LECTURE NOTES

ON

COMPUTATIONAL AERODYNAMICS

VI Semester

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UNIT I

INTRODUCTION TO COMPUTATIONAL AERODYNAMICS

→ continuous (without any ext. force)

› Displacement occurs due to external agent (i.e., force)

› Study related to motion of body → kinematics.

› Study related to motion and reason behind it → kinetics.

(kinetics + ~~kyn~~ kinematics) → Dynamics.

→ Body is rigid.

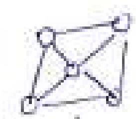
→ Body consists of its own number of molecules.

 – body

→ mass
↳ volume } ⇒ $\rho = \frac{\text{mass}}{\text{Volume}}$

molecules

from gravitational law



$F \propto \frac{m_1 m_2}{d^2}$

intermolecular force.

› high ⇒ solid.

› less ⇒ fluid (allows continuous deformation)

fluid → liquid
↳ air

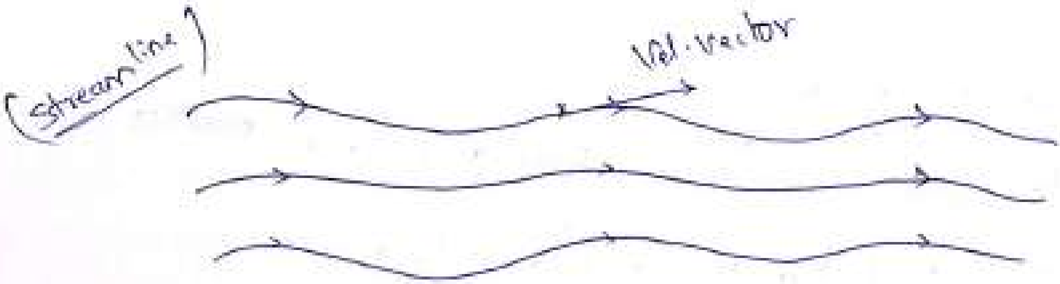
› Study of fluid's motion and the reason behind it is called as fluid dynamics.

Computational fluid dynamics:

- A computational approach for the study of fluid motion and its reason is called computational fluid dynamics.

- A study of moving object and moving air is called aerodynamics.

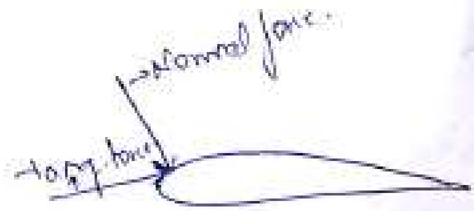
Fluid flow: (It is represented by streamline, pathline, streakline)



(path/streak)

Fluid variables

- pressure = F/A ← Normal force



- Shear stress = Shear force/Area ← Tangential

- Pressure difference always leads to velocity (vector) – another fluid variable ⟹ $v = \frac{disp}{time}$

- $P_2 > P_1$

Bernoulli's principle:

- Steady flow.
- Incompressible flow.
- Irrotational flow.
- Inviscid fluid flow

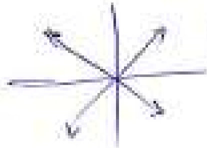
→ $P + \frac{1}{2}\rho v^2 = const$

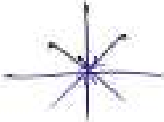
> density (scalar quantity) = $\frac{\text{mass}}{\text{volume}}$

> Temperature

$\underbrace{(P, V, T \text{ and } \rho)}_{\text{(used in isentropic flow.}}$ - basic fluid variables

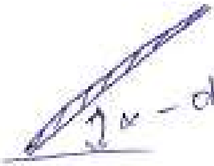
- There are many types of flows

>  (source flow)

>  (sink flow)

- rotational flow, - Irrotational flow, > Subsonic, > Sonic,

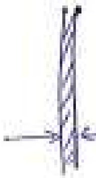
> Supersonic, > Hypersonic, > transonic flow.



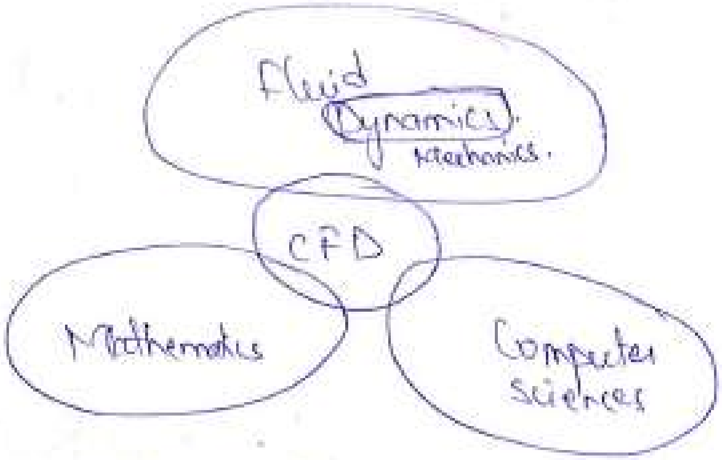
α - oblique angle.

(Oblique shock)

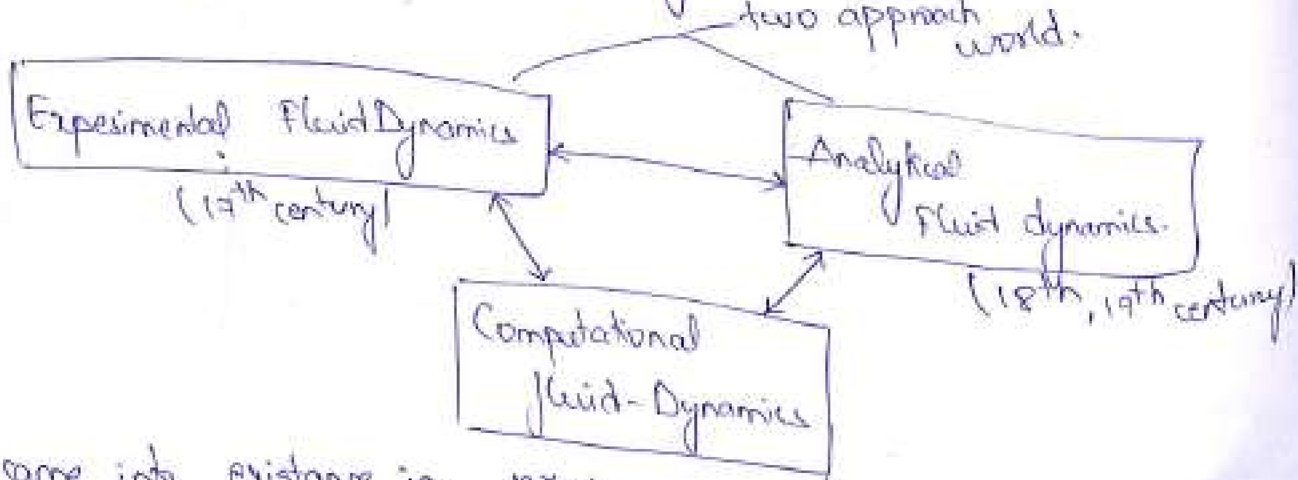
N-shock.



10^{-3} mm.



> CFD is a combination of Fluid Mechanics with Mathematical algorithms to form a computer interface.

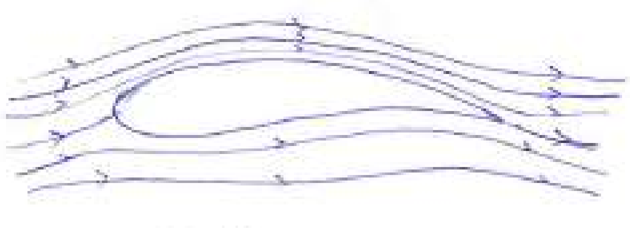


> CFD came into existance in 1970's.

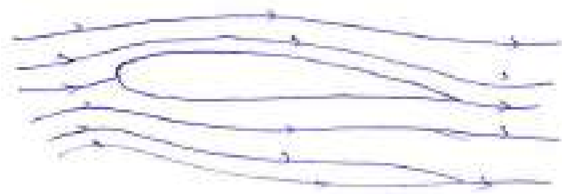
Advantages of CFD:

> Saves time.
> Easy to attain required results
> Easy to perform many iterations.

CFD as a research tool:



(a) laminar, $Re = 10^5$



(b) turbulent $Re = 10^5$

- In CFD laminar, turbulence can be visualised by changing the flow pattern over the airfoil.
- In experimental techniques laminar flow or turbulance flow can be obtained with diff. inlets, geometric properties of wind tunnel.

1. Write some advantages of CFD.
2. What are the three ~~disciplines~~ dimensions that CFD is ~~derived from~~ formulated as.
3. What are the 3-disciplines that CFD is derived from.

1.Ans: Advantages of CFD:

- key advantage of CFD is that it is a very compelling, non intrusive, virtual modelling technique with powerful visualization capabilities.
- This technology has widely been applied to various engineering applications i.e., aircraft design, automobile design, civil engg etc,

Practical advantages of CFD:

- CFD predicts performance before modifying or installing systems:
  - Without modifying and/or installing actual system or a proto type, CFD can predict which design changes are most crucial to enhance performance.
- CFD provides exact and detailed information about design parameters:
  - The advances in technology require broader and more detailed information about the flow within an occupied zone, and CFD meets this goal better than any other method.

> CFD ~~saves cost and time~~ is reliable:

– The numerical schemes and methods upon which CFD is based are improving rapidly, so CFD results are increasingly reliable. CFD is a dependable tool for design and analysis

× CFD saves Cost and Time:

– It costs much less than experiments because physical modifications are not necessary.

~~3 Ans~~ Other striking advantages:

→ It presents the perfect opportunity to study specific terms in the governing equations in a more detailed fashion.

→ CFD has the capacity to simulate flow conditions that are not ~~reproducible~~ reproducible in experimental tests found in geophysical and biological fluid dynamics.

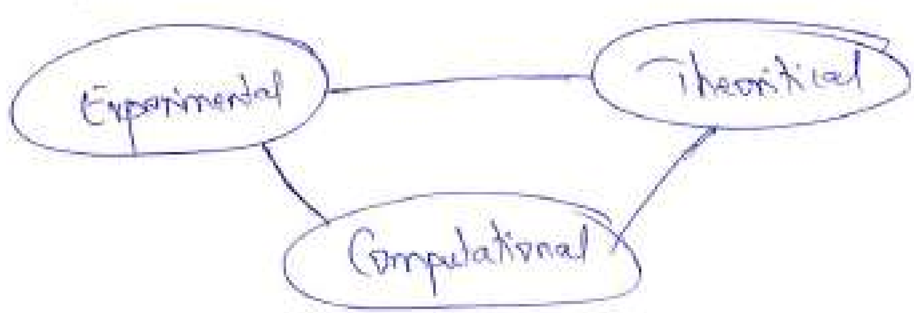
→ It can provide detailed visualisation and comprehensive information when compared to analytical and experimental fluid dynamics.

2 Ans:

> Initially Fluid dynamics or Aerodynamics are dealt in two ways they are theoritical and experimental approaches.

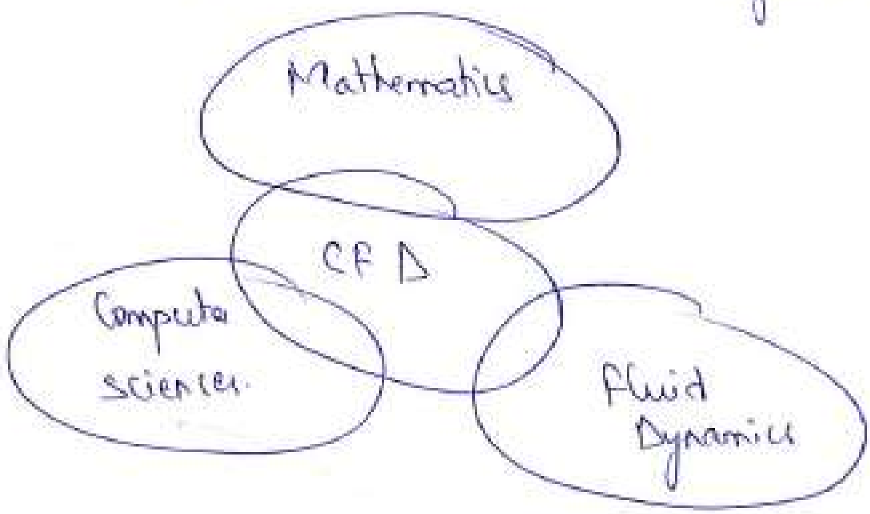
× In and around seventeenth century there was the basic required initation of the experimental approach, since then it ~~shaped up~~ started to shape up.

- For the details of theoritical approach's initation towards Fluid or Aero dynamics one has to get back to eighteenth, a nineteenth centuries.
- Based on the experimental details available till that date, many theories were formulated.
- Thus from then, it has become a two dimensional approach. This continued in the ~~poor~~ inital parts of twentith century as well.
- In the later half of 1960's there was an additional approach added to the existing methods i.e., through use of computers and is known as Computational Fluid Dynamics / Computational aerodynamics.
- Since then there was 3 dimensional approach for a problem persisting in fluid Mechanics.
- All these 3 approaches goes hand-in hand. One can't completely eliminate any one of these approaches.
- The following depicted figure illustrates the three dimensional approach of fluid dynamics.



3 Ans:

- Computational fluid Dynamics can be clearly ~~separated~~ illustrated as a combination of computation and fluid Dynamics.
- Apart from this this is also related with Mathematics also
- Thus on a whole this forms a three ~~dimensional~~ disciplinary approach
- these all the three disciples ~~of~~ viz., computer science (for programming etc.,), Mathematics (for solving), Fluid Dynamics (for understanding the fluid properties) are regarded.



- The above depicted figure illustrates how CFD is ~~a~~ interdisciplinary.

Physical principles

Mass conserved | Newton's Second law | Energy conserved

Models of flow

- → Fixed Finite Control Volume. (F.C.V)
- → Moving F.C.V.
- → Fixed Infinitesimal fluid element.
- → Moving Infinitesimal fluid element.

Continuity
Momentum
Energy

Boundary conditions.

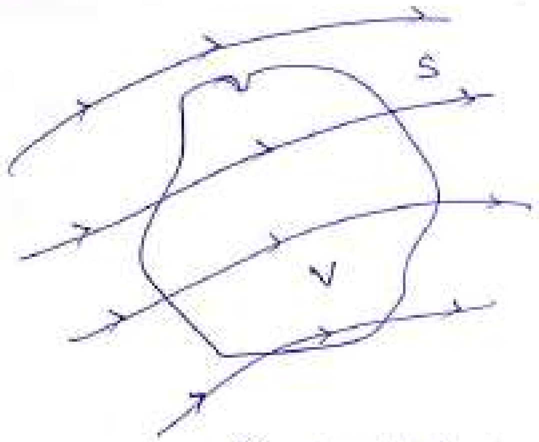
Forms of governing eq^ns suitable for CFD

- Three principles are applied to the models.
- For fixed models if the principles are applied we get conservative form equations.
- For moving models if the principles are applied we get non conservative form of equations.

<u>3 steps</u>:
1. Sel. a fundamental principle.
2. Applying for a particular model.
3. Equations are derived.

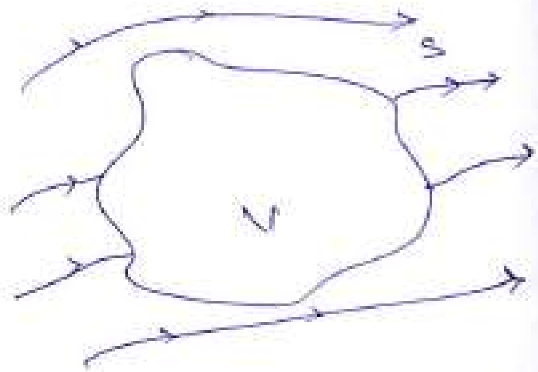
- Final equations are cumbersome and the no. of terms are reduced which is called as forms of governing equations.
- Then B.C are to be applied to forms of governing equations.

Control Volume:



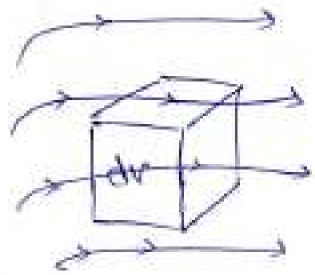
Fixed control volume.

(we get conservative eqn either in diff. or integral form)

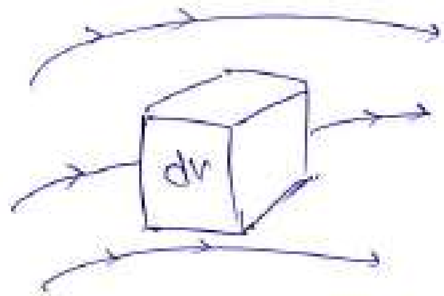


Moving control volume

(we get non-conservative eqn)

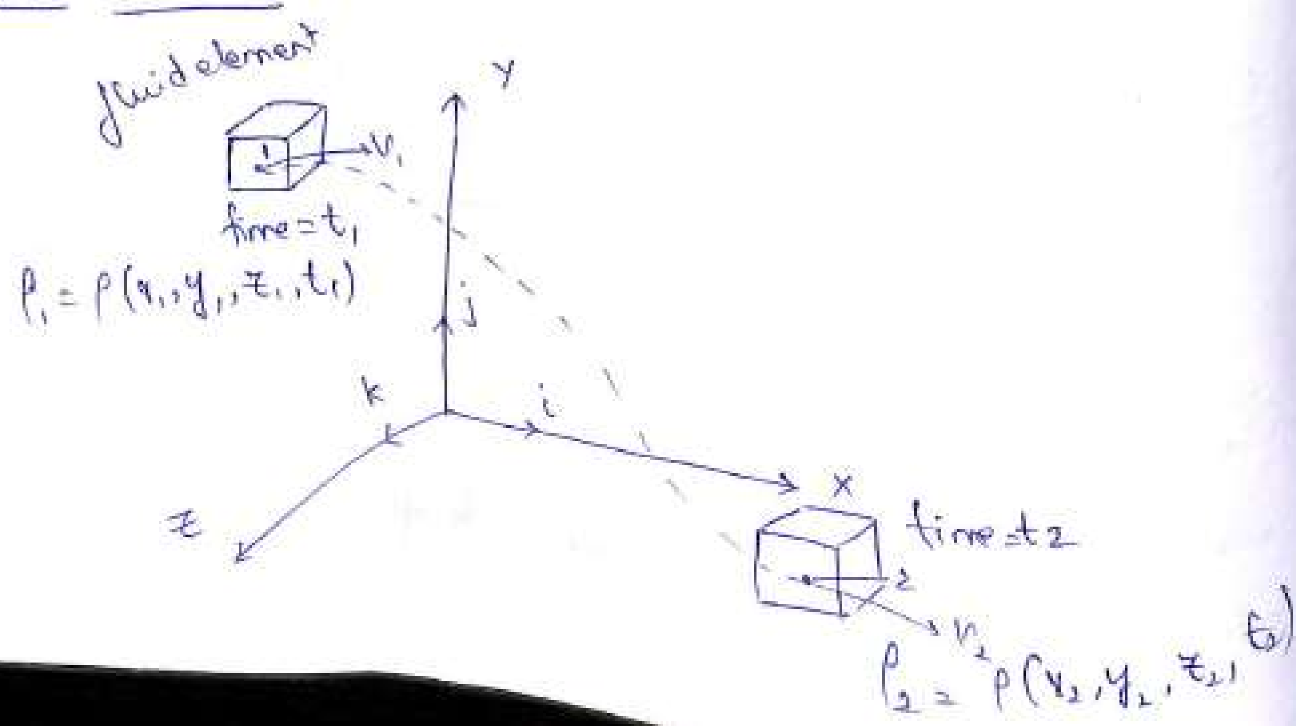


Fixed Infinitesimally small fluid element.



Moving Infinitesimally small fluid element.

Substantial Derivatives:



* Consider infitesimally small fluid element moving with the fluid flow. ~~The~~

* The fluid element is moving through cartesian space, unit vectors along x, y, z axes are i, j, k respectively.

(here we are considering unsteady flow)

$$p = p(x, y, z, t) \quad — 1(a)$$

$$\rho = \rho(x, y, z, t) \quad — 1(b)$$

$$T = T(x, y, z, t) \quad — 1(c)$$

$$V = u\bar{i} + v\bar{j} + w\bar{k} \quad — 2(a)$$

$$u = u(x, y, z, t) \quad — 2(b)$$

$$v = v(x, y, z, t) \quad — 2(c)$$

$$w = w(x, y, z, t) \quad — 2(d)$$

The flow we are considering is an unsteady flow, where u, v, w are the functions of both space and time.

At time t_1 the fluid element is located at point 1, at this point the density of fluid element is $\rho_1 = \rho(x_1, y_1, z_1, t_1)$

$|||^{ly}$ at $t = t_2$ same fluid ele. is at point 2.

$$\Rightarrow \rho_2 = \rho(x_2, y_2, z_2, t_2)$$

Since $\rho = \rho(x, y, z, t)$

using taylor series expansion.

$$\partial\rho = \frac{\partial\rho}{\partial x_1} \cdot dx_1 + \frac{\partial\rho}{\partial y_1} \cdot dy_1 + \frac{\partial\rho}{\partial z_1} \cdot dz_1 + \frac{\partial\rho}{\partial t_1} \cdot dt_1$$

$\Rightarrow \rho_2 = \rho_1 + \left(\frac{\partial \rho}{\partial x}\right)_1 (x_2 - x_1) + \left(\frac{\partial \rho}{\partial y}\right)_1 (y_2 - y_1) + \left(\frac{\partial \rho}{\partial z}\right)_1 (z_2 - z_1) +$

$$\left(\frac{\partial \rho}{\partial t}\right)_1 (t_2 - t_1) + \text{H.O.T} \quad - (3)$$

divide (3) with $(t_2 - t_1)$

$$\Rightarrow \frac{\rho_2 - \rho_1}{(t_2 - t_1)} = \left(\frac{\partial \rho}{\partial x}\right)_1 \left(\frac{x_2 - x_1}{t_2 - t_1}\right) + \left(\frac{\partial \rho}{\partial y}\right)_1 \left(\frac{y_2 - y_1}{t_2 - t_1}\right) + \left(\frac{\partial \rho}{\partial z}\right)_1 \left(\frac{z_2 - z_1}{t_2 - t_1}\right)$$

$$+ \left(\frac{\partial \rho}{\partial t}\right)_1 \quad - (4)$$

L.H.S of (4) is the avg. time rate of change of density of fluid ele. as it moves from point '1' to '2'.

$$\lim_{t_2 \to t_1} \frac{\rho_2 - \rho_1}{t_2 - t_1} = \frac{D\rho}{Dt}$$

(here $\frac{D}{Dt}$ - Instantaneous time rate of change is the substantial derivative)

$\Rightarrow$ '$\frac{D\rho}{Dt}$' is instantaneous time rate of change of density of the fluid element as it moves from point 1.

$$\Rightarrow \frac{D\rho}{Dt} = \left(\frac{\partial \rho}{\partial t}\right)_1 + \left(\frac{\partial \rho}{\partial x}\right)_1 u + \left(\frac{\partial \rho}{\partial y}\right)_1 \cdot v + \left(\frac{\partial \rho}{\partial z}\right)_1 \cdot w \quad - (5)$$

$$\left[\because \lim_{t_2 \to t_1} \frac{x_2 - x_1}{t_2 - t_1} = u \right.$$

$$\lim_{t_2 \to t_1} \frac{y_2 - y_1}{t_2 - t_1} = v$$

$$\left. \lim_{t_2 \to t_1} \frac{z_2 - z_1}{t_2 - t_1} = w \right]$$

~~Now~~ from ⑤ we can obtain an eqn for substantial derivatives $\frac{D\vec{R}}{Dt}$ in cartesian coordinates.

$$\Rightarrow \boxed{\frac{D}{Dt} = \frac{\partial}{\partial t} + \left(\frac{\partial}{\partial x}\right)u + \left(\frac{\partial}{\partial y}\right)v + \left(\frac{\partial}{\partial z}\right)w} \quad ⑥$$

$$\Rightarrow \boxed{\frac{D}{Dt} = \frac{\partial}{\partial t} + V.\nabla} \quad ⑦ \quad \left(\begin{aligned} V &= u\bar{i} + v\bar{j} + w\bar{k} \\ \nabla &= \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z} \end{aligned} \right)$$

further in cartesian coordinates ⑥ can be written as ⑦.

here

$\frac{D}{Dt}$ - time rate of change of a fluid element when it is moving.

$\frac{\partial}{\partial t}$ - time rate of change for a fixed point or local derivative.

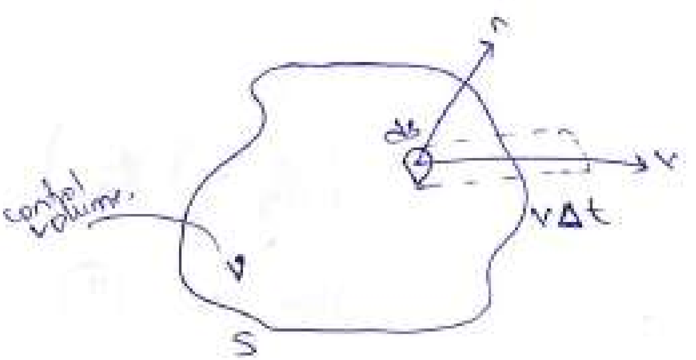
$V.\nabla$ - Convective derivative.

time rate of change due to the movement of fluid element from one location to another in a flow field where the flow properties are different.

- The substantial derivative applies to any flow field variable

viz, $\frac{DP}{Dt}$, $\frac{D\rho}{Dt}$, $\frac{DT}{Dt}$.

Divergence of Velocity :



* Consider a control volume, moving with the fluid.
* Mass is fixed invariant of time.

* Its volume 'V' and control surface 'S' changes with time, as it moves to diff. regions of flow where diff. values of density exist
* Considering infinitesimally small element 'ds' moving with velocity
* Change in the volume of the control volume is 'ΔV' due to the movement of 'ds' over a time increment Δt.

$$\Rightarrow \Delta V = (v\Delta t \cdot n)\, ds \quad \text{— (1)}$$
$$= (v\Delta t) \cdot ds$$

* Over the time increment 'Δt' the total change in volume of whole control volume is equal to summation of total control surface

$$\Rightarrow \quad \Delta V = \iint_S (v \cdot \Delta t) \cdot ds .$$

divide this eqn with Δt. The result is time rate of change of control volume

$$\Rightarrow \frac{DV}{Dt} = \frac{1}{\Delta t} \iint_S (v \cdot \Delta t) \cdot ds \Rightarrow \boxed{\frac{DV}{Dt} = \iint_S v \cdot ds}$$

→ left hand side of ③ is the substantial derivative of 'V', as the time rate of change of control volume moves with the flow.

→ Applying the divergence theorem to ③

$$\Rightarrow \boxed{\frac{DV}{Dt} = \iiint_V (\nabla \cdot V)\,dV} \quad — ④$$

Moving control volume shrunk to a very small volume δV, essentially becomes infinitesimally moving element.

$$\Rightarrow \boxed{\frac{D(\delta V)}{Dt} = \iiint_{\delta V} (\nabla \cdot V)\,\delta V} \quad — ⑤$$

$$\Rightarrow \frac{D(\delta V)}{Dt} = (\nabla \cdot V)\,\delta V \quad — ⑥$$

$$\Rightarrow \boxed{(\nabla \cdot V) = \frac{1}{\delta V}\frac{D}{Dt}(\delta V)} \quad — ⑦$$

'$\nabla \cdot V$' is time rate of change of volume of a moving fluid element by unit volume.

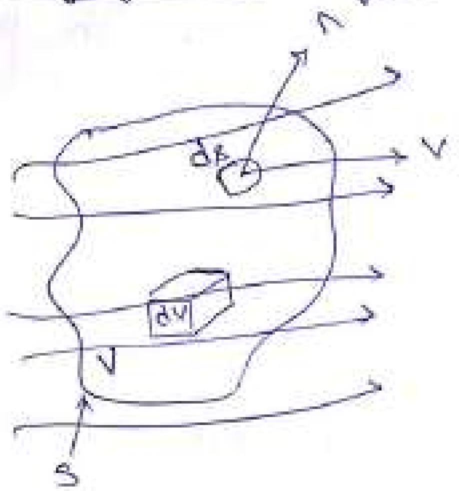
Continuity Equation:

Step 1: Write down a fundamental physical principle

Step 2: Applying to a suitable model of flow.

Step 3: Obtain an eqn which represents fundamental physical principle.

Model of finite C.V fixed in space:



→ Consider a flow model, the volume is fixed in space. The surface which bounds the control volume is control surface.

At a point on control surface, the flow velocity 'v' and 'ds' is ele. surface area. and 'dv' be elemental volume.

Applying the fundamental physical principle. that mass is conserved

⇒ $\left.\begin{array}{l}\text{Net Mass flow out of} \\ \text{C.V through surface 'S'}\end{array}\right\} = \left\{\begin{array}{l}\text{time rate of decrease of mass inside} \\ \text{the control volume.}\end{array}\right.$

B = C.

expression for B:

density × Area × Velocity.

ele. mass flow rate across the area 'ds'

$= \rho . V . ds.$

$B = \iint_S \rho . v . ds$

expression for C:

for ele. volume time rate of ~~decreasing~~ mass is

$\rho . dv.$

$\Rightarrow \quad C = \frac{-\partial}{\partial t} \iiint_V \rho . dv \quad [\because$ time rate of dec. of mass$]$

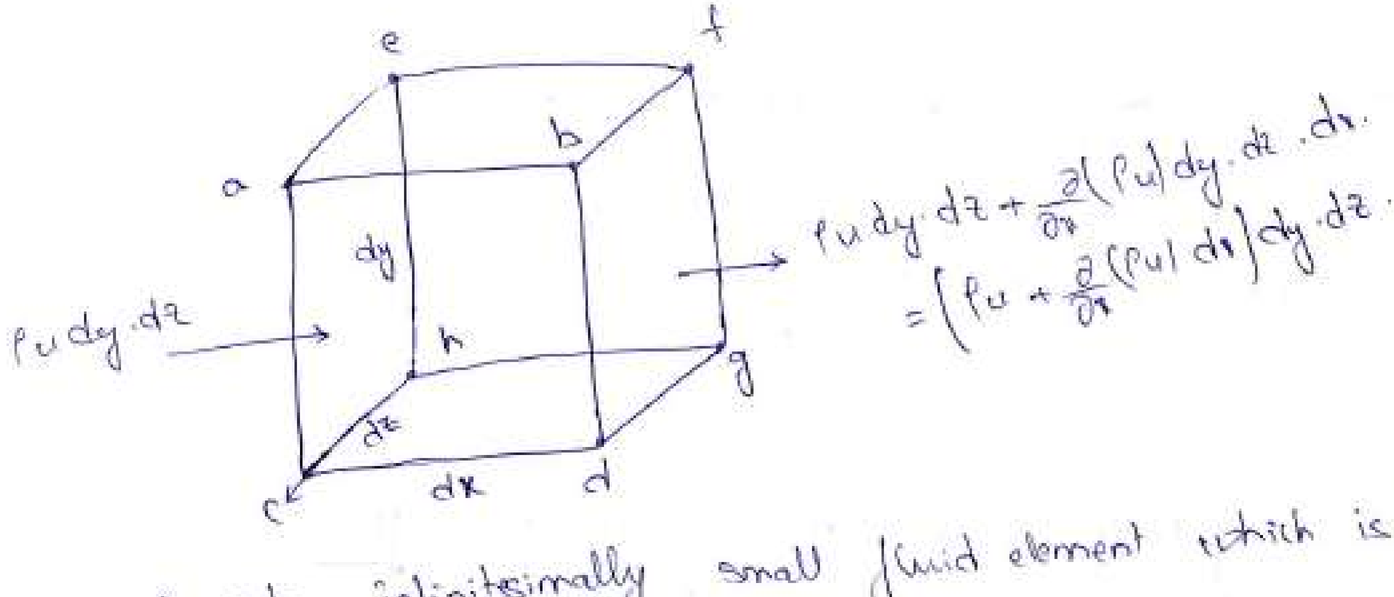
Now,

$B = C.$

$$\Rightarrow \iint_S \rho\, v . ds = \frac{-\partial}{\partial t} \iiint_V \rho . dv$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} \iiint_V \rho . dv + \iint_S \rho . v . ds = 0}$$

Model of infinitesimal small fluid ele. fixed in space:



Consider infinitesimally small fluid element which is fixed in space.

$\underline{x}$ aech and bfdg.

$\rho u\, dy\, dz \qquad \rho u . dy\, dz + \frac{\partial}{\partial x}(\rho u)\, dy . dz . dx$

Net out flow in x-direction is

$$\frac{\partial(\rho u)}{\partial x} . dx\, dy\, dz .$$

ll^{ly} for y and z directions we have

$$\frac{\partial}{\partial y}(\rho v)\,dx\,dy\,dz\ ,\ \frac{\partial}{\partial z}(\rho w)\,dx\,dy\,dz \quad \text{respectively}$$

total inlet mass flow

$$\frac{\partial}{\partial x}(\rho u)\,dx\,dy\,dz + \frac{\partial}{\partial y}(\rho v)\,dx\,dy\,dz + \frac{\partial}{\partial z}(\rho w)\,dx\,dy\,dz \quad -① $$

rime rate of dec. of mass inside $c.v = -\frac{\partial \rho}{\partial t}\,dx\,dy\,dz$ $-②$

① = ②

$$\Rightarrow \left(\frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \frac{\partial}{\partial z}\rho w\right) \cancel{dx\,dy\,dz} = -\frac{\partial \rho}{\partial t}\,\cancel{dx\,dy\,dz}$$

$$\Rightarrow \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t} = 0 .$$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + (\nabla \cdot \rho v) = 0}$$

Mass of fluid element $= \rho . dv$

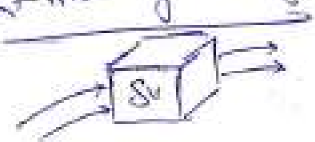
$$\text{Mass} = \underbrace{\iiint_V \rho . dv}_{}\quad \text{const.}$$

applying substantial derivative

$$\Rightarrow \boxed{\frac{D}{Dt}\iiint_V \rho . dv = 0}$$

Infinitesimal

for a small element moving with fluid :



$$\Rightarrow \delta m = \rho . \delta v$$

$$\Rightarrow \frac{D}{Dt}(\delta m) = \frac{D}{Dt}(\rho . \delta v)$$

$$\Rightarrow \frac{D}{Dt}(\rho . \delta v) = 0 \quad [\because \text{Mass is fixed}]$$

$$\Rightarrow \delta v \frac{D\rho}{Dt} + \rho \frac{D(\delta v)}{Dt} = 0$$

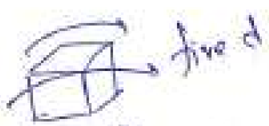
$$\Rightarrow \frac{D\rho}{Dt} + \left(\frac{\rho}{\delta v}\right)\frac{D(\delta v)}{Dt} = 0$$

$$\Rightarrow \boxed{\frac{D\rho}{Dt} + \rho \nabla . v = 0}$$

I $\qquad\qquad\qquad\qquad\qquad\qquad$ II

$$\frac{\partial}{\partial t}\iiint_V \rho . dv + \iint_S \rho v . ds = 0 \quad \xleftrightarrow{A} \quad \frac{D}{Dt}\iiint_V \rho . dv = 0$$

C $\qquad\qquad\qquad\qquad\qquad\qquad$ D



$\xleftrightarrow{B}$

$$\text{III}\quad \frac{\partial \rho}{\partial t} + \nabla(\rho v) = 0 \qquad\qquad \frac{D\rho}{Dt} + \rho \nabla . v = 0 \quad \text{IV}$$

Path C:

$$\mathrm{I} \rightarrow \frac{\partial}{\partial t} \iiint_V \rho . dv + \iint_S \rho v . ds = 0.$$

$$\Rightarrow \iiint_V \frac{\partial \rho}{\partial t} . dv + \iint_S \rho v . ds = 0.$$

$$\Rightarrow \iiint_V \frac{\partial \rho}{\partial t} . dv + \iiint_V \nabla . (\rho v) . dv = 0.$$

$$\Rightarrow \iiint_V \left(\frac{\partial \rho}{\partial t} + \nabla . (\rho v) \right) . dv = 0.$$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla . (\rho v) = 0} \quad - \mathrm{III}.$$

Path B:

$$\mathrm{III} \rightarrow \frac{\partial \rho}{\partial t} + \nabla . (\rho v) = 0.$$

$$\Rightarrow \nabla . (\rho v) = \rho (\nabla . v) + v . \nabla \rho$$

$$\Rightarrow \underbrace{\frac{\partial \rho}{\partial t} + v . \nabla \rho} + \rho (\nabla . v) = 0.$$

$$\Rightarrow \boxed{\frac{D \rho}{D t} + \rho (\nabla . v) = 0} \quad - \mathrm{IV}.$$

Path A:

$$\mathrm{II} \rightarrow \frac{D}{Dt} \iiint_V \rho . dv = 0.$$

$$\Rightarrow \iiint_V \frac{D}{Dt} (\rho . dv) = 0.$$

$$\Rightarrow \iiint_V \frac{D \rho}{D t} dv + \iiint_V \rho . \frac{D (dv)}{D t} = 0.$$

div. and mul. 2^{nd} term with 'dv'

$$\Rightarrow \iiint_V \frac{D\rho}{Dt}\, dv + \iiint_V \rho \underbrace{\left[\frac{1}{dv}\frac{D}{Dt}(dv)\right]}_{\nabla \cdot V} dv = 0.$$

$$\Rightarrow \iiint_V \frac{D\rho}{Dt}\, dv + \iiint_V \rho(\nabla \cdot V)\, dv = 0.$$

$$\Rightarrow \iiint_V \left(\frac{\partial \rho}{\partial t} + V\cdot\nabla\rho\right) + \iiint_V \rho(\nabla \cdot V)\cdot dv = 0.$$

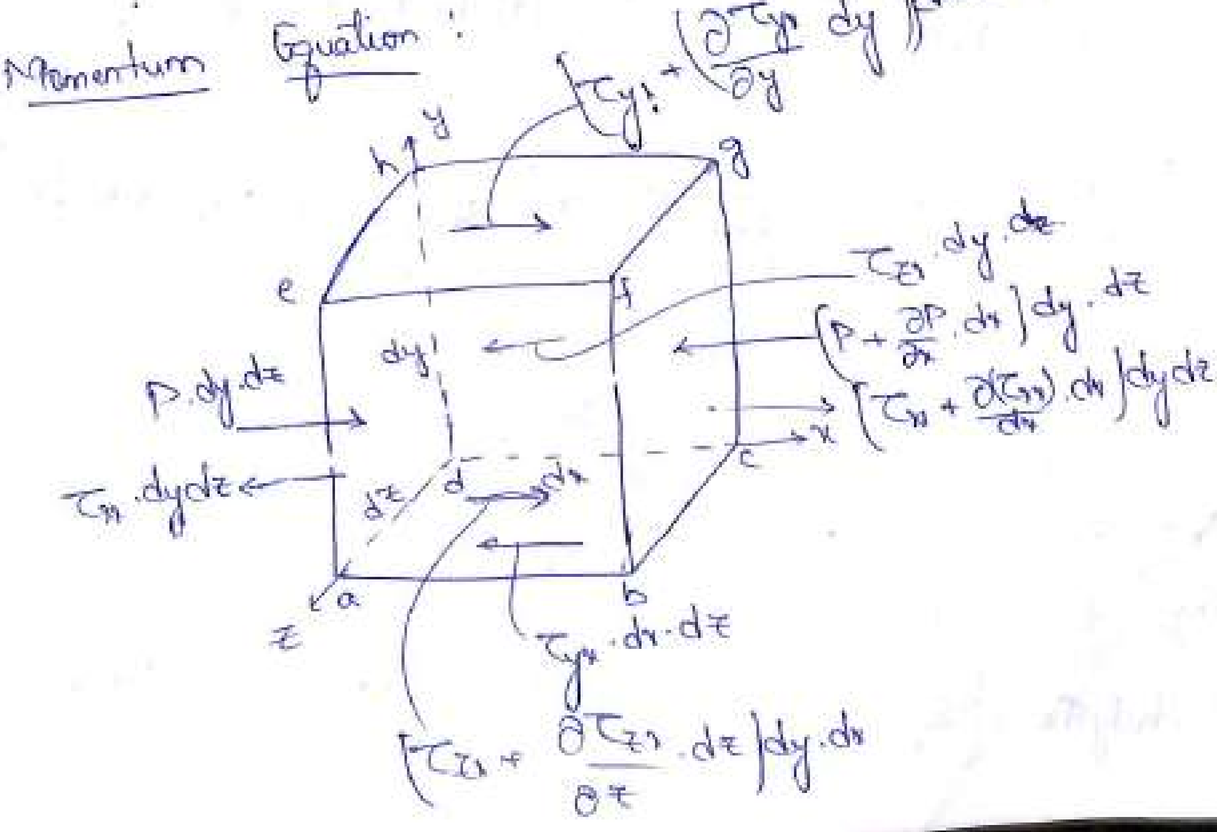
$$\Rightarrow \iiint_V \left(\frac{\partial \rho}{\partial t} + V\cdot\nabla\rho + \rho\,\nabla\cdot V\right) dv = 0.$$

$$\Rightarrow \iiint_V \left(\frac{\partial \rho}{\partial t} + \nabla\cdot \rho V\right)\cdot dv$$

$$\Rightarrow \iiint_V \frac{\partial \rho}{\partial t}\cdot dv + \underbrace{\iiint_V (\nabla\cdot(\rho V))\cdot dv}_{\text{replace with surface integral.}}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\iiint_V (\rho\, dv) + \iint_S \rho . V . dS} \quad \text{--- } \underline{I}$$

<u>Momentum</u> <u>Equation</u> :



Consider an infinitesimally small fluid element moving with fluid and apply fundamental physical principle for that fluid element.

$\Rightarrow F_x = m a_x$.

'F' is the force exerted on a body of mass 'm' and 'a' is the acceleration.

LHS:

$F = (\text{Body forces}) + (\text{Surface forces})$

Body force per unit mass acting on a fluid element is 'f_x'

~~Surfa~~ B.F acting on fluid elements $= \rho f_x \, dx\, dy\, dz$.

Surface force in x-direction

Net surfaces in x-direction $= \left[p - \left(p + \frac{\partial p}{\partial x} \cdot dx\right)\right] dy \cdot dz +$

$\left[\left(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x} \cdot dx\right) - \tau_{xx}\right] dy \cdot dz +$

$\left[\left(\tau_{yx} + \frac{\partial}{\partial y}(\tau_{yx}) dy\right) - \tau_{yx}\right] dx\, dz + \left[\tau_{zx} + \frac{\partial(\tau_{zx})}{\partial z} dz - \tau_{zx}\right] \cdot dx\, dy$

$F_x = \left(-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dx\, dy\, dz + \rho f_x \, dx\, dy\, dz$

R.H.S:

$m = \rho \cdot dx\, dy\, dz$

$a_x = \frac{Du}{Dt}$

equating LHS, RHS.

$\Rightarrow \rho \frac{Du}{Dt} \cancel{dx\, dy\, dz} = \left[-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x\right] \cancel{dx\, dy\, dz}$

$$\Rightarrow \boxed{\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x} \quad (1)$$

Ily

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y \quad (2)$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \rho f_z \quad (3)$$

called as N-Stokes eqns.

Eqn (1), (2), (3) are in non conservative form

$$\rho \frac{Du}{Dt} = \rho \frac{\partial u}{\partial t} + \rho V \cdot \nabla u \quad (a)$$

$$\frac{\partial \rho u}{\partial t} = \rho \frac{\partial u}{\partial t} + u \frac{\partial \rho}{\partial t}$$

$$\boxed{\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(\rho u) - u \frac{\partial \rho}{\partial t}}$$

$$\nabla \cdot (\rho u V) = u \nabla \cdot (\rho V) + (\rho V) \cdot \nabla u$$

$$\boxed{\rho V \cdot \nabla u = \nabla \cdot (\rho u V) - u \nabla (\rho V)}$$

sub. in (a)

$$\Rightarrow \rho \frac{Du}{Dt} = \frac{\partial}{\partial t}(\rho u) - u \Big[\underbrace{\frac{\partial \rho}{\partial t} + \nabla \cdot \rho V}_{0}\Big] + \nabla \cdot (\rho u V).$$

$$\Rightarrow \boxed{\rho \frac{Du}{Dt} = \frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u V)}$$

Now,

$$(1) \Rightarrow \boxed{\frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \vec{V}) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x} \quad (A)$$

III^{ly}

$$\frac{\partial(\rho v)}{\partial t} + \nabla\cdot(\rho v \vec{V}) = \frac{-\partial p}{\partial y} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y \quad \text{(B)}$$

$$\frac{\partial(\rho w)}{\partial t} + \nabla(\rho w \vec{V}) = \frac{-\partial p}{\partial z} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \rho f_z \quad \text{(C)}$$

(A), (B), (C) are in conservative form.

where

$$\tau_{xx} = \lambda(\nabla\cdot V) + 2\mu\frac{\partial u}{\partial x}$$

$$\tau_{yy} = \lambda(\nabla\cdot V) + 2\mu\frac{\partial v}{\partial y}$$

$$\tau_{zz} = \lambda(\nabla\cdot V) + 2\mu\frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

Sub. these, we get.

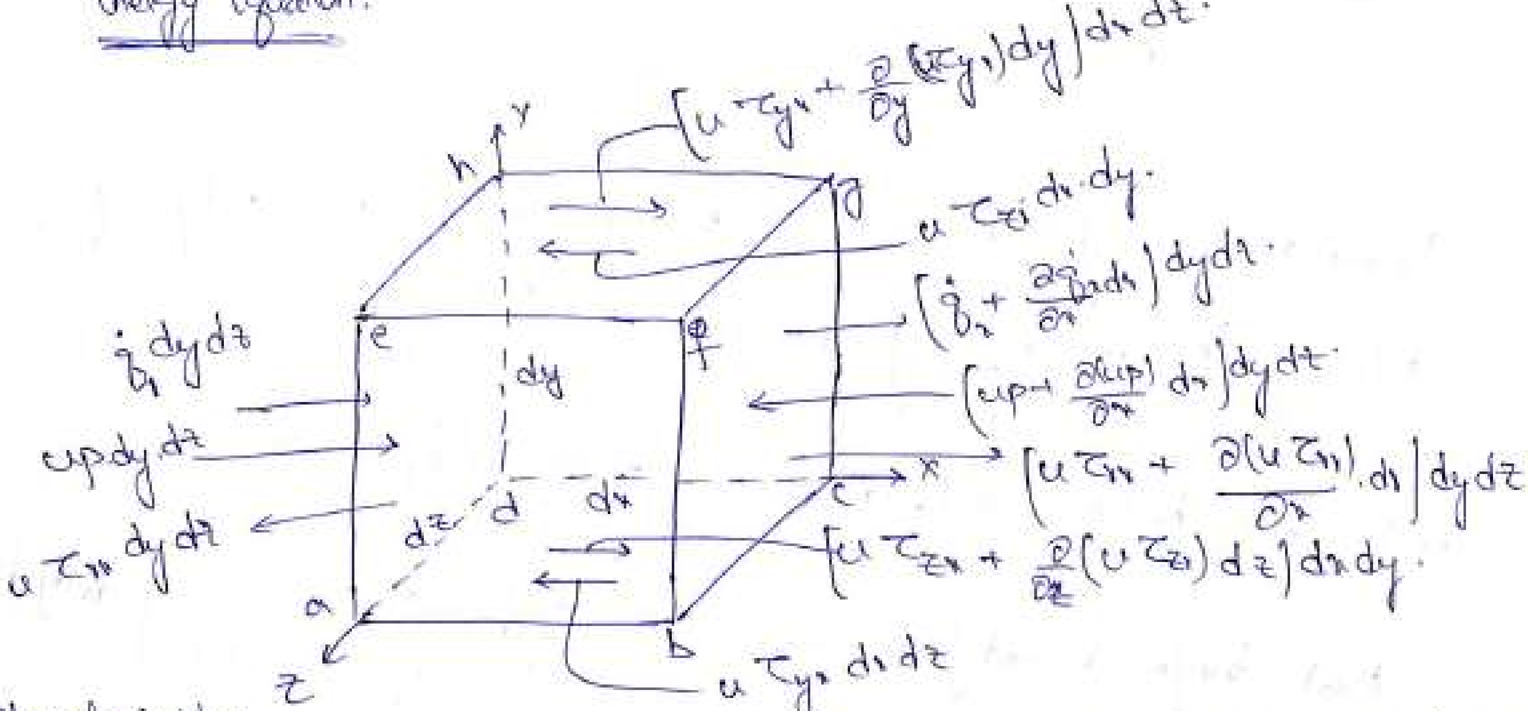
$$\frac{\partial(\rho u)}{\partial t} + \nabla(\rho u V) = \frac{-\partial p}{\partial x} + \frac{\partial}{\partial x}(\lambda(\nabla\cdot V)) + 2\mu\frac{\partial^2 u}{\partial x^2} + \mu\left[\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\right] + \mu\left[\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z}\right] + \rho f_x.$$

III^{ly}

$$\frac{\partial(\rho v)}{\partial t} + \nabla(\rho v V) = \frac{-\partial p}{\partial y} + \frac{\partial}{\partial y}(\lambda(\nabla\cdot V)) + 2\mu\frac{\partial^2 v}{\partial y^2} + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y}\right] + \mu\left[\frac{\partial^2 w}{\partial y \partial z} + \frac{\partial^2 v}{\partial z^2}\right] + \rho f_y$$

$$\frac{\partial(\rho w)}{\partial t} + \nabla\cdot(\rho w \vec{V}) = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left(\lambda(\nabla\cdot\vec{V})\right) + 2\mu\frac{\partial^2 w}{\partial z^2} + \mu\left[\frac{\partial^2 u}{\partial x\partial z} + \frac{\partial^2 w}{\partial x^2}\right] + \mu\left[\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 v}{\partial z\partial y}\right] + \rho f_z$$

Energy Equation:



Physical principle:

Rate of change of energy inside the fluid element = Net flux of heat into the element + Rate of work done on element due to body and surface force.

A = B + C.

C

B.F + Surface Force
(Pressure + Shear + Normal).

$$\boxed{C = \left[-\left(\frac{\partial(up)}{\partial x} + \frac{\partial(vp)}{\partial y} + \frac{\partial(wp)}{\partial z}\right) + \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial v\tau_{xy}}{\partial x} + \frac{\partial v\tau_{yy}}{\partial y} + \frac{\partial v\tau_{zy}}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} + \rho\vec{f}\cdot\vec{V}\right]dx\,dy\,dz}$$

B

(1) Volumetric heating of fluid element.

(2) Heat transfer to and from element.

$$B = (1) + (2)$$

Volumetric rate of heat addition per unit mass denoted by $\dot{q}$

Mass of the moving fluid element $= \rho \cdot dx\,dy\,dz$.

⇒ Volumetric heating of fluid element $= \rho \dot{q}\, dx\,dy\,dz$.

Heat transfer to and from element $= -\left(\frac{\partial \dot{q}_x}{\partial x} + \frac{\partial \dot{q}_y}{\partial y} + \frac{\partial \dot{q}_z}{\partial z} \right) dx\,dy\,dz$

adding both the expressions.

$$\rightarrow B = \left[\rho \dot{q} - \frac{\partial \dot{q}_x}{\partial x} - \frac{\partial \dot{q}_y}{\partial y} - \frac{\partial \dot{q}_z}{\partial z} \right] dx\,dy\,dz.$$

where $\dot{q}_x = -k \frac{\partial T}{\partial x}$; $\dot{q}_y = -k \frac{\partial T}{\partial y}$; $\dot{q}_z = -k \frac{\partial T}{\partial z}$

$$\Rightarrow \cancel{B = \rho \dot{q} + k \frac{\partial^2 T}{\partial x^2} + k \frac{\partial^2 T}{\partial y^2} + k \frac{\partial^2 T}{\partial z^2}}$$

$$\Rightarrow \cancel{B = \left[\rho \dot{q} + k \frac{\partial}{\partial x} \left(\frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} + \frac{\partial T}{\partial z} \right) \right] dx\,dy\,dz}$$

$$\Rightarrow \boxed{B = \left[\rho \dot{q} + k \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) + k \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right) + k \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right) \right] dx\,dy\,dz}$$

$\underline{A}$ $B = \left[\rho \dot{q} + k \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) + k \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right) + k \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right) \right] dx\,dy\,dz$

↳ The time rate of change of energy of a fluid element

depends on

① Translation + Rotational + Vibration $= e$.

② K.E of fluid element $= \frac{V^2}{2}$.

① Translation + Rotat. + Vibr. $= e$

② K.E of fluid element $= \frac{V^2}{2}$

$A = \rho \frac{D}{Dt}\left(e + \frac{V^2}{2}\right) dx\,dy\,dt$.

$$\Rightarrow \boxed{A = \rho \frac{D}{Dt}\left(e + \frac{V^2}{2}\right) dx\,dy\,dz}$$

$$\Rightarrow \boxed{\begin{aligned} \rho \frac{D}{Dt}\left(e + \frac{V^2}{2}\right) dx\,\cancel{dy}\,dz = \Big[& \rho \dot{q} + \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) - \\ & \frac{\partial (up)}{\partial x} - \frac{\partial (vp)}{\partial y} - \frac{\partial (wp)}{\partial z} + \frac{\partial}{\partial x}(u \tau_{xx}) + \frac{\partial}{\partial y}(u \tau_{yx}) \\ & + \frac{\partial (u \tau_{zx})}{\partial z} + \frac{\partial}{\partial x}(v \tau_{xy}) + \frac{\partial}{\partial y}(v \tau_{yy}) + \frac{\partial}{\partial z}(v \tau_{zy}) + \frac{\partial}{\partial x}(w \tau_{xz}) + \\ & \frac{\partial}{\partial y}(w \tau_{yz}) + \frac{\partial}{\partial z}(w \tau_{zz}) + \rho \cdot f \cdot V \Big] \cancel{dx\,dy\,dz} \end{aligned}}$$

Continuity equation: (for Viscous flow) (also known as N-stokes eq)

→ Non conservative form: $\frac{D\rho}{Dt} + \rho \nabla \cdot V = 0$.

→ Conservative form: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0$.

Momentum equation:

→ Non conservative form:

x-direction → $\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x$

y-direction → $\frac{\rho Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y$

z-direction → $\frac{\rho Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z$

→ Conservative form:

x-direction → $\frac{\partial}{\partial t}(\rho u) + \nabla(\rho u V) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x$

y-direction → $\frac{\partial}{\partial t}(\rho v) + \nabla(\rho v V) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y$

z-direction → $\frac{\partial}{\partial t}(\rho w) + \nabla(\rho w V) = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z$

Energy equation:

→ Non Conservative form:

$$\rho \frac{D}{Dt}\left(e + \frac{V^2}{2}\right) = \left(\rho \dot{q} + \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) - \frac{\partial (up)}{\partial x} - \frac{\partial (vp)}{\partial y} - \frac{\partial (wp)}{\partial z} + \frac{\partial}{\partial x}(u \tau_{xx}) + \frac{\partial}{\partial y}(u \tau_{yx}) + \frac{\partial}{\partial y}(v \tau_{yy}) + \frac{\partial}{\partial z}(v \tau_{zy}) + \frac{\partial}{\partial z}(u \tau_{zx}) + \frac{\partial}{\partial x}(v \tau_{xy}) + \frac{\partial}{\partial x}(w \tau_{xz}) + \frac{\partial}{\partial y}(w \tau_{yz}) + \frac{\partial}{\partial z}(w \tau_{zz})\right.$$

$$+ \rho f \cdot V$$

→ Conservative form: (In terms of Internal Energy)

$$\frac{\partial(\rho e)}{\partial t} + \nabla\cdot(\rho e V) = \rho\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

$$- p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 +$$

$$\mu\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\right]$$

(In terms of total energy)

$$\frac{\partial}{\partial t}\left[\rho\left(e + \frac{V^2}{2}\right)\right] + \nabla\cdot\left[\rho\left(e + \frac{V^2}{2}\right)V\right] = \rho\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) +$$

$$\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) - \frac{\partial(up)}{\partial x} - \frac{\partial(vp)}{\partial y} - \frac{\partial(wp)}{\partial z} + \frac{\partial(u\tau_{xx})}{\partial x} +$$

$$\frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} +$$

$$\frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} + \rho f\cdot V.$$

Equations for Inviscid flow: (also known as Euler Equations)

Continuity: Same as viscous flow.

Momentum: Non-conservative:

x-comp: $\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho f_x$

y-comp: $\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho f_y$

z-comp: $\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho f_z$

Conservative form:

x-comp: $\frac{\partial(\rho u)}{\partial t} + \nabla\cdot(\rho u V) = -\frac{\partial p}{\partial x} + \rho f_x$

y-comp: $\frac{\partial(\rho v)}{\partial t} + \nabla\cdot(\rho v V) = -\frac{\partial p}{\partial y} + \rho f_y$

z-comp: $\frac{\partial(\rho w)}{\partial t} + \nabla\cdot(\rho w V) = -\frac{\partial p}{\partial z} + \rho f_z$

Energy: Non conservative:

$$\rho\frac{D}{Dt}\left(e + \frac{V^2}{2}\right) = \rho\dot{q} - \frac{\partial(up)}{\partial x} - \frac{\partial(vp)}{\partial y} - \frac{\partial(wp)}{\partial z} + \rho f\cdot V$$

Conservative form:

$$\frac{\partial}{\partial t}\left[\rho\left(e + \frac{V^2}{2}\right)\right] + \nabla\cdot\left[\rho\left(e + \frac{V^2}{2}\right)V\right] = \rho\dot{q} - \frac{\partial(up)}{\partial x} - \frac{\partial(vp)}{\partial y} - \frac{\partial(wp)}{\partial z} + \rho f\cdot V$$

1. Derive the energy equation interms of
(i) Internal Energy (ii) Flowfield variable (iii) Non conservative to conservative

Ans: (i) We know that energy equation is

$$\rho \frac{D}{Dt}\left(e+\frac{V^2}{2}\right) = \rho\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) - \frac{\partial(up)}{\partial x} - \frac{\partial(vp)}{\partial y}$$
$$-\frac{\partial(wp)}{\partial z} + \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} +$$
$$\frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} + \rho f.V.$$

> Above equation is the non-conservative form, it is in terms of tot. energy $\left(e+\frac{V^2}{2}\right)$

> L.H.S can be expressed in terms of internal energy 'e' alone.

Multiplying Non conservative momentum equation with x, y, z respectively.

$$\Rightarrow \rho\frac{D(u^2/2)}{Dt} = -u\frac{\partial P}{\partial x} + u\frac{\partial \tau_{xx}}{\partial x} + u\frac{\partial \tau_{yx}}{\partial y} + u\frac{\partial \tau_{zx}}{\partial z} + \rho u f_x$$
$$\rho\frac{D(v^2/2)}{Dt} = -v\frac{\partial P}{\partial y} + v\frac{\partial \tau_{xy}}{\partial x} + v\frac{\partial \tau_{yy}}{\partial y} + v\frac{\partial \tau_{zy}}{\partial z} + \rho v f_y$$
$$\rho\frac{D(w^2/2)}{Dt} = -w\frac{\partial P}{\partial z} + w\frac{\partial \tau_{xz}}{\partial x} + w\frac{\partial \tau_{yz}}{\partial y} + w\frac{\partial \tau_{zz}}{\partial z} + \rho w f_z$$

adding these three, noting $u^2+v^2+w^2 = V^2$

$$\Rightarrow \rho\frac{DV^2}{Dt} = -u\frac{\partial P}{\partial x} - v\frac{\partial P}{\partial y} - w\frac{\partial P}{\partial z} + u\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)$$
$$+ v\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + w\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right)$$
$$+ \rho(u f_x + v f_y + w f_z)$$

Substracting this from Non-conservative energy equation and noting $\rho f \cdot V = \rho(u f_x + v f_y + w f_z)$

$$\Rightarrow \rho \frac{De}{Dt} = \rho \dot{q} + \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right)$$

$$- p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yx}\frac{\partial u}{\partial y} + \tau_{zx}\frac{\partial u}{\partial z} +$$

$$\tau_{xy}\frac{\partial v}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zy}\frac{\partial v}{\partial z} + \tau_{xz}\frac{\partial w}{\partial x} + \tau_{yz}\frac{\partial w}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z}$$

The above equation is in non-conservative form but strictly in terms of Internal Energy only.

(ii) taking down the Non-conservative form of energy equation interms of Energy.

and substituting $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$; $\tau_{yz} = \tau_{zy}$.

$$\Rightarrow \rho \frac{De}{Dt} = \rho \dot{q} + \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) - p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$+ \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z} + \tau_{yx}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \tau_{zx}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$+ \tau_{zy}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

expressing viscous stresses interm of velocity gradients.

$$\Rightarrow \rho \frac{De}{Dt} = \rho \dot{q} + \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) - p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$+ \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 + \mu\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right.$$

$$\left. + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\right]$$

Above equation is a Non-conservative energy equation but completely interms of flow-field variables.

(iii) Consider LHS of non conservative energy eqn interms of flow field variables,

from def. of substantial derivative

$$\rho \frac{De}{Dt} = \rho \frac{\partial e}{\partial t} + \rho V \cdot \nabla e.$$

however

$$\frac{\partial(\rho e)}{\partial t} = \rho \frac{\partial e}{\partial t} + e \frac{\partial \rho}{\partial t}$$

$$\rho \frac{\partial e}{\partial t} = \frac{\partial(\rho e)}{\partial t} - e \frac{\partial \rho}{\partial t}$$

from vector identity concerning the divergence of the product of a scalar times a vector,

$$\nabla \cdot (\rho e V) = e \nabla \cdot (\rho V) + \rho V \cdot \nabla e.$$

or $$\rho V \cdot \nabla e = \nabla \cdot (\rho e V) - e \nabla \cdot (\rho V)$$

Subs. these into def. of substantial derivative

$$\Rightarrow \rho \frac{De}{Dt} = \frac{\partial(\rho e)}{\partial t} - e\left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V)\right] + \nabla \cdot (\rho e V) \quad (\because \text{from cont. eq}^n)$$

$$\Rightarrow \rho \frac{De}{Dt} = \frac{\partial(\rho e)}{\partial t} + \nabla \cdot (\rho e V)$$

Sub. this into non-conservative energy eqn interms of flow field variables.

$$\Rightarrow \frac{\partial(\rho e)}{\partial t} + \nabla \cdot (\rho e V) = \rho \dot{q} + \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right)$$
$$- p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 +$$
$$\mu\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\right]$$

This is a conservative form of energy equation, written interms of Internal energy.

Illy

In substantial derivative using $e+\frac{V^2}{2}$ instead of 'e'

$$\Rightarrow \rho \frac{D(e+\frac{V^2}{2})}{Dt} = \frac{\partial}{\partial t}\left[\rho\left(e+\frac{V^2}{2}\right)\right] + \nabla\cdot\left[\rho\left(e+\frac{V^2}{2}\right)V\right]$$

Subs. this into Non-conservative energy equation interms of total energy.

$$\Rightarrow \frac{\partial}{\partial t}\left[\rho\left(e+\frac{V^2}{2}\right)\right] + \nabla\cdot\left[\rho\left(e+\frac{V^2}{2}\right)V\right] = \rho\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) +$$

$$\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) - \frac{\partial(up)}{\partial x} - \frac{\partial(vp)}{\partial y} - \frac{\partial(wp)}{\partial z} + \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} +$$

$$\frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} +$$

$$\frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} + \rho f\cdot V.$$

The above equation is the conservative form of the energy equation, written in terms of total energy $\left(e+\frac{V^2}{2}\right)$.

forms of governing equations:

(1) generic form is as follows

$$\boxed{\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = J} \quad \text{— (1)}$$

(for viscous flow)

$$U = \begin{Bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho\left(e + \frac{V^2}{2}\right) \end{Bmatrix}$$

$$F = \begin{Bmatrix} \rho u \\ \rho u^2 + p - \tau_{xx} \\ \rho vu - \tau_{xy} \\ \rho wu - \tau_{xz} \\ \rho\left(e + \frac{V^2}{2}\right)u + pu - k\frac{\partial T}{\partial x} - u\tau_{xx} - v\tau_{xy} - w\tau_{xz} \end{Bmatrix}$$

$$G = \begin{Bmatrix} \rho v \\ \rho uv - \tau_{yx} \\ \rho v^2 + p - \tau_{yy} \\ \rho wv - \tau_{yz} \\ \rho\left(e + \frac{V^2}{2}\right)v + pv - k\frac{\partial T}{\partial y} - u\tau_{yx} - v\tau_{yy} - w\tau_{yz} \end{Bmatrix}$$

$$H = \begin{Bmatrix} \rho w \\ \rho uw - \tau_{zx} \\ \rho vw - \tau_{zy} \\ \rho w^2 + p - \tau_{zz} \\ \rho\left(e + \frac{V^2}{2}\right)w + pw - k\frac{\partial T}{\partial z} - u\tau_{zx} - v\tau_{zy} - w\tau_{zz} \end{Bmatrix}$$

$$J = \begin{Bmatrix} 0 \\ \rho f_x \\ \rho f_y \\ \rho f_z \\ \rho(uf_x + vf_y + wf_z) + \rho\dot{q} \end{Bmatrix}$$

F, G, H are flux terms and J is a source term and U is solution vector.

* Eq^n (1) is written with time derivative $\frac{\partial U}{\partial t}$. It applies to unsteady flow

$$\frac{\partial U}{\partial t} = J - \left[\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z}\right]$$

where 'U' is a solution vector, elements in U $\rho, \rho u, \rho v, \rho w, \rho\left(e+\frac{V^2}{2}\right)$ are dependent variables which usually obtained numerically with time.

$$\rho = \rho$$

$$u = \frac{\rho u}{\rho}$$

$$v = \frac{\rho v}{\rho}$$

$$w = \frac{\rho w}{\rho}$$

$$e = \frac{\rho\left(e+\frac{V^2}{2}\right)}{\rho} - \left(\frac{u^2+v^2+w^2}{2}\right)$$

For Inviscid flow:

$$U = \begin{Bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho\left(e+\frac{V^2}{2}\right) \end{Bmatrix} \qquad H = \begin{Bmatrix} \rho w \\ \rho u w \\ \rho v w \\ \rho w^2 + p \\ \rho\left(e+\frac{V^2}{2}\right)w + pw \end{Bmatrix}$$

$$F = \begin{Bmatrix} \rho u \\ \rho u^2 + p \\ \rho v u \\ \rho w u \\ \rho\left(e+\frac{V^2}{2}\right)u + pu \end{Bmatrix}$$

$$G = \begin{Bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho w v \\ \rho\left(e+\frac{V^2}{2}\right)v + pv \end{Bmatrix}$$

Marching with space in x-direction:

$$\Rightarrow \boxed{\frac{\partial F}{\partial x} = J - \frac{\partial G}{\partial y} - \frac{\partial H}{\partial z}}$$

$$\rho u = C_1$$

$$\rho u^2 + p = C_2$$

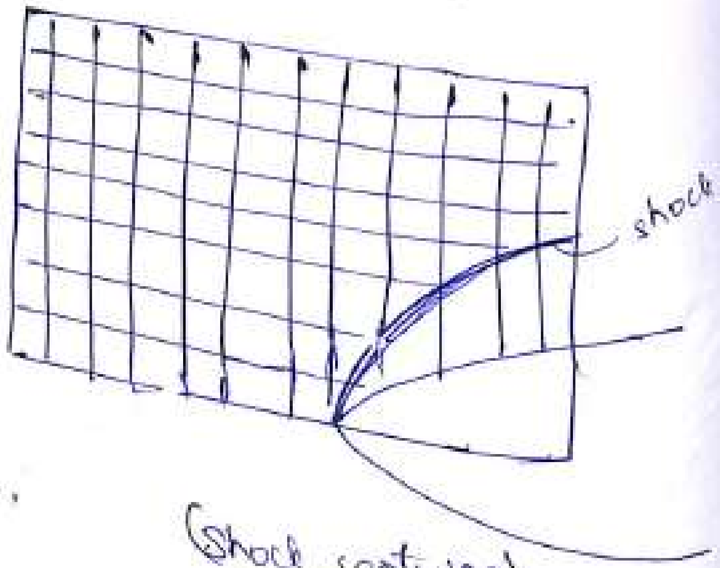
$$\rho u v = C_3$$

$$\rho u w = C_4$$

$$\rho u\left(e + \frac{u^2+v^2+w^2}{2}\right) + p u = C_5 .$$

(2) Shock capturing:

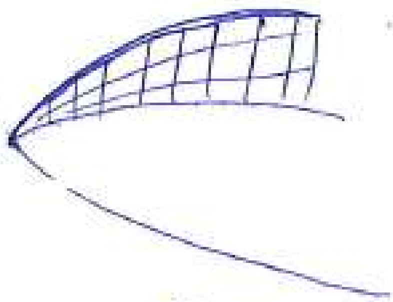
Many computational flows which shocks are designed to have the shock waves naturally within the computational space as a direct result of overall flow field solution.



(Shock capturing)

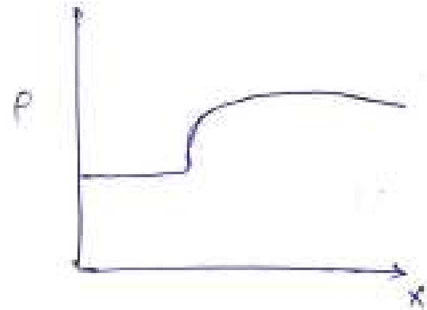
Shock fitting:

Shock waves are explicitly introduced in a flow field solution. The exact rankine and Hugoniot relation for changes across a shock.

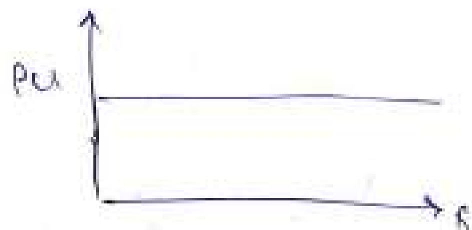


(Shock fitting)

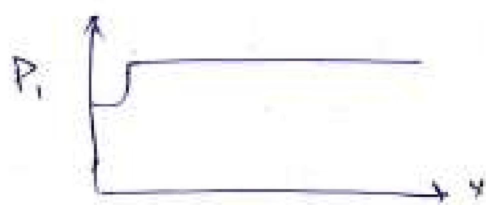
(for shock capturing)



(Non-conservative)



(conservative)



(Non-conservative)

(conservative)

UNIT II

Mathematical Behavior of Partial Differential Equations and Their Impact on Computational Aerodynamics

Classification of PDE

Cramer rule | Eigen Value Method

Hyperbolic | Parabolic | elliptical | Mixed.

(Cramers method)

<u>Classification of Quasi linear Partial Differential equations</u>:

$$a_1\frac{\partial u}{\partial x} + b_1\frac{\partial u}{\partial y} + c_1\frac{\partial v}{\partial x} + d_1\frac{\partial v}{\partial y} = f_1 \quad \text{--- (1)}$$

$$a_2\frac{\partial u}{\partial x} + b_2\frac{\partial u}{\partial y} + c_2\frac{\partial v}{\partial x} + d_2\frac{\partial v}{\partial y} = f_2 \quad \text{--- (2)}$$

$$du = \frac{\partial u}{\partial x}\cdot dx + \frac{\partial u}{\partial y}\cdot dy \quad \text{--- (3)}$$

$$dv = \frac{\partial v}{\partial x}\cdot dx + \frac{\partial v}{\partial y}\cdot dy \quad \text{--- (4)}$$

The four linear equations with 4 unknown $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

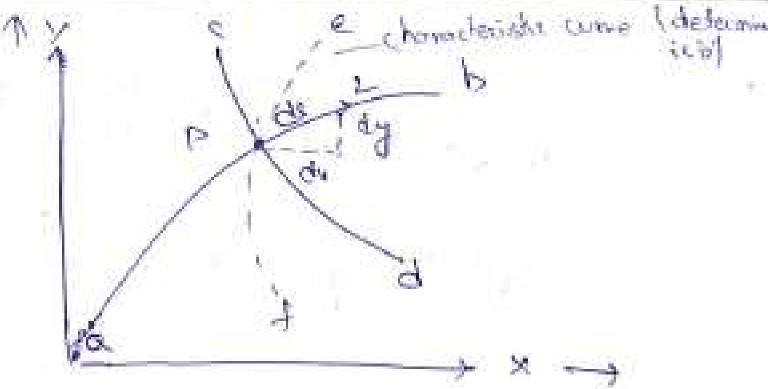
$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{bmatrix} \begin{pmatrix} \partial u/\partial x \\ \partial u/\partial y \\ \partial v/\partial x \\ \partial v/\partial y \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ du \\ dv \end{pmatrix}$$

$$(A) = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{pmatrix} \qquad (B) = \begin{bmatrix} f_1 & b_1 & c_1 & d_1 \\ f_2 & b_2 & c_2 & d_2 \\ du & dy & 0 & 0 \\ dv & 0 & dx & dy \end{bmatrix}$$

(using cramer rule)

$$\frac{\partial u}{\partial x} = \frac{|B|}{|A|}$$

(a)



$$dx = x_2 - x_p$$
$$dy = y_2 - y_p$$
$$du = u_2 - u_p$$
$$dv = v_2 - v_p$$

$|A| = 0$

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{vmatrix} = 0.$$

$$\Rightarrow (a_1 c_2 - a_2 c_1)(dy)^2 - (a_1 d_2 - a_2 d_1 + b_1 c_2 - b_2 c_1)\, dx\, dy + (b_1 d_2 - b_2 d_1)(dx)^2 = 0.$$

divide the eqn with $(dx)^2$.

$$\Rightarrow \underbrace{(a_1 c_2 - a_2 c_1)}_{a} \frac{(dy)^2}{(dx)^2} - \underbrace{(a_1 d_2 - a_2 d_1 + b_1 c_2 - b_2 c_1)}_{b} \frac{dy}{dx} + \underbrace{(b_1 d_2 - b_2 d_1)}_{c} = 0.$$

(for simplicity)

$$\Rightarrow a\left(\frac{dy}{dx}\right)^2 + b\left(\frac{dy}{dx}\right) + c = 0$$

$$\frac{dy}{dx} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ (quadratic eqn expression)

if $\Delta > 0$ — hyperbolic
$\Delta = 0$ — parabolic
$\Delta < 0$ — elliptical

(Δ is the discriminant i.e., $b^2 - 4ac$)

(Eigen value method)

$$a_1 \frac{\partial u}{\partial x} + b_1 \frac{\partial u}{\partial y} + c_1 \frac{\partial v}{\partial x} + d_1 \frac{\partial v}{\partial y} = 0.$$

$$a_2 \frac{\partial u}{\partial x} + b_2 \frac{\partial u}{\partial y} + c_2 \frac{\partial v}{\partial x} + d_2 \frac{\partial v}{\partial y} = 0.$$

$$w = \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} \frac{\partial w}{\partial x} + \begin{pmatrix} b_1 & d_1 \\ b_2 & d_2 \end{pmatrix} \frac{\partial w}{\partial y}$$

$$[K] \frac{\partial w}{\partial x} + [M] \frac{\partial w}{\partial y} = 0.$$

$$\frac{\partial w}{\partial x} + \underbrace{[K]^{-1}[M]}_{[N]} \frac{\partial w}{\partial y} = 0. \Rightarrow \underline{\frac{\partial w}{\partial x} + [N] \frac{\partial w}{\partial y} = 0.}$$

~~Problems:~~

~~Consider irrotational 2-D inviscid~~ steady flow

1. Calculate eigen values for $(1-M_\infty^2) \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$ — (1)

$$\frac{\partial u'}{\partial y} - \frac{\partial v'}{\partial x} = 0 \quad \text{— (2)}$$

__Sol:__ __Given:__

$$(1-M_\infty^2) \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$$

$$\frac{\partial u'}{\partial y} - \frac{\partial v'}{\partial x} = 0$$

using Eigen value method, writing the matrix.

$$\Rightarrow \underbrace{\begin{bmatrix} (1-M_\infty^2) & 0 \\ 0 & -1 \end{bmatrix}}_{[K]} \frac{\partial w}{\partial x} + \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{[M]} \frac{\partial w}{\partial y} = 0$$

Now, finding inverse for matrix __[K]__

$$\Rightarrow [k]^{-1} = \frac{1}{-(1-M_\infty^2)} \begin{pmatrix} -1 & 0 \\ 0 & 1-M_\infty^2 \end{pmatrix} \quad \left(\because \text{transpose of } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ is } \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$$

$$\Rightarrow [k]^{-1} = \begin{pmatrix} \frac{1}{(1-M_\infty^2)} & 0 \\ 0 & -1 \end{pmatrix}$$

Now,

$$\Rightarrow [N] = [k]^{-1}[M] = \begin{pmatrix} \frac{1}{1-M_\infty^2} & 0 \\ 0 & -1 \end{pmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(N) = \begin{pmatrix} 0 & \frac{1}{1-M_\infty^2} \\ -1 & 0 \end{pmatrix}$$

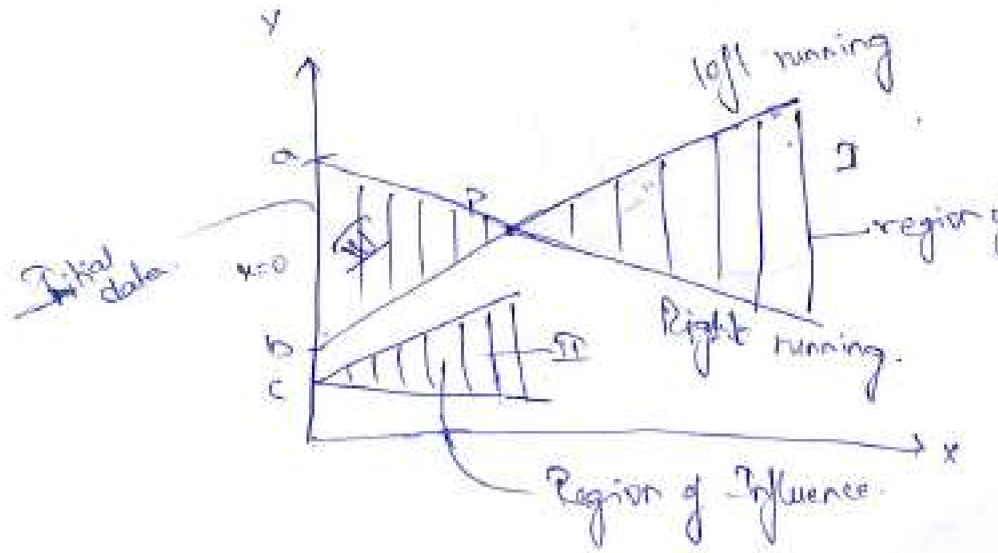
finding eigen values for (N)

$$\Rightarrow |(N) - \lambda(I)| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda & \frac{1}{1-M_\infty^2} \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + \frac{1}{1-M_\infty^2} = 0 \quad \Rightarrow \boxed{\lambda = \pm\sqrt{\frac{1}{M_\infty^2 - 1}}}$$ — eigen values.

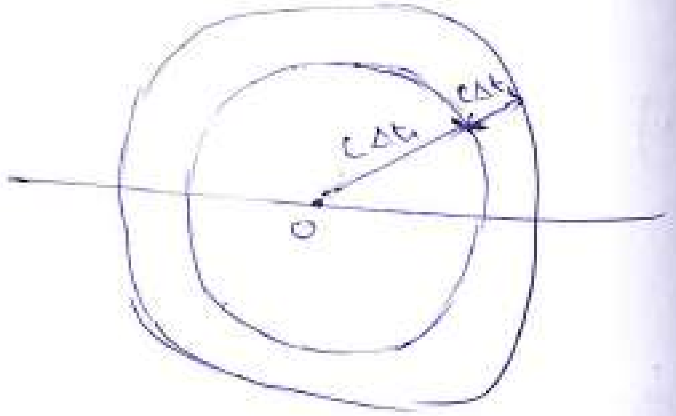
Hyperbolic equations: $D > 0$ — (two real and distinct)



c = Sonic speed.
u = speed of flow.

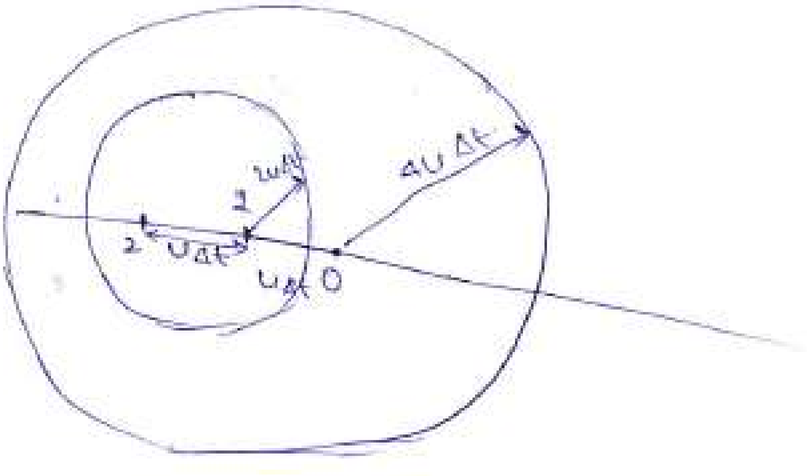
$M_a = u/c$

eg.1) $M_a = 0$.

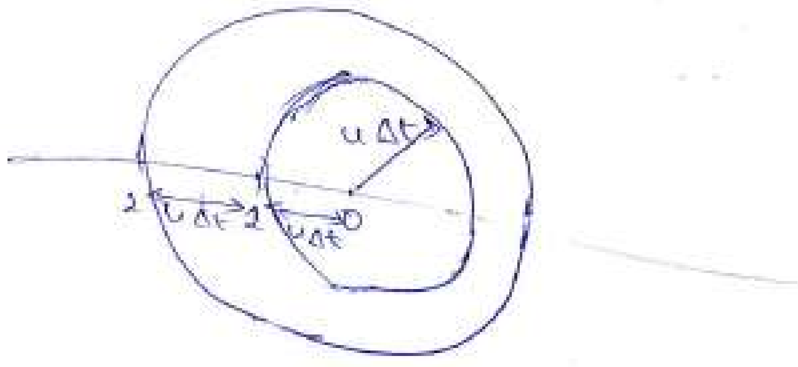


eg.2) $M_a < 1$

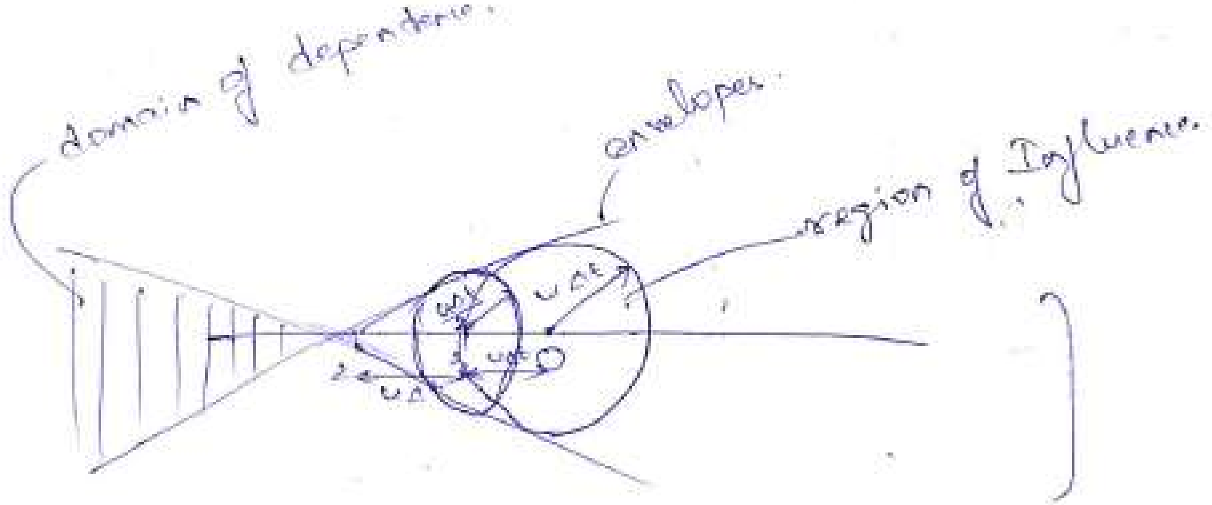
$M_a = 1/2$ ~~u=~~ $c = 2u$



$M_a = 1$, $u = c$.



$M_a > 1$, $M_a = 2$ $u = 2c$.



I - Region of Influences.

Info. at pt. 'p' influces only b/w left and right characteristics.

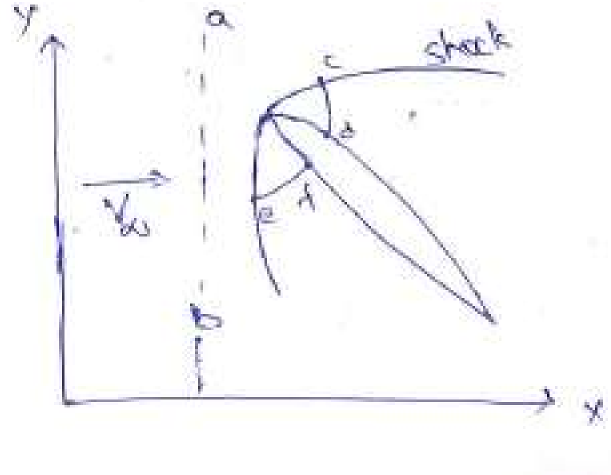
III - Region

Assume the B.C are specified on the y-axis. It is called as domain of dependance.

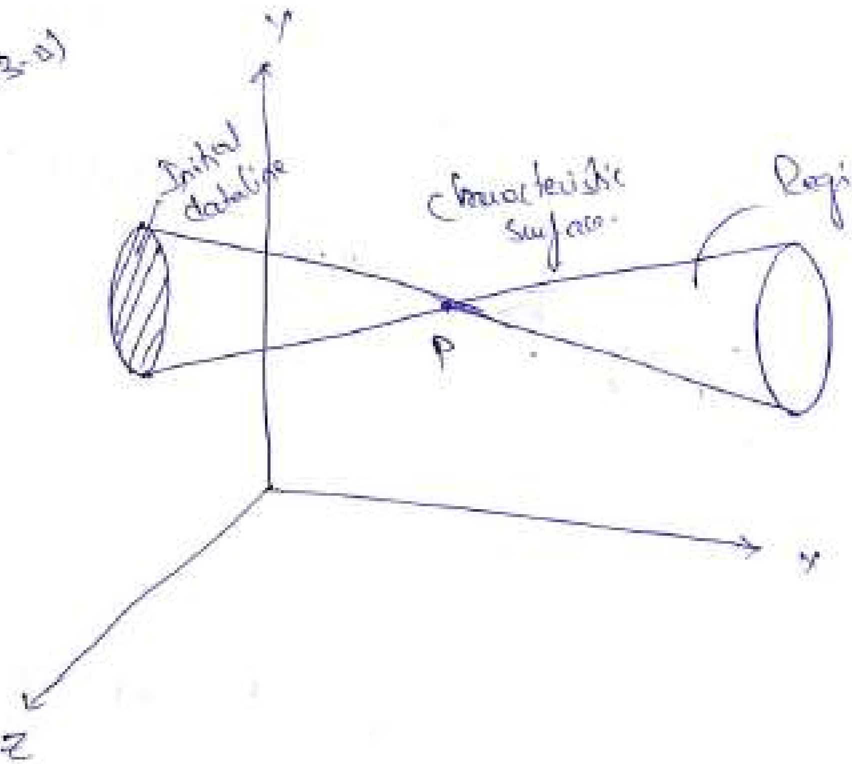
II - Region

The info. at pt. 'c' which is outside, the interval a,b is propogated along the characteristic through 'c' and influences only the region - III.

Steady Inviscid Supersonic flows:

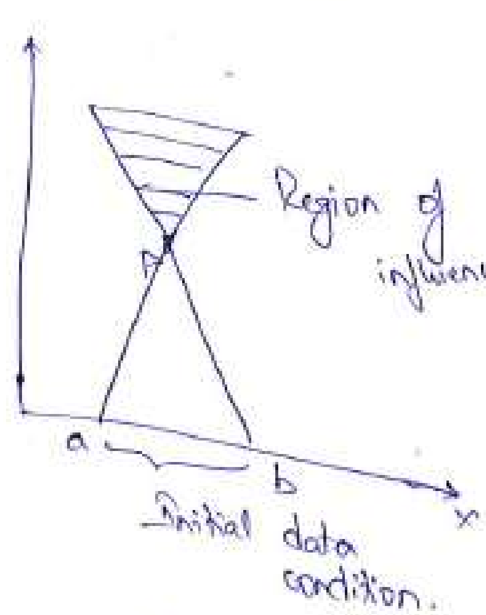


(3-D)

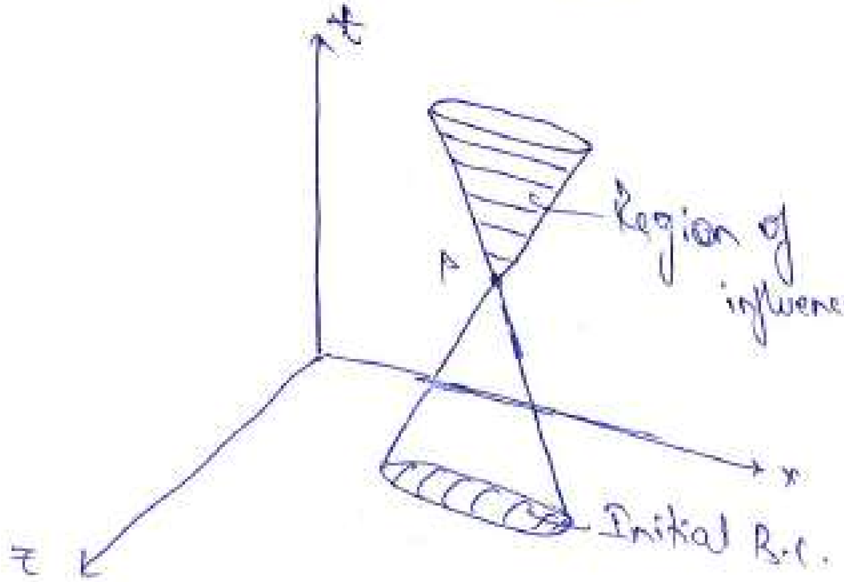


unsteady, Inviscid Supersonic flows.

(2-D)

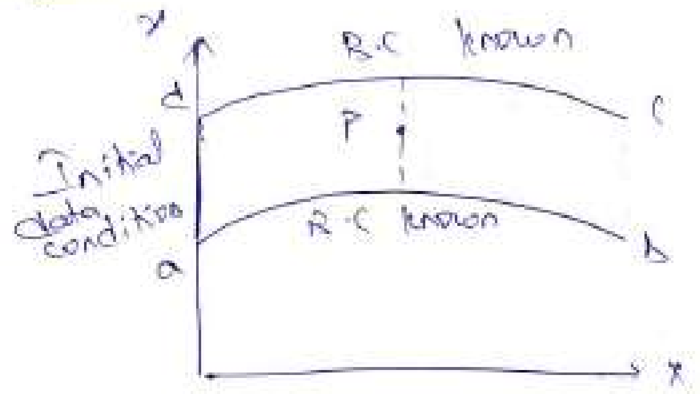


(3-D)

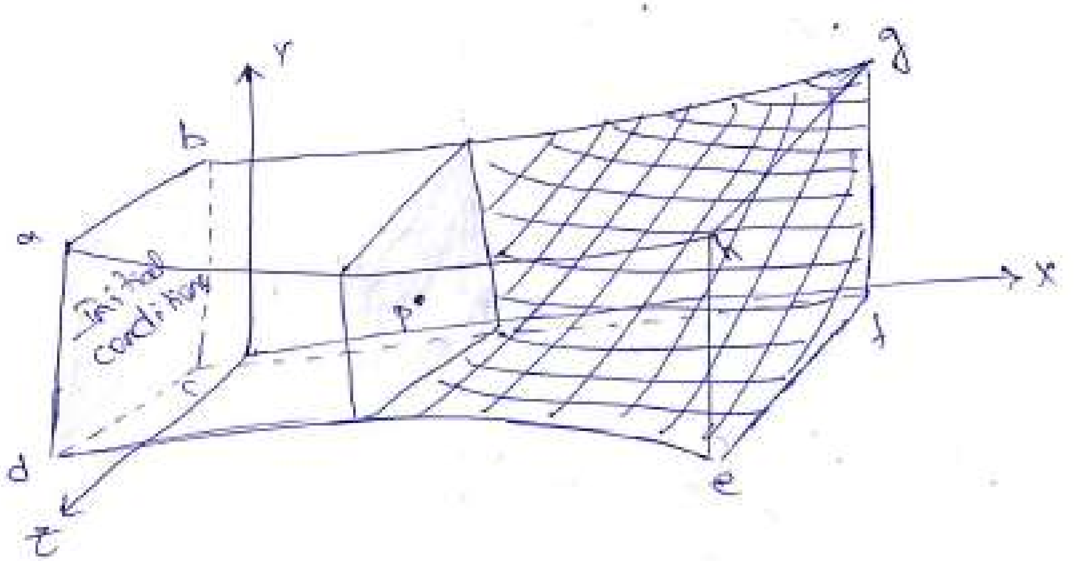


Parabolic equations:

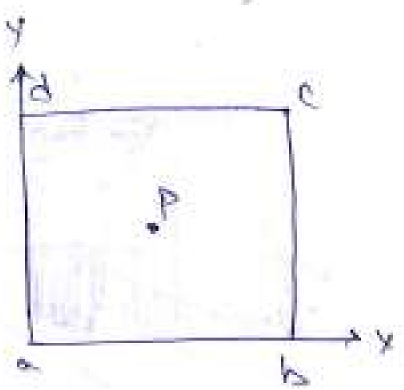
(2-D)



(3-D)



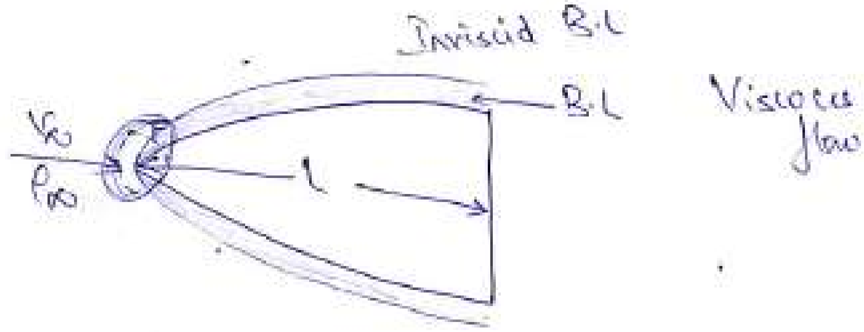
Elliptical equation:

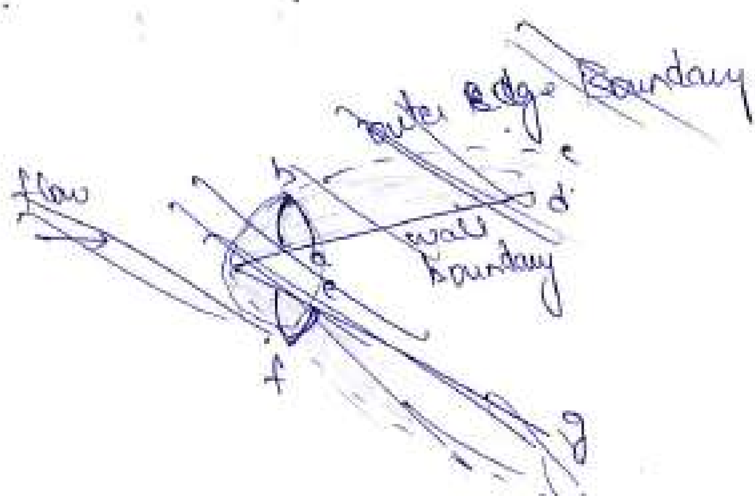


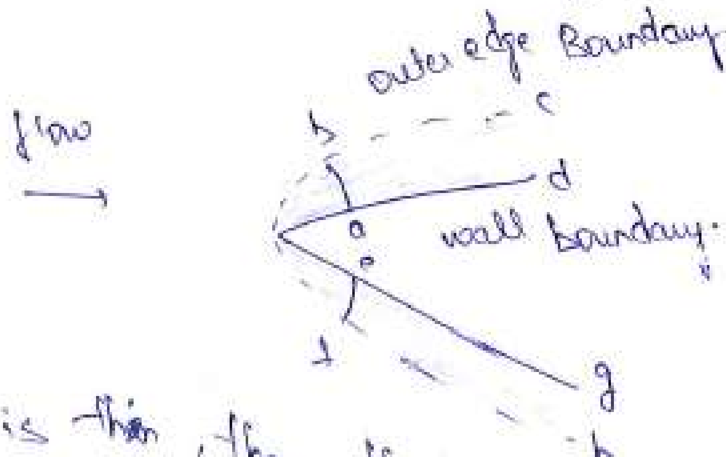
Consider pt. 'p' in x-y plane. Assume that domain of the problem is □ ^{lo} and pt. 'p' is located somewhere inside the domain. The mathematical characteristic of elliptic eqn is that any disturbance felt everywhere throughout the domain i.e. at point 'P' it is influenced by the entire closed boundary ~~[illegible]~~ abcd.

Parabolic equations:

Steady Boundary layer flows:







→ If B.L is thin, then the eqn used are called B.L eqn

The concept of dividing the general flow field in 2 regions

1. The thin layer adjacent to the solid surface where all the viscous effects are contained.
2. An inviscid flow outside the thin viscous layer.

The B.L is thin and Reynolds number is based on body length i.e., 'L' the N-S equations are reduced to set of equations ~~the~~ ~~are~~ called B.L equations. These eq^ns are solved by marching technique.

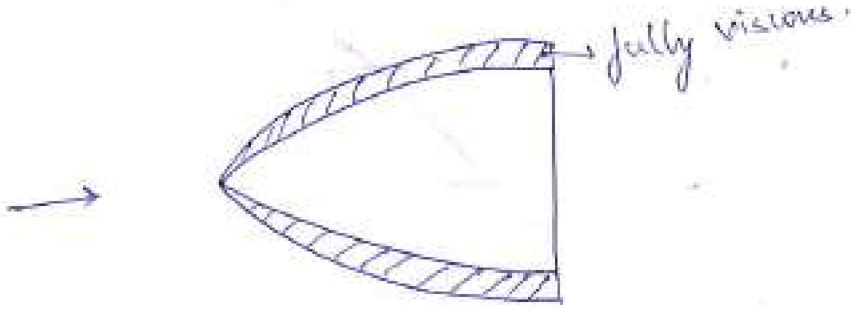
Initial Boundary condition lines:

At nose initial conditions are given along 'ab' and 'ef'.

Boundary condition lines:

'ad' and 'eg' i.e., on surface of body, 'bc' and 'fh' on 'ad' and 'eg' there is a surface of Boundary. At ~~ab~~ ~~on~~ 'ad' and 'eg' represents the no slip Boundary conditions. At 'bc' and 'fh' represents the outer edge at which known inviscid conditions are applied.

Parabolised Viscous flow:

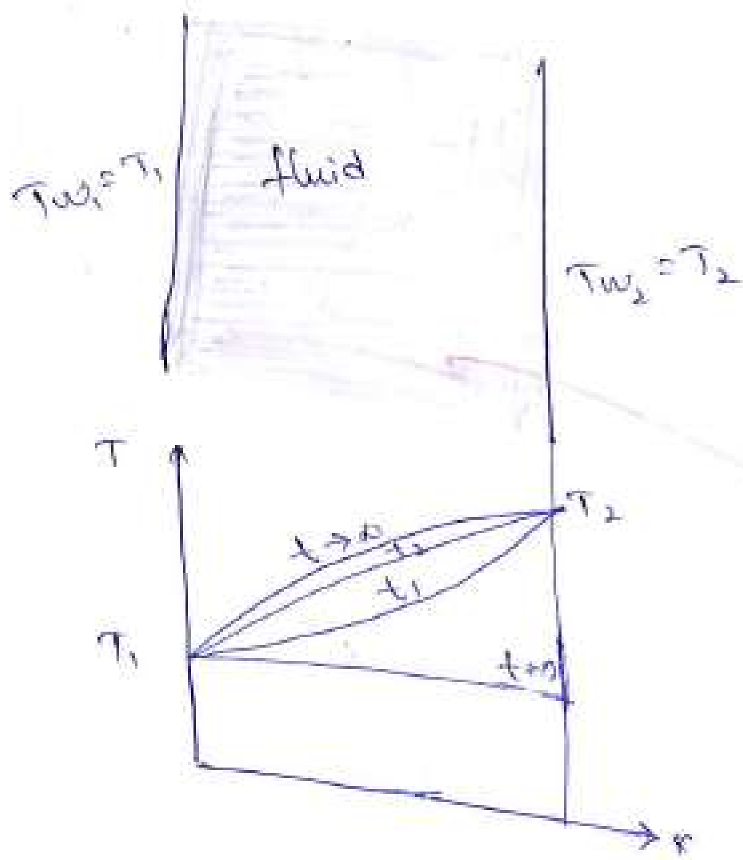


If reynolds number is low and the B.L is thick. for this type of case B.L eqns are not valid for the flow. If all the viscous terms in momentum, energy eqns that involve derivatives are neglected the resulting eqns are called PNS equations or parabolised N-S equations.

Advantages of PNS equations:

- They are simpler i.e., contains less terms than the N-S equations.
- They can be solved by downstream marching process.

Unsteady thermal condition:



In energy eqn put

$\vec{V} = 0$.

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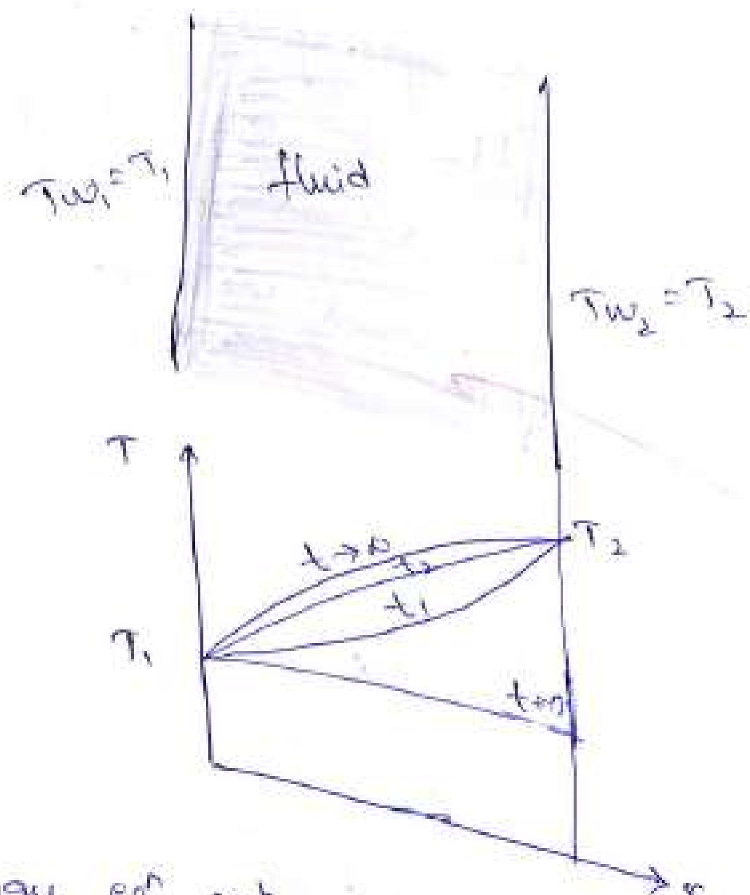


In energy eq^n put

$\vec{V} = 0$.

$\Rightarrow$ we get,

$$\rho \frac{\partial e}{\partial t} = \rho \dot{q} + \frac{\partial}{\partial x} k\left(\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right)$$

Now,

$$\dot{q} = 0, \quad e = C_v T$$

$$\Rightarrow \frac{\partial T}{\partial t} = \frac{1}{\rho C_v}\left[\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right)\right]$$

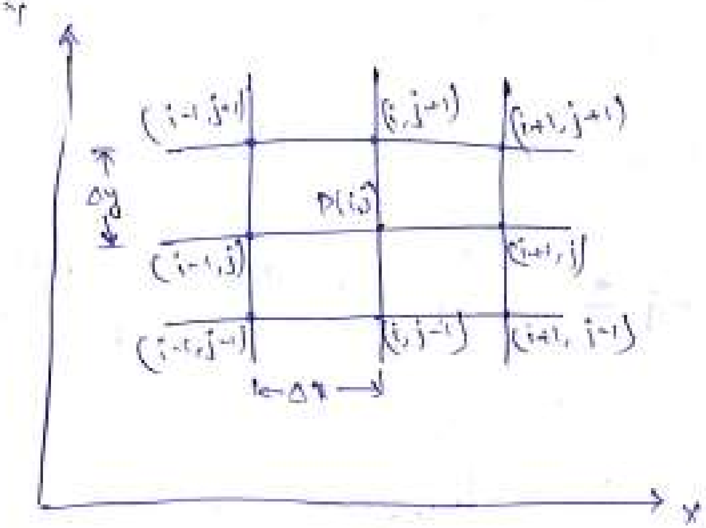
taking 'k' as const.

$$\Rightarrow \boxed{\frac{\partial T}{\partial t} = \alpha \nabla^2 T} \qquad \left(\text{where } \alpha = \frac{k}{\rho C_v}\right)$$

UNIT III

BASIC ASPECTS OF DISCRETIZATION

Basic aspects of Discretization:



- Discretization of P.D.E is called finite difference technique or finite differences.

- Discretization of Integral equations is called finite volume.

Finite difference Method:

If $U_{i,j}$ denotes x-component of velocity at (i,j) then the velocity $U_{i+1,j}$ at pt. $(i+1,j)$ can be expressed in terms of taylor series.

$$\Rightarrow U_{i+1,j} = U_{i,j} + \left(\frac{\partial U}{\partial x}\right)_{i,j}\Delta x + \left(\frac{\partial^2 U}{\partial x^2}\right)\frac{(\Delta x)^2}{2!} + \cdots \quad \text{—} (1)$$

$$\left(\frac{\partial U}{\partial x}\right)_{i,j} = \underbrace{\left[\frac{U_{i+1,j} - U_{i,j}}{\Delta x}\right]}_{\text{finite difference representation}} - \underbrace{\left(\frac{\partial^2 U}{\partial x^2}\right)_{i,j}\frac{\Delta x}{2!} - \cdots}_{\text{truncation error}}$$

$$\Rightarrow \left(\frac{\partial U}{\partial x}\right)_{i,j} = \left[\frac{U_{i+1,j} - U_{i,j}}{\Delta x}\right] + O(\Delta x)$$ (first order forward difference)

Ill^ly for first order backward difference

$$\Rightarrow U_{i-1,j} = U_{i,j} - \left(\frac{\partial U}{\partial x}\right)_{i,j}\Delta x + \left(\frac{\partial^2 U}{\partial x^2}\right)_{i,j}\frac{(\Delta x)^2}{2!} - \cdots \quad \text{—} (2)$$

$$\Rightarrow \left(\frac{\partial U}{\partial x}\right)_{i,j} = \left(\frac{U_{i,j} - U_{i-1,j}}{\Delta x}\right) + \left(\frac{\partial^2 U}{\partial x^2}\right)_{i,j}\frac{\Delta x}{2} \cdots$$

$$\Rightarrow \boxed{\left(\frac{\partial U}{\partial x}\right)_{i,j} = \left[\frac{U_{i,j} - U_{i-1,j}}{\Delta x}\right] + O(\Delta x)}$$ (first order backward difference eq.)

(1) - (2)

$$\Rightarrow \boxed{\left(\frac{\partial U}{\partial x}\right)_{i,j} = \left[\frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}\right] + O(\Delta x)^2}$$ (sec. order central diff equation)

In x-direction

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \left[\frac{u_{i+1,j} - u_{i,j}}{\Delta x}\right] + O(\Delta x) \longleftarrow \text{forward difference.}$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \left[\frac{u_{i,j} - u_{i-1,j}}{\Delta x}\right] + O(\Delta x) \text{ — Backward difference.}$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \left(\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}\right) + O(\Delta x)^2 \text{ — Central difference.}$$

lly In y-direction

$$\left(\frac{\partial u}{\partial y}\right)_{i,j} = \left(\frac{u_{i,j+1} - u_{i,j}}{\Delta y}\right) + O(\Delta y) \text{ — forward difference.}$$

$$\left(\frac{\partial u}{\partial y}\right)_{i,j} = \left(\frac{u_{i,j} - u_{i,j-1}}{\Delta y}\right) + O(\Delta y) \text{ — Backward difference.}$$

$$\left(\frac{\partial u}{\partial y}\right)_{i,j} = \left[\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}\right] + O(\Delta y)^2 \text{ — Central difference.}$$

① + ②

$$\Rightarrow u_{i+1,j} + u_{i-1,j} = 2\left[u_{i,j} + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{(\Delta x)^2}{2!} + \cdots\cdots\right]$$

$$\Rightarrow \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + O(\Delta x)^2$$

sec. order central diff. for second derivative.

Mixed derivatives

diff. ①, ② w.r.t y:

$$\Rightarrow \left(\frac{\partial u}{\partial y}\right)_{i+1,j} = \left(\frac{\partial u}{\partial y}\right)_{i,j} + \left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} \Delta x + \dots \quad — ③$$

$$\left(\frac{\partial u}{\partial y}\right)_{i-1,j} = \left(\frac{\partial u}{\partial y}\right)_{i,j} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} \Delta x + \dots \quad — ④$$

③ − ④

$$\Rightarrow \left(\frac{\partial u}{\partial y}\right)_{i+1,j} - \left(\frac{\partial u}{\partial y}\right)_{i-1,j} = 2\left[\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} \Delta x + \left(\frac{\partial^4 u}{\partial x^3 \partial y}\right)_{i,j} \frac{(\Delta x)^3}{3!} + \dots\right.$$

$$\Rightarrow \left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} = \frac{\left(\frac{\partial u}{\partial y}\right)_{i+1,j} - \left(\frac{\partial u}{\partial y}\right)_{i-1,j}}{2\Delta x} - \frac{\partial^4 u}{\partial x^3 \partial y}\frac{(\Delta x)^2}{6} + \dots$$

$$\Rightarrow \left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} = \frac{\left(\frac{\partial u}{\partial y}\right)_{i+1,j} - \left(\frac{\partial u}{\partial y}\right)_{i-1,j}}{2\Delta x} + O(\Delta x)^2 \quad — ⑤$$

Now,

$$\left(\frac{\partial u}{\partial y}\right)_{i+1,j} = \left(\frac{u_{i+1,j+1} - u_{i+1,j-1}}{2\Delta y}\right) + O(\Delta y)^2$$

$$\left(\frac{\partial u}{\partial y}\right)_{i-1,j} = \left(\frac{u_{i-1,j+1} - u_{i-1,j-1}}{2\Delta y}\right) + O(\Delta y)^2$$

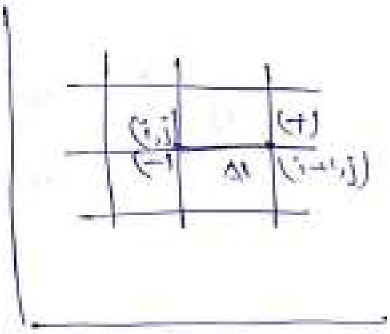
Sub. above both eq in ⑤

$$\Rightarrow \boxed{\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} = \frac{u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}}{4\Delta x \Delta y} + O[(\Delta x)^2, (\Delta y)^2]}$$

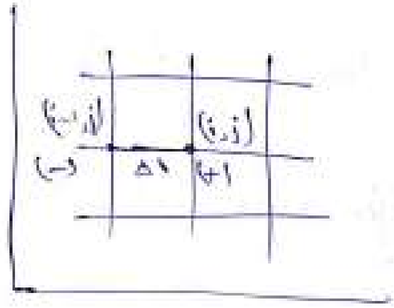
(sec. order central diff for mixed derivative.

x direction

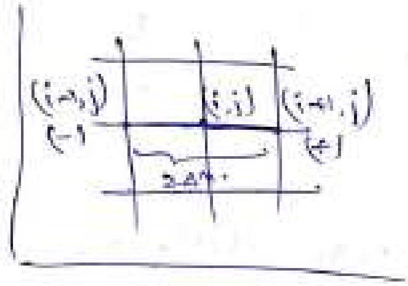
$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} + O(\Delta x)$$



$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{\Delta x} + O(\Delta x)$$



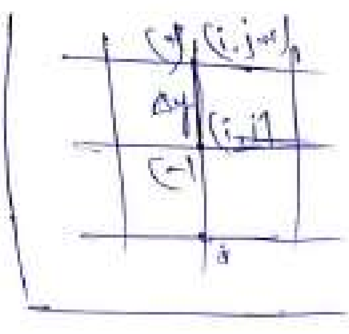
$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \left[\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}\right] + O(\Delta x)^2$$



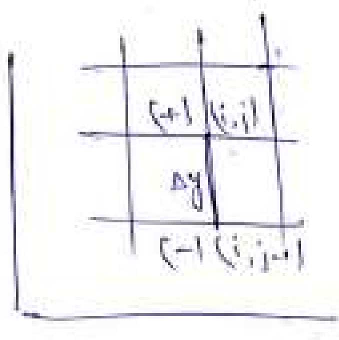
ully

y-direction:

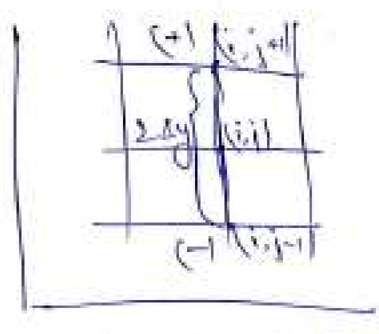
forward



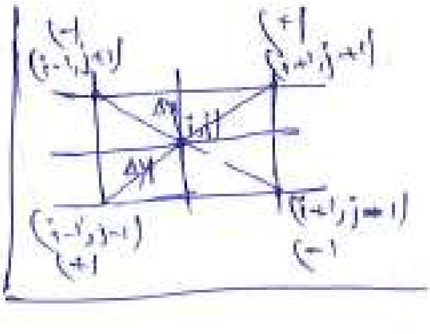
Backward



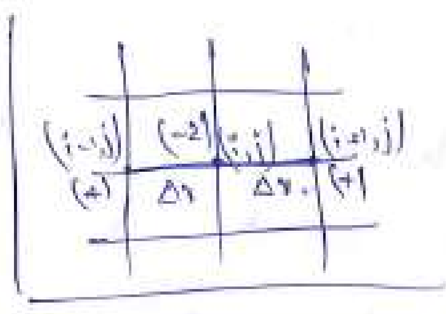
Central



Mixed derivative



Sec. derivative (x-direction)



Eg: $$\frac{\partial^2 U}{\partial y^2} = \frac{-U_{i+2,j} + 16U_{i+1,j} - 30U_{i,j} + 16U_{i-1,j} - U_{i-2,j}}{12(\Delta y)^2} + O(\Delta y)^4$$

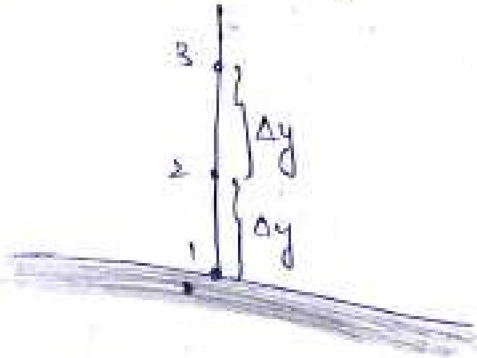
(4th order accuracy) — (1)

Pros and Cons of Higher order accuracy: (pg (130))

- The Higher order accurate difference quotient as in (1) which require more No. of grid points which results in more computer time for each time wise or spatial step is called a con.

- Higher order difference equations may require small No. of grid points in a flow solution to obtain compatible overall accuracy is called a pro.

Types of differencing at the Boundary or polynomial approach or alternate approach



- Consider a portion of the boundary to a flow field & the Y-axis ⊥ to the boundary.
- let grid point 1 be on the boundary and grid points 2, 3 are at a dist. of Δy, 2Δy from the Boundary
- Consider a first order forward difference

$$\left(\frac{\partial U}{\partial y}\right)_1 = \frac{U_2 - U_1}{\Delta y} + O(\Delta y)$$

→ To obtain a sec. order accuracy the central difference fails because it requires another pt. below the boundary say 2'.

→ In early days of CFD this problem can be calculated by assuming $U_2' = U_2$. This is called as a reflection boundary condition.

$U = a + by + cy^2$

at grid pt. 1 : $U_1 = a$.

at grid pt. 2 : $U_2 = a + b(\Delta y) + c(\Delta y)^2$

at grid pt. 3 : $U_3 = a + b(2\Delta y) + c(2\Delta y)^2$

solving U_2, U_3 equations

$$\left.\begin{aligned} c(\Delta y)^2 &= U_2 - a - b(\Delta y) \\ c(\Delta y)^2 &= \frac{U_3 - a - b(2\Delta y)}{4} \end{aligned}\right] \text{equating.}$$

$$\Rightarrow \quad U_2 - a - b(\Delta y) = \frac{U_3 - a - b(2\Delta y)}{4}$$

$$4U_2 - U_3 - 3a - b(2\Delta y) = 0.$$

$$b(2\Delta y) = -U_3 + 4U_2 - 3a$$

$$b = \frac{-U_3 + 4U_2 - 3a}{2\Delta y}$$

$$\Rightarrow \boxed{b = \frac{-3U_1 + 4U_2 - U_3}{2\Delta y}}$$

or $U = a + by + cy^2$

diff. w.r.t y:

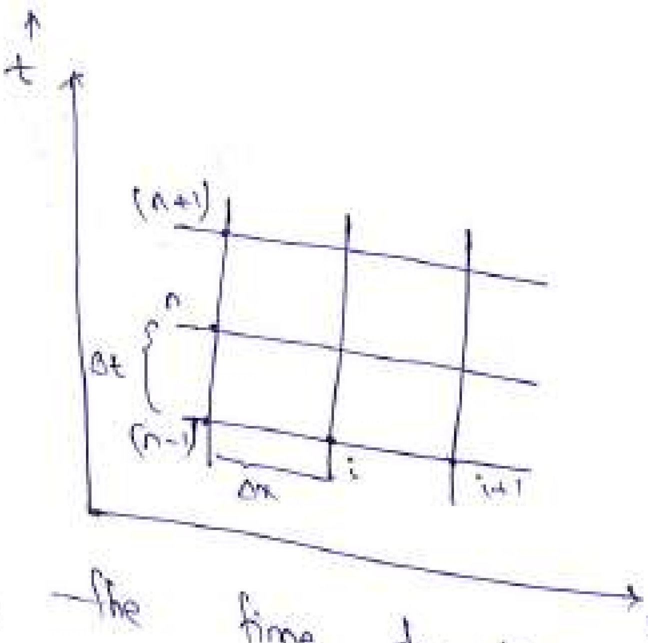
$$\Rightarrow \frac{\partial U}{\partial y} = b + 2cy.$$

$$\Rightarrow \left(\frac{\partial U}{\partial y}\right)_1 = b.$$

$$\Rightarrow \boxed{\left(\frac{\partial U}{\partial y}\right) = \frac{-3U_1 + 4U_2 - U_3}{2\Delta y} + O(\Delta y)^2}$$ — One side finite difference expression

Difference Equations:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$



Let us replace the time derivative with a forward difference

$$\left(\frac{\partial T}{\partial t}\right)_i^n = \frac{T_i^{n+1} - T_i^n}{\Delta t} - \left(\frac{\partial^2 T}{\partial t^2}\right)^n \frac{\Delta t}{2} + \cdots\cdots$$

$$\left(\frac{\partial^2 T}{\partial x^2}\right) = \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2} - \left(\frac{\partial^2 T}{\partial x^2}\right)_i^n \frac{(\Delta x)^2}{6} + \cdots\cdots$$

$$\Rightarrow \frac{T_i^{n+1} - T_i^n}{\Delta t} - \left(\frac{\partial^2 T}{\partial t^2}\right)^n \frac{\Delta t}{2} + \cdots\cdots = \alpha\left(\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2} - \left(\frac{\partial^2 T}{\partial x^2}\right)_i^n \frac{(\Delta x)^2}{6}\right.$$

$$\Rightarrow \frac{T_i^{n+1} - T_i^n}{\Delta t} - \alpha\left[\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2}\right] - \left(\frac{\partial^2 T}{\partial t^2}\right)_i^n \frac{\Delta t}{2} + \alpha\left[\left(\frac{\partial^4 T}{\partial x^4}\right)_i^n \frac{(\Delta x)^2}{6}\right] = 0$$

$$\Rightarrow \boxed{\frac{T_i^{n+1} - T_i^n}{\Delta t} - \alpha\left[\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2}\right] - O\left(\Delta t, (\Delta x)^2\right) = 0.}$$

(or)

$$\boxed{\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha\left[\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2}\right] + O(\Delta t, (\Delta x)^2)}$$

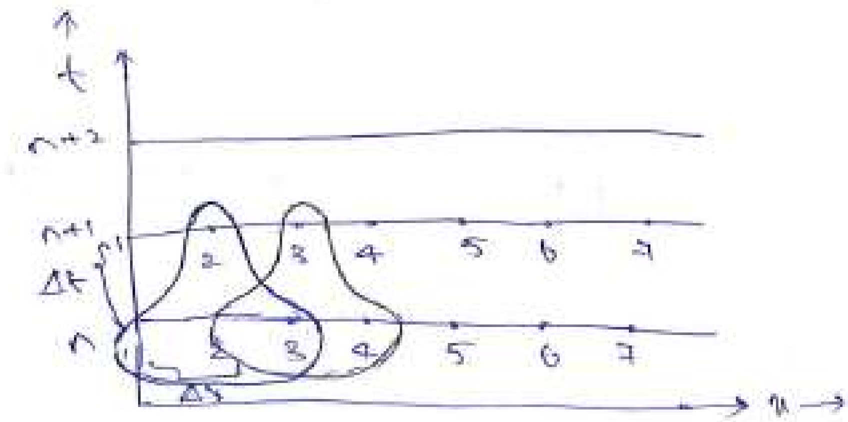
this is called difference eqn.

Explicit approach:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$\Rightarrow \frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha\left(\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2}\right)$$

$$\Rightarrow T_i^{n+1} = T_i^n + \frac{\alpha \Delta t}{(\Delta x)^2}\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right)$$



Assume that 't' is known at all grid points at time level 'n'.

at grid point $i=2$.

$$\boxed{T_2^{n+1} = T_2^n + \frac{\alpha \Delta t}{(\Delta x)^2}\left(T_3^n - 2T_2^n + T_1^n\right)}$$

at grid pt. $i=3$

$$\boxed{T_3^{n+1} = T_3^n + \frac{\alpha \Delta t}{(\Delta x)^2}\left(T_4^n - 2T_3^n + T_2^n\right)}$$

"In an explicit approach each difference equation contains one unknown and therefore can be solved explicitly for the unknown in a straight forward manner."

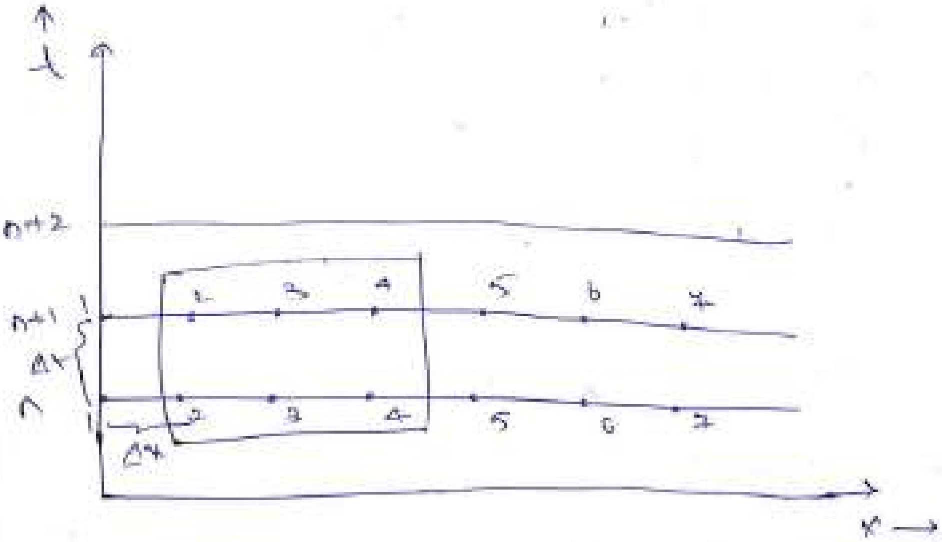
Implicit approach:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \frac{\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right)}{(\Delta x)^2}$$

let us replace the heat conduction eq^n R.H.S terms by average properties b/w time levels 'n' and 'n+1'

$$\Rightarrow \frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \underbrace{\left(\frac{\frac{1}{2}\left(T_{i+1}^{n+1} + T_{i+1}^n\right) + \frac{1}{2}\left(-2T_i^{n+1} - 2T_i^n\right) + \frac{1}{2}\left(T_{i-1}^{n+1} + T_{i-1}^n\right)}{(\Delta x)^2}\right)}_{\text{crank nicolson formula}}$$



$$\Rightarrow \boxed{\underbrace{\frac{\alpha \Delta t}{2(\Delta x)^2}}_{A} T_{i-1}^{n+1} - \underbrace{\left[1 + \frac{\alpha \Delta t}{(\Delta x)^2}\right]}_{B} T_i^{n+1} + \frac{\alpha \Delta t}{2(\Delta x)^2} T_{i+1}^{n+1} = \underbrace{-T_i^n - \frac{\alpha \Delta t}{2(\Delta x)^2}\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right)}_{k_i}}$$

$$\Rightarrow \boxed{A \cdot T_{i-1}^{n+1} - B T_i^{n+1} + A T_{i+1}^{n+1} = k_i}$$

$i=2$: $A T_1^{n+1} - B T_2^{n+1} + A T_3^{n+1} = k_2 \longrightarrow -BT_2 + AT_3 = \underbrace{k_2 - AT_1}_{k_2'}$ (we can omit writing of (n+1))

$i=3$: $A T_2^{n+1} - B T_3^{n+1} + A T_4^{n+1} = k_3$

$i=4$: $A T_3^{n+1} - B T_4^{n+1} + A T_5^{n+1} = k_4$

$i=5$: $A T_4^{n+1} - B T_5^{n+1} + A T_6^{n+1} = k_5$

$i=6$: $A T_5^{n+1} - B T_6^{n+1} + A T_7^{n+1} = k_6 \Rightarrow AT_5 - BT_6 = \underbrace{k_6 - AT_7}_{k_6'}$

keeping (n+1) apart because of same time level

from the known B.C at grid points 1 and 7. The term involving T_1 which is known can be transferred to the R.H.S and denote $k_2 - AT_1$ by $\underline{k_2'}$ (known number)

Ully at $i=6$, T_7 is known thus taking to R.H.S and writing as k_6' (known value)

$$\Longrightarrow \begin{pmatrix} -B & A & 0 & 0 & 0 \\ A & -B & A & 0 & 0 \\ 0 & A & -B & A & 0 \\ 0 & 0 & A & -B & A \\ 0 & 0 & 0 & A & -B \end{pmatrix} \begin{pmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{pmatrix} = \begin{pmatrix} k_2' \\ k_3 \\ k_4 \\ k_5 \\ k_6' \end{pmatrix}$$

"An Implicit approach is one, way where the unknowns must be obtained by means of a simultaneous solution of the difference eqn applied at all grid points arranged at a given time level."

$$\frac{\partial T}{\partial t} = \alpha(T)\frac{\partial^2 T}{\partial x^2}$$

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha(T_i^n)\left(\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2}\right)$$

Exp

$$T_i^{n+1} = \alpha(T_i^n)\Delta t\left[\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2}\right] + T_i^n$$

It has one unknown in the next time level that means non linearity has no effect on the explicit approach.

$$\frac{T_i^{n+1}-T_i^n}{\Delta t}=\alpha\left[\frac{\frac{1}{2}\left(T_{i+1}^{n+1}+T_{i+1}^{n}\right)+\frac{1}{2}\left[2T_i^{n+1}-2T_i^n\right]+\frac{1}{2}\left(T_{i-1}^{n+1}+T_{i-1}^{n}\right)}{(\Delta x)^2}\right]$$

Linear P.D.E

~~Lin P.D.E~~

Now for Non linear P.D.E, we get

$$\frac{T_i^{n+1}-T_i^n}{\Delta t}=\alpha\,\frac{1}{2}\left(T_i^{n+1}+T_i^n\right)\left(\frac{\frac{1}{2}\left(T_{i+1}^{n+1}+T_{i+1}^{n}\right)+\frac{1}{2}\left(2T_i^{n+1}-2T_i^n\right)+\frac{1}{2}\left(T_{i-1}^{n+1}+T_{i-1}^{n}\right)}{(\Delta x)^2}\right)$$

$$\Rightarrow T_i^{n+1}=\left[\frac{\alpha}{2}\left(T_i^{n+1}+T_i^n\right)\left(\frac{\frac{1}{2}\left(T_{i+1}^{n+1}+T_{i+1}^{n}\right)+\frac{1}{2}\left(2T_i^{n+1}-2T_i^n\right)+\frac{1}{2}\left(T_{i-1}^{n+1}+T_{i-1}^{n}\right)}{(\Delta x)^2}\right)\right]\cdot\Delta t+T_i^n$$

Clearly the new difference eqⁿ involves

$\left[\alpha(T_i^{n+1})\right]T_i^{n+1}$, $\left(\alpha(T_i^{n+1})\right)T_{i+1}^{n+1}$, $\left[\alpha\left(T_i^{n+1}\right)\right]T_{i-1}^{n+1}$.

> *** Explicit and Implicit adv. and disadv.

<u>Errors and Stability Analysis of Stability</u>:

$$\frac{\partial T}{\partial t}=\alpha\frac{\partial^2 T}{\partial x^2}$$

$$\frac{T_i^{n+1}-T_i^n}{\Delta t}=\alpha\frac{\left(T_{i+1}^n-2T_i^n+T_{i-1}^n\right)}{(\Delta x)^2}$$

<u>Discretization Error</u>:

The difference b/w the analytical sol. of partial D.E and the solution of the final D.E. So the error is simply the truncation error for the difference equation.

Roundoff error:

The numerical error introduced after a repetetive number of calculations in which the computer is constantly rounding the numbers to some significant figure.

let A = Analytical sol. of partial D.E.

D = Exact sol. of a difference equation.

N = Numerical solution with finite accuracy.

⇒ Discretization error = $A - D$

Roundoff error $(\epsilon) = N - D$.

$$\boxed{N = D + \epsilon}$$

Sub. in difference equation.

$$\Rightarrow \boxed{\frac{(D+\epsilon)_i^{n+1} - (D+\epsilon)_i^n}{\Delta t} = \alpha \left[\frac{(D+\epsilon)_{i+1}^n - 2(D+\epsilon)_i^n + (D+\epsilon)_{i-1}^n}{(\Delta x)^2} \right]}$$

$$\Rightarrow \boxed{\frac{D_i^{n+1} + \epsilon_i^{n+1} - D_i^n - \epsilon_i^n}{\alpha \Delta t} = \frac{D_{i+1}^n + \epsilon_{i+1}^n - 2D_i^n - 2\epsilon_i^n + D_{i-1}^n + \epsilon_{i-1}^n}{(\Delta x)^2}}$$

'D' is the exact solution of difference eqn, hence it exactly satisfies the difference equation.

$$\Rightarrow \frac{D_i^{n+1} - D_i^n}{\Delta t} = \alpha \left(\frac{D_{i+1}^n - 2D_i^n + D_{i-1}^n}{(\Delta x)^2} \right) \quad \text{— (b)}$$

(a) - (b)

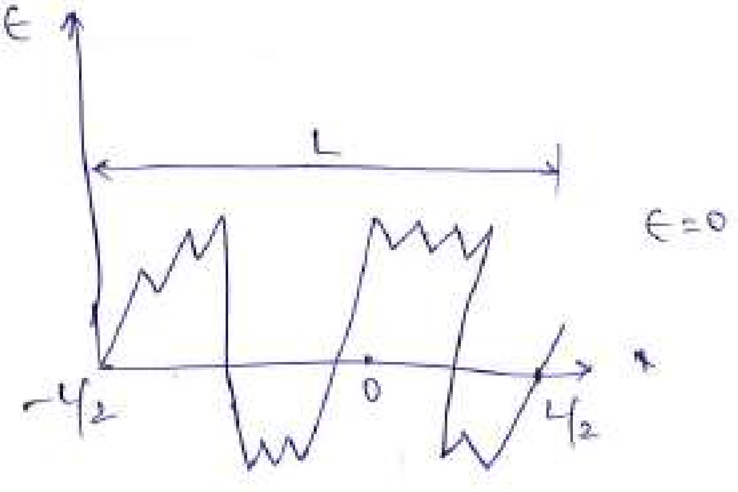
$$\Rightarrow \frac{\epsilon_i^{n+1} - \epsilon_i^n}{\Delta t} = \alpha \left(\frac{\epsilon_{i+1}^n - 2\epsilon_i^n + \epsilon_{i-1}^n}{(\Delta x)^2} \right) \quad \text{— (c)}$$

The error 'ϵ' also satisfies the difference equation

~~Stable~~ Stable:
The solution will be stable if the errors shrink or at best they stay the same as a solution progress from step 'n' to '$n+1$'.

Unstable: If the error grow large during the progression of the solution from step 'n' to '$n+1$' then the solution is unstable.

The condition for the solution to be stable is $\left| \frac{\epsilon_i^{n+1}}{\epsilon_i^n} \right| \leq 1$.



$\Rightarrow$ For conveniance let us trace the origin at mid point of the domain. left boundary is at $-L/2$ and right boundary at $L/2$.

- $\epsilon = 0$ at $x = -L/2, L/2$ because they are specified boundary values at both ends of the domain and hence no error is introduced.

- At any given time the random variation of ϵ with 'x' can be expressed by fourier series

$$\epsilon(x) = \sum_m A_m e^{ik_m x}$$

where $e^{ik_m x} = \cos \underbrace{k_m}_{\text{wave number}} x + i \sin k_m x.$

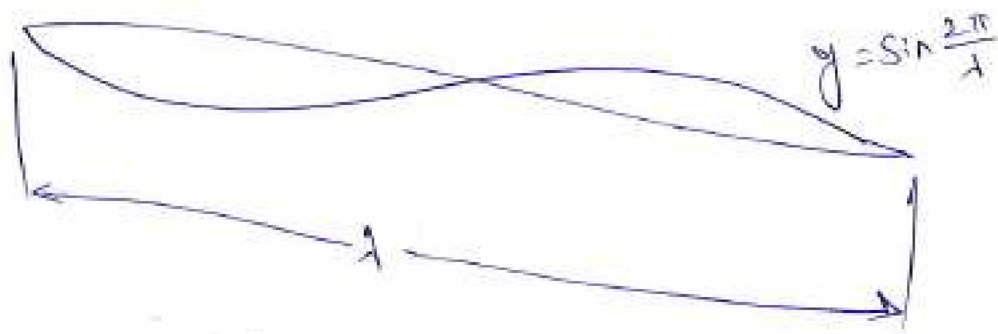
Meaning of wave number:

$y = \sin \frac{2\pi}{\lambda} \times x.$

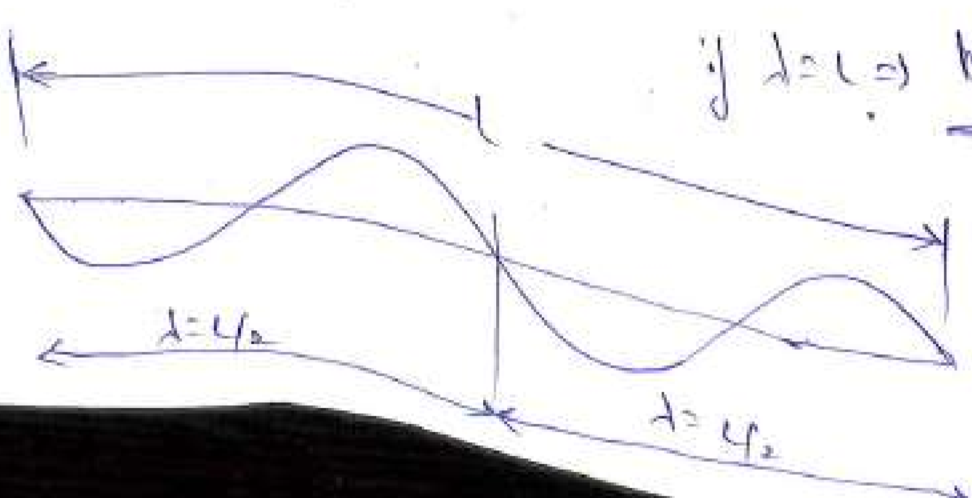
$k_m \rightarrow$ No. waves fitted in an interval.

where $k_m = \frac{2\pi}{\lambda}$

subscript 'm' denotes the No. of waves that are fitted inside a given interval.



if $\lambda = L \Rightarrow k_m = \frac{2\pi}{L}$



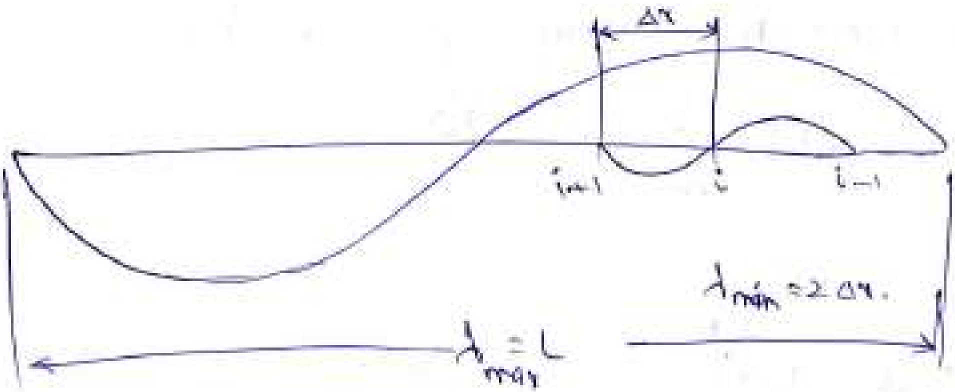
Two sine waves are fitted within an interval 'L'

$$k_m = \frac{2\pi}{\lambda} = \frac{2\pi}{(L/2)} = \left(\frac{2\pi}{L}\right)2.$$

If three waves are fitted within an interval 'L'

$$k_m = \frac{2\pi}{\lambda} = \left(\frac{2\pi}{L}\right)3$$

∴ The wave no. for the various sine waves of diff. wave lengths is $k_m = \left(\frac{2\pi}{L}\right)m.$



For sake of numerical calculations let us consider the largest wavelength $\lambda_{max} = L$, the smallest possible wavelength is $\lambda_{min} = 2\Delta x$.

$$\lambda_{min} = 2\Delta x.$$

$$\Delta x = L/N$$

$$\Rightarrow \lambda_{min} = \frac{2L}{N}$$

$$k_n = \frac{2\pi}{\lambda} = \left(\frac{2\pi}{L}\right)\frac{N}{2}.$$

$$\cancel{E(x,t) = \sum_{m=1}^{N/2} A_m(t) \cdot e^{ik_m x}}$$

$$E(x) = \sum_{m=1}^{N/2} A_m \cdot e^{ik_m x}$$

This gives a spatial variation at a given time level 'n' but for numerical stability we are interested in variation of roundoff error with time.

~~Assume~~ A_m is a function of time.

$$E(x,t) = \sum_{m=1}^{N/2} A_m(t) e^{ik_m x}$$

~~Assume~~ an exp. variation with time i.e., errors tend to grow or diminish exponentially with time.

$$E(x,t) = \sum_{m=1}^{N/2} e^{at} \cdot e^{ik_m x}$$

sub. in (C)

$$\Rightarrow \cancel{(e^{at} \cdot e^{ik_m x})^{n+1}_i - e^{at} \cdot e^{ik_m x}}$$

$$\frac{e^{a(t+\Delta t)} \cdot e^{ik_m x} - e^{at} \cdot e^{ik_m x}}{\alpha \cdot \Delta t} = \frac{e^{at} \cdot e^{ik_m (x+\Delta x)} - 2e^{at} \cdot e^{ik_m x} + e^{at} \cdot e^{ik_m (x-\Delta x)}}{(\Delta x)^2}$$

$\div$ by $e^{at} \cdot e^{ik_m x}$

$$\Rightarrow \frac{e^{a\Delta t} - 1}{\alpha \Delta t} = \frac{e^{ik_m \Delta x} - 2 + e^{-ik_m \Delta x}}{(\Delta x)^2}$$

$$\Rightarrow e^{a\Delta t} = \frac{\alpha \Delta t}{(\Delta x)^2} \left[e^{ik_m \Delta x} - 2 + e^{-ik_m \Delta x} \right] + 1.$$

$$= 1 + \frac{2\alpha\Delta t}{(\Delta x)^2}\left[\frac{e^{ik_m\Delta x} + e^{-ik_m\Delta x}}{2} - 1\right]$$

$$= 1 + \frac{2\alpha\Delta t}{(\Delta x)^2}\left[\cos(k_m\Delta x) - 1\right]$$

mul. and div. by '2':

$$\Rightarrow e^{a\Delta t} = 1 + \frac{2(2\alpha\Delta t)}{(\Delta x)^2}\left(\frac{\cos(k_m\Delta x) - 1}{2}\right]$$

$$\frac{\epsilon_i^{n+1}}{\epsilon_i^n} = e^{a\Delta t} = \underbrace{1 - \frac{4\alpha\Delta t}{(\Delta x)^2}\sin^2\left(\frac{k_m\Delta x}{2}\right)}_{\text{amplification factor (G)}} \le 1.$$

$$\Rightarrow \frac{\epsilon_i^{n+1}}{\epsilon_i^n} = e^{a\Delta t} = G \le 1.$$

~~Case 1:~~

$$1 - \frac{4\alpha\Delta t \sin^2 k_m\Delta'}{(\Delta x)^2}$$

~~Cast~~ we have two possible situations.

Case (i):

$$1 - \frac{4\alpha\Delta t}{(\Delta x)^2}\sin^2\left(\frac{k_m\Delta x}{2}\right) \le 1.$$

$$\Rightarrow \frac{4\alpha\Delta t}{(\Delta x)^2}\cancel{\sin^2\left(\frac{k_m\Delta x}{2}\right)} \ge 0.$$

Case (ii)

$$1 - \frac{4\alpha\Delta t}{(\Delta x)^2}\sin^2\left(\frac{k_m\Delta x}{2}\right) \ge -1.$$

$$\Rightarrow \frac{\alpha\Delta t}{(\Delta x)^2}\cancel{\sin^2\left(\frac{k_m\Delta x}{2}\right)} \le 1/2.$$

This type of analysis is called "Von-Neumann ~~Network and~~ Stability analysis"

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0.$$

$$\Rightarrow \frac{u_i^{n+1} - u_i^n}{\Delta t} = -c\,\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}$$ — this type of eqns are called "Euler explicit form".

to be replaced

[***** Imp. Question:

1. Hyperbolic eqn and PNS equations.
2. What is discretization, and graphical representation of finite difference module.
3. Polynomial approach.
4. Explicit and Implicit approaches.
5. Von-Neumann stability analysis
6. Physical principles used for governing equations.
7. Manipulations from conservative to non-conservative form.]

$$\Rightarrow u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - \frac{c\Delta t}{\Delta x}\left(\frac{u_{i+1}^n - u_{i-1}^n}{2}\right)$$

(after replacing with avg) [called as Lax Method

after carrying analysis, we get

$$e^{at} = \cos(k_m \Delta x) - i\,C\sin(k_m \Delta x)$$

Courant Number.

$$\boxed{C = \frac{c\Delta t}{\Delta x} \le 1.}$$ — CFL condition.

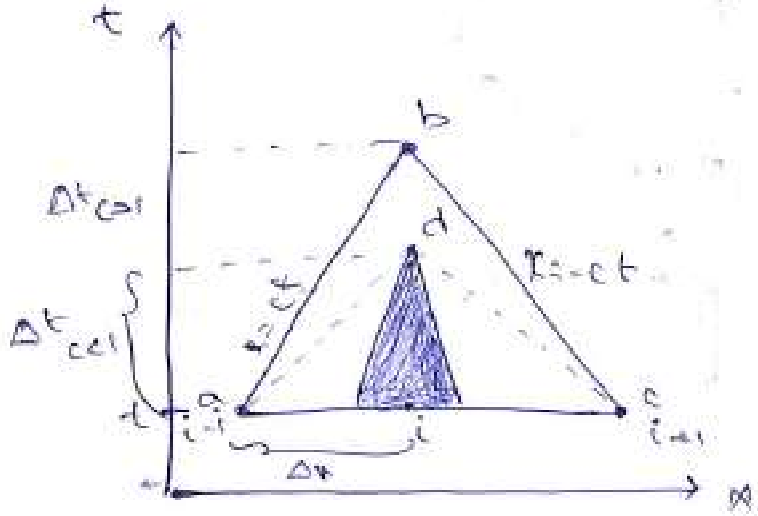
$$\Rightarrow \Delta t \le \frac{\Delta x}{c}$$ (stable condition)

The eqn says that $\Delta t \le \frac{\Delta x}{c}$ for the numerical solution of the eqn to be stable.

(CFL - Courant Friedrich's Lewy condition.)

Stability condition for second order wave eqn:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$



The characteristic lines are given by $x=ct$ which is right running and $x=-ct$, left running. Denoting right running characteristic at '$i-1$' and left running at '$i+1$'. $\Delta t_{c=1}$ denotes the value of 'Δt' given by CFL condition where $c=1$.

$$\boxed{\Delta t_{c=1} = \frac{\Delta x}{c}}$$

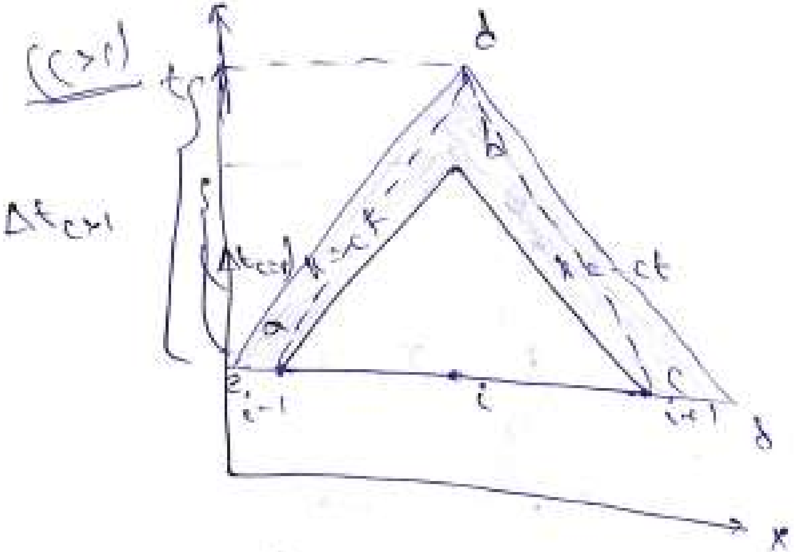
Case (i) More distance $\Delta t_{c=1}$

Case (ii) $\Delta t_{c<1}$.

$$\Rightarrow \boxed{\Delta t_{c<1} < \Delta t_{c=1}.}$$

The ~~pt.~~ 'd' prop. at pt. 'd' are calc. numerically from the diff. eqn using the info. at grid points '$i-1$' and '$i+1$'. So the numerical domain for pt. 'd' is 'adc'.

Analytical domain for pt. 'd' is shaded triangle. Therefore ~~we can give~~ the physical meaning of the CFL condition for al stability the numerical domain must include all the analytical domain.



UNIT IV

CFD TECHNIQUES

Lax - Wendroff technique:

Eulers eqn in Non-conservative form:

(2D)

Continuity: $\frac{D\rho}{Dt} + \rho\nabla \cdot V = 0.$

Energy ~~conservation~~: $\rho \frac{D}{Dt}\left(e + \frac{V^2}{2}\right) = \rho\dot{q} - \frac{\partial(up)}{\partial x} - \frac{\partial(vp)}{\partial y} - \cancel{\frac{\partial(wp)}{\partial z}} + \rho f \cdot V.$

~~Both~~ Momentum: $\rho\frac{Du}{Dt} = \frac{-\partial p}{\partial x} + \rho f_x$

$$\rho\frac{Dv}{Dt} = \frac{-\partial p}{\partial y} + \rho f_y$$

$$\cancel{\rho\frac{Dw}{Dt} = \frac{-\partial p}{\partial z}} + \cancel{\rho f_z}$$

→ Continuity: $\frac{\partial \rho}{\partial t} = -\left[\rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial y} + v\frac{\partial \rho}{\partial y}\right]$ — (C)

Momentum: $\rho\left[\frac{\partial u}{\partial t} + V \cdot \nabla u\right] = -\frac{\partial p}{\partial x} + \rho . f_x$

$$\rightarrow \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \rho . f_x$$

$$\text{or}\quad \frac{\partial u}{\partial t} = -\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\cdot\frac{\partial p}{\partial x}\right] \quad (2) \qquad [\because \text{here } f_x, f_y = 0],$$

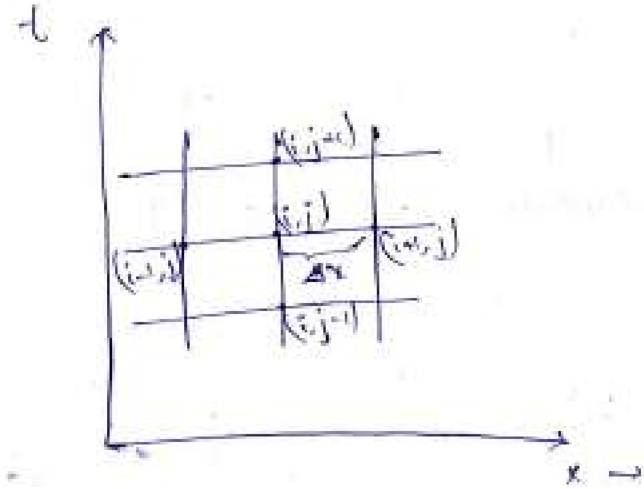
Ill^ly

$$\frac{\partial v}{\partial t} = -\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y}\right] \quad (3)$$

Energy:

(here $f \cdot v = 0$, $\rho\dot{q} = 0$ $\because$ neg. volume forces, body forces, volumetric heating)

$$\frac{\partial e}{\partial t} = -\left[u\frac{\partial e}{\partial x} + v\frac{\partial e}{\partial y} + \frac{p}{\rho}\frac{\partial u}{\partial x} + \frac{p}{\rho}\frac{\partial v}{\partial y}\right] \quad (4)$$



Choose any independent flow variable, let us choose density (ρ)

let $\rho_{i,j}^{t}$ denotes the density at grid pt. i,j at time 't',

then density at same grid pt. at time $t+\Delta t$ is

$$\rho_{i,j}^{t+\Delta t} = \rho_{i,j}^{t} + \left(\frac{\partial \rho}{\partial t}\right)_{i,j}^{t}\Delta t + \left(\frac{\partial^2 \rho}{\partial t^2}\right)_{i,j}^{t}\frac{(\Delta t)^2}{2!} + \dots \quad (5)$$

(given by taylor series)

> In eq^n (5), we assume that flow field at time 't' is known.

> If we find numbers for $\left(\frac{\partial \rho}{\partial t}\right)_{i,j}^{t}$, $\left(\frac{\partial^2 \rho}{\partial t^2}\right)_{i,j}^{t}$ then the value of density at next stepping time $\rho_{i,j}^{t+\Delta t}$ can be calculated explicitly from eq^n (5).

lly

$$u_{i,j}^{t+\Delta t} = u_{i,j}^{t} + \left(\frac{\partial u}{\partial t}\right)_{i,j}^{t} \Delta t + \left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j}^{t} \frac{(\Delta t)^2}{2!} + \dots \quad (6)$$

$$v_{i,j}^{t+\Delta t} = v_{i,j}^{t} + \left(\frac{\partial v}{\partial t}\right)_{i,j}^{t} \Delta t + \left(\frac{\partial^2 v}{\partial t^2}\right)_{i,j}^{t} \frac{(\Delta t)^2}{2!} + \dots \quad (7)$$

$$e_{i,j}^{t+\Delta t} = e_{i,j}^{t} + \left(\frac{\partial e}{\partial t}\right)_{i,j}^{t} \Delta t + \left(\frac{\partial^2 e}{\partial t^2}\right)_{i,j}^{t} \frac{(\Delta t)^2}{2!} + \dots \quad (8)$$

let us concentrate on calculation of density at time $t+\Delta t$ given by eq^n (5).

A number for $\left(\frac{\partial \rho}{\partial t}\right)_{i,j}^{t}$ in eq^n (5) obtained from the cont. eq^n where the spatial derivatives are given by sec. order central differences.

$$\Rightarrow \left(\frac{\partial \rho}{\partial t}\right)_{i,j}^{t} = -\left(\rho_{i,j}^{t} \frac{u_{i+1,j}^{t} - u_{i-1,j}^{t}}{2\Delta x} + u_{i,j}^{t} \frac{\rho_{i+1,j}^{t} - \rho_{i-1,j}^{t}}{2\Delta x} + \cancel{\rho_{i,j}^{t}} \cancel{\rho_{i+1,j}^{t}} \rho_{i,j}^{t} \frac{v_{i,j+1}^{t} - v_{i,j-1}^{t}}{2\Delta y} + v_{i,j}^{t} \frac{\rho_{i,j+1}^{t} - \rho_{i,j-1}^{t}}{2\Delta y} \right) \quad (9)$$

diff (1) w.r.t 't'

$$\Rightarrow \left(\frac{\partial^2 \rho}{\partial t^2}\right)_{i,j}^{t} = -\left[\rho \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial u}{\partial x}\cdot\frac{\partial \rho}{\partial t} + u\frac{\partial^2 \rho}{\partial x \partial t} + \frac{\partial \rho}{\partial x}\frac{\partial u}{\partial t} + \rho\frac{\partial^2 v}{\partial y \partial t} + \frac{\partial v}{\partial y}\frac{\partial \rho}{\partial t} + v\frac{\partial^2 \rho}{\partial y \partial t} + \frac{\partial \rho}{\partial y}\frac{\partial v}{\partial t}\right] \quad (10)$$

~~diff (2) w.r.t x, (3) w.r.t y and (4) w.r.t y and~~

~~substituting the~~

Now,

$$\frac{\partial^2 u}{\partial x \partial t} = -\left[u \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial^2 p}{\partial x^2} - \frac{1}{\rho^2}\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial x}\right] \quad (11)$$

$$\left(\frac{\partial^2 u}{\partial x \partial t}\right)^t_{i,j} = -\left[u^t_{i,j} \cdot \frac{u^t_{i+1,j} - 2u^t_{i,j} + u^t_{i-1,j}}{(\Delta x)^2} + \left(\frac{u^t_{i+1,j} - u^t_{i-1,j}}{2\Delta x}\right)^2 + v^t_{i,j} \cdot \frac{u^t_{i+1,j+1} + u^t_{i-1,j-1} - u^t_{i-1,j+1} - u^t_{i+1,j-1}}{4(\Delta x)(\Delta y)} + \cancel{\frac{u^t_{i,j+1} - u^t_{i,j-1}}{2\Delta y}} \right.$$

$$\left. \frac{u^t_{i,j+1} - u^t_{i,j-1}}{2\Delta y} \cdot \frac{v^t_{i+1,j} - v^t_{i-1,j}}{2\Delta x} + \frac{1}{\rho^t_{i,j}} \frac{p^t_{i+1,j} - 2p^t_{i,j} + p^t_{i-1,j}}{(\Delta x)^2} - \frac{1}{(\rho^t_{i,j})^2} \frac{p^t_{i+1,j} - p^t_{i-1,j}}{2\Delta x} \cdot \frac{\rho^t_{i+1,j} - \rho^t_{i-1,j}}{2\Delta x}\right] \quad (12)$$

from eq^n (12) all terms on R.H.S are known from the known flow field at time 't', this provides a number for the L.H.S i.e., a number for $\left(\frac{\partial^2 u}{\partial x \partial t}\right)^t_{i,j}$.

Similarly calc. for $\frac{\partial^2 \rho}{\partial x \partial t}$ is found by diff. (1) w.r.t 'x' and replacing all derivatives on right side with sec. order central difference is similar to (12)

Similarly cal. for $\frac{\partial^2 v}{\partial y \partial t}$ and $\frac{\partial^2 \rho}{\partial y \partial t}$ in (10).

Remaining derivatives on R.H.S of (10) are first ~~order~~ spatial derivatives namely $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$, $\frac{\partial \rho}{\partial x}$ and $\frac{\partial p}{\partial y}$ are replaced by

Second order central differences

$$\left(\frac{\partial u}{\partial x}\right)^t_{i,j} = \frac{u^t_{i+1,j} - u^t_{i-1,j}}{2\Delta x} \quad — (13)$$

first time derivatives $\frac{\partial \rho}{\partial t}$, $\frac{\partial u}{\partial t}$, $\frac{\partial v}{\partial t}$ has already been obtain from eqn (1) to (4).

All eqn's subst. in (5) where $\frac{\partial t}{\partial t}$ was obtained earlier from eqn (9)

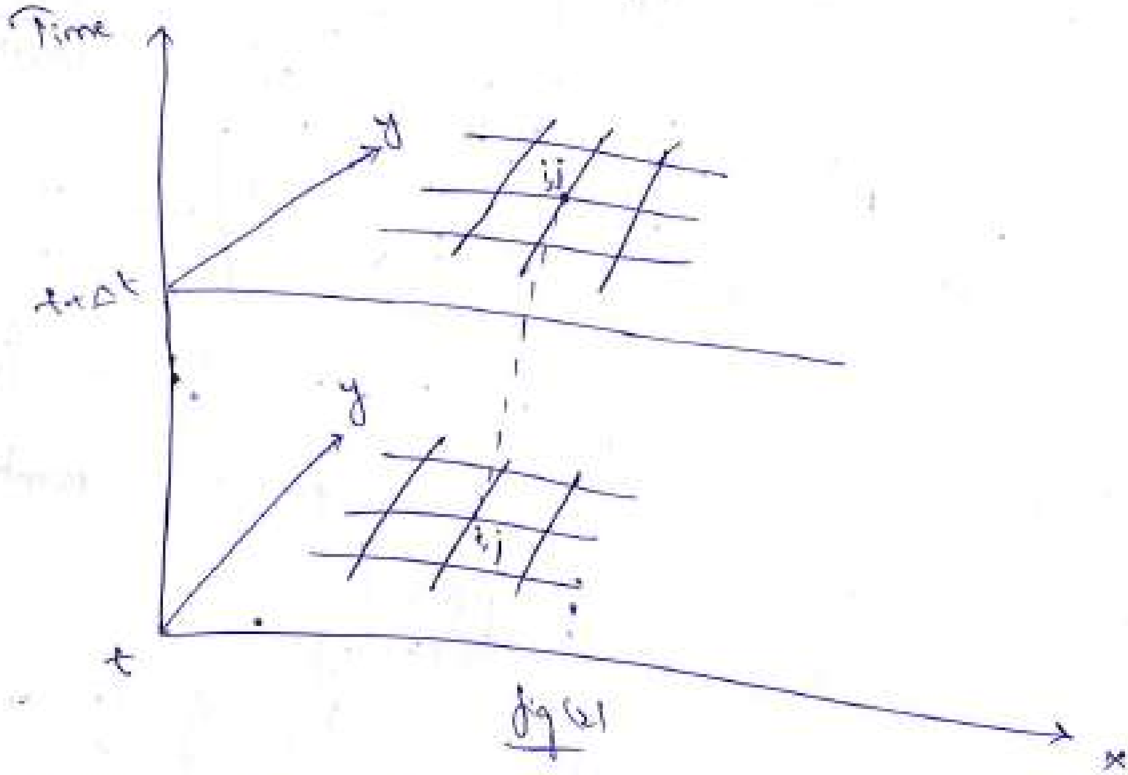


fig (2)

Now, we have a known values at time 't' for all three terms on right hand side of eqn (5) this allows a calc. of density at time 't+Δt'.

Clearly lax-Wendroff method allows us to obtain explicitly the flow field variables at grid pt. i,j at time 't+Δt' from the known flow field variables at grid pt. i, j at time t.

The idea is st. forward but the algebra is lengthy.

Maccormack's technique:

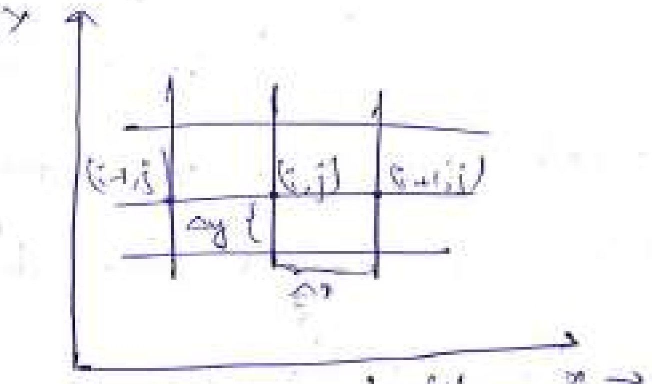


fig (1)

Maccormack's technique is variant of Lax-wendroff approxin but simpler in its application. Maccormack's method is very student friendly and it is easy to understand the program. The results obtained by using Maccormack's method are perfectly satisfactory for many fluid flow applications.

Consider 2-D grid as shown in figure, assume the flow field at grid pts is known at time 't' and proceed to calculate the flow field variables at same grid points at time 't+Δt' as shown in fig (2).

$$\rho_{i,j}^{t+\Delta t} = \rho_{i,j}^{t} + \left(\frac{\partial \rho}{\partial t}\right)_{avg} \Delta t \quad \text{--- (1a)}$$

First consider the density at grid point i, j at time 't+Δt'. this can be obtained from above given In Maccormack's method equation. Compare the equation ① and ⑤. In eqⁿ ⑤ the time derivatives are evaluated at time 't' and the carry of second derivative $\left(\frac{\partial^2 \rho}{\partial t^2}\right)_{i,j}^{t}$ is necessary to obtain second order accuracy.

In contrast eqⁿ (1a) the value $\left(\frac{\partial \rho}{\partial t}\right)$ is calculated so as to preserve the second order ~~accu~~ accuracy without the need to calculate the values of second time derivative $\left(\frac{\partial^2 \rho}{\partial t^2}\right)_{i,j}^{t}$ which inturn involves a lot of algebra.

Ully

$$u_{i,j}^{t+\Delta t} = u_{i,j}^{t} + \left(\frac{\partial u}{\partial t}\right)_{avg} \cdot \Delta t \quad \text{— (2a)}$$

$$v_{i,j}^{t+\Delta t} = v_{i,j}^{t} + \left(\frac{\partial v}{\partial t}\right)_{avg} \cdot \Delta t \quad \text{— (3a)}$$

$$e_{i,j}^{t+\Delta t} = e_{i,j}^{t} + \left(\frac{\partial e}{\partial t}\right)_{avg} \cdot \Delta t \quad \text{— (4a)}$$

Value can be obtained from predictor-corrector step.

Predictor step:

In cont. eqⁿ ① replace the spatial derivatives in R.H.S with forward differences.

$$\left(\frac{\partial \rho}{\partial t}\right)_{i,j}^{t} = -\left[\rho_{i,j}^{t}\frac{u_{i+1,j}^{t} - u_{i,j}^{t}}{\Delta x} + u_{i,j}^{t}\frac{\rho_{i+1,j}^{t} - \rho_{i,j}^{t}}{\Delta x} + \rho_{i,j}^{t}\frac{v_{i,j+1}^{t} - v_{i,j}^{t}}{\Delta y} + v_{i,j}^{t}\frac{\rho_{i,j+1}^{t} - \rho_{i,j}^{t}}{\Delta y}\right] \quad \text{— (6)}$$

In eqⁿ (5a) all flow variables at time 't' are known values i.e., R.H.S is known.

Now, obtain a predicted value of density $(\bar{\rho})^{t+\Delta t}_{i,j}$ from the first two terms of taylor series.

$$\Rightarrow (\bar{\rho})^{t+\Delta t}_{i,j} = \rho^{t}_{i,j} + \left(\frac{\partial \rho}{\partial t}\right)^{t}_{i,j} \Delta t \quad \text{— (6a)}$$

In eqⁿ (6a) $(\rho)^{t}_{i,j}$ is known and $\left(\frac{\partial \rho}{\partial t}\right)^{t}_{i,j}$ is a known number from eqⁿ (5a). Hence $(\bar{\rho})^{t+\Delta t}_{i,j}$ is obtained.

llly

$$(\bar{u})^{t+\Delta t}_{i,j} = u^{t}_{i,j} + \left(\frac{\partial u}{\partial t}\right)^{t}_{i,j} \Delta t \quad \text{— (7a)}$$

$$(\bar{v})^{t+\Delta t}_{i,j} = v^{t}_{i,j} + \left(\frac{\partial v}{\partial t}\right)^{t}_{i,j} \Delta t \quad \text{— (8a)}$$

$$(\bar{e})^{t+\Delta t}_{i,j} = e^{t}_{i,j} + \left(\frac{\partial e}{\partial t}\right)^{t}_{i,j} \Delta t \quad \text{— (9a)}$$

from eqⁿ (7a) to (9a) numbers for the time derivatives on R.H.S are obtained from eqⁿ (1) to (4) with forward difference used for spatial derivatives.

<u>Corrector Step</u>:

$$\Rightarrow \left(\frac{\overline{\partial \rho}}{\partial t}\right)^{t+\Delta t}_{i,j} = -\left[(\bar{\rho}^{t+\Delta t}_{i,j}) \frac{\bar{u}^{t+\Delta t}_{i,j} - \bar{u}^{t+\Delta t}_{i-1,j}}{\Delta x} + \bar{u}^{t+\Delta t}_{i,j} \frac{\bar{\rho}^{t+\Delta t}_{i,j} - \bar{\rho}^{t+\Delta t}_{i-1,j}}{\Delta x} + \bar{\rho}^{t+\Delta t}_{i,j} \frac{\bar{v}^{t+\Delta t}_{i,j} - \bar{v}^{t+\Delta t}_{i,j-1}}{\Delta y} + \bar{v}^{t+\Delta t}_{i,j} \frac{\bar{\rho}^{t+\Delta t}_{i,j} - \bar{\rho}^{t+\Delta t}_{i,j-1}}{\Delta y}\right] \quad \text{— (10a)}$$

The avg. value of time derivative of density which appears in eq^n (1a) is obtained from arithmetic mean of $\left(\frac{\partial \rho}{\partial t}\right)^t_{i,j}$ i.e., from eq^n (5a) and $\left(\overline{\frac{\partial \rho}{\partial t}}\right)^{t+\Delta t}_{i,j}$ from eq^n (10 a)

$$\Rightarrow \left(\frac{\partial \rho}{\partial t}\right)_{avg} = \frac{1}{2}\left[\left(\frac{\partial \rho}{\partial t}\right)^{t}_{i,j} + \left(\overline{\frac{\partial \rho}{\partial t}}\right)^{t+\Delta t}_{i,j}\right] \quad - (11a)$$

Eq^n (11a) allows us to obtain the final corrected value of density at time '$t+\Delta t$' from eq^n (1a).

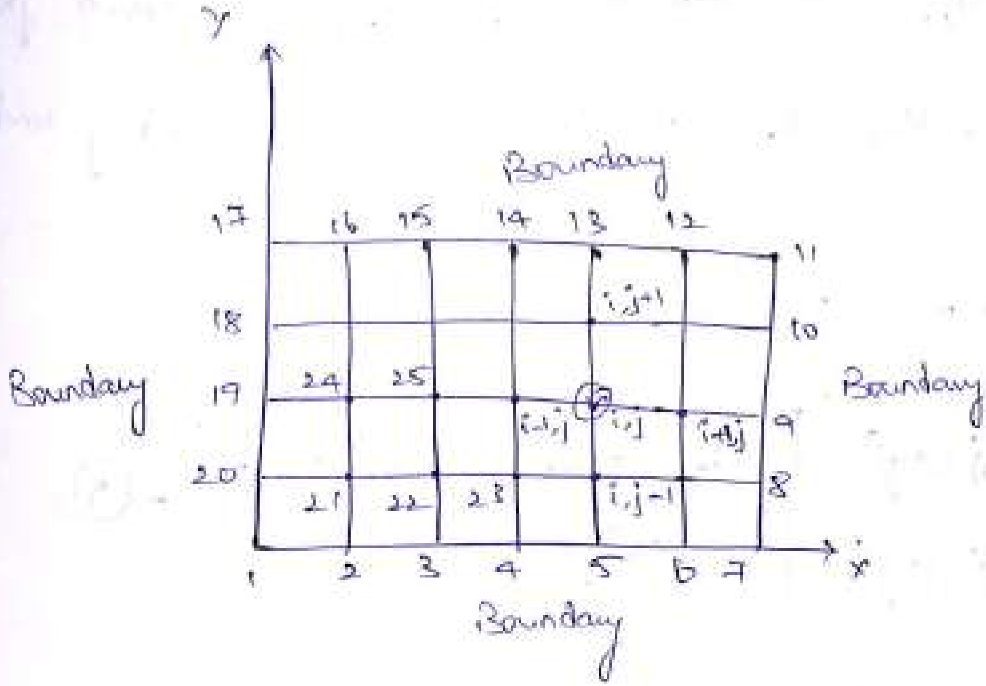
<u>Relaxation Method</u>/<u>Technique</u>: (mainly used for elliptical equations)

$$\cancel{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad - (1)}$$

Relaxation technique is a finite difference method suited for solution of elliptic P.D.E. ~~They~~ Relaxation techniques can be either explicit or implicit. Let us consider Laplace equation in terms of a scalar vel. potential 'ϕ'

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad - (1)$$

$$\Rightarrow \frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{(\Delta x)^2} + \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{(\Delta y)^2} = 0$$



$$\Rightarrow \left[(\Delta y)^2 + (\Delta x)^2\right]\left(\phi_{i,j}^{n+1}\right) = \frac{\left(\phi_{i+1,j}^{n} + \phi_{i-1,j}^{n}\right)(\Delta y)^2 + \left(\phi_{i,j+1}^{n} + \phi_{i,j-1}^{n}\right)(\Delta x)^2}{2}$$

$$\Rightarrow \phi_{i,j}^{n+1} = \frac{\left(\phi_{i+1,j}^{n} + \phi_{i-1,j}^{n}\right)(\Delta y)^2 + \left(\phi_{i,j+1}^{n} + \phi_{i,j-1}^{n}\right)(\Delta x)^2}{2\left[(\Delta x)^2 + (\Delta y)^2\right]}$$

$$\Rightarrow \phi_{i,j}^{n+1} = \frac{(\Delta x)^2(\Delta y)^2}{2\left[(\Delta x)^2 + (\Delta y)^2\right]}\left(\frac{\phi_{i+1,j}^{n} + \phi_{i-1,j}^{n}}{(\Delta x)^2} + \frac{\phi_{i,j+1}^{n} + \phi_{i,j-1}^{n}}{(\Delta y)^2}\right) \quad \text{--- (3)}$$

Consider eqn (3) applied at grid pt. '21', assume that they are already carried out 'n' iterations then for 'n+1' iteration re-write the eqn (3) for grid pt. '21'.

$$\Rightarrow \phi_{21}^{n+1} = \frac{(\Delta x)^2(\Delta y)^2}{2\left[(\Delta x)^2 + (\Delta y)^2\right]}\left[\frac{\phi_{22}^{n} + \phi_{20}^{\bullet}}{(\Delta x)^2} + \frac{\phi_{24}^{n} + \phi_{2}^{\bullet}}{(\Delta y)^2}\right] \quad \text{--- (4)}$$

In eqn ④ ϕ_{21}^{n+1} is unknown, ϕ_{22}^{n}, ϕ_{24}^{n} are known from previous iterations, ϕ_{20}, ϕ_{2} are known from Boundary condition.

Illy for grid pt. '22'.

$$\phi_{22}^{n+1} = \frac{(\Delta x)^2(\Delta y)^2}{2[(\Delta x)^2+(\Delta y)^2]}\left[\frac{\phi_{23}^{n}+\phi_{21}^{n+1}}{(\Delta x)^2}+\frac{\phi_{25}^{n}+\phi_{3}}{(\Delta y)^2}\right] \quad (5)$$

In eqn ⑤ ϕ_{22}^{n+1} is unknown, ϕ_{23}^{n}, ϕ_{25}^{n} are known from previous iterations ϕ_{3} is known from B.C.

The unknown ϕ's at iterations 'n+1' are progressively calculated along the given horizontal line sweeping from left to right. This approach is called "Gauss Seidel method."

Alternating Direction Implicit Technique (ADI):

Consider a model eqn based on unsteady heat conduction eqn with two spatial dimensions.

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \quad (1)$$

$$\Rightarrow \frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\Delta t} = \alpha \cdot \frac{\frac{1}{2}\left(T_{i+1,j}^{n+1} + T_{i+1,j}^{n}\right) + \frac{1}{2}\left(-2T_{i,j}^{n+1} - 2T_{i,j}^{n}\right) + \frac{1}{2}\left(T_{i-1,j}^{n+1} + T_{i-1,j}^{n}\right)}{(\Delta x)^2}$$

$$+ \alpha \cdot \frac{\frac{1}{2}\left(T_{i,j+1}^{n+1} - T_{i,j-1}^{n}\right) + \frac{1}{2}\left(-2T_{i,j}^{n+1} - 2T_{i,j}^{n}\right) + \frac{1}{2}\left(T_{i,j-1}^{n+1} + T_{i,j-1}^{n}\right)}{(\Delta y)^2} \quad \text{—— (2)}$$

Eqⁿ (2) contains '5' unknowns namely $T_{i+1,j}^{n+1}$, $T_{i,j}^{n+1}$, $T_{i-1,j}^{n+1}$, $T_{i,j+1}^{n+1}$, $T_{i,j-1}^{n+1}$.

Where the last two unknowns prevent the tridiagonal form hence thomas algoritham cannot be used. Although matrix method exist which can solve eqⁿ (2) the computer time is much longer than that for a tridiagonal system.

Developing a scheme will allow eqⁿ (1) to be solved by means of tridiagonal forms, such schemes namely ADI scheme (Alternating Direction Implicit Scheme)

(Case (i)) Step 1: In first step over a time interval $\frac{\Delta t}{2}$ replace the spatial derivatives in eqⁿ (1) with the central differences,

$$\Rightarrow \frac{T_{i,j}^{n+\frac{1}{2}} - T_{i,j}^{n}}{\left(\frac{\Delta t}{2}\right)} = \alpha \cdot \frac{T_{i+1,j}^{n+\frac{1}{2}} - 2T_{i,j}^{n+\frac{1}{2}} + T_{i-1,j}^{n+\frac{1}{2}}}{(\Delta x)^2} + \alpha \cdot \frac{T_{i,j+1}^{n} - 2T_{i,j}^{n} + T_{i,j-1}^{n}}{(\Delta y)^2} \quad \text{—— (3)}$$

where only the x derivatives treated implicitly i.e., from (1)

~~Fr~~ eq 1

$$\Rightarrow T_{i,j}^{n+\frac{1}{2}} - T_{i,j}^{n} = \alpha \cdot \frac{\Delta t}{2}\left(\frac{T_{i+1,j}^{n+\frac{1}{2}} - 2T_{i,j}^{n+\frac{1}{2}} + T_{i-1,j}^{n+\frac{1}{2}}}{(\Delta x)^2}\right) + \alpha \cdot \frac{\Delta t}{2}\left(\frac{T_{i,j+1}^{n} - 2T_{i,j}^{n} + T_{i,j-1}^{n}}{(\Delta y)^2}\right)$$

$$\Rightarrow T_{i,j}^{n+\frac{1}{2}} - \alpha \cdot \frac{\Delta t}{2}\left(\frac{T_{i+1,j}^{n+\frac{1}{2}} - 2T_{i,j}^{n+\frac{1}{2}} + T_{i-1,j}^{n+1/2}}{(\Delta x)^2}\right) = T_{i,j}^{n} + \alpha \cdot \frac{\Delta t}{2}\left(\frac{T_{i,j+1}^{n} - 2T_{i,j}^{n} + T_{i,j-1}^{n}}{(\Delta y)^2}\right)$$

$$\cancel{\frac{\alpha \Delta t}{2(\Delta x)^2} T_{i-1,j}^{n+1/2}}$$

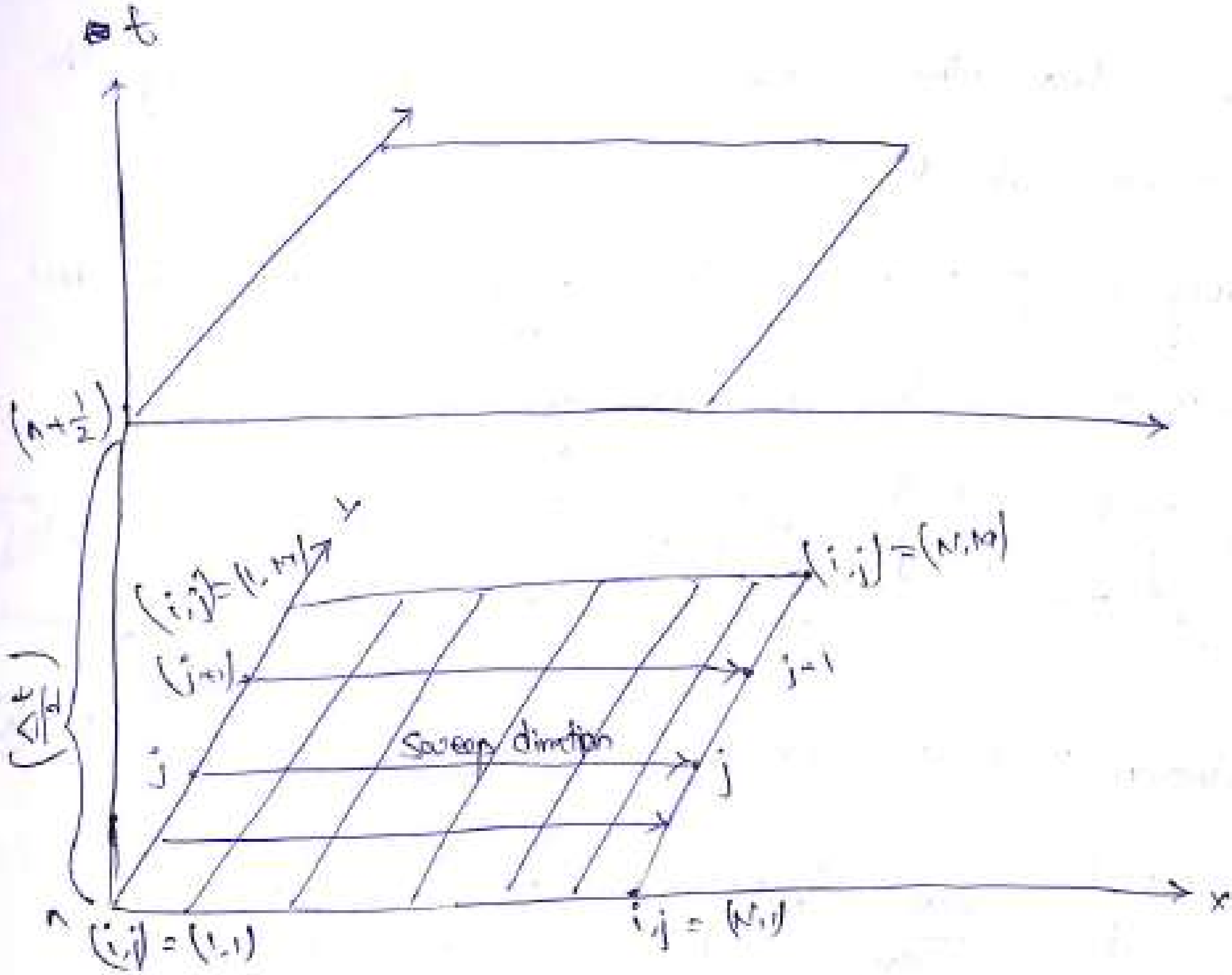
$$\Rightarrow \frac{\alpha \cdot \Delta t}{2(\Delta x)^2} T_{i-1,j}^{n+1/2} - \left(1 + \frac{\alpha \cdot \Delta t}{(\Delta x)^2}\right) T_{i,j}^{n+1/2} + \frac{\alpha \cdot \Delta t}{2(\Delta x)^2} T_{i+1,j}^{n+1/2} = -T_{i,j}^{n} - \frac{\alpha \cdot \Delta t}{2(\Delta y)^2}\left(T_{i,j+1}^{n} - 2T_{i,j}^{n} + T_{i,j-1}^{n}\right)$$

let $\frac{\alpha \cdot \Delta t}{2(\Delta x)^2} = A$.

$\left(1 + \frac{\alpha \cdot \Delta t}{(\Delta x)^2}\right) = B$

and R.H.S as k_i.

$$\Rightarrow A\, T_{i-1,j}^{n+\frac{1}{2}} - B\, T_{i,j}^{n+\frac{1}{2}} + A\, T_{i+1,j}^{n+\frac{1}{2}} = k_i \quad \text{— (4)}$$



From figure at fixed value of 'j' we sweep in the x-direction using eqⁿ (4) to solve for $T_{i,j}^{n+\frac{1}{2}}$ for all values of 'i'. This field utilizes the thomas algorithm once. This calcⁿ is repeated at next row of grid points designated by 'j+1'.

At the end of the step the values of 'T' at the intermediate time '$t+\frac{\Delta t}{2}$' are known at all grid points i.e., $t_{i,j}^{n+\frac{1}{2}}$ is known at i,j

Sec. step:

Here it takes the solution to time $t+\Delta t$, using the known values at $t+\frac{\Delta t}{2}$.

Replacing in eqn ① with central difference eqn, and where 'y' derivative is treated implicitly.

$$\Rightarrow \frac{T_{i,j}^{n+1} - T_{i,j}^{n+1/2}}{(\Delta t/2)} = \alpha \cdot \frac{T_{i+1,j}^{n+1/2} - 2T_{i,j}^{n+1/2} + T_{i-1,j}^{n+1/2}}{(\Delta x)^2} + \alpha \cdot \frac{T_{i,j+1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j-1}^{n+1}}{(\Delta y)^2} \quad —⑤$$

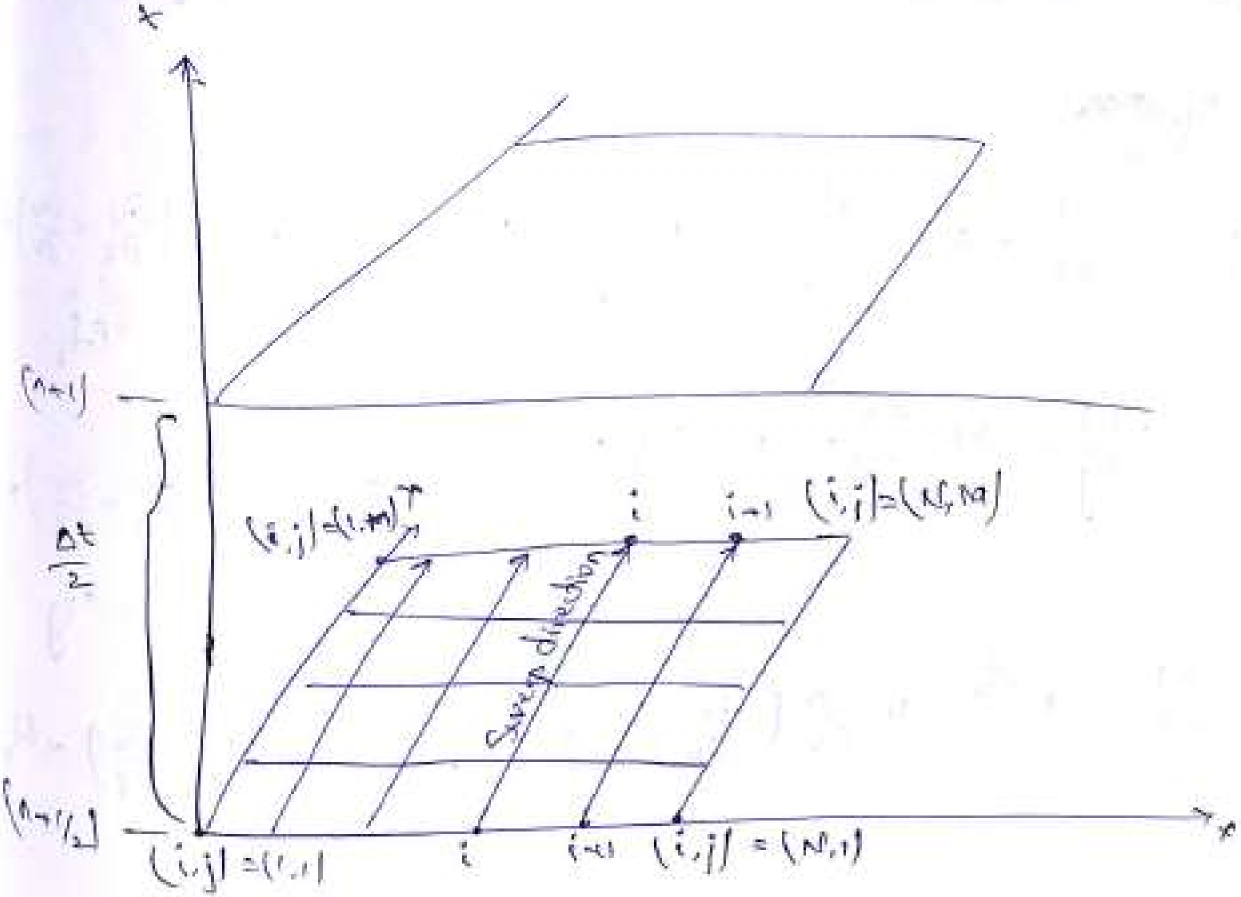
unknowns to L.H.S and knowns to R.H.S

$$\Rightarrow T_{i,j}^{n+1} - T_{i,j}^{n+1/2} = \frac{\alpha \cdot \Delta t}{2(\Delta x)^2}\left(T_{i+1,j}^{n+1/2} - 2T_{i,j}^{n+1/2} + T_{i-1,j}^{n+1/2}\right) + \frac{\alpha \cdot \Delta t}{2(\Delta y)^2}\left(T_{i,j+1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j-1}^{n+1}\right)$$

$$\Rightarrow T_{i,j}^{n+1} - \frac{\alpha \cdot \Delta t}{2(\Delta y)^2}\left(T_{i,j+1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j-1}^{n+1}\right) = T_{i,j}^{n+\frac{1}{2}} + \left(T_{i+1,j}^{n+1/2} - 2T_{i,j}^{n+1/2} + T_{i-1,j}^{n+1/2}\right)$$

$$\Rightarrow \frac{\alpha \cdot \Delta t}{2(\Delta y)^2} T_{i,j+1}^{n+1} - \left(1 + \frac{\alpha \cdot \Delta t}{(\Delta y)^2}\right) T_{i,j}^{n+1} + \frac{\alpha \cdot \Delta t}{2(\Delta y)^2} \cdot T_{i,j-1}^{n+1} = -T_{i,j}^{n+1/2} - \left(T_{i+1,j}^{n+1/2} - 2T_{i,j}^{n+1/2} + T_{i-1,j}^{n+1/2}\right)$$

$$\Rightarrow \boxed{C \cdot T_{i,j+1}^{n+1} - D\, T_{i,j}^{n+1} + C\, T_{i,j-1}^{n+1} = L_j}$$



Pressure Correction Technique:

A numerical technique for the solⁿ of compressible flow was relaxation technique for solving elliptic problems. The viscous incompressible flow is governed by incompressible N-S equations which exhibit a mixed elliptic and parabolic behaviour and hence relaxation technique is not particularly helpful.

Pressure correction technique which has found wide spread application a numerical solution of incompressible N-S equations.

Continuity eqn: $\nabla \cdot v = 0.$ | $\nabla \cdot v = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$

Momentum equations:

$$\rho \frac{Du}{Dt} = -\frac{\partial P}{\partial x} + 2\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \mu \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \rho f_x$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial P}{\partial y} + 2\mu \frac{\partial^2 v}{\partial y^2} + \mu \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \mu \frac{\partial}{\partial z}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + \rho f_y$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + 2\mu \frac{\partial^2 w}{\partial z^2} + \mu \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + \mu \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + \rho f_z$$

Now,

as $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$

$$\Rightarrow \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 w}{\partial x \partial z}$$

$$\boxed{\rho \frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \mu \nabla^2 u + \rho f_x}$$

$$\Rightarrow \cancel{2\mu\frac{\partial^2 u}{\partial x^2}} = -\mu\frac{\partial^2 v}{\partial x\partial y} - \mu\frac{\partial^2 w}{\partial x\partial z} \cancel{+\mu\frac{\partial^2 u}{\partial x^2}}$$

$$\Rightarrow \rho\frac{Du}{Dt} = \frac{-\partial P}{\partial x} + \mu\left[-2\frac{\partial^2 v}{\partial x\partial y} - 2\frac{\partial^2 w}{\partial x\partial z} + \frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x\partial z}\right] + \rho f_x$$

$$\Rightarrow \rho\frac{Du}{Dt} = \frac{-\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 v}{\partial x\partial y} - \frac{\partial^2 w}{\partial x\partial z}\right] + \rho f_x$$

$$= \frac{-\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial x^2}\right] + \rho f_x$$

$$\Rightarrow \rho\frac{Du}{Dt} = \frac{-\partial P}{\partial x} + \mu\nabla^2.u + \rho f_x.$$

lll^y

$$\rho\frac{Dv}{Dt} = \frac{-\partial P}{\partial y} + \mu\nabla^2.v + \rho f_y.$$

$$\rho\frac{Dw}{Dt} = \frac{-\partial P}{\partial z} + \mu\nabla^2.w + \rho f_z.$$

∴ All incompressible N-S eqn are as follows

continuity: $\nabla.v = 0.$

x-Momentum: $\rho\frac{Du}{Dt} = \frac{-\partial P}{\partial x} + \mu\nabla^2.u + \rho f_x$

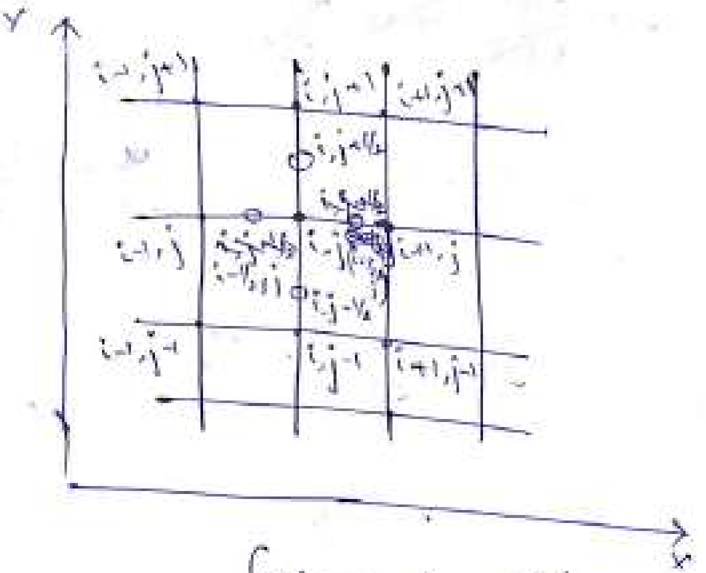
Y-Momentum: $\rho\frac{Dv}{Dt} = \frac{-\partial P}{\partial y} + \mu\nabla^2.v + \rho f_y$

Z-Momentum: $\rho\frac{Dw}{Dt} = \frac{-\partial P}{\partial z} + \mu\nabla^2.w + \rho f_z.$

Staggered grid:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

$$\frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x} + \frac{V_{i,j+1} - V_{i,j-1}}{2\Delta y} = 0.$$



(Staggered grid)

The pressures are calc. at solid grid points (i,j), $(i+1,j)$, $(i-1,j)$ and velocities are calc. at open grid points $(i+\frac{1}{2},j)$, $(i-\frac{1}{2},j)$, $(i,j+\frac{1}{2})$, $(i,j-\frac{1}{2})$.

'u' is calc. at $(i+\frac{1}{2},j)$, $(i-\frac{1}{2},j)$ etc., 'v' is calc. at $(i,j+\frac{1}{2})$, $(i,j-\frac{1}{2})$

Steps of pressure correction formulae:

Step 1: Start the iterative process by guessing the pressure field i.e., p^*.

Step 2: Use the values of p^* to solve for u, v, w from the momentum equations, since the vel. are associated with values of p^* denote them by u^*, v^*, w^*.

Step 3: Since they were obtained from guessed values of p^* the values u^*, v^* and w^* when substituted into cont. eq^n will not satisfy that equation. Hence construct the pressure correction p' when added to p^* then the corrected pressure $p = p^* + p'$

lly

$u = u^* + u'$; $v = v^* + v'$; $w = w^* + w'$.

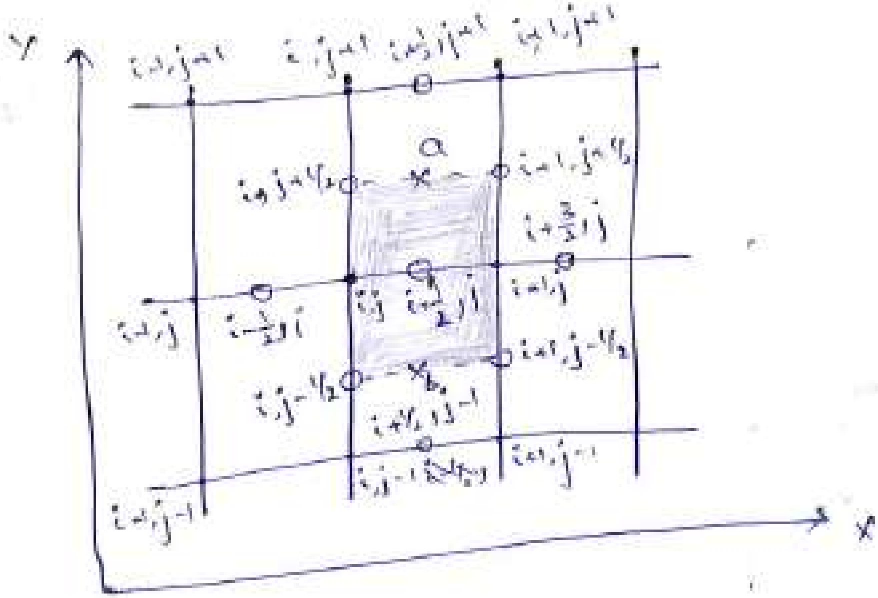
Step 4: Repeat the process steps until the velocity field satisfies the continuity eqn.

Pressure correction formula:

$$x \rightarrow \frac{\partial(\rho u)}{\partial t} + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho u v) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \text{---} (1)$$

lly

$$y \rightarrow \frac{\partial(\rho v)}{\partial t} + \frac{\partial}{\partial x}(\rho u v) + \frac{\partial}{\partial y}(\rho v^2) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad \text{---} (2)$$



Consider a region in a staggered grid, where pressure are evaluated at solid grid points, velocities at open grid points

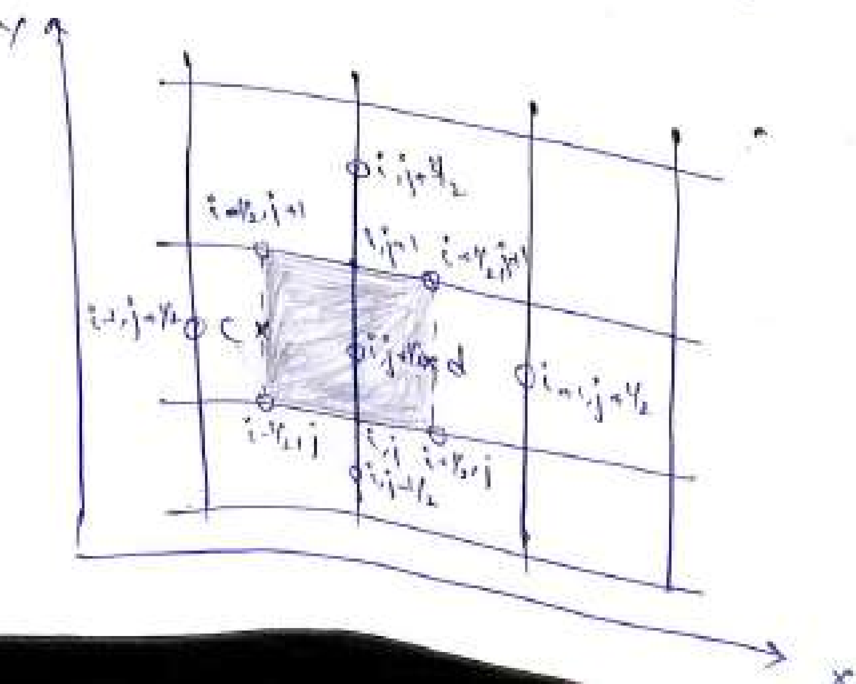
write difference eqⁿ ① centered around the point $(i+\frac{1}{2}, j)$.

$$\cancel{\frac{\partial \rho u}{\partial t} \approx \frac{(\rho u)^{n+1}_{i+1/2,j} - (\rho u)^n_{i,j}}{\Delta t}}$$

$$\frac{(\rho u)^{n+1}_{i+1/2,j} - (\rho u)^n_{i+1/2,j}}{\Delta t} = -\left[\frac{(\rho u^2)^n_{i+\frac{3}{2},j} - (\rho u^2)^n_{i-1/2,j}}{2\Delta x} + \frac{(\rho u \bar{v})^n_{i+\frac{1}{2},j+1} - (\rho u v)^n_{i+1/2,j-1}}{2\Delta y}\right]$$

$$- \frac{p^n_{i+1,j} - p^n_{i,j}}{\Delta x} + \mu\left[\frac{u^n_{i+3/2,j} - 2u^n_{i+1/2,j} + u^n_{i-1/2,j}}{(\Delta x)^2} + \frac{u^n_{i+1/2,j+1} - 2u^n_{i+1/2,j} + u^n_{i+1/2,j-1}}{(\Delta y)^2}\right] \quad — (3)$$

$$(\rho u)^{n+1}_{i+1/2,j} = (\rho u)^n_{i+\frac{1}{2},j} + A(\Delta t) - \frac{\Delta t}{\Delta x}\left(p^n_{i+1,j} - p^n_{i,j}\right) \quad — (4)$$



~~Ily~~ follow similar procedure, that is applied for x-direction. we get.

$$(\rho v)^{n+1}_{i,j+1/2} = (\rho v)^{n}_{i,j+1/2} + B\Delta t - \frac{\Delta t}{\Delta y}\left(p^{n}_{i,j+1} - p^{n}_{i,j}\right) \quad (5)$$

take $p = p^*$ then ~~becomes~~
$u = u^*, v = v^*$

$\therefore$ (4), (5) becomes.

$$(\rho u^*)^{n+1}_{i+1/2,j} = (\rho u^*)^{n}_{i+1/2,j} + A^*(\Delta t) - \frac{\Delta t}{\Delta x}\left(p^{*n}_{i+1,j} - p^{*n}_{i,j}\right) \quad (6)$$

$$(\rho v^*)^{n+1}_{i,j+1/2} = (\rho v^*)^{n}_{i,j+1/2} + B^*(\Delta t) - \frac{\Delta t}{\Delta y}\left(p^{*n}_{i,j+1} - p^{*n}_{i,j}\right) \quad (7)$$

Subtract(8)

$$(6) - (4) \Rightarrow (\rho u')^{n+1}_{i+\frac{1}{2},j} = (\rho u')^{n}_{i+\frac{1}{2},j} + A'(\Delta t) - \frac{\Delta t}{\Delta x}\left(p'^{n}_{i+1,j} - p'^{n}_{i,j}\right) \quad (8)$$

$$(7) - (5) \Rightarrow (\rho v')^{n+1}_{i,j+1/2} = (\rho v')^{n}_{i,j+1/2} + B'(\Delta t) - \frac{\Delta t}{\Delta y}\left(p'^{n}_{i,j+1} - p'^{n}_{i,j}\right) \quad (9)$$

where

$$(\rho u')^{n+1}_{i+1/2,j} = (\rho u)^{n+1}_{i+1/2,j} - (\rho u^*)^{n+1}_{i+1/2,j}$$

$$(\rho v')^{n+1}_{i,j+1/2} = (\rho v)^{n+1}_{i,j+1/2} - (\rho u^*)^{n+1}_{i,j+1/2}$$

$$A' = A - A^* \qquad p'^{n}_{i+1,j} = p^{n}_{i+1,j} - p^{*n}_{i+1,j}$$

$$B' = B - B^* \qquad p'^{n}_{i,j} = p^{n}_{i,j} - p^{*n}_{i,j}$$

$$p'^{n}_{i,j+1} = p^{n}_{i,j+1} - p^{*n}_{i,j+1} \; ; \; \cancel{p_{i,j}}$$

eq^n (8) and (9) are 'x' and 'y' momentum equations expressed in pressure and velocity corrections p', u' and v'.

arbitrarily setting $A', B', (\rho u')^n, (\rho v')^n$ as zero in (8), (9)

$$\Rightarrow \; (\rho u')^{n+1}_{i+1/2,j} = -\frac{\Delta t}{\Delta x}\left(p'_{i+1,j} - p'_{i,j}\right)^n \quad \text{— (10)}$$

$$(\rho v')^{n+1}_{i,j+1/2} = -\frac{\Delta t}{\Delta y}\left(p'_{i,j+1} - p'_{i,j}\right)^n \quad \text{— (11)}$$

Sub. $(\rho u')^{n+1}_{i+1/2,j} = (\rho u)^{n+1}_{i+1/2,j} - (\rho u^*)^{n+1}_{i+1/2,j}$

and $(\rho v')^{n+1}_{i,j+1/2} = (\rho v)^{n+1}_{i,j+1/2} - (\rho v^*)^{n+1}_{i,j+1/2}$ in (10) (11)

respectively.

$$\Rightarrow \; (\rho u)^{n+1}_{i+1/2,j} = (\rho u^*)^{n+1}_{i+1/2,j} - \frac{\Delta t}{\Delta x}\left(p'_{i+1,j} - p'_{i,j}\right)^n \quad \text{— (12)}$$

$$(\rho v)^{n+1}_{i,j+1/2} = (\rho v^*)^{n+1}_{i,j+1/2} - \frac{\Delta t}{\Delta y}\left(p'_{i,j+1} - p'_{i,j}\right)^n \quad \text{— (13)}$$

w.k.t cont. eq^n as

$$\cancel{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)} \; \text{so.} \; \left(\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y}\right) = 0.$$

$$\Rightarrow \frac{\rho u_{i+1/2,j} - \rho u_{i-1/2,j}}{2\Delta x} + \frac{\rho v_{i,j+1/2} - \rho v_{i,j-1/2}}{2\Delta y} = 0. \quad \text{—(14)}$$

Sub. (12), (13) in (14) by dropping the superscript.

$$\Rightarrow \frac{(\rho u^*)_{i+1/2,j} - \frac{\Delta t}{\Delta x}(p'_{i+1,j} - p'_{i,j}) - (\rho u^*)_{i-1/2,j} + \frac{\Delta t}{\Delta x}(p'_{i,j} - p'_{i-1,j})}{\Delta x}$$

$$+ \frac{(\rho v^*)_{i,j+1/2} - \frac{\Delta t}{\Delta y}(p'_{i,j+1} - p'_{i,j}) - (\rho v^*)_{i,j-1/2} + \frac{\Delta t}{\Delta y}(p'_{i,j} - p'_{i,j-1})}{\Delta y} = 0.$$

rearranging, we get

$$\boxed{a p'_{i,j} + b p'_{i+1,j} + b p'_{i-1,j} + c\, p'_{i,j+1} + c\, p'_{i,j-1} + d = 0.} \quad \text{—(15)}$$

where

$$a = 2\left(\frac{\Delta t}{(\Delta x)^2} + \frac{\Delta t}{(\Delta y)^2}\right); \quad b = \frac{-\Delta t}{(\Delta x)^2}; \quad c = \frac{-\Delta t}{(\Delta y)^2};$$

$$d = \frac{1}{\Delta x}\left[(\rho u^*)_{i+1/2,j} - (\rho u^*)_{i-1/2,j}\right] + \frac{1}{\Delta y}\left[(\rho v^*)_{i,j+1/2} - (\rho v^*)_{i,j-1/2}\right]$$

Equation (15) is a pressure correction formula.

Equation (15) :

Term 'd' in eqn (15) is a central difference formulation of the L.H.S of continuity eqn expressed in terms of u^* and v^* define a velocity field that doesnot satisfy the continuity equation i.e., $d \neq 0$. for all but the last iteration.

The last iteration, the velocity field has converged that satisfies the continuity eqn i.e., $d=0$.

SIMPLE Algorithm:

SIMPLE - Semi Implicit Method for Pressure Linked Equations

Steps:

1> Guess the value of $(p^*)^n$ at all the pressure grid points, and $(\rho u^*)^n$, $(\rho v^*)^n$ at velocity grid points.

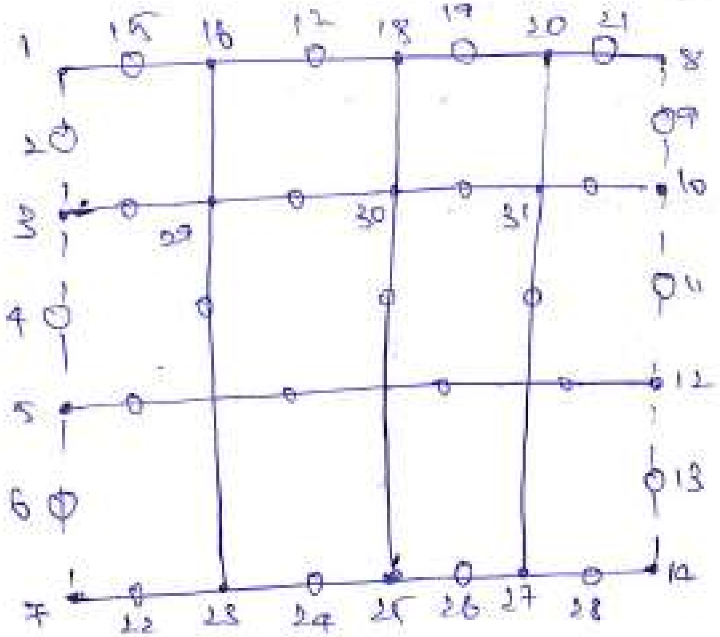
2> Solve for $(\rho u^*)^{n+1}$ and $(\rho v^*)^{n+1}$ at all internal grid points.

3> Substitute these values of $(\rho u^*)^{n+1}$ and $(\rho v^*)^{n+1}$ in (15) and solve for p'.

4> Calculate p^{n+1} at all internal grid points. $\left(p^{n+1} = (p^*)^n + p'\right)$

5> Return to step 2 and repeat steps 2 to 5 until convergence is achieved.

Boundary conditions for pressure correction Method



At inflow boundary p and v are specified ($p' = 0$ at inflow boundary)

i.e., $p'_1 = p'_3 = p'_5 = p'_7 = 0$.

likewise for out flow

$$p'_8 = p'_{10} = p'_{12} = p'_{14} = 0.$$

at walls → No slip condition holds ⇒ Vel. at walls are zero.

$$u'_{15} = u'_{17} = u'_{19} = u'_{21} = u'_{22} = u'_{24} = u'_{26} = u'_{28} = 0.$$

UNIT V

FINITE VOLUME METHODS

Finite Volume Method :

- Introduced into the field of Numerical fluid dynamics by McDonald (1971), McCormack and Paullay (1972)

  for 2-D, time dependent Euler equations

→ extented by Rizzi and Inove (1973) to 3-D flows.

FVM:

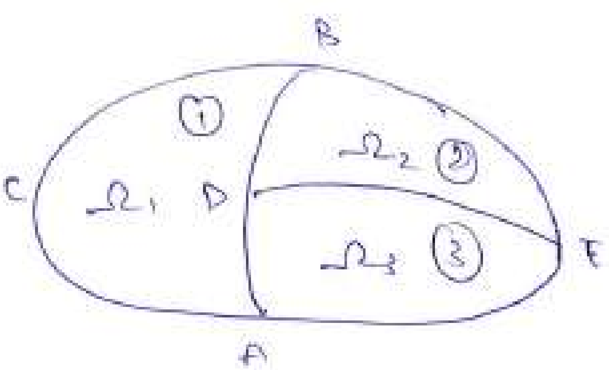
Technique

Integral formulation of conservative laws is discretized in physical space.

- Simple concept
- Easy to implement (structured / unstructured)
- Based on cell averaged values.
- Conservative discretization.

Conservative discretization

Arbitrary sub volume. (v)



Volume source – v
– flux term depends on surface integral.

$$\boxed{\frac{\partial}{\partial t}\int_{\Omega} U\, d\Omega + \oint_{S} \bar{F}\cdot d\bar{S} = \int_{\Omega} Q\cdot d\Omega}$$ ← Integral conservative form.

Arbitrary volume

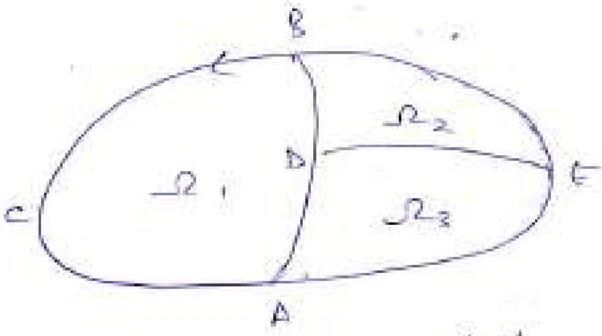
$$\frac{\partial}{\partial t}\int_{\Omega_1} U d\Omega + \int_{ADBC} \bar{F}\, ds = \int_{\Omega} Q\cdot d\Omega$$

$$\frac{\partial}{\partial t}\int_{\Omega_2} U d\Omega + \int_{BDE} \bar{F}\, ds = \int_{\Omega} Q\cdot d\Omega$$

$$\frac{\partial}{\partial t}\int_{\Omega_3} U d\Omega + \int_{AED} \bar{F}\, ds = \int_{\Omega} Q d\Omega.$$

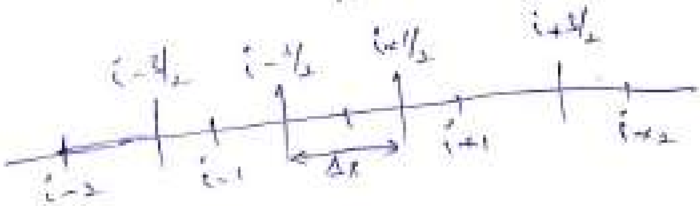
1-D form of conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = q$$



cell size

$\Delta x_i = \Delta x_{i-1} = \Delta x_{i+1}$



cell - difference method

i : $$\frac{\partial u_i}{\partial t} + \frac{f_{i+1/2} - f_{i-1/2}}{\Delta x} = q_i$$

$i+1$: $$\frac{\partial u_{i+1}}{\partial t} + \frac{f_{i+3/2} - f_{i+1/2}}{\Delta x} = q_{i+1}$$

$i-1$

$$\frac{\partial u_{i-1}}{\partial t} + \frac{f_{i-1/2} - f_{i-3/2}}{\Delta x} = q_{i-1}$$

flux contribution internal canceled out

$$\text{or} \quad \boxed{\frac{\partial}{\partial t}\left(\frac{u_i + u_{i+1} + u_{i-1}}{3}\right) + \frac{f_{i+3/2} - f_{i-3/2}}{3 \cdot \Delta x} = \frac{q_i + q_{i-1} + q_{i+1}}{3}}$$

telescopic property

conservative form

for cell AB associated pt 'i'

$$\boxed{\frac{\partial u_i}{\partial t} + \frac{f_{i+3/2} - f_{i-3/2}}{3 \cdot \Delta x} = q_i}$$

Non-conservative form

$$\boxed{\frac{\partial u}{\partial x} + a(u)\frac{\partial u}{\partial x} = q}$$

Formal expression for

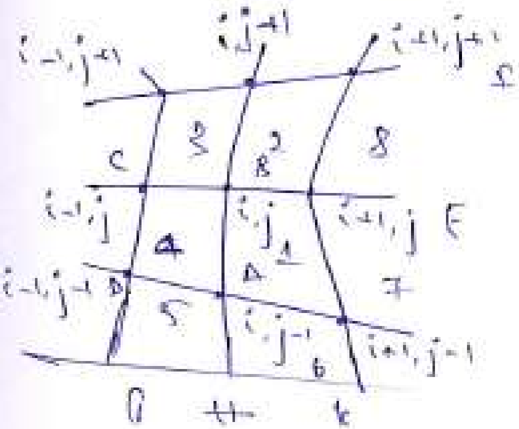
conservative discretization

$$\frac{\partial u}{\partial t} + \frac{f_{i+1/2} - f_{i-1/2}}{\Delta x} = q_i$$

Theorem: If the solution u_i of the discretized eqn converges boundedly almost everywhere to some function $u(x,t)$ when $\Delta x, \Delta t$ tends to zero then $u(x,t)$ is a weak solution

Conditions of Finite Volume Selection:

1. Cell centered approach.

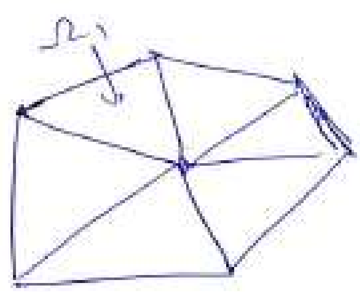


Structured

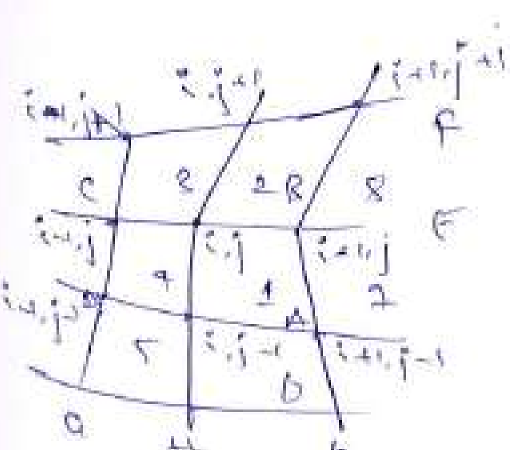
- unknowns are at the centre of mesh cells and grid lines ~~deform~~ define finite volume and surface.

~~Unstructured~~

Unstructured



2. Cell vertex approach



Structured

- unknowns are at the corners
- Mesh variable attached to mesh points i.e., cell vertex.

Unstructured