MATHEMATICAL TRANSFORM TECHNIQUES

Course code:AHSB11

I. B.Tech II semester

Regulation: IARE R-18)

BY

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CO’s	Course outcomes
CO1	Analyzing real roots of algebraic and transcendental equations by Bisection method, False position and Newton -Raphson method. Applying Laplace transform and evaluating given functions using shifting theorems, derivatives, multiplications of a variable and periodic function.
CO2	Understanding symbolic relationship between operators using finite differences. Applyiing Newton’s forward, Backward, Gauss forward and backward for equal intervals and Lagrange’s method for unequal interval to obtain the unknown value. Evaluating inverse Laplace transform using derivatives, integrals, convolution method. Finding solution to linear differential equation .

COs	Course Outcome
CO3	Applying linear and nonlinear curves by method of least squares. Understanding Fourier integral, Fourier transform, sine and cosine Fourier transforms, finite and infinite and inverse of above said transforms.
CO4	Using Numericals methods such as Taylors, Eulers, Modified Eulers and Runge-Kutta methods to solve ordinary differential equations.
CO5	Analyzing order and degree of partial differential equation, formation of PDE by eliminating arbitrary constants and functions, evaluating linear equation b Lagrange’s method. Applying the heat equation and wave equation in subject to boundary conditions.

MODULE– I
ROOTS FINDING TECHNIQUES AND LAPLACE TRANSFORMS

CLOs	Course Learning Outcome
CLO1	Evaluate the real roots of algebraic and transcendental equations by Bisection method, False position and Newton -Raphson method
CLO2	Apply the nature of properties to Laplace transform of the given function.
CLO3	Solving Laplace transforms of a given function using shifting theorems.
CLO4	Evaluate Laplace transforms using derivatives and integrals of a given function.

CLOs	Course Learning Outcome
CLO5	Evaluate Laplace transforms using multiplication and division of a variable to a given function
CLO6	Apply Laplace transforms to periodic functions

PROBLEMS

1). Find a root of the equation $x^3 - 5x + 1 = 0$ using the bisection method in 5 – stages

Sol Let $f(x) = x^3 - 5x + 1$. We note that $f(0) > 0$ and $f(1) < 0$

$\therefore$ One root lies between 0 and 1

Consider $x_0 = 0$ *and* $x_1 = 1$

By Bisection method the next approximation is

$$x_2 = \frac{x_0 + x_1}{2} = \frac{1}{2}(0+1) = 0.5$$

$$\Rightarrow f(x_2) = f(0:5) = -1.375 < 0 \text{ and } f(0) > 0$$

We have the root lies between 0 and 0.5

Now $x_3 = \frac{0+0.5}{2} = 0.25$

We find $f(x_3) = -0.234375 < 0 \quad and \quad f(0) > 0$

Since $f(0) > 0$, we conclude that root lies between $x_0 \; and \; x_3$

The third approximation of the root is

$$x_4 = \frac{x_0 + x_3}{2} = \frac{1}{2}(0 + 0.25) = 0.125$$

We have $f(x_4) = 0.37495 > 0$

Since $f(x_4) > 0 \; and \; f(x_3) < 0$, the root lies between

$x_4 = 0.125 \; and \; x_3 = 0.25$

Considering the 4th approximation of the roots

$$x_5 = \frac{x_3 + x_4}{2} = \frac{1}{2}(0.125 + 0.25) = 0.1875$$

$f(x_5) = 0.06910 > 0$, since $f(x_5) > 0$ *and* $f(x_3) < 0$ the root must lie between

$x_5 = 0.18758$ *and* $x_3 = 0.25$

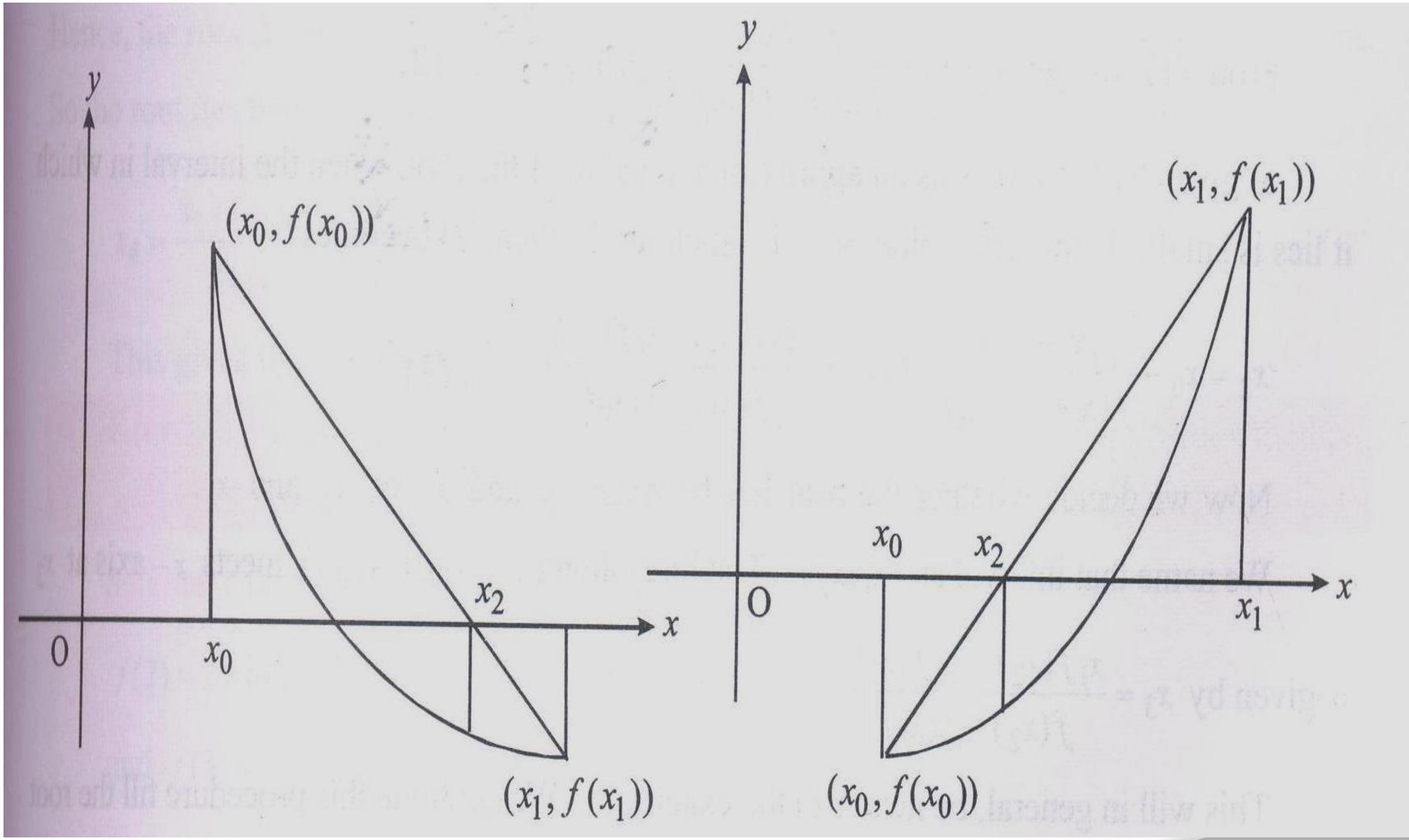
Here the fifth approximation of the root is

$$\begin{aligned} x_6 &= \frac{1}{2}(x_5 + x_3) \\ &= \frac{1}{2}(0.1875 + 0.25) \\ &= 0.21875 \end{aligned}$$

We are asked to do up to 5 stages

We stop here 0.21875 is taken as an approximate value of the root and it lies between 0 and 1

REGULAR-FLASE POSITION METHOD



1. **By using Regula - Falsi method, find an approximate root of the equation $x^4 - x - 10 = 0$ that lies between 1.8 and 2. Carry out three approximations**

Sol. Let us take $f(x) = x^4 - x - 10$ and $x_0 = 1.8, x_1 = 2$

Then $f(x_0) = f(1.8) = -1.3 < 0$ and $f(x_1) = f(2) = 4 > 0$

Since $f(x_0)$ and $f(x_1)$ are of opposite signs, the equation $f(x) = 0$ has a

root between x_0 *and* x_1

The first order approximation of this root is

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 1.8 - \frac{2 - 1.8}{4 + 1.3} \times (-1.3)$$

$$= 1.849$$

We find that $f(x_2) = -0.161$ so that $f(x_2)\ and\ f(x_1)$ are of opposite signs. Hence the root lies between $x_2\ and\ x_1$ and the second order approximation of the root is

$$x_3 = x_2 - \left[\frac{x_1 - x_2}{f(x_1) - f(x_2)}\right].f(x_2)$$

$$= 1.8490 - \left[\frac{2 - 1.849}{0.159}\right] \times (-0.159)$$

$$= 1.8548$$

we find that $f(x_3) = f(1.8548)$

$$= -0.019$$

REGULAR-FLASE POSITION METHOD

So that $f(x_3)$ *and* $f(x_2)$ are of the same sign. Hence, the root does not lie between x_2 *and* x_3. But $f(x_3)$ *and* $f(x_1)$ are of opposite signs. So the root lies between x_3 *and* x_1 and the third order approximate value of the root is

$$x_4 = x_3 - \left[\frac{x_1 - x_3}{f(x_1) - f(x_3)}\right] f(x_3)$$

$$= 1.8548 - \frac{2 - 1.8548}{4 + 0.019} \times (-0.019)$$

$$= 1.8557$$

This gives the approximate value of x.

NEWTON-RAPHSON METHOD

Let the given equation be $f(x)=0$

Find $f^1(x)$ and initial approximation x_0

The first approximation is $x_1 = x_0 - f(x_0)/ f^1(x_0)$

The second approximation is $x_2 = x_1 - f(x_1)/ f^1(x_1)$

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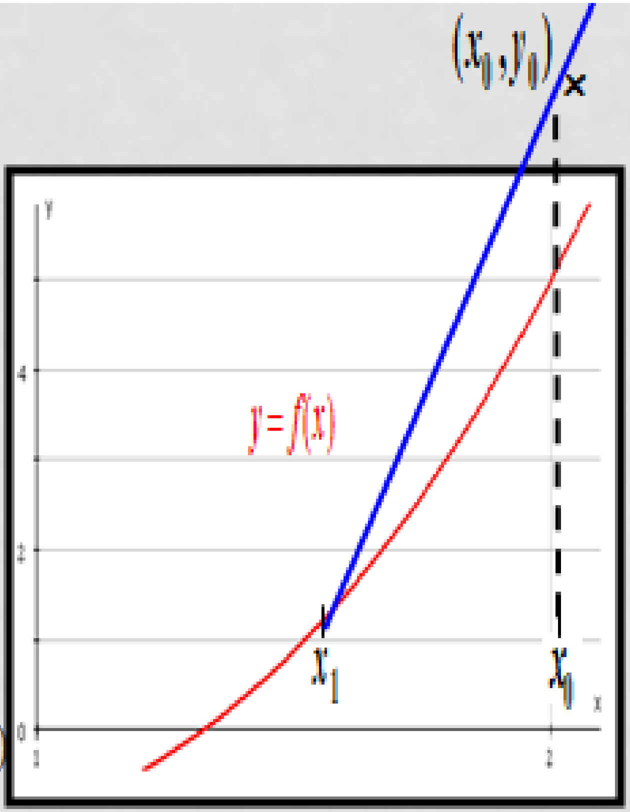
The n^{th} approximation is $x_n = x_{n-1} - f(x_n)/ f^1(x_n)$



1. Apply Newton – Raphson method to find an approximate root, correct to three decimal places, of the equation $x^3 - 3x - 5 = 0$, which lies near $x = 2$

Sol:- Here $f(x) = x^3 - 3x - 5 = 0$ *and* $f^1(x) = 3(x^2 - 1)$

$\therefore$ The Newton – Raphson iterative formula

$$x_{i+1} = x_i - \frac{x_i^3 - 3x_i - 5}{3(x_i^2 - 1)} = \frac{2x_i^3 + 5}{3(x_i^2 - 1)}, i = 0, 1, 2 \ldots . (1)$$

To find the root near $x = 2$, we take $x_0 = 2$ then (1) gives

$$x_1 = \frac{2x_0^3 + 5}{3(x_0^2 - 1)} = \frac{16 + 5}{3(4 - 1)} = \frac{21}{9} = 2.3333$$

$$x_2 = \frac{2x_1^3 + 5}{3(x_1^2 - 1)} = \frac{2 \times (2.3333)^3 + 5}{3\left[(2.3333)^2 - 1\right]} = 2.2806$$

NEWTON-RAPHSON METHOD

$$x_3 = \frac{2x_2^3 + 5}{3(x_2^3 - 1)} = \frac{2 \times (2.2806)^3 + 5}{3[(2.2806)^2 - 1]} = 2.2790$$

$$x_4 = \frac{2 \times (2.2790)^3 + 5}{3[(2.2790)^2 - 1]} = 2.2790$$

Since x_3 and x_4 are identical up to 3 places of decimal, we take $x_4 = 2.279$ as the required root, correct to three places of the decimal

Let f(t) be a given function which is defined for all positive values of t, if

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

exists, then F(s) is called Laplace transform of f(t) and is denoted by

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

The inverse transform, or inverse of L{f(t)} or F(s), is

$$f(t) = L^{-1}\{F(s)\}$$

where s is real or complex value.

Laplace Transform of Basic Functions

1. $\mathcal{L}\ [1] = \int_0^\infty e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s}$

2. $\mathcal{L}\ [t^a] = \int_0^\infty t^a e^{-st} dt = \int_0^\infty (\frac{u}{s})^a e^{-u} \frac{du}{s} = \frac{1}{s^{a+1}} \int_0^\infty u^a e^{-u} du = \frac{\Gamma(a+1)}{s^{a+1}}$

3. $\mathcal{L}\ [e^{at}] = \int_0^\infty e^{at} e^{-st} dt = \frac{e^{-(s-a)t}}{-(s-a)} \Big|_0^\infty = \frac{1}{s-a}$

4. $\mathcal{L}\ [e^{iat}] = \frac{1}{s-ia} \Rightarrow \mathcal{L}\ [\cos at + i \sin at] = \frac{s}{s^2+a^2} + i \frac{a}{s^2+a^2}$

$\therefore \mathcal{L}\ [\cos at] = \frac{s}{s^2+a^2}$, and $\mathcal{L}\ [\sin at] = \frac{a}{s^2+a^2}$

5. $\mathcal{L}\ [\sinh at] = \mathcal{L}\ [\frac{e^{at} - e^{-at}}{2}] = \frac{1}{2}(\frac{1}{s-a} - \frac{1}{s+a}) = \frac{a}{s^2-a^2}$

$\mathcal{L}\ [\cosh at] = \mathcal{L}\ [\frac{e^{at} + e^{-at}}{2}] = \frac{1}{2}(\frac{1}{s-a} + \frac{1}{s+a}) = \frac{s}{s^2-a^2}$

EX: Find the Laplace transform of $\cos 2t$.

$$\text{Solution} : \because L\ [\cos t] = \frac{s}{s^2+1}$$

$$\therefore L\ [\cos 2t] = \frac{1}{2}\frac{\frac{s}{2}}{(\frac{s}{2})^2+1} = \frac{s}{s^2+4}$$

EX: Find the Laplace transform of te^t.

$$\text{Solution:}\quad L(e^t) = \frac{1}{s-1} \Rightarrow L(te^t) = -\frac{d}{ds}\left(\frac{1}{s-1}\right) = \frac{1}{(s-1)^2}$$

EX: $f(t)=\begin{cases} t^2, & 0\le t\le 1 \\ 0, & t>1 \end{cases}$, find $\mathcal{L}[f'(t)]$.

Solution : $f(t)=t^2[u(t)-u(t-1)]$

$$\mathcal{L}[f(t)]=\mathcal{L}[t^2u(t)]-\mathcal{L}[t^2u(t-1)]=\frac{2!}{s^3}-\mathcal{L}\{[(t-1)+1]^2u(t-1)\}$$

$$=\frac{2}{s^3}-\mathcal{L}\{[(t-1)^2+2(t-1)+1]u(t-1)\}$$

$$=\frac{2}{s^3}-e^{-s}(\frac{2}{s^3}+2\frac{1}{s^2}+\frac{1}{s})$$

$$\mathcal{L}[f'(t)]=sF(s)-f(0)-e^{-s}[f(1^+)-f(1^-)]$$

$$=[\frac{2}{s^2}-e^{-s}(\frac{2}{s^2}+\frac{2}{s}+1)]-0-e^{-s}(0-1)=\frac{2}{s^2}-e^{-s}(\frac{2}{s^2}+\frac{2}{s})$$

EX: Find $(a)\ L\ [\frac{1-e^{-t}}{t}]$ $(b)\ L\ [\frac{1-e^{-t}}{t^2}]$.

$$\text{Solution}: (a) L\ [1-e^{-t}] = \frac{1}{s} - \frac{1}{s+1}$$

$$L\ [\frac{1-e^{-t}}{t}] = \int_s^{\infty} (\frac{1}{s} - \frac{1}{s+1}) ds = \ln s - \ln(s+1)\Big|_s^{\infty} = \ln \frac{s}{s+1}\Big|_s^{\infty}$$

$$= 0 - \ln \frac{s}{s+1} = \ln \frac{s+1}{s}$$

$$(b) L\ [\frac{1-e^{-t}}{t^2}] = \int_s^{\infty} \ln \frac{s+1}{s} ds = s \ln \frac{s+1}{s}\Big|_s^{\infty} - \int_s^{\infty} s(\frac{1}{s+1} - \frac{1}{s}) ds$$

$$= s \ln \frac{s+1}{s}\Big|_s^{\infty} + \int_s^{\infty} \frac{1}{s+1} ds = \left[s \ln \frac{s+1}{s} + \ln(s+1) \right]_s^{\infty}$$

$$= \left[(s+1)\ln(s+1) - s \ln s \right]_s^{\infty} = s \ln s - (s+1)\ln(s+1)$$

EX: Find (a) $\int_0^\infty \frac{\sin kt\, e^{-st}}{t}\, dt$ (b) $\int_{-\infty}^\infty \frac{\sin x}{x}\, dx$.

$$\text{Solution : } (a) \int_0^\infty \frac{\sin kt e^{-st}}{t}\, dt = L\ [\frac{\sin kt}{t}]$$

$$\because L\ [\sin kt] = \frac{k}{s^2 + k^2}$$

$$L\ [\frac{\sin kt}{t}] = \int_s^\infty \frac{k}{s^2 + k^2}\, ds = \frac{1}{k} \int_s^\infty \frac{1}{(\frac{s}{k})^2 + 1}\, ds$$

$$= \tan^{-1} \frac{s}{k}\Big|_s^\infty = \frac{\pi}{2} - \tan^{-1} \frac{s}{k}$$

$$(b) \int_{-\infty}^\infty \frac{\sin x}{x}\, dx = 2\int_0^\infty \frac{\sin x}{x}\, dx$$

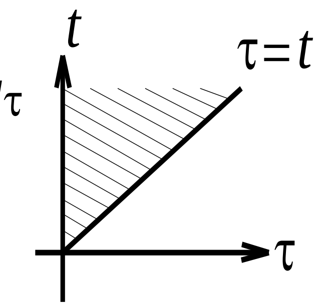
$$= 2 \lim_{\substack{k \to 1 \\ s \to 0}} \int_0^\infty \frac{\sin kte^{-st}}{t}\, dt$$

$$= 2 \lim_{\substack{k \to 1 \\ s \to 0}} (\frac{\pi}{2} - \tan^{-1} \frac{s}{k}) = \pi$$

Convolution theorem

$$\mathcal{L}\left[\int_0^t f(\tau)g(t-\tau)d\tau\right] = \int_0^\infty \int_0^t f(\tau)g(t-\tau)d\tau e^{-st}dt$$

$$= \int_0^\infty \int_\tau^\infty f(\tau)g(t-\tau)e^{-st}dtd\tau = \int_0^\infty f(\tau)\int_\tau^\infty g(t-\tau)e^{-st}dtd\tau$$



Let $u = t - \tau$, $du = dt$, then

$$\mathcal{L}\left[\int_0^t f(\tau)g(t-\tau)d\tau\right] = \int_0^\infty f(\tau)\int_0^\infty g(u)e^{-s(u+\tau)}dud\tau$$

$$= \int_0^\infty f(\tau)e^{-s\tau}d\tau\int_0^\infty g(u)e^{-su}du = F(s)G(s)$$

Find the Laplace transform of $\int_0^t e^{t-\tau} \sin 2\tau \, d\tau$.

Solution : $\because L\ [e^t] = \dfrac{1}{s-1}$, $L\ [\sin 2t] = \dfrac{2}{s^2+4}$

$$\therefore L\ [\int_0^t e^{t-\tau} \sin 2t \, d\tau] = L\ [e^t * \sin 2t] = L\ [e^t] \cdot L\ [\sin 2t]$$

$$= \frac{1}{s-1} \cdot \frac{2}{s^2+4} = \frac{2}{(s-1)(s^2+4)}$$

Periodic Function: *f* (*t* + *T*) =*f* (*t*)

$$L\ [f(t)] = \int_0^\infty f(t)e^{-st}dt = \int_0^T f(t)e^{-st}dt + \int_T^{2T} f(t)e^{-st}dt + \cdots\cdots$$

$$\text{and } \int_T^{2T} f(t)e^{-st}dt = \int_0^T f(u+T)e^{-s(u+T)}du = e^{-sT}\int_0^T f(u)e^{-su}du$$

Similarly,

$$\int_{2T}^{3T} f(t)e^{-st}dt = e^{-2sT}\int_0^T f(u)e^{-su}du$$

$$\therefore L\ [f(t)] = (1 + e^{-sT} + e^{-2sT} + \cdots\cdots)\int_0^T f(t)e^{-st}dt$$

$$= \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt$$

Find the Laplace transform of $f(t)=\frac{k}{p}t,\ 0<t<p,\ f(t+p)=f(t)$.

$$\text{Solution : L }[f(t)]=\frac{1}{1-e^{-ps}}\int_0^p \frac{k}{p}te^{-st}dt$$

$$=\frac{1}{1-e^{-ps}}\frac{k}{p}[\frac{1}{-s}(te^{-st}\Big|_0^p-\int_0^p e^{-st}dt)]$$

$$=\frac{-k}{ps(1-e^{-ps})}(te^{-st}+\frac{1}{s}e^{-st})\Big|_0^p$$

$$=\frac{-k}{ps(1-e^{-ps})}(pe^{-sp}+\frac{e^{-sp}}{s}-\frac{1}{s})$$

Initial Value Theorem:

$$\because L\ [f'(t)] = sF(s) - f(0) \Rightarrow \lim_{s\to\infty}\int_0^\infty f'(t)e^{-st}dt = \lim_{s\to\infty} sF(s) - f(0) \Rightarrow 0 = \lim_{s\to\infty} sF(s) - f(0)$$

we get initial value theorem $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Deduce general initial value theorem : $\lim_{t\to 0}\frac{f(t)}{g(t)} = \lim_{s\to\infty}\frac{F(s)}{G(s)}$

Final Value Theorem:

$$L\ [f'(t)] = sF(s) - f(0) \Rightarrow \lim_{s \to 0} \int_0^{\infty} f'(t) e^{-st} dt = \lim_{s \to 0} sF(s) - f(0) \Rightarrow$$

$$\lim_{t \to \infty} f(t) - f(0) = \lim_{s \to 0} sF(s) - f(0) \Rightarrow \text{final value theorem}: \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

$$\text{General final value theorem}: \lim_{t \to \infty} \frac{f(t)}{g(t)} = \lim_{s \to 0} \frac{F(s)}{G(s)}$$

Find $\mathcal{L}\left[\int_t^{\infty} \frac{e^{-x}}{x}\,dx\right]$.

Solution : Let $f(t) = \int_x^{\infty} \frac{e^{-x}}{x}\,dx \Rightarrow f'(t) = -\frac{e^{-t}}{t}, \lim_{t\to\infty} f(t) = 0$

$$\mathcal{L}\,[tf'(t)] = \mathcal{L}\,[-e^{-t}] = -\frac{1}{s+1}$$

$$-\frac{d}{ds}[sF(s) - f(0)] = -\frac{1}{s+1}$$

$$\frac{d}{ds}[sF(s)] = \frac{1}{s+1}$$

$$sF(s) = \ln(s+1) + C$$

From the final value theorem : $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$0 = 0 + C \Rightarrow C = 0, \text{ and } F(s) = \frac{\ln(s+1)}{s}$$

MODULE II
INTERPOLATION AND INVERSE LAPLACE TRANSFORMS

CLOs	Course Learning Outcome
CLO 7	Apply the symbolic relationship between the operators using finite differences.
CLO 8	Apply the Newtons forward and Backward, Gauss forward and backward Interpolation method to determine the desired values of the given data at equal intervals, also unequal intervals.
CLO 9	Solving inverse Laplace transform using derivatives and integrals.
CLO 10	Evaluate inverse Laplace transform by the method of convolution.

CLOs	Course Learning Outcome
CLO11	Solving the linear differential equations using Laplace transform.
CLO 12	Understand the concept of Laplace transforms to the real-world problems of electrical circuits, harmonic oscillators, optical devices, and mechanical systems

If we consider the statement $y = f(x) x_0 \le x \le x_n$ we understand that we can find the value of y, corresponding to every value of x in the range $x_0 \le x \le x_n$. If the function $f(x)$ is single valued and continuous and is known explicitly then the values of $f(x)$ for certain values of x like $x_0, x_1, ...x_n$ can be calculated. The problem now is if we are given the set of tabular values

Forward Differences:-

Consider a function $y=f(x)$ of an independent variable x. let $y_0, y_1, y_2, \ldots . y_r$ be the values of y corresponding to the values $x_0, x_1, x_2 \ldots . x_r$ of x respectively. Then the differences y_1-y_0, y_2-y_1----- are called the first forward differences of y, and we denote them by $\Delta y_0, \Delta y_1, \ldots \ldots$ that is

$$\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \Delta y_2 = y_3 - y_2 \ldots \ldots \ldots$$

In general $\Delta y_r = y_{r+1} - y_r \therefore r = 0,1,2-----$

Here, the symbol Δ is called the forward difference operator

The first forward differences of the first forward differences are called second forward differences and are denoted by $\Delta^2 y_0, \Delta^2 y_1 \ldots \ldots$ that is

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$
$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

In general $\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r \quad r = 0,1,2.......$ similarly, the n[th] forward differences are defined by the formula.

$$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r \quad r = 0,1,2.......$$

While using this formula for $n=1$, use the notation $\Delta^0 y_r = y_r$ and we have $\Delta^n y_r = 0 \forall n = 1,2...... \; and \; r = 0,2,.........$ the symbol Δ^n is referred as the n[th] forward difference operator.

Backward Differences:-

As mentioned earlier, let $y_0, y_1 \ldots\ldots y_r \ldots\ldots$ be the values of a function $y = f(x)$ corresponding to the values $x_0, x_1, x_2 \ldots\ldots\ldots\ldots x_r \ldots\ldots$ of x respectively. Then, $\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \nabla y_3 = y_3 - y_2, \ldots.$ are called the first backward differences

In general $\nabla y_r = y_r - y_{r-1},\ r = 1, 2, 3 \ldots\ldots\ldots \rightarrow (1)$

The symbol ∇ is called the backward difference operator, like the operator Δ, this operator is also a linear operator

Comparing expression (1) above with the expression (1) of section we immediately note that $\nabla y_r = \nabla y_{r-1}, r = 0, 1, 2 \ldots\ldots \rightarrow (2)$

The first backward differences of the first background differences are called second differences and are denoted by $\nabla^2 y_2, \nabla^2 y_3 --- \nabla^2{}_r ----$ i.e.,..

$\nabla^2 y_2 = \nabla y_2 - \nabla y_1, \nabla^2 y_3 = \nabla y_3 - \nabla y_2$ ....

In general $\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}, r = 2,3..... \rightarrow (3)$ similarly, the n^{th} backward differences are defined by the formula $\nabla^n y_r = \nabla^{n-1} y_r - \nabla^{n-1} y_{r-1}, r = n, n+1..... \rightarrow (4)$ While using this formula, for n = 1 we employ the notation $\nabla^0 y_r = y_r$

If $y = f(x)$ is a constant function , then y = c is a constant, for all x, and we get $\nabla^n y_r = 0 \forall n$ the symbol ∇^n is referred to as the n^{th} backward difference operator

Central Differences:-

With $y_0, y_1, y_2 \ldots y_r$ as the values of a function $y = f(x)$ corresponding to the values $x_1, x_2 \ldots x_r \ldots$ of x, we define the first central differences

$\delta y_{1/2}, \delta y_{3/2}, \delta y_{5/2} ----$ as follows

$$\delta y_{1/2} = y_1 - y_0, \delta y_{3/2} = y_2 - y_1, \delta y_{5/2} = y_3 - y_2 ----$$

$$\delta y_{r-1/2} = y_r - y_{r-1} \to (1)$$

The symbol δ is called the central differences operator. This operator is a linear operator

Comparing expressions (1) above with expressions earlier used on forward and backward differences we get

$$\delta y_{1/2} = \Delta y_0 = \nabla y_1, \delta y_{3/2} = \Delta y_1 = \nabla y_2 \ldots..$$

In general $\delta y_{n+1/2} = \Delta y_n = \nabla y_{n+1}, n = 0, 1, 2 \ldots\ldots \to (2)$

In general $\delta y_{n+1/2} = \Delta y_n = \nabla y_{n+1}, n = 0,1,2...... \to (2)$

The first central differences of the first central differences are called the second central differences and are denoted by $\delta^2 y_1, \delta^2 y_2 ...$

Thus $\delta^2 y_1 = \delta_{3/2} - \delta y_{1/2}, \delta^2 y_2 = \delta_{5/2} - \delta_{3/2}.......$

$$\delta^2 y_n = \delta y_{n+1/2} - \delta y_{n-1/2} \to (3)$$

Higher order central differences are similarly defined. In general the n^th central differences are given by

i) for odd $n : \delta^n y_{r-1/2} = \delta^{n-1} y_r - \delta^{n-1} y_{r-1}, r = 1,2.... \to (4)$

Given $f(-2)=12, f(-1)=16, f(0)=15, f(1)=18, f(2)=20$ from the central difference table and write down the values of $\delta y_{3/2}, \delta^2 y_0$ *and* $\delta^3 y_{7/2}$ by taking $x_0 = 0$ Sol. The central difference table is

x	$y=f(x)$	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
-2	12				
		4			
-1	16		-5		
		-1		9	
0	15		4		-14
		3		-5	
1	18		-1		
		2			
2	20				

Symbolic Relations and Separation of symbols:

We will define more operators and symbols in addition to Δ, ∇ and δ already defined and establish difference formulae by symbolic methods

Definition:- The averaging operator μ is defined by the equation $\mu y_r = \frac{1}{2}[y_{r+1/2} + y_{r-1/2}]$

Definition:- The shift operator E is defined by the equation $Ey_r = y_{r+1}$. This shows that the effect of E is to shift the functional value y_r to the next higher value y_{r+1}. A second operation with E gives $E^2 y_r = E(Ey_r) = E(y_{r+1}) = y_{r+2}$

Generalizing $E^n y^r = y_{r+n}$

Relationship Between Δ *and* E

We have

$$\Delta y_0 = y_1 - y_0$$
$$= Ey_0 - y_0 = (E-1)\,y_0$$
$$\Rightarrow \Delta = E - y\,(or)\,E = 1 + \Delta$$

Some more relations

$$\Delta^3 y_0 = (E-1)^3\, y_0 = \left(E^3 - 3E^2 + 3E - 1\right) y_0$$
$$= y_3 - 3y_2 + 3y_1 - y_0$$

Inverse operator E^{-1} is defined as $E^{-1}y_r = y_{r-1}$

In general $E^{-n}y_n = y_{r-n}$

We can easily establish the following relations

i) $\nabla \equiv 1 - E^{-1}$

ii) $\delta \equiv E^{1/2} - E^{-1/2}$

iii) $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$

iv) $\Delta = \nabla E = E^{1/2}$

v) $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$

The operator D is defined as $Dy(x) = \frac{\partial}{\partial x}[y(x)]$

Relation Between The Operators D And E

Using Taylor's series we have,

$$y(x+h) = y(x) + hy^1(x) + \frac{h^2}{2!}y^{11}(x) + \frac{h^3}{3!}y^{111}(x) + ----$$

This can be written in symbolic form

$$Ey_x = \left[1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + ----\right]y_x = e^{hD}.y_x$$

We obtain in the relation $E = e^{hD} \rightarrow (3)$

- If $f(x)$ is a polynomial of degree n and the values of x are equally spaced then $\Delta^n f(x)$ is constant

Evaluate

$$(i)\,\Delta \cos x$$

$$(ii)\,\Delta^2 \sin(px+q)$$

$$(iii)\,\Delta^n e^{ax+b}$$

Sol.Let h be the interval of differencing

$$(i)\,\Delta \cos x = \cos(x+h) - \cos x$$

$$= -2\sin\left(x+\frac{h}{2}\right)\sin\frac{\mathrm{h}}{2}$$

$$(ii)\,\Delta \sin(px+q) = \sin\left[p(x+h)+q\right] - \sin(px+q)$$

$$= 2\cos\left(px+q+\frac{ph}{2}\right)\sin\frac{ph}{2}$$

$$= 2\sin\frac{ph}{2}\sin\left(\frac{\pi}{2}+px+q+\frac{ph}{2}\right)$$

$$\Delta^2 \sin(px+q) = 2\sin\frac{ph}{2}\Delta\left[\sin(px+q)+\frac{1}{2}(\pi+ph)\right]$$

$$= \left[2\sin\frac{ph}{2}\right]^2 \sin\left[px+q+\frac{1}{2}(\pi+ph)\right]$$

Using the method of separation of symbols show that

$$\Delta^n \mu_{x-n} = \mu_{x-n} - n\mu_{x-1} + \frac{n(n-1)}{2}\mu_{x-2} + -----+(-1)^n \mu_{x-n}$$

Sol.To prove this result, we start with the right hand side. Thus

$$\mu x - n\mu x - 1 + \frac{n(n-1)}{2}\mu x - 2 + ------+(-1)^n \mu x - n$$

$$= \mu x - nE^{-1}\mu x + \frac{n(n-1)}{2}E^{-2}\mu x + ------+(-1)^n E^{-n}\mu x$$

$$= \left[1 - nE^{-1} + \frac{n(n-1)}{2}E^{-2} + ------+(-1)^n E^{-n}\right]\mu x = (1-E^{-1})^n \mu x$$

$$= \left(1-\frac{1}{E}\right)^n \mu n = \frac{(E-1)^n}{E}\mu n$$

$$= \frac{\Delta^n}{E^n}\mu x = \Delta^n E^{-n}\mu x$$

$=\Delta^n \mu_{x-n}$ which is left hand side

Find the melting point of the alloy containing 54% of lead, using appropriate interpolation formula

Percentage of lead(p)	50	60	70	80
Temperature $(Q°c)$	205	225	248	274

Sol.The difference table is

x	Y	Δ	Δ^2	Δ^3
50	205			
		20		
60	225		3	
		23		0
70	248		3	
		26		
80	274			

Let temperature = $f(x)$

$$x_0 + ph = 24, x_0 = 50, h = 10$$
$$50 + p(10) = 54\ (or)\ p = 0.4$$

By Newton's forward interpolation formula

$$f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{n!}\Delta^3 y_0 + ---$$
$$f(54) = 205 + 0.4(20) + \frac{0.4(0.4-1)}{2!}(3) + \frac{(0.4)(0.4-1)(0.4-2)}{3!}(0)$$
$$= 205 + 8 - 0.36$$
$$= 212.64$$

Melting point = 212.64

The population of a town in the decimal census was given below. Estimate the population for the 1895

Year x	1891	1901	1911	1921	1931
Population of y	46	66	81	93	101

Sol.

X	Y	Δ	Δ^2	Δ^3	Δ^4
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3

$$y(1895) = 46 + (0.4)(20) + \frac{(0.4)(0.4-1)}{6} - (-5)$$
$$+ \frac{(0.4-1)0.4(0.4-2)}{6}(2)$$
$$+ \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{24}$$
$$= 54.45 \; thousands$$

Gauss's Interpolation Formula:- We take x_0 as one of the specified of x that lies around the middle of the difference table and denote $x_0 - rh$ by $x - r$ and the corresponding value of y by $y - r$. Then the middle part of the forward difference table will appear as shown in the next page

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_{-4}	y_{-4}					
x_{-3}	y_{-3}	Δy_{-4}				
x_{-2}	y_{-2}	Δy_{-3}	$\Delta^2 y_{-4}$			
x_{-1}	y_{-1}	Δy_{-2}	$\Delta^2 y_{-3}$	$\Delta^3 y_{-4}$		
x_0	y_0	Δy_{-1}	$\Delta^2 y_{-2}$	$\Delta^3 y_{-3}$	$\Delta^4 y_{-4}$	
x_1	y_1	Δy_0	$\Delta^2 y_{-1}$	$\Delta^3 y_{-2}$	$\Delta^4 y_{-3}$	$\Delta^5 y_{-4}$
x_2	y_2	Δy_1	$\Delta^2 y_0$	$\Delta^3 y_{-1}$	$\Delta^4 y_{-2}$	$\Delta^5 y_{-3}$

$$\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$$
$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$
$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$
$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} ------(1) \quad and$$
$$\Delta y_{-1} = \Delta y_{-2} + \Delta^2 y_{-2}$$
$$\Delta^2 y_{-1} = \Delta^2 y_{-2} + \Delta^3 y_{-2}$$
$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$
$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2} ------(2)$$

By using the expressions (1) and (2), we now obtain two versions of the following Newton's forward interpolation formula

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_0) + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$+ \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 +$$ -----------------3

Here y_p is the value of y at $x = x_p = x_0 + ph$

Gauss Forward Interpola$\Delta^2 y_0, \Delta^3 y_0, \ldots$tion Formula:-

Substituting for from (1)in the formula (3), we get

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{-1}$$
$$+ \Delta^4 y_{-1} + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_{-1} + \Delta^5 y_{-1} + ----]$$

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1}) + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{p(p+1)(p-1)(p-2)}{4!}(\Delta^4 y_{-1}) + ----]$$

Substituting $\Delta^4 y_{-1}$ *from* (2), this becomes

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(p+1)(p-1)p(p-2)}{4!}(\Delta^4 y_{-2}) + ---]$$ -----------------4

Using Lagrange's formula calculate $f(3)$ from the following table

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

Sol. Given $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 4, x_5 = 6, x_4 = 5$

$f(x_0) = 1, f(x_1) = 14, f(x_2) = 15, f(x_3) = 5, f(x_4) = 6, f(x_5) = 19$

From langrange's interpolation formula

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)} f(x_2)$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)} f(x_5)$$

Here $x=3$ then

$$f(3)=\frac{(3-1)(3-2)(3-4)(3-5)(3-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)}\times 1+$$

$$\frac{(3-0)(3-2)(3-4)(3-5)(3-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)}\times 14+$$

$$\frac{(3-0)(3-1)(3-4)(3-5)(3-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)}\times 15+$$

$$\frac{(3-0)(3-1)(3-2)(3-5)(3-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)}\times 5+$$

$$\frac{(3-0)(3-1)(3-2)(3-4)(3-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)}\times 6+$$

$$\frac{(3-0)(3-1)(3-2)(3-4)(3-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)}\times 19=\frac{12}{240}-\frac{18}{60}\times 14+\frac{36}{48}\times 15+\frac{36}{48}\times 5-\frac{18}{60}\times 6+\frac{12}{40}\times 19$$

$$=0.05-4.2+11.25+3.75-1.8+0.95$$

$$=10$$

$$f\left(x_3\right)=10$$

Linearity

Ex. 1.

$$(a)\mathcal{L}^{-1}[\frac{2s+1}{s^2+4}] \qquad (b)\mathcal{L}^{-1}[\frac{4(s+1)}{s^2-16}].$$

$$\text{Solution}:(a)\mathcal{L}^{-1}[\frac{2s+1}{s^2+4}]=\mathcal{L}^{-1}[2\frac{s}{s^2+2^2}+\frac{1}{2}\frac{2}{s^2+2^2}]=2\cos 2t+\frac{1}{2}\sin 2t$$

$$(b)\mathcal{L}^{-1}[\frac{4(s+1)}{s^2-16}]=\mathcal{L}^{-1}[4\frac{s}{s^2-4^2}+\frac{4}{s^2-4^2}]=4\cosh 4t+\sinh 4t$$

Shifting

Ex. 1.

$(a)\ \mathcal{L}^{-1}[\frac{e^{-\pi s}}{s^2+2s+2}]$ $(b)\ \mathcal{L}^{-1}[\frac{2s+3}{s^2+3s+2}]$.

$$\text{Solution}: (a)\ \mathcal{L}^{-1}[\frac{e^{-\pi s}}{s^2+2s+2}] = \mathcal{L}^{-1}[\frac{e^{-\pi s}}{(s+1)^2+1}]$$

$$\because \mathcal{L}^{-1}[\frac{1}{(s+1)^2+1}] = e^{-t}\sin t$$

$$\text{and } \mathcal{L}[f(t-a)u(t-a)] = e^{-as}F(s)$$

$$\therefore \mathcal{L}^{-1}[\frac{e^{-\pi s}}{(s+1)^2+1}] = e^{-(t-\pi)}\sin(t-\pi)u(t-\pi) = -e^{-(t-\pi)}\sin t\, u(t-\pi)$$

$$(b)\ \mathcal{L}^{-1}[\frac{2s+3}{s^2+3s+2}] = \mathcal{L}^{-1}[\frac{2(s+\frac{3}{2})}{(s+\frac{3}{2})^2-(\frac{1}{2})^2}] = 2e^{-\frac{3}{2}t}\cosh\frac{t}{2}$$

Scaling

Ex. 1.

$$\mathcal{L}^{-1}\left[\frac{4s}{16s^2-4}\right].$$

Solution : $\mathcal{L}^{-1}\left[\frac{4s}{16s^2-4}\right] = \mathcal{L}^{-1}\left[\frac{4s}{(4s)^2-2^2}\right] = \frac{1}{4}\cosh 2\cdot\frac{1}{4}t = \frac{1}{4}\cosh\frac{t}{2}$

Derivative

Ex. 1.

$(a)\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}]$ $(b)\mathcal{L}^{-1}[\ln\frac{s+a}{s+b}]$.

$$\text{solution}:(a)\mathcal{L}[\sin\omega t]=\frac{\omega}{s^2+\omega^2}\Rightarrow\mathcal{L}[t\sin\omega t]=-\frac{d}{ds}(\frac{\omega}{s^2+\omega^2})=\frac{2\omega s}{(s^2+\omega^2)^2}$$

$$\text{Let } F(t)=t\sin\omega t\Rightarrow\mathcal{L}[F'(t)]=s\cdot\frac{2\omega s}{(s^2+\omega^2)^2}-F(0)$$

$$\mathcal{L}[F'(t)]=2\omega\frac{s^2}{(s^2+\omega^2)^2}=2\omega[\frac{(s^2+\omega^2)-\omega^2}{(s^2+\omega^2)^2}]=2\omega[\frac{1}{s^2+\omega^2}-\frac{\omega^2}{(s^2+\omega^2)^2}]$$

$$=2\mathcal{L}[\sin\omega t]-\frac{2\omega^3}{(s^2+\omega^2)^2}$$

$$\frac{1}{(s^2+\omega^2)^2}=\frac{1}{2\omega^3}\cdot\mathcal{L}[2\sin\omega t-F'(t)]$$

$$\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}]=\frac{1}{2\omega^3}\cdot[2\sin\omega t-F'(t)]=\frac{1}{2\omega^3}(\sin\omega t-\omega t\cos\omega t)$$

Integration

Ex. 1.

$$(a)\mathcal{L}^{-1}[\frac{1}{s^2}(\frac{s-1}{s+1})] \qquad (b)\mathcal{L}^{-1}[\ln\frac{s+a}{s+b}].$$

$$\text{Solution}:(a)\mathcal{L}^{-1}[\frac{1}{s^2}(\frac{s-1}{s+1})]=\mathcal{L}^{-1}[\frac{1}{s(s+1)}-\frac{1}{s^2(s+1)}]=\int_0^t e^{-t}dt-\int_0^t\int_0^t e^{-t}dtdt$$

$$=-(e^{-t}-1)+\int_0^t(e^{-t}-1)dt=-(e^{-t}-1)-(e^{-t}-1)-t=2-2e^{-t}-t$$

$$(b)\mathcal{L}[e^{-bt}-e^{-at}]=\frac{1}{s+b}-\frac{1}{s+a}$$

$$\mathcal{L}[\frac{e^{-bt}-e^{-at}}{t}]=\int_s^\infty(\frac{1}{s+b}-\frac{1}{s+a})ds=\ln\frac{s+b}{s+a}\Big|_s^\infty=\ln\frac{s+a}{s+b}$$

$$\therefore\mathcal{L}^{-1}[\ln\frac{s+a}{s+b}]=\frac{e^{-bt}-e^{-at}}{t}$$

Convolution

Ex. 1.

$$(a)\mathcal{L}^{-1}\left[\frac{1}{(s^2+\omega^2)^2}\right] \qquad (b)\mathcal{L}^{-1}\left[\frac{s}{(s^2+\omega^2)^2}\right].$$

$$\text{Solution}: (a)\mathcal{L}[\sin\omega t] = \frac{\omega}{s^2+\omega^2} \Rightarrow \mathcal{L}\left[\frac{1}{\omega}\sin\omega t\right] = \frac{1}{s^2+\omega^2}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+\omega^2)^2}\right] = \frac{1}{\omega^2}\int_0^t \sin\omega\tau\sin\omega(t-\tau)d\tau$$

$$= \frac{1}{\omega^2}\int_0^t \frac{1}{2}[\cos(\omega\tau-\omega t+\omega\tau)-\cos(\omega\tau+\omega t-\omega\tau)]d\tau$$

$$= \frac{1}{2\omega^2}\int_0^t [\cos(2\omega\tau-\omega t)-\cos\omega t]d\tau = \frac{1}{2\omega^2}\left[\frac{1}{2\omega}\sin(2\omega\tau-\omega t)-\tau\cos\omega t\right]_0^t$$

$$= \frac{1}{2\omega^2}\{[\frac{1}{2\omega}(\sin\omega t-\sin(-\omega t)]-t\cos\omega t\} = \frac{1}{2\omega^3}(\sin\omega t-\omega t\cos\omega t)$$

$$L^{-1}\left[\frac{s+1}{s^3+s^2-6s}\right].$$

$$\text{Solution}: \frac{s+1}{s^3+s^2-6s} = \frac{s+1}{s(s-2)(s+3)} = \frac{A_1}{s} + \frac{A_2}{s-2} + \frac{A_3}{s+3}$$

$$A_1 = \lim_{s\to 0} \frac{s+1}{(s-2)(s+3)} = -\frac{1}{6}$$

$$A_2 = \lim_{s\to 2} \frac{s+1}{s(s+3)} = \frac{3}{10}$$

$$A_3 = \lim_{s\to -3} \frac{s+1}{s(s-2)} = \frac{-2}{15}$$

$$L^{-1}\left[\frac{s+1}{s^3+s^2-6s}\right] = \frac{-\frac{1}{6}}{s} + \frac{\frac{3}{10}}{s-2} + \frac{\frac{-2}{15}}{s+3} = -\frac{1}{6} + \frac{3}{10}e^{2t} - \frac{2}{15}e^{-3t}$$

INVERSE LAPLACE TRANSFORM

$$L^{-1}\left[\frac{s^2}{s^4+4}\right].$$

$$\text{Solution}: \frac{s^2}{s^4+4} = \frac{s^2}{(s^2)^2+2\cdot s^2\cdot 2+2^2-2\cdot s^2\cdot 2} = \frac{s^2}{(s^2+2)^2-(2s)^2}$$

$$= \frac{s^2}{(s^2+2s+2)(s^2-2s+2)} = \frac{A_1 s+B_1}{(s+1)^2+1} + \frac{A_2 s+B_2}{(s-1)^2+1}$$

$$\lim_{s\to -1+i} \frac{s^2}{(s-1)^2+1} = A_1(-1+i)+B_1 \Rightarrow \frac{-2i}{4-4i} = (-A_1+B_1)+iA_1$$

$$\frac{8-8i}{32} = (-A_1+B_1)+iA_1 \Rightarrow A_1 = -\frac{1}{4}, B_1 = 0$$

$$\lim_{s\to 1+i} \frac{s^2}{(s+1)^2+1} = A_2(1+i)+B_2 \Rightarrow \frac{2i}{4+4i} = (A_2+B_2)+iA_2$$

$$\frac{8+8i}{32} = (A_2+B_2)+iA_2 \Rightarrow A_2 = \frac{1}{4}, B_2 = 0$$

$$L^{-1}\left[\frac{s^2}{s^4+4}\right] = L^{-1}\left[\frac{-\frac{1}{4}(s+1)+\frac{1}{4}}{(s+1)^2+1} + \frac{\frac{1}{4}(s-1)+\frac{1}{4}}{(s-1)^2+1}\right]$$

$$= \frac{e^{-t}}{4}(-\cos t+\sin t) + \frac{e^{t}}{4}(\cos t+\sin t)$$

INVERSE LAPLACE TRANSFORM

$$L^{-1}\left[\frac{1}{(s^2+\omega^2)^2}\right].$$

$$\text{Solution}: \frac{d}{d\omega}\left(\frac{1}{s^2+\omega^2}\right) = \frac{-2\omega}{(s^2+\omega^2)^2} \Rightarrow L^{-1}\left[\frac{d}{d\omega}\left(\frac{1}{s^2+\omega^2}\right)\right] = L^{-1}\left[\frac{-2\omega}{(s^2+\omega^2)^2}\right]$$

$$-2\omega L^{-1}\left[\frac{1}{(s^2+\omega^2)^2}\right] = \frac{d}{d\omega}L^{-1}\left[\frac{1}{s^2+\omega^2}\right] = \frac{d}{d\omega}\left(\frac{1}{\omega}\sin\omega t\right) = -\frac{1}{\omega^2}\sin\omega$$

$$L^{-1}\left[\frac{1}{(s^2+\omega^2)^2}\right] = \frac{1}{2\omega^3}(\sin\omega t - \omega t\cos\omega t)$$

L $^{-1}[e^{-\sqrt{s}}]$.

$$\text{Solution: } \bar{y} = e^{-\sqrt{s}} \Rightarrow \bar{y}' = -\frac{e^{-\sqrt{s}}}{2\sqrt{s}},\ \bar{y}'' = \frac{e^{-\sqrt{s}}}{4s} + \frac{e^{-\sqrt{s}}}{4\sqrt{s^3}}$$

$$\text{we get the equation } 4s\bar{y}'' + 2\bar{y}' - \bar{y} = 0 \Rightarrow 4L[\frac{d}{dt}(t^2 y)] + 2L[-ty] - L[y] = 0$$

$$4\frac{d}{dt}(t^2 y) - 2ty - y = 0 \Rightarrow 4t^2 y' + (6t-1)y = 0 \Rightarrow \frac{dy}{y} + \frac{6t-1}{4t^2}dt = 0$$

$$\ln y + \frac{3}{2}\ln t + \frac{1}{4t} = c_1 \Rightarrow y = ct^{-\frac{3}{2}} e^{-\frac{1}{4t}}$$

$$\because L[t^{-\frac{1}{2}}] = \frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{\sqrt{s}},\ \text{and } L[ty] = L[ct^{-\frac{1}{2}} e^{-\frac{1}{4t}}]$$

$$\text{while } L[ty] = -\bar{y}' = \frac{e^{-\sqrt{s}}}{2\sqrt{s}} \Rightarrow L[ct^{-\frac{1}{2}} e^{-\frac{1}{4t}}] = \frac{e^{-\sqrt{s}}}{2\sqrt{s}}$$

$$\text{Apply general final value theorem } \lim_{t\to\infty} \frac{ct^{-\frac{1}{2}} e^{-\frac{1}{4t}}}{t^{-\frac{1}{2}}} = \lim_{s\to 0} \frac{\frac{e^{-\sqrt{s}}}{2\sqrt{s}}}{\frac{\sqrt{\pi}}{\sqrt{s}}} \Rightarrow c = \frac{1}{2\sqrt{\pi}}$$

$$\therefore y = \frac{1}{2\sqrt{\pi}t^{3/2}} e^{-\frac{1}{4t}}$$

$$y''+y'+y = g(x),\ y(0)=1,\ y'(0)=0, \text{ where } g(x)=\begin{cases}1 & 0<x<3\\ 3 & x>3\end{cases}.$$

Solution : $g(x) = u(x) + 2u(x-3)$

$$[s^2Y - sy(0) - y'(0)] + [sY - y(0)] + Y = \frac{1}{s} + 2\frac{e^{-3s}}{s}$$

$$(s^2+s+1)Y = s+1+\frac{1}{s}+2\frac{e^{-3s}}{s}$$

$$Y = \frac{s+1}{s^2+s+1} + \frac{1}{s(s^2+s+1)} + \frac{2e^{-3s}}{s(s^2+s+1)}$$

$$= \frac{s+1}{s^2+s+1} + \left(\frac{1}{s} - \frac{s+1}{s^2+s+1}\right) + 2e^{-3s}\left(\frac{1}{s} - \frac{s+1}{s^2+s+1}\right)$$

$$\frac{s+1}{s^2+s+1} = \frac{(s+\frac{1}{2}) + \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \Rightarrow L^{-1}\left[\frac{s+1}{s^2+s+1}\right] = e^{-\frac{x}{2}}\left(\cos\frac{\sqrt{3}}{2}x + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}x\right)$$

$$y(x) = u(x) + 2u(x-3)\{1 - e^{-\frac{x-3}{2}}[\cos\frac{\sqrt{3}}{2}(x-3) + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}(x-3)]\}$$

CURVE FITTING AND FOURIER TRANSFORM

CLOs	Course Learning Outcome
CLO 13	Ability to curve fit data using several linear and non linear curves by method of least squares.
CLO 14	Understand the nature of the Fourier integral.
CLO 15	Ability to compute the Fourier transforms of the given function.
CLO 16	Ability to compute the Fourier sine and cosine transforms of the function

CLOs	Course Learning Outcome
CLO 17	Evaluate the inverse Fourier transform, Fourier sine and cosine transform of the given function.
CLO 18	Evaluate finite and infinite Fourier transforms.
CLO 19	Understand the concept of Fourier transforms to the real-world problems of circuit analysis, control system design

Suppose that a data is given in two variables x & y the problem of finding an analytical expression of the form $y = f(x)$ which fits the given data is called curve fitting

Let $(x_1, y_1), (x_2, y_2) \ldots\ldots (x_n, y_n)$ be the observed set of values in an experiment and $y = f(x)$ be the given relation $x \& y, Let\ E_1, E_2, \ldots\ldots E_x$ are the error of approximations then we have

$$E_1 = y_1 - f(x_1)$$
$$E_2 = y_2 - f(x_2)$$
$$E_3 = y_3 - f(x_3)$$

$E_n = y_n - f(x_n)$ where $f(x_1), f(x_2) \ldots\ldots\ldots f(x_n)$ are called the expected values of y corresponding to $x = x_1, x = x_2 \ldots\ldots x = x_n$

CURVE FITTING

$y_1, y_2 \ldots\ldots y_n$ are called the observed values of y corresponding to $x = x_1, x = x_2 \ldots\ldots\ldots x = x_n$ the differences $E_1, E_2 \ldots\ldots E_n$ between expected values of y and observed values of y are called the errors, of all curves approximating a given set of points, the curve for which

$E = E_1^2 + E_2^2 + \ldots\ldots E_n^2$ is a minimum is called the best fitting curve (or) the least square curve

This is called the method of least squares (or) principles of least squares

1. FITTING OF A STRAIGHT LINE:-

Let the straight line be $y = a + bx \rightarrow (1)$

Let the straight line (1) passes through the data points

$(x_1, y_1), (x_2, y_2) ......(x_n, y_n)$ $i.e., (x_i, y_i), i = 1, 2....n$

So we have $yi = a + bxi \rightarrow (2)$

The error between the observed values and expected values of $y = yi$ is defined as

$Ei = y_1 - (a + bxi).i = 1, 2.......n \rightarrow (3)$

The sum of squares of these error is

$E = \sum_{i=1}^{n} Ei^2 = \sum_{i=1}^{n} [yi - (a + bxi)]^2$ now for E to be minimum

$$\frac{\partial E}{\partial a} = 0; \frac{\partial E}{\partial b} = 0$$

These equations will give normal equations

$$\sum_{i=1}^{n} yi = na + b\sum_{i=1}^{n} xi$$

$$\sum_{i=1}^{n} xiyi = a\sum_{i=1}^{n} xi + b\sum_{i=1}^{n} xi^2$$

The normal equations can also be written as

$$\sum y = na + b\sum x$$

$$\sum xy = a\sum x + b\sum x^2$$

Solving these equation for a, b substituting in (1) we get required line of best fit to the given data.

1. Let the equation of the parabola to be fit
   The parabola (1) passes through the data points
   $$\left(x_1, y_1\right),\left(x_2, y_2\right)............\left(x_n, y_n\right), i.e.,\left(x_i, y_i\right); i=1,2......x$$
   We have $yi=a+bx_i+cx_i^2 \rightarrow (2)$
   $$y=a+bx+cx^2 \rightarrow (1)$$
   The error Ei between the observed an expected value of $y=y_i$ is defined as
   $$Ei=yi-\left(a+bxi+cxi^2\right), i=1,2,3......n \rightarrow (3)$$
   The sum of the squares of these error is
   $$E=\varepsilon_{i=1}^{n}\ Ei^2=\varepsilon_{i=1}^{n}\ \left(yi-a-bxi-cxi^2\right)^2 \rightarrow (4)$$

For E to be minimum, we have

$$\frac{\partial E}{\partial a}=0, \frac{\partial E}{\partial b}=0, \frac{\partial E}{\partial c}=0$$

The normal equations can also be written as

$$\varepsilon y = na + b\varepsilon x + c\varepsilon x^2$$

$$\varepsilon xy = a\varepsilon x + b\varepsilon x^2 + c\varepsilon x^3 \qquad use \sum instead\ of\ \varepsilon$$

$$\varepsilon x^2 y = a\varepsilon x^2 + b\varepsilon x^3 + c\varepsilon x^4$$

Solving these equations for a, b, c and satisfying (1) we get required parabola of best fit

1. POWER CURVE:-

The power curve is given by $y = ax^b \rightarrow (1)$

Taking logarithms on both sides

$$\log_{10}{}^{y} = \log_{10}{}^{a} + b\log_{10}{}^{X}$$

$$(or)\, y = A + bX \rightarrow (2)$$

$$where\;\; y = \log_{10}{}^{y}, A = \log_{10}{}^{a} \;\; and \;\; X = \log_{10}{}^{X}$$

Equation (2) is a linear equation in $X \,\&\, y$

$\therefore$ The normal equations are given by

$$\varepsilon y = nA + b\varepsilon X$$

$$\varepsilon xy = A\varepsilon X + b\varepsilon X^{2} \qquad use\ \ \Sigma\, symbol$$

From these equations, the values A and b

can be calculated then a = antilog (A)

substitute a & b in (1) to get the required

curve of best fit

1. **<u>EXPONENTIAL CURVE</u> :-**

$$(1)\, y = ae^{bx} \quad (2)\, y = ab^{x}$$

$$y = ae^{bx} \rightarrow (1)$$

Taking logarithms on both sides

$$\log_{10} y = \log_{10} a + bx \log_{10} e$$

$$(or)\, y = A + BX \rightarrow (2)$$

Where $y = \log_{10} y, A = \log_{10} a \,\&\, B = b \log_{10} e$

Equation (2) is a linear equation in X and Y

So the normal equation are given by

$$\Sigma Y = nA + B\Sigma X$$

$$\Sigma xy = A\Sigma X + B\Sigma X^2$$

Solving the equation for A & B, we can find

$$a = anti\log A \,\&\, b = \frac{B}{\log_{10} e}$$

Substituting the values of a and b so obtained in (1) we get The curve of best fir to the given data.

2. $y = ab^x \rightarrow (1)$ Taking log on both sides

$$\log_{10} y = \log_{10} a + x\log_{10} b \; (or) Y = A + Bx$$
$$Y = \log_{10} y, A = \log_{10} a, B = \log_{10} b$$

The normal equation (2) are given by

$$\Sigma y = nA + B\Sigma X$$
$$\Sigma xy = A\Sigma X + B\Sigma X^2$$

Solving these equations for A and B we can find $a = anti\log A, b = anti\log B$

Substituting a and b in (1)

1. **By the method of least squares, find the straight line that best fits the following data**

X	1	2	3	4	5
Y	14	27	40	55	68

$$y = a + bx$$

Ans. The values of $\varepsilon x, \varepsilon y, \varepsilon x^2$ *and* εxy are calculated as follows

xi	yi	xi^2	$xiyi$
1	14	1	14
2	27	4	54
3	40	9	120
4	55	16	220
5	68	25	340

Replace xi, yi by x_i, y_i and use Σ *instead of* ε

$$\varepsilon xi = 15; \varepsilon yi = 204, \varepsilon xi^2 = 55 \text{ and } \varepsilon xiyi = 748$$

The normal equations are

$$\varepsilon y = na + b\varepsilon x \rightarrow (1)$$

$$\varepsilon xy = a\varepsilon x + b\varepsilon x^2 \rightarrow (2)$$

$$204 = 15a + 5b$$

$$748 = 55a + 15b$$

Solving we get $a = 0, b = 13.6$

Substituting these values a & b we get

$$y = 0 + 13.6x \Rightarrow y = 13.6x$$

1. Fit a second degree parabola to the following data

X	0	1	2	3	4
Y	1	5	10	22	38

$$y = a + bx + cx^2$$

Ans. Equation of parabola

$$y = a + bx + cx^2 \rightarrow (1)$$

Normal equations $\varepsilon y = na + b\varepsilon x + c\varepsilon x^2$

$$\varepsilon xy = a\varepsilon x + b\varepsilon x^2 + c\varepsilon x^3$$

$$\varepsilon x^2 y = a\varepsilon x^2 + b\varepsilon x^3 + c\varepsilon x^4 \rightarrow (2)$$

x	y	xy	x^2	x^2y	x^3	x^4
0	**1**	**0**	**0**	**0**	**0**	**0**
1	**5**	**5**	**1**	**5**	**1**	**1**
2	**10**	**20**	**4**	**40**	**8**	**16**
3	**22**	**66**	**9**	**198**	**27**	**81**
4	**38**	**152**	**16**	**608**	**64**	**256**

1. Fit a curve $y = ax^b$ to the following data

X	1	2	3	4	5	6
Y	2.98	4.26	5.21	6.10	6.80	7.50

Ans. Let the equation of the curve be

$$y = ax^b \rightarrow (1)$$

Taking log on both sides

$$\log y = \log a + b \log x$$

$$y = A + bX \rightarrow (2)$$

$$y = \log y, A = \log a, X = \log x$$

$$\varepsilon y = nA + b\varepsilon X$$

$$\varepsilon xy = A\varepsilon x + b\varepsilon x^2 \rightarrow (3)$$

1. Fit a curve $y = ab^x \rightarrow (1)$

X	2	3	4	5	6
Y	144	172.8	207.4	248.8	298.5

Ans.

$$\log y = \log a + x \log b \rightarrow (I)$$

$$y = A + xB \rightarrow (2)$$

$$y = \log y, A = \log a, B = \log b$$

$$\Sigma y = nA + B\varepsilon x$$

$$\varepsilon xy = A\varepsilon x + B\varepsilon x^2 \rightarrow (3)$$

x	y	x^2	$Y = \log y$	xy
2	**144.0**	**4**	**2.1584**	**4.3168**
3	**172.8**	**9**	**2.2375**	**6.7125**
4	**207.4**	**16**	**2.3168**	**9.2672**
5	**248.8**	**25**	**2.3959**	**11.9795**
6	**298.5**	**36**	**2.4749**	**14.8494**

Equation of parabola $y = a + bx + cx^2 \rightarrow (1)$

Normal equations $\varepsilon y = na + b\varepsilon x + c\varepsilon x^2$

$$\varepsilon xy = a\varepsilon x + b\varepsilon x^2 + c\varepsilon x^3$$

x	y	xy	x^2	$x^2 y$	x^3	x^4
0	**1**	**0**	**0**	**0**	**0**	**0**
1	**1.8**	**1.8**	**1**	**1.8**	**1**	**1**
2	**1.3**	**2.6**	**4**	**5.2**	**8**	**16**
3	**2.5**	**7.5**	**9**	**22.5**	**27**	**81**
4	**6.3**	**25.2**	**16**	**100.8**	**64**	**256**

$\sum x_i = 10, \sum y_i = 12.9, \sum x^2 = 30, \sum x_i^3 = 100,$
$\sum x_i^4 = 354, \sum x_i^2 y_i = 130.3$

$\sum x_i\, y_i, = 37.1$

$\sum x_i\, y_i$, = 37.1

Normal equations

5a + 10b + 30c = 12.9

10a + 30b +100c = 37.1

30a + 100b +354c = 130.3

Solving a= 1.42 b = -1.07 c= .55

Substitute in (1) $y = 1.42 - 1.07x + .55x^2$

Fourier integral theorem

If f(x) is a given function defined in (-*l*,*l*) and satisfies the Dirichlet conditions then

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\,d\lambda$$

Fourier Sine Integral

If f (t) is an odd function

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin\lambda x\int_0^\infty f(t)\sin\lambda t\,dt\,d\lambda$$

Fourier Cosine Integral

If f (t) is an even function

$$f(x)=\frac{2}{\pi}\int_0^\infty \cos\lambda x\int_0^\infty f(t)\cos\lambda t\,dt\,d\lambda$$

Fourier Sine Integral

If f (t) is an odd function

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\infty f(t)\sin\lambda t\,dt\,d\lambda$$

Fourier Cosine Integral

If f (t) is an even function

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty f(t)\cos\lambda t\,dt\,d\lambda$$

Express $f(x) = \{ \begin{matrix} 1, |x| < 1 \\ 0, |x| > 1 \end{matrix}$ **as a Fourier integral. Hence evaluate** $\int_0^\infty \frac{\sin\lambda \cos\lambda x}{\lambda}$ **and also find the value of** $\int_0^\infty \frac{\sin\lambda}{\lambda}$

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\lambda(t - x)\, d\lambda dt$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-1}^1 \cos\lambda(t - x)\, d\lambda dt$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{2}{\lambda}\sin\lambda \cos\lambda\, x d\lambda$$

$$\int_0^\infty \frac{\sin\lambda \cos\lambda\, x}{\lambda} d\lambda = \frac{\pi}{2} f(x) = \{ \begin{matrix} \frac{\pi}{2}, |x| < 1 \\ 0, |x| > 1 \end{matrix}$$

$$|x| = 1$$

$$\int_0^\infty \frac{\sin\lambda \cos\lambda\, x}{\lambda} d\lambda = \frac{\pi}{2}\left[\frac{1 + 0}{2}\right] = \frac{\pi}{4}$$

$$x = 0$$

$$\int_0^\infty \frac{\sin\lambda}{\lambda} d\lambda = \frac{\pi}{2}$$

Using Fourier Integral show that

$$e^{-x}\cos x = \frac{2}{\pi}\int_0^\infty \frac{\lambda^2+2}{\lambda^4+2}\cos\lambda x\, d\lambda$$

$$f(x) = e^{-x}\cos x$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x\left[\int_0^\infty f(t)\cos\lambda t\, dt\right]d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x\left[\int_0^\infty e^{-t}\cos t\cos\lambda t\, dt\right]d\lambda$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \cos\lambda x\left[\int_0^\infty e^{-t}(\cos(\lambda+1)t+\cos(\lambda-1)t\, dt\right]d\lambda$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \cos\lambda x\left[\frac{1}{(\lambda+1)^2+1}+\frac{1}{(\lambda-1)^2+1}\right]d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\lambda^2+2}{\lambda^4+2}\cos\lambda x\, d\lambda$$

FOURIER TRANSFORMS

The complex form of Fourier integral of any function f(x) is in the form

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\int_{-\infty}^{\infty} f(t)e^{i\lambda t}\,dt\,d\lambda$$

Replacing λ by s

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\,ds\int_{-\infty}^{\infty} f(t)e^{ist}\,dt$$

Let

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{ist}\,dt$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}\,ds$$

Fourier Cosine Transform

Infinite

$$F_C[f(t)] = F_C(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t)\cos st\,dt$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_C[f(t)]\cos sx\,ds$$

Finite

$$F_C[f(t)] = F_C(s) = \sqrt{\frac{2}{l}} \int_0^{l} f(t)\cos(\frac{n\pi t}{l})dt$$

$$f(x) = \frac{1}{l}F_C(0) + \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} F_C(s)\cos(\frac{n\pi x}{l})$$

Fourier Sine Transform

Infinite

$$F_S[f(t)] = F_S(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin st dt$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_S[f(t)] \sin sx ds$$

Finite

$$F_S[f(t)] = F_S(s) = \sqrt{\frac{2}{l}} \int_0^{l} f(t) \sin(\frac{n\pi t}{l}) dt$$

$$f(x) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} F_S(s) \sin(\frac{n\pi x}{l})$$

Linear

$$F[af_1(x)+bf_2(x)]=aF_1(s)+bF_2(s)$$

$$F[af_1(x)+bf_2(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[af_1(x)+bf_2(x)]e^{ist}dt$$

$$F[af_1(x)+bf_2(x)]=\frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f_1(x)e^{ist}dt+\frac{b}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f_2(x)e^{ist}dt$$

$$F[af_1(x)+bf_2(x)]=aF_1(s)+bF_2(s)$$

Shifting Theorem: (a) $F[f(x-a)] = e^{ias} F(s)$

(b) $F[e^{iax} f(x)] = F(s+a)$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t-a) e^{ist} dt$$

$$t - a = z$$

$$dt = dz$$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z) e^{isz} e^{ias} dz$$

$$F[f(x-a)] = e^{ias} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z) e^{isz} dz$$

$$F[f(x-a)] = e^{ias} F(s)$$

Change of scale property:

$$F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right) (a > 0)$$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(at) e^{ist} dt$$

$$at = z$$

$$dt = \frac{1}{a} dz$$

$$F[f(ax)] = \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z) e^{i\left(\frac{s}{a}\right)z} dz$$

$$F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Multiplication Property:

$$F[x^n f(x)] = (-i)^n \frac{d^n F}{ds^n}$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt$$

$$\frac{dF}{ds} = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t.f(t) e^{ist} dt$$

$$\frac{d^2F}{ds^2} = \frac{i^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^2.f(t) e^{ist} dt$$

continuing

$$\frac{d^nF}{ds^n} = \frac{i^n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^n.f(t) e^{ist} dt$$

$$F[x^n f(x)] = (-i)^n \frac{d^n F}{ds^n}$$

Modulation

$$F[f(x)\cos ax] = \frac{1}{2}\left[F(s+a)+F(s-a)\right], F[s] = F[f(x)]$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\cos at\, e^{ist}dt$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\left[\frac{e^{iat}+e^{-iat}}{2}\right]e^{ist}dt$$

$$F[f(x)] = \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i(s+a)t}dt + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i(s-a)t}dt\right]$$

$$F[f(x)\cos ax] = \frac{1}{2}\left[F(s+a)+F(s-a)\right]$$

Find the Fourier transform of $f(x)=\{\begin{matrix}1-x^2, |x|\le 1\\ 0, |x|>1\end{matrix}$

Hence evaluate $\int_0^\infty \frac{x\cos x-\sin x}{x^3}\cos\frac{x}{2}dx$

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F[f(x)] = \int_{-1}^{1} (1-x^2)e^{isx}dx$$

$$F[f(x)] = \left|(1-x^2)\frac{e^{isx}}{is} - 2x\frac{e^{isx}}{(is)^2} + 2\frac{e^{isx}}{(is)^3}\right|_{-1}^{1}$$

$$F[f(x)] = 2\left(\frac{e^{is}+e^{-is}}{-s^2}\right) - 2\left(\frac{e^{is}-e^{-is}}{-is^3}\right)$$

$$F[f(x)] = \frac{-4}{s^3}(s\cos s - \sin s)$$

$$f(x) = \frac{1}{2\Pi}\int_{-\infty}^{\infty} F[s]e^{-isx}ds$$

$$f(x) = \frac{1}{2\Pi}\int_{-\infty}^{\infty} \frac{-4}{s^3}(s\cos s - \sin s)e^{-isx}ds$$

$$\frac{1}{2\Pi}\int_{-\infty}^{\infty} \frac{-4}{s^3}(s\cos s - \sin s)e^{-isx}ds = \{\begin{matrix}1-x^2, |x|\le 1\\ 0, |x|>1\end{matrix}$$

$x = 1/2$

$$\frac{1}{2\Pi}\int_{-\infty}^{\infty} \frac{-4}{s^3}(s\cos s - \sin s)e^{-isx}ds = \frac{3}{4}$$

$$\int_{-\infty}^{\infty} \frac{(s\cos s - \sin s)}{s^3}[\cos\frac{s}{2} - i\sin\frac{s}{2}]ds = -\frac{3\Pi}{8}$$

$$\int_{0}^{\infty} \frac{(s\cos s - \sin s)}{s^3}\cos\frac{s}{2}ds = -\frac{3\Pi}{16}$$

Find the Fourier cosine transform e^{-x^2} **.**

$$F_c(e^{-x^2}) = \int_0^\infty e^{-x^2} \cos sx dx = I$$

$$\frac{dI}{ds} = -\int_0^\infty xe^{-x^2} \sin s x dx = \frac{1}{2}\int_0^\infty (-2xe^{-x^2}) \sin s x dx$$

$$\frac{dI}{ds} = \frac{-s}{2}\int_0^\infty e^{-x^2} \cos sx dx = \frac{-s}{2} I$$

$$\frac{dI}{I} = \frac{-s}{2} ds$$

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$$\log I = \int \frac{-s}{2} ds + \log c = \frac{-s^2}{4} + \log c = \log(ce^{-s^2/4})$$

$$I = ce^{-s^2/4}$$

$$\int_0^\infty e^{-x^2} \cos sx dx = ce^{-s^2/4}$$

$$s = 0$$

$$c = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\Pi}}{2}$$

$$\int_0^\infty e^{-x^2} \cos sx dx = \frac{\sqrt{\Pi}}{2} e^{-s^2/4}$$

Find the Fourier sine transform $e^{-|x|}$ .Hence show that

$$\int_0^\infty \frac{x\sin mx}{1+x^2}dx = \frac{\Pi e^{-m}}{2}$$ **,m>0**

x being positive in the interval $(0,\infty)$

$$e^{-|x|} = e^{-x}$$

$$F_s(e^{-x}) = \int_0^\infty e^{-x}\sin sx\,dx = \frac{s}{1+s^2}$$

$$f(x) = \frac{2}{\Pi}\int_0^\infty F_s(e^{-x})\sin sx\,ds$$

$$f(x) = \int_0^\infty \frac{s}{1+s^2}\sin sx\,ds$$

$$e^{-x} = \int_0^\infty \frac{s}{1+s^2}\sin sx\,ds$$

Replace x by m

$$e^{-m} = \frac{2}{\Pi}\int_0^\infty \frac{s}{1+s^2}\sin sm\,ds$$

$$\int_0^\infty \frac{s}{1+s^2}\sin sm\,ds = \frac{\Pi}{2}e^{-m}$$

$$\int_0^\infty \frac{x}{1+x^2}\sin mx\,ds = \frac{\Pi}{2}e^{-m}$$

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Find the Fourier cosine transform

$$f(x) = \begin{cases} x, 0 < x < 1 \\ 2 - x, 1 < x < 2 \\ 0, x > 2 \end{cases}.$$

$$F_c(f(x)) = \int_0^\infty f(x)\cos sx\,dx$$

$$F_c(f(x)) = \int_0^1 x\cos sx\,dx + \int_1^2 (2-x)\cos sx\,dx + \int_2^\infty 0.\cos sx\,dx$$

$$F_c(f(x)) = \left(\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2}\right) + \left(-\frac{\sin s}{s} - \frac{\cos 2s}{s^2} + \frac{\cos s}{s^2}\right)$$

$$F_c(f(x)) = \frac{2\cos s}{s^2} - \frac{1}{s^2} - \frac{\cos 2s}{s^2}$$

If the Fourier sine transform of

$$f(x)=\frac{1-\cos n\Pi}{(n\Pi)^2}$$ **then find f(x).**

$$f(x)=\frac{2}{\Pi}\sum_{n=1}^{\infty}F_s(n)\sin nx$$

$$F_s(n)=\frac{1-\cos n\Pi}{(n\Pi)^2}$$

$$f(x)=\frac{2}{\Pi}\sum_{n=1}^{\infty}\frac{1-\cos n\Pi}{(n\Pi)^2}\sin nx$$

$$f(x)=\frac{2}{\Pi^3}\sum_{n=1}^{\infty}\frac{1-\cos n\Pi}{n^2}\sin nx$$

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATION

CLOs	Course Learning Outcome
CLO 20	Apply numerical methods to obtain approximate solutions to Taylors, Eulers, Modified Eulers
CLO 21	Runge-Kutta methods of ordinary differential equations.

The important methods of solving ordinary differential equations of first order numerically are as follows

1) Taylors series method
2) Euler's method
3) Modified Euler's method of successive approximations
4) Runge- kutta method

To describe various numerical methods for the solution of ordinary differential eqn's,we consider the general 1^{st} order differential eqn $dy/dx=f(x,y)$-------(1) with the initial condition $y(x_0)=y_0$

The methods will yield the solution in one of the two forms:

i) A series for y in terms of powers of x,from which the value of y can be obtained by direct substitution

Using Taylor's expansion evaluate the integral of

$y' - 2y = 3e^x, y(0) = 0$ **, at a)** $x = 0.2$

b) compare the numerical solution obtained with exact solution .

Sol: Given equation can be written as $2y + 3e^x = y', y(0) = 0$

Differentiating repeatedly w.r.t to 'x' and evaluating at $x = 0$

$$y'(x) = 2y + 3e^x, y'(0) = 2y(0) + 3e^0 = 2(0) + 3(1) = 3$$

$$y''(x) = 2y' + 3e^x, y''(0) = 2y'(0) + 3e^0 = 2(3) + 3 = 9$$

$$y'''(x) = 2.y''(x) + 3e^x, y'''(0) = 2y''(0) + 3e^0 = 2(9) + 3 = 21$$

$$y^{iv}(x) = 2.y'''(x) + 3e^x, y^{iv}(0) = 2(21) + 3e^0 = 45$$

$$y^{v}(x) = 2.y^{iv} + 3e^x, y^{v}(0) = 2(45) + 3e^0 = 90 + 3 = 93$$

In general, $y^{(n+1)}(x) = 2.y^{(n)}(x) + 3e^x$ or $y^{(n+1)}(0) = 2.y^{(n)}(0) + 3e^0$

The Taylor's series expansion of $y(x)$ about $x_0 = 0$ is

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y''''(0) + \frac{x^5}{5!}y'''''(0) + \ldots.$$

Substituting the values of

$y(0), y'(0), y''(0), y'''(0), \ldots\ldots\ldots$

$$y(x) = 0 + 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{45}{24}x^4 + \frac{93}{120}x^5 + \ldots\ldots\ldots$$

$$y(x) = 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \frac{15}{8}x^4 + \frac{31}{40}x^5 + \ldots\ldots\ldots \rightarrow \text{equ1}$$

Now put $x = 0.1$ in equ1

$$y(0.1) = 3(0.1) + \frac{9}{2}(0.1)^2 + \frac{7}{2}(0.1)^3 + \frac{15}{8}(0.1)^4 + \frac{31}{40}(0.1)^5$$

$$= 0.34869$$

Now put $x = 0.2$ in equal

$$y(0.2) = 3(0.2) + \frac{9}{2}(0.2)^2 + \frac{7}{2}(0.2)^3 + \frac{15}{8}(0.2)^4 + \frac{31}{40}(0.2)^5 = 0.811244$$

$$y(0.3) = 3(0.3) + \frac{9}{2}(0.3)^2 + \frac{7}{2}(0.3)^3 + \frac{15}{8}(0.3)^4 + \frac{31}{40}(0.3)^5$$

$$= 1.41657075$$

Using Taylor's series method, solve the equation $\frac{dy}{dx} = x^2 + y^2$ **for** $x = 0.4$ **given that** $y = 0$ **when** $x = 0$

Sol:Given that $\frac{dy}{dx} = x^2 + y^2$ and $y = 0$ when $x = 0$ i.e. $y(0) = 0$

Here $y_0 = 0$, $x_0 = 0$

Differentiating repeatedly w.r.t 'x' and evaluating at $x = 0$

$$y'(x) = x^2 + y^2, y'(0) = 0 + y^2(0) = 0 + 0 = 0$$
$$y''(x) = 2x + y'.2y, y''(0) = 2(0) + y'(0)2.y = 0$$
$$y'''(x) = 2 + 2yy'' + 2y'.y', y'''(0) = 2 + 2.y(0).y''(0) + 2.y'(0)^2 = 2$$
$$y''''(x) = 2.y.y''' + 2.y''.y' + 4.y''.y', y''''(0) = 0$$

The Taylor's series for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y''''(0) + \ldots$$

Substituting the values of $y(0), y'(0), y''(0), \ldots\ldots$

$$y(x) = 0 + x(0) + 0 + \frac{2x^3}{3!} + 0 + \ldots\ldots\ldots = \frac{x^3}{3} +$$ (Higher order terms are neglected)

$$\therefore y(0.4) = \frac{(0.4)^3}{3} = \frac{0.064}{3} = 0.02133$$

Solve $y' = x - y^2, y(0) = 1$ **using Taylor's series method and compute y(0.1),y(0.2)**

Sol:Given that $y' = x - y^2, y(0) = 1$

Here $y_0 = 1$, $x_0 = 0$

Differentiating repeatedly w.r.t 'x' and evaluating at x=0

$$y'(x) = x - y^2, y'(0) = 0 - y(0)^2 = 0 - 1 = -1$$
$$y''(x) = 1 - 2y.y', y''(0) = 1 - 2.y(0)y'(0) = 1 - 2(-1) = 3$$
$$y'''(x) = 1 - 2yy' - 2(y')^2, y'''(0) = -2.y(0).y''(0) - 2.(y'(0))^2 = -6 - 2 = -8$$
$$y''''(x) = -2.y.y''' - 2.y''.y' - 4.y''.y', y''''(0) = -2.y(0).y'''(0) - 6.y''(0).y'(0) =$$

The Taylor's series for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + \frac{x}{1!}y^{1}(0) + \frac{x^2}{2!}y^{11}(0) + \frac{x^3}{3!}y^{111}(0) + ...$$

Substituting the value of $y(0)$, $y^{1}(0)$, $y^{11}(0)$,.....

$$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{8}{6}x^3 + \frac{34}{24}x^4 + ...$$

$$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 + \frac{17}{12}x^4 + ... \rightarrow (1)$$

now put x = 0.1 in (1)

$$y(0.1) = 1 - 0.1 + \frac{3}{2}(0.1)^2 + \frac{4}{3}(0.1)^3 + \frac{17}{12}(0.1)^4 + ...$$

$= 0.91380333 \simeq 0.91381$

Similarly put x = 0.2 in (1)

$$y(0.2) = 1 - 0.2 + \frac{3}{2}(0.2)^2 - \frac{4}{3}(0.2)^3 + \frac{17}{12}(0.2)^4 + ... =$$

0.8516.

Solve $y^1 = x^2 - y$, $y(0) = 1$, using Taylor's series method and compute y(0.1), y(0.2), y(0.3) and y(0.4) (correct to 4 decimal places).

Sol. Given that $y^I = x^2 - y$ and $y(0) = 1$

Here $x_0 = 0$, $y_0 = 1$ or y =1 when x=0

Differentiating repeatedly w.r.t 'x' and evaluating at x = 0.

$Y^I(x) = x^2 - y, \quad y^I(0) = 0 - 1 = -1$

$y^{II}(x) = 2x - y^I, \quad y^{II}(0) = 2(0) - y^I(0) = 0 - (-1) = 1$

$y^{III}(x) = 2 - y^{II}, \quad y^{III}(0) = 2 - y^{II}(0) = 2 - 1 = 1,$

$y^{IV}(x) = - y^{III}, \quad y^{IV}(0) = -y^{III}(0) = -1.$

The Taylor's servies for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + \frac{x}{1!}y^{I}(0) + \frac{x^2}{2!}y^{II}(0) + \frac{x^3}{3!}y^{III}(0) + \frac{x^4}{4!}y^{IV}(0) + \ldots\ldots$$

substituting the values of $y(0)$, $y^{1}(0)$, $y^{11}(0)$, $y^{111}(0)$,......

$$y(x) = 1 + x\,(-1) + \frac{x^2}{2}(1) + \frac{x^3}{6}(1) + \frac{x^4}{24}(-1) + \ldots\ldots$$

$$y(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + \ldots\ldots \quad \rightarrow (1)$$

Now put x = 0.1 in (1),

$$y(0.1) = 1 - 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{6} - \frac{(0.1)^4}{24} + \ldots.$$

$$= 1 - 0.1 + 0.005 + 0.01666 - 0.0000416 -$$

0.905125 ~ 0.9051

(4 decimal places)

Now put x = 0.2 in eq (1),

$$y(0.2) = 1 - 0.2 + \frac{(0.2)^2}{2} + \frac{(0.2)^3}{6} - \frac{(0.2)^4}{64}$$

$$= 1 - 0.2 + 0.02 + 0.001333 - 0.000025$$

$$= 1.021333 - 0.200025$$

$$= 0.821308 \sim 0.8213 \text{ (4 decimals)}$$

Similarly y(0.3) = 0.7492 and y (0.4) = 0.6897 (4 decimal places).

Solve $\frac{dy}{dx}$ -1 = xy and y(0) = 1 using Taylor's series method and ompute y(0.1).

Sol. Given that $\frac{dy}{dx} - 1 = xy$ and $y(0) = 1$

Here $\frac{dy}{dx} = 1 + xy$ and $y_0 = 1$, $x_0 = 0$.

Differentiating repeatedly w.r.t 'x' and evaluating at $x_0 = 0$

$y^{I}(x) = 1 + xy,$ $\quad y^{I}(0) = 1+0(1) = 1$.

$y^{II}(x) = x.y' + y,$ $\quad y^{II}(0) = 0+1=1$

$y^{III}(x) = x.y'' + y^{I} + y^{I},$ $\quad y^{III}(0) = 0.(1) + 2(1) =2$

$y^{IV}(x) = xy^{III} + y^{II} + 2y^{II},$ $\quad y^{IV}(0) = 0+3(1) =3.$

$y^{V}(x) = xy^{IV} + y^{III} +2y^{III},$ $\quad y^{V}(0) = 0 + 2 + 2(3) = 8$

The Taylor series for f(x) about $x_0 = 0$ is

1. **Using Euler's method solve for x = 2 from $\frac{dy}{dx} = 3x^2 + 1, y(1) = 2$, taking step size** (I) h = 0.5 and (II) h=0.25

Sol: here $f(x,y) = 3x^2 + 1$, $x_0 = 1, y_0 = 2$

Euler's algorithm is $y_{n+1} = y_n + h\ f(x_n,y_n)$, n = 0,1,2,3,.....

→(1)

h = 0.5 $\therefore x_1 = x_0 + h = 1+0.5 = 1.5$

aking n = 0 in (1) , we have $x_2 = x_1 + h = 1.5 + 0.5 = 2$

$y_1 = y_0 + h\ f(x_0,y_0)$

i.e. $y_1 = y(0.5) = 2 + (0.5)\ f(1,2) = 2 + (0.5)\ (3 + 1) = 2 + (0.5)(4)$

Here $x_1 = x_0 + h = 1 + 0.5 = 1.5$

$\therefore y(1.5) = 4 = y_1$

Taking n = 1 in (1),we have

$y_2 = y_1 + h\, f(x_1,y_1)$ i.e. $y(x_2) = y_2 = 4 + (0.5)\, f(1.5,4) = 4 + (0.5)[3(1.5)^2 + 1] = 7.875$ Here $x_2 = x_4 + h = 1.5 + 0.5 = 2$

$\therefore y(2) = 7.875$

(I) $h = 0.25$ $\therefore x_1 = 1.25,\ x_2 = 1.50,\ x_3 = 1.75,\ x_4 = 2$

Taking n = 0 in (1), we have

$y_1 = y_0 + h\, f(x_0,y_0)$

i.e. $y(x_1) = y_1 = 2 + (0.25)\, f(1,2) = 2 + (0.25)\,(3 + 1) = 3$ $y(x_2) = y_2 = y_1 + h\, f(x_1,y_1)$

i.e. $y(x_2) = y_2 = 3 + (0.25)\, f(1.25,3)$

$= 3 + (0.25)[3(1.25)^2 + 1]$

$= 4.42188$

Here $x_2 = x_1 + h = 1.25 + 0.25 = 1.5$

$\therefore y(1.5) = 5.42188$

Taking n = 2 in (1), we have

i.e. $y(x_3) = y_3 = h\, f(x_2,y_2)$

$= 5.42188 + (0.25)\, f(1.5,2)$

$= 5.42188 + (0.25)\, [3(1.5)^2 + 1]$

$= 6.35938$ Here $x_3 = x_2 + h = 1.5 + 0.25 = 1.75$

$\therefore y(1.75) = 7.\ 35938$

Taking n = 4 in (1),we have $y(x_4) = y_4 = y_3 + h\, f(x_3,y_3)$

i.e. $y(x_4) = y_4 = 7.35938 + (0.25)\, f(1.75,2)$

$= 7.35938 + (0.25)[3(1.75)^2 + 1]$

$= 8.90626$

3. **Given that $\frac{dy}{dx}$ = xy ,y(0) = 1 determine y(0.1),using Euler's method.** h =0.1

Sol: The given differentiating equation is $\frac{dy}{dx}$= xy, y(0) = 1

a = 0

Here f(x,y) = xy , $x_0 = 0$ and $y_0 = 1$

Since h is not given much better accuracy is obtained by breaking up the interval (0,0.1) in to five steps.

i.e. h = $\frac{b-a}{5}$ = $\frac{0.1}{5}$ = 0.02

Euler's algorithm is $y_{n+1} = y_n + h\, f(x_n,y_n)$
→(1)

∴From (1) form = 0, we have

$$y_1 = y_0 + h\, f(x0,y_0)$$
$$= 1 + (0.02)\, f(0,1)$$
$$= 1 + (0.02)\,(0)$$
$$= 1$$

Next we have $x_1 = x_0 + h = 0 + 0.02 = 0.02$

∴From (1), form = 1,we have

$$y_2 = y_1 + h\, f(x_1,y_1)$$

$$= 1 + (0.02)\, f(0.02,1)$$

$$= 1 + (0.02)\,(0.02)$$

$$= 1.0004$$

Next we have $x_2 = x_1 + h = 0.02 + 0.02 =0.04$

∴From (1), form = 2,we have

$$y_3 = y_2 + h\, f(x_2,y_2)$$

$$= 1.004 + (0.02)\,(0.04)\,(1.0004)$$

$$= 1.0012$$

Next we have $x_3 = x_2 + h = 0.04 + 0.02 = 0.06$

∴From (1), form = 3,we have

$y_4 = y_3 + h\, f(x_3, y_3)$

$= 1.0012 + (0.02)\,(0.06)\,(1.00012)$

$= 1.0024.$

Next we have $x_4 = x_3 + h = 0.06 + 0.02 = 0.08$

∴From (1), form = 4,we have

$y_5 = y_4 + h\, f(x_4, y_4)$

$= 1.0024 + (0.02)\,(0.08)\,(1.00024)$

$= 1.0040.$

Next we have $x_5 = x_4 + h = 0.08 + 0.02 = 0.1$

When $x = x_5$, $y \simeq y_5$

∴$y = 1.0040$ when $x = 0.1$

using modified Euler's method find the approximate value of x when $x = 0.3$ given that $dy/dx = x + y \; and \; y(0) = 1$

sol: Given $dy/dx = x + y \; and \; y(0) = 1$

Here $f(x, y) = x + y, x_0 = 0, \; and \; y_0 = 1$

Take h = 0.1 which is sufficiently small

Here

$$x_0 = 0, x_1 = x_0 + h = 0.1, x_2 = x_1 + h = 0.2, x_3 = x_2 + h = 0.3$$

The formula for modified Euler's method is given by

$$y_{k+1}{}^{(i)} = y_k + h/2\left[f(x_k + y_k) + f\left(x_{k+1}, y_{k+1}{}^{(i-1)}\right)\right] \rightarrow (1)$$

$$y_{k+1}^{(i)} = y_k + h/2\left[f\left(x_k + y_k\right) + f\left(x_{k+1}, y_{k+1}^{(i-1)}\right)\right] \rightarrow (1)$$

Step1: To find $y_1 = y(x_1) = y(0.1)$

Taking k = 0 in eqn(1)

$$y_{k+1}^{(i)} = y_0 + h/2\left[f\left(x_0 + y_0\right) + f\left(x_1, y_1^{(i-1)}\right)\right] \rightarrow (2)$$

when i = 1 in eqn (2)

$$y_1^{(i)} = y_0 + h/2\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(0)}\right)\right]$$

First apply Euler's method to calculate $y_1^{(0)}$ = y_1

$$\therefore\ y_1^{(0)} = y_0 + h\, f\left(x_0, y_0\right)$$

= 1+(0.1)f(0.1)

= 1+(0.1)

= 1.10

$$y_{k+1}^{(i)} = y_k + h/2\left[f\left(x_k + y_k\right) + f\left(x_{k+1}, y_{k+1}^{(i-1)}\right)\right] \rightarrow (1)$$

Step1: To find $y_1 = y(x_1) = y\ (0.1)$

Taking k = 0 in eqn(1)

$$y_{k+1}^{(i)} = y_0 + h/2\left[f\left(x_0 + y_0\right) + f\left(x_1, y_1^{(i-1)}\right)\right] \rightarrow (2)$$

when i = 1 in eqn (2)

$$y_1^{(i)} = y_0 + h/2\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(0)}\right)\right]$$

First apply Euler's method to calculate $y_1^{(0)}$ = y_1

$$\therefore\ y_1^{(0)} = y_0 + h\, f\left(x_0, y_0\right)$$

= 1+(0.1)f(0.1)

= 1+(0.1)

= 1.10

$$\therefore y_1^{(1)} = y_0 + 0.1/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$

= 1+0.1/2[f(0,1) + f(0.1,1.10)

= 1+0.1/2[(0+1)+(0.1+1.10)]

= 1.11

When i=2 in eqn (2)

$$y_1^{(2)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

= 1+0.1/2[f(0.1)+f(0.1,1.11)]

= 1 + 0.1/2[(0+1)+(0.1+1.11)]

= 1.1105

$$y_1^{(3)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right]$$

= 1+0.1/2[f(0,1)+f(0.1 , 1.1105)]

= 1+0.1/2[(0+1)+(0.1+1.1105)]

= 1.1105

Since $y_1^{(2)} = y_1^{(3)}$ $\therefore$ $y_1 = 1.1105$

Step:2 To find $y_2 = y(x_2) = y(0.2)$

Taking k = 1 in eqn (1) , we get

$$y_2^{(i)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2, y_2^{(i-1)}\right)\right] \rightarrow (3)$$

i = 1,2,3,4,.....

For i = 1

$$y_2^{(1)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2, y_2^{(0)}\right)\right]$$

$y_2^{(0)}$ is to be calculate from Euler's method

$$y_2^{(0)} = y_1 + h\ f(x_1, y_1)$$

= 1.1105 + (0.1) f(0.1 , 1.1105)

= 1.1105+(0.1)[0.1+1.1105]

= 1.2316

ORDINARY DIFFERENTIAL EQUATION

$\therefore y_2^{(1)} = 1.1105 + 0.1/2\left[f(0.1, 1.1105) + f(0.2, 1.2316)\right]$

= 1.1105 +0.1/2[0.1+1.1105+0.2+1.2316] = 1.2426

$$y_2^{(2)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2 y_2^{(1)}\right)\right]$$

= 1.1105 + 0.1/2[f(0.1 , 1.1105) , f(0.2 . 1.2426)]

= 1.1105 + 0.1/2[1.2105 + 1.4426]

= 1.1105 + 0.1(1.3266) = 1.2432

$$y_2^{(3)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2 y_2^{(2)}\right)\right]$$

= 1.1105+0.1/2[f(0.1,1.1105)+f(0.2 , 1.2432)]

= 1.1105+0.1/2[1.2105+1.4432)] = 1.1105 + 0.1(1.3268) = 1.2432 Since $y_2^{(3)} = y_2^{(3)}$ Hence y_2 = 1.2432

Step:3

To find $y_3 = y(x_3) = y\ y(0.3)$

Taking k =2 in eqn (1) we get

$$y_3^{(1)} = y_2 + h/2\left[f\left(x_2, y_2\right) + f\left(x_3, y_3^{(i-1)}\right)\right] \to (4)$$

For i = 1 ,

$$y_3^{(1)} = y_2 + h/2\left[f\left(x_2, y_2\right) + f\left(x_3, y_3^{(0)}\right)\right]$$

$y_3^{(0)}$ is to be evaluated from Euler's method .

$$y_3^{(0)} = y_2 + h\, f\left(x_2, y_2\right)$$

= 1.2432 +(0.1) f(0.2 , 1.2432)

= 1.2432+(0.1)(1.4432)

= 1.3875

$\therefore\ y_3^{(1)}$ = 1.2432+0.1/2[f(0.2 , 1.2432)+f(0.3, 1.3875)]

= 1.2432 + 0.1/2[1.4432+1.6875]

= 1.2432+0.1(1.5654) = 1.3997

$$y_3^{(2)} = y_2 + h/2\left[\ f\left(x_2, y_2\right) + f\left(x_3, y_3^{(1)}\right)\right]$$

= 1.2432+0.1/2[1.4432+(0.3+1.3997)]

= 1.2432+ (0.1) (1.575) = 1.4003

$$y_3^{(3)} = y_2 + h/2\left[f\left(x_2, y_2\right) + f\left(x_3, y_3^{(2)}\right)\right]$$

= 1.2432+0.1/2[f(0.2 , 1.2432)+f(0.3 , 1.4003)]

= 1.2432 + 0.1(1.5718)

= 1.4004

$$y_3^{(4)} = y_2 + h/2\left[f\left(x_2, y_2\right) + f\left(x_3, y_3^{(3)}\right)\right]$$

= 1.2432 + 0.1/2[1.4432+1.7004]

= 1.2432+(0.1)(1.5718)

= 1.4004

Since $y_3^{(3)} = y_3^{(4)}$

Hence $y_3 = 1.4004$ $\therefore$ The value of y at x = 0.3 is 1.4004

PARTIAL DIFFERENTIAL EQUATION AND APPLICATIONS

CLOs	Course Learning Outcome
CLO 22	Understand the concept of order and degree with reference to partial differential equation
CLO 23	Formulate and solve partial differential equations by elimination of arbitrary constants and functions
CLO 24	Understand partial differential equation for solving linear equations by Lagrange method.

CLOs	Course Learning Outcome
CLO 25	Learning method of separation of variables
CLO 26	Solving the heat equation and wave equation in subject to boundary conditions
CLO 27	Understand the concept of partial differential equations to the real-world problems of electromagnetic and fluid dynamics

similarly differentiating (1) w.r.t. y, we get

$\frac{\partial z}{\partial y} = f^1(u)\frac{\partial u}{\partial y} = f^1(u)(-2y)$

i.e. $q = f^1(u)(-2y)$..........(3)

$\therefore$ (2) $\div$ (3) gives $\frac{p}{q} = \frac{f^1(u)2x}{f^1(u)(-2y)} = -\frac{x}{y}$

i.e., $py + qx = 0$

This is the required partial differential equation.

2) Form a partial differential equation by eliminating the arbitrary functions from Z=yf(x)+xf(y)

given $Z = yf(x) + xf(y)$ ....(1)

Differentiating (1) partially with respect to x and y, we have

$P = yf^1(x) + g(y)$ ....(2) and

$q = f(x) + xg^1(y)$ ....(3)

Since the reletions (1),(2) and (3) are not sufficient to eliminate f,g,f^1,g^1

So we find the second order partial derivatives

$\frac{\partial^2 z}{\partial x^2} = r = yf^{11}(x)$ ....(4)

$\frac{\partial^2 z}{\partial x\ \partial y}=s=f^1(x) + g^1(y)$ ....(5)

$\frac{\partial^2 z}{\partial y^2}=t=xg^{11}(y)$ ....(6)

From (2) and (3) ,we have

$f^1(x) = \frac{1}{y}[p - g(y)]$ and $g^1(y) = \frac{1}{x}[q - f(x)]$ ....(7)

From (5), we have

$S=f^1(x) + g^1(y)$

$\therefore s = \frac{1}{y}[p - g(y)] + \frac{1}{x}[q - f(x)]$, [using (7)]
i.e., $xys=x[p - g(y)] + y[q - f(x)]$

i.e., $xys=px+qy-[yf(x)+xg(y)]$ or $xys= px+qy-z$ [using (1)]

this is the required partial differential equation.

Let us consider a Partial Differential Equation of the form F (x,y,z, p,
q)=0. If it is Linear in p and q, it is called a Linear Partial Differential Equation (i.e. Order and Degree of p and q is one) If it is Not Linear in p and q, it is called as nonlinear Partial Differential Equation (i.e. Order and Degree of p and q is other than one) Consider a relation of the type F (x,y,z,a,b) =0 By eliminating the arbitrary constants a and b from this equation, we get F(x,y,z,p,q) =0, which is called a **complete integral** or **complete solution** of the PDE. A solution of (,y,z,p,z) =0 obtained by giving particular values to a and b in the complete Integral is called a **particular Integral.**

LAGRANGE'S LINEAR EQUATION

A linear Partial Differential Equation of order one, involving a dependent variable z and two independent variables x and y, and is of the form $Pp+Qq=R$, where P,Q,R are functions of x,y,z is called Lagrange's Linear Equation.

Solution of the Linear Equation

Consider $Pp+Qq=R$ Now, $\frac{dx}{p}=\frac{dy}{Q}=\frac{dz}{R}$

Case 1: If it is possible to separate variables then, consider any two equations, solve them by integrating. Let the solutions of these equations are $u=a,=b$

$\therefore (,v) =0$ is the required solution of given equation.

Case 2: If it is not possible to separate variables then $dxP(x,y,z)=dyQ(x,y,z)=dzR(x,y,z)$ To solve above type of problems we have following methods

Lagrange's solution:

The partial differential equation of the Pp+Qq=R where P,Q and R are functions of x,y,z is the standard form of linear partial differential equation of first order .therefore ,is called Lagrange's linear equation

Lagrange's method of solving the linear partial differential equation of order one, namely, Pp+Qq=R

The general solution of the linear differential equation

Pp+Qq=R ----(1)is, Ø(u,v)=0----(2) Where Ø is an arbitrary function u(x,y,z)=C_1 and v(x,y,z)=C_2 from a solution of the equations its auxiliary equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$solution for these auxiliary equations .There are two methods for solving the above auxiliary equations

Method 1

Take two members and solve the equation, then take two other members and solve that equation. Then proceed to step (2)

Method2

Method of multipliers.

Let $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{ldx+mdy+ndz}{lP+mQ+nR}$ and

$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{l_1dx+m_1dy+n_1dz}{l_1P+m_1Q+n_1R}$ Where l, m, n and l_1, m_1, n_1 are chosen such that ldx+mdy+ndz=0 and l1dx+m1dy+n1dz=0 and solve these Then proceed to step (2)

Method of grouping:

In some problems, it is possible to solve any two of the equations,

$\frac{dx}{p}=\frac{dy}{Q}$ (or) $\frac{dy}{Q}=\frac{dz}{R}$ (or) $\frac{dx}{p}=\frac{dz}{R}$

In such cases, solve the differential equation, get the solution and then substitute in the other differential equation

Example:

Find the general solution of $y^2zp+x^2zq=y^2x$.

Solution:

Given equation is $y^2zp+x^2zq=y^2x$..................(1)

The auxiliary equations are $dx/y^2z = dy/x^2z = dz/y^2x$

From $dx/y^2z = dy/x^2z$

Or $x^2dx = y^2\,dy$

We have $x^3/3 - y^3/3 = a$

Or $x^3 - y^3 = c_1$

And from $dx/y^2z = dz/y^2x$

$x^2/2 - y^2/2 = b$ (or) $x^2 - y^2 = c_2$

Thus the general solution is $\phi(x^3 - y^3, x^2 - y^2) = 0$

Or $x^3 - y^3 = f(x^2 - y^2)$ Where f is arbitrary function.

Method of Multipliers

Consider $\frac{dx}{p}=\frac{dy}{Q}=\frac{dz}{R}=l\,dx+m\,dy+n\,dz$

In this, we have to choose l,, so that denominator=0.

That will give us solution by integrating $l\,dx+m\,dy+n\,dz$

1. Solve $(x^2 - yz)\,p + (y^2 - xz)q = z^2 - xy$

Solution;

The equation is $(x^2 - yz)\,p + (y^2 - xz)q = z^2 - xy$

Here $P = x^2 - yz$, $Q = y^2 - xz$, $R = z^2 - xy$

The auxiliary equations are $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

i,e,. $\frac{dx}{(x^2-yz)}=\frac{dy}{(y^2-xz)}=\frac{dz}{z^2-xy}$ .... (1)

Taking 1, -1, 0 and 0, 1,-1 as multifliers, we get

Each fraction $=\frac{dx-dy}{(x^2-yz)-(y^2-xz)}$ and also $=\frac{dy-dz}{(y^2-xz)-(z^2-xy)}$

Therefore $\frac{dx-dy}{(x^2-y^2)+z(x-y)} = \frac{dy-dz}{(y^2-z^2)+x(y-z)}$

i.e., $\frac{d(x-y)}{(x-y)(x+y+z)} = \frac{d(y-z)}{(y-z)(x+y+Z)}$

or $\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$

Integrating both sides we get

$\int \frac{d(x-y)}{(x-y)} = \int \frac{d(y-z)}{(y-z)} + c$

i.e., log(x-y) =log(y-z) +logc1

Or $\frac{x-y}{y-z} = c1$………………. (2)

Again taking x,y,z are multipliers, we have

Each fraction = $\frac{xdx+ydy+zdz}{x(x^2-yz)+y(y^2-zx)+z(z^2-xy)} = \frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz}$

$= \frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$ ………………. (3)

Now 1,1,1 as multipliers , we get

Each fraction = $\frac{xdx+ydy+zdz}{(x^2-yz)+(y^2-zx)+(z^2-xy)} = \frac{xdx+ydy+zdz}{x^2+y^2+z^2-xy-yz-zx}$ ……………….(4)

Equating (3) and (4) and on simplification we get

$$\frac{xdx+ydy+zdz}{x+y+z}=dx+dy+dz$$

i.e. $(x+y+z)d(x+y+z)=xdx+ydy+zdz$

Integrating, we get

$$\frac{(x+y+z)^2}{2}=\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}+c_2$$

Or $xy+yz+zx=c_2$.................. (5)

Hence the general solution is $\phi\left(\frac{x-y}{y-z}, xy+yz+zx\right)=0$ ....(6) [from(2)and(5)]

Problem 1:

Find the general solution of $y^2zp+x^2zq=y^2x$.

Solution:

Given equation is $y^2zp+x^2zq=y^2x$……………….(1)

The auxiliary equations are $dx/y^2z = dy/x^2z = dz/y^2x$

From $dx/y^2z = dy/x^2z$

Or $x^2dx = y^2 dy$

We have $x^3/3 - y^3/3 = a$

Or $x^3 - y^3 = c_1$

And from $dx/y^2z = dz/y^2x$

$x^2/2 - y^2/2 = b$ (or) $x^2 - y^2 = c_2$

Thus the general solution is $\phi(x^3 - y^3, x^2 - y^2) = 0$

Or $x^3 - y^3 = f(x^2 - y^2)$ Where f is arbitrary function.

Problem 2:

Find the general solution of $p+q=1$

Solution:

The given equation is $p+q=1$

The subsidiary equations are $dx/1=dy/1=dz/1$ ....(1)

Taking the first two members, we get $dx=dy$

By integrating we get $x= y+a$ i.e, $x-y=a$

By taking the last two members we get $dy=dz$

By integrating we get $y=z+b$ i.e, $y-z=b$

Hence the general solution of (1) is $\Phi(a,b)=0$

i.e, $\Phi(x-y,y-z)=0$ where Φ is arbitrary

Problem 3:

Solve $px+qy=z$

Solution:

The subsidiary equations are $dx/x=dy/y=dz/z$

Now taking the first two members,

we have $dx/x=dy/y$

Integrating and simplifying we get

$$\log x=\log y+\log c_1 \text{ or } x/y=c_1$$

taking the last two members we have $dy/y=dz/z$

Integrating and simplifying we get

$$\log y=\log z+\log c_2 \text{ or } y/z=c_2$$

Hence the general solution is $f(c_1,c_2)=0$

i.e, $f(x/y,y/z)=0$ where f is arbitrary.

Problem 4:

Solve (mz-ny)p+(nx-lz)q=ly-mx

Solution:

The given equation is

(mz-ny)p+(nx-lz)q=ly-mx

Here P=mz-ny,Q=nx-lz,R+ly-mx

The auxiliary equations are

dx/P=dy/Q=dz/R

i.e, dx/mz-ny=dy/nx-lz=dz/ly-mx

Choosing x,y,z as multipliers,we get

Each fraction=xdx+ydy+zdz/0,which gives xdx+ydy+zdz=0

By integrating we get $x^2+y^2+z^2=c_1$

Again by choosing l,m,n as multipliers we obtain

Each fraction=ldx+mdy+ndz/0,which gives

ldx+mdy+ndz=0

Integrating,$lx+my+nz=c_2$

Hence the general solution is $f(x^2+y^2+z^2,lx+my+nz)=0$

Problem 5:

Solve $z(z^2+xy)(px-qy)=x^4$

Solution:

Given equation is

$xz(z^2+xy)p-yz(z^2+xy)q=x^4$

the auxiliary equations are

$dx/xz(z^2+xy)=dy/-z(z^2+xy)=dz/x^4$.....(1)

From first two ratios we get

$dx/x=dy/-y$

By integrating we get $\log x=-\log y+c_1$ (or)

$xy+c_1$...............(2)

By taking first and last ratios and by integrating we get

$x^4-z^4-2xyz^2=c^2$

Therefore the required general solution is $f(xy,x^4-y^4-2xyz^2)=0$

Method of Separation of variables

This method involves a solution which breaks up into product of functions, each of which contains only one of the independent variables.

Procedure:

For the given PDE, let us consider the solution to be $z=(x).\,Y(\,y\,)$

$$\Rightarrow \frac{\partial z}{\partial x}=\frac{\partial X}{\partial x}Y=X'Y,\ \frac{\partial z}{\partial y}=X\frac{\partial y}{\partial y}=X\,Y'$$

Substitute these values in the given equation, from which we can separate variables.

Write the equation such that X', and x terms are on one side and

similarly Y',Y and y terms are on the other side.

Let it be $F\ X',,=G\ Y',Y,y=\lambda \Rightarrow F\ X',X,x=\lambda$ and $G\ Y',Y,y=\lambda$

Solve these equations; finally substitute in $z=X\ (x)\,(\ \ )$

which gives the required solution.

PROBLEMS

1) Solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ where u(x,0) = $6e^{-3x}$.

Solution; Given that $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$.........(1)

Subject to the condition u(x,0) = $6e^{-3x}$ ....(2)

Using the method of separation of variables , we seek a solution of (1) in the form

U(x,t) = X(x) T(t) ....(3)

If (3) is a solution of (1),(3) must satisfy the equation.

We have $\frac{\partial u}{\partial x} = X^1(x)T(t)$; $\frac{\partial u}{\partial t} = X(x)T^1(t)$ ....(4)

Using (3) and (4) in (1) , we get

$$X^1(x)T(t) = 2X(x)T^1(t) + X(x)T(t)$$

i.e.,

$$X^1(x)T(t) = X(x)[2T^1(t) + T(t)]$$

or $\frac{X^1(x)}{X(x)} = \frac{2T^1(t)+T(t)}{T(t)}$

Since L.H.S is a function of x and R.H.S is a function of t, the equality is valid for all x and t if and only if each is equal to the same constant λ for all x and t.

$\therefore \quad \frac{X^1(x)}{X(x)} = \frac{2T^1(t)+T(t)}{T(t)} = \lambda$

i.e., $X^1(x) - \lambda X(x) = 0$

i.e., $X(x) = Ae^{\lambda x}$

and $2T^1(t) + T(t) == \lambda T(t)$

i.e., $T^1(t) + \frac{(1-\lambda)}{2} T(t) = 0$

$$\therefore T(t) = Be^{\frac{(\lambda-1)t}{2}}$$

$$\therefore u(x,t) = Ae^{\lambda x}.Be^{\frac{(\lambda-1)t}{2}}$$

i.e., $u(x,t) = Ce^{\lambda x}.e^{\frac{(\lambda-1)t}{2}}$

Using condition (2), $u(x,0) = 6e^{-3x}$

$$Ce^{\lambda x} = 6e^{-3x}$$

$\therefore$ λ= - 3 , C = 6

Hence the required solution is

$$u(x,t) = 6e^{-3x}e^{-2t}$$

i.e.,

$$u(x,t) = 6e^{-(3x+2t)}$$

EXERCISE

1.Solve by the method of separation of variables $2xz_x - 3yz_y = 0$

2. Solve by the method of separation of variables $4u_x + u_y = 3u$ *and* $u(0,y) = e^{-5y}$

3.Solve by the method of separation of variables , Solve $u_{xt} = e^{-t} \cos x$ with u(x,0)=0 and u(0,t)=0

4- Solve $\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$, Given that u=o when t=0 and $\frac{\partial u}{\partial t} = 0$ when x=o .

5. Solve by the method of separation of variables $u_x - 4u_y = 0$ *and* $u(0,y) = 8e^{-3y}$

one-dimensional heat equation: $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Before we start to solve this equation, let's mention a few more conditions that we will need to know to nail down a specific solution. If the metal bar that we're studying has a specific length, *l*, then we need to know the temperatures at the ends of the bars. These temperatures will give us boundary conditions similar to the ones we worked with for the wave equation. To make life a bit simpler for us as we solve the heat equation, let's start with the case when the ends of the bar, at $x = 0$ and $x = l$ both have temperature equal to 0 for all time (you can picture this situation as a metal bar with the ends stuck against blocks of ice, or some other cooling apparatus keeping the ends exactly at 0 degrees). Thus we will be working with the same boundary conditions as before, namely

(2) $u(0,t) = 0$ and $u(l,t) = 0$ for all values of *t*

let's start by writing

(3) $u(x,t) = F(x)G(t)$

where *F*, and *G*, are single variable functions. Differentiating this equation for $u(x,t)$ with respect to each variable yields

(4) $\frac{\partial^2 u}{\partial x^2} = F''(x)G(t)$ and $\frac{\partial u}{\partial t} = F(x)G'(t)$

When we substitute these two equations back into the original heat equation

(7) $\frac{G'(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)}$

Left-hand side only depends on the variable *t*, and the right-hand side just depends on *x*. As a result, to have these two be equal can only mean one thing, that they are both equal to the same constant, *k*:

(8) $\frac{G'(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k$

let's first take a look at the implications for $F(x)$ as the boundary conditions will again limit the possible solution functions. From (8) we get that $F(x)$ has to satisfy

These solutions are just the same as before, namely the general solution is:

where again *A* and *B* are constants and now we have $\omega = \sqrt{-k}$. Next, let's consider the boundary conditions $u(0,t) = 0$ and $u(l,t) = 0$. These are equivalent to stating that $F(0) = F(l) = 0$. Substituting in 0 for *x* in (11) leads to $F(0) = A\cos(0) + B\sin(0) = A = 0$

so that $F(x) = B\sin(\omega x)$. Next, consider $F(l) = B\sin(\omega l) = 0$. As before, we check that *B* can't equal 0, otherwise $F(x) = 0$ which would then mean that $u(x,t) = F(x)G(t) = 0 \cdot G(t) = 0$, the trivial solution, again. With $B \neq 0$, then it must be the case that $\sin(\omega l) = 0$ in order to have $B\sin(\omega l) = 0$. Again, the only way that this can happen is for ωl to be a multiple of π. This means that once again

Heat Equation with Non-Zero Temperature Boundaries

In this section we want to expand one of the cases from the previous section a little bit. In the previous section we look at the following heat problem.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \qquad u(0,t) = 0 \qquad u(L,t) = 0$$

What we'd like to do in this section is instead look at the following problem.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \qquad u(0,t) = T_1 \qquad u(L,t) = T_2 \qquad \textbf{(1)}$$

In this case we'll allow the boundaries to be any fixed temperature, T_1 or T_2 . The problem here is that separation of variables will no longer work on this problem because the boundary conditions are no longer homogeneous. Recall that separation of variables will only work if both the partial differential equation and the boundary conditions are linear and homogeneous. So, we're going to need to deal with the boundary conditions in some way before we actually try and solve this

It makes some sense that we should expect that as $t \to \infty$ our temperature distribution, $u(x,t)$ should behave as,

$$\lim_{t \to \infty} u(x,t) = u_E(x)$$

where $u_E(x)$ is called the **equilibrium temperature**. Note as well that is should still satisfy the heat equation and boundary conditions. It won't satisfy the initial condition however because it is the temperature distribution as $t \to \infty$ whereas the initial condition is at $t = 0$. So, the equilibrium temperature distribution should satisfy,

$$0 = \frac{d^2 u_E}{dx^2} \qquad u_E(0) = T_1 \qquad u_E(L) = T_2 \qquad \textbf{(2)}$$

This is a really easy 2nd order ordinary differential equation to solve. If we integrate twice we get,

$$u_E(x) = c_1 x + c_2$$

and applying the boundary conditions (we'll leave this to you to verify) gives us,

$$u_E(x) = T_1 + \frac{T_2 - T_1}{L} x$$

let's define the function,

$$v(x,t) = u(x,t) - u_E(x) \qquad (3)$$

where $u(x,t)$ is the solution to (1) and $u_E(x)$ is the equilibrium temperature for (1).

Now let's rewrite this as,

$$u(x,t) = v(x,t) + u_E(x)$$

and let's take some derivatives.

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial u_E}{\partial t} = \frac{\partial v}{\partial t} \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u_E}{\partial x^2} = \frac{\partial^2 v}{\partial x^2}$$

In both of these derivatives we used the fact that $u_E(x)$ is the equilibrium temperature and so is independent of time *t* and must satisfy the differential equation in (2).

What this tells us is that both $u(x,t)$ and $v(x,t)$ must satisfy the same partial differential equation. Let's see what the initial conditions and boundary conditions would need to be for $v(x,t)$.

$$v(x,0) = u(x,0) - u_E(x) = f(x) - u_E(x)$$
$$v(0,t) = u(0,t) - u_E(0) = T_1 - T_1 = 0$$
$$v(L,t) = u(L,t) - u_E(L) = T_2 - T_2 = 0$$

So, the initial condition just gets potentially messier, but the boundary conditions are now homogeneous! The partial differential equation that $v(x,t)$ must satisfy is,

We know how to solve this in the previous section and so we the solution is,

$$v(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

where the coefficients are given by,

$$B_n = \frac{2}{L}\int_0^L \left(f(x) - u_E(x)\right)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = 1,2,3,\ldots$$

The solution to (1) is then,

$$u(x,t) = u_E(x) + v(x,t)$$
$$= T_1 + \frac{T_2 - T_1}{L}x + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

and the coefficients are given above.

Example 1:

Find a solution to the following partial differential equation that will also satisfy the boundary conditions.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \qquad u(0,t) = 0 \qquad u(L,t) = 0$$

Solution:

First, we assume that the solution will take the form,

$$u(x,t) = \varphi(x)G(t)$$

and we plug this into the partial differential equation and boundary conditions. We separate the equation to get a function of only *t* on one side and a function of only *x* on the other side and then introduce a separation constant. This leaves us with two ordinary differential equations.

The two ordinary differential equations are,

$$\frac{dG}{dt} = -k\lambda G \qquad \frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$$

$$\varphi(0) = 0 \qquad \varphi(L) = 0$$

The time dependent equation can really be solved at any time, but since we don't know what λ λ is yet let's hold off on that one. Also note that in many problems only the boundary value problem can be solved at this point so don't always expect to be able to solve either one at this point.

Now, we actually solved the spatial problem,

$$\frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$$

$$\varphi(0) = 0 \qquad \varphi(L) = 0$$

We've got three cases to deal with so let's get going.

$\lambda > 0$

In this case we know the solution to the differential equation is,

$$\varphi(x) = c_1 \cos\left(\sqrt{\lambda}\, x\right) + c_2 \sin\left(\sqrt{\lambda}\, x\right)$$

Applying the first boundary condition gives,

$$0 = \varphi(0) = c_1$$

Now applying the second boundary condition, and using the above result of course, gives,

$$0 = \varphi(L) = c_2 \sin\left(L\sqrt{\lambda}\right)$$

Now, we are after non-trivial solutions and so this means we must have,

$$\sin\left(L\sqrt{\lambda}\right) = 0 \qquad \Rightarrow \qquad L\sqrt{\lambda} = n\pi \quad n = 1, 2, 3, \ldots$$

The positive eigenvalues and their corresponding eigenfunctions of this boundary value problem are then,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$

Note that we don't need the c_2 c_2 in the eigenfunction as it will just get absorbed into another constant that we'll be picking up later on.

$\lambda = 0$

The solution to the differential equation in this case is,

$$\varphi(x) = c_1 + c_2 x$$

Applying the boundary conditions gives,

$$0 = \varphi(0) = c_1 \qquad\qquad 0 = \varphi(L) = c_2 L \qquad \Rightarrow \qquad c_2 = 0$$

So, in this case the only solution is the trivial solution and so $\lambda = 0$ is not an eigenvalue for this boundary value problem.

$\underline{\lambda < 0}$

Here the solution to the differential equation is,

$$\varphi(x) = c_1 \cosh\left(\sqrt{-\lambda}\, x\right) + c_2 \sinh\left(\sqrt{-\lambda}\, x\right)$$

Applying the first boundary condition gives,

$$0 = \varphi(0) = c_1$$

and applying the second gives,

$$0 = \varphi(L) = c_2 \sinh\left(L\sqrt{-\lambda}\right)$$

So, we are assuming $\lambda < 0$ and so $L\sqrt{-\lambda} \neq 0$ and this means $\sinh\left(L\sqrt{-\lambda}\right) \neq 0$. We therefore we must have $c_2 = 0$ and so we can only get the trivial solution in this case.

Therefore, there will be no negative eigenvalues for this boundary value problem.

The complete list of eigenvalues and eigenfunctions for this problem are then,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1,2,3,\ldots$$

Now let's solve the time differential equation,

$$\frac{dG}{dt} = -k\lambda_n G$$

and note that even though we now know λ we're not going to plug it in quite yet to keep the mess to a minimum. We will however now use λ_n to remind us that we actually have an infinite number of possible values here.

This is a simple linear (and separable for that matter) 1st order differential equation and so we'll let you verify that the solution is,

$$G(t) = ce^{-k\lambda_n t} = ce^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Okay, now that we've gotten both of the ordinary differential equations solved we ca n finally write down a solution. Note however that we have in fact found infinitely many solutions since there are infinitely many solutions (*i.e.* eigenfunctions) to the spatial problem.

Our product solution are then,

$$u_n(x,t) = B_n \sin\left(\frac{n\pi x}{L}\right)\mathrm{e}^{-k\left(\frac{n\pi}{L}\right)^2 t} \qquad n = 1,2,3,\ldots$$

We've denoted the product solution u_n to acknowledge that each value of *n* will yield a different solution. Also note that we've changed the *c* in the solution to the time problem to B_n B_n to denote the fact that it will probably be different for each value of *n* as well and because had we kept the c_2 with the eigen function we'd have absorbed it into the *c* to get a single constant in our solution.

Example 2:

Solve the following BVP.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = 20 \qquad u(0,t) = 0 \qquad u(L,t) = 0$$

Solution:

There isn't really all that much to do here as we've done most of it in the examples and discussion above.

First, the solution is,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

The coefficients are given by,

$$B_n = \frac{2}{L}\int_0^L 20\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left(\frac{20L(1-\cos(n\pi))}{n\pi}\right) = \frac{40\left(1-(-1)^n\right)}{n\pi}$$

If we plug these in we get the solution,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{40\left(1-(-1)^n\right)}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

One-dimensional wave equation is given by $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Solution of the Wave Equation by Separation of Variables:

There are several approaches to solving the wave equation. The first one we will work with, using a technique called separation of variables.

First, note that for a specific wave equation situation, in addition to the actual PDE, we will also have boundary conditions arising from the fact that the endpoints of the string are attached solidly, at the left end of the string, when $x = 0$ and at the other end of the string, which we suppose has overall length l. Let's start the process of solving the PDE by first figuring out what these boundary conditions imply for the solution function, $u(x,t)$.

Answer: for all values of t, the time variable, it must be the case that the vertical displacement at the endpoints is 0, since they don't move up and down at all, so that

(1) $u(0,t) = 0$ and $u(l,t) = 0$ for all values of t

are the ***boundary conditions*** for our wave equation. These will be key when we later on need to sort through possible solution functions for functions that satisfy our particular vibrating string set-up.

Note that we probably need to specify what the shape of the string is right when time $t = 0$, and you're right - to come up with a particular solution function, we would need to know $u(x,0)$. In fact we would also need to know the initial velocity of the string, which is just $u_t(x,0)$. These two requirements are called the ***initial conditions*** for the wave equation, and are also necessary to specify a particular vibrating string solution. For instance, as the simplest example of initial conditions, if no one is plucking the string, and it's perfectly flat to start with, then the initial conditions would just be $u(x,0)=0$ (a perfectly flat string) with initial velocity, $u_t(x,0)=0$. Here, then, the solution function is pretty unenlightening – it's just $u(x,t)=0$, i.e. no movement of the string through time.

To start the separation of variables technique we make the key assumption that whatever the solution function is, that it can be written as the product of two independent functions, each one of which depends on just one of the two variables, *x* or *t*. Thus, imagine that the solution function, $u(x,t)$ can be written as

(2) $$u(x,t)=F(x)G(t)$$

where *F*, and *G*, are single variable functions of *x* and *t* respectively. Differentiating this equation for $u(x,t)$ twice with respect to each variable yields

(3) $$\frac{\partial^2 u}{\partial x^2}=F''(x)G(t) \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2}=F(x)G''(t)$$

Thus when we substitute these two equations back into the original wave equation, which is

(4) $$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$$

then we get

(5) $$\frac{\partial^2 u}{\partial t^2}=F(x)G''(t)=c^2\frac{\partial^2 u}{\partial x^2}=c^2F''(x)G(t)$$

Using separation of variables assumption we get

(6) $\frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)}$

(7) $\frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k$

where *k* is a constant. First let's examine the possible cases for *k*.

Case One: k = 0

Suppose *k* equals 0. Then the equations in (7) can be rewritten as

(8) $G''(t) = 0 \cdot c^2 G(t) = 0$ and $F''(x) = 0 \cdot F(x) = 0$

yielding with very little effort two solution functions for *F* and *G:*

(11) $F(0) = F(l) = 0$.

But how can a linear function have two roots? Only by being identically equal to 0, thus it must be the case that $F(x)=0$. Sigh, then we still get that $u(x,t)=0$, and we end up with the dull solution again, the only possible solution if we start with k = 0.

Case Two: k > 0

So now if k is positive, then from equation (7) we again start with

(12) $$G''(t)=kc^2G(t)$$

and (13) $$F''(x)=kF(x)$$

where now A and B are constants and $\omega=\sqrt{k}$. Knowing that $F(0)=F(l)=0$, then unfortunately the only possible values of A and B that work are $A=B=0$, i.e. that $F(x)=0$. Thus, once again we end up with $u(x,t)=F(x)G(t)=0\cdot G(t)=0$, i.e. the dull solution once more.

Case Three: k < 0

So now we go back to equations (12) and (13) again, but now working with *k* as a negative constant. So, again we have

(12) $G''(t) = kc^2 G(t)$

and (13) $F''(x) = kF(x)$

Exponential functions won't satisfy these two ODEs, but now the sine and cosine functions will. The general solution function for (13) is now

(15) $F(x) = A\cos(\omega x) + B\sin(\omega x)$

where again *A* and *B* are constants and now we have $\omega^2 = -k$. Again, we consider the boundary conditions that specified that $F(0) = F(l) = 0$. Substituting in 0 for *x* in (15) leads to

(16) $F(0) = A\cos(0) + B\sin(0) = A = 0$

so that $F(x) = B\sin(\omega x)$. Next, consider $F(l) = B\sin(\omega l) = 0$. We can assume that *B* isn't equal to 0, otherwise $F(x) = 0$ which would mean that $u(x,t) = F(x)G(t) = 0 \cdot G(t) = 0$, again, the trivial unplucked string solution. With $B \neq 0$, then it must be the case that $\sin(\omega l) = 0$ in order to have $B\sin(\omega l) = 0$. The only way that this can happen is for ωl to be a multiple of π. This means that

(17) $\omega l = n\pi$ or $\omega = \frac{n\pi}{l}$ (where *n* is an integer)

This means that there is an infinite set of solutions to consider (letting the constant *B* be equal to 1 for now), one for each possible integer *n*.

(18) $F(x) = \sin\left(\frac{n\pi}{l} x\right)$

Well, we would be done at this point, except that the solution function $u(x,t)=F(x)G(t)$ and we've neglected to figure out what the other function, $G(t)$, equals. So, we return to the ODE in (12):

$$(12) \quad G''(t)=kc^2G(t)$$

where, again, we are working with *k,* a negative number. From the solution for $F(x)$ we have determined that the only possible values that end up leading to non-trivial solutions are with

$k=-\omega^2=-\left(\frac{n\pi}{l}\right)^2$ for *n* some integer. Again, we get an infinite set of solutions for (12) that can be written in the form

$$(19) \quad G(t)=C\cos(\lambda_n t)+D\sin(\lambda_n t)$$

where *C* and *D* are constants and $\lambda_n=c\sqrt{-k}=c\omega=\frac{cn\pi}{l}$, where *n* is the same integer that showed up in the solution for $F(x)$ in (18) (we're labeling λ with a subscript "*n*" to identify which value of *n* is used).

Now we really are done, for all we have to do is to drop our solutions for $F(x)$ and $G(t)$ into $u(x,t)=F(x)G(t)$, and the result is

$$(20) \quad u_n(x,t)=F(x)G(t)=\left(C\cos(\lambda_n t)+D\sin(\lambda_n t)\right)\sin\left(\frac{n\pi}{l}x\right)$$

where the integer *n* that was used is identified by the subscript in $u_n(x,t)$ and λ_n, and *C* and *D* are arbitrary constants.

Example:

Find a solution to the following partial differential equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$

$$u(0,t) = 0 \qquad u(L,t) = 0$$

Solution

So, let's start off with the product solution.

$$u(x,t) = \varphi(x)h(t)$$

Plugging this into the two boundary conditions gives,

$$\varphi(0) = 0 \qquad \varphi(L) = 0$$

Plugging the product solution into the differential equation, separating and introducing a separation constant gives,

$$\frac{\partial^2}{\partial t^2}\left(\varphi(x)h(t)\right) = c^2 \frac{\partial^2}{\partial x^2}\left(\varphi(x)h(t)\right)$$

$$\varphi(x)\frac{d^2 h}{dt^2} = c^2 h(t)\frac{d^2 \varphi}{dx^2}$$

$$\frac{1}{c^2 h}\frac{d^2 h}{dt^2} = \frac{1}{\varphi}\frac{d^2 \varphi}{dx^2} = -\lambda$$

We moved the c^2 to the left side for convenience and chose $-\lambda$ for the separation constant so the differential equation for φ would match a known (and solved) case.

The two ordinary differential equations we get from separation of variables are then,

$$\frac{d^2h}{dt^2}+c^2\lambda h=0 \qquad \frac{d^2\varphi}{dx^2}+\lambda\varphi$$

$$\varphi(0)=0 \qquad \varphi(L)=0$$

We solved the boundary value problem above in example 1 of the Solving the Heat Equation section of this chapter and so the eigen values and eigen functions for this problem are,

$$\lambda_n=\left(\frac{n\pi}{L}\right)^2 \qquad \varphi_n(x)=\sin\left(\frac{n\pi x}{L}\right) \qquad n=1,2,3,\ldots$$

The first ordinary differential equation is now,

$$\frac{d^2h}{dt^2}+\left(\frac{n\pi c}{L}\right)^2 h=0$$

and because the coefficient of the *h* is clearly positive the solution to this is,

$$h(t)=c_1\cos\left(\frac{n\pi c\,t}{L}\right)+c_2\sin\left(\frac{n\pi c\,t}{L}\right)$$

Because there is no reason to think that either of the coefficients above are zero we then get two product solutions,

$$u_n(x,t)=A_n\cos\left(\frac{n\pi c\,t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

$$u_n(x,t)=B_n\sin\left(\frac{n\pi c\,t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \qquad n=1,2,3,\ldots$$

The solution is then,

$$u(x,t)=\sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right)+B_n\sin\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\right]$$

Now, in order to apply the second initial condition we'll need to differentiate this with respect to *t* so,

$$\frac{\partial u}{\partial t}=\sum_{n=1}^{\infty}\left[-\frac{n\pi c}{L}A_n\sin\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right)+\frac{n\pi c}{L}B_n\cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\right]$$

If we now apply the initial conditions we get,

$$u(x,0)=f(x)=\sum_{n=1}^{\infty}\left[A_n\cos(0)\sin\left(\frac{n\pi x}{L}\right)+B_n\sin(0)\sin\left(\frac{n\pi x}{L}\right)\right]=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{L}\right)$$

$$\frac{\partial u}{\partial t}(x,0)=g(x)=\sum_{n=1}^{\infty}\frac{n\pi c}{L}B_n\sin\left(\frac{n\pi x}{L}\right)$$

Both of these are Fourier sine series. The first is for $f(x)$ on $0\le x\le L$ while the second is for $g(x)$ on $0\le x\le L$ with a slightly messy coefficient. As in the last few sections we're faced with the choice of either using the orthogonality of the sines to derive formulas for A_n and B_n or we could reuse formula from previous work.

It's easier to reuse formulas so using the formulas form the Fourier sine series we get,

$$A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n=1,2,3,\ldots$$

$$\frac{n\pi c}{L}B_n=\frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n=1,2,3,\ldots$$

Upon solving the second one we get,

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = 1, 2, 3, \ldots$$

$$B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = 1, 2, 3, \ldots$$

So, there is the solution to the 1-D wave equation and with that we've solved the final partial differential equation