LECTURE NOTES

ON

LINEAR ALGEBRA AND CALCULUS

I B. Tech I semester

Ms. L Indira
Associate Professor

FRESHMAN ENGINEERING
INSTITUTE OF AERONAUTICAL ENGINEERING
(Autonomous)
Dundigal, Hyderabad - 500 043

MODULE-I
THEORY OF MATRICES

Solution for linear systems

Matrix : A system of mn numbers real (or) complex arranged in the form of an ordered set of 'm' rows, each row consisting of an ordered set of 'n' numbers between [ ] (or) ( ) (or) || || is called a matrix of order m xn.

Eg: $\begin{bmatrix} a_{11} & a_{12} \ldots\ldots\ldots a_{1n} \\ a_{21} & a_{12} \ldots\ldots\ldots a_{2n} \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \ldots\ldots \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \ldots \\ a_{m1} & a_{m2} \ldots\ldots\ldots a_{mn} \end{bmatrix}_{m \times n}$ $= [a_{ij}]_{mxn}$ where $1 \le i \le m$, $1 \le j \le n$.

Some types of matrices:

1. **square matrix :** A square matrix A of order n x n is sometimes called as a n- rowed matrix A (or) simply a square matrix of order n

   eg : $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is 2^{nd} order matrix

2. **Rectangular matrix:** A matrix which is not a square matrix is called a rectangular matrix,

   $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$ is a 2x3 matrix

3. **Row matrix:** A matrix of order 1xm is called a row matrix

   eg: $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1 \times 3}$

4. **Column matrix:** A matrix of order nx1 is called a column matrix

   Eg: $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}_{3 \times 1}$

5. **Unit matrix:** if A= $[a_{ij}]_{nxn}$ such that a_{ij} = 1 for i = j and a_{ij} = 0 for i≠j, then A is called a unit matrx.

   Eg:I_2 = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ I_3= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6. **Zero matrix :** it A = $[a_{ij}]_{mxn}$ such that $a_{ij} = 0$ $\forall$ I and j then A is called a zero matrix (or) null matrix

   Eg: O_{2x3}= $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

7. **Diagonal elements in a matrix:** A= $[a_{ij}]_{nxn}$, the elements a_{ij} of A for which i = j. i.e. (a_{11}, a_{22}....a_{nn}) are called the diagonal elements of A

   Eg: A= $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ diagonal elements are 1,5,9

   Note: the line along which the diagonal elements lie is called the principle diagonal of A

8. **Diagonal matrix:** A square matrix all of whose elements except those in leading diagonal are zero is called diagonal matrix.

If $d_1, d_2 \ldots\ldots d_n$ are diagonal elements of a diagonal matrix A, then A is written as A = diag $(d_1, d_2 \ldots . d_n)$

E.g. : A = diag (3, 1,-2)= $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

9. **Scalar matrix:** A diagonal matrix whose leading diagonal elements are equal is called a scalar matrix.

Eg : A= $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

10. **Equal matrices :** Two matrices A = $[a_{ij}]$ and b= $[b_{ij}]$ are said to be equal if and only if (i) A and B are of the same type(order) (ii) $a_{ij} = b_{ij}$ for every i & j

11. **The transpose of a matrix:** The matrix obtained from any given matrix A, by interchanging its rows and columns is called the transpose of A. It is denoted by A^1 (or) A^T.

If A = $[a_{ij}]_{m \times n}$ then the transpose of A is $A^1 = [b_{ji}]_{n \times m}$, where $b_{ji} = a_{ij}$ Also $(A^1)^1 = A$

Note: A^1 and B^1 be the transposes of A and B respectively, then

(i) $(A^1)^1 = A$

(ii) $(A+B)^1 = A^1+B^1$

(iii) $(KA)^1 = KA^1$, K is a scalar

(iv) $(AB)^1 = B^1A^1$

12. **The conjugate of a matrix:** The matrix obtained from any given matrix A, on replacing its elements by corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\overline{A}$

Note: if $\bar{A}$ and $\bar{B}$ be the conjugates of A and B respectively then,

(i) $\overline{(\overline{A})} = A$

(ii) $\overline{(A+B)} = \overline{A}+\overline{B}$

(iii) $\overline{(KA)} = \overline{K}\,\overline{A}$, K is a any complex number

(iv) $\overline{(AB)} = \overline{B}\,\overline{A}$

Eg ; if A= $\begin{bmatrix} 2 & 3i & 2-5i \\ -i & 0 & 4i+3 \end{bmatrix}_{2 \times 3}$ *then* $\overline{A} = \begin{bmatrix} 2 & -3i & 2+5i \\ i & 0 & -4i+3 \end{bmatrix}_{2 \times 3}$

13. The conjugate Transpose of a matrix

The conjugate of the transpose of the matrix A is called the conjugate transpose of A and is denoted by A^θ

Thus $A^\theta = (\bar{A^1})$ where A^1 is the transpose of A. Now A = $[a_{ij}]_{m \times n} \Rightarrow A^\theta = [b_{ij}]_{n \times m}$, where bij = $\bar{a}$ ij i.e. the $(i,j)^{th}$ element of A^θ conjugate complex of the $(j, i)^{th}$ element of A

Eg: if $A = \begin{bmatrix} 5 & 3-i & -2i \\ 0 & 1+i & 4-i \end{bmatrix}_{2X3}$ then $A^{\theta} = \begin{bmatrix} 5 & 0 \\ 3+i & 1-i \\ 2i & 4+i \end{bmatrix}_{3x2}$

Note: $A^{\theta} = \bar{A}^{1} = (\bar{A})^{1}$ and $(A^{\theta})^{\theta} = A$

14.

(i) Upper Triangular matrix: A square matrix all of whose elements below the leading diagonal are zero is called an Upper triangular matrix. i.e, $a_{ij=0}$ for $i>j$

Eg; $\begin{bmatrix} 1 & 3 & 8 \\ 0 & 4 & -5 \\ 0 & 0 & 2 \end{bmatrix}$ is an **Upper triangular matrix**

(ii) **Lower triangular matrix:** A square matrix all of whose elements above the leading diagonal are zero is called a lower triangular matrix. i.e, $a_{ij=0}$ for $i<j$

Eg: $\begin{bmatrix} 4 & 0 & 0 \\ 5 & 2 & 0 \\ 7 & 3 & 6 \end{bmatrix}$ is an **Lower triangular matrix**

(iii) **Triangular matrix:** A matrix is said to be triangular matrix it is either an upper triangular matrix or a lower triangular matrix

15. Symmetric matrix: A square matrix $A = [a_{ij}]$ is said to be symmetric if $a_{ij} = a_{ji}$ for every i and j
Thus A is a symmetric matrix if $A^T = A$

Eg: $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix

16. Skew – Symmetric: A square matrix $A = [a_{ij}]$ is said to be skew – symmetric if $a_{ij} = -a_{ji}$ for every i and j.

E.g. : $\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$ is a skew – symmetric matrix

Thus A is a skew – symmetric iff $A = -A^1$ (or) $-A = A^1$

Note: Every diagonal element of a skew – symmetric matrix is necessarily zero.

Since $a_{ij} = -a_{ij} \Rightarrow a_{ij} = 0$

7. Multiplication of a matrix by a scalar.

Let 'A' be a matrix. The matrix obtain by multiplying every element of A by a scalar K, is called the product of A by K and is denoted by KA (or) AK

Thus: $A + [a_{ij}]_{m \times n}$ then $KA = [ka_{ij}]_{m \times n} = k[a_{ij}]_{m \times n}$

18. Sum of matrices:

Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$ be two matrices. The matrix $C = [c_{ij}]_{m \times n}$ where $c_{ij} = a_{ij}+b_{ij}$ is called the sum of the matrices A and B.

The sum of A and B is denoted by A+B. Thus $[a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [a_{ij}+b_{ij}]_{m\times n}$ and $[a_{ij}+b_{ij}]_{m\times n} = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}$

19. The difference of two matrices: If A, B are two matrices of the same type then A+(-B) is taken as A – B

20. Matrix multiplication: Let A = $[a_{ik}]_{m\times n}$, B = $[b_{kj}]_{nxp}$ then the matrix C = $[c_{ij}]_{mxp}$ where c_{ij} is called the product of the matrices A and B in that order and we write C = AB.

The matrix A is called the pre-factor & B is called the post – factor

Note: If the number of columns of A is equal to the number of rows in B then the matrices are said to be conformable for multiplication in that order.

21. Positive integral powers of a square matrix:

Let A be a square matrix. Then A^2 is defined A.A

Now, by associative law $A^3 = A^2.A = (AA)A$

$= A(AA) = AA^2$

Similarly we have $A^{m-1}A = A\,A^{m-1} = A^m$ where m is a positive integer

Note: $I^n = I$

$O^n = 0$

Note 1: Multiplication of matrices is distributive w.r.t. addition of matrices.

i.e, A(B+C) = AB + AC

(B+C)A = BA+CA

Note 2: If A is a matrix of order mxn then $AI_n = I_nA = A$

22. **Trace of A square matrix :** Let A = $[a_{ij}]_{n\times n}$ the trace of the square matrix A is defined as $\sum_{i=1}^{n} a_{ii}$. And is denoted by 'tr A'

Thus trA = $\sum_{i=1}^{n} a_{ii} = a_{11}+a_{22}+ ....a_{nn}$

Eg : A = $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ then trA = a+b+c

Properties: If A and B are square matrices of order n and λ is any scalar, then

(i) tr (λ A) = λ tr A

(ii) tr (A+B) = trA + tr B

(iii) tr(AB) = tr(BA)

23. Idempotent matrix: If A is a square matrix such that $A^2 = A$ then 'A' is called idempotent matrix

24. Nilpotent Matrix: If A is a square matrix such that $A^m=0$ where m is a +ve integer then A is called nilpotent matrix.

Note: If m is least positive integer such that $A^m = 0$ then A is called nilpotent of index m

25. Involutary : If A is a square matrix such that $A^2 = I$ then A is called involuntary matrix.

26. Orthogonal Matrix: A square matrix A is said to be orthogonal if $AA^1 = A^1A = I$

Examples:

1. Show that A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Sol: Given A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Consider $A.A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\therefore$ A is orthogonal matrix.

2. Prove that the matrix $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ is orthogonal.

Sol: Given A = $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

Then $A^T = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

Consider $A .A^T = \frac{1}{9}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A.A^T = I$

Similarly $A^T.A = I$

Hence A is orthogonal Matrix

3. Determine the values of a, b, c when $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal.

Sol: - For orthogonal matrix $AA^T = I$

So $AA^T = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix} \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = I$

$$\Rightarrow \begin{bmatrix} 4b^2 + c^2 & 2b^2 - c^2 & -2b^2 + c^2 \\ 2b^2 - c^2 & a^2 + b^2 + c^2 & a^2 - b^2 - c^2 \\ -2b^2 + c^2 & a^2 - b^2 - c^2 & a^2 + b^2 + c^2 \end{bmatrix} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving $2b^2-c^2 = 0$, $a^2-b^2-c^2 = 0$

We get $c = \pm\sqrt{2}b$ $a^2 = b^2+2b^2 = 3b^2$

$\Rightarrow a = \pm\sqrt{3}b$

From the diagonal elements of I

$4b^2+c^2 = 1 \Rightarrow 4b^2+2b^2=1$ $(c^2=2b^2)$

$\Rightarrow b = \pm\frac{1}{\sqrt{6}}$

$a = \pm\sqrt{3b}$

$= \pm\frac{1}{\sqrt{2}}$

$b = \pm\frac{1}{\sqrt{6}}$

$c = \pm\sqrt{2b}$

$= \pm\frac{1}{\sqrt{3}}$

27. Determinant of a square matrix:

If A = $\begin{bmatrix} a_{11} & a_{12} \ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22} \ldots\ldots\ldots a_{2n} \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \;\ldots\ldots \\ a_{i1} & a_{i2} \ldots\ldots\ldots a_{in} \\ a_{n1} & a_{n2} \ldots\ldots\ldots a_{nn} \end{bmatrix}_{nxn}$ then $|A| = \begin{vmatrix} a_{11} & a_{12} \ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22} \ldots\ldots\ldots a_{2n} \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \;\ldots\ldots \\ a_{i1} & a_{i2} \ldots\ldots\ldots a_{in} \\ a_{n1} & a_{n2} \ldots\ldots\ldots a_{nn} \end{vmatrix}$

28. Minors and cofactors of a square matrix

Let A =$[a_{ij}]_{n\times n}$ be a square matrix when form A the elements of i^{th} row and j^{th} column are deleted the determinant of (n-1) rowed matrix [Mij] is called the minor of aij of A and is denoted by $|M_{ij}|$

The signed minor $(-1)^{i+j}\,|M_{ij}|$ is called the cofactor of a_{ij} and is denoted by A_{ij}..

If A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$|A| = a_{11}\,|M_{11}| + a_{12}\,|M_{12}| + a_{13}\,|M_{13}|$ (or)

$= a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$

E.g.: Find Determinant of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ by using minors and co-factors.

Sol: A = $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

det A = $1\begin{vmatrix} 3 & -3 \\ -4 & -4 \end{vmatrix} - 1\begin{vmatrix} 1 & -3 \\ -2 & -4 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix}$

=1(-12-12)-1(-4-6)+3(-4+6)

= -24+10+6 = -8

Similarly we find det A by using co-factors also.

Note 1: If A is a square matrix of order n then $|KA| = K^n\,|A|$, where k is a scalar.

Note 2: If A is a square matrix of order n, then $|A| = |A^T|$

Note 3: If A and B be two square matrices of the same order, then $|AB| = |A|\,|B|$

29. Inverse of a Matrix: Let A be any square matrix, then a matrix B, if exists such that AB = BA =I then B is called inverse of A and is denoted by A^{-1}.

Note:1 $(A^{-1})^{-1} = A$

Note 2: $I^{-1} = I$

30. Adjoint of a matrix:

Let A be a square matrix of order n. The transpose of the matrix got from A

By replacing the elements of A by the corresponding co-factors is called the adjoint of A and is denoted by adj A.

Note: For any scalar k, adj(kA) = k^{n-1} adj A

Note: The necessary and sufficient condition for a square matrix to posses' inverse is that $|A| \neq 0$

Note: if $|A| \neq 0$ then $A^{-1} = \frac{1}{|A|}(adj\ A)$

3. Singular and Non-singular Matrices:

A square matrix A is said to be singular if $|A| = 0$. If $|A| \neq 0$ then ' A' is said to be non-singular.

Note: 1. only non-singular matrices possess inverses.

2. The product of non-singular matrices is also non-singular.

Theorem 9: If A, B are invertible matrices of the same order, then

(i). $(AB)^{-1} = B^{-1}A^{-1}$

(ii). $(A^1)^{-1} = (A^{-1})^1$

Proof: (i). we have $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$

$= B^{-1}(I\ B)$

$= B^{-1}B$

$= I$

$(AB)^{-1} = B^{-1}A^{-1}$

(ii). $A^{-1}A = AA^{-1} = I$

Consider $A^{-1}A = I$

⇨ $(A^{-1}A)^1 = I^1$

⇨ $A^1.(A^{-1})^1 = I$

⇨ $(A^1)^{-1} = (A^{-1})^1$

Unitary matrix:

A square matrix A such that $(\bar{A})^T = A^{-1}$

i.e $(\bar{A})^T$ A=A$(\bar{A})^T$ =I

If A^{θ} A=I then A is called Unitary matrix

Theorem: The Eigen values of a Hermitian matrix are real.

Note: The Eigen values of a real symmetric are all real

Corollary: The Eigen values of a skew-Hermitian matrix are either purely imaginary (or) Zero

Theorem 3: The Eigen values of an unitary matrix have absolute value l.

Note 1: From the above theorem, we have "The characteristic root of an orthogonal matrix is unit modulus".

2. The only real Eigen values of unitary matrix and orthogonal matrix can be $\pm$ 1

Theorem 4: Prove that transpose of a unitary matrix is unitary.

PROBLEMS

1) Find the eigen values of A= $\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

Sol: we have A= $\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

So $\overline{A} = \begin{bmatrix} -3i & 2-i \\ -2-i & i \end{bmatrix}$ and $A^T = \begin{bmatrix} 3i & -2+i \\ 2+i & -i \end{bmatrix}$

$\Rightarrow \overline{A} = -A^T$

Thus A is a skew-Hermitian matrix.

$\therefore$ The characteristic equation of A is $|A - \lambda I| = 0$

$\Rightarrow A^T = \begin{vmatrix} 3i-\lambda & -2+i \\ -2+i & -i-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 2i\lambda + 8 = 0$

$\Rightarrow \lambda = 4i, -2i$ are the Eigen values of A

2) Find the Eigen values of $A = \begin{bmatrix} \frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}i \end{bmatrix}$

$Now \ \overline{A} = \begin{bmatrix} -\frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}i \end{bmatrix} and$

$(\overline{A})^T = \begin{bmatrix} -\frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}i \end{bmatrix}$

We can see that $\overline{A}^T.A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Thus A is a unitary matrix

$\therefore$ The characteristic equation is $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} \frac{1}{2}i-\lambda & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}i-\lambda \end{vmatrix} = 0$

Which gives $\lambda = \frac{\sqrt{3}}{2} + i\frac{1}{2} \ and \ \frac{-\sqrt{3}}{2} + \frac{1}{2}i$ and

$\lambda = 1/2\sqrt{3} + 1/2i$

Hence above λ values are Eigen values of A.

3) If A= $\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ **then show that**

A is Hermitian and iA is skew-Hermitian.

Sol: Given A= $\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ then

$\overline{A} = \begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3-i \\ -2+5i & 3+i & 4 \end{bmatrix}$ And $\overline{(A)}^T = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$

$\therefore\ A = \left(\overline{A}\right)^T$ Hence A is Hermitian matrix.

Let B= iA

i.e B= $\begin{bmatrix} 3i & 4+7i & -5-2i \\ -4+7i & -2i & -1+3i \\ 5-2i & 1+3i & 4i \end{bmatrix}$ then

$$\overline{B} = \begin{bmatrix} -3i & 4-7i & -5+2i \\ -4-7i & 2i & -1-3i \\ 5+2i & 1-3i & -4i \end{bmatrix}$$

$$\left(\overline{B}\right)^T = \begin{bmatrix} -3i & -4-7i & 5+2i \\ 4-7i & 2i & 1-3i \\ -5+2i & -1-3i & -4i \end{bmatrix} = (-1)\begin{bmatrix} 3i & 4+7i & -5-2i \\ -4+7i & -2i & -1+3i \\ 5-2i & 1+3i & 4i \end{bmatrix} = -B$$

$\therefore\ \left(\overline{B}\right)^T$ =-B

$\therefore$ B= iA is a skew Hermitian matrix.

4) If A and B are Hermitian matrices, prove that AB-BA is a skew-Hermitian matrix.

Sol: Given A and B are Hermitan matrices

$\therefore\ \left(\overline{A}\right)^T = A$ And $\left(\overline{B}\right)^T = B$ ------------- (1)

Now $\overline{(AB-BA)}^T = \left(\overline{AB} - \overline{BA}\right)^T$

$= \left(\overline{A}\,\overline{B} - \overline{B}\,\overline{A}\right)^T$

$= \left(\overline{A}\,\overline{B}\right)^T - \left(\overline{B}\,\overline{A}\right)^T = \left(\overline{B}\right)^T \left(\overline{A}\right)^T - \left(\overline{A}\right)^T \left(\overline{B}\right)^T$

$= BA - AB$ (By (1))

$= -(AB - BA)$

Hence AB-BA is a skew-Hemitian matrix.

5) Show that A=$\begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$ is unitary if and only if $a^2+b^2+c^2+d^2=1$

Sol: Given A=$\begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$

Then $\overline{A} = \begin{bmatrix} a-ic & -b-id \\ b-id & a+ic \end{bmatrix}$

Hence $A^\theta = \left(\overline{A}\right)^T = \begin{bmatrix} a-ic & b-id \\ -b-id & a+ic \end{bmatrix}$

$\therefore AA^\theta = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}\begin{bmatrix} a-ic & b-id \\ -b-id & a+ic \end{bmatrix}$

$= \begin{pmatrix} a^2+b^2+c^2+d^2 & 0 \\ 0 & a^2+b^2+c^2+d^2 \end{pmatrix}$

$\therefore AA^\theta = I$ if and only if $a^2+b^2+c^2+d^2=1$

6) **Show that every square matrix is uniquely expressible as the sum of a Hermitian matrix and a skew- Hermitian matrix.**

Sol. Let A be any square matrix

Now $\left(A+A^\theta\right)^\theta = A^\theta + \left(A^\theta\right)^\theta$

$= A^\theta + A$

$\left(A+A^\theta\right)^\theta = A + A^\theta \Rightarrow A + A^\theta$ is a Hermitian matrix.

$\therefore \frac{1}{2}\left(A+A^\theta\right)$ is also a Hermitian matrix

Now $\left(A-A^\theta\right)^\theta = A^\theta - \left(A^\theta\right)^\theta$

$= A^\theta - A = -\left(A-A\right)^\theta$

Hence $A - A^\theta$ is a skew-Hermitian matrix

$\therefore \frac{1}{2}\left(A-A^\theta\right)$ is also a skew –Hermitian matrix.

Uniqueness:

Let A =R+S be another such representation of A

Where R is Hermitian and

S is skew-Hermitian

Then $A^{\theta} = (R+S)^{\theta}$

$$= R^{\theta} + S^{\theta}$$
$$= R - S \quad \left(\because R^{\theta} = R, S^{\theta} = -S\right)$$

$$\therefore R = \frac{1}{2}\left(A + A^{\theta}\right) = P \ \ and \ \ S = \frac{1}{2}\left(A - A^{\theta}\right) = Q$$

Hence P=R and Q=S

Thus the representation is unique.

7) Given that A=$\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, show that $(I-A)(I+A)^{-1}$ is a unitary matrix.

Sol: we have $I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$ And

$$I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$$\therefore (I+A)^{-1} = \frac{1}{1-\left(4i^2-1\right)}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

Let $B = (I-A)(I+A)^{-1}$

$$B = \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1+(1-2i)(-1-2i) & -1-2i-1-2i \\ 1-2i+1-2i & (-1-2i)(1-2i)+1 \end{bmatrix}$$

$$B = \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$$

Now $\overline{B} = \frac{1}{6}\begin{bmatrix} -4 & -2+4i \\ 2+4i & -4 \end{bmatrix}$ and $\overline{(B)}^{T} = \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$

$$B\left(\overline{B}\right)^{T} = \frac{1}{36}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$$

$$= \frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\left(\overline{B}\right)^{T} = B^{-1}$$

i.e., B is unitary matrix.

$\therefore (I - A)(I + A)^{-1}$ is a unitary matrix.

8) Show that the inverse of a unitary matrix is unitary.

Sol: Let A be a unitary matrix. Then $AA^{\theta} = I$

i.e $\left(AA^{\theta}\right)^{-1} = I^{-1}$

$\Rightarrow \left(A^{\theta}\right)^{-1} A^{-1} = I$

$\Rightarrow \left(A^{-1}\right)^{\theta} A^{-1} = I$

Thus A^{-1} is unitary.

Problems

1). Express the matrix A as sum of symmetric and skew – symmetric matrices. Where

$$A = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$$

Sol: Given $A = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$

Then $A^T = \begin{bmatrix} 3 & 2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{bmatrix}$

Matrix A can be written as A = ½ $(A+A^T)$+ ½ $(A-A^T)$

⇨P = ½ $(A+A^T) = \frac{1}{2}\left\{\begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & +2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{bmatrix}\right\}$

$$= \frac{1}{2}\begin{bmatrix} 6 & 0 & 11 \\ 0 & 14 & 3 \\ 11 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 11/2 \\ 0 & 7 & 3/2 \\ 11/2 & 3/2 & 0 \end{bmatrix}$$

Q= ½ (A-A^T)

$$= \frac{1}{2}\left\{\begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2 & 1/2 \\ 2 & 0 & -5/2 \\ -1/2 & 5/2 & 0 \end{bmatrix}$$ s

A = P+Q where 'P' is symmetric matrix
'Q' is skew-symmetric matrix.

Sub – Matrix: Any matrix obtained by deleting some rows or columns or both of a given matrix is called is sub matrix.

E.g.: Let A = $\begin{bmatrix} 1 & 5 & 6 & 7 \\ 8 & 9 & 10 & 5 \\ 3 & 4 & 5 & -1 \end{bmatrix}$. Then $\begin{bmatrix} 8 & 9 & 10 \\ 3 & 4 & 5 \end{bmatrix}_{2x3}$ is a sub matrix of A obtained by deleting first row and 4th column of A.

Minor of a Matrix: Let A be an m x n matrix. The determinant of a square sub matrix of A is called a minor of the matrix.

Note: If the order of the square sub matrix is't' then its determinant is called a minor of order is't'.

Eg: A = $\begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}_{4X3}$ be a matrix

$\rightarrow B = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ is a sub-matrix of order '2'

$|B|$ = 2-3 = -1 is a minor of order '2'

$\rightarrow C = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 5 & 6 & 7 \end{bmatrix}$ is a sub-matrix of order '3'

detc= 2(7-12)-1(21-10)+(18-5)
= 2(-5)-1(11)+1(13)
= -10-11+13 = -8 is a minor of order '3'

*Rank of a Matrix:

Let A be m x n matrix. If A is a null matrix, we define its rank to be ‘0’. If A is a non-zero matrix, we say that r is the rank of A if

(i) Every $(r+1)^{th}$ order minor of A is ‘0’ (zero) &

(ii) At least one r^{th} order minor of A which is not zero.

Note: 1. It is denoted by ρ (A)

2. Rank of a matrix is unique.

3. Every matrix will have a rank.

4. If A is a matrix of order mxn,

Rank of A $\leq$ min (m,n)

5. If $\rho(A) = r$ then every minor of A of order r+1, or more is zero.

6. Rank of the Identity matrix I_n is n.

7. If A is a matrix of order n and A is non-singular then $\rho(A) = n$

Important Note:

1. The rank of a matrix is $\leq r$ if all minors of $(r+1)^{th}$ order are zero.
2. The rank of a matrix is $\geq r$, if there is at least one minor of order ‘r’ which is not equal to zero.

PROBLEMS

1. Find the rank of the given matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

Sol: Given matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

→ det A = 1(48-40)-2(36-28)+3(30-28)

= 8-16+6 = -2 ≠ 0

We have minor of order 3

$\rho(A) = 3$

2. Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$

Sol: Given the matrix is of order 3x4

Its Rank ≤ min(3,4) = 3

Highest order of the minor will be 3.

Let us consider the minor $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{bmatrix}$

Determinant of minor is 1(-49)-2(-56)+3(35-48)

= -49+112-39 = 24 ≠ 0.

Hence rank of the given matrix is '3'.

*** Elementary Transformations on a Matrix:**

i). Interchange of i^{th} row and j^{th} row is denoted by $R_i \leftrightarrow R_j$

(ii). If i^{th} row is multiplied with k then it is denoted by $R_i \rightarrow K R_i$

(iii). If all the elements of i^{th} row are multiplied with k and added to the corresponding elements of j^{th} row then it is denoted by $R_j \rightarrow R_j + KR_i$

Note: 1. The corresponding column transformations will be denoted by writing 'c'. i.e

$c_i \leftrightarrow c_j$, $c_i \rightarrow k\, c_j$ $c_j \rightarrow c_j + kc_i$

2. The elementary operations on a matrix do not change its rank.

Equivalence of Matrices: If B is obtained from A after a finite number of elementary transformations on A, then B is said to be equivalent to A.

It is denoted as B~A.

Note : 1. If A and B are two equivalent matrices, then rank A = rank B.

2. If A and B have the same size and the same rank, then the two matrices are equivalent.

Echelon form of a matrix:

A matrix is said to be in Echelon form, if

(i). Zero rows, if any exists, they should be below the non-zero row.

(ii). the first non-zero entry in each non-zero row is equal to '1'.

(iii). the number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

Note: 1. the number of non-zero rows in echelon form of A is the rank of 'A'.

2. The rank of the transpose of a matrix is the same as that of original matrix.
3. The condition (ii) is optional.

E.g.: 1. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a row echelon form.

3. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is a row echelon form.

PROBLEMS

1. Find the rank of the matrix A = $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$ **by reducing it to Echelon form.**

sol: Given A = $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

Applying row transformations on A.

A ~ $\begin{bmatrix} 1 & -3 & -1 \\ 3 & -2 & 4 \\ 2 & 3 & 7 \end{bmatrix}$ $R_1 \leftrightarrow R_3$

~ $\begin{bmatrix} 1 & -3 & -1 \\ 0 & 7 & 7 \\ 0 & 9 & 9 \end{bmatrix}$ $R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

~ $\begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $R_2 \to R_2/7, R_3 \to R_3/9$

~ $\begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 - R_2$

This is the Echelon form of matrix A.

The rank of a matrix A.

= Number of non – zero rows =2

2. For what values of k the matrix $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$ **has rank '3'.**

Sol: The given matrix is of the order 4x4

If its rank is 3 ⇨ det A =0

A = $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$

Applying $R_2 \to 4R_2 - R_1$, $R_3 \to 4R_3 - kR_1$, $R_4 \to 4R_4 - 9R_1$

We get A ~ $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 8-4k & 8+3k & 8-k \\ 0 & 0 & 4k+27 & 3 \end{bmatrix}$

Since Rank A = 3 ⇨ det A =0

⇨ 4 $\begin{vmatrix} 0 & -1 & -1 \\ 8-4k & 8+3k & 8-k \\ 0 & 4k+27 & 3 \end{vmatrix} = 0$

⇨ 1[(8-4k)3]-1(8-4k)(4k+27)] = 0

⇨ (8-4k) (3-4k-27) = 0

⇨ (8-4k)(-24-4k) =0

⇨ (2-k)(6+k)=0

⇨ k =2 or k = -6

Normal Form:

Every mxn matrix of rank r can be reduced to the form $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ (or) (I_r) (or) $\begin{pmatrix} I_r \\ 0 \end{pmatrix}$ (or) $\begin{pmatrix} I_r & 0 \end{pmatrix}$ by a finite number of elementary transformations, where I_r is the r – rowed unit matrix.

Note: 1. If A is an mxn matrix of rank r, there exists non-singular matrices P and Q such that PAQ = $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$

Normal form another name is "canonical form"

e.g.: By reducing the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ into normal form, find its rank.

Sol: Given A = $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & 5 \\ 0 & -6 & -4 & -22 \end{bmatrix} R_2 \to R_2 - 2R_1$$

$R_3 \to R_3 - 3R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 3 & 2 & 11 \end{bmatrix} R_3 \to R_3/-2$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix} R_3 \to R_3+R_2$$

$$A \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix} c_2 \to c_2 - 2c_1,\ c_3 \to c_3 - 3c_1,\ c_4 \to c_4 - 4c_1$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 18 \end{bmatrix} c_3 \to 3c_3 - 2c_2,\ c_4 \to 3c_4 - 5c_2$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} c_2 \to c_2/-3,\ c_4 \to c_4/18$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} c_4 \leftrightarrow c_3$$

This is in normal form $[I_3\ 0]$

Hence Rank of A is '3'.

Gauss – Jordan method

- The inverse of a matrix by elementary Transformations: **(Gauss – Jordan method)**

1. suppose A is a non-singular matrix of order 'n' then we write $A = I_n A$
2. Now we apply elementary row-operations only to the matrix A and the pre-factor I_n of the R.H.S
3. We will do this till we get $I_n = BA$ then obviously B is the inverse of A.

1. Find the inverse of the matrix A using elementary operations where $A = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

Sol:

Given A = $\begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

We can write A = I_3 A

$$\begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \rightarrow 2R_3 - R_2$, we get

$$\begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} A$$

Applying $R_1 \rightarrow R_1 - 3R_2$, we get

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} A$$

Applying $R_1 \rightarrow R_1 + 5R_3$, $R_2 \rightarrow R_2 - 3R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -8 & 10 \\ 0 & 4 & -6 \\ 0 & -1 & 2 \end{bmatrix} A$$

Applying $R_2 \rightarrow R_2/2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -8 & 10 \\ 0 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix} A \Rightarrow I_3 = BA$$

B is the inverse of A.

Cayley - Hamilton Theorem:

Statement:

Every square matrix satisfies its own characteristic equation

PROBLEMS

1. Show that the matrix A = $\begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ satisfies its characteristic equation Hence find A^{-1}

Sol: Characteristic equation of A is det $(A-\lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 & 2 \\ 1 & -2-\lambda & 3 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0 \quad C2 \rightarrow C2+C3$$

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1-\lambda & 3 \\ 0 & 1-\lambda & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$

By Cayley – Hamilton theorem, we have A^3-A^2+A-I=0

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} \quad A^2 = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \quad A^3 = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$A^3 - A^2 + A - I = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Multiplying with A^{-1} we get $A^2 - A + I = A^{-1}$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

2. Using Cayley - Hamilton Theorem find the inverse and A^4 of the matrix A = $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

Sol: Let A = $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

The characteristic equation is given by |A-λI|=0

$$i.e., \begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 6 & 2 & -(1+\lambda) \end{vmatrix} = 0$$

$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By Cayley – Hamilton theorem we have $A^3-5A^2+7A-3I=0$.....(1)

Multiply with A^{-1} we get

$A^{-1} = \frac{1}{3}[A^2 - 5A + 7I]$

$$A^2 = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} \quad A^3 = \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$$

Multiply (1)with A,we get

$A^4 - 5A^3 + 7A^2 - 3A = 0$

$$= \begin{bmatrix} 395 & 130 & -130 \\ -390 & -125 & 130 \\ 390 & 130 & -125 \end{bmatrix} - \begin{bmatrix} 175 & 56 & -56 \\ -168 & -49 & 56 \\ 168 & 56 & -69 \end{bmatrix} + \begin{bmatrix} 21 & 6 & -6 \\ -18 & -3 & 6 \\ 18 & 6 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 241 & 80 & -80 \\ -240 & -79 & 80 \\ 240 & 80 & -79 \end{bmatrix}$$

Problems

1. Diagonalize the matrix (i) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

1. Verify Cayley – Hamilton Theorem for A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$. Hence find A^{-1}.

Linear dependence and independence of Vectors:

1. Show that the vectors (1,2,3), (3,-2,1), (1,-6,-5) from a linearly dependent set.

Sol. The Given Vector $X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} X_2 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} X_3 = \begin{bmatrix} 1 \\ -6 \\ -5 \end{bmatrix}$

The Vectors X_1, X_2, X_3 from a square matrix.

Let $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -2 & -6 \\ 3 & 1 & -5 \end{bmatrix}$

Then $|A| = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -2 & -6 \\ 3 & 1 & -5 \end{bmatrix}$

= 1(10+6)-2(15-1)+3(-18+2)

=16+32-48=0

The given vectors are linearly dependent $\because$ |A|=0

2. Show that the Vector X_1=(2,2,1), X_2=(1,4,-1) and X_3=(4,6,-3) are linearly independent.

Sol. Given Vectors X_1=(2,-2,1) X_2=(1,4,-1) and X_3=(4,6,-3) The Vectors X_1, X_2, X_3 form a square matrix.

$$A = \begin{bmatrix} 2 & 1 & 4 \\ -2 & 4 & 6 \\ 1 & -1 & -3 \end{bmatrix}$$

Then $|A| = \begin{bmatrix} 2 & 1 & 4 \\ -2 & 4 & 6 \\ 1 & -1 & -3 \end{bmatrix}$

=2(-12+6)+2(-3+4)+1(6-16)

=-20$\neq$ 0

$\therefore$ The given vectors are linearly independent

$\therefore$ |A|$\neq$0

Eigen Values & Eigen Vectors

Def: Characteristic vector of a matrix:

Let A= [a_{ij}] be an n x n matrix. A non-zero vector X is said to be a Characteristic Vector of A if there exists a scalar such that AX=λX.

Note: If AX=λX (X≠0), then we say 'λ' is the Eigen value (or) characteristic root of 'A'.

Eg: Let A= $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ X = $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

AX = $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ = $\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1.\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$= 1.X$

Here Characteristic vector of A is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and Characteristic root of A is "1".

Note: We notice that an Eigen value of a square matrix A can be 0. But a zero vector cannot be an Eigen vector of A.

Method of finding the Eigen vectors of a matrix.

Let A = $[a_{ij}]$ be a nxn matrix. Let X be an Eigen vector of A corresponding to the Eigen value λ.

Then by definition AX = λX.

⇨ AX = λIX

⇨ AX –λIX = 0

⇨ (A-λI)X = 0 ------- (1)

This is a homogeneous system of n equations in n unknowns.

(1) Will have a non-zero solution X if and only |A-λI| = 0

- A-λI is called characteristic matrix of A
- |A-λI| is a polynomial in λ of degree n and is called the characteristic polynomial of A
- |A-λI|=0 is called the characteristic equation

Solving characteristic equation of A, we get the roots, $\lambda_1, \lambda_2, \lambda_3, \ldots \ldots . \lambda_n,$ these are called the characteristic roots or Eigen values of the matrix.

- Corresponding to each one of these n Eigen values, we can find the characteristic vectors.

Procedure to find Eigen values and Eigen vectors

Let A = $\begin{bmatrix} a_{11} & a_{12} \ldots \ldots & a_{1n} \\ a_{21} & a_{22} \ldots .. & a_{2n} \\ \ldots \ldots . & \ldots \ldots . & \ldots \ldots . \\ a_{n1} & a_{n2} \ldots \ldots . & a_{nn} \end{bmatrix}$ *be a given matrix*

$$Characteristic\ matrix\ of\ A\ is\ A - \lambda I$$

$$i.e., A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

Then the characteristic polynomial is $|A - \lambda I|$

$$say\ \phi(\lambda) = |A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix}$$

The characteristic equation is $|A-\lambda I| = 0$ we solve the $\emptyset(\lambda) = |A - \lambda I| = 0$, we get n roots, these are called

Eigen values or latent values or proper values.

Let each one of these Eigen values say λ their Eigen vector X corresponding the given value λ is obtained by solving Homogeneous system

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and determining the non-trivial solution.

PROBLEMS

1. **Find the Eigen values and the corresponding Eigen vectors of** $\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

$$sol: Let\ A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$$

$$Characteristic\ matrix = A - \lambda I$$

$$= \begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix}$$

$$Characteristic\ equation\ of\ A\ is\ |A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda)(2-\lambda) + 8 = 0$$

$$\Rightarrow 16 + \lambda^2 - 10\lambda + 8 = 0$$

$$\Rightarrow \lambda^2 - 10\,\lambda + 24 = 0$$

$$\Rightarrow (\lambda - 6)(\lambda - 4) = 0$$

$$\Rightarrow \lambda = 6, 4 \text{ are eigen values of A}$$

$$Consider\ system\ \begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Eigen vector corresponding to $\lambda = 4$

Put $\lambda = 4$ in the above system, we get

$$\begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow 4x_1 - 4x_2 = 0 --- (1)$$

$$2x_1 - 2x_2 = 0 --- (2)$$

$$from\ (1) and\ (2) we\ have\ x_1 = x_2$$

Let $x_1 = \alpha$

$$Eigen\ vector\ is \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ is a Eigen vector of matrix } A, \text{corresponding eigen value } \lambda = 4$$

Eigen Vector corresponding to $\lambda = 6$

$$put\ \lambda = 6 \text{ in the above system, we get}$$

$$\begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow 2x_1 - 4x_2 = 0 - - - (1)$$

$$2x_1 - 4x_2 = 0 - - - (2)$$

from (1) and (2) we have $x_1 = 2x_2$

Say $x_2 = \alpha \Rightarrow x_1 = 2\alpha$

$$Eigen\ vector = \begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ is eigen vector of matrix } A \text{ corresponding eigen value } \lambda = 6$$

2. **Find the eigen values and the corresponding eigen vectors of matrix** $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Sol: Let A = $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

The characteristic equation is $|A-\lambda I|=0$

i.e. $|A-\lambda I| = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$

$$\Rightarrow (2-\lambda)(2-\lambda)^2 - 0 + [-(2-\lambda)] = 0$$

$$\Rightarrow (2-\lambda)^3 — (\lambda - 2) = 0$$

$$\Rightarrow \lambda - 2\ [-(\lambda-2)^2 - 1] = 0$$

$$\Rightarrow \lambda - 2\ \left[-\lambda^2 + 4\lambda - 3\right] = 0$$

$$\Rightarrow (\lambda-2)(\lambda-3)(\lambda-1) = 0$$

⇨ λ=1,2,3

The eigen values of A is 1,2,3.

$$For\ finding\ eigen\ vector\ the\ system\ is\ (A-\lambda I)X=0$$

$$\Rightarrow \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$Eigen\ vector\ corresponding\ to\ \lambda=1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1+x_3=0$

$x_2=0$

$x_1+x_3=0$

$x_1=-x_3, x_2=0$

$say\ x_3=\alpha$

$x_1=-\alpha \quad x_2=0, \quad x_3=\alpha$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -\alpha \\ 0 \\ \alpha \end{bmatrix}=\alpha\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\ is\ Eigen\ vector$$

$Eigen\ vector\ corresponding\ to\ \lambda=2$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$Here\ x_1=0\ and\ x_3=0\ and\ we\ can\ take\ any\ arbitary\ value\ x_2\ i.e\ x_2=\alpha\ (say)$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix}=\alpha\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$Eigen\ vector\ is\ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$Eigen\ vector\ corresponding\ to\ \lambda=3$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1+x_3=0$

$-x_2=0$

$x_1-x_3=0$

$here\ by\ solving\ we\ get\ x_1=x_3, x_2=0\ say\ x_3=\propto$

$x_1=\propto \ , \quad x_2=0 \quad , x_3=\propto$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$Eigen\ vector\ is\ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Properties of Eigen Values:

Theorem 1: The sum of the eigen values of a square matrix is equal to its trace and product of the eigen values is equal to its determinant.

Example: if A= $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 2 & -1 & 1 \end{bmatrix}$ then trace=1+2+1=4 and determinant=15

Theorem 2: If λ is an Eigen value of A corresponding to the Eigen vector X, then λ^n is Eigen value A^n corresponding to the Eigen vector X.

Example: if A= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then Eigen values of $A^3$are 1,8,1

Theorem 3: A Square matrix A and its transpose A^T have the same Eigen values.

Example: if A= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then Eigen values of A^T are 1,2,1.

Theorem 4: If A and B are n-rowed square matrices and If A is invertible show that $A^{-1}B$ and B A^{-1} have same Eigen values.

Theorem 5: If $\lambda_1, \lambda_2, \ldots\ldots\ldots \ldots \lambda_n$ are the Eigen values of a matrix A then k λ_1, k λ_2, ... k λ_n are the Eigen value of the matrix KA, where K is a non-zero scalar.

Example:

If 1,2,3 are eigen values of A then eigen values of 3A are 3,3,9

Theorem 6: If λ is an Eigen values of the matrix A then λ+K is an Eigen value of the matrix A+KI

Example:

If 1,2,3 are eigen values of A then eigen values of 3+A are 4,5,6

Theorem 7: If λ_1, λ_2 ... λ_n are the Eigen values of A, then $\lambda_1 - K$, $\lambda_2 - K$, ... $\lambda_n - K$,

are the eigen values of the matrix $(A - KI)$, *where* K *is a non* $-$ *zero scalar*

Example:

If 1,2,3 are eigen values of A then eigen values of 3-A are 2,1,0

Theorem 8: If $\lambda_1, \lambda_2 \ldots \lambda_n$ are the Eigen values of A, find the Eigen values of the matrix $(A-\lambda I)^2$

Theorem 9: If λ is an Eigen value of a non-singular matrix A corresponding to the Eigen vector X, then λ^{-1} *is an Eigen value of* A^{-1} *and corresponding Eigen vector X itself.*

Theorem 10: If

λ is an eigen value of a non − singular matrix A, then $\frac{|A|}{\lambda}$ is an eigen value of the matrix Adj A

Theorem 11: If λ is an eigen value of an orthogonal matrix then $\frac{1}{\lambda}$ is also an eigen value

Theorem 12: If λ is Eigen value of A then prove that the Eigen value of B = $a_0A^2+a_1A+a_2I$ is $a_0\lambda^2+a_1\lambda+a_2$

Theorem 14: Suppose that A and P be square matrices of order n such that P is non singular. Then A and $P^{-1}AP$ have the same Eigen values.

Corollary 1: If A and B are square matrices such that A is non-singular, then $A^{-1}B$ and BA^{-1} have the same Eigen values.

Corollary 2: If A and B are non-singular matrices of the same order, then AB and BA have the same Eigen

Theorem 15: The Eigen values of a triangular matrix are just the diagonal elements of the matrix.

Theorem 16: The Eigen values of a real symmetric matrix are always real.

Theorem 17: For a real symmetric matrix, the Eigen vectors corresponding to two distinct Eigen values are orthogonal.

PROBLEMS

1. Find the Eigen values and Eigen vectors of the matrix A and its inverse, where

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

Sol: Given $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

The characteristic equation of A is given by $|A-\lambda I| = 0$

$$\Longrightarrow \begin{vmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$\Longrightarrow (1-\lambda)[(2-\lambda)(3-\lambda)] = 0$

$\Longrightarrow \lambda = 1, 2, 3$

$Characteristic\ roots\ are\ 1, 2, 3$

$Characteristic\ vector\ for\ \lambda = 1$

$For\ \lambda = 1, becomes \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Longrightarrow 3x_2 + 4x_3 = 0$

$x_2 + 5x_3 = 0$

$2x_3 = 0$

$x_2 = 0, x_3 = 0$ and $x_1 = \alpha$

$$X = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ is the solution where α is arbitrary constant

$$\therefore X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ Is the Eigen vector corresponding to $\lambda = 1$

Characteristic vector for $\lambda = 2$

For $\lambda = 2$, *becomes* $\begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Longrightarrow -x_1 + 3x_2 + 4x_3 = 0$

$5x_3 = 0 \Rightarrow x_3 = 0$

$-x_1 + 3x_2 = 0 \Rightarrow x_1 = 3x_2$

Let $x_2 = k$

$x_1 = 3k$

$$X = \begin{bmatrix} 3k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

is the solution $\therefore X = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

Is the Eigen vector corresponding to $\lambda = 2$

Hence the characteristic vector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Characteristic vector for $\lambda = 3$

For $\lambda = 3$, *becomes* $\begin{bmatrix} -2 & 3 & 4 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Longrightarrow -2x_1 + 3x_2 + 4x_3 = 0$

$-x_2 + 5x_3 = 0$

Say $x_3 = K \Longrightarrow x_2 = 5K$

$x_1 = \frac{19}{2}K$

$$X = \begin{bmatrix} \frac{19}{2}K \\ 5K \\ K \end{bmatrix} = \frac{K}{2}\begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix}$$

$\therefore X = \begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix}$ is the Eigen vector corresponding to $\lambda = 3$

Eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$

$\Rightarrow$ *Eigen values of* A^{-1} *are* $1, \frac{1}{2}, \frac{1}{3}$

We know Eigen vectors of A^{-1} are same as Eigen vectors of A.

2. *Find the Eigen values of* $3A^3 + 5A^2 - 6A + 2I$ *where* $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

Sol: The characteristic equation of A is $|A - \lambda I| = 0$

$$i.e, \begin{vmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$\Longrightarrow [(1-\lambda)(3-\lambda)(-2-\lambda) - 0] = 0$

$\Longrightarrow (1-\lambda)(3-\lambda)(2+\lambda) = 0 \qquad \lambda = 1, 3, -2$

Eigen values of A are $1, 3, -2$

We know that if λ *is an eigen value of A and* $f(A)$ *is a polynomial in A.*

then the eigen value of $f(A)$ *is* $f(\lambda)$

Let f(A) = $3A^3 + 5A^2 - 6A + 2I$

Then eigen values of f(A) are f(1), f(3) and f(-2)

$f(1) = 3(1)^3+5(1)^2-6(1)+2(1) = 4$

$f(3) = 3(3)^3+5(3)^2-6(3)+2(1) = 110$

$f(-2) = 3(-2)^3+5(-2)^2-6(-2)+2(1) = 10$

Eigen values of $3A^3 + 5A^2 - 6A + 2I$ are 4,110,10

Diagonalization of a matrix:

Theorem: If a square matrix A of order n has n linearly independent eigen vectors ($X_1, X_2 \ldots X_n$) corresponding to the n eigen values $\lambda_1, \lambda_2 \ldots \lambda_n$ respectively then a matrix P can be found such that

$P^{-1}AP$ is a diagonal matrix.

Proof: Given that ($X_1, X_2 \ldots X_n$) be eigen vectors of A corresponding to the eigen values $\lambda_1, \lambda_2 \ldots \lambda_n$ respectively and these eigen vectors are linearly independent Define P = ($X_1, X_2 \ldots X_n$)

Since the n columns of P are linearly independent $|P| \neq 0$

Hence P^{-1} exists

Consider AP = A[$X_1, X_2 \ldots X_n$]

$= [AX_1, AX_2.....AX_n]$

$= [\lambda X_1, \lambda_2 X_2....\lambda_n X_n]$

$$[X_1,X_2...X_n]\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

$= PD$

Where D = diag $(\lambda_1, \lambda_2, \lambda_3, \dots\dots\dots \; ..\lambda_n)$

AP=PD

$P^{-1}(AP) = P^{-1}(PD) \Rightarrow P^{-1}AP = (P^{-1}P)D$

$\Rightarrow P^{-1}AP = (I)D$

$= D$

= diag $(\lambda_1, \lambda_2, \lambda_3, \dots\dots\dots \; ..\lambda_n)$

Hence the theorem is proved.

Modal and Spectral matrices:

The matrix P in the above result which diagonalizable the square matrix A is called modal matrix of A and the resulting diagonal matrix D is known as spectral matrix.

Note 1: If $X_1,X_2...X_n$ are not linearly independent this result is not true.

2: Suppose A is a real symmetric matrix with n pair wise distinct Eigen values $\lambda_1, \lambda_2 \cdots \lambda_n$ then the corresponding Eigen vectors $X_1,X_2\dots X_n$ are pair wise orthogonal.

Hence if $P = (e_1,e_2\dots e_n)$

Where $e_1 = (X_1 / ||X_1||)$, $e_2 = (X_2 / ||X_2||)\dots.e_n = (X_n)/ ||X_n||$

then P will be an orthogonal matrix.

i.e, $P^TP=PP^T=I$

Hence $P^{-1} = P^T$

$P^{-1}AP = D \Longrightarrow P^TAP=D$

Calculation of powers of a matrix:

We can obtain the power of a matrix by using diagonalization

Let A be the square matrix then a non-singular matrix P can be found such that $D = P^{-1}AP$

$D^2=(P^{-1}AP)(P^{-1}AP)$

$= P^{-1}A(PP^{-1})AP$

$= P^{-1}A^2P$ (since $PP^{-1}=I$)

Similarly $D^3 = P^{-1}A^3P$

In general $D^n = P^{-1}A^nP$……..(1)

To obtain A^n, Pre-multiply (1) by P and post multiply by P^{-1}

Then $PD^nP^{-1} = P(P^{-1}A^nP)P^{-1}$

$= (PP^{-1})A^n (PP^{-1}) = A^n \Rightarrow A^n = PD^nP^{-1}$

Hence $A^n = P \begin{bmatrix} \lambda_1^n & 0 & 0\cdots & 0 \\ 0 & \lambda_2^n & 0\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \lambda_n^n \end{bmatrix} P^{-1}$

PROBLEMS

2. Determine the modal matrix P of $= \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. Verify that $P^{-1}AP$ is a diagonal matrix.

Sol: The characteristic equation of A is $|A-\lambda I| = 0$

i.e, $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

$which\ gives\ (\lambda-5)(\lambda+3)^2 = 0$

Thus the eigen values are λ=5, λ=-3 and λ=-3

when λ=5 $\Longrightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By solving above we get $X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Similarly, for the given eigen value λ=-3 we can have two linearly independent eigen vectors $X_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ $and\ X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

$P = (X_1\ X_2\ \ X_3)$

$P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = modal\ matrix\ of\ A$

$Now\ \det P = 1(-1) - 2(2) + 3(0-1) = -8$

$P^{-1} = \frac{adj\ P}{\det P} = -\frac{1}{8} \begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix}$

$= -\frac{1}{8} \begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

$= -\frac{1}{8} \begin{bmatrix} -5 & -10 & 15 \\ 6 & -12 & -18 \\ 3 & 6 & 15 \end{bmatrix}$

$P^{-1}AP = -\frac{1}{8} \begin{bmatrix} -40 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}$

$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = diag\ (5, -3, -3)$

$Hence\ P^{-1}AP$ is a diagonal matrix.

3. Find a matrix P which transform the matrix A = $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ *to diagonal form. Hence calculate* A^4

Sol: Characteristic equation of A is given by |A-λI| = 0

i.e, $\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda)-2]-0-1[2-2(2-\lambda 0\} = 0$

$\Rightarrow (\lambda-1)(\lambda-2)(\lambda-3) = 0$

$\Rightarrow \lambda = 1, \lambda = 2, \lambda = 3$

Thus the eigen values of A are 1, 2, 3

If x_1, x_2, x_3 be the components of an Eigen vector corresponding to the Eigen value λ, we have

[A-λI]X = $\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

For $\lambda = 1$, *eigen vectors are given by*

$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ i.e, $0.x_1+0.x_2+0.x_3=0$ and $x_1+x_2+x_3=0$

$x_3=0$ and $x_1+x_2+x_3=0$

$x_3=0$, $x_1=-x_2$

$x_1=1$, $x_2=-1$, $x_3=0$

Eigen vector is $[1,-1,0]^T$

Also every non-zero multiple of this vector is an Eigen vector corresponding to λ=1

For λ=2, λ=3 we can obtain Eigen vector $[-2,1,2]^T$ and $[-1,1,2]^T$

P = $\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

The Matrix P is called modal matrix of A

$P^{-1} = -\frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}$

$$Now \quad P^{-1}AP = \begin{bmatrix} 0 & -1 & \frac{1}{2} \\ -1 & -1 & 0 \\ 1 & 1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - D \ (say)$$

$A^4 = PD^4P^{-1}$

$$= \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix} \begin{bmatrix} 0 & -1 & \frac{-1}{2} \\ -1 & 1 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$$

Diagonalization of Symmetric Matrices:

NOTE:

a matrix A is diagonalizable if and only if there is an invertible matrix P such that A = P DP $^{-1}$ where D is a diagonal matrix.

A matrix A is orthogonally diagonalizable if and only if there is an orthogonal matrix P such that A = P DP $^{-1}$ where D is a diagonal matrix.

Remark : Recall that any orthogonal matrix A is invertible and also that $A^{-1} = A^T$. Thus we can say that A matrix A is orthogonally diagonalizable if there is a square matrix P such that A = P DP^T where D is a diagonal matrix.

Remark: The formula for transpose of a product: $(MN)^T = N^T M^T$. Using this we can see that any orthogonally diagonalizable A must be symmetric. This is because $A^T = (P DP^T)^T = ((P^T)^T D^T P^T = P DP^T =$ A.

If A is symmetric then any two Eigen values from different Eigen spaces are orthogonal

Proposition: (The Spectral Theorem) An n × n symmetric matrix has the following properties:

1. A has n real Eigen values if we count multiplicity
2. For each Eigen values the dimension of the corresponding Eigen spaces is equal to the algebraic multiplicity of that Eigen values
3. The Eigen spaces are mutually orthogonal.
4. A is orthogonally diagonalizable.

NOTE:

All Eigen values (all roots of the characteristic polynomial) of a symmetric matrix are real.

Eigenvectors of a symmetric matrix corresponding to different Eigen values are orthogonal.

Problems:

1) Find an orthogonal matrix P which diagonalizes $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

Sol: Eigen systems:

Eigen values and Eigenvector are 3 ,3,0 and (−1, 0, 1) ,(−1, 1, 0), (1, 1, 1)

Using the Gram-Schmidt process we find that an orthonormal basis for the eigenspace of A corresponding to $\lambda_1 = 3$ is

$$\mathbf{p}_1 = \frac{(-1,0,1)}{\|(-1,0,1)\|} = (-1/\sqrt{2}, 0, 1/\sqrt{2})$$

$$\mathbf{u}_2 = (-1,1,0) - \langle(-1,1,0), \mathbf{p}_1\rangle \mathbf{p}_1 = (-1/2, 1, -1/2)$$

$$\mathbf{p}_2 = \frac{(-1/2,1,-1/2)}{\|(-1/2,1,-1/2)\|} = (-1/\sqrt{6}, \sqrt{2/3}, -1/\sqrt{6})$$

$$\mathbf{p}_3 = \frac{(1,1,1)}{\|(1,1,1)\|} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$$

Let orthogonal matrix $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ then

$$P^T A P$$
$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2. Find an orthogonal matrix P which diagonalizes $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$

Sol:

Let

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{bmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{bmatrix}$$

$= (3 - \lambda)[(3 - \lambda)(5 - \lambda) - 1] - 1 [5 - \lambda - 1] - 1 [-1 - 3 - \lambda]$
$= [(3 - \lambda)(3 - \lambda)(5 - \lambda) - (3 - \lambda)] - (4 - \lambda) - (2 - \lambda)$
$= [(3 - \lambda)(3 - \lambda)(5 - \lambda) - (3 - \lambda)] - 2 (3 - \lambda)$
$= (3 - \lambda)[(3 - \lambda)(5 - \lambda) - 1 - 2]$
$= (3 - \lambda)[15 - 5\lambda - 3\lambda - \lambda^2 - 3]$
$= (3 - \lambda)(\lambda^2 - 8\lambda - 12)$
$= (3 - \lambda)(\lambda - 6)(\lambda - 2)$

Thus, $\lambda = 2, 3, 6$ are the eigenvalues of A .Let us find an eigenvector corresponding to each eigenvalue . For the eigenvalue $\lambda = 2$, since

$$A - 2I = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

A vector $\boldsymbol{X}^t = (x_1, x_2, x_3)$ will be an eigenvector for eigenvalue $\lambda = 2$ if

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0,$$

i.e., $x_3 = 0$

$x_1 - x_2 = x_3 = 0.$

If we choose x_2=1, then x_1 = -1. Hence

$$\boldsymbol{X}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

is an eigenvector for $\lambda = 2$ For the eigenvalue $\lambda = 3$,

$$A - 3I = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$\boldsymbol{X}_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

will be an eigenvector for the eigenvalue $\lambda = 3$, if

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0},$$

i.e., $x_2 = x_3$ and $x_1 - x_3 = 0$. Hence

$$\boldsymbol{X}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ is an eigenvector for } \lambda = 3.$$

Finally, for $\lambda = 6$

$$A - 4I = \begin{bmatrix} -3 & 1 & -1 \\ 1 & -3 & -1 \\ -1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & -1 \\ 0 & -4 & -2 \\ 0 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & 1 \\ 0 & -4 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus ,

$$\boldsymbol{X}_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ will be an eigenvector for eigenvalue } \lambda = 6 \text{ if}$$

$-4x_2 - 2x_3 = 0, \quad -x_1 - x_2 - x_3 = 0.$

Thus, if we take $x_3 = 2$, then $x_2 = -1$ and $x_1 = -x_2 - x_3 = -1$.
Hence

$$\boldsymbol{X}_3 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \text{ is an eigenvector for the eigenvalue } \lambda = 6.$$

Note that for

$$P := \begin{bmatrix} \boldsymbol{X}_1 & \boldsymbol{X}_2 & \boldsymbol{X}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}, \text{ the columns of } P \text{ are orthogonal.}$$

To make P orthogonal, we normalize each x_1, x_2 and x_3, and define

$$\tilde{P} := \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix},$$

It is easy to verify that $\tilde{P} A \tilde{P}^{-1} = D$, where

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

In fact, $\tilde{P}^{-1} = \tilde{P}^{t}$. Thus, we checks $\tilde{P}^{-1} A \tilde{P}^{-t} = D$.

3) Find an orthogonal matrix P which diagonalizes $A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$

Sol:

Let

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}.$$

Then A is a real symmetric matrix with eigenvalues given by

$$0 = \det(A - \lambda I) = \begin{bmatrix} -\lambda & 2 & 2 \\ 2 & -\lambda & 2 \\ 2 & 2 & -\lambda \end{bmatrix}$$

$$= -\lambda(\lambda^2 - 4) - 2(-2\lambda - 4) - 2(4 + 2\lambda)$$
$$= -\lambda(\lambda - 2)(\lambda - 2) - 4(\lambda - 2) + 4(\lambda + 2)$$
$$= (\lambda - 2)(-\lambda^2 - 2\lambda - 8)$$
$$= -(\lambda - 2)(\lambda - 2)(\lambda - 4).$$

Hence, λ = -2 , λ = 4 are two distinct eigenvalues of A. To find eigenvectors for λ = -2, since

$$\left(A+2I\right)=\begin{bmatrix}+2 & 2 & 2\\ 2 & +2 & 2\\ 2 & 2 & +2\end{bmatrix}\sim\begin{bmatrix}+2 & 2 & 2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$E_{\lambda=-2}$ is given by $\left\{\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}\middle|\ x_1+x_2+x_3=0\right\}$.

If we choose $x_3 = 0$ and $x_2 = 1$ then $x_1 = -1$. And for $x_3 = 1$ and $x_2 = 0$ we get $x_1 = -1$.
Thus

$$X_1=\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix},\ X_2=\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$$

are two mutually orthogonal eigenvector for λ = -2 .

For the eigenvalue $\lambda = 4$, since

$$A-4I=\begin{bmatrix}-4 & 2 & 2\\ 2 & -4 & 2\\ 2 & 2 & -4\end{bmatrix}\sim\begin{bmatrix}-4 & 2 & 2\\ 0 & -3 & 3\\ 0 & 3 & -3\end{bmatrix}\sim\begin{bmatrix}-4 & 2 & 2\\ 0 & -3 & 3\\ 0 & 0 & 0\end{bmatrix},$$

$$E_{\lambda=4}=\left\{\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}\middle|\ -3x_2+3x_3=0,-4x_1+2x_2+2x_3=0\right\}.$$

$$=\left\{\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}\middle|\ x_2=x_3=x_1\right\}.$$

Thus, if we choose

$x_3 = x_2 = x_1 = 1$, then

$$X_3=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

is an eigenvector for the eigenvalue $\lambda = 4$. Thus, the eigenvector for A are

$$X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

To make $\{X_1, X_2\}$ orthonormal, we use the Gram-Schmidt process.
Define

$$\tilde{X}_1 := X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},$$

$$\tilde{X}_2 := X_2 - \frac{\langle X_2, X_1 \rangle}{\langle X_1, X_1 \rangle} X_1$$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ +1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} +\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}.$$

We normalize X_3 also to get

$$\tilde{X}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \tilde{X}_2 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}, \tilde{X}_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix},$$

are orthonormal basis of $\mathbb{R}^3$ of eigenvectors of A. Thus, for

$$P := \begin{bmatrix} \tilde{X}_1 & \tilde{X}_2 & \tilde{X}_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix},$$

one checks that

$$PAP^t = D$$

where

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

MODULE-II

FUNCTIONS
OF SINGLE AND SEVERAL
VARIABLES

MEAN VALUE THEOREMS

I Rolle's Theorem:

Let f(x) be a function such that

(i). It is continuous in closed interval [a,b]

(ii). It is differentiable in open interval (a,b) and

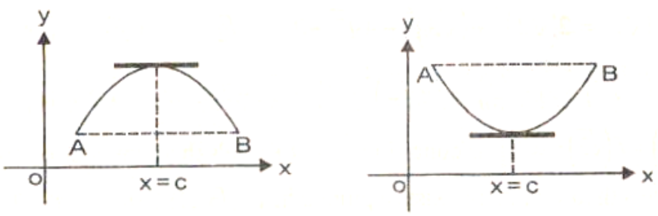
(iii). f(a) = f(b).

Then there exists at least one point 'c' in (a,b) such that

$f^1(c) = 0$.

Geometrical Interpretation of Rolle's Theorem:

Let $f:[a,b] \rightarrow R$ be a function satisfying the three conditions of Rolle 's Theorem. Then the graph.



1. y=f(x) in a continuous curve in [a,b].
2. There exist a unique tangent line at every point x=c, where a<c<b
3. The ordinates f(a), f(b) at the end points A,B are equal so that the points A and B are equidistant from the X-axis.
4. By Rolle's Theorem, There is at least one point x=c between A and B on the curve at which the tangent line is parallel to the x-axis and also it is parallel to chord of the curve.

1. Verify Rolle's theorem for the function $f(x) = \sin x/e^x$ or $e^{-x} \sin x$ in [0,π]

Sol: i) Since sinx and e^x are both continuous functions in [0, π].

Therefore, $\sin x/e^x$ is also continuous in [0,π].

ii) Since sinx and e^x be derivable in (0,π), then f is also derivable in (0,π).

iii) $f(0) = \sin 0/e^0 = 0$ and $f(\pi)= \sin \pi/e^{\pi} =0$

$\therefore$ f(0) = f(π)

Thus all three conditions of Rolle 's Theorem are satisfied.

$\therefore$ There exists c ϵ(0, π) such that $f^1(c)=0$

Now $f^1(x) = \dfrac{e^x \cos x - \sin x\, e^x}{(e^x)^2} = \dfrac{\cos x - \sin x}{e^x}$

$f^1(c)= 0 \Rightarrow \dfrac{\cos c - \sin c}{e^c} = 0$

cos c = sin c => tan c = 1

c = π/4 ϵ(0,π)

Hence Rolle's theorem is verified.

2. Verify Rolle's theorem for the functions $\log\left(\frac{x^2+ab}{x(a+b)}\right)$ **in[a,b] , a>0, b>0,**

Sol: Let f(x)= $\log\left(\frac{x^2+ab}{x(a+b)}\right)$

= log(x^2+ab) – log x –log(a+b)

(i). Since f(x) is a composite function of continuous functions in [a,b], it is continuous in [a,b].

(ii). $f^1(x) = \frac{1}{x^2+ab}.2x - \frac{1}{x} = \frac{x^2-ab}{x(x^2+ab)}$

$f^1(x)$ exists for all xϵ (a,b)

(iii). $f(a) = \log\left[\frac{a^2+ab}{a^2+ab}\right] = \log 1 = 0$

$f(b) = \log\left[\frac{b^2+ab}{b^2+ab}\right] = \log 1 = 0$

f(a) = f(b)

Thus f(x) satisfies all the three conditions of Rolle 's Theorem.

So, $\exists$ c $\in$ (a, b) $\Rightarrow$ $f^1(c) = 0$,

$f^1(c) = 0, \Rightarrow \frac{c^2-ab}{c(c^2+ab)} = 0 \Rightarrow c^2 = ab$

$\Rightarrow c = \sqrt{ab} \in (a,b)$

Hence Rolle's theorem verified.

3. Verify whether Rolle 's Theorem can be applied to the following functions in the intervals.

i) f(x) = tan x in[0 , π] and ii) f(x) = 1/x^2 in [-1,1]

(i) f(x) is discontinuous at x = π/2 as it is not defined there. Thus condition (i) of Rolle 's Theorem is not satisfied. Hence we cannot apply Rolle 's Theorem here.

$\therefore$ Rolle's theorem cannot be applicable to f(x) = tan x in [0,π].

(ii). f(x) = 1/x^2 in [-1,1]

f(x) is discontinuous at x= 0.Hence Rolle 's Theorem cannot be applied.

4. Verify Rolle's theorem for the function f(x) = $(x-a)^m(x-b)^n$ where m,n are positive integers in [a,b].

Sol: (i). Since every polynomial is continuous for all values, f(x) is also continuous in[a,b].

(ii) f(x) = $(x-a)^m(x-b)^n$

$f^1(x) = m(x-a)^{m-1}(x-b)^n+(x-a)^m.n(x-b)^{n-1}$

$= (x-a)^{m-1}(x-b)^{n-1}[m(x-b)+n(x-a)]$

$=(x-a)^{m-1}(x-b)^{n-1}[(m+n)x-(mb+na)]$

Which exists

Thus f(x) is derivable in (a,b)

(iii) f(a) = 0 and f(b) = 0

$\therefore$ f(a) =f(b)

Thus three conditions of Rolle's theorem are satisfied.

$\therefore$ There exists cϵ(a,b) such that $f^1(c)=0$

$(c-a)^{m-1}(c-b)^{n-1}[(m+n)c-(mb+na)]=0$

$\Rightarrow$ (m+n)c-(mb+na)=0 => (m+n)c = mb+na

$\Rightarrow c = \dfrac{mb+na}{m+n} \epsilon(a,b)$

$\therefore$ Rolle 's Theorem verified.

5. Using Rolle 's Theorem, show that $g(x) = 8x^3-6x^2-2x+1$ has a zero between 0 and 1.

Sol: $g(x) = 8x^3-6x^2-2x+1$ being a polynomial, it is continuous on [0,1] and differentiable on (0,1)

Now g(0) = 1 and g(1)= 8-6-2+1 = 1

Also g(0)=g(1)

Hence, all the conditions of Rolle's theorem are satisfied on [0,1].

Therefore, there exists a number cϵ(0,1) such that $g^1(c)=0$.

Now $g^1(x) = 24x^2-12x-2$

$\therefore$ $g^1(c)= 0 \Rightarrow 24c^2-12c-2 =0$

$\Rightarrow c= \dfrac{3 \pm \sqrt{21}}{12}$ *ie* c= 0.63 or -0.132

only the value c = 0.63 lies in (0,1)

Thus there exists at least one root between 0 and 1.

6. Verify Rolle's theorem for $f(x) = x^{2/3} -2x^{1/3}$ in the interval (0,8).

Sol: Given $f(x) = x^{2/3} -2x^{1/3}$

f(x) is continuous in [0,8]

$f^1(x) = 2/3 . 1/x^{1/3} -2/3 . 1/x^{2/3} = 2/3(1/x^{1/3} – 1/x^{2/3})$

Which exists for all x in the interval (0,8)

$\therefore$ f is derivable (0,8).

Now f(0) = 0 and $f(8) = (8)^{2/3}-2(8)^{1/3} = 4-4 =0$

i.e., f(0) = f(8)

Thus all the three conditions of Rolle's Theorem are satisfied.

$\therefore$ There exists at least one value of c in(0,8) such that $f^1(c)=0$

ie. $\frac{1}{c^{\frac{1}{3}}} - \frac{1}{c^{\frac{2}{3}}} = 0$ => c = 1 ϵ (0,8)

Hence Rolle's Theorem is verified.

7. Verify Rolle's theorem for $f(x) = x(x+3)e^{-x/2}$ in [-3,0].

Sol: - (i). Since x(x+3) being a polynomial is continuous for all values of x and $e^{-x/2}$ is also continuous for all x, their product $x(x+3)e^{-x/2}$ = f(x) is also continuous for every value of x and in particular f(x) is continuous in the [-3,0].

(ii). we have $f^1(x) = x(x+3)(-1/2\ e^{-x/2})+(2x+3)e^{-x/2}$

$= e^{-x/2}\left[2x+3-\frac{x^2+3x}{2}\right]$

$=e^{-x/2}[6+x-x^2/2]$

Since $f^1(x)$ doesnot become infinite or indeterminate at any point of the interval(-3,0).

f(x) is derivable in (-3,0)

(iii) Also we have f(-3) = 0 and f(0) =0

$\therefore$ f (-3)=f(0)

Thus f(x) satisfies all the three conditions of Rolle's theorem in the interval [-3,0].

Hence there exist at least one value c of x in the interval (-3,0) such that $f^1(c)=0$

i.e., $\frac{1}{2} e^{-c/2}(6+c-c^2)=0 \Rightarrow 6+c-c^2=0$ ($e^{-c/2}\neq 0$ for any c)

$\Rightarrow c^2+c-6 = 0 \Rightarrow (c-3)(c+2)=0$

c=3,-2

Clearly, the value c= -2 lies within the (-3,0) which verifies Rolle's theorem.

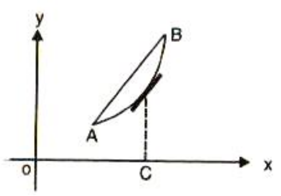
II. Lagrange's mean value Theorem

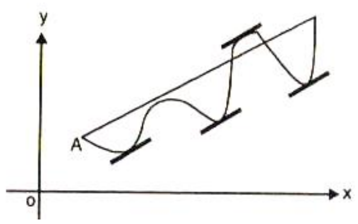
Let f(x) be a function such that (i) it is continuous in closed interval [a,b] & (ii) differentiable in (a,b). Then ∃ at least one point c in (a,b) such that

$f^1(c) = \frac{f(b)-f(a)}{b-a}$

Geometrical Interpretation of Lagrange's Mean Value theorem:

Let $f:[a,b]\to R$ be a function satisfying the two conditions of Lagrange's theorem. Then the graph.





1. y=f(x) is continuous curve in [a,b]

2. At every point x=c, when a<c<b, on the curve y=f(x), there is unique tangent to the curve. By Lagrange's theorem there exists at least one point $c \in (a,b) \ni f^1(c) = \frac{f(b) - f(a)}{b - a}$

Geometrically there exist at least one point c on the curve between A and B such that the tangent line is parallel to the chord $\overleftrightarrow{AB}$

1. Verify Lagrange's Mean value theorem for f(x)= x^3-x^2-5x+3 in [0,4]

Sol: Let f(x)= x^3-x^2-5x+3 is a polynomial in x.

$\therefore$ It is continuous & derivable for every value of x.

In particular, f(x) is continuous [0,4] & derivable in (0,4)

Hence by Lagrange's Mean value theorem $\exists$ c$\in$ (0,4) $\ni$

$f^1(c) = \frac{f(4) - f(0)}{4 - 0}$

i.e., $3c^2-2c-5 = \frac{f(4) - f(0)}{4}$ ……………………(1)

Now $f(4) = 4^3-4^2-5.4+3$ =64-16-20-3=67-36= 31 & f(0)=3

$$\frac{f(4) - f(0)}{4} = \frac{(31 - 3)}{4} = 7$$

From equation (1), we have

$3c^2-2c-5 =7 \Rightarrow 3c^2-2c-12 =0$

$$c = \frac{2 \pm \sqrt{4 + 144}}{6} = \frac{2 \pm \sqrt{148}}{6} = \frac{1 \pm \sqrt{37}}{3}$$

We see that $\frac{1 + \sqrt{37}}{3}$ lies in open interval (0,4) & thus Lagrange's Mean value theorem is verified.

2. Verify Lagrange's Mean value theorem for f(x) = $\log_e x$ in [1,e]

Sol: - f(x) = $\log_e x$

This function is continuous in closed interval [1,e] & derivable in (1,e). Hence L.M.V.T is applicable here. By this theorem, $\exists$ a point c in open interval (1,e) such that

$$f^1(c) = \frac{f(e) - f(1)}{e - 1} = \frac{1 - 0}{e - 1} = \frac{1}{e - 1}$$

But $f^1(c) = \frac{1}{e - 1} \Longrightarrow \frac{1}{c} = \frac{1}{e - 1}$

$\therefore$ c = e - 1

Note that (e-1) is in the interval (1,e).

Hence Lagrange's mean value theorem is verified.

4. **Give an example of a function that is continuous on [-1, 1] and for which mean value theorem does not hold with explanations**.

Sol:- The function f(x) = $|x|$ is continuous on [-1,1]

But Lagrange Mean value theorem is not applicable for the function f(x) as its derivative does not exist in (-1,1) at x=0.

4. If a<b, P.T $\frac{b-a}{1+b^2} < Tan^{-1}b - Tan^{-1}a < \frac{b-a}{1+a^2}$ **using Lagrange's Mean value theorem. Deduce the following.**

i). $\frac{\pi}{4} + \frac{3}{25} < Tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

ii). $\frac{5\pi + 4}{20} < Tan^{-1}2 < \frac{\pi + 2}{4}$

Sol: consider f(x) = Tan^{-1} x in [a,b] for 0<a<b<1

Since f(x) is continuous in closed interval [a,b] & derivable in open interval (a,b).

We can apply Lagrange's Mean value theorem here.

Hence there exists a point c in (a,b)∋

$f^1(c) = \frac{f(b) - f(a)}{b-a}$

Here $f^1(x) = \frac{1}{1+x^2}$ & hence $f^1(c) = \frac{1}{1+c^2}$

Thus ∃ c, a<c<b ∋

$\frac{1}{1+c^2} = \frac{Tan^{-1}b - Tan^{-1}a}{b-a}$ ------- (1)

We have $1+a^2<1+c^2<1+b^2$

$\therefore \frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$ .... (2)

From (1) and (2), we have

$\frac{1}{1+a^2} > \frac{Tan^{-1}b - Tan^{-1}a}{b-a} > \frac{1}{1+b^2}$

or

$\frac{b-a}{1+a^2} < Tan^{-1}b - Tan^{-1}a < \frac{b-a}{1+b^2}$ ....(3)

Hence the result

Deductions: -

(i) We have $\frac{b-a}{1+b^2} < Tan^{-1}b - Tan^{-1}a < \frac{b-a}{1+a^2}$

Take $b = \frac{4}{3}$ & a=1, we get

$$\frac{\frac{4}{3}-1}{1+\frac{16}{9}} < Tan^{-1}(\frac{4}{3}) - Tan^{-1}(1) < \frac{\frac{4}{3}-1}{1+1^2} ==> \frac{\frac{4-3}{3}}{\frac{25}{9}} < Tan^{-1}(\frac{4}{3}) - \frac{\pi}{4} < \frac{\frac{4-3}{3}}{2}$$

$$\Rightarrow \frac{3}{25} + \frac{\pi}{4} < Tan^{-1}(\frac{4}{3}) < \frac{\pi}{4} + \frac{1}{6}$$

(ii) Taking b=2 and a=1, we get

$$\frac{2-1}{1+2^2} < Tan^{-1}2 - Tan^{-1}1 < \frac{2-1}{1+1^2} \Rightarrow \frac{1}{5} < Tan^{-1}2 - \frac{\pi}{4} < \frac{1}{2}$$

$$\Rightarrow \frac{1}{5} + \frac{\pi}{4} < Tan^{-1}2 < \frac{2+\pi}{4}$$

$$\Rightarrow \frac{4+5\pi}{20} + < Tan^{-1}2 < \frac{2+\pi}{4}$$

5. **Show that for any x > 0, 1 + x < e^x < 1 + xe^x.**

Sol: - Let f(x) = e^x defined on [0,x]. Then f(x) is continuous on [0,x] & derivable on (0,x).

By Lagrange's Mean value theorem ∃ a real number c ϵ(0,x) such that

$$\frac{f(x)-f(0)}{x-0} = f^1(c)$$

$$\Rightarrow \frac{e^x - e^0}{x-0} = e^c \Rightarrow \frac{e^x - 1}{x} = e^c \quad \text{............(1)}$$

Note that 0<c<x => $e^0<e^c<e^x$ (e^x is an increasing function)

=> $1 < \frac{e^x - 1}{x} < e^x$ From (1)

=> $x<e^x-1<xe^x$

=> $1+x<e^x<1+xe^x$.

6. Calculate approximately $\sqrt[5]{245}$ by using L.M.V.T.

Sol:- Let f(x) = $\sqrt[5]{x}$ =$x^{1/5}$ & a=243 , b=245

Then $f^1(x) = 1/5\, x^{-4/5}$ & $f^1(c) = 1/5c^{-4/5}$

By L.M.V.T, we have

$$\frac{f(b)-f(a)}{b-a} = f^1(c)$$

=> $\frac{f(245)-f(243)}{245-243} = \frac{1}{5}c^{\frac{-4}{5}}$

=> $f(245) = f(243)+2/5c^{-4/5}$

=> c lies b/w 243 & 245 take c= 243

=> $\sqrt[5]{245} = (243)^{1/5} + 2/5(243)^{-4/5} = (3^5)^{\frac{1}{5}} + \frac{2}{5}(3^5)^{\frac{-4}{5}}$

$= 3 + (2/5)(1/81) = 3 + 2/405 = 3.0049$

7. Find the region in which $f(x) = 1-4x-x^2$ is increasing & the region in which it is decreasing using M.V.T.

Sol: - Given $f(x) = 1-4x-x^2$

f(x) being a polynomial function is continuous on [a,b] & differentiable on (a,b) $\forall$ a,b $\in$R

$\therefore$ f satisfies the conditions of L.M.V.T on every interval on the real line.

$f^1(x) = -4-2x = -2(2+x) \forall x \in R$

$f^1(x) = 0$ if $x = -2$

for $x<-2$, $f^1(x) > 0$ & for $x>-2$, $f^1(x) < 0$

Hence f(x) is strictly increasing on $(-\infty, -2)$ & strictly decreasing on $(-2, \infty)$

8. Using Mean value theorem prove that Tan x > x in $0<x<\pi/2$

Sol:- Consider f(x) = Tan x in $[\xi, x]$ where $0<\xi<x<\pi/2$

Apply L.M.V.T to f(x)

$\exists$ a points c such that $0<\xi<c<x<\pi/2$ such that

$$\frac{Tan\ x - Tan\ \xi}{x - \xi} = \sec^2 c \implies$$

$$\text{Tan x} - \text{Tan } \xi = (x - \xi)\sec^2 c$$

$$Take\ \xi \to 0+0\ then\ Tan\ x = x \sec^2 x$$

But $\sec^2 c > 1$.

Hence Tan x > x

9. If $f^1(x) = 0$ Through out an interval [a,b], prove using M.V.T f(x) is a constant in that interval.

Sol:- Let f(x) be function defined in [a,b] & let $f^1(x) = 0$ $\forall$ x in [a,b].

Then $f^1(t)$ is defined & continuous in [a,x] where $a \le x \le b$.

& f(t) exist in open interval (a,x).

By L.M.V.T $\exists$ a point c in open interval (a,x) ∍

$$\frac{f(x) - f(a)}{x - a} = f^1(c)$$

But it is given that $f^1(c) = 0$

$\therefore f(x) - f(a) = 0$

$\therefore f(x) = f(a) \ \forall \ x$

Hence f(x) is constant.

10 Using mean value theorem

S.T i) $x > \log(1+x) > \frac{x}{1+x}$ $x > 0$

ii) $\pi/6 + (\sqrt{3}/15) < \sin^{-1}(0.6) < \pi/6 + (1/6)$

iii) $1+x < e^x < 1+xe^x \quad \forall\, x > 0$

iv) $\frac{v-u}{1+v^2} < \tan^{(-1)}v - \tan^{(-1)}u < \frac{v-u}{1+u^2}$ where $0 < u < v$ hence deduce

a) $\pi/4 + (3/25) < \tan^{(-1)}(4/3) < \pi/4 + (1/6)$

III. Cauchy's Mean Value Theorem

If f: [a,b] →R, g:[a,b] →R ∍ (i) f,g are continuous on [a,b] (ii) f,g are differentiable on (a,b)

$(iii)\ g^1(x) \neq 0 \ \forall x \in (a,b),$ *then*

$$\exists a \text{ point } c \in (a,b) \ni \frac{f^1(c)}{g^1(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

1. Find c of Cauchy's mean value theorem for

$f(x) = \sqrt{x} \ \ \& \ \ g(x) = \frac{1}{\sqrt{x}}$ in [a,b] where 0<a<b

Sol: - Clearly f, g are continuous on $[a,b] \subseteq R^+$

We have $f^1(x) = \frac{1}{2\sqrt{x}}$ and $g^1(x) = \frac{-1}{2x\sqrt{x}}$ which exits on (a,b)

$\therefore$ f, g are differentiable on $(a, b) \subseteq R^+$

Also $g^1(x) \neq 0, \forall x \in (a,b) \subseteq R^+$

Conditions of Cauchy's Mean value theorem are satisfied on (a,b) so ∃c∈(a,b) ∍

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f^1(c)}{g^1(c)}$$

$$\frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}} = \frac{\frac{1}{2\sqrt{c}}}{\frac{-1}{2c\sqrt{c}}} \Longrightarrow \frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}} = \frac{-2c\sqrt{c}}{2\sqrt{c}} \Rightarrow \sqrt{ab} = c$$

Since a,b >0 , $\sqrt{ab}$ is their geometric mean and we have $a<\sqrt{ab}<b$

c∈(a,b) which verifies Cauchy's mean value theorem.

2. Verify Cauchy's Mean value theorem for $f(x) = e^x$ & $g(x) = e^{-x}$ in [3,7] & find the value of c.

Sol: We are given $f(x) = e^x$ & $g(x) = e^{-x}$

f(x) & g(x) are continuous and derivable for all values of x.

=>f & g are continuous in [3,7]

=> f & g are derivable on (3,7)

Also $g^1(x) = e^{-x} \neq 0 \ \forall\, x \in (3,7)$

Thus f & g satisfies the conditions of Cauchy's mean value theorem.

Consequently, ∃ a point c ∈(3,7) such that

$$\frac{f(7)-f(3)}{g(7)-g(3)} = \frac{f^1(c)}{g^1(c)} \implies \frac{e^7-e^3}{e^{-7}-e^{-3}} = \frac{e^c}{-e^{-c}} \implies \frac{e^7-e^3}{\frac{1}{e^7}-\frac{1}{e^3}} = -e^{2c}$$

=> $-e^{7+3} = -e^{2c}$

=> 2c = 10

=> c = 5∈(3,7)

Hence C.M.T. is verified

Partial Differentiation

Partial differential coefficients : The Partial differential coefficient of f(x,y) with respect to x is the ordinary differential coefficient of f(x,y) when y is regarded as a constant. It is written as

$\frac{\partial f}{\partial x}$ or $\partial f/\partial x$ or $D_x f$

Thus $\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$

Again, the partial differential coefficient $\partial f/\partial y$ of f(x,y) with respect to y is the ordinary differential coefficient of f(x,y) when x is regarded as a constant.

Thus $\frac{\partial f}{\partial y} = \lim_{k\to 0}\frac{f(x.y+k)-f(x,y)}{k}$

Similarly, if f is a function of the n variables $x_1, x_2, \ldots\ldots\ldots x_n$, the partial differential coefficient of f with respect to x_1 is the ordinary differential coefficient of f when all the variables except x_1 are regarded as constants and is written as $\partial f/\partial x_1$.

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are also denoted by f_x and f_y respectively.

The partial differential coefficients of f_x and f_y are f_{xx}, f_{xy}, f_{yx}, f_{yy}

or $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x\partial y}, \frac{\partial^2 f}{\partial y\partial x}, \frac{\partial^2 f}{\partial y^2}$, respectively.

It should be specially noted that $\frac{\partial^2 f}{\partial y\partial x}$ means $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ and $\frac{\partial^2 f}{\partial x\partial y}$ means $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$.

The student will be able to convince himself that in all ordinary cases

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial^2 f}{\partial x\partial y}$$

PROBLEMS

Example 1 : If u = log ($x^3 + y^3 + z^3 - 3xyz$) show that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+x)^2}$$

Solution : The given relation is

$u = \log(x^3 + y^3 + x^3 - 3xyz)$

Differentiate it w.r.t. x partially, we get

$$\frac{\partial u}{\partial x}=\frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

similarly $\frac{\partial u}{\partial y}=\frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$

and $\frac{\partial u}{\partial z}=\frac{3z^2-3xyz}{x^3+y^3+z^3-3xyz}$

$$\therefore \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^2+y^2+z^2-yz-zx-xy)}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3(x^2+y^2+z^2-yz-xz-xy)}{(x+y+z)(x^2+y^2+z^2-y^z-zx-xy)}$$

$$=\frac{3}{x+y+z}$$

Now $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$=3\left[\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right)\right]$$

$$=3\left[-\frac{1}{(x+y+z)^2}+\frac{-1}{(x+y+z)^2}+\frac{-1}{(x+y+z)^2}\right]$$

$$=3\left[\frac{-3}{(x+y+z)^2}\right]$$

$=-\frac{9}{(x+y+z)^2}$ Hence Proved.

Example 2: If $u = e^{xyz}$, show that

$$\frac{\partial^3 u}{\partial x\partial y\partial z} = (1+3xyz+x^2y^2z^2)\, e^{xyz}$$

Solution : Given $u = e^{xyz}$

$$\therefore \frac{\partial u}{\partial z} = xy\, e^{xyz}$$

$$\frac{\partial^2 u}{\partial y\partial z}=\frac{\partial}{\partial y}(xy\,e^{xyz})=x\frac{\partial}{\partial y}ye^{xyz}$$

$= x[y\ xz\ e^{xyz} + e^{xyz}]$

$= e^{xyz}(x^2yz + x)$

Hence $\frac{\partial^3 u}{\partial x\partial y\partial z}=\frac{\partial}{\partial x}[e^{xyz}(x^2yz+x)]$

$= e^{xyz}(2xyz+1) + yz\, e^{xyz}(x^2yz+x)$

$= e^{xyz}[2xyz + 1 + x^2y^2z^2 + xyz]$

$= e^{xyz}(1+3xyz + x^2y^2z^2)$ Hence Proved.

Example 3 : If $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$, prove that ,

$u_x^2 + u_y^2 + u_z^2 = 2\,(xu_x + yu_y + zu_z)$

Solution : Given $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$ ....(i)

where u is a function of x,y and z,

Differentiating (i) partially with respect to x, we get

$$\frac{(a^2+u).2x - x^2\frac{\partial u}{\partial x}}{\left(a^2+u\right)^2}+\frac{(b^2+u).0 - y^2\frac{\partial u}{\partial x}}{\left(b^2+u\right)^2}+\frac{(c^2+u).0 - z^2\frac{\partial u}{\partial x}}{\left(c^2+u\right)^2}=0$$

or $\frac{2x}{a^2+u}-\left[\frac{x^2}{\left(a^2+u\right)^2}+\frac{y^2}{\left(b^2+u\right)^2}+\frac{z^2}{\left(c^2+u\right)^2}\right]\frac{\partial u}{\partial x}=0$

or $\frac{\partial u}{\partial x}=\frac{2x/\left(a^2+u\right)}{\left[x^2/\left(a^2+u\right)^2+y^2/\left(b^2+u\right)^2+z^2/\left(c^2+u\right)^2\right]}$

$=\frac{2x/a^2+u}{\sum\left[x^2/\left(a^2+u\right)^2\right]}$

Similarly $\frac{\partial u}{\partial y}=\frac{2y/\left(b^2+u\right)}{\sum\left[x^2/\left(a^2+u\right)^2\right]}$

and $\frac{\partial u}{\partial z}=\frac{2z/\left(c^2+u\right)}{\sum\left[x^2/\left(a^2+u\right)^2\right]}$

$$\therefore \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 4\frac{\left[x^2/(a^2+u)^2 + y^2/(b^2+u)^2 + z^2/(c^2+u)^2\right]}{\left[\sum\left\{x^2/(a^2+u)^2\right\}\right]^2}$$

$$\text{or } u_x^2 + u_y^2 + u_z^2 = \frac{4}{\sum\left[\left\{x^2/(a^2+u)^2\right\}\right]} \text{.....................(ii)}$$

Also $xu_x + yu_y + zu_z = = x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) + z\left(\frac{\partial u}{\partial z}\right)$

$$= \frac{1}{\sum\left[x^2/(a^2+u)^2\right]}\left[\frac{2x^2}{(a^2+u)} + \frac{2y^2}{(b^2+u)} + \frac{2z^2}{(c^2+u)}\right]$$

$$= \frac{2}{\sum\left[x^2/(a^2+u)^2\right]}[1] \text{..................(iii)}$$

From (i), (ii) (iii) and we have

$u_x^2 + u_y^2 + u_z^2 = 2(xu_x + yu_y + zu_z)$ Hence Proved.

Example 4 : If $u = f(r)$ and $x = r\cos\theta$, $y = r\sin\theta$ i.e. $r^2 = x^2+y^2$, Prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$$

Solution : Given $u = f(r)$.................(i)

Differentiating (i) partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = f'(r).\frac{\partial r}{\partial x}$$

$$= f'(r).\ \frac{x}{r}$$

$\because\ r^2 = x^2 + y^2$

$$\Rightarrow 2r\ \frac{\partial r}{\partial x} = 2x$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$

Differentiating above once again, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{xf'(r)}{r}\right]$$

$$= \frac{r\left[f'(r).1 + xf''(r)(\partial r/\partial x)\right] - xf'(r)(\partial r/\partial x)}{r^2}$$

or $\frac{\partial^2 u}{\partial x^2} = \frac{1}{r^2}\,[rf'(r) + x^2 f''(r) - \frac{x^2}{r} f'(r)]$ (ii)

Similarly, $\frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\,[rf'(r) + y^2 f''(r) - \frac{y^2}{r} f'(r)]$ (iii)

Adding (ii) and (iii), we get

$$\frac{\partial u^2}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left[2rf'(r) + \left(x^2 + y^2\right)f''(r) - \frac{\left(x^2 + y^2\right)}{r}f'(r)\right]$$

$$= \frac{1}{r^2}\,[2r\,f'(r) + r^2\,f''(r) - r\,f'(r)]$$

$= \frac{1}{r}f'(r) + f''(r)$, Hence proved.

Example 5 : If $x^x y^y z^z = c$, show that at $x = y = z$,

$$\frac{\partial^2 z}{\partial x \partial y} = -\left(x \log ex\right)^{-1}$$

Solution : Given $x^x y^y z^z = c$, where z is a function of x and y

Taking logarithms, $x \log x + y \log y + z \log z = \log c$ (i)

Differentiating (i) partially with respect to x, we get

$$\left[x\left(\frac{1}{x}\right) + (\log x)1\right] + \left[z\left(\frac{1}{z}\right) + (\log z)1\right]\frac{\partial z}{\partial x} = 0$$

or $\frac{\partial z}{\partial x} = -\frac{(1 + \log x)}{(1 + \log z)}$ (ii)

Similarly from (i) we have

$\frac{\partial z}{\partial y} = -\frac{(1 + \log y)}{(1 + \log z)}$ (iii)

$$\therefore \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$= \frac{\partial}{\partial x}\left[-\left(\frac{1 + \log y}{1 + \log z}\right)\right]$ From (iii) or $\frac{\partial^2 z}{\partial x \partial y} = -(1+\log y).\frac{\partial}{\partial x}\left[(1+\log z)^{-1}\right]$

$= -(1+\log y).\left[-(1+\log z)^{-2}.\frac{1}{z}.\frac{\partial z}{\partial x}\right]$

or $\frac{\partial^2 z}{\partial x \partial y} = \frac{(1+\log y)}{z(1+\log z)^2}.\left\{-\left(\frac{1+\log x}{1+\log z}\right)\right\}$, using (ii)

At x = y = z, we have $\frac{\partial^2 z}{\partial x \partial y} = -\frac{(1+\log x)^2}{x(1+\log x)^3}$

Substituting x for y and z

i.e. $\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x(1+\log x)}$

$= -\frac{1}{x(\log e + \log x)}$ $\quad \therefore \log e = 1$

$= -\frac{1}{x \log(ex)}$

$= -\{x \log(ex)\}^{-1}$ Hence Proved.

Chain rule of Partial Differentiation

Change of Variables : If u is a function of x, y and x, y are functions of t and r, then u is called a composite function of t and r.

Let u =f(x, y) and x = g(t, r), y = h(t, r) then the continuous first order partial derivatives are

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial t}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial r}$$

This is called as Chain rule of Partial Differentiation.

Problems

Example 1:

If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$ show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

Solution : Here given $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$

= u (r, s)

where $r = \frac{y-x}{xy}$ and $s = \frac{z-x}{zx}$

$\Rightarrow r = \frac{1}{x} - \frac{1}{y}$ and $s = \frac{1}{x} - \frac{1}{z}$ ....(i)

we know that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}$$

$$= \frac{\partial u}{\partial r}\left(-\frac{1}{x^2}\right) + \frac{\partial u}{\partial s}\left(-\frac{1}{x^2}\right) \qquad \because r = \frac{1}{x} - \frac{1}{y}$$

$$\Rightarrow \frac{\partial r}{\partial x} = -\frac{1}{x^2}$$

$$= -\frac{1}{x^2}\frac{\partial u}{\partial r} - \frac{1}{x^2}\frac{\partial u}{\partial s} \qquad \because s = \frac{1}{x} - \frac{1}{z}$$

$$\Rightarrow \frac{\partial s}{\partial x} = -\frac{1}{x^2}$$

or $x^2 \frac{\partial u}{\partial x} = -\frac{\partial u}{\partial r} - \frac{\partial u}{\partial s}$ ....(ii)

Similarly $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}$

$= \frac{\partial u}{\partial y}\cdot\frac{1}{y^2} + \frac{\partial u}{\partial s}.0$ from (i)

or $y^2 \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}$ ....(iii)

and $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}$

$= \frac{\partial u}{\partial r}.0 + \frac{\partial u}{\partial s}\cdot\frac{1}{z^2}$

$\Rightarrow z^2 \frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}$ ....(iv)

Adding (i) (ii) and (iii) we get

$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$ Hence Proved.

Example 2:

If $u = u(y - z, z - x, x - y)$ Prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Solution : Here given $u = u(y - z, z - x, x - y)$

Let $X = y - z$, $Y = z - x$ and $Z = x - y$................(i)

Then $u = u(X,Y,Z)$, where X, Y, Z are function of x,y and z.

Then

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial X}\frac{\partial X}{\partial x}+\frac{\partial u}{\partial Y}\frac{\partial Y}{\partial x}+\frac{\partial u}{\partial Z}\frac{\partial Z}{\partial x}\ \ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial X}\frac{\partial X}{\partial y}+\frac{\partial u}{\partial Y}\frac{\partial Y}{\partial y}+\frac{\partial u}{\partial Z}\frac{\partial Z}{\partial y}\ \ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

and $\frac{\partial u}{\partial z}=\frac{\partial u}{\partial X}\frac{\partial X}{\partial z}+\frac{\partial u}{\partial Y}\frac{\partial Y}{\partial z}+\frac{\partial u}{\partial Z}\frac{\partial Z}{\partial z}\ \ldots\ldots\ldots\text{(iv)}$

with the help of (i), equations (ii), (iii) and (iv) gives.

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial X}0+\frac{\partial u}{\partial Y}(-1)+\frac{\partial u}{\partial Z}(1)=-\frac{\partial u}{\partial Y}+\frac{\partial u}{\partial Z}\ \ldots\ldots\ldots\ldots\ldots\ldots\text{(v)}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial X}.1+\frac{\partial u}{\partial Y}.0+\frac{\partial u}{\partial Z}(-1)=\frac{\partial u}{\partial X}-\frac{\partial u}{\partial Z}\ \ldots\ldots\ldots\ldots\ldots\ldots\text{(vi)}$$

and $\frac{\partial u}{\partial z}=\frac{\partial u}{\partial X}(-1)+\frac{\partial u}{\partial Y}(1)+\frac{\partial u}{\partial Z}(0)=-\frac{\partial u}{\partial X}+\frac{\partial u}{\partial Y}\ \ldots\ldots\ldots\ldots\text{(vii)}$

Adding (v), (vi) and (vii) we get $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0.$ Hence Proved.

Example 3: If z is a function of x and y and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$,

Prove that $\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}$

Solution : Here z is a function of x and y, where x and y are functions of u and v.

$$\therefore \frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\ \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

and $\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}.\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}\ \ldots\ldots\ldots\text{(ii)}$

Also given that

$x = e^u + e^{-v}$ and $y = e^{-u} - e^v$

$$\therefore \frac{\partial x}{\partial u}=e^u,\ \frac{\partial x}{\partial v}=-e^{-v},\ \frac{\partial y}{\partial u}=-e^{-u},\ \frac{\partial y}{\partial v}=-e^{v}$$

$\therefore$ From (i) we get

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}(e^u)+\frac{\partial z}{\partial y}(-e^{-u})\ \ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

and from (ii) we get

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}(-e^{-v})+\frac{\partial z}{\partial y}(-e^{v})\ \ldots\ldots\ldots\ldots\text{(iv)}$$

Subtracting (iv) from (iii) we get

$$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=(e^u+e^{-v})\frac{\partial z}{\partial x}-(e^{-u}-e^{v})\frac{\partial z}{\partial y}$$

$=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}.$ Hence Proved.

Example 4:

If V = f(2x -3y, 3y -4z, 4z -2x), compute the value of $6V_x + 4V_y + 3V_z$.

Solution : Here given V = f(2x −3y, 3y −4z, 4z −2x)

~~Let X = 2x-3y, Y =3y-4z and Z = 4z-2x~~............(i)

Then u = f(X, Y, Z), where X, Y, Z are function of x, y and z.

Then $V_x = \frac{\partial V}{\partial x} = \frac{\partial V}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial V}{\partial Y}\frac{\partial Y}{\partial x} + \frac{\partial V}{\partial Z}\frac{\partial Z}{\partial x}$ ....(ii)

$V_y = \frac{\partial V}{\partial y} = \frac{\partial V}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial V}{\partial Y}\frac{\partial Y}{\partial y} + \frac{\partial V}{\partial Z}\frac{\partial Z}{\partial y}$ ....(iii)

and $V_z = \frac{\partial V}{\partial z} = \frac{\partial V}{\partial X}\frac{\partial X}{\partial z} + \frac{\partial V}{\partial Y}\frac{\partial Y}{\partial z} + \frac{\partial V}{\partial Z}\frac{\partial Z}{\partial z}$ ....(iv)

with the help of (i), equations (ii), (iii) and (iv) gives

$$V_x = \frac{\partial V}{\partial X}(2) + \frac{\partial V}{\partial Y}(0) + \frac{\partial V}{\partial Z}(-2)$$

or $V_x = 2\left(\frac{\partial V}{\partial X} - \frac{\partial V}{\partial Z}\right)$

$\Rightarrow 6V_x = 12\left(\frac{\partial V}{\partial X} - \frac{\partial V}{\partial Z}\right)$ ....(v)

Now $V_y = \frac{\partial V}{\partial X}(-3) + \frac{\partial V}{\partial Y}(3) + \frac{\partial V}{\partial Z}(0)$

or $V_y = 3\left(-\frac{\partial V}{\partial X} + \frac{\partial V}{\partial Y}\right)$

$\Rightarrow 4V_y = 12\left(-\frac{\partial V}{\partial X} + \frac{\partial V}{\partial Y}\right)$ ....(vi)

and $V_z = \frac{\partial V}{\partial X}(0) + \frac{\partial V}{\partial Y}(-4) + \frac{\partial V}{\partial Z}(4)$

or $V_z = 4\left(-\frac{\partial V}{\partial Y} + \frac{\partial V}{\partial Z}\right)$

$\Rightarrow 3V_z = 12\left(-\frac{\partial V}{\partial Y} + \frac{\partial V}{\partial Z}\right)$ ....(vii)

Adding (v), (vi) and (vii) we get

$6V_x + 4V_y + 3V_z = 0$ Answer.

Total Differentiation

Introduction : In partial differentiation of a function of two or more variables, only one variable varies. But in total differentiation, increments are given in all the variables.

Total differential Coefficient : If u = f(x,y)

where x = ϕ(t), and y = Ψ(t) then we can find the value of u in terms of t by substituting from the last two equations in the first equation. Hence we can regard u as a function of the single variable t, and find the ordinary differential coefficient $\frac{du}{dt}$.

Then $\frac{du}{dt}$ is called the total differential coefficient of u, to distinguish it from the partial differential coefficient $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

Hence

$$\frac{du}{dt} = \frac{\partial f}{\partial x}.\frac{dx}{dt} + \frac{\partial f}{\partial y}.\frac{dy}{dt}$$

i.e. $$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

Similarly, if $u = f(x_1, x_2 \ldots\ldots x_n)$ and $x_1, x_2 \ldots\ldots, x_n$ are all functions of t, we can prove that

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1}.\frac{dx_1}{dt} + \frac{\partial u}{\partial x_2}.\frac{dx_2}{dt} + \ldots\ldots + \frac{\partial u}{\partial x_n}.\frac{dx_n}{dt}$$

An important case : By supposing t to be the same, as x in the formula for two variables, we get the following proposition :

When f(x,y) is a function of x and y, and y is a function of x, the total (i.e., the ordinary) differential coefficient of f with respect to x is given by

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{dy}{dx}$$

Now, if we have an implicit relation between x and y of the form f(x,y) = C where C is a constant and y is a function of x, the above formula becomes

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{dy}{dx}$$

Which gives the important formula

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}$$

Again, if f is a function of n variables $x_1, x_2, x_3, \ldots\ldots x_n$, and $x_2, x_3 \ldots\ldots x_n$ are all functions of x_1, the total (i.e. the ordinary) differential coefficient of f with respect to x_1 is given by

$$\frac{df}{dx_1} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2}.\frac{dx_2}{dx_1} + \frac{\partial f}{\partial x_3}.\frac{dx_3}{dx_1} + \ldots\ldots + \frac{\partial f}{\partial x_n}.\frac{dx_n}{dx_1}$$

Example 1:

If $u = x \log xy$, where $x^3+y^3+3xy = 1$, find $\frac{du}{dx}$.

Solution : Given $u = x \log xy$……………..(i)

we know $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$ ……………….(ii)

Now from (i) $\frac{\partial u}{\partial x} = x.\frac{1}{xy} y + \log x y$

$= 1 + \log x y$

and $\frac{\partial u}{\partial y} = x\frac{1}{xy}x = \frac{x}{y}$

Again, we are given $x^3+y^3+3xy = 1$, whence differentiating, we get

$3x^2 + 3y^2 \frac{dy}{dx} + 3\left(x\frac{dy}{dx} + y.1\right) = 0$

or $\frac{dy}{dx} = -\frac{(x^2+y)}{(y^2+x)}$

Substituting these values in (ii) we get

$\frac{du}{dx} = (1+\log xy) + \frac{x}{y}\left[-\frac{(x^2+y)}{(y^2+x)}\right]$ Answer.

Example 2:

If $f(x, y) = 0$, $\phi(y, z) = 0$ show that $\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial z}\frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{\partial \phi}{\partial y}$

Solution : If $f(x, y) = 0$ then $\frac{dy}{dx} = -\left(\frac{\partial f}{\partial x}\right)/\left(\frac{\partial f}{\partial y}\right)$………….(i)

if $\phi(y, z) = 0$, then $\frac{dz}{dy} = -\left(\frac{\partial \phi}{\partial y}\right)/\left(\frac{\partial \phi}{\partial z}\right)$…………………..(ii)

Multiplying (i) and (ii), we have

$\frac{dy}{dx}.\frac{dz}{dy} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial \phi}{\partial y}\right)/\left(\frac{\partial f}{\partial y}\right).\left(\frac{\partial \phi}{\partial z}\right)$

or $\left(\frac{\partial f}{\partial y}\right).\left(\frac{\partial \phi}{\partial z}\right)\frac{dz}{dx} = \left(\frac{\partial f}{\partial x}\right)/\left(\frac{\partial \phi}{\partial y}\right)$. Hence Proved

Example 3:

If the curves $f(x,y) = 0$ and $\phi(x, y) = 0$ touch, show that at the point of contact $\frac{\partial f}{\partial x}.\frac{\partial \phi}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial \phi}{\partial x} = 0$

Solution : For the curve f(x, y) =0, we have

$$\frac{dy}{dx} = -\left(\frac{\partial f}{\partial x}\right) / \left(\frac{\partial f}{\partial y}\right)$$ and for the curve $\phi(x, y) = 0$, $$\frac{dy}{dx} = -\left(\frac{\partial \phi}{\partial x}\right) / \left(\frac{\partial \phi}{\partial y}\right)$$

Also if two curves touch each other at a point then at that point the values of (dy/dx) for the two curves must be the same,

Hence at the point of contact

$$-\left(\frac{\partial f}{\partial x}\right) / \left(\frac{\partial f}{\partial y}\right) = -\left(\frac{\partial \phi}{\partial x}\right) / \left(\frac{\partial \phi}{\partial y}\right)$$

or $$\left(\frac{\partial f}{\partial x}\right) \cdot \left(\frac{\partial \phi}{\partial y}\right) - \left(\frac{\partial \phi}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right) = 0.$$ Hence Proved

Example 4:

If $\phi(x,y,z) = 0$ show that $$\left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z = -1$$

Solution : The given relation defines y as a function of x and z. treating x as constant

$$\left(\frac{\partial y}{\partial z}\right)_x = -\frac{\partial \phi / \partial z}{\partial \phi / \partial y} \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

The given relation defines z as a function of x and y. Treating y as constant

$$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\partial \phi / \partial x}{\partial \phi / \partial z} \ldots\ldots\ldots\text{(ii)}$$

Similarly, $$\left(\frac{\partial x}{\partial z}\right)_z = -\frac{\partial \phi / \partial y}{\partial \phi / \partial x} \ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

Multiplying (i), (ii) and (iii) we get

$$\left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z = -1$$ Hence Proved.

Euler's Theorem on Homogeneous Functions :

Statement : If f(x,y) is a homogeneous function of x and y of degree n, then

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$$

Proof : Since f(x,y) is a homogeneous function of degree n, it can be expressed in the form

$f(x,y) = x^n F(y/x)$………………(i)

$$\therefore \frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\{x^n F(y/x)\} = nx^{n-1} F(y/x) + x^n F'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

or $x\frac{\partial f}{\partial x} = n\,x^n F\left(\frac{y}{x}\right) - yx^{n-1} F'\left(\frac{y}{x}\right)$……………..(ii)

Again from (i), we have

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\{x^n F(y/x)\}$$

$$= x^n F'(y/x).\ \frac{1}{x}$$

or $y\frac{\partial f}{\partial y} = yx^{n-1} F'(y/x)$…………..(iii)

Adding (ii) and (iii), we get

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = nx^n F(y/x)$$

= nf using(i) Hence Proved.

Note. In general if f $(x_1, x_2 \ldots\ldots\ldots\ldots x_n)$ be a homogeneous function of degree n, then by Euler's theorem, we have

$$x_1\frac{\partial f}{\partial x_1}+x_2\frac{\partial f}{\partial x_2}+\ldots\ldots\ldots\ldots+x_n\frac{\partial f}{\partial x_n}=nf$$

Example 1: If $u = \log\left(\frac{x^2+y^2}{x+y}\right)$, Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$

Solution : We are given that

$$u = \log\left(\frac{x^2+y^2}{x+y}\right)$$

$$\therefore e^u = \frac{x^2+y^2}{x+y} = f(\text{say})$$

Clearly f is a homogeneous function in x and y of degree 2-1 i.e. 1

$\therefore$ By Euler's theorem for f, we should have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f$$

$$x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = e^u \qquad \because f = e^u$$

$$\text{or } xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = e^u$$

or $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$ Hence Proved.

Example 2: If $u = \sin^{-1}\left\{\frac{x+y}{\sqrt{x}+\sqrt{y}}\right\}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$

Solution : Here $u = \sin^{-1}\left\{\frac{x+y}{\sqrt{x}+\sqrt{y}}\right\}$

$$\Rightarrow \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = f\,(\text{say})$$

Here f is a homogeneous function in x and y of degree $\left(1-\frac{1}{2}\right)$ i.e $\frac{1}{2}$

$\therefore$ By Euler's theorem for f, we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{1}{2}f$$

$$\text{or } x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \frac{1}{2}\sin u$$

$\therefore f = \sin u$

$$\text{or } x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

or $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u.$ Hence Proved

Example 3:

If $u = \tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u$

Solution : Here $\tan u = \frac{x^2+y^2}{x+y} = f$ (say)

Then for $\frac{x^2+y^2}{x+y}$ is a homogeneous function in x and y of degree 2-1 i.e 1.

$\therefore$ By Euler's theorem for f, we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 1.f$$

or $x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = \tan u$

$\because f = \tan u$

or $x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = \tan u$

or $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\tan u}{\sec^2 u} = \sin u\cos u = \frac{1}{2}\sin 2u.$ Hence Proved

Example 4:

If u be a homogeneous function of degree n, then prove that

(i) $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = (n-1)\frac{\partial u}{\partial x}$

(ii) $x\frac{\partial^2 u}{\partial x\partial y} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}$ (iii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$

Solution : Since u is a homogenous function of degree n, therefore by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \text{ ....(1)}$$

Differentiating (i) partially w.r.t. x, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}.1 + y\frac{\partial^2 u}{\partial x\partial y} = n\frac{\partial u}{\partial x}$$

or $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = n\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}$

or $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = (n-1)\frac{\partial u}{\partial x}$ ....(2)

which prove the result (i)

Now differentiating (i) partially w.r.t. y, we get

$$x\frac{\partial^2 u}{\partial y\partial x} + y\frac{\partial^2 u}{\partial y^2} + 1.\frac{\partial u}{\partial y} = n\frac{\partial u}{\partial y}$$

or $x\frac{\partial^2 u}{\partial y\partial x} + y\frac{\partial^2 u}{\partial y^2} = n\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}$

or $x\frac{\partial^2 u}{\partial y\partial x} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}$ ....(3)

Which proves the result (ii)

Multiplying (2) by x and (3) by y and then adding, we get

$$x\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}+xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\left\{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right\}$$

or $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\,nu$

which proves the result (iii). Hence Proved

Example 5:

If u(x,y,z) = log (tan x + tan y + tan z) Prove that

$$\sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}+\sin 2z\frac{\partial u}{\partial z}=2$$

Solution : we have

u(x,y,z) = log (tan x + tan y + tan z)................(i)

Differentiating (i) w.r.t. 'x' partially, we get

$$\frac{\partial u}{\partial x}=\frac{\sec^2 x}{\tan x+\tan y+\tan z}\text{ ....(ii)}$$

Differentiating (i) w.r.t. 'y' partially we get

$$\frac{\partial u}{\partial y}=\frac{\sec^2 y}{\tan x+\tan y+\tan z}\text{ ....(iii)}$$

Again differentiating (i) w.r.t 'z' partially we get

$$\frac{\partial u}{\partial z}=\frac{\sec^2 z}{\tan x+\tan y+\tan z}\text{ ....(iv)}$$

Multiplying (ii), (iii) and (iv) by sin 2x, sin 2y and sin 2z respectively and adding them, we get

$$\sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}+\sin 2z\frac{\partial u}{\partial z}=\frac{\sin 2x\sec^2 x+\sin 2y\sec^2 y+\sin 2z\sec^2 z}{\tan x+\tan y+\tan z}$$

$$=\frac{2\sin x\cos x.\sec^2 x+2\sin y\cos y.\sec^2 y+2\sin z\cos z.\sec^2 z}{\tan x+\tan y+\tan z}$$

$$=\frac{2(\tan x+\tan y+\tan z)}{\tan x+\tan y+\tan z}$$

$=2$

$$\Rightarrow \sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}+\sin 2z\frac{\partial u}{\partial z}=2.\text{ Hence Proved}$$

** Maximum & Minimum for function of a single Variable:

To find the Maxima & Minima of f(x) we use the following procedure.

(i) Find $f^1(x)$ and equate it to zero

(ii) Solve the above equation we get x_0,x_1 as roots.

(iii) Then find $f^{11}(x)$.

If $f^{11}(x)_{(x=x0)} > 0$, then f(x) is minimum at x_0

If $f^{11}(x)_{(x=x0)}, < 0$, f(x) is maximum at x_0. Similarly we do this for other stationary points.

PROBLEMS:

1. Find the max & min of the function $f(x) = x^5 - 3x^4 + 5$ ('08 S-1)

Sol: Given $f(x) = x^5 - 3x^4 + 5$

$f^1(x) = 5x^4 - 12x^3$

for maxima or minima $f^1(x) = 0$

$5x^4 - 12x^3 = 0$

$x = 0, x = 12/5$

$f^{11}(x) = 20x^3 - 36x^2$

At $x = 0 \Rightarrow f^{11}(x) = 0$. So f is neither maximum nor minimum at x = 0

At $x = (12/5) \Rightarrow f^{11}(x) = 20(12/5)^3 - 36(12/5)$

$= 144(48-36)/25 = 1728/25 > 0$

So f(x) is minimum at x = 12/5

The minimum value is $f(12/5) = (12/5)^5 - 3(12/5)^4 + 5$

** Maxima & Minima for functions of two Variables:

Working procedure:

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ Equate each to zero. Solve these equations for x & y we get the pair of values $(a_1, b1)$ (a_2, b_2) (a_3, b_3) ……………..

2. Find $l = \frac{\partial^2 f}{\partial x^2}, m = \frac{\partial^2 f}{\partial x \partial y}$, $n = \frac{\partial^2 f}{\partial y^2}$

3. i. If $ln - m^2 > 0$ and $l < 0$ at (a_1, b_1) then f(x ,y) is maximum at (a_1, b_1) and maximum value is $f(a_1, b_1)$

ii. If $ln - m^2 > 0$ and $l > 0$ at (a_1, b_1) then f(x ,y) is minimum at (a_1, b_1) and minimum value is $f(a_1, b_1)$.

iii. If $ln - m^2 < 0$ and at $(a_1, b1)$ then f(x, y) is neither maximum nor minimum at (a_1, b_1). In this case $(a_1, b1)$ is saddle point.

iv. If $ln - m^2 = 0$ and at (a_1, b_1) , no conclusion can be drawn about maximum or minimum and needs further investigation. Similarly we do this for other stationary points.

PROBLEMS:

1. **Locate the stationary points & examine their nature of the following functions.**

$u = x^4 + y^4 - 2x^2 + 4xy - 2y^2$, $(x > 0, y > 0)$

Sol: Given $u(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

For maxima & minima $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 0$

$\frac{\partial u}{\partial x} = 4x^3 - 4x + 4y = 0 \Rightarrow x^3 - x + y = 0$ -------------------> (1)

$\frac{\partial u}{\partial y} = 4y^3 + 4x - 4y = 0 \Rightarrow y^3 + x - y = 0$ -------------------> (2)

Adding (1) & (2),

$x^3 + y^3 = 0$

$\Rightarrow x = -y$ -------------------> (3)

(1) $\Rightarrow x^3 - 2x \Rightarrow x = 0, \sqrt{2}, -\sqrt{2}$

Hence (3) $\Rightarrow y = 0, -\sqrt{2}, \sqrt{2}$

$l = \frac{\partial^2 u}{\partial x^2} = 12x^2 - 4$, $m = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial u}{\partial y}) = 4$ & $n = \frac{\partial^2 u}{\partial y^2} = 12y^2 - 4$

$ln - m^2 = (12x^2 - 4)(12y^2 - 4) - 16$

At $(-\sqrt{2}, \sqrt{2})$, $ln - m^2 = (24 - 4)(24 - 4) - 16 = (20)(20) - 16 > 0$ and $l = 20 > 0$

The function has minimum value at $(-\sqrt{2}, \sqrt{2})$

At (0,0), $ln - m^2 = (0 - 4)(0 - 4) - 16 = 0$

(0,0) is not a extreme value.

2. **Investigate the maxima & minima, if any, of the function $f(x) = x^3y^2(1-x-y)$.**

Sol: Given $f(x) = x^3y^2(1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$

$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$ $\frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$

For maxima & minima $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

⇨ $3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0 \Rightarrow x^2y^2(3 - 4x - 3y) = 0$ ---------------> (1)

⇨ $2x^3y - 2x^4y - 3x^3y^2 = 0 \Rightarrow x^3y(2 - 2x - 3y) = 0$ ----------------> (2)

From (1) & (2) $4x + 3y - 3 = 0$

$2x + 3y - 2 = 0$

$2x = 1 \Rightarrow x = ½$

$4(½) + 3y - 3 = 0 \Rightarrow 3y = 3 - 2$, $y = (1/3)$

$l = \frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$

$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(1/2,1/3)} = 6(1/2)(1/3)^2 - 12\,(1/2)^2(1/3)^2 - 6(1/2)(1/3)^3 = 1/3 – 1/3 - 1/9 = -1/9$

$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 6x^2y - 8\,x^3y – 9x^2y^2$

$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(1/2\,,1/3)} = 6(1/2)^2(1/3) - 8\,(1/2)^3(1/3) - 9(1/2)^2(1/3)^3 = \frac{6-4-3}{12} = \frac{-1}{12}$

$n = \frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y$

$\left(\frac{\partial^2 f}{\partial y^2}\right)_{(1/2,1/3)} = 2(1/2)^3 - 2(1/2)^4 - 6(1/2)^3(1/3) = \frac{1}{4} - \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}$

$ln - m^2 = (-1/9)(-1/8) – (-1/12)^2 = \frac{1}{72} - \frac{1}{144} = \frac{2-1}{144} = \frac{1}{144} > 0$ and $l = \frac{-1}{9} < 0$

The function has a maximum value at (1/2 , 1/3)

$\therefore$ Maximum value is $f\left(\frac{1}{2},\frac{1}{3}\right) = \left(\frac{1}{8}\times\frac{1}{9}\right)\left(1-\frac{1}{2}-\frac{1}{3}\right) = \frac{1}{72}\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{1}{432}$

3. **Find three positive numbers whose sum is 100 and whose product is maximum.**

Sol: Let x ,y ,z be three +ve numbers.

Then $x + y + z = 100$

⇨ $z = 100 – x – y$

Let $f(x,y) = xyz = xy(100 – x – y) = 100xy – x^2y - xy^2$

For maxima or minima $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 100y – 2xy - y^2 = 0 \Rightarrow y(100 - 2x – y) = 0$ ----------------> (1)

$\frac{\partial f}{\partial y} = 100x – x^2 - 2xy = 0 \Rightarrow x(100 – x - 2y) = 0$ ------------------> (2)

From (1) & (2)

$100 - 2x – y = 0$

$200 - 2x - 4y = 0$

$-100 + 3y = 0 \Rightarrow 3y = 100 \Rightarrow y = 100/3$

$100 – x – (200/3) = 0 \Rightarrow x = 100/3$

$l = \frac{\partial^2 f}{\partial x^2} = -2y$

$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(100/3\,,\,100/3)} = -200/3$

$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 100 - 2x - 2y$

$\left(\frac{\partial^2 f}{\partial x \partial y}\right)$ (100/3 , 100/3) = 100 –(200/3) –(200/3) = -(100/3)

$n = \frac{\partial^2 f}{\partial y^2} = -2x$

$\left(\frac{\partial^2 f}{\partial y^2}\right)$ (100/3 , 100/3) = - 200/3

$ln - m^2 = (-200/3)(-200/3) - (-100/3)^2 = (100)^2/3$

The function has a maximum value at (100/3 , 100/3)

i.e. at x = 100/3, y = 100/3 $\therefore z = 100 - \frac{100}{3} - \frac{100}{3} = \frac{100}{3}$

The required numbers are x = 100/3, y = 100/3, z = 100/3

4. Find the maxima & minima of the function $f(x) = 2(x^2 - y^2) - x^4 + y^4$

Sol: Given $f(x) = 2(x^2 - y^2) - x^4 + y^4 = 2x^2 - 2y^2 - x^4 + y^4$

For maxima & minima $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 4x - 4x^3 = 0 \Rightarrow 4x(1-x^2) = 0 \Rightarrow x = 0 , x = \pm 1$

$\frac{\partial f}{\partial y} = -4y + 4y^3 = 0 \Rightarrow -4y(1-y^2) = 0 \Rightarrow y = 0, y = \pm 1$

$l = \left(\frac{\partial^2 f}{\partial x^2}\right) = 4 - 12x^2$

$m = \left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 0$

$n = \left(\frac{\partial^2 f}{\partial y^2}\right) = -4 + 12y^2$

we have $ln - m^2 = (4-12x^2)(-4+12y^2) - 0$

$= -16 + 48x^2 + 48y^2 - 144x^2y^2$

$= 48x^2 + 48y^2 - 144x^2y^2 - 16$

i) At (0 , ± 1)

$ln - m^2 = 0 + 48 - 0 - 16 = 32 > 0$

l = 4-0 = 4 > 0

f has minimum value at (0 , ± 1)

$f(x, y) = 2(x^2 - y^2) - x^4 + y^4$

f (0 , ± 1) = 0 – 2 – 0 + 1 = -1

The minimum value is '-1 '.

ii) At (± 1 ,0)

$ln - m^2 = 48 + 0 - 0 - 16 = 32 > 0$

$l = 4-12 = -8 < 0$

f has maximum value at (± 1 ,0)

$f(x,y) = 2(x^2 - y^2) - x^4 + y^4$

$f(\pm 1, 0) = 2 - 0 - 1 + 0 = 1$

The maximum value is '1 '.

iii) At (0,0) , (± 1 , ± 1)

$ln - m^2 < 0$

$l = 4 - 12x^2$

(0 , 0) & (± 1 , ± 1) are saddle points.

f has no max & min values at (0 , 0) , (± 1 , ± 1).

***Extremum** : A function which have a maximum or minimum or both is called 'extremum'

***Extreme value** :- The maximum value or minimum value or both of a function is Extreme value.

***Stationary points:** - To get stationary points we solve the equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ i.e the pairs $(a_1, b_1), (a_2, b_2)$ ………….. are called Stationary.

*Maxima & Minima for a function with constant condition :Lagranges Method

Suppose $f(x, y, z) = 0$ ------------(1)

$\emptyset(x, y, z) = 0$ ------------- (2)

$F(x, y, z) = f(x, y, z) + \gamma\, \emptyset(x, y, z)$ where γ is called Lagrange's constant.

1. $\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \gamma \frac{\partial \emptyset}{\partial x} = 0$ --------------- (3)

   $\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + \gamma \frac{\partial \emptyset}{\partial y} = 0$ --------------- (4)

   $\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{\partial f}{\partial z} + \gamma \frac{\partial \emptyset}{\partial z} = 0$ --------------- (5)

2. Solving the equations (2) (3) (4) & (5) we get the stationary point (x, y, z).
3. Substitute the value of x , y , z in equation (1) we get the extremum.

Problem:

1. **Find the minimum value of $x^2 + y^2 + z^2$, given $x + y + z = 3a$ ('08 S-2)**

Sol: $u = x^2 + y^2 + z^2$

$\varnothing = x + y + z - 3a = 0$

Using Lagrange's function

$F(x, y, z) = u(x, y, z) + \gamma\, \varnothing(x, y, z)$

For maxima or minima

$$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x} + \gamma \frac{\partial \varnothing}{\partial x} = 2x + \gamma = 0 \text{ ------------ (1)}$$

$$\frac{\partial F}{\partial y} = \frac{\partial u}{\partial y} + \gamma \frac{\partial \varnothing}{\partial y} = 2y + \gamma = 0 \text{ ------------ (2)}$$

$$\frac{\partial F}{\partial z} = \frac{\partial u}{\partial z} + \gamma \frac{\partial \varnothing}{\partial z} = 2z + \gamma = 0 \text{ ------------ (3)}$$

From (1) , (2) & (3)

$\gamma = -2x = -2y = -2z$

$x = y = z$

$\varnothing = x + x + x - 3a = 0 \;\; x = a$

$x = y = z = a$

Minimum value of $u = a^2 + a^2 + a^2 = 3a^2$

MODULE-III

HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS

LINEAR DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER

Definition: An equation of the form $\frac{d^n y}{dx^n}$ + $P_1(x)$ · $\frac{d^{n-1} y}{dx^{n-1}}$ + $P_2(x)$ · $\frac{d^{n-2} y}{dx^{n-2}}$ + --------+ $P_n(x)$ .y = Q(x) Where $P_1(x)$, $P_2(x)$, $P_3(x)$… …..$P_n(x)$ and Q(x) (functions of x) continuous is called a linear differential equation of order n.

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Def: An equation of the form $\frac{d^n y}{dx^n}$ + P_1 · $\frac{d^{n-1} y}{dx^{n-1}}$ + P_2 · $\frac{d^{n-2} y}{dx^{n-2}}$ + --------+ P_n .y = Q(x) where P_1, P_2, P_3,…..P_n, are real constants and Q(x) is a continuous function of x is called an linear differential equation of order ' n' with constant coefficients.

Note:

1. Operator D $=\frac{d}{dx}$; $D^2 = \frac{d^2}{dx^2}$; …………………… $D^n = \frac{d^n}{dx^n}$

   $Dy = \frac{dy}{dx}$; $D^2 y = \frac{d^2 y}{dx^2}$; …………………… $D^n y = \frac{d^n y}{dx^n}$

2. Operator $\frac{1}{D}Q = \int Q$ i e $D^{-1}Q$ is called the integral of Q.

To find the general solution of f(D).y = 0 :

Where $f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2}$ +-----------+P_n is a polynomial in D.

Now consider the auxiliary equation : f(m) = 0

i.e $f(m) = m^n + P_1 m^{n-1} + P_2 m^{n-2}$ +-----------+P_n = 0

where p_1, p_2, p_3 ……………p_n are real constants.

Let the roots of f(m) =0 be m_1, m_2, m_3,…..m_n.

Depending on the nature of the roots we write the complementary function as follows:

Consider the following table

S.No	Roots of A.E f(m) =0	Complementary function(C.F)
1.	$m_1, m_2, ..m_n$ are real and distinct.	$y_c = c_1e^{m_1x} + c_2e^{m_2x} + \dots + c_ne^{m_nx}$
2.	$m_1, m_2, ..m_n$ are and two roots are equal i.e., m_1, m_2 are equal and real(i.e repeated twice) &the rest are real and different.	$y_c = (c_1+c_2x)e^{m_1x} + c_3e^{m_3x} + \dots + c_ne^{m_nx}$
3.	$m_1, m_2, ..m_n$ are real and three roots are equal i.e., m_1, m_2, m_3 are equal and real(i.e repeated thrice) &the rest are real and different.	$y_c = (c_1+c_2x+c_3x^2)e^{m_1x} + c_4e^{m_4x} + \dots + c_ne^{m_nx}$
4.	Two roots of A.E are complex say α+iβ α -iβ and rest are real and distinct.	$y_c = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x) + c_3e^{m_3x} + \dots + c_ne^{m_nx}$
5.	If $\alpha\pm i\beta$ are repeated twice & rest are real and distinct	$y_c = e^{\alpha x}[(c_1+c_2x)\cos\beta x + (c_3+c_4x)\sin\beta x] + c_5e^{m_5x} + \dots + c_ne^{m_nx}$
6.	If $\alpha\pm i\beta$ are repeated thrice & rest are real and distinct	$y_c = e^{\alpha x}[(c_1+c_2x+c_3x^2)\cos\beta x + (c_4+c_5x+c_6x^2)\sin\beta x] + c_7e^{m_7x} + \dots\dots\dots + c_ne^{m_nx}$
7.	If roots of A.E. irrational say $\alpha \pm \sqrt{\beta}$ and rest are real and distinct.	$y_c = e^{\alpha x}\left[c_1\cosh\sqrt{\beta}\,x + c_2\sinh\sqrt{\beta}\,x\right] + c_3e^{m_3x} + \dots\dots + c_ne^{m_nx}$

Solve the following Differential equations :

1. Solve $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$

Sol: Given equation is of the form f(D).y = 0

Where $f(D) = (D^3 - 3D + 2)\,y = 0$

Now consider the auxiliary equation f(m) = 0

$f(m) = m^3 - 3m + 2 = 0 \Rightarrow (m-1)(m-1)(m+2) = 0$

$\Rightarrow m = 1, 1, -2$

Since m_1 and m_2 are equal and m_3 is -2

We have $y_c = (c_1+c_2x)e^x + c_3e^{-2x}$

2. Solve $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0$

Sol: Given $f(D) = (D^4 - 2D^3 - 3D^2 + 4D + 4)\,y = 0$

⇨ A.equation $f(m) = (m^4 - 2m^3 - 3m^2 + 4m + 4) = 0$

⇨ $(m+1)^2(m-2)^2 = 0$

⇨ m= -1 , -1 , 2 , 2

⇨ $y_c = (c_1+c_2x)e^{-x} +(c_3+c_4x)e^{2x}$

3. Solve $(D^4 +8D^2 + 16)\, y = 0$

Sol: Given $f(D) = (D^4 +8D^2 + 16)\, y = 0$

Auxiliary equation $f(m) = (m^4 +8\, m^2 + 16) = 0$

⇨ $(m^2 + 4)^2 = 0$

⇨ $(m+2i)^2\, (m+2i)^2 = 0$

⇨ $m= 2i\, ,2i\, , -2i\, , -2i$

$Y_c = e^{0x}\, [(c_1+c_2x)\cos 2x + (c_3+c_4x)\sin 2x)]$

4. Solve $y^{11}+6y^1+9y = 0$; $y(0) = -4$, $y^1(0) = 14$

Sol: Given equation is $y^{11}+6y^1+9y = 0$

Auxiliary equationf(D) y = 0 ⇒ $(D^2 +6D +9)\, y = 0$

A.equation $f(m) = 0$ ⇒ $(m^2 +6m +9) = 0$

⇒ $m = -3\, ,-3$

$y_c = (c_1+c_2x)e^{-3x}$ -------------------> (1)

Differentiate of (1) w.r.to x ⇒ $y^1 =(c_1+c_2x)(-3e^{-3x}) + c_2(e^{-3x})$

Given $y_1\,(0) =14$ ⇒ $c_1 = -4$ & $c_2 =2$

Hence we get $y =(-4 + 2x)\,(e^{-3x})$

5. Solve $4y^{111} + 4y^{11} +y^1 = 0$

Sol: Given equation is $4y^{111} + 4y^{11} +y^1 = 0$

That is $(4D^3+4D^2+D)y=0$

Auxiliary equation f(m) = 0

$4m^3 +4m^2 + m = 0$

$m(4m^2 +4m + 1) = 0$

$m(2m + 1)^2 = 0$

$m = 0\, , -1/2\, ,-1/2$

$y =c_1+ (c_2+ c_3x)\, e^{-x/2}$

6. Solve $(D^2 - 3D +4)\, y = 0$

Sol: Given equation $(D^2 - 3D +4)\, y = 0$

A.E. f(m) = 0

$m^2-3m + 4 = 0$

$$m = \frac{3 \pm \sqrt{9-16}}{2} = \frac{3 \pm i\sqrt{7}}{2}$$

$$\alpha \pm i\beta = \frac{3 \pm i\sqrt{7}}{2} = \frac{3}{2} \pm i\,\frac{\sqrt{7}}{2}$$

$$y = e^{\frac{3}{2}x}\,(c_1 \cos\frac{\sqrt{7}}{2}x + c_2\sin\frac{\sqrt{7}}{2}x)$$

General solution of $f(D)\, y = Q(x)$

Is given by $y = y_c + y_p$

i.e. y = C.F+P.I

Where the P.I consists of no arbitrary constants and P.I of f (D) y = Q(x)

Is evaluated as P.I = $\frac{1}{f(D)}$. Q(x)

Depending on the type of function of Q(x).

P.I is evaluated as follows:

1. P.I of f (D) y = Q(x) where Q(x) =e^{ax} for (a) ≠ 0

Case1: P.I = $\frac{1}{f(D)}$. Q(x) = $\frac{1}{f(D)} e^{ax}$ = $\frac{1}{f(a)} e^{ax}$

Provided f(a) ≠ 0

Case 2: If f(a) = 0 then the above method fails. Then

if f(D) = $(D-a)^k \emptyset(D)$

(i.e ‘ a’ is a repeated root k times).

Then P.I = $\frac{1}{\emptyset(a)} e^{ax} \cdot \frac{1}{k!} x^k$ provided $\emptyset$ (a) ≠ 0

2. P.I of f(D) y =Q(x) where Q(x) = sin ax or Q(x) = cos ax where ‘ a ‘ is constant then P.I = $\frac{1}{f(D)}$. Q(x).

Case 1: In f(D) put $D^2 = -a^2 \ni f(-a^2) \neq 0$ then P.I = $\frac{\sin ax}{f(-a^2)}$

Case 2: If $f(-a^2) = 0$ then $D^2 + a^2$ is a factor of $\emptyset(D^2)$ and hence it is a factor of f(D). Then let $f(D) = (D^2 + a^2) .\Phi(D^2)$.

Then $\frac{\sin ax}{f(D)} = \frac{\sin ax}{(D^2+a^2)\Phi(D^2)} = \frac{1}{\Phi(-a^2)} \frac{\sin ax}{D^2+a^2} = \frac{1}{\Phi(-a^2)} \frac{-x\cos ax}{2a}$

$$\frac{\cos ax}{f(D)} = \frac{\cos ax}{(D^2+a^2)\Phi(D^2)} = \frac{1}{\Phi(-a^2)} \frac{\cos ax}{D^2+a^2} = \frac{1}{\Phi(-a^2)} \frac{x\sin ax}{2a}$$

3. **P.I for f(D) y = Q(x) where Q(x) = x^k where k is a positive integer f(D) can be express as f(D) =[1± $\emptyset(D)$]**

Express $\frac{1}{f(D)} = \frac{1}{1\pm\emptyset(D)}$ = $[1\pm \emptyset(D)]^{-1}$

Hence P.I = $\frac{1}{1\pm\emptyset(D)}$ Q(x).

= $[1\pm \emptyset(D)]^{-1} .x^k$

4. **P.I of f(D) y = Q(x) when Q(x) = e^{ax} V where ‘a’ is a constant and V is function of x. where V =sin ax or cos ax or x^k**

Then P.I =$\frac{1}{f(D)}$ Q(x)

=$\frac{1}{f(D)} e^{ax}$ V

= $e^{ax} [\frac{1}{f(D+a)}(V)]$

& $\frac{1}{f(D+a)}$ V is evaluated depending on V.

5. **P.I of f(D) y = Q(x) when Q(x) = x V where V is a function of x.**

Then P.I = $\frac{1}{f(D)}$ **Q(x)**

$= \frac{1}{f(D)}$ x V

$= [x - \frac{1}{f(D)} f^1(D)] \frac{1}{f(D)} V$

6. **i. P.I. of f(D)y=Q(x) where Q(x)=x^mv where v is a function of x.**

Then P.I. $= \frac{1}{f(D)} \times Q(x) = \frac{1}{f(D)} x^m v = I.P.of \frac{1}{f(D)} x^m (\cos ax + i \sin ax)$

$= I.P.of \frac{1}{f(D)} x^m e^{iax}$

ii. P.I. $= \frac{1}{f(D)} x^m \cos ax = R.P.of \frac{1}{f(D)} x^m e^{iax}$

Formulae

1. $\frac{1}{1-D} = (1-D)^{-1} = 1 + D + D^2 + D^3 +$ ------------------
2. $\frac{1}{1+D} = (1+D)^{-1} = 1 - D + D^2 - D^3 +$ ------------------
3. $\frac{1}{(1-D)^2} = (1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 +$ ------------------
4. $\frac{1}{(1+D)^2} = (1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 +$ ------------------
5. $\frac{1}{(1-D)^3} = (1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 +$ ------------------
6. $\frac{1}{(1+D)^3} = (1+D)^{-3} = 1 - 3D + 6D^2 - 10D^3 +$ ------------------

I. HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS:

1. Find the Particular integral of f(D) y = e^{ax} when f(a) ≠0
2. Solve the D.E $(D^2 + 5D + 6)\, y = e^x$
3. Solve $y^{11}+4y^1+4y = 4e^{3x}$; y(0) = -1 , $y^1(0) = 3$
4. Solve $y^{11} + 4y^1 + 4y = 4\cos x + 3\sin x$, y(0) = 1 , $y^1(0) = 0$
5. Solve $(D^2+9)\, y = \cos 3x$
6. Solve $y^{111} + 2y^{11} - y^1 - 2y = 1-4x^3$
7. Solve the D.E $(D^3 - 7D^2 + 14D - 8)\, y = e^x \cos 2x$
8. Solve the D.E $(D^3 - 4D^2 - D + 4)\, y = e^{3x} \cos 2x$
9. Solve $(D^2 - 4D + 4)\, y = x^2 \sin x + e^{2x} + 3$

10. Apply the method of variation parameters to solve $\frac{d^2y}{dx^2} + y = \text{cosec}x$

11. Solve $\frac{dx}{dt} = 3x + 2y$, $\frac{dy}{dt} + 5x + 3y = 0$

12. Solve $(D^2 + D - 3)\, y = x^2e^{-3x}$

13. Solve $(D^2 - D - 2)\, y = 3e^{2x}$, $y(0) = 0$, $y^1(0) = -2$

SOLUTIONS:

1) Particular integral of f(D) y = e^{ax} when f(a) ≠0

Working rule:

Case (i):

In f(D), put D=a and Particular integral will be calculated.

Particular integral= $\frac{1}{f(D)}\ e^{ax} = \frac{1}{f(a)}\ e^{ax}$ provided f(a) ≠0

Case (ii) :

If f(a)= 0 , then above method fails. Now proceed as below.

If $f(D) = (D-a)^K\phi(D)$

i.e. 'a' is a repeated root k times, then

Particular integral= $\frac{e^{ax}}{\phi(a)} \cdot \frac{x^k}{k!}$ provided ϕ(a) ≠0

2. Solve the Differential equation$(D^2+5D+6)y=e^x$

Sol : Given equation is $(D^2+5D+6)y=e^x$

Here $Q(x) = e^x$

Auxiliary equation is $f(m) = m^2+5m+6=0$

$m^2+3m+2m+6=0$

$m(m+3)+2(m+3)=0$

m=-2 or m=-3

The roots are real and distinct

C.F = $y_c = c_1e^{-2x} + c_2\, e^{-3x}$

Particular Integral = $y_p = \frac{1}{f(D)} \cdot Q(x)$

$= \frac{1}{D2+5D+6}\, e^x \quad = \frac{1}{(D+2)(D+3)}\, e^x$

Put D = 1 in f(D)

P.I. = $\frac{1}{(3)(4)}$ e^x

Particular Integral = $y_p = \frac{1}{12} \cdot e^x$

General solution is $y = y_c + y_p$

$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{e^x}{12}$

3) Solve $y^{11} - 4y^1 + 3y = 4e^{3x}$, $y(0) = -1$, $y^1(0) = 3$

Sol : Given equation is $y^{11} - 4y^1 + 3y = 4e^{3x}$

i.e. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 4e^{3x}$

it can be expressed as

$D^2y - 4Dy + 3y = 4e^{3x}$

$(D^2 - 4D + 3)y = 4e^{3x}$

Here $Q(x) = 4e^{3x}$; $f(D) = D^2 - 4D + 3$

Auxiliary equation is $f(m) = m^2 - 4m + 3 = 0$

$m^2 - 3m - m + 3 = 0$

$m(m-3) - 1(m-3) = 0 \Rightarrow m = 3$ or 1

The roots are real and distinct.

C.F= $y_c = c_1 e^{3x} + c_2 e^x$ ----→ (2)

P.I.= $y_p = \frac{1}{f(D)} \cdot Q(x)$

$= y_p = \frac{1}{D^2 - 4D + 3} \cdot 4e^{3x}$

$= y_p = \frac{1}{(D-1)(D-3)} \cdot 4e^{3x}$

Put D=3

$$y_p = \frac{4e^{3x}}{(3-1)(D-3)} = \frac{4}{2}\frac{e^{3x}}{(D-3)} = 2\frac{x^1}{1!}e^{3x} = 2xe^{3x}$$

General solution is $y = y_c + y_p$

$y = c_1 e^{3x} + c_2 e^x + 2xe^{3x}$ -------------------→ (3)

Equation (3) differentiating with respect to ‘x’

$y^1 = 3c_1 e^{3x} + c_2 e^x + 2e^{3x} + 6xe^{3x}$ -----------→ (4)

By data, $y(0) = -1$, $y^1(0) = 3$

From (3), $-1 = c_1 + c_2$ -------------------→ (5)

From (4), $3 = 3c_1 + c_2 + 2$

$3c_1 + c_2 = 1$ -------------------→ (6)

Solving (5) and (6) we get $c_1 = 1$ and $c_2 = -2$

$y = -2e^x + (1+2x)e^{3x}$

(4). Solve $y^{11}+4y^{1}+4y= 4\cos x + 3\sin x$, $y(0) = 0$, $y^{1}(0) = 0$

Sol: Given differential equation in operator form

$(D^2 + 4D + 4)y= 4\cos x +3\sin x$

A.E is $m^2+4m+4 = 0$

$(m+2)^2=0$ then $m=-2, -2$

∴ C.F is $y_c= (c_1 + c_2x)e^{-2x}$

P.I is = $y_p=\frac{4\cos x+3\sin x}{(D^2+4D+4)}$ put $D^2 = -1$

$y_p=\frac{4\cos x+3\sin x}{(4D+3)} = \frac{(4D-3)(4\cos x+3\sin x)}{(4D-3)(4D+3)}$

$=\frac{(4D-3)(4\cos x+3\sin x)}{16D^2-9}$

Put $D^2 = -1$

∴$y_p=\frac{(4D-3)(4\cos x+3\sin x)}{-16-9}$

$=\frac{-16\sin x+12\cos x-12\cos x-9\sin x)}{-25} = \frac{-25\sin x}{-25} = \sin x$

∴General equation is $y = y_c+ y_p$

$y = (c_1 + c_2x)e^{-2x} + \sin x$ ------------ (1)

By given data, $y(0) = 0$ ∴ $c_1 = 0$ and

Diff (1) w.r.. t. $y^1 = (c_1 + c_2x)(-2)e^{-2x} + e^{-2x}(c_2) +\cos x$ ------------ (2)

given $y^1(0) = 0$

(2) $\Rightarrow -2c_1 + c_2+1=0$ ∴ $c_2 = -1$

∴Required solution is $y = -xe^{-2x}+\sin x$

5. Solve $(D^2+9)y = \cos 3x$

Sol:Given equation is $(D^2+9)y = \cos 3x$

A.E is $m^2+9 = 0$

$\therefore$ m = $\pm$ 3i

y_c = C.F = c_1 cos3x+ $c_2$sin3x

y_c =P.I = $\frac{\cos 3x}{D^2+9} = \frac{\cos 3x}{D^2+3^2}$

= $\frac{x}{2(3)}$ sin3x = $\frac{x}{6}$ sin3x

General equation is y = y_c+ y_p

y = $c_1$cos3x + $c_2$cos3x + $\frac{x}{6}$ sin3x

6. Solve $y^{111}+2y^{11} - y^1-2y= 1-4x^3$

Sol:Given equation can be written as

$(D^3+2D^2-D-2)y = 1-4x^3$

A.E is $(m^3+2m^2-m-2) = 0$

$(m^2-1)(m+2) = 0$

$m^2 = 1$ or m=- 2

m = 1, -1, -2

C.F =$c_1e^{x} + c_2e^{-x} + c_3e^{-2x}$

P.I = $\frac{1}{(D^3+2D^2-D-2)}(1-4x^3)$

$= \frac{-1}{2[1-\frac{(D^3+2D^2-D)}{2}]}(1-4x^3)$

$= \frac{-1}{2}[1-\frac{(D^3+2D^2-D)}{2}]^{-1}(1-4x^3)$

$= \frac{-1}{2}[1+\frac{(D^3+2D^2-D)}{2}+\frac{(D^3+2D^2-D)^2}{4}+\frac{(D^3+2D^2-D)^3}{8}+ .....](1-4x^3)$

$= \frac{-1}{2}\left[1+\frac{1}{2}(D^3+2D^2-D)+\frac{1}{4}(D^2-4D^3)+\frac{1}{8}(-D^3)\right](1-4x^3)$

$= \frac{-1}{2}[1-\frac{5}{8}(D^3)+\frac{5}{4}(D^2)-\frac{1}{2}D](1-4x^3)$

$= \frac{-1}{2}[(1-4x^3) - \frac{5}{8}(-24) + \frac{5}{4}(-24x) - \frac{1}{2}(-12x^2)$

$= \frac{-1}{2}[-4x^3+6x^2 -30x +16] =$

$= [2x^3-3x^2 +15x -8]$

The general solution is

y= C.F + P.I

$y= c_1e^{x} + c_2e^{-x} + c_3e^{-2x} + [2x^3-3x^2 +15x -8]$

7. Solve $(D^3 - 7D^2 + 14D - 8)y = e^x \cos 2x$

Given equation is

$(D^3 - 7D^2 + 14D - 8)y = e^x \cos 2x$

A.E is $(m^3 - 7m^2 + 14m - 8) = 0$

(m-1) (m-2)(m-4) = 0

Then m = 1,2,4

$C.F = c_1e^{x} + c_2e^{2x} + c_3e^{4x}$

$P.I = \frac{e^x \cos 2x}{(D^3-7D^2+14D-8)}$

$= e^x . \frac{1}{(D+1)^3-7(D+1)^2+14(D+1)-8} . \text{Cos}2x$

$\left[\because P.I = \frac{1}{f(D)} e^{ax} v = e^{ax} \frac{1}{f(D+a)} v \right]$

$= e^x . \frac{1}{(D^3-4D^2+3D)} .\cos 2x$

$= e^x . \frac{1}{(-4D+3D+16)} .\cos 2x$ (Replacing D^2 with -2^2)

$= e^x . \frac{1}{(16-D)} .\cos 2x$

$= e^x . \frac{16+D}{(16-D)(16+D)} .\cos 2x$

$= e^x . \frac{16+D}{256-D^2} .\cos 2x$

$= e^{x} \cdot \frac{16+D}{256-(-4)}$ .cos2x

$= \frac{e^{x}}{260}$ (16cos2x – 2sin2x)

$= \frac{2e^{x}}{260}(8\cos 2x - \sin 2x)$

$= \frac{e^{x}}{130}(8\cos 2x - \sin 2x)$

General solution is y = $y_c + y_p$

$$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{4x} + \frac{e^{x}}{130}(8\cos 2x - \sin 2x)$$

8. Solve $(D^2 - 4D + 4)y = x^2 sinx + e^{2x} + 3$

Sol:Given $(D^2 - 4D + 4)y = x^2 sinx + e^{2x} + 3$

A.E is $(m^2 - 4m + 4) = 0$

$(m-2)^2 = 0$ then m=2,2

C.F. = $(c_1 + c_2x)e^{2x}$

P.I = $\frac{x^2 sinx + e^{2x} + 3}{(D-2)^2} = \frac{1}{(D-2)^2}(x^2 sinx) + \frac{1}{(D-2)^2}e^{2x} + \frac{1}{(D-2)^2}(3)$

Now $\frac{1}{(D-2)^2}(x^2 sinx) = \frac{1}{(D-2)^2}(x^2)$ (I.P of e^{ix})

= I.P of $\frac{1}{(D-2)^2}(x^2)(e^{ix})$

= I.P of $(e^{ix}) . \frac{1}{(D+i-2)^2}(x^2)$

On simplification, we get

$\frac{1}{(D+i-2)^2}(x^2\ sinx) = \frac{1}{625}$ [(220x+244)cosx+(40x+33)sinx]

and $\frac{1}{(D-2)^2}(e^{2x}) = \frac{x^2}{2}(e^{2x})$,

$\frac{1}{(D-2)^2}(3) = \frac{3}{4}$

P.I = $\frac{1}{625}$ [(220x+244)cosx+(40x+33)sinx] + $\frac{x^2}{2}(e^{2x}) + \frac{3}{4}$

y = $y_c + y_p$

$y = (c_1 + c_2x)e^{2x} + \frac{1}{625}[(220x+244)\cos x+(40x+33)\sin x] + \frac{x^2}{2}(e^{2x}) + \frac{3}{4}$

Variation of Parameters :

Working Rule :

1. Reduce the given equation of the form $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R$
2. Find C.F.
3. Take P.I. $y_p = Au+Bv$ where $A = -\int \frac{vRdx}{uv^1 - vu^1}$ and $B = \int \frac{uRdx}{uv^1 - vu^1}$
4. Write the G.S. of the given equation $y = y_c + y_p$

9. Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2}$ + y = cosecx

Sol: Given equation in the operator form is $(D^2 + 1)y = cosecx$----------------(1)

A.E is $(m^2 + 1) = 0$

$\therefore m = \pm i$

The roots are complex conjugate numbers.

∴ C.F. is $y_c = c_1\cos x + c_2\sin x$

Let $y_p = A\cos x + B\sin x$ be P.I. of (1)

$u\frac{dv}{dx} - v\frac{du}{dx} = \cos^2 x + \sin^2 x = 1$

A and B are given by

$A = -\int \frac{vRdx}{uv^1 - vu^1} = -\int \frac{\sin x \operatorname{cosec} x}{1} dx = -\int dx = -x$

$B = \int \frac{uRdx}{uv^1 - vu^1} = \int \cos x . \operatorname{cosec} x \, dx = \int \cot x \, dx = \log(\sin x)$

∴ $y_p = -x\cos x + \sin x . \log(\sin x)$

∴ General solution is $y = y_c + y_p$.

$y = c_1\cos x + c_2\sin x - x\cos x + \sin x . \log(\sin x)$

10. Solve $(4D^2 - 4D + 1)y = 100$

Sol: A.E is $(4m^2 - 4m + 1) = 0$

$(2m-1)^2 = 0$ then $m = \frac{1}{2}, \frac{1}{2}$

C.F = $(c_1 + c_2x)\, e^{\frac{x}{2}}$

$$P.I = \frac{100}{(4D^2-4D+1)} = \frac{100\, e^{0.x}}{(2D-1)^2} = \frac{100}{(0-1)^2} = 100$$

Hence the general solution is y = C.F +P.I

$$y = (c_1+c_2x)\, e^{\frac{x}{2}} + 100$$

Applications of Differential Equations:

11. The differential equation satisfying a beam uniformly loaded (w kg/meter) with one end fixed and the second end subjected to tensile force p is given by

$$EI\frac{d^2y}{dx^2} = py - \frac{1}{2}wx^2$$

Show that the elastic curve for the beam with conditions $y=0=\frac{dy}{dx}$ at x=0 is given by $y = \frac{w}{n^2p}(1-\cosh nx) + \frac{wx^2}{2p}$ where $n^2 = \frac{p}{EI}$

Sol:The given differential equation can be written as

$$\frac{d^2y}{dx^2} - \frac{p}{EI}y = \frac{-1}{2EI}wx^2 \text{ (or)}$$

$$\frac{d^2y}{dx^2} - n^2y = \frac{-w}{2EI}x^2 \text{ (or)}$$

$$(D^2 - n^2)y = \frac{-w}{2EI}x^2 \text{ -----------(1)}$$

The auxiliary equation is $(m^2 - n^2) = 0 \Rightarrow m = n$ and $m= -n$

$\therefore$ C.F = $y_c = c_1e^{nx} + c_2e^{-nx}$

$$P.I = \frac{1}{(D^2-n^2)}\left(\frac{-w}{2EI}x^2\right)$$

$$= \frac{w}{2EI}\left(\frac{1}{(n^2-D^2)}x^2\right)$$

$$= \frac{w}{2EI}\left(\frac{1}{\left(n^2(1-\frac{D^2}{n^2})\right)}x^2\right)$$

$$= \frac{w}{2EI.n^2}\left(1-\frac{D^2}{n^2}\right)^{-1}.x^2$$

$$= \frac{w}{2EI.n^2}\left(1+\frac{D^2}{n^2}+ -----\right).x^2$$

$= \frac{w}{2EI.n^2}\left(\chi^2 + \frac{2}{n^2}\right)$

∴ The general solution of equation (1) is given by y= C.F + P.I

$y = c_1 e^{nx} + c_2 e^{-nx} + \frac{w}{2EI.n^2}\left(\chi^2 + \frac{2}{n^2}\right)$

12. A condenser of capacity 'C' discharged through an inductance L and resistance R in series and the charge q at time t satisfies the equation $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$. Given that L=0.25H, R = 250ohms, $c = 2 * 10^{-6}$farads, and that when t =0, change q is 0.002 coulombs and the current $\frac{dq}{dt} = 0$, obtain the value of 'q' in terms of t.

Sol:

Given differential equation is

$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$ or $\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0$ -------------(1)

Substituting the given values in (1), we get

$\frac{d^2q}{dt^2} + \frac{250}{0.25}\frac{dq}{dt} + \frac{q}{0.25*2*10^{-6}} = 0$ or

$\frac{d^2q}{dt^2} + 1000\frac{dq}{dt} + 2 * 10^6 q = 0$ or

$(D^2 + 1000D + 2 * 10^6)q = 0$

Its A.E is $m^2 + 1000m + 2 * 10^6 = 0$

∴ $m = \frac{-1000 \pm \sqrt{10^6 - 8 \times 10^6}}{2} = \frac{-1000 \pm 1000\sqrt{7}i}{2}$

$= -500 \pm 1323i$

Thus the solution is $q = e^{-500t}(c_1\cos 1323t + c_2\sin 1323t)$

When t=0, q=0.002 since $c_1 = 0.002$

Now $\frac{dq}{dt} = -500e^{-500t}(c_1\cos 1323t + c_2\sin 1323t) + e^{-500t} \times 1323(-c_1\sin 1323t + c_2\cos 1323t)$

When $t = 0, \frac{dq}{dt} = 0$

There fore $c_2 = 0.0008$

Hence the required solution is $q = e^{-500t}(0.002 \cos 1323t + 0.0008 \sin 1323t)$

13. A particle is executing S.H.M, with amplitude 5 meters and time 4 seconds. Find the time required by the particle in passing between points which are at distances 4 and 2 meters from the Centre of force and are on the same side of it.

Sol: The equation of S.H.M is $\frac{d^2x}{dt^2} = -\mu^2 x$-------------(1)

Give time period = $\frac{2\pi}{\mu}$ =4

$\mu = \frac{\pi}{2}$

We have the solution of (1) is x=acos μt

a =5, $\mu = \frac{\pi}{2}$

x = 5cos $\frac{\pi}{2}t$-------------(2)

Let the times when the particle is at distances of 4 meters and 2 meters from the centre of motion respectively be t_1 sec and t_2 sec

$\therefore t_1 = \frac{2}{\pi}\cos^{-1}\left(\frac{4}{5}\right)$ since [4= 5cos($\frac{\pi}{2}t_1$)]

and $t_2 = \frac{2}{\pi}\cos^{-1}\left(\frac{2}{5}\right)$ since [2= 5cos($\frac{\pi}{2}t_2$)]

time required in passing through these points

$t_2 - t_1 = \frac{2}{\pi}\left[\cos^{-1}\left(\frac{2}{5}\right) - \cos^{-1}\left(\frac{4}{5}\right)\right]$ = 0.33sec

differentiating (2) w.r.to 't'

$\frac{dx}{dt} = \frac{-5\pi}{2}\sin\frac{\pi}{2}t$

$= \frac{-5\pi}{2}\sqrt{1 - \frac{x^2}{25}}$

$\frac{dx}{dt} = \frac{-\pi}{2}\sqrt{25 - x^2}$

When x=4 meters v = $\frac{\pi}{2}\sqrt{5^2 - 4^2}$ = 4.71 m/sec

When x=2 meters v=$\frac{\pi}{2}\sqrt{21}$ m/sec

14. A body weighing 10kgs is hung from a spring. A pull of 20kgs will stretch the spring to 10cms. The body is pulled down to 20cms below the static equilibrium position and then released. Find the displacement of the body from its equilibrium position at time t seconds the maximum velocity and the period of oscillation.

Sol:Let 0 be the fixed end and A be the other end of the spring. Since load of 20kg attached to A stretches the spring by 0.1m.

Let e(AB) be the elongation produced by the mass 'm' hanging in equilibrium.

If 'k' be the restoring force per unit stretch of the spring due to elasticity, then for the equilibrium at B

$Mg = T = ke$

$20 = T_0 = k * 0.1$

$K = 200$kg/m

Let B be the equilibrium position when 10kg weight is

$10 = T_B = k * AB \Rightarrow AB = \frac{10}{200} = 0.05$m

Now the weight is pulled down to c, where BC=0.2. After any time t of its release from c, let the weight be at p, where BP=x.

Then the tension $T = k * AP$

$= 200(0.05+x) = 10 + 200x$

∴ The equation of motion of the body is

$\frac{w}{g}\frac{d^2x}{dt^2} = w - T$ where $g = 9.8$m/sec^2

$= \frac{10}{9.8}\frac{d^2x}{dt^2}$

$= 10 - (10+200x)$

$\Rightarrow \frac{d^2x}{dt^2} = -\mu^2 x$ where $\mu = 14$

This shows that the motion of the body in simple harmonic about B as centre and the period of oscillation = $\frac{2\pi}{\mu}$= 0.45sec

Also the amplitude of motion being B C=0.2m, the displacement of the body from B at time t is given by x = 0.2cosect

X = 0.2cosect = 0.2cos14t m.

Maximum velocity = μ (amplitude) = 14 * 0.2 = 2.8m/sec

MODULE -IV

Multiple Integrals

Multiple Integrals

Double Integral :

I. When y_1, y_2 are functions of x and x_1 and x_2 are constants. f(x,y)is first integrated w.r.t y keeping 'x' fixed between limits y_1, y_2 and then the resulting expression is integrated w.r.t 'x' with in the limits x_1, x_2 i.e.,

$$\iint_R f(x,y)\,dx\,dy = \int_{x=x_1}^{x=x_2}\int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x,y)\,dy\,dx$$

II. When x_1, x_2 are functions of y and y_1, y_2 are constants, f(x,y)is first integrated w.r.t 'x' keeping 'y' fixed, with in the limits x_1, x_2 and then resulting expression is integrated w.r.t 'y' between the limits y_1, y_2 i.e.,

$$\iint_R f(x,y)\,dx\,dy = \int_{y=y_1}^{y=y_2}\int_{x=\phi_1(y)}^{x=\phi_2(y)} f(x,y)\,dx\,dy$$

III. When x_1, x_2, y_1, y_2 are all constants. Then

$$\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)\,dx\,dy = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dy\,dx$$

Problems

1. Evaluate $\int_1^2\int_1^3 xy^2\,dx\,dy$

Sol. $\int_1^2\left[\int_1^3 xy^2\,dx\right]dy$

$$= \int_1^2\left[y^2.\frac{x^2}{2}\right]_1^3 dy = \int_1^2 \frac{y^2}{2}dy\,[9-1]$$

$$= \frac{8}{2}\int_1^2 y^2 dy = 4.\int_1^2 y^2 dy$$

$$= 4.\left[y^3/3\right]_1^2 = \frac{4}{3}[8-1] = \frac{4.7}{3} = \frac{28}{3}$$

2. Evaluate $\int_0^2\int_0^x y\,dy\,dx$

Sol. $\int_{x=0}^2\int_{y=0}^x y\,dy\,dx = \int_{x=0}^2\left[\int_{y=0}^x y\,dy\right]dx$

$$= \int_{x=0}^2\left[\frac{y^2}{2}\right]_0^x dx = \int_{x=0}^2 \frac{1}{2}\left(x^2-0\right)dx = \frac{1}{2}\int_{x=0}^2 x^2 dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{1}{6}\left(8-0\right) = \frac{8}{6} = \frac{4}{3}$$

3. Evaluate $\int_0^5 \int_0^{x^2} x\left(x^2+y^2\right)dx\,dy$

Sol.

$$\int_{x=0}^{5}\int_{y=0}^{x^2} x\left(x^2+y^2\right)dy\,dx = \int_{x=0}^{5}\left[x^3y+\frac{xy^3}{3}\right]_{y=0}^{x^2}dx$$

$$= \int_{x=0}^{5}\left[x^3.x^2+\frac{x(x^2)^3}{3}\right]dx = \int_{x=0}^{5}\left(x^5+\frac{x^7}{3}\right)dx = \left[\frac{x^6}{6}+\frac{1}{3}.\frac{x^8}{8}\right]_0^5 = \frac{5^6}{6}+\frac{5^8}{24}$$

4. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2}$

Sol: $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2} = \int_{x=0}^{1}\left[\int_{y=0}^{\sqrt{1+x^2}}\frac{1}{\left(1+x^2\right)+y^2}dy\right]dx$

$$= \int_{x=0}^{1}\left[\int_{y=0}^{\sqrt{1+x^2}}\frac{1}{\left(\sqrt{1+x^2}\right)^2+y^2}dy\right]dx = \int_{x=0}^{1}\frac{1}{\sqrt{1+x^2}}\left[Tan^{-1}\frac{y}{\sqrt{1+x^2}}\right]_{y=0}^{\sqrt{1+x^2}}dx \quad [\because \int\frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}(x/a)]$$

$$= \int_{x=0}^{1}\frac{1}{\sqrt{1+x^2}}\left[Tan^{-1}1-Tan^{-1}0\right]dx \quad or \quad \frac{\pi}{4}(\sinh^{-1}x)_0^1 = \frac{\pi}{4}(\sinh^{-1}1)$$

$$= \frac{\pi}{4}\int_{x=0}^{1}\frac{1}{\sqrt{1+x^2}}dx = \frac{\pi}{4}\left[\log(x+\sqrt{x^2+1})\right]_{x=0}^{1}$$

$$= \frac{\pi}{4}\log(1+\sqrt{2})$$

5. Evaluate $\int_0^4 \int_0^{x^2} e^{y/x}dydx$

Answer: $3e^4-7$

6. Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)dxdy$

Answer: 3/35

. Evaluate $\int_0^2 \int_0^x e^{(x+y)}dydx$

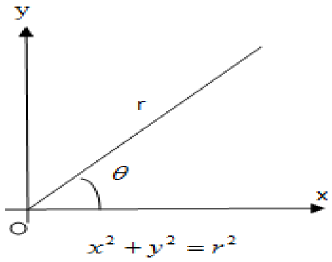
Ans: $\frac{e^4-e^2}{2}$

8. Evaluate $\int_0^{\frac{\pi}{2}} \int_{-1}^{1} x^2y^2dxdy$

Ans: $\frac{\pi^3}{36}$

9. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy$

Sol: $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy = \int_0^\infty e^{-y^2}\left[\int_0^\infty e^{-x^2}dx\right]dy$



$= \int_0^\infty e^{-y^2}\frac{\sqrt{\pi}}{2}dy \qquad \because \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

$= \frac{\sqrt{\pi}}{2}\int_0^\infty e^{-y^2}dy = \frac{\sqrt{\pi}}{2}.\frac{\sqrt{\pi}}{2} = \frac{\pi}{4}$

Alter:

$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy = \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{\infty} e^{-r^2}rdrd\theta \qquad (\because x^2+y^2=r^2)$

(changing to polar coordinates taking $x=r\cos\theta, y=r\sin\theta$)

$= \int_0^{\pi/2}\left[\frac{e^{-r^2}}{-2}\right]_0^\infty d\theta = \int_0^{\pi/2}\left[\frac{0-1}{-2}\right]d\theta$

$= \frac{1}{2}(\theta)_0^{\pi/2} = \frac{1}{2}\left(\pi/2 - 0\right)$

$= \frac{\pi}{4}$

10. Evaluate $\int\int xy(x+y)dxdy$ over the region R bounded by y=x^2 and y=x

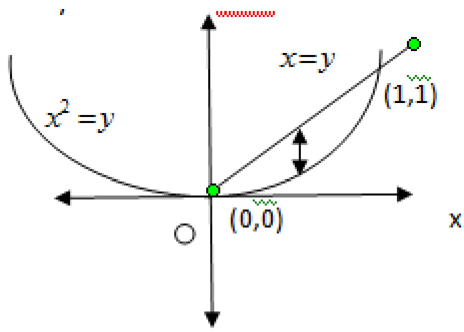
Sol: y= x^2 is a parabola through (0, 0) symmetric about y-axis y=x is a straight line through (0,0) with slope1. Let us find their points of intersection solving y= x^2 , y=x we get x^2 =x $\Rightarrow$ x=0,1Hence y=0, 1

$\therefore$ The point of intersection of the curves are (0,0), (1,1)

Consider $\iint_R xy(x+y)dxdy$

For the evaluation of the integral, we first integrate w.r.t 'y' from y=x^2 to y=x and then w.r.t. 'x' from x=0 to x=1

$\int_{x=0}^{1}\left[\int_{y=x^2}^{x} xy(x+y)dy\right]dx = \int_{x=0}^{1}\left[\int_{y=x^2}^{x}\left(x^2y+xy^2\right)dy\right]dx$



$= \int_{x=0}^{1}\left(x^2\frac{y^2}{2}+\frac{xy^3}{3}\right)_{y=x^2}^{x}dx$

$= \int_{x=0}^{1}\left(\frac{x^4}{2}+\frac{x^4}{3}-\frac{x^6}{2}-\frac{x^7}{3}\right)dx$

$$= \int_{x=0}^{1} \left(\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3} \right) dx$$

$$= \left(\frac{5}{6} \cdot \frac{x^5}{5} - \frac{x^7}{14} - \frac{x^8}{24} \right)_0^1$$

$$= \frac{1}{6} - \frac{1}{14} - \frac{1}{24} = \frac{28-12-7}{168} = \frac{28-19}{168} = \frac{9}{168} = \frac{3}{56}$$

11. Evaluate $\iint_R xy\,dx\,dy$ where R is the region bounded by x-axis and x=2a and the curve $x^2=4ay$.

Sol. The line x=2a and the parabola $x^2=4ay$ intersect at B(2a,a)

∴The given integral = $\iint_R xy\ dx\ dy$

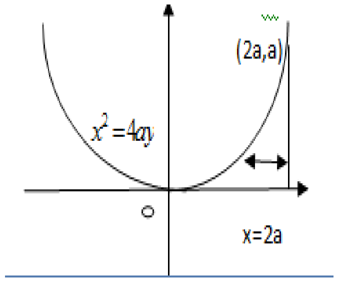
Let us fix 'y'

For a fixed 'y', x varies from $2\sqrt{ay}$ to 2a. Then y varies from 0 to a.

Hence the given integral can also be written as

$$\int_{y=0}^{a} \int_{x=2\sqrt{ay}}^{x=2a} xy\ dx\ dy = \int_{y=0}^{a} \left[\int_{x=2\sqrt{ay}}^{x=2a} x\,dx \right] y\,dy$$

$$= \int_{y=0}^{a} \left[\frac{x^2}{2} \right]_{x=2\sqrt{ay}}^{2a} y\,dy$$



$$= \int_{y=0}^{a} \left[2a^2 - 2ay \right] y\ dy$$

$$= \left[\frac{2a^2y^2}{2} - \frac{2ay^3}{3} \right]_0^a = a^4 - \frac{2a^4}{3} = \frac{3a^4 - 2a^4}{3} = \frac{a^4}{3}$$

12. Evaluate $\int_0^1 \int_0^{\pi/2} r \sin\theta\, d\theta\ dr$

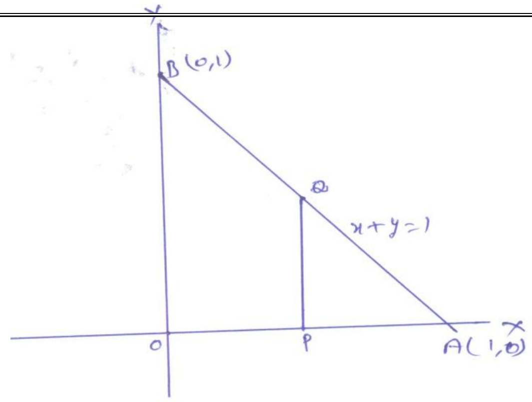
Sol. $\int_{r=0}^{1} r \left[\int_{\theta=0}^{\pi/2} \sin\theta\, d\theta \right] dr$

$$= \int_{r=0}^{1} r\,(-\cos\theta)_{\theta=0}^{\pi/2}\ dr$$

$$= \int_{r=0}^{1} -r\left(\cos \pi/2 - \cos 0\right) dr$$

$$= \int_{r=0}^{1} -r\,(0-1)\,dr = \int_0^1 r\,dr = \left(\frac{r^2}{2} \right)_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

13. Evaluate $\iint (x^2+y^2)dx\,dy$ in the positive quadrant



For

Which $x+y\le 1$

Sol. $\iint_R (x^2+y^2)dx\,dy = \int_{x=0}^{1} dx \int_{y=0}^{y=1-x} (x^2+y^2)\,dy$

$$= \int_{x=0}^{1} \left(x^2 y + \frac{y^3}{3} \right)_0^{1-x} dx$$

$$= \int_{x=0}^{1} \left(x^2 - x^3 + \frac{1}{3}(1-x)^3 \right) dx = \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{1}{12}(1-x)^4 \right]_0^1$$

$$= \frac{1}{3} - \frac{1}{4} - 0 + \frac{1}{12} = \frac{1}{6}$$

14. Evaluate $\iint (x^2+y^2)dx\,dy$ over the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Sol. Given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

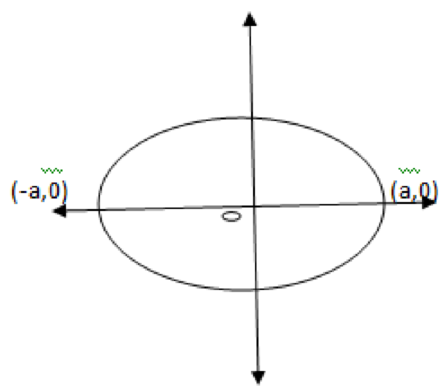
$$i.e., \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{1}{a^2}(a^2-x^2)\ (or)\ y^2 = \frac{b^2}{a^2}(a^2-x^2)$$

$$\therefore\ y = \pm \frac{b}{a}\sqrt{a^2-x^2}$$

Hence the region of integration R can be expressed as

$$-a \le x \le a, \frac{-b}{a}\sqrt{a^2-x^2} \le y \le \frac{b}{a}\sqrt{a^2-x^2}$$

$$\therefore \iint_R (x^2+y^2)\,dx\,dy = \int_{x=-a}^{a} \int_{y=-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} (x^2+y^2)\,dx\,dy$$



$$= 2\int_{x=-a}^{a} \int_{y=0}^{b/a\sqrt{a^2-x^2}} (x^2+y^2)\,dx\,dy = 2\int_{-a}^{a} \left(x^2 y + y^3/3 \right)_0^{b/a\sqrt{a^2-x^2}}$$

$$= 2\int_{-a}^{a} \left[x^2 . b/a \sqrt{a^2-x^2} + \frac{b^3}{3a^3}(a^2-x^2)^{3/2} \right] dx$$

$$= 4\int_{0}^{a} \left[b/a\, x^2 \sqrt{a^2-x^2} + \frac{b^3}{3a^3}(a^2-x^2)^{3/2} \right] dx$$

Changing to polar coordinates

$putting\ x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

$\frac{x}{a} = \sin\theta \Rightarrow \theta = \sin^{-1}\frac{x}{a}$

$x \to 0, \theta \to 0$

$x \to a, \theta \to \frac{\pi}{2}$

$$= 4\int_0^{\pi/2}\left[b/a .a^2\sin^2\theta .a\cos\theta + \frac{b^3}{3a^3}.a^3\cos^3\theta \right] a\cos\theta\, d\theta$$

$$= 4\int_0^{\pi/2}\left[a^3 b\sin^2\theta\cos^2\theta + \frac{ab^3}{3}\cos^4\theta \right] d\theta = 4\left[a^3 b.\frac{1}{4}\frac{1}{2}.\frac{\pi}{2} + \frac{ab^3}{3}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} \right]$$

$$\left[\because \int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\, d\theta = \frac{n-1}{m+n}.\frac{n-3}{m+n-2}........\frac{\frac{1}{2}.\frac{\pi}{2}}{m} \right]$$

$$= \frac{4\pi}{16}\left(a^3 b + ab^3 \right) = \frac{\pi ab}{4}\left(a^2 + b^2 \right)$$

Double integrals in polar co-ordinates:

1. Evaluate $\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r dr d\theta}{\sqrt{a^2 - r^2}}$

Sol. $\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r dr d\theta}{\sqrt{a^2 - r^2}} = \int_0^{\pi/4}\left\{ \int_0^{a\sin\theta} \frac{r}{\sqrt{a^2 - r^2}} dr \right\} d\theta = -1/2\int_0^{\pi/4}\left\{ \int_0^{a\sin\theta} \frac{-2r}{\sqrt{a^2 - r^2}} dr \right\} d\theta$

$$= \frac{-1}{2}\int_0^{\pi/4} 2\left(\sqrt{a^2 - r^2} \right)_0^{a\sin\theta} d\theta = (-1)\int_0^{\pi/4} 2\left[\sqrt{a^2 - a^2\sin^2\theta} - \sqrt{a^2 - 0} \right] d\theta$$

$$= (-a)\int_0^{\pi/4} (\cos\theta - 1)\, d\theta = (-a)(\sin\theta - \theta)_0^{\pi/4}$$

$$= (-a)\left[\left[\sin \pi/4 - \pi/4 \right] - (0 - 0) \right]$$

$$= (-a)\left[1/\sqrt{2} - \pi/4 \right] = 2\left[\pi/4 - 1/\sqrt{2} \right]$$

2. Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\, dr\, d\theta$ Ans: $\frac{a^2\pi}{4}$

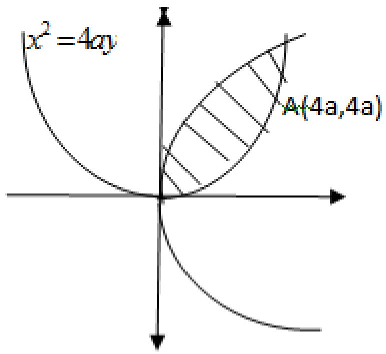
3. Evaluate $\int_0^{\infty}\int_0^{\pi/2} e^{-r^2} r\, d\theta\, dr$ Ans: $\frac{\pi}{4}$

4. Evaluate $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\ dr\ d\theta$ Ans: $\frac{3\pi a^2}{4}$

Change of order of Integration:

1. Change the order of Integration and evaluate $\int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}} dy\,dx$

Sol. In the given integral for a fixed x, y varies from $\frac{x^2}{4a}$ to $2\sqrt{ax}$ and then x varies from 0 to 4a. Let us draw the curves $y=\frac{x^2}{4a}$ and $y=2\sqrt{ax}$



he region of integration is the shaded region in diagram.

The given integral is $=\int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}} dy\,dx$

Changing the order of integration, we must fix y first, for a fixed y, x varies from $\frac{y^2}{4a}$ to $\sqrt{4ay}$ and then y varies from 0 to 4a.

Hence the integral is equal to

$$\int_{y=0}^{4a}\int_{x=y^2/4a}^{2\sqrt{ay}} dx\,dy = \int_{y=0}^{4a}\left[\int_{x=y^2/4a}^{2\sqrt{ay}} dx\right]dy$$

$$=\int_{y=0}^{4a}[x]_{x=y^2/4a}^{2\sqrt{ay}}\,dy = \int_{y=0}^{4a}\left[2\sqrt{ay}-y^2/4a\right]dy$$

$$=\left[2\sqrt{a}.\frac{y^{3/2}}{3/2}-\frac{1}{4a}.\frac{y^3}{3}\right]_0^{4a}$$

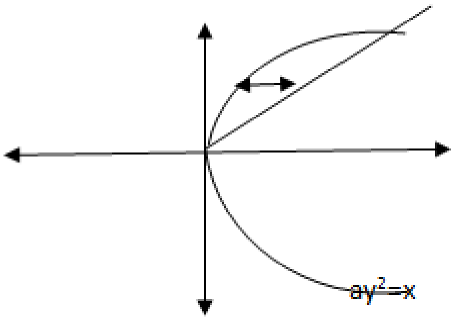
$$=\frac{4}{3}.\sqrt{a}.4a\sqrt{4a}-\frac{1}{12a}.64a^3$$

$$=\frac{32}{3}a^2-\frac{16}{3}a^2=\frac{16}{3}a^2$$

2. Change the order of integration and evaluate $= \int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2 + y^2)\, dx\, dy$

Sol. In the given integral for a fixed x, y varies from $\frac{x}{a}$ to $\sqrt{\frac{x}{a}}$ and then x varies from 0 to a

Hence we shall draw the curves $y = \frac{x}{a}$ and $y = \sqrt{\frac{x}{a}}$



i.e. ay=x and ay^2=x

we get $ay = ay^2$

$\Rightarrow ay - ay^2 = 0$

$\Rightarrow ay(1 - y) = 0$

$\Rightarrow y = 0, y = 1$

If y=0, x=0 if y=1, x=a

The shaded region is the region of integration. The given integral is $\int_{x=0}^{a} \int_{y=x/a}^{y=\sqrt{x/a}} (x^2 + y^2)dx\, dy$

Changing the order of integration, we must fix y first. For a fixed y, x varies from ay^2 to ay and then y varies from 0 to 1.

Hence the given integral, after change of the order of integration becomes

$\int_{y=0}^{1} \int_{x=ay^2}^{ay} (x^2 + y^2)dx\, dy$

$= \int_{y=0}^{1} \left[\int_{x=ay^2}^{ay} (x^2 + y^2)\, dx \right] dy$

$= \int_{y=0}^{1} \left(\frac{x^3}{3} + xy^2 \right)_{x=ay^2}^{ay} dy$

$= \int_{y=0}^{1} \left(\frac{a^3y^3}{3} + ay^3 - \frac{a^3y^6}{3} - ay^4 \right) dy$

$$= \left(\frac{a^3 y^4}{12} + \frac{ay^4}{4} - \frac{a^3 y^7}{21} - \frac{ay^5}{5} \right)_{y=0}^{1}$$

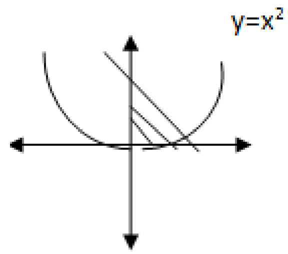
$$= \frac{a^3}{12} + \frac{a}{4} - \frac{a^3}{21} - \frac{a}{5} = \frac{a^3}{28} + \frac{a}{20}$$

3. Change the order of integration in $\int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy$ and hence evaluate the double integral.

Sol.In the given integral for a fixed x,y varies from x^2 to 2-x and then x varies from 0 to 1. Hence we shall draw the curves $y=x^2$ and y=2-x

The line y=2-x passes through (0,2), (2,0

Solving $y=x^2$,y=2-x



Then we get $x^2 = 2 - x$

$\Rightarrow x^2 + x - 2 = 0$

$\Rightarrow x^2 + 2x - x - 2 = 0$

$\Rightarrow x(x+2) - 1(x+2) = 0$

$\Rightarrow (x-1)(x+2) = 0$

$\Rightarrow x = 1, -2$

$If\ x = 1, y = 1$

$If\ x = -2, y = 4$

Hence the points of intersection of the curves are (-2,4) (1,1)

The Shaded region in the diagram is the region of intersection.

Changing the order of integration, we must fix y, for the region with in OACO for a fixed y, x varies from

0 to $\sqrt{y}$

Then y varies from 0 to 1

For the region within CABC, for a fixed y, x varies from 0 to 2-y ,then y varies from 1 to 2

Hence $\int_0^1 \int_{x^2}^{2-x} xy\ dy\ dx = \iint_{OACO} xy\ dx\ dy + \iint_{CABC} xy\ dx\ dy$

$$= \int_{y=0}^{1} \left[\int_{x=0}^{\sqrt{y}} x dx \right] y\ dy + \int_{y=1}^{2} \left[\int_{x=0}^{2-y} x\ dx \right] y\ dy$$

$$= \int_{y=0}^{1} \left(\frac{x^2}{2} \right)_{x=0}^{\sqrt{y}} y\,dy + \int_{y=1}^{2} \left(\frac{x^2}{2} \right)_{x=0}^{2-y} y\,dy$$

$$= \int_{y=0}^{1} \frac{y}{2}.y\,dy + \int_{y=1}^{2} \frac{(2-y)^2}{2} y\,dy$$

$$= \frac{1}{2}\int_{y=0}^{1} y^2 dy + \frac{1}{2}.\int_{y=1}^{2} \left(4y - 4y^2 + y^3\right) dy$$

$$= \frac{1}{2}.\left(\frac{y^3}{3} \right)_0^1 + \frac{1}{2}.\left[\frac{4y^2}{2} - \frac{4y^3}{3} + \frac{y^4}{4} \right]_1^2$$

$$= \frac{1}{2}.\frac{1}{3} + \frac{1}{2}\left[2.4 - 2.1 - \tfrac{4}{3}(8-1) + \tfrac{1}{4}(16-1)\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[6 - \frac{28}{3} + \frac{15}{4}\right] = \frac{1}{6} + \frac{1}{2}\left[\frac{72 - 112 + 45}{12}\right] = \frac{1}{6} + \frac{1}{2}\left[\frac{5}{12}\right] = \frac{4+5}{24} = \frac{9}{24} = \frac{3}{8}$$

4. Changing the order of integration $\int_0^a \int_{x^2/a}^{2a-x} xy^2 dy\,dx$

5. Change of the order of integration $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dx\,dy \qquad Ans : \frac{\pi}{16}$

Hint : Now limits are $y = 0\ to\ 1$ and $x = 0\ to\ \sqrt{1-y^2}$

$put\ y = \sin\theta$

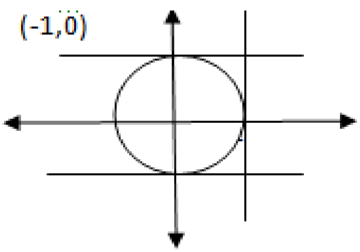
$\sqrt{1-y^2} = \cos\theta$

$dy = \cos\theta\,d\theta$

$$= \int_0^1 y^2 \sqrt{1-y^2}\,dy$$

$$= \int_0^{\pi/2} \sin^2\theta \cos^2\theta\,d\theta = \int_0^{\pi/2} \sin^2\theta\,d\theta - \int_0^{\pi/2} \sin^4\theta\,d\theta$$

$$= \frac{1}{2}\left(\pi/2\right) - \frac{3}{4}.\frac{1}{2}\left(\pi/2\right) = \pi/16$$



Change of variables:

The variables x,y in $\iint_R f(x,y)dx\,dy$ are changed to u,v with the help of the relations $x = f_1(u,v), y = f_2(u,v)$

then the double integral is transferred into

$$\iint_{R^1} f\left[f_1(u,v), f_2(u,v)\right] \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

Where R^1 is the region in the uv plane, corresponding to the region R in the xy-plane.

Changing from Cartesian to polar co-ordinates

$x = r\cos\theta, y = r\sin\theta$

$$\partial\left(\frac{(x,y)}{(r,\theta)}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\left(\cos^2\theta + \sin^2\theta\right) = r \therefore \iint_R f(x,y)dx\,dy = \iint_{R_1} f(r\cos\theta, r\sin\theta)r\,dr\,\,d\theta$$

Note : In polar form dx dy is replaced by $r\,dr\,\,d\theta$

Problems:

1. Evaluate the integral by changing to polar co-ordinates $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy$

Sol.The limits of x and y are both from 0 to ∞ .

$\therefore$ The region is in the first quadrant where r varies from 0 to ∞ and θ varies from 0 to $\pi/2$

Substituting $x = r\cos\theta, y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$

Hence $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}r\,dr\,d\theta$

$Put\,r^2 = t$

$\Rightarrow 2rdr = dt$

$\Rightarrow r\,dr = dt/2$

$Where\;\; r = 0 \Rightarrow t = 0\;\; and\;\; r = \infty \Rightarrow t = \infty$

$$\therefore \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy = \int_{\theta=0}^{\pi/2}\int_{t=0}^{\infty}\frac{1}{2}e^{-t}dt\,d\theta$$

$$= \int_0^{\pi/2}\frac{-1}{2}\left(e^{-t}\right)_0^\infty d\theta$$

$$= \frac{-1}{2}\int_0^{\pi/2}(0-1)\,d\theta \Rightarrow \frac{1}{2}(\theta)_0^{\pi/2} = \frac{1}{2}\pi/2 = \pi/4$$

2. Evaluate the integral by changing to polar co-ordinates $\int_0^a\int_0^{\sqrt{a^2-y^2}}\left(x^2+y^2\right)dx\,dy$

Sol.The limits for x are x=0 to $x = \sqrt{a^2-y^2}$

$\Rightarrow x^2 + y^2 = a^2$

$\therefore$ The given region is the first quadrant of the circle.

By changing to polar co-ordinates

$x = r\cos\theta, y = r\sin\theta, dx\,dy = r\,dr\,d\theta$

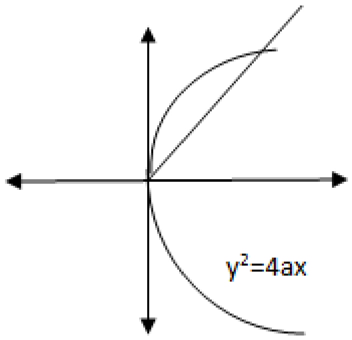
Here 'r' varies from 0 to a and 'θ' varies from 0 to $\pi/2$

$$\therefore \int_0^a\int_0^{\sqrt{a^2-y^2}}\left(x^2+y^2\right)dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^2\,r\,dr\,d\theta = \int_0^{\pi/2}\left(\frac{r^4}{4}\right)_0^a d\theta = \frac{a^4}{4}(\theta)_0^{\pi/2}$$

$= \pi/8 \, a^4$

3. Show that $\int_0^{4a} \int_{y^2/4a}^{y} \frac{x^2 - y^2}{x^2 + y^2} dx\, dy = 8a^2\left(\frac{\pi}{2} - \frac{5}{3}\right)$

Sol. The region of integration is given by $x = y^2/4a$, $x = y$ and y=0, y=4a



i.e., The region is bounded by the parabola $y^2=4ax$ and the straight line x=y.

Let $x = r\cos\theta, y = r\sin\theta. Then\, dx\, dy = r dr d\theta$

The limits for r are r=0 at O and for P on the parabola

$r^2 \sin^2\theta = 4a(r\cos\theta) \Rightarrow r = \frac{4a\cos\theta}{\sin^2\theta}$

For the line y=x, slope m=1 i.e., $Tan\theta = 1, \theta = \pi/4$

The limits for $\theta : \pi/4 \to \pi/2$

Also $x^2 - y^2 = r^2(\cos^2\theta - \sin^2\theta)$ and $x^2 + y^2 = r^2$

$$\therefore \int_0^{4a} \int_{y^2/4a}^{y} \frac{x^2 - y^2}{x^2 + y^2} dx\, dy = \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{4a\cos\theta/\sin^2\theta} (\cos^2\theta - \sin^2\theta) r dr d\theta$$

$$= \int_{\theta=\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta)\left(\frac{r^2}{2}\right)_0^{4a\cos\theta/\sin^2\theta} d\theta$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta)\frac{\cos^2\theta}{\sin^4\theta} d\theta = 8a^2 \int_{\pi/4}^{\pi/2} (\cos^4\theta - \cot^2\theta) d\theta = 8a^2\left[\frac{3\pi - 8}{12} + \frac{\pi}{4} - 1\right] = 8a^2\left(\frac{\pi}{2} - \frac{5}{3}\right)$$

Triple integrals:

If x_1, x_2 are constants. y_1, y_2 are functions of x and z_1, z_2 are functions of x and y, then f(x,y,z) is first integrated w.r.t. ‘z’ between the limits z_1 and z_2 keeping x and y fixed. The resulting expression is integrated w.r.t ‘y’ between the limits y_1 and y_2 keeping x constant. The resulting expression is integrated w.r.t. ‘x’ from x_1 to x_2

$$2.\ \iiint_V f(x,y,z)\,dx\,dy\,dz = \int_{x=a}^{b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x,y,z)\,dz\ dy\ dx$$

Problems

1. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\,dx\,dy\,dz$

Sol. $\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\,dx\,dy\,dz$

$$= \int_{x=0}^{1} dx \int_{y=0}^{\sqrt{1-x^2}} dy \int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\,dz$$

$$= \int_{x=0}^{1} dx \int_{y=0}^{\sqrt{1-x^2}} xy\left(\frac{z^2}{2}\right)_{z=0}^{\sqrt{1-x^2-y^2}} dy$$

$$= \frac{1}{2}\int_{x=0}^{1} dx \int_{y=0}^{\sqrt{1-x^2}} xy\left(1-x^2-y^2\right)dy$$

$$= \frac{1}{2}\int_{x=0}^{1} dx \int_{y=0}^{\sqrt{1-x^2}} x\left[\left(1-x^2\right)y - y^3\right]dy$$

$$= \frac{1}{2}\int_{x=0}^{1} x\left[\left(1-x^2\right)\frac{y^2}{2} - \frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2}.\int_{x=0}^{1} x\left[\frac{y^2}{2} - \frac{x^2y^2}{2} - \frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{8}.\int_{x=0}^{1} x\left[2\left(1-x^2\right) - 2x^2\left(1-x^2\right) - \left(1-x^2\right)^2\right]dx$$

$$= \frac{1}{8}\int_{x=0}^{1}\left(x - 2x^3 + x^5\right)dx = \frac{1}{8}\left[\frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6}\right]_0^1$$

$$= \frac{1}{8}\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) = \frac{1}{48}$$

2. Evaluate $\int_{-1}^{1}\int_0^{z}\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$

$$Sol: \int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+y)\,dx\,dydz$$

$$= \int_{-1}^{1}\int_0^{z}\left[\left(xy + \frac{y^2}{2} + zy\right)_{x-z}^{x+z}\right]dx\,dz$$

$$= \int_{-1}^{1}\int_0^{z} x(x+z) - x(x-z) + \left[\frac{x+z}{2}\right]^2 - \left[\frac{x-z}{2}\right]^2 + z(x+z) - z(x-z)\,dx\,dz$$

$$= \int_{-1}^{1}\int_0^{z}\left[2z(x+z) + \frac{1}{2}4xz\right]dx\,dz$$

$$= 2\int_{-1}^{1}\left[z.\frac{x^2}{2} + z^2 x + z.\frac{x^2}{2}\right]_0^z dz = 2.\int_{-1}^{1}\left[\frac{z^3}{2} + z^3 + \frac{z^3}{2}\right]dz = 4.\left(\frac{z^4}{4}\right)_{-1}^{1} = 0$$

Definition of an double Integral

Just as we can take partial derivative by considering only one of the variables a true variable and holding the rest of the variables constant, we can take a "partial integral". We indicate which the true variable is by writing "dx", "dy", etc. Also as with partial derivatives, we can take two "partial integrals" taking one variable at a time. In practice, we will either take x first then y or y first then x. We call this an *iterated integral* or a *double integral*.

Let f(x,y) be a function of two variables defined on a region R bounded below and above by

$y = g_1(x)$ and $y = g_2(x)$

and to the left and right by

$x = a$ and $x = b$

then the double integral (or iterated integral) of f(x,y) over R is defined by

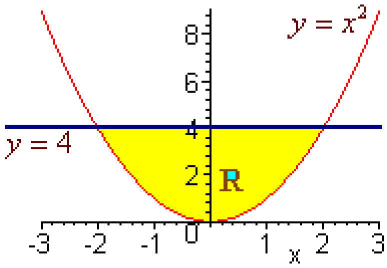
$$\iint_R f(x,y)\,dydx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dydx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right]dx$$

Example 1

Find the double integral of $f(x,y) = 6x^2 + 2y$ over R where R is the region between $y = x^2$ and $y = 4$.

Solution

First we have that the inside limits of integration are x^2 and 4. The region is bounded from the left by $x = -2$ and from the right by $x = 2$ as indicated by the picture below.



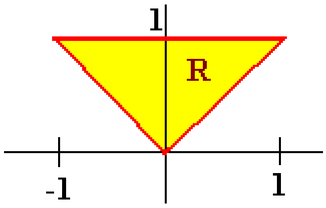
We now integrate

$$\int_{-2}^{2}\int_{x^2}^{4}\left(6x^2+2y\right)dydx \;=\; \int_{-2}^{2}\left[6x^2y+y^2\right]_{x^2}^{4} dx$$

$$= \int_{-2}^{2}(24x^2+16)-(6x^4+x^4)\; dx \;=\; \left[8x^3+16x-\frac{7}{5}x^5\right]^{2} \;=\; 102.4$$

Example 2

Find the double integral of f(x,y) = 3y over the triangle with vertices (-1,1), (0,0), and (1,1).



Solution

If we try to integrate with respect y first, we will have to cut the region into two pieces and perform two iterated integrals. Instead we integrate with respect to x first. The region is bounded on the left and the right by x = -y and x = y. The lowest the region gets is y = 0 and the highest is y = 1. The integral is

$$\int_{0}^{1}\int_{-y}^{y} 3y\; dxdy \;=\; \int_{0}^{1}\left[3xy\right]_{-y}^{y} dy$$

$$= \int_{0}^{1} 6y^2\, dy \;=\; \left[2y^3\right]_0^1 = \; 2$$

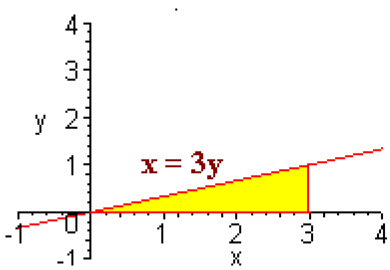
Example 3

Evaluate the integral

$$\int_{0}^{1}\int_{3y}^{3} e^{x^2}\; dxdy$$

Solution

Try as you may, you will not find an antiderivative of e^{x^2} and we don't want to get into power series expansions. We have another choice. The picture below shows the region.



We can switch the order of integration. The region is bounded above and below by $y = 1/3\,x$ and $y = 0$. The double integral with respect to y first and then with respect to x is

$$\int_0^3 \int_0^{x/3} e^{x^2}\, dy\,dx$$

The integrand is just a constant with respect to y so we get

$$\int_0^3 \left[e^{x^2} y\right]_0^{x/3} dx = \int_0^3 \frac{x}{3} e^{x^2} dx$$

This integral can be performed with simple u-substitution.

$u = x^2 \qquad du = 2x\,dx$

and the integral becomes

$$\frac{1}{6}\int_0^9 e^u\,du = \left[\frac{1}{6}e^u\right]_0^9 = \frac{1}{6}e^9 - \frac{1}{6}$$

Area and Double Integrals

If a region R is bounded below by $y = g_1(x)$ and above by $y = g_2(x)$, and $a \le x \le b$, then the area is given by

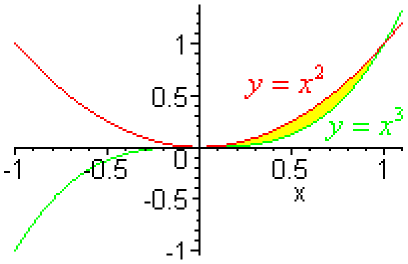
$$\text{Area} = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx$$

Example

Set up the double integral that gives the area between $y = x^2$ and $y = x^3$. Then use a computer or calculator to evaluate this integral.

Solution

The picture below shows the region



We set up the integral

$$\int_0^1 \int_{x^3}^{x^2} dy\,dx$$

A computer gives the answer of 1/12.

Calculation of Volumes Using Triple Integrals

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Example 1 *Evaluate* the following integral.

$$\iiint_B 8xyz\,dV\ ,$$

$$B = [2,3]\times[1,2]\times[0,1]$$

Solution

Just to make the point that order doesn't matter let's use a different order from that listed above. We'll do the integral in the following order.

$$\begin{aligned}\iiint_B 8xyz\,dV &= \int_1^2 \int_2^3 \int_0^1 8xyz\,dz\,dx\,dy \\ &= \int_1^2 \int_2^3 4xyz^2\Big|_0^1\,dx\,dy \\ &= \int_1^2 \int_2^3 4xy\,dx\,dy \\ &= \int_1^2 2x^2y\Big|_2^3\,dy \\ &= \int_1^2 10y\,dy = 15\end{aligned}$$

Example 1 Evaluate $\iiint_E y\,dV$ where *E* is the region that lies below the plane $z = x + 2$ above the *xy*-plane and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution

There really isn't too much to do with this one other than do the conversions and then evaluate the integral.

We'll start out by getting the range for *z* in terms of cylindrical coordinates.

$$0 \le z \le x+2 \qquad \Rightarrow \qquad 0 \le z \le r\cos\theta + 2$$

Remember that we are above the *xy*-plane and so we are above the plane $z = 0$

Next, the region *D* is the region between the two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ in the *xy*-plane and so the ranges for it are,

$$0 \le \theta \le 2\pi \qquad\qquad 1 \le r \le 2$$

Here is the integral.

$$\iiint_E y\,dV = \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2}(r\sin\theta)r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 r^2\sin\theta(r\cos\theta+2)dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 \frac{1}{2}r^3\sin(2\theta)+2r^2\sin\theta\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{1}{8}r^4\sin(2\theta)+\frac{2}{3}r^3\sin\theta\right)\Bigg|_1^2 d\theta$$

$$= \int_0^{2\pi}\frac{15}{8}\sin(2\theta)+\frac{14}{3}\sin\theta\,d\theta$$

$$= \left(-\frac{15}{16}\cos(2\theta)-\frac{14}{3}\cos\theta\right)\Bigg|_0^{2\pi}$$

$$= 0$$

MODULE-V

VECTOR CALCULUS

Vector Calculus and Vector Operators

INTRODUCTION

In this chapter, vector differential calculus is considered, which extends the basic concepts of differential calculus, such as, continuity and differentiability to vector functions in a simple and natural way. Also, the new concepts of gradient, divergence and curl are introduced.

DIFFERENTIATION OF A VECTOR FUNCTION

Let S be a set of real numbers. Corresponding to each scalar t ε S, let there be associated a unique vector $\bar{f}$. Then $\bar{f}$ is said to be a vector (vector valued) function. S is called the domain of $\bar{f}$. We write $\bar{f}$ = $\bar{f}$ (t).

Let $\bar{i}, \bar{j}, \bar{k}$ be three mutually perpendicular unit vectors in three dimensional space. We can write $\bar{f}$ = $\bar{f}$ (t)= $f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$, where *$f_1(t)$, $f_2(t)$, $f_3(t)$* are real valued functions (which are called components of $\bar{f}$). (we shall assume that $\bar{i}, \bar{j}, \bar{k}$ are constant vectors).

1. Derivative:

Let $\bar{f}$ be a vector function on an interval *I* and a ϵ *I*. Then $\underset{t\to a}{Lt}\dfrac{\bar{f}(t)-\bar{f}(a)}{t-a}$, if exists, is called the derivative of $\bar{f}$ at a and is denoted by $\bar{f}^{1}$(a) or $\left(\dfrac{d\bar{f}}{dt}\right)$ at t = a. We also say that $\bar{f}$ is differentiable at t =a if $\bar{f}^{1}$(a) exists.

2. Higher order derivatives

Let $\bar{f}$ be differentiable on an interval *I* and $\bar{f}^{1}=\dfrac{d\bar{f}}{dt}$ be the derivative of $\bar{f}$. If $\underset{t\to a}{Lt}\dfrac{\bar{f}^{1}(t)-\bar{f}^{1}(a)}{t-a}$ exists for every a ∈ $I_1 \subset I$. It is denoted by $\bar{f}^{11}=\dfrac{d^2\bar{f}}{dt^2}$.

Similarly we can define $\bar{f}^{111}(t)$ etc.

We now state some properties of differentiable functions (without proof)

(1) Derivative of a constant vector is $\bar{a}$.

If $\bar{a}$ and $\bar{b}$ are differentiable vector functions, then

(2). $\dfrac{d}{dt}(\bar{a}\pm\bar{b})=\dfrac{d\bar{a}}{dt}\pm\dfrac{d\bar{b}}{dt}$

(3). $\dfrac{d}{dt}(\bar{a}.\bar{b})=\dfrac{d\bar{a}}{dt}.\bar{b}+\bar{a}.\dfrac{d\bar{b}}{dt}$

(4). $\dfrac{d}{dt}(\bar{a}\times\bar{b})=\dfrac{d\bar{a}}{dt}\times\bar{b}+\bar{a}\times\dfrac{d\bar{b}}{dt}$

(5). If $\bar{f}$ is a differentiable vector function and ϕ is a scalar differential function, then $\frac{d}{dt}(\phi\,\bar{f}) = \phi\frac{d\bar{f}}{dt} + \frac{d\phi}{dt}\bar{f}$

(6). If $\bar{f} = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$ where $f_1(t), f_2(t), f_3(t)$ are cartesian components of the vector $\bar{f}$, then $\frac{d\bar{f}}{dt} = \frac{df_1}{dt}\bar{i} + \frac{df_2}{dt}\bar{j} + \frac{df_3}{dt}\bar{k}$

(7). The necessary and sufficient condition for $\bar{f}$ (t) to be constant vector function is $\frac{d\bar{f}}{dt} = \bar{0}$.

3. Partial Derivatives

Partial differentiation for vector valued functions can be introduced as was done in the case of functions of real variables. Let $\bar{f}$ be a vector function of scalar variables *p, q, t*. Then we write $\bar{f} = \bar{f}$ (*p,q,t*). Treating *t* as a variable and *p,q* as constants, we define

$$\underset{\delta t \to 0}{Lt} \frac{\bar{f}(p,q,t+\delta t) - \bar{f}(p,q,t)}{\delta t}$$

if exists, as partial derivative of $\bar{f}$ w.r.t. *t* and is denote by $\frac{\partial \bar{f}}{\partial t}$

Similarly, we can define $\frac{\partial \bar{f}}{\partial p}$, $\frac{\partial \bar{f}}{\partial q}$ also. The following are some useful results on partial differentiation.

4. Properties

1) $\frac{\partial}{\partial t}(\phi\bar{a}) = \frac{\partial \phi}{\partial t}\bar{a} + \phi\frac{\partial \bar{a}}{\partial t}$

2). If λ is a constant, then $\frac{\partial}{\partial t}(\lambda\bar{a}) = \lambda\frac{\partial \bar{a}}{\partial t}$

3). If $\bar{c}$ is a constant vector, then $\frac{\partial}{\partial t}(\phi\bar{c}) = \bar{c}\frac{\partial \phi}{\partial t}$

4). $\frac{\partial}{\partial t}(\bar{a} \pm \bar{b}) = \frac{\partial \bar{a}}{\partial t} \pm \frac{\partial \bar{b}}{\partial t}$

5). $\frac{\partial}{\partial t}(\bar{a}.\bar{b}) = \frac{\partial \bar{a}}{\partial t}.\bar{b} + \bar{a}.\frac{\partial \bar{b}}{\partial t}$

6). $\frac{\partial}{\partial t}(\bar{a} \times \bar{b}) = \frac{\partial \bar{a}}{\partial t} \times \bar{b} + \bar{a} \times \frac{\partial \bar{b}}{\partial t}$

7). Let $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$, where f_1, f_2, f_3 are differential scalar functions of more than one variable, Then $\frac{\partial \bar{f}}{\partial t} = \bar{i}\frac{\partial f_1}{\partial t} + \bar{j}\frac{\partial f_2}{\partial t} + \bar{k}\frac{\partial f_3}{\partial t}$ (treating $\bar{i}, \bar{j}, \bar{k}$ as fixed directions)

5. Higher order partial derivatives

Let $\bar{f} = \bar{f}$ (*p,q,t*). Then $\frac{\partial^2 \bar{f}}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \bar{f}}{\partial t}\right)$, $\frac{\partial^2 \bar{f}}{\partial p \partial t} = \frac{\partial}{\partial p}\left(\frac{\partial \bar{f}}{\partial t}\right)$ etc.

6.Scalar and vector point functions: Consider a region in three dimensional space. To each point $p(x,y,z)$, suppose we associate a unique real number (called scalar) say ϕ. This $\phi(x,y,z)$ is called a scalar point function. Scalar point function defined on the region. Similarly if to each point $p(x,y,z)$we associate a unique

vector $\bar{f}(x,y,z)$, $\bar{f}$ is called a ***vector point function.***

Examples:

For example take a heated solid. At each point p(x,y,z)of the solid, there will be temperature T(x,y,z). This T is a scalar point function.

Suppose a particle (or a very small insect) is tracing a path in space. When it occupies a position p(x,y,z) in space, it will be having some speed, say, v. This **speed**v is a scalar point function.

Consider a particle moving in space. At each point P on its path, the particle will be having a velocity $\bar{v}$ which is vector point function. Similarly, the acceleration of the particle is also a vector point function.

In a magnetic field, at any point P(x,y,z) there will be a magnetic force $\bar{f}(x,y,z)$. This is called magnetic force field. This is also an example of a vector point function.

7. Tangent vector to a curve in space.

Consider an interval [a,b].

Let x = x(t),y=y(t),z=z(t)be continuous and derivable for a$\leq$ t $\leq$b.

Then the set of all points (x(t),y(t),z(t)) is called a curve in a space.

Let A = (x(a),y(a),z(a)) and B = (x(b),y(b),z(b)). These A,B are called the end points of the curve. If A =B, the curve in said to be a closed curve.

Let P and Q be two neighbouring points on the curve.

Let $\overline{OP} = \bar{r}(t), \overline{OQ} = \bar{r}(t+\delta t) = \bar{r} + \delta\bar{r}. Then\ \delta\bar{r} = \overline{OQ} - \overline{OP} = \overline{PQ}$

Then $\frac{\delta\bar{r}}{\delta t}$ is along the vector $\overline{PQ}$. As Q→P, $\overline{PQ}$ and hence $\frac{\overline{PQ}}{\delta t}$ tends to be along the tangent to the curve at P.

Hence $\lim_{\delta t\to 0} \frac{\delta\bar{r}}{\delta t} = \frac{d\bar{r}}{dt}$ will be a tangent vector to the curve at P. (This $\frac{d\bar{r}}{dt}$ may not be a unit vector)

Suppose arc length AP = s. If we take the parameter as the arc length parameter, we can observe that $\frac{d\bar{r}}{ds}$ is unit tangent vector at P to the curve.

VECTOR DIFFERENTIAL OPERATOR

Def. The vector differential operator ∇(read as del) is defined as

$\nabla \equiv \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$. This operator possesses properties analogous to those of ordinary vectors as well as differentiation operator. We will define now some quantities known as "**gradient**", "**divergence**" and "**curl**" involving this operator ∇. We must note that this operator has no meaning by itself unless it operates on some function suitably.

GRADIENT OF A SCALAR POINT FUNCTION

Let $\phi(x,y,z)$ be a scalar point function of position defined in some region of space. Then the vector function $\bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$ is known as the gradient of ϕ or $\nabla\phi$

$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$$

Properties:

(1) If f and g are two scalar functions then grad(f ±g)= grad f ± grad g

(2) The necessary and sufficient condition for a scalar point function to be constant is that $\nabla f = \bar{0}$

(3) grad(fg) = *f*(grad g)+g(grad f)

(4) If c is a constant, grad (c*f*) = *c*(grad *f*)

(5) grad $\left(\frac{f}{g}\right) = \frac{g(grad\ f) - f(grad\ g)}{g^2}, (g \neq 0)$

(6) Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$. Then $d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$ if ϕ is any scalar point function, then

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = \left(\bar{i}\frac{\partial \Phi}{\partial x} + \bar{j}\frac{\partial \Phi}{\partial y} + \bar{k}\frac{\partial \Phi}{\partial z}\right).\left(\bar{i}dx + \bar{j}dy + \bar{k}dz\right) = \nabla\Phi.d\bar{r}$$

DIRECTIONAL DERIVATIVE

Let ϕ(x,y,z) be a scalar function defined throughout some region of space. Let this function have a value ϕ at a point P whose position vector referred to the origin O is $\overline{OP} = \bar{r}$. Let $\phi+\Delta\phi$ be the value of the function at neighboring point Q. If $\overline{OQ} = \bar{r} + \Delta\bar{r}$. Let Δr be the length of $\Delta\bar{r}$

$\frac{\Delta\phi}{\Delta r}$ gives a measure of the rate at which ϕ change when we move from P to Q. The limiting value of $\frac{\Delta\phi}{\Delta r}$ $as\ \Delta r \to 0$ is called the derivative of ϕ in the direction of $\overline{PQ}$ or simply directional derivative of ϕ at P and is denoted by $d\phi/dr$.

Theorem 1: The directional derivative of a scalar point function ϕ at a point P(x,y,z) in the direction of a unit vector $\bar{e}$ is equal to $\bar{e}$. grad $\phi = \bar{e} . \nabla\phi$.

Level Surface

If a surface ϕ(x,y,z)= c be drawn through any point *P*($\bar{r}$), such that at each point on it, function has the same value as at P, then such a surface is called a level surface of the function ϕ through *P*.

e.g. : equipotential or isothermal surface.

Theorem 2: $\nabla\phi$ at any point is a vector normal to the level surface $\phi(x,y,z)=c$ through that point, where c is a constant.

The physical interpretation of $\nabla\phi$

The gradient of a scalar function $\phi(x,y,z)$ at a point $P(x,y,z)$ is a vector along the normal to the level surface $\phi(x,y,z) = c$ at P and is in increasing direction. Its magnitude is equal to the greatest rate of increase of ϕ. Greatest value of directional derivative of $\overline{\Phi}$ **at a point P = |grad ϕ|** at that point.

SOLVED PROBLEMS

1: If $a=x+y+z$, $b= x^2+y^2+z^2$, $c = xy+yz+zx$, prove that [grad a, grad b, grad c] = 0.

Sol:- Given $a=x+y+z$

There fore $\frac{\partial a}{\partial x} = 1, \ \frac{\partial a}{\partial y} = 1, \frac{\partial a}{\partial z} = 1$

Grad a = $\nabla a = \sum \bar{i}\frac{\partial a}{\partial x} = \bar{i} + \bar{j} + \bar{k}$

Given $b= x^2+y^2+z^2$

Therefore $\frac{\partial b}{\partial x} = 2x, \ \frac{\partial b}{\partial y} = 2y, \frac{\partial b}{\partial z} = 2z$

Grad b = $\nabla b = \bar{i}\frac{\partial b}{\partial x} + \bar{j}\frac{\partial b}{\partial y} + \bar{z}\frac{\partial b}{\partial z} = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$

Again $c = xy+yz+zx$

Therefore $\frac{\partial c}{\partial x} = y+z, \ \frac{\partial c}{\partial y} = z+x, \frac{\partial c}{\partial z} = y+x$

Grad c = $\bar{i}\frac{\partial c}{\partial x} + \bar{j}\frac{\partial c}{\partial y} + \bar{z}\frac{\partial c}{\partial z} = (y+z)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}$

[grad a, grad b, grad c] = $\begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix} = 0, (on\ simplification)$

[grad a, grad b, grad c] =0

2: Show that $\nabla[f(r)] = \frac{f^{i}(r)}{r}\bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

Sol:- Since $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, we have $r^2= x^2+y^2+z^2$

Differentiating w.r.t. 'x' partially, we get

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}.Similarly \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$

$\nabla[f(r)] = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)f(r) = \sum \bar{i}f^{1}(r)\frac{\partial r}{\partial x} = \sum \bar{i}f^{1}(r)\frac{x}{r}$

$= \frac{f^{1}(r)}{r}\sum \bar{i}x = \frac{f^{1}(r)}{r}.\bar{r}$

Note : From the above result, $\nabla(\log r) = \frac{1}{r^2}\bar{r}$

3: Prove that $\nabla(r^n) = nr^{n-2}\bar{r}$.

Sol:- Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$. Then we have $r^2 = x^2+y^2+z^2$ Differentiating w.r.t. x partially, we have

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}.Similarly \quad = \frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla(r^n) = \sum \bar{i}\frac{\partial}{\partial x}(r^n) = \sum \bar{i}nr^{n-1}\frac{\partial r}{\partial x} = \sum \bar{i}nr^{n-1}\frac{x}{r} = nr^{n-2}\sum \bar{i}x = nr^{n-2}(\bar{r})$$

Note : From the above result, we can have

(1). $\nabla\left(\frac{1}{r}\right) = -\frac{\bar{r}}{r^3}$, taking n = -1 (2) grad r = $\frac{\bar{r}}{r}$, taking n = 1

4: Find the directional derivative of $f = xy+yz+zx$ in the direction of vector $\bar{i} + 2\bar{j} + 2\bar{k}$ at the point (1,2,0).

Sol:- Given $f = xy+yz+zx$.

$$\text{Grad } f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{z}\frac{\partial f}{\partial z} = (y+z)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}$$

If $\bar{e}$ is the unit vector in the direction of the vector $\bar{i} + 2\bar{j} + 2\bar{k}$, then

$$\bar{e} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{1}{3}(\bar{i} + 2\bar{j} + 2\bar{k})$$

Directional derivative of f along the given direction $= \bar{e}.\nabla f$

$$= \frac{1}{3}(\bar{i} + 2\bar{j} + 2\bar{k}).[(y+z)\bar{i} + (z+x)\bar{j} + (x+y\bar{k})] at\ (1,2,0)$$

$$= \frac{1}{3}[(y+z) + 2(z+x) + 2(x+y)](1,2,0) = \frac{10}{3}$$

5: Find the directional derivative of the function $xy^2+yz^2+zx^2$ along the tangent to the curve x =t, y = t^2, z = t^3 at the point (1,1,1).

Sol: - Here $f = xy^2+yz^2+zx^2$

$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = (y^2 + 2xz)\bar{i} + (z^2 + 2xy)\bar{j} + (x^2 + 2yz)\bar{k}$$

At (1,1,1) , $\nabla f = 3\bar{i} + 3\bar{j} + 3\bar{k}$

Let $\bar{r}$ be the position vector of any point on the curve x =t , y = t^2, z = t^3. Then

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} = t\bar{i} + t^2\bar{j} + t^3\bar{k}$$

$$\frac{\partial \bar{r}}{\partial t} = \bar{i} + 2t\bar{j} + 3t^2\bar{k} = (\bar{i} + 2\bar{j} + 3\bar{k}) \text{ at } (1,1,1)$$

We know that $\frac{\partial \bar{r}}{\partial t}$ is the vector along the tangent to the curve.

Unit vector along the tangent = $\bar{e}$ $\bar{e} = \frac{\bar{i} + 2\bar{j} + 3\bar{k}}{\sqrt{1 + 2^2 + 3^2}} = \frac{\bar{i} + 2\bar{j} + 3\bar{k}}{\sqrt{14}}$

Directional derivative along the tangent = $\nabla f.\bar{e}$

$$= \frac{1}{\sqrt{14}}(\bar{i} + 2\bar{j} + 3\bar{k}).3(\bar{i} + \bar{j} + \bar{k})\ \frac{3}{\sqrt{14}}(1 + 2 + 3) = \frac{18}{\sqrt{14}}$$

6: Find the directional derivative of the function $f = x^2-y^2+2z^2$ at the point P =(1,2,3) in the direction of the line $\overline{PQ}$ where Q = (5,0,4).

Sol:- The position vectors of P and Q with respect to the origin are $\overline{OP} = \bar{i} + 2\bar{j} + 3\bar{k}$ and $\overline{OQ} = 5\bar{i} + 4\bar{k}$

$$\overline{PQ} = \overline{OQ} - \overline{OP} = 4\bar{i} - 2\bar{j} + \bar{k}$$

Let $\bar{e}$ be the unit vector in the direction of $\overline{PQ}$. Then $\bar{e} = \dfrac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{21}}$

$$\text{grad } f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = 2x\bar{i} - 2y\bar{j} + 4z\bar{k}$$

The directional derivative of $\bar{f}$ at P (1,2,3) in the direction of $\overline{PQ} = \bar{e}.\nabla f$

$$= \frac{1}{\sqrt{21}}(4\bar{i} - 2\bar{j} + \bar{k}).(2x\bar{i} - 2y\bar{j} + 4z\bar{k})\ \frac{1}{\sqrt{21}}(8x + 4y + 4z)_{at\,(1,2,3)} = \frac{1}{\sqrt{21}}(28)$$

7: Find the greatest value of the directional derivative of the function $f = x^2yz^3$ at (2,1,-1).

Sol: we have

$$\text{grad } f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k} = -4\bar{i} - 4\bar{j} + 12\bar{k} \text{ at } (2,1,-1).$$

Greatest value of the directional derivative of $f = |\nabla f| = \sqrt{16 + 16 + 144} = 4\sqrt{11}$.

8: Find the directional derivative of xyz^2+xz at (1, 1 ,1) in a direction of the normal to the surface $3xy^2+y= z$ at (0,1,1).

Sol:- Let $f(x, y, z) \equiv 3xy^2+y- z = 0$

Let us find the unit normal e to this surface at (0,1,1). Then

$$\frac{\partial f}{\partial x} = 3y^2, \quad \frac{\partial f}{\partial y} = 6xy + 1, \quad \frac{\partial f}{\partial z} = -1.$$

$\nabla f = 3y^2i+(6xy+1)j-k$

$(\nabla f)_{(0,1,1)} = 3i+j-k = \bar{n}$

$$\bar{e} = \frac{\bar{n}}{|\bar{n}|} = \frac{3i + j - k}{\sqrt{9 + 1 + 1}} = \frac{3i + j - k}{\sqrt{11}}$$

Let $g(x,y,z) = xyz^2+xz$,then

$$\frac{\partial g}{\partial x} = yz^2 + z, \quad \frac{\partial g}{\partial y} = xz^2, \quad \frac{\partial g}{\partial z} = 2xy + x$$

$\nabla g=(yz^2+z)i+xz^2j+(2xyz+x)k$

And $[\nabla g]_{(1,1,1)} = 2i+j+3k$

Directional derivative of the given function in the direction of $\bar{e}$ at (1,1,1) = $\nabla g.\bar{e}$

$$=(2i+j+3k).\left(\frac{3i + j - k}{\sqrt{11}}\right) = \frac{6 + 1 - 3}{\sqrt{11}} = \frac{4}{\sqrt{11}}$$

9: Find the directional derivative of $2xy+z^2$ at (1,-1,3) in the direction of $\bar{i} + 2\bar{j} + 3\bar{k}$.

Sol: Let $f = 2xy+z^2$ then $\frac{\partial f}{\partial x} = 2y, \frac{\partial f}{\partial y} = 2x, \frac{\partial f}{\partial z} = 2z.$

grad f = $\sum \bar{i}\frac{\partial f}{\partial x} = 2y\bar{i} + 2x\bar{j} + 2z\bar{k}$ and (grad f)at (1,-1,3)= $-2\bar{i} + 2\bar{j} + 6\bar{k}$

given vector is $\bar{a} = \bar{i} + 2\bar{j} + 3\bar{k} \Rightarrow |\bar{a}| = \sqrt{1+4+9} = \sqrt{14}$

Directional derivative of f in the direction of $\bar{a}$ is

$$\frac{\bar{a}.\nabla f}{|\bar{a}|} = \frac{(\bar{i} + 2\bar{j} + 3\bar{k})(-2\bar{i} + 2\bar{j} + 6\bar{k}).}{\sqrt{14}} = \frac{-2+4+18}{\sqrt{14}} = \frac{20}{\sqrt{14}}$$

10: Find the directional derivative of $\phi = x^2yz+4xz^2$ at (1,-2,-1) in the direction 2i-j-2k.

Sol:- Given $\phi = x^2yz+4xz^2$

$$\frac{\partial \phi}{\partial x} = 2xyz + 4z^2, \frac{\partial \phi}{\partial y} = x^2z, \frac{\partial \phi}{\partial z} = x^2y + 8xz.$$

Hence $\nabla\phi = \sum \bar{i}\frac{\partial \phi}{\partial x} = \bar{i}(2xyz + 4z^2) + \bar{j}x^2z + \bar{k}(x^2y + 8xz)$

$\nabla\phi$ at (1,-2,-1) = i(4+4)+j(-1)+k(-2-8)= 8i-j-10k.

The unit vector in the direction 2i-j-2k is

$$\bar{a} = \frac{2i - j - 2k.}{\sqrt{4+1+4}} = \frac{1}{3}(2i - j - 2k)$$

Required directional derivative along the given direction = $\nabla\phi.\ \bar{a}$

= (8i-j-10k). 1/3 (2i-j-2k)

= 1/3(16+1+20) = 37/3.

11: If the temperature at any point in space is given by t = xy+yz+zx, find the direction in which temperature changes most rapidly with distance from the point (1,1,1) and determine the maximum rate of change.

Sol:- The greatest rate of increase of t at any point is given in magnitude and direction by ∇t.

We have $\nabla t = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(xy + yz + zx)$

$= \bar{i}(y+z) + \bar{j}(z+x) + \bar{k}(x+y) = 2\bar{i} + 2\bar{j} + 2\bar{k}$ at (1,1,1)

Magnitude of this vector is $\sqrt{2^2 + 2^2 + 2^2} = \sqrt{12} = 2\sqrt{3}$

Hence at the point (1,1,1) the temperature changes most rapidly in the direction given by the vector $2\bar{i} + 2\bar{j} + 2\bar{k}$ and greatest rate of increase = $2\sqrt{3}$.

12: Findthe directional derivative of $\phi(x,y,z) = x^2yz+4xz^2$ at the point (1,-2,-1) in the direction of the normal to the surface $f(x,y,z) = x \log z-y^2$ at (-1,2,1).

Sol:- Given $\phi(x,y,z) = x^2yz+4xz^2$ at (1,-2,-1) and $f(x,y,z) = x \log z-y^2$ at (-1,2,1)

Now $\nabla\phi = \frac{\partial \phi}{\partial x}\bar{i} + \frac{\partial \phi}{\partial y}\bar{j} + \frac{\partial \phi}{\partial z}\bar{k}$

$= (2xyz + 4z^2)\bar{i} + (x^2 z)\bar{j} + (x^2 y + 8xz)\bar{k}$

$(\nabla\phi)_{(1,-2,-1)} = [2(1)(-2)(-1) + 4(-1)^2]i + [(1)^2(-1)\bar{j}] + [(1^2)(-2) + 8(-1)]\bar{k} ----(1)$

$= 8\bar{i} - \bar{j} - 10\bar{k}$

Unit normal to the surface $f(x,y,z)= x \log z- y^2$ is $\frac{\nabla f}{|\nabla f|}$

Now $\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = \log z\,\bar{i} + (-2y)\bar{j} + \frac{x}{z}\bar{k}$

At (-1,2,1), $\nabla f = \log(1)\bar{i} - 2(2)\bar{j} + \frac{-1}{1}\bar{k} = -4\bar{j} - \bar{k}$

$\frac{\nabla f}{|\nabla f|} = \frac{-4\bar{j} - \bar{k}.}{\sqrt{16+1}} = \frac{-4\bar{j} - \bar{k}.}{\sqrt{17}}$

Directional derivative = $\nabla\phi.\frac{\nabla f}{|\nabla f|}$

$= (8\bar{i} - \bar{j} - 10\bar{k}).\frac{-4\bar{j} - \bar{k}.}{\sqrt{17}} = \frac{4+10}{\sqrt{17}} = \frac{14}{\sqrt{17}}.$

13: Find a unit normal vector to the given surface $x^2y+2xz = 4$ at the point (2,-2,3).

Sol:- Let the given surface be $f = x^2y+2xz – 4$

On differentiating,

$\frac{\partial f}{\partial x} = 2xy + 2z,\ \frac{\partial f}{\partial y} = x^2,\ \frac{\partial f}{\partial z} = 2x.$

grad $f = \sum \bar{i}\frac{\partial f}{\partial x} = \bar{i}(2xy + 2z) + \bar{j}x^2 + 2x\bar{k}$

(grad f) at (2,-2,3) = $\bar{i}(-8+6) + 4\bar{j} + 4\bar{k} = 2\bar{i} + 4\bar{j} + 4\bar{k}$

grad (f) is the normal vector to the given surface at the given point.

Hence the required unit normal vector $\frac{\nabla f}{|\nabla f|} = \frac{2(-\bar{i} + 2\bar{j} + 2\bar{k}).}{2\sqrt{1+2^2+2^2}} = \frac{-\bar{i} + 2\bar{j} + 2\bar{k}}{3}$

14: Evaluate the angle between the normal to the surface $xy= z^2$ at the points (4,1,2) and (3,3,-3).

Sol:- Given surface is $f(x,y,z) = xy- z^2$

Let $\bar{n}_1$ and $\bar{n}_2$ be the normal to this surface at (4,1,2) and (3,3,-3) respectively.

Differentiating partially, we get

$\frac{\partial f}{\partial x} = y,\ \frac{\partial f}{\partial y} = x,\ \frac{\partial f}{\partial z} = -2z.$

grad f = $y\bar{i} + x\bar{j} - 2z\bar{k}$

$\bar{n}_1$ = (grad f) at (4,1,2) = $\bar{i} + 4\bar{j} - 4\bar{k}$

$\bar{n}_2$ = (grad f) at (3,3,-3) = $3\bar{i} + 3\bar{j} + 6\bar{k}$

Let θ be the angle between the two normal.

cos θ = $\dfrac{\overline{n_1}.\overline{n_2}}{|\overline{n_1}||\overline{n_2}|} = \dfrac{(i+4j-4k)}{\sqrt{1+16+16}}.\dfrac{(3i+3j+6k)}{\sqrt{9+9+36}}$

$$\frac{(3+12-24)}{\sqrt{33}\sqrt{54}} = \frac{-9}{\sqrt{33}\sqrt{54}}$$

15: Find a unit normal vector to the surface $x^2+y^2+2z^2 = 26$ at the point (2, 2 ,3).

Sol:- Let the given surface be $f(x,y,z) \equiv x^2+y^2+2z^2 - 26=0$. Then

$$\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 2y, \frac{\partial f}{\partial z} = 4z.$$

grad f = $\sum \bar{i}\dfrac{\partial f}{\partial x}$ = 2xi+2yj+4zk

Normal vector at(2,2,3) = $[\nabla f]_{(2,2,3)}$ = $4\bar{i} + 4\bar{j} + 12\bar{k}$

Unit normal vector = $\dfrac{\nabla f}{|\nabla f|} = \dfrac{4(\bar{i}+\bar{j}+3\bar{k})}{4\sqrt{11}} = \dfrac{\bar{i}+\bar{j}+3\bar{k}}{\sqrt{11}}$

16: Find the values of a and b so that the surfaces $ax^2-byz = (a+2)x$ and $4x^2y+z^3= 4$ may intersect orthogonally at the point (1, -1,2).

(or) Find the constants a and b so that surface $ax^2-byz=(a+2)x$ will orthogonal to $4x^2y+z^3=4$ at the point (1,-1,2).

Sol:- Let the given surfaces be $f(x,y,z) = ax^2-byz - (a+2)x$-------------(1)

And $g(x,y,z) = 4x^2y+z^3- 4$------------(2)

Given the two surfaces meet at the point (1,-1,2).

Substituting the point in (1), we get

a+2b-(a+2) = 0 ⇒ b=1

Now $\dfrac{\partial f}{\partial x} = 2ax - (a+2), \dfrac{\partial f}{\partial y} = -bz$ and $\dfrac{\partial f}{\partial z} = -by$.

$\nabla f = \sum \bar{i}\dfrac{\partial f}{\partial x}$ = [(2ax-(a+2)]i-bz+bk = (a-2)i-2bj+bk

= (a-2)i-2j+k = $\bar{n}_1$, normal vector to surface 1.

Also $\dfrac{\partial g}{\partial x} = 8xy, \dfrac{\partial g}{\partial y} = 4x^2, \dfrac{\partial g}{\partial z} = 3z^2.$

$\nabla g = \sum \bar{i}\frac{\partial g}{\partial x} =$ 8xyi+4x²j+3z²k

$(\nabla g)_{(1,-1,2)}$ = -8i+4j+12k = $\bar{n}_2$, normal vector to surface 2.

Given the surfaces f(x,y,z), g(x,y,z) are orthogonal at the point (1,-1,2).

$[\nabla f].[\nabla g] = 0 \Rightarrow$ ((a-2)i-2j+k). (-8i+4j+12k)=0

$\Rightarrow$-8a+16-8+12 $\Rightarrow$ a =5/2

Hence a = 5/2 and b=1.

17: Find a unit normal vector to the surface $z= x^2+y^2$ at (-1,-2,5)

Sol:- Let the given surface be $f = x^2+y^2-z$

$$\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 2y, \frac{\partial f}{\partial z} = -1.$$

grad f = $\nabla f = \sum \bar{i}\frac{\partial f}{\partial x} =$ 2xi+2yj-k

(∇f) at (-1,-2,5)= -2i-4j-k

∇f is the normal vector to the given surface.

Hence the required unit normal vector = $\frac{\nabla f}{|\nabla f|}$ =

$$\frac{-2i-4j-k}{\sqrt{(-2)^2+(-4)^2+(-1)^2}} = \frac{-2i-4j-k}{\sqrt{21}} = -\frac{1}{\sqrt{21}}(2i+4j+k)$$

18: Find the angle of intersection of the spheres $x^2+y^2+z^2 =29$ and $x^2+y^2+z^2 +4x-6y-8z-47 =0$ at the point (4,-3,2).

Sol:- Let $f = x^2+y^2+z^2 -29$ and $g = x^2+y^2+z^2 +4x-6y-8z-47$

Then grad f= $\bar{i}\frac{\partial f}{\partial x}+\bar{j}\frac{\partial f}{\partial y}+\bar{k}\frac{\partial f}{\partial z} = 2x\bar{i}+2y\bar{j}+2z\bar{k}$ and

grad g = $(2x+4)\bar{i}+(2y-6)\bar{j}+(2z-8)\bar{k}$

The angle between two surfaces at a point is the angle between the normal to the surfaces at that point.

Let $\bar{n}_1$ = (grad f) at (4,-3,2) =$8\bar{i}-6\bar{j}+4\bar{k}$

$\bar{n}_2$ = (grad f) at (4,-3,2) = $12\bar{i}-12\bar{j}-4\bar{k}$

The vectors $\bar{n}_1$ and $\bar{n}_2$ are along the normal to the two surfaces at (4,-3,2). Let θ be the angle between the surfaces. Then

Cos θ= $\frac{\overline{n_1}.\overline{n_2}}{|\overline{n_1}||\overline{n_2}|} = .\frac{152}{\sqrt{116}\ \sqrt{304}}$

$$\therefore \theta = \cos^{-1}\left(\sqrt{\frac{19}{29}}\right)$$

19: Find the angle between the surfaces $x^2+y^2+z^2=9$, and $z = x^2+y^2-3$ at point (2,-1,2).

Sol:- Let $\phi_1 = x^2+y^2+z^2-9=0$ and $\phi_2= x^2+y^2-z-3=0$ be the given surfaces. Then

$\nabla\phi_1$= 2xi+2yj+2zk and $\nabla\phi_2$ = 2xi+2yj-k

Let $\overline{n_1} = \nabla\phi_1$ at(2,-1,2)= 4i-2j+4k and

$\overline{n_2} = \nabla\phi_2$ at (2,-1,2) = 4i-2j-k

The vectors $\overline{n_1}$ and $\overline{n_2}$ are along the normals to the two surfaces at the point (2,-1,2). Let θ be the angle between the surfaces. Then

$$\text{Cos } \theta = \frac{\overline{n_1}\ \overline{n_2}}{|\overline{n_1}\ \overline{n_2}|} = \frac{(4i-2j+4k)}{\sqrt{16+4+16}}.\frac{(4i-2j-k)}{\sqrt{16+4+16}} = \frac{16+4-4}{6\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

20: If $\overline{a}$ is constant vector then prove that grad ($\overline{a}.\overline{r}$)= $\overline{a}$

Sol: Let $\overline{a} = a_1\overline{i} + a_2\overline{j} + a_3\overline{k}$, where a_1,a_2,a_3 are constants.

$\overline{a}.\overline{r} = (a_1\overline{i} + a_2\overline{j} + a_3\overline{k}).(x\overline{i} + y\overline{j} + z\overline{k}) = a_1x + a_2y + a_3z$

$$\frac{\partial}{\partial x}(\overline{a}.\overline{r}) = a_1, \frac{\partial}{\partial y}(\overline{a}.\overline{r}) = a_2, \frac{\partial}{\partial z}(\overline{a}.\overline{r}) = a_3$$

grad ($\overline{a}.\overline{r}$)= $a_1\overline{i} + a_2\overline{j} + a_3\overline{k} = \overline{a}$

21: If $\nabla\phi = yz\overline{i} + zx\overline{j} + xy\overline{k}$, find ϕ.

Sol:- We know that $\nabla\phi = \overline{i}\frac{\partial f}{\partial x} + \overline{j}\frac{\partial f}{\partial y} + \overline{k}\frac{\partial f}{\partial z}$

Given that $\nabla\phi = yz\overline{i} + zx\overline{j} + xy\overline{k}$

Comparing the corresponding coefficients, we have $\frac{\partial\phi}{\partial x} = yz, \frac{\partial\phi}{\partial y} = zx, \frac{\partial\phi}{\partial z} = xy$

Integrating partially w.r.t. x,y,z, respectively, we get

ϕ= xyz + a constant independent of x.

ϕ= xyz + a constant independent of y.

ϕ= xyz + a constant independent of z.

Here a possible form of ϕ is ϕ= xyz+a constant.

DIVERGENCE OF A VECTOR

Let $\bar{f}$ be any continuously differentiable vector point function. Then $\bar{i}.\frac{\partial \bar{f}}{\partial x}+\bar{j}.\frac{\partial \bar{f}}{\partial y}+\bar{k}.\frac{\partial \bar{f}}{\partial z}$ is called the divergence of $\bar{f}$ and is written as div $\bar{f}$.

i.e., div $\bar{f}=\bar{i}.\frac{\partial \bar{f}}{\partial x}+\bar{j}.\frac{\partial \bar{f}}{\partial y}+\bar{k}.\frac{\partial \bar{f}}{\partial z}=\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right).\bar{f}$

Hence we can write div $\bar{f}$ as

div $\bar{f}=\nabla.\bar{f}$

This is a scalar point function.

Theorem 1: If the vector $\bar{f}=f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$, then div $\bar{f}=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$

Prof: Given $\bar{f}=f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$

$$\frac{\partial \bar{f}}{\partial x}=\bar{i}\frac{\partial f_1}{\partial x}+\bar{j}\frac{\partial f_2}{\partial x}+\bar{k}\frac{\partial f_3}{\partial x}$$

Also $\bar{i}.\frac{\partial \bar{f}}{\partial x}=\frac{\partial f_1}{\partial x}$. Similarly $\bar{j}.\frac{\partial \bar{f}}{\partial y}=\frac{\partial f_2}{\partial y}$ and $\bar{k}.\frac{\partial \bar{f}}{\partial z}=\frac{\partial f_3}{\partial z}$

We have div $\bar{f}=\sum \bar{i}.\left(\frac{\partial \bar{f}}{\partial x}\right)=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$

Note : If $\bar{f}$ is a constant vector then $\frac{\partial f_1}{\partial x},\frac{\partial f_2}{\partial y},\frac{\partial f_3}{\partial z}$ are zeros.

$\therefore$ div $\bar{f}=0$ for a constant vector $\bar{f}$.

Theorem 2: div $(\bar{f}\pm\bar{g})=div\ \bar{f}\pm div\ \bar{g}$

Proof: div $(\bar{f}\pm\bar{g})=\sum \bar{i}.\frac{\partial}{\partial x}(\bar{f}\pm\bar{g})=\sum \bar{i}.\frac{\partial}{\partial x}(\bar{f})\pm\sum \bar{i}.\frac{\partial}{\partial x}(\bar{g})=div\ \bar{f}\pm div\ \bar{g}$.

Note: If ϕ is a scalar function and $\bar{f}$ is a vector function, then

(i). $(\bar{a}.\nabla)\phi=\left[\bar{a}.\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\right]\phi$

$=\left[(\bar{a}.\bar{i})\frac{\partial}{\partial x}+(\bar{a}.\bar{j})\frac{\partial}{\partial y}+(\bar{a}.\bar{k})\frac{\partial}{\partial z}\right]\phi$

$=\left[(\bar{a}.\bar{i})\frac{\partial \phi}{\partial x}+(\bar{a}.\bar{j})\frac{\partial \phi}{\partial y}+(\bar{a}.\bar{k})\frac{\partial \phi}{\partial z}\right]$

$=\sum(\bar{a}.\bar{i})\frac{\partial \phi}{\partial x}$. and

(ii). $(\bar{a}.\nabla)\bar{f}=\sum(\bar{a}.\bar{i})\frac{\partial \bar{f}}{\partial x}$. by proceeding as in (i) [simply replace ϕ by $\bar{f}$ in (i)].

SOLENOIDAL VECTOR

A vector point function $\bar{f}$ is said to be solenoidal if div $\bar{f}$ =0.

Physical interpretation of divergence:

Depending upon $\bar{f}$ in a physical problem, we can interpret div $\bar{f}$ (= $\nabla . \bar{f}$).

Suppose $\bar{F}$ (x,y,z,t) is the velocity of a fluid at a point(x,y,z) and time 't'. Though time has no role in computing divergence, it is considered here because velocity vector depends on time.

Imagine a small rectangular box within the fluid as shown in the figure. We would like to measure the rate per unit volume at which the fluid flows out at any given time. The divergence of $\bar{F}$ measures the outward flow or expansions of the fluid from their point at any time. This gives a physical interpretation of the divergence.

Similar meanings are to be understood with respect to divergence of vectors $\bar{f}$ from other branches. A detailed elementary interpretation can be seen in standard books on fluid dynamics, electricity and magnetism etc.

SOLVED PROBLEMS

1: If $\bar{f}$ **=** $xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ find div $\bar{f}$ at(1, -1, 1).

Sol:- Given $\bar{f}$ **=** $xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$.

Thendiv $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2) = y^2+2x^2z-6yz$

(div $\bar{f}$) at (1, -1, 1) = 1+2+6 =9

2: Find div $\bar{f}$ when grad($x^3+y^3+z^3-3xyz$)

Sol:- Let $\phi = x^3+y^3+z^3-3xyz$.

Then $\frac{\partial \phi}{\partial x} = 3x^2 - 3yz, \frac{\partial \phi}{\partial y} = 3y^2 - 3zx, \frac{\partial \phi}{\partial z} = 3z^2 - 3xy$

grad $\phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z} = 3[(x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}]$

div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}[3(x^2 - yz)] + \frac{\partial}{\partial y}[3(y^2 - zx)] + \frac{\partial}{\partial z}[3(z^2 - xy)]$

= 3(2x)+3(2y)+3(2z) = 6(x+y+z)

3: If $\bar{f} = (x + 3y)\bar{i} + (y - 2z)\bar{j} + (x + pz)\bar{k}$ is solenoid, find *P*.

Sol:- Let $\bar{f} = (x + 3y)\bar{i} + (y - 2z)\bar{j} + (x + pz)\bar{k} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

We have $\frac{\partial f_1}{\partial x} = 1, \frac{\partial f_2}{\partial y} = 1, \frac{\partial f_3}{\partial z} = p$

div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$ = 1+1+p =2+p since $\bar{f}$ is solenoid, we have div $\bar{f}$ = 0 $\Rightarrow 2 + p = 0 \Rightarrow p = -2$

4: Find div $\bar{f} = r^n \bar{r}$. Find n if it is solenoid?

Sol: Given $\bar{f} = r^n \bar{r}$. where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$

We have $r^2 = x^2+y^2+z^2$

Differentiating partially w.r.t. x , we get

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$\bar{f} = r^n(x\bar{i} + y\bar{j} + z\bar{k})$

div $\bar{f} = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$

$= nr^{n-1}\frac{\partial r}{\partial x}x + r^n + nr^{n-1}\frac{\partial r}{\partial y}y + r^n + nr^{n-1}\frac{\partial r}{\partial z}z + r^n$

$= nr^{n-1}\left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right] + 3r^n = nr^{n-1}\frac{(r^2)}{r} + 3r^n = nr^n + 3r^n = (n+3)r^n$

Let $\bar{f} = r^n\bar{r}$ be solenoid. Then div $\bar{f}$ = 0

$(n+3)r^n = 0 \Rightarrow n = -3$

5: Evaluate $\nabla.\left(\frac{\bar{r}}{r^3}\right)$ where $\bar{r} = xi + yj + zk$ and $r = |\bar{r}|$.

Sol:- We have

$\bar{r}$ = xi+yj+zk and r = $\sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \text{and } \frac{\partial r}{\partial z} = \frac{z}{r}$$

$\therefore \frac{\bar{r}}{r^3} = \bar{r}.\ r^{-3} = r^{-3}xi + r^{-3}yj + r^{-3}zk = f_1 i + f_2 j + f_3 k$

Hence $\nabla.\left(\frac{\bar{r}}{r^3}\right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

We have $f_1 = r^{-3}x \Rightarrow \frac{\partial f_1}{\partial x} = r^{-3}.1 + x(-3)r^{-4}.\frac{\partial r}{\partial x}$

$$\therefore \frac{\partial f_1}{\partial x} = r^{-3} - 3xr^{-4}\frac{x}{y} = r^{-3} - 3x^2r^{-5}$$

$$\nabla.\left(\frac{\bar{r}}{r^3}\right) = \sum \frac{\partial f_1}{\partial x} = 3r^{-3} - 3r^{-5}\sum x^2$$

$= 3r^{-3} - 3r^{-5}r^2 = 3r^{-3} - 3r^{-3} = 0$

6: Find div $\bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Sol:- We have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\text{div } \bar{r} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$$

CURL OF A VECTOR

Def: Let $\bar{f}$ be any continuously differentiable vector point function. Then the vector function defined by $\bar{i} \times \frac{\partial \bar{f}}{\partial x} + \bar{j} \times \frac{\partial \bar{f}}{\partial y} + \bar{k} \times \frac{\partial \bar{f}}{\partial z}$ is called curl of $\bar{f}$ and is denoted by curl $\bar{f}$ or (∇x $\bar{f}$).

$$\text{Curl } \bar{f} = \bar{i} \times \frac{\partial \bar{f}}{\partial x} + \bar{j} \times \frac{\partial \bar{f}}{\partial y} + \bar{k} \times \frac{\partial \bar{f}}{\partial z} = \sum \left(\bar{i} \times \frac{\partial \bar{f}}{\partial x} \right)$$

Theorem 1: If $\bar{f}$ is differentiable vector point function given by $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ then curl $\bar{f}$ =

$$\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right)\bar{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right)\bar{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right)\bar{k}$$

Proof : $\text{curl } \bar{f} = \sum \bar{i} \times \frac{\partial}{\partial x}(\bar{f}) = \sum \bar{i} \times \frac{\partial}{\partial x}(f_1\bar{i} + f_2\bar{j} + f_3\bar{k}) = \sum \left(\frac{\partial f_2}{\partial x}\bar{k} - \frac{\partial f_3}{\partial x}\bar{j} \right)$

$$= \left(\frac{\partial f_2}{\partial x}\bar{k} - \frac{\partial f_3}{\partial x}\bar{j} \right) + \left(\frac{\partial f_3}{\partial y}\bar{i} - \frac{\partial f_1}{\partial y}\bar{k} \right) + \left(\frac{\partial f_1}{\partial z}\bar{j} - \frac{\partial f_2}{\partial z}\bar{i} \right)$$

$$= \bar{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) + \bar{j}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) + \bar{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right)$$

Note: (1) the above expression for curl $\bar{f}$ can be remembered easily through the representation.

$$\textbf{curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \nabla \textbf{x } \bar{f}$$

Note (2) : If $\bar{f}$ is a constant vector then curl $\bar{f} = \bar{o}$.

Theorem 2: curl $(\bar{a} \pm \bar{b}) = curl\ \bar{a} \pm curl\ \bar{b}$

Proof: $\text{curl}(\bar{a} \pm \bar{b}) = \sum \bar{i} \times \frac{\partial}{\partial x}(\bar{a} \pm \bar{b})$

$$= \sum i \times \left(\frac{\partial \bar{a}}{\partial x} \pm \frac{\partial \bar{b}}{\partial x} \right) = \sum i \times \frac{\partial \bar{a}}{\partial x} \pm \sum ix \frac{\partial \bar{b}}{\partial x}$$

$$= curl\ \bar{a} \pm curl\ \bar{b}$$

1. Physical Interpretation of curl

If $\overline{w}$ is the angular velocity of a rigid body rotating about a fixed axis and $\overline{v}$ is the velocity of any point P(x,y,z) on the body, then $\overline{w}$ = ½ curl $\overline{v}$. Thus the angular velocity of rotation at any point is equal to half the curl of velocity vector. This justifies the use of the word "curl of a vector".

2. Irrotational Motion, Irrotational Vector

Any motion in which curl of the velocity vector is a null vector i.e curl $\overline{v} = \overline{0}$ is said to be Irrotational.

Def: A vector $\overline{f}$ is said to be Irrotational if curl $\overline{f} = \overline{0}$.

If $\overline{f}$ is Irrotational, there will always exist a scalar function φ(x,y,z) such that $\overline{f}$ =grad ϕ. Thisϕ is called scalar potential of $\overline{f}$.

It is easy to prove that, if $\overline{f}$ = grad ϕ, then curl $\overline{f}$ = 0.

Hence ∇x $\overline{f}$ = 0 ⇔ there exists a scalar function ϕ such that $\overline{f}$ = ∇ϕ.

This idea is useful when we study the "work done by a force" later.

SOLVED PROBLEMS

1: If $\overline{f} = xy^2\overline{i} + 2x^2yz\,\overline{j} - 3yz^2\,\overline{k}$ find curl $\overline{f}$ at the point (1,-1,1).

Sol:- Let $\overline{f} = xy^2\overline{i} + 2x^2yz\,\overline{j} - 3yz^2\,\overline{k}$. Then

$$\text{curl } \overline{f} = \nabla \text{x } \overline{f} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= \overline{i}\left(\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right) + \overline{j}\left(\frac{\partial}{\partial z}(xy^2) - \frac{\partial}{\partial x}(-3yz^2)\right) + \overline{k}\left(\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right)$$

$= \overline{i}\left(-3z^2 - 2x^2z\right) + \overline{j}(0-0) + \overline{k}(4xyz - 2xy) = -\left(3z^2 + 2x^2y\right)\overline{i} + (4xyz - 2xy)\overline{k}$

= curl $\overline{f}$ at (1,-1,1) = $-\overline{i} - 2\overline{k}$.

2: Find curl $\overline{f}$ where $\overline{f}$ = grad($x^3+y^3+z^3-3xyz$)

Sol:- Let ϕ= $x^3+y^3+z^3-3xyz$. Then

grad ϕ= $\sum \overline{i}\frac{\partial \phi}{\partial x} = 3(x^2 - yz)\overline{i} + 3(y^2 - zx)\overline{j} + 3(z^2 - xy)\overline{k}$

curl grad ϕ= ∇x grad ϕ= $3\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$

$= 3[\bar{i}(-x + x) - \bar{j}(-y + y) + \bar{k}(-z + z)] = \bar{0}$

$\therefore$ curl $\bar{f} = \bar{0}$.

Note: We can prove in general that curl (grad ϕ)= $\bar{0}$.(i.e) grad ϕ is always irrotational.

3: Prove that if $\bar{r}$ is the position vector of an point in space, then $r^n\bar{r}$ is Irrotational. (or) Show that curl$(r^n\bar{r}) = 0$

Sol:- Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and r = $|\bar{r}|$ $\therefore$ $r^2 = x^2+y^2+z^2$.

Differentiating partially w.r.t. 'x', we get

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$,

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$, *and* $\frac{\partial r}{\partial z} = \frac{z}{r}$

We have $r^n\bar{r} = r^n(x\bar{i} + y\bar{j} + z\bar{k})$

$\nabla \times (r^n\bar{r}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xr^n & yr^n & zr^n \end{vmatrix}$

$= \bar{i}\left(\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right) + \bar{j}\left(\frac{\partial}{\partial z}(r^n x) - \frac{\partial}{\partial x}(r^n z)\right) + \bar{k}\left(\frac{\partial}{\partial x}(r^n y) - \frac{\partial}{\partial y}(r^n x)\right)$

$= \sum \bar{i}\left\{znr^{n-1}\frac{\partial r}{\partial y} - ynr^{n-1}\frac{\partial r}{\partial z}\right\} = nr^{n-1}\sum \bar{i}\left\{z\left(\frac{y}{r}\right) - y\left(\frac{z}{r}\right)\right\}$

$= nr^{n-2}[(zy - yz)\bar{i} + (xz - zx)\bar{j} + (xy - yz)\bar{k}]$

$= nr^{n-2}[0\bar{i} + 0j + 0\bar{k}] = nr^{n-2}[\bar{0}] = \bar{0}$

Hence $r^n\bar{r}$ is Irrotational.

4: Prove that curl $\bar{r} = \bar{0}$

Sol:- Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

curl $\bar{r} = \sum \bar{i} \times \frac{\partial}{\partial x}(\bar{r}) = \sum(\bar{i}x\bar{i}) = \bar{0} + \bar{0} + \bar{0} = \bar{0}$

$\therefore$ $\bar{r}$ is Irrotational vector.

5: If $\bar{a}$ is a constant vector, prove that curl $\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = -\frac{\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{a}.\bar{r})$.

Sol:- We have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\frac{\partial \bar{r}}{\partial x} = \bar{i}, \frac{\partial \bar{r}}{\partial y} = \bar{j}, \frac{\partial \bar{r}}{\partial z} = \bar{k}$$

If $|\bar{r}| = r$ then $r^2 = x^2+y^2+z^2$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}$$

curl $\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = \sum \bar{i}\times\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right)$

Now $\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = \bar{a}\times\frac{\partial}{\partial x}\left(\frac{\bar{r}}{r^3}\right) = \bar{a}\times\left[\frac{1}{r^3}\frac{\partial \bar{r}}{\partial x} - \frac{3}{r^4}\frac{\partial r}{\partial x}\bar{r}\right]$

$$= \bar{a}\times\left[\frac{1}{r^3}\bar{i} - \frac{3}{r^5}x\bar{r}\right] = \frac{\bar{a}\times\bar{i}}{r^3} - \frac{3x(\bar{a}.\times\bar{r})}{r^5}.$$

$$\therefore\ i\times\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = \bar{i}\times\left[\frac{\bar{a}\times\bar{i}}{r^3} - \frac{3x}{r^5}(\bar{a}\times\bar{r})\right] = \frac{\bar{i}\times(\bar{a}\times\bar{i})}{r^3} - \frac{3x}{r^5}\bar{i}\times(\bar{a}\times\bar{r})$$

$$= \frac{(\bar{i}.\bar{i})\bar{a} - (\bar{i}.\bar{a})i}{r^3} - \frac{3x}{r^5}[(\bar{i}.\bar{r})\bar{a} - (i.a)\bar{r}]$$

Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$. Then $\bar{i}.\bar{a} = a_1$, etc.

$$\therefore\ i\times\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = \sum\frac{(\bar{a} - a_1 i)}{r^3} - \frac{3x}{r^3}(x\bar{a} - a_1\bar{r})$$

$$\therefore\ \sum i\times\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = \sum\frac{\bar{a} - a_1\bar{i}}{r^3} - \frac{3}{r^5}\sum(x^2\bar{a} - a_1 x\bar{r})$$

$$= \frac{3\bar{a} - \bar{a}}{r^3} - \frac{3\bar{a}}{r^5}(r^2) + \frac{3\bar{r}}{r^5}(a_1x + a_2y + a_3z)$$

$$= \frac{2\bar{a}}{r^3} - \frac{3\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{r}.\bar{a}) = -\frac{\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{r}.\bar{a})$$

6: Show that the vector $(x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$ is irrotational and find its scalar potential.

Sol: let $\bar{f} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$

$$\text{Then curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix} = \sum\bar{i}(-x + x) = \bar{0}$$

$\therefore$ $\bar{f}$ is Irrotational. Then there exists ϕ such that $\bar{f} = \nabla\phi$.

$$\Rightarrow \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$$

Comparing components, we get

$$\frac{\partial\phi}{\partial x} = x^2 - yz \Rightarrow \phi = \int (x^2 - yz)dx = \frac{x^3}{3} - xyz + f_1(y,z) \quad (1)$$

$$\frac{\partial\phi}{\partial y} = y^2 - zx \Rightarrow \phi = \frac{y^3}{3} - xyz + f_2(z,x)......(2)$$

$$\frac{\partial\phi}{\partial z} = z^2 - xy \Rightarrow \phi = \frac{z^3}{3} - xyz + f_3(x,y)......(3)$$

From (1), (2),(3), $\phi = \dfrac{x^3 + y^3 + z^3}{3} - xyz$

$$\therefore \quad \phi = \frac{1}{3}(x^3 + y^3 + z^3) - xyz + constant$$

Which is the required scalar potential.

7: Find constants a,b and c if the vector $\bar{f} = (2x + 3y + az)\bar{i} + (bx + 2y + 3z)\bar{j} + (2x + cy + 3z)\bar{k}$ is Irrotational.

Sol:- Given $\bar{f} = (2x + 3y + az)\bar{i} + (bx + 2y + 3z)\bar{j} + (2x + cy + 3z)\bar{k}$

$$\text{Curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + 3y + az & bx + 2y + 3z & 2x + cy + 3z \end{vmatrix} = (c-3)\bar{i} - (2-a)\bar{j} + (b-3)\bar{k}$$

If the vector is Irrotational then curl $\bar{f} = \bar{0}$

$$\therefore 2 - a = 0 \Rightarrow a = 2, b - 3 = 0 \Rightarrow b = 3, c - 3 = 0 \Rightarrow c = 3$$

8: If f(r) is differentiable, show that curl { $\bar{r}$ f(r)} = $\bar{0}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

Sol: r = $|\bar{r}|$ = $\sqrt{x^2 + y^2 + z^2}$

$\therefore$ $r^2 = x^2+y^2+z^2$

$$\Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \text{ similarly } \frac{\partial r}{\partial y} = \frac{y}{r}, and \ \frac{\partial r}{\partial z} = \frac{z}{r}$$

curl{$\bar{r}$ f(r)}= curl{f(r)($x\bar{i} + y\bar{j} + z\bar{k}$)}= curl $(x.f(r)\bar{i} + y.f(r)\bar{j} + z.f(r)\bar{k})$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix} = \sum \bar{i}\left[\frac{\partial}{\partial y}[zf(r)] - \frac{\partial}{\partial z}[yf(r)]\right]$$

$$\sum \bar{i}\left[zf^1(r)\frac{\partial r}{\partial y} - yf^1(r)\frac{\partial r}{\partial z}\right] = \sum \bar{i}\left[zf^1(r)\frac{y}{r} - yf^1(r)\frac{z}{r}\right]$$

= $\bar{0}$.

9: If $\overline{A}$ is irrotational vector, evaluate div($\overline{A}$ x $\overline{r}$) where $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$.

Sol: We have $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$

Given $\overline{A}$ is an irrotational vector

∇x $\overline{A} = \overline{0}$

div ($\overline{A}$ x $\overline{r}$) = ∇.($\overline{A}$ x $\overline{r}$)

= $\overline{r}$.(∇x $\overline{A}$)- $\overline{A}$.(∇x $\overline{r}$)

= $\overline{r}$.($\overline{0}$)- $\overline{A}$.(∇x $\overline{r}$) [using (1)]

= - $\overline{A}$.(∇x $\overline{r}$)…..(2)

Now ∇x $\overline{r}$ = $\begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \overline{i}\left(\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y\right) - \overline{j}\left(\frac{\partial}{\partial x}z - \frac{\partial}{\partial z}x\right) + \overline{k}\left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right) = \overline{0}$

$\therefore$ $\overline{A}$.(∇x $\overline{r}$)=0 …(3)

Hence div ($\overline{A}$ x $\overline{r}$)=0. [using (2) and (3)]

10: Find constants a,b,c so that the vector $\overline{A} = (x + 2y + az)\overline{i} + (bx - 3y - z)\overline{j} + (4x + cy + 2z)\overline{k}$ is Irrotational. Also find ϕ such that $\overline{A} = \nabla\phi$.

Sol: Given vector is $\overline{A} = (x + 2y + az)\overline{i} + (bx - 3y - z)\overline{j} + (4x + cy + 2z)\overline{k}$

Vector $\overline{A}$ is Irrotational $\Rightarrow$ curl $\overline{A} = \overline{0}$

$$\Rightarrow \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = \overline{0}$$

$\Rightarrow (c + 1)\overline{i} + (a - 4)\overline{j} + (b - 2)\overline{k} = \overline{0}$

$\Rightarrow (c + 1)\overline{i} + (a - 4)\overline{j} + (b - 2)\overline{k} = 0\overline{i} + 0\overline{j} + 0\overline{k}$

Comparing both sides,

c+1=0, a-4=0, b-2=0

c= -1, a=4,b=2

Now $\overline{A} = (x + 2y + 4z)\overline{i} + (2x - 3y - z)\overline{j} + (4x - y + 2z)\overline{k}$, on substituting the values of a,b,c

we have $\overline{A} = \nabla\phi$.

$\Rightarrow \overline{A} = (x + 2y + 4z)\overline{i} + (2x - 3y - z)\overline{j} + (4x - y + 2z)\overline{k} = \overline{i}\frac{\partial\phi}{\partial x} + \overline{j}\frac{\partial\phi}{\partial y} + \overline{k}\frac{\partial\phi}{\partial z}$

Comparing both sides, we have

$\frac{\partial \phi}{\partial x}$ = x+2y+4z $\Rightarrow \phi = x^2/2+2xy+4zx+f_1(y,z)$

$\frac{\partial \phi}{\partial y}$ = 2x-3y-z $\Rightarrow \phi = 2xy-3y^2/2-yz+f_2(z,x)$

$\frac{\partial \phi}{\partial z}$ = 4x-y+2z $\Rightarrow \phi = 4xz-yz+z^2+f_3(x,y)$

Hence $\phi = x^2/2 - 3y^2/2+z^2+2xy+4zx-yz+c$

11: If ω is a constant vector, evaluate curl V where V = ωx$\bar{r}$.

Sol: curl (ωx$\bar{r}$) = $\sum \bar{i} \times \frac{\partial}{\partial x}(\varpi \times \bar{r}) = \sum \bar{i} \times \left[\frac{\partial \varpi}{\partial x} \times \bar{r} + \varpi \times \frac{\partial \bar{r}}{\partial x}\right]$

$= \sum \bar{i} \times [\bar{0} + \omega \times \bar{i}]$ $[\because\ \bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b}).\bar{c}]$

$= \sum \bar{i} \times (\omega \times \bar{i}) = \sum [(\bar{i}.\bar{i})\omega - (\bar{i}.\omega)\bar{i}] = \sum \omega - \sum (\bar{i}.\omega)\bar{i} = 3\omega - \omega = 2\omega$

Assignments

1. If $\bar{f}$ = $e^{x+y+z}(\bar{i} + \bar{j} + \bar{k})$ find curl $\bar{f}$.

2. Prove that $\bar{f} = (y+z)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}$ is irrotational.

3. Prove that ∇.($\bar{a} \times \bar{f}$)= – $\bar{a}$. curl $\bar{f}$ where $\bar{a}$ is a constant vector.

4. Prove that curl ($\bar{a} \times \bar{r}$)=2$\bar{a}$ where $\bar{a}$ is a constant vector.

5. If $\bar{f} = x^2 y\bar{i} - 2zx\bar{j} + 2yz\bar{k}$ find (i) curl $\bar{f}$ (ii) curl curl $\bar{f}$.

OPERATORS

Vector differential operator ∇

The operator ∇ = $\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$ is defined such that ∇ϕ= $\bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$ where ϕ is a scalar point function.

Note: If ϕ is a scalar point function then ∇ϕ= grad ϕ= $\sum \bar{i}\frac{\partial \phi}{\partial x}$

(2) Scalar differential operator $\bar{a}$.∇

The operator $\bar{a}$.∇ = $(\bar{a}.\bar{i})\frac{\partial \phi}{\partial x} + (\bar{a}.\bar{j})\frac{\partial \phi}{\partial y} + (\bar{a}.\bar{k})\frac{\partial \phi}{\partial z}$ is defined such that

($\bar{a}$.∇)ϕ=$(\bar{a}.\bar{i})\frac{\partial \phi}{\partial x} + (\bar{a}.\bar{j})\frac{\partial \phi}{\partial y} + (\bar{a}.\bar{k})\frac{\partial \phi}{\partial z}$

And ($\bar{a}$.∇) $\bar{f}$ =$(\bar{a}.\bar{i})\frac{\partial \bar{f}}{\partial x} + (\bar{a}.\bar{j})\frac{\partial \bar{f}}{\partial y} + (\bar{a}.\bar{k})\frac{\partial \bar{f}}{\partial z}$

(3). Vector differential operator $\bar{a}$ x∇

The operator $\bar{a}$ x∇= $(\bar{a} \times \bar{i})\frac{\partial}{\partial x} + (\bar{a} \times \bar{j})\frac{\partial}{\partial y} + (\bar{a} \times \bar{k})\frac{\partial}{\partial z}$ is defined such that

(i). ($\bar{a}$ x∇)ϕ=$(\bar{a} \times \bar{i})\frac{\partial \phi}{\partial x} + (\bar{a} \times \bar{j})\frac{\partial \phi}{\partial y} + (\bar{a} \times \bar{k})\frac{\partial \phi}{\partial z}$

(ii). $(\bar{a} \times \nabla).\bar{f} = (\bar{a}\times \bar{i})\frac{\partial \bar{f}}{\partial x} + (\bar{a}\times \bar{j})\frac{\partial \bar{f}}{\partial y} + (\bar{a}\times \bar{k})\frac{\partial \bar{f}}{\partial z}$

(iii). $(\bar{a} \times \nabla)\times \bar{f} = (\bar{a}\times \bar{i})\times \frac{\partial \bar{f}}{\partial x} + (\bar{a}\times \bar{j})\times \frac{\partial \bar{f}}{\partial y} + (\bar{a}\times \bar{k})\times \frac{\partial \bar{f}}{\partial z}$

(4). Scalar differential operator ∇.

The operator $\nabla = \bar{i}.\frac{\partial}{\partial x} + \bar{j}.\frac{\partial}{\partial y} + \bar{k}.\frac{\partial}{\partial z}$ is defined such that $\nabla.\bar{f} = \bar{i}.\frac{\partial \bar{f}}{\partial x} + \bar{j}.\frac{\partial \bar{f}}{\partial y} + \bar{k}.\frac{\partial \bar{f}}{\partial z}$

Note: $\nabla.\bar{f}$ is defined as div $\bar{f}$. It is a scalar point function.

(5). Vector differential operator ∇ x

The operator $\nabla \times = \bar{i}\times \frac{\partial}{\partial x} + \bar{j}\times \frac{\partial}{\partial y} + \bar{k}\times \frac{\partial}{\partial z}$ is defined such that

$$\nabla \times \bar{f} = \bar{i}\times \frac{\partial \bar{f}}{\partial x} + \bar{j}\times \frac{\partial \bar{f}}{\partial y} + \bar{k}\times \frac{\partial \bar{f}}{\partial z}$$

Note : $\nabla \times \bar{f}$ is defined as curl $\bar{f}$. It is a vector point function.

(6). Laplacian Operator ∇^2

$$\nabla.\nabla\phi = \sum \bar{i}.\frac{\partial}{\partial x}\left(\bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}\right) = \sum \frac{\partial^2 \phi}{\partial x^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi$$

Thus the operator $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called Laplacian operator.

Note : (i). $\nabla^2\phi = \nabla.(\nabla\phi)$ = div(grad ϕ)

(ii). If $\nabla^2\phi = 0$ then ϕ is said to satisfy Laplacian equation. This ϕ is called a harmonic function.

SOLVED PROBLEMS

1: Prove that div.(grad r^m)= $m(m+1)r^{m-2}$ (or) $\nabla^2(r^m) = m(m+1)r^{m-2}$ (or) $\nabla^2(r^n) = n(n+1)r^{n-2}$

Sol: Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$ then $r^2 = x^2+y^2+z^2$.

Differentiating w.r.t. 'x' partially, wet get $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$.

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now grad(r^m) = $\sum \bar{i}\frac{\partial}{\partial x}(r^m) = \sum \bar{i}mr^{m-1}\frac{\partial r}{\partial x} = \sum \bar{i}mr^{m-1}\frac{x}{r} = \sum \bar{i}mr^{m-2}x$

$\therefore$ div (grad r^m) = $\sum \frac{\partial}{\partial x}[mr^{m-2}x] = m\sum\left[(m-2)r^{m-3}\frac{\partial r}{\partial x}x + r^{m-2}\right]$

$= m\sum\left[(m-2)r^{m-4}x^2 + r^{m-2}\right] = m\left[(m-2)r^{m-4}\sum x^2 + \sum r^{m-2}\right]$

$= m[(m-2)r^{m-4}(r^2)+3r^{m-2}]$

$= m[(m-2)\, r^{m-2}+3r^{m-2}] = m[(m-2+3)r^{m-2}] = m(m+1)r^{m-2}$.

Hence $\nabla^2(r^m) = m(m+1)r^{m-2}$

2: Show that $\nabla^2[f(r)]= \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} = f^{11}(r) + \frac{2}{r} f^1(r)$ where r = $|\bar{r}|$.

Sol: grad [f(r)] = $\nabla f(r)= \sum i \frac{\partial}{\partial x}[f(r)] = \sum i\, f^1(r)\frac{\partial r}{\partial x} = \sum i\, f^1(r)\frac{x}{r}$

$\therefore$ div [grad f(r)] = $\nabla^2[f(r)] = \nabla.\nabla f(r)= \sum \frac{\partial}{\partial x}\left[f^1(r)\frac{x}{r}\right]$

$$= \sum \frac{r\frac{\partial}{\partial x}[f^1(r)x] - f^1(r)x\frac{\partial}{\partial x}(r)}{r^2}$$

$$= \sum \frac{r\left(f^{11}(r)\frac{\partial r}{\partial x}x + f^1(r)\right) - f^1(r)x\left(\frac{x}{r}\right)}{r^2}$$

$$= \sum \frac{rf^{11}(r)\frac{x}{r}x + rf^1(r) - f^1(r)x\left(\frac{x}{r}\right)}{r^2}$$

$$= \frac{\sum rf^{11}(r)\frac{x}{r}x + rf^1(r) - x^2}{r^2}.\frac{f^1(r)}{r}$$

$$= \frac{f^{11}(r)}{r^2}\sum x^2 + \frac{1}{r}\sum f^1(r) - \frac{1}{r^3}f^1(r)\sum x^2$$

$$= \frac{f^{11}(r)}{r^2}(r^2) + \frac{3}{r}f^1(r) - \frac{1}{r^3}f^1(r)r^2$$

$$= f^{11}(r) + \frac{2}{r}f^1(r)$$

3: If ϕ satisfies Laplacian equation, show that $\nabla\phi$ is both solenoidal and irrotational.

Sol: Given $\nabla^2\phi = 0 \Rightarrow$div(grad ϕ)= 0 $\Rightarrow$ grad ϕ is solenoidal

We know that curl (grad ϕ) = $\bar{0}$ $\Rightarrow$grad ϕ is always irrotational.

2.Show that (i) $(\bar{a}.\nabla)\phi = \bar{a}.\nabla\phi$ (ii) $(\bar{a}.\nabla)\bar{r} = \bar{a}$.

Sol: (i). Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$. Then

$$\bar{a}.\nabla = (a_1\bar{i} + a_2\bar{j} + a_3\bar{k}).\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) = a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}$$

$$\therefore (\bar{a}.\nabla)\phi = a_1\frac{\partial\phi}{\partial x} + a_2\frac{\partial\phi}{\partial y} + a_3\frac{\partial\phi}{\partial z}$$

Hence $(\bar{a}.\nabla)\phi = \bar{a}.\nabla\phi$

(ii). $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\therefore \frac{\partial\bar{r}}{\partial x} = \bar{i},\ \frac{\partial\bar{r}}{\partial y} = \bar{j},\ \frac{\partial\bar{r}}{\partial z} = \bar{k}$$

$$(\bar{a}.\nabla)\bar{r} = \sum a_1\frac{\partial}{\partial x}(\bar{r}) = \sum a_1\frac{\partial\bar{r}}{\partial x} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k} = \bar{a}$$

5: Prove that (i) $(\bar{f}\times\nabla).\bar{r}=0$ (ii). $(\bar{f}\times\nabla)\times\bar{r}=-2\bar{f}$

Sol: (i) $(\bar{f}\times\nabla).\bar{r}=\sum(\bar{f}\times\bar{i}).\frac{\partial\bar{r}}{\partial x}=\sum(\bar{f}\times\bar{i}).\bar{i}=0$

(ii) $(\bar{f}\times\nabla)=(\bar{f}\times\bar{i})\frac{\partial}{\partial x}\times(\bar{f}\times\bar{j})\frac{\partial}{\partial y}\times(\bar{f}\times\bar{k})\frac{\partial}{\partial z}$

$$(\bar{f}\times\nabla)\times\bar{r}=(\bar{f}\times\bar{i})\times\frac{\partial\bar{r}}{\partial x}+(\bar{f}\times\bar{j})\times\frac{\partial\bar{r}}{\partial y}+(\bar{f}\times\bar{k})\times\frac{\partial\bar{r}}{\partial z}=\sum(\bar{f}\times\bar{i})\times\bar{i}=\sum\left[(\bar{f}.i)i-\bar{f}\right]$$

$$=(\bar{f}.i)\bar{i}+(\bar{f}.\bar{j})\bar{j}+(\bar{f}.\bar{k})\bar{k}-3\bar{f}=\bar{f}-3\bar{f}=-2\bar{f}$$

6: Find div $\bar{F}$, where $\bar{F}$ = grad $(x^3+y^3+z^3-3xyz)$

Sol: Let $\phi=x^3+y^3+z^3-3xyz$. Then

$\bar{F}$ = grad ϕ

$$=\sum i\frac{\partial\phi}{\partial x}=3(x^2-yz)\bar{i}+3(y^2-zx)\bar{j}+3(x^2-xy)\bar{k}=F_1i+F_2j+F_3k\ (say)$$

$$\therefore \text{ div } \bar{F}=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}=6x+6y+6z=6(x+y+z)$$

i.e div[grad$(x^3+y^3+z^3-3xyz)$]= $\nabla^2(x^3+y^3+z^3-3xyz)=6(x+y+z)$.

7: If $f=(x^2+y^2+z^2)^{-n}$ then find div grad f and determine n if div grad $f=0$.

Sol: Let $f=(x^2+y^2+z^2)^{-n}$ and $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$

$r=|\bar{r}|\Rightarrow r^2=x^2+y^2+z^2$

$\Rightarrow f(r)=(r^2)^{-n}=r^{-2n}$

$\therefore f^1(r)=-2n\,r^{-2n-1}$

and $f^{11}(r)=(-2n)(-2n-1)r^{-2n-2}=2n(2n+1)r^{-2n-2}$

We have div grad f = $\nabla^2 f(r)=f^{11}(r)+{}^2/_r f^1(r)=(2n)(2n+1)r^{-2n-2}-4n\,r^{-2n-2}$

$=r^{-2n-2}[2n(2n+1-2)]=(2n)(2n-1)r^{-2n-2}$

If div grad $f(r)$ is zero, we get n = 0 or n = ½ .

8: Prove that $\nabla\times\left(\frac{\bar{A}\times\bar{r}}{r^n}\right)=\frac{(2-n)\bar{A}}{r^n}+\frac{n(\bar{r}.\bar{A})\bar{r}}{r^{n+2}}$.

Sol: We have $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$ and $r=|\bar{r}|=\sqrt{x^2+y^2+z^2}$

$\therefore \frac{\partial\bar{r}}{\partial x}=\bar{i},\ \frac{\partial\bar{r}}{\partial y}=\bar{j},\ \frac{\partial\bar{r}}{\partial z}=\bar{k}$ and

$r^2=x^2+y^2+z^2$.....(1)

Diff. (1) partially,

$2r\frac{\partial r}{\partial x}=2x\Rightarrow\frac{\partial r}{\partial x}=\frac{x}{r}$, similarly $\frac{\partial r}{\partial y}=\frac{y}{r}$ and $\frac{\partial r}{\partial z}=\frac{z}{r}$

$$\nabla \times \left(\frac{\bar{A}\times\bar{r}}{r^n}\right) = \sum \bar{i}\times\frac{\partial}{\partial x}\left(\frac{(\bar{A}\times\bar{r})}{r^n}\right)$$

Now $\frac{\partial}{\partial x}\left(\frac{(\bar{A}\times\bar{r})}{r^n}\right) = \bar{A}\times\frac{\partial}{\partial x}\left(\frac{\bar{r}}{r^n}\right) = \bar{A}\times\left[\frac{r^n\bar{i} - \bar{r}\,n\,r^{n-1}}{r^{2n}}\right]\frac{\partial r}{\partial x}$

$$= \bar{A}\times\left[\frac{r^n\bar{i} - n\,r^{n-2}x\bar{r}}{r^{2n}}\right] = \bar{A}\times\left[\frac{1}{r^n}\bar{i} - \frac{n}{r^{n+2}}.x\bar{r}\right]$$

$$= \frac{\bar{A}\times\bar{i}}{r^n} - \frac{n}{r^{n+2}}.x(\bar{A}\times\bar{r})$$

$$\therefore\ \bar{i}\times\frac{\partial}{\partial x}\left(\frac{(\bar{A}\times\bar{r})}{r^n}\right) = \frac{\bar{i}\times(\bar{A}\times\bar{i})}{r^n} - \frac{nx}{r^{n+2}}.\bar{i}\times(\bar{A}\times\bar{r})$$

$$= \frac{(\bar{i}.\bar{i})\bar{A} - (\bar{i}.\bar{A})\bar{i}}{r^n} - \frac{nx}{r^{n+2}}[(\bar{i}.\bar{r})\bar{A} - (\bar{i}.\bar{A})\bar{r}]$$

Let $A_1\bar{i} + A_2\bar{j} + A_3\bar{k}$. Then $\bar{i}.\bar{A} = A_1$

$$\therefore\ \bar{i}\times\frac{\partial}{\partial x}\left(\frac{(\bar{A}\times\bar{r})}{r^n}\right) = \left(\frac{\bar{A} - A_1\bar{i}}{r^n}\right) - \frac{nx}{r^{n+2}}[x\bar{A} - A_1\bar{r}]$$

and $\sum \bar{i}\times\frac{\partial}{\partial x}\left(\frac{(\bar{A}\times\bar{r})}{r^n}\right) = \sum\left(\frac{\bar{A} - A_1\bar{i}}{r^n}\right) - \frac{nx}{r^{n+2}}[x\bar{A} - A_1\bar{r}]$

$$= \frac{3\bar{A} - \bar{A}}{r^n} - \frac{n}{r^{n+2}}[r^2\bar{A}] + \frac{n\bar{r}}{r^{n+2}}(A_1x + A_2y + A_3z)$$

$$= \frac{2\bar{A}}{r^n} - \frac{n}{r^n}\bar{A} + \frac{n\bar{r}}{r^{n+2}}(\bar{A}.\bar{r}) = \frac{(2-n)\bar{A}}{r^n} + \frac{n\bar{r}}{r^{n+2}}(\bar{A}.\bar{r})$$

Hence the result.

VECTOR IDENTITIES

Theorem 1: If $\bar{a}$ is a differentiable function and ϕ is a differentiable scalar function, then prove that div($\phi\bar{a}$)= (grad ϕ). $\bar{a}$ +ϕ div $\bar{a}$ or $\nabla.(\phi\bar{a}) = (\nabla\phi).\bar{a} + \phi(\nabla.\bar{a})$

Proof: div($\phi\bar{a}$)=$\nabla.(\phi\bar{a})$= $\sum i.\frac{\partial}{\partial x}(\phi\bar{a})$

$$= \sum \bar{i}.\left(\frac{\partial\phi}{\partial x}\bar{a} + \phi\frac{\partial\bar{a}}{\partial x}\right) = \sum\left(i.\frac{\partial\phi}{\partial x}\bar{a}\right) + \sum\left(i\frac{\partial\bar{a}}{\partial x}\right)\phi$$

$$= \sum\left(\bar{i}\frac{\partial\phi}{\partial x}\right).\bar{a} + \left(\sum\bar{i}\frac{\partial\bar{a}}{\partial x}\right)\phi = (\nabla\phi).\bar{a} + \phi(\nabla.\bar{a})$$

Theorem 2: Prove that curl ($\phi\bar{a}$)= (grad ϕ)x$\bar{a}$ +ϕ curl $\bar{a}$

Proof : curl ($\phi\bar{a}$)=∇x($\phi\bar{a}$)= $\sum i\times\frac{\partial}{\partial x}(\phi\bar{a})$

$$= \sum\bar{i}\times\left(\frac{\partial\phi}{\partial x}\bar{a} + \phi\frac{\partial\bar{a}}{\partial x}\right) = \sum\left(\bar{i}\frac{\partial\phi}{\partial x}\right)\times\bar{a} + \sum\left(\bar{i}\times\frac{\partial\bar{a}}{\partial x}\right)\phi$$

$$= \nabla\phi\times\bar{a} + (\nabla\times\bar{a})\phi = (\text{grad }\phi)\times\bar{a} + \phi\text{ curl }\bar{a}$$

Theorem 3: Prove that grad $(\bar{a}.\bar{b}) = (\bar{b}.\nabla)\bar{a} + (\bar{a}.\nabla)\bar{b} + \bar{b}\times curl\ \bar{a} + \bar{a}\times curl\ \bar{b}$

Proof: Consider

$$\bar{a}\times curl(\bar{b}) = \bar{a}\times(\nabla\times\bar{b}) = a\times\sum\left(\bar{i}\times\frac{\partial\bar{b}}{\partial x}\right)$$

$$=\sum\bar{a}\times\left(\bar{i}\times\frac{\partial\bar{b}}{\partial x}\right)$$

$$=\sum\left\{\left(\bar{a}.\frac{\partial\bar{b}}{\partial x}\right)\bar{i}-(\bar{a}.\bar{i})\frac{\partial\bar{b}}{\partial x}\right\}=\sum\bar{i}\left\{\bar{a}.\frac{\partial\bar{b}}{\partial x}\right\}-\left\{\bar{a}.\sum i\frac{\partial}{\partial x}\right\}\bar{b}$$

$$\therefore\bar{a}\times curl\ \bar{b}=\sum\bar{i}\left(\bar{a}.\frac{\partial\bar{b}}{\partial x}\right)-(\bar{a}.\nabla)\bar{b}\quad....(1)$$

Similarly, $\bar{b}\times curl\ \bar{b}=\sum\bar{i}\left(\bar{b}.\frac{\partial\bar{a}}{\partial x}\right)-(\bar{b}.\nabla)\bar{a}$ ....(2)

(1)+(2) gives

$$\bar{a}\times curl\ \bar{b}+\bar{b}\times curl\ \bar{a}=\sum\bar{i}\left(\bar{a}.\frac{\partial\bar{b}}{\partial x}\right)-(\bar{a}.\nabla)\bar{b}+\sum\bar{i}\left(\bar{b}.\frac{\partial\bar{a}}{\partial x}\right)-(\bar{b}.\nabla)\bar{a}$$

$$\Rightarrow\bar{a}\times curl\ \bar{b}+\bar{b}\times curl\ \bar{a}+(\bar{a}.\nabla)\bar{b}+(\bar{b}.\nabla)\bar{a}=\sum\bar{i}\left(\bar{a}.\frac{\partial\bar{b}}{\partial x}+\bar{b}.\frac{\partial\bar{a}}{\partial x}\right)$$

$$=\sum\bar{i}\frac{\partial}{\partial x}(\bar{a}.\bar{b})$$

$$=\nabla(\bar{a}.\bar{b})=\text{grad}\,(\bar{a}.\bar{b})$$

Theorem 4: Prove that div $(\bar{a}\times\bar{b})=\bar{b}.curl\ \bar{a}-\bar{a}.curl\ \bar{b}$

Proof: div $(\bar{a}\times\bar{b})=\sum\bar{i}.\frac{\partial}{\partial x}(\bar{a}\times\bar{b})=\sum\bar{i}.\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b}+\bar{a}\times\frac{\partial\bar{b}}{\partial x}\right)$

$$=\sum\bar{i}.\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b}\right)+\sum\bar{i}.\left(\bar{a}\times\frac{\partial\bar{b}}{\partial x}\right)=\sum\left(\bar{i}\times\frac{\partial\bar{a}}{\partial x}\right).\bar{b}-\sum\left(\bar{i}\times\frac{\partial\bar{b}}{\partial x}\right).\bar{a}$$

$$=(\nabla\times\bar{a}).\bar{b}-(\nabla\times\bar{b}).\bar{a}=\bar{b}.curl\ \bar{a}-\bar{a}.curl\ \bar{b}$$

Theorem 5 : Prove that $curl(\bar{a}\times\bar{b})=\bar{a}\,div\,\bar{b}-\bar{b}\,div\,\bar{a}+(\bar{b}.\nabla)\bar{a}-(\bar{a}.\nabla)\bar{b}$

Proof : $curl(\bar{a}\times\bar{b})=\sum\bar{i}\times\frac{\partial}{\partial x}(\bar{a}\times\bar{b})=\sum\bar{i}\times\left[\frac{\partial\bar{a}}{\partial x}\times\bar{b}+\bar{a}\times\frac{\partial\bar{b}}{\partial x}\right]$

$$\sum\bar{i}\times\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b}\right)+\sum\bar{i}\times\left(\bar{a}\times\frac{\partial\bar{b}}{\partial x}\right)$$

$$=\sum\left\{(\bar{i}.\bar{b})\frac{\partial\bar{a}}{\partial x}-\left(\bar{i}.\frac{\partial\bar{a}}{\partial x}\right)\bar{b}\right\}+\sum\left\{\left(\bar{i}.\frac{\partial\bar{b}}{\partial x}\right)\bar{a}-(\bar{i}.\bar{a})\frac{\partial\bar{b}}{\partial x}\right\}$$

$$= \sum (\bar{b}.\bar{i})\frac{\partial \bar{a}}{\partial x} - \sum \left(\bar{i}.\frac{\partial \bar{a}}{\partial x}\right)\bar{b} + \sum \left(\bar{i}.\frac{\partial \bar{b}}{\partial x}\right)\bar{a} - \left(\bar{a}.\sum \bar{i}\frac{\partial}{\partial x}\right)\bar{b}$$

$$= (\bar{b}.\nabla)\bar{a} - (\nabla.\bar{a})\bar{b} + (\nabla.\bar{b})\bar{a} - (\bar{a}.\nabla)\bar{b}$$

$$= (\nabla.\bar{b})\bar{a} - (\nabla.\bar{a})\bar{b} + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$$

$$= \bar{a}\ div\ \bar{b} - \bar{b}\ div\ \bar{a} + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$$

Theorem 6: Prove that curl grad $\phi = 0$.

Proof: Let ϕ be any scalar point function. Then

$$.grad\ \phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$$

$$curl(grad\,\phi) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right) - \bar{j}\left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x}\right) - \bar{k}\left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right) = \bar{0}$$

Note : Since $Curl(grad\,\phi) = \bar{0}$, we have $grad\ \phi$ is always irrotational.

7. Prove that div $curl\ \bar{f} = 0$

$Proof\ :\ Let\ \bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\therefore\ curl\ \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\bar{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\bar{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\bar{k}$$

$$\therefore\ div\ curl\ \bar{f} = \nabla.(\nabla \times \bar{f}) = \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial x \partial z} - \frac{\partial^2 f_3}{\partial y \partial x} + \frac{\partial^2 f_1}{\partial y \partial z} + \frac{\partial^2 f_2}{\partial z \partial x} - \frac{\partial^2 f_1}{\partial z \partial y} = 0$$

Note : Since $div(curl\,\bar{f}) = 0$, we have $curl\,\bar{f}$ is always solenoidal.

Theorem 8: If *f* and g are two scalar point functions, prove that $div(f\nabla g) = f\nabla^2 g + \nabla f.\nabla g$

Sol: Let *f* and g be two scalar point functions. Then

$$\nabla g = \bar{i}\frac{\partial g}{\partial x} + \bar{j}\frac{\partial g}{\partial y} + \bar{k}\frac{\partial g}{\partial z}$$

Now $f\nabla g = \bar{i} f \frac{\partial g}{\partial x} + \bar{j} f \frac{\partial g}{\partial y} + \bar{k} f \frac{\partial g}{\partial z}$

$$\therefore \nabla.(f\nabla g) = \frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(f\frac{\partial g}{\partial z}\right)$$

$$= f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial x^2}\right) + \left(\frac{\partial f}{\partial x}.\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}.\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}.\frac{\partial g}{\partial z}\right)$$

$$= f\nabla^2 g + \left(\bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}\right).\left(\bar{i}\frac{\partial g}{\partial x} + \bar{j}\frac{\partial g}{\partial y} + \bar{k}\frac{\partial g}{\partial z}\right).$$

$$= f\nabla^2 g + \nabla f.\ \nabla g$$

Theorem 9: Prove that $\nabla \times (\nabla \times \bar{a}) = \nabla(\nabla.\bar{a}) - \nabla^2 \bar{a}$.

Proof: $\nabla \times (\nabla \times \bar{a}) = \sum \bar{i} \times \frac{\partial}{\partial x}(\nabla \times \bar{a})$

Now $\bar{i} \times \frac{\partial}{\partial x}(\nabla \times \bar{a}) = i \times \frac{\partial}{\partial x}\left(\bar{i} \times \frac{\partial \bar{a}}{\partial x} + \bar{j} \times \frac{\partial \bar{a}}{\partial y} + \bar{k} \times \frac{\partial \bar{a}}{\partial z}\right)$

$$= \bar{i} \times \left(\bar{i} \times \frac{\partial^2 \bar{a}}{\partial x^2} + \bar{j} \times \frac{\partial^2 \bar{a}}{\partial x \partial y} + \bar{k} \times \frac{\partial^2 \bar{a}}{\partial x \partial z}\right)$$

$$= \bar{i} \times \left(\bar{i} \times \frac{\partial^2 \bar{a}}{\partial x^2}\right) + \bar{i} \times \left(\bar{j} \times \frac{\partial^2 \bar{a}}{\partial x \partial y}\right) + \bar{i} \times \left(\bar{k} \times \frac{\partial^2 \bar{a}}{\partial x \partial z}\right)$$

$$= \left(\bar{i}.\frac{\partial^2 \bar{a}}{\partial x^2}\right)\bar{i} - \frac{\partial^2 \bar{a}}{\partial x^2} + \left(\bar{i}.\frac{\partial^2 \bar{a}}{\partial x \partial y}\right)\bar{j} + \left(\bar{i}.\frac{\partial^2 \bar{a}}{\partial x \partial z}\right)\bar{k} \qquad [\because i.i = 1, i.j = i.k = 0]$$

$$= \bar{i}\frac{\partial}{\partial x}\left(\bar{i}.\frac{\partial \bar{a}}{\partial x}\right) + j\frac{\partial}{\partial y}\left(\bar{i}.\frac{\partial \bar{a}}{\partial y}\right) + k\frac{\partial}{\partial z}\left(\bar{i}.\frac{\partial \bar{a}}{\partial x}\right) - \frac{\partial^2 \bar{a}}{\partial x^2} = \nabla\left(\bar{i}.\frac{\partial \bar{a}}{\partial x}\right) - \frac{\partial^2 \bar{a}}{\partial x^2}$$

$$\therefore \sum \bar{i} \times \frac{\partial}{\partial x}(\nabla \times \bar{a}) = \nabla \sum \bar{i}.\frac{\partial \bar{a}}{\partial x} - \sum \frac{\partial^2 \bar{a}}{\partial x^2} = \nabla(\nabla.\bar{a}) - \left(\frac{\partial^2 \bar{a}}{\partial x^2} + \frac{\partial^2 \bar{a}}{\partial y^2} + \frac{\partial^2 \bar{a}}{\partial z^2}\right)$$

$$\therefore \nabla \times (\nabla \times \bar{a}) = \nabla(\nabla.\bar{a}) - \nabla^2 \bar{a}$$

i.e., $curl\,curl\,\bar{a} = grad\ div\,\bar{a} - \nabla^2 \bar{a}$

SOLVED PROBLEMS

1: Prove that $(\nabla f \times \nabla g)$ is solenoidal.

Sol: We know that div $(\bar{a} \times \bar{b}) = \bar{b}.curl\ \bar{a} - \bar{a}.curl\ \bar{b}$

Take $a=\nabla f$ and $b=\nabla g$

Then div $(\nabla f \times \nabla g) = \nabla g.\ curl(\nabla f) - \nabla f.\ curl(\nabla g) = 0 \left[\because curl(\nabla f) = \bar{0} = curl(\nabla g)\right]$

$\therefore \nabla f \times \nabla g$ is solenoidal.

3. Prove that (i) $div\left\{(\bar{r} \times \bar{a})\bar{b}\right\} = -2(\bar{b}.\bar{a})$ (ii) $curl\left\{(\bar{r}.\bar{a}) \times \bar{b}\right\} = \bar{b} \times \bar{a}$ where $\bar{a}$ and $\bar{b}$ are constant vectors.

Sol: (i)

$$div\left\{(\bar{r} \times \bar{a}) \times \bar{b}\right\} = div[(\bar{r}.\bar{b})\bar{a} - (\bar{a}..\bar{b})\bar{r}]$$

$$= div\ (\bar{r}.\bar{b})\bar{a} - (\bar{a}..\bar{b})\bar{r}$$

$$= \left[\left(\bar{r}.\bar{b}\right) div\,\bar{a} + \bar{a}.grad\left(\bar{r}.\bar{b}\right)\right] - \left[\left(\bar{a}.\bar{b}\right) div\,\bar{r} + \bar{r}.grad\left(\bar{a}.\bar{b}\right)\right]$$

We have $div\ \bar{a} = 0, div\ \bar{r} = 3, grad\ (\bar{a}.\bar{b}) = 0$

$$div\left\{\left(\bar{r} \times \bar{a}\right) \times \bar{b}\right\} = 0 + \bar{a}.grad\left(\bar{r}.\bar{a}\right) - 3\left(\bar{a}.\bar{a}\right)$$

$$= \bar{a}.\sum \frac{i\partial}{\partial x}\left(\bar{r}.\bar{b}\right) - 3\left(\bar{a}.\bar{b}\right)$$

$$= \bar{a}.\sum \bar{i}\frac{\partial \bar{r}}{\partial x}.\bar{b} - 3\left(\bar{a}.\bar{b}\right)$$

$$= \bar{a}.\sum \bar{i}\left(\bar{i}.\bar{b}\right) - 3\left(\bar{a}.\bar{b}\right)$$

$$= \bar{a}.\bar{b} - 3\left(\bar{a}.\bar{b}\right) = -2\left(\bar{a}.\bar{b}\right)$$

$$= -2\left(\bar{b}.\bar{a}\right)$$

(ii) $curl\left\{\left(\bar{r} \times \bar{a}\right) \times \bar{b}\right\} = curl\left[\left(\bar{r}.\bar{b}\right)\bar{a} - \left(\bar{a}.\bar{b}\right)\bar{r}\right]$

$$= curl\left(\bar{r}.\bar{b}\right)\bar{a} - curl\left(\bar{a}.\bar{b}\right)\bar{r}$$

$$= \left(\bar{r}.\bar{b}\right)curl\,\bar{a} + grad\left(\bar{r}.\bar{b}\right) \times \bar{a}$$

$$= \bar{0} + \nabla\left(\bar{r}.\bar{b}\right) \times \bar{a}\left(\because curl\,\bar{a} = \bar{0}\right)$$

$= \bar{b} \times \bar{a}$ Since $grad\left(\bar{r}.\bar{b}\right) = \bar{b}$

3: Prove that $\nabla\left[\nabla.\frac{\bar{r}}{r}\right] = \frac{-2}{r^3}\bar{r}.$

Sol: We have $\nabla.\left(\frac{\bar{r}}{r}\right) = \sum i.\frac{\partial}{\partial x}\left(\frac{\bar{r}}{r}\right)$

$$= \sum i.\left[\frac{1}{r}\frac{\partial \bar{r}}{\partial x} + \bar{r}\left(\frac{-1}{r^2}\right)\left(\frac{x}{r}\right)\right] = \sum i.\left(\frac{1}{r}i - \frac{\bar{r}}{r^3}x\right)$$

$$= \frac{1}{r}\sum i.i - \frac{1}{r^3}r^2 = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

$$\therefore \nabla\left[\nabla.\left(\frac{\bar{r}}{r}\right)\right] = \sum i\left(\frac{\partial}{\partial x}\left(\frac{2}{r}\right)\right) = \sum i\left(\frac{-2}{r^2}\right)\left(\frac{x}{r}\right) = \frac{-2}{r^3}\sum xi = \frac{-2\bar{r}}{r^3}.$$

4: Find $(A\times\nabla)\phi$, if $A = yz^2\bar{i} - 3xz^2\bar{j} + 2xyz\bar{k}$ and $\phi = xyz$.

Sol : We have

$$A\times\nabla = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ yz^2 & -3xz^2 & 2xyz \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial x}(-3xz^2) - \frac{\partial}{\partial y}(2xyz)\right] - \bar{j}\left[\frac{\partial}{\partial z}(yz^2) - \frac{\partial}{\partial x}(2xyz)\right] + \bar{k}\left[\frac{\partial}{\partial y}(yz^2) - \frac{\partial}{\partial x}(-3xz^2)\right]$$

$= \bar{i}(-6xz-2xz) - \bar{j}(2yz-2yz) + \bar{k}(z^2+3z^2) = -8xz\,\bar{i} - 0\,\bar{j} + 4z^2\bar{k}$

$\therefore (A\times\nabla)\phi, = (-8xz\,\bar{i} + 4z^2\bar{k})xyz = -8x^2yz^2\bar{i} + 4xyz^3\bar{k}$

Vector Integration

Line integral:- (i) $\int_c \bar{F}.d\bar{r}$ is called Line integral of $\bar{F}$ along c

Note : Work done by $\bar{F}$ along a curve c is $\int_c \bar{F}.d\bar{r}$

PROBLEMS

1. If $\bar{F}$ $(x^2-27)\bar{i} - 6yz\bar{j} + 8xz^2\bar{k}$, evaluate $\int \bar{F}.d\bar{r}$ from the point (0,0,0) to the point (1,1,1) along the Straight line from (0,0,0) to (1,0,0), (1,0,0) to (1,1,0) and (1,1,0) to (1,1,1).

Solution: Given $\bar{F} = (x^2-27)\bar{i} - 6yz\bar{j} + 8xz^2\bar{k}$

Now $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\therefore \quad \bar{F}.d\bar{r} = (x^2-27)dx - (6yz)dy + 8xz^2dz$

(i) Along the straight line from O = (0,0,0) to A = (1,0,0)

Here y =0 =z and dy=dz=0. Also x changes from 0 to 1.

$$\therefore \int_{OA} \bar{F}.d\bar{r} = \int_0^1 (x^2-27)dx = \left[\frac{x^3}{3} - 27x\right]_0^1 = \frac{1}{3} - 27 = \frac{-80}{3}$$

(ii) Along the straight line from A = (1,0,0) to B = (1,1,0)

Here x =1, z=0 $\Rightarrow$ dx=0, dz=0. y changes from 0 to 1.

$$\therefore \int_{AB} \bar{F} \cdot d\bar{r} = \int_{y=0}^{1} (-6yz)\, dy = 0$$

(iii) Along the straight line from B = (1,1,0) to C = (1,1,1)

x =1 =y $\Rightarrow$ dx=dy=0 and z changes from 0 to 1.

$$\therefore \int_{BC} \bar{F} \cdot d\bar{r} = \int_{z=0}^{1} 8xz^2\, dz = \int_{z=0}^{1} 8xz^2\, dz = \left[\frac{8z^3}{3}\right]_0^1 = \frac{8}{3}$$

$$(i) + (ii) + (iii) \Rightarrow \int_C \bar{F} \cdot d\bar{r} = \frac{88}{3}$$

2. If $\bar{F}$ =(5xy-6x^2)$\bar{i}$ +(2y-4x)$\bar{j}$, evaluate $\int_C \bar{F} \cdot d\bar{r}$ Along the curve C in xy-plane y=x^3 from (1,1) to (2,8).

Solution : Given $\bar{F}$ =(5xy-6x^2)$\bar{i}$ +(2y-4x)$\bar{j}$,-------(1)

Along the curve y=x^3, dy =3x^2 dx

$\therefore$ $\bar{F}$ =(5x^4-6x^2)$\bar{i}$ +(2x^3-4x)$\bar{j}$, [Putting y=x^3 in (1)]

$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} = dx\,\bar{i}$ +3x^{2}dx $\bar{j}$

$\therefore$ $\bar{F} \cdot d\bar{r}$ = [(5x^4-6x^2)$\bar{i}$ +(2x^3-4x)$\bar{j}$].$\left[dx\,\bar{i} + 3x^2 dx\,\bar{j}\right]$

= (5x^4 – 6x^2) dx+(2x^3 – 4x)3x^{3}dx

= (6x^5+5x^4-12x^3 -6x^2)dx

Hence $\int_{y=x^3} \bar{F} \cdot d\bar{r} = \int_1^2 (6x^5 + 5x^4 - 12x^3 - 6x^2)dx$

$$= \left(6.\frac{x^6}{6} + 5.\frac{x^5}{5} - 12.\frac{x^4}{4} - 6.\frac{x^3}{4}\right) = \left(x^6 + x^5 - 3x^4 - 2x^3\right)_1^2$$

= 16(4+2-3-1) – (1+1-3-2) = 32+3 = 35

3. Find the work done by the force $\bar{F} = z\bar{i} + x\bar{j} + y\bar{k}$, when it moves a particle along the arc of the curve $\bar{r}$ = cost $\bar{i}$ + sint $\bar{j}$ -t $\bar{k}$ from t = 0 to t = 2π

Solution : Given force $\bar{F}$ = z$\bar{i}$ + x $\bar{j}$ +y $\bar{k}$ and the arc is $\bar{r}$ = cost $\bar{i}$ + sin t $\bar{j}$ -t$\bar{k}$

i.e., x = cost, y= sin t, z = -t

$\therefore$ d$\bar{r}$ = (-sin t $\bar{i}$ +cost $\bar{j}$ - $\bar{k}$)dt

$\therefore$ $\bar{F} \cdot d\bar{r}$ = (-t $\bar{i}$ +cost $\bar{j}$ +sin t $\bar{k}$). (-sin t $\bar{i}$ + cost $\bar{j}$ - $\bar{k}$)dt = (t sin t + cos^2 t – sin t)dt

Hence work done = $\int_0^{2\pi} \bar{F} \cdot d\bar{r} = \int_0^{2\pi} (t \sin t + \cos^2 t - \sin t)\, dt$

$$= \left[t(-\cos t)\right]_0^{2\pi} - \int_0^{2\pi} (-\sin t)dt + \int_0^{2\pi} \frac{1+\cos 2t}{2} dt - \int_0^{2\pi} \sin t \, dt$$

$$= -2\pi - (\cos t)_0^{2\pi} + \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right)_0^{2\pi} + (\cos t)_0^{2\pi}$$

$$= -2\pi - (1-1) + \frac{1}{2}(2\pi) + (1-1) = -2\pi + \pi = -\pi$$

PROBLEMS

1 : Evaluate $\int \bar{F}.n\,dS$ where $\bar{F}$ = zi + xj – $3y^2zk$ and S is the surface $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5.

Sol. The surface S is $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5.

Let $\phi = x^2 + y^2 = 16$

Then $\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = 2x\bar{i} + 2y\bar{j}$

$\therefore$ unit normal $\bar{n} = \frac{\nabla\phi}{|\nabla\varphi|} = \frac{x\bar{i} + y\bar{j}}{4}$ $(\because x^2 + y^2 = 16)$

Let R be the projection of S on yz-plane

Then $\int_S \bar{F}.n\,dS = \iint_R \bar{F}.\bar{n}\frac{dydz}{|\bar{n}\,.\,\bar{i}|}$ .... *

Given $\bar{F}$ = zi + xj – $3y^2zk$

$\therefore$ $\bar{F}\,.\,\bar{n} = \frac{1}{4}(xz + xy)$

and $\bar{n}\,.\,\bar{i} = \frac{x}{4}$

In yz-plane, x = 0, y = 4

In first octant, y varies from 0 to 4 and z varies from 0 to 5.

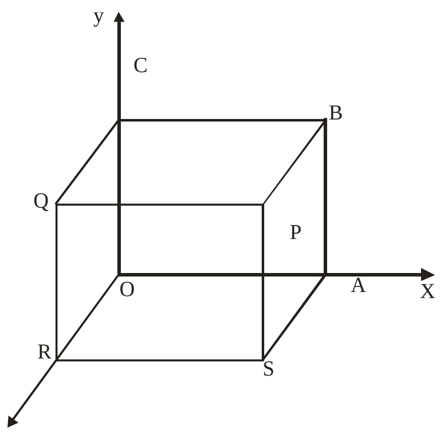
$$\int_S \bar{F}.n\,dS = \int_{y=0}^{4}\int_{z=0}^{5}\left(\frac{xz + xy}{4}\right)\frac{dydz}{\left|\frac{x}{4}\right|}$$

$$= \int_{y=0}^{4} \int_{z=0}^{5} (y + z)\,dz\,dy$$

$$= 90.$$

2 : If $\overline{F} = zi + xj - 3y^2zk$, evaluate $\int_S \overline{F}.\overline{n}dS$ where S is the surface of the cube bounded by x = 0, x = a, y = 0, y= a, z = 0, z = a.

Sol. Given that S is the surface of the x = 0, x = a, y = 0, y = a, z = 0, z = a, and $\overline{F} = zi + xj - 3y^2zk$ we need to evaluate $\int_S \overline{F}.\overline{n}dS$.



(i) For OABC

Eqn is z = 0 and dS = dxdy

$$\overline{n} = -\overline{k}$$

$$\int_{S_1} \overline{F}.\overline{n}dS = -\int_{x=0}^{a} -\int_{y=0}^{a} (yz)\,dxdy = 0$$

(ii) For PQRS

Eqn is z = a and dS = dxdy

$$\overline{n} = \overline{k}$$

$$\int_{S_2} \overline{F}.ndS = \int_{x=0}^{a} \left(\int_{y=0}^{a} y(a)dy \right) dx = \frac{a^4}{2}$$

(iii) For OCQR

Eqn is x = 0, and $\overline{n} = -\overline{i}$, dS = dydz

$$\int_{S_3} \overline{F}.ndS = \int_{y=0}^{a} \int_{z=0}^{a} 4xzdydz = 0$$

(iv) For ABPS

Eqn is x = a, and $\overline{n} = -\overline{i}$, dS = dydz

$$\int_{S_3} \bar{F}.\bar{n}dS = \int_{y=0}^{a} \left(\int_{z=0}^{a} 4azdz \right) dy = 2a^4$$

(v) For OASR

Eqn is y = 0, and $\bar{n} = -\bar{j}$, dS = dxdz

$$\int_{S_5} \bar{F}.ndS = \int_{y=0}^{a} \int_{z=0}^{a} y^2 dzdx = 0$$

(vi) For PBCQ

Eqn is y = a, and $\bar{n} = -\bar{j}$, dS = dxdz

$$\int_{S_6} \bar{F}.\bar{n}dS = -\int_{y=0}^{a} \int_{z=0}^{a} y^2 dzdx = 0$$

From (i) – (vi) we get

$$\int_{S_6} \bar{F}.\bar{n}dS = 0 + \frac{a^4}{2} + 0 + 2a^4 + 0 - a4 = \frac{3a^4}{2}$$

VOLUME INTEGRALS

Let V be the volume bounded by a surface $\bar{r} = \bar{f}$ (u,v). Let $\bar{F}(\bar{r})$ be a vector point function define over V. Divide V into m sub-regions of volumes $\delta V_1, \delta V_2, ... \delta V_p .... \delta V_m$

Let P_i ($\bar{r}_i$) be a point in δV_i .Then form the sum I_m = $\sum_{i=1}^{m} \bar{F}(r_i)\delta V_i$. Let m $\rightarrow \infty$ in such a way that δV_i

shrinks to a point,. The limit of I_m if it exists, is called the volume integral of $\bar{F}(\bar{r})$ in the region V is

denoted by $\int_V \bar{F}(\bar{r})\, dv$ or $\int_V \bar{F}\, dv$.

Cartesian form : Let $\bar{F}(r) = F_1 \bar{i} + F_2 \bar{i} + F_3 \bar{k}$ where F_1, F_2, F_3 are functions of x,y,z. We know that

dv = dx dy dz. The volume integral given by

$$\int_v \bar{F}\, dv = \iiint (F_1 \bar{i} + F_2 \bar{i} + F_3 \bar{k})\, dx\, dy\, dz = \bar{i} \iiint F_1\, dxdydz + \bar{j} \iiint F_2\, dxdydz + \bar{k} \iiint F_3\, dxdydz$$

SOLVED EXAMPLES

Example 1 : If $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$ evaluate $\int_V \bar{F} dv$ where V is the region bounded by the surfaces $x = 0, x = 2, y = 0, y = 6, z = x^2, z = 4$.

Solution : Given $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$. $\therefore$ The volume integral is

$$\int_V \bar{F} dv = \iiint (2xz\bar{i} - x\bar{j} + y^2\bar{k}) dx\, dy\, dz$$

$$= \bar{i} \int_{x=0}^{2} \int_{y=0}^{6} \int_{z=x^2}^{4} 2xz\, dx\, dy\, dz - \bar{j} \int_{x=0}^{2} \int_{y=0}^{6} \int_{z=x^2}^{4} x\, dx\, dy\, dz + \bar{k} \int_{x=0}^{2} \int_{y=0}^{6} \int_{z=x^2}^{4} y^2\, dx\, dy\, dz$$

$$= \bar{i} \int_{x=0}^{2} \int_{y=0}^{6} [xz^2]_{x^2}^{4}\, dx\, dy - \bar{j} \int_{x=0}^{2} \int_{y=0}^{6} (xz)_{x^2}^{4}\, dx\, dy + \bar{k} \int_{x=0}^{2} \int_{y=0}^{6} y^2 (z)_{x^2}^{4}\, dx\, dy$$

$$= \bar{i} \int_{x=0}^{2} \int_{y=0}^{6} x(16 - x^4) dx\, dy - \bar{j} \int_{x=0}^{2} \int_{y=0}^{6} x(4 - x^2) dx\, dy - \bar{k} \int_{x=0}^{2} \int_{y=0}^{2} y^2 (x^2 - 4) dx\, dy$$

$$= \bar{i} \int_{x=0}^{2} (16x - x^5)(y)_0^6 dx - \bar{j} \int_{x=0}^{2} (4x - x^3)(y)_0^6 dx - \bar{k} \int_{x=0}^{2} (x^2 - 4)\left(\frac{y^3}{3}\right)_0^6 dx$$

$$= \bar{i}\left(8x^2 - \frac{x^6}{6}\right)_0^2 (6) - \bar{j}\left(2x^2 - \frac{x^4}{4}\right)_0^2 (6) - \bar{k}\left(4x - \frac{x^3}{3}\right)_0^2 \left(\frac{211}{3}\right)$$

$$= 128\bar{i} - 24\bar{j} - 384\bar{k}$$

Vector Integral Theorems

Introduction

In this chapter we discuss three important vector integral theorems: (i) Gauss divergence theorem, (ii) Green's theorem in plane and (iii) Stokes theorem. These theorems deal with conversion of

(i) $\int_S \bar{F}.\bar{n}$ ds into a volume integral where S is a closed surface.

(ii) $\int_C \bar{F}.d\bar{r}$ into a double integral over a region in a plane when C is a closed curve in the plane and.

(iii) $\int_S (\nabla \times \bar{A}).\bar{n}$ ds into a line integral around the boundary of an open two sided surface.

I. GAUSS'S DIVERGENCE THEOREM

(Transformation between surface integral and volume integral)

Let S be a closed surface enclosing a volume V. If $\bar{F}$ is a continuously differentiable vector point function, then

$$\int_V div\, F\, dv = \int_S \bar{F}.\bar{n}\, dS$$

When $\bar{n}$ is the outward drawn normal vector at any point of S.

SOLVED PROBLEMS

1) Verify Gauss Divergence theorem for $\bar{F} = (x^3 - yz)\bar{\imath} - 2x^2 y\bar{\jmath} + z\bar{k}$ taken over the surface of the cube bounded by the planes x = y = z = a and coordinate planes.

Sol: By Gauss Divergence theorem we have

$$\int_S \overline{F}.\overline{n}dS = \int_V div\,\overline{F}\,dv$$

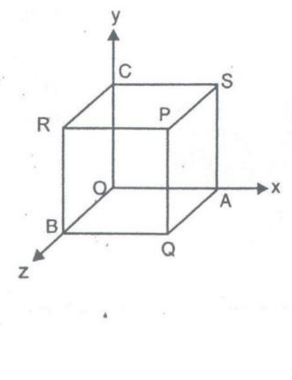
$$RHS = \int_0^a\int_0^a\int_0^a (3x^2 - 2x^2 + 1)dx\,dy\,dz = \int_0^a\int_0^a\int_0^a (x^2+1)\,dx\,dy\,dz = \int_0^a\int_0^a \left(\frac{x^3}{3}+x\right)_0^a dy\,dz$$

$$\int_0^a\int_0^a\left[\frac{a^3}{3}+a\right]dydz = \int_0^a\left[\frac{a^3}{3}+a\right](y)_0^a dz = \left(\frac{a^3}{3}+a\right)a\int_0^a dz = \left(\frac{a^3}{3}+a\right)(a^2) = \frac{a^5}{3}+a^3 \text{ ......(1)}$$

Verification: We will calculate the value of $\int_S \overline{F}.\overline{n}dS$ over the six faces of the cube.

(i) For S_1 = PQAS; unit outward drawn normal $\bar{n} = \bar{\imath}$

x=a; ds=dy dz; 0≤y≤a, 0≤z≤a



$$\therefore\ \overline{F}.\overline{n} = x^3 - yz = a^3 - yz \sin ce\ x = a$$

$$\therefore \iint_{S_1} \overline{F}.\overline{n}dS = \int_{z=0}^{a}\int_{y=0}^{a} (a^3 - yz)dydz$$

$$= \int_{z=0}^{a}\left[a^3 y - \frac{y^2}{2}z\right]_{y=0}^{a} dz$$

$$= \int_{z=0}^{a} \left(a^4 - \frac{a^2}{2} z \right) dz$$

$$= a^5 - \frac{a^4}{4} \ldots (2)$$

(ii) For S_2 = OCRB; unit outward drawn normal $\bar{n} = -\bar{\imath}$

x=0; ds=dy dz; 0≤y≤a, y≤z≤a

$$\bar{F}.\bar{n} = -(x^3 - yz) = yz \ since\ x = 0$$

$$\int_{S_2}\int \bar{F}.\bar{n} dS = \int_{z=0}^{a} \int_{y=0}^{a} yz\, dy\, dz = \int_{z=0}^{a} \left[\frac{y^2}{2}\right]_{y=0}^{a} zdz$$

$$= \frac{a^2}{2} \int_{z=0}^{a} zdz = \frac{a^4}{4} \ldots (3)$$

(iii) For S_3 = RBQP; Z = a; ds = dxdy; $\bar{n} = \bar{k}$

0≤x≤a, 0≤y≤a

$$\bar{F}.\bar{n} = z = a \ \ since\ z = a$$

$$\therefore \int_{S_3}\int \bar{F}\, \bar{n} dS = \int_{y=0}^{a} \int_{x=0}^{a} a\, dx\, dy = a^3 \ldots\ldots (4)$$

(iv) For S_4 = OASC; z = 0; $\bar{n} = -\bar{k}$, ds = dxdy;

0≤x≤a, 0≤y≤a

$$\bar{F}.\bar{n} = -z = 0 \ \ since\ z = 0$$

$$\int_{S_4}\int \bar{F}.\bar{n} dS = 0 \ldots (5)$$

(v) For S_5 = PSCR; y = a; $\bar{n} = \bar{\jmath}$, ds = dzdx;

0≤x≤a, 0≤z≤a

$$\bar{F}.\bar{n} = -2x^2 y = -2ax^2 \ \ since\ y = a$$

$$\int_{S_5}\int \bar{F}.\bar{n} dS = \int_{x=0}^{a} \int_{z=0}^{a} (-2ax^2) dzdx$$

$$\int_{x=0}^{a} (-2ax^2 z)_{z=0}^{a} dx$$

$$= -2a^2 \left(\frac{x^3}{3}\right)_0^a = \frac{-2a^5}{3} \ldots (6)$$

(vi) For S_6 = OBQA; $y = 0$; $\bar{n} = -\bar{j}$, $ds = dzdx$;

$0 \le x \le a$, $0 \le y \le a$

$\bar{F}.\bar{n} = 2x^2y = 0 \; since \; y = 0$

$$\int_{S_6}\int \bar{F}.\bar{n}dS = 0 \int_S \int \bar{F}.\bar{n}dS = \int_{S_1}\int + \int_{S_2}\int + \int_{S_3}\int + \int_{S_4}\int + \int_{S_5}\int + \int_{S_6}\int$$

$$= a^5 - \frac{a^4}{4} - \frac{a^4}{4} + a^3 + 0 - \frac{2a^5}{3} + 0$$

$$= \frac{a^5}{3} + a^3 = \int\int_V\int \bar{\nabla}.\bar{F}\, dv \; using \; (1)$$

Hence Gauss Divergence theorem is verified

2. Compute $\int(\boldsymbol{ax^2 + by^2 + cz^2})\boldsymbol{dS}$ over the surface of the sphere $x^2+y^2+z^2 = 1$

Sol: By divergence theorem $\int_S \bar{F}.\bar{n}dS = \int_V \bar{\nabla}.\bar{F}\; dv$

Given $\bar{F}.\bar{n} = ax^2 + by^2 + cz^2$. *Let* $\phi = x^2 + y^2 + z^2 - 1$

$\therefore$ *Normal vector* $\bar{n}$ *to the surface* ϕ *is*

$$\bar{\nabla}\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial y}\right)\left(x^2 + y^2 + z^2 - 1\right) = 2(x\bar{i} + y\bar{j} + z\bar{k})$$

$\therefore$ Unit normal vector = $\bar{n} = \dfrac{2(x\bar{i} + y\bar{j} + z\bar{k})}{2\sqrt{x^2 + y^2 + z^2}} = x\bar{i} + y\bar{j} + z\bar{k}$ Since $x^2 + y^2 + z^2 = 1$

$\therefore \bar{F}.\bar{n} = \bar{F}.(x\bar{i} + y\bar{j} + z\bar{k}) = (ax^2 + by^2 + cz^2) = (a\,x\bar{i} + by\bar{j} + cz\bar{k}).(x\bar{i} + y\bar{j} + z\bar{k})$

i.e., $\bar{F} = ax\bar{i} + by\bar{j} + cz\bar{k}$ $\nabla.\bar{F} = a + b + c$

Hence by Gauss Divergence theorem,

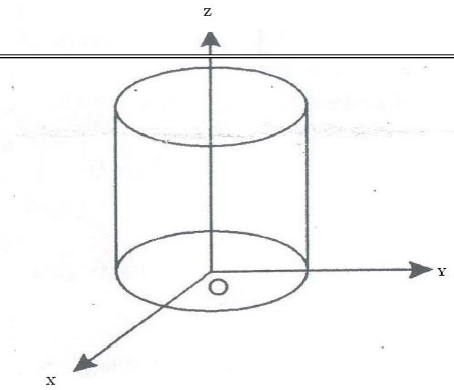
$$\int_S (ax^2 + by^2 + cz^2)dS = \int_V (a + b + c)dv = (a + b + c)V = \frac{4\pi}{3}(a + b + c)$$

$$\left[SInce\; V = \frac{4\pi}{3} \; is\; the\; volume\; of\; the\; sphere\; of\; unit\; radius\right]$$

3) By transforming into triple integral, evaluate $\int\int \boldsymbol{x^3\; dy\; dz + x^2y\; dz\; dx + x^2dx\; dy}$ where S is the closed surface consisting of the cylinder $x^2+y^2 = a^2$ and the circular discs $z= 0$, $z= b$.

Sol: Here $F_1 = x^3, F_2 = x^2y, F_3 = x^2z \; and \; \bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

$$\frac{\partial F_1}{\partial x} = 3x^2, \frac{\partial F_2}{\partial y} = x^2, \frac{\partial F_3}{\partial z} = x^2$$



$$\nabla.\overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3x^2 + x^2 + x^2 = 5x^2$$

By Gauss Divergence theorem,

$$\int\int F_1 dydz + F_2 dzdx + F_3 dxdy = \int\int\int\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz$$

$$\therefore \int\int_s (x^3 dydz + x^2 ydzdx + x^2 zdxdy = \int\int\int 5x^2 dxdydz$$

$$= 5\int_{-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{z=0}^{b} x^2 dxdydz$$

$$= 20\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_{z=0}^{b} x^2 dxdydz$$ [Integrand is even function]

$$= 20\int_0^a\int_0^{\sqrt{a^2-x^2}} x^2 (z)_0^b dxdy = 20b\int_{x=0}^{a}\int_0^{\sqrt{a^2-x^2}} x^2 dxdy$$

$$= 20b\int_{x=0}^{a} x^2 (y)_0^{\sqrt{a^2-x^2}} dx = 20b\int_0^a x^2\sqrt{a^2-x^2}\, dx$$

$$= 20b\int_0^{\frac{\pi}{2}} a^2 \sin^2\theta\sqrt{a^2 - a^2\sin^2\theta}\,(a\cos\theta d\theta)$$

[Put $x = a\sin\theta \Longrightarrow dx = a\cos\theta d\theta$ when $x = a \Rightarrow \theta = \frac{\pi}{2}$ and $x = 0 \Rightarrow \theta = 0$]

$$= 20a^4 b\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta d\theta = 5a^4 b\int_0^{\frac{\pi}{2}}(2\sin\theta\cos\theta)^2\, d\theta = 5a^4 b\int_0^{\frac{\pi}{2}}\frac{1-\cos 4\theta}{2} d\theta$$

$$= \frac{5a^4 b}{2}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2} = \frac{5a^4 b}{2}\left[\frac{\pi}{2}\right] = \frac{5}{4}\pi a^4 b$$

4: Applying Gauss divergence theorem, Prove that $\int \overline{r}\,.\overline{n}dS = 3V$ or $\int \overline{r}\,.d\overline{s} = 3V$

Sol: Let $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$ we know that div $\overline{r} = 3$

By Gauss divergence theorem, $\int \bar{F}.\bar{n}dS = \int_v div\bar{F}dv$

$Take\ \bar{F} = \bar{r} \Rightarrow \int_S \bar{r}.\bar{n}dS = \int_V 3\ dV = 3V. Hence\ the\ result$

5: Show that $\int_S (ax\bar{i} + by\bar{j} + cz\bar{k}).\bar{n}dS = \frac{4\pi}{3}(a+b+c)$, where S is the surface of the sphere $x^2+y^2+z^2=1$.

Sol: Take $\bar{F} = ax\bar{i} + by\bar{j} + cz\bar{k}$

$$div\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = a + b + c$$

By Gauss divergence theorem, $\int_S \bar{F}.\bar{n}dS = \int_V \bar{\nabla}.\bar{F}dV = (a+b+c)\int_V dV = (a+b+c)V$

$We\ have\ V = \frac{4}{3}\pi r^3\ for\ the\ sphere. Here\ r = 1$

$$\therefore \int_s \bar{F}.\bar{n}dS = (a+b+c)\frac{4\pi}{3}$$

6: Using Divergence theorem, evaluate

$\int\int_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy), where\ S$: $x^2+y^2+z^2=a^2$

Sol: We have by Gauss divergence theorem, $\int_s \bar{F}.\bar{n}dS = \int_v div\bar{F}dv$

L.H.S can be written as $\int (F_1 dydz + F_2 dzdx + F_3 dxdy)$ in Cartesian form

Comparing with the given expression, we have $F_1=x$, $F_2=y$, $F_3=z$

Then $div\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3$

$$\therefore \int_v div\bar{F}dv = \int_v 3dv = 3V$$

Here V is the volume of the sphere with radius a.

$$\therefore V = \frac{4}{3}\pi a^3$$

Hence $\int\int(x\,dy\,dz + y\,dz\,dx + z\,dx\,dy) = 4\pi a^3$

7: Apply divergence theorem to evaluate $\int\int_s (x+z)dydz + (y+z)dzdx + (x+y)dxdy$ S is the surface of the sphere $x^2+y^2+z^2=4$

Sol: Given $\int\int_s (x+z)dydz + (y+z)dzdx + (x+y)dxdy$

Here $F_1 = x+z$, $F_2 = y+z$, $F_3 = x+y$

$$\frac{\partial F_1}{\partial x} = 1, \frac{\partial F_2}{\partial y} = 1, \frac{\partial F_3}{\partial z} = 0 \text{ and } \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 1 + 1 + 0 = 2$$

By Gauss Divergence theorem,

$$\int\int_s F_1 dydz + F_2 dzdx + F_3 dxdy = \int\int\int_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz$$

$$= \int\int\int 2dxdydz = 2\int_V dv = 2V$$

$$= 2\left[\frac{4}{3}\pi(2)^3\right] = \frac{64\pi}{3} \; [for\ the\ sphere, radius = 2]$$

8: Evaluate $\int_S \bar{F}.\bar{n}ds, if\ F = xy\bar{\imath} + z^2\bar{\jmath} + 2yz\bar{k}$ over the tetrahedron bounded by x=0, y=0, z=0 and the plane x+y+z=1.

Sol: Given F = $xy\bar{\imath} + z^2\bar{\jmath} + 2yz\bar{k}$, then div. F = y+2y = 3y

$$\therefore \int_s \bar{F}.\bar{n}dS = \int_v div\bar{F}dv = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} 3ydxdydz$$

$$= 3\int_{x=0}^{1}\int_{y=0}^{1-x} y[z]_0^{1-x-y} dxdy = 3\int_{x=0}^{1}\int_{y=0}^{1-x} y(1-x-y)dx\,dy$$

$$= 3\int_{x=0}^{1}\left[\frac{y^2}{2} - \frac{xy^2}{2} - \frac{y^3}{3}\right]_0^{1-x} dx = 3\int_0^1\left[\frac{(1-x)^2}{2} - \frac{x(1-x)^2}{2} - \frac{(1-x)^3}{3}\right]dx$$

$$= 3\int_0^1\left[\frac{(1-x)^3}{2} - \frac{(1-x)^3}{3}\right]dx = 3\int_0^1 \frac{(1-x)^3}{6}dx = \frac{3}{6}\left[\frac{-(1-x)^4}{4}\right]_0^1 = \frac{1}{8}$$

9: Use divergence theorem to evaluate $\int\int_s \bar{F}.d\bar{S}$ where $\bar{F} = x^3i+y^3j+z^3k$ and S is the surface of the sphere $x^2+y^2+z^2 = r^2$

Sol: We have

$$\bar{V}.\bar{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = 3(x^2+y^2+z^2)$$

∴By divergence theorem,

$$\bar{V}.\bar{F}dV = \int\int_V \int \bar{V}.\bar{F}dV = \int\int\int_v 3(x^2+y^2+z^2)dxdydz$$

$$= 3\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2(r^2\sin\theta\,dr\,d\theta\,d\phi)$$

$[Changing\ into\ spherical\ polar\ coordinates\ x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta]$

$$\int_S\int \bar{F}.d\bar{S} = 3\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^4 \sin\theta \left[\int_{\phi=0}^{2\pi} d\phi\right] dr\, d\theta$$

$$= 3\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^4 \sin\theta (2\pi - 0) dr d\theta = 6\pi \int_{r=0}^{a} r^4 \left[\int_0^{\pi} \sin\theta d\theta\right] dr$$

$$= 6\pi \int_{r=0}^{a} r^4 (-\cos\theta)_0^{\pi}\, dr = -6\pi \int_0^{a} r^4 (\cos\pi - \cos 0)\, dr$$

$$= 12\pi \int_0^{a} r^4 dr = 12\pi \left[\frac{r^5}{5}\right]_0^a = \frac{12\pi a^5}{5}$$

10: Use divergence theorem to evaluate $\int\int_S \ \bar{\boldsymbol{F}}.\boldsymbol{dS}$ where $\bar{\boldsymbol{F}} = 4x\boldsymbol{i} - 2y^2\boldsymbol{j} + z^2\boldsymbol{k}$ and S is the surface bounded by the region $x^2+y^2=4$, z=0 and z=3.

Sol: We have

$$div\bar{F} = \nabla.\bar{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2) = 4 - 4y + 2z$$

By divergence theorm,

$$\int_S\int \bar{F}.d\bar{S} = \int\int_V\int \bar{\nabla}.\bar{F} dV$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3} (4 - 4y + 2z) dx\, dy\, dz$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [(4-4y)z + z^2]_0^3\, dx\, dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [12(1-y) + 9]\, dx\, dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y) dx\, dy$$

$$= \int_{-2}^{2}\left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 21\, dy - 12\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y dy\right] dx$$

$$= \int_{-2}^{2}\left[21\times 2\int_{0}^{\sqrt{4-x^2}} dy - 12(0)\right] dx$$

[Since the integrans in forst integral is even and in 2nd integral it is on add function]

$$= 42\int_{-2}^{2}(y)_0^{\sqrt{4-x^2}}\, dx$$

$$= 42\int_{-2}^{2}\sqrt{4-x^2}\, dx = 42\times 2\int_{0}^{2}\sqrt{4-x^2}\, dx$$

$$= 84\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 84\left[0 + 2.\frac{\pi}{2} - 0\right] = 84\pi$$

11: Verify divergence theorem for $\overline{\boldsymbol{F}} = \boldsymbol{x^2 i + y^2 j + z^2 k}$ over the surface S of the solid cut off by the plane x+y+z=a in the first octant.

Sol: By Gauss theorem, $\int_s \overline{F}.\overline{n} dS = \int_v div \overline{F} dv$

Let $\phi = x + y + z - a$ be the given plane then

$$\frac{\partial\phi}{\partial x} = 1, \frac{\partial\phi}{\partial y} = 1, \frac{\partial\phi}{\partial z} = 1$$

$$\therefore\ grad\,\phi = \sum \bar{i}\frac{\partial\phi}{\partial x} = \bar{i} + \bar{j} + \bar{k}$$

$$Unit\ normal = \frac{grad\ \phi}{|grad\ \phi|} = \frac{\bar{i} + \bar{j} + \bar{k}}{\sqrt{3}}$$

Let R be the projection of S on xy-plane

Then the equation of the given plane will be x+y=a $\Rightarrow$ y=a-x

Also when y=0, x=a

$$\therefore \int_s \overline{F}.\overline{n} dS = \iint_R \frac{\overline{F}.\overline{n} dx dy}{|\overline{n}.\overline{k}|}$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x}\frac{\frac{x^2+y^2+z^2}{\sqrt{3}}}{1/\sqrt{3}}dx\, dy = \int_0^a\int_{y=0}^{a-x}[x^2+y^2+(a-x-y)^2]dx\, dy\ [\,since\ x+y+z=a]$$

$$= \int_0^a\int_0^{a-x}[2x^2+2y^2-2ax+2xy-2ay+a^2]dx\, dy$$

$$= \int_{x=0}^{a}\left[2x^2y + \frac{2y^3}{3} + xy^2 - 2axy - ay^2 + a^2y\right]_0^{a-x} dx$$

$$= \int_{x=0}^{a} [2x^2(a-x) + \frac{2}{3}(a-x)^3 + x(a-x)^2 - 2ax(a-x) - a(a-x)^2 + a^2(a-x)]dx$$

$$\therefore \int_s \overline{F}.\overline{n}dS = \int_0^a \left(-\frac{5}{3}x^3 + 3ax^2 - 2a^2x + \frac{2}{3}a^3 \right) dx = \frac{a^4}{4},$$ on simplification...(1)

Given $\overline{F} = x^2\overline{i} + y^2\overline{j} + z^2\overline{k}$

$$\therefore div\ \overline{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) = 2(x+y+z)$$

$$Now \iiint div\overline{F}.dv = 2\int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y}(x+y+z)dxdydz$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{a-x}\left[z(x+y) + \frac{z^2}{2}\right]_0^{a-x-y} dx\ dy$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{a-x}(a-x-y)\left[x+y+\frac{a-x-y}{2}\right]dx\ dy$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x}(a-x-y)[a+x+y]dx\ dy$$

$$= \int_0^a\int_0^{a-x}[a^2-(x+y)^2]\ dy\ dx = \int_0^a\int_0^{a-x}(a^2-x^2-y^2-2xy)dx\ dy$$

$$= \int_0^a\left[a^2y - x^2y - \frac{y^3}{3} - xy^2\right]_0^{a-x} dx$$

$$= \int_0^a (a-x)(2a^2-x^2-ax)dx = \frac{a^4}{4} \ldots\ldots (2)$$

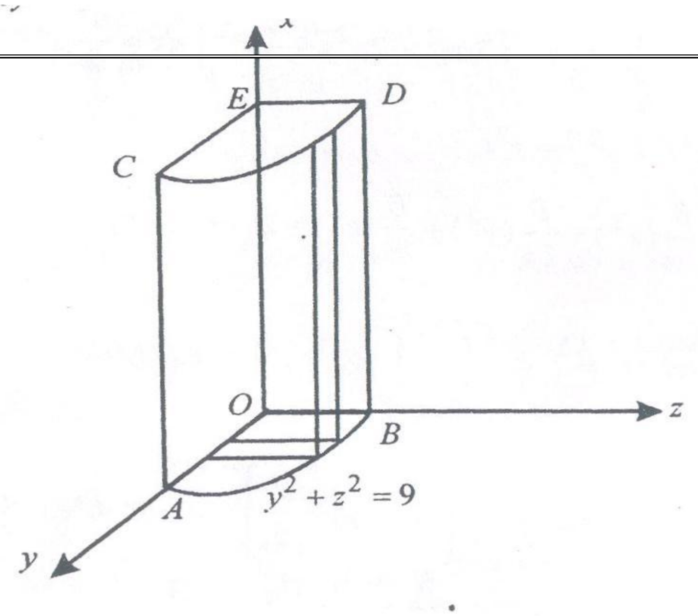
Hence from (1) and (2), the Gauss Divergence theorem is verified.

12: Verify divergence theorem for $2x^2y\overline{i} - y^2\overline{j} + 4xz^2\overline{k}$ taken over the region of first octant of the cylinder $y^2+z^2=9$ and $x=2$.

(or) Evaluate $\int\int_s \overline{F}.\overline{n}dS$, where $\overline{F} = 2x^2y\overline{i} - y^2\overline{j} + 4xz^2\overline{k}$ and S is the closed surface of the region in the first octant bounded by the cylinder $y^2+z^2 = 9$ and the planes x=0, x=2, y=0, z=0

Sol: Let $\overline{F} = 2x^2y\overline{i} - y^2\overline{j} + 4xz^2\overline{k}$

$$\therefore \nabla.\overline{F} = \frac{\partial}{\partial x}(2x^2) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(4xz^2) = 4xy - 2y + 8xz$$



$$\iiint_V \bar{\nabla}.\bar{F}\,dv = \int_{x=0}^{2}\int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} (4xy-2y+8xz)dz\,dy\,dx$$

$$= \int_0^2\int_0^3 \left[(4xy-2y)z+8x\,\frac{z^2}{2}\right]_{z=0}^{\sqrt{9-y^2}} dy\,dx$$

$$= \int_0^2\int_0^3 \left[(4xy-2y)\sqrt{9-y^2}+4x(9-y^2)\right]dy\,dx$$

$$= \int_0^2\int_0^3 [(1-2x)(-2y)\sqrt{9-y^2}+4x(9-y^2)]\,dy\,dx$$

$$= \int_0^2 \left\{\left[(1-2x)\frac{(9-y^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3 + 4x\left(9y-\frac{y^3}{3}\right)_0^3\right\}dx$$

$$= \int_0^2 \{\tfrac{2}{3}(1-2x)[0-27]+4x\,[27-9]\}\,dx = \int_0^2 [-18(1-2x)+72x]dx$$

$$\left[-18(x-x^2)+72\,\frac{x^2}{2}\right]_0^2 = -18(2-4)+36(4) = 36+144 = 180\ldots(1)$$

Now we sall calculate $\int_S \bar{F}.\bar{n}\,ds$ *for all the five faces.*

$$\int_s \overline{F}.\overline{n}dS = \int_{s_1} \overline{F}.\overline{n}dS + \int_{s_2} \overline{F}.\overline{n}dS + \ldots + \int_{s_5} \overline{F}.\overline{n}dS$$

Where S_1 is the face OAB, S_2 is the face CED, S_3 is the face OBDE, S_4 is the face OACE and S_5 is the curved surface ABDC.

(i) On $S_1 : x = 0, \overline{n} = -i \therefore \overline{F}.\overline{n} = 0$ $Hence \int_{s_1} \overline{F}.\overline{n}dS$

(ii) On $S_2 : x = 2, \overline{n} = i \therefore \overline{F}.\overline{n} = 8y$

$$\therefore \int_{s_2} \overline{F}.\overline{n}dS = \int_0^3 \int_0^{\sqrt{9-z^2}} 8y\,dy\,dz = \int_0^3 8\left(\frac{y^2}{2}\right)_0^{\sqrt{9-z^2}} dz$$

$$= 4\int_0^3 (9 - z^2)dz = 4\left(9z - \frac{z^3}{3}\right)_0^3 = 4(27 - 9) = 72$$

(iii) On $S_3 : y = 0, \overline{n} = -j. \therefore \overline{F}.\overline{n} = 0$ $Hence \int_{s_3} \overline{F}.\overline{n}dS$

$$(iv) On\ S_4 : z = 0, \overline{n} = -k. \quad \overline{F}.\overline{n} = 0. \quad Hence \int_{S_4} \overline{F}.\overline{n}ds = 0$$

(v) On $S_5 : y^2 + z^2 = 9, \overline{n} = \frac{\nabla(y^2+z^2)}{|\nabla(y^2+z^2)|} = \frac{2y\overline{j} + 2z\overline{k}}{\sqrt{4y^2+4z^2}} = \frac{y\overline{j} + z\overline{k}}{\sqrt{4\times 9}} = \frac{y\overline{j} + z\overline{k}}{3}$

$\overline{F}.\overline{n} = \frac{-y^3 + 4xz^3}{3}$ and $\overline{n}.\overline{k} = \frac{z}{3} = \frac{1}{3}\sqrt{9 - y^2}$

Hence $\int_{S_5} \overline{F}.\overline{n}ds = \int\int_R \overline{F}.\overline{n}\frac{dx\,dy}{|\overline{n}.\overline{k}|}$ $Where\ R\ is\ the\ projection\ of\ S_5\ on\ xy-plane.$

$$= \int_R\int \frac{4xz^3 - y^3}{\sqrt{9-y^2}}dx\,dy = \int_{x=0}^{2}\int_{y=0}^{3} \left[4x(9-y^2) - y^3\left(9 - y^2\right)^{-\frac{1}{2}}\right]dy\,dx$$

$$= \int_0^2 72x\,dx - 18\int_0^2 dx = 72\left(\frac{x^2}{2}\right)_0^2 - 18(x)_0^2 = 144 - 36 = 108$$

$Thus \int_S \overline{F}.\overline{n}ds = 0 + 72 + 0 + 0 + 108 = 180 \ldots\ldots (2)$

Hence the Divergence theorem is verified from the equality of (1) and (2).

13: Use Divergence theorem to evaluate $\int\int (x\overline{i} + y\overline{j} + z^2\overline{k}).\overline{n}ds$. Where S is the surface bounded by the cone $x^2+y^2=z^2$ in the plane $z = 4$.

Sol: Given $\int\int(x\bar{i}+y\bar{j}+z^2\bar{k}).\bar{n}.ds$ *Where* S is the surface bounded by the cone $x^2+y^2=z^2$ in the plane z = 4.

Let $\bar{F} = x\bar{i}+y\bar{j}+z^2\bar{k}$

By Gauss Divergence theorem,we have

$$\int\int(x\bar{i}+y\bar{j}+z^2\bar{k}).\bar{n}.ds = \int\int_V\int \bar{\nabla}.\bar{F}\,dv$$

$$Now\ \nabla.\overline{F} = \frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z^2) = 1+1+2z = 2(1+z)$$

On the cone, $x^2+y^2=z^2$ and z=4 $\Rightarrow x^2+y^2=16$

The limits are $z = 0\ to 4, y = o\ to\ \sqrt{16-x^2}, x = 0\ to\ 4.$

$$\int\int_V\int \bar{\nabla}.\bar{F}\,dv = \int_0^4\int_0^{\sqrt{16-x^2}}\int_0^4 2(1+z)dx\,dy\,dz$$

$$= 2\int_0^4\int_0^{\sqrt{16-x^2}}\left\{[z]_0^4+\left[\frac{z^2}{2}\right]_0^4\right\}dx\,dy$$

$$= 2\int_0^4\int_0^{\sqrt{16-x^2}}[4+8]dxdy = 2\times 12\int_0^4[y]_0^{\sqrt{16-x^2}}dx$$

$$= 24\int_0^4\sqrt{16-x^2}\,dx = 24\int_0^{\frac{\pi}{2}}\sqrt{16-16\,sin^2\,\theta}\,.4\cos\theta d\theta$$

$$[\,put x = 4\sin\theta \Rightarrow dx = 4\cos\theta d\theta.\ Also\ x = 0 \Rightarrow \theta = 0\ and\ x = 4 \Rightarrow \theta = \frac{\pi}{2}]$$

$$\therefore \int\int_V\int\nabla.\overline{F}dv = 96\times 4\int_0^{\frac{\pi}{2}}4\sqrt{1-\sin^2\theta}\cos\theta d\theta = 96\times 4\int_0^{\frac{\pi}{2}}\cos^2\theta d\theta$$

$$\iiint_V \bar{\nabla}.\bar{F}\, dv = 96 X 4 \int_0^{\frac{\pi}{2}} 4\sqrt{1-\sin^2\theta}\ \cos\theta d\theta = 96 X 4 \int_0^{\frac{\pi}{2}} \cos^2\theta\ d\theta$$

$$= 96 X 4 \int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} d\theta = 96X4 \int_0^{\frac{\pi}{2}} \left[\frac{1}{2} + \frac{\cos 2\theta}{2}\right] d\theta$$

$$= 384 \left[\frac{1}{2}\theta + \frac{1}{2}\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} = 96\pi$$

14: Use Gauss Divergence theorem to evaluate $\int\int_S (yz^2\bar{\imath} + zx^2\bar{\jmath} + 2z^2\bar{k}).ds,\ where$ S is the closed surface bounded by the xy-plane and the upper half of the sphere $x^2+y^2+z^2=a^2$ above this plane.

Sol: Divergence theorem states that

$$\iint_S \bar{F}.ds = \iiint_V \bar{\nabla}.\bar{F}\ dv$$

Here $\nabla.\overline{F} = \frac{\partial}{\partial x}(yz^2) + \frac{\partial}{\partial y}(zx^2) + \frac{\partial}{\partial z}(2z^2) = 4z$

$$\therefore \iint_s \overline{F}.ds = \iiint_V 4z dx dy dz$$

Introducing spherical polar coordinates $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,$

$z = r\cos\theta$ then $dxdydz = r^2 dr d\theta d\phi$

$$\therefore \iint_s \overline{F}.ds = 4\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}(r\cos\theta)(r^2\sin\theta dr d\theta d\phi)$$

$$= 4\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^3 \sin\theta\cos\theta \left[\int_{\phi=0}^{2\pi} d\phi\right] dr\ d\theta$$

$$= 4.\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^3 \sin\theta\cos\theta\ (2\pi - 0) dr\ d\theta$$

$$= 4\pi \int_{r=0}^{a} r^3 \left[\int_0^{\pi} \sin 2\theta\ d\theta\right] dr = 4\pi \int_{r=0}^{a} r^3 \left(-\frac{\cos 2\theta}{2}\right)_0^{\pi} dr$$

$$= (-2\pi)\int_0^a r^3(1-1)dr = 0$$

15: Verify Gauss divergence theorem for $\bar{F} = x^3\bar{\imath} + y^3\bar{\jmath} + z^3\bar{k}$ taken over the cube bounded by x = 0, x = a, y= 0, y = a, z = 0, z = a.

Sol: We have $\bar{F} = x^3\bar{\imath} + y^3\bar{\jmath} + z^3\bar{k}$

$$\nabla \cdot \overline{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = 3x^2 + 3y^2 + 3z^2$$

$$\int\int_V\int \bar{\nabla}.\bar{F}\,dv = \int\int_V\int (3x^2 + 3y^2 + 3z^2)dx\,dy\,dz$$

$$= 3\int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a}(x^2 + y^2 + z^2)dx\,dy\,dz$$

$$= 3\int_{z=0}^{a}\int_{y=0}^{a}\left(\frac{x^3}{3} + xy^2 + z^2x\right)_0^a dy\,dz$$

$$= 3\int_{z=0}^{a}\int_{y=0}^{a}\left(\frac{a^3}{a} + ay^2 + az^2\right)dy\,dz$$

$$= 3\int_{z=0}^{a}\left(\frac{a^3}{3}y + a\frac{y^3}{3} + az^2y\right)_0^a dz$$

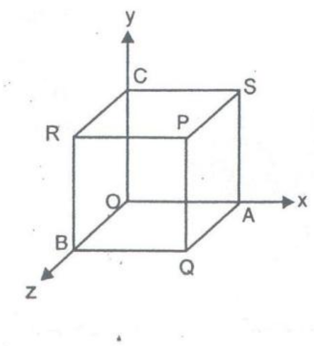
$$= 3\int_0^a\left(\frac{a^4}{3} + \frac{a^4}{3} + a^2z^2\right)dz = 3\int_0^a\left(\frac{2}{3}a^4 + a^2z^2\right)dz$$

$$= 3\left(\frac{2}{3}a^4z + a^2.\frac{z^3}{3}\right)_0^a = 3\left(\frac{2}{3}a^5 + \frac{1}{3}a^5\right)$$

$$= 3a^5$$

To evaluate the surface integral divide the closed surface S of the cube into 6 parts.

i.e., S_1 : The face DEFA ; S_4 : The face OBDC



S_2 : The face AGCO ; S_5 : The face GCDE

S_3 : The face AGEF ; S_6: The face AFBO

$$\int\int_S \bar{F}.\bar{n}ds = \int\int_{S_1} \bar{F}.\bar{n}ds + \int\int_{S_2} \bar{F}.\bar{n}ds + \cdots + \int\int_{S_6} \bar{F}.\bar{n}ds$$

On S_1, *we have* $\bar{n} = \bar{\imath}, x = a$

$$\therefore \int\int_{s_1} \overline{F}.\overline{n}ds = \int_{z=0}^{a}\int_{y=0}^{a}\left(a^3\overline{i} + y^3\overline{j} + z^3\overline{k}\right).\overline{i}dydz$$

$$\int\int_{S_1} \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{y=0}^{a}(a^3\bar{\imath} + y^3\bar{\jmath} + z^3\bar{k}).\bar{\imath}\,dy\,dz$$

$$= \int_{z=0}^{a}\int_{y=0}^{a} a^3dy\,dz = a^3\int_0^a (y)_0^a\,dz$$

$$= a^4(z)_0^a = a^5$$

On S_2, *we have* $\bar{n} = -\bar{\imath}, x = 0$

$$\iint_{S_2} \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{y=0}^{a} \left(y^3\bar{j} + z^3\bar{k}\right).\left(-\bar{i}\right)dydz = 0$$

$On\ S_3, we\ have\ \bar{n} = \bar{j}, y = a$

$$\iint_{S_3} \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{x=0}^{a} \left(x^3\bar{i} + a^3\bar{j} + z^3\bar{k}\right).\bar{j}dxdz = a^3\int_{z=0}^{a}\int_{x=0}^{a} dxdz = a^3\int_{0}^{a} adz = a^4\left(z\right)_0^a$$

$$= a^5$$

$On\ S_4, we\ have\ \bar{n} = -\bar{j}, y = 0$

$$\int_{S_4}\int \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{x=0}^{a} (x^3\bar{i} + z^3\bar{k}).(-\bar{j})dx\,dz = 0$$

$On\ S_5, we\ have\ \bar{n} = \bar{k}, z = a$

$$\int_{S_5}\int \bar{F}.\bar{n}ds = \int_{y=0}^{a}\int_{x=0}^{a} (x^3\bar{i} + y^3\bar{j} + a^3\bar{k}).\bar{k}\,dx\,dy$$

$$= \int_{y=0}^{a}\int_{x=0}^{a} a^3 dx\,dy = a^3\int_{0}^{a} (x)_0^a dy = a^4(y)_0^a = a^5$$

$On\ S_6, we\ have\ \bar{n} = -\bar{k}, z = 0$

$$\int_{S_6}\int \bar{F}.\bar{n}ds = \int_{y=0}^{a}\int_{x=0}^{a} (x^3\bar{i} + y^3\bar{j}).(-\bar{k})dx\,dy = 0$$

$$Thus \int_{S}\int \bar{F}.\bar{n}ds = a^5 + 0 + a^5 + 0 + a^5 + 0 = 3a^5$$

$$Hence \int_{S}\int \bar{F}.\bar{n}ds = \int_{V}\int \bar{\nabla}.\bar{F}\,dv$$

$\therefore$ *The* Gauss divergence theorem is verified.

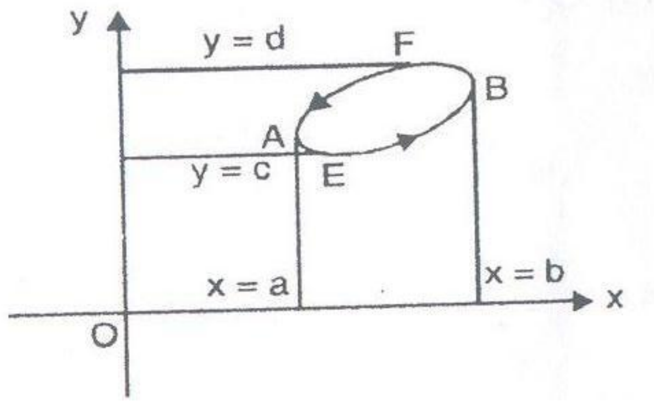
II. GREEN'S THEOREM IN A PLANE

(Transformation Between Line Integral and Surface Integral) [JNTU 2001S].

If S is Closed region in xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy.$$

Where C is traversed in the positive(anti clock-wise) direction

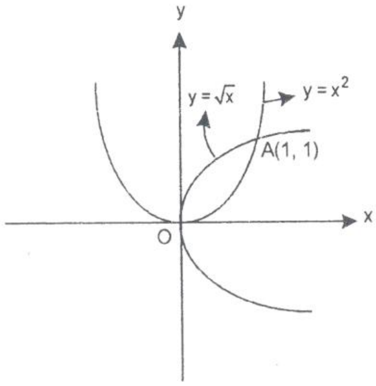


SOLVED PROBLEMS

1: Verify Green's theorem in plane for $\oint(3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the region bounded by y=$\sqrt{x}$ and y=x^2 .

Solution: Let M=3x^2-8y^2 and N=4y-6xy. Then

$\frac{\partial M}{\partial y} = -16y, \frac{\partial N}{\partial x} = -6y$



We have by Green's theorem,

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy.$$

Now $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \iint_R (16y - 6y)dxdy$

$$=10\iint_R ydxdy = 10\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} ydydx = 10\int_{x=0}^{1}\left(\frac{y^2}{2}\right)_{x^2}^{\sqrt{x}}dx$$

$$=5\int_0^1 (x - x^4)dx = 5\left(\frac{x^2}{2} - \frac{x^5}{5}\right)_0^1 = 5\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{2} \qquad ....(1)$$

Verification:

We can write the line integral along c

=[line integral along y=x^2(from O to A) + [line integral along y^2=x(from A to O)]

=I_1+I_2(say)

Now $I_1 = \int_{x=0}^{1}\{[3x^2 - 8(x^2)^2]dx + [4x^2 - 6x(x^2)]2xdx\}$ $\left[\because y = x^2 \Rightarrow \frac{dy}{dx} = 2x\right]$

$=\int_0^1 (3x^3 + 8x^3 - 20x^4)dx = -1$

And $I_2 = \int_1^0 \left[(3x^2 - 8x)\,dx + \left(4\sqrt{x} - 6x^{3/2}\right) \frac{1}{2\sqrt{x}}\,dx \right] = \int_1^0 (3x^2 - 11x + 2)\,dx = \frac{5}{2}$

$\therefore\ I_1 + I_2 = -1 + 5/2 = 3/2.$

From(1) and (2), we have $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy.$

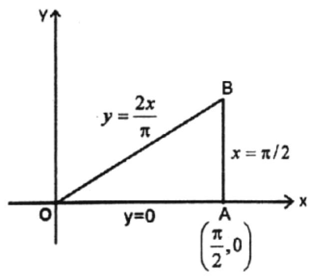
hence the verification of the Green's theorem.

2: Evaluate by Green's theorem $\int_c (y - \sin x)dx + \cos x\,dy$ where C is the triangle enclosed by the lines y=0, x=$\frac{\pi}{2}$, $\pi y = 2x$.

Solution: Let M=y-$\sin x$ $and\ N = \cos x$ Then

$\frac{\partial M}{\partial y}$=1 and $\frac{\partial N}{\partial x}$=-$\sin x$

$\therefore$ By Green's theorem $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy.$



$\Rightarrow \int_c (y - \sin x)dx + \cos x dy = \iint_R (-1 - \sin x)dx dy$

$= -\int_{x=0}^{\pi/2} \int_{y=0}^{\frac{2x}{\pi}} (1 + \sin x)\,dx dy$

$= -\int_{x=0}^{\pi/2} (\sin x + 1)\,[y]_0^{2x/\pi}\,dx$

$= \frac{-2}{\pi} \int_{x=0}^{\pi/2} x(\sin x + 1)dx$

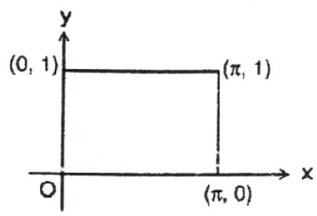
$= \frac{-2}{\pi} \left[x(-\cos x + x) \right]_0^{\pi} - \int_0^{\pi/2} 1(-\cos x + x)dx$

$= \frac{-2}{\pi} \left[x(-\cos x + x) + \sin x - \frac{x^2}{2} \right]_0^{\pi/2}$

$= \frac{-2}{\pi} \left[-x\cos x + \frac{x^2}{2} + \sin x \right]_0^{\pi/2} = \frac{-2}{\pi}\left[\frac{\pi^2}{8} + 1\right] = -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$

3: Evaluate by Green's theorem for $\oint_c (x^2 - \cosh y)dx + (y + \sin x)dy$ where C is the rectangle with vertices$(0,0)$, $(\pi, 0)$, $(\pi, 1)$, $(0,1)$.

Solution: Let M=$x^2 - \cosh y$,$N = y + \sin x$



$\therefore\ \frac{\partial M}{\partial y} = -\sinh y\ and\ \frac{\partial N}{\partial x} = \cos x$

By Green's theorem, $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$.

$\Rightarrow \oint_c (x^2 - \cosh y)dx + (y + \sin x)dy = \int_R\int (\cos x + \sinh y)\,dx\,dy$

$$\Rightarrow \oint_c (x^2 - \cosh y)dx + (y + \sin x)dy = \int_s\int (\cos x + \sinh y)\,dx\,dy$$

$= \int_{x=0}^{\pi}\int_{y=0}^{1} (\cos x + \sinh y)\,dy\,dx = \int_{x=0}^{\pi} (y\cos x + \cosh y)_0^1\,dx$

$= \int_{x=0}^{\pi} (\cos x + \cosh 1 - 1)\,dx$

$= \pi(\cosh 1 - 1)$

4: A Vector field is given by $\overline{F} = (\sin y)\bar{i} + x(1 + \cos y)\,\bar{j}$

Evaluate the line integral over the circular path $x^2 + y^2 = a^2$, z=0

(i) Directly (ii) By using Green's theorem

Solution: (i) Using the line integral

$\oint_c \overline{F}.d\overline{r} = \oint_c F_1\,dx + F_2\,dy = \oint_c \sin y\,dx + x(1 + \cos y)dy$

$= \oint_c \sin y\,dx + x\cos y\,dy + x\,dy = \oint_c d(x\sin y) + x\,dy$

Given Circle is $x^2 + y^2 = a^2$. Take x=a $\cos\theta$ and y=a $\sin\theta$ so that dx=-a $\sin\theta\, d\theta$ and dy=a $\cos\theta\, d\theta$ and $\theta = 0 \to 2\pi$

$\therefore \oint \overline{F}.d\overline{r} = \int_0^{2\pi} d[a\cos\theta \sin(a\sin\theta)] + \int_0^{2\pi} a(\cos\theta)a\cos\theta\,d\theta$

$= [a\cos\theta\sin(a\sin\theta)]_0^{2\pi} + 4a^2 \int_0^{\pi/2} \cos^2\theta\,d\theta$

$= 0 + 4a^2 . \frac{1}{2} . \frac{\pi}{2} = \pi a^2$

(ii)Using Green's theorem

Let M=$\sin y$ and N=x$(1 + \cos y)$. Then

$\frac{\partial M}{\partial y} = \cos y$ and $\frac{\partial N}{\partial x} = (1 + \cos y)$

By Green's theorem,

$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$

$\therefore \oint_c \sin y\,dx + x(1 + \cos y)dy = \iint_R (-\cos y + 1 + \cos y)\,dx\,dy == \iint dx\,dy$

$= \int_R\int dA = A = \pi a^2 (\because$ area of circle$= \pi a^2)$

We observe that the values obtained in (i) and (ii) are same to that Green's theorem is verified.

5: Show that area bounded by a simple closed curve C is given by $\frac{1}{2}\oint xdy - ydx$ and hence find the area of

(i)The ellipse x=$a\cos\theta$, $y = b\sin\theta$ $(i.e)\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(ii)The Circle x=$a\cos\theta$, $y = a\sin\theta$ $(i.e) x^2 + y^2 = a^2$

<u>Solution:</u> We have by Green's theorem $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here M=-y and N=x so that $\frac{\partial M}{\partial y} = -1$ $and \frac{\partial N}{\partial x} = 1$

$\oint_c xdy - ydx = 2\int_R dxdy = 2A$ where A is the area of the surface.

$\therefore \frac{1}{2}\int xdy - ydx = A$

(i)For the ellipse x=$a\cos\theta$ and y=$b\sin\theta$ and $\theta = 0 \to 2\pi$

$\therefore Area, A$=$\frac{1}{2}\oint xdy - ydx = \frac{1}{2}\int_0^{2\pi}[(a\ \cos\theta)(b\cos\theta) - (b\ \sin\theta\,(-a\ \sin\theta))]d\,\theta$

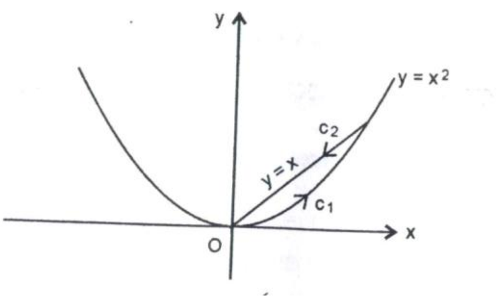
=$\frac{1}{2}ab\int_0^{2\pi}(\cos^2\theta + sin^2\theta)\,d\theta = \frac{1}{2}ab(\theta)_0^{2\pi} = \frac{ab}{2}(2\pi - 0) = \pi ab$

(ii)Put a=b to get area of the circle A=πa^2

6: Verify Green's theorem for $\int_c[(xy + y^2)dx + x^2dy]$, where C is bounded by y=x and y=x^2

Solution: By Green's theorem, we have $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here M=xy +y^2 and N=x^2



The line y=x and the parabola y=x^2 intersect at O$(0,0)$ and A$(1,1)$

Now $\oint_c M\ dx + N\,dy = \int_{c_1} M\,dx + N\,dy + \int_{c_2} M\,dx + N\,dy......(1)$ .....(1)

Along C_1 $(i.e.\ y = x^2)$, the line integral is

$\int_{c_1} M\,dx + N\,dy = \int_{c_1}[x(x^2) + x^4]dx + x^2 d(x^2)\int_c (x^3 + x^4 + 2x^3)dx = \int_0^1 (3x^3 + x^4)dx$

=$\left(3.\frac{x^4}{4} + \frac{x^5}{5}\right)_0^1$ =$\frac{3}{4} + \frac{1}{5} = \frac{19}{20}$ ....(2)

Along C_2 $(i.e. y = x)$ from $(1,1)$ to $(0,0)$, the line integral is

$$\int_{c_2} M\,dx + N\,dy = \int_{c_2} (x.x + x^2)dx + x^2 dx \; [\because dy = dx]$$

$$= \int_{c_2} 3x^2 dx = 3 \int_1^0 x^2 dx = 3\left(\frac{x^3}{3}\right)_1^0 = (x^3)_1^0 = 0-1=-1 \quad ....(3)$$

From (1), (2) and (3), we have

$$\int_c M dx + N dy = \frac{19}{20} - 1 = \frac{-1}{20} \quad ...(4)$$

Now

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R (2x - x - 2y)\,dxdy$$

$$= \int_0^1 [(x^2 - x^2) - (x^3 - x^4)]dx = \int_0^1 (x^4 - x^3)dx$$

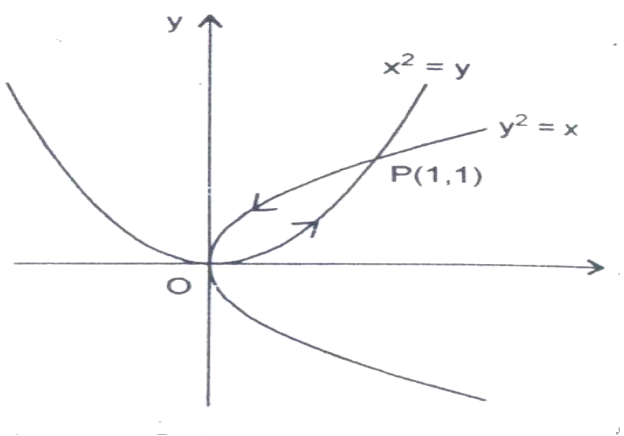
$$= \left(\frac{x^5}{5} + \frac{x^4}{4}\right)_0^1 = \frac{1}{5} - \frac{1}{4} = \frac{-1}{20} \quad ....(5)$$

From (4) and (5), $We\ have$ $\oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ dxdy

Hence the verification of the Green's theorem.

7: Using Green's theorem evaluate $\int_c (2xy - x^2)dx + (x^2 + y^2)dy$, Where "C" is the closed curve of the region bounded by y=x^2 and $y^2 = x$

Solution:



The two parabolas $y^2 = x\ and\ y = x^2$ are intersecting at O$(0,0)$, and P(1,1)

Here M=2xy-x^2 and N=x^2 +y^2

$\therefore \frac{\partial M}{\partial y} = 2x\ and\ \frac{\partial N}{\partial x} = 2x$

Hence $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - 2x = 0$

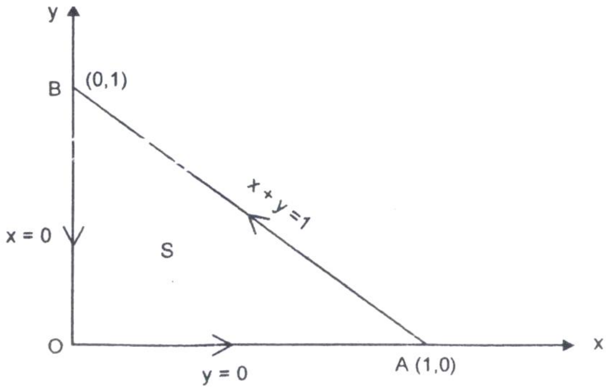
By Green's theorem $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ dxdy i.e., $\int_c (2xy - x^2)dx + (x^2 + y^2)dy = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (0)\,dx\,dy = 0$

8: Verify Green's theorem for $\int_c [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$ where c is the region bounded by x=0, y=0 and x+y=1.

Solution : By Green's theorem, we have

$$\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

Here M=3$x^2 - 8y^2$ and N=4y-6xy



\$\therefore \frac{\partial M}{\partial y} = -16y$ and $\frac{\partial N}{\partial x} = -6y$

Now $\int_c M\,dx + N\,dy = \int_{OA} M\,dx + N\,dy + \int_{AB} M\,dx + N\,dy + \int_{BC} M\,dx + N\,dy \ldots(1)$

Along OA, y=0 $\therefore dy = 0$

$\int_{OA} Mdx + Ndy = \int_0^1 3x^2 dx = \left(\frac{x^3}{3}\right)_0^1 = 1$

Along AB, x+y=1 $\therefore dy = -dx$ and x=1-y and y varies from 0 to 1.

$\int_{AB} M\,dx + N\,dy = \int_0^1 [3(y-1)^2 - 8y^2](-dy) + [4y + 6y(y-1)]dy$

$= \int_0^1 (-5y^2 - 6y + 3)(-dy) + (6y^2 - 2y)dy$

$= \int_0^1 (11y^2 + 4y - 3)dy = \left(11\frac{y^3}{3} + 4\frac{y^2}{2} - 3y\right)_0^1$

$= \frac{11}{3} + 2 - 3 = \frac{8}{3}$

Along BO, x=0 $\therefore dx = 0$ and limits of y are from 1 to 0

. $\int_{BO} Mdx + Ndy = \int_1^0 4ydy = \left(4\frac{y^2}{2}\right)_1^0 = (2y^2)_0^1 = -2$

from (1), we have $\int_c M dx + N dy = 1 + \frac{8}{3} - 2 = \frac{5}{3}$

Now $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \int_{x=0}^{1} \int_{y=0}^{1-x} (-6y + 16y) dx dy$

$=10 \int_{x=0}^{1} \left[\int_{y=0}^{1-x} y dy \right] dx = 10 \int_0^1 \left(\frac{y^2}{2} \right)_0^{1-x} dx$

$=5 \int_0^1 (1-x)^2 dx = 5 \left[\frac{(1-x)^3}{-3} \right]_0^1$

$=-\frac{5}{3}[(1-1)^3 - (1-0)^3] = \frac{5}{3}$

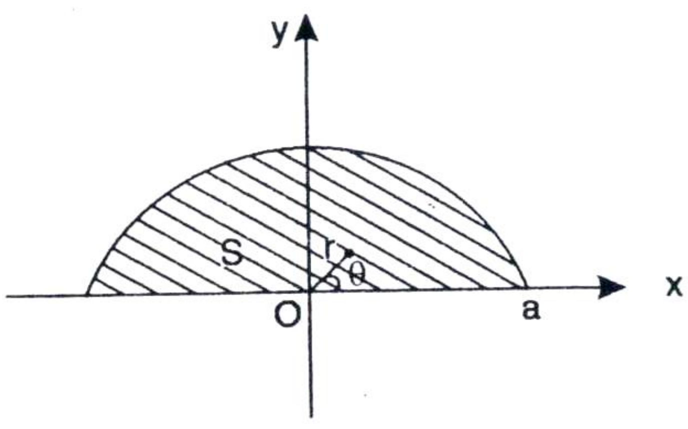
From (2) and (3), we have $\int_c M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$

Hence the verification of the Green's Theorem.

9: Apply Green's theorem to evaluate $\oint_c (2x^2 - y^2) dx + (x^2 + y^2) dy, where\ c\ is$ the boundary of the area enclosed by the x-axis and upper half of the circle $x^2 + y^2 = a^2$

Solution : Let M=$2x^2 - y^2$ and N=$x^2 + y^2$ Then

$\frac{\partial M}{\partial y} = -2y \ and \ \frac{\partial N}{\partial x} = 2x$



Figure

$\therefore \ By\ Green's Theorem, \int_c M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$

$\oint_c [(2x^2 - y^2)dx + (x^2 + y^2)dy] = \int\int_R (2x + 2y)dxdy$

$=2\int\int_R (x + y)dy$

$$=2\int_0^a \int_0^\pi r(\cos\theta + \sin\theta).rd\theta dr$$

[Changing to polar coordinates (r,θ), r varies from 0 to a and θ varies from 0 to π]

$\therefore \oint_c [(2x^2 - y^2)dx + (x^2 + y^2)dy] = 2\int_0^a r^2 dr \int_0^\pi (\cos\theta + \sin\theta)d\theta$

$=2.\frac{a^3}{3}(1+1) = \frac{4a^3}{3}$

10: Find the area of the Folium of Descartes $x^3 + y^3 = 3axy(a > 0) using\ Green's$ Theorem.

Solution: from Green's theorem, we have

$\int Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$

By Green's theorem, Area = $\frac{1}{2}\oint (xdy - ydx)$

Considering the loop of folium Descartes(a>0)

Let x= $\frac{3at}{1+t^3}, y = \frac{3at^2}{1+t^3}, Then\ dx = \left[\frac{d}{dt}\left(\frac{3at}{1+t^3}\right)\right]dt$ and $dy = \left[\frac{d}{dt}\left(\frac{3at^2}{1+t^3}\right)\right]dt$

The point of intersection of the loop is $\left(\frac{3a}{2}, \frac{3a}{2}\right) \Rightarrow t = 1$

Along OA, t varies from 0 to1.

$\therefore \frac{1}{2}\oint(xdy - ydx) = \frac{1}{2}\int_0^1 \left(\frac{3at}{1+t^3}\right)\left[\frac{d}{dt}\left(\frac{3at^2}{1+t^3}\right)\right]dt - \left(\frac{3at^2}{1+t^3}\right)\left[\frac{d}{dt}\left(\frac{3at}{1+t^3}\right)\right]dt$

$$= \frac{1}{2}\int_0^1 \left\{\frac{3at}{1+t^3}\left[\frac{3at(2-t^3)}{(1+t^3)^2}\right] - \frac{3at^2}{1+t^3}\left[\frac{3a(1-2t^3)}{(1+t^3)^2}\right]\right\}dt$$

$$=\frac{9a^2}{2}\int_0^1 \left[\frac{t^2(2-t^3)}{(1+t^3)^3} - \frac{t^2(1-2t^3)}{(1+t^3)^3}\right]dt = \frac{9a^2}{2}\int_0^1 \frac{2t^2 - t^5 - t^2 + 2t^5}{(1+t^3)^3}dt$$

$= \frac{9a^2}{2}\int_0^1 \frac{t^2 + t^5}{(1+t^3)^3}dt = \frac{9a^2}{2}\int_0^1 \frac{t^2(1+t^3)}{(1+t^3)^3}dt$

$=\frac{9a^2}{2}\int_0^1 \frac{t^2}{(1+t^3)^2}dt$ [Put 1+$t^3 = x \Longrightarrow 3t^2\, dt = dx$

L.L. : x=1, U.L.:x=2]

$=\frac{9a^2}{2}\int_1^2 \frac{t^2}{x^2}\cdot\frac{dx}{3t^2} - \frac{9a^2}{6}\int_1^2 \frac{1}{x^2}dx = \frac{3a^2}{4}$ sq. units(a>0).

11: Verify Green's theorem in the plane for $\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy$

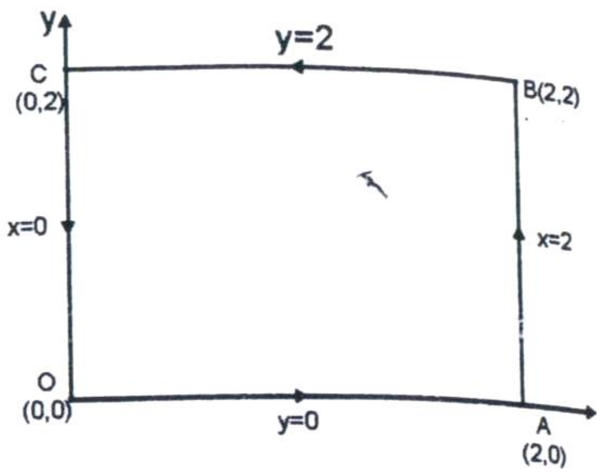
Where C is square with vertices (0,0), (2,0), (2,2), (0,2).

Solution: The Cartesian form of Green's theorem in the plane is

$$\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Here M=$x^2 - xy^3$ and N=$y^2 - 2xy$

$\therefore\ \frac{\partial M}{\partial y}$ = - 3 xy^2 and $\frac{\partial N}{\partial x} = -2y$



Evaluation of $\int_C (Mdx + Ndy)$

To Evaluate $\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy$, we shall take C in four different segments viz (i) along OA(y=0) (ii) along AB(x=2) (iii) along BC(y=2) (iv) along CO(x=0).

(i)Along OA(y=0)

$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_0^2 x^2 dx = \left(\frac{x^3}{3}\right)_0^2 = \frac{8}{3}$ .....(1)

(ii)Along AB(x=2)

$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_0^2 (y^2 - 4y)\,dy$ $[\because x = 2, dx = 0]$

$=\left(\frac{y^3}{3} - 2y^2\right)_0^2 = \left(\frac{8}{3} - 8\right) = 8\left(-\frac{2}{3}\right) = -\frac{16}{3}$ ....(2)

(iii)Along BC(y=2)

$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_2^0 (x^2 - 8x)\,dx$ $[\because y = 2, dy = 0]$

$=\left(\frac{x^3}{3} - 4x^2\right)_0^2 = -\left(\frac{8}{3} - 16\right) = \frac{40}{3}$ ......(3)

(iv)Along CO(x=0)

$\int_C (x^2 - xy^3)\, dx + (y^2 - 2xy)dy = \int_2^0 y^2\, dx\ \ [\because x = 0, dx = 0] = \left(\frac{y^3}{3}\right)_2^0 = -\frac{8}{3}$ .....(4)

Adding(1),(2),(3) and (4), we get

$\int_c (x^2 - xy^3)dx + (y^2 - 2xy)dy = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = \frac{24}{3} = 8$ ...(5)

Evaluation of $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

Here x ranges from 0 to 2 and y ranges from 0 to 2.

$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_0^2 \int_0^2 (-2y + 3xy^2)dxdy$

$= \int_0^2 \left(-2xy + \frac{3x^2}{2} y^2\right)_0^2 dy$

$= \int_0^2 (-4y + 6y^2)dy = (-2y^2 + 2y^3)_0^2$

=-8+16=8 ...(6)

From (5) and (6), we have

$\int_c M\, dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

Hence the Green's theorem is verified.

III. STOKE'S THEOREM

(Transformation between Line Integral and Surface Integral) [JNTU 2000]

Let S be a open surface bounded by a closed, non intersecting curve C. If $\bar{F}$ is any differentieable vector point function then $\oint_C \bar{F}.d\,\bar{r} = \int_S curl\ \bar{F}.\bar{n}\, ds$ *where c is traversed in the positive* direction and $\bar{n}$ *is unit outward drawn normal at any point of the surface.*

PROBLEMS:

1: Prove by Stokes theorem, Curl grad $\phi = \bar{0}$

Solution: Let S be the surface enclosed by a simple closed curve C.

$\therefore$ *By Stokes theorem*

$\int_s (curl\ grand\phi).\bar{n}\, ds = \int_s (\nabla x \nabla\phi).\bar{n}\, dS = \oint_C \nabla\phi.d\bar{r} = \oint_C \nabla\phi.d\bar{r}$

$= \oint_c \left(\bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}\right).\left(\bar{i}dx + \bar{j}dy + \bar{k}dz\right)$

$= \oint_c \left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz\right) = \int d\phi = [\phi]_p$ where P is any point on C.

$\therefore \int curl\ grad\phi.\bar{n}\ ds = \bar{0} \Rightarrow curl\ grad\phi = \bar{0}$

2: prove that $\int_s \phi curl\ \bar{f}.dS = \int_c \phi \bar{f}.d\bar{r} - \int_s curl\ grad\phi \times \bar{f} dS$

Solution: Applying Stokes theorem to the function $\phi\ \bar{f}$

$$\int_c \phi \bar{f}.d\bar{r} = \int curl(\phi \bar{f}).\bar{n}ds = \int_s (grad\phi \times \bar{f} + \phi curl\ \bar{f})ds$$

$$\therefore \int_c \phi curl\ \bar{f}.ds = \int_c \phi \bar{f}.d\bar{r} - \int \nabla\phi \times \bar{f}.ds$$

3: Prove that $\oint_c f\nabla f.d\bar{r} = 0$.

Solution: By Stokes Theorem,

$$\oint_c (f\nabla f).d\bar{r} = \int_s curl f\nabla \bar{f}.\bar{n}\ ds = \int_s [fcurl\nabla f + \nabla f \times \nabla f].\bar{n}\ ds$$

$$= \int \bar{0}.\bar{n}ds = 0[\because curl\nabla f = \bar{0} \text{ and } \nabla f \times \nabla f = \bar{0}]$$

4: Prove that $\oint_c f\nabla g.d\bar{r} = \int(\nabla f \times \nabla g).\bar{n}ds$

Solution: By Stokes Theorem,

$$\oint_c (f\nabla g.d\bar{r}) = \int_s [\nabla \times (f\nabla g)]\bar{n}ds = \int_s [\nabla f \times \nabla g + fcurl\ grad g].\bar{n}ds$$

$$= \int [\nabla f \times \nabla g].\bar{n}ds \left[\because curl(grad g) = \bar{0}\right]$$

5: Verify Stokes theorem for $\bar{F} = -y^3\bar{\imath} + x^3\bar{\jmath}$, Where S is the circular disc $x^2 + y^2 \le 1, z = 0$.

Solution: Given that $\bar{F} = -y^3\bar{\imath} + x^3\bar{\jmath}$. The boundary of C of S is a circle in xy plane. $x^2 + y^2 \le 1, z = 0$. We use the parametric co-ordinates x=cos$\theta, y = sin\theta, z = 0, 0 \le \theta \le 2\pi$;

dx=-sin$\theta\ d\theta$ and dy =cos$\theta\ d\theta$

$$\therefore \oint_c \bar{F}.dr = \int_c F_1dx + F_2dy + F_3dz = \int_c -y^3dx + x^3dy$$

$$= \int_0^{2\pi}[-sin^3\theta(-sin\theta) + cos^3\theta cos\theta]d\theta = \int_0^{2\pi}(cos^4\theta + sin^4\theta)d\theta$$

$$= \int_0^{2\pi}(1 - 2sin^2\theta\ cos^2\theta)d\theta = \int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi}(2sin\theta\ cos\theta)^2\ d\theta$$

$$= \int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi} sin^2 2d\theta = (2\pi - 0) - \frac{1}{4}\int_0^{2\pi}(1 - cos4\theta)d\theta$$

$$= 2\pi + \left[-\frac{1}{4}\theta + \frac{1}{16}sin4\theta\right]_0^{2\pi} = 2\pi - \frac{2\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$$

Now $\nabla \times \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = \bar{k}(3x^2 + 3y^2)$

$\therefore \int_s (\nabla \times \bar{F}).\bar{n}ds = 3\int_s (x^2 + y^2)\bar{k}.\bar{n}ds$

We have $(\bar{k}.\bar{n})ds = dxdy$ and R is the region on xy-plane

$\therefore \iint_s (\nabla \times \bar{F}).\bar{n}ds = 3\iint_R (x^2 + y^2)\, dx\, dy$

Put x=r cos$\emptyset$, $y = r\, sin\emptyset \therefore dxdy = rdr\, d\emptyset$

r is varying from 0 to 1 and $0 \leq \emptyset \leq 2\pi$.

$\therefore \int (\nabla \times \bar{F}).\, \bar{n}ds = 3\int_{\emptyset=0}^{2\pi} \int_{r=0}^{1} r^2.rdr\, d\emptyset = \frac{3\pi}{2}$

L.H.S=R.H.S.Hence the theorem is verified.

6: If $\bar{F} = y\bar{i} + (x - 2xz)\bar{j} - xy\bar{k}$, evaluate $\int_s (\nabla \times \bar{F}).\bar{n}ds$. Where S is the surface of sphere $x^2 + y^2 + z^2 = a^2, above\ the\ xy - plane.$

Solution: Given $\bar{F} = y\bar{\imath} + (x - 2xz)\bar{\jmath} - xy\bar{k}.$

By Stokes Theorem,

$\int_s (\nabla \times \bar{F}).\bar{n}ds = \int_c \bar{F}.d\bar{r} = \int_c F_1 dx + F_2 dy + F_3 dz = \int_c ydx + (x - 2xz)dy - xydz$

Above the xy plane the sphere is $x^2 + y^2 += a^2, z = 0$

$\therefore \int_c \bar{F}.d\bar{r} = \int_c ydx + xdy.$

Put x=a cos θ,y=asinθ $so\ that\ dx = -a\ sin\theta d\theta, dy = acos\theta d\theta\ and \theta = 0 \to 2\pi$

$\int_c \bar{F}.\, d\bar{r} = \int_0^{2\pi} (a\ sin\theta)(-a\ sin\theta)\, d\theta + (acos\theta)(acos\theta)d\theta$

$= a^2 \int_0^{2\pi} cos2\theta\ d\theta = a^2 \left[\frac{sin2\theta}{2}\right]_0^{2\pi} = \frac{a^2}{2}(0) = 0$

7: Verify Stokes theorem for $\bar{F} = (2x - y)\bar{\imath} - \dot{y}z^2\bar{\jmath} - y^2z\bar{k}$ over the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ bounded by the projection of the xy-plane.

Solution: The boundary C of S is a circle in xy plane i.e $x^2 + y^2$=1, z=0

The parametric equations are x=$cos\theta, y = sin\theta, \theta = 0 \to 2\pi$

$\therefore\ dx = -sin\theta\ d\theta, dy = cos\theta\ d\theta$

$\int_c \bar{F}.d\bar{r} = \int_c \bar{F_1}dx + \bar{F_2}dy + \bar{F_3}dz = \int_c (2x - y)dx - yz^2dy - y^2zdz$

$= \int_c (2x - y)dx (since\ z = 0\ and\ dz = 0)$

$$= -\int_0^{2\pi} (2\cos\theta - \sin\theta)\sin\theta\, d\theta = \int_0^{2\pi} \sin^2\theta\, d\theta - \int_0^{2\pi} \sin 2\theta\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{1-\cos 2\theta}{2} d\theta - \int_0^{2\pi} \sin 2\theta\ d\theta = \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{2}.\cos 2\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}(2\pi - 0) + 0 + \frac{1}{2}.(\cos 4\pi - \cos 0) = \pi$$

Again $\nabla \times \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = \bar{\imath}(-2yz + 2yz) - \bar{\jmath}(0-0) + \bar{k}(0+1) = \bar{k}$

$\therefore \int_S (\nabla \times \bar{F}).\bar{n}ds = \int_S \bar{k}.\bar{n}ds = \int_R \int dxdy$

Where R is the projection of S on xy plane and $\bar{k}.\bar{n}ds = dxdy$

Now $\int\int_R dxdy = 4\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} dydx = 4\int_{x=0}^{1}\sqrt{1-x^2}\, dx = 4\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1$

$= 4\left[\frac{1}{2}\sin^{-1}1\right] = 2\frac{\pi}{2} = \pi$

$\therefore$ *The* Stokes theorem is verified.

8: Verify Stokes theorem for the function $\bar{F} = x^2\,\bar{\imath} + xy\,\bar{\jmath}$ integrated round the square in the plan z=0 whose sides are along the lines x=0, y=0, x=a, y=a.

Solution: Given $\bar{F} = x^2\,\bar{\imath} + xy\,\bar{\jmath}$

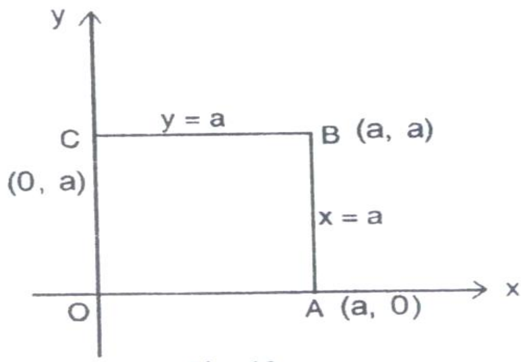


Fig. 13

By Stokes Theorem, $\int_S (\nabla \times \bar{F}).\bar{n}ds = \int_c \bar{F}.d\bar{r}$

Now $\nabla \times \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix} = \bar{k}$y

L.H.S.= $\int_S (\nabla \times \bar{F}).\bar{n}ds = \int_s y(\bar{n}.\bar{k})ds = \int_s ydxdy$

$\therefore \bar{n}.\bar{k}.ds = dxdy$ and R is the region bounded for the square.

$\therefore \int_S (\nabla \times \bar{F}).\bar{n}ds = \int_0^a \int_0^a ydydx = \frac{a^3}{2}$

R.H.S. = $\int_C \bar{F}.d\bar{r} = \int_C (x^2dx + xydy)$

But $\int \bar{F}.d\bar{r} = \int_{OA} \bar{F}.d\bar{r} + \int_{AB} \bar{F}.d\bar{r} + \int_{BC} \bar{F}.d\bar{r} + \int_{CO} \bar{F}.d\bar{r}$

(i)Along OA: y=0, z=0, dy=0, dz=0

$\therefore \int_{OA} \bar{F}.d\bar{r} = \int_0^a x^2dx = \frac{a^3}{3}$

(ii)Along AB:x=a, z=0,dx=0,dz=0

$$\int_{AB} \bar{F}.d\bar{r} = \int_0^a aydy = \frac{1}{2}a^3$$

(iii)Along BC: y=a,z=0,dy=0,dz=0

$\therefore \int_{BC} \bar{F}.d\bar{r} = \int_a^0 0dx = \frac{1}{3}a^3$

(iv)Along CO: x=0, z=0, dx=0, dz=0

$\therefore \int_{CO} \bar{F}.d\bar{r} = \int_a^0 0dy = 0$

Adding $\int_c \bar{F}.d\bar{r} = \frac{1}{3}a^3 + \frac{1}{2}a^3 + \frac{1}{3}a^3 + 0 = \frac{1}{2}a^3$

Hence the verification.

9: Apply Stokes theorem, to evaluate $\oint_c (ydx + zdy + xdz)$ where c is the curve of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and x+z=a.

Solution : The intersection of the sphere $x^2 + y^2 + z^2 = a^2$ *and* the plane x+z=a. is a circle in the plane x+z=a. with AB as diameter.

Equation of the plane is x+z=a$\Longrightarrow \frac{x}{a} + \frac{z}{a} = 1$

$\therefore OA = OB = a$ i.e., $A = (a,0,0)$ and B=(0,0,a)

$\therefore$ Length of the diameter AB= $\sqrt{a^2 + a^2 + 0}$ =a$\sqrt{2}$

Radius of the circle, r=$\frac{a}{\sqrt{2}}$

Let $\bar{F}.d\bar{r} = ydx + zdy + xdz \Longrightarrow \bar{F}.d\bar{r} = \bar{F}.(\bar{i}dx + \bar{j}dy + \bar{k}dz) = ydx + zdy + xdz$

$\Longrightarrow \bar{F} = y\,\bar{i} + z\,\bar{j} + x\,\bar{k}$

$\therefore curl\ \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -(\bar{i} + \bar{j} + \bar{k})$

Let $\bar{n}$ be the unit normal to this surface. $\bar{n} = \frac{\nabla s}{|\nabla s|}$

Then s=x+z-a, $\nabla s = \bar{i} + \bar{k}$ $\therefore$ $\bar{n} = \frac{\nabla s}{|\nabla s|} = \frac{\bar{i}+\bar{k}}{\sqrt{2}}$

Hence $\oint_c \bar{F}.d\bar{r} = \int curl\ \bar{F}.\bar{n}\, ds$ $(by\ Stokes\ Theorem)$

$$=-\int(\bar{i}+\bar{j}+\bar{k}).\left(\frac{\bar{i}+\bar{k}}{\sqrt{2}}\right)ds = -\int\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)ds$$

$$=-\sqrt{2}\int_S ds = -\sqrt{2}S = -\sqrt{2}\left(\frac{\pi a^2}{2}\right) = \frac{\pi a^2}{\sqrt{2}}$$

10: Apply the Stoke's theorem and show that $\int_S\int curl\ \bar{F}.\bar{n}d\bar{s} = 0\ where\ \bar{F}$ is any vector and S = $x^2+y^2+z^2=1$

Solution: Cut the surface if the Sphere $x^2+y^2+z^2=1\ by\ any\ plane,$ Let $S_1\ and\ S_2$ denotes its upper and lower portions a C, be the common curve bounding both these portions.

$$\therefore \int_s curl\bar{F}.d\bar{s} = \int_{s_1}\bar{F}.d\bar{s} + \int_{s_2}\bar{F}.d\bar{s}$$

Applying Stoke's theorem,

$$\int_s curl\bar{F}.d\bar{s} = \int_{s_1}\bar{F}.d\bar{R} + \int_{s_2}\bar{F}.d\bar{R} = 0$$

The 2nd integral curl $\bar{F}.d\bar{s}$ is negative because it is traversed in opposite direction to first integral.

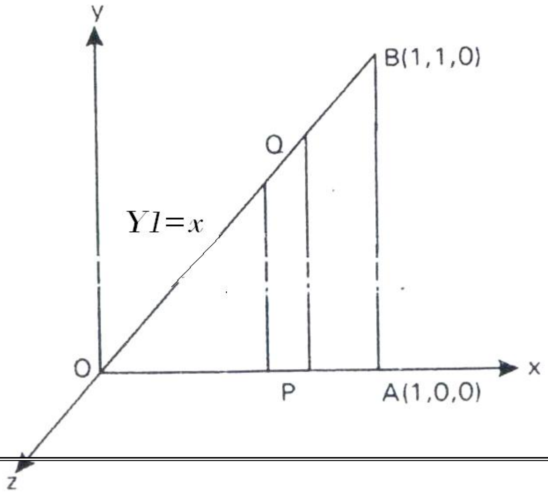
The above result is true for any closed surface S.

11: Evaluate by Stokes theorem $\oint_c (x+y)dx + (2x-z)dy + (y+z)dz$ where C is the boundary of the triangle with vertices (0,0,0), (1,0,0) and (1,1,0).

Solution: Let $\bar{F}.d\bar{r} = \bar{F}.(\bar{i}dx + \bar{j}dy + \bar{k}dz) = (x+y)dx + (2x-z)dy + (y+z)dz$

Then $\bar{F} = (x+y)\bar{i} + (2x-z)\bar{j} + (y+z)\bar{k}$

By Stokes theorem, $\oint_C \bar{F}.d\bar{r} = \int\int_S curl\ \bar{F}.\bar{n}\, ds$



Where S is the surface of the triangle OAB which lies

in the xy plane. Since the z Co-ordinates of O,A and B

Are zero. Therefore $\bar{n} = \bar{k}$. Equation of OA is y=0 and

that of OB, y=x in the xy plane.

$$\therefore\ curl\ \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = 2\bar{i} + \bar{k}$$

$\therefore\ curl\ \bar{F}.\bar{n}$ds=curl $\bar{F}.\bar{K}\ dx\ dy = dx\ dy$

$$\therefore \oint_c \bar{F}.d\bar{r} = \int\int_s dx\ dy = \int\int_s dA = A = area\ of \text{ the } \Delta\ OAB$$

$$= \frac{1}{2} OA \times AB = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$$

12: Use Stoke's theorem to evaluate $\int\int_S curl\ \bar{F}.\bar{n}\ dS$ over the surface of the paraboloid $z + x^2 + y^2 = 1, z \geq 0$ where $\bar{F} = y\,\bar{i} + z\,\bar{j} + x\,\bar{k}$.

Solution : By Stoke's theorem

$$\int_s curl\bar{F}.d\bar{s} = \oint_c \bar{F}.d\bar{r} = \int_c (y\bar{i} + z\bar{j} + x\bar{k}).(\bar{i}dx + \bar{j}dy + \bar{k}dz)$$

$$= \int_c y\,dx \text{ (Since z=0,dz=0) ......(1)}$$

Where C is the circle $x^2 + y^2 = 1$

The parametric equations of the circle are x=$cos\theta, y = sin\theta$

$\therefore dx = -\ sin\theta\ d\theta$

Hence (1) becomes

$$\int_s curl\bar{F}.ds = \int_{\theta=0}^{2\pi} \sin\theta(-\sin\theta)d\theta = -\int_{\theta=0}^{2\pi} \sin^2\theta d\theta = -4\int_0^{\frac{\pi}{2}} \sin^2\theta d\theta = -4 \times \frac{1}{2} \times \frac{\pi}{2} = -\pi$$

13: Verify Stoke's theorem for $\bar{F} = (x^2 + y^2)\bar{i} - 2xy\bar{j}$ taken round the rectangle bounded by the lines x= $\pm a, y = 0, y = b$.

Solution: Let ABCD be the rectangle whose vertices are (a,0), (a,b), (-a,b) and (-a,0).

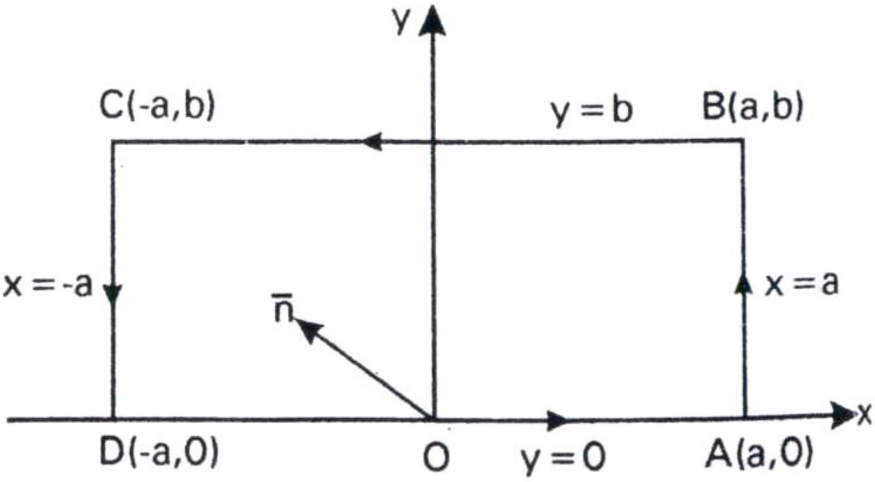
Equations of AB, BC, CD and DA are x=a, y=b, x=-a and y=0.

We have to prove that $\oint_c \bar{F}.d\bar{r} = \int_S curl\ \bar{F}.\bar{n}ds$

$\oint_c \bar{F}.d\bar{r} = \oint_c \{(x^2+y^2)\bar{\imath} - 2xy\bar{\jmath}\}.\{\bar{\imath}dx + \bar{\jmath}dy\}$

$= \oint_c (x^2+y^2)\, dx - 2xydy$

$= \int_{AB} + \int_{Bc} + \int_{CD} + \int_{DA}$ .....(1)



(i) Along AB, x=a, dx=0

from (1), $\int_{AB} = \int_{y=0}^{b} -2ay\ dy = -2a\left[\frac{y^2}{2}\right]_0^b = -ab^2$

(ii)Along BC, y=b, dy=0

from (1), $\int_{BC} = \int_{x=a}^{x=-a} (x^2+b^2)dx = \left[\frac{x^3}{3} + b^2x\right]_{x=a}^{-a} = \frac{-2a^3}{3} - 2ab^2$

(iii) Along CD, x=-a, dx=0

from (1), $\int_{CD} = \int_{y=b}^{0} 2aydy = \cancel{2}a\left[\frac{y^2}{\cancel{2}}\right]_{y=b}^{0} = -ab^2$

(iv)Along DA, y=0, dy=0

from (1), $\int_{DA} = \int_{x=-a}^{x=a} x^2dx = \left[\frac{x^3}{3}\right]_{x=-a}^{a} = \frac{2a^3}{3}$

(i)+(ii)+(iii)+(iv) gives

$\therefore \oint_c \bar{F}.d\bar{r} = -ab^2 - \frac{-2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3} = -4ab^2$ ....(2)

Consider $\int_S curl\ \bar{F}.\bar{n}\,dS$

Vector Perpendicular to the xy-plane is $\bar{n} = k$

$\therefore\ curl\ \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2+y^2) & -2xy & 0 \end{vmatrix} = 4y\bar{k}$

Since the rectangle lies in the xy plane,

$\bar{n} = \bar{k}$ and ds =dx dy

$\int_S curl\ \bar{F}.\bar{n}dS = \int_S -4y\bar{k}.\bar{k}dx\ dy = \int_{x=-a}^{a}\int_{y=0}^{b} -4y\ dx\ dy$

$=\int_{y=0}^{b}\int_{x=-a}^{a} -4y\ dx\ dy = 4\int_{y=0}^{b} y[x]_{-a}^{a}\ dy = -4\int_{y=0}^{b} 2ay dy$

$=-4a[y^2]_{y=0}^{b} = -4ab^2$ .....(3)

Hence from (2) and (3), the Stoke's theorem is verified.

14: Verify Stoke's theorem for $\bar{F} = (y - z + 2)\bar{\imath} + (yz + 4)\bar{\jmath} - xz\bar{k}$ where S is the surface of the cube x =0, y=0, z=0, x=2, y=2,z=2 above the xy plane.

Solution: Given $\bar{F} = (y - z + 2)\bar{\imath} + (yz + 4)\bar{\jmath} - xz\bar{k}$ where S is the surface of the cube.

x=0, y=0, z=0, x=2, y=2, z=2 above the xy plane.

By Stoke's theorem, we have $\int curl\ \bar{F}.\bar{n}ds = \int \bar{F}.d\bar{r}$

$\nabla \times \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z + 2 & y + 4 & -xz \end{vmatrix} = \bar{\imath}(0 + y) - \bar{\jmath}(-z + 1) + \bar{k}(0 - 1) = y\bar{\imath} - (1 - z)\bar{\jmath} - \bar{k}$

$\therefore\ \nabla \times \bar{F}.\bar{n} = \nabla \times \bar{F}.k = (y\bar{\imath} - (1 - z)\bar{\jmath} - \bar{k}).k = -1$

$\therefore \int \nabla \times \bar{F}.\bar{n}.ds = \int_0^2\int_0^2 -1\ dx\ dy\ \ (\because z = 0, dz = 0) = -4$ .....(1)

To find $\int \bar{F}.d\bar{r}$

$\int \bar{F}.d\bar{r} = \int \left((y - z + 2)\bar{\imath} + (yz + 4)\bar{\jmath} - xz\bar{k}\right).(dx\bar{\imath} + dy\bar{\jmath} + dz\bar{k})$

$= \int[(y - z + 2)dx + (yz + 4)dy - (xz)dz]$

Sis the surface of the cube above the xy-plane

$\therefore z = 0 \Rightarrow dz = 0$

$\therefore \int \bar{F}.d\bar{r} = \int(y + 2)dx + \int 4dy$

Along $\overline{OA}, y = 0, z = 0, dy = 0, dz = 0, x\ change\ from\ 0\ to\ 2.$

$\int_0^2 2dx = 2[x]_0^2 = 4$ ....(2)

Along $\overline{BC}, y = 2, z = 0, dy = 0, dz = 0, x\ change\ from\ 2\ to\ 0.$

$\int_2^0 4dx = 4[x]_2^0 = -8$ ....(3)

Along $\overline{AB}, x = 2, z = 0, dx = 0, dz = 0, y\ change\ from\ 0\ to\ 2.$

$\int \bar{F}.d\bar{r} = \int_0^2 4dy = [4y]_0^2 = 8$ ......(4)

Along $\overline{CO}, x = 0, z = 0, dx = 0, dz = 0, y\ change\ from\ 2\ to\ 0.$

$.\int_2^0 4dy = -8$ .....(5)

Above the surface When z=2

Along $0'A'$, $\int_0^2 \bar{F}\,\overline{dr} = 0$ ....(6)

Along $A'B', x = 2, z = 2, dx = 0, dz = 0,$ y changes from 0 to 2

$$\int_0^2 \bar{F}.d\bar{r} = \int_0^2 (2y+4)dy = 2\left[\frac{y^2}{2}\right]_0^2 + 4[y]_0^2 = 4 + 8 = 12 \qquad ....(7)$$

Along $B'C', y = 2, z = 2, dy = 0, dz = 0,$ x changes from 2 to 0

$\int_0^2 \bar{F}.\overline{dr} = 0$ ....(8)

Along $C'D', x = 0, z = 2, dx = 0, dz = 0,$ y changes from 2 to 0.

$$\int_2^0 (2y+4) = 2\left[\frac{y^2}{2}\right]_2^0 + 4[y]_2^0 = -12 \qquad .....(9)$$

(2)+(3)+(4)+(5)+(6)+(7)+(8)+(9) gives

$$\int_C \bar{F}.d\bar{r} = 4 - 8 + 8 - 8 + 0 + 12 + 0 - 12 = -4 \qquad .....(10)$$

By Stokes theorem, We have

$\int \bar{F}.d\bar{r}$=$\int curl\ \bar{F}.\bar{n}$ds=-4

Hence Stoke's theorem is verified.

15: Verify the Stoke's theorem for $\bar{F} = y\bar{\imath} + z\bar{\jmath} + x\bar{k}$ and surface is the part of the sphere $x^2 + y^2 + z^2 = 1$ *above the xy plane.*

Solution: Given $\bar{F} = y\bar{\imath} + z\bar{\jmath} + x\bar{k}$ over the surface $x^2 + y^2 + z^2 = 1$ *is xy plane.*

We have to prove $\int_C \bar{F}.d\bar{r}$=$\int\int_s Curl\ \bar{F}.\bar{n}ds$

$\bar{F}.d\bar{r}$ =. $(y\bar{\imath} + z\bar{\jmath} + x\bar{k}).(dx\bar{\imath} + dy\bar{\jmath} + dz\bar{k})$=ydx + zdy + xdz

$\int_C(ydx + zdy + xdz) = \int ydx \quad (in\ xy\ plane\ z = 0, dz = 0)$

Let x= $\cos\theta, y = \sin\theta \implies dx = -\sin\theta\ d\theta, dy = \cos\theta\ d\theta$

$\therefore \int_C \bar{F}.d\bar{r} = \int_C y.dx = \int_0^{2\pi} ydx \qquad [\because x^2 + y^2 = 1, z = 0]$

$$=\int_0^{2\pi} \sin\theta\,(-\sin\theta)d\theta = -4\int_0^{\pi/2} \sin^2\theta\, d\theta$$

$$=-4\int_0^{\pi/2} \frac{1-\cos 2\theta}{2} d\theta = -4\left[\left(\frac{1}{2}.\frac{\pi}{2}\right) - \frac{1}{4}(\sin\pi)\right]$$

$$=-4\left[\left(\frac{1}{2}.\frac{\pi}{2}\right) - 0\right] = -4\left[\frac{\pi}{4}\right] = -\pi \qquad .....(1)$$

$$Curl\bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = -(\bar{\imath} + \bar{\jmath} + \bar{k})$$

Unit normal vector $\bar{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\bar{\imath}+2y\bar{\jmath}+2z\bar{k}}{\sqrt{4x^2+4y^2+4z^2}} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$

Substituting the spherical polar coordinates, we get

$\bar{n} = \sin\theta\cos\phi\,\bar{\imath} + \sin\theta\sin\phi\,\bar{\jmath} + \cos\theta\,\bar{k}$

$\therefore Curl\,\bar{F}.\bar{n} = -(\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta)$

$$\iint curl\,\bar{F}.\bar{n}ds = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}(\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta)\sin\theta\, d\theta\, d\phi$$

$$= -\int_0^{\pi/2}[\sin\theta\sin\phi - \sin\theta\cos\phi + \phi\cos\theta]_0^{2\pi}\ \sin\theta d\theta$$

$$= -2\pi\int_0^{\pi/2}\cos\theta\sin\theta d\theta = -\pi\int_0^{\pi/2}\sin 2\theta d\theta = (-\pi)\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{\pi}{2}(-1-1) = -\pi \quad .....(2)$$

From (1) and (2), we have

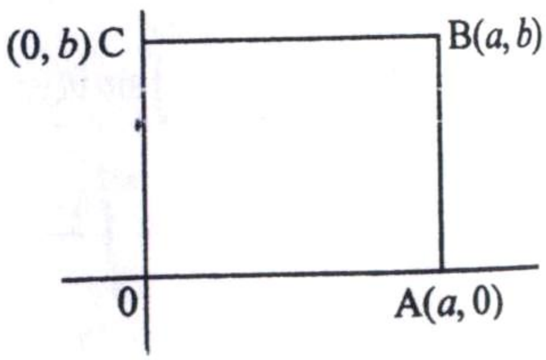
$$\int_C \bar{F}.d\bar{r} = \int\int_s Curl\,\bar{F}.\bar{n}ds = -\pi$$

∴ $Stoke$'s theorem is verified.

16: Verify Stoke's theorem for $\bar{F} = (x^2 - y^2)\bar{i} + 2xy\,\bar{j}$ over the box bounded by the planes

x=0,x=a,y=0,y=b.

Solution :



Stoke"s theorem states that $\int_c \bar{F}.d\bar{r} = \int_s Curl\,\bar{F}.\bar{n}ds$

Given $\bar{F} = (x^2 - y^2)\bar{\imath} + 2xy\bar{\jmath}$

$$\text{Curl}\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = \bar{i}(0,0) - \bar{j}(0,0) + \bar{k}(2y+2y) = 4y\bar{k}$$

R.H.S= $\int_s Curl\,\bar{F}.\bar{n}ds = \int_s 4y(\bar{k}.\bar{n})ds$

Let R be the region bounded by the rectangle

$(\bar{k}.\bar{n})ds = dx\, dy$

$$\int_s Curl\,\bar{F}.\bar{n}ds = \int_{x=0}^{a}\int_{y=0}^{b} 4y\,dx\,dy = \int_{x=0}^{a}\left[4\frac{y^2}{2}\right]_0^b dx = 2b^2\int_{x=0}^{a} 1dx$$

$$=2b^2(x)_0^a = 2ab^2$$

To Calculate L.H.S

$\bar{F}.d\bar{r} = (x^2 - y^2)dx + 2xy\, dy$

Let O=$(0,0), A = (a,0), B = (a,b)$ and

C=(0,b) are the vertices of the rectangle.

(i)Along the line OA

y=0; dy=0, x ranges from 0 to a.

$$\int_{OA} \bar{F}.d\bar{r} = \int_{x=0}^{a} x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

(ii)Along the line AB

x=a; dx=0, y ranges from 0 to b.

$$\int_{AB} \bar{F}.d\bar{r} = \int_{y=0}^{b}(2xy)\,dy = \left[2a\frac{y^3}{2}\right]_0^b = ab^2$$

(iii)Along the line BC

y=b; dy=0, x ranges from a to 0

$$\int_{BC} \bar{F}.d\bar{r} = \int_{x=a}^{0}(x^2 - y^2)dx = \left[\frac{x^3}{3} - b^2x\right]_a^0 = 0 - \left(\frac{a^3}{3} - b^2a\right)$$

$$= ab^2 - \frac{a^3}{3}$$

(iv) Along the line CO

x=0,dx=0,y changes from b to 0

$$\int_C \bar{F}.d\bar{r} = \int_{y=b}^{0} 2xy\,dy = 0$$

Adding these four values

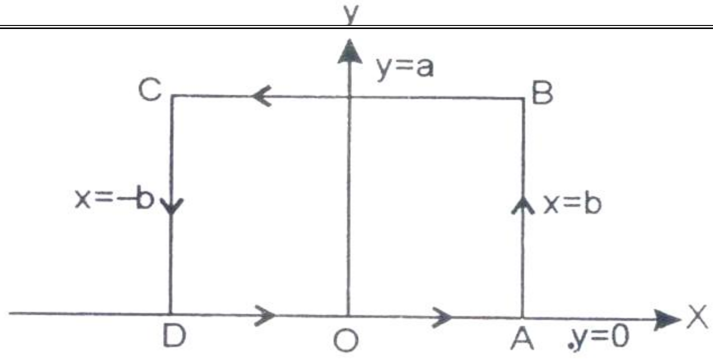
$$\int_{co} \bar{F}.d\bar{r} = \frac{a3}{3} + ab^2 + ab^2 - \frac{a3}{3} = 2ab^2$$

L.H.S = R.H.S

Hence the verification of the stoke's theorem.

17: Verify Stoke's theorem for $\bar{F}=y^2\,\bar{i} - 2xy\bar{j}$ taken round the rectangle bounded by x=$\pm b$, y=0,y=a.

Solution:



Curl $\bar{A}$ = $\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & -2xy & 0 \end{vmatrix}$ = -4y$\bar{k}$

For the given surface S, $\bar{n}=\bar{k}$

$\therefore (Curl\ \bar{F}).\bar{n} = -4y$

Now $\iint_s (Curl\ \bar{F}).\bar{n}dS$ = $\iint_s -4ydxdy$

$$= \int_{y=0}^{a}\left[\int_{x=-b}^{b} -4ydx\right]dy$$

$$= \int_0^a [-4xy]_{-b}^{b}\ dy$$

$$= \int_0^a -8bydy = \left[-4by^2\right]_0^a = -4a^2b........(1)$$

$\int_c \bar{F}.d\bar{r} = \int_{DA} + \int_{AB} + \int_{BC} + \int_{CD}$

$\int \bar{F}.d\bar{r} = y^2\,dx - 2xydy$

Along DA , y=0,dy=0 $\Rightarrow \int_{DA} \bar{F}.d\bar{r}$ =0 ($\because \overline{F}.dr = 0$)

Along AB, x=b,dx=0

$\int_{AB} \bar{F}.d\bar{r} = \int_{y=0}^{a} -2bydy = \left[-by^2\right]_0^a = -a^2b$

Along BC,y=a,dy=0

$\int_{BC} \bar{F}.d\bar{r} = \int_b^{-b} a^2\,dx = -2a^2b$

Along CD, x=-b,dx=0

$\int_{CD} \bar{F}.d\bar{r} = \int_a^0 2bydy = \left[-by^2\right]_a^0 = -a^2b$.

$\int_C \bar{F}.d\bar{r} = 0 - a^2b - 2a^2b - a^2b = -4a^2b$ -------(2)

From (1),(2) $\int_C \bar{F}.d\bar{r} = \iint_s (Curl\ \bar{F}).\bar{n}dS$

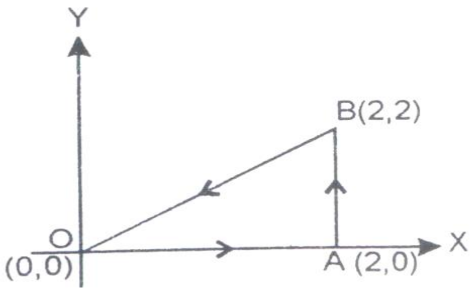
Hence the theorem is verified.

19: Using Stroke's theorem evaluate the integral $\int_C \bar{F}.d\bar{r}$ where

$\bar{F}$=2y^2 $\bar{\imath}$+3x^2 $\bar{\jmath}$ -(2x+z)$\bar{k}$ and C is the boundary of the triangle whose vertices are (0,0,0),(2,0,0),(2,2,0).

Solution:

$$\text{Curl } \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y^2 & 3x^2 & -2x - z \end{vmatrix} = 2\bar{\jmath} + (6x-4y)\,\bar{k}$$



Since the z-coordinate of each vertex of the triangle is zero , the triangle lies in the xy-plane .

$\therefore$ $\bar{n}$ =k

$\therefore$ (Curl $\bar{F}$).$\bar{n}$= 6x-4y

Consider the triangle in xy-plane .

Equation of the straight line OB is y=x.

By Stroke's theorem

$$\int_c \bar{F}.d\bar{r} = \int\int_s (curl\,\bar{F}).\bar{n}ds$$

$$= \int_{x=0}^{2}\int_{y=0}^{y=x}(6x-4y)dxdy \quad = \int_{x=0}^{2}\left[\int_{y=0}^{x}(6x-4y)dy\right]dx$$

$$= \int_{x=0}^{2}\left[6xy - 2y^2\right]_0^x dx = \int_0^2(6x^2-2x^2)dx$$

$$= 4\left[\frac{x^3}{3}\right]_0^2 = \frac{32}{3}$$