LECTURE NOTES

ON

COMPLEX ANALYSIS AND SPECIAL FUNCTIONS

Course Code: AHSB05

Regulation: IARE-R18

II B. Tech I semester

Ms.L.Indira
Assistant Professor

ELECTRONICS AND COMMUNICATION ENGINEERING

INSTITUTE OF AERONAUTICAL ENGINEERING
(Autonomous)
Dundigal, Hyderabad - 500 043

SYLLABUS

MODULE-I	COMPLEX FUNCTIONS AND DIFFERENTIATION
Complex functions differentiation and integration: Complex functions and its representation on argand plane, concepts of limit, continuity, differentiability, analyticity, Cauchy-Riemann conditions and harmonic functions; Milne-Thomson method, Bilinear Transformation.	
MODULE -II	**COMPLEX INTEGRATION**
Line integral: Evaluation along a path and by indefinite integration; Cauchy's integral theorem; Cauchy's integral formula; Generalized integral formula; Power series expansions of complex functions and contour Integration: Radius of convergence.	
MODULE -III	**POWER SERIES EXPANSION OF COMPLEX FUNCTION**
Expansion in Taylor's series, Maclaurin's series and Laurent series. Singular point; Isolated singular point; Pole of order m; Essential singularity; Residue: Cauchy Residue Theorem. Evaluation of Residue by Laurent Series and Residue Theorem. Evaluation of integrals of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$ and $\int_{-\infty}^{\infty} f(x)dx$	
MODULE -IV	**SPECIAL FUNCTIONS-I**
Improper integrals; Beta and Gamma functions: Definitions; Properties of Beta and Gamma function; Standard forms of Beta functions; Relationship between Beta and Gamma functions.	
MODULE -V	**SPECIAL FUNCTIONS-II**
Bessel's Differential equation: Bessel function, properties of Bessel function, Recurrence relations of Bessel function, Generating function and Orthogonality of Bessel function, Trigonometric expansions involving Bessel function.	

TEXT BOOKS:

1	Kreyszig, "Advanced Engineering Mathematics", John Wiley & Sons Publishers, 10th Edition, 2010.
2	B. S. Grewal, "Higher Engineering Mathematics", Khanna Publishers, 43rd Edition, 2015.

REFERENCES:

1	T.K.V Iyengar, B.Krishna Gandhi, "Engineering Mathematics - III", S. Chand & Co., 12th Edition, 2015.
2	Churchill, R.V. and Brown, J.W, "Complex Variables and Applications", Tata Mc Graw-Hill, 8th Edition, 2012.

MODULE-I

COMPLEX FUNCTIONS AND DIFFERENTIATION

COMPLEX FUNCTIONS

Complex number

For a complex number $z = x + iy$, the number Re $z = x$ is called the real part of z and the number Im $z = y$ is said to be the its imaginary part. If $x = 0$, z is said to be a purely imaginary number.

Definition : Let $z = x + iy \in C$. The complex number $\bar{z} = x - iy$ is called the complex conjugate of z and $|z| = \sqrt{x^2 + y^2}$ is said to be the absolute value or the modulus of the complex number z.

Functions of a Complex Variable :

Let D be a nonempty set in C. A single-valued complex function or, simply, a complex function $f : D \to C$ is a map that assigns to each complex argument $z = x + iy$ in D a unique complex number $w = u + iv$. We write $w = f(z)$.

The set D is called the domain of the function f and the set f(D) is the range or the image of f. So, a complex-valued function f of a complex variable z is a rule that assigns to each complex number z in a set D one and only one complex number w. We call w the image of z under f.

If $z = x + iy \in D$, we shall write $f(z) = u(x, y) + iv(x, y)$ or $f(z) = u(z) + iv(z)$. The real functions u and v are called the real and, respectively, the imaginary part of the complex function f. Therefore, we can describe a complex function with the aid of two real functions depending on two real variables.

Example 1.The function $f : C \to C$, defined by $f(z) = z^3$, can be written as $f(z) = u(x, y) + iv(x, y)$, with $u, v : R^2 \to R$ given by $u(x, y) = x^3 - 3xy^2$, $v(x, y) = 3x^2 y - y^3$.

Example 2.For the function $f : C \to C$, defined by $f(z) = e^z$, we have $u(x, y) = e^x \cos y$, $v(x, y) = e^x \sin y$, for any $(x, y) \in R^2$.

Limits of Functions : Let $D \subseteq C$, $a \in D'$ and $f : D \to C$. A number $l \in C$ is called a limit of the function f at the point a if for any $V \in V(l)$, there exists $U \in V(a)$ such that, for any $z \in U \cap D \setminus \{a\}$, it follows that $f(z) \in V$. We shall use the notation $l = \lim_{z \to z_0} f(z)$.

Remark : If a complex function $f : D \to C$ possesses a limit l at a given point a, then this limit is unique.

Exercise 1: Prove that $\lim_{z \to z_0} \frac{\bar{z}}{z}$ does not exist.

Solution: To prove that the above limit does not exist, we compute this limit as $z \to 0$ on the real and on the imaginary axis, respectively. In the first situation, i.e. for $z = x \in R$, the value of the limit is 1. In the second situation, i.e. for z = i y, with y ∈ R, the limit is −1. Thus, the limit depends on the direction from which we approach 0, which implies that the limit does not exist.

Differentiability of complex function:

Let w = f(z) be a given function defined for all z in a neighbourhood of z_0.If $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$ exists,the function f(z) is said to be derivable at z_0 and the limit is denoted by $f'(z_0)$. $f'(z_0)$ if exists is called the derivative of f(z) at z_0.

Exercise: f(z)= $|z^2|$ is a function which is continuous at all z but not derivable at any z $\neq 0$

Solution: Let f(z)=$|z^2|$ $= z\bar{z}$

Then f(z)= $z_0\bar{z}_0$

We have to prove that $\underset{z\to z_0}{lt}$ z=z_0 and $\underset{z\to z_0}{lt}\ \bar{z} = \bar{z}_0$ Thus $\underset{z\to z_0}{lt}\ z\bar{z} = z_0\bar{z}_0$

$\therefore \underset{z\to z_0}{lt}$ f(z)=f(z_0)

$\therefore$ The function is continuous at all z

$\therefore$ f(z_0+ Δz) $= (z_0+\Delta z)(\bar{z}+\Delta\bar{z}) = z_0\bar{z}_0 + z_0\Delta\bar{z} + \Delta z\bar{z}_0 + \Delta z\Delta\bar{z}$

Now $\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{z_0\Delta\bar{z} + \Delta z\bar{z}_0 + \Delta z\Delta\bar{z}}{\Delta z}$

Consider the limit as $\Delta z \to 0$

Case 1: let $\Delta z \to 0$ along x-axis then $\Delta x = \Delta x, \Delta y = 0 \Rightarrow \Delta z = \Delta x$

$$\therefore \underset{\Delta z\to 0}{lt} \frac{f(z_0+\Delta z) - f(z)}{\Delta z} = \underset{\Delta x\to 0}{lt} \frac{z_0\Delta x + \Delta x\bar{z}_0 + \Delta x\Delta x}{\Delta x} = z_0 + \bar{z}_0 \qquad \to (1)$$

Case 2: Let $\Delta z \to 0$ along y-axis then $\Delta x = 0, \Delta y = \Delta y \Rightarrow \Delta z = i\Delta y$

$$\underset{\Delta z\to 0}{lt} \frac{f(z_0+\Delta z) - f(z)}{\Delta z} = \underset{\Delta y\to 0}{lt} \frac{z_0(-i\Delta y) + i\Delta y\bar{z}_0 + (i\Delta y)(-i\Delta y)}{i\Delta y} = -z_0 + \bar{z}_0 \qquad \to (2)$$

Thus , from (1) and (2) for f'(z_0) to exists i.e., $z_0 = -z_0 \Rightarrow 2z_0 = 0 \Rightarrow z_0 \neq 0$

$\therefore$ f'(z) does not exists though f(z)= $\left|z^2\right|$ is continuous at all z.

polar form of Cauchy-Riemann equation:

Theorem:

If $f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ and f(z) is derivable at $z_0 = r_0 e^{i\theta_0}$ then

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Proof: Let $z = re^{i\theta}$ Then $f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$

Differentiating it with respect to r partially,

$$\frac{\partial}{\partial r} f(z) = f'(z)\frac{\partial z}{\partial r} = f'(z)e^{i\theta}$$

$$\therefore f'(z) = \frac{1}{e^{i\theta}}\frac{\partial f}{\partial r} = \frac{1}{e^{i\theta}}(u_r + iv_r) \qquad \rightarrow (1)$$

Similarly differentiating partially with respect to θ

$$\frac{\partial f}{\partial \theta} = f'(z)\frac{\partial z}{\partial \theta} = f'(z).rie^{i\theta}$$

$$\therefore f'(z) = \frac{1}{rie^{i\theta}}(u_\theta + iv_\theta) \qquad \rightarrow (2)$$

From (1) and (2) we have

$$\frac{1}{e^{i\theta}}(u_r + iv_r) = \frac{1}{rie^{i\theta}}(u_\theta + iv_\theta)$$

$$\therefore u_r + iv_r = \frac{1}{r}\frac{\partial v}{\partial \theta} - i\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Equating real and imaginary parts ,we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Analytic function:

A complex function is said to be analytic on a region R if it is complex differentiable at every point in R . The terms holomorphic function, differentiable function, and complex differentiable function are sometimes used interchangeably with "analytic function". Many mathematicians prefer the term "holomorphic function" (or "holomorphic map") to "analytic function" .

If a complex function is analytic on a region R , it is infinitely differentiable in R.

Singularities:

A complex function may fail to be analytic at one or more points through the presence of singularities, or along lines or line segments through the presence of branch cuts.

Eg. $f(z)=\frac{1}{z}$ is analytic every where except at z=0.

At z=0 $f'(z)$ does not exist.

So z=0 is an isolated singular point.

Entire function:

A complex function that is analytic at all finite points of the complex plane is said to be entire. A single-valued function that is analytic in all but possibly a discrete subset of its domain, and at those singularities goes to infinity like a polynomial (i.e., these exceptional points must be poles and not essential singularities), is called a meromorphic function.

Cauchy–Riemann equations:

The Cauchy–Riemann equations on a pair of real-valued functions of two real variables $u(x,y)$ and $v(x,y)$ are the two equations:

1. $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$
2. $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

Typically *u* and *v* are taken to be the real and imaginary parts respectively of a complex-valued function of a single complex variable $z = x + iy$, $f(x + iy) = u(x,y) + iv(x,y)$

Relation with harmonic functions:

Analytic functions are intimately related to harmonic functions. We say that a real-valued function h(x, y) on the plane is harmonic if it obeys Laplace’s equation:

$$\frac{\partial^2 h}{\partial^2 x}+\frac{\partial^2 h}{\partial^2 y}=0\,.$$

In fact, as we now show, the real and imaginary parts of an analytic function are harmonic. Let f = u + i v be analytic in some open set of the complex plane.

Then, $\frac{\partial^2 u}{\partial^2 x}+\frac{\partial^2 u}{\partial^2 y}=\frac{\partial}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial}{\partial y}\frac{\partial u}{\partial y}$

$= \frac{\partial}{\partial x}\frac{\partial u}{\partial y}-\frac{\partial}{\partial y}\frac{\partial u}{\partial x}$ (using Cauchy–Riemann)

$=\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 u}{\partial y\partial x}$

$=0$

A similar calculation shows that v is also har monic. This result is important in applications because it shows that one can obtain solutions of a second order partial differential equation by solving a system of first order partial differential equations. It is particularly important in this case because we will be able to obtain solutions of the Cauchy–Riemann equations without really solving these equations.

Given a harmonic function u we say that another harmonic function v is its harmonic conjugate if the complex-valued function f = u+i v is analytic.

Conjugate harmonic function:

If two harmonic functions u and v satisfy the Cauchy-Reimann equations in a domain D and they are real and imaginary parts of an analytic function f in D then v is said to be a conjugate harmonic function of u in D.If f(z)=u+iv is an analytic function and if u and v satisfy Laplace's equation ,then u and v are called conjugate harmonic functions.

Polar form of cauchys Riemann equations:

The Cauchy-Riemann equations can be written in other coordinate systems. For instance, it is not difficult to see that in the system of coordinates given by the polar representation z = r $e^{i\theta}$ these equations take the following form:

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Problem: Show that the function f : C → C, defined by f(z) = $\bar{z}$ does not satisfy the Cauchy-Riemann equations.

Solution: Indeed, since u(x, y) = x, v(x, y) = −y, it follows that ∂u /∂x = 1, while ∂v/ ∂y = −1. So, this function, despite the fact that it is continuous everywhere on C, it is R differentiable on C, is nowhere C-derivable.

Problem: Show that the function $f(z) = e^z$ satisfies the Cauchy-Riemann equations.
Solution:
Since $e^z = e^x($ cosy+i siny),

Indeed it follows that
u(x, y) =e^xcosy, v(x, y) =e^x siny
And $\frac{\partial u}{\partial x}$ = e^xcosy = $\frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y}$ = e^xsiny = $-\frac{\partial v}{\partial x}$;

Moreover, e^z is complex derivable and it follows immediately that its complex derivative is e^z.

Holomorphic functions:

Holomorphic functions are complex functions, defined on an open subset of the complex plane, that are differentiable. In the context of complex analysis, the derivative of f at z_0 is defined to be

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \ z \in c.$$

Construction of analytic function whose real or imaginary part is known:

Suppose f(z)=u+iv is an analytic function ,whose real part u is known .We can find v, the imaginary part and also the function f(z).

Problem: Showthat $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0$ where f(z) is an analytic function.

Solution: Taking $x = \frac{z + \bar{z}}{2}, y = \frac{z - \bar{z}}{2} = \frac{-i}{2}(z - \bar{z})$

We have $\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial x}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial y}{\partial z}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$

And $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

$$\therefore \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

Hence $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\log|f'(z)| = 4\ \frac{\partial^2}{\partial z \partial \bar{z}}\left(\frac{1}{2}\log|f'(z)|^2\right)$

$$= 2\frac{\partial^2}{\partial z \partial \bar{z}}\ [(\log f'(z) f'(\bar{z}))] \qquad (\because |z|^2 = z\bar{z})$$

$$= 2\frac{\partial^2}{\partial z \partial \bar{z}}\ [(\log f'(z) + f'(\bar{z}))]$$

$$= 2\left[\frac{\partial}{\partial \bar{z}}\frac{f''(z)}{f'(z)} + \frac{\partial}{\partial z}\frac{f''(z)}{f'(z\_)}\right]$$

= 2(0+0)=0

Since f(z) is analytic , f(z) is analytic, $f'(\bar{z})$ is also analytic and $\frac{\partial f'(z)}{\partial \bar{z}} = 0, \frac{\partial f'(\bar{z})}{\partial z} = 0$

Problem: Show that f(z)= $\begin{cases} \frac{xy(x+iy)}{x^2+y^4} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$ is not analytic at z=0 although C-R equations satisified at origin.

Solution: $\frac{f(z) - f(0)}{z - 0} = \frac{f(z) - 0}{z} = \frac{f(z)}{z}$

$$= \frac{xy^2(x+iy)}{(x^2+y^4).z} = \frac{xy^2(z)}{(x^2+y^4).z} = \frac{xy^2}{(x^2+y^4)}$$

Clearly $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy^2}{(x^2+y^4)} = \lim_{\substack{y \to 0 \\ x \to 0}} \frac{xy^2}{(x^2+y^4)}$ =0

Along path y=mx

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{x \to 0} \frac{x(m^2.x^2)}{x^2 + m^4.x^4} = \lim_{x \to 0} \frac{m^2.x^2}{1 + m^4.x^2} = 0$$

Along path x=my^2

$$\lim_{z \to 0} \frac{f(z)-f(0)}{z-0} = \lim_{y \to 0} \frac{y^2(m.y^2)}{y^4+m^2.y^4} = \lim_{y \to 0} \frac{m}{1+m^2} \neq 0$$

Limit value depends on m i.e on the path of approach and its different for the different paths

Followed and therefore limit does not exists.

Hence f(z) is not differentiable at z=0.Thus f(z) is not analytic at z=0

To prove that C-R conditions are satisified at origin

Let $f(z) = u + iv = \dfrac{xy^2(x+iy)}{(x^2+y^4)}$

Then u(x,y)= $\dfrac{x^2y^2}{(x^2+y^4)}$ and v(x,y)= $\dfrac{xy^3}{(x^2+y^4)}$ for $z \neq 0$

Also u(0,0)=0 and v(0,0)= 0 [$\because$ f(z)=0 at z=0]

Now $\dfrac{\partial u}{\partial x} = \lim_{x \to 0} \dfrac{u(x,0)-u(0,0)}{x} = \lim_{x \to 0} \dfrac{0}{x} = 0$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{x \to 0} \frac{0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x \to 0} \frac{0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{x \to 0} \frac{0}{y} = 0$$

Thus C-R equations are satisified are satisified at the origin

Hence f(z) is not analytic at z=0 even C-R equations are satisified at origin.

Milne Thomson method:

Problem: Find the regular function whose imaginary part is $\log(x^2+y^2)+x-2y$.

Solution: Given $v = \log(x^2+y^2)+x-2y$

$\therefore \dfrac{\partial v}{\partial x} = \dfrac{2x}{x^2+y^2}+1$ →(1) and $\dfrac{\partial v}{\partial y} = \dfrac{2y}{x^2+y^2}-2$ →(2)

f'(z) = $\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}$ (Using C-R equation)

= $\frac{2y}{x^2+y^2}-2+\left(\frac{2x}{x^2+y^2}+1\right)$ (using (1) ,(2))

By Milne Thomson method ,f'(z) is expressed in terms of z by replacing x z and y by 0.

Hence $f'(z)=-2+i\left(\frac{2z}{z^2}+1\right)=-2+i\left(\frac{2}{z}+1\right)$

On integrating, $f(z)=\int\left[-2+i\left(\frac{2}{z}+1\right)\right]dz+c$

$$=-2z+i(2\log z+z)+c=2i\log z-(2-i)z+c\,.$$

Problem: Show that the function $u=4xy-3x+2$ is harmonic .construct the corresponding analytic function f(z)=u+iv in terms of z.

Solution: Given $u=4xy-3x+2$ →(1)

Differentiating (1) partially w.r.t .x, $\frac{\partial u}{\partial x}=4y-3$

Again differentiating $\frac{\partial^2 u}{\partial x^2}=0$

Again differentiating (1) partially w.r.t .y, $\frac{\partial u}{\partial x}=4x$

Again differentiating $\frac{\partial^2 u}{\partial y^2}=0$

$\therefore\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

Hence u is Harmonic.

Now $f'(z)=\frac{\partial u}{\partial x}+\text{i}\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}\Rightarrow f'(z)=4y-3-i.4x$

Using Milne Thomson method

$f'(z) = -3 - i4z$ (Putting x=z and y=0)

Integrating, $f(z) = -3z - i2z^2 + c$

Problem: Find the imaginary part of an analytic function whose real part is $e^x(x\cos y - y\sin y)$.

Solution: Let $f(z) = u + iv$ where u=$e^x(x\cos y - y\sin y)$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \text{(using C-R equ)}$$

$$= [e^x(x\cos y - y\sin y) + e^x\cos y] - i[e^x(-x\sin y - \sin y - y\cos y)]$$

By Milne's method $f'(z) = (ze^z + e^z) - i(0) = ze^z + e^z$

Integrating, we get

$$f(z) = \int (ze^z + e^z)dz + c = (z-1)e^z + e^z + c = ze^z + c$$

i.e., $u + iv = (x+iy)e^{x+iy} + c$

$$= (x+iy)e^x.e^{iy} + c$$

$$= e^x(x+iy)(\cos y + i\sin y) + c$$

$$= e^x(x\cos y + ix\sin y + iy\cos y - y\sin y) + c$$

$$= e^x[(x\cos y - y\sin y) + i(x\sin y + y\cos y)] + c$$

Bilinear Transformation-Mobius Transformations:

Another important class of elementary mappings was studied by August Ferdinand Möbius (1790-1868). These mappings are conveniently expressed as the quotient of two linear expressions and are commonly known as linear fractional or bilinear transformations. They arise naturally in mapping problems involving the function arc tan(z). In this section, we show how they are used to map a disk one-to-one and onto a half-plane. An important property is that these transformations are conformal in the entire complex plane except at one point.

Let $a, b, c,$ and d denote four complex constants with the restriction that $ad \neq bc$. Then the function

(10-13) $$w = S(z) = \frac{az + b}{cz + d}$$

is called a bilinear transformation, a Möbius transformation, or a linear fractional transformation. If the expression for S(z) in Equation (10-13) is multiplied through by the quantity $cz + d$, then the resulting expression has the bilinear form $cwz - az + dw - b = 0$.

We collect terms involving z and write $z(cw - a) = -dw + b$. Then, for values of $w \neq \frac{a}{c}$ the inverse transformation is given by

(10-14) $$z = S^{-1}(w) = \frac{-dw + b}{cw - a}.$$

We can extend $S(z)$ and $S^{-1}(w)$ to mappings in the extended complex plane. The value $S(\infty)$ should be chosen to equal the limit of $S(z)$ as $z \to \infty$. Therefore we define

$$S(\infty) = \lim_{z\to\infty} S(z) = \lim_{z\to\infty} \frac{a + \frac{b}{z}}{c + \frac{d}{z}} = \frac{a}{c},$$

and the inverse is $S^{-1}\left(\frac{a}{c}\right) = \infty$. Similarly, the value $S^{-1}(\infty)$ is obtained by

$$S^{-1}(\infty) = \lim_{w\to\infty} S^{-1}(w) = \lim_{w\to\infty} \frac{-d + \frac{b}{w}}{c - \frac{a}{w}} = \frac{-d}{c},$$

and the inverse is $S\left(\frac{-d}{c}\right) = \infty$. With these extensions we conclude that the transformation $w = S(z)$ is a one-to-one mapping of the extended complex z-plane onto the extended complex w-plane.

We now show that a bilinear transformation carries the class of circles and lines onto itself. If S(z) is an arbitrary bilinear transformation given by Equation (10-13) and $c = 0$, then S(z) reduces to a linear transformation, which carries lines onto lines and circles onto circles. If $c \neq 0$, then we can write S(z) in the form

$$S(z) = \frac{az+b}{cz+d}$$

$$= \frac{c(az+b)}{c(cz+d)}$$

$$= \frac{acz+bc}{c(cz+d)}$$

$$= \frac{acz+ad-ad+bc}{c(cz+d)}$$

$$= \frac{a(cz+d)-ad+bc}{c(cz+d)}$$

(10-15) $$= \frac{a}{c} + \frac{bc-ad}{c}\frac{1}{cz+d}$$

The condition $ad \neq bc$ precludes the possibility that S(z) reduces to a constant. Equation (10-15) indicates that S(z) can be considered as a composition of functions.

It is a linear mapping $\xi = cz + d$, followed by the reciprocal transformation $Z = \frac{1}{\xi}$, followed by $w = \frac{a}{c} + \frac{bc-ad}{c}Z$. In Section 2.1 we showed that each function in this composition maps the class of circles and lines onto itself; it follows that the bilinear transformation S(z) has this property. A half-plane can be considered to be a family of parallel lines and a disk as a family of circles. Therefore we conclude that a bilinear transformation maps the class of half-planes and disks onto itself. Example 10.3 illustrates this idea.

The general formula for a bilinear transformation (Equation (10-13)) appears to involve four independent coefficients: $a, b, c,$ and d. But as S(z) is not identically constant, either $a \neq 0$ or $c \neq 0$, we can express the transformation with three unknown coefficients and write either

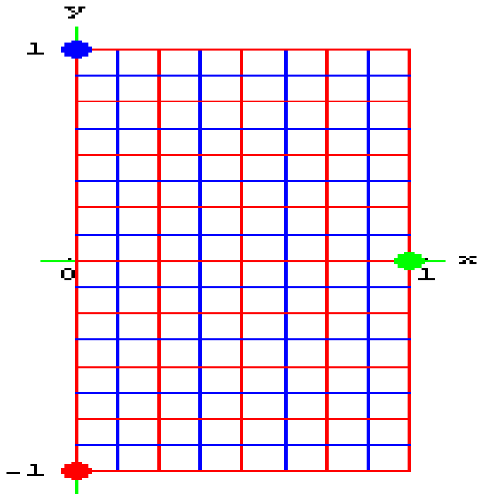
$$S(z) = \frac{z+\frac{b}{a}}{\frac{c}{a}z+\frac{d}{a}}$$ or $$S(z) = \frac{\frac{a}{c}z+\frac{b}{c}}{z+\frac{d}{c}},$$

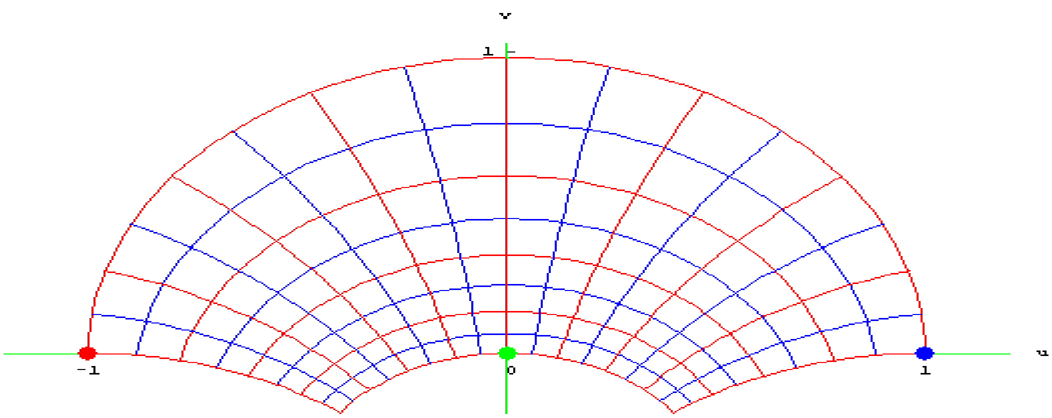
respectively. Doing so permits us to determine a unique a bilinear transformation if three distinct image values $S(z_1) = w_1$, $S(z_2) = w_2$, and $S(z_3) = w_3$ are specified. To determine such a mapping, we can conveniently use an implicit formula involving z and w.

Theorem 10.3 (The Implicit Formula). There exists a unique bilinear transformation that maps three distinct points $z_1, z_2,$ and z_3 onto three distinct points $w_1, w_2,$ and w_3, respectively. An implicit formula for the mapping is given by the equation

(10-18) $$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}.$$

Example 1. Construct the bilinear transformation $w = S(z)$ that maps the points $z_1 = -i,\ z_2 = 1,\ z_3 = i$ onto the points $w_1 = -1,\ w_2 = 0,\ w_3 = 1$, respectively.





Solution. We use the implicit formula, Equation (10-18), and write

$$\frac{(z-(-i))\,(1-i)}{(z-i)\,(1-(-i))} = \frac{(w-(-1))\,(0-1)}{(w-1)\,(0-(-1))}$$

$$\frac{(z+i)\,(1-i)}{(z-i)\,(1+i)} = \frac{(w+1)\,(0-1)}{(w-1)\,(0+1)}$$

$$\frac{(z+i)\,(1-i)}{(z-i)\,(1+i)} = \frac{w+1}{-w+1}.$$

Expanding this equation, collecting terms involving w and zw on the left and then simplify.

$$(z-i)\,(1+i)\,(w+1) = (z+i)\,(1-i)\,(-w+1)$$

$$(1+i)zw + (1-i)w + (1+i)z + (1-i)$$
$$=$$
$$(-1+i)zw + (-1-i)w + (1-i)z + (1+i)$$

$$zw + izw + w - iw + z + iz + 1 - i$$
$$=$$
$$-zw + izw - w - iw + z - iz + 1 + i$$

$$2zw + 2w = -2iz + 2i$$

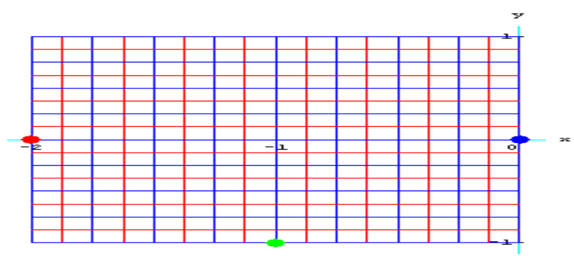
$$zw + w = -iz + i$$

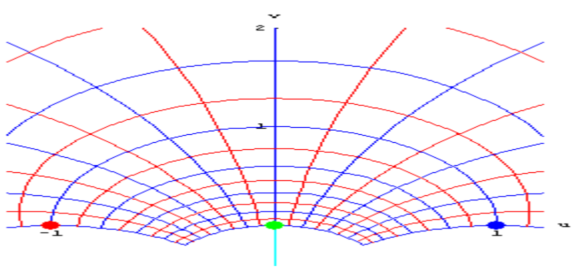
$$w(1+z) = i(1-z)$$

Therefore the desired bilinear transformation is

$$w = S(z) = \frac{i(1-z)}{1+z}.$$

Example 2. Find the bilinear transformation w = S(z) that maps the points $z_1 = -2,\ z_2 = -1 - i,\ z_3 = 0$ onto the points $w_1 = -1,\ w_2 = 0,\ w_3 = 1$, respectively.





Solution: Again, we use the implicit formula, Equation (10-18), and write

$$\frac{(z-(-2))((-1-i)-0)}{(z-0)((-1-i)-(-2))} = \frac{(w-(-1))(0-1)}{(w-1)(0-(-1))}$$

$$\frac{(z+2)(-1-i)}{(z)(-1-i+2)} = \frac{(w+1)(-1)}{(w-1)(1)}$$

$$\frac{z+2}{z}\,\frac{-1-i}{1-i} = \frac{1+w}{1-w}$$

Using the fact that $\frac{-1-i}{1-i} = \frac{1}{i}$, we rewrite this equation as

$$\frac{z+2}{i\,z} = \frac{1+w}{1-w}.$$

We now expand the equation and obtain

$$(z+2)(1-w) = i\,z\,(1+w)$$

$$z+2-z\,w-2\,w = i\,z + i\,z\,w$$

$$z - i\,z + 2 = z\,w + i\,z\,w + 2\,w$$

$$(1-i)\,z + 2 = w\,(z + i\,z + 2)$$

$$(1-i)\,z + 2 = w\,((1+i)\,z + 2)$$

which can be solved for w in terms of z, giving the desired solution

$$w = S(z) = \frac{(1-i)\,z+2}{(1+i)\,z+2}.$$

Corollary (The Implicit Formula with a point at Infinity). In equation (10-18) the point at infinity can be introduced as one of the prescribed points in either the z plane or the w plane.

Proof:

Case 1. If $z_3 = \infty$, then we can write $\frac{(z_2 - z_3)}{(z - z_3)} = \frac{(z_2 - \infty)}{(z - \infty)} = 1$ and substitute this expression into Equation (10-18) to obtain $\frac{(z - z_1)(z_2 - \infty)}{(z - \infty)(z_2 - z_1)} = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}$ which can be rewritten as $\frac{(z - z_1)(z_2 - \infty)}{(z_2 - z_1)(z - \infty)} = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}$ and simplifies to obtain

$$\frac{z - z_1}{z_2 - z_1} = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}.$$

Case 2. If $w_3 = \infty$, then we can write $\frac{(w_2 - w_3)}{(w - w_3)} = \frac{(w_2 - \infty)}{(w - \infty)} = 1$ and substitute this expression into

Equation (10-18) to obtain $\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = \frac{(w - w_1)(w_2 - \infty)}{(w - \infty)(w_2 - w_1)}$ which can be rewritten as $\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = \frac{(w - w_1)(w_2 - \infty)}{(w_2 - w_1)(w - \infty)}$ and simplifies to obtain

(10-21) $$\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = \frac{w - w_1}{w_2 - w_1}.$$

Example 1: Find the bilinear transformation $w = S(z)$ that maps the points

$z_1 = 0,\ z_2 = i, \text{ and } z_3 = -i$ onto $w_1 = -1,\ w_2 = 1, \text{ and } w_3 = 0$, respectively.

Solution: **Method I.** Use the implicit formula $\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}$.

Substitute the values given above and get

$$\frac{(z - 0)(i + i)}{(z + i)(i - 0)} = \frac{(w + 1)(1 - 0)}{(w - 0)(1 + 1)},$$

then simplify and get

$$\frac{2z}{z + i} = \frac{w + 1}{2w}.$$

Solving for w we obtain

$$(2z)(2w) = (z + i)(w + 1)$$

$$4wz = w(z + i) + z + i$$

$$w(4z - (z + i)) = z + i$$

$$w(3z - i) = z + i$$

$$w = \frac{z + i}{3z - i}$$

Therefore, $w = S(z) = \frac{z + i}{3z - i}$

Solution: Method II. The general form of a bilinear transformation is

$$w = S(z) = \frac{az+b}{cz+d},$$ and it is not the case that both $a = 0$ and $c = 0$.

So the desired formula must have one of the following two forms:

either $$S(z) = \frac{1z + \frac{b}{a}}{\frac{c}{a}z + \frac{d}{a}} = \frac{z+b_1}{c_1 z + d_1}$$ or $$S(z) = \frac{\frac{a}{c}z + \frac{b}{c}}{\frac{c}{c}z + \frac{d}{c}} = \frac{a_2 z + b_2}{z + d_2}.$$

Let us assume that the first form $S(z) = \frac{z+b}{cz+d}$ is the one that works out.

Then we can set up three equations to solve $\frac{z_k+b}{c z_k + d} = w_k$ for $k = 1, 2, 3$:

$$\frac{0+b}{c*0+d} = -1, \quad \frac{i+b}{c*i+d} = 1, \quad \frac{-i+b}{c*(-i)+d} = 0,$$

then simplify these equations get

$$b = -d, \quad i+b = ic+d, \quad -i+b = 0.$$

The last equation is easy to solve and we get $b = i$ and then the first equation yields $d = -b = -i$.

Use these values to rewrite the second equation as $i + i = ic - i$ and then obtain $c = 3$.

Substituting these into $S(z) = \frac{z+b}{cz+d}$ produces the desired result:

$$w = S(z) = \frac{z+i}{3z-i}.$$

Example 2: Find the bilinear transformation $w = S(z)$ that maps the points

$z_1 = -i$, $z_2 = 0$, and $z_3 = i$ onto $w_1 = -1$, $w_2 = i$, and $w_3 = 1$, respectively.

Solution: **Method I.** Use the implicit formula $$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}.$$

Substitute the values given above and get

$$\frac{(z-(-i))(0-i)}{(z-i)(0-(-i))} = \frac{(w-(-1))(i-1)}{(w-1)(i-(-1))},$$

$$\frac{(z+i)(0-i)}{(z-i)(0+i)} = \frac{(w+1)(i-1)}{(w-1)(i+1)},$$

then simplify and get

$$-\frac{z+i}{z-i} = \frac{i w + i}{w-1}.$$

Solving for w we obtain

$$-(z+i)(w-1) = (z-i)(i w + i)$$

$$-w z - i w + z + i = w + i w z + i z + 1$$

$$-w z - i w z - w - i w = -z + i z + 1 - i$$

$$w(-z - i z - 1 - i) = -z + i z + 1 - i$$

$$w = \frac{-z + i z + 1 - i}{-z - i z - 1 - i}$$

$$w = \frac{z(-1+i) + 1 - i}{z(-1-i) - 1 - i}$$

$$w = \frac{(-1+i)(z-1)}{(-1-i)(z+1)}$$

$$w = \left(\frac{-1+i}{-1-i}\right)\frac{(z-1)}{(z+1)}$$

$$w = (-i)\frac{(z-1)}{(z+1)}$$

$$w = \frac{-i(z-1)}{z+1}$$

Therefore, $w = S(z) = \dfrac{-iz + i}{z+1}$.

Solution: **Method II.** The general form of a bilinear transformation is

$w = S(z) = \dfrac{az+b}{cz+d}$, and it is not the case that both $a = 0$ and $c = 0$.

So the desired formula must have one of the following two forms:

either $S(z) = \dfrac{1z + \frac{b}{a}}{\frac{c}{a}z + \frac{d}{a}} = \dfrac{z + b_1}{c_1 z + d_1}$ or $S(z) = \dfrac{\frac{a}{c}z + \frac{b}{c}}{\frac{c}{c}z + \frac{d}{c}} = \dfrac{a_2 z + b_2}{z + d_2}$.

Let us assume that the first form $S(z) = \dfrac{z+b}{cz+d}$ is the one that works out.

Then we can set up three equations to solve $\frac{z_k + b}{c\, z_k + d} = w_k$ for $k = 1, 2, 3$:

$$\frac{-i + b}{c*(-i) + d} = -1, \quad \frac{0 + b}{c*0 + d} = i, \quad \frac{i + b}{c*i + d} = 1,$$

then simplify these equations get the system of equations

$$\begin{aligned} b - i\,c + d &= i \\ b \quad\quad - i\,d &= 0 \\ b - i\,c - d &= -i \end{aligned}$$

Add row 1 to row 3 and get

$$\begin{aligned} b - i\,c + d &= i \\ b \quad\quad - i\,d &= 0 \\ 2b - i\,2c \quad\quad &= 0 \end{aligned}$$

Divide row 2 by 1 and subtract it from row 1 to get

$$\begin{aligned} d &= i \\ b \quad\quad - i\,d &= 0 \\ b - i\,c \quad\quad &= 0 \end{aligned}$$

Use $d = i$ to rewrite the second equation as $b - i*i = 0$ and then obtain $b = -1$.

Use $b = -1$ to rewrite the third equation as $-1 - i\,c = 0$ and then obtain $c = i$.

Substituting these into $S(z) = \frac{z + b}{c\,z + d}$ produces the desired result:

$$w = S(z) = \frac{z - 1}{i\,z + i} = \frac{-i\,z + i}{z + 1}.$$

Example 3: Find the bilinear transformation $w = S(z)$ that maps the points

$z_1 = 0,\ z_2 = 1,$ and $z_3 = 2$ on to $w_1 = 0,\ w_2 = 1,$ and $w_3 = \infty$, respectively.

Solution: **Method I.** Use the implicit formula $\frac{(z - z_1)\,(z_2 - z_3)}{(z - z_3)\,(z_2 - z_1)} = \frac{(w - w_1)}{(w_2 - w_1)}$.

Substitute the values given above and get $\frac{(z - 0)\,(1 - 2)}{(z - 2)\,(1 - 0)} = \frac{(w - 0)}{(1 - 0)}$, then simplify and obtain $\frac{z}{2-z} = \frac{w}{1}$.

Therefore, $w = S(z) = \frac{z}{2-z}$.

Solution: **Method II.** The general form of a bilinear transformation is

$w = S(z) = \frac{a z + b}{c z + d}$, and it is not the case that both $a = 0$ and $c = 0$.

So the desired formula must have one of the following two forms:

either $S(z) = \frac{1 z + \frac{b}{a}}{\frac{c}{a} z + \frac{d}{a}} = \frac{z + b_1}{c_1 z + d_1}$ or $S(z) = \frac{\frac{a}{c} z + \frac{b}{c}}{\frac{c}{c} z + \frac{d}{c}} = \frac{a_2 z + b_2}{z + d_2}$.

Let us assume that the first form $S(z) = \frac{z + b}{c z + d}$ is the one that works out.

Then we can set up three equations to solve $\frac{z_k + b}{c z_k + d} = w_k$ for $k = 1, 2, 3$:

$$\frac{0 + b}{c * 0 + d} = 0, \quad \frac{1 + b}{c * 1 + d} = 1, \quad \frac{2 + b}{c * 2 + d} = \infty,$$

In the third equation we will take reciprocals and write it as $\frac{c * 2 + d}{2 + b} = 0$, then we have

$$\frac{0 + b}{c * 0 + d} = 0, \quad \frac{1 + b}{c * 1 + d} = 1, \quad \frac{c * 2 + d}{2 + b} = 0,$$

then simplify these equations get

$$b = 0, \quad 1 + b = c + d, \quad 2 c + d = 0.$$

Use $b = 0$ to rewrite the second equation as $1 = c + d$ then solve the system of two equations

$$c + d = 1$$
$$2 c + d = 0$$

subtracting the first equation from the second equation and get $c = -1$.

Use $c = -1$ in the first equation and get $d = 2$.

Substituting these into $S(z) = \frac{z + b}{c z + d}$ produces the desired result:

$$w = S(z) = \frac{z + 0}{-z + 2} = \frac{z}{2 - z}.$$

Example 4: Find the bilinear transformation $w = S(z)$ that maps the points

$z_1 = 1,\ z_2 = i$, and $z_3 = -1$ on to $w_1 = 0,\ w_2 = 1$, and $w_3 = \infty$, respectively.

Solution: **Method I.** Use the implicit formula $\dfrac{(z - z_1)\,(z_2 - z_3)}{(z - z_3)\,(z_2 - z_1)} = \dfrac{(w - w_1)}{(w_2 - w_1)}$.

Substitute the values given above and get

$$\frac{(z - 1)\,(i - (-1))}{(z - (-1))\,(i - 1)} = \frac{(w - 0)}{(1 - 0)}$$

$$\frac{(z - 1)}{(z + 1)}\,\frac{(i + 1)}{(i - 1)} = \frac{w}{1}$$

$$\frac{(z - 1)}{(z + 1)}\,(-i) = w$$

$$\frac{(-i)\,(z - 1)}{(z + 1)} = w$$

$$\frac{i\,(1 - z)}{1 + z} = w$$

Therefore, $w = S(z) = \dfrac{i - i\,z}{1 + z}$.

Solution: **Method II.** The general form of a bilinear transformation is

$w = S(z) = \dfrac{a\,z + b}{c\,z + d}$, and it is not the case that both $a = 0$ and $c = 0$.

So the desired formula must have one of the following two forms:

either $S(z) = \dfrac{1\,z + \frac{b}{a}}{\frac{c}{a}\,z + \frac{d}{a}} = \dfrac{z + b_1}{c_1\,z + d_1}$ or $S(z) = \dfrac{\frac{a}{c}\,z + \frac{b}{c}}{\frac{c}{c}\,z + \frac{d}{c}} = \dfrac{a_2\,z + b_2}{z + d_2}$.

Let us assume that the first form $S(z) = \dfrac{z + b}{c\,z + d}$ is the one that works out.

Then we can set up three equations to solve $\dfrac{z_k + b}{c\,z_k + d} = w_k$ for $k = 1, 2, 3$:

$$\frac{1 + b}{c * 1 + d} = 0,\quad \frac{i + b}{c * i + d} = 1,\quad \frac{-1 + b}{c * (-1) + d} = \infty,$$

In the third equation we will take reciprocals and write it as $\dfrac{-c + d}{-1 + b} = 0$, then we have

$$\frac{1+b}{c*1+d} = 0, \quad \frac{i+b}{c*i+d} = 1, \quad \frac{-c+d}{-1+b} = 0,$$

then simplify these equations get

$$1+b = 0, \quad i+b = c*i+d, \quad -c+d = 0.$$

The first equation is easy to solve and we get $b = -1$.

Use $b = -1$ to rewrite the second equation as $i - 1 = i\,c + d$ then solve the system of two equations

$$i\,c + d = -1 + i$$

$$-c + d = 0$$

Subtract the second equation from the first equation obtain $(1+i)\,c = -1 + i$ and get $c = i$.

Use $c = i$ in the second equation and get $d = i$.

Substituting these into $S(z) = \frac{z+b}{c\,z+d}$ produces the desired result:

$$w = S(z) = \frac{z-1}{i\,z+i} = \frac{i - i\,z}{1+z}.$$

Fixed Point:

A fixed point of a mapping $w = f(z)$ is a point z_0 such that $f(z_0) = z_0$.

Example 1: Show that a bilinear transformation, $w = f(z) = \frac{a\,z+b}{c\,z+d}$, can have at most two fixed points.

Solution:

The equation $z = \frac{a\,z+b}{c\,z+d}$ can be written as

$$c\,z^2 + (d-a)\,z - b = 0,$$

and this quadratic equation has, at most, two distinct solutions:

$$z = \frac{a - d + \sqrt{(d-a)^2 + 4\,b\,c}}{2\,c}$$

and

$$z = \frac{a - d - \sqrt{(d-a)^2 + 4bc}}{2c}.$$

Example 2: Find the fixed points of **(a).** $w = S(z) = \frac{z-1}{z+1}$. **(b).** $w = S(z) = \frac{4z+3}{2z-1}$.

Solution: **(a).** Solve the equation $z = \frac{z-1}{z+1}$ for z and get

$$z = \frac{z-1}{z+1}$$

$$z(z+1) = z - 1$$

$$z^2 + z = z - 1$$

$$z^2 = -1$$

$$z = \pm i$$

Therefore, the fixed points of $S(z) = \frac{z-1}{z+1}$ are $z = +i$ and $z = -i$.

Just for fun, we can substitute $z = \pm i$ into the formula $S(z) = \frac{z-1}{z+1}$.

$$\frac{-i-1}{-i+1} = \frac{-1-i}{1-i} = \frac{(-1-i)(1+i)}{(1-i)(1+i)} = \frac{-2i}{2} = -i,$$ and

$$\frac{i-1}{i+1} = \frac{-1+i}{1+i} = \frac{(-1+i)(1-i)}{(1+i)(1-i)} = \frac{2i}{2} = i.$$

(b). **Solution:** Solve the equation $z = \frac{4z+3}{2z-1}$ for z and get

$$z = \frac{4z+3}{2z-1}$$

$$z(2z-1) = 4z + 3$$

$$2z^2 - z = 4z + 3$$

$$2z^2 - 5z - 3 = 0$$

$$(2z+1)(z-3) = 0$$

$$2\left(z + \frac{1}{2}\right)(z-3) = 0$$

Therefore, the fixed points of $S(z) = \frac{4z+3}{2z-1}$ are $z = -\frac{1}{2}$ and $z = 3$.

Just for fun, we can substitute $z = -\frac{1}{2}$ and $z = 3$ into the formula $S(z) = \frac{4z+3}{2z-1}$.

$$\frac{4\left(-\frac{1}{2}\right)+3}{2\left(-\frac{1}{2}\right)-1} = \frac{-2+3}{-1-1} = -\frac{1}{2}$$, and

$$\frac{4(3)+3}{2(3)-1} = \frac{12+3}{6-1} = \frac{15}{5} = 3$$.

EXCERCISE PROBLEMS:

1) Show that the real part of an analytic function f (z) where u = $e^{-2xy}\sin(x^2-y^2)$ is a harmonic function. Hence find its harmonic conjugate.

2) Prove that the real part of analytic function f (z) where u = $\log|z|^2$ is harmonic function. If so find the analytic function by Milne Thompson method.

3) Obtain the regular function f (z) whose imaginary part of an analytic function is $\frac{x-y}{x^2+y^2}$

4) Find an analytic function f (z) whose real part of an analytic function is u = $\frac{sin2x}{cosh2y-cos2x}$ by Milne-Thompson method.

5) Find an analytic function f (z) = u +iv if the real part of an analytic function is u = a (1+cosθ) using Cauchy-Riemann equations in polar form.

6) Prove that if $u = x^2-y^2, v = -\frac{y}{x^2+y^2}$ both u and v satisfy Laplace's equation, but u + iv is not a regular (analytic) function of z.

7) Show that the function f (z) = $\sqrt{|xy|}$ is not analytic at the origin although Cauchy –Riemann equations are satisfied at origin.

8) If $w = \emptyset + i\varphi$ represents the complex potential for an electric field where $\varphi = x^2-y^2+\frac{x}{x^2+y^2}$

then determine the function φ.

9) State and Prove the necessary condition for f (z) to be an analytic function in Cartesian form.

10) If u and v are conjugate harmonic functions then show that uv is also a harmonic function.

11) Find the orthogonal trajectories of the family of curves $r^2 cos2\theta$ = c

12) Find an analytic function whose real part is u = $\frac{sin2x}{cosh2y-cos2x}$

13) Find an analytic function whose imaginary part is v = $e^x(xsiny + ycosy)$

14) Find an analytic function whose real part is (i) u = $\frac{x}{x^2+y^2}$ (ii) u = $\frac{y}{x^2+y^2}$

15) Find an analytic function whose imaginary part is v =$\frac{2sinxsiny}{cosh2x+cosh2y}$

16) Find an analytic function f(z) = u +iv if u = a(1+cosθ)

17) Find the conjugate harmonic of u = $e^{x^2-y^2}$cos2xyand find f(z) in terms of z.

18) If f(z) is an analytic function of z and if u - v = $e^x(cosy - siny)$ find f(z) in terms of z.

19) If f(z) is an analytic function of z and if u - v = (x-y)$(x^2 + 4xy + y^2)$) find f(z) in terms of z.

20) Find the orthogonal trajectories of the family of curves $x^3y - xy^3$= C = constant

MODULE-II

COMPLEX INTEGRATION

LINE INTEGRAL

Defination: In mathematics, a **line integral** is an integral where the function to be integrated is evaluated along a curve. The terms **path integral**, **curve integral**, and **curvilinear integral** are also used; contour integral as well, although that is typically reserved for line integrals in the complex plane.

The function to be integrated may be a scalar field or a vector field. The value of the line integral is the sum of values of the field at all points on the curve, weighted by some scalar function on the curve (commonly arc length or, for a vector field, the scalar product of the vector field with a differential vector in the curve). This weighting distinguishes the line integral from simpler integrals defined on intervals. Many simple formulae in physics (for example, $W = \mathbf{F} \cdot \mathbf{s}$) have natural continuous analogs in terms of line integrals ($W = \int_C \mathbf{F} \cdot d\mathbf{s}$). The line integral finds the work done on an object moving through an atomic or gravitational field.

In complex analysis, the line integral is defined in terms of multiplication and addition of complex numbers.

Let us consider F(t)= u(t)+i v(t) , $a \le t \le b$. Where u and v are real valued continuous functions of t in [a,b].

we define $\int_a^b F(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$

Thus, $\int_a^b F(t)\,dt$ is a complex number such that real part of $\int_a^b F(t)\,dt$ is $\int_a^b u(t)\,dt$ and imaginary part of $\int_a^b F(t)\,dt$ is $\int_a^b v(t)\,dt$.

Problem: Evaluate $\int_0^{1+i} (x^2 - iy)dz$ along the paths 1)y=x 2)$y=x^2$

Solution: 1) along the line y=x, dy= dx so that dz = dx+idx=(1+i) dx

$\therefore \int_0^{1+i} (x^2 - iy)dz = \int_0^1 (x^2 - ix)(1+i)dx,$ Since y=x

$$=(1+i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1$$

$$=(1+i)\left[\frac{1}{3} - \frac{1}{2}i\right]$$

$$=\frac{5}{6}-\frac{1}{6}i$$

2) alongtheparabola y=x^2,dy=2xdx sothatdz=dx+2ixdx

dz=(1+2ix)dx and xvaries from 0to1

$$\therefore \int_0^{1+i}(x^2-iy)dz = \int_0^1 (x^2-ix^2)(1+2ix)dx$$

$$= (1\text{-i}) \int_0^1 x^2(1+2ix)dx$$

$$= (1\text{-i}) \left(\frac{1}{3}+\frac{1}{2}i\right)$$

$$=\frac{(1-i)(2+3i)}{6}$$

$$=\frac{5}{6}+\frac{1}{6}i$$

Problem: Evaluate $\int_{z=0}^{z=1+i}(x^2+2xy+i(y^2-x))dz$ along y=x^2

Solution: Given f(z)=$x^2+2xy+i(y^2-x))dz$

Z=x+iy, dz=dx+idy

$$\therefore thecurvey = x^2, dy = 2xdx$$

$$\therefore dz = dx+2xidx = (1+2ix)dx$$

f(z)=x^2+2x(x^2)+i(x^4-x)

=x^2+2x^3 +i(x^4-x)

f(z) dz=(x^2 +2x^3)+i(x^4-x)(1+2ix))dx

=x^2+2x^3+i(x^4-x)+2ix^3+4ix^4-2x^5+2x^2

$$\therefore \int_c f(z)dz = \int_{z=0}^{1+i} x^2+2xy+i(y^2-x)dz$$

$$=\int_0^1 (-2x^5+3x^2+2x^3+i(5x^4-x+2x^3))dx)$$

$$=\left[-\frac{x^6}{3}+x^3+\frac{x^4}{2}+i(\frac{5x^5}{5}-\frac{x^2}{2}+\frac{x^4}{2}\right]_0^1$$

$$=\left[\left(\frac{-1}{3}+1+\frac{1}{2}\right)+\left(\frac{5}{5}-\frac{1}{2}+\frac{1}{2}\right)\right]-0$$

$$=\frac{7}{6}+\frac{5}{5}i=\frac{7}{6}+i$$

$$\int_c f(z)dz=\frac{7}{6}+i$$

Cauchy-Goursat Theorem: Let f(z) be analytic in a simply connected domain D. If C is a simple closed contour that lies in D, then

$$\int_C f(z)\,dz = 0$$

Let us recall that e^z, $\cos(z)$, and z^n (where n is a positive integer) are all entire functions and have continuous derivatives. The Cauchy-Goursat theorem implies that, for any simple closed contour,

(a) $\int_C e^z\,dz = 0$,

(b) $\int_C \cos(z)\,dz = 0$, and

(c) $\int_C z^n\,dz = 0$.

Cauchy integral formula:

STATEMENT : let F(z)=u(x,y)+iv(x,y) be analytic on and within a simple closed contour (or curve) ‘c’ and let f ′(z) be continuous there,then $\int f(z)dz = 0$

Proof: f (z)=u(x,y)+iv(x,y)

And dz=dx+idy

$\Rightarrow$ f(z).dz = (u(x,y)+iv(x,y))dx+idy

$f(z).dz = u(x,y)dx+i\, u(x,y)dy+iv(x,y)dx+i^2\, v(x,y)dy$

$f(z).dz= u(x,y)dx- v(x,y)dy+i(u(x,y)dy+ v(x,y)dx$

Integrate both sides, we get

$$\int f(z)dz = \int (\mathrm{udx} - \mathrm{vdy}) + i(\mathrm{udy} + \mathrm{vdx})$$

By greens theorem, we have

$$\int Mdx + Ndy = \iint \frac{\partial N}{\partial x} - \frac{\partial M}{\partial Y} \mathrm{dxdy}$$

Now $\int f(z)dz = \iint (-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial Y})\mathrm{dxdy} + i(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial Y})\mathrm{dxdy}$

Since $f'(z)$ is continuous &four partial derivatives i.e $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial Y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial Y}$ are also continuous in the region R enclosed by C, Hence we can apply Green’s Theorem.

Using Green’s Theorem in plane, assuming that R is the region bounded by C.

It is given that f (z)=u(x,y)+iv(x,y) is analytic on and within c.

Hence $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Using this we have

$$\int_c f(z)dz = \iint_R 0 \quad dxdy + i\iint_R 0 \quad dxdy \quad = 0$$

Hence the theorem.

Cauchy's integral formula:

Cauchy's integral formula states that

$$f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)\,dz}{z - z_0}, \qquad (1)$$

Where the integral is a contour integral along the contour γ enclosing the point z_0.

It can be derived by considering the contour integral

$$\oint_\gamma \frac{f(z)\,dz}{z-z_0}, \tag{2}$$

Defining a path γ_r as an infinitesimal counterclockwise circle around the point z_0, and defining the path γ_0 as an arbitrary loop with a cut line (on which the forward and reverse contributions cancel each other out) so as to go around z_0. The total path is then

$$\gamma = \gamma_0 + \gamma_r, \tag{3}$$

so

$$\oint_\gamma \frac{f(z)\,dz}{z-z_0} = \oint_{\gamma_0} \frac{f(z)\,dz}{z-z_0} + \oint_{\gamma_r} \frac{f(z)\,dz}{z-z_0}. \tag{4}$$

From the Cauchy integral theorem, the contour integral along any path not enclosing a pole is 0. Therefore, the first term in the above equation is 0 since γ_0 does not enclose the pole, and we are left with

$$\oint_\gamma \frac{f(z)\,dz}{z-z_0} = \oint_{\gamma_r} \frac{f(z)\,dz}{z-z_0}. \tag{5}$$

Now, let $z \equiv z_0 + r e^{i\theta}$, so $dz = i r e^{i\theta}\,d\theta$. Then

$$\oint_\gamma \frac{f(z)\,dz}{z-z_0} = \oint_{\gamma_r} \frac{f\left(z_0 + r e^{i\theta}\right)}{r e^{i\theta}}\, i r e^{i\theta}\,d\theta \tag{6}$$

$$= \oint_{\gamma_r} f\left(z_0 + r e^{i\theta}\right) i\,d\theta. \tag{7}$$

But we are free to allow the radius r to shrink to 0, so

$$\oint_\gamma \frac{f(z)\,dz}{z-z_0} = \lim_{r\to 0} \oint_{\gamma_r} f\left(z_0 + r e^{i\theta}\right) i\,d\theta \tag{8}$$

$$= \oint_{\gamma_r} f(z_0)\, i\,d\theta \tag{9}$$

$$= i\,f(z_0) \oint_{\gamma_r} d\theta \tag{10}$$

$$= 2\pi i\,f(z_0), \tag{11}$$

giving (1).

If multiple loops are made around the point z_0, then equation (11) becomes

$$n(\gamma, z_0) f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)\,dz}{z - z_0}, \tag{12}$$

Where $n(\gamma, z_0)$ is the contour winding number.

A similar formula holds for the derivatives of $f(z)$,

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{13}$$

$$= \lim_{h \to 0} \frac{1}{2\pi i h} \left[\oint_\gamma \frac{f(z)\,dz}{z - z_0 - h} - \oint_\gamma \frac{f(z)\,dz}{z - z_0} \right] \tag{14}$$

$$= \lim_{h \to 0} \frac{1}{2\pi i h} \oint_\gamma \frac{f(z)\,[(z - z_0) - (z - z_0 - h)]\,dz}{(z - z_0 - h)(z - z_0)} \tag{15}$$

$$= \lim_{h \to 0} \frac{1}{2\pi i h} \oint_\gamma \frac{h\, f(z)\,dz}{(z - z_0 - h)(z - z_0)} \tag{16}$$

$$= \frac{1}{2\pi i} \oint_\gamma \frac{f(z)\,dz}{(z - z_0)^2}. \tag{17}$$

Iterating again,

$$f''(z_0) = \frac{2}{2\pi i} \oint_\gamma \frac{f(z)\,dz}{(z - z_0)^3}. \tag{18}$$

Continuing the process and adding the contour winding number n,

$$n(\gamma, z_0) f^{(r)}(z_0) = \frac{r!}{2\pi i} \oint_\gamma \frac{f(z)\,dz}{(z - z_0)^{r+1}}.$$

Problem: Evaluate using cauchy's integral formula $\int_c \frac{e^{2z}}{(z-1)(z-2)} dz$ where c is the circle $|z| = 3$

Solution: Given $\int_c \frac{e^{2z}}{(z-1)(z-2)} dz$ ……………(1)

Both the points z=1,z=2 line inside $|z| = 3$

Resolving into partial fractions

$$\frac{1}{(z-1)(z-2)}=\frac{A}{(z-1)}+\frac{B}{(z-2)}$$

A=-1, B=1

From (1)

$$\int_c \frac{e^{2z}}{(z-1)(z-2)}dz=\int_c \frac{-e^{2z}}{(z-1)}dz+\int_c \frac{e^{2z}}{(z-2)}dz$$ (by cauchy's integral formula)

$=-2\pi if(1)+2\pi if(2)$

$=-2\pi ie^{2.1}+2\pi ie^{2.2}$

$=-2\pi ie^{2}+2\pi ie^{4}=2\pi i(e^{4}-e^{2})$

Problem: Using cauchy'sintegralformula to evaluate $\int_c \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)z-2)}dz$, where c is the circle $|z|=3$

Solution: $\int_c \frac{f(z)}{(z-1)z-2)}dz=(\int_c \frac{1}{(z-2)}dz+\int_c \frac{1}{(z-1)}dz$)f(z)dz

$=\int_c \frac{f(z)}{(z-2)}dz+\int_c \frac{f(z)}{(z-1)}dz$

$=2\Pi if(2)-2\prod if(1)$

$=2\Pi i(\sin 4\Pi+\cos 4\Pi)-(\sin\Pi+\cos\Pi))$

$=2\Pi i(1-(-1))=4\Pi i$

$$\int_c \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)z-2)}dz=4\Pi i$$

Problem: Evaluate $\int_c \frac{(z-1)}{(z+1)^2(z-2)}dz$ whrere c is $|Z-i|=2$

Solution: the singularities of $\frac{(z-1)}{(z+1)^2(z-2)}$ are given by

$(z+1)^2(z-2)=0$

$\Rightarrow$ Z=-1 and z=2

Z=-1 lies inside the circle since $|-1-i|-2<0$

Z=2 lies outside the circle sinceI2-iI-2>0 $|2-i|-2>0$

The given line integral can be written as

$$\int_c \frac{(z-1)}{(z+1)^2(z-2)}dz = \int_c \frac{\frac{(z-1)}{(z-2)}}{(z+1)^2} \text{ ---------------------------- (1)}$$

The derivative of analytic function is given by

$$\int_c \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i f^n(a)}{n!} \text{------------------------------------- (2)}$$

From (1) and (2) f(z)=$\frac{(z-1)}{(z-2)}$, a=-1,n=1

$$f^1(z) = \frac{1(z-2)-1(z-1)}{(z-2)^2} = \frac{1}{(z-2)^2}$$

$$f^1(-1) = \frac{1}{-9}$$

Substituting in (2),we get

$$\int_c \frac{(z-1)}{(z+1)^2(z-2)}dz = \frac{2\pi i}{1}(-\frac{1}{9})$$

$$=\frac{-2}{9}\Pi i$$

Problem: Evaluate $\int_c \frac{e^{2z}}{(z+1)^4}dz$ where c: $|z-1|=1$

Solution: the singular points of $\frac{e^{2z}}{(z+1)^4}dz$ are givenby7

$(z+1)^4=0 \Rightarrow z = -1$

The singular point z=-1 lies insidethecirclec: $|z-1| = 3$

Applying cauchy's integral formula for derivatives

$$\int_c \frac{f(z)}{(z-a)^{n+1}}dz = \int_c \frac{2\pi i f^n(-1)}{n!}dz \text{ -------------------------- } (1)$$

F (z)=e^{2z},n=3,a=-1

f (z)=$2e^{2z}$

f^1 (z)=$4e^{2z}$

f^{11} (z)=$8e^{2z}$

$f^{111(}$z)=$16e^{2z}$

f^{111} (-1)=$16e^{-2}$

Substituting in (1)

$$\int_c \frac{e^{2z}}{(z+1)^4}dz = \int_c \frac{2\pi i f^{111}(-1)}{n!}$$

$$= \frac{2\pi i 16e^{-2}}{2!}$$

=16Πie^{-2}

Problem: Use cauchy's integral formula to evaluate $\int_c \frac{e^{-2z}}{(z+1)^3}dz$ with c: $|z| = 2$

Solution: Given $\int_c \frac{e^{-2z}}{(z+1)^3}dz$

$f(z)=e^{-2z}$

The singular point z=-1 lies inside the given circle $|z|=2$

Apply Cauchy's integral formula for derivatives

$$\int_c \frac{e^{-2z}}{(z+1)^3}dz = \frac{2\pi i f^1(-1)}{2!} \qquad \left[\because \int_c \frac{f(z)}{(z-a)^3} = \frac{2\pi i f^1(a)}{2!}\right]$$

Where $f(z)=e^{-2z}$

$f^1(z)=-2e^{-2z}$

$f^{11}(z)=4e^{-2z}$

$f^{11}(-1)=4e^2$

$$\therefore \int_c \frac{e^{-2z}}{(z+1)^3}dz = \frac{2\pi i 4e^2}{2} = 4\pi i e^2$$

Problem: Evaluate $\int_c \frac{dz}{z^8(z+4)}dz$ withc: $|z|=2$

Solution: The singularities of $\int_c \frac{dz}{z^8(z+4)}dz$ are given by

$$Z^8(z+4)=0 \Rightarrow z=0, z=-4$$

The point z=0 lie inside and the z=-4 lies outside the circle $|z|=2$

By the derivative of analytic function.

Problem: Evaluate using integral formula $\oint_c \frac{e^{2z}dz}{(z-1)(z-2)}$ where c is the circle $|z|=3$

Solution: Let (z)= e^z which is analytic within the circle c: $|z|=3$ and the two singular points a=1,a=2 lie inside c.

$$\oint_c \frac{e^{2z}dz}{(z-1)(z-2)} = \frac{\oint e^{2z}\left(\frac{1}{z-2}-\frac{1}{z-1}\right)dz}{\oint_c \frac{e^{2z}}{z-2}dz - \oint_c \frac{e^{2z}}{z-1}dz}$$

Now using cauchy's integral formula ,we obtain

$$\oint_c \frac{e^{2z}dz}{(z-1)(z-2)} = 2\pi i e^4 - 2\pi i e^2$$

$$= 2\pi i(e^4 - e^2)$$

$$\oint_c \frac{e^{2z}dz}{(z-1)(z-2)} = 2\pi i(e^4 - e^2)$$

Problem: Evaluate $\oint_c \frac{3z^2+z}{z^2-1}dz$ where cisthe circle $|z-1| = 1$

Solution: Given f(z)= $3z^2+z$

Z=a=+1or -1

The circle $|z-1| = 1$ has centre at z=1 and radius 1 and includes the point z=1,f(z)=$3z^2+z$ is an analytic function

Also $\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$

$$\oint \frac{3z^2+z}{z^2-1} = \frac{1}{2}\left[\int_c \frac{3z^2+z}{z-1}dz\right] - \frac{1}{2}\left[\int_c \frac{3z^2+z}{z+1}dz\right] \text{---------------- (1)}$$

Since z=1 lies inside c, we have by cauchy's integralformula

$$\oint_c \frac{3z^2+z}{z^2-1}dz = 2\pi i f(i)$$

$$= 2\pi i * 4$$

Bycauchy'sintegral theorem, since z=-1 lies out side c, we have

$$\oint_c \frac{3z^2+z}{z-1}dz = 0$$

From equation(1) we have

$$\oint_c \frac{3z^2+z}{z^2-1}dz = \frac{1}{2}(8\pi i)-0$$

$$= 4\pi i$$

EXCERCISE PROBLEMS:

1) Evaluate $\int \frac{dz}{z-z_0}$ where c: $|z-z_0|$ = r

2) Evaluate $\int_{(1,1)}^{(2,2)} (x+y)dx+(y-x)dy$ along the parabola $y^2 = x$

3) Evaluate $\int_c \frac{z^2+4}{z^2-1}dz$ where C: $|z|=2$ using Cauchy's Integral formula

4) Evaluate $\int_c \frac{e^{2z}}{(z-1)(z-2)}dz$ where C: $|z|=4$ using Cauchy's integral formula

5) Evaluate $\int_c \frac{z^3-z}{(z-2)^3}$ where $C:|z|=3$ using Cauchy's integral formula

6) Expand f(z) = $\int_c \frac{e^{2z}}{(z-1)^3}$ at a point z=1

7) Expand f(z)= $\int_c \frac{1}{z^2-4z+3}$ for $1<|z|<3$

8) Evaluate $\int (y^2+z^2)dx+(z^2+x^2)dy+(x^2+y^2)dz$ from (0,0,0) to (1,1,1) , where

C is the curve $x=t, x=t^2, x=t^3$

9) Evaluate $\int_{(0,0)}^{(1,1)} (3x^2+4xy+ix^2)dz$ along $y=x^2$

10) Evaluate $\int_0^{1+i} (x-y+ix^2)$ dz

(i) Along the straight from z = 0 to z = 1+i .

(ii) along the real axis from z = 0 to z = 1 and then along a line parallel to real axis from z = 1 to z = 1+i

(iii) Along the imaginary axis from z = 0 to z = I and then along a line parallel to real axis z = i to z = 1+ i .

11) Evaluate $\int_{1-i}^{2+i}(2x+1+iy)$ dz along (1-i) to (2+i)

12) Evaluate $\int_{c}(y^2+2xy)dx+(x^2-2xy)dy$ where c is boundary of the region y=x ^{2} and x=y ^{2}

MODULE-III

POWER SERIES EXPANSION OF COMPLEX FUNCTION

Power series:

A series expansion is a representation of a particular function as a sum of powers in one of its variables, or by a sum of powers of another (usually elementary) function f(z).

A power series in a variable z is an infinite sum of the form

$$\sum a_i z^i$$

A series of the form $\sum a_n z^n$ is called as power series.

That is $\sum a_n z^n = a_1 z + a_2 z^2 + \ldots\ldots\ldots + a_n z^n + \ldots\ldots$

Taylor's series:

Taylor's theorem states that any function satisfying certain conditions may be represented by a Taylor series.

The Taylor series is an infinite series, whereas a Taylor polynomial is a polynomial of degree n and has a finite number of terms. The form of a Taylor polynomial of degree n for a function

f (z) at x = a is

$$f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{(z-a)^2}{2!} + f'''(a)\frac{(z-a)^3}{3!} + \ldots\ldots\ldots + f^n(a)\frac{(z-a)^n}{n!} + \ldots\ldots,$$
$$|z-a| < r$$

Maclaurin series:

A Maclaurin series is a Taylor series expansion of a function about x=0,

$$f(z) = f(0) + f'(0)(z) + f''(0)\frac{(z)^2}{2!} + f'''(0)\frac{(z)^3}{3!} + \ldots\ldots\ldots + f^n(0)\frac{(z)^n}{n!} + \ldots\ldots$$

This series is called as maclurins series expansion of f(z).

Some important result:

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots \quad \text{for } -1 < x \le 1$$
$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \ldots \quad \text{for } -\infty < x < \infty$$
$$\cos x = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \tfrac{1}{720}x^6 + \ldots \quad \text{for } -\infty < x < \infty$$
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots \quad \text{for } -1 < x < 1$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Problems

Problem: Determine the first four terms of the power series for sin 2x using Maclaurin's series.
Solution:

Let

f(x) = sin 2x f(0) = sin 0 = 0
f′(x)= 2 cos 2x f′(0) = 2 cos 0 = 2
f″(x)= –4 sin 2x f″(0) = –4 sin 0 = 0
f‴(x) = –8 cos 2x f‴(0) = –8 cos 0 = –8
f^{iv}(x)= 16 sin 2x f^{iv}(0)= 16 sin 0 = 0
f^{v} (x)= 32 cos 2x(0) f^{v} (0)= 32 cos 0 = 32
f^{vi} (x)= –64 sin 2x f^{vi} (0)= –64 sin 0 = 0
$f^{vi\,i}$(x)= –128 cos 2x f^{vii} (0)= –128 cos 0 = –128

$$f(x) = \sin 2x = 0+2x+0x^2+(-8)\frac{x^3}{3!}+0.x^4+32\frac{x^5}{5!}$$

$$= 2x - \frac{4x^3}{3} + \frac{4x^5}{15}$$

Problem : Find the Taylor series about z = -1 for f (x) = 1/z. Express your answer in sigma notation.
Solution:

Let f (z) = z^{-1} f(-1) = -1
f ' = - z^{-2} f '(-1) = -1
f " = $2z^{-3}$ f " (-1) = -2
f "' = $-6z^{-4}$ f "' (-1) = -6
f "" = $24z^{-5}$ f "" (-1) = -24

$$f(z) = -1-1(z+1) - \frac{2}{2!}(z+1)^2 - \frac{6}{3!}(z+1)^3 - \frac{24}{4!}(z+1)^4 - \ldots\ldots\ldots$$

$$= \sum_{n=0}^{\infty} -1(z+1)^n$$

Problem : Find the Maclaurin series for f (z) = z e^z Express your answer in sigma notation.
Solution:

Let f (z) = z e^z f (0) = 0
f ' = e^z+ z e^z f '(0) = 1 + 0 = 1
f " = e^z+ e^z+ z e^z f "(0) = 1 + 1 + 0 = 2
f "' = e^z+ e^z+ e^z+ z e^z f "'(0) = 1 + 1 + 1 + 0 = 3
f "" = e^z+ e^z+ e^z+ e^z+ z e^z f ""(0) = 1 + 1 + 1 + 1 + 0 = 4

$$f(z) = 0 + 1z + \frac{2}{2!}z^2 + \frac{3}{3!}z^3 + \frac{4}{4!}z^4 + \ldots\ldots\ldots..$$

$$= z + z^2 + \frac{1}{2}z^3 + \frac{1}{6}z^4 + \ldots\ldots\ldots..$$

$$= \sum_{n=1}^{\infty} \frac{z^n}{(n-1)!}$$

Problem: Expand log z by taylor's series about z=1.

Solution:

Let f(z) = log z

Put z-1= w

z= 1+w

log z = log (1+w)

f(z)=log z = log (1+w)

$$= w-\frac{w^2}{2}+\frac{w^3}{3}-........+(-1)^n\frac{w^n}{n!}+........;|w|<1$$

$$f(z) = (z-1)-\frac{(z-1)^2}{2}+\frac{(z-1)^3}{3}-........+(-1)^n\frac{(z-1)^n}{n!}+........;|z-1|<1$$

Laurent series:

In mathematics, the **Laurent series** of a complex function f(*z*) is a representation of that function as a power series which includes terms of negative degree. It may be used to express complex functions in cases where a Taylor series expansion cannot be applied.

The Laurent series for a complex function *f*(*z*) about a point *c* is given by:

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-a)^n$$

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}b_n\frac{1}{(z-a)^n}$$

where the a_n *and a* are constants.

Laurent polynomials:

A Laurent polynomial is a Laurent series in which only finitely many coefficients are non-zero. Laurent polynomials differ from ordinary polynomials in that they may have terms of negative degree.

Principal part:

The principal part of a Laurent series is the series of terms with negative degree, that is

$$f(z)=\sum_{K=-\infty}^{-1}a_K(z-a)^K$$

If the principal part of *f* is a finite sum, then *f* has a pole at *c* of order equal to (negative) the degree of the highest term; on the other hand, if *f* has an essential singularity at *c*, the principal part is an infinite sum (meaning it has infinitely many non-zero terms).

Two Laurent series with only *finitely* many negative terms can be multiplied: algebraically, the sums are all finite; geometrically, these have poles at *c*, and inner radius of convergence 0, so they both converge on an overlapping annulus.

Thus when defining formal Laurent series, one requires Laurent series with only finitely many negative terms.

Similarly, the sum of two convergent Laurent series need not converge, though it is always defined formally, but the sum of two bounded below Laurent series (or any Laurent series on a punctured disk) has a non-empty annulus of convergence.

Zero's of an analytic function:

A zero of an analytic function f(z) is a value of z such that f(z)=0 .Particularly a point a is called a zero of an analytic function f (z) if f(a) = 0.

Eg: $f(z) = \frac{(z+1)^2}{(z^2+1)^2}$

Now, $(z+1)^2 = 0$

Z = -1, z = -1 are zero's of an analytic function.

Zero's of m^th^ order:

If an analytic function f(z) can be expressed in the form $f(z) = (z-a)^m \Phi(z)$ where $\Phi(z)$ is analytic function and $\Phi(a) \neq 0$ then z=a is called zero of m^{th} order of the function f(z).

- A simple zero is a zero of order 1.

Eg: 1. $f(z) = (z-1)^3$

$\Rightarrow (z-1)^3 = 0$

z=1 is a zero of order 3 of the function f(z).

2. $f(z) = \frac{1}{1-z}$

i.e $z = \infty$ is a simple zero of f(z).

3. $f(z) = \sin z$

i.e $z = n\pi \quad \forall n = 0,1,2,3,\ldots\ldots$ are simple zero's of f(z).

Problems

Problem: Find the first four terms of the Taylor's series expansion of the complex function

$f(z) = \frac{z+1}{(z-3)(z-4)}$ About z =2.Find the region of convergence.

Solution:

The singularities of the function $f(z) = \frac{z+1}{(z-3)(z-4)}$ are z = 3 and z = 4

Draw a circle with centre at z=2 and radius 1 .Then the distance of singularities from the centre are 1 and 2.

Hence within the circle $|z-2| = 1$, the given function is analytic .Hence ,it can be extended in Taylor's series within the circle $|z-2| = 1$.

Hence $|z-2| = 1$ is the circle of convergence.

Now $f(z) = \frac{5}{z-4} - \frac{4}{z-3}$ (partial fraction) , f(2)= 3/2

$f'(z) = -\frac{5}{(z-4)^2} + \frac{4}{(z-3)^2}$, $f'(2) = \frac{11}{4}$

$$f''(z) = -\frac{8}{(z-3)^3} + \frac{10}{(z-4)^3} \quad , \quad f''(2) = \frac{27}{4}$$

$$f'''(z) = \frac{24}{(z-3)^4} - \frac{30}{(z-4)^4} \quad , \quad f'''(2) = \frac{177}{8}$$

Taylor's series expansion for f(z) at z=a is

$$f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{(z-a)^2}{2!} + f'''(a)\frac{(z-a)^3}{3!} + \ldots\ldots\ldots + f^n(a)\frac{(z-a)^n}{n!} + \ldots\ldots$$

$$\frac{z+1}{(z-3)(z-4)} = \frac{3}{2} + (z-2)\frac{11}{4} + \frac{(z-2)^2}{2!}\left(\frac{27}{4}\right) + \frac{(z-2)^3}{3!}\left(\frac{177}{8}\right)$$

$$f(z) = \frac{3}{2} + (z-2)\frac{11}{4} + (z-2)^2\left(\frac{27}{8}\right) + (z-2)^3\left(\frac{59}{16}\right).$$

Problem: Obtain Laurent series for $f(z) = \frac{e^{2z}}{(z-1)^3}$ about z = 1.

Solution:

Given $f(z) = \frac{e^{2z}}{(z-1)^3}$

Put z-1= w so that z = w+1

$$f(z) = \frac{e^{2(1+w)}}{w^3}$$

$$f(z) = \frac{e^2 e^{2w}}{w^3}$$

$$= \frac{e^2}{w^3}\left[1 + 2w + \frac{(2w)^2}{2!} + \frac{(2w)^3}{3!} + \ldots\ldots\right] \text{ if } w \neq 0$$

$$= e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!} w^{n-3}$$

$$= e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!} (z-1)^{n-3} \quad , \text{ if } z-1 \neq 0$$

$$= e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!} (z-1)^{n-3} \quad , \text{ if } |z-1| \neq 0$$

$$f(z) = e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!} (z-1)^{n-3} \quad , \text{ if } |z-1| > 0$$

Since points $|z-1| \le 0$ will be singular points.

Singular point of an analytic function: A point at which an analytic function f(z) is not analytic, i.e. at which f '(z) fails to exist, is called a **singular point** or **singularity** of the function.

There are different types of singular points:

Isolated and non-isolated singular points: A singular point z_0 is called an **isolated singular point** of an analytic function f(z) if there exists a deleted ε-spherical neighborhood of z_0 that contains no singularity. If no such neighborhood can be found, z_0 is called a non-isolated singular point. Thus an isolated singular point is a singular point that stands completely by itself, embedded in regular points. In fig 1a where z_1, z_2 and z_3 are isolated singular points. Most singular points are isolated singular points. A non-isolated singular point is a singular point such that every deleted ε-spherical neighborhood of it contains singular points. See Fig. 1b where z_0 is the limit point of a set of singular points. Isolated singular points include poles, removable singularities, essential singularities and branch points.

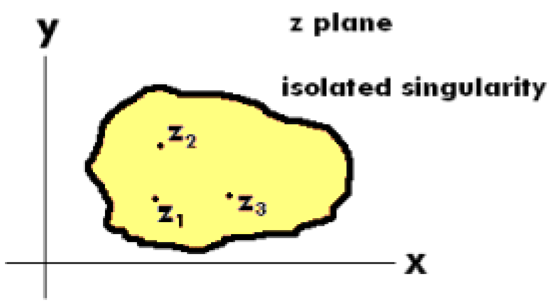


Fig. 1a

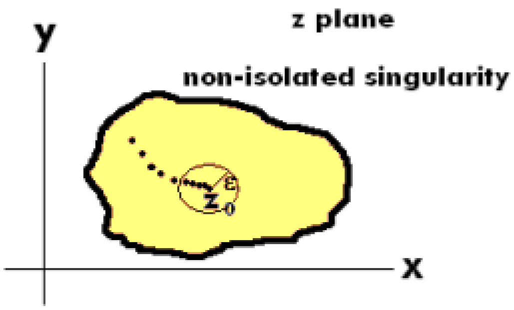


Fig. 1b

Types of isolated singular points:

Pole: An isolated singular point z_0 such that f(z) can be represented by an expression that is of the form

$$f(z) = \frac{\phi(z)}{(z-z_0)^n}$$

Where n is a positive integer, □(z) is analytic at z_0, and □(z_0) ≠ 0. The integer n is called the **order** of the pole. If n = 1, z_0 is called a simple pole.

Example: 1. The function

$$f(z) = \frac{5z+1}{(z-2)^3(z+3)(z-2)}$$

has a pole of order 3 at z = 2 and simple poles at z = -3 and z = 2.

1. A point z is a pole for f if f blows up at z (f goes to infinity as you approach z). An example of a pole is z=0 for f(z) = 1/z.

Simple pole: A pole of order 1 is called a simple pole whilst a pole of order 2 is called a double pole.

If the principal part of the Laurent series has an infinite number of terms then z = z_0 is called an isolated essential singularity of f(z). The function f(z) = i/ z(z − i) ≡ 1/(z – i) – (1/ z) has a simple pole at z = 0 and another simple pole at z = i.

The function $e^{\frac{1}{z-2}}$ has an isolated essential singularity at z = 2. Some complex functions have non-isolated singularities called branch points. An example of such a function is √z.

Removable singular point: An isolated singular point z_0 such that f can be defined, or redefined, at z_0 in such a way as to be analytic at z_0. A singular point z_0 is removable if

$\lim_{z \to z_0} f(z)$ Exist.

Example: 1.The singular point z = 0 is a removable singularity of f(z) = (sin z)/z since $\lim_{z \to 0} \frac{\sin z}{z} = 1$

A point z is a removable singularity for f if f is defined in a neighborhood of the point z, but not at z, but f can be defined at z so that f is a continuous function which includes z. Here is an example of this: if f(z) = z is defined in the punctured disk, the disk minus 0, then f is not defined at z=0, but it can certainly be extended continuously to 0 by defining f(0) = 0. This means at z=0 is a removable singularity.

Essential singular point: A singular point that is not a pole or removable singularity is called an essential singular point.

Example: 1. f(z) = $e^{1/(z-3)}$ has an essential singularity at z = 3.

2. A point z is an essential singularity if the limit as f approaches z takes on different values as you approach z from different directions. An example of this is exp(1/z) at z=0. As z approaches 0 from the right, exp(1/z) blows up and as z approaches 0 from the left, exp(1/z) goes to 0.

Singular points at infinity: The type of singularity of f(z) at $z = \infty$ is the same as that of f(1/w) at w = 0. Consult the following example.

Example: The function $f(z) = z^2$ has a pole of order 2 at $z = \infty$, since f(1/w) has a pole of order 2 at w = 0.

Using the transformation w = 1/z the point z = 0 (i.e. the origin) is mapped into $w = \infty$, called the point at infinity in the w plane. Similarly, we call $z = \infty$ the point at infinity in the z plane. To consider the behavior of f(z) at $z = \infty$, we let z = 1/w and examine the behavior of f(1/w) at w = 0.

Residues:

The constant a_{-1} in the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \qquad (1)$$

of about a point z_0 is called the residue of f(z). If f is analytic at z_0, its residue is zero, but the converse is not always true (for example, $\frac{1}{z^2}$ has residue of 0 at z=0 but is not analytic at z=0 . The residue of a function f at a point z_0 may be denoted $\operatorname*{Res}_{Z \to Z_0} f(z)$.

Residue: Let f(z) have a nonremovable isolated singularity at the point z_0. Then f(z) has the Laurent series representation for all z in some disk $D_R^*(z_0)$ given by

$$. \quad f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \qquad (1)$$

The coefficient a_{-1} of $\frac{1}{z - z_0}$ is called the residue of f(z) at z_0 and we use the notation

$\text{Res}[f, z_0] = a_{-1}$

Example: If $f(z) = e^{\frac{2}{z}}$, then the Laurent series of f about the point $z_0 = 0$ has the form

$$f(z) = 1 + 2\frac{1}{z} + \frac{2^2}{2!\,z^2} + \frac{2^3}{3!\,z^3} + \frac{2^4}{4!\,z^4} + \frac{2^5}{5!\,z^5} + \cdots$$, and

$\text{Res}[f, 0] = a_{-1} = 2$

The residue of a function f around a point z_0 is also defined by

$$\operatorname{Re} sf = \frac{1}{2\pi i}\int_c f(z)dz \tag{2}$$

Where C is counterclockwise simple closed contour, small enough to avoid any other poles of f. In fact, any counterclockwise path with contour winding number 1 which does not contain any other poles gives the same result by the Cauchy integral formula. The above diagram shows a suitable contour for which to define the residue of function, where the poles are indicated as black dots.

It is more natural to consider the residue of a meromorphic one-form because it is independent of the choice of coordinate. On a Riemann surface, the residue is defined for a meromorphic one-form α at a point p by writing $\alpha = f\,d\,z$ in a coordinate z around p. Then

$$\operatorname*{Res}_{p} \alpha = \operatorname*{Res}_{z=p} f. \tag{3}$$

The sum of the residues of $\int f\,d\,z$ is zero on the Riemann sphere. More generally, the sum of the residues of a meromorphic one-form on a compact Riemann surface must be zero.

The residues of a function $f(z)$ may be found without explicitly expanding into a Laurent series as follows. If $f(z)$ has a pole of order m at z_0, then $a_n = 0$ for $n < -m$ and $a_{-m} \neq 0$. Therefore,

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} a_{-m+n} (z - z_0)^{-m+n} \tag{4}$$

$$(z - z_0)^m f(z) = \sum_{n=0}^{\infty} a_{-m+n} (z - z_0)^n \tag{5}$$

$$\frac{d}{d\,z}[(z - z_0)^m f(z)] = \sum_{n=0}^{\infty} n\, a_{-m+n} (z - z_0)^{n-1} \tag{6}$$

$$= \sum_{n=1}^{\infty} n\, a_{-m+n} (z - z_0)^{n-1} \tag{7}$$

$$= \sum_{n=0}^{\infty} (n+1)\, a_{-m+n+1} (z - z_0)^n \tag{8}$$

$$\frac{d^2}{d\,z^2}[(z - z_0)^m f(z)] = \sum_{n=0}^{\infty} n\,(n+1)\, a_{-m+n+1} (z - z_0)^{n-1} \tag{9}$$

$$= \sum_{n=1}^{\infty} n\,(n+1)\, a_{-m+n+1} (z - z_0)^{n-1} \tag{10}$$

$$= \sum_{n=0}^{\infty} (n+1)(n+2)\, a_{-m+n+2}\, (z-z_0)^n . \tag{11}$$

Iterating,

$$\frac{d^{m-1}}{d\,z^{m-1}} [(z-z_0)^m f(z)] = \sum_{n=0}^{\infty} (n+1)(n+2)\cdots(n+m-1)\, a_{n-1}\, (z-z_0)^n$$
$$= (m-1)!\, a_{-1} + \sum_{n=1}^{\infty} (n+1)(n+2)\cdots(n+m-1)\, a_{n-1}\, (z-z_0)^{n-1} . \tag{12}$$

So

$$\lim_{z\to z_0} \frac{d^{m-1}}{d\,z^{m-1}} [(z-z_0)^m f(z)] = \lim_{z\to z_0} (m-1)!\, a_{-1} + 0 \tag{13}$$

$$= (m-1)!\, a_{-1} , \tag{14}$$

And the residue is

$$a_{-1} = \frac{1}{(m-1)!} \frac{d^{m-1}}{d\,z^{m-1}} [(z-z_0)^m f(z)]_{z=z_0} . \tag{15}$$

The residues of a holomorphic function at its poles characterize a great deal of the structure of a function, appearing for example in the amazing residue theorem of contour integration.

If f(z) has a removable singularity at z_0 then $a_{-1} = 0$ for n=1,2,……. Therefore, $\mathrm{Res}[f, z_0]=0$.

Residues at Poles:

(i) If f(z) has a simple pole at z_0, then $\mathrm{Res}[f, z_0] = \lim_{Z\to Z_0} (z-z_0) f(z)$

(ii) If f(z) has a pole of order 2 at z_0, then $\mathrm{Res}[f, z_0] = \lim_{Z\to Z_0} \frac{d}{dz} (z-z_0)^2 f(z)$

(iii) If f(z) has a pole of order 3 at z_0 , then $\operatorname{Res}[f, z_0] = \frac{1}{2!} \lim_{Z \to Z_0} \frac{d^2}{dz^2}((z-z_0)^3 f(z))$

(v) If f(z) has a pole of order k at z_0 z_0 , then $\operatorname{Res}[f, z_0] = \frac{1}{(k-1)!} \lim_{Z \to Z_0} \frac{d^{k-1}}{dz^{k-1}}((z-z_0)^k f(z))$

Cauchy's Residue Theorem:

An analytic function f(z) whose Laurent series is given by $f(z) = \lim_{Z \to Z_0} (z - z_0) f(z)$ (1)

Can be integrated term by term using a closed contour C encircling z_0,

$$\int_c f(z)dz = \sum_{n=-\infty}^{\infty} a_n \int_c (z-z_0)^n dz$$
$$= \sum_{n=-\infty}^{-2} a_n \int_c (z-z_0)^n dz + a_{-1} \int_c \frac{dz}{(z-z_0)} + \sum_{n=0}^{\infty} a_n \int_c (z-z_0)^n dz \quad (2)$$

The Cauchy integral theorem requires that the first and last terms vanish, so we have

$$\int_c f(z)dz = a_{-1} \int_c \frac{dz}{z-z_0} \quad (3)$$

Where a_{-1} is the complex residue. Using the contour $z=c(t)=e^{it}+z_0$ gives

$$\int_c \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{ie^{it}}{e^{it}} dt = 2\pi i \quad (4)$$

so we have

$$\int_c f(z)dz = a_{-1} 2\pi i \quad (5)$$

If the contour C encloses multiple poles, then the theorem gives the general result

$$\int_c f(z)dz = 2\pi i \sum_{a \in A} \operatorname{Res}_{z=a_i} f(z) \quad (6)$$

Where A is the set of poles contained inside the contour. This amazing theorem therefore says that the value of a contour integral for *any* contour in the complex plane depends *only* on the properties of a few very special points *inside* the contour.

Residue at infinity:

The residue at infinity is given by:

$$\text{Re}\,s[f(z)]_{Z=\infty} = -\frac{1}{2\pi i}\int_C f(z)dz$$

Where f is an analytic function except at finite number of singular points and C is a closed countour so all singular points lie inside it.

Problem: Determine the poles of the function $f(z) = \frac{z+2}{(z+1)^2(z-2)}$ and the residue at each pole.

Solution: The poles of f(z) are given by $(z+1)^2(z-2)=0$

Here z=2 is a simple pole and z= -1 is a pole of order 2 .

Residue at z=2 is

$$\lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}(z-2)\frac{z+2}{(z+1)^2(z-2)} = \frac{4}{9}$$

Residue at z=-1 is

$$\lim_{z\to -1}\frac{d}{dz}(z+1)^2 f(z) = \lim_{z\to -1}\frac{d}{dz}(z+1)^2\frac{z+2}{(z+1)^2(z-2)}$$

$$\lim_{z\to -1}\frac{d}{dz}\frac{(z+2)}{(z-2)} = \lim_{z\to -1}\frac{-4}{(z-2)^2} = \frac{-4}{9}$$

Problem: Find the residue of the function $f(z) = \frac{1-e^{2z}}{z^4}$ at the poles.

Solution: Let $f(z) = \frac{1-e^{2z}}{z^4}$

z =0 is a pole of order 4

Residue of f(z) at z=0 is

$$= \frac{1}{3!}\lim_{z\to 0}\frac{d^3}{dz^3}(z-0)^4\frac{(1-e^{2z})}{z^4}$$

$$= \frac{1}{3!}\lim_{z\to 0}\frac{d^3}{dz^3}(1-e^{2z})$$

$$= \frac{1}{3!}\lim_{z\to 0}\frac{d^2}{dz^2}(-2e^{2z})$$

$$=\frac{1}{3!}\lim_{z\to 0}\frac{d}{dz}(-4e^{2z})$$

$$=\frac{1}{3!}\lim_{z\to 0}(-8e^{2z})$$

$$=\frac{-8}{3!}=\frac{-4}{3}.$$

Problem: Find the residue of the function $f(z)=z^3\cos(\frac{1}{z})$ at $z=\infty$.

Solution: Let $f(z)=z^3\cos(\frac{1}{z})$

$$g(t)=f(\frac{1}{t})=\frac{1}{t^3}\cos t$$

$$=\frac{1}{t^3}\left[1-\frac{t^2}{2!}+\frac{t^4}{4!}-.........\right]$$

$$=\left[\frac{1}{t^3}-\frac{1}{2t}+\frac{t}{24}-.........\right]$$

There fore $\operatorname{Res}_{z\to\infty} f(z)$ = - coefficient ot t in the eapansion of g(t) about t=0

= -1/24.

Problem: Evaluate $\int_c \frac{4-3z}{z(z-1)(z-2)}dz$ where c is the circle $|z|=\frac{3}{2}$. Using Residue theorem.

Solution: Let $\int_c \frac{4-3z}{z(z-1)(z-2)}dz$

The poles of f(z) are z (z-1)(z-2)=0

z=0, z= 1, z=2

These poles are simple poles.

The poles z=0 and z=1 lie within the circle c: $|z|=\frac{3}{2}$

Residue of f(z) at z=0 is $R_1=\lim_{z\to 0}(z-0)f(z)=\lim_{z\to 0}z\frac{4-3z}{z(z-1)(z-2)}=\frac{4}{2}=2$

Residue of f(z) at z=1 is $R_2=\lim_{z\to 1}(z-1)f(z)=\lim_{z\to 1}(z-1)\frac{4-3z}{z(z-1)(z-2)}=\frac{4-3}{1-2}=-1$

By Residue theorem, $\int_c \frac{4-3z}{z(z-1)(z-2)}dz=2\pi i(R_1+R_2)=2\pi i(2-1)=2\pi i$.

Problem: Evaluate $\int_0^{2\pi}\frac{d\theta}{5+4\sin\theta}$ Using Residue theorem.

Solution: **Let** $I = \int_{-\pi}^{\pi} \frac{d\theta}{5+4\sin\theta}$

Put $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$ and $\sin\theta = \frac{1}{2i}(z - \frac{1}{z})$

$z = e^{i\theta}$ Unit circle c: $|z| = 1$

$$I = \int_c \frac{1}{5+4\frac{1}{2i}(z-\frac{1}{z})} \frac{dz}{iz}$$

$$= \int_c \frac{dz}{z^2+5iz+2i^2} = \int_c \frac{dz}{(2z+i)(z+2i)}$$

$$= \int_c \frac{dz}{(z+\frac{i}{2})(z+2i)}$$

$$= \frac{1}{2}\int_c f(z)dz$$

Where $f(z) = \left(z+\frac{i}{2}\right)(z+2i)$

The poles of f(z) are z=-i/2 and z=-2i

The pole z=-i/2 lies inside the unit circle.

Residue of f(z) at z=-i/2 is

$$= \lim_{z\to -i/2}(z+\frac{i}{2})f(z)$$

$$= \lim_{z\to -i/2}(z+\frac{i}{2})\frac{1}{(z+\frac{i}{2})(z+2i)}$$

$$= \lim_{z\to -i/2}\frac{1}{(z+2i)}$$

$$= \frac{1}{\frac{-i}{2}+2i} = \frac{2}{3i}$$

By cauchy's residue theorem

$$I = \frac{1}{2}\int_c f(z)dz = \frac{1}{2}2\pi i\left(\frac{2}{3i}\right) = \frac{2\pi}{3}$$

$$\int_0^{2\pi}\frac{d\theta}{5+4\sin\theta} = \frac{2\pi}{3}.$$

Problem: Prove that $\int_{-\infty}^{\infty}\frac{x^2dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b}$ $\quad (a>0, b>0, a\neq b)$

Solution: To evaluate the given integral, consider $\int_c \frac{z^2 dz}{(z^2+a^2)(z^2+b^2)} = \int_c f(z)dz$

Where c is the contour consisting of the semi circle C_R of radius R together with the real part of the real axis from –R to R.

The poles of $f(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)}$ are $z = \pm ai \quad ; z = \pm bi$

But z=ia and z=ib are the only two poles lie in the upper half of the plane .

$[\operatorname{Res} f(z)]_{z=ia} = \underset{z \to ai}{Lt}(z-ia) f(z)$

$$= \underset{z \to ai}{Lt} \frac{z^2}{(z+ia)(z^2+b^2)} = \frac{-a^2}{2ia(-a^2+b^2)} = \frac{a}{2i(a^2-b^2)}$$

Also $[\operatorname{Res} f(z)]_{z=ib} = \underset{z \to bi}{Lt}(z-ib) f(z)$

$$= \underset{z \to bi}{Lt} \frac{z^2}{(z+ib)(z^2+a^2)} = \frac{-b^2}{2ib(a^2-b^2)} = \frac{-b}{2i(a^2-b^2)}$$

By Cauchy’s Residue theorem, we have $\int_c f(z)dz = 2\pi i$ (sum of the residues with in C)

$$\int_c f(z)dz = 2\pi i \left[\frac{a}{2i(a^2-b^2)} - \frac{b}{2i(a^2-b^2)} \right] = \pi \left[\frac{a-b}{(a^2-b^2)} \right] = \frac{\pi}{a+b}$$

We have $$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = \frac{\pi}{a+b}$$

But $\int_{C_R} f(z)dz \to 0 \quad as \quad z = \operatorname{Re}^{i\theta} \; and \quad R \to \infty$

Hence $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b}$

EXCERCISE PROBLEMS:

1) Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$ where C: $|z|=1$

2) Prove that $\int_{-\infty}^{\infty} \frac{dx}{a^2+x^2} = \frac{\pi}{a}$

3) Show that $\int_{-\infty}^{\infty} \frac{\mathrm{dx}}{(\mathrm{x}+1)^3} = \frac{3\pi}{8}$

4) Prove that $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{ab(a+b)}$

5) Evaluate $\int (y^2+z^2)dx+(z^2+x^2)dy+(x^2+y^2)dz$ from (0,0,0) to (1,1,1) , where C is the curve $x=t, x=t^2, x=t^3$

6) Evaluate $\int_{(0,0)}^{(1,1)} (3x^2+4xy+ix^2)dz$ along $y=x^2$

7) Obtain the Taylor series expansion of f(z) = $\frac{1}{z}$ about the point z = 1

8) Obtain the Taylor series expansion of f(z) = e^z about the point z = 1

9) Expand f(z) = $\frac{z-1}{z+1}$ in Taylor's series about the point (i) z = 0 (ii) z = 1

10) Expand f(z) = $\frac{1}{z^2}$ in Taylor's series in powers of z +1

11) Obtain Laurent's series expansion of f(z) = $\frac{z^2-4}{z^2+5z+4}$ valid in 1< z < 2

12) Give two Laurent's series expansions in powers of Z for f(z) = $\frac{1}{z^2(1-z)}$

13) Expand f(z)= $\frac{1}{(1-z)(z-2)}$

14) Maclaurin's series expansion of f(z)

15) Laurent's series expansion in the annulus region in

a) $1<|z|<2$

16) Find the residue of the function $f(z)=\frac{z^3}{(z^2-1)} at\ z=\infty$

17) Find the residue of $\frac{z^2}{z^4+1}$ at these singular points which lie inside the circle $|z|=2$

18) Find the residue of the function $f(z)=\frac{z^2-2z}{(z^2+1)(z+1)^2}$ at each pole

MODULE-IV

SPECIAL FUNCTIONS-I

- **Improper Integrals: Beta and Gamma functions**
- **Definitions**
- **Properties of Beta and Gamma functions**
- **Standard forms of Beta functions**
- **Relationship between Beta and Gamma function**

DEFINITION:

IMPROPER INTEGRAL:

The integral $\int_a^b f(x)\,dx$ for which

i) Either the interval of integration is not finite i. e. $a=-\infty\ or b=\infty$ or both

ii) The function f(x) is unbounded at one or more point in [a, b] is called das improper integral.

NOTE: Integral of (i) and (ii) are called the improper integrals of first and second kinds respectively**.**

Examples:

1. $\int_0^\infty \frac{1}{1+x^4}\,dx$ And $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ are improper integrals of the first kind.
2. $\int_0^1 \frac{1}{1-x^2}\,dx$ is an improper integral of the second kind.

DEFINITION:

BETA FUNCTION:

The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is called the Beta function and is denoted by $\beta(m,n)$. The integral converges for m>0,n>0.

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

NOTE:

Beta function is also called as Eulerian integral of first kind

.

PROPERTIES OF BETA FUNCTION:

i) SYMMETRY PROPERTY OF BETA FUNCTION

i.e., $\beta(m,n)=\beta(n,m)$

Proof:

By definition, we have

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\ dx$$

Put 1-x=y so that dx=-dy

$$\therefore \beta(m,n)=\int_1^0 (1-y)^{m-1} y^{n-1}\ (-dy)$$

$$=\int_0^1 y^{n-1}(1-y)^{m-1}\ dy$$

$$=\int_0^1 x^{n-1}(1-x)^{m-1}\ dx$$

$$\beta(n,m)\left[\because \int_a^b f(t)\,dt=\int_a^b f(x)\,dx\right]$$

Hence $\beta(m,n)=\beta(n,m)$

ii) **Prove that**

$$\beta(m,n)=\int_1^0 \sin^{2m-1}\theta \cos^{2n-1}\theta\ d\theta$$

Proof:

By definition, we have

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\ dx$$

put $x=\sin^2\theta$ so that $dx=\sin 2\theta\ d\theta$

$$\therefore \beta(m,n)=\int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\ \sin 2\theta\ d\theta$$

$$=2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\ d\theta$$

Hence proved

ii) $\beta(m,n)=\beta(m+1,n)+\beta(m,n+1)$

Proof:

By definition, we have

$$\beta(m+1,n)+\beta(m,n+1)=\int_0^1 x^m(1-x)^{n-1}\ dx+\int_0^1 x^{m-1}(1-x)^n\ dx$$

$$=\int_0^1 x^m(1-x)^{n-1}\ dx+\int_0^1 x^{m-1}(1-x)^n\ dx$$

$$=\int_0^1 x^{m-1}(1-x)^{n-1}[x+(1-x)]\,dx$$

$$=\int_0^1 x^{m-1}(1-x)^{n-1}\ dx$$

$$=\beta(m,n)$$

Hence $\beta(m,n)=\beta(m+1,n)+\beta(m,n+1)$

Iv)

If m and n are positive integers, then $\beta(m,n)=\dfrac{(m-1)!(n-1)!}{(m+n-1)!}$

Proof:

We have $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\ dx$

Integrating by parts

$$\left[x^{m-1}\frac{(1-x)^n}{n(-1)}\right]_0^1-\int_0^1\frac{(1-x)^n}{n(-1)}(m-1)x^{m-2}dx$$

$$=\frac{m-1}{n}\Big|_0^1-\int_0^1 x^{m-2}(1-x)^n dx=\frac{m-1}{n}\beta(m-1,n+1)\text{………….(1)}$$

Now we have to find $\beta(m-1, n+1)$.

To obtain this put m=m-1 and n=n+1 in (1). Then, we have

$$\beta(m-1, n+1) = \frac{m-2}{n+1}\beta(m-2, n+2)$$

Putting this value of $\beta(m-1, n+1)$ in (1) we have

$$\beta(m, n) = \frac{m-1}{n}.\frac{m-2}{n+1}\beta(m-2, n+2)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\textbf{(2)}$$

Changing m to m-2 and n to n-2 from (1) we have

$$\beta(m-2, n+2) = \frac{m-3}{n+2}.\frac{m-2}{n+1}\beta(m-3, n+3)$$

From (2)

$$\beta(m, n) = \frac{m-1}{n}.\frac{m-2}{n+1}.\frac{m-3}{n+2}\beta(m-3, n+3)$$

Proceeding like this we get

$$\beta(m, n) = \frac{m-1}{n}.\frac{m-2}{n+1}.\frac{m-3}{n+2}\ldots\ldots\ldots\ldots\frac{[m-(m-1)]}{[n+(m-2)]}\beta(m-(m-1), n+(m-1))$$

$$= \frac{m-1}{n}.\frac{m-2}{n+1}.\frac{m-3}{n+2}\ldots\ldots\ldots\ldots\frac{1}{(n+m-2)}\beta(1, n+m-1)\ldots\ldots\textbf{(3)}$$

From (3)

$$\beta(m, n) = \frac{m-1}{n}.\frac{m-2}{n+1}.\frac{m-3}{n+2}\ldots\ldots\ldots\ldots\frac{1}{(n+m-2)}..\frac{1}{(n+m-1)}$$

$$= \frac{(m-1)!}{(n+m-1)(n+m-2)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(n+2)(n+1)n}$$

Multiplying the numerator and denominator by (n-1)!, we have

$$\beta(m, n) = \frac{(m-1)!(n-1)!}{(n+m-1)!}$$

STANDARD FORMS OF BETA FUNCTIONS

FORM I:

To show

$$\beta(m,n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Proof:

We have

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad \textbf{..............................(1)}$$

put $x=\frac{1}{1+y}$ so that $dx=\frac{dy}{(1+y)^2}$

From (1)

We have

$$\beta(m,n)=\int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(1-\frac{1}{1+y}\right)^{n-1}.-\frac{dy}{(1+y)^2}$$

$$=\int_0^\infty \frac{y^{n-1}dy}{(1+y)^{m+1}(1+y)^{n-1}}$$

$$=\int_0^\infty \frac{y^{n-1}dy}{(1+y)^{m+n}}$$

$$\therefore \beta(m,n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

Hence proved.

FORM II:

To show that

$$\beta(m,n)=\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$$

Proof:

From form we have

$$\beta(m,n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx+\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

Now putting $x=\frac{1}{y}$ and $dx=-\frac{1}{y^2}dy$ in the second integral, we get

$$\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx=\int_\infty^0 \frac{\left(\frac{1}{y}\right)^{m-1}}{\left(1+\frac{1}{y}\right)^{m+n}}\cdot\frac{-1}{y^2}\,dy$$

$$\int_0^1 \frac{y^{m+n}}{(1+y)^{m+n}}\cdot\frac{-1}{y^{m+1}}\,dy$$

$$=\int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

$$\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Hence

$$\beta(m,n)=\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$$

FORM III:

$$\beta(m,n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}\,dx$$

Proof:

We have

$$a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}\,dx = a^m b^n \int_0^\infty \frac{x^{m-1}}{b^{m+n}\left(\frac{ax}{b}+1\right)^{m+n}}\,dx$$

Put

$\frac{ax}{b}$ =t then $\frac{a\,dx}{b} = dt$

$$\frac{a^m b^n}{b^{m+n}} \int_0^\infty \frac{\frac{b^{m-1}}{a^{m-1}} t^{m-1}}{(t+1)^{m+n}} \frac{b}{a}\,dt$$

$$= \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\,dt = \beta(m,n)$$

Hence proved.

FORM IV:

To show $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}\,dx = \frac{\beta(m,n)}{a^n(1+a)^m}$

PROOF:

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Put $x = \frac{(1+a)t}{t+a}$ then $dx = (1+a)\left[\frac{(t+a)1 - t(1+0)}{(t+a)^2}\right]$

$$= \frac{a(1+a)}{(t+a)^2}$$

$$dx = \frac{a(1+a)}{(t+a)^2}\,dt$$

Also when x=0, t=0 and x=1,t=1.

Now (1) become

$$\beta(m,n)=\int_0^1 \frac{(1+a)^{m-1}t^{n-1}}{(t+a)^{m-1}}\left(1-\frac{(1+a)t^1}{(t+a)^1}\right)^{n-1}\frac{a(1+a)}{(t+a)^2}dt$$

$$\beta(m,n)=\int_0^1 \frac{(1+a)^{m-1}t^{n-1}}{(t+a)^{m-1}}\left(\frac{a-at}{t+a}\right)^{n-1} a\ dt$$

Also we have $\beta(m,n)=\frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$

$$\therefore \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx=\frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$$

Taking m + n=1 so that m=n-1, we get

$$\therefore \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx=\frac{\gamma(1-n)\gamma(n)}{\gamma(1)}$$

Or

$$\therefore \gamma(1-n)\gamma(n)=\int_0^\infty \frac{x^{n-1}}{1+x}dx.......(1)$$

We have

$$\therefore \int_0^\infty \frac{x^{2n}}{1+x^{2n}}dx=\frac{\pi}{2n}\operatorname{cosec}\frac{(2m+1)\pi}{2n}$$ Where m>0, n>0 and m>n

Put x^{2n}=t and $\frac{(2m+1)}{2n}=s$, we have

$$\therefore \int_0^\infty \frac{t^{\left(2m/2n\right)}t^{1/2n}}{(2n)(1+t)t}dt=\frac{\pi}{2n}\operatorname{cosec} s\pi$$

$$\therefore \int_0^\infty \frac{t^{\left(2m/2n\right)}t^{\frac{1}{2n}-1}}{(1+t)}dt=\frac{\pi}{2n}\operatorname{cosec} s\pi$$

Or $$\therefore \int_0^\infty \frac{t^{\left[(2m+1)/2n\right]-1}t^{\frac{1}{2n}-1}}{(1+t)}dt=\frac{\pi}{2n}\operatorname{cosec} s\pi$$

$$\therefore \int_0^\infty \frac{t^{s-1}}{(1+t)}dt=\frac{\pi}{2n\sin s\pi}$$

$$\therefore \int_0^\infty \frac{x^{s-1}}{(1+x)}dt=\frac{\pi}{2n\sin s\pi}\ \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From (1) and (2) we have

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}\,dx = \frac{\beta(m,n)}{a^n(1+a)^m}$$

Hence Proved

PROBLEMS:

1. Show that

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2},\frac{n+1}{2}\right)$$

Solution:

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \int_0^{\pi/2} \sin^{m-1}\theta \cos^{n-1}\theta(\sin\theta\cos\theta)\,d\theta$$

$$= \int_0^{\pi/2} (\sin^2\theta)^{\frac{m-1}{2}} (\cos^2\theta)^{\frac{n-1}{2}} (\sin\theta\cos\theta)\,d\theta$$

Put $\sin^2\theta = x$ so that $(\sin\theta\cos\theta)\,d\theta = \frac{dx}{2}$

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \int_0^1 x^{\frac{m-1}{2}}(1-x)^{\frac{n-1}{2}}\,dx$$

$$= \int_0^1 x^{\left(\frac{m+1}{2}\right)-1}(1-x)^{\left(\frac{n-1}{2}\right)-1}\,dx$$

$$= \frac{1}{2}\beta\left(\frac{m+1}{2},\frac{n+1}{2}\right)$$

$$\therefore \int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2},\frac{n+1}{2}\right)$$

Hence proved

2. Express the following integrals in terms of Beta function:

i. $\int_0^1 \frac{x}{\sqrt{1-x^2}}dx$

ii. $\int_0^4 \frac{x}{\sqrt{9-x^2}}dx$ Answer: $\frac{1}{2}\beta\left(\frac{1}{2},\frac{1}{2}\right)$

Solution: Put $x^2 = y$ so that $dx = \frac{1}{2}y^{-1/2}dy$

When x=0, y=0 when x=1, y=1.

$$\int_0^1 \frac{x}{\sqrt{1-x^2}}dx = \int_0^1 \frac{y^{1/2}}{\sqrt{1-y}}\frac{1}{2}y^{-1/2}dy$$

$$= \frac{1}{2}\int_0^1 (1-y)^{-1/2}dy$$

$$= \frac{1}{2}\int_0^1 y^{1-1}(1-y)^{\frac{1}{2}-1}dy$$

$$= \frac{1}{2}\beta\left(1,\frac{1}{2}\right)$$

Exercise Problems:

1. Prove that $\int_0^a (a-x)^{m-1}x^{n-1}dx = a^{m+n-1}\beta(m,n)$

   Hint: put x=ay

2. Show that $\int_0^a x^{m-1}(1-x^n)^p dx = \frac{1}{n}\beta\left(\frac{m+1}{n},\ p+1\right)$

   Hint: put x^n =y

3. Show that $\int_0^a (1+x)^{m-1}(1-x\ )^{n-1}dx = 2^{m+n-1}\beta(m,n)$

   Hint: put $x = \frac{1+y}{2}$

4. Show that

i. $\int_0^\infty \frac{x^{m-1} - x^{n-1}}{(1+x)^{m+n}} dx = 0$

ii. $\int_0^\infty \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx = 2\beta(m, n)$

5. Prove that $\frac{\beta(p, q+1)}{q} = \frac{\beta(p+1, q)}{p} = \frac{\beta(p, q)}{p+q}$, where p>0, q>0.

6. Show that $\int_a^b (x-a)^m (b-x)^n dx = (b-a)^{m+n+1} \beta(m+1, n+1)$

GAMMA FUNCTION:

- The Gamma function and Beta functions belong to the category of the special transcendental functions and are defined in terms of improper definite integrals.

Definition:

The definite integral $\int_0^\infty e^{-x} x^{n-1} dx$ is called the Gamma function and is denoted by

$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$ And read as "gamma n".

NOTE:

1. The integral converges for n>0.
2. Gamma function is also called Eulerian integral of the second kind.
3. The integral Gamma function does not converges if n≤0.

PROPERTIES OF GAMMA FUNCTIONS:

I. To show that $\Gamma(1) = 1$

Proof: By definition of Gamma function, we have

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$$\therefore \Gamma(n) = \int_0^\infty e^{-x} x^0 dx = \int_0^\infty e^{-x} dx = \left(e^{-x}\right)_0^\infty = 1$$

II. To show that $\Gamma(n) = (n-1)\Gamma(n-1)$ where n > 1.

Proof: By definition of Gamma function, we have

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx = \left[x^{n-1} \frac{e^{-x}}{(-1)} \right]_0^\infty - \int_0^\infty (n-1) x^{n-2} . \left(\frac{e^{-x}}{-1} \right) dx \text{ Integrate by parts}$$

$$= (n-1) \int_0^\infty e^{-x} x^{n-2} dx = (n-1)\Gamma(n-1)$$

Note:

1. $\Gamma(n+1) = (n)\Gamma(n)$
2. If n is a positive fraction, then we write
   $\Gamma(n) = (n-1)(n-2)(n-3)(n-4)\ldots\ldots\ldots\ldots\Gamma(n-r)$
   Where $(n-r) > 0$
3. If n is a non-negative integer, then $\Gamma(n+1) = (n)!$

Properties of Gamma function :

1) $\Gamma(m+1) = m\Gamma m$
2) $\Gamma(m+1) = m!$ When m is a positive integer.
3) $\Gamma(m+a) = (m+a-1)(m+a-2)\ldots\ldots\ldots a\Gamma a$, when n is a positive integer.
4) $\Gamma m = 2\int_0^\infty e^{-x^2} x^{2m-1}\, dx \quad (m > 0)$
5) $\frac{\Gamma m}{t^m} = \int_0^\infty e^{-tx} x^{m-1}\, dx \quad (m > 0)$
6) $\Gamma \frac{1}{2} = \sqrt{\pi}$
7) $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$
8) $\int_0^1 x^n (\log x)^m\, dx = \frac{(-1)^m}{(n+1)^{m+1}} \Gamma(m+1)$

Example 1: **Evaluate** $\mathbf{\Gamma(-\frac{1}{2})}$.

Solution: We know that $\Gamma(m+1) = m\Gamma m$

$$\Rightarrow \Gamma\left(-\tfrac{1}{2}+1\right) = -\tfrac{1}{2}\Gamma\left(-\tfrac{1}{2}\right)$$
$$\Rightarrow \Gamma\left(\tfrac{1}{2}\right) = -\tfrac{1}{2}\Gamma\left(-\tfrac{1}{2}\right)$$
$$\Rightarrow \sqrt{\pi} = -\tfrac{1}{2}\Gamma\left(-\tfrac{1}{2}\right)$$
$$\therefore \quad \mathbf{\Gamma\left(-\tfrac{1}{2}\right) = -2\sqrt{\pi}}.$$

RELATION BETWEEN BETA AND GAMMA FUNCTIONS

1. $\beta(m,n) = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$ Where m>0, n>0

Proof:

: By definition of Gamma function, we have

$\Gamma(m) = \int_0^\infty e^{-x} x^{m-1} dx$ ……………………….. (1)

Put x = yt so that dx =y dt then (1) gives

$$\Gamma(m) = \int_0^\infty e^{-yt} yt^{m-1} t^{m-1} ydt = \int_0^\infty e^{-yt} y^m t^{m-1} dt = \int_0^\infty e^{-yx} y^m x^{m-1} dx \text{ ……………………….(2)}$$

Or $\dfrac{\Gamma(m)}{y^m} = \int_0^\infty e^{-yx} x^{m-1} dx$ ……………………….(3)

Multiplying both sides of (3)

$$\Gamma(m)\int_0^\infty e^{-y} y^{n-1} dy = \int_0^\infty \left\{ \int_0^\infty e^{-y(x+1)} y^{m+n-1} x^{m-1} dx \right\} dy \text{ ……………..(4)}$$

$\Gamma(m)\Gamma(n) = \int_0^\infty \left\{ \int_0^\infty e^{-y(x+1)} y^{m+n-1} dy \right\} x^{m-1} dx$, by interchanging the order of integration

$$\Gamma(m)\Gamma(n) = \int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}}\ x^{m-1} dx$$

$$\Gamma(m)\Gamma(n) = dx\int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}}\ x^{m-1} dx = \Gamma(m+n)\beta(m, n)$$

$$\therefore \beta(m,n) = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$$

Hence proved

2. To prove that $\Gamma(n)\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$

Proof:

By Form I of Beta function

$$\beta(m,n) == \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

Also we have $\therefore \beta(m,n) = \dfrac{\gamma(m)\gamma(n)}{\gamma(m+n)}$

$$\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$$

Taking m + n=1 so that m=1-n, we get

$$\int_0^\infty \frac{x^{n-1}}{(1+x)}dx = \frac{\gamma(1-n)\gamma(n)}{\gamma(1)}$$

$$\gamma(1-n)\gamma(n) = \int_0^\infty \frac{x^{n-1}}{(1+x)}dx$$

We have

$$\int_0^\infty \frac{x^{2m}}{(1+x^{2n})}dx = \frac{\pi}{2n}\operatorname{cosec}\frac{(2m+1)\pi}{2n}$$, where m>0, n>0 and n>m

Put $x^{2m} = t$ and $\frac{(2m+1)}{2n} = s$, we have

$$\int_0^\infty \frac{t^{(2m/2n)}t^{1/2n}}{(2n)(1+t)t}dt = \frac{\pi}{2n}\operatorname{cosec} s\pi$$

Or $$\int_0^\infty \frac{t^{(2m/2n)}t^{\frac{1}{2n}-1}}{(1+t)}dt = \frac{\pi}{2n}\operatorname{cosec} s\pi$$

$$\int_0^\infty \frac{t^{(2m+1/2n)-1}}{(1+t)}dt = \pi\operatorname{cosec} s\pi$$

Or $$\int_0^\infty \frac{t^{s-1}}{(1+t)}dt = \frac{\pi}{\sin n\pi}$$

$$\therefore \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

Hence proved

3. To show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof: we know that $$\beta(m,n) = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$$

Taking m=n= $\frac{1}{2}$, we have

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\gamma\left(\frac{1}{2}\right)\gamma\left(\frac{1}{2}\right)}{\gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \left[\Gamma\left(\frac{1}{2}\right)\right]^2 \quad [\because \gamma(1) = 1]$$ ……………………………..(1)

$$\beta\left(\frac{1}{2},\frac{1}{2}\right)=\int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}dx \quad =\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}dx$$

$x=\sin^2\theta$ so that $dx = 2\sin\theta\cos\theta d\theta$

Also when x=0, θ =0: when x=1, $\theta = \pi/2$

$$\therefore \beta\left(\frac{1}{2},\frac{1}{2}\right)=\int_0^{\pi/2} dx \quad =\int_0^1 \frac{1}{\sin\theta}\frac{1}{\cos\theta}2\sin\theta\cos\theta\, d\theta$$

$$=2\int_0^{\pi/2} d\theta = \pi$$

………………………………………………….(2)

From (1) and (2) we have

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

4. To show that $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

Proof: we have

$$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx$$

Taking n=$\frac{1}{2}$, we have $\Gamma\left(\frac{1}{2}\right)=\int_0^\infty e^{-x}x^{-1/2}dx$

Put x =t^2 so that d x=2t d t

Also when x=0, t=0: when $x\to\infty, t\to\infty$

$$\therefore \Gamma\left(\frac{1}{2}\right)=\int_0^\infty e^{-t^2}\left(t^2\right)^{-1/2}2dt = 2\int_0^\infty e^{-t^2}dt$$

Or $2\int_0^\infty e^{-x^2}dx = \sqrt{\pi}$

$$\therefore \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

PROBLEMS

1. Compute

   i) $\Gamma\left(\frac{11}{2}\right)$

   ii) $\Gamma\left(-\frac{1}{2}\right)$

   iii) $\Gamma\left(-\frac{7}{2}\right)$

Solutions: i)

We have $\Gamma(n+1) = (n)\Gamma(n)$

Taking $n = \frac{7}{2}$

$$\Gamma\left(\frac{11}{2}\right) = \frac{9}{2}\Gamma\left(\frac{9}{2}\right)$$

$$= \frac{9}{2}\frac{7}{2}\Gamma\left(\frac{7}{2}\right)$$

$$= \frac{9}{2}\frac{7}{2}\frac{5}{2}\Gamma\left(\frac{5}{2}\right)$$

$$= \frac{9}{2}\frac{7}{2}\frac{5}{2}\frac{3}{2}\Gamma\left(\frac{3}{2}\right)$$

$$= \frac{9}{2}\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{9}{2}\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}$$

<u>Solution: ii)</u>

We have $\Gamma(n+1) = (n)\Gamma(n)$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

Taking $n = -\frac{1}{2}$

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = -2\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$$

2.

Evaluate each of the following:

(a) $$\frac{\Gamma(6)}{2\Gamma(3)} = \frac{5!}{2\cdot 2!} = \frac{5\cdot 4\cdot 3\cdot 2}{2\cdot 2} = 30$$

(b) $$\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{3}{4}$$

(c) $$\frac{\Gamma(3)\Gamma(2.5)}{\Gamma(5.5)} = \frac{2!(1.5)(0.5)\Gamma(0.5)}{(4.5)(3.5)(2.5)(1.5)(0.5)\Gamma(0.5)} = \frac{16}{315}$$

(d) $$\frac{6\Gamma\left(\frac{8}{3}\right)}{5\Gamma\left(\frac{2}{3}\right)} = \frac{6\left(\frac{5}{3}\right)\left(\frac{2}{3}\right)\Gamma\left(\frac{2}{3}\right)}{5\Gamma\left(\frac{2}{3}\right)} = \frac{4}{3}$$

3. Evaluate

i. $\int_0^1 x^5(1-x)^3\,dx$

ii. $\int_0^1 x^4(1-x)^2\,dx$ Answer: 1/105

iii. $\int_0^1 x\,(1-x)^{1/3}\,dx$ Answer: $\frac{16\sqrt{\pi}}{9\sqrt{3}}$

iv. $\int_0^1 x^{5/2}\left(1-x^2\right)^{3/2}\,dx$ Answer: $\frac{8}{65}\frac{\Gamma\left(\frac{3}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)}$

Solution: i)

$$\int_0^1 x^5(1-x)^3\,dx = \int_0^1 x^{6-1}(1-x)^{4-1}\,dx$$

$$\beta(6,4) = \frac{\Gamma(6)\Gamma(4)}{\Gamma(6+4)}$$

$$\frac{5!3!}{9!} = \frac{1}{504}$$

4.

Evaluate $\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx$

Solution: Let $I = \int_0^\infty x^{\frac{1}{4}} e^{-\sqrt{x}}\, dx$ ___________(i)

Putting $\sqrt{x} = t \implies x = t^2$ so that $dx = 2t$ in (i), we get

$$I = \int_0^\infty t^{1/2} e^{-t}\, 2t\, dt$$

$$= 2\int_0^\infty t^{3/2} e^{-t}\, dt$$

$$= 2\int_0^\infty t^{\frac{5}{2}-1}\, e^{-t}\, dt$$

$$= 2\Gamma\left(\frac{5}{2}\right)$$

$$= \left(2 \times \frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)$$

$$= \left(2 \times \frac{3}{2} \times \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{3}{2}\sqrt{\pi}$$

$$\therefore \quad \int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx = \frac{3}{2}\sqrt{\pi}$$

5. Evaluate

i) $\int_0^\infty x^6 e^{-2x} dx$

ii) $\int_0^\infty x^{3/2} e^{-4x} dx$

iii) $\int_0^\infty x^2 e^{-x^2} dx$

iv) $\int_0^\infty \sqrt{x} e^{-x^2} dx$

Solution: Put $2x = y$ so that $dx = \frac{1}{2} dy$

$$\int_0^\infty x^6 e^{-2x} dx = \int_0^\infty \left(\frac{y}{2}\right)^6 e^{-y} \frac{1}{2} dy$$

$$= \frac{1}{2}\int_0^\infty y^6 e^{-y} \frac{1}{2} dy$$

$$= \frac{1}{2}\int_0^\infty y^{7-1} e^{-y} \frac{1}{2} dy = \frac{1}{2^7} 6!$$

Evaluate

i. $\int_0^{\pi/2} \sin^5\theta \cos^{7/2}\theta \, d\theta$

ii. $\int_0^{\pi/2} \sin^7\theta \, d\theta$

iii. $\int_0^{\pi/2} \cos^{11}\theta \, d\theta$

iv. $\int_0^{\pi/2} \sqrt{\cot\theta} \, d\theta$

<u>Solution: i)</u> we have $\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2}\beta(m,n)$

Put 2 m-1=5 and 2n-1=1/2 so that m=3, n=9/4

Therefore $\int_0^{\pi/2} \sin^5\theta \cos^{7/2}\theta \, d\theta = \frac{1}{2}\beta(3, 9/4)$

$$\frac{1}{2}\frac{\Gamma(3)}{\Gamma\left(3+\frac{9}{4}\right)} = \frac{\Gamma\left(\frac{9}{4}\right)}{\Gamma\left(21/4\right)} = \frac{64}{1989}$$

vi.

Solution: We know that,

$$\int_0^{\pi/2} sin^p\ \theta\ cos^p\ \theta\ d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right).\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

$$\int_0^{\frac{\pi}{2}} \sqrt{cot\theta}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{cos^{1/2}\theta}{sin^{1/2}\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} sin^{-1/2}\theta\ \ cos^{1/2}\theta\, d\theta$$

On applying formula (1), we have

$$\int_0^{\frac{\pi}{2}} \sqrt{cot\theta}\, d\theta = \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{2\,\Gamma(1)}$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

EXERCISE:

1) Evaluate $\int_0^1 (1-x^3)^{-1/2}\ dx$
2) Evaluate $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\ dx$
3) Evaluate $\int_0^1 \left(\frac{x^3}{1-x^3}\right)^{\frac{1}{2}}\ dx$
4) Prove that $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}$
5) Show that $\beta(p,q) = \beta(p+1,q) + (p,q+1)$

6. Evaluate

1. $\int_0^\infty 3^{-4x^2}\, dx$

2. $\int_0^\infty a^{-bx^2}\,dx$

3. $\int_0^\infty x^4\left(\log\frac{1}{x}\right)^3 dx$

4. **Prove that** $\int_0^1 (x\,log x)^4\,dx = \frac{4!}{5^5}$

5. $\int_0^\infty x^2\left(\log\frac{1}{x}\right)^3 dx$

Example 4: Prove that $\int_0^1 (x\,log x)^4\,dx = \frac{4!}{5^5}$

Solution: We know that

$$\int_0^1 x^n (log\,x)^m\,dx = \frac{(-1)^m}{(n+1)^{m+1}}\Gamma(m+1)$$

Now, $\int_0^1 (x\,log x)^4\,dx = \int_0^1 x^4\,(log x)^4\,dx$

Putting $n = m = 4$ in (i), we get

$$\int_0^1 x^4\,(log x)^4\,dx = \frac{(-1)^4}{(4+1)^{4+1}}\Gamma(4+1)$$

$$= \frac{\Gamma 5}{5^5}$$

$$= \frac{4!}{5^5}$$

MODULE-V

SPECIAL FUNCTIONS-II

Bessel's equation

$x^2 y''+x y'+(x^2-v^2)y=0$ is called Bessel's equation.

Solution of Bessel's Equation:

Because x=0 is a regular singular point of Bessel's equation we know that there exists at least one solution of the form $y=\sum_{n=o}^{\infty} c_n x^{n+r}$. Substituting the last expression into (6.10) gives

$$x^2 y''+x y'+(x^2-v^2)y=\sum_{n=o}^{\infty} c_n(n+r)(n+r-1)x^{n+r}+\sum_{n=o}^{\infty} c_n(n+r)x^{n+r}+\sum_{n=o}^{\infty} c_n x^{n+r+2}$$

$$-v^2\sum_{n=o}^{\infty} c_n x^{n+r} = c_0(r^2-r+r-v^2)x^r$$

$$+x^r\sum_{n=1}^{\infty} c_n[(n+r)(n+r-1)+(n+r)-v^2]x^n+x^r\sum_{n=o}^{\infty} c_n x^{n+2}$$

$$=c_0(r^2-v^2)x^r+x^r\sum_{n=1}^{\infty} c_n[(n+r)^2]-v^2]x^n+x^r\sum_{n=o}^{\infty} c_n x^{n+2}$$

From (6.11) we see that the indicial equation is $r^2-v^2=0$, so the indicial roots are $r_1=v$ and $r_2 = -v$. When $r_1=v$, (6.11) becomes

$$x^v\sum_{n=1}^{\infty} c_n n(n+2v)x^n+x^v\sum_{n=o}^{\infty} c_n x^{n+2}$$

$$=x^v\left[(1+2v)c_1x+\sum_{n=2}^{\infty} c_n n(n+2v)x^n+\sum_{n=0}^{\infty} c_n x^{n+2}\right]$$

$$=x^v\left[(1+2v)c_1x+\sum_{k=0}^{\infty}[(k+2)(k+2+2v)c_{k+2}+c_k]x^{k+2}\right]=0$$

Therefore by the usual argument we can write $(1+2v)c_1=0$ and
$(k+2)(k+2+2v)c_{k+2}+c_k=0$

or $c_{k+2}=\dfrac{-c_k}{(k+2)(k+2+2v)},k=0,1,2,----$

The choice $c_1=0$ in (6.12) implies $c_3=c_5=c_7=$ - - - - = 0, so for k=0,2,4, - - - - we find, after letting k +2 = 2n,

n = 1,2,3, - - - - that

$$c_{2n} = -\frac{c_{2n-2}}{2^2 n(n+v)}$$

Thus $c_2 = -\dfrac{c_0}{2^2.1(1+v)}$

$$c_4 = -\frac{c_2}{2^2 2(2+v)}=\frac{c_0}{2^4.2.1(1+v)(2+v)}$$

$$c_6 = -\frac{c_4}{2^2.3(3+v)}=-\frac{c_0}{2^6.1.2.3(1+v)(2+v)(3+v)}$$

:

$$c_{2n} = \frac{(-1)^n c_0}{2^2 n!(1+v)(2+v)\ldots(n+v)}, n = 1,2,3,- - - - \qquad (6.14)$$

It is standard practice to choose c_0 to be specific value – namely.

$$c_0 = \frac{1}{2^v \Gamma(1+v)}$$

where Γ (1+v) is the gamma function. (See Appendix) Since this latter function possesses the convenient property Γ (1+α) = αΓ(α), we can reduce the indicated product in the denominator of (6.14) to one term.

For example:

Γ (1+v+1)= (1+v) Γ (1+v)

Γ (1+v+2)= (2+v) Γ (2+v)= (2+v)(1+v)Γ(1+v).

Hence we can write (6.14) as

$$c_{2n} = \frac{(-1)^n}{2^{2n+v} n!(1+v)(2+v)\ldots(n+v)\Gamma(1+v)} = \frac{(-1)^n}{2^{2n+v} n!\Gamma(1+v+n)}$$

for n=0,1,2, - - - -

Bessel Function of the First Kind:

Using the coefficients c_{2n} just obtained and r=v, a series solution of (6.10) is $y=\sum_{n=0}^{\infty} c_{2n} x^{2n+v}$ This solution is usually denoted by $J_v(x)$:

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+v+n)} \left(\frac{x}{2}\right)^{2n+v}.$$

If v≥0, the series converges at least on the interval $[o,\infty)$. Also, for the second exponent r_2= -v we obtain, in exactly the same manner,

$$J_{-v}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1-v+n)} \left(\frac{x}{2}\right)^{2n-v}.$$

The functions $J_v(x)$ and $J_{-v}(x)$ are called Bessel functions of the first kind of order v and –v, respectively. Depending on the value of v, (6.16) may contain negative powers of x and hence converge on $(0, \infty)$.*

Many Differential equations arising from physical problems are linear but have variable coefficients and do not permit a general analytical solution in terms of known functions. Such equations can be solved by numerical methods (Unit – I), but in many cases it is easier to find a solution in the form of an infinite convergent series. The series solution of certain differential equations gives rise to special functions such as Bessel's function, Legendre's polynomial. These special functions have many applications in engineering.Series solution of the Bessel Differential Equation

Consider the Bessel Differential equation of order n in the form

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \qquad \text{(i)}$$

Where n is a non negative real constant or parameter.

We assume the series solution of (i) in the form

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} \quad \text{Where } a_0 \neq 0 \qquad \text{(ii)}$$

Hence, $\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1}$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2}$$

Substituting these in (i) we get,

$$x^2 \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2} + x\sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1} + \left(x^2 - n^2\right)\sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

i.e., $\sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r} + \sum_{r=0}^{\infty} a_r (k+r) x^{k+r} + \sum_{r=0}^{\infty} a_r x^{k+r+2} - n^2 \sum_{r=0}^{\infty} a_r x^{k+r} = 0$

Grouping the like powers, we get

$$\sum_{r=0}^{\infty} a_r \left[(k+r)(k+r-1) + (k+r) - n^2\right] x^{k+r} + \sum_{r=0}^{\infty} a_r x^{k+r+2} = 0$$

$$\sum_{r=0}^{\infty} a_r \left[(k+r)^2 - n^2\right] x^{k+r} + \sum_{r=0}^{\infty} a_r x^{k+r+2} = 0 \qquad \text{(iii)}$$

Now we shall equate the coefficient of various powers of x to zero

Equating the coefficient of x^k from the first term and equating it to zero, we get

$a_0\left[k^2 - n^2\right] = 0.$ Since $a_0 \neq 0$, we get $k^2 - n^2 = 0, \quad \therefore k = \pm n$

Coefficient of x^{k+1} is got by putting r = 1 in the first term and equating it to zero, we get

i.e., $a_1\left[(k+1)^2 - n^2\right] = 0.$ This gives $a_1 = 0,$ since $(k+1)^2 - n^2 = 0$ gives, $k+1 = \pm n$

Which is a contradiction to $k = \pm n$.

Let us consider the coefficient of x^{k+r} from (iii) and equate it to zero.

i.e, $a_r\left[(k+r)^2 - n^2\right] + a_{r-2} = 0.$

$$\therefore a_r = \frac{-a_{r-2}}{\left[(k+r)^2 - n^2\right]} \qquad \text{(iv)}$$

If $k = +n$, (iv) becomes

$$a_r = \frac{-a_{r-2}}{\left[(n+r)^2 - n^2\right]} = \frac{-a_{r-2}}{\left[r^2 + 2nr\right]}$$

Now putting r = 1,3,5, ….., (odd vales of n) we obtain,

$$a_3 = \frac{-a_1}{6n+9} = 0, \quad \because a_1 = 0$$

Similarly a_5, a_7, are equal to zero.

i.e., $a_1 = a_5 = a_7 = \ldots\ldots = 0$

Now, putting r = 2,4,6, ……(even values of n) we get,

$$a_2 = \frac{-a_0}{4n+4} = \frac{-a_0}{4(n+1)}; \qquad a_4 = \frac{-a_2}{8n+16} = \frac{a_0}{32(n+1)(n+2)};$$

Similarly we can obtain a_6, a_8, …

We shall substitute the values of $a_1, a_2, a_3, a_4, \cdots\cdots$ in the assumed series solution, we get

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} = x^k(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots\cdots)$$

Let y_1 be the solution for $k = +n$

$$\therefore \quad y_1 = x^n\left[a_0 - \frac{a_0}{4(n+1)}x^2 + \frac{a_0}{32(n+1)(n+2)}x^4 - \cdots\cdots\right]$$

$$i.e., \quad y_1 = a_0 x^n \left[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^5(n+1)(n+2)} - \cdots\cdots \right] \qquad \text{(v)}$$

This is a solution of the Bessel's equation.

Let y_2 be the solution corresponding to k = - n. Replacing n be – n in (v) we get

$$y_2 = a_0 x^{-n} \left[1 - \frac{x^2}{2^2(-n+1)} + \frac{x^4}{2^5(-n+1)(-n+2)} - \cdots\cdots \right] \qquad \text{(vi)}$$

The complete or general solution of the Bessel's differential equation is $y = c_1 y_1 + c_2 y_2$, where c_1, c_2 are arbitrary constants.

Now we will proceed to find the solution in terms of Bessel's function by choosing $a_0 = \frac{1}{2^n \Gamma(n+1)}$

and let us denote it as Y_1.

$$i.e., \quad Y_1 = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \left(\frac{x}{2}\right)^2 \frac{1}{(n+1)} + \left(\frac{x}{2}\right)^4 \frac{1}{(n+1)(n+2)\cdot 2} - \cdots\cdots \right]$$

$$= \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2 \frac{1}{(n+1)\Gamma(n+1)} + \left(\frac{x}{2}\right)^4 \frac{1}{(n+1)(n+2)\Gamma(n+1)\cdot 2} - \cdots\cdots \right]$$

We have the result Γ(n) = (n – 1) Γ(n – 1) from Gamma function

Hence, Γ(n + 2) = (n + 1) Γ(n + 1) and

Γ(n + 3) = (n + 2) Γ(n + 2) = (n + 2) (n + 1) Γ(n + 1)

Using the above results in Y_1, we get

$$Y_1 = \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma(n+2)} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma(n+3)\cdot 2} - \cdots\cdots \right]$$

Which can be further put in the following form?

$$Y_1 = \left(\frac{x}{2}\right)^n \left[\frac{(-1)^0}{\Gamma(n+1)\cdot 0!}\left(\frac{x}{2}\right)^0 + \frac{(-1)^1}{\Gamma(n+2)\cdot 1!}\left(\frac{x}{2}\right)^2 + \frac{(-1)^2}{\Gamma(n+3)\cdot 2!}\left(\frac{x}{2}\right)^4 + \cdots\cdots \right]$$

$$= \left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(n+r+1) \cdot r!} \left(\frac{x}{2}\right)^{2r}$$

$$= \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1) \cdot r!}$$

This function is called the Bessel function of the first kind of order n and is denoted by $J_n(x)$.

Thus $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1) \cdot r!}$

Further the particular solution for $k = -n$ (replacing n by –n) be denoted as $J_{-n}(x)$. Hence the general solution of the Bessel's equation is given by $y = AJ_n(x) + BJ_{-n}(x)$, where A and B are arbitrary constants.

Properties of Bessel's function

1. $J_{-n}(x) = (-1)^n J_n(x)$, where n is a positive integer.

Proof: By definition of Bessel's function, we have

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1) \cdot r!} \qquad \text{……….(1)}$$

Hence, $J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{-n+2r} \cdot \frac{1}{\Gamma(-n+r+1) \cdot r!}$ ……….(2)

But gamma function is defined only for a positive real number. Thus we write (2) in the following from

$$J_{-n}(x) = \sum_{r=n}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{-n+2r} \cdot \frac{1}{\Gamma(-n+r+1) \cdot r!} \qquad \text{………..(3)}$$

Let r – n = s or r = s + n. Then (3) becomes

$$J_{-n}(x) = \sum_{s=0}^{\infty} (-1)^{s+n} \cdot \left(\frac{x}{2}\right)^{-n+2s+2n} \cdot \frac{1}{\Gamma(s+1) \cdot (s+n)!}$$

We know that Γ(s+1) = s! and (s + n)! = Γ(s+n+1)

$$= \sum_{s=0}^{\infty} (-1)^{s+n} \cdot \left(\frac{x}{2}\right)^{n+2s} \cdot \frac{1}{\Gamma(s+n+1) \cdot s!}$$

$$= (-1)^n \sum_{s=0}^{\infty} (-1)^{s} \cdot \left(\frac{x}{2}\right)^{n+2s} \cdot \frac{1}{\Gamma(s+n+1) \cdot s!}$$

Comparing the above summation with (1), we note that the RHS is $J_n(x)$.

Thus, $\boldsymbol{J_{-n}(x) = (-1)^n J_n(x)}$

2. $J_n(-x) = (-1)^n J_n(x) = J_{-n}(x)$, *where n is a positive integer*

Proof : By definition, $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1) \cdot r!}$

$$\therefore \qquad J_n(-x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(-\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1) \cdot r!}$$

i.e., $$= \sum_{r=0}^{\infty} (-1)^r \cdot (-1)^{n+2r} \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1) \cdot r!}$$

$$= (-1)^n \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1) \cdot r!}$$

Thus, $\boldsymbol{J_n(-x) = (-1)^n J_n(x)}$

Since, $(-1)^n J_n(x) = J_{-n}(x)$, we have $\boldsymbol{J_n(-x) = (-1)^n J_n(x) = J_{-n}(x)}$

<u>Recurrence Relations:</u>

Recurrence Relations are relations between Bessel's functions of different order.

Recurrence Relations 1: $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

From definition,

$$x^n J_n(x) = x^n \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1)\cdot r!} \;=\; \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{2(n+r)} \cdot \frac{1}{\Gamma(n+r+1)\cdot r!}$$

$$\therefore \quad \frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{r=0}^{\infty} (-1)^r \cdot \frac{2(n+r)x^{2(n+r)-1}}{2^{n+2r}\,\Gamma(n+r+1)\cdot r!}$$

$$= x^n \sum_{r=0}^{\infty} (-1)^r \cdot \frac{(n+r)x^{n+2r-1}}{2^{n+2r-1}(n+r)\,\Gamma(n+r)\cdot r!}$$

$$= x^n \sum_{r=0}^{\infty} (-1)^r \cdot \frac{(x/2)^{(n-1)+2r}}{\Gamma(n-1+r+1)\cdot r!} \;=\; x^n J_{n-1}(x)$$

Thus, $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$ --------(1)

Recurrence Relations 2: $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

From definition,

$$x^{-n} J_n(x) = x^{-n} \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1)\cdot r!}$$

$$= \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{2r} \cdot \frac{1}{\Gamma(n+r+1)\cdot r!}$$

$$\therefore \quad \frac{d}{dx}\left[x^{-n} J_n(x)\right] = \sum_{r=0}^{\infty} (-1)^r \cdot \frac{2r\, x^{2r-1}}{2^{n+2r}\,\Gamma(n+r+1)\cdot r!}$$

$$= -x^{-n} \sum_{r=1}^{\infty} (-1)^{r-1} \cdot \frac{x^{n+1+2(r-1)}}{2^{n+1+2(r-1)}\,\Gamma(n+r+1)\cdot (r-1)!}$$

Let $k = r - 1$

$$= -x^{-n} \sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^{n+1+2k}}{2^{n+1+2k}\,\Gamma(n+1+k+1)\cdot k!} \;=\; -x^{-n} J_{n+1}(x)$$

Thus, $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$ --------(2)

Recurrence Relations 3: $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

We know that $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

Applying product rule on LHS, we get $x^n J_n^{/}(x) + nx^{n-1} J_n(x) = x^n J_{n-1}(x)$

Dividing by x^n we get $J_n^{/}(x) + (n/x) J_n(x) = J_{n-1}(x)$ --------(3)

Also differentiating LHS of $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$, we get

$$x^{-n} J_n^{/}(x) - nx^{-n-1} J_n(x) = -x^{-n} J_{n+1}(x)$$

Dividing by $-x^{-n}$ we get $-J_n^{/}(x) + (n/x) J_n(x) = J_{n+1}(x)$ --------(4)

Adding (3) and (4), we obtain $2nJ_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$

i.e., $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

Recurrence Relations 4: $J_n^{/}(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

Subtracting (4) from (3), we obtain $2J_n^{/}(x) = \left[J_{n-1}(x) - J_{n+1}(x)\right]$

i.e., $J_n^{/}(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

Recurrence Relations 5: $J_n^{/}(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$

This recurrence relation is another way of writing the Recurrence relation 2.

Recurrence Relations 6: $J_n^{/}(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)$

This recurrence relation is another way of writing the Recurrence relation 1.

Recurrence Relations 7: $J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$

This recurrence relation is another way of writing the Recurrence relation 3.

Problems:

Prove that $(a)\quad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x \quad (b)\quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$

By definition,

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1) \cdot r!}$$

Putting n = ½, we get

$$J_{1/2}(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{1/2+2r} \cdot \frac{1}{\Gamma(r+3/2) \cdot r!}$$

$$J_{1/2}(x) = \sqrt{\frac{x}{2}}\left[\frac{1}{\Gamma(3/2)} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma(5/2)1!} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma(7/2)2!} - \cdots\cdots\right] \quad \text{--------(1)}$$

Using the results Γ(1/2) = √π and Γ(n) = (n – 1) Γ(n–1), we get

$\Gamma(3/2) = \dfrac{\sqrt{\pi}}{2}, \Gamma(5/2) = \dfrac{3\sqrt{\pi}}{4}, \Gamma(7/2) = \dfrac{15\sqrt{\pi}}{8}$ and so on.

Using these values in (1), we get

$$J_{1/2}(x) = \sqrt{\frac{x}{2}}\left[\frac{2}{\sqrt{\pi}} - \frac{x^2}{4}\frac{4}{3\sqrt{\pi}} + \frac{x^4}{16}\frac{8}{15\sqrt{\pi}.2} - \cdots\cdots\right]$$

$$= \sqrt{\frac{x}{2\pi}} \cdot \frac{2}{x}\left[x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\cdots\right] = \sqrt{\frac{2}{x\pi}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots\right]$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

Putting n = - 1/2, we get

$$J_{-1/2}(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{-1/2+2r} \cdot \frac{1}{\Gamma(r+1/2) \cdot r!}$$

$$J_{-1/2}(x) = \sqrt{\frac{x}{2}}\left[\frac{1}{\Gamma(1/2)} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma(3/2)1!} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma(5/2)2!} - \cdots\cdots\right] \quad \text{--------(2)}$$

Using the results Γ(1/2) = √π and Γ(n) = (n – 1) Γ(n–1) in (2), we get

$$J_{-1/2}(x) = \sqrt{\frac{2}{x}}\left[\frac{1}{\sqrt{\pi}} - \frac{x^2}{4}\frac{2}{\sqrt{\pi}} + \frac{x^4}{16}\frac{4}{3\sqrt{\pi}.2} - \cdots\cdots\right]$$

$$= \sqrt{\frac{2}{x\pi}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\cdots\right]$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

2. Prove the following results:

$$(a) \quad J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right] \text{ And}$$

$$(b) \quad J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x^2}\cos x + \frac{3}{x}\sin x\right]$$

Solution:

We prove this result using the recurrence relation $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$ ------ (1).

Putting n = 3/2 in (1), we get $J_{1/2}(x) + J_{5/2}(x) = \frac{3}{x}J_{3/2}(x)$

$$\therefore \quad J_{5/2}(x) = \frac{3}{x}J_{3/2}(x) - J_{1/2}(x)$$

$$i.e., \quad J_{5/2}(x) = \frac{3}{x}\sqrt{\frac{2}{\pi x}}\left[\frac{\sin x - x\cos x}{x}\right] - \sqrt{\frac{2}{\pi x}}\sin x$$

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3\sin x - 3x\cos x - x^2 \sin x}{x^2}\right] = \sqrt{\frac{2}{\pi x}}\left[\frac{(3-x^2)}{x^2}\sin x - \frac{3}{x}\cos x\right]$$

Also putting n = - 3/2 in (1), we get $J_{-5/2}(x) + J_{-1/2}(x) = -\frac{3}{x}J_{-3/2}(x)$

$$\therefore \quad J_{-5/2}(x) = -\frac{3}{x}J_{-3/2}(x) - J_{-1/2}(x) = \left(\frac{-3}{x}\right)\left(-\sqrt{\frac{2}{\pi x}}\right)\left[\frac{x\sin x + \cos x}{x}\right] - \sqrt{\frac{2}{\pi x}}\cos x$$

$$i.e., \quad J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3x\sin x + 3\cos x - x^2 \cos x}{x^2}\right] = \sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\sin x + \frac{3-x^2}{x^2}\cos x\right]$$

3. Show that $\frac{d}{dx}\left[J_n^2(x) + J_{n+1}^2(x)\right] = \frac{2}{x}\left[nJ_n^2(x) - (n+1)J_{n+1}^2(x)\right]$

Solution:

L.H.S = $\frac{d}{dx}\left[J_n^2(x)+J_{n+1}^2(x)\right]=2J_n(x)J_n'(x)+2J_{n+1}(x)J_{n+1}'(x)$ ------- (1)

We know the recurrence relations

$$xJ_n'(x)=nJ_n(x)-xJ_{n+1}(x) \qquad \text{------- (2)}$$

$$xJ_{n+1}'(x)=xJ_n(x)-(n+1)J_{n+1}(x) \qquad \text{------- (3)}$$

Relation (3) is obtained by replacing ***n*** by ***n+1*** in $xJ_n'(x)=xJ_{n-1}(x)-nJ_n(x)$

Now using (2) and (3) in (1), we get

$$\text{L.H.S} = \frac{d}{dx}\left[J_n^2(x)+J_{n+1}^2(x)\right]=2J_n(x)\left[\frac{n}{x}J_n(x)-J_{n+1}(x)\right]+2J_{n+1}(x)\left[J_n(x)-\frac{n+1}{x}J_{n+1}(x)\right]$$

$$=\frac{2n}{x}J_n^2(x)-2J_n(x)J_{n+1}(x)+2J_{n+1}(x)J_n(x)-2\frac{n+1}{x}J_{n+1}^2(x)$$

Hence, $\frac{d}{dx}\left[J_n^2(x)+J_{n+1}^2(x)\right]=\frac{2}{x}\left[nJ_n^2(x)-(n+1)J_{n+1}^2(x)\right]$

4. Prove that $J_0''(x)=\frac{1}{2}\left[J_2(x)-J_0(x)\right]$

Solution:

We have the recurrence relation $J_n'(x)=\frac{1}{2}\left[J_{n-1}(x)-J_{n+1}(x)\right]$ ------- (1)

Putting n = 0 in (1), we get $J_0'(x)=\frac{1}{2}\left[J_{-1}(x)-J_1(x)\right]=\frac{1}{2}\left[-J_1(x)-J_1(x)\right]=-J_1(x)$

Thus, $J_0'(x)=-J_1(x)$. Differentiating this w.r.t. x we get, $J_0''(x)=-J_1'(x)$ ----- (2)

Now, from (1), for n = 1, we get $J_1'(x)=\frac{1}{2}\left[J_0(x)-J_2(x)\right]$.

Using (2), the above equation becomes

$$-J_0''(x)=\frac{1}{2}\left[J_0(x)-J_2(x)\right] \text{or} J_0''(x)=\frac{1}{2}\left[J_2(x)-J_0(x)\right].$$

Thus we have proved that, $J_0''(x)=\frac{1}{2}\left[J_2(x)-J_0(x)\right]$

5. Show that (a) $\int J_3(x)dx = c - J_2(x) - \frac{2}{x}J_1(x)$

(b) $\int xJ_0^2(x)dx = \frac{1}{2}x^2\left[J_0^2(x) + J_1^2(x)\right]$

Solution :

(a) We know that $\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$ or $\int x^{-n}J_{n+1}(x)dx = -x^{-n}J_n(x)$ ------ (1)

Now, $\int J_3(x)dx = \int x^2 \cdot x^{-2}J_3(x)dx + c = x^2 \cdot \int x^{-2}J_3(x)dx - \int 2x\left[\int x^{-2}J_3(x)dx\right]dx + c$

$= x^2 \cdot \left[-x^{-2}J_2(x)\right] - \int 2x\left[-x^{-2}J_2(x)\right]dx + c$ (from (1) when n = 2)

$= c - J_2(x) - \int \frac{2}{x}J_2(x)dx = c - J_2(x) - \frac{2}{x}J_1(x)$ (from (1) when n = 1)

Hence, $\int J_3(x)dx = c - J_2(x) - \frac{2}{x}J_1(x)$

(b) $\int xJ_0^2(x)dx = J_0^2(x)\cdot\frac{1}{2}x^2 - \int 2J_0(x)\cdot J_0^{/}(x).\frac{1}{2}x^2dx$ (Integrate by parts)

$= \frac{1}{2}x^2J_0^2(x) + \int x^2J_0(x)\cdot J_1(x)dx$ (From (1) for n = 0)

$= \frac{1}{2}x^2J_0^2(x) + \int xJ_1(x)\cdot\frac{d}{dx}\left[xJ_1(x)\right]dx \left[\because \frac{d}{dx}\left[xJ_1(x)\right] = xJ_0(x) \text{ from recurrence relation (1)}\right]$

$= \frac{1}{2}x^2J_0^2(x) + \frac{1}{2}\left[xJ_1(x)\right]^2 = \frac{1}{2}x^2\left[J_0^2(x) + J_1^2(x)\right]$

<u>Generating Function for $J_n(x)$</u>

To prove that $e^{\frac{x}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} t^nJ_n(x)$

or

If n is an integer then $J_n(x)$ is the coefficient of t^n in the expansion of $e^{\frac{x}{2}(t-1/t)}$.

Proof:

We have $e^{\frac{x}{2}(t-1/t)} = e^{xt/2} \times e^{-x/2t}$

$$= \left[1 + \frac{(xt/2)}{1!} + \frac{(xt/2)^2}{2!} + \frac{(xt/2)^3}{3!} + \cdots\cdots\right] \bullet \left[1 + \frac{(-xt/2)}{1!} + \frac{(-xt/2)^2}{2!} + \frac{(-xt/2)^3}{3!} + \cdots\cdots\right]$$

(using the expansion of exponential function)

$$= \left[1 + \frac{xt}{2 \cdot 1!} + \frac{x^2 t^2}{2^2 2!} + \cdots + \frac{x^n t^n}{2^n n!} + \frac{x^{n+1} t^{n+1}}{2^{n+1}(n+1)!} + \cdots\cdots\right] \bullet \left[1 - \frac{x}{2t \cdot 1!} + \frac{x^2}{2^2 t^2 2!} - \cdots + \frac{(-1)^n x^n}{2^n t^n n!} + \frac{(-1)^{n+1} x^{n+1}}{2^{n+1} t^{n+1} (n+1)!} + \cdots\cdots\right]$$

If we collect the coefficient of t^n in the product, they are

$$= \frac{x^n}{2^n n!} - \frac{x^{n+2}}{2^{n+2}(n+1)!1!} + \frac{x^{n+4}}{2^{n+4}(n+2)!2!} - \cdots\cdots$$

$$= \frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!1!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{(n+2)!2!}\left(\frac{x}{2}\right)^{n+4} - \cdots\cdots = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)r!} = J_n(x)$$

Similarly, if we collect the coefficients of t^{-n} in the product, we get $J_{-n}(x)$.

Thus, $e^{\frac{x}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$

<u>**Result:**</u> $e^{\frac{x}{2}(t-1/t)} = J_0(x) + \sum_{n=1}^{\infty}\left[t^n + (-1)^n t^{-n}\right] J_n(x)$

Proof :

$$e^{\frac{x}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(x) = \sum_{n=-\infty}^{-1} t^n J_n(x) + \sum_{n=0}^{\infty} t^n J_n(x)$$

$$= \sum_{n=1}^{\infty} t^{-n} J_{-n}(x) + J_0(x) + \sum_{n=1}^{\infty} t^n J_n(x) = J_0(x) + \sum_{n=1}^{\infty} t^{-n}(-1)^n J_n(x) + \sum_{n=1}^{\infty} t^n J_n(x) \quad \{\because J_{-n}(x) = (-1)^n J_n(x)\}$$

Thus, $e^{\frac{x}{2}(t-1/t)} = J_0(x) + \sum_{n=1}^{\infty}\left[t^n + (-1)^n t^{-n}\right] J_n(x)$

Problem 6: Show that

(a) $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta$ *, n being an integer*

(b) $J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\theta)d\theta$

(c) $J_0^2 + 2J_1^2 + 2J_2^2 + J_3^2 + \cdots\cdots = 1$

Solution :

We know that $e^{\frac{x}{2}(t-1/t)} = J_0(x) + \sum_{n=1}^{\infty}\left[t^n + (-1)^n t^{-n}\right]J_n(x)$

$= J_0(x) + tJ_1(x) + t^2 J_2(x) + t^3 J_3(x) + \cdots\cdots + t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + t^{-3}J_{-3}(x) + \cdots\cdots$

Since $J_{-n}(x) = (-1)^n J_n(x)$, we have

$e^{\frac{x}{2}(t-1/t)} = J_0(x) + J_1(x)(t - 1/t) + J_2(x)(t^2 + 1/t^2) + J_3(x)(t^3 - 1/t^3) + \cdots\cdots$ ----- (1)

Let t = cosθ + i sinθ so that t^p = cospθ + i sinpθ and $1/t^p$ = cospθ - i sinpθ.

From this we get, $t^p + 1/t^p$ = 2cospθ and $t^p – 1/t^p$ = 2i sinpθ

Using these results in (1), we get

$$e^{\frac{x}{2}(2i\sin\theta)} = e^{ix\sin\theta} = J_0(x) + 2\left[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \cdots\right] + 2i\left[J_1(x)\sin\theta + J_3(x)\sin 3\theta + \cdots\right]$$

-----(2)

Since $e^{ixsin\theta} = cos(xsin\theta) + i\ sin(xsin\theta)$, equating real and imaginary parts in (2) we get,

$\cos(x\sin\theta) = J_0(x) + 2\left[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \cdots\right]$ ----- (3)

$\sin(x\sin\theta) = 2\left[J_1(x)\sin\theta + J_3(x)\sin 3\theta + \cdots\right]$ ----- (4)

These series are known as ***Jacobi Series***.

Now multiplying both sides of (3) by $\cos n\theta$ and both sides of (4) by $\sin n\theta$ and integrating each of the resulting expression between 0 and π, we obtain

$$\frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\cos n\theta d\theta = \begin{cases} J_n(x), & \text{n is even or zero} \\ 0, & \text{n is odd} \end{cases}$$

and $\frac{1}{\pi}\int_0^{\pi} sin(x\sin\theta)\sin n\theta d\theta = \begin{cases} 0, & \text{n is even} \\ J_n(x), & \text{n is odd} \end{cases}$

Here we used the standard result $\int_0^{\pi}\cos p\theta\cos q\theta d\theta = \int_0^{\pi}\sin p\theta\sin q\theta d\theta = \begin{cases} \frac{\pi}{2}, & \text{if } p = q \\ 0, & \text{if } p \neq q \end{cases}$

From the above two expression, in general, if n is a positive integer, we get

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi}\left[cos(x\sin\theta)\cos n\theta + sin(x\sin\theta)\sin n\theta\right]d\theta = \frac{1}{\pi}\int_0^{\pi}cos(n\theta - x\sin\theta)d\theta$$

(b) Changing θ to (π/2) θ in (3), we get

$$cos(x\cos\theta) = J_0(x) + 2\left[J_2(x)cos(\pi - 2\theta) + J_4(x)cos(\pi - 4\theta) + \cdots\right]$$

$$cos(x\cos\theta) = J_0(x) - 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta - \cdots$$

Integrating the above equation w.r.t θ from 0 to π, we get

$$\int_0^{\pi}cos(x\cos\theta)d\theta = \int_0^{\pi}\left[J_0(x) - 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta - \cdots\right]$$

$$\int_0^{\pi}cos(x\cos\theta)d\theta = \left|J_0(x)\cdot\theta - 2J_2(x)\frac{\sin 2\theta}{2} + 2J_4(x)\frac{\sin 4\theta}{4} - \cdots\right|_0^{\pi} = J_0(x)\cdot\pi$$

Thus, $J_0(x) = \frac{1}{\pi}\int_0^{\pi}cos(x\cos\theta)d\theta$

(c) Squaring (3) and (4) and integrating w.r.t. θ *from 0 to* π and noting that *m* and *n* being integers

$$\int_0^{\pi}cos^2(x\sin\theta)d\theta = \left[J_0(x)\right]^2\cdot\pi + 4\left[J_2(x)\right]^2\frac{\pi}{2} + 4\left[J_4(x)\right]^2\frac{\pi}{2} + \cdots$$

$$\int_0^{\pi}sin^2(x\sin\theta)d\theta = 4\left[J_1(x)\right]^2\frac{\pi}{2} + 4\left[J_3(x)\right]^2\frac{\pi}{2} + \cdots$$

Adding, $\int_0^{\pi}d\theta = \pi = \pi\left[J_0^2(x) + 2J_1^2(x) + 2J_2^2(x) + J_3^2(x) + \cdots\cdots\right]$

Hence, $J_0^2 + 2J_1^2 + 2J_2^2 + J_3^2 + \cdots\cdots = 1$

Orthogonality of Bessel Functions

If α *and* β *are the two distinct roots of* $\boldsymbol{J_n(x) = 0}$, *then*

$$\int_0^{\pi} xJ_n(\alpha x)J_n(\beta x)dx = \begin{cases} 0, & \text{if } \alpha \neq \beta \\ \frac{1}{2}\left[J_n^{\prime}(\alpha)\right]^2 = \frac{1}{2}\left[J_{n+1}(\alpha)\right]^2, & \text{if } \alpha = \beta \end{cases}$$

Proof:

We know that the solution of the equation

$$x^2u^{\prime\prime} + xu^{\prime} + (\alpha^2x^2 - n^2)u = 0 \qquad \text{-------- (1)}$$

$$x^2v^{\prime\prime} + xv^{\prime} + (\beta^2x^2 - n^2)v = 0 \qquad \text{-------- (2)}$$

are $u = J_n(\alpha x)$ *and* $v = J_n(\beta x)$ respectively.

Multiplying (1) by ***v/x*** and (2) by ***u/x*** and subtracting, we get

$$x(u^{\prime\prime} v - u v^{\prime\prime}) + (u^{\prime} v - uv^{\prime}) + (\beta^2 - \alpha^2)xuv = 0$$

or $$\frac{d}{dx}\left\{x\left(u^{\prime} v - uv^{\prime}\right)\right\} = \left(\beta^2 - \alpha^2\right)xuv$$

Now integrating both sides from 0 to 1, we get

$$\left(\beta^2 - \alpha^2\right)\int_0^1 xuv dx = \left[x\left(u^{\prime} v - uv^{\prime}\right)\right]_0^1 = \left(u^{\prime} v - uv^{\prime}\right)_{x=1} \qquad \text{------- (3)}$$

Since $u = J_n(\alpha x)$, $u^{\prime} = \frac{d}{dx}\left[J_n(\alpha x)\right] = \frac{d}{d(\alpha x)}\left[J_n(\alpha x)\right] \cdot \frac{d(\alpha x)}{dx} = \alpha J_n^{\prime}(\alpha x)$

Similarly $v = J_n(\beta x)$ *gives* $v^{\prime} = \frac{d}{dx}\left[J_n(\beta x)\right] = \beta J_n^{\prime}(\beta x)$. Substituting these values in (3), we get

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \frac{\alpha J_n^{\prime}(\alpha)J_n(\beta) - \beta J_n(\alpha)J_n^{\prime}(\beta)}{\beta^2 - \alpha^2} \qquad \text{------- (4)}$$

If α and β are the two distinct roots of $\mathbf{J_n(x) = 0}$, then $J_n(\alpha) = 0$ and $J_n(\beta) = 0$, and hence (4) reduces to

$$\int_0^{\pi} xJ_n(\alpha x)J_n(\beta x)dx = 0 .$$

This is known as Orthogonality relation of Bessel functions.

When $\beta = \alpha$, the RHS of (4) takes 0/0 form. Its value can be found by considering α as a root of $J_n(x) = 0$ and β as a variable approaching to α. Then (4) gives

$$\underset{\beta\to\alpha}{Lt}\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \underset{\beta\to\alpha}{Lt}\frac{\alpha J_n^{/}(\alpha)J_n(\beta)}{\beta^2-\alpha^2}$$

Applying L'Hospital rule, we get

$$\underset{\beta\to\alpha}{Lt}\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \underset{\beta\to\alpha}{Lt}\frac{\alpha J_n^{/}(\alpha)J_n^{/}(\beta)}{2\beta} = \frac{1}{2}\left\{J_n^{/}(\alpha)\right\}^2 \text{ --------(5)}$$

We have the recurrence relation $J_n^{/}(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$.

$\therefore \quad J_n^{/}(\alpha) = \frac{n}{\alpha}J_n(\alpha) - J_{n+1}(\alpha)$. Since $J_n(\alpha) = 0$, we have $J_n^{/}(\alpha) = -J_{n+1}(\alpha)$

Thus, (5) becomes $\underset{\beta\to\alpha}{Lt}\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \frac{1}{2}\left\{J_n^{/}(\alpha)\right\}^2 = \frac{1}{2}\left\{J_{n+1}(\alpha)\right\}^2$