^2 + a^2 + a^2 = 3a^2$

Applications of Method of Lagrange Multipliers

Example 2.

Find the minimum value of $x^2 + y^2 + z^2$ *subject to the condition* $ax + by + cz = p$.

Solution

Let $$F = (x^2 + y^2 + z^2) + \lambda (ax + by + cz)$$

We form the equations $F_x = 0,\ F_y = 0,\ F_z = 0$

i.e., $$2x + \lambda a = 0,\ 2y + \lambda b = 0,\ 2z + \lambda c = 0$$

or $$\lambda = \frac{-2x}{a},\ \lambda = \frac{-2y}{b},\ \lambda = \frac{-2z}{c}$$

$$\Rightarrow \quad \frac{-2x}{a} = \frac{-2y}{b} = \frac{-2z}{c}$$

or $$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = k \text{ (say)}$$

$\therefore$ $$x = ak,\ y = bk,\ z = ck$$

But $ax + by + cz = p$ and hence, we have

$$a^2k + b^2k + c^2k = p$$

$\therefore$ $$k = \frac{p}{a^2 + b^2 + c^2}$$

Hence, the required minimum value of $x^2 + y^2 + z^2$ is

$$a^2k^2 + b^2k^2 + c^2k^2 = k^2 (a^2 + b^2 + c^2)$$

i.e., $$\frac{p^2 (a^2 + b^2 + c^2)}{(a^2 + b^2 + c^2)^2} = \frac{p^2}{a^2 + b^2 + c^2}$$

thus, the required minimum value is $\dfrac{p^2}{a^2 + b^2 + c^2}$.

Applications of Method of Lagrange Multipliers

Example

3. *If* $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 1$, *show that the minimum value of the function* $a^3x^2 + b^3y^2 + c^3z^2$ *is* $(a + b + c)^3$.

Solution

Let $$F = (a^3x^2 + b^3y^2 + c^3z^2) + \lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

we form the equations $F_x = 0,\ F_y = 0,\ F_z = 0$

i.e.,
$$2a^3x + \lambda\left(\frac{-1}{x^2}\right) = 0 \text{ or } \lambda = 2a^3x^3$$
$$2b^3y + \lambda\left(\frac{-1}{y^2}\right) = 0 \text{ or } \lambda = 2b^3y^3$$
$$2c^3z + \lambda\left(\frac{-1}{z^2}\right) = 0 \text{ or } \lambda = 2c^3z^3$$

Now $$2a^3x^3 = 2b^3y^3 = 2c^3z^3$$
$$\Rightarrow \quad a^3x^3 = b^3y^3 = c^3z^3$$
$$\Rightarrow \quad ax = by = cz$$
$$\therefore \quad y = \frac{ax}{b},\quad z = \frac{ax}{c}$$

But $$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 1 \text{ i.e., } \frac{1}{x}+\frac{b}{ax}+\frac{c}{ax} = 1$$
$$\frac{a+b+c}{ax} = 1$$
$$\therefore \quad x = \frac{a+b+c}{a}$$

Applications of Method of Lagrange Multipliers

Also $$y = \frac{a+b+c}{b}, \quad z = \frac{a+b+c}{z}$$

Required minimum value of the function $a^3x^2 + b^3y^2 + c^3z^2$ is given by

$$= a^3 \cdot \left(\frac{a+b+c}{a}\right)^2 + b^3\left(\frac{a+b+c}{b}\right)^2 + c^3\left(\frac{a+b+c}{c}\right)^2$$

$$= (a+b+c)^2\,(a+b+c) = (a+b+c)^3$$

Thus, the required minimum value is $(a+b+c)^3$.

Applications of Method of Lagrange Multipliers

Example 4.

Find the minimum value of $x^2 + y^2 + z^2$ subject to the conditions $xy + yz + zx = 3a^2$.

Solution

Let $\quad F = (x^2 + y^2 + z^2) + \lambda\,(xy + yz + zx) = 0$

We form the equations $\quad F_x = 0,\ F_y = 0,\ F_z = 0$

i.e., $\quad 2x + \lambda\,(y + z) = 0,\ 2y + \lambda\,(x + z) = 0,\ 2z + \lambda\,(x + y) = 0$

$$\Rightarrow \quad \lambda = \frac{-2x}{y+z},\ \lambda = \frac{-2y}{x+z},\ \lambda = \frac{-2z}{x+y}.$$

Equating the R.H.S. of these, we have

$$\frac{2x}{y+z} = \frac{2y}{x+z} = \frac{2z}{x+y} \qquad \text{...(1)}$$

Consider,

$$\frac{x}{y+z} = \frac{y}{x+z}$$

i.e., $\quad x^2 + xz = y^2 + yz$ or $(x^2 - y^2) + z\,(x - y) = 0$

or $\quad (x - y)\,(x + y + z) = 0$

$\Rightarrow \quad x = y$ or $x + y + z = 0$

Applications of Method of Lagrange Multipliers

we must have $x = y$, since $x + y + z$ cannot be zero.

Suppose $x + y + z = 0$, then by squaring, we get

$$(x^2 + y^2 + z^2) + 2\,(xy + yz + zx) = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 + 2\,(3a^2) = 0$$

or $$x^2 + y^2 + z^2 = -\,6a^2 < 0$$

which is not possible. Similarly by equating the other two pairs in (1), we get

$$y = z,\ z = x \text{ thus } x = y = z$$

But $xy + yz + zx = 3a^2$, putting $y = z = x$, we get

$$3x^2 = 3a^2 \Rightarrow x = a$$

Thus, $x = a = y = z$ and the minimum value of $x^2 + y^2 + z^2$ is

$$a^2 + a^2 + a^2 = 3a^2.$$

Applications of Method of Lagrange Multipliers

Example 5.

Find the volume of largest parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ using Lagrange's method of Multipliers.

Solution: Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$$\phi(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \qquad \text{(i)}$$

Let 2x, 2y, and 2z be the length, breadth and height, respectively of the rectangular parallelopiped inscribed in the ellipsoid. Then

$V = (2x)(2y)(2z) = 8xyz$

Applications of Method of Lagrange Multipliers

Therefore, we have

$$\frac{\partial V}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0 \Rightarrow 8yz+\lambda\frac{2x}{a^2}=0 \qquad \text{(ii)}$$

$$\frac{\partial V}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0 \Rightarrow 8xz+\lambda\frac{2x}{b^2}=0 \qquad \text{(iii)}$$

$$\frac{\partial V}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0 \Rightarrow 8xy+\lambda\frac{2z}{c^2}=0 \qquad \text{(iv)}$$

Multiplying (ii), (iii) and (iv) be x, y and z respectively, and adding, we get

$$24\,xyz+2\lambda\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right]=0$$

Applications of Method of Lagrange Multipliers

$24\,xyz + 2\lambda\,(1) = 0$

$\Rightarrow \lambda = -\,12\,xyz$

putting the value of λ in (ii) we have

$$8yz + (-12\,xyz)\,\frac{2x}{a^2} = 0$$

$$\Rightarrow 1 - \frac{3x^2}{a} = 0$$

$$\Rightarrow x = \frac{a}{\sqrt{3}}$$

Similarly, on putting $\lambda = -\,12\,xyz$ in equation (iii) and (iv) we get

$$y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$$

Hence, the volume of the largest parallelopiped = 8xyz

$$= 8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right)$$

$$= \frac{8abc}{3\sqrt{3}}$$ Answer.

MODULE-III

HIGHER ORDER DIFFERENTIAL EQUATIONS AND ITS APPLICATIONS

- **Definition:** An equation of the form + $P_1(x)$ + $P_2(x)$ + --------+
- $P_n(x)$ .y = Q(x) Where $P_1(x)$, $P_2(x)$, $P_3(x)$... .....$P_n(x)$ and Q(x) (functions of x) continuous is called a linear differential equation of order n.
- **LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS**
- Def: An equation of the form + P_1 + P_2 + --------+ P_n .y = Q(x) where P_1, P_2, $P_{3,}$.....P_n, are real constants and Q(x) is a continuous function of x is called an linear differential equation of order ‘ n’ with constant coefficients.

Note:

1. Operator D $=\frac{d}{dx}$; $D^2 = \frac{d^2}{dx^2}$; .... $D^n = \frac{d^n}{dx^n}$

   $Dy = \frac{dy}{dx}$; $D^2 y = \frac{d^2 y}{dx^2}$; .... $D^n y = \frac{d^n y}{dx^n}$

2. Operator $\frac{1}{D}Q = \int Q$ i e $D^{-1}Q$ is called the integral of Q.

To find the general solution of f(D).y = 0 :

Where $f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2}$ +-----------+P_n is a polynomial in D.

Now consider the auxiliary equation : f(m) = 0

i.e $f(m) = m^n + P_1 m^{n-1} + P_2 m^{n-2}$ +-----------+$P_n = 0$

where $p_1, p_2, p_3 \ldots\ldots\ldots\ldots\ldots p_n$ are real constants.

Let the roots of f(m) =0 be m_1, m_2, m_3,…..m_n.

Depending on the nature of the roots we write the complementary function as follows:

Consider the following table

S.No	Roots of A.E f(m) =0	Complementary function(C.F)
1.	$m_1, m_2, ..m_n$ are real and distinct.	$y_c = c_1e^{m_1x} + c_2e^{m_2x} + \ldots + c_ne^{m_nx}$
2.	$m_1, m_2, ..m_n$ are and two roots are equal i.e., m_1, m_2 are equal and real(i.e repeated twice) &the rest are real and different.	$y_c = (c_1+c_2x)e^{m_1x} + c_3e^{m_3x} + \ldots + c_ne^{m_nx}$
3.	$m_1, m_2, ..m_n$ are real and three roots are equal i.e., m_1, m_2 , m_3 are equal and real(i.e repeated thrice) &the rest are real and different.	$y_c = (c_1+c_2x+c_3x^2)e^{m_1x} + c_4e^{m_4x} + \ldots + c_ne^{m_nx}$
4.	Two roots of A.E are complex say α+iβ α -iβ and rest are real and distinct.	$y_c = e^{\alpha x}(c_1 \cos\beta x + c_2\sin\beta x) + c_3e^{m_3x} + \ldots + c_ne^{m_nx}$
5.	If $\alpha \pm i\beta$ are repeated twice & rest are real and distinct	$y_c = e^{\alpha x}[(c_1+c_2x)\cos\beta x + (c_3+c_4x)\sin\beta x)] + c_5e^{m_5x} + \ldots + c_ne^{m_nx}$
6.	If $\alpha \pm i\beta$ are repeated thrice & rest are real and distinct	$y_c = e^{\alpha x}[(c_1+c_2x+ c_3x^2)\cos\beta x + (c_4+c_5x+ c_6x^2) \sin\beta x)] + c_7e^{m_7x} + \ldots\ldots\ldots + c_ne^{m_nx}$
7.	If roots of A.E. irrational say $\alpha \pm \sqrt{\beta}$ and rest are real and distinct.	$y_c = e^{\alpha x}\left[c_1 \cosh \sqrt{\beta}\, x + c_2 \sinh \sqrt{\beta}\, x\right] + c_3e^{m_3x} + \ldots\ldots + c_ne^{m_nx}$

1. Solve $\frac{d^3y}{dx^3}$ **- 3$\frac{dy}{dx}$ + 2y = 0**

Sol: Given equation is of the form f(D).y = 0

Where $f(D) = (D^3 - 3D + 2)\, y = 0$

Now consider the auxiliary equation f(m) = 0

$f(m) = m^3 - 3m + 2 = 0 \Rightarrow (m-1)(m-1)(m+2) = 0$

$\Rightarrow m = 1, 1, -2$

Since m_1 and m_2 are equal and m_3 is -2

We have $y_c = (c_1 + c_2x)e^x + c_3e^{-2x}$

2. **Solve $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0$**

Sol: Given $f(D) = (D^4 - 2D^3 - 3D^2 + 4D + 4)\, y = 0$

- ⇨ A.equation $f(m) = (m^4 - 2m^3 - 3m^2 + 4m + 4) = 0$
- ⇨ $(m + 1)^2 (m - 2)^2 = 0$
- ⇨ $m = -1, -1, 2, 2$
- ⇨ $y_c = (c_1 + c_2x)e^{-x} + (c_3 + c_4x)e^{2x}$

3. Solve $(D^4 + 8D^2 + 16)\, y = 0$

Sol: Given $f(D) = (D^4 + 8D^2 + 16)\, y = 0$

Auxiliary equation $f(m) = (m^4 + 8\, m^2 + 16) = 0$

⇨ $(m^2 + 4)^2 = 0$

⇨ $(m+2i)^2\, (m+2i)^2 = 0$

⇨ $m = 2i, 2i, -2i, -2i$

$Y_c = e^{0x}\, [(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x)]$

4. Solve $y^{11}+6y^{1}+9y = 0$; $y(0) = -4$, $y^{1}(0) = 14$

Sol: Given equation is $y^{11}+6y^{1}+9y = 0$

Auxiliary equationf(D) y = 0 $\Rightarrow$ $(D^2 +6D +9)\, y = 0$

A.equation $f(m) = 0 \Rightarrow (m^2 +6m +9) = 0$

$\Rightarrow m = -3 ,-3$

$y_c = (c_1+c_2x)e^{-3x}$ --------------------> (1)

Differentiate of (1) w.r.to x $\Rightarrow$ $y^1 =(c_1+c_2x)(-3e^{-3x}) + c_2(e^{-3x})$

Given $y_1 (0) =14$ $\Rightarrow$ $c_1 = -4$ & $c_2 =2$

Hence we get $y =(-4 + 2x) (e^{-3x})$

5. Solve $4y^{111} + 4y^{11} +y^{1} = 0$

Sol: Given equation is $4y^{111} + 4y^{11} +y^{1} = 0$

That is $(4D^3+4D^2+D)y=0$

Auxiliary equation f(m) = 0

$4m^3 +4m^2 + m = 0$

$m(4m^2 +4m + 1) = 0$

$m(2m + 1)^2 = 0$

$m = 0 , -1/2 ,-1/2$

$y =c_1+ (c_2+ c_3x)\, e^{-x/2}$

General solution of f(D) y = Q(x)

Is given by $y = y_c + y_p$

i.e. y = C.F+P.I

Where the P.I consists of no arbitrary constants and P.I of f (D) y = Q(x)

Is evaluated as $P.I = \frac{1}{f(D)} . Q(x)$

Depending on the type of function of Q(x).

P.I is evaluated as follows:

1. P.I of f (D) y = Q(x) where Q(x) = e^{ax} for (a) $\neq$ 0

Case1: $\text{P.I} = \frac{1}{f(D)} \cdot Q(x) = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$

Provided $f(a) \neq 0$

Case 2: If f(a) = 0 then the above method fails. Then

if $f(D) = (D-a)^k \varnothing(D)$

(i.e ' a' is a repeated root k times).

Then $\text{P.I} = \frac{1}{\varnothing(a)} e^{ax} \cdot \frac{1}{k!} x^k$ provided $\varnothing(a) \neq 0$

Express $\frac{1}{f(D)} = \frac{1}{1 \pm \varnothing(D)} = [1 \pm \varnothing(D)]^{-1}$

Hence $\text{P.I} = \frac{1}{1 \pm \varnothing(D)} Q(x)$.

$= [1 \pm \varnothing(D)]^{-1} . x^k$

1) Particular integral of f(D) y = e^{ax} when f(a) $\neq 0$

Working rule:

Case (i):

In f(D), put D=a and Particular integral will be calculated.

Particular integral= $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ provided f(a) $\neq 0$

Case (ii) :

If f(a)= 0 , then above method fails. Now proceed as below.

If f(D)= $(D-a)^K \phi(D)$

i.e. 'a' is a repeated root k times, then

Particular integral= $\frac{e^{ax}}{\phi(a)} \cdot \frac{x^k}{k!}$ provided ϕ(a) $\neq 0$

Solve the Differential equation$(D^2+5D+6)y=e^x$

Sol : Given equation is $(D^2+5D+6)y=e^x$

Here $Q(x) = e^x$

Auxiliary equation is $f(m) = m^2+5m+6=0$

$m^2+3m+2m+6=0$

$m(m+3)+2(m+3)=0$

$m=-2$ or $m=-3$

The roots are real and distinct

C.F $= y_c= c_1e^{-2x} +c_2 e^{-3x}$

Particular Integral = $y_p = \frac{1}{f(D)} \cdot Q(x)$

$= \frac{1}{D2+5D+6} e^x \qquad = \frac{1}{(D+2)(D+3)} e^x$

Put D = 1 in f(D)

P.I. = $\frac{1}{(3)(4)} e^x$

Particular Integral = $y_p = \frac{1}{12} \cdot e^x$

General solution is $y = y_c + y_p$

$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{e^x}{12}$

Solve $y^{11}-4y^{1}+3y=4e^{3x}$, $y(0) = -1$, $y^{1}(0) = 3$

Sol : Given equation is $y^{11}-4y^{1}+3y=4e^{3x}$

i.e. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 4e^{3x}$

it can be expressed as

$D^2y-4Dy+3y=4e^{3x}$

$(D^2-4D+3)y=4e^{3x}$

Here $Q(x)=4e^{3x}$; $f(D)= D^2-4D+3$

Auxiliary equation is $f(m)=m^2-4m+3 = 0$

$m^2-3m-m+3 = 0$

$m(m-3) -1(m-3)=0 \Rightarrow m=3$ or 1

The roots are real and distinct.

C.F$= y_c=c_1e^{3x}+c_2e^{x}$ ----→ (2)

P.I.= $y_p = \frac{1}{f(D)} . Q(x)$

$= y_p = \frac{1}{D^2-4D+3} . 4e^{3x}$

$= y_p = \frac{1}{(D-1)(D-3)} . 4e^{3x}$

Put D=3

$$y_p = \frac{4e^{3x}}{(3-1)(D-3)} = \frac{4}{2}\frac{e^{3x}}{(D-3)} = 2\frac{x^1}{1!}e^{3x} = 2xe^{3x}$$

General solution is $y=y_c+y_p$

$y=c_1e^{3x}+c_2 e^x+2xe^{3x}$ --------------------→ (3)

Equation (3) differentiating with respect to 'x'

$y^1=3c_1e^{3x}+c_2e^x+2e^{3x}+6xe^{3x}$ -----------→ (4)

By data, $y(0) = -1$, $y^1(0)=3$

From (3), $-1=c_1+c_2$ --------------------→ (5)

From (4), $3=3c_1+c_2+2$

$3c_1+c_2=1$ --------------------→ (6)

Solving (5) and (6) we get $c_1=1$ and $c_2 = -2$

$y=-2e^{x}+(1+2x)e^{3x}$

P.I of f(D) y =Q(x) where Q(x) = sin ax or Q(x) = cos ax where ‘ a ‘ is constant then P.I = $\frac{1}{f(D)}$. Q(x).

Case 1: In f(D) put $D^2 = -a^2 \ni f(-a^2) \neq 0$ then P.I = $\frac{\sin ax}{f(-a^2)}$

Case 2: If $f(-a^2) = 0$ then $D^2 + a^2$ is a factor of $\emptyset(D^2)$ and hence it is a factor of f(D). Then let $f(D) = (D^2 + a^2).\Phi(D^2)$.

Then $\frac{\sin ax}{f(D)} = \frac{\sin ax}{(D^2+a^2)\Phi(D^2)} = \frac{1}{\Phi(-a^2)}\frac{\sin ax}{D^2+a^2} = \frac{1}{\Phi(-a^2)}\frac{-x\cos ax}{2a}$

$$\frac{\cos ax}{f(D)} = \frac{\cos ax}{(D^2+a^2)\Phi(D^2)} = \frac{1}{\Phi(-a^2)}\frac{\cos ax}{D^2+a^2} = \frac{1}{\Phi(-a^2)}\frac{x\sin ax}{2a}$$

Solve $y^{11}+4y^{1}+4y= 4\cos x + 3\sin x$, $y(0) = 0$, $y^{1}(0) = 0$

Sol: Given differential equation in operator form

$(D^2 + 4D + 4)y = 4\cos x + 3\sin x$

A.E is $m^2+4m+4 = 0$

$(m+2)^2=0$ then $m=-2, -2$

$\therefore$ C.F is $y_c = (c_1 + c_2 x)e^{-2x}$

P.I is $= y_p = \frac{4\cos x+3\sin x}{(D^2+4D+4)}$ put $D^2 = -1$

$y_p = \frac{4\cos x+3\sin x}{(4D+3)} = \frac{(4D-3)(4\cos x+3\sin x)}{(4D-3)(4D+3)}$

$= \frac{(4D-3)(4\cos x+3\sin x)}{16D^2-9}$

Put D^2 = -1

$\therefore y_p = \dfrac{(4D-3)(4cosx+3sinx)}{-16-9}$

$= \dfrac{-16sinx+12cosx-12cosx-9sinx)}{-25} = \dfrac{-25sinx}{-25}$ = sinx

$\therefore$General equation is $y = y_c + y_p$

$y = (c_1 + c_2x)e^{-2x}$ + sinx ------------ (1)

By given data, y(0) = 0 $\therefore c_1 = 0$ and

Diff (1) w.r.. t. $y^1 = (c_1 + c_2x)(-2)e^{-2x} + e^{-2x}(c_2)$ +cosx ------------ (2)

given $y^1(0) = 0$

(2) $\Rightarrow$ $-2c_1 + c_2+1=0$ $\therefore c_2 = -1$

$\therefore$Required solution is $y = -xe^{-2x}$+sinx

Solve $(D^2+9)y = \cos 3x$

Sol:Given equation is $(D^2+9)y = \cos 3x$

A.E is $m^2+9 = 0$

$\therefore$ $m = \pm 3i$

y_c = C.F = $c_1 \cos 3x + c_2 \sin 3x$

y_c =P.I = $\frac{\cos 3x}{D^2+9} = \frac{\cos 3x}{D^2+3^2}$

$= \frac{x}{2(3)} \sin 3x = \frac{x}{6} \sin 3x$

General equation is $y = y_c + y_p$

$y = c_1 \cos 3x + c_2 \cos 3x + \frac{x}{6} \sin 3x$

P.I for f(D) y = Q(x) where Q(x) = x^k

1. **P.I for f(D) y = Q(x) where Q(x) = x^k where k is a positive integer f(D) can be express as f(D) =[1± $\emptyset(D)$]**

   Express $\frac{1}{f(D)} = \frac{1}{1\pm\emptyset(D)}$ = [1± $\emptyset(D)$] $^{-1}$

   Hence P.I = $\frac{1}{1\pm\emptyset(D)}$ Q(x).

   = [1± $\emptyset(D)$] $^{-1}$ .x^k

Formulae

1. $\frac{1}{1-D} = (1-D)^{-1} = 1 + D + D^2 + D^3 +$ -------------------
2. $\frac{1}{1+D} = (1+D)^{-1} = 1 - D + D^2 - D^3 +$ -------------------
3. $\frac{1}{(1-D)^2} = (1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 +$ -------------------
4. $\frac{1}{(1+D)^2} = (1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 +$ -------------------
5. $\frac{1}{(1-D)^3} = (1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 +$ -------------------
6. $\frac{1}{(1+D)^3} = (1+D)^{-3} = 1 - 3D + 6D^2 - 10D^3 +$ -------------------

Solve $y^{111}+2y^{11} - y^{1}-2y= 1-4x^3$

Sol:Given equation can be written as

$(D^3 + 2D^2 - \mathrm{D} - 2)\mathrm{y}$ = $1-4x^3$

A.E is $(m^3 + 2m^2 - \mathrm{m} - 2)$ = 0

$(m^2 - 1)(m+2) = 0$

$m^2 = 1$ *or* m=- 2

m = 1, -1, -2

C.F $= c_1 e^{x} + c_2 e^{-x} + c_3 e^{-2x}$

$$\text{P.I} = \frac{1}{(D^3+2D^2-D-2)}(1-4x^3)$$

$$= \frac{-1}{2[\,1-\frac{(D^3+2D^2-D)}{2}\,]}\;(1-4x^3)$$

$$= \frac{-1}{2}\left[\,1-\frac{(D^3+2D^2-D)}{2}\,\right]^{-1}(1-4x^3)$$

$$= \frac{-1}{2}[\,1+\frac{(D^3+2D^2-D)}{2}+\frac{(D^3+2D^2-D)^2}{4}+\frac{(D^3+2D^2-D)^3}{8}+\ldots..]\;(1-4x^3)$$

$$= \frac{-1}{2}\left[1+\frac{1}{2}(D^3+2D^2-D)+\frac{1}{4}(D^2-4D^3)+\frac{1}{8}(-D^3)\right](1-4x^3)$$

$$= \frac{-1}{2}[\,1-\frac{5}{8}(D^3)+\frac{5}{4}(D^2)-\frac{1}{2}D]\;(1\text{-}4x^3)$$

$$= \frac{-1}{2}\,[(1\text{-}4x^3)-\frac{5}{8}(-24)+\frac{5}{4}(-24x)-\frac{1}{2}(-12x^2)$$

$$= \frac{-1}{2}[-4x^3+6x^2-30x+16] =$$

$$= [2x^3-3x^2+15x-8]$$

The general solution is

y= C.F + P.I

$$y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{-2x} + [2x^3 - 3x^2 + 15x - 8]$$

P.I of f(D) y = Q(x) when Q(x) = e^{ax} V

P.I of f(D) y = Q(x) when Q(x) = e^{ax} V where 'a' is a constant and V is function of x. where V =sin ax or cos ax or x^k

Then P.I $= \frac{1}{f(D)} Q(x)$

$= \frac{1}{f(D)} e^{ax} V$

$= e^{ax} \left[\frac{1}{f(D+a)}(V)\right]$

& $\frac{1}{f(D+a)}$ V is evaluated depending on V.

Solve $(D^3 - 7D^2 + 14\mathrm{D}$ -8)y = e^x cos2x

Given equation is

$(D^3 - 7D^2 + 14\mathrm{D}$ -8)y = e^x cos2x

A.E is $(m^3 - 7m^2 + 14\mathrm{m} - 8)$ = 0

(m-1) (m-2)(m-4) = 0

Then m = 1,2,4

C.F = $c_1 e^x + c_2 e^{2x} + c_3 e^{4x}$

$$\text{P.I} = \frac{e^{x}\cos 2x}{(D^3 - 7D^2 + 14D - 8)}$$

$$= e^{x} \cdot \frac{1}{(D+1)^3 - 7(D+1)^2 + 14(D+1) - 8} \cdot \text{Cos}2x$$

$$\left[\because\ P.I = \frac{1}{f(D)} e^{ax} v = e^{ax} \frac{1}{f(D+a)} v \right]$$

$$= e^{x} \cdot \frac{1}{(D^3 - 4D^2 + 3D)} \cdot \cos 2x$$

$$= e^{x} \cdot \frac{1}{(-4D + 3D + 16)} \cdot \cos 2x$$ (Replacing D^2 with -2^2)

$$= e^{x} \cdot \frac{1}{(16-D)} . \cos 2x$$

$$= e^{x} \cdot \frac{16+D}{(16-D)(16+D)} . \cos 2x$$

$$= e^{x} \cdot \frac{16+D}{256-D^2} . \cos 2x$$

$$= e^{x} \cdot \frac{16+D}{256-(-4)} . \cos 2x$$

$$= \frac{e^{x}}{260} (16\cos 2x - 2\sin 2x)$$

$$= \frac{2e^{x}}{260} (8\cos 2x - \sin 2x)$$

$$= \frac{e^{x}}{130} (8\cos 2x - \sin 2x)$$

General solution is $y = y_c + y_p$

$$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{4x} + \frac{e^{x}}{130} (8\cos 2x - \sin 2x)$$

Solve $(D^2 - 4D + 4)y = x^2 sinx + e^{2x} + 3$

Sol:Given $(D^2 - 4D + 4)y = x^2 sinx + e^{2x} + 3$

A.E is $(m^2 - 4m + 4) = 0$

$(m-2)^2 = 0$ then m=2,2

C.F. = $(c_1 + c_2x)e^{2x}$

P.I = $\frac{x^2 sinx + e^{2x} + 3}{(D-2)^2} = \frac{1}{(D-2)^2}(x^2 sinx) + \frac{1}{(D-2)^2} e^{2x} + \frac{1}{(D-2)^2}(3)$

Now $\frac{1}{(D-2)^2}(x^2 sinx) = \frac{1}{(D-2)^2}(x^2)$ (I.P of e^{ix})

$= \text{I.P of } \frac{1}{(D-2)^2}(x^2)(e^{ix})$

$= \text{I.P of } (e^{ix}).\frac{1}{(D+i-2)^2}(x^2)$

On simplification, we get

$$\frac{1}{(D+i-2)^2}(x^2\ sinx) = \frac{1}{625}[(220x+244)cosx+(40x+33)sinx]$$

and $\frac{1}{(D-2)^2}(e^{2x}) = \frac{x^2}{2}(e^{2x})$,

$$\frac{1}{(D-2)^2}(3) = \frac{3}{4}$$

$$P.I = \frac{1}{625}[(220x+244)cosx+(40x+33)sinx] + \frac{x^2}{2}(e^{2x}) + \frac{3}{4}$$

$y = y_c + y_p$

$$y = (c_1 + c_2x)e^{2x} + \frac{1}{625}[(220x+244)cosx+(40x+33)sinx] + \frac{x^2}{2}(e^{2x}) + \frac{3}{4}$$

P.I of f(D) y = Q(x) when Q(x) = e^{ax} V

P.I of f(D) y = Q(x) when Q(x) = e^{ax} V where 'a' is a constant and V is function of x. where

V =sin ax or cos ax or x^k

Then P.I $= \frac{1}{f(D)}$ Q(x)

$= \frac{1}{f(D)} e^{ax} V$

$= e^{ax} [\frac{1}{f(D+a)}(V)]$

& $\frac{1}{f(D+a)}$ V is evaluated depending on V.

Solve $(D^3 - 7D^2 + 14\mathrm{D}$ -8)y = e^x cos2x

Given equation is

$(D^3 - 7D^2 + 14\mathrm{D}$ -8)y = e^x cos2x

A.E is $(m^3 - 7m^2 + 14\mathrm{m} - 8)$ = 0

(m-1) (m-2)(m-4) = 0

Then m = 1,2,4

C.F = $c_1 e^x + c_2 e^{2x} + c_3 e^{4x}$

MODULE-IV

Multiple Integrals

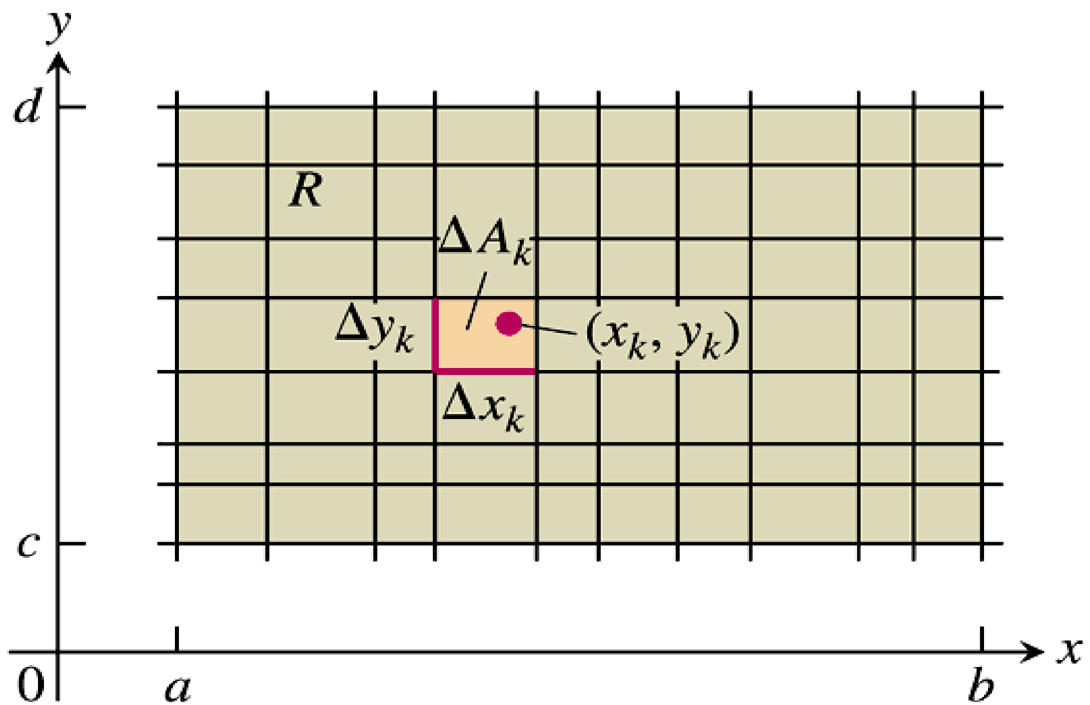


FIGURE 15.1 Rectangular grid partitioning the region R into small rectangles of area $\Delta A_k = \Delta x_k \, \Delta y_k$.

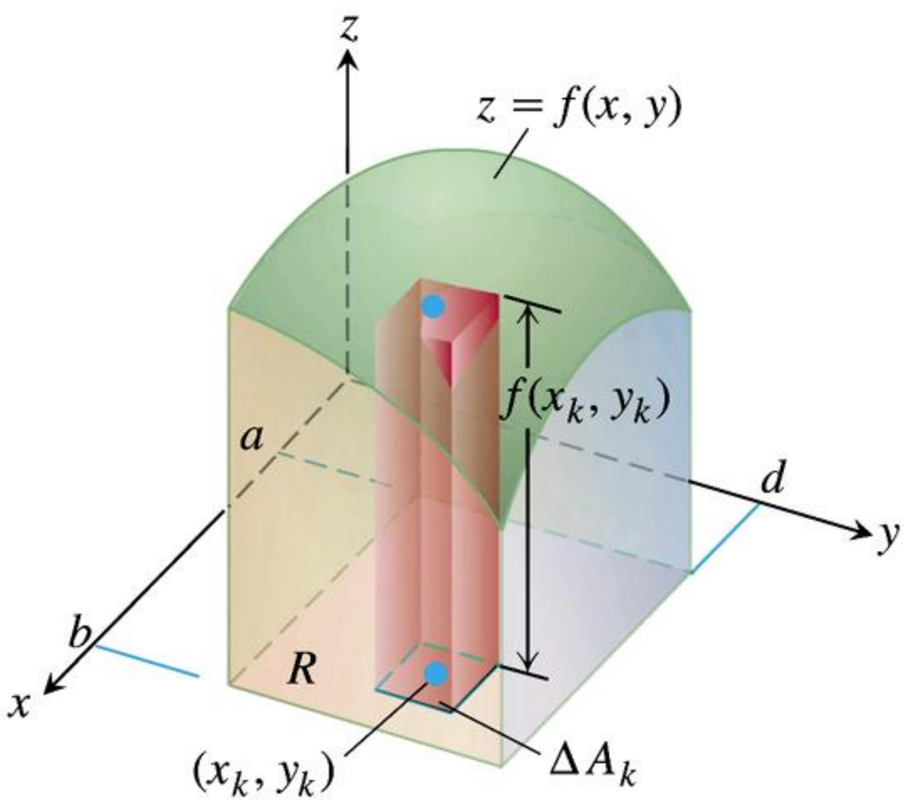
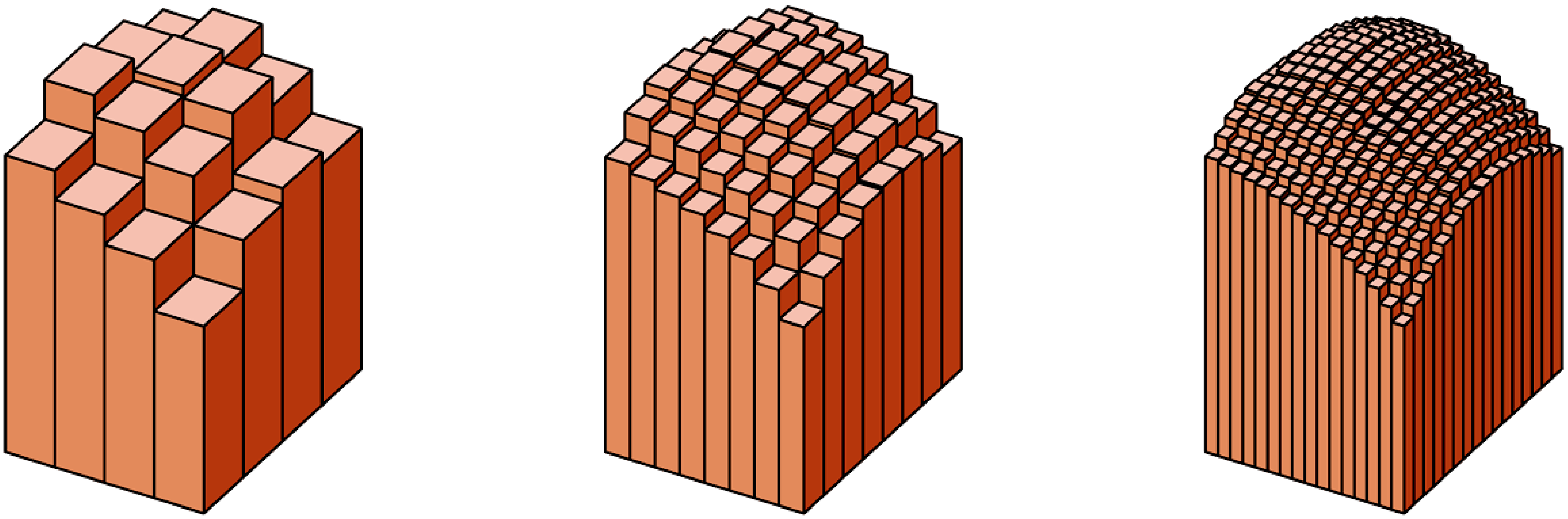


FIGURE 15.2 Approximating solids with rectangular boxes leads us to define the volumes of more general solids as double integrals. The volume of the solid shown here is the double integral of $f(x, y)$ over the base region R.

$$\text{Volume} = \lim_{n \to \infty} S_n = \iint_R f(x, y)\, dA,$$

where $\Delta A_k \to 0$ as $n \to \infty$.



(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

FIGURE 15.3 As n increases, the Riemann sum approximations approach the total volume of the solid shown in Figure 15.2.

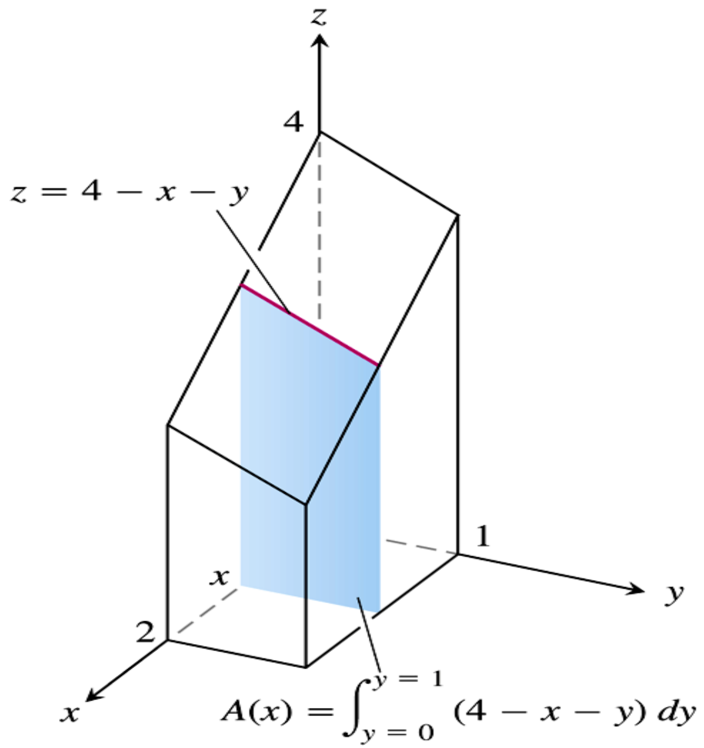


FIGURE 15.4 To obtain the cross-sectional area $A(x)$, we hold x fixed and integrate with respect to y.

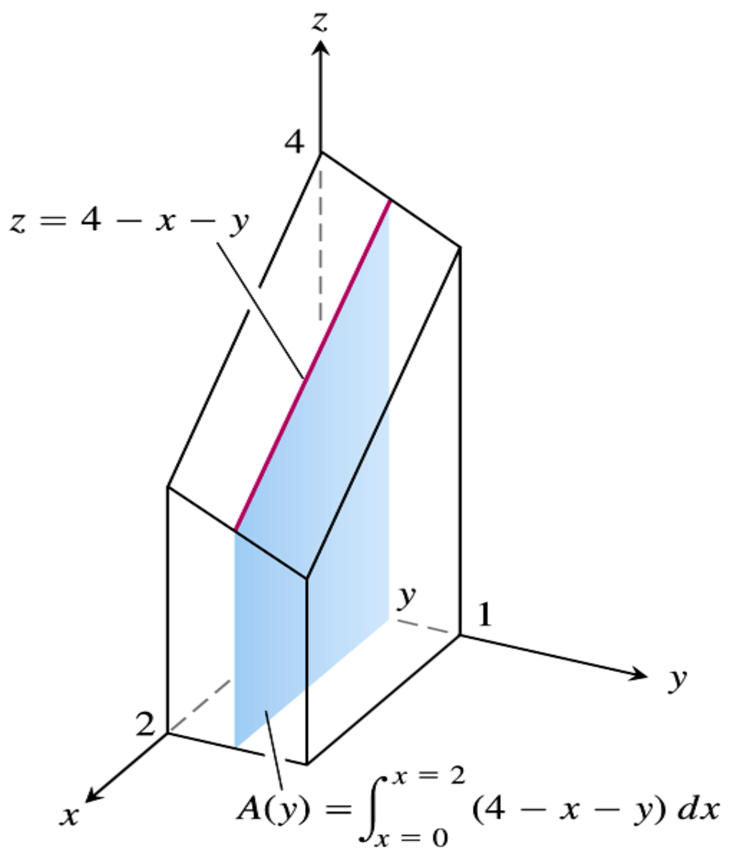


FIGURE 15.5 To obtain the cross-sectional area $A(y)$, we hold y fixed and integrate with respect to x.

THEOREM 1 Fubini's Theorem (First Form)

If $f(x, y)$ is continuous throughout the rectangular region $R: a \leq x \leq b$, $c \leq y \leq d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

EXAMPLE 1 Evaluating a Double Integral

Calculate $\iint_R f(x, y)\, dA$ for

$$f(x, y) = 1 - 6x^2y \qquad \text{and} \qquad R: \quad 0 \le x \le 2, \quad -1 \le y \le 1.$$

Solution By Fubini's Theorem,

$$\iint_R f(x, y)\, dA = \int_{-1}^{1}\int_{0}^{2}(1 - 6x^2y)\, dx\, dy = \int_{-1}^{1}\left[x - 2x^3y\right]_{x=0}^{x=2} dy$$

$$= \int_{-1}^{1}(2 - 16y)\, dy = \left[2y - 8y^2\right]_{-1}^{1} = 4.$$

Reversing the order of integration gives the same answer:

$$\int_0^2\int_{-1}^{1}(1 - 6x^2y)\, dy\, dx = \int_0^2\left[y - 3x^2y^2\right]_{y=-1}^{y=1} dx$$

$$= \int_0^2[(1 - 3x^2) - (-1 - 3x^2)]\, dx$$

$$= \int_0^2 2\, dx = 4.$$

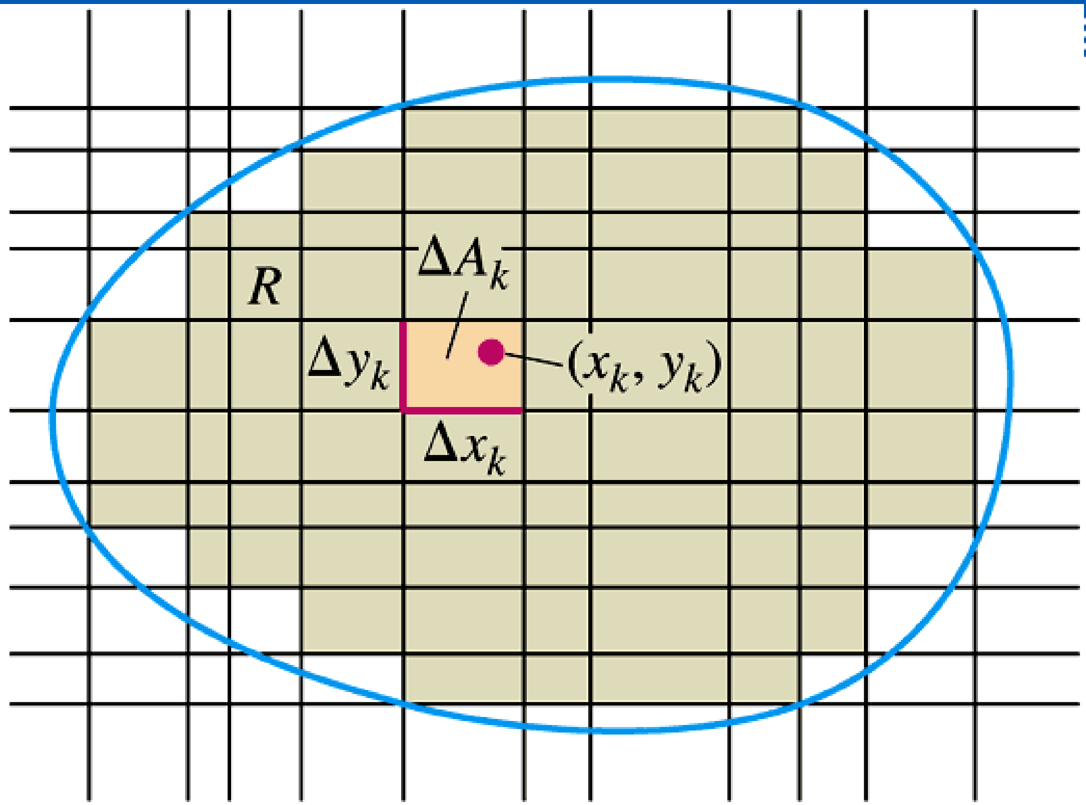
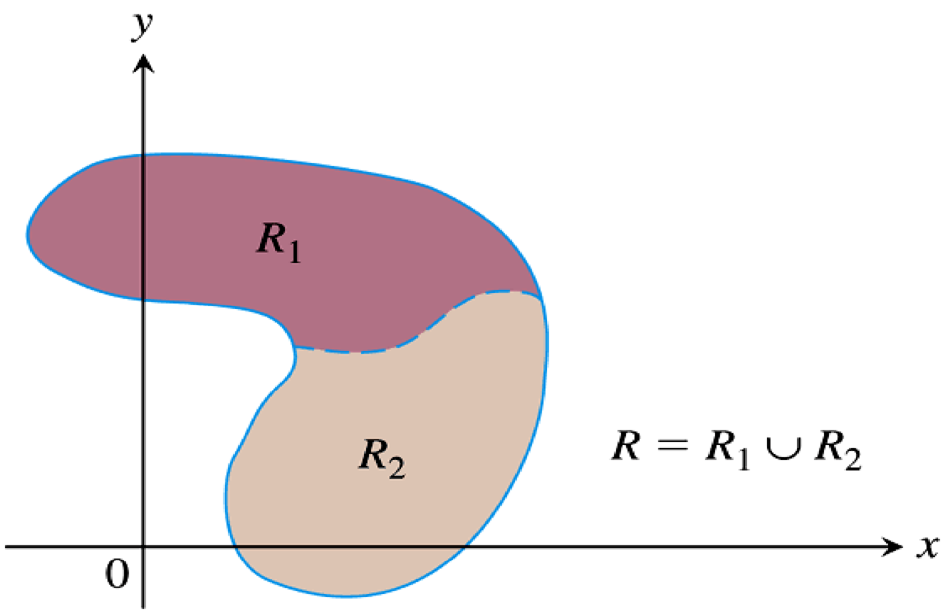
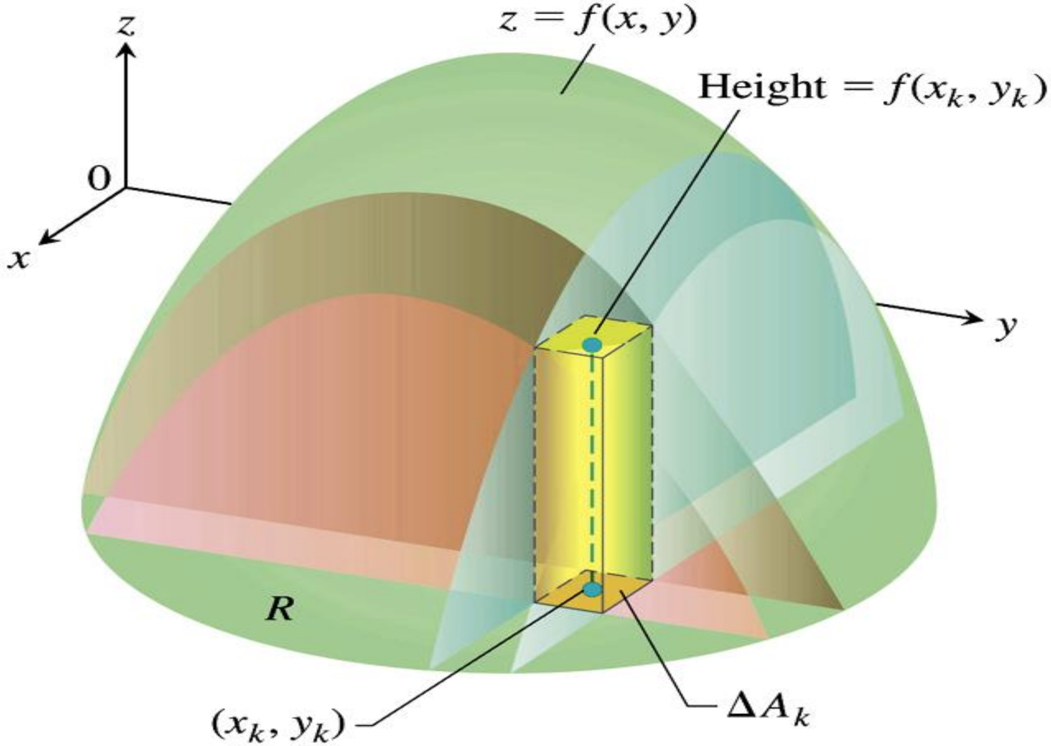


FIGURE 15.6 A rectangular grid partitioning a bounded nonrectangular region into rectangular cells.



$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$$

FIGURE 15.7 The Additivity Property for rectangular regions holds for regions bounded by continuous curves.



$$\text{Volume} = \lim \sum f(x_k, y_k)\, \Delta A_k = \iint_R f(x, y)\, dA$$

FIGURE 15.8 We define the volumes of solids with curved bases the same way we define the volumes of solids with rectangular bases.

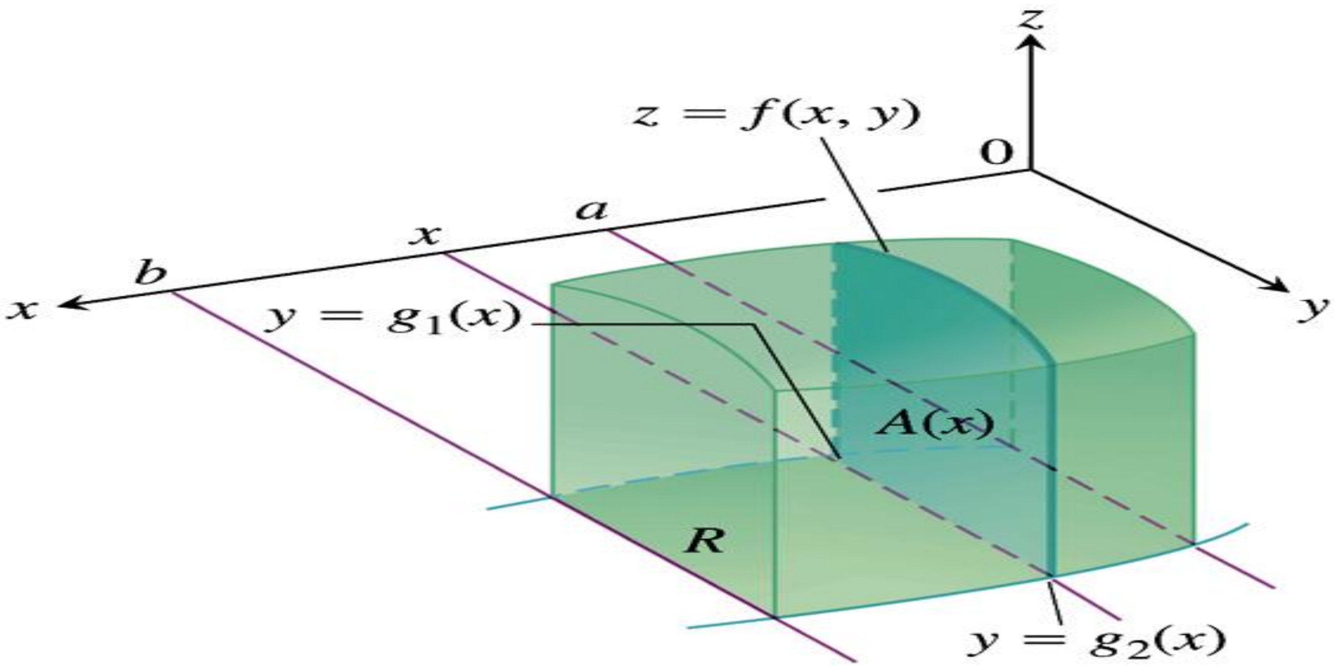


FIGURE 15.9 The area of the vertical slice shown here is

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy.$$

To calculate the volume of the solid, we integrate this area from $x = a$ to $x = b$.

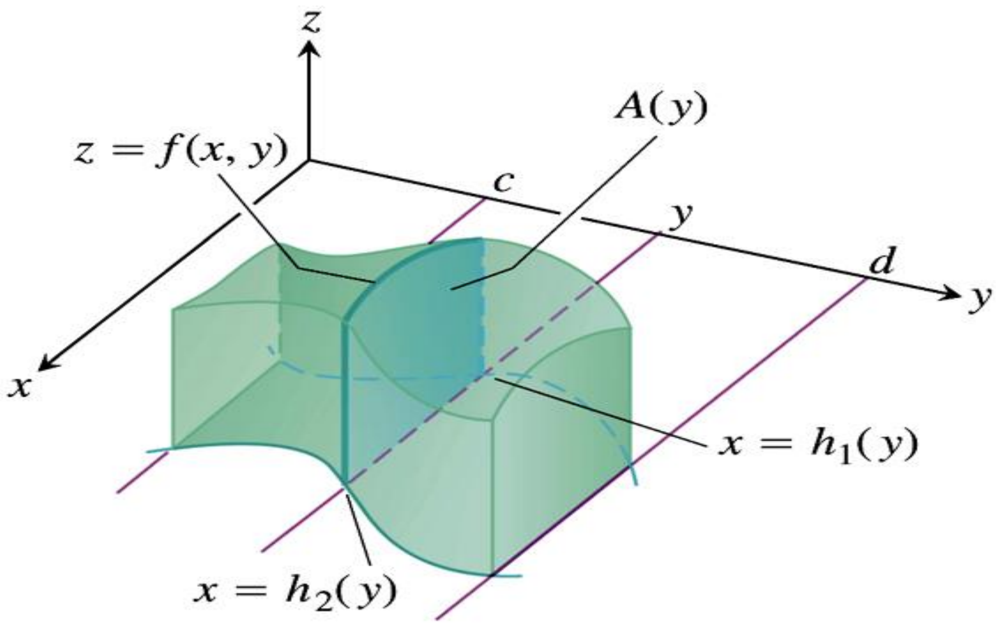


FIGURE 15.10 The volume of the solid shown here is

$$\int_c^d A(y)\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy.$$

If $f(x, y)$ is positive and continuous over R we define the volume of the solid region between R and the surface $z = f(x, y)$ to be $\iint_R f(x, y)\, dA$, as before (Figure 15.8).

If R is a region like the one shown in the xy-plane in Figure 15.9, bounded "above" and "below" by the curves $y = g_2(x)$ and $y = g_1(x)$ and on the sides by the lines $x = a$, $x = b$, we may again calculate the volume by the method of slicing. We first calculate the cross-sectional area

$$A(x) = \int_{y=g_1(x)}^{y=g_2(x)} f(x, y)\, dy$$

and then integrate $A(x)$ from $x = a$ to $x = b$ to get the volume as an iterated integral:

$$V = \int_a^b A(x)\, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx. \tag{5}$$

Similarly, if R is a region like the one shown in Figure 15.10, bounded by the curves $x = h_2(y)$ and $x = h_1(y)$ and the lines $y = c$ and $y = d$, then the volume calculated by slicing is given by the iterated integral

$$\text{Volume} = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy. \tag{6}$$

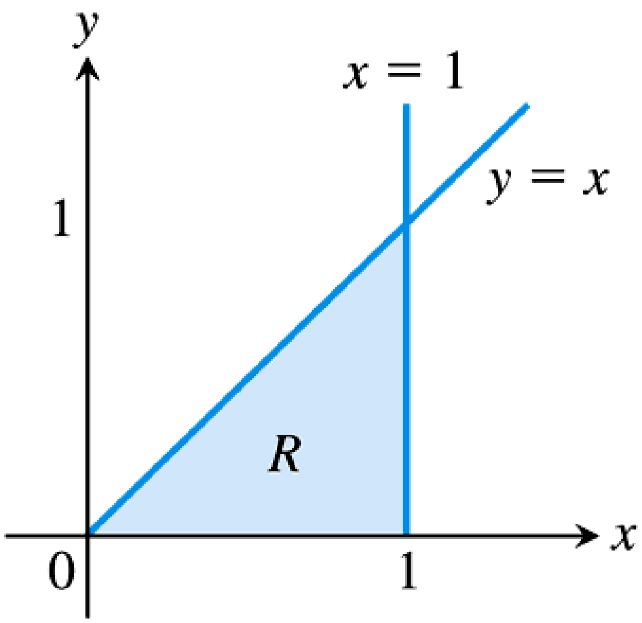


FIGURE 15.12 The region of integration in Example 3.

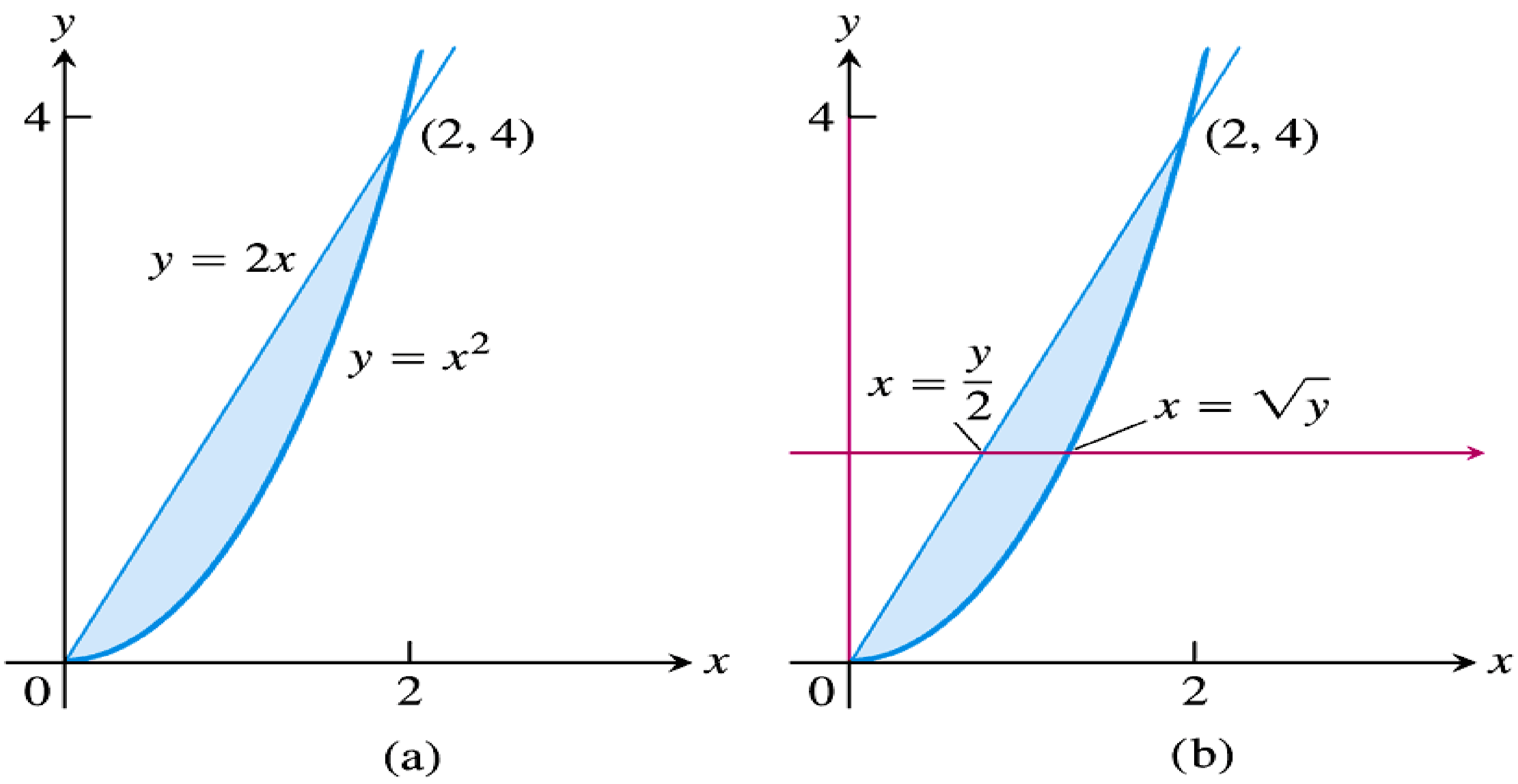


FIGURE 15.13 Region of integration for Example 4.

Properties of Double Integrals

If $f(x, y)$ and $g(x, y)$ are continuous, then

1. *Constant Multiple:* $\displaystyle\iint_R cf(x, y)\, dA = c\iint_R f(x, y)\, dA$ (any number c)

2. *Sum and Difference:*

$$\iint_R (f(x, y) \pm g(x, y))\, dA = \iint_R f(x, y)\, dA \pm \iint_R g(x, y)\, dA$$

3. *Domination:*

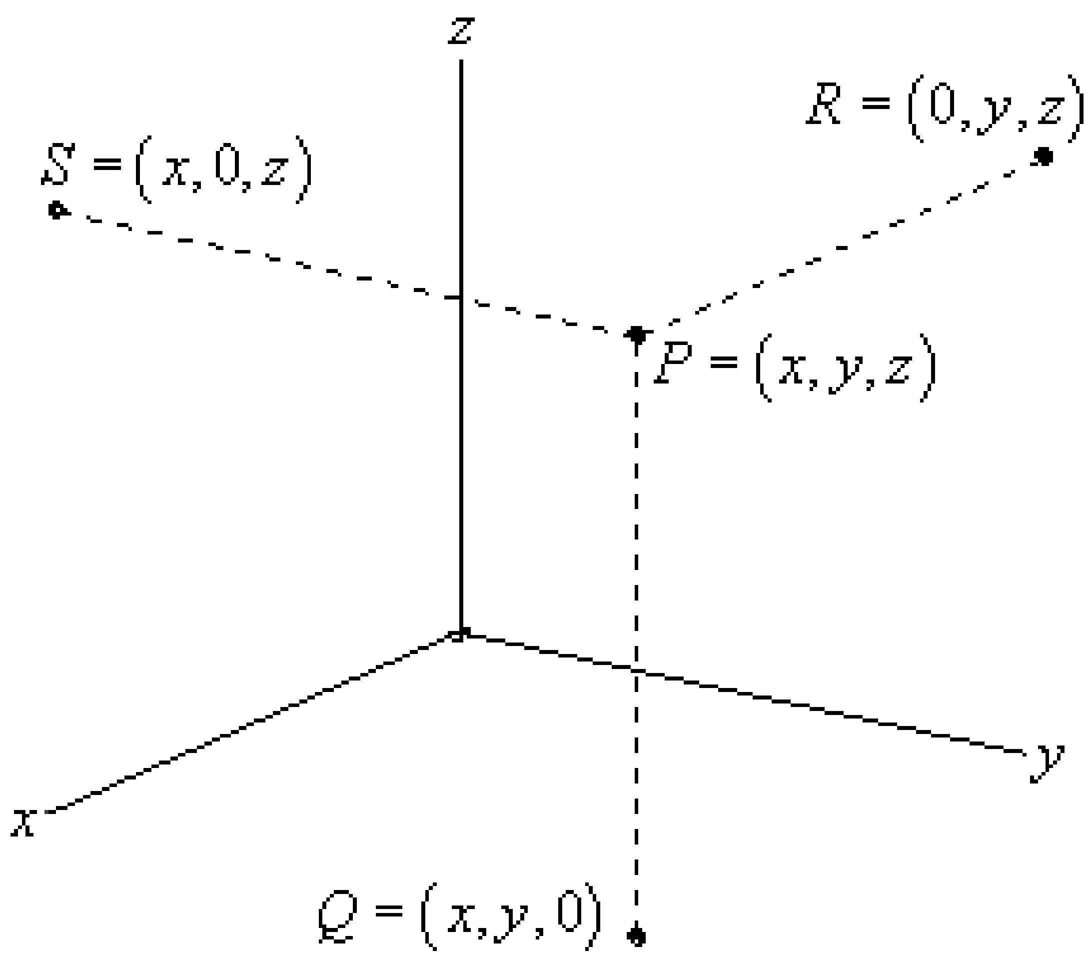
(a) $\displaystyle\iint_R f(x, y)\, dA \geq 0 \quad \text{if} \quad f(x, y) \geq 0 \text{ on } R$

(b) $\displaystyle\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA \quad \text{if} \quad f(x, y) \geq g(x, y) \text{ on } R$

4. *Additivity:* $\displaystyle\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$

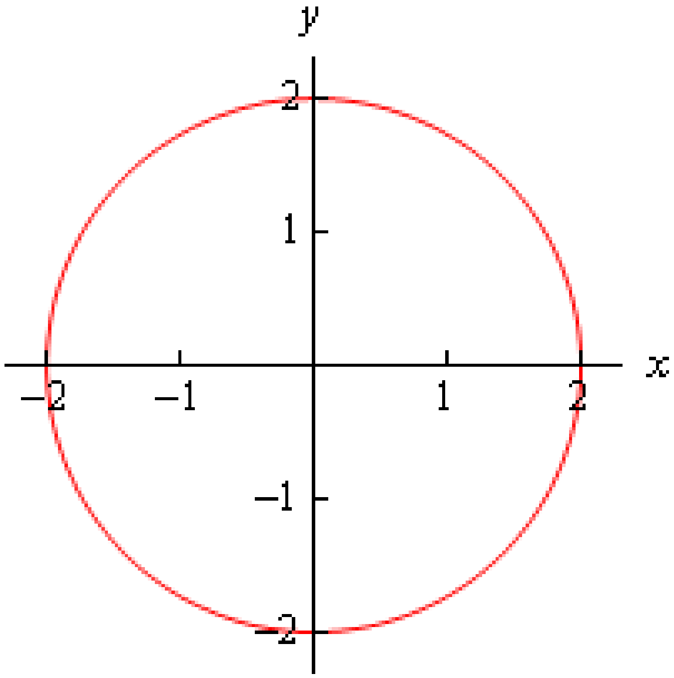
if R is the union of two nonoverlapping regions R_1 and R_2 (Figure 15.7).

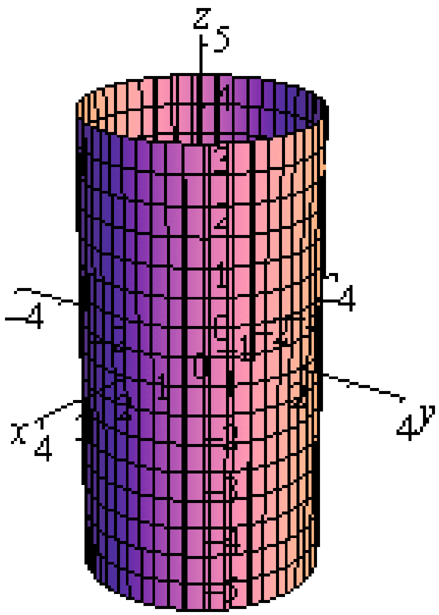
Three Dimensional Space



In Two-Dimensional Space, you have a circle

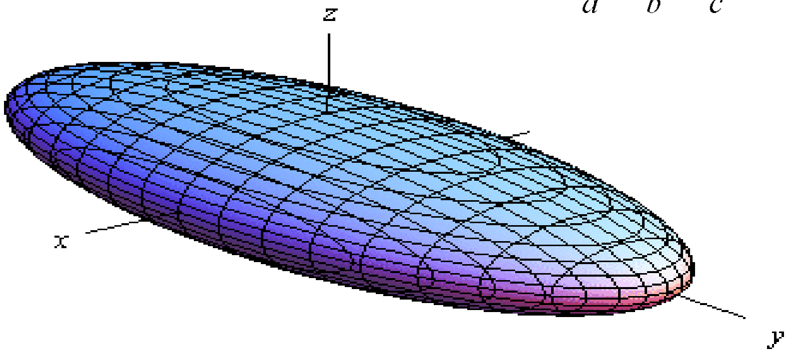
In Three-Dimensional space, you have a ______________!!!!!!!!!!



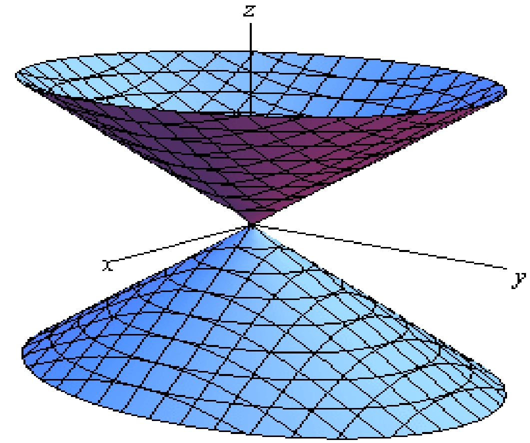


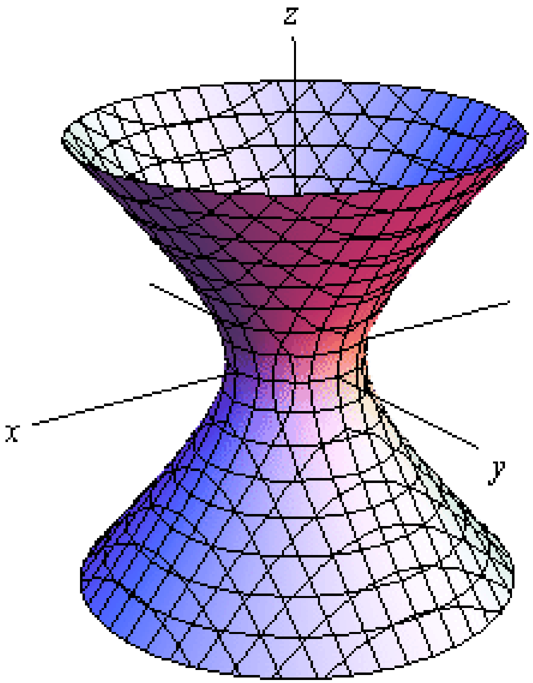
More 3-D graphs

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

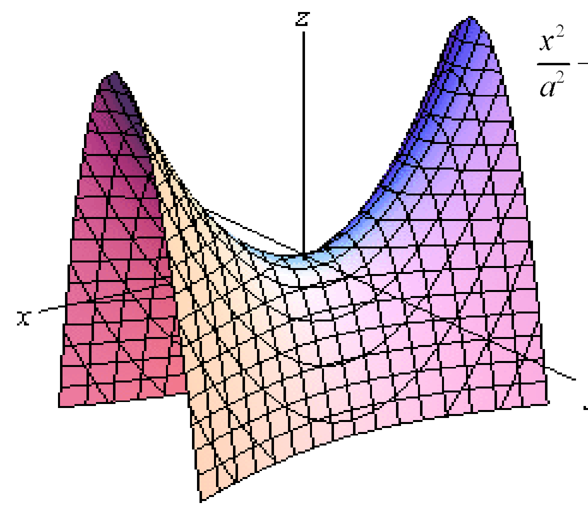


$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$$





$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

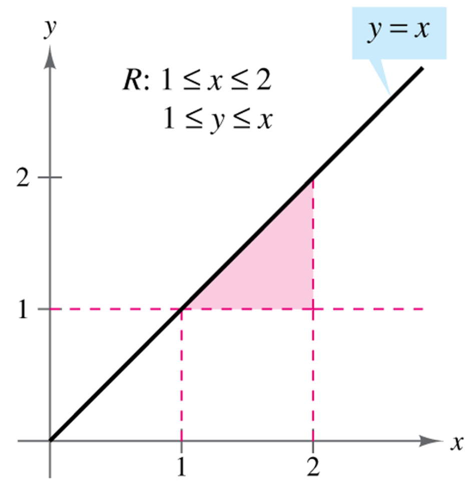


$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{z}{c}$$

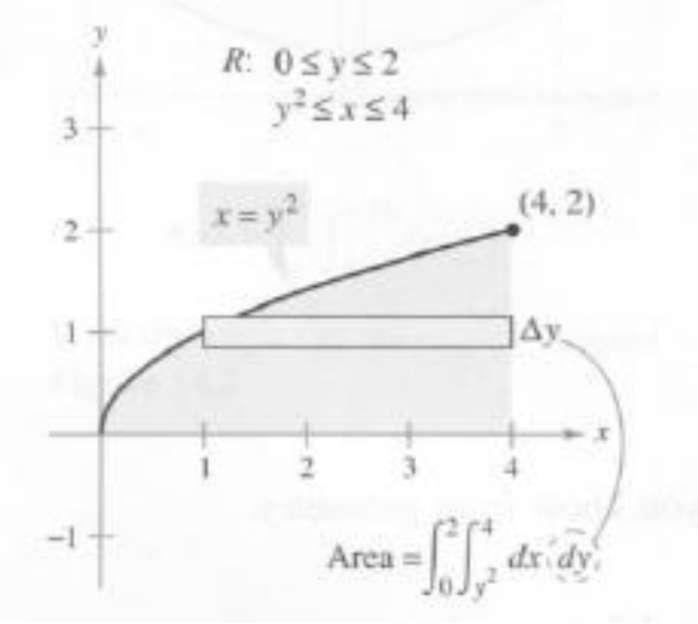
The Iterated Integral

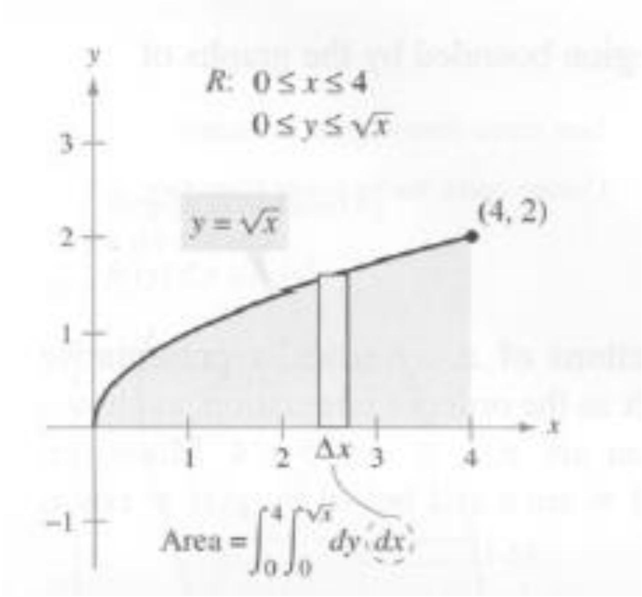
$$\int_{1}^{2y} 2xy\,dx$$

$$\int_{1}^{2}\int_{1}^{x} 2x^{2}y^{-2} + 2y\,dy\,dx$$

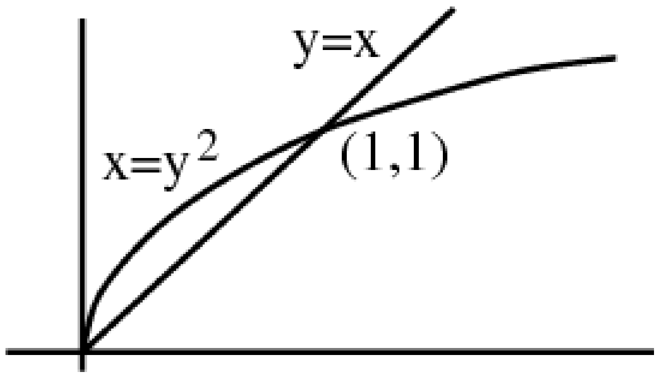


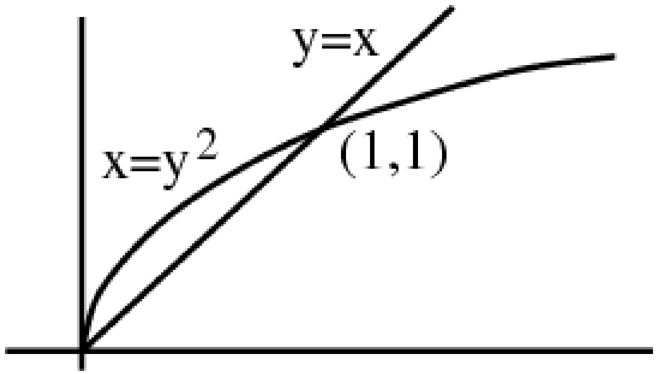
Setting up the Double Integral





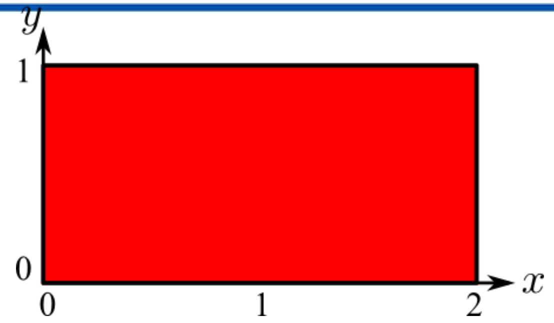
Finding Area using Double Integrals

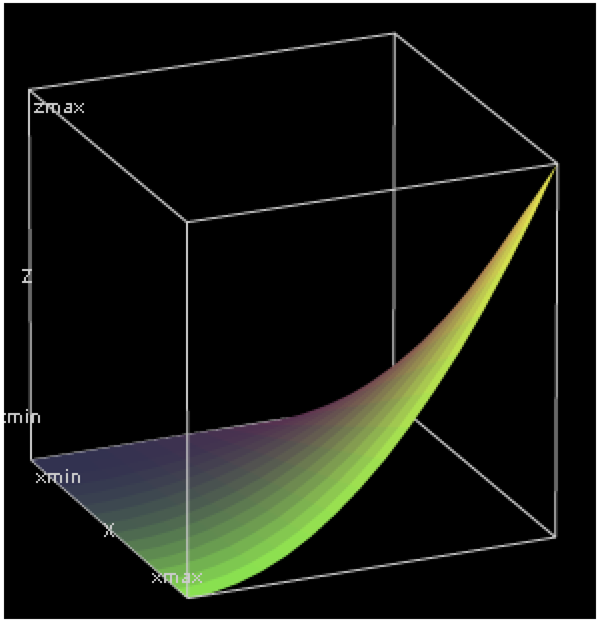




Compute the integral on the pictured region

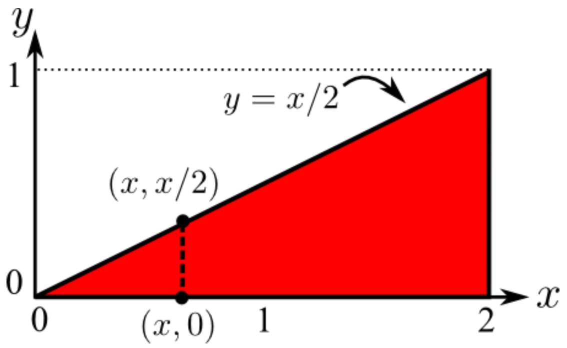
$$\iint_R x y^2 \, dA$$



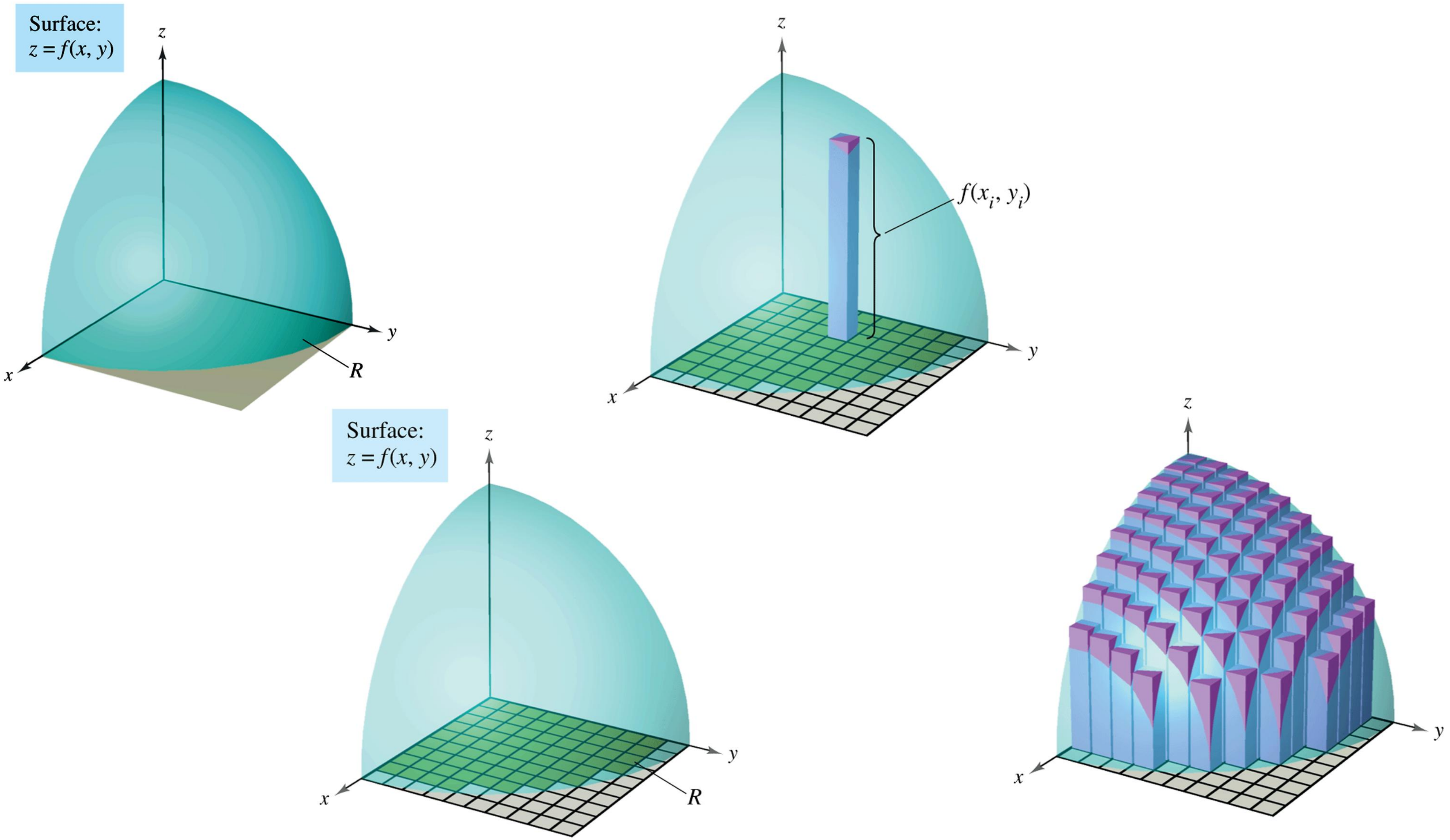


Compute the integral on the pictured region

$$\iint_R xy^2\, dA$$



Finding Volume using the Double Integral

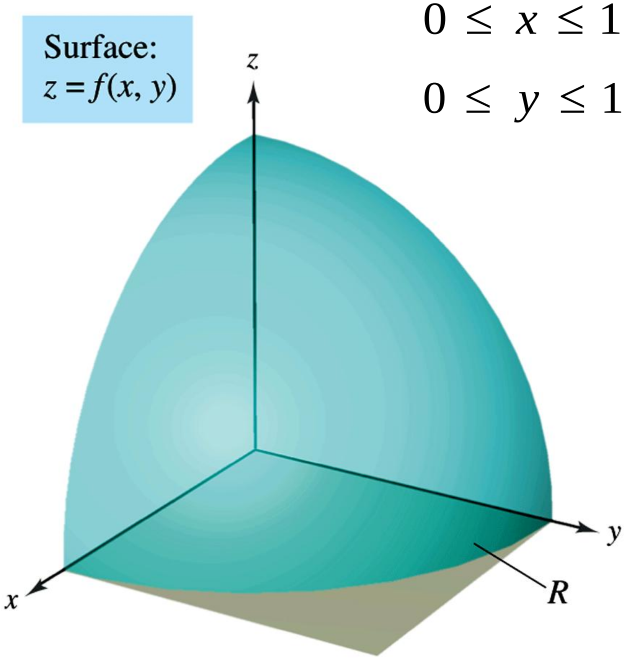


Evaluate the volume using the region

$$\iint_R 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2 \, dA$$

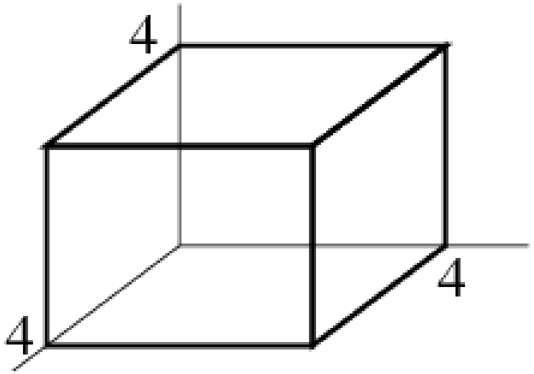
$0 \le x \le 1$

$0 \le y \le 1$



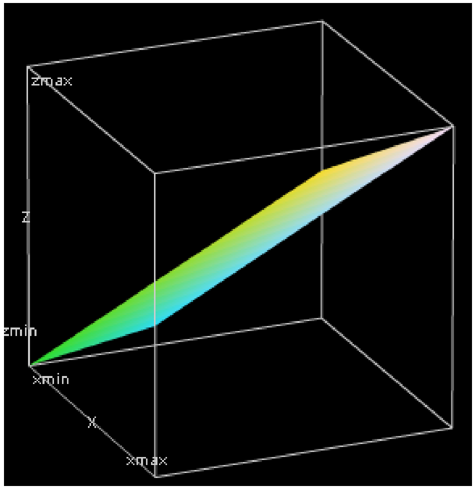
Volume using the Triple Integral

$$\int_0^4 \int_0^4 \int_0^4 dV$$



The cubes density is proportional to its distance away from the Xy-plane. Find its mass.

$$\int_0^2 \int_0^x \int_0^{1+x+y} dV$$



Double integrals

Definition of Double Integral

If f is defined on a closed, bounded region R in the xy-plane, then the **double integral of *f* over *R*** is given by

$$\int_R\int f(x, y)\, dA = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i)\, \Delta A_i$$

provided the limit exists. If the limit exists, then f is **integrable** over R.

Definition of Double Integral

The expression:

$$\int_{y=y_1}^{y_2} \int_{x=x_1}^{x_2} f(x, y)\, dx.dy$$

is called a *double integral* and indicates that $f(x, y)$ is first integrated with respect to x and the result is then integrated with respect to y

If the four limits on the integral are all constant the order in which the integrations are performed does not matter.

If the limits on one of the integrals involve the other variable then the order in which the integrations are performed is crucial.

Multiple Integrals

Double Integral :

I. When y_1,y_2 are functions of x and x_1 and x_2 are constants. f(x,y)is first integrated w.r.t y keeping 'x' fixed between limits y_1,y_2 and then the resulting expression is integrated w.r.t 'x' with in the limits x_1,x_2 i.e.,

$$\iint_R f(x,y)dx\,dy = \int_{x=x_1}^{x=x_2} \int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x,y)dydx$$

II. When x_1,x_2 are functions of y and y_1, y_2 are constants, f(x,y)is first integrated w.r.t 'x' keeping 'y' fixed, with in the limits x_1,x_2 and then resulting expression is integrated w.r.t 'y' between the limits y_1,y_2 i.e.,

$$\iint_R f(x,y)dx\,dy = \int_{y=y_1}^{y=y_2} \int_{x=\phi_1(y)}^{x=\phi_2(y)} f(x,y)dx\ dy$$

III. When x_1,x_2, y_1,y_2 are all constants. Then

$$\iint_R f(x,y)dx\,dy = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x,y)dx\ dy = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y)dy\ dx$$

$$\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)\,dx\;dy = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dy\;dx$$

Problems

1. Evaluate $\int_1^2\int_1^3 xy^2\,dx\,dy$

Sol. $\int_1^2\left[\int_1^3 xy^2\,dx\right]dy$

$$= \int_1^2\left[y^2.\frac{x^2}{2}\right]_1^3 dy = \int_1^2 \frac{y^2}{2}dy\,[9-1]$$

$$= \frac{8}{2}\int_1^2 y^2\,dy = 4.\int_1^2 y^2\,dy$$

$$= 4.\left[y^3/3\right]_1^2 = \frac{4}{3}[8-1] = \frac{4.7}{3} = \frac{28}{3}$$

2. Evaluate $\int_0^2\int_0^x y\,dy\,dx$

Sol. $\int_{x=0}^{2}\int_{y=0}^{x} y\,dy\,dx = \int_{x=0}^{2}\left[\int_{y=0}^{x} y\,dy\right]dx$

$$= \int_{x=0}^{2}\left[\frac{y^2}{2}\right]_0^x dx = \int_{x=0}^{2}\frac{1}{2}\left(x^2-0\right)dx = \frac{1}{2}\int_{x=0}^{2}x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{1}{6}\left(8-0\right) = \frac{8}{6} = \frac{4}{3}$$

Double integrals

2. Evaluate $\int_0^2 \int_0^x y\,dy\,dx$

Sol. $\int_{x=0}^{2} \int_{y=0}^{x} y\,dy\,dx = \int_{x=0}^{2} \left[\int_{y=0}^{x} y\,dy \right] dx$

$$= \int_{x=0}^{2} \left[\frac{y^2}{2} \right]_0^x dx = \int_{x=0}^{2} \frac{1}{2}\left(x^2 - 0\right) dx = \frac{1}{2}\int_{x=0}^{2} x^2 dx = \frac{1}{2}\left[\frac{x^3}{3} \right]_0^2 = \frac{1}{6}(8-0) = \frac{8}{6} = \frac{4}{3}$$

3. Evaluate $\int_0^5 \int_0^{x^2} x\left(x^2 + y^2\right) dx\,dy$

Sol.

$$\int_{x=0}^{5} \int_{y=0}^{x^2} x\left(x^2 + y^2\right) dy\,dx = \int_{x=0}^{5} \left[x^3 y + \frac{xy^3}{3} \right]_{y=0}^{x^2} dx$$

$$= \int_{x=0}^{5} \left[x^3.x^2 + \frac{x(x^2)^3}{3} \right] dx = \int_{x=0}^{5} \left(x^5 + \frac{x^7}{3} \right) dx = \left[\frac{x^6}{6} + \frac{1}{3}.\frac{x^8}{8} \right]_0^5 = \frac{5^6}{6} + \frac{5^8}{24}$$

Double integrals

4. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

Sol: $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2} = \int_{x=0}^1 \left[\int_{y=0}^{\sqrt{1+x^2}} \frac{1}{(1+x^2)+y^2} dy \right] dx$

$$= \int_{x=0}^1 \left[\int_{y=0}^{\sqrt{1+x^2}} \frac{1}{\left(\sqrt{1+x^2}\right)^2 + y^2} dy \right] dx = \int_{x=0}^1 \frac{1}{\sqrt{1+x^2}} \left[Tan^{-1} \frac{y}{\sqrt{1+x^2}} \right]_{y=0}^{\sqrt{1+x^2}} dx \quad [\because \int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}(x/a)]$$

$$= \int_{x=0}^1 \frac{1}{\sqrt{1+x^2}} \left[Tan^{-1} 1 - Tan^{-1} 0 \right] dx \quad or \quad \frac{\pi}{4} (\sinh^{-1} x)_0^1 = \frac{\pi}{4} (\sinh^{-1} 1)$$

$$= \frac{\pi}{4} \int_{x=0}^1 \frac{1}{\sqrt{1+x^2}} dx = \frac{\pi}{4} \left[\log(x + \sqrt{x^2+1}) \right]_{x=0}^1$$

$$= \frac{\pi}{4} \log(1+\sqrt{2})$$

Double integrals

5. Evaluate $\int_0^4 \int_0^{x^2} e^{y/x} dy dx$

Ans: $3e^4-7$

6. Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) dx dy$

Ans: 3/35

7. Evaluate $\int_0^2 \int_0^x e^{(x+y)} dy dx$

Ans: $\frac{e^4 - e^2}{2}$

8. Evaluate $\int_0^{\frac{\pi}{2}} \int_{-1}^1 x^2 y^2 dx dy$

Ans: $\frac{\pi^3}{36}$

9. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$

Sol: $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy = \int_0^\infty e^{-y^2} \left[\int_0^\infty e^{-x^2} dx \right] dy$

$= \int_0^\infty e^{-y^2} \frac{\sqrt{\pi}}{2} dy \qquad \because \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

$= \frac{\sqrt{\pi}}{2} \int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2} . \frac{\sqrt{\pi}}{2} = \frac{\pi}{4}$

Double integrals

10. Evaluate $\int\int xy(x+y)dxdy$ over the region R bounded by y=x^2 and y=x

Sol: y= x^2 is a parabola through (0,0) symmetric about y-axis y=x is a straight line through (0,0) with slope1.

Let us find their points of intersection solving y= x^2 , y=x we get x^2 =x $\Rightarrow$ x=0,1Hence y=0,1

$\therefore$ The point of intersection of the curves are (0,0), (1,1)

Consider $\iint_R xy(x+y)dxdy$

the evaluation of the integral, we first integrate w.r.t 'y' from y=x^2 to y=x and then w.r.t. 'x' from x=0 to

$$\left[\int_{y=x^2}^{x} xy(x+y)\,dy\right]dx = \int_{x=0}^{1}\left[\int_{y=x^2}^{x}(x^2y+xy^2)\,dy\right]dx$$

$$\int_{x=0}^{1}\left(x^2\frac{y^2}{2}+\frac{xy^3}{3}\right)_{y=x^2}^{x}dx$$

$$\int_{x=0}^{1}\left(\frac{x^4}{2}+\frac{x^4}{3}-\frac{x^6}{2}-\frac{x^7}{3}\right)dx$$

$$\int_{x=0}^{1}\left(\frac{5x^4}{6}-\frac{x^6}{2}-\frac{x^7}{3}\right)dx$$

$$\left(\frac{5}{6}.\frac{x^5}{5}-\frac{x^7}{14}-\frac{x^8}{24}\right)_0^1$$

$$=\frac{1}{6}-\frac{1}{14}-\frac{1}{24}=\frac{28-12-7}{168}=\frac{28-19}{168}=\frac{9}{168}=\frac{3}{56}$$

Double integrals

Evaluate $\int\int_R xy\,dx\,dy$ where R is the region bounded by x-axis and x=2a and the curve $x^2=4ay$.

The line x=2a and the parabola $x^2=4ay$ intersect at B(2a,a)

ıe given integral = $\int\int_R xy\ dx\ dy$

us fix 'y'

a fixed 'y', x varies from $2\sqrt{ay}$ to 2a. Then y varies from 0 to a.

ıce the given integral can also be written as

$$\int_{x=2\sqrt{ay}}^{x=2a} xy\ dx\ dy = \int_{y=0}^{a}\left[\int_{x=2\sqrt{ay}}^{x=2a} x dx\right] y dy$$

$$_{y=0}^{a}\left[\frac{x^2}{2}\right]_{x=2\sqrt{ay}}^{2a} y dy$$

$$_{y=0}^{a}\left[2a^2-2ay\right] y\ dy$$

$$\left.\frac{2a^2y^2}{2}-\frac{2ay^3}{3}\right]_0^a = a^4-\frac{2a^4}{3}=\frac{3a^4-2a^4}{3}=\frac{a^4}{3}$$

Evaluate $\int_0^1\int_0^{\pi/2} r\sin\theta\, d\theta\, dr$

$$\int_{r=0}^{1} r\left[\int_{\theta=0}^{\pi/2}\sin\theta\, d\theta\right] dr$$

$$_{=0}^{1} r\,(-\cos\theta)_{\theta=0}^{\pi/2}\ dr$$

$$_{=0}^{1} -r\left(\cos\pi/2-\cos 0\right) dr$$

Double integrals

Double integrals in polar co-ordinates:

1. Evaluate $\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}$

Sol. $\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}} = \int_0^{\pi/4}\left\{\int_0^{a\sin\theta} \frac{r}{\sqrt{a^2-r^2}}\,dr\right\}d\theta = -1/2\int_0^{\pi/4}\left\{\int_0^{a\sin\theta} \frac{-2r}{\sqrt{a^2-r^2}}\,dr\right\}d\theta$

$= \frac{-1}{2}\int_0^{\pi/4} 2\left(\sqrt{a^2-r^2}\right)_0^{a\sin\theta} d\theta = (-1)\int_0^{\pi/4} 2\left[\sqrt{a^2-a^2\sin^2\theta} - \sqrt{a^2-0}\right]d\theta$

$= (-a)\int_0^{\pi/4}(\cos\theta - 1)\,d\theta = (-a)(\sin\theta - \theta)_0^{\pi/4}$

$= (-a)\left[\left[\sin \pi/4 - \pi/4\right] - (0-0)\right]$

$= (-a)\left[1/\sqrt{2} - \pi/4\right] = 2\left[\pi/4 - 1/\sqrt{2}\right]$

2. Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$ Ans: $\frac{a^2\pi}{4}$

3. Evaluate $\int_0^{\infty}\int_0^{\pi/2} e^{-r^2}\, r\, d\theta\, dr$ Ans: π

4. Evaluate $\int_0^{\pi}\int_0^{a(1+co}$

5. Evaluate $\int_0^1 \int_0^{\pi/2} r\sin\theta\, d\theta\, dr$

Sol. $\int_{r=0}^{1} r\left[\int_{\theta=0}^{\pi/2} \sin\theta\, d\theta\right] dr$

$$= \int_{r=0}^{1} r\left(-\cos\theta\right)_{\theta=0}^{\pi/2} dr$$

$$= \int_{r=0}^{1} -r\left(\cos \pi/2 - \cos 0\right) dr$$

$$= \int_{r=0}^{1} -r\left(0-1\right) dr = \int_0^1 r dr = \left(\frac{r^2}{2}\right)_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

Double integrals

Example 3.

Evaluate the integral $\iint_R \sin\theta d r d\theta$, where the region of integration R is enclosed by the cardioid $r=1+\cos\theta$

Solution.

In polar coordinates, the integral can be written as

$$\iint_R \sin\theta dr d\theta = \int_0^{2\pi}\int_0^{1+\cos\theta} \sin\theta dr d\theta$$

$$= \int_0^{2\pi}\left[\int_0^{1+\cos\theta} dr\right]\sin\theta d\theta$$

$$= \int_0^{2\pi}\left[r|_0^{1+\cos\theta}\right]\sin\theta d\theta$$

$$= \int_0^{2\pi}(1+\cos\theta)\sin\theta d\theta$$

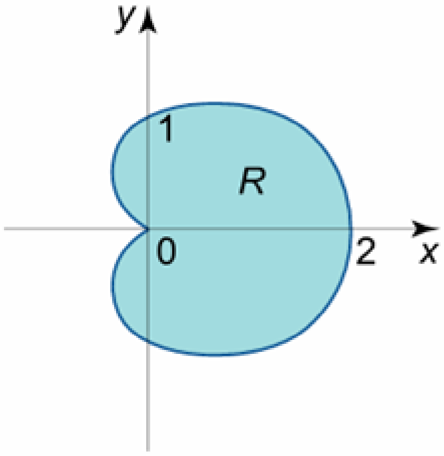


Figure 6.

Double integrals

$$= \int_0^{2\pi} (\sin\theta + \cos\theta \sin\theta)\, d\theta = \int_0^{2\pi} \sin\theta d\theta + \int_0^{2\pi} \frac{\sin 2\theta}{2} d\theta = (-\cos\theta)\Big|_0^{2\pi}$$

$$+ \frac{1}{2}\left(-\frac{\cos 2\theta}{2}\right)\Bigg|_0^{2\pi} = -\cos 2\pi + \cos 0 - \frac{1}{4}\cos 4\pi + \frac{1}{4}\cos 0$$

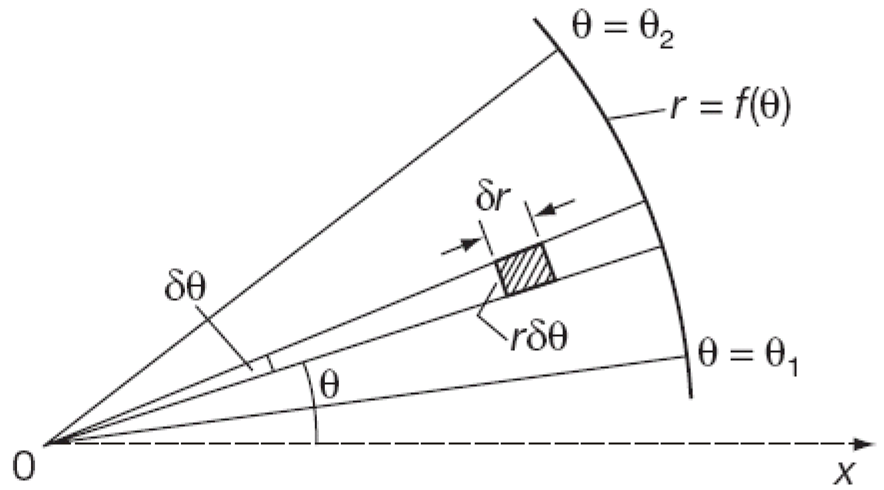
$$= -\cancel{1} + \cancel{1} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} = 0.$$

Double integrals

$$A = \int_{\theta=\theta_1}^{\theta_2} \int_{r=0}^{r_1} r.dr.d\theta$$

$$= \int_{\theta=\theta_1}^{\theta_2} \left[\frac{r^2}{2} \right]_0^{r_1} .d\theta$$

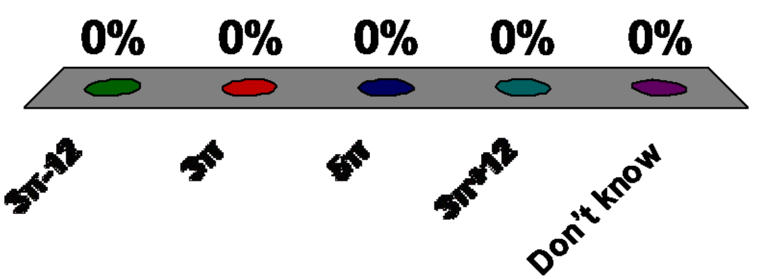
$$= \int_{\theta=\theta_1}^{\theta_2} \frac{1}{2} r_1^2 .d\theta$$



Evaluate

$$I = \int_{1}^{4} \int_{0}^{\pi} (1 - 2\cos\theta)\, d\theta\, dr$$

1. 3π-12
2. 3π5π
3. 3π+12
4. Don’t know



Change of order of integration

Change of order of Integration:

1. Change the order of Integration and evaluate $\int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}} dy\,dx$

Sol. In the given integral for a fixed x, y varies from $\frac{x^2}{4a}$ to $2\sqrt{ax}$ and then x varies from 0 to 4a. Let us draw the curves $y=\frac{x^2}{4a}$ and $y=2\sqrt{ax}$

The region of integration is the shaded region in diagram.

The given integral is $=\int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}} dy\,dx$

Change of order of integration

Changing the order of integration, we must fix y first, for a fixed y, x varies from $\frac{y^2}{4a}$ to $\sqrt{4ay}$ and then y varies from 0 to 4a.

Hence the integral is equal to

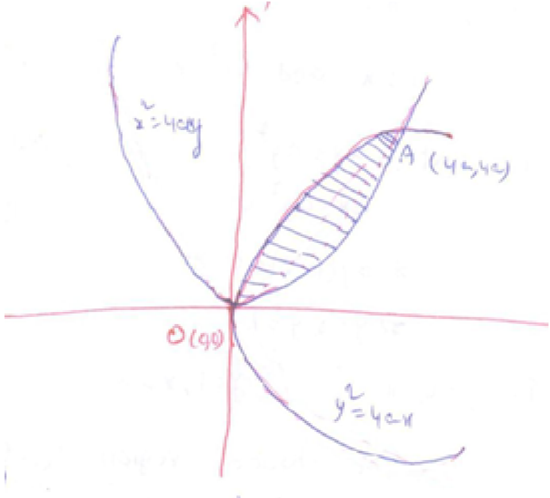
$$\int_{y=0}^{4a}\int_{x=y^2/4a}^{2\sqrt{ay}} dx\,dy = \int_{y=0}^{4a}\left[\int_{x=y^2/4a}^{2\sqrt{ay}} dx\right]dy$$

$$= \int_{y=0}^{4a}[x]_{x=y^2/4a}^{2\sqrt{ay}}\, dy = \int_{y=0}^{4a}\left[2\sqrt{ay} - {y^2}/{4a}\right]dy$$

$$= \left[2\sqrt{a}.\frac{y^{3/2}}{3/2} - \frac{1}{4a}.\frac{y^3}{3}\right]_0^{4a}$$

$$= \frac{4}{3}.\sqrt{a}.4a\sqrt{4a} - \frac{1}{12a}.64a^3$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2$$



Change of order of integration

2. Change the order of integration and evaluate $= \int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2 + y^2)\, dx\, dy$

Sol. In the given integral for a fixed x, y varies from $\frac{x}{a}$ to $\sqrt{\frac{x}{a}}$ and then x varies from 0 to a

Hence we shall draw the curves $y = \frac{x}{a}$ and $y = \sqrt{\frac{x}{a}}$

i.e. ay=x and ay²=x

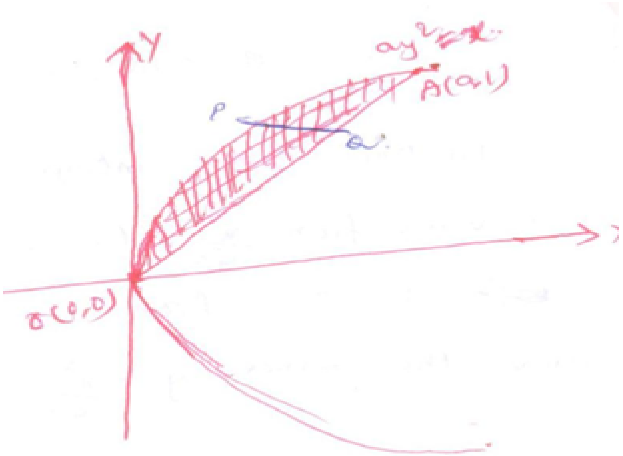
we get $ay = ay^2$

$\Rightarrow ay - ay^2 = 0$

$\Rightarrow ay(1-y) = 0$

$\Rightarrow y = 0, y = 1$

If y=0, x=0 if y=1, x=a



The shaded region is the region of integration.

The given integral is $\int_{x=0}^{a} \int_{y=x/a}^{y=\sqrt{x/a}} (x^2 + y^2) dx\, dy$

Changing the order of integration, we must fix y first. For a fixed y, x varies from ay^2 to ay and then y

Change of order of integration

varies from 0 to 1.

Hence the given integral, after change of the order of integration becomes

$$\int_{y=0}^{1}\int_{x=ay^2}^{ay}\left(x^2+y^2\right)dx\,dy$$

$$=\int_{y=0}^{1}\left[\int_{x=ay^2}^{ay}\left(x^2+y^2\right)dx\right]dy$$

$$=\int_{y=0}^{1}\left(\frac{x^3}{3}+xy^2\right)_{x=ay^2}^{ay}dy$$

$$=\int_{y=0}^{1}\left(\frac{a^3y^3}{3}+ay^3-\frac{a^3y^6}{3}-ay^4\right)dy$$

$$=\left(\frac{a^3y^4}{12}+\frac{ay^4}{4}-\frac{a^3y^7}{21}-\frac{ay^5}{5}\right)_{y=0}^{1}$$

$$=\frac{a^3}{12}+\frac{a}{4}-\frac{a^3}{21}-\frac{a}{5}=\frac{a^3}{28}+\frac{a}{20}$$

Change of order of integration

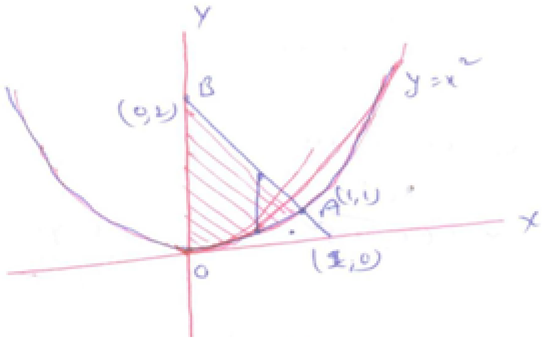
3.Change the order of integration in $\int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy$ and hence evaluate the double integral.

Sol.In the given integral for a fixed x,y varies from x^2 to 2-x and then x varies from 0 to 1. Hence we shall draw the curves $y=x^2$ and y=2-x.

The line y=2-x passes through (0,2), (2,0)

Solving $y=x^2$,y=2-x

Then we get $x^2 = 2-x$



$$\Rightarrow x^2 + x - 2 = 0$$

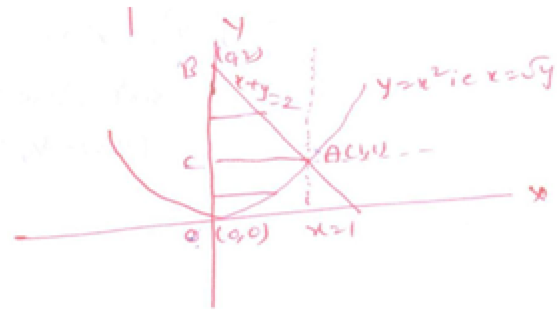
$$\Rightarrow x^2 + 2x - x - 2 = 0$$

$$\Rightarrow x(x+2) - 1(x+2) = 0$$

Change of order of integration

$$\Rightarrow (x-1)(x+2)=0$$

$$\Rightarrow x=1,-2$$



IARE MULTIPLE INTEGRALS

Change of order of integration

$If\ x=1, y=1$

$If\ x=-2, y=4$

Hence the points of intersection of the curves are (-2,4) (1,1)

The Shaded region in the diagram is the region of intersection.

Changing the order of integration, we must fix y, for the region with in OACO for a fixed y, x varies from 0 to $\sqrt{y}$

Then y varies from 0 to 1

For the region within CABC, for a fixed y, x varies from 0 to 2-y, then y varies from 1 to 2

Hence $\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx = \iint_{OACO} xy\,dx\,dy + \iint_{CABC} xy\,dx\,dy$

$$= \int_{y=0}^{1} \left[\int_{x=0}^{\sqrt{y}} x dx \right] y\,dy + \int_{y=1}^{2} \left[\int_{x=0}^{2-y} x\,dx \right] y\,dy$$

$$= \int_{y=0}^{1} \left(\frac{x^2}{2} \right)_{x=0}^{\sqrt{y}} y\,dy + \int_{y=1}^{2} \left(\frac{x^2}{2} \right)_{x=0}^{2-y} y\,dy$$

Change of order of integration

$$=\int_{y=0}^{1}\frac{y}{2}.y\,dy+\int_{y=1}^{2}\frac{(2-y)^{2}}{2}y\,dy$$

$$=\frac{1}{2}\int_{y=0}^{1}y^{2}dy+\frac{1}{2}.\int_{y=1}^{2}\left(4y-4y^{2}+y^{3}\right)dy$$

$$=\frac{1}{2}.\left(\frac{y^{3}}{3}\right)_{0}^{1}+\frac{1}{2}.\left[\frac{4y^{2}}{2}-\frac{4y^{3}}{3}+\frac{y^{4}}{4}\right]_{1}^{2}$$

$$=\frac{1}{2}.\frac{1}{3}+\frac{1}{2}\left[2.4-2.1-4/3(8-1)+1/4(16-1)\right]$$

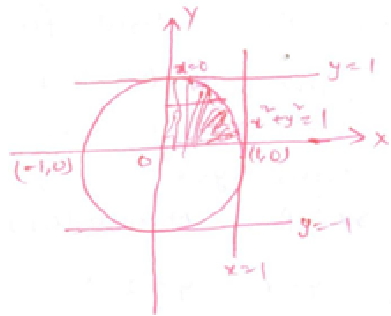
Change of order of integrals

$$=\frac{1}{6}+\frac{1}{2}\left[6-\frac{28}{3}+\frac{15}{4}\right]=\frac{1}{6}+\frac{1}{2}\left[\frac{72-112+45}{12}\right]=\frac{1}{6}+\frac{1}{2}\left[\frac{5}{12}\right]=\frac{4+5}{24}=\frac{9}{24}=\frac{3}{8}$$

4. Changing the order of integration $\int_0^a \int_{x^2/a}^{2a-x} xy^2 dy\, dx$

5. Change of the order of integration $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dx\, dy \qquad Ans: \frac{\pi}{16}$

Hint: Now limits are $y = 0\ to\ 1$ and $x = 0\ to\ \sqrt{1-y^2}$



$$put\ y = \sin\theta$$
$$\sqrt{1-y^2} = \cos\theta$$
$$dy = \cos\theta\, d\theta$$

$$=\int_0^1 y^2\sqrt{1-y^2}\,dy$$

$$=\int_0^{\pi/2}\sin^2\theta\cos^2\theta\, d\theta=\int_0^{\pi/2}\sin^2\theta\, d\theta-\int_0^{\pi/2}\sin^4\theta\, d\theta$$

$$=\frac{1}{2}\left(\pi/2\right)-\frac{3}{4}\cdot\frac{1}{2}\left(\pi/2\right)=\pi/16$$

Change of order of integrals

Change of order of Integration:

Change the order of integration in the following integral

$$\int_0^1 \int_1^{e^y} f(x, y) dx\, dy.$$

Solution: In the original integral, the integration order is $dx\, dy$. This integration order corresponds to integrating first with respect to x (i.e., summing along rows in the picture below), and afterwards integrating with respect to y (i.e., summing up the values for each row). Our task is to change the integration to be $dy\, dx$, which means integrating first with respect to y.

We begin by transforming the limits of integration into the domain D. The limits of the outer dy integral mean that $0 \le y \le 1$, and the limits on the inner dx integral mean that for each value of y the range of x is $1 \le x \le e^y$. The region D is shown in the following figure.

Change of order of integrals

Change of order of Integration:

Change the order of integration in the following integral

$$\int_0^1 \int_1^{e^y} f(x,y)dx\,dy.$$

Solution: In the original integral, the integration order is $dx\,dy$. This integration order corresponds to integrating first with respect to x (i.e., summing along rows in the picture below), and afterwards integrating with respect to y (i.e., summing up the values for each row). Our task is to change the integration to be $dy\,dx$, which means integrating first with respect to y.

We begin by transforming the limits of integration into the domain D. The limits of the outer dy integral mean that $0 \le y \le 1$, and the limits on the inner dx integral mean that for each value of y the range of x is $1 \le x \le e^y$. The region D is shown in the following figure.

Change of order of integrals

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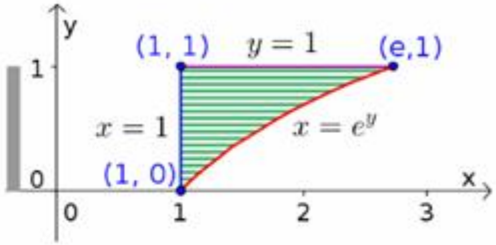
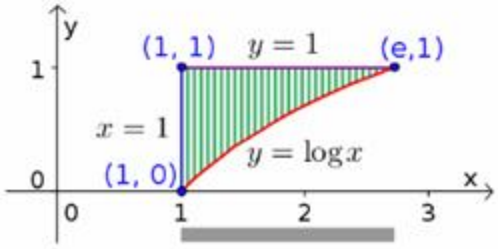


Change of order of integrals

The maximum range of y over the region is from 0 to 1, as indicated by the gray bar to the left of the figure. The horizontal hashing within the figure indicates the range of x for each value of y, beginning at the left edge $x = 1$ (blue line) and ending at the right curve edge $x = e^y$ (red curve).

We have also labeled all the corners of the region. The upper-right corner is the intersection of the line $y = 1$ with the curve $x = e^y$. Therefore, the value of x at this corner must be $e = e^1 = e$, and the point is $(e, 1)$.

To change order of integration, we need to write an integral with order $dy\,dx$. This means that x is the variable of the outer integral. Its limits must be constant and correspond to the total range of x over the region D. The total range of x is $1 \leq x \leq e$, as indicated by the gray bar below the region in the following figure.



Change of order of integrals

Since y will be the variable for the inner integration, we need to integrate with respect to y first. The vertical hashing indicates how, for each value of x, we will integrate from the lower boundary (red curve) to the upper boundary (purple line). These two boundaries determine the range of y. Since we can rewrite the equation $x = e^y$ for the red curve as $y = \log x$, the range of y is $\log x \le y \le 1$. (The function $\log x$ indicates the natural logarithm, which sometimes we write as $\ln x$.)

In summary, the region D can be described not only by

$$\begin{gathered} 0 \le y \le 1 \\ 1 \le x \le e^y \end{gathered}$$

as it was for the original $dx\,dy$ integral, but also by

$$\begin{gathered} 1 \le x \le e \\ \log x \le y \le 1, \end{gathered}$$

which is the description we need for the new $dy\,dx$ integration order. This latter pair of inequalites determine the bounds for integral.

We conclude that the integral $\int_0^1 \int_1^{e^y} f(x,y)dx\,dy$ with integration order reversed is

$$\int_1^e \int_{\log x}^1 f(x,y)dy\,dx.$$

Example 2

Sometimes you need to change the order of integration to get a tractable integral. For example, if you tried to evaluate

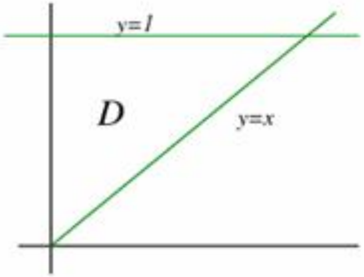
$$\int_0^1 \int_x^1 e^{y^2}\, dy\, dx$$

directly, you would run into trouble. There is no antiderivative of e^{y^2}, so you get stuck trying to compute the integral with respect to y. But, if we change the order of integration, then we can integrate with respect to x first, which is doable. And, it turns out that the integral with respect to y also becomes possible after we finish integrating with respect to x.

According to the limits of integration of the given integral, the region of integration is

$$\begin{gathered} 0 \le x \le 1 \\ x \le y \le 1, \end{gathered}$$

which is shown in the following picture.



Since we can also describe the region by

$$\begin{aligned} 0 \le y \le 1 \\ 0 \le x \le y, \end{aligned}$$

the integral with the order changed is

$$\int_0^1 \int_x^1 e^{y^2}\,dy\,dx = \int_0^1 \int_0^y e^{y^2}\,dx\,dy$$

With this new $dx\,dy$ order, we integrate first with respect to x

$$\int_0^1 \int_0^y e^{y^2}\,dx\,dy = \int_0^1 xe^{y^2}\Big|_{x=0}^{x=y} dy = \int_0^1 ye^{y^2}\,dy.$$

Since the integration with respect to x gave us an extra factor of y, we can compute the integral with respect to y by using a u-substitution, $u = y^2$, so $du = 2y\,dy$. With this substitution, u rannges from 0 to 1, and we calculate the integral as

$$\begin{aligned} \int_0^1 \int_0^y e^{y^2}\,dx\,dy &= \int_0^1 ye^{y^2}\,dy \\ &= \int_0^1 \frac{1}{2}e^u\,du = \frac{1}{2}e^u\Big|_0^1 = \frac{1}{2}(e-1). \end{aligned}$$

Change of Variables

Change of variables:

The variables x,y in $\iint_R f(x,y)\,dx\,dy$ are changed to u,v with the help of the relations

$x = f_1(u,v), y = f_2(u,v)$ then the double integral is transferred into

$$\iint_{R^1} f\left[f_1(u,v), f_2(u,v)\right]\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Where R^1 is the region in the uv plane, corresponding to the region R in the xy-plane.

Change of Variables

Changing from Cartesian to polar co-ordinates

$x = r\cos\theta, y = r\sin\theta$

$$\partial\left(\frac{(x,y)}{(r,\theta)}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\left(\cos^2\theta + \sin^2\theta\right) = r \therefore \iint_R f(x,y)\,dx\,dy = \iint_{R_1} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Note : In polar form dx dy is replaced by $r\,dr\,d\theta$

Change of Variables

Problems:

1. Evaluate the integral by changing to polar co-ordinates $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$

Sol.The limits of x and y are both from 0 to ∞.

$\therefore$ The region is in the first quadrant where r varies from 0 to ∞ and θ varies from 0 to $\pi/2$

Substituting $x = r\cos\theta, y = r\sin\theta$ and $dx\, dy = r\, dr\, d\theta$

Hence $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2}\, r\, dr\, d\theta$

Inverse of a matrix by Gauss-Jordan method

$$Put\, r^2 = t$$
$$\Rightarrow 2rdr = dt$$
$$\Rightarrow r\,dr = {dt}/{2}$$
$$Where\ r = 0 \Rightarrow t = 0\ and\ r = \infty \Rightarrow t = \infty$$
$$\therefore \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy = \int_{\theta=0}^{\pi/2} \int_{t=0}^\infty \frac{1}{2} e^{-t} dt\, d\theta$$
$$= \int_0^{\pi/2} \frac{-1}{2} \left(e^{-t}\right)_0^\infty d\theta$$
$$= \frac{-1}{2} \int_0^{\pi/2} (0-1)\, d\theta \Rightarrow \frac{1}{2} (\theta)_0^{\pi/2} = \frac{1}{2}\, {\pi}/{2} = {\pi}/{4}$$

Inverse of a matrix by Gauss-Jordan method

2. Evaluate the integral by changing to polar co-ordinates $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy$

Sol.The limits for x are x=0 to $x=\sqrt{a^2-y^2}$

$$\Rightarrow x^2+y^2=a^2$$

$\therefore$ The given region is the first quadrant of the circle.

By changing to polar co-ordinates

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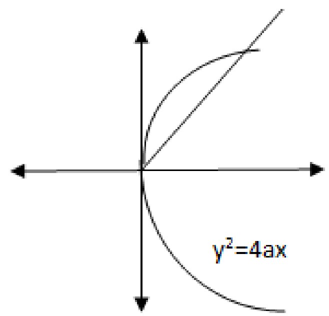
$$= \pi/8\, a^4$$

Change of Variables

1. Show that $\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy = 8a^2\left(\frac{\pi}{2}-\frac{5}{3}\right)$

Sol. The region of integration is given by $x = {y^2}/{4a}$, $x = y$ and y=0, y=4a



i.e., The region is bounded by the parabola $y^2=4ax$ and the straight line x=y.

Let $x = r\cos\theta, y = r\sin\theta. Then\ dx\,dy = r\,dr\,d\theta$

The limits for r are r=0 at O and for P on the parabola

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$$= 8a^2\int_{\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta)\frac{\cos^2\theta}{\sin^4\theta}\,d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2} (\cos^4\theta - \cot^2\theta)\,d\theta = 8a^2\left[\frac{3\pi-8}{12} + \frac{\pi}{4} - 1\right] = 8a^2\left(\frac{\pi}{2} - \frac{5}{3}\right)$$

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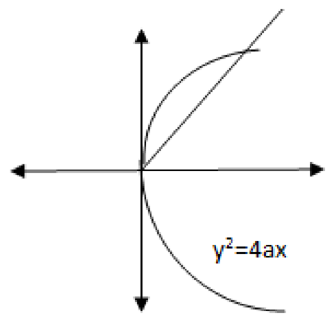

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Example1

Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dxdy$ by changing to polar coordinates.

Hence show that $\int_0^\infty e^{-x}\,dx = \sqrt{\dfrac{\pi}{2}}$

Solution: Since both x and y vary from 0 to ∞

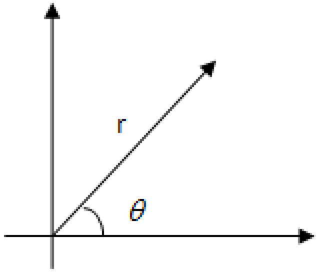
The region of integration is the 1st quadrant of the xy plane.
Change into polar coordinates, by putting $x = r\cos\theta, y = r\sin\theta$,

we have dxdy = rdrdθ

Transformation of coordinate systems

And $x^2 + y^2 = r^2$ .in the region of integration r varies

from 0 to ∞ and θ various from 0 to $\frac{\pi}{2}$



$$\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}dxdy = \int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-(r^2)}drd\theta$$ (put t=r^2,dt=2rdr.)

$$= \int_0^{\frac{\pi}{2}}\int_0^{\infty} \frac{1}{2}e^{-(t)}dtd\theta$$ r varies from 0 to ∞, t varies from 0 to ∞

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}}(e^{-t})_0^{\infty}\,d\theta = \sqrt{\frac{\pi}{2}}$$

Example 2:

Evaluate the double integral

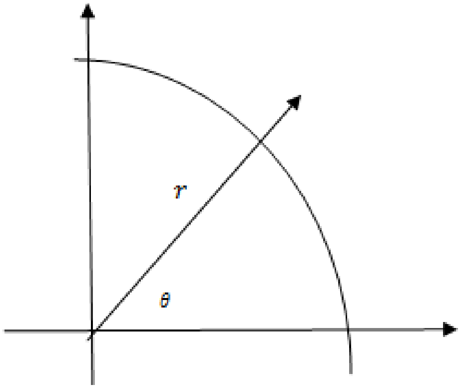
$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dydx$$

Solutions:

The region R is bounded by the circle $x^2+y^2=a^2$ lies in the first quadrant; change into polar coordinates, by putting

$x = r\cos\theta, y = r\sin\theta$,

We have dxdy =rdrdθ



$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dydx$$

$$= \int_0^{\frac{\pi}{2}} \int_0^a r^2\, r dr d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{r^4}{4}\right)_0^a d\theta = \frac{\pi a^4}{8}$$

Example 3:

By changing into polar coordinates, evaluate $\iint \frac{x^2 y^2}{x^2+y^2} dxdy$ over the annular region between the circle $x^2+y^2=a^2$ and $x^2+y^2=b^2$ (b>a)

Solution: change to polar coordinates buy putting

$x = r\cos\theta, y = r\sin\theta$, dxdy =rdrdθ

$x^2+y^2=a^2 \rightarrow r^2(\cos^2\theta+\sin^2\theta)=a^2 \rightarrow$ r=a

$x^2+y^2=b^2 \rightarrow r^2(\cos^2\theta+\sin^2\theta)=b^2 \rightarrow$ r=b

θ various from 0 to 2π

Hence $\iint \frac{x^2y^2}{x^2+y^2}dxdy \quad = \quad \int_0^{2\pi}\int_a^b \frac{(r^2r^2(\cos^2\theta\sin^2\theta))}{(r^2(\cos^2\theta+\sin^2\theta)}rdrd\theta$

$$= \int_0^{2\pi}\int_0^b r^3\cos^2\theta\sin^2\theta drd\theta$$

$$= \int_0^{2\pi}\cos^2\theta\sin^2\theta(\frac{r^4}{4})_a{}^b d\theta$$

$$= \frac{b^2-a^4}{4}\int_0^{2\pi}\cos^2\theta\sin^2\theta d\theta$$

$$= \frac{b^2-a^4}{16}\int_0^{2\pi}\sin^2 2\theta d\theta$$

Transformation of coordinate systems

$$= \frac{b^2 - a^4}{4}\int_0^{2\pi} \cos^2\theta \sin^2\theta d\theta$$

$$= \frac{b^2 - a^4}{16}\int_0^{2\pi} \sin^2 2\theta d\theta$$

$$= \frac{b^2 - a^4}{32}\int_0^{2\pi} 1 - \cos 4\theta d\theta$$

$$= \frac{b^2 - a^4}{32}(\theta - \frac{\sin 4\theta}{4})_0^{2\pi}$$

$$= \frac{\pi}{16}(b^4 - a^4)$$

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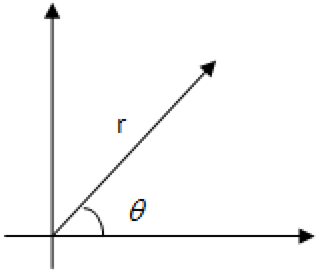
from 0 to ∞ and θ various from 0 to $\frac{\pi}{2}$



$$\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}dxdy = \int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-(r^2)}drd\theta$$ (put t=r^2,dt=2rdr.)

$$= \int_0^{\frac{\pi}{2}}\int_0^{\infty} \frac{1}{2}e^{-(t)}dtd\theta$$ r varies from 0 to ∞, t varies from 0 to ∞

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}}(e^{-t})_0^{\infty}d\theta = \sqrt{\frac{\pi}{2}}$$

Example 2:

Evaluate the double integral

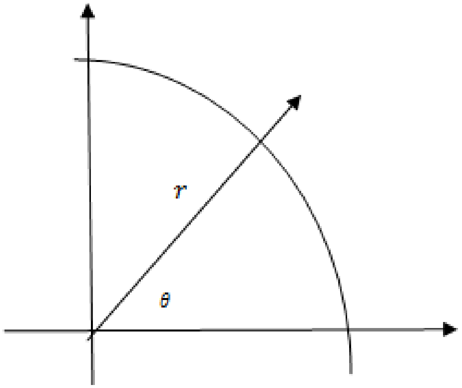
$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dy\,dx$$

Solutions:

The region R is bounded by the circle $x^2+y^2=a^2$ lies in the first quadrant; change into polar coordinates, by putting

$x = r\cos\theta, y = r\sin\theta$,

We have dxdy =rdrdθ



$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dy\,dx$$

$$= \int_0^{\frac{\Pi}{2}} \int_0^a r^2\, r\,dr\,d\theta = \int_0^{\frac{\Pi}{2}} \left(\frac{r^4}{4}\right)_0^a d\theta = \frac{\pi a^4}{8}$$

Example 3:

By changing into polar coordinates, evaluate $\iint \frac{x^2y^2}{x^2+y^2}dxdy$ over the annular region between the circle $x^2+y^2=a^2$ and $x^2+y^2=b^2$ (b>a)

Solution: change to polar coordinates buy putting

$x = r\cos\theta, y = r\sin\theta$, dxdy =rdrdθ

$x^2+y^2=a^2 \rightarrow r^2(\cos^2\theta+\sin^2\theta)=a^2 \rightarrow$ r=a

$x^2+y^2=b^2 \rightarrow r^2(\cos^2\theta+\sin^2\theta)=b^2 \rightarrow$ r=b

Transformation of coordinate systems

θ various from 0 to 2π

Hence $\iint \frac{x^2y^2}{x^2+y^2}dxdy \quad = \quad \int_0^{2\pi}\int_a^b \frac{(r^2r^2(\cos^2\theta \sin^2\theta))}{(r^2(\cos^2\theta + \sin^2\theta)}rdrd\theta$

$$= \int_0^{2\pi}\int_0^b r^3\cos^2\theta\sin^2\theta drd\theta$$

$$= \int_0^{2\pi}\cos^2\theta\sin^2\theta(\frac{r^4}{4})_a{}^b d\theta$$

$$= \frac{b^2-a^4}{4}\int_0^{2\pi}\cos^2\theta\sin^2\theta d\theta$$

$$= \frac{b^2-a^4}{16}\int_0^{2\pi}\sin^2 2\theta d\theta$$

Transformation of coordinate systems

$$= \frac{b^2 - a^4}{4}\int_0^{2\pi} \cos^2\theta \sin^2\theta d\theta$$

$$= \frac{b^2 - a^4}{16}\int_0^{2\pi} \sin^2 2\theta d\theta$$

$$= \frac{b^2 - a^4}{32}\int_0^{2\pi} 1 - \cos 4\theta d\theta$$

$$= \frac{b^2 - a^4}{32}(\theta - \frac{\sin 4\theta}{4})_0^{2\pi}$$

$$= \frac{\pi}{16}(b^4 - a^4)$$

Area using double integral

AREA ENCLOSED BY A PLANE CURVE

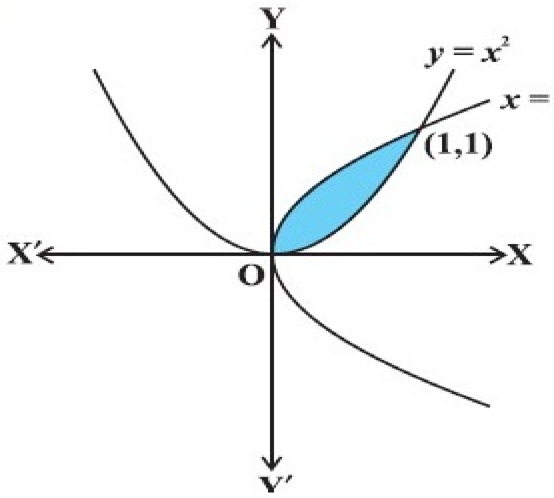
Consider the area enclosed by the curves y= f(x) , y = g(x), x = a , x = b in the xy plane.

The area of the region R bounded by the given curves is given by $\iint_R dxdy \; or \iint_R dydx = \int_{x=a}^{b} \int_{y=f(x)}^{y=g(x)} dydx$

If the region is represented through polar coordinates then the area is given by $\iint_R rdrd\theta$

1) **Find the area of the region bounded by the two parabolas $y = x^2$ and $y^2 = x$.**

Solution The point of intersection of these two parabolas are O (0, 0) and A (1, 1) as shown in the Fig 8.15.



Areas using double integrals

The area of the region R bounded by the given curves is given by $\iint_R dxdy$

$$A = \iint_R dxdy \text{ or } \iint_R dydx$$

$$= \int_{y=0}^{1} \int_{x=y^2}^{\sqrt{y}} dxdy$$

$$\int_{y=0}^{1} \left(\sqrt{y} - y^2\right) dy$$

$$\left(\frac{2}{3} y^{\frac{3}{2}} - \frac{y^3}{3}\right)_0^1$$

$$\frac{2}{3} - \frac{1}{3} = \frac{1}{3} \text{ sq.units}$$

Area of a Region in the Plane, Figure 14.2 and Figu

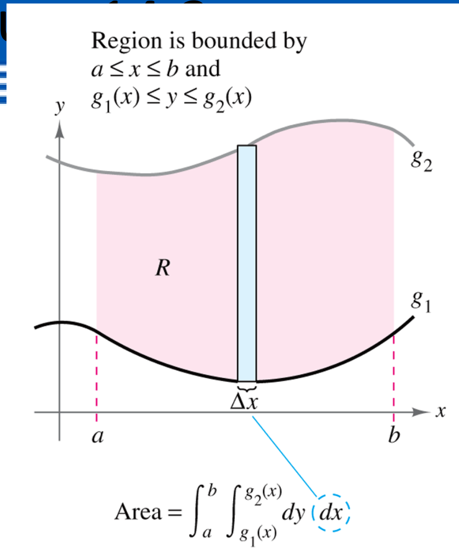
Area of a Region in the Plane

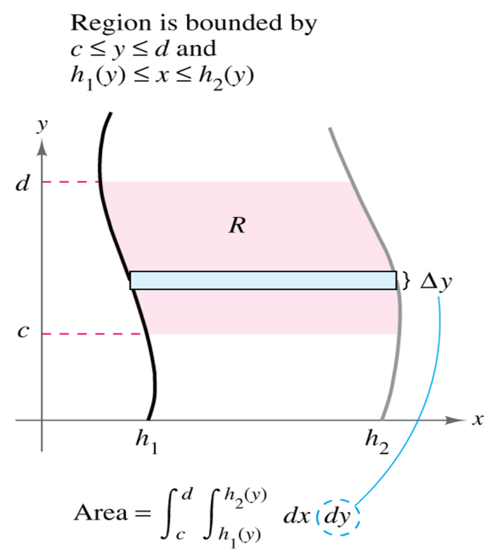
1. If R is defined by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous on $[a, b]$, then the area of R is given by

$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx. \qquad \text{Figure 14.2 (vertically simple)}$$

2. If R is defined by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous on $[c, d]$, then the area of R is given by

$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy. \qquad \text{Figure 14.3 (horizontally simple)}$$

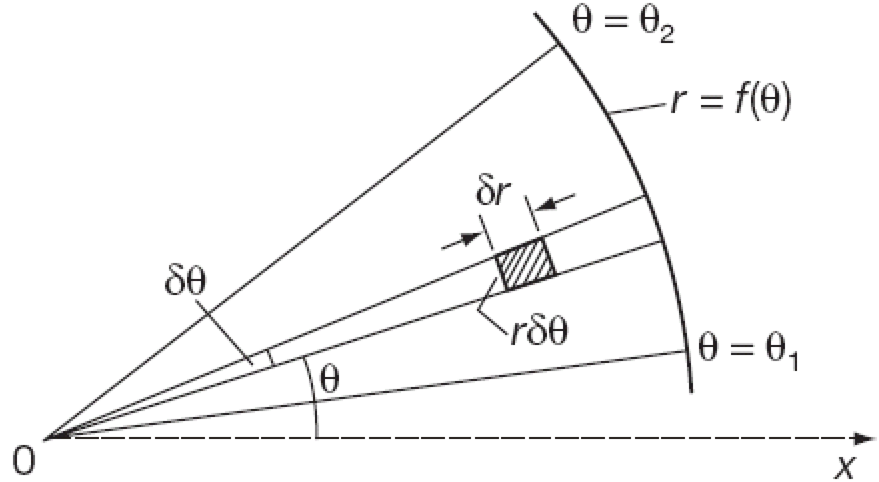




$$A = \int_{\theta=\theta_1}^{\theta_2} \int_{r=0}^{r_1} r.dr.d\theta$$

$$= \int_{\theta=\theta_1}^{\theta_2} \left[\frac{r^2}{2} \right]_0^{r_1} .d\theta$$

$$= \int_{\theta=\theta_1}^{\theta_2} \frac{1}{2} r_1^2 .d\theta$$



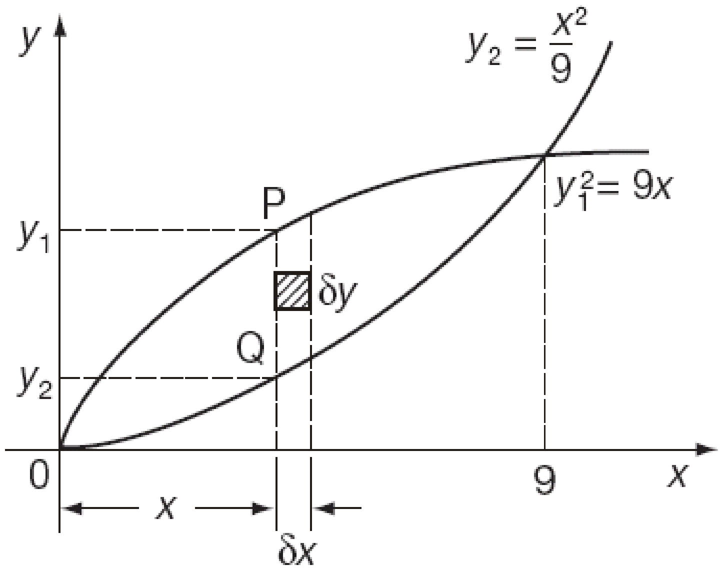
Example 2: To find the area enclosed by the curves $y^2 = 9x$ and $y = \frac{x^2}{9}$

$$A \cong \sum_{x=0}^{x=9} \sum_{y=x^2/9}^{y=3x^{1/2}} dy.dx \quad \text{so} \quad A = \int_{x=0}^{9} \int_{y=x^2/9}^{3x^{1/2}} dydx$$

$$= \int_{x=0}^{9} \left(3x^{\frac{1}{2}} - \frac{x^2}{9} \right) dx$$

$$= \left[2x^{\frac{3}{2}} - \frac{x^3}{27} \right]_{x=0}^{9}$$

$$= 27 \text{ units}^2$$



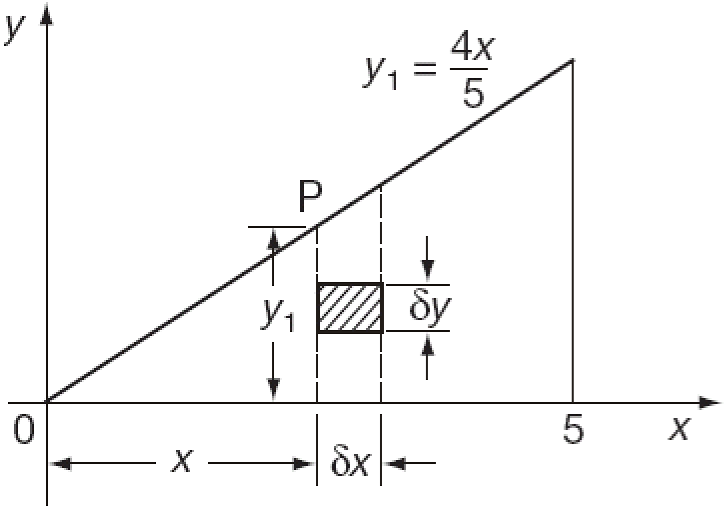
Example 3: To find the area bounded by $y = \dfrac{4x}{5}$ the x-axis and the ordinate at $x = 5$.

$$A \cong \sum_{x=0}^{x=5} \sum_{y=0}^{y=4x/5} \delta y . \delta x \quad \text{so} \quad A = \int_{x=0}^{5} \int_{y=0}^{4x/5} dy dx$$

$$= \int_{x=0}^{5} (4x/5) dx$$

$$= \left[\frac{2x^2}{5} \right]_{x=0}^{5}$$

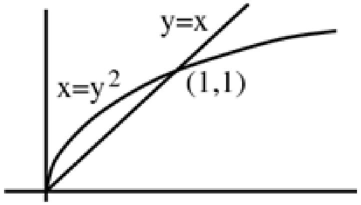
$$= 10 \text{ units}^2$$



Area using double integral

3)Find area of region bounded by parabola $x^2 = y$ and the line $y=x$

The region is pictured below.



Solution: Let y varies from 0 to 1

Then x varies from y^2 to y

The area is

$$\int_0^1 \int_{y^2}^{y} dx dy = \int_0^1 [x]_{y^2}^{y} dy$$

$$= \int_0^1 (y - y^2) dy$$

$$\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$

$$\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Area using double integral

4) Using double integral find the area of the cardioid $r = a(1-\cos\theta)$

If the region is represented through polar coordinates then the area is given by $\iint_R r\,dr\,d\theta$

$$A = 2\int_0^{\pi}\int_{r=0}^{a(1-\cos\theta)} r\,dr\,d\theta$$

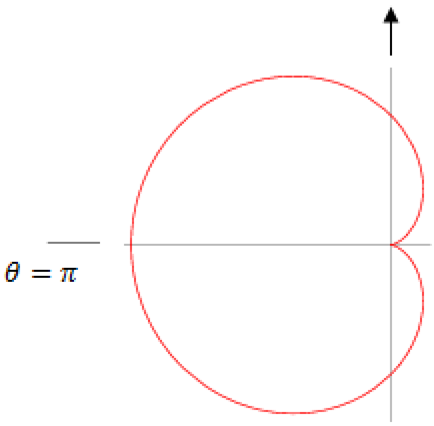
$$2\int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{a(1-\cos\theta)} d\theta$$

$$a^2\int_0^{\pi}(1-\cos\theta)^2\,d\theta$$

$$4a^2\int_0^{\pi}\sin^4\frac{\theta}{2}\,d\theta$$

$$4a^2\int_0^{\frac{\pi}{2}}\sin^4\phi\,2d\varphi\left[putting\,\frac{\theta}{2}=\varphi\right]$$

$$8a^2\int_0^{\frac{\pi}{2}}\sin^4\phi\,d\varphi = 8a^2.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{3\pi a^2}{2}$$



Volume using Triple integral

Calculation of Volumes Using Triple Integrals

$$\iiint\limits_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Example 1 *Evaluate* the following integral.

$$\iiint\limits_B 8xyz\,dV,$$

$$B = [2,3]\times[1,2]\times[0,1]$$

Solution

Just to make the point that order doesn't matter let's use a different order from that listed above. We'll do the integral in the following order.

$$\begin{aligned}\iiint\limits_B 8xyz\,dV &= \int_1^2 \int_2^3 \int_0^1 8xyz\,dz\,dx\,dy \\ &= \int_1^2 \int_2^3 4xyz^2\Big|_0^1\,dx\,dy \\ &= \int_1^2 \int_2^3 4xy\,dx\,dy \\ &= \int_1^2 2x^2y\Big|_2^3\,dy \\ &= \int_1^2 10y\,dy = 15\end{aligned}$$

Volume using Triple integral

Example 1 Evaluate $\iiint_E y\,dV$ where E is the region that lies below the plane $z = x + 2$ above the xy-plane and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution

There really isn't too much to do with this one other than do the conversions and then evaluate the integral.

We'll start out by getting the range for z in terms of cylindrical coordinates.

$$0 \le z \le x + 2 \qquad \Rightarrow \qquad 0 \le z \le r\cos\theta + 2$$

Remember that we are above the xy-plane and so we are above the plane $z = 0$

Volume using Triple integral

Next, the region D is the region between the two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ in the xy-plane and so the ranges for it are,

$$0 \le \theta \le 2\pi \qquad\qquad 1 \le r \le 2$$

Here is the integral.

$$\begin{aligned}
\iiint_E y\,dV &= \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2} (r\sin\theta)\,r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_1^2 r^2\sin\theta\,(r\cos\theta+2)\,dr\,d\theta \\
&= \int_0^{2\pi}\int_1^2 \frac{1}{2}r^3\sin(2\theta)+2r^2\sin\theta\,dr\,d\theta \\
&= \int_0^{2\pi}\left(\frac{1}{8}r^4\sin(2\theta)+\frac{2}{3}r^3\sin\theta\right)\Big|_1^2\,d\theta \\
&= \int_0^{2\pi}\frac{15}{8}\sin(2\theta)+\frac{14}{3}\sin\theta\,d\theta \\
&= \left(-\frac{15}{16}\cos(2\theta)-\frac{14}{3}\cos\theta\right)\Big|_0^{2\pi} \\
&= 0
\end{aligned}$$

Volume using Triple integral

Example 1 Evaluate the following integral.

$$\iiint\limits_B 8xyz\, dV \quad , \qquad B = [2,3]\times[1,2]\times[0,1]$$

Solution

Just to make the point that order doesn't matter let's use a different order from that listed above. We'll do the integral in the following order.

$$\iiint\limits_B 8xyz\, dV = \int_1^2 \int_2^3 \int_0^1 8xyz\, dz\, dx\, dy$$

$$= \int_1^2 \int_2^3 4xyz^2 \Big|_0^1 \, dx\, dy$$

$$= \int_1^2 \int_2^3 4xy\, dx\, dy$$

$$= \int_1^2 2x^2 y \Big|_2^3 \, dy$$

$$= \int_1^2 10y\, dy = 15$$

Volume using Triple integral

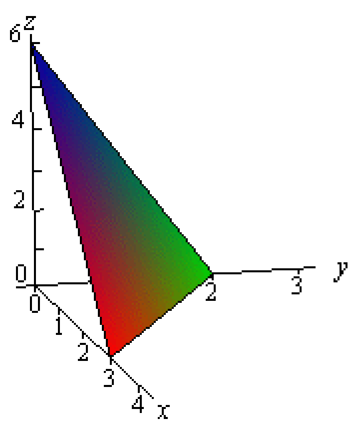
Example 2 Evaluate $\iiint_E 2x\,dV$ where E is the region under the plane $2x + 3y + z = 6$ that lies in the first octant.

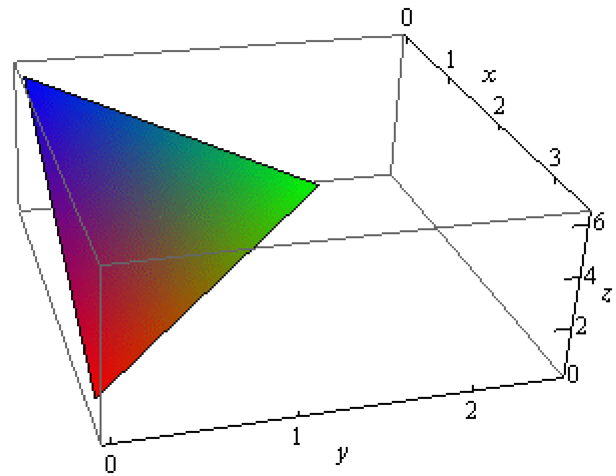
Solution

We should first define *octant*. Just as the two-dimensional coordinates system can be divided into four quadrants the three-dimensional coordinate system can be divided into eight octants. The first octant is the octant in which all three of the coordinates are positive.

Here is a sketch of the plane in the first octant.

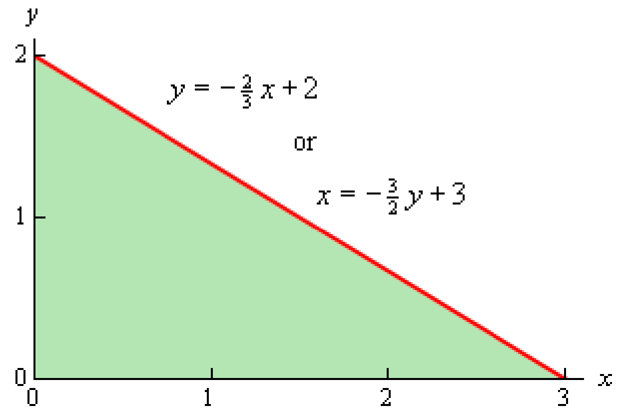
Volume using triple integrals





We now need to determine the region D in the xy-plane. We can get a visualization of the region by pretending to look straight down on the object from above. What we see will be the region D in the xy-plane. So D will be the triangle with vertices at $(0,0)$, $(3,0)$, and $(0,2)$. Here is a sketch of D.

Volume using triple integrals



Now we need the limits of integration. Since we are under the plane and in the first octant (so we're above the plane $z = 0$) we have the following limits for z.

$$0 \le z \le 6 - 2x - 3y$$

We can integrate the double integral over D using either of the following two sets of inequalities.

$$0 \le x \le 3 \qquad\qquad 0 \le x \le -\frac{3}{2}y + 3$$

$$0 \le y \le -\frac{2}{3}x + 2 \qquad\qquad 0 \le y \le 2$$

Volume using triple integrals

Since neither really holds an advantage over the other we'll use the first one. The integral is then,

$$\begin{aligned}\iiint_E 2x\,dV &= \iint_D \left[\int_0^{6-2x-3y} 2x\,dz\right] dA \\ &= \iint_D 2xz\Big|_0^{6-2x-3y}\,dA \\ &= \int_0^3 \int_0^{-\frac{2}{3}x+2} 2x(6-2x-3y)\,dy\,dx \\ &= \int_0^3 \left(12xy-4x^2y-3xy^2\right)\Big|_0^{-\frac{2}{3}x+2}\,dx \\ &= \int_0^3 \frac{4}{3}x^3-8x^2+12x\,dx \\ &= \left(\frac{1}{3}x^4-\frac{8}{3}x^3+6x^2\right)\Big|_0^3 \\ &= 9\end{aligned}$$

.

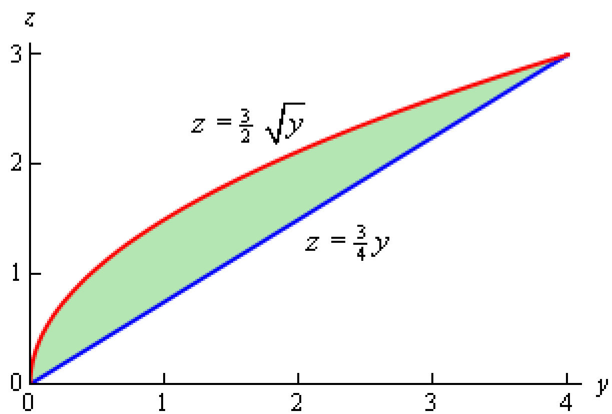
Volume using triple integrals

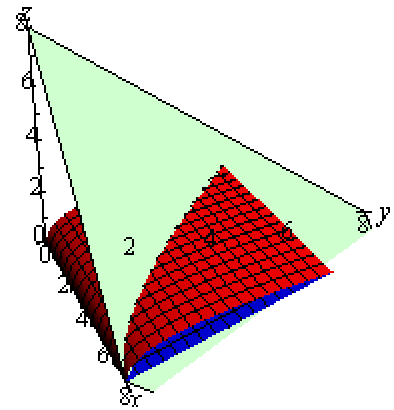
= 9

Example 3 Determine the volume of the region that lies behind the plane $x + y + z = 8$ $x+y+z=8$ and in front of the region in the yz-plane that is bounded by $z = \frac{3}{2}\sqrt{y}$ $z = \frac{3}{2}\sqrt{y}$ and $z = \frac{3}{4}y$ $z = \frac{3}{4}y$.

Solution

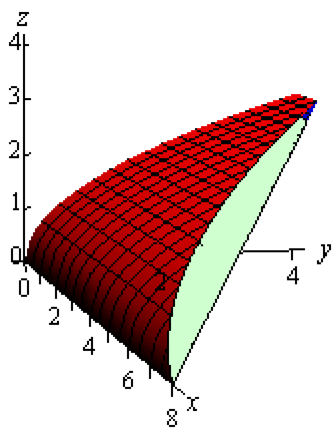
In this case we've been given D and so we won't have to really work to find that. Here is a sketch of the region D as well as a quick sketch of the plane and the curves defining D projected out past the plane so we can get an idea of what the region we're dealing with looks like.

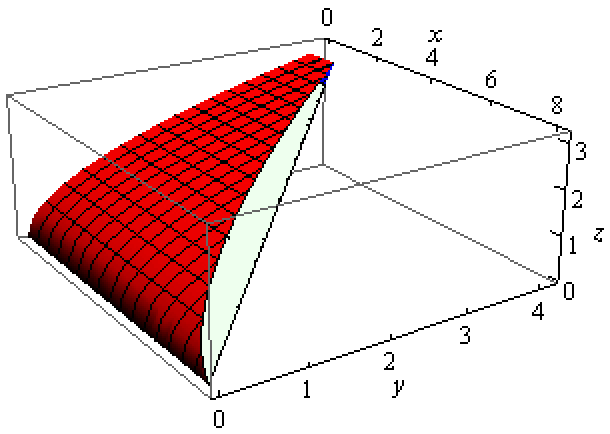




Volume using triple integrals

Now, the graph of the region above is all okay, but it doesn't really show us what the region is. So, here is a sketch of the region itself.





Here are the limits for each of the variables.

$$0 \le y \le 4$$

$$\frac{3}{4}y \le z \le \frac{3}{2}\sqrt{y}$$

$$0 \le x \le 8 - y - z$$

MODULE-V

VECTOR CALCULUS

CONTENT

INTRODUCTION OF SCALAR AND VECTOR POINT FUNCTIONS

OBJECTIVE:

Definitions of Gradient, divergent and curl

OUTCOME:

Students get to understand the concept of Vector functions and its application on solving Problems.

DIFFERENTIATION OF A VECTOR FUNCTION

Let S be a set of real numbers. Corresponding to each scalar t ε S, let there be associated a unique vector $\bar{f}$. Then $\bar{f}$ is said to be a vector (vector valued) function. S is called the domain of $\bar{f}$. We write $\bar{f}$ = $\bar{f}$ (t).

Let $\bar{i}, \bar{j}, \bar{k}$ be three mutually perpendicular unit vectors in three dimensional space. We can write $\bar{f}$ = $\bar{f}$ (t)= $f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$, where $f_1(t), f_2(t), f_3(t)$ are real valued functions (which are called components of $\bar{f}$). (we shall assume that $\bar{i}, \bar{j}, \bar{k}$ are constant vectors).

PROPERTIES

1) $\frac{\partial}{\partial t}(\phi \bar{a}) = \frac{\partial \phi}{\partial t}\bar{a} + \phi \frac{\partial \bar{a}}{\partial t}$

2). If λ is a constant, then $\frac{\partial}{\partial t}(\lambda \bar{a}) = \lambda \frac{\partial \bar{a}}{\partial t}$

3). If $\bar{c}$ is a constant vector, then $\frac{\partial}{\partial t}(\phi \bar{c}) = \bar{c}\frac{\partial \phi}{\partial t}$

4). $\frac{\partial}{\partial t}(\bar{a} \pm \bar{b}) = \frac{\partial \bar{a}}{\partial t} \pm \frac{\partial \bar{b}}{\partial t}$

5). $\frac{\partial}{\partial t}(\bar{a}.\bar{b}) = \frac{\partial \bar{a}}{\partial t}.\bar{b} + \bar{a}.\frac{\partial \bar{b}}{\partial t}$

6). $\frac{\partial}{\partial t}(\bar{a} \times \bar{b}) = \frac{\partial \bar{a}}{\partial t} \times \bar{b} + \bar{a} \times \frac{\partial \bar{b}}{\partial t}$

7). Let $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$, where f_1, f_2, f_3 are differential scalar functions of more than one variable, Then $\frac{\partial \bar{f}}{\partial t} = \bar{i}\frac{\partial f_1}{\partial t} + \bar{j}\frac{\partial f_2}{\partial t} + \bar{k}\frac{\partial f_3}{\partial t}$ (treating $\bar{i}, \bar{j}, \bar{k}$ as fixed directions)

VECTOR DIFFERENTIAL OPERATOR

Def. The vector differential operator ∇(read as del) is defined as

$\nabla \equiv \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$. This operator possesses properties analogous to those of ordinary vectors as well as differentiation operator. We will define now some quantities known as "**gradient", "divergence"** and **"curl"** involving this operator ∇.

GRADIENT OF A SCALAR POINT FUNCTION

Let $\phi(x,y,z)$ be a scalar point function of position defined in some region of space. Then the vector function $\bar{i}\frac{\partial \phi}{\partial x}+\bar{j}\frac{\partial \phi}{\partial y}+\bar{k}\frac{\partial \phi}{\partial z}$ is known as the gradient of ϕ or $\nabla\phi$

$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\phi = \bar{i}\frac{\partial \phi}{\partial x}+\bar{j}\frac{\partial \phi}{\partial y}+\bar{k}\frac{\partial \phi}{\partial z}$$

PROPERTIES OF GRADIENT FUNCTION

1) If f and g are two scalar functions then grad(f $\pm$g)= grad f $\pm$ grad g
2) The necessary and sufficient condition for a scalar point function to be constant is that $\nabla f = \bar{0}$
3) grad(fg) = *f*(grad g)+g(grad f)
4) If c is a constant, grad (c*f*) = *c*(grad *f*)
5) grad $\left(\frac{f}{g}\right) = \frac{g(grad\ f) - f(grad\ g)}{g^2}, (g \neq 0)$
6) Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$. Then $d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$ if ϕ is any scalar point function,

then $d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = \left(\bar{i}\frac{\partial \Phi}{\partial x} + \bar{j}\frac{\partial \Phi}{\partial y} + \bar{k}\frac{\partial \Phi}{\partial z}\right).\left(\bar{i}dx + \bar{j}dy + \bar{k}dz\right) = \nabla\Phi . d\bar{r}$

DIRECTIONAL DERIVATIVE

Let $\phi(x,y,z)$ be a scalar function defined throughout some region of space. Let this function have a value ϕ at a point P whose position vector referred to the origin O is $\overline{OP} = \bar{r}$. Let $\phi+\Delta\phi$ be the value of the function at neighboring point Q. If $\overline{OQ} = \bar{r} + \Delta\bar{r}$. Let Δr be the length of $\Delta\bar{r}$

$\frac{\Delta\phi}{\Delta r}$ gives a measure of the rate at which ϕ change when we move from P to Q. The limiting value of $\frac{\Delta\phi}{\Delta r}$ $as\ \Delta r \to 0$ is called the derivative of ϕ in the direction of $\overline{PQ}$ or simply directional derivative of ϕ at P and is denoted by $d\phi/dr$.

CONTENT

- SCALAR AND VECTOR POINT FUNCTIONS
- GRADIENT AND DIVERGENCE
- CURL OF A VECTOR
- DIRECTIONAL DERIVATIVE
- SOLENOIDAL VECTOR
- IRROTATIONAL VECTOR

SCALAR AND VECTOR POINT FUNCTIONS

Scalar and vector point functions: Consider a region in three dimensional space. To each point p(*x,y,z*), suppose we associate a unique real number (called scalar) say ϕ. This ϕ(*x,y,z*) is called a scalar point function on the region. Similarly if to each point *p*(*x,y,z*) we associate a unique vector $\bar{f}(x, y, z)$ then $\bar{f}$ is called a ***vector point function***.

SCALAR AND VECTOR POINT FUNCTIONS

For example take a heated solid. At each point p(x,y,z)of the solid, there will be temperature T(x,y,z). This T is a scalar point function. Suppose a particle (or a very small insect) is tracing a path in space. When it occupies a position p(x,y,z) in space, it will be having some speed, say, v. This **speed** v is a scalar point function.

Consider a particle moving in space. At each point P on its path, the particle will be having a velocity $\overline{v}$ which is vector point function. Similarly, the acceleration of the particle is also a vector point function.

In a magnetic field, at any point P(x,y,z) there will be a magnetic force $\bar{f}(x, y, z)$ This is called magnetic force field. This is also an example of a vector point function.

VECTOR DIFFERENTIAL OPERATOR

Def. The vector differential operator ∇(read as del) is defined as $\nabla \equiv \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$

GRADIENT OF A SCALAR POINT FUNCTION

Let $\phi(x,y,z)$ be a scalar point function of position defined in some region of space. Then the vector function $\bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$ is known as the gradient of ϕ or $\nabla\phi$

DIRECTIONAL DERIVATIVE

Find the directional derivative of xyz^2+xz at (1, 1 ,1) in a direction of the normal to the surface $3xy^2+y= z$ at (0,1,1).

Sol:- Let $f(x, y, z) \equiv 3xy^2+y- z = 0$

Let us find the unit normal e to this surface at (0,1,1).

Then

$$\frac{\partial f}{\partial x} = 3y^2, \quad \frac{\partial f}{\partial y} = 6xy + 1, \quad \frac{\partial f}{\partial z} = -1.$$

$\nabla f = 3y^2 i+(6xy+1)j-k$

$(\nabla f)_{(0,1,1)} = 3\bar{i} + \bar{j} - \bar{k} = \bar{n}$

$$\bar{e} = \frac{\bar{n}}{|\bar{n}|} = \frac{3i + j - k}{\sqrt{9+1+1}} = \frac{3i + j - k}{\sqrt{11}}$$

DIVERGENCE OF A VECTOR

Let $\bar{f}$ be any continuously differentiable vector point function. Then $\bar{i}.\frac{\partial \bar{f}}{\partial x}+\bar{j}.\frac{\partial \bar{f}}{\partial y}+\bar{k}.\frac{\partial \bar{f}}{\partial z}$ is called the divergence of $\bar{f}$ and is written as div $\bar{f}$.

i.e., div $\bar{f}$ $=\bar{i}.\frac{\partial \bar{f}}{\partial x}+\bar{j}.\frac{\partial \bar{f}}{\partial y}+\bar{k}.\frac{\partial \bar{f}}{\partial z}=\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right).\bar{f}$

Hence we can write div $\bar{f}$ as

div $\bar{f}=\nabla.\bar{f}$ This is a scalar point function.

Theorem 1: If the vector $\bar{f}=f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$, then div $\bar{f}$ = $\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$

Prof: Given $\bar{f}=f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$

$\frac{\partial \bar{f}}{\partial x}=\bar{i}\frac{\partial f_1}{\partial x}+\bar{j}\frac{\partial f_2}{\partial x}+\bar{k}\frac{\partial f_3}{\partial x}$

Also $\bar{i}.\frac{\partial \bar{f}}{\partial x}=\frac{\partial f_1}{\partial x}$. Similarly $\bar{j}.\frac{\partial \bar{f}}{\partial y}=\frac{\partial f_2}{\partial y}$ and $\bar{k}.\frac{\partial \bar{f}}{\partial z}=\frac{\partial f_3}{\partial z}$

We have div $\bar{f}=\sum \bar{i}.\left(\frac{\partial \bar{f}}{\partial x}\right)=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$

Note : If $\bar{f}$ is a constant vector then $\frac{\partial f_1}{\partial x},\frac{\partial f_2}{\partial y},\frac{\partial f_3}{\partial z}$ are zeros.

$\therefore$ div $\bar{f}$ =0 for a constant vector $\bar{f}$.

PHYSICAL INTERPRETATION OF DIVERGENCE

Depending upon $\bar{f}$ in a physical problem, we can interpret $div\ \bar{f} = \nabla . \bar{f}$

Suppose $\bar{F}$(x,y,z,t) is the velocity of a fluid at a point(x,y,z) and time 't'. Though time has no role in computing divergence, it is considered here because velocity vector depends on time.

Imagine a small rectangular box within the fluid as shown in the figure. We would like to measure the rate per unit volume at which the fluid flows out at any given time. The divergence of $\bar{F}$ measures the outward flow or expansions of the fluid from their point at any time. This gives a physical interpretation of the divergence.

SOLENOIDAL VECTOR

A vector point function $\bar{f}$ is said to be solenoidal if div $\bar{f}$ =0.

Find div $\bar{f} = r^n \bar{r}$ Find n if it is solenoidal?

Sol: Given $\bar{f} = r^n \bar{r}$. where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$

We have $r^2 = x^2+y^2+z^2$

Differentiating partially w.r.t. x , we get

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$\bar{f} = r^n (x\bar{i} + y\bar{j} + z\bar{k})$

div $\bar{f} = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$

$= nr^{n-1}\frac{\partial r}{\partial x}x + r^n + nr^{n-1}\frac{\partial r}{\partial y}y + r^n + nr^{n-1}\frac{\partial r}{\partial z}z + r^n$

$= nr^{n-1}\left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right] + 3r^n = nr^{n-1}\frac{(r^2)}{r} + 3r^n = nr^n + 3r^n = (n+3)r^n$

Let $\bar{f} = r^n \bar{r}$ be solenoidal. Then div $\bar{f} = 0$

$(n+3)r^n = 0 \Rightarrow n = -3$

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Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ *and* $\frac{\partial r}{\partial z} = \frac{z}{r}$

$\bar{f} = r^n \left(x\bar{i} + y\bar{j} + z\bar{k} \right)$

div $\bar{f} = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$

$= nr^{n-1}\frac{\partial r}{\partial x}x + r^n + nr^{n-1}\frac{\partial r}{\partial y}y + r^n + nr^{n-1}\frac{\partial r}{\partial z}z + r^n$

$= nr^{n-1}\left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right] + 3r^n = nr^{n-1}\frac{(r^2)}{r} + 3r^n = nr^n + 3r^n = (n+3)r^n$

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CURL OF A VECTOR

CURL OF A VECTOR

Def: Let $\bar{f}$ be any continuously differentiable vector point function. Then the vector function defined by $\bar{i} \times \frac{\partial \bar{f}}{\partial x} + \bar{j} \times \frac{\partial \bar{f}}{\partial y} + \bar{k} \times \frac{\partial \bar{f}}{\partial z}$ is called curl of $\bar{f}$ and is denoted by curl $\bar{f}$ or $(\nabla \times \bar{f})$.

Curl $\bar{f}$ = $\bar{i} \times \frac{\partial \bar{f}}{\partial x} + \bar{j} \times \frac{\partial \bar{f}}{\partial y} + \bar{k} \times \frac{\partial \bar{f}}{\partial z} = \sum \left(\bar{i} \times \frac{\partial \bar{f}}{\partial x} \right)$

Theorem 1: If $\bar{f}$ is differentiable vector point function given by $\bar{f} = f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}$ then curl $\bar{f}$ =

$$\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \bar{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \bar{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \bar{k}$$

CURL OF A VECTOR

Theorem 1: If $\bar{f}$ is differentiable vector point function given by $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ then curl $\bar{f}$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\bar{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\bar{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\bar{k}$$

Proof : curl $\bar{f} = \sum \bar{i} \times \frac{\partial}{\partial x}(\bar{f}) = \sum \bar{i} \times \frac{\partial}{\partial x}(f_1\bar{i} + f_2\bar{j} + f_3\bar{k}) = \sum \left(\frac{\partial f_2}{\partial x}\bar{k} - \frac{\partial f_3}{\partial x}\bar{j}\right)$

$$= \left(\frac{\partial f_2}{\partial x}\bar{k} - \frac{\partial f_3}{\partial x}\bar{j}\right) + \left(\frac{\partial f_3}{\partial y}\bar{i} - \frac{\partial f_1}{\partial y}\bar{k}\right) + \left(\frac{\partial f_1}{\partial z}\bar{j} - \frac{\partial f_2}{\partial z}\bar{i}\right)$$

$$= \bar{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \bar{j}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \bar{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

Note : (1) The above expression for curl $\bar{f}$ can be remembered easily through the representation.

$$\textbf{curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \nabla \mathbf{x}\ \bar{f}$$

Note (2) : If $\bar{f}$ is a constant vector then curl $\bar{f} = \bar{o}$.

CURL OF A VECTOR

CURL OF A VECTOR

Def: Let $\bar{f}$ be any continuously differentiable vector point function. Then the vector function defined by $\bar{i}\times\frac{\partial \bar{f}}{\partial x}+\bar{j}\times\frac{\partial \bar{f}}{\partial y}+\bar{k}\times\frac{\partial \bar{f}}{\partial z}$ is called curl of $\bar{f}$ and is denoted by curl $\bar{f}$ or (∇x $\bar{f}$).

Curl $\bar{f}$ = $\bar{i}\times\frac{\partial \bar{f}}{\partial x}+\bar{j}\times\frac{\partial \bar{f}}{\partial y}+\bar{k}\times\frac{\partial \bar{f}}{\partial z}=\sum\left(\bar{i}\times\frac{\partial \bar{f}}{\partial x}\right)$

Theorem 1: If $\bar{f}$ is differentiable vector point function given by $\bar{f}=f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$ then curl $\bar{f}$ =

$$\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)\bar{i}+\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)\bar{j}+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)\bar{k}$$

Proof : curl $\bar{f}$ = $\sum \bar{i}\times\frac{\partial}{\partial x}(\bar{f})=\sum \bar{i}\times\frac{\partial}{\partial x}(f_1\bar{i}+f_2\bar{j}+f_3\bar{k})=\sum\left(\frac{\partial f_2}{\partial x}\bar{k}-\frac{\partial f_3}{\partial x}\bar{j}\right)$

$$=\left(\frac{\partial f_2}{\partial x}\bar{k}-\frac{\partial f_3}{\partial x}\bar{j}\right)+\left(\frac{\partial f_3}{\partial y}\bar{i}-\frac{\partial f_1}{\partial y}\bar{k}\right)+\left(\frac{\partial f_1}{\partial z}\bar{j}-\frac{\partial f_2}{\partial z}\bar{i}\right)$$

$$=\bar{i}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)+\bar{j}\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)+\bar{k}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)$$

Note : (1) The above expression for curl $\bar{f}$ can be remembered easily through the representation.

$$\textbf{curl } \bar{f}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_1 & f_2 & f_3\end{vmatrix}=\nabla\text{x } \bar{f}$$

Note (2) : If $\bar{f}$ is a constant vector then curl $\bar{f}=\bar{o}$.

PHYSICAL INTERPRETATION OF CURL

Physical Interpretation of curl

If $\overline{w}$ is the angular velocity of a rigid body rotating about a fixed axis and $\overline{v}$ is the velocity of any point P(x,y,z) on the body, then $\overline{w} = \frac{1}{2}$ curl $\overline{v}$. Thus the angular velocity of rotation at any point is equal to half the curl of velocity vector. This justifies the use of the word "curl of a vector".

2. Irrotational Motion, Irrotational Vector

Any motion in which curl of the velocity vector is a null vector i.e curl $\overline{v} = \overline{o}$ is said to be Irrotational.

Def: A vector $\overline{f}$ is said to be Irrotational if curl $\overline{f} = \overline{0}$.

SCALAR POTENTIAL

If $\bar{f}$ is Irrotational, there will always exist a scalar function $\varphi(x,y,z)$ such that $\bar{f}$ =grad ϕ. This ϕ is called scalar potential of $\bar{f}$.

It is easy to prove that, if $\bar{f}$ = grad ϕ, then curl $\bar{f}$ = 0.

Hence $\nabla x \bar{f} = 0 \Leftrightarrow$ there exists a scalar function ϕ such that $\bar{f} = \nabla\phi$.

This idea is useful when we study the "work done by a force" later.

PROBLEM

If $\bar{f} = xy^2\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$ find curl $\bar{f}$ at the point (1,-1,1).

Sol:- Let $\bar{f} = xy^2\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$. Then

$$\text{curl } \bar{f} = \nabla \text{x } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

=

$$\bar{i}\left(\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right) + \bar{j}\left(\frac{\partial}{\partial z}(xy^2) - \frac{\partial}{\partial x}(-3yz^2)\right) + \bar{k}\left(\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right)$$

$$= \bar{i}\left(-3z^2 - 2x^2z\right) + \bar{j}(0-0) + \bar{k}\,(4xyz - 2xy) = -\left(3z^2 + 2x^2y\right)\bar{i} + (4xyz - 2xy)\bar{k}$$

= curl $\bar{f}$ at (1,-1,1) = $-\bar{i} - 2\bar{k}$.

VECTOR IDENTITY

Prove that div $curl\,\overline{f} = 0$

$Proof$: Let $\overline{f} = f_1\overline{i} + f_2\overline{j} + f_3\overline{k}$

$$\therefore\ curl\,\overline{f} = \nabla \times \overline{f} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\overline{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\overline{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\overline{k}$$

$$\therefore\quad div\ curl\,\overline{f} = \nabla.(\nabla \times \overline{f}) = \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial x \partial z} - \frac{\partial^2 f_3}{\partial y \partial x} + \frac{\partial^2 f_1}{\partial y \partial z} + \frac{\partial^2 f_2}{\partial z \partial x} - \frac{\partial^2 f_1}{\partial z \partial y} = 0$$

Note : Since $div(curl\,\overline{f}) = 0,$ we have $curl\,\overline{f}$ is always solenoidal.

PROBLEM

Find constants a,b,c so that the vector

$\overline{A} = (x+2y+az)\overline{i} + (bx-3y-z)\overline{j} + (4x+cy+2z)\overline{k}$ is Irrotational.

Also find ϕ such that $\overline{A} = \nabla\phi$.

Sol: Given vector is

$\overline{A} = (x+2y+az)\overline{i} + (bx-3y-z)\overline{j} + (4x+cy+2z)\overline{k}$

Vector $\overline{A}$ is Irrotational $\Rightarrow$ curl $\overline{A} = \overline{0}$

$$\Rightarrow \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \overline{0}$$

$$\Rightarrow (c+1)\overline{i} + (a-4)\overline{j} + (b-2)\overline{k} = \overline{0}$$

$$\Rightarrow (c+1)\overline{i} + (a-4)\overline{j} + (b-2)\overline{k} = 0\overline{i} + 0\overline{j} + 0\overline{k}$$

Comparing both sides,

c+1=0, a-4=0, b-2=0

c= -1, a=4,b=2

Now $\overline{A} = (x+2y+4z)\overline{i} + (2x-3y-z)\overline{j} + (4x-y+2z)\overline{k}$, on

substituting the values of a,b,c

PROBLEM

we have $\bar{A} = \nabla\phi$.

$$\Rightarrow \bar{A} = (x+2y+4z)\bar{i} + (2x-3y-z)\,\bar{j} + (4x-y+2z)\,\bar{k} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

Comparing both sides, we have

$\frac{\partial\phi}{\partial x} = x+2y+4z \Rightarrow \phi = x^2/2+2xy+4zx+f_1(y,z)$

$\frac{\partial\phi}{\partial y} = 2x-3y-z \Rightarrow \phi = 2xy-3y^2/2-yz+f_2(z,x)$

$\frac{\partial\phi}{\partial z} = 4x-y+2z \Rightarrow \phi = 4xz-yz+z^2+f_3(x,y)$

Hence $\phi = x^2/2 - 3y^2/2 + z^2 + 2xy + 4zx - yz + c$

LAPLACIAN OPERATOR

Laplacian Operator ∇^2

$\nabla.\nabla\phi = \sum \bar{i}.\frac{\partial}{\partial x}\left(\bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}\right) = \sum \frac{\partial^2 \phi}{\partial x^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2 \phi$

Thus the operator $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called Laplacian operator.

Note : (i). $\nabla^2\phi = \nabla.(\nabla\phi)$ = div(grad ϕ)

(ii). If $\nabla^2\phi=0$ then ϕ is said to satisfy Laplacian equation. This ϕ is called a harmonic function

Find div $\bar{F}$, where $\bar{F}$ = grad $(x^3+y^3+z^3-3xyz)$

Sol: Let $\phi = x^3+y^3+z^3-3xyz$. Then

$\bar{F}$ = grad ϕ

$= \sum i\frac{\partial \phi}{\partial x} = 3(x^2 - yz)\bar{i} + 3(y^2 - zx)\bar{j} + 3(x^2 - xy)\bar{k} = F_1 i + F_2 j + F_3 k \ (say)$

$\therefore$ div $\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$ = 6x+6y+6z= 6(x+y+z)

i.e div[grad$(x^3+y^3+z^3-3xyz)$]= $\nabla^2(x^3+y^3+z^3-3xyz)$= 6($x$+$y$+$z$).

VECTOR IDENTITY

Prove that div $curl\,\overline{f} = 0$

$\Pr oof\ :\ Let\ \ \overline{f} = f_1\overline{i} + f_2\overline{j} + f_3\overline{k}$

$$\therefore\ curl\ \overline{f} = \nabla \times \overline{f} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\overline{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\overline{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\overline{k}$$

$$\therefore\ \ div\ \ curl\ \overline{f} = \nabla.(\nabla \times \overline{f}) = \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \frac{\partial^2 f_3}{\partial x\partial y} - \frac{\partial^2 f_2}{\partial x\partial z} - \frac{\partial^2 f_3}{\partial y\partial x} + \frac{\partial^2 f_1}{\partial y\partial z} + \frac{\partial^2 f_2}{\partial z\partial x} - \frac{\partial^2 f_1}{\partial z\partial y} = 0$$

Note : Since $div(curl\,\overline{f}) = 0,$ we have $curl\,\overline{f}$ is always solenoidal.

PROBLEM

If $\bar{F} = (x^2-27)\,\bar{i} - 6yz\,\bar{j} + 8xz^2\,\bar{k}$, evaluate $\int_C \bar{F}\cdot d\bar{r}$ from the point (0,0,0) to the point (1,1,1) along the Straight line from (0,0,0) to (1,0,0), (1,0,0) to (1,1,0) and (1,1,0) to (1,1,1).

Solution : Given $\bar{F} = (x^2-27)\,\bar{i} - 6yz\,\bar{j} + 8xz^2\,\bar{k}$

Now $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

$\therefore \quad \bar{F}\cdot d\bar{r} = (x^2-27)dx - (6yz)dy + 8xz^2dz$

(i) Along the straight line from O = (0,0,0) to A = (1,0,0)

Here y =0 =z and dy=dz=0. Also x changes from 0 to 1.

$$\therefore \int_{OA} \bar{F}\cdot d\bar{r} = \int_o^1 (x^2-27)dx = \left[\frac{x^3}{3} - 27x\right]_0^1 = \frac{1}{3} - 27 = \frac{-80}{3}$$

PROBLEM

(i) Along the straight line from A = (1,0,0) to B = (1,1,0)

Here x =1, z=0 $\Rightarrow$ dx=0, dz=0. y changes from 0 to 1.

$$\therefore \int_{AB} \bar{F} \cdot d\bar{r} = \int_{y=0}^{1} (-6yz)\,dy = 0$$

(ii)Along the straight line from B = (1,1,0) to C = (1,1,1)

x =1 =y $\Rightarrow$ dx=dy=0 and z changes from 0 to 1.

$$\therefore \int_{BC} \bar{F} \cdot d\bar{r} = \int_{z=0}^{1} 8xz^2\,dz = \int_{z=0}^{1} 8xz^2\,dz = \left[\frac{8z^3}{3}\right]_0^1 = \frac{8}{3}$$

$$(i) + (ii) + (iii) \Rightarrow \int_{C} \bar{F} \cdot d\bar{r} = \frac{88}{3}$$

PROBLEM

If $\bar{F}=(5xy-6x^2)\bar{i}+(2y-4x)\bar{j}$, evaluate $\int_C \bar{F}\cdot d\bar{r}$

along the curve C in xy-plane $y=x^3$ from

(1,1) to (2,8).

Solution : Given $\bar{F}=(5xy-6x^2)\bar{i}+(2y-4x)\bar{j}$,-------(1)

Along the curve $y=x^3$, $dy=3x^2\,dx$

$\therefore \quad \bar{F}=(5x^4-6x^2)\bar{i}+(2x^3-4x)\bar{j}$, [Putting $y=x^3$ in (1)]

$d\bar{r}=dx\bar{i}+dy\bar{j}=dx\bar{i}+3x^2dx\,\bar{j}$

$\therefore \quad \bar{F}.d\bar{r}=[(5x^4-6x^2)\bar{i}+(2x^3-4x)\bar{j}].\left[dx\,\bar{i}+3x^2dx\,\bar{j}\right]$

PROBLEM

$= (5x^4 - 6x^2)\,dx + (2x^3 - 4x)3x^3 dx$

$= (6x^5 + 5x^4 - 12x^3 - 6x^2)dx$

Hence $\int_{y=x^3} \bar{F} \cdot d\bar{r} = \int_1^2 (6x^5 + 5x^4 - 12x^3 - 6x^2)\,dx$

$= \left(6.\frac{x^6}{6} + 5.\frac{x^5}{5} - 12.\frac{x^4}{4} - 6.\frac{x^3}{4}\right) = \left(x^6 + x^5 - 3x^4 - 2x^3\right)_1^2$

$= 16(4+2-3-1) - (1+1-3-2) = 32+3 = 35$

PROBLEM

$= (5x^4 - 6x^2)\,dx + (2x^3 - 4x)3x^3dx$

$= (6x^5+5x^4-12x^3 -6x^2)dx$

Hence $\displaystyle\int_{y=x^3} \bar{F}\cdot d\bar{r} = \int_1^2 (6x^5 + 5x^4 - 12x^3 - 6x^2)\,dx$

$= \left(6.\frac{x^6}{6} + 5.\frac{x^5}{5} - 12.\frac{x^4}{4} - 6.\frac{x^3}{4}\right) = \left(x^6 + x^5 - 3x^4 - 2x^3\right)_1^2$

$= 16(4+2-3-1) - (1+1-3-2) = 32+3 = 35$

PROBLEM

Hence work done = $\int_0^{2\pi} \bar{F} . d\bar{r}$ = $\int_0^{2\pi}$ (t sin t + cos^2 t – sin t) dt

$$\left[t(-\cos t)\right]_0^{2\pi} - \int_0^{2\pi} (-\sin t)\,dt + \int_0^{2\pi} \frac{1+\cos 2t}{2}\,dt - \int_0^{2\pi} \sin t \text{ dt}$$

$$= -2\pi - (\cos t)_0^{2\pi} + \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right)_0^{2\pi} + (\cos t)_0^{2\pi}$$

=

$$-2\pi - (1-1) + \frac{1}{2}(2\pi) + (1-1) = -2\pi + \pi = -\pi$$

Surface integral

$\int_S \bar{F}\cdot\bar{n}\, ds$ is called surface integral

PROBLEM

Evaluate $\int \bar{F}.\bar{n}\,dS$ where $\bar{F}$ = zi + xj – $3y^2zk$ and S is the surface $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5.

Sol. The surface S is $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5.

Let $x^2 + y^2 = 16$

Then $\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = 2x\bar{i} + 2y\bar{j}$

unit normal $\bar{n} = \frac{\nabla\phi}{|\nabla\varphi|} = \frac{x\bar{i} + y\bar{j}}{4}\ (\because\ x^2 + y^2 = 16)$

Let R be the projection of S on yz-plane

Then $\int_S \bar{F}.\bar{n}\,dS = \iint_R \bar{F}.\bar{n}\frac{dydz}{|\bar{n}.\bar{i}|}$ …………….. *

PROBLEM

Given $\bar{F} = zi + xj - 3y^2zk$

$\therefore \quad \bar{F} \cdot \bar{n} = \frac{1}{4}(xz + xy)$

and $\bar{n} \cdot \bar{i} = \frac{x}{4}$

In yz-plane, x = 0, y = 4

In first octant, y varies from 0 to 4 and z varies from 0 to 5.

$$\int_S \bar{F}.n\,dS = \int_{y=0}^{4} \int_{z=0}^{5} \left(\frac{xz + xy}{4} \right) \frac{dy\,dz}{\left| \frac{x}{4} \right|}$$

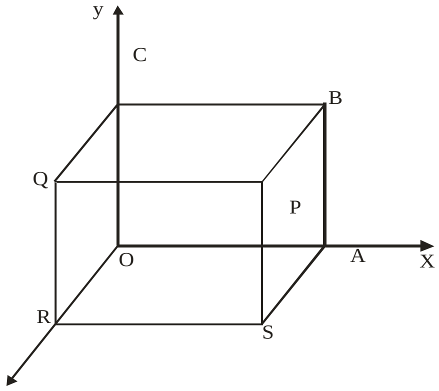
$$= \int_{y=0}^{4} \int_{z=0}^{5} (y + z)\,dz\,dy$$

$$= 90.$$

PROBLEM

If $\bar{F} = zi + xj - 3y^2zk$, evaluate $\int_S \bar{F}.\bar{n}\,dS$ where S is the surface of the cube bounded by x = 0, x = a, y = 0, y= a, z = 0, z = a.

Sol. Given that S is the surface of the x = 0, x = a, y = 0, y = a, z = 0, z = a, and $\bar{F} = zi + xj - 3y^2zk$ we need to evaluate $\int_S \bar{F}.\bar{n}\,dS$.



PROBLEM

(i) For OABC

Eqn is z = 0 and dS = dxdy

$$\bar{n} = -\bar{k}$$

$$\int_{S_1} \bar{F}.\bar{n}\,dS = -\int_{x=0}^{a} -\int_{y=0}^{a} (yz)\,dxdy = 0$$

(ii) For PQRS

Eqn is z = a and dS = dxdy

$$\bar{n} = \bar{k}$$

$$\int_{S_2} \bar{F}.n\,dS = \int_{x=0}^{a} \left(\int_{y=0}^{a} y(a)\,dy \right) dx = \frac{a^4}{2}$$

PROBLEM

(i) For OCQR

Eqn is x = 0, and $\bar{n} = -\bar{i}$, dS = dydz

$$\int_{S_3} \bar{F}.n\,dS = \int_{y=0}^{a} \int_{z=0}^{a} 4xz\,dy\,dz = 0$$

(ii) For ABPS

Eqn is x = a, and $\bar{n} = -\bar{i}$, dS = dydz

$$\int_{S_3} \bar{F}.\bar{n}\,dS = \int_{y=0}^{a} \left(\int_{z=0}^{a} 4az\,dz \right) dy = 2a^4$$

(iii) For OASR

Eqn is y = 0, and $\bar{n} = -\bar{j}$, dS = dxdz

$$\int_{S_5} \bar{F}.n\,dS = \int_{y=0}^{a} \int_{z=0}^{a} y^2\,dz\,dx = 0$$

PROBLEM

For PBCQ

Eqn is y = a, and $\bar{n} = -\bar{j}$, dS = dxdz

$$\int_{S_6} \bar{F}.\bar{n}dS = -\int_{y=0}^{a}\int_{z=0}^{a} y^2 dzdx = 0$$

From (i) – (vi) we get

$$\int_{S_6} \bar{F}.\bar{n}dS = 0 + \frac{a^4}{2} + 0 + 2a^4 + 0 - a4 = \frac{3a^4}{2}$$

GAUSS DIVERGENCE THEOREM

GAUSS'S DIVERGENCE THEOREM

(Transformation between surface integral and volume integral)

Let S be a closed surface enclosing a volume V. If $\bar{F}$ is a continuously differentiable vector point function, then

$$\int_V div\, F\, dv = \int_s \bar{F} \cdot \bar{n}\, dS$$

When $\bar{n}$ is the outward drawn normal vector at any point of S.

PROBLEM

Verify Gauss Divergence theorem for

$\bar{F} = (x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}$ taken over the surface of the cube

bounded by the planes x = y = z = a and coordinate

planes.

Sol: By Gauss Divergence theorem we have

$$\int_S \bar{F}.\bar{n}dS = \int_V div\,\bar{F}\,dv$$

$$RHS = \int_0^a\int_0^a\int_0^a (3x^2 - 2x^2 + 1)dx\,dy\,dz = \int_0^a\int_0^a\int_0^a (x^2 + 1)\,dx\,dy\,dz = \int_0^a\int_0^a \left(\frac{x^3}{3} + x\right)_0^a dy\,dz$$

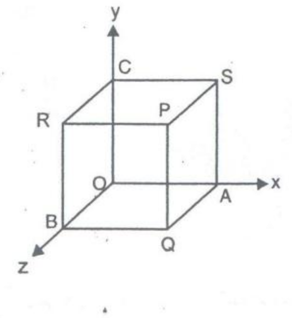
$$\int_0^a\int_0^a \left[\frac{a^3}{3} + a\right]dydz = \int_0^a \left[\frac{a^3}{3} + a\right](y)_0^a dz = \left(\frac{a^3}{3} + a\right)a\int_0^a dz = \left(\frac{a^3}{3} + a\right)\left(a^2\right) = \frac{a^5}{3} + a^3 \text{......(1)}$$

PROBLEM

Verification: We will calculate the value of $\int_S \bar{F}.\bar{n}dS$ over the six faces of the cube.

(i) For S_1 = PQAS; unit outward drawn normal $\bar{n} = \bar{i}$

x=a; ds=dy dz; $0 \le y \le a$, $0 \le z \le a$



$$\therefore \bar{F}.\bar{n} = x^3 - yz = a^3 - yz \text{ since } x = a$$

$$\therefore \int\int_{S_1} \bar{F}.\bar{n}dS = \int_{z=0}^{a}\int_{y=0}^{a} (a^3 - yz)dydz$$

$$= \int_{z=0}^{a} \left[a^3 y - \frac{y^2}{2} z\right]_{y=0}^{a} dz$$

$$= \int_{z=0}^{a} \left(a^4 - \frac{a^2}{2} z\right) dz$$

$$= a^5 - \frac{a^4}{4} \ldots(2)$$

PROBLEM

For S_2 = OCRB; unit outward drawn normal

$$\bar{n} = -\bar{i}$$

x=0; ds=dy dz; 0≤y≤a, y≤z≤a

$$\bar{F}.\bar{n} = -(x^3 - yz) = yz \; since \; x = 0$$

$$\int_{S_2}\int \bar{F}.\bar{n} dS = \int_{z=0}^{a}\int_{y=0}^{a} yz \, dy \, dz = \int_{z=0}^{a}\left[\frac{y^2}{2}\right]_{y=0}^{a} z dz$$

$$= \frac{a^2}{2}\int_{z=0}^{a} z dz = \frac{a^4}{4} \dots (3)$$

PROBLEM

For S_3 = RBQP; Z = a; ds = dxdy; $\bar{n} = \bar{k}$

0≤x≤a, 0≤y≤a

$\bar{F}.\bar{n} = z = a \;\; since\, z = a$

$$\therefore \int\int_{S_3} \overline{F}\,\overline{n} dS = \int_{y=0}^{a}\int_{x=0}^{a} a\,dx\,dy = a^3 .....(4)$$

PROBLEM

Verify divergence theorem for $\bar{F} = x^2 i + y^2 j + z^2 k$ over the surface S of the solid cut off by the plane x+y+z=a in the first octant.

Sol; By Gauss theorem, $\int_s \overline{F}.\overline{n}dS = \int_v div\overline{F}dv$

$Let\ \phi = x + y + z - a\ be\ the\ given\ plane\ then$

$$\frac{\partial \phi}{\partial x} = 1, \frac{\partial \phi}{\partial y} = 1, \frac{\partial \phi}{\partial z} = 1$$

$$\therefore\ grad\,\phi = \sum \bar{i}\frac{\partial \phi}{\partial x} = \bar{i} + \bar{j} + \bar{k}$$

$$Unit\ normal = \frac{grad\ \phi}{|grad\ \phi|} = \frac{\bar{i} + \bar{j} + \bar{k}}{\sqrt{3}}$$

PROBLEM

Let R be the projection of S on xy-plane

Then the equation of the given plane will be x+y=a $\Rightarrow$ y=a-x

Also when y=0, x=a

$$\therefore \int_s \overline{F}.\overline{n}\,dS = \iint_R \frac{\overline{F}.\overline{n}\,dx\,dy}{\left|\overline{n}.\overline{k}\right|}$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x} \frac{\dfrac{x^2+y^2+z^2}{\sqrt{3}}}{1/\sqrt{3}}\,dx\,dy \quad = \int_{0}^{a}\int_{y=0}^{a-x} \left[x^2+y^2+(a-x-y)^2\right]dx\,dy\,[\text{since } x+y+z=a]$$

PROBLEM

$$= \int_0^a \int_0^{a-x} [2x^2 + 2y^2 - 2ax + 2xy - 2ay + a^2] dx\, dy$$

$$= \int_{x=0}^{a} \left[2x^2 y + \frac{2y^3}{3} + xy^2 - 2axy - ay^2 + a^2 y\right]_0^{a-x} dx$$

$$= \int_{x=0}^{a} [2x^2(a-x) + \frac{2}{3}(a-x)^3 + x(a-x)^2 - 2ax(a-x) - a(a-x)^2 + a^2(a-x) dx$$

$$\therefore \int_s \overline{F}.\overline{n} dS = \int_0^a \left(-\frac{5}{3}x^3 + 3ax^2 - 2a^2x + \frac{2}{3}a^3\right) dx = \frac{a^4}{4}, \text{ on simplification}\ldots(1)$$

PROBLEM

Given $\overline{F} = x^2 \overline{i} + y^2 \overline{j} + z^2 \overline{k}$

$$\therefore\ div\ \overline{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) = 2(x + y + z)$$

$$Now \iiint div\,\overline{F}.dv = 2\int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y}(x + y + z)dxdydz$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{a-x}\left[z(x+y) + \frac{z^2}{2}\right]_0^{a-x-y} dx\ dy$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{a-x}(a-x-y)\left[x + y + \frac{a-x-y}{2}\right] dx\ dy$$

PROBLEM

$$= \int_{x=0}^{a}\int_{y=0}^{a-x}(a-x-y)[a+x+y]dx\,dy$$

$$= \int_{0}^{a}\int_{0}^{a-x}[a^2-(x+y)^2]\ dy\,dx = \int_{0}^{a}\int_{0}^{a-x}(a^2-x^2-y^2-2xy)dx\,dy$$

$$= \int_{0}^{a}[a^2y-x^2y-\frac{y^3}{3}-xy^2]_{0}^{a-x}\ dx$$

$$= \int_{0}^{a}(a-x)(2a^2-x^2-ax)dx = \frac{a^4}{4}\dots\dots(2)$$

Hence from (1) and (2), the Gauss Divergence theorem is verified.

II. GREEN'S THEOREM IN A PLANE

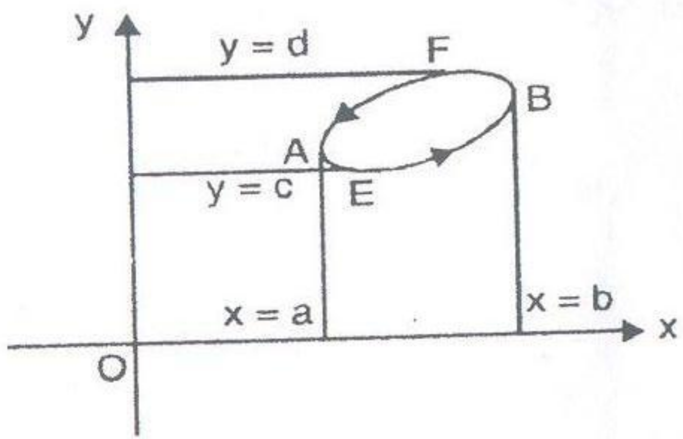
(Transformation Between Line Integral and Surface Integral)

If S is Closed region in xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy.$$

Where C is traversed in the positive(anti clock-wise) direction

GREENS THEOREM IN A PLANE



PROBLEM

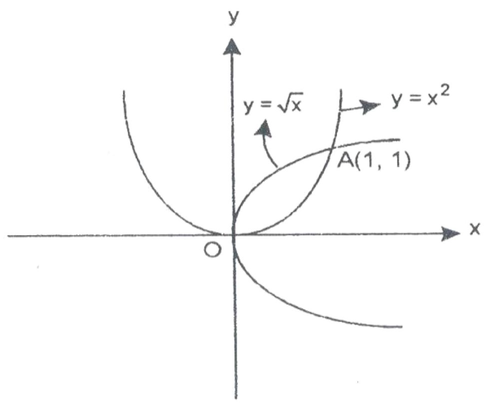
Verify Green's theorem in plane for

$\oint(3x^2-8y^2)dx+(4y-6xy)dy$ where C is the region bounded

by y=$\sqrt{x}$ and y=x^2 .

Solution: Let M=3x^2-$8y^2$ and N=4y-6xy. Then

$\frac{\partial M}{\partial y}=-16y$, $\frac{\partial N}{\partial x}=-6y$



PROBLEM

We have by Green's theorem,

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy.$$

Now $\displaystyle \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy = \iint_R (16y - 6y)\,dx\,dy$

$$= 10 \iint_R y\,dx\,dy = 10 \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} y\,dy\,dx = 10 \int_{x=0}^{1} \left(\frac{y^2}{2} \right)_{x^2}^{\sqrt{x}} dx$$

PROBLEM

Verification:

We can write the line integral along c

=[line integral along y=x^2(from O to A) + [line integral along y^2=x(from A to O)]

=I_1+I_2(say)

Now $I_1 = \int_{x=0}^{1} \{[3x^2 - 8(x^2)^2]dx + [4x^2 - 6x(x^2)]2xdx\}$ $\left[\because y = x^2 \Rightarrow \frac{dy}{dx} = 2x\right]$

PROBLEM

$$=\int_0^1 (3x^3 + 8x^3 - 20x^4)dx = -1$$

And

$$I_2 = \int_1^0 \left[(3x^2 - 8x)dx + \left(4\sqrt{x} - 6x^{3/2}\right)\frac{1}{2\sqrt{x}}dx \right] = \int_1^0 (3x^2 - 11x + 2)dx = \frac{5}{2}$$

$$\therefore \quad I_1 + I_2 = -1 + 5/2 = 3/2 .$$

From(1) and (2), we have

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy.$$

Hence the verification of the Green's theorem.

PROBLEM

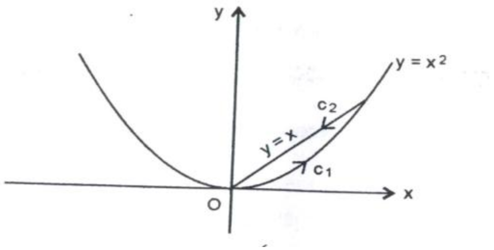
Verify Green's theorem for $\int_c [(xy + y^2)dx + x^2 dy],$ where C is bounded by y=x and

y=x^2

Solution: By Green's theorem, we have

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

Here M=xy +y^2 and N=x^2



PROBLEM

The line y=x and the parabola y=x^2 intersect at O$(0,0)$

and A$(1,1)$

Now $\oint_c M\,dx + N\,dy = \int_{c_1} M\,dx + N\,dy + \int_{c_2} M\,dx + N\,dy \ldots\ldots(1)$ .....(1)

Along c_1 $(i.e.\, y = x^2),$ the line integral is

$$\int_{c_1} M\,dx + N\,dy = \int_{c_1} [x(x^2) + x^4]dx + x^2 d(x^2) \int_c (x^3 + x^4 + 2x^3)dx = \int_0^1 (3x^3 + x^4)dx$$

$$=\left(3.\frac{x^4}{4} + \frac{x^5}{5}\right)_0^1$$

$$=\frac{3}{4} + \frac{1}{5} = \frac{19}{20} \qquad \ldots\ldots.(2)$$

PROBLEM

Along $C_2\ (i.e.\ y = x)$ from $(1,1)$ to $(0,0)$, the line integral is

$$\int_{c_2} M\,dx + N\,dy = \int_{c_2} (x.x + x^2)\,dx + x^2\,dx \quad [\because dy = dx]$$

$$= \int_{c_2} 3x^2 dx = 3\int_1^0 x^2 dx = 3\left(\frac{x^3}{3}\right)_1^0 = (x^3)_1^0 \qquad = 0-1 = -1 \qquad ....(3)$$

PROBLEM

From (1), (2) and (3), we have

$$\int_c M\,dx + N\,dy = \frac{19}{20} - 1 = \frac{-1}{20}$$

...(4)

Now

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (2x - x - 2y)\,dx\,dy$$

$$= \int_0^1 [(x^2 - x^2) - (x^3 - x^4)]\,dx = \int_0^1 (x^4 - x^3)\,dx$$

$$= \left(\frac{x^5}{5} + \frac{x^4}{4}\right)_0^1 = \frac{1}{5} - \frac{1}{4} = \frac{-1}{20}$$

....(5)

From (4) and (5), We have $\oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ dxdy

Hence the verification of the Green's theorem.

PROBLEM

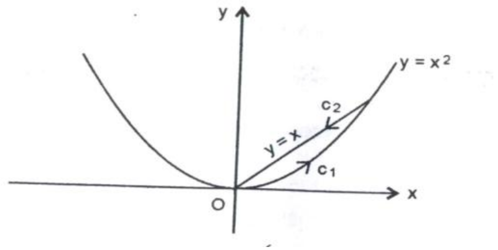
Verify Green's theorem for $\int_c [(xy+y^2)dx + x^2dy]$, where C is bounded by y=x and

y=x^2

Solution: By Green's theorem, we have

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Here M=xy +y^2 and N=x^2



PROBLEM

The line y=x and the parabola y=x^2 intersect at O$(0,0)$

and A$(1,1)$

Now $\oint_c M\,dx + N\,dy = \int_{c_1} M\,dx + N\,dy + \int_{c_2} M\,dx + N\,dy......(1)$ .....(1)

Along c_1 $(i.e.\, y = x^2)$, the line integral is

$$\int_{c_1} M\,dx + N\,dy = \int_{c_1} [x(x^2) + x^4]dx + x^2 d(x^2) \int_c (x^3 + x^4 + 2x^3)dx = \int_0^1 (3x^3 + x^4)dx$$

$=\left(3.\frac{x^4}{4} + \frac{x^5}{5}\right)_0^1$

$=\frac{3}{4} + \frac{1}{5} = \frac{19}{20}$ ....(2)

PROBLEM

Along C_2 $(i.e.\ y = x)$ from $(1,1)$ to $(0,0)$, the line integral is

$$\int_{c_2} M\,dx + N\,dy = \int_{c_2} (x.x + x^2)\,dx + x^2\,dx \quad [\because dy = dx]$$

$$= \int_{c_2} 3x^2\,dx = 3\int_1^0 x^2\,dx = 3\left(\frac{x^3}{3}\right)_1^0 = (x^3)_1^0 \qquad = 0\text{-}1 = -1 \qquad ....(3)$$

PROBLEM

From (1), (2) and (3), we have

$\int_c M\,dx + N\,dy = \frac{19}{20} - 1 = \frac{-1}{20}$

…(4)

Now

$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy = \iint_R (2x - x - 2y)\,dx\,dy$

$= \int_0^1 [(x^2 - x^2) - (x^3 - x^4)]\,dx = \int_0^1 (x^4 - x^3)\,dx$

$= \left(\frac{x^5}{5} + \frac{x^4}{4} \right)_0^1 = \frac{1}{5} - \frac{1}{4} = \frac{-1}{20}$

….(5)

From $(4) and (5), We\ have \oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$ dxdy

Hence the verification of the Green's theorem.

STOKES THEOREM

III. STOKE'S THEOREM

(Transformation between Line Integral and Surface Integral)

Let S be a open surface bounded by a closed, non intersecting curve C. If $\bar{F}$ is any differentieable vector point function then

$\oint_C \bar{F} \cdot d\bar{r} = \int_S curl\ \bar{F} . \bar{n}\, ds$ where c is traversed in the positive direction and

$\bar{n}$ is unit outward drawn normal at any point of the surface.

PROBLEM

Verify Stokes theorem for $F = -y^3 \bar{\imath} + x^3 \bar{\jmath}$,

Where S is the circular disc

$x^2 + y^2 \leq 1, z = 0.$

Solution: Given that $F = -y^3 \bar{\imath} + x^3 \bar{\jmath}$. The boundary of C of S is a circle in xy plane.

$x^2 + y^2 \leq 1, z = 0.$ We use the parametric co-ordinates x=cos$\theta, y = sin\theta, z = 0, 0 \leq \theta \leq 2\pi;$

dx=-sin$\theta\, d\theta$ and dy =cos$\theta\, d\theta$

$$\therefore \oint_c \bar{F}.dr = \int_c F_1 dx + F_2 dy + F_3 dz = \int_c -y^3 dx + x^3 dy$$

$$= \int_0^{2\pi} [-sin^3\theta(-sin\theta) + cos^3\theta cos\theta] d\theta = \int_0^{2\pi} (cos^4\theta + sin^4\theta) d\theta$$

$$= \int_0^{2\pi} (1 - 2sin^2\theta\ cos^2\theta) d\theta$$

$$= \int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi} (2sin\theta\ cos\theta)^2 d\theta$$

$$= \int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi} sin^2 2 d\theta = (2\pi - 0) - \frac{1}{4}\int_0^{2\pi} (1 - cos4\theta) d\theta$$

$$= 2\pi + \left[-\frac{1}{4}\theta + \frac{1}{16} sin4\theta\right]_0^{2\pi} = 2\pi - \frac{2\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$$

PROBLEM

Now $\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = \bar{k}(3x^2 + 3y^2)$

$\therefore \int_s (\nabla \times \bar{F}).\bar{n}ds = 3\int_s (x^2 + y^2)\bar{k}.\bar{n}ds$

We have $(\bar{k}.\bar{n})ds = dxdy$ and R is the region on xy-plane

$\therefore \iint_s (\nabla \times \bar{F}).\bar{n}ds = 3\iint_R (x^2 + y^2)\, dx\, dy$

Put x=r cos$\emptyset, y = r\ sin\emptyset \therefore dxdy = rdr\ d\emptyset$

r is varying from 0 to 1 and 0$\leq \emptyset \leq 2\pi.$

$\therefore \int (\nabla \times \bar{F}).\ \bar{n}ds = 3\int_{\emptyset=0}^{2\pi}\int_{r=0}^{1} r^2$.rdr d$\emptyset = \frac{3\pi}{2}$

L.H.S=R.H.S.Hence the theorem is verified.

PROBLEM

Verify Stokes theorem for $\bar{F} = (2x - y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$ over the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ bounded by the projection of the xy-plane.

Solution: The boundary C of S is a circle in xy plane

i.e $x^2 + y^2$=1, z=0

The parametric equations are x=$\cos\theta, y = \sin\theta, \theta = 0 \to 2\pi$

$$\therefore \quad dx = -\sin\theta \, d\theta, dy = \cos\theta \, d\theta$$

$$\int_c \overline{F}.d\overline{r} = \int_c \overline{F_1}dx + \overline{F_2}dy + \overline{F_3}dz = \int_c (2x - y)dx - yz^2dy - y^2zdz$$

$$= \int_c (2x - y)dx (since\ z = 0\ and\ dz = 0)$$

$$= -\int_0^{2\pi} (2\cos\theta - \sin\theta)\sin\theta \, d\theta = \int_0^{2\pi} \sin^2\theta \, d\theta - \int_0^{2\pi} \sin 2\theta \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{1 - \cos2\theta}{2} d\theta - \int_0^{2\pi} \sin2\theta \, d\theta = \left[\frac{1}{2}\theta - \frac{1}{4}\sin2\theta + \frac{1}{2}.\cos2\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}(2\pi - 0) + 0 + \frac{1}{2}.(\cos4\pi - \cos0) = \pi$$

PROBLEM

Again $\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = \bar{i}(-2yz + 2yz) - \bar{j}(0 - 0) + \bar{k}(0 + 1) = \bar{k}$

$\therefore \int_S (\nabla \times \bar{F}).\bar{n} ds = \int_S \bar{k}.\bar{n} ds = \int_R \int dx dy$

Where R is the projection of S on xy plane and

$\bar{k}.\bar{n} ds = dx dy$

Now

$$\int \int_R dx dy = 4 \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} dy dx = 4 \int_{x=0}^{1} \sqrt{1 - x^2}\, dx = 4 \left[\frac{x}{2} \sqrt{1 - x^2} + \frac{1}{2} \sin^{-1} x \right]_0^1$$

$$= 4 \left[\frac{1}{2} \sin^{-1} 1 \right] = 2 \frac{\pi}{2} = \pi$$

$\therefore The$ Stokes theorem is verified.

PROBLEM

III. <u>STOKE'S THEOREM</u>

(Transformation between Line Integral and Surface Integral)

Let S be a open surface bounded by a closed, non intersecting curve C.

If $\bar{F}$ is any differentieable vector point function then

$\oint_c \bar{F}.d\bar{r} = \int_s curl\ \bar{F}.\bar{n}\, ds$ *where c is traversed in the positive* **direction and**

$\bar{n}$ *is unit outward drawn normal at any point of the surface.*

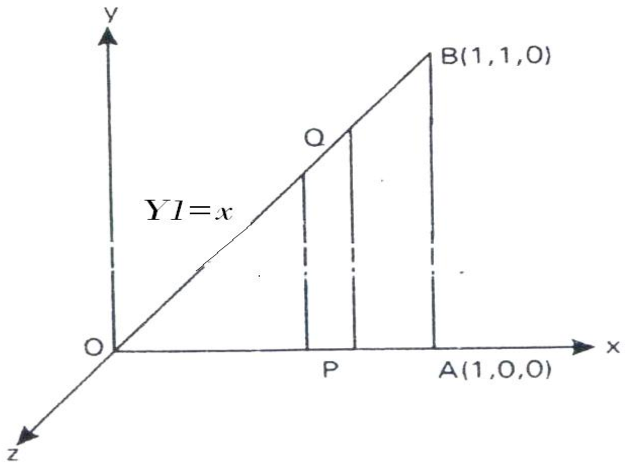
PROBLEM

Evaluate by Stokes theorem $\oint_c (x+y)dx + (2x-z)dy + (y+z)dz$ where C is the boundary of the triangle with vertices (0,0,0), (1,0,0) and (1,1,0).

Solution: Let $\bar{F}.d\bar{r} = \bar{F}.(\bar{\imath}dx + \bar{\jmath}dy + \bar{k}dz) = (x+y)dx + (2x-z)dy + (y+z)dz$

Then $\bar{F} = (x+y)\bar{\imath} + (2x-z)\bar{\jmath} + (y+z)\bar{k}$

By Stokes theorem, $\oint_C \bar{F}.d\bar{r} = \int\int_S curl\,\bar{F}.\bar{n}\,ds$



Where S is the surface of the triangle OAB which lies in the xy plane. Since the z Co-ordinates of O,A and B Are zero. Therefore$\bar{n} = \bar{k}$. Equation of OA is y=0 and that of OB, y=x in the xy plane.

PROBLEM

Now $\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = \bar{k}(3x^2 + 3y^2)$

$\therefore \int_s (\nabla \times \bar{F}).\bar{n} ds = 3 \int_s (x^2 + y^2)\bar{k}.\bar{n} ds$

We have $(\bar{k}.\bar{n}) ds = dx dy$ and R is the region on xy-plane

$\therefore \iint_s (\nabla \times \bar{F}).\bar{n} ds = 3 \iint_R (x^2 + y^2)\, dx\, dy$

Put x=r cos$\emptyset$, $y = r \sin\emptyset \therefore dxdy = r dr\, d\emptyset$

r is varying from 0 to 1 and 0$\leq \emptyset \leq 2\pi$.

$\therefore \int (\nabla \times \bar{F}).\, \bar{n} ds = 3 \int_{\emptyset=0}^{2\pi} \int_{r=0}^{1} r^2$.rdr d$\emptyset = \frac{3\pi}{2}$

L.H.S=R.H.S.Hence the theorem is verified.

PROBLEM

$$\therefore \ curl\ \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = 2\bar{\imath} + \bar{k}$$

$$\therefore \ curl\ \bar{F}.\bar{n}ds = \text{curl } \bar{F}.\bar{K}\ dx\ dy = dx\ dy$$

$$\therefore \oint_c \bar{F}.d\bar{r} = \int\int_s dx\ dy = \int\int_s dA = A = area\ of \text{ the } \Delta\ OAB$$

$$= \frac{1}{2} OA \times AB = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$$