COMPLEX ANALYSIS AND SPECIAL FUNCTIONS

Course Code: AHSB05

B . TECH: III SEM ECE

Regulation: IARE-R18

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2020- 2021

COMPLEX ANALYSIS AND SPECIAL FUNCTIONS

Complex analysis and special functions

Unit-I: Complex Functions and differentiation

Unit-II: Complex integration

Unit-III: Power series expansions of complex functions

Unit-IV: Special functions-I (Beta and Gamma functions)

Unit-IV: Special functions-II (Bessel Differential Equations)

MODULE-I
COMPLEX FUNCTIONS AND DIFFERENTIATION

Second slide: Course outcome / Topic learning outcome

List the course outcome / Topic outcome

Name of the Topic covered	Topic Learning Outcome	Course Outcome
Complex Functions and differentiation	**Identify** the fundamental concepts of analyticity and differentiability for calculus of complex functions and their role in applied context.	**Utilize** the concepts of analyticity for finding complex conjugates and their role in applied contexts.

Complex number:

For a complex number z = x + iy, the number Re z = x is called the real part of z and the number Im z = y is said to be the its imaginary part. If x = 0, z is said to be a purely imaginary number.

Definition : Let z = x + iy ∈ C. The complex number z = x − iy is called the complex conjugate of z and |z| = $\sqrt{x^2 + y^2}$ is said to be the absolute value or the modulus of the complex number z.

Let D be a nonempty set in C. A single-valued complex function or, simply, a complex function f : D → C is a map that assigns to each complex argument z = x + iy in D a unique complex number w = u + iv. We write w = f(z).

The set D is called the domain of the function f and the set f(D) is the range or the image of f. So, a complex-valued function f of a complex variable z is a rule that assigns to each complex number z in a set D one and only one complex number w. We call w the image of z under f.

If $z = x + iy \in D$, we shall write $f(z) = u(x, y) + iv(x, y)$ or $f(z) = u(z) + iv(z)$. The real functions u and v are called the real and, respectively, the imaginary part of the complex function f. Therefore, we can describe a complex function with the aid of two real functions depending on two real variables.

Example 1.The function f : C → C, defined by $f(z) = z^3$, can be written as $f(z) = u(x, y) + iv(x, y)$, with $u, v : R^2 \to R$ given by $u(x, y) = x^3 - 3xy^2$, $v(x, y) = 3x^2 y - y^3$.

Example 2.For the function f : C → C, defined by $f(z) = e^z$, we have

$u(x, y) = e^x \cos y$, $v(x, y) = e^x \sin y$, for any $(x, y) \in R^2$.

Exercise 1: Prove that $\lim_{z \to z_0} \frac{\bar{z}}{z}$ does not exist.

Solution : To prove that the above limit does not exist, we compute this limit as z → 0 on the real and on the imaginary axis, respectively. In the first situation, i.e. for z = x ∈ R, the value of the limit is 1. In the second situation,

i.e. for z = i y, with y ∈ R, the limit is −1. Thus, the limit depends on the direction from which we approach 0, which implies that the limit does not exist.

Let w = f(z) be a given function defined for all z in a neighbourhood of z_0.If $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ exists,the function f(z) is said to be derivable at z_0 and the limit is denoted by $f'(z_0)$. $f'(z_0)$ if exists is called the derivative of f(z) at z_0

Exercise : f(z)= $|z^2|$ is a function which is continuous at all z but not derivable at any

$z \neq 0$

Solution: Let f(z)=$|z^2|$ $= z\bar{z}$

Then f(z)=$z_0\bar{z}_0$

We have to prove that $\underset{z\to z_0}{lt}$ z=z_0 and $\underset{z\to z_0}{lt}\ \bar{z}=\bar{z}_0$ Thus $\underset{z\to z_0}{lt}\ z\bar{z}=z_0\bar{z}_0$

$\therefore \underset{z\to z_0}{lt}$ f(z)=f(z_0)

$\therefore$The function is continuous at all z

The Cauchy–Riemann equations on a pair of real-valued functions of two real variables $u(x,y)$ and $v(x,y)$ are the two equations:

1. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$
2. $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Typically *u* and *v* are taken to be the real and imaginary parts respectively of a complex-valued function of a single complex variable $z = x + iy$, $f(x + iy) = u(x,y) + iv(x,y)$

COMPLEX FUNCTIONS AND DIFFERENTIATION

- We found

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \frac{df}{dz} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

- For a unique derivative, these expressions must be equal. That is, a *necessary* condition for the existence of a derivative of function of a complex variable is that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{Cauchy-Riemann conditions}$$

- We've proved that if $\frac{df}{dz}$ exists,

(implies) $\Rightarrow$ Cauchy-Riemann conditions (necessity).

polar form of Cauchy-Riemann equation:

Theorem:

If $f(z)=f(re^{i\theta})=u(r,\theta)+iv(r,\theta)$ and f(z) is derivable at $z_0=r_0e^{i\theta_0}$ then

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Proof: Let $z=re^{i\theta}$ Then $f(z)=f(re^{i\theta})=u(r,\theta)+iv(r,\theta)$

Differentiating it with respect to r partially,

$$\frac{\partial}{\partial r}f(z)=f'(z)\frac{\partial z}{\partial r}=f'(z)e^{i\theta}$$

$\therefore$ f'(z)= $\frac{1}{e^{i\theta}}\frac{\partial f}{\partial r}=\frac{1}{e^{i\theta}}(u_r+iv_r)$

$\rightarrow (1)$

Similarly differentiating partially with respect to θ

$$\frac{\partial f}{\partial \theta} = f'(z)\frac{\partial z}{\partial \theta} = f'(z).rie^{i\theta}$$

$$\therefore\ f'(z) = \frac{1}{rie^{i\theta}}(u_\theta + iv_\theta) \qquad \rightarrow (2)$$

From (1) and (2) we have

$$\frac{1}{e^{i\theta}}(u_r + iv_r) = \frac{1}{rie^{i\theta}}(u_\theta + iv_\theta)$$

$$\therefore\ u_r + iv_r = \frac{1}{r}\frac{\partial v}{\partial \theta} - i\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Equating real and imaginary parts ,we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}\ and\ \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Analytic function

A complex function is said to be analytic on a region R if it is complex differentiable at every point in R. The terms holomorphic function, differentiable function, and complex differentiable function are sometimes used interchangeably with "analytic function"

If a complex function is analytic on a region R , it is infinitely differentiable in R.

Singularities:

A complex function may fail to be analytic at one or more points through the presence of singularities, or along lines or line segments through the presence of branch cuts.

Eg. f(z)= $\frac{1}{z}$ is analytic every where except at z=0.

At z=0 $f'(z)$ does not exist.

So z=0 is an isolated singular point.

Entire function

A complex function that is analytic at all finite points of the complex plane is said to be entire. A single-valued function that is analytic in all but possibly a discrete subset of its domain, and at those singularities goes to infinity like a polynomial (i.e., these exceptional points must be poles and not essential singularities), is called a meromorphic function.

Cauchy–Riemann equations

The Cauchy–Riemann equations on a pair of real-valued functions of two real variables $u(x,y)$ and $v(x,y)$ are the two equations:

1. $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
2. $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Typically *u* and *v* are taken to be the real and imaginary parts respectively of a complex-valued function of a single complex variable $z = x + iy$, $f(x + \mathrm{i}y) = u(x,y) + \mathrm{i}v(x,y)$

Conjugate harmonic function:

If two harmonic functions u and v satisfy the Cauchy-Reimann equations in a domain D and they are real and imaginary parts of an analytic function f in D then v is said to be a conjugate harmonic function of u in D.If f(z)=u+iv is an analytic function and if u and v satisfy Laplace's equation ,then u and v are called conjugate harmonic functions.

C-R equations in polar form

The Cauchy-Riemann equations can be written in other coordinate systems. For instance, it is not difficult to see that in the system of coordinates given by the polar representation $z = re^{i\theta}$ these equations take the following form:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Problem: Show that the function $f(z)=e^z$ ***satisfies the Cauchy-Riemann equations.***

Solution:

since $e^z = e^x(\cos y + i \sin y),$

Indeed it follows that

$u(x,y)=e^x\cos y, \quad v(x,y)=e^x \sin y \;, u(x,y)=e^x\cos y, \quad v(x,y)=e^x\text{si}$

and $\frac{\partial u}{\partial x} = e^x\cos y = \frac{\partial v}{\partial y} \;; \quad \frac{\partial u}{\partial y} = e^x\sin y = -\frac{\partial v}{\partial x} \;;$

Moreover, e^z is complex derivable and it follows immediately that its complex derivative is e^z.

Construction of analytic function whose real or imaginary part is known:

Suppose f(z)=u+iv is an analytic function ,whose real part u is known .We can find v, the imaginary part and also the function f(z).

Problem: Showthat $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)|=0$ where f(z) is an analytic function.

Solution: Taking $x=\frac{z+\bar{z}}{2}, y=\frac{z-\bar{z}}{2}=\frac{-i}{2}(z-\bar{z})$

We have $\frac{\partial}{\partial z}=\frac{\partial}{\partial x}\left(\frac{\partial x}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial y}{\partial z}\right)=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)$

And $\frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$

$$\therefore \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \cdot \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

Hence $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\log|f'(z)| = 4\frac{\partial^2}{\partial z \partial \bar{z}}\left(\frac{1}{2}\log|f'(z)|^2\right)$

$$= 2\frac{\partial^2}{\partial z \partial \bar{z}}$$

$[(\log f'(z) f'(\bar{z}))]$ $\quad (\because |z|^2 = z\bar{z})$

$$= 2\frac{\partial^2}{\partial z \partial \bar{z}}$$

$[(\log f'(z) + f'(\bar{z}))]$

$$= 2$$

$$\left[\frac{\partial}{\partial \bar{z}}\frac{f''(z)}{f'(z)} + \frac{\partial}{\partial z}\frac{f''(z)}{f'(z\_)}\right]$$

$$= 2(0+0)=0$$

Since f(z) is analytic , f(z) is analytic, $f'(\bar{z})$ is also analytic and $\frac{\partial f'(z)}{\partial \bar{z}} = 0, \frac{\partial f'(\bar{z})}{\partial z} = 0$

Problem: Show that f(z)= $\begin{cases} \frac{xy(x+iy)}{x^2+y^4} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$ is not analytic at z=0 although C-R equations satisified at origin.

Solution: $\frac{f(z)-f(0)}{z-0}=\frac{f(z)-0}{z}=\frac{f(z)}{z}$

$$=\frac{xy^2(x+iy)}{(x^2+y^4).z}=\frac{xy^2(z)}{(x^2+y^4).z}=\frac{xy^2}{(x^2+y^4)}$$

Clearly $\lim_{\substack{x\to 0\\ y\to 0}}\frac{xy^2}{(x^2+y^4)} = \lim_{\substack{y\to 0\\ x\to 0}}\frac{xy^2}{(x^2+y^4)} = 0$

Along path y=mx

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{x\to 0}\frac{x(m^2.x^2)}{x^2+m^4.x^4}=\lim_{x\to 0}\frac{m^2.x^2}{1+m^4.x^2}=0$$

Along path $x=my^2$

$$\lim_{z \to 0} \frac{f(z)-f(0)}{z-0} = \lim_{y \to 0} \frac{y^2(m.y^2)}{y^4+m^2.y^4} = \lim_{y \to 0} \frac{m}{1+m^2} \neq 0$$

Limit value depends on m i.e on the path of approach and its different for the different paths

Followed and therefore limit does not exists.

Hence f(z) is not differentiable at z=0.Thus f(z) is not analytic at z=0

To prove that C-R conditions are satisified at origin

Let $f(z) = u + iv = \frac{xy^2(x+iy)}{(x^2+y^4)}$

Then u(x,y)= $\frac{x^2y^2}{(x^2+y^4)}$ and v(x,y)= $\frac{xy^3}{(x^2+y^4)}$ for z $\neq 0$

Also u(0,0)=0 and v(0,0)= 0

[$\because$f(z)=0 at z=0]

Now $\frac{\partial u}{\partial x} = x \xrightarrow{\lim} 0 \frac{u(x,0) - u(0,0)}{x} = x \xrightarrow{\lim} 0 \frac{0}{x} = 0$

$$\frac{\partial u}{\partial y} = y \xrightarrow{\lim} 0 \frac{u(0,y) - u(0,0)}{y} = x \xrightarrow{\lim} 0 \frac{0}{y} = 0$$

$$\frac{\partial v}{\partial x} = x \xrightarrow{\lim} 0 \frac{v(x,0) - v(0,0)}{x} = x \xrightarrow{\lim} 0 \frac{0}{x} = 0$$

$$\frac{\partial v}{\partial y} = y \xrightarrow{\lim} 0 \frac{v(0,y) - v(0,0)}{y} = x \xrightarrow{\lim} 0 \frac{0}{y} = 0$$

Thus C-R equations are satisified are satisified at the origin

Hence f(z) is not analytic at z=0 even C-R equations are satisified at origin.

Milne Thomson method:

Problem : Find the regular function whose imaginary part is $\log(x^2+y^2)+x-2y$.

Solution: Given $v=\log(x^2+y^2)+x-2y$

$$\therefore \frac{\partial v}{\partial x}=\frac{2x}{x^2+y^2}+1 \quad \text{-}\!\!\rightarrow(1) \qquad \text{and} \quad \frac{\partial v}{\partial y}=\frac{2y}{x^2+y^2}-2 \quad \text{-}\!\!\rightarrow(2)$$

$$f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}$$ (Using C-R equation)

$$=\frac{2y}{x^2+y^2}-2+i\left(\frac{2x}{x^2+y^2}+1\right)$$ (using (1) ,(2))

By Milne Thomson method ,f'(z) is expressed in terms of z by replacing x z and y by 0.

Hence $f'(z) = -2 + i\left(\frac{2z}{z^2} + 1\right) = -2 + i\left(\frac{2}{z} + 1\right)$

On integrating, $f(z) = \int \left[-2 + i\left(\frac{2}{z} + 1\right)\right] dz + c$

$$= -2z + i(2\log z + z) + c = 2i\log z - (2 - i)z + c.$$

Problem: Show that the function $u = 4xy - 3x + 2$ is harmonic .construct the corresponding analytic function f(z)=u+iv in terms of z.

Solution: Given $u = 4xy - 3x + 2 \rightarrow (1)$

Differentiating (1) partially w.r.t .x, $\frac{\partial u}{\partial x} = 4y - 3$

Again differentiating $\frac{\partial^2 u}{\partial x^2} = 0$

Again differentiating (1) partially w.r.t .y, $\frac{\partial u}{\partial x} = 4x$

Again differentiating $\frac{\partial^2 u}{\partial y^2} = 0$

Let a,b,c,d denote four complex constants with the restriction

$ad \neq bc$ that . Then the function

$$w = s(z) = \frac{az+b}{cz+d}$$

is called a bilinear transformation, a Möbius transformation, or a linear fractional transformation.

There exists a unique bilinear transformation that maps three distinct points $z_1, z_2 \ and \ z_3$ onto three distinct points $w_1, w_2 \ and w_3$ respectively. An implicit formula for the mapping is given by the equation

$$\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}$$

Problem: Construct the bilinear transformation w = S(z) that maps the points $z_1 = -i, \ z_2 = 1, \ z_3 = i$ onto the points $w_1 = -1, \ w_2 = 0, \ w_3 = 1$, respectively.

Solution: We use the implicit formula and write

$$\frac{(z - (-i))(1 - i)}{(z - i)(1 - (-i))} = \frac{(w - (-1))(0 - 1)}{(w - 1)(0 - (-1))}$$

$$\frac{(z + i)(1 - i)}{(z - i)(1 + i)} = \frac{(w + 1)(0 - 1)}{(w - 1)(0 + 1)}$$

$$\frac{(z + i)(1 - i)}{(z - i)(1 + i)} = \frac{w + 1}{-w + 1}.$$

Expanding this equation, collecting terms involving w and zw on the left and then simplify.

$$(z - i)(1 + i)(w + 1) = (z + i)(1 - i)(-w + 1)$$

$$(1 + i)zw + (1 - i)w + (1 + i)z + (1 - i)$$

$$=$$

$$(-1 + i)zw + (-1 - i)w + (1 - i)z + (1 + i)$$

$$zw + izw + w - iw + z + iz + 1 - i$$

$$=$$

$$-zw + izw - w - iw + z - iz + 1 + i$$

$$2zw + 2w = -2iz + 2i$$

$$zw + w = -iz + i$$

$$w(1+z) = i(1-z)$$

Therefore the desired bilinear transformation is

$$w = S(z) = \frac{i(1-z)}{1+z}.$$

Problem: Find the bilinear transformation $w = S(z)$ that maps the points $z_1 = -2,\ z_2 = -1 - i,\ z_3 = 0$ onto the points $w_1 = -1,\ w_2 = 0,\ w_3 = 1$, respectively.

Solution. Again, we use the implicit formula and write

$$\frac{(z-(-2))((-1-i)-0)}{(z-0)((-1-i)-(-2))} = \frac{(w-(-1))(0-1)}{(w-1)(0-(-1))}$$

$$\frac{(z+2)(-1-i)}{(z)(-1-i+2)} = \frac{(w+1)(-1)}{(w-1)(1)}$$

$$\frac{z+2}{z}\,\frac{-1-i}{1-i} = \frac{1+w}{1-w}$$

Using the fact that $\frac{-1-i}{1-i}=\frac{1}{i}$, we rewrite this equation as

$$\frac{z+2}{iz}=\frac{1+w}{1-w}.$$

We now expand the equation and obtain

$$(z+2)(1-w)=iz(1+w)$$
$$z+2-zw-2w=iz+izw$$
$$z-iz+2=zw+2w+izw$$
$$z(1-i)+2=w(z+iz+2)$$
$$(1-i)z+2=w((1+i)z+2)$$

which can be solved for w in terms of z, giving the desired solution

$$w = s(z) = \frac{(1-i)z+2}{(1+i)z+2}$$

EXCERCISE PROBLEMS:

1)Show that the real part of an analytic function f (z) where u = $e^{-2xy}\sin(x^2-y^2)$ is a harmonic function. Hence find its harmonic conjugate.

2) Prove that the real part of analytic function f (z) where u = log$|z|^2$ is harmonic function. If so find the analytic function by Milne Thompson method.

3)Obtain the regular function f (z) whose imaginary part of an analytic function is $\frac{x-y}{x^2+y^2}$

4) Find an analytic function f (z) whose real part of an analytic function is u = $\frac{\sin 2x}{\cosh 2y-\cos 2x}$ by Milne-Thompson method.

5) Find an analytic function f (z) = u +iv if the real part of an analytic function is u = a (1+cosθ) using Cauchy-Riemann equations in polar form.

6) Prove that if $u = x^2 - y^2, v = -\frac{y}{x^2+y^2}$ both u and v satisfy Laplace's equation, but u + iv is not a regular (analytic) function of z.

7)Show that the function f (z) = $\sqrt{|xy|}$ is not analytic at the origin although Cauchy –Riemann equations are satisfied at origin.

8) If $w = \emptyset + i\varphi$ represents the complex potential for an electric field where $\varphi = x^2 - y^2 + \frac{x}{x^2+y^2}$ then determine the function φ.

9)State and Prove the necessary condition for f (z) to be an analytic function in Cartesian form.

10)If u and v are conjugate harmonic functions then show that uv is also a harmonic function.

11)Find the orthogonal trajectories of the family of curves $r^{2} cos2\theta$ = c

12)Find an analytic function whose real part is $u = \frac{sin\,2x}{cos\,h2y - cos\,2x}$

13)Find an analytic function whose imaginary part is v = $e^x(xsiny + ycosy)$

14)Find an analytic function whose real part is (i) u = $\frac{x}{x^2+y^2}$ (ii) u = $\frac{y}{x^2+y^2}$

15)Find an analytic function whose imaginary part is v =$\frac{2sinxsiny}{cosh2x+cosh2y}$

16)Find an analytic function f(z) = u +iv if u = a(1+cosθ)

17)Find the conjugate harmonic of u = $e^{x^2-y^2}$cos2xyand find f(z) in terms of z.

18)If f(z) is an analytic function of z and if u - v = $e^x(cosy - siny)$ find f(z) in terms of z.

19)If f(z) is an analytic function of z and if u - v = (x-y)$(x^2 + 4xy + y^2)$) find f(z) in terms of z.

20) Find the orthogonal trajectories of the family of curves $x^3y - xy^3$= C = constant

MODULE-II

COMPLEX INTEGRATION

Defination: In mathematics, a line integral is an integral where the function to be integrated is evaluated along a curve. The terms path integral, curve integral, and curvilinear integral are also used; contour integral as well, although that is typically reserved for line integrals in the complex plane.

The function to be integrated may be a scalar field or a vector field. The value of the line integral is the sum of values of the field at all points on the curve, weighted by some scalar function on the curve (commonly arc length or, for a vector field, the scalar product of the vector field with a differential vector in the curve). This weighting distinguishes the line integral from simpler integrals defined on intervals. Many simple formulae in physics (for example, $W = F \cdot s$) have natural continuous analogs in terms of line integrals ($W = \int_C F \cdot ds$). The line integral finds the work done on an object moving through an atomic or gravitational field.

COMPLEX INTEGRATION

In complex analysis, the line integral is defined in terms of multiplication and addition of complex numbers.

Let us consider F(t)= u(t)+i v(t) , $a \le t \le b$. Where u and v are real valued continuous functions of t in [a,b].

we define $\int_a^b F(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$

Thus, $\int_a^b F(t)\,dt$ is a complex number such that real part of $\int_a^b F(t)\,dt$ is $\int_a^b u(t)\,dt$ and imaginary part of $\int_a^b F(t)\,dt$ is $\int_a^b v(t)\,dt$.

Problem: Evaluate $\int_0^{1+i}(x^2 - iy)dz$ along the paths

1)y=x 2)y=x^2

Solution: 1)along the line y=x, dy= dx so that dz = dx+idx=(1+i) dx

$$\therefore \int_0^{1+i}(x^2 - iy)dz = \int_0^1 (x^2 - ix)(1+i)dx,$$ since y=x

$$=(1+i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1$$

$$=(1+i)\left[\frac{1}{3} - \frac{1}{2}i\right]$$

$$=\frac{5}{6} - \frac{1}{6}i$$

2) alongtheparabola y=x^2,dy=2xdx sothatdz=dx+2ixdx

dz=(1+2ix)dx and xvaries from 0to1

$$\therefore \int_0^{1+i} (x^2 - iy)dz = \int_0^1 (x^2 - ix^2)(1+2ix)dx$$

$$=(1-i)\int_0^1 x^2(1+2ix)dx$$

$$=(1-i)\left(\frac{1}{3}+\frac{1}{2}i\right)$$

$$=\frac{(1-i)(2+3i)}{6}$$

$$=\frac{5}{6}+\frac{1}{6}i$$

Problem: Evaluate $\int_{z=0}^{z=1+i}(x^2+2xy+i(y^2-x))dz$ along $y=x^2$

Solution: Given $f(z)=x^2+2xy+i(y^2-x))dz$
$z=x+iy, dz=dx+idy$

$$\therefore thecurve, y=x^2, dy=2xdx$$

$$\therefore dz = dx+2xidx=(1+2ix)dx$$

$$F(z)=x^2+2xx^2+i(x^4-x)$$

$$=x^2+2x^3+i(x^4-x)$$

$$F(Z)dz=(x^2+2x^3)+i((x^4-x)(1+2ix))dx$$
$$=x^2+2x^3+i(x^4-x)+2ix^3+4ix^4-2x^5+2x^2$$

$$\therefore \int_c f(z)dz = \int_{z=0}^{1+i} x^2 + 2xy + i(y^2 - x)dz$$

$$= \int_0^1 (-2x^5 + 3x^2 + 2x^3 + i(5x^4 - x + 2x^3))dx)$$

$$= \left[-\frac{x^6}{3} + x^3 + \frac{x^4}{2} + i(\frac{5x^5}{5} - \frac{x^2}{2} + \frac{x^4}{2} \right]_0^1$$

$$= \left[\left(\frac{-1}{3} + 1 + \frac{1}{2} \right) + \left(\frac{5}{5} - \frac{1}{2} + \frac{1}{2} \right) \right] - 0$$

$$= \frac{7}{6} + \frac{5}{5} i = \frac{7}{6} + i$$

$$\int_c f(z)dz = \frac{7}{6} + i$$

CAUCHY INTEGRAL THEOREM

STATEMENT : let F(z)=u(x,y)+iv(x,y) be analytic on and within a simple closed contour (or curve) 'c' and let $f'(z)$ be continuous there,then

$$\int f(z)dz = 0$$

Proof: f (z)=u(x,y)+iv(x,y)

And dz=dx+idy

$\Rightarrow$f(z).dz = (u(x,y)+iv(x,y))dx+idy

f(z).dz = u(x,y)dx+i u(x,y)dy+iv(x,y)dx+i^2 v(x,y)dy

f(z).dz= u(x,y)dx- v(x,y)dy+i(u(x,y)dy+ v(x,y)dx

Integrate both sides, we get

$$\int f(z)dz = \int (\text{udx} - \text{vdy}) + i(\text{udy} + \text{vdx})$$

By greens theorem ,we have

$$\int Mdx + Ndy = \iint \frac{\partial N}{\partial x} - \frac{\partial M}{\partial Y} dxdy$$

Now $\int f(z)dz = \iint (-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial Y})dxdy + i(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial Y})dxdy$

Since $f'(z)$ is continuous &four partial derivatives i.e $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial Y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial Y}$ are also continuous in the region R enclosed by C, Hence we can apply Green's Theorem.

Using Green's Theorem in plane ,assuming that R is the region bounded by C.

It is given that f (z)=u(x,y)+iv(x,y) is analytic on and within c.

Hence $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Using this we have

$$\int_c f(z)dz = \iint_R 0 \quad dxdy + i\iint_R 0 \quad dxdy \quad = 0$$

Hence the theorem.

Cauchy's integral formula

Cauchy's integral formula states that

$$f(z_0) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)}dz$$

where the integral is a contour integral along the contour c enclosing the point z_0.

Generalization of Cauchy's integral formula:

Generalization of Cauchy's integral formula states that

$$f^{n}(z_0) = \frac{n!}{2\pi i}\oint_{c} \frac{f(z)}{(z-a)^{n+1}} dz$$

where the integral is a contour integral along the contour c enclosing the point z_0

Problem: Evaluate using cauchy's integral formula $\int_c \frac{e^{2z}}{(z-1)(z-2)} dz$ where c is the circle $|z| = 3$

Solution: Given $\int_c \frac{e^{2z}}{(z-1)(z-2)} dz$ ....(1)

Both the points z=1,z=2 line inside $|z| = 3$

Resolving into partial fractions

$$\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

A=-1, B=1

From(1)

$$\int_c \frac{e^{2z}}{(z-1)(z-2)} dz = \int_c \frac{-e^{2z}}{(z-1)} dz + \int_c \frac{e^{2z}}{(z-2)} dz$$

(by cauchy's integral formula)

$$=-2\pi i f(1)+2\pi i f(2)$$

$$=-2\pi i e^{2.1}+2\pi i e^{2.2}$$

$$=-2\pi i e^{2}+2\pi i e^{4}=2\pi i(e^{4}-e^{2})$$

Problem: Using cauchy'sintegralformula to evaluate $\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)z-2)} dz$, where c is the circle $|z| = 3$

Solution: $\int_c \frac{f(z)}{(z-1)z-2)} dz$ =($\int_c \frac{1}{(z-2)} dz + \int_c \frac{1}{(z-1)} dz$)f(z)dz

$$= \int_c \frac{f(z)}{(z-2)} dz + \int_c \frac{f(z)}{(z-1)} dz$$

=2Πif(2)- 2∏if(1)

=2Πi(sin4Π+cos4Π)-(sinΠ+cosΠ))

=2Πi(1-(-1))=4Πi

$\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)z-2)} dz$ =4Πi

Problem: Evaluate $\int_c \frac{(z-1)}{(z+1)^2(z-2)}dz$ whrere c is $|Z-i|=2$

Solution: the singularities of $\frac{(z-1)}{(z+1)^2(z-2)}$ are given by

$(z+1)^2(z-2)=0$

$\Rightarrow$ Z=-1 and z=2

Z=-1 lies inside the circle since $|-1-i|-2<0$

Z=2 lies outside the circle sinceI2-iI-2>0 $|2-i|-2>0$

The given line integral can be written as

$$\int_c \frac{(z-1)}{(z+1)^2(z-2)}dz = \int_c \frac{\frac{(z-1)}{(z-2)}}{(z+1)^2} \text{-----------------------------}(1)$$

The derivative of analytic function is given by

$$\int_c \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i f^n(a)}{n!} \text{--(2)}$$

From (1) and (2) f(z)=$\frac{(z-1)}{(z-2)}$, a=-1,n=1

$$f^1(z) = \frac{1(z-2)-1(z-1)}{(z-2)^2} = \frac{1}{(z-2)^2}$$

$$f^1(-1) = \frac{1}{-9}$$

Substituting in (2),we get

$$\int_c \frac{(z-1)}{(z+1)^2(z-2)}dz = \frac{2\pi i}{1}(-\frac{1}{9})$$

$$=\frac{-2}{9}\Pi i$$

Problem: Evaluate $\int_c \frac{e^{2z}}{(z+1)^4}dz$ where c: $|z-1|=1$

Solution: the singular points of $\frac{e^{2z}}{(z+1)^4}dz$ are givenby7

$(z+1)^4=0 \Rightarrow z=-1$

The singular point z=-1 lies insidethecirclec: $|z-1|=3$

Applying cauchy's integral formula for derivatives

$$\int_c \frac{f(z)}{(z-a)^{n+1}}dz = \int_c \frac{2\pi i f^n(-1)}{n!}dz \text{ -------------------------(1)}$$

f(z)=e^{2z},n=3,a=-1

f(z)=$2e^{2z}$

$f^{1}(z)=4e^{2z}$

$f^{11}(z)=8e^{2z}$

$f^{111})z)=16e^{2z}$

$f^{111}(-1)=16e^{-2}$

substituting in(1)

$$\int_c \frac{e^{2z}}{(z+1)^4}dz = \int_c \frac{2\pi i f^{111}(-1)}{n!}$$

$$= \frac{2\pi i 16 e^{-2}}{2!}$$

$=16\Pi ie^{-2}$

Problem: Use cauchy's integral formula to evaluate $\int_c \frac{e^{-2z}}{(z+1)^3} dz \quad with\ c: |z| = 2$

Solution:

Given $\int_c \frac{e^{-2z}}{(z+1)^3} dz$

$f(z)=e^{-2z}$

the singular poin z=-1 lies inside the given circle $|z| = 2$

apply Cauchy's integral formula for derivatives

$$\int_c \frac{e^{-2z}}{(z+1)^3}dz = \frac{2\pi i f^1(-1)}{2!} \qquad \left[\because \int_c \frac{f(z)}{(z-a)^3} = \frac{2\pi i f^1(a)}{2!}\right]$$

Where $f(z)=e^{-2z}$

$f^1(z)=-2\ e^{-2z}$

$f^{11}(z)=4\ e^{-2z}$

$f^{11}(-1)= 4\ e^2$

$$\therefore \int_c \frac{e^{-2z}}{(z+1)^3}dz = \frac{2\pi i 4e^2}{2} = 4\pi i e^2$$

Problem: Evaluate $\int_c \frac{dz}{z^8(z+4)}dz$ *with c:* $|z| = 2$

Solution: The singularities of $\int_c \frac{dz}{z^8(z+4)}dz$ are given by

$Z^8(z+4) = 0 \Rightarrow z = 0, z = -4$ The point z=0 lie inside and the z=-4 lies outside the circle $|z| = 2$

By the derivative of analytic function.

Problem: Evaluate using integral formula $\oint_c \frac{e^{2z}dz}{(z-1)(z-2)}$ where c is the circle $|z|=3$

Solution: Let (z)= e^z which is analytic within the circle c: $|z|=3$ and the two singular points a=1,a=2 lie inside c.

$$\oint_c \frac{e^{2z}dz}{(z-1)(z-2)} = \oint e^{2z}\left(\frac{1}{z-2}-\frac{1}{z-1}\right)dz$$

$$\oint_c \frac{e^{2z}}{z-2}dz - \oint_c \frac{e^{2z}}{z-1}dz$$

Now using cauchy's integral formula ,we obtain

$$\oint_c \frac{e^{2z}dz}{(z-1)(z-2)} = 2\pi i e^4 - 2\pi i e^2$$

$$= 2\pi i(e^4 - e^2)$$

$$\oint_c \frac{e^{2z}dz}{(z-1)(z-2)} = 2\pi i(e^4 - e^2)$$

Problem : Evaluate $\oint_c \frac{3z^2+z}{z^2-1}dz$ where cisthe circle $|z-1|=1$

Solution: Given f(z)= $3z^2+z$

Z=a=+1or -1

The circle $|z-1|=1$ has centre at z=1 and radius 1 and includes the point z=1,f(z)=$3z^2+z$ is an analytic function

Also $\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$

$$\oint \frac{3z^2+z}{z^2-1} = \frac{1}{2}\left[\int_c \frac{3z^2+z}{z-1}dz\right] - \frac{1}{2}\left[\int_c \frac{3z^2+z}{z+1}dz\right] \text{-----------------}(1)$$

Since z=1 lies inside c,we have by cauchy's integralformula

$$\oint_c \frac{3z^2+z}{z^2-1}dz = 2\pi i f(i)$$

$$= 2\pi i * 4$$

Bycauchy'sintegral theorem ,since z=-1 lies out side c,we have

$$\oint_c \frac{3z^2+z}{z-1}dz = 0$$

From equation(1) we have

$$\oint_c \frac{3z^2+z}{z^2-1}dz = \frac{1}{2}(8\pi i) - 0$$

$$= 4\pi i$$

EXCERCISE PROBLEMS:

1) Evaluate $\int \frac{dz}{z - z_0}$ where c: $|z - z_0| = r$

2) Evaluate $\int_{(1,1)}^{(2,2)} (x + y)dx + (y - x)dy$ along the parabola $y^2 = x$

3) Evaluate $\int_c \frac{z^2 + 4}{z^2 - 1} dz$ where C: $|z| = 2$ using Cauchy's Integral formula

4)Evaluate $\int_c \frac{e^{2z}}{(z-1)(z-2)}dz$ where C: $|z|=4$ using Cauchy's integral formula

5) Evaluate $\int_c \frac{z^3-z}{(z-2)^3}$ where $C:|z|=3$ using Cauchy's integral formula

6) Expand f(z) = $\int_c \frac{e^{2z}}{(z-1)^3}$ at a point z=1

7) Expand f(z)= $\int_c \frac{1}{z^2-4z+3}$ for

$1<|z|<3$

8)Evaluate $\int (y^2+z^2)dx+(z^2+x^2)dy+(x^2+y^2)dz$ from (0,0,0) to (1,1,1) , where

C is the curve $x=t, x=t^2, x=t^3$

9) Evaluate $\int_{(0,0)}^{(1,1)} (3x^2+4xy+ix^2)dz$ along $y=x^2$

10) Evaluate $\int_{0}^{1+i}(x-y+ix^2)\,dz$

(i) along the straight from z = 0 to z = 1+i .

(ii) along the real axis from z = 0 to z = 1 and then along a line parallel to real axis from z = 1 to z = 1+i

(iii) along the imaginary axis from z = 0 to z = I and then along a line parallel to real axis z = i to z = 1+ i .

11) Evaluate $\int_{1-i}^{2+i}(2x+1+iy)\,dz$ along (1-i) to (2+i)

12) Evaluate $\int_{c}(y^2+2xy)dx+(x^2-2xy)dy$ where c is boundary of the region $y=x^2$ and $x=y^2$

MODULE-III

POWER SERIES EXPANSION OF COMPLEX FUNCTION

POWER SERIES EXPANSION OF COMPLEX FUNCTION

Power series:

A series expansion is a representation of a particular function as a sum of powers in one of its variables, or by a sum of powers of another (usually elementary) function f(z).

A power series in a variable z is an infinite sum of the form

$$\sum a_i z^i$$

A series of the form $\sum a_n z^n$ is called as power series.

That is $\sum a_n z^n = a_1 z + a_2 z^2 + ...+ a_n z^n + ...$

Taylor's series:

Taylor's theorem states that any function satisfying certain conditions may be represented by a Taylor series.

The Taylor series is an infinite series, whereas a Taylor polynomial is a polynomial of degree n and has a finite number of terms. The form of a Taylor polynomial of degree n for a function

f (z) at x = a is

$$f(z) = f(a) + f^{'}(a)(z-a) + f^{''}(a)\frac{(z-a)^2}{2!} + f^{'''}(a)\frac{(z-a)^3}{3!} + \ldots\ldots\ldots + f^{n}(a)\frac{(z-a)^n}{n!} + \ldots\ldots, \; |z-a| < r$$

A Maclaurin series is a Taylor series expansion of a function about x=0,

$$f(z) = f(0) + f^{'}(0)(z) + f^{''}(0)\frac{(z)^2}{2!} + f^{'''}(0)\frac{(z)^3}{3!} + \ldots\ldots\ldots + f^{n}(0)\frac{(z)^n}{n!} + \ldots\ldots\ldots$$

This series is called as maclurins series expansion of f(z).

Some important result:

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots \quad \text{for } -1 < x \le 1$$

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \ldots \quad \text{for } -\infty < x < \infty$$

$$\cos x = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \tfrac{1}{720}x^6 + \ldots \quad \text{for } -\infty < x < \infty$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots \quad \text{for } -1 < x < 1$$

Problem: Determine the first four terms of the power series for sin 2x using Maclaurin's series.

Solution:

Let

$f(x) = \sin 2x \qquad f(0) = \sin 0 = 0$

$f'(x) = 2\cos 2x \qquad f'(0) = 2\cos 0 = 2$

$f''(x) = -4\sin 2x \qquad f''(0) = -4\sin 0 = 0$

$f'''(x) = -8\cos 2x \qquad f'''(0) = -8\cos 0 = -8$

$f^{iv}(x) = 16\sin 2x \qquad f^{iv}(0) = 16\sin 0 = 0$

$f^{v}(x) = 32\cos 2x(0) \qquad f^{v}(0) = 32\cos 0 = 32$

$f^{vi}(x) = -64\sin 2x \qquad f^{vi}(0) = -64\sin 0 = 0$

$f^{vi\,i}(x) = -128\cos 2x \qquad f^{vii}(0) = -128\cos 0 = -128$

$$f(x) = \sin 2x = 0 + 2x + 0x^2 + (-8)\frac{x^3}{3!} + 0.x^4 + 32\frac{x^5}{5!}$$

$$= 2x - \frac{4x^3}{3} + \frac{4x^5}{15}$$

Problem : Find the Maclaurin series for $f(z) = z e^z$ Express your answer in sigma notation.

Solution:

Let $f(z) = z e^z$ $\quad f(0) = 0$

$f' = e^z + z e^z$ $\quad f'(0) = 1 + 0 = 1$

$f'' = e^z + e^z + z e^z$ $\quad f''(0) = 1 + 1 + 0 = 2$

$f''' = e^z + e^z + e^z + z e^z$ $\quad f'''(0) = 1 + 1 + 1 + 0 = 3$

$f'''' = e^z + e^z + e^z + e^z + z e^z$ $\quad f''''(0) = 1 + 1 + 1 + 1 + 0 = 4$

$$f(z) = 0 + 1z + \frac{2}{2!}z^2 + \frac{3}{3!}z^3 + \frac{4}{4!}z^4 + \ldots\ldots\ldots\ldots$$

$$= z + z^2 + \frac{1}{2}z^3 + \frac{1}{6}z^4 + \ldots\ldots\ldots\ldots$$

$$= \sum_{n=1}^{\infty} \frac{z^n}{(n-1)!}$$

Laurent series:

In mathematics, the **Laurent series** of a complex function $f(z)$ is a representation of that function as a power series which includes terms of negative degree. It may be used to express complex functions in cases where a Taylor series expansion cannot be applied.

The Laurent series for a complex function $f(z)$ about a point c is given by:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z-a)^n}$$

where the a_n *and* a are constants.

Zero's of an analytic function:

A zero of an analytic function f(z) is a value of z such that f(z)=0 .Particularly a point a is called a zero of an analytic function f (z) if f(a) = 0.

Eg: $f(z) = \frac{(z+1)^2}{(z^2+1)^2}$

Now, $(z+1)^2 = 0$

Z = -1, z = -1 are zero's of an analytic function.

Zero's of m^th order:

If an analytic function f(z) can be expressed in the form $f(z) = (z-a)^m \Phi(z)$ where $\Phi(z)$ is analytic function and $\Phi(a) \neq 0$ then z=a is called zero of m^{th} order of the function f(z).

- A simple zero is a zero of order 1.

Eg: 1. $f(z) = (z-1)^3$

$\Rightarrow (z-1)^3 = 0$

z=1 is a zero of order 3 of the function f(z).

2. $f(z) = \frac{1}{1-z}$

i.e $z = \infty$ is a simple zero of f(z).

3. $f(z) = \sin z$

i.e $z = n\pi \quad \forall n = 0,1,2,3,......$ are simple zero's of f(z).

Problem: Find the first four terms of the Taylor's series expansion of the complex function

$f(z)=\frac{z+1}{(z-3)(z-4)}$ about z =2.Find the region of convergence.

Solution:

The singularities of the function $f(z)=\frac{z+1}{(z-3)(z-4)}$ are z = 3 and z = 4

Draw a circle with centre at z=2 and radius 1 .Then the distance of singularities from the centre are 1 and 2.

Hence within the circle $|z-2|=1$,the given function is analytic .Hence ,it can be extended in Taylor's series within the circle $|z-2|=1$.

Hence $|z-2|=1$ is the circle of convergence.

POWER SERIES EXPANSION OF COMPLEX FUNCTION

Now $f(z) = \frac{5}{z-4} - \frac{4}{z-3}$ (partial fraction) , f(2)= 3/2

$$f'(z) = -\frac{5}{(z-4)^2} + \frac{4}{(z-3)^2} \quad , \quad f'(2) = \frac{11}{4}$$

$$f''(z) = -\frac{8}{(z-3)^3} + \frac{10}{(z-4)^3} \quad , \quad f''(2) = \frac{27}{4}$$

$$f'''(z) = \frac{24}{(z-3)^4} - \frac{30}{(z-4)^4} \quad , \quad f'''(2) = \frac{177}{8}$$

Taylor's series expansion for f(z) at z=a is

$$f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{(z-a)^2}{2!} + f'''(a)\frac{(z-a)^3}{3!} + \ldots\ldots\ldots + f^n(a)\frac{(z-a)^n}{n!} + \ldots\ldots$$

$$\frac{z+1}{(z-3)(z-4)} = \frac{3}{2} + (z-2)\frac{11}{4} + \frac{(z-2)^2}{2!}\left(\frac{27}{4}\right) + \frac{(z-2)^3}{3!}\left(\frac{177}{8}\right)$$

$$f(z) = \frac{3}{2} + (z-2)\frac{11}{4} + (z-2)^2\left(\frac{27}{8}\right) + (z-2)^3\left(\frac{59}{16}\right).$$

POWER SERIES EXPANSION OF COMPLEX FUNCTION

Problem: Obtain Laurent series for $f(z)=\dfrac{e^{2z}}{(z-1)^3}$ about z = 1.

Solution:

Given $f(z)=\dfrac{e^{2z}}{(z-1)^3}$

Put z-1= w so that z = w+1

$$f(z)=\frac{e^{2(1+w)}}{w^3}$$

$$f(z)=\frac{e^2e^{2w}}{w^3}$$

$$=\frac{e^2}{w^3}\left[1+2w+\frac{(2w)^2}{2!}+\frac{(2w)^3}{3!}+\ldots\ldots\right] \text{ if } w\neq 0$$

$$=e^2\sum_{n=0}^{\infty}\frac{2^n}{n!}w^{n-3}$$

$$=e^2\sum_{n=0}^{\infty}\frac{2^n}{n!}(z-1)^{n-3} \quad \text{, if } z-1\neq 0$$

$$=e^2\sum_{n=0}^{\infty}\frac{2^n}{n!}(z-1)^{n-3} \quad \text{, if } |z-1|\neq 0$$

$$f(z)=e^2\sum_{n=0}^{\infty}\frac{2^n}{n!}(z-1)^{n-3} \quad \text{, if } |z-1|>0$$

Since points $|z-1|\leq 0$ will be singular points.

Singular point of an analytic function: A point at which an analytic function f(z) is not analytic, i.e. at which f '(z) fails to exist, is called a **singular point** or **singularity** of the function.

There are different types of singular points:

Isolated and non-isolated singular points: A singular point z_0 is called an **isolated singular point** of an analytic function f(z) if there exists a deleted ε-spherical neighborhood of z_0 that contains no singularity. If no such neighborhood can be found, z_0 is called a non-isolated singular point. Thus an isolated singular point is a singular point that stands completely by itself, embedded in regular points. In fig 1a where z_1, z_2 and z_3 are isolated singular points. Most singular points are isolated singular points. A non-isolated singular point is a singular point such that every deleted ε-spherical neighborhood of it contains singular points. See Fig. 1b where z_0 is the limit point of a set of singular points. Isolated singular points include poles, removable singularities, essential singularities and branch points.

POWER SERIES EXPANSION OF COMPLEX FUNCTION

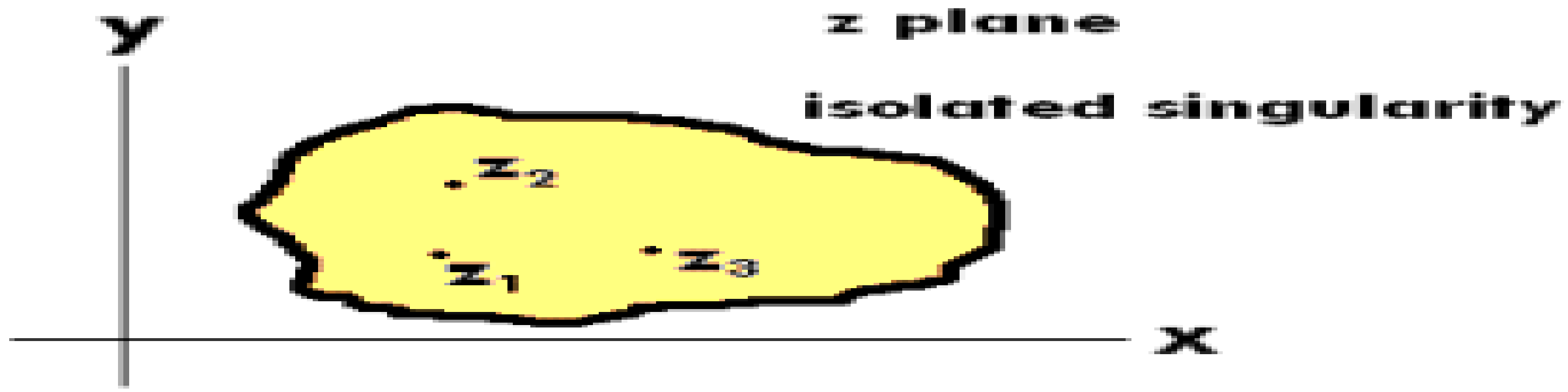


Fig. 1a

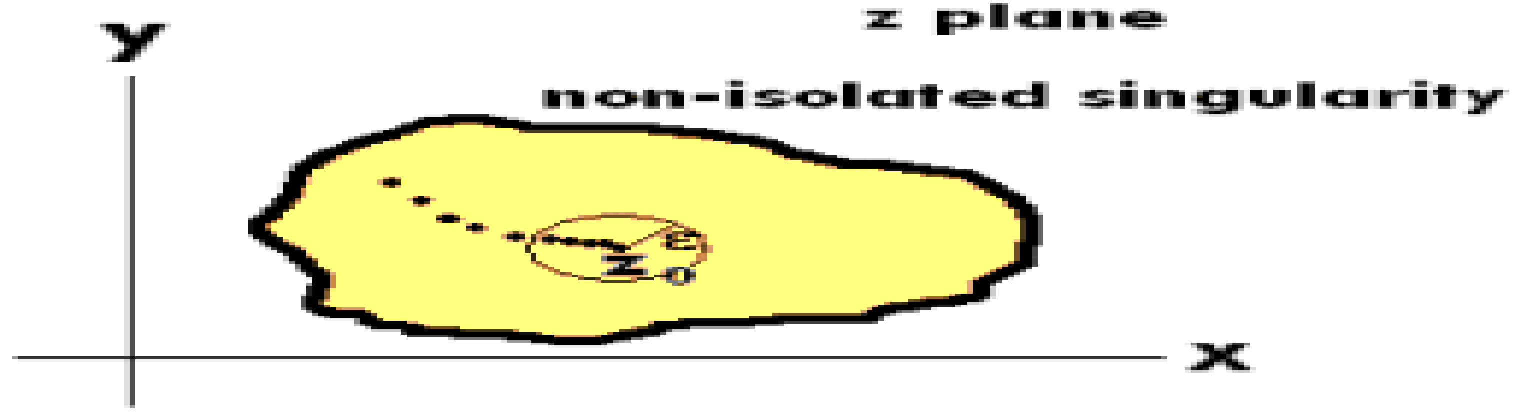


Fig. 1b

The Taylors Series

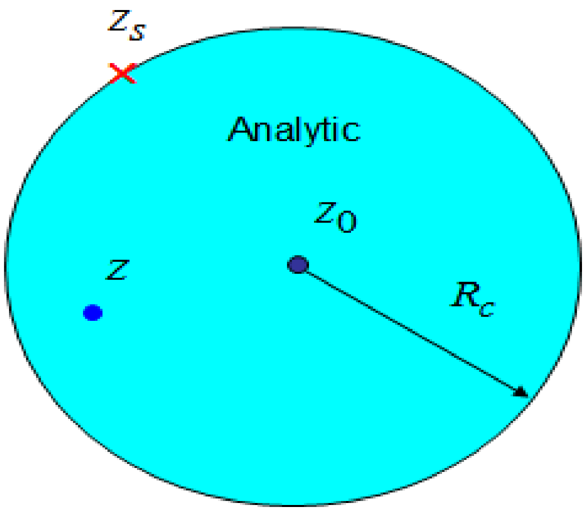
Taylor's theorem is named after the mathematician Brook Taylor, who stated a version of it in 1712. Yet an explicit expression of the error was not provided until much later on by Joseph-Louis Lagrange. An earlier version of the result was already mentioned in 1671 by James Gregory.

Brook Taylor (1685-1731)

The Taylors Series

This expansion assumes we have a function that is analytic in a disk.

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n$$



$$a_n=\frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}}\,dz'$$

$$a_n=\frac{f^{(n)}(z_0)}{n!}$$

"derivative formula"

Here z_s is the closest singularity.

Note: Both forms are useful.

The path C is any counterclockwise closed path that encircles the point z_0.

R_c = radius of convergence of the Taylor series

The Taylor series will converge within the radius of convergence, and diverge outside.

Taylors Series Expansion

- Write the Cauchy integral formula in the form

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')}{z'-z}dz'$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)-(z-z_0)}dz'$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)\left[1-\frac{z-z_0}{z'-z_0}\right]}dz' \quad (|z'-z_0| > |z-z_0|)$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n dz'$$

uniform convergence

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \oint_C \frac{f(z')}{(z'-z_0)^{n+1}}dz'$$

derivative formulas

$$= \sum_{n=0}^{\infty}(z-z_0)^n \frac{f^{(n)}(z_0)}{n!} \qquad \left(\text{recall } f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}}dz'\right)$$

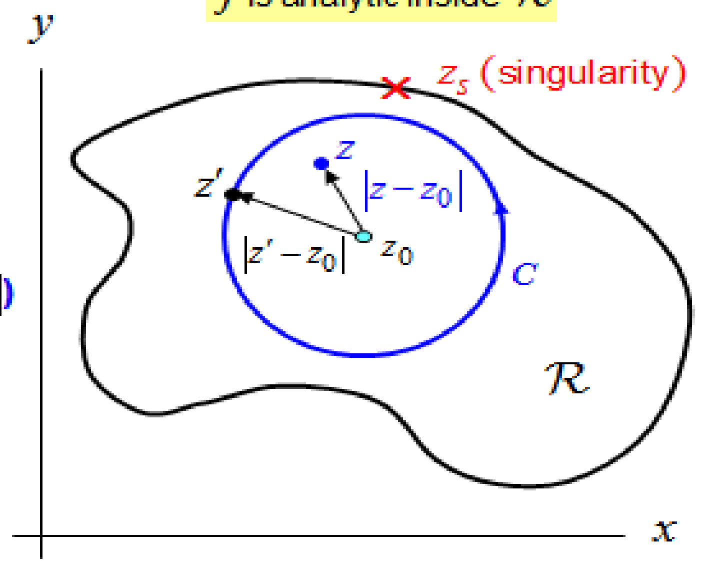
$$\Rightarrow \boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n} \Leftrightarrow$$ **Taylor series expansion of $f(z)$ about z_0**

where

$$\boxed{a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}}dz' = \frac{f^{(n)}(z_0)}{n!}}$$

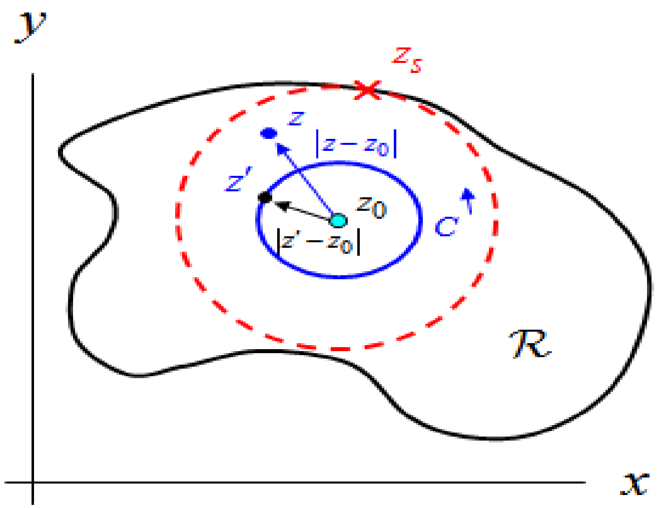
(both forms are useful!)

f is analytic inside $\mathcal{R}$



$$|z-z_0| < |z'-z_0| < |z_s - z_0|$$

POWER SERIES EXPANSION OF COMPLEX FUNCTION



Note that in the result for a_n, the integrand is analytic away from z_0, and hence the path is now arbitrary, as long as it encircles z_0.

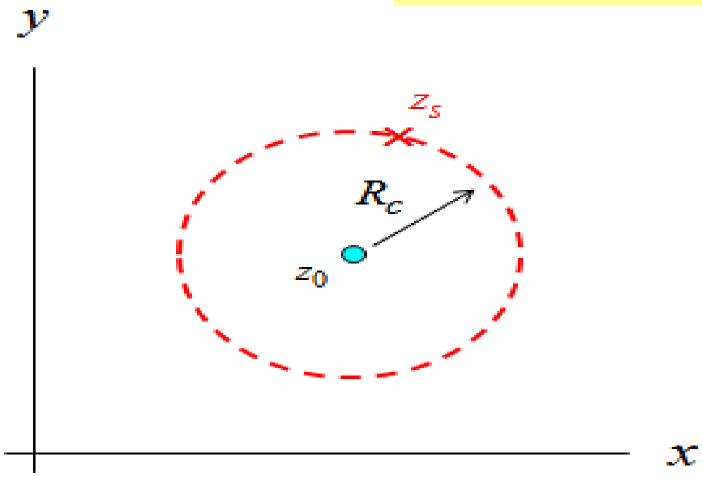
$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}}\,dz'$$

- Note that the construction is valid for any $|z-z_0| < |z_s - z_0|$ where z_s is the singularity nearest z_0 ; hence the series will converge if

$$|z-z_0| < |z_s - z_0|$$

Note: It can also be shown that the series will diverge for $|z-z_0| > |z_s - z_0|$

The radius of convergence of a Taylor series is the distance out to the <u>closest</u> singularity.



Key point:
The point z_0 about which the expansion is made is *arbitrary*, but It determines the region of convergence of the Taylor series.

Converges for : $|z - z_0| < R_c \equiv |z_s - z_0|$

Properties of Taylor Series

R_c = radius of convergence = distance to closest singularity

- A Taylor series will converge for $|z-z_0| < R_c$ (i.e., inside the radius of convergence).
- A Taylor series will diverge for $|z-z_0| > R_c$ (i.e., outside the radius of convergence).
- A Taylor series may be differentiated or integrated term-by-term within the radius of convergence. This does not change the radius of convergence.
- A Taylor series *converges absolutely* inside the radius of convergence (i.e., the series of absolute values converges).
- A Taylor series *converges uniformly* for $|z-z_0| \leq R < R_c$.
- When a Taylor series converges, the resulting function is an analytic function.
- Within the common region of convergence, we can add and multiply Taylor series, collecting terms to find the resulting Taylor series.

Maclaurine's Series Expansion

Maclaurin series:

A Maclaurin series is a Taylor series expansion of a function about x=0,

$$f(z) = f(0) + f'(0)(z) + f''(0)\frac{(z)^2}{2!} + f'''(0)\frac{(z)^3}{3!} + \ldots\ldots\ldots + f^n(0)\frac{(z)^n}{n!} + \ldots\ldots$$

This series is called as maclurins series expansion of f(z).

Some important result:

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots \quad \text{for } -1 < x \le 1$$

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \ldots \quad \text{for } -\infty < x < \infty$$

$$\cos x = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \tfrac{1}{720}x^6 + \ldots \quad \text{for } -\infty < x < \infty$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots \quad \text{for } -1 < x < 1$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Problem on Maclaurine's Series

Problem: Determine the first four terms of the power series for sin 2x using Maclaurin's series.

Solution:

Let

$f(x) = \sin 2x \qquad f(0) = \sin 0 = 0$

$f'(x) = 2 \cos 2x \qquad f'(0) = 2 \cos 0 = 2$

$f''(x) = -4 \sin 2x \qquad f''(0) = -4 \sin 0 = 0$

$f'''(x) = -8 \cos 2x \qquad f'''(0) = -8 \cos 0 = -8$

$f^{iv}(x) = 16 \sin 2x \qquad f^{iv}(0) = 16 \sin 0 = 0$

$f^{v}(x) = 32 \cos 2x(0) \qquad f^{v}(0) = 32 \cos 0 = 32$

$f^{vi}(x) = -64 \sin 2x \qquad f^{vi}(0) = -64 \sin 0 = 0$

$f^{vii}(x) = -128 \cos 2x \qquad f^{vii}(0) = -128 \cos 0 = -128$

$$f(x) = \sin 2x = 0 + 2x + 0x^2 + (-8)\frac{x^3}{3!} + 0.x^4 + 32\frac{x^5}{5!}$$

$$= 2x - \frac{4x^3}{3} + \frac{4x^5}{15}$$

The Laurent's Series

The Laurent series was named after and first published by Pierre Alphonse Laurent in 1843. Karl Weierstrass may have discovered it first in a paper written in 1841, but it was not published until after his death.

Pierre Alphonse Laurent (1813 -1854)

The Laurent's Series

This generalizes the concept of a Taylor series to include cases where the function is analytic in an annulus.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

or

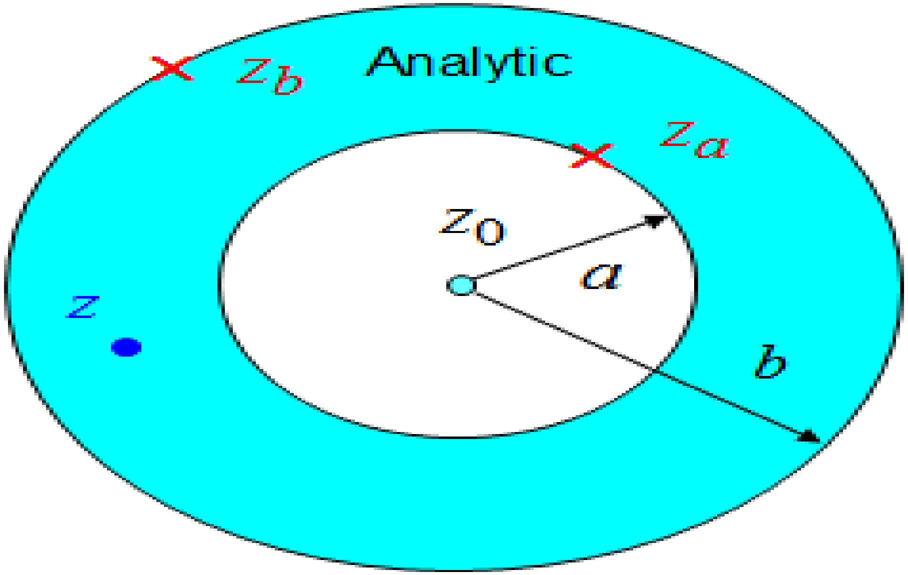
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z - z_0)^n}$$

where $b_n = a_{-n}$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz'$$

(derived later)

POWER SERIES EXPANSION OF COMPLEX FUNCTION



Here z_a and z_b are two singularities.

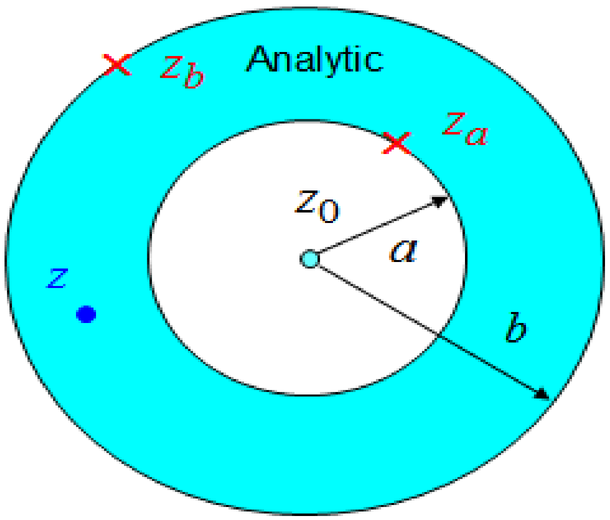
Note:
The point z_b may be at infinity.

The path C is any counterclockwise closed path that stays inside the annulus an encircles the point z_0.

Note: We no longer have the "derivative formula" as we do for a Taylor series.

Laurent series:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$



The Laurent series converges inside the region

$$a < |z - z_0| < b$$

The Laurent series diverges outside this region if there are singularities at

$$|z - z_0| = a, b$$

The Laurent's Series

This is particularly useful for functions that have poles.

Examples of functions with poles, and how we can choose a Laurent series:

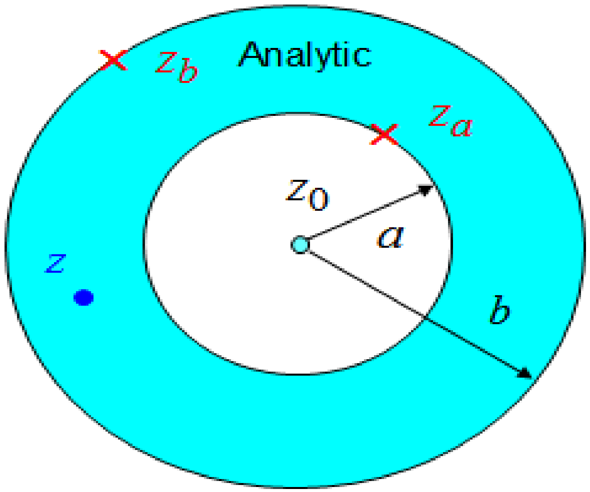
$$f(z) = \frac{1}{z} \quad \left(\text{Choose } z_0 = 0:\ a = 0,\ b < \infty\right)$$

$$f(z) = \frac{z}{z-1} \quad \left(\text{Choose } z_0 = 0:\ a = 1,\ b < \infty\right)$$

$$f(z) = \frac{z}{z-1} \quad \left(\text{Choose } z_0 = 1:\ a = 0,\ b < \infty\right)$$

$$f(z) = \frac{z}{(z-1)(z-2)} \quad \left(\text{Choose } z_0 = 0:\ a = 1,\ b = 2\right)$$

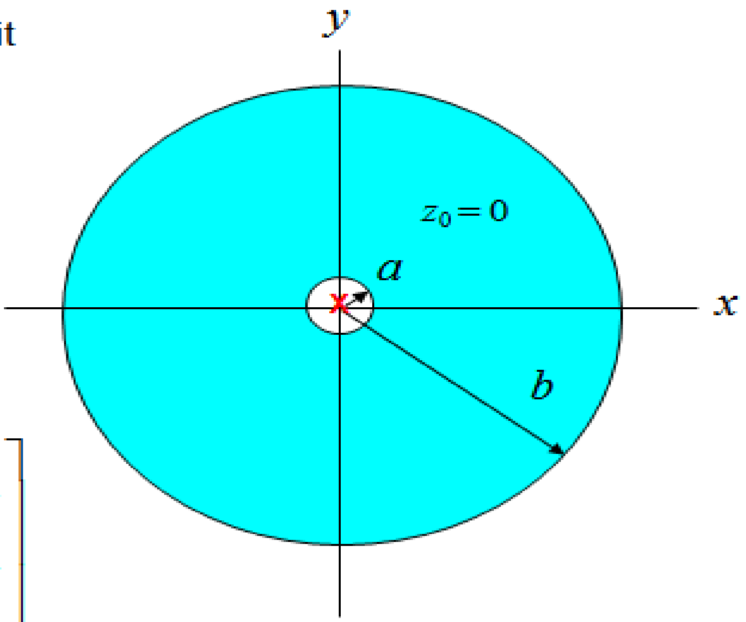
$$f(z) = \frac{z}{(z-1)(z-2)} \quad \left(\text{Choose } z_0 = 0:\ a = 2,\ b < \infty\right)$$



Theorem:
The Laurent series expansion in the annulus region is unique.

(So it doesn't matter how we get it; once we obtain it by any series of valid steps, it is correct!)

This is justified by our Laurent series expansion formula, derived later.



Example:

$$f(z) = \frac{\cos(z)}{z} \qquad (z_0 = 0,\ a = 0,\ b < \infty)$$

$$\Rightarrow f(z) = \underbrace{\frac{1}{z}}_{\text{analytic for } |z|>0} \left[\underbrace{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots}_{\text{valid for } |z|<\infty} \right]$$

Hence $f(z) = \dfrac{1}{z} - \dfrac{z}{2!} + \dfrac{z^3}{4!} - \dfrac{z^5}{6!} \ldots, \quad 0 < |z| < \infty$

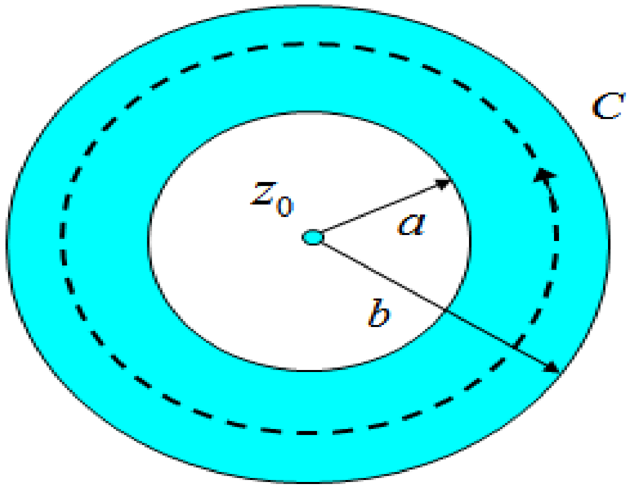
POWER SERIES EXPANSION OF COMPLEX FUNCTION

A Taylor series is a special case of a Laurent series.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz'$$

$$a_n = 0, \ \ n = -1, -2, -3 \ldots$$



Here f is assumed to be analytic within C.

If $f(z)$ is analytic within C, the integrand is analytic for negative values of n. Hence, all coefficients a_n for negative n become zero (by Cauchy's theorem).

Derivation of the Laurent's Series

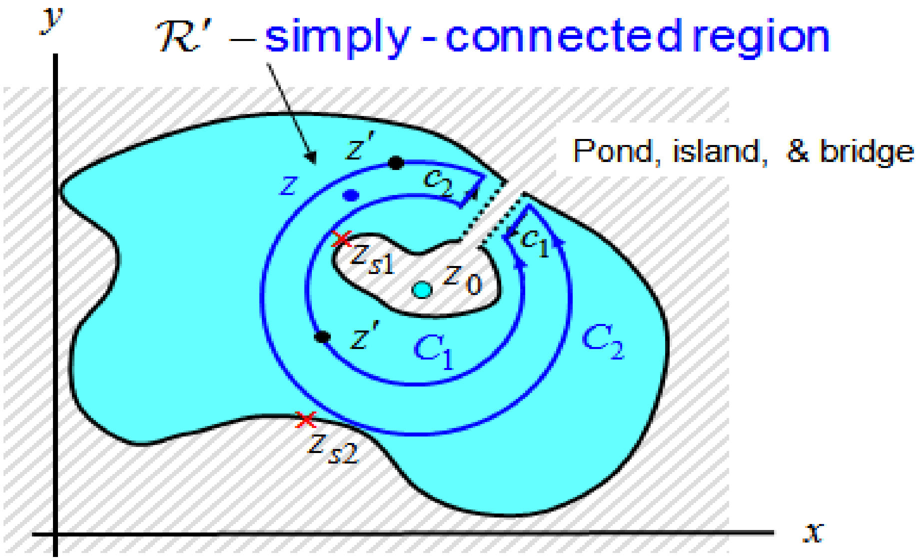
Derivation of Laurent Series

We use the "bridge" principle again

Pond, island, & bridge

Pond: Domain of analyticity
Island: Region containing singularities
Bridge: Region connecting island and boundary of pond

Contributions from the paths c_1 and c_2 cancel!



- **By Cauchy's Integral Formula,**

$$f(z) = \frac{1}{2\pi i} \oint_{C_2 + \cancel{c_1} - \cancel{c_2} - C_1} \frac{f(z')}{z'-z} dz'$$

$$= \frac{1}{2\pi i} \oint_{C_2} \frac{f(z')}{z'-z} dz' - \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')}{z'-z} dz'$$

where on C_2, $|z'-z_0| > |z-z_0|$,

$$\Rightarrow \frac{1}{z'-z} = \frac{1}{(z'-z_0)-(z-z_0)} = \frac{1}{(z'-z_0)\left(1-\frac{z-z_0}{z'-z_0}\right)} = \frac{1}{z'-z_0}\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(z'-z_0)^n} = \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(z'-z_0)^{n+1}}$$

and on C_1, $|z'-z_0| < |z-z_0|$ (note the convergence regions for C_1, C_2 overlap!)

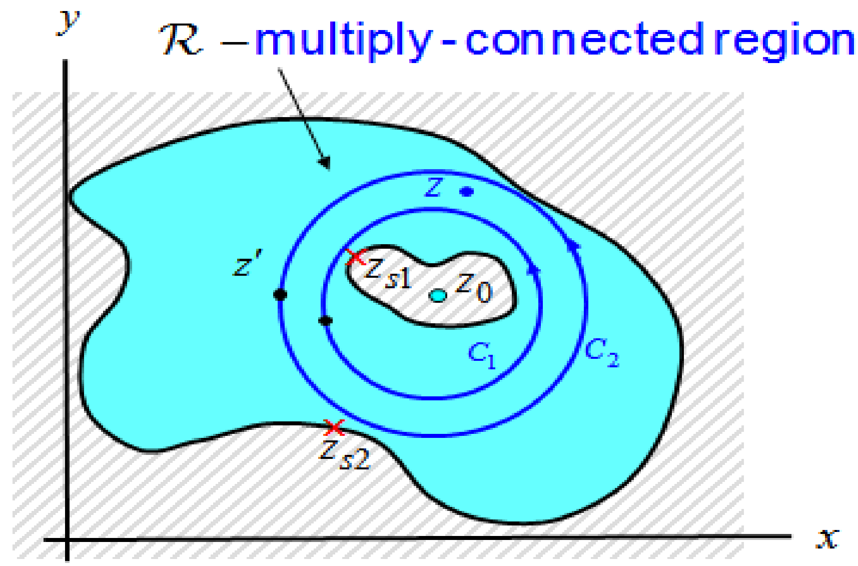
$$\Rightarrow \frac{1}{z'-z} = \frac{1}{(z'-z_0)-(z-z_0)} = \frac{-1}{(z-z_0)\left(1-\frac{z'-z_0}{z-z_0}\right)} = -\sum_{n=0}^{\infty}\frac{(z'-z_0)^n}{(z-z_0)^{n+1}} \overset{\substack{n\to -n'-1,\\ n'\to n}}{=} -\sum_{n=-1}^{-\infty}\frac{(z-z_0)^n}{(z'-z_0)^{n+1}}$$

Hence,

$$f(z)=\frac{1}{2\pi i}\oint_{C_2+c_1-c_2-C_1}\frac{f(z')}{z'-z}\,dz'$$

uniform convergence

$$=\frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\oint_{C_2}\frac{f(z')}{(z'-z_0)^{n+1}}\,dz'+\frac{1}{2\pi i}\sum_{n=-1}^{-\infty}(z-z_0)^n\oint_{C_1}\frac{f(z')}{(z'-z_0)^{n+1}}\,dz'$$



Let $C_1 \to C_2$ (call them path C).

We thus have

$$f(z)=\frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\oint_{C}\frac{f(z')}{(z'-z_0)^{n+1}}\,dz'+\frac{1}{2\pi i}\sum_{n=-1}^{-\infty}(z-z_0)^n\oint_{C}\frac{f(z')}{(z'-z_0)^{n+1}}\,dz'$$

POWER SERIES EXPANSION OF COMPLEX FUNCTION

Because the integrand for the coefficient is analytic with $\mathcal{R}$, the path C is arbitrary as long as it stays within $\mathcal{R}$.

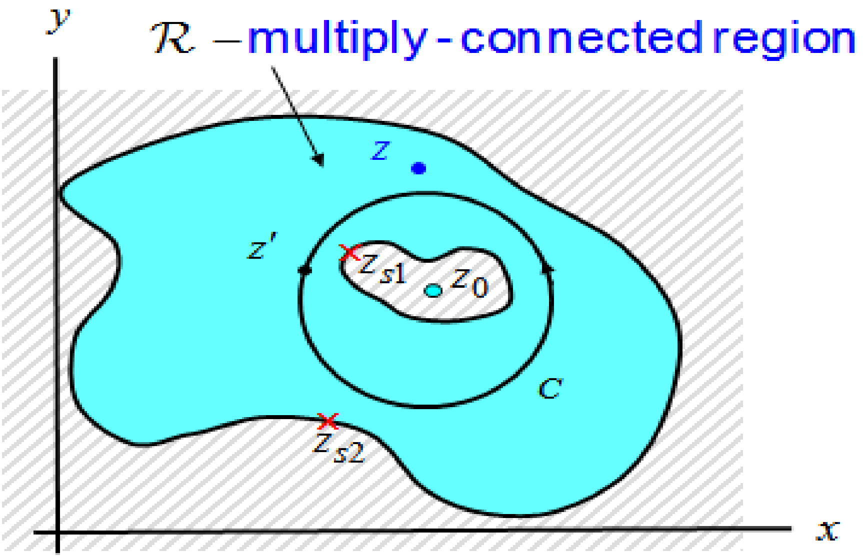
$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \oint_C \frac{f(z')}{(z'-z_0)^{n+1}}dz' + \frac{1}{2\pi i}\sum_{n=-1}^{-\infty}(z-z_0)^n \oint_C \frac{f(z')}{(z'-z_0)^{n+1}}dz'$$

We thus have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

where

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}}dz'$$



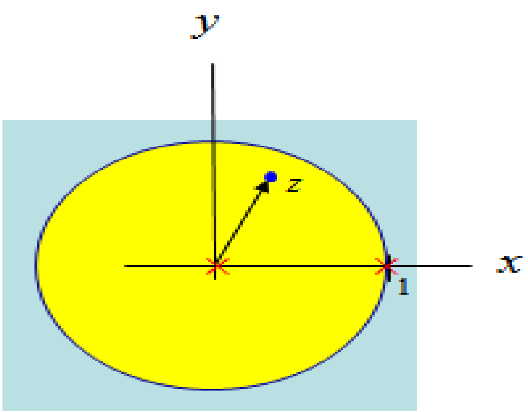
The path C is now arbitrary, as long as it stays in the analytic (blue) region.

Problem on Taylor's and Laurent's Series

Example 1: Obtain all expansions of $f(z) = \frac{1}{z(z-1)}$ about the origin.

Use the integral formula for the a_n coefficients.

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz'$$



The path C can be inside the yellow region or outside of it (parts (a) and (b)).

POWER SERIES EXPANSION OF COMPLEX FUNCTION

a) Laurent series with $a=0,\ b=1$

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')}{z'^{\,n+1}}dz' = \frac{1}{2\pi i}\oint_C \frac{1}{(z'-1)z'^{\,n+2}}dz' = \frac{1}{2\pi i}\oint_C \frac{-1}{z'^{\,n+2}}\sum_{m=0}^{\infty} z'^m dz', \quad (|z'|<1)$$

$$= \frac{1}{2\pi i}\oint_C \sum_{m=0}^{\infty}\frac{-1}{z'^{\,n-m+2}}dz' = \frac{-1}{2\pi i}\sum_{m=0}^{\infty}\oint_C \frac{1}{z'^{\,n-m+2}}dz' = \frac{-1}{2\pi i}\cdot\begin{cases}2\pi i, & m=n+1\\ 0, & m\neq n+1\end{cases}$$

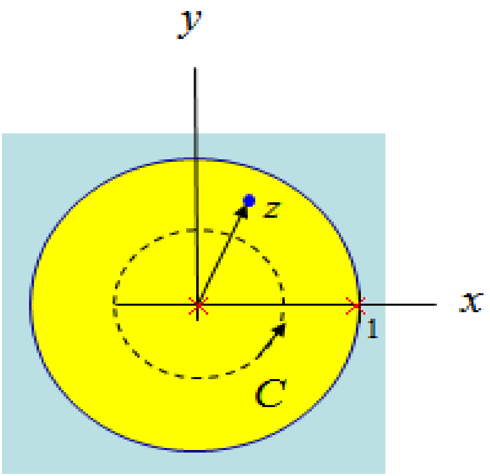
$= -1$ (for $m = n+1 \geq 0$)

From uniform convergence

From previous example in Notes 3

$$a_n = -1, \quad n \geq -1$$

$$f(z) = \frac{1}{z(z-1)}$$



The path C is inside the yellow region.

Hence

$$f(z) = -\frac{1}{z} - 1 - z - z^2 - z^3 - \cdots, \quad 0<|z|<1$$

b) Laurent series with $a=1,\ b=\infty$

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')}{z'^{\,n+1}}dz' = \frac{1}{2\pi i}\oint_C \frac{1}{(z'-1)z'^{\,n+2}}dz' = \frac{1}{2\pi i}\oint_C \frac{1}{\left(1-\frac{1}{z'}\right)z'^{\,n+3}}dz'$$

$$= \frac{1}{2\pi i}\oint_C \frac{1}{z'^{\,n+3}}\sum_{m=0}^{\infty}\frac{1}{z'^{\,m}}dz', \quad (|z'|>1)$$

From previous example in Notes 3

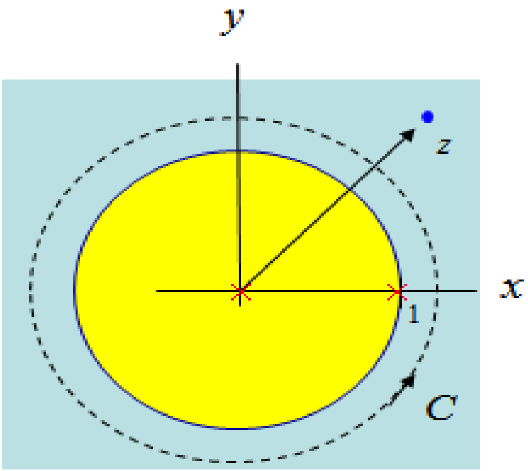
$$= \frac{1}{2\pi i}\oint_C \sum_{m=0}^{\infty}\frac{1}{z'^{\,n+m+3}}dz' = \frac{1}{2\pi i}\sum_{m=0}^{\infty}\oint_C \frac{1}{z'^{\,n+m+3}}dz' = \frac{1}{2\pi i}\cdot\begin{cases}2\pi i, & m=-n-2\\ 0, & m\neq -n-2\end{cases}$$

From uniform convergence

$$=1 \quad (\text{for } m=-n-2\geq 0)$$

$$\Rightarrow \quad a_n = 1, \quad n\leq -2$$

$$f(z) = \frac{1}{z(z-1)}$$



The path C is outside the yellow region.

Hence

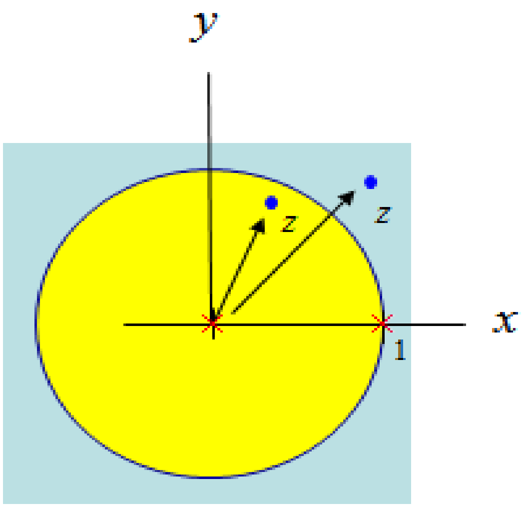
$$f(z) = \frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots, \quad |z|>1$$

Summary of results for the example:

$$f(z) = \frac{1}{z(z-1)}$$

$$f(z) = -\frac{1}{z} - 1 - z - z^2 - z^3 - \cdots, \quad 0 < |z| < 1$$

$$f(z) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots, \quad |z| > 1$$



Note:

Often it is easier to directly use the geometric series (GS) formula together with some algebra, instead of the contour integral approach, to determine the coefficients of the Laurent expanson.

This is illustrated next (using the same example as in Example 1).

Problem on the Laurent's series

◇ **Example 1**

Expand $f(z)=\dfrac{1}{z(z-1)}$ about the origin (we use partial fractions and GS):

$$f(z)=\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$$

$$A=\lim_{z\to 0} z\,f(z)=\lim_{z\to 0}\frac{\cancel{z}}{\cancel{z}(z-1)}=-1$$

$$B=\lim_{z\to 1}(z-1)\,f(z)=\lim_{z\to 1}\frac{\cancel{z-1}}{z\cancel{(z-1)}}=1$$

$$\Rightarrow f(z)=\frac{1}{z(z-1)} = \frac{-1}{z} + \frac{1}{z-1} = \frac{-1}{z} - \frac{1}{1-z}$$

$$= \frac{-1}{z} - \left(1+z+z^2+\ldots\right)$$

Hence

$$f(z) = -\frac{1}{z}-1-z-z^2-z^3-\cdots,\quad 0<|z|<1$$

Alternative expansion:

$$f(z) = \frac{1}{z(z-1)} = \frac{-1}{z} + \frac{1}{z-1} = \frac{-1}{z} + \frac{1}{z}\left(\frac{1}{1-1/z}\right)$$

$$\Rightarrow f(z) = \frac{-1}{z} + \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right)$$

Hence

$$f(z) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots, \quad |z| > 1$$

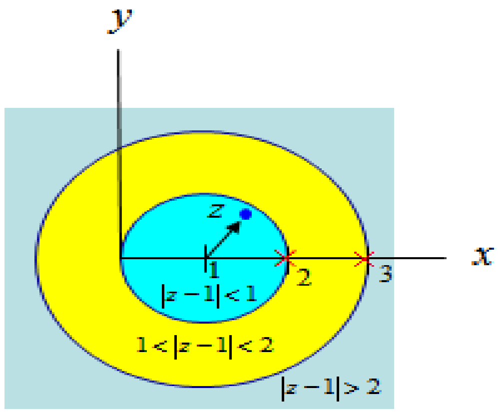
◊ **Example 2**

Expand $f(z)=\dfrac{1}{(z-2)(z-3)}$ in a Taylor / Laurent series about $z_0=1$, valid following in the annular regions :

(a) $0\le|z-1|<1$,

(b) $1<|z-1|<2$,

(c) $|z-1|>2$.



a) For $0\le|z-1|<1$:

Using partial fraction expansion and GS,

$$f(z)=\frac{1}{(z-2)(z-3)}=\frac{1}{z-3}-\frac{1}{z-2}$$

$$=\frac{1}{(z-1)-2}-\frac{1}{(z-1)-1}=\frac{-1}{2\left[1-(z-1)/2\right]}+\frac{1}{1-(z-1)}$$

$$=-\frac{1}{2}\left[1+\frac{(z-1)}{2}+\frac{(z-1)^2}{2^2}+\cdots\right]+\left[1+(z-1)+(z-1)^2+\cdots\right]$$

Hence

$$f(z)=\frac{1}{2}+\frac{3}{4}(z-1)+\frac{7}{8}(z-1)^2+\frac{15}{16}(z-1)^3+\cdots,\quad 0\le|z-1|<1$$ (Taylor series)

$$f(z)=\frac{1}{(z-2)(z-3)}$$

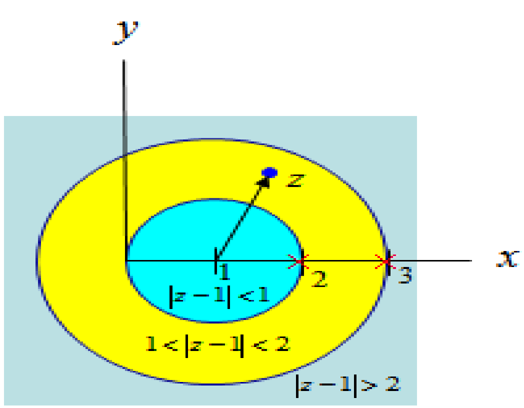
b) For $1<|z-1|<2$:

$$f(z)=\frac{1}{(z-1)-2}-\frac{1}{(z-1)-1}=\frac{-1}{2[1-(z-1)/2]}-\frac{1}{(z-1)[1-1/(z-1)]}$$

so

$$f(z)=-\frac{1}{2}\left[1+\frac{(z-1)}{2}+\frac{(z-1)^2}{2^2}+\cdots\right]-\frac{1}{(z-1)}\left[1+\frac{1}{(z-1)}+\frac{1}{(z-1)^2}+\cdots\right]$$

(Laurent series)



$$f(z) = \frac{1}{(z-2)(z-3)}$$

c) For $|z-1| > 2$:

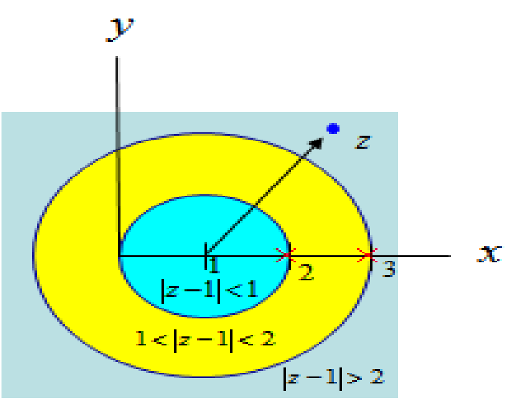
$$f(z) = \frac{1}{(z-1)-2} - \frac{1}{(z-1)-1} = \frac{1}{(z-1)[1-2/(z-1)]} - \frac{1}{(z-1)[1-1/(z-1)]}$$

$$= \frac{1}{(z-1)}\left[1 + \frac{2}{(z-1)} + \frac{2^2}{(z-1)^2} + \cdots\right] - \frac{1}{(z-1)}\left[1 + \frac{1}{(z-1)} + \frac{1}{(z-1)^2} + \cdots\right]$$

so

$$f(z) = \frac{1}{(z-1)^2} + \frac{3}{(z-1)^3} + \frac{7}{(z-1)^4} + \cdots$$

(Laurent series)



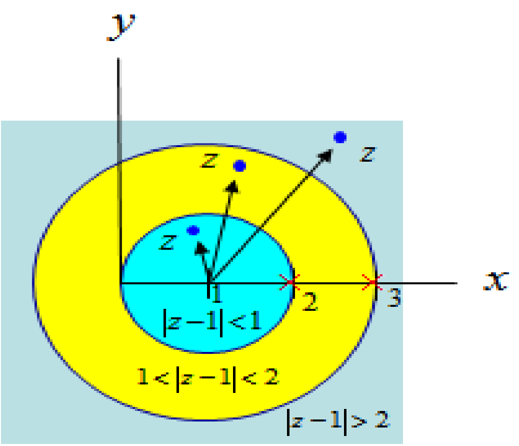
Summary of results for example

$$f(z)=\frac{1}{(z-2)(z-3)}$$

$$f(z)=\frac{1}{2}+\frac{3}{4}(z-1)+\frac{7}{8}(z-1)^2+\frac{15}{16}(z-1)^3+\cdots,\quad 0\le|z-1|<1$$

$$f(z)=-\frac{1}{2}\left[1+\frac{(z-1)}{2}+\frac{(z-1)^2}{2^2}+\cdots\right]-\frac{1}{(z-1)}\left[1+\frac{1}{(z-1)}+\frac{1}{(z-1)^2}+\cdots\right],\quad 1<|z-1|<2$$

$$f(z)=\frac{1}{(z-1)^2}+\frac{3}{(z-1)^3}+\frac{7}{(z-1)^4}+\cdots,\quad |z-1|>2$$



Problem on the Laurent's series

◇ **Example 3**

Find the series expansion about $z = 0$:

$$f(z) = \begin{cases} \dfrac{1-\cos z}{z^2}, & z \neq 0 \\ 1/2, & z = 0 \end{cases} \qquad (z = 0 \text{ is a "removable" singularity})$$

$$1 - \cos z = 1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) = \frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots$$

Hence

$$f(z) = \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots, \quad |z| < \infty$$

Similarly, we have $f(z) = \dfrac{\sin z}{z} = 1 - \dfrac{z^2}{3!} + \dfrac{z^4}{5!} - \cdots \quad |z| < \infty$

Note: $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots \quad |z| < \infty$

◊ **Example 4**

Note: $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \quad |z| < \infty$

Find the series for $\sin z$ about $z = \pi$:

$$f(z) = \sin z = \sin\left[(z-\pi)+\pi\right]$$

$$= \sin(z-\pi)\cos\pi + \cos(z-\pi)\sin\pi = -\sin(z-\pi)$$

$$f(z) = -(z-\pi) + \frac{(z-\pi)^3}{3!} - \frac{(z-\pi)^5}{5!} + \cdots, \quad |z| < \infty$$

Alternatively, directly use the derivative formula for Taylor series :

$$f(\pi) = \sin\pi = 0$$

$$f'(\pi) = \cos\pi = -1$$

$$f''(\pi) = -\sin\pi = 0$$

$$f'''(\pi) = -\cos\pi = +1$$

$$f^{(iv)}(\pi) = \sin\pi = 0$$

$$f^{(v)}(\pi) = \cos\pi = -1$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}}\,dz' = \frac{f^{(n)}(z_0)}{n!}$$

$$f(z) = -(z-\pi) + \frac{(z-\pi)^3}{3!} - \frac{(z-\pi)^5}{5!} + \cdots, \quad |z| < \infty$$

◊ **Example 5**

Find the first three terms of the series for $\sin^2 z \ln(1-z)$ about $z=0$.

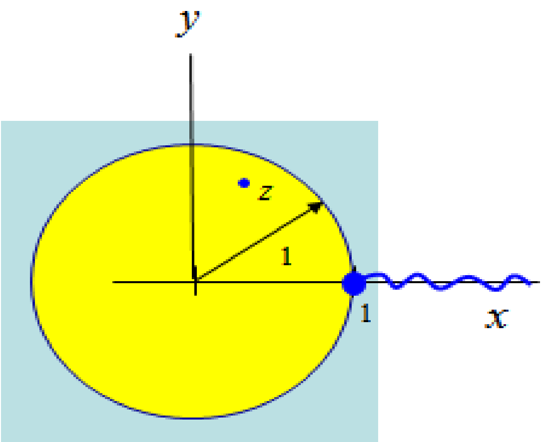
Since $\dfrac{1}{1-z} = 1+z+z^2+\cdots,\ |z|<1$ then

$$\int_0^z \frac{1}{1-z}dz = -\ln(1-z) = z+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\cdots,\ |z|<1$$

$$\Rightarrow \ln(1-z) = -z-\frac{z^2}{2}-\frac{z^3}{3}-\frac{z^4}{4}-\cdots,\ |z|<1$$

Also $$\sin^2 z = \left[z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right]\left[z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right]$$

$$= z^2-\frac{z^4}{3}+\frac{2}{45}z^6+\cdots$$



The branch cut is chosen away from the yellow region.

Hence

$$\sin^2 z \ln(1-z) = -\left(z^2-\frac{z^4}{3}+\frac{2}{45}z^6+\cdots\right)\left(z+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\cdots\right)$$

$$= -z^3-\frac{z^4}{2}+0\cdot z^5+\cdots,\quad |z|<1$$

Summary of the methods for generating Taylors and Laurent's Series

Summary of Methods

- Taylor (*not* Laurent) series, $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, can be generated using $a_n = \frac{f^{(n)}(z_0)}{n!}$

- Taylor *and* Laurent series, $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$, can be generated using $a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz'$

- To expand about $z = z_0$, first write $f(z)$ in the form $f\left[(z - z_0) + z_0\right]$, rearrange and expand using geometric series or other methods.

- Use partial fraction expansion and geometric series to generate series for rational functions (ratios of polynomials, degree of numerator less than degree of denominator).

- Laurent / Taylor series can be integrated or differentiated term - by - term within their region of convergence.

Summary of Methods

- Note that for two Taylor or Laurent series,

$$f(z)=\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n, \quad g(z)=\sum_{n=-\infty}^{\infty} b_n (z-z_0)^n$$

then $f(z)+g(z)=\sum_{n=-\infty}^{\infty}(a_n+b_n)(z-z_0)^n$ *in their common region of convergence.*

- Two Taylor series can be multiplied term - by - term *within their common region of convergence* :

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n, \qquad g(z)=\sum_{m=0}^{\infty} b_m (z-z_0)^m$$

$$\Rightarrow f(z)g(z)=\left(\sum_{n=0}^{\infty} a_n (z-z_0)^n\right)\left(\sum_{m=0}^{\infty} b_m (z-z_0)^m\right)=\sum_{p=0}^{\infty} c_p (z-z_0)^p \text{ where } c_p=\sum_{n=0}^{p} a_n b_{p-n}$$

Some more Examples on Taylor's and Laurent's series

1. Expand $\frac{1}{z-2}$ at x = 1 in Taylor series

Solution: Given f (z) = $\frac{1}{z-2}$ f(1) = -1

$$f'(z) = \frac{-1}{(z-2)^2}, f'(1) = -1$$

$$f''(z) = \frac{2}{(z-2)^3}, f''(1) = -2$$

$$f'''(z) = \frac{-6}{(z-2)^4}, f'''(1) = -6$$

∴ The Taylor series at z = 1 is given by

$$f(z) = f(a) + (z-a)\frac{f'(a)}{1!} + (z-a)^2\frac{f''(a)}{2!} + \ldots\ldots\ldots\ldots\ldots$$

$$= -1 + (z-1)(-1) + \frac{(z-1)^2}{2!}(-2) + \frac{(z-1)^3}{3!}(-6) + \ldots\ldots\ldots$$

$$= -\left[1 + (z-1) + (z-1)^2 + (z-1)^3 + \ldots\ldots\ldots\ldots\ldots\ldots\right]$$

2. Expand $f(z)=\dfrac{z^2-1}{(z+2)(z+3)}$ **as a Laurent's series if (i)** $|z|<2$ **(ii)** $|z|>3$

(iii) $2<|z|<3$

Solution:

$$\text{Let } f(z)=\frac{z^2-1}{(z+2)(z+3)}$$

$$=1+\frac{-5z-7}{(z+2)(z+3)}$$

Now $\dfrac{-5z-7}{(z+2)(z+3)}=\dfrac{A}{z+2}+\dfrac{B}{z+3}$

$$-5z-7=A(z+3)+B(z+2)$$

Put z = -3 $\Rightarrow 15-7=-B \Rightarrow -B=8 \Rightarrow B=-8$

Put $z=-2 \Rightarrow 10-7=A \Rightarrow A=3$

$$\therefore f(z)=1+\frac{3}{z+2}-\frac{8}{z+3}$$

$$\text{(i) } |z|<2 \Rightarrow \frac{|z|}{2}<1 \text{ also } \frac{|z|}{3}<1$$

$$\therefore f(z)=1+\frac{3}{2\left(1+\frac{z}{2}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$=1+\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=1+\frac{3}{2}\left(1-\frac{z}{2}+\left(\frac{z}{2}\right)^{2}-\ldots\ldots\right)-\frac{8}{3}\left(1-\frac{z}{3}+\left(\frac{z}{3}\right)^{2}-\ldots\ldots\right)$$

(ii) $|z| > 3 \Rightarrow 3 < |z| \Rightarrow \frac{3}{|z|} < 1, 2 < 3 < |z|, \frac{2}{|z|} < 1$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \ldots\ldots\right) - \frac{8}{z}\left(1 - \frac{3}{z} + \left(\frac{3}{z}\right)^2 - \ldots\ldots\right)$$

(iii) $2 < |z| < 3$

$$\Rightarrow 2 < |z|, |z| < 3$$

$$\Rightarrow \frac{2}{|z|} < 1, \frac{|z|}{3} < 1$$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \dots\dots\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \dots\dots\right)$$

3. Find Laurent's series expansion of f (z) = $\frac{7z-2}{z(z-2)(z+1)}$ **in** $1 < |z+1| < 3$

Solution: Let f (z) = $\frac{7z-2}{z(z-2)(z+1)}$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$$7z-2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$$

Put $z=0, A=1$

$z=2, B=2$

$z=-1, C=-3$

$$F\,(z) = \frac{7z-2}{z(z-2)(z+1)}$$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

Given $1 < |z+1| < 3$

Now put t = z + 1, then $1 < |t| < 3$

$$\Rightarrow 1 < |t|, |t| < 3$$

$$\Rightarrow \frac{1}{|t|} < 1, \frac{|t|}{3} < 1$$

POWER SERIES EXPANSION OF COMPLEX FUNCTION

$$\therefore f(z) = \frac{1}{t-1} + \frac{2}{t-3} - \frac{3}{t}$$

$$= -\frac{3}{t} + \frac{1}{t\left(1-\frac{1}{t}\right)} - \frac{2}{3\left(1-\frac{t}{3}\right)}$$

$$= -\frac{3}{t} + \frac{1}{t}\left(1-\frac{1}{t}\right)^{-1} - \frac{2}{3}\left(1-\frac{t}{3}\right)^{-1}$$

$$= -\frac{3}{t} + \frac{1}{t}\left(1 + \frac{1}{t} + \left(\frac{1}{t}\right)^2 + \ldots\ldots\right) - \frac{2}{3}\left(1 + \frac{t}{3} + \left(\frac{t}{3}\right)^2 + \ldots\ldots\right)$$

$$= -\frac{3}{z+1} + \frac{1}{z+1}\left(1 + \frac{1}{z+1} + \left(\frac{1}{z+1}\right)^2 + \ldots\ldots\right) - \frac{2}{3}\left(1 + \frac{z+1}{3} + \left(\frac{z+1}{3}\right)^2 + \ldots\ldots\right)$$

Definition

Definition: Zero of an analytic function

A point z = a is said to be a zero of an analytic function f (z) if f(z) is zero at z = a.

If f (a) = 0 and f' (a) $\neq 0$ then z = a is called a simple zero of f (z) (or) a zero of the first order.

If $f(a) = f'(a) = ... = f^{n-1}(a) = 0$ & $f^{n}(a) \neq 0$ then z = a is a zero of order n.

Example; Let f (z) = z^2

Then $f'(z) = 2z, f''(z) = 2$

$f(0), f'(0) = 0, f''(0) = 2 \neq 0$

$\therefore z = 0$ is a zero of order 2.

Definition

Singular point:

A point z = a is said to be a singular point (or) singularity of f (z) is not analytic at z =a.

Types of Singular Point:

Isolated Singular Point:

A point z = a is said to be an isolated singular point of f (z) if

(i) f (z) is not analytic at z = a

(ii) f (z) is analytic at all points for some neighbourhood of z = a

Example: $f(z) = \dfrac{z}{(z-1)(z-2)}$

Then z = 1, 2 are isolated points.

Definition

Pole:

A point z = a is said to be a pole of f(z) of order n if we can find a positive integer such that $\lim_{z \to a}(z-a)^n f(z) \neq 0$

Essential singular point:

A singular point z = a is said to be an essential point f (z) if the Laurent's series of f (z) about z = a possesses the infinite number of terms in the principal part (terms containing negative powers).

Example

Let $f(z) = e^{1/z}$

Clearly z = 0 is a singular point

Also $f(z) = e^{1/z} = 1 + \frac{1/z}{1!} + \frac{\left(1/z\right)^2}{2!} + \ldots\ldots$

$$= 1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z^2}\right) + \ldots\ldots$$

∴ z = 0 is an essential singular point

Definition

Removable singular point:

A singular point z = a is said to be a removable singular point of f(z) if the Laurent's series of f(z) about z = a does not contain the principal part.

Example

Let $f(z) = \frac{\sin z}{z}$

Clearly z = 0 is a singular point

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\ldots\right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots\ldots$$

Z = 0 is a removable singular point.

Problem on Definitions

1. Classify the nature of the singular point of $f(z)=\dfrac{\tan z}{z}$

Solution : $f(z) = \dfrac{\tan z}{z}$

$$=\frac{1}{z}\left(z+\frac{z^3}{3}+\ldots\ldots\right)$$

$$=1+\frac{z^2}{3}+\ldots..$$

This is the Laurent's series of $f(z)$ about z=0 and there is no principal part.

$\therefore$ Z=0 is a removable singular point.

Also $f(z) = \dfrac{\tan z}{z}=\dfrac{\sin z}{z\cos z}$

Poles of f(z) are $z\cos z = 0$

$\Rightarrow z=0, z=n\pi, n=0,\pm1,\ldots.$

$\Rightarrow z=0, n\pi$ are simple poles (pole of order 1)

Problem on Definitions

2. Consider the function f $(z)=\dfrac{\sin z}{z^4}$ **. Find the pole and its order.**

Solution: $f(z)=\dfrac{\sin z}{z^4}$

$$=\frac{1}{z^4}\left[z-\frac{z^3}{3!}+\frac{z^5}{5!}-\ldots\ldots\right]$$

$$=\frac{1}{z^3}-\frac{1}{6z}+\frac{z}{120}-\ldots\ldots$$

$\therefore z=0$ **is a pole of order 3.**

Definition

Definition

The residue of a function $f(z)$ at a singular point $z = a$ is the coefficien

b_1 **of** $\frac{1}{z-a}$ **in the Laurent's series of** $f(z)$ **about the point** $z = a$

EVALUATION OF RESIDUES

1. Suppose z = a is a pole of order 1

Then $\{\text{Re}\, sf(z)\}_{z=a} = \lim_{z \to a} (z-a) f(z)$

2. Suppose z = a is a pole of order n

Then $\{\text{Re}\, sf(Z)\}_{z=a} = \lim_{z=a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z)$

3. Suppose z =a is a pole of order 1 and $f(z) = \frac{P(z)}{Q(z)}$

Then { Res $f(z)$ **}** $_{z=a} = \frac{P(a)}{Q'(a)}$

Problems on Residues of a function by definition, Laurent's Series and by formulae method

Ex : Find the Laurent series of $f(z)=\dfrac{1}{z(z-2)^3}$ about the singularities $z=0$ and $z=2$. Hence verify that $z=0$ is a pole of order 1 and $z=2$ is a pole of order 3, and find the residue of $f(z)$ at each pole.

(1) point $z=0$

$$f(z)=\frac{-1}{8z(1-z/2)^3}=\frac{-1}{8z}[1+(-3)(\frac{-z}{2})+\frac{(-3)(-4)}{2!}(\frac{-z}{2})^2+\frac{(-3)(-4)(-5)}{3!}(\frac{-z}{2})^3+\ldots]$$

$$=-\frac{1}{8z}-\frac{3}{16}-\frac{3}{16}z-\frac{5z^2}{32}-\ldots \quad z=0 \text{ is a pole of order 1}$$

(2) point $z=2 \Rightarrow \text{set } z-2=\xi \Rightarrow z(z-2)^3=(2+\xi)\xi^3=2\xi^3(1+\xi/2)$

$$f(z)=\frac{1}{2\xi^3(1+\xi/2)}=\frac{1}{2\xi^3}[1-(\frac{\xi}{2})+(\frac{\xi}{2})^2-(\frac{\xi}{2})^3+(\frac{\xi}{2})^4-\ldots]$$

$$=\frac{1}{2\xi^3}-\frac{1}{4\xi^2}+\frac{1}{8\xi}-\frac{1}{16}+\frac{\xi}{32}-\ldots=\frac{1}{2(z-2)^3}-\frac{1}{4(z-2)^2}+\frac{1}{8(z-2)}-\frac{1}{16}+\frac{z-2}{32}-$$

$z=2$ is a pole of order 3, the residue of $f(z)$ at $z=2$ is $1/8$.

Ex : Suppose that $f(z)$ has a pole of order m at the point $z = z_0$. By considering the Laurent series of $f(z)$ about z_0, deriving a general expression for the residue $R(z_0)$ of $f(z)$ at $z = z_0$. Hence evaluate the residue of the function $f(z) = \dfrac{\exp iz}{(z^2+1)^2}$ at the point $z = i$.

$$f(z) = \frac{\exp iz}{(z^2+1)^2} = \frac{\exp iz}{(z+i)^2(z-i)^2} \quad \text{poles of order 2 at } z = i \text{ and } z = -i$$

for pole at $z = i$:

$$\frac{d}{dz}[(z-i)^2 f(z)] = \frac{d}{dz}\left[\frac{\exp iz}{(z+i)^2}\right] = \frac{i}{(z+i)^2}\exp iz - \frac{2}{(z+i)^3}\exp iz$$

$$R(i) = \frac{1}{1!}\left[\frac{i}{(2i)^2}e^{-1} - \frac{2}{(2i)^3}e^{-1}\right] = \frac{-i}{2e}$$

Problem

Example 1 Find the residue of f(z) $= \dfrac{z+2}{(z-2)(z+1)^2}$ **about each singularity.**

Solution: The poles of f (z) are given by

$$(z-2)=0, z+1=0$$

$$\Rightarrow z=2, z=-1$$

∴ The poles of f (z) are z = 2 is a simple poles and z = -1 is a pole of order 2.

$$\therefore [\operatorname{Res} f(z)]_{z=2} = \lim_{z\to 2}(z-2)f(z)$$

$$= \lim_{z\to 2}(z-2)\frac{z+2}{(z-2)(z+1)^2} = \frac{4}{9}$$

$$\therefore [\operatorname{Res} f(z)]_{z=-1} = \lim_{z\to -1}\frac{d}{dz}[(z-2)^2 f(z)]$$

$$= \lim_{z\to -1}\frac{d}{dz}[(z-2)^2\frac{z+2}{(z-2)(z+1)^2}]$$

$$= \lim_{z\to -1}\frac{d}{dz}\left[\frac{z+2}{z-2}\right]$$

$$= \lim_{z\to -1}\left[\frac{(z-2)(1)-(z+2)(1)}{(z-2)^2}\right] = -\frac{4}{9}$$

Problem

Problem: Determine the poles of the function $f(z)=\dfrac{z+2}{(z+1)^2(z-2)}$ and the residue at each pole.

Solution: The poles of f(z) are given by $(z+1)^2(z-2)=0$

Here z=2 is a simple pole and z= -1 is a pole of order 2 .

Residue at z=2 is

$$\lim_{z\to 2}(z-2)f(z)=\lim_{z\to 2}(z-2)\frac{z+2}{(z+1)^2(z-2)}=\frac{4}{9}$$

Residue at z=-1 is

$$\lim_{z\to -1}\frac{d}{dz}(z+1)^2 f(z)=\lim_{z\to -1}\frac{d}{dz}(z+1)^2\frac{z+2}{(z+1)^2(z-2)}$$

$$\lim_{z\to -1}\frac{d}{dz}\frac{(z+2)}{(z-2)}=\lim_{z\to -1}\frac{-4}{(z-2)^2}=\frac{-4}{9}$$

Problem

Problem: Find the residue of the function $f(z)=\frac{1-e^{2z}}{z^4}$ at the poles.

Solution: Let $f(z)=\frac{1-e^{2z}}{z^4}$

z =0 is a pole of order 4

Residue of f(z) at z=0 is

$$=\frac{1}{3!}\lim_{z\to 0}\frac{d^3}{dz^3}(z-0)^4\frac{(1-e^{2z})}{z^4}$$

$$=\frac{1}{3!}\lim_{z\to 0}\frac{d^3}{dz^3}(1-e^{2z})$$

$$=\frac{1}{3!}\lim_{z\to 0}\frac{d^2}{dz^2}(-2e^{2z})$$

$$=\frac{1}{3!}\lim_{z\to 0}\frac{d}{dz}(-4e^{2z})$$

$$=\frac{1}{3!}\lim_{z\to 0}(-8e^{2z})$$

$$=\frac{-8}{3!}=\frac{-4}{3}.$$

Problem

Example:

$f(z)=\dfrac{1}{\sin z}$ has simple poles at $z=n\pi,\ n=0,\pm 1,\pm 2,\cdots$

$$\operatorname{Res} f(n\pi) = \lim_{z\to n\pi}(z-n\pi)\times\frac{1}{\sin z} \overset{\text{L'Hospital's rule}}{=} \lim_{z\to n\pi}\frac{1}{\cos z}=(-1)^n$$

Alternatively,

$$\operatorname{Res} f(n\pi) = \lim_{z\to n\pi}(z-n\pi)\frac{1}{\sin z} = \lim_{z\to n\pi}(z-n\pi)\frac{1}{\sin(z-n\pi+n\pi)}$$

$$=\lim_{z\to n\pi}(z-n\pi)\frac{1}{\sin(z-n\pi)\cos n\pi+\cos(z-n\pi)\sin n\pi}$$

$$=\lim_{z\to n\pi}\frac{(z-n\pi)}{\sin(z-n\pi)\cos n\pi} = (-1)^n$$ since we already know $\displaystyle\lim_{z\to 0}\frac{z}{\sin z}=1$

$$\operatorname{Res} f(n\pi) = (-1)^n$$

Problem

Example: $f(z) = \tan z$

$$\tan z = \frac{\sin z}{\cos z}$$ has simple poles at $z = \frac{(2n+1)\pi}{2}$,

for $n = 0, \pm 1, \pm 2, \cdots$

$$\Rightarrow \text{Res}\,(\tan z) = \frac{\sin\frac{(2n+1)\pi}{2}}{-\sin\frac{(2n+1)\pi}{2}} = -1$$

$$\text{Res}\,(\tan z) = -1, \quad z = \frac{(2n+1)\pi}{2}$$

Problem

Example:

$f(z)=\dfrac{1}{\sin^2 z}$ has poles of order 2 at $z=n\pi,\ n=0,\pm1,\pm2,\cdots$

$$\Rightarrow \operatorname{Res} f(n\pi) = \frac{d}{dz}\left[\frac{(z-n\pi)^2}{\sin^2 z}\right]\Bigg|_{z=n\pi} = \left[\frac{2(z-n\pi)\sin^2 z-2(z-n\pi)^2\sin z\cos z}{\sin^4 z}\right]_{z=n\pi}$$

$$= \left[\frac{2(z-n\pi)\sin z-2(z-n\pi)^2\cos z}{\sin^3 z}\right]_{z=n\pi}$$

After three applications of L'Hospital's rule:

$$\operatorname{Res} f(n\pi)=0$$

Alternative calculation:

$$f(z) = \frac{1}{\sin^2 z} = \frac{1}{\sin^2(z-n\pi+n\pi)} = \frac{1}{\sin^2(z-n\pi)} = \frac{1}{\sin^2 u} \quad (u = z-n\pi)$$

$$= \frac{1}{\left(u-\frac{u^3}{3!}+\frac{u^5}{5!}-\cdots\right)^2} = \frac{1}{u^2\left(1-\frac{u^2}{3!}+\frac{u^4}{5!}-\cdots\right)^2}$$

Geometric Series

$$= \frac{1}{u^2}\left[1+\left(\frac{u^2}{3!}-\frac{u^4}{5!}+\frac{u^6}{7!}\cdots\right)+\left(\frac{u^2}{3!}-\frac{u^4}{5!}+\frac{u^6}{7!}\cdots\right)^2+\cdots\right]^2$$

$$= \frac{1}{u^2}\left[1+\frac{u^2}{3!}+\left(-\frac{1}{5!}+\frac{1}{(3!)^2}\right)u^4+\cdots\right]^2 = \frac{1}{u^2}+\frac{2}{3!}+\cdots$$

missing $\frac{a_{-1}}{u}$ term $\Rightarrow$ $\text{Res}\, f(n\pi) = 0$

Residue at Infinity

Residue at infinity:

The residue at infinity is given by:

$$\operatorname{Res}[f(z)]_{Z=\infty} = -\frac{1}{2\pi i}\oint_C f(z)dz$$

Where f is an analytic function except at finite number of singular points and C is a closed countour so all singular points lie inside it.

Problem

Problem: Find the residue of the function $f(z) = z^3 \cos(\frac{1}{z})$ at $z = \infty$.

Solution: Let $f(z) = z^3 \cos(\frac{1}{z})$

$$g(t) = f(\frac{1}{t}) = \frac{1}{t^3}\cos t$$

$$= \frac{1}{t^3}\left[1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots\dots\dots\right]$$

$$= \left[\frac{1}{t^3} - \frac{1}{2t} + \frac{t}{24} - \dots\dots\dots\right]$$

There fore $\operatorname{Res}_{z\to\infty} f(z) =$ - coefficient ot t in the eapansion of g(t) about t=0

= -1/24.

CAUCHY'S RESIDUE THEOREM

If $f(z)$ is analytic at all point inside and on a simple closed curve C Except at a finite number of point $z_1, z_2, z_3 \ldots z_n$ inside C

Then

$$\int_c f(z)dz = 2\pi i \text{ [Sum of residues of f(z) at } z_1, z_2, z_3, \ldots z_n \text{]}$$

Proof

Given that $f(z)$ is not analytic Only at $z_1, z_2, z_3, \ldots z_n$

Draw the non intersecting small Circles $c_1, c_2, c_3 \ldots c_n$ with centre at $z_1, z_2, z_3, \ldots z_n$ and radii $\rho_1, \rho_2, \rho_3 \ldots \rho_n$

Then $f(z)$ is analytic in the region Between c and $c_1, c_2, c_3 \ldots c_n$

$$\int_c f(z)dz = \int_{c_1} f(z)dz + \int_{c_2} f(z)dz + \ldots + \int_{c_n} f(z)dz \qquad \ldots..(1)$$

Now $z_1, z_2, z_3 \ldots z_n$ **are the singular points of** $f(z)$.

$\therefore$ **Res** $f(z)_{z=z_i}$ = **the coefficient of** $\frac{1}{z-z_i}$ **in the Laurent's series of** $f(z)$ **about** $z = z_i$ **(by definition of residues)**

$$= b_1 = \frac{1}{2\pi i}\int_{c_1} \frac{f(z)}{(z-z_1)^{1-n}}dz$$

Since $\left(b_n = \frac{1}{2\pi i}\int_{c_1} \frac{f(z)}{(z-z_1)^{z-n}}dz \right)$

$$= \frac{1}{2\pi i} \int_{c_1} \frac{f(z)}{(z - z_1)^0} dz$$

$$= \frac{1}{2\pi i} \int_{c_1} f(z) dz$$

$$\Rightarrow \int_{c_1} f(z) dz = 2\pi i \left|\operatorname{Re} s f(z)\right|_{z=z_i} \quad \text{.....(2)}$$

From (1) and (2)

$$\int_{c_1} f(z) dz = 2\pi \left|\operatorname{Re} s f(z)\right|_{z=z_1} + 2\pi \left|\operatorname{Re} s f(z)\right|_{z=z_2} + 2\pi \left|\operatorname{Re} s f(z)\right|_{z=z_n}$$

$$= 2\pi \left|\operatorname{Re} s f(z)\right|_{z=z_1} + \left|\operatorname{Re} s f(z)\right|_{z=z_2} + \ldots + \left|\operatorname{Re} s f(z)\right|_{z=z_n}$$

$= 2\pi i$ {**Sum of residues of** $f(z)$ **at** $z = z_1, z_2, z_3 \ldots z_n$}

Problem

Example 1 Find the residue of f(z) $= \dfrac{z+2}{(z-2)(z+1)^2}$ about each singularity.

Solution: The poles of f (z) are given by

$$(z-2)=0, z+1=0$$

$$\Rightarrow z=2, z=-1$$

∴ The poles of f (z) are z = 2 is a simple poles and z = -1 is a pole of order 2.

$$\therefore [\operatorname{Re} s f(z)]_{z=2} = \lim_{z\to 2}(z-2) f(z)$$

$$= \lim_{z\to 2}(z-2)\frac{z+2}{(z-2)(z+1)^2} = \frac{4}{9}$$

$$\therefore [\operatorname{Re} s f(z)]_{z=-1} = \lim_{z\to -1}\frac{d}{dz}[(z-2)^2 f(z)]$$

$$= \lim_{z\to -1}\frac{d}{dz}[(z-2)^2\frac{z+2}{(z-2)(z+1)^2}]$$

$$= \lim_{z\to -1}\frac{d}{dz}\left[\frac{z+2}{z-2}\right]$$

$$= \lim_{z\to -1}\left[\frac{(z-2)(1)-(z+2)(1)}{(z-2)^2}\right] = -\frac{4}{9}$$

Problem

Example 2. Evaluate $\int_c \frac{dz}{(z^2+4)^2}$ **where c is the circle** $|z-i|=2$

Solution: Given $f(z)=\frac{1}{(z^2+4)^2}$

$z=\pm 2i$ **is a pole of order 2**

Here $z=2i$ **lies inside the circle** $|z-i|=2$

And $z=-2i$ **lies outside the circle and is of order 2.**

$$\therefore [\operatorname{Res} f(z)]_{z=2i} = \lim_{z\to 2i} \frac{d}{dz}(z-2i)^2 f(z)$$

$$=\lim_{z\to 2i} \frac{d}{dz}(z-2i)^2 \frac{1}{(z-2i)^2(z+2i)^2}$$

$$= \lim_{z \to 2i} \frac{d}{dz}(z-2i)^2$$

$$= \lim_{z \to 2i} \frac{(z+2i)^2(0) - 2(z+2i)}{(z+2i)^4}$$

$$= -\frac{8i}{256}$$

$\therefore$ **By Residue theorem**

$$\int_c \frac{dz}{(z^2+4)^4} = 2\pi i\left(\frac{-8i}{256}\right) = \frac{\pi}{6}$$

Problem

Problem: Evaluate $\int_c \frac{4-3z}{z(z-1)(z-2)}dz$ where c is the circle $|z| = \frac{3}{2}$. Using Residue theorem.

Solution: Let $\int_c \frac{4-3z}{z(z-1)(z-2)}dz$

The poles of f(z) are z (z-1)(z-2)=0

z=0, z= 1, z=2

These poles are simple poles.

The poles z=0 and z=1 lie within the circle c: $|z| = \frac{3}{2}$

Residue of f(z) at z=0 is $R_1 = \lim_{z \to 0}(z-0)f(z) = \lim_{z \to 0} z\frac{4-3z}{z(z-1)(z-2)} = \frac{4}{2} = 2$

Residue of f(z) at z=1 is $R_2 = \lim_{z \to 1}(z-1)f(z) = \lim_{z \to 1}(z-1)\frac{4-3z}{z(z-1)(z-2)} = \frac{4-3}{1-2} = -1$

By Residue theorem, $\int_c \frac{4-3z}{z(z-1)(z-2)}dz = 2\pi i(R_1 + R_2) = 2\pi i(2-1) = 2\pi i$.

Problem

Evaluate $\int_C \frac{z \sec z}{(1-z^2)} dz$ where C is the ellipse $4x^2 + 9y^2 = 9$, using Cauchy's residue theorem.

Solution:

Equation of ellipse is

$$4x^2 + 9y^2 = 9$$

$$\frac{x^2}{9/4} + \frac{y^2}{1} = 1$$

i.e., $\frac{x^2}{\left(3/2\right)^2} + \frac{y^2}{1} = 1$

$\therefore$ Major axis is $\frac{3}{2}$, Minor axis is 1.

The ellipse meets the x axis at $\pm\frac{3}{2}$ and the y axis at ± 1

Given $f(z) = \dfrac{z \sec z}{1-z^2}$

$$= \frac{z}{(1+z)(1-z)\cos z}$$

The poles are the solutions of $(1+z)(1-z)\cos z = 0$

i.e., $z = -1,\ z = 1$ are simple poles and $z = (2n+1)\dfrac{\pi}{2}$

Out of these poles $z \pm 1$ lies inside the ellipse

$z = \pm \dfrac{\pi}{4},\ \pm 3\dfrac{\pi}{4}$ lies outside the ellipse

$$\left[\operatorname{Re}s\, f(z)\right]_{z=1} = \underset{z \to 1}{Lt}(z-1)\frac{z}{(1+z)(1-z)\cos z}$$

$$= \underset{z \to 1}{Lt}\frac{-z}{(1+z)\cos z} = \frac{-1}{2\cos 1}$$

$$\left[\operatorname{Res} f(z)\right]_{z=-1} = \underset{z \to -1}{Lt} (z+1) \frac{z}{(1+z)(1-z)\cos z}$$

$$= \underset{z \to -1}{Lt} \frac{z}{(1-z)\cos z}$$

$$= \frac{-1}{2\cos 1} = \frac{-1}{2\cos 1}$$

$$\therefore \int_C \frac{z \sec z}{1-z^2}\, dz = 2\pi i \left[\text{sum of the residues }\right]$$

$$= 2\pi i \left[\frac{-1}{2\cos 1} - \frac{1}{2\cos 1}\right]$$

$$= -2\pi i \left[\sec 1\right].$$

Contour Integration

TYPE: I

Integrals of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$ **where** $f(\cos\theta, \sin\theta)$ **is**

A rational function of $\sin\theta$ & $\cos\theta$. **In this case we take unit circle** $|z| = 1$ **as the contour. On** $|z| = 1$.

$$z = e^{i\theta}, \cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$dz = ie^{i\theta}d\theta, \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\Rightarrow d\theta = \frac{dz}{iz}$$

Also θ **various from 0 to 2**π

$$\therefore \int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta = \int_c f\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right)\frac{dz}{iz}$$

Now applying Cauchy's residue theorem, we can evaluate the Integral on the right side.

Problem on Type-1

Example 1. Evaluate $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$ **by using contour Integration.**

Solution: $I = \int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$ **limit : 0 to** 2π

Contour: $|z| = 1$

Put $z = e^{i\theta}$

$$dz = ie^{i\theta}d\theta, \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\Rightarrow d\theta = \frac{dz}{iz}$$

$$\mathbf{I} = \int_c \frac{dz}{iz\left(13+5\left(\frac{z^2-1}{2iz}\right)\right)}$$

$$= 2\int_c \frac{dz}{5z^2+26iz-5}$$

Where $f(z) = \frac{1}{5z^2+26iz-5}$

$$\therefore I = 2\int_c f(z)dz \quad \ldots\ldots\ldots\ldots\ldots(1)$$

To find Residue

The poles of f (z) are $5z^2+26iz-5$

$$Z=\frac{-26\pm\sqrt{(26i)^2+100}}{10}$$

$$=\frac{-26\pm 24}{10}=\frac{-i}{5},-5i$$

The pole z = $\frac{-i}{5}$ lies inside the circle $|z|=1$ and z = -5i lies outside the circle $|z|=1$

Now $\therefore [\operatorname{Re}s f(z)]_{z=-i/5}=\lim_{z\to -i/5}\left(z+\frac{i}{5}\right)f(z)$

$$=\lim_{z\to -i/5}\left(z+\frac{i}{5}\right)\frac{1}{5\left(z+\frac{i}{5}\right)(z+5i)}$$

$$=\frac{1}{5\left(\frac{-i}{5}+5i\right)}=\frac{1}{24i}$$

$\therefore$ By Cauchy's Residue theorem

$$\int_c f(z)dz=2\pi i\sum R$$

$$=2\pi i\frac{1}{24i}=\frac{\pi}{12}$$

(1) becomes

$$I=\frac{2\pi}{12}=\frac{\pi}{6}$$

Problem on Type-1

Example 2: Evaluate $\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}d\theta, a>b>0$ **using contour integration.**

Solution: Let $I = \int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}d\theta,$

$$= \int_0^{2\pi} \frac{1-\cos 2\theta}{2a+2b\cos\theta}d\theta$$

We can write $\cos 2\theta =$ **Real part of** $e^{2i\theta}, \because e^{2i\theta} = \cos 2\theta + i\sin 2\theta$

$$\therefore I = R.P\int_0^{2\pi} \frac{1-e^{2i\theta}}{2a+2b\cos\theta}d\theta$$

Put $z = e^{i\theta}$

$$d\theta = \frac{dz}{iz}, \cos\theta = \frac{z^2+1}{z}$$

$$\therefore I = R.P\int_0^{2\pi} \frac{1-z^2}{2a+b\left(\frac{z^2+1}{z}\right)}\frac{dz}{iz}$$

$$= R.P\frac{1}{i}\int_0^{2\pi} \frac{1-z^2}{bz^2+2az+b}dz$$

$$= R.P\frac{1}{i}\int_0^{2\pi} f(z)dz \,..........(1)$$

Where $f(z) = \dfrac{1-z^2}{bz^2+2az+b}$

To find Residues:

Poles of f(z) are given by $bz^2+2az+b=0$

$$\mathbf{Z} = \frac{-2a \pm \sqrt{4a^2-4b^2}}{2b}$$

$$= \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

Let $\alpha = \dfrac{-a+\sqrt{a^2-b^2}}{b}$ **and** $\beta = \dfrac{-a-\sqrt{a^2-b^2}}{b}$**, since a>b,**

$$|\alpha| < 1 \,\&\, |\beta| > 1$$

$\therefore$ **The simple pole z =** α **lies inside C,**

$$\therefore [\operatorname{Res} f(z)]_{z=\alpha} = \lim_{z\to\alpha}(z-\alpha)f(z)$$

$$= \lim_{z\to\alpha}(z-\alpha)\frac{1-z^2}{b(z-\alpha)(z-\beta)}$$

$$= \frac{1-\alpha^2}{b(\alpha-\beta)} = \frac{a-\sqrt{a^2-b^2}}{b^2}$$

$$\therefore \sum R = \frac{a-\sqrt{a^2-b^2}}{b^2}$$

$\therefore$ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i \sum R$$

$$= 2\pi i\left(\frac{a-\sqrt{a^2-b^2}}{b^2}\right)$$

(1) becomes

$$\mathbf{I} = R.P\frac{1}{i}\int_0^{2\pi} f(z)dz$$

$$= R.P\,2\pi\left(\frac{a-\sqrt{a^2-b^2}}{b^2}\right)$$

$$= \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right)$$

Problem on Type-1

Prove that $\int_0^{2\pi} \frac{d\theta}{a^2 - 2a\cos\theta + 1} = \frac{2\pi}{1-a^2}$, given $a^2 < 1$.

Solution: Let $I = \int_0^{2\pi} \frac{d\theta}{a^2 - 2a\cos\theta + 1}$

Put $z = e^{i\theta}$

Then $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$$\therefore\ I = \int_C \frac{dz/iz}{a^2 - a\left(z + \frac{1}{z}\right) + 1} \text{ where C is } |z| = 1.$$

$$= \frac{1}{ai} \int_C \frac{dz}{\left(a + \frac{1}{a}\right)z - z^2 - 1}$$

$$= \frac{i}{a} \int_C \frac{dz}{z^2 - \left(a + \frac{1}{a}\right)z + 1}$$

$$= \int_C f(z)dz \text{ where } f(z) = \left(\frac{i}{a}\right)\frac{1}{z^2 - \left(a + \frac{1}{a}\right)z + 1}$$

$$= \left(\frac{i}{a}\right)\frac{1}{(z-a)\left(z-\frac{1}{a}\right)}$$

The singularities of f(z) are simple poles at a and $\frac{1}{a}$. $a^2 < 1$ implies $|a| < 1$ and $\frac{1}{|a|} > 1$

$\therefore$ The pole that lies inside C is z = a.

$$\text{Res}[f(z);\, a] = \lim_{z \to a}(z-a).\left(\frac{i}{a}\right)\frac{1}{(z-a)\left(z-\frac{1}{a}\right)}$$

$$= \left(\frac{i}{a}\right)\frac{1}{\left(a-\frac{1}{a}\right)}$$

$$= \frac{i}{a^2-1}$$

$$\text{Hence } I = 2\pi i.\frac{i}{a^2-1} = \frac{2\pi}{1-a^2}$$

Contour Integration and Problem on Type-2

Type II Integration around semi-circular contour

Consider the integral

Improper integrals of the form $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$ **,where P(x) and Q(x)**

Are polynomials in x such that the degree of Q exceeds that of P atleast by two and Q(x) does not vanish for any x.

Example 1: Prove that $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx = \frac{5\pi}{12}$ **using contour integration.**

Solution: Let $f(z) = \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}$

Consider $\int_C f(z)dz$ **where C is the closed contour consisting of** Γ**, semi- large circle of radius R and the real axis from –R to R.**

Then $\int f(z)dz = \int f(z)dz + \int^{R} f(x)dx$ **.........(1)**

Now $f(z)=\dfrac{z^2-z+2}{z^4+10z^2+9}\to 0$ as $z\to\infty$

$$\therefore \lim_{z\to\infty} zf(z)=0$$

Hence from (1) $\int_{-\infty}^{\infty} f(x)dx=\int_C f(z)dz$

By using residue theorem, $\int_C f(z)dz=2\pi i\sum R$

$$\therefore \int_{-\infty}^{\infty} f(x)dx=2\pi i \operatorname{Re} s f(z)$$

The poles of f(z) are given by

$$z^4+10z^2+9=0$$

$$(z^2+9)(z^2+1)=0$$

$$z=\pm i,\ z=\pm 3i$$

The poles z = 3i, z = i lies in the upper half of the z – plane

$$[\operatorname{Res} f(z)]_{z=i} = \lim_{z\to i}(z-i)f(z)$$

$$= \lim_{z\to i}(z-i)\frac{z^2-z+2}{(z+i)(z-i)(z^2+9)}$$

$$= \frac{1-i}{8(2i)} = \frac{1-i}{16i}$$

$$[\operatorname{Res} f(z)]_{z=3i} = \lim_{z\to 3i}(z-3i)f(z)$$

$$= \lim_{z\to 3i}(z-3i)\frac{z^2-z+2}{(z-3i)(z+3i)(z^2+9)}$$

$$= \lim_{z\to 3i}\frac{z^2-z+2}{(z+3i)(z^2+9)}$$

$$= \frac{-7-3i}{(6i)(-8)} = \frac{7+3i}{48i}$$

$$\therefore \sum \operatorname{Res} f(z) = \frac{1-i}{16i} + \frac{7+3i}{48i}$$

$$= \frac{10}{48i}$$

$$= \frac{5}{24i}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\left(\frac{5}{24i}\right) = \frac{5\pi}{12}$$

Problem on Type-2

Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$ **by using contour**

Solution: Consider the integral $\int_C f(z)dz$ **where**

$$f(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)}$$

And C is the closed contour consisting of Γ**, the upper semi large Circle** $|z| = R$ **and the real axis from –R to R**

$$\int_c f(z)dz = \int_\Gamma f(z)dz + \int_{-R}^{R} f(x)dx$$

When R $\rightarrow \infty, \int_\Gamma f(z)dz$

Hence $\int_{-\infty}^{\infty} f(x)dx = \int_C f(z)dz$

By using residue theorem, $\int_C f(z)dz = 2\pi i \sum R$

Poles of f(z) are given by $z^2 + a^2 = 0, z^2 + b^2 = 0$

$$z = \pm ai, z = \pm bi$$

The pole z = ai and z = bi lies in the upper half plane

$$[\operatorname{Re} s f(z)]_{z=ai} = \lim_{z \to ai} (z - ai) f(z)$$

$$= \lim_{z \to ai} (z - ai) \frac{z^2}{(z + ai)(z - ai)(z^2 + b^2)}$$

$$= \lim_{z \to ai} \frac{z^2}{(z + ai)(z^2 + b^2)}$$

$$= \frac{(ai)^2}{2ai(-a^2 + b^2)}$$

$$= \frac{a}{2i(a^2 - b^2)}$$

$$[\operatorname{Re} s f(z)]_{z=bi} = \lim_{z \to bi}(z-bi)f(z)$$

$$= \lim_{z \to bi}(z-bi)\frac{z^2}{(z+bi)(z-bi)(z^2+a^2)}$$

$$= \frac{-b^2}{2bi(-b^2+a^2)}$$

$$= \frac{-b}{2i(a^2-b^2)}$$

$$\therefore \int_{-\infty}^{\infty} f(z)dz = \frac{2\pi i}{2i}\left[\frac{a}{a^2-b^2} - \frac{b}{a^2-b^2}\right]$$

$$= \pi\left[\frac{a-b}{(a+b)(a-b)}\right]$$

$$= \pi\left[\frac{1}{(a+b)}\right]$$

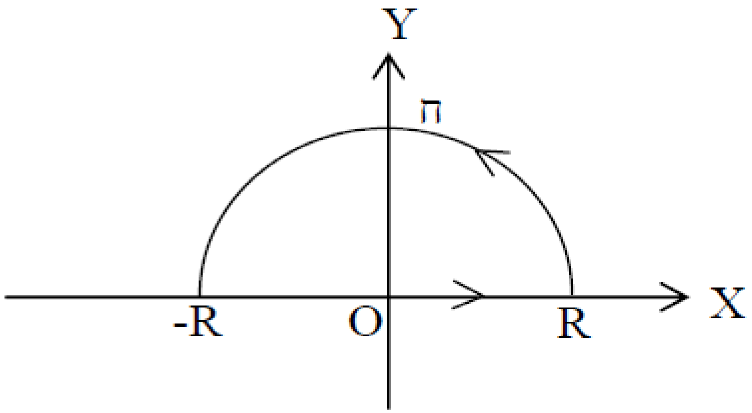
Problem Prove that $\int_0^{\infty} \frac{dx}{\left(x^2+1\right)^2} = \frac{\pi}{4}$

Solution:

Let $\int_C \phi(z)\,dz = \int_C \frac{dz}{\left(z^2+1\right)^2}$

Where $\phi(z) = \frac{1}{\left(z^2+1\right)^2}$

Here C is the semicircle Γ bounded by the diameter $[-R, R]$



By Cauchy residue theorem,

$$\int_C \phi(z)\,dz = \int_{-R}^{R} \phi(x)\,dx + \int_\Gamma \phi(z)\,dz \ldots\ldots(1)$$

To evaluate of $\int_C \phi(z)\,dz$

The poles of $\phi(z) = \dfrac{1}{\left(z^2+1\right)^2}$ is the solution of $\left(z^2+1\right)^2 = 0$

i.e., $(z+i)^2(z-i)^2 = 0$

i.e., the poles are $z = i,\ z = -i$

$z = i$ lies with inside the semi circle

$z = -i$ lies outside the semi circle

Now $\left[Res\,\phi(z)\right]_{z=i} = \underset{z \to i}{Lt} \dfrac{1}{1!}\dfrac{d}{dz}(z-i)^2\,\phi(z)$

$$= \underset{z \to i}{Lt} \frac{1}{1!}\left[(z-i)^2 \frac{1}{(z^2+1)^2}\right]$$

$$= \underset{z \to i}{Lt} \frac{1}{1!}\frac{d}{dz}\left[\frac{1}{(z+i)^2}\right] \qquad \because z^2+1=(z+i)(z-i)$$

$$= \underset{z \to i}{Lt} \frac{-2}{(z+i)^3}$$

$$= \frac{-2}{i+i} = \frac{-2}{(2i)^3} = \frac{1}{4i}$$

$$\therefore \int_C \phi(z)\, dz = 2\pi i\left[Sum\ of\ residues\ of\ \phi(z)\ \text{at}\ its\ poles\ which\ lies\ in\ C\right]$$

$$= 2\pi i\left[\frac{1}{4i}\right] = \frac{\pi}{2} \text{..........}(2)$$

Let $R \to \infty$, then $|z| \to \infty$ so that $\phi(z) = 0$

$$\therefore \underset{|z| \to \infty}{Lt} \int_{\Gamma} \phi(z) dz = 0 \ldots\ldots\ldots(3)$$

Sub (2) and (3) in (1)

$$\int_{C} \phi(z)\, dz = \int_{-\infty}^{\infty} \phi(x)\, dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2}$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2}$$

$$\Rightarrow \int_{0}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}.$$

MODULE IV- SPECIAL FUNCTIONS-

MODULE - IV
SPECIAL FUNCTIONS-I

MODULE IV- SPECIAL FUNCTIONS-I

- **Beta and Gamma functions:**
- **Improper Integrals: Beta and Gamma functions**
- **Definitions**
- **Properties of Beta and Gamma functions**
- **Standard forms of Beta functions**
- **Relationship between Beta and Gamma function**

IMPROPER INTEGRAL:

The integral $\int_a^b f(x)\,dx$ for which

i) Either the interval of integration is not finite i. e. $a = -\infty \, or b = \infty$ or both

ii) The function f(x) is unbounded at one or more point in [a, b] is called das improper integral.

<u>NOTE</u>: Integral of (i) and (ii) are called the improper integrals of first and second kinds respectively.

<u>Examples:</u>

1. $\int_{0}^{\infty}\frac{1}{1+x^{4}}\,dx$ And $\int_{-\infty}^{\infty}\frac{1}{1+x^{2}}\,dx$ are improper integrals of the first kind.
2. $\int_{0}^{1}\frac{1}{1-x^{2}}\,dx$ is an improper integral of the second kind.

Beta functions

DEFINITION:

BETA FUNCTION:

The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is called the Beta function and is denoted by $\beta(m,n)$. The integral converges for m>

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

NOTE:

Beta function is also called as Eulerian integral of first kind

PROPERTIES OF BETA FUNCTION:

i) SYMMETRY PROPERTY OF BETA FUNCTION

i.e., $\beta(m,n)=\beta(n,m)$

Proof:

By definition, we have

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Put 1-x=y so that dx=-dy

$$\therefore \beta(m,n)=\int_1^0 (1-y)^{m-1}\,y^{n-1}\,(-dy)$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\, dy$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}\, dx$$

$$\beta(n,m)\left[\because \int_a^b f(t)\,dt = \int_a^b f(x)\,dx\right]$$

Hence $\beta(m,n) = \beta(n,m)$

ii) Prove that

$$\beta(m,n) = \int_1^0 \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

Proof:

By definition, we have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

put $x = \sin^2\theta$ so that $dx = \sin 2\theta \, d\theta$

$$\therefore \beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} \sin 2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

Hence proved

i) $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$

Proof:

By definition, we have

$$\beta(m+1,n) + \beta(m,n+1) = \int_0^1 x^m (1-x)^{n-1} \, dx + \int_0^1 x^{m-1}(1-x)^n \, dx$$

$$= \int_0^1 x^m (1-x)^{n-1} \, dx + \int_0^1 x^{m-1}(1-x)^n \, dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}[x + (1-x)] \, dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1} \, dx$$

$= \beta(m,n)$

Hence $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$

Iv)

If m and n are positive integers, then

$$\beta(m,n)=\frac{(m-1)!(n-1)!}{(m+n-1)!}$$

<u>Proof:</u>

We have $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Integrating by parts

$$\left[x^{m-1}\frac{(1-x)^n}{n(-1)}\right]_0^1-\int_0^1\frac{(1-x)^n}{n(-1)}(m-1)x^{m-2}dx$$

$$=\frac{m-1}{n}{}_0^1-\int_0^1 x^{m-2}(1-x)^n dx=\frac{m-1}{n}\beta(m-1,n+1)\text{..............(1)}$$

Now we have to find $\beta(m-1,n+1)$.

To obtain this put m=m-1 and n=n+1 in (1). Then, we have

Beta and Gamma functions

$$\beta(m,n)=\frac{m-1}{n}.\frac{m-2}{n+1}\beta(m-2,n+2)$$ **......................(2)**

Changing m to m-2 and n to n-2 from (1) we have

$$\beta(m-2,n+2)=\frac{m-3}{n+2}.\frac{m-2}{n+1}\beta(m-3,n+3)$$

From (2)

$$\beta(m,n)=\frac{m-1}{n}.\frac{m-2}{n+1}.\frac{m-3}{n+2}\beta(m-3,n+3)$$

Proceeding like this we get

$$\beta(m,n)=\frac{m-1}{n}.\frac{m-2}{n+1}.\frac{m-3}{n+2}..........\frac{[m-(m-1)]}{[n+(m-2)]}\beta(m-(m-1),n+(m-1))$$

$$=\frac{m-1}{n}.\frac{m-2}{n+1}.\frac{m-3}{n+2}..........\frac{1}{(n+m-2)}\beta(1,n+m-1)$$ **......(3)**

From (3)

$$\beta(m,n)=\frac{m-1}{n}.\frac{m-2}{n+1}.\frac{m-3}{n+2}..........\frac{1}{(n+m-2)}..\frac{1}{(n+m-1)}$$

$$= \frac{(m-1)!}{(n+m-1)(n+m-2)\ldots\ldots\ldots\ldots(n+2)(n+1)n}$$

Multiplying the numerator and denominator by (n-1)!, we have

$$\beta(m,n) = \frac{(m-1)!\,(n-1)!}{(n+m-1)!}$$

STANDARD FORMS OF BETA FUNCTIONS

FORM I:

To show

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Proof:

We have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

put $x = \frac{1}{1+y}$ so that $dx = \frac{dy}{(1+y)^2}$

From (1)

We have

$$\beta(m,n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(1-\frac{1}{1+y}\right)^{n-1} \cdot -\frac{dy}{(1+y)^2}$$

$$= \int_0^\infty \frac{y^{n-1} dy}{(1+y)^{m+n}}$$

$$\therefore \beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

Hence proved.

FORM II:

To show that

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$$

Proof:

From form we have

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

Now putting $x = \frac{1}{y}$ and $dx = -\frac{1}{y^2} dy$ in the second integral,

we get

Beta and Gamma functions

$$\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = \int_\infty^0 \frac{\left(\frac{1}{y}\right)^{m-1}}{\left(1+\frac{1}{y}\right)^{m+n}} \cdot \frac{-1}{y^2}dy$$

$$\int_0^1 \frac{y^{m+n}}{(1+y)^{m+n}} \cdot \frac{-1}{y^{m+1}}dy$$

$$= \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}dy$$

$$\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

Hence

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$$

FORM III:

$$\beta(m,n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}\,dx$$

Proof:

We have

$$a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}\,dx = a^m b^n \int_0^\infty \frac{x^{m-1}}{b^{m+n}\left(\frac{ax}{b}+1\right)^{m+n}}\,dx$$

Put

$\frac{ax}{b}$ =t then $\frac{a\,dx}{b} = dt$

$$\frac{a^m b^n}{b^{m+n}} \int_0^\infty \frac{\frac{b^{m-1}}{a^{m-1}} t^{m-1}}{(t+1)^{m+n}} \frac{b}{a}\,dt$$

$$= \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\,dt = \beta(m,n)$$

Hence proved.

FORM IV:

To show $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}\,dx = \frac{\beta(m,n)}{a^n(1+a)^m}$

PROOF:

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Put $x = \frac{(1+a)t}{t+a}$ then $dx = (1+a)\left[\frac{(t+a)1 - t(1+0)}{(t+a)^2}\right]$

$$= \frac{a(1+a)}{(t+a)^2}$$

$$dx = \frac{a(1+a)}{(t+a)^2}dt$$

Also when x=0, t=0 and x=1,t=1.

Now (1) become

$$\beta(m,n) = \int_0^1 \frac{(1+a)^{m-1}t^{n-1}}{(t+a)^{m-1}}\left(1 - \frac{(1+a)t^1}{(t+a)^1}\right)^{n-1} \frac{a(1+a)}{(t+a)^2}dt$$

Beta functions

$$\beta(m,n)=\int_0^1 \frac{(1+a)^{m-1}t^{n-1}}{(t+a)^{m-1}}\left(\frac{a-at}{t+a}\right)^{n-1} a\ dt$$

Also we have $\beta(m,n)=\frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$

$$\therefore \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx=\frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$$

Taking m + n=1 so that m=n-1, we get

$$\therefore \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx=\frac{\gamma(1-n)\gamma(n)}{\gamma(1)}$$

Or

$$\therefore \gamma(1-n)\gamma(n)=\int_0^\infty \frac{x^{n-1}}{1+x}dx.......(1)$$

We have

$$\therefore \int_0^\infty \frac{x^{2n}}{1+x^{2n}}dx=\frac{\pi}{2n}\operatorname{cosec}\frac{(2m+1)\pi}{2n}$$ Where m>0, n>0 and m>n

Put x^{2n}=t and $\frac{(2m+1)}{2n}=s$, we have

$$\therefore \int_0^\infty \frac{t^{\left(2m/2n\right)}t^{1/2n}}{(2n)(1+t)t}dt=\frac{\pi}{2n}\operatorname{cosec}s\pi$$

$$\therefore \int_0^\infty \frac{t^{\left(2m/2n\right)}t^{\frac{1}{2n}-1}}{(1+t)}dt=\frac{\pi}{2n}\operatorname{cosec}s\pi$$

Or $\therefore \int_0^\infty \frac{t^{\left[\left(2m+1\right)/2n\right]-1} t^{\frac{1}{2n}-1}}{(1+t)} dt = \frac{\pi}{2n} \operatorname{cosec} s\pi$

$$\therefore \int_0^\infty \frac{t^{s-1}}{(1+t)} dt = \frac{\pi}{2n \sin s\pi}$$

$\therefore \int_0^\infty \frac{x^{s-1}}{(1+x)} dt = \frac{\pi}{2n \sin s\pi}$ ....(2)

From (1) and (2) we have

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}} dx = \frac{\beta(m,n)}{a^n(1+a)^m}$$

Hence Proved

PROBLEMS:

1. Show that

$$\int_0^{\pi/2} \sin^m\theta\cos^n\theta\,d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2},\frac{n+1}{2}\right)$$

Solution:

$$\int_0^{\pi/2} \sin^m\theta\cos^n\theta\,d\theta = \int_0^{\pi/2} \sin^{m-1}\theta\cos^{n-1}\theta(\sin\theta\cos\theta)d\theta$$

$$= \int_0^{\pi/2} (\sin^2\theta)^{\frac{m-1}{2}}(\cos^2\theta)^{\frac{n-1}{2}}(\sin\theta\cos\theta)d\theta$$

Put $\sin^2\theta = x$ *so* that $(\sin\theta\cos\theta)d\theta = \frac{dx}{2}$

$$\int_0^{\pi/2} \sin^m\theta\cos^n\theta\,d\theta = \int_0^1 x^{\frac{m-1}{2}}(1-x)^{\frac{n-1}{2}}\,dx$$

$$= \int_0^1 x^{\left(\frac{m+1}{2}\right)-1}(1-x)^{\left(\frac{n-1}{2}\right)-1}\,dx$$

$$= \frac{1}{2}\beta\left(\frac{m+1}{2},\frac{n+1}{2}\right)$$

$$\therefore \int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

Hence proved

1. Express the following integrals in terms of Beta function:
2. $\int_0^1 \frac{x}{\sqrt{1-x^2}}dx$
3. $\int_0^4 \frac{x}{\sqrt{9-x^2}}dx$

   Answer: $\frac{1}{2}\beta\left(\frac{1}{2}, \frac{1}{2}\right)$

Solution: Put $x^2 = y$ so that $dx = \frac{1}{2} y^{-1/2} dy$

When x=0, y=0 when x=1, y=1.

$$\int_0^1 \frac{x}{\sqrt{1-x^2}}dx = \int_0^1 \frac{y^{1/2}}{\sqrt{1-y}} \frac{1}{2} y^{-1/2} dy$$

Gamma functions

$$= \frac{1}{2}\int_0^1 (1-y)^{-1/2} dy$$

$$= \frac{1}{2}\int_0^1 y^{1-1}(1-y)^{\frac{1}{2}-1} dy$$

$$= \frac{1}{2}\beta\left(1, \frac{1}{2}\right)$$

Exercise Problems:

1. Prove that $\int_0^a (a-x)^{m-1} x^{n-1} dx = a^{m+n-1}\beta(m, n)$

   Hint: put x=ay

2. Show that $\int_0^a x^{m-1}(1-x^n)^p dx = \frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right)$

   Hint: put x^n =y

3. Show that $\int_0^a (1+x)^{m-1}(1-x\ )^{n-1} dx = 2^{m+n-1}\beta(m, n)$

   Hint: put $x = \frac{1+y}{2}$

Gamma functions

1. Show that

   i. $\int_0^\infty \frac{x^{m-1} - x^{n-1}}{(1+x)^{m+n}} dx = 0$

   ii. $\int_0^\infty \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx = 2\beta(m, n)$

2. Prove that $\frac{\beta(p, q+1)}{q} = \frac{\beta(p+1, q)}{p} = \frac{\beta(p, q)}{p+q}$, where p>0, q>0.

Show that $\int_a^b (x-a)^m (b-x)^n \, dx = (b-a)^{m+n+1} \beta(m+1, n+1)$

Gamma functions

GAMMA FUNCTION:

- The Gamma function and Beta functions belong to the category of the special transcendental functions and are defined in terms of improper definite integrals.

Definition:

The definite integral $\int_0^\infty e^{-x}x^{n-1}dx$ is called the Gamma function and is denoted by

$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx$ And read as “gamma n”.

NOTE:

1. The integral converges for n>0.
2. Gamma function is also called Eulerian integral of the second kind.
3. The integral Gamma function does not converges if n≤0.

Gamma functions

PROPERTIES OF GAMMA FUNCTIONS:

I. To show that $\Gamma(1) = 1$

Proof: By definition of Gamma function, we have

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$$\therefore \Gamma(n) = \int_0^\infty e^{-x} x^0 dx = \int_0^\infty e^{-x} dx = \left(e^{-x}\right)_0^\infty = 1$$

II. To show that $\Gamma(n) = (n-1)\Gamma(n-1)$ where n > 1.

Proof: By definition of Gamma function, we have

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx = \left[x^{n-1} \frac{e^{-x}}{(-1)} \right]_0^\infty - \int_0^\infty (n-1) x^{n-2} . \left(\frac{e^{-x}}{-1} \right) dx$$ Integrate by parts

$$= (n-1)\int_0^\infty e^{-x} x^{n-2} dx = (n-1)\Gamma(n-1)$$

Gamma functions

Note:

1. $\Gamma(n+1)=(n)\Gamma(n)$
2. If n is a positive fraction, then we write

   $\Gamma(n)=(n-1)(n-2)(n-3)(n-4)..............\Gamma(n-r)$

   Where $(n-r)>0$
3. If n is a non-negative integer, then $\Gamma(n+1)=(n)!$

<u>Properties of Gamma function</u> :

1) $\Gamma(m+1)=m\Gamma m$
2) $\Gamma(m+1)=m!$ When m is a positive integer.
3) $\Gamma(m+a)=(m+a-1)(m+a-2)\ldots\ldots\ldots a\Gamma a$, when n is a positive integer.
4) $\Gamma m=2\int_0^\infty e^{-x^2}x^{2m-1}\,dx \quad (m>0)$
5) $\frac{\Gamma m}{t^m}=\int_0^\infty e^{-tx}x^{m-1}\,dx \quad (m>0)$
6) $\Gamma\frac{1}{2}=\sqrt{\pi}$
7) $\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$
8) $\int_0^1 x^n(\log x)^m\,dx=\frac{(-1)^m}{(n+1)^{m+1}}\Gamma(m+1)$

Example 1: Evaluate $\Gamma(-\frac{1}{2})$.

Solution: We know that $\Gamma(m+1) = m\Gamma m$

$$\Rightarrow \Gamma\left(-\tfrac{1}{2}+1\right) = -\tfrac{1}{2}\Gamma\left(-\tfrac{1}{2}\right)$$
$$\Rightarrow \Gamma\left(\tfrac{1}{2}\right) = -\tfrac{1}{2}\Gamma\left(-\tfrac{1}{2}\right)$$
$$\Rightarrow \sqrt{\pi} = -\tfrac{1}{2}\Gamma\left(-\tfrac{1}{2}\right)$$
$$\therefore \quad \Gamma\left(-\tfrac{1}{2}\right) = -2\sqrt{\pi}.$$

RELATION BETWEEN BETA AND GAMMA FUNCTIONS

1. $\beta(m,n) = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$ Where m>0, n>0

Proof:

: By definition of Gamma function, we have

$$\Gamma(m) = \int_0^\infty e^{-x} x^{m-1} dx \text{ .... } (1)$$

Put x = yt so that dx =y dt then (1) gives

$$\Gamma(m) = \int_0^\infty e^{-yt} yt^{m-1} t^{m-1} y dt = \int_0^\infty e^{-yt} y^m t^{m-1} dt = \int_0^\infty e^{-yx} y^m x^{m-1} dx \text{ ....} (2)$$

Or $\frac{\Gamma(m)}{y^m} = \int_0^\infty e^{-yx} x^{m-1} dx$ ....(3)

Gamma functions

Multiplying both sides of (3)

$$\Gamma(m)\int_0^\infty e^{-y}y^{n-1}dy = \int_0^\infty \left\{\int_0^\infty e^{-y(x+1)}y^{m+n-1}x^{m-1}dx\right\}dy$$ ....(4)

$$\Gamma(m)\Gamma(n) = \int_0^\infty \left\{\int_0^\infty e^{-y(x+1)}y^{m+n-1}dy\right\}x^{m-1}dx$$, by interchanging the order of integration

$$\Gamma(m)\Gamma(n) = \int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}}x^{m-1}dx$$

$$\Gamma(m)\Gamma(n) = dx\int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}}x^{m-1}dx = \Gamma(m+n)\beta(m, n)$$

$$\therefore \beta(m,n) = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$$

Hence proved

1. To prove that $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

Proof:

By Form I of Beta function

$$\beta(m,n) == \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

Also we have $\therefore \beta(m,n) = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$

Gamma functions

$$\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$$

Taking m + n=1 so that m=1-n, we get

$$\int_0^\infty \frac{x^{n-1}}{(1+x)}dx = \frac{\gamma(1-n)\gamma(n)}{\gamma(1)}$$

$$\gamma(1-n)\gamma(n) = \int_0^\infty \frac{x^{n-1}}{(1+x)}dx$$

We have

$$\int_0^\infty \frac{x^{2m}}{(1+x^{2n})}dx = \frac{\pi}{2n}\operatorname{cosec}\frac{(2m+1)\pi}{2n}$$, where m>0, n>0 and n>m

Put $x^{2m} = t$ and $\frac{(2m+1)}{2n} = s$, we have

Gamma functions

$$\int_0^\infty \frac{t^{\left(2m/2n\right)} t^{1/2n}}{(2n)(1+t)t} dt = \frac{\pi}{2n} \operatorname{cosec} s\pi$$

Or $\displaystyle\int_0^\infty \frac{t^{\left(2m/2n\right)} t^{\frac{1}{2n}-1}}{(1+t)} dt = \frac{\pi}{2n} \operatorname{cosec} s\pi$

$$\int_0^\infty \frac{t^{\left(2m+1/2n\right)-1}}{(1+t)} dt = \pi \operatorname{cosec} s\pi$$

Or $\displaystyle\int_0^\infty \frac{t^{s-1}}{(1+t)} dt = \frac{\pi}{\sin n\pi}$

$$\therefore \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

Hence proved

1. To show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof: we know that $\beta(m,n) = \dfrac{\gamma(m)\gamma(n)}{\gamma(m+n)}$

Taking m=n=$\frac{1}{2}$, we have

Gamma functions

$$\beta\left(\frac{1}{2},\frac{1}{2}\right)=\frac{\gamma\left(\frac{1}{2}\right)\gamma\left(\frac{1}{2}\right)}{\gamma\left(\frac{1}{2}+\frac{1}{2}\right)}=\left[\Gamma\left(\frac{1}{2}\right)\right]^2 \quad [\because \gamma(1)=1]$$ ……………………………(1)

$$\beta\left(\frac{1}{2},\frac{1}{2}\right)=\int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}dx \quad =\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}dx$$

$x=\sin^2\theta$ so that $dx=2\sin\theta\cos\theta d\theta$

Also when x=0, θ=0: when x=1, $\theta=\pi/2$

$$\therefore\ \beta\left(\frac{1}{2},\frac{1}{2}\right)=\int_0^{\pi/2} dx \quad =\int_0^1 \frac{1}{\sin\theta}\frac{1}{\cos\theta}2\sin\theta\cos\theta\, d\theta$$

$$=2\int_0^{\pi/2} d\theta=\pi$$ ……………(2)

From (1) and (2) we have

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

1. To show that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Proof: we have

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

Taking n=$\frac{1}{2}$, we have $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x} x^{-1/2} dx$

Put x =t^2 so that d x=2t d t

Also when x=0, t=0: when $x \to \infty, t \to \infty$

$$\therefore \Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t^2} \left(t^2\right)^{-1/2} 2dt = 2\int_0^\infty e^{-t^2} dt$$

Or $2\int_0^\infty e^{-x^2} dx = \sqrt{\pi}$

$$\therefore \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

PROBLEMS

1. Compute

i) $\Gamma\left(\frac{11}{2}\right)$

ii) $\Gamma\left(-\frac{1}{2}\right)$

iii) $\Gamma\left(-\frac{7}{2}\right)$

Solutions: i)

We have $\Gamma(n+1)=(n)\Gamma(n)$

Taking $n=\frac{7}{2}$

$$\Gamma\left(\frac{11}{2}\right)=\frac{9}{2}\Gamma\left(\frac{9}{2}\right)$$

$$=\frac{9}{2}\frac{7}{2}\Gamma\left(\frac{7}{2}\right)$$

$$=\frac{9}{2}\frac{7}{2}\frac{5}{2}\Gamma\left(\frac{5}{2}\right)$$

$$=\frac{9}{2}\frac{7}{2}\frac{5}{2}\frac{3}{2}\Gamma\left(\frac{3}{2}\right)$$

$$=\frac{9}{2}\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{9}{2}\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}$$

Solution: ii)

We have $\Gamma(n+1) = (n)\Gamma(n)$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

Taking $n = -\frac{1}{2}$

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = -2\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$$

Gamma functions

Evaluate each of the following:

(a) $\dfrac{\Gamma(6)}{2\Gamma(3)} = \dfrac{5!}{2\cdot 2!} = \dfrac{5\cdot 4\cdot 3\cdot 2}{2\cdot 2} = 30$

(b) $\dfrac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \dfrac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \dfrac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \dfrac{3}{4}$

(c) $\dfrac{\Gamma(3)\Gamma(2.5)}{\Gamma(5.5)} = \dfrac{2!(1.5)(0.5)\Gamma(0.5)}{(4.5)(3.5)(2.5)(1.5)(0.5)\Gamma(0.5)} = \dfrac{16}{315}$

(d) $\dfrac{6\Gamma\left(\frac{8}{3}\right)}{5\Gamma\left(\frac{2}{3}\right)} = \dfrac{6\left(\frac{5}{3}\right)\left(\frac{2}{3}\right)\Gamma\left(\frac{2}{3}\right)}{5\Gamma\left(\frac{2}{3}\right)} = \dfrac{4}{3}$

Gamma functions

1. Evaluate

i. $\int_0^1 x^5 (1-x)^3 \, dx$

ii. $\int_0^1 x^4 (1-x)^2 \, dx$

Answer: 1/105

iii. $\int_0^1 x \; (1-x)^{1/3} \, dx$

Answer: $\frac{16\sqrt{\pi}}{9\sqrt{3}}$

iv. $\int_0^1 x^{5/2} (1-x^2)^{3/2} \, dx$

Answer: $\frac{8}{65} \frac{\Gamma\left(\frac{3}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)}$

Solution: i)

$$\int_0^1 x^5(1-x)^3\,dx = \int_0^1 x^{6-1}(1-x)^{4-1}\,dx$$

$$\beta(6,4) = \frac{\Gamma(6)\Gamma(4)}{\Gamma(6+4)}$$

$$\frac{5!3!}{9!} = \frac{1}{504}$$

Evaluate $\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx$

Solution: Let $I = \int_0^\infty x^{\frac{1}{4}} e^{-\sqrt{x}}\, dx$ __________(i)

Putting $\sqrt{x} = t \implies x = t^2$ so that $dx = 2t$ in (i), we get

$$I = \int_0^\infty t^{1/2} e^{-t}\, 2t\, dt$$

$$= 2\int_0^\infty t^{3/2} e^{-t}\, dt$$

$$= 2\int_0^\infty t^{\frac{5}{2}-1}\, e^{-t}\, dt$$

$$= 2\Gamma\left(\frac{5}{2}\right)$$

$$= \left(2\times\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)$$

$$= \left(2\times\frac{3}{2}\times\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$$

Gamma functions

1. Evaluate

i) $\int_0^\infty x^6 e^{-2x}dx$

ii) $\int_0^\infty x^{3/2} e^{-4x}dx$

iii) $\int_0^\infty x^2 e^{-x^2}dx$

iv) $\int_0^\infty \sqrt{x} e^{-x^2}dx$

Solution: Put $2x = y$ so that $dx = \frac{1}{2}dy$

$$\int_0^\infty x^6 e^{-2x}dx = \int_0^\infty \left(\frac{y}{2}\right)^6 e^{-y}\frac{1}{2}dy$$

$$= \frac{3}{2}\sqrt{\pi}$$

$$\therefore \quad \int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx = \frac{3}{2}\sqrt{\pi}$$

$$= \frac{1}{2}\int_0^\infty y^6 e^{-y}\ \frac{1}{2}\,dy$$

$$= \frac{1}{2}\int_0^\infty y^{7-1} e^{-y}\ \frac{1}{2}\,dy = \frac{1}{2^7}\,6!$$

Evaluate

i. $\int_0^{\pi/2} \sin^5\theta \cos^{7/2}\theta\ d\theta$

ii. $\int_0^{\pi/2} \sin^7\theta\ d\theta$

iii. $\int_0^{\pi/2} \cos^{11}\theta\ d\theta$

iv. $\int_0^{\pi/2} \sqrt{\cot\theta}\ d\theta$

Solution: i) we have $\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\ d\theta = \frac{1}{2}\beta(m,n)$

Put 2 m-1=5 and 2n-1=1/2 so that m=3, n=9/4

Therefore $\int_0^{\pi/2} \sin^5\theta \cos^{7/2}\theta\ d\theta = \frac{1}{2}\beta(3, 9/4)$

$$\frac{1}{2}\,\frac{\Gamma(3)}{\Gamma\left(3+\frac{9}{4}\right)} = \frac{\Gamma\left(\frac{9}{4}\right)}{\Gamma\left(21/4\right)} = \frac{64}{1989}$$

Solution: We know that,

$$\int_0^{\pi/2} \sin^p\theta \cos^p\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right).\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{\cos^{1/2}\theta}{\sin^{1/2}\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^{-1/2}\theta \ \cos^{1/2}\theta \, d\theta$$

On applying formula (1), we have

$$\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta = \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{2\,\Gamma(1)}$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

Gamma functions

EXERCISE:

1) Evaluate $\int_0^1 (1-x^3)^{-1/2}\, dx$
2) Evaluate $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\, dx$
3) Evaluate $\int_0^1 \left(\frac{x^3}{1-x^3}\right)^{\frac{1}{2}} dx$
4) Prove that $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}$
5) Show that $\beta(p,q) = \beta(p+1,q) + (p,q+1)$

1. Evaluate

1. $\int_0^\infty 3^{-4x^2}\, dx$

2. $\int_0^\infty a^{-bx^2}\, dx$

3. $\int_0^\infty x^4 \left(\log \frac{1}{x}\right)^3 dx$

4. Prove that $\int_0^1 (x \log x)^4\, dx = \frac{4!}{5^5}$

Gamma functions

- $\int_0^\infty x^2 \left(\log \frac{1}{x}\right)^3 dx$

Example 4: **Prove that** $\int_0^1 (x\log x)^4\, dx = \frac{4!}{5^5}$

Solution: We know that

$$\int_0^1 x^n (\log x)^m\, dx = \frac{(-1)^m}{(n+1)^{m+1}}\Gamma(m+1)$$

Now, $\int_0^1 (x\log x)^4\, dx = \int_0^1 x^4 (\log x)^4\, dx$

Putting $n = m = 4$ in (i), we get

$$\int_0^1 x^4 (\log x)^4\, dx = \frac{(-1)^4}{(4+1)^{4+1}}\Gamma(4+1)$$

$$= \frac{\Gamma 5}{5^5}$$

$$= \frac{4!}{5^5}$$

MODULE - V
SPECIAL FUNCTIONS-II

Bessel's equation

$x^2 y''+x y'+(x^2-v^2)y=0$ is called Bessel's equation.

Solution of Bessel's Equation:

Because x=0 is a regular singular point of Bessel's equation we know

that there exists at least one solution of the form $y=\sum_{n=o}^{\infty} c_n x^{n+r}$

.

Substituting the last expression into

Solution of Bessel's Equation:

$$x^2 y''+x y'+(x^2-v^2)y=\sum_{n=o}^{\infty} c_n(n+r)(n+r-1)x^{n+r}+\sum_{n=o}^{\infty} c_n(n+r)x^{n+r}+\sum_{n=o}^{\infty} c_n x^{n+r+2}$$

$$-v^2\sum_{n=o}^{\infty} c_n x^{n+r} = c_0(r^2-r+r-v^2)x^r$$

$$+x^r \sum_{n=1}^{\infty} c_n[(n+r)(n+r-1)+(n+r)-v^2]x^n + x^r\sum_{n=o}^{\infty} c_n x^{n+2}$$

$$=c_0(r^2-v^2)x^r + x^r\sum_{n=1}^{\infty} c_n[(n+r)^2]-v^2]x^n + x^r\sum_{n=o}^{\infty} c_n x^{n+2}$$

From , we see that the indicial equation is $r^2-v^2=0$, so the indicial roots are $r_1=v$ and $r_2 = -v$. When $r_1=v$, becomes

$$x^v\sum_{n=1}^{\infty} c_n n(n+2v)x^n + x^v\sum_{n=o}^{\infty} c_n x^{n+2}$$

$$=x^v\left[(1+2v)c_1x+\sum_{n=2}^{\infty} c_n n(n+2v)x^n+\sum_{n=0}^{\infty} c_n x^{n+2}\right]$$

$$=x^v\left[(1+2v)c_1x+\sum_{k=0}^{\infty}[(k+2)(k+2+2v)c_{k+2}+c_k]x^{k+2}\right]=0$$

Therefore by the usual argument we can write $(1+2v)c_1=0$ and
$(k+2)(k+2+2v)c_{k+2}+c_k=0$

or $C_{K+2} = \frac{-C_k}{(k+2)(k+2+2v)}, k=0,1,2,----$

The choice $c_1=0$ in implies $c_3=c_5=c_7=----=0$, so for $k=0,2,4,----$ we find, after letting $k+2=2n$,

$n = 1,2,3,----$ that

$$C_{2n} = -\frac{c_{2n-2}}{2^2 n(n+v)}$$

Thus $C_2 = -\frac{c_0}{2^2.1(1+v)}$

$$C_4 = -\frac{c_2}{2^2 2(2+v)} = \frac{c_0}{2^4.2.1(1+v)(2+v)}$$

$$C_6 = -\frac{C_4}{2^2.3(3+v)} = -\frac{C_0}{2^6.1.2.3(1+v)(2+v)(3+v)}$$

…………………………

…………………………

$$C_{2n} = \frac{(-1)^n c_0}{2^2 n!(1+v)(2+v).....(n+v)}, n = 1,2,3,----$$

It is standard practice to choose c_0 to be specific value – namely.

$$C_0 = \frac{1}{2^v \Gamma(1+v)}$$

where Γ (1+v) is the gamma function. (See Appendix) Since this latter function possesses the convenient property Γ (1+α) = $\alpha\Gamma(\alpha)$, we can reduce the indicated product in the denominator of to one term.

For example:

Γ (1+v+1)= (1+v) Γ (1+v)

Γ (1+v+2)= (2+v) Γ (2+v)= (2+v)(1+v)Γ(1+v).

Hence we can write as

$$c_{2n} = \frac{(-1)^n}{2^{2n+v} n!(1+v)(2+v)\ldots(n+v)\Gamma(1+v)} = \frac{(-1)^n}{2^{2n+v} n!\Gamma(1+v+n)}$$

for n=0,1,2, - - - -

Bessel Function of the First Kind:

Using the coefficients c_{2n} just obtained and r=v, a series solution of (6.10) is y= $\sum_{n=0}^{\infty} c_{2n} x^{2n+v}$ This solution is usually denoted by $J_v(x)$:

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+v+n)} \left(\frac{x}{2}\right)^{2n+v}.$$

If v≥0, the series converges at least on the interval [o,∞). Also, for the second exponent r_2= -v we obtain, in exactly the same manner,

$$J_{-v}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1-v+n)} \left(\frac{x}{2}\right)^{2n-v}.$$

The functions $J_v(x)$ and $J_{-v}(x)$ are called Bessel functions of the first kind of order v and –v, respectively. Depending on the value of v, (6.16) may contain negative powers of x and hence converge on (0, ∞).*

Bessel Function of the First Kind:

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$$J_v(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+v+n)}\left(\frac{x}{2}\right)^{2n+v}.$$

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The functions $J_v(x)$ and $J_{-v}(x)$ are called Bessel functions of the first kind of order v and –v, respectively. Depending on the value of v, (6.16) may contain negative powers of x and hence converge on (0, ∞).*

SPECIAL FUNCTIONS

Introduction

Many Differential equations arising from physical problems are linear but have variable coefficients and do not permit a general analytical solution in terms of known functions. Such equations can be solved by numerical methods (Unit – I), but in many cases it is easier to find a solution in the form of an infinite convergent series. The series solution of certain differential equations give rise to special functions such as Bessel’s function, Legendre’s polynomial. These special functions have many applications in engineering.

Series solution of the Bessel Differential Equation

Consider the Bessel Differential equation of order n in the form

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0 \qquad \text{(i)}$$

where n is a non negative real constant or parameter.

We assume the series solution of (i) in the form

$$y=\sum_{r=0}^{\infty}a_r x^{k+r} \quad \text{where } a_0 \neq 0 \qquad \text{(ii)}$$

Hence, $\dfrac{dy}{dx}=\sum_{r=0}^{\infty}a_r(k+r)x^{k+r-1}$

$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(k+r)(k+r-1)x^{k+r-2}$$

Substituting these in (i) we get,

$$x^2\sum_{r=0}^{\infty}a_r(k+r)(k+r-1)x^{k+r-2}+x\sum_{r=0}^{\infty}a_r(k+r)x^{k+r-1}+\left(x^2-n^2\right)\sum_{r=0}^{\infty}a_r x^{k+r} =$$

i.e.,

$$\sum_{r=0}^{\infty}a_r(k+r)(k+r-1)x^{k+r}+\sum_{r=0}^{\infty}a_r(k+r)x^{k+r}+\sum_{r=0}^{\infty}a_r x^{k+r+2}-n^2\sum_{r=0}^{\infty}a_r x^{k+r}$$

Grouping the like powers, we get

$$\sum_{r=0}^{\infty} a_r \left[(k+r)(k+r-1)+(k+r)-n^2\right] x^{k+r} + \sum_{r=0}^{\infty} a_r x^{k+r+2} = 0$$

$$\sum_{r=0}^{\infty} a_r \left[(k+r)^2 - n^2\right] x^{k+r} + \sum_{r=0}^{\infty} a_r x^{k+r+2} = 0 \qquad \text{(iii)}$$

Now we shall equate the coefficient of various powers of x to 0

Equating the coefficient of x^k from the first term and equating it to zero, we get

$a_0\left[k^2-n^2\right]=0.$ Since $a_0 \neq 0$, we get $k^2-n^2=0, \quad \therefore k=\pm n$

Coefficient of x^{k+1} is got by putting r = 1 in the first term and equating it to zero, we get

$$a_1\left[(k+1)^2-n^2\right]=0.$$

This gives $a_1=0,$

i.e., since $(k+1)^2-n^2=0$ gives, $k+1=\pm n$

which is a contradiction to $k=\pm n$.

Let us consider the coefficient of x^{k+r} from (iii) and equate it to zero.

i.e, $a_r\left[(k+r)^2-n^2\right]+a_{r-2}=0.$

$$\therefore a_r = \frac{-a_{r-2}}{\left[(k+r)^2 - n^2\right]} \qquad \text{(iv)}$$

If *k = +n*, (iv) becomes

$$a_r = \frac{-a_{r-2}}{\left[(n+r)^2 - n^2\right]} = \frac{-a_{r-2}}{\left[r^2 + 2nr\right]}$$

Now putting r = 1,3,5, ….., (odd vales of n) we obtain,

$$a_3 = \frac{-a_1}{6n+9} = 0, \quad \because a_1 = 0$$

Similarly a_5, a_7, ….. are equal to zero.

i.e., $a_1 = a_5 = a_7 = \ldots\ldots = 0$

Now, putting r = 2,4,6, ……(even values of n) we get,

$$a_2 = \frac{-a_0}{4n+4} = \frac{-a_0}{4(n+1)}; \qquad a_4 = \frac{-a_2}{8n+16} = \frac{a_0}{32(n+1)(n+2)};$$

Similarly we can obtain a_6, a_8, …

We shall substitute the values of $a_1, a_2, a_3, a_4, \ldots\ldots$ in the assumed series solution, we get

$$y=\sum_{r=0}^{\infty}a_r x^{k+r}=x^k(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots\cdots)$$

Let y_1 be the solution for $k = +n$

$$\therefore \quad y_1=x^n\left[a_0-\frac{a_0}{4(n+1)}x^2+\frac{a_0}{32(n+1)(n+2)}x^4-\cdots\cdots\right]$$

$$i.e., \quad y_1=a_0x^n\left[1-\frac{x^2}{2^2(n+1)}+\frac{x^4}{2^5(n+1)(n+2)}-\cdots\cdots\right] \qquad \text{(v)}$$

This is a solution of the Bessel's equation.

Let y_2 be the solution corresponding to k = - n. Replacing n be – n in (v) we get

$$y_2=a_0x^{-n}\left[1-\frac{x^2}{2^2(-n+1)}+\frac{x^4}{2^5(-n+1)(-n+2)}-\cdots\cdots\right] \qquad \text{(vi)}$$

The complete or general solution of the Bessel's differential equation is $y = c_1y_1 + c_2y_2$, where c_1, c_2 are arbitrary constants.

Now we will proceed to find the solution in terms of Bessel's function by choosing $a_0=\frac{1}{2^n\overline{)(n+1)}}$ and let us denote it as Y_1.

$$i.e., \quad Y_1 = \frac{x^n}{2^n \Gamma(n+1)}\left[1 - \left(\frac{x}{2}\right)^2 \frac{1}{(n+1)} + \left(\frac{x}{2}\right)^4 \frac{1}{(n+1)(n+2)\cdot 2} - \cdots\cdots\right]$$

$$= \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2 \frac{1}{(n+1)\Gamma(n+1)} + \left(\frac{x}{2}\right)^4 \frac{1}{(n+1)(n+2)\Gamma(n+1)\cdot 2} - \cdots\cdots\right]$$

We have the result $\Gamma(n) = (n - 1)\, \Gamma(n - 1)$ from Gamma function

Hence, $\Gamma(n + 2) = (n + 1)\, \Gamma(n + 1)$ and

$\Gamma(n + 3) = (n + 2)\, \Gamma(n + 2) = (n + 2)\,(n + 1)\, \Gamma(n + 1)$

Using the above results in Y_1, we get

$$Y_1 = \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma(n+2)} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma(n+3)\cdot 2} - \cdots\cdots\right]$$

which can be further put in the following form

$$Y_1 = \left(\frac{x}{2}\right)^n \left[\frac{(-1)^0}{\Gamma(n+1)\cdot 0!}\left(\frac{x}{2}\right)^0 + \frac{(-1)^1}{\Gamma(n+2)\cdot 1!}\left(\frac{x}{2}\right)^2 + \frac{(-1)^2}{\Gamma(n+3)\cdot 2!}\left(\frac{x}{2}\right)^4 + \cdots\cdots\right]$$

SPECIAL FUNCTIONS-II

$$= \left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(n+r+1)\cdot r!}\left(\frac{x}{2}\right)^{2r}$$

$$= \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1)\cdot r!}$$

This function is called the Bessel function of the first kind of order n and is denoted by $J_n(x)$.

Thus $$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1)\cdot r!}$$

Further the particular solution for $k = -n$ (replacing n by –n) be denoted as $J_{-n}(x)$. Hence the general solution of the Bessel's equation is given by $y = AJ_n(x) + BJ_{-n}(x)$, where A and B are arbitrary constants

Properties of Bessel's function

1. $J_{-n}(x) = (-1)^n J_n(x)$, where n is a positive integer.

Proof: By definition of Bessel's function, we have

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1) \cdot r!} \quad \text{..........(1)}$$

Hence, $$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{-n+2r} \cdot \frac{1}{\Gamma(-n+r+1) \cdot r!} \quad \text{..........(2)}$$

But gamma function is defined only for a positive real number. Thus we write (2) in the following from

But gamma function is defined only for a positive real number. Thus we write (2) in the following from

$$J_{-n}(x)=\sum_{r=n}^{\infty}(-1)^r\cdot\left(\frac{x}{2}\right)^{-n+2r}\cdot\frac{1}{\overline{)(-n+r+1)}\cdot r!}\qquad ....(3)$$

Let r – n = s or r = s + n. Then (3) becomes

$$J_{-n}(x)=\sum_{s=0}^{\infty}(-1)^{s+n}\cdot\left(\frac{x}{2}\right)^{-n+2s+2n}\cdot\frac{1}{\overline{)(s+1)}\cdot(s+n)!}$$

We know that Γ(s+1) = s! and (s + n)! = Γ(s+n+1)

$$=\sum_{s=0}^{\infty}(-1)^{s+n}\cdot\left(\frac{x}{2}\right)^{n+2s}\cdot\frac{1}{\overline{)(s+n+1)}\cdot s!}$$

$$=(-1)^n\sum_{s=0}^{\infty}(-1)^{s}\cdot\left(\frac{x}{2}\right)^{n+2s}\cdot\frac{1}{\overline{)(s+n+1)}\cdot s!}$$

Comparing the above summation with (1), we note that the RHS is $J_n(x)$.

Thus, $\boldsymbol{J_{-n}(x)=(-1)^n J_n(x)}$

2)

$$J_n(-x) = (-1)^n J_n(x) = J_{-n}(x)$$ *, where n is a positive integer*

Proof : By definition,

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1) \cdot r!}$$

$$\therefore \quad J_n(-x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(-\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1) \cdot r!}$$

i.e., $$= \sum_{r=0}^{\infty} (-1)^r \cdot (-1)^{n+2r} \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1) \cdot r!}$$

$$= (-1)^n \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1) \cdot r!}$$

Thus, $$\boldsymbol{J_n(-x) = (-1)^n J_n(x)}$$

Since, $(-1)^n J_n(x) = J_{-n}(x)$, we have $\boldsymbol{J_n(-x) = (-1)^n J_n(x) = J_{-n}(x)}$

Recurrence Relations:

Recurrence Relations are relations between Bessel's functions of different order.

Recurrence Relations 1: $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

From definition,

$$x^n J_n(x) = x^n \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1) \cdot r!} = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{2(n+r)} \cdot \frac{1}{\Gamma(n+r+1)}$$

$$\therefore \quad \frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{r=0}^{\infty} (-1)^r \cdot \frac{2(n+r)x^{2(n+r)-1}}{2^{n+2r}\,\Gamma(n+r+1) \cdot r!}$$

$$= x^{n} \sum_{r=0}^{\infty} (-1)^{r} \cdot \frac{(n+r)x^{n+2r-1}}{2^{n+2r-1}(n+r)\sqrt{(n+r)} \cdot r!}$$

$$= x^{n} \sum_{r=0}^{\infty} (-1)^{r} \cdot \frac{(x/2)^{(n-1)+2r}}{\sqrt{(n-1+r+1)} \cdot r!} = x^{n} J_{n-1}(x)$$

Thus, $$\frac{d}{dx}\left[x^{n} J_{n}(x)\right] = x^{n} J_{n-1}(x)$$ --------(1)

SPECIAL FUNCTIONS-II

Recurrence Relations 2: $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

From definition,

$$x^{-n} J_n(x) = x^{-n} \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1)\cdot r!}$$

$$= \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{2r} \cdot \frac{1}{\Gamma(n+r+1)\cdot r!}$$

$$\therefore \quad \frac{d}{dx}\left[x^{-n} J_n(x)\right] = \sum_{r=0}^{\infty} (-1)^r \cdot \frac{2r\, x^{2r-1}}{2^{n+2r}\,\Gamma(n+r+1)\cdot r!}$$

$$= -x^{-n} \sum_{r=1}^{\infty} (-1)^{r-1} \cdot \frac{x^{n+1+2(r-1)}}{2^{n+1+2(r-1)}\,\Gamma(n+r+1)\cdot (r-1)!}$$

Let $k = r - 1$

$$= -x^{-n} \sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^{n+1+2k}}{2^{n+1+2k}\,\Gamma(n+1+k+1)\cdot k!} = -x^{-n} J_{n+1}(x)$$

Thus, $$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$ ....(2))

Recurrence Relations 3:

$$J_n(x) = \frac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$$

We know that $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

Applying product rule on LHS, we get

$$x^n J_n^{/}(x) + nx^{n-1} J_n(x) = x^n J_{n-1}(x)$$

Dividing by x^n we get $J_n^{/}(x) + (n/x)J_n(x) = J_{n-1}(x)$ --------(3)

Also differentiating LHS of $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$, we get

$$x^{-n} J_n^{/}(x) - nx^{-n-1} J_n(x) = -x^{-n} J_{n+1}(x)$$

Dividing by $-x^{-n}$ we get $-J_n^{/}(x) + (n/x)J_n(x) = J_{n+1}(x)$ --------(4)

Adding (3) and (4), we obtain $2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$

i.e., $J_n(x) = \frac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$

Recurrence Relations 4:

$$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$$

Subtracting (4) from (3), we obtain

$$2J_n'(x) = [J_{n-1}(x) - J_{n+1}(x)]$$

i.e., $J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

Recurrence Relations 5:

$$J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$$

This recurrence relation is another way of writing the Recurrence relation 2.

Recurrence Relations 7: $J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$

This recurrence relation is another way of writing the Recurrence relation 3.

Prove that $(a) \quad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \quad (b) \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

By definition,

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\overline{)(n+r+1)} \cdot r!}$$

Putting n = ½, we get

$$J_{1/2}(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{1/2+2r} \cdot \frac{1}{\overline{)(r+3/2)} \cdot r!}$$

$$J_{1/2}(x) = \sqrt{\frac{x}{2}} \left[\frac{1}{\Gamma(3/2)} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma(5/2)1!} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma(7/2)2!} - \cdots\cdots \right] \qquad \text{--------(1)}$$

Using the results $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(n) = (n - 1)\, \Gamma(n-1)$, we get

$\Gamma(3/2) = \frac{\sqrt{\pi}}{2}, \Gamma(5/2) = \frac{3\sqrt{\pi}}{4}, \Gamma(7/2) = \frac{15\sqrt{\pi}}{8}$ and so on.

Using these values in (1), we get

$$\Gamma(3/2)=\frac{\sqrt{\pi}}{2},\Gamma(5/2)=\frac{3\sqrt{\pi}}{4},\Gamma(7/2)=\frac{15\sqrt{\pi}}{8}$$ and so on.

Using these values in (1), we get

$$J_{1/2}(x)=\sqrt{\frac{x}{2}}\left[\frac{2}{\sqrt{\pi}}-\frac{x^2}{4}\frac{4}{3\sqrt{\pi}}+\frac{x^4}{16}\frac{8}{15\sqrt{\pi}.2}-\cdots\cdots\right]$$

$$=\sqrt{\frac{x}{2\pi}}\cdot\frac{2}{x}\left[x-\frac{x^3}{6}+\frac{x^5}{120}-\cdots\cdots\right]=\sqrt{\frac{2}{x\pi}}\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\cdots\right]$$

$$J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x$$

Putting n = - 1/2, we get

$$J_{-1/2}(x)=\sum_{r=0}^{\infty}(-1)^r\cdot\left(\frac{x}{2}\right)^{-1/2+2r}\cdot\frac{1}{\Gamma(r+1/2)\cdot r!}$$

$$J_{-1/2}(x)=\sqrt{\frac{x}{2}}\left[\frac{1}{\Gamma(1/2)}-\left(\frac{x}{2}\right)^2\frac{1}{\Gamma(3/2)1!}+\left(\frac{x}{2}\right)^4\frac{1}{\Gamma(5/2)2!}-\cdots\cdots\right]$$ --------(2)

Using the results Γ(1/2) = √π and Γ(n) = (n – 1) Γ(n–1) in (2), we get

$$J_{-1/2}(x)=\sqrt{\frac{2}{x}}\left[\frac{1}{\sqrt{\pi}}-\frac{x^2}{4}\frac{2}{\sqrt{\pi}}+\frac{x^4}{16}\frac{4}{3\sqrt{\pi}.2}-\cdots\cdots\right]$$

$$=\sqrt{\frac{2}{x\pi}}\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\cdots\right]$$

$$J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x$$

3. Show that $\frac{d}{dx}\left[J_n^2(x)+J_{n+1}^2(x)\right]=\frac{2}{x}\left[nJ_n^2(x)-(n+1)J_{n+1}^2(x)\right]$

Solution:

L.H.S = $\frac{d}{dx}\left[J_n^2(x)+J_{n+1}^2(x)\right]=2J_n(x)J_n^{/}(x)+2J_{n+1}(x)J_{n+1}^{/}(x)$ ------- (1)

We know the recurrence relations

$$xJ_n^{/}(x)=nJ_n(x)-xJ_{n+1}(x) \qquad \text{------- (2)}$$

$$xJ_{n+1}^{/}(x)=xJ_n(x)-(n+1)J_{n+1}(x) \qquad \text{------- (3)}$$

Relation (3) is obtained by replacing ***n*** by ***n+1*** in $xJ_n^{/}(x)=xJ_{n-1}(x)-nJ_n(x)$

Now using (2) and (3) in (1), we get

L.H.S = $\frac{d}{dx}\left[J_n^2(x)+J_{n+1}^2(x)\right]=2J_n(x)\left[\frac{n}{x}J_n(x)-J_{n+1}(x)\right]+2J_{n+1}(x)\left[J_n(x)-\frac{n+1}{x}J_{n+1}(x)\right]$

$$=\frac{2n}{x}J_n^2(x)-2J_n(x)J_{n+1}(x)+2J_{n+1}(x)J_n(x)-2\frac{n+1}{x}J_{n+1}^2(x)$$

Hence, $\frac{d}{dx}\left[J_n^2(x)+J_{n+1}^2(x)\right]=\frac{2}{x}\left[nJ_n^2(x)-(n+1)J_{n+1}^2(x)\right]$

4. Prove that $J_0^{//}(x) = \frac{1}{2}[J_2(x) - J_0(x)]$

Solution :

We have the recurrence relation $J_n^{/}(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$ -------(1)

Putting n = 0 in (1), we get $J_0^{/}(x) = \frac{1}{2}[J_{-1}(x) - J_1(x)] = \frac{1}{2}[-J_1(x) - J_1(x)] = -J_1(x)$

Thus, $J_0^{/}(x) = -J_1(x)$. Differentiating this w.r.t. x we get, $J_0^{//}(x) = -J_1^{/}(x)$ ----- (2)

Now, from (1), for n = 1, we get $J_1^{/}(x) = \frac{1}{2}[J_0(x) - J_2(x)]$.

Using (2), the above equation becomes

$-J_0^{//}(x) = \frac{1}{2}[J_0(x) - J_2(x)]$ or $J_0^{//}(x) = \frac{1}{2}[J_2(x) - J_0(x)]$.

Thus we have proved that, $J_0^{//}(x) = \frac{1}{2}[J_2(x) - J_0(x)]$

5. Show that (a) $\int J_3(x)dx = c - J_2(x) - \frac{2}{x} J_1(x)$

(b) $\int xJ_0^2(x)dx = \frac{1}{2} x^2 \left[J_0^2(x) + J_1^2(x)\right]$

Solution :

(a) We know that $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$ or $\int x^{-n} J_{n+1}(x)dx = -x^{-n} J_n(x)$ ------ (1)

Now, $\int J_3(x)dx = \int x^2 \cdot x^{-2} J_3(x)dx + c = x^2 \cdot \int x^{-2} J_3(x)dx - \int 2x\left[\int x^{-2} J_3(x)dx\right]dx + c$

$= x^2 \cdot \left[-x^{-2} J_2(x)\right] - \int 2x\left[-x^{-2} J_2(x)\right]dx + c$ (from (1) when n = 2)

$= c - J_2(x) - \int \frac{2}{x} J_2(x)dx = c - J_2(x) - \frac{2}{x} J_1(x)$ (from (1) when n = 1)

Hence, $\int J_3(x)dx = c - J_2(x) - \frac{2}{x} J_1(x)$

(b) $\int xJ_0^2(x)dx = J_0^2(x)\cdot\frac{1}{2}x^2 - \int 2J_0(x)\cdot J_0'(x).\frac{1}{2}x^2 dx$ (Integrate by parts)

$= \frac{1}{2}x^2 J_0^2(x) + \int x^2 J_0(x)\cdot J_1(x)dx$ (From (1) for n = 0)

$= \frac{1}{2}x^2 J_0^2(x) + \int xJ_1(x)\cdot\frac{d}{dx}\left[xJ_1(x)\right]dx$ $\left[\because \frac{d}{dx}\left[xJ_1(x)\right] = xJ_0(x) \text{ from recurrence relation (1)}\right]$

$= \frac{1}{2}x^2 J_0^2(x) + \frac{1}{2}\left[xJ_1(x)\right]^2 = \frac{1}{2}x^2\left[J_0^2(x) + J_1^2(x)\right]$

Generating Function for $J_n(x)$

To prove that $$e^{\frac{x}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

or

If n is an integer then $J_n(x)$ is the coefficient of t^n in the expansion of

$$e^{\frac{x}{2}(t-1/t)}.$$

Proof:

We have $$e^{\frac{x}{2}(t-1/t)} = e^{xt/2} \times e^{-x/2t}$$

$$= \left[1 + \frac{(xt/2)}{1!} + \frac{(xt/2)^2}{2!} + \frac{(xt/2)^3}{3!} + \cdots\cdots\right] \bullet \left[1 + \frac{(-xt/2)}{1!} + \frac{(-xt/2)^2}{2!} + \frac{(-xt/2)^3}{3!} + \cdots\cdots\right]$$

(using the expansion of exponential function)

$$= \left[1 + \frac{xt}{2\cdot 1!} + \frac{x^2t^2}{2^2 2!} + \cdots + \frac{x^n t^n}{2^n n!} + \frac{x^{n+1}t^{n+1}}{2^{n+1}(n+1)!} + \cdots\cdots\right] \bullet \left[1 - \frac{x}{2t\cdot 1!} + \frac{x^2}{2^2 t^2 2!} - \cdots + \frac{(-1)^n x^n}{2^n t^n n!} + \frac{(-1)^{n+1} x^{n+1}}{2^{n+1} t^{n+1}(n+1)!} + \cdots\cdots\right]$$

If we collect the coefficient of t^n in the product, they are

$$=\frac{x^n}{2^n n!}-\frac{x^{n+2}}{2^{n+2}(n+1)!1!}+\frac{x^{n+4}}{2^{n+4}(n+2)!2!}-\cdots\cdots$$

$$=\frac{1}{n!}\left(\frac{x}{2}\right)^n-\frac{1}{(n+1)!1!}\left(\frac{x}{2}\right)^{n+2}+\frac{1}{(n+2)!2!}\left(\frac{x}{2}\right)^{n+4}-\cdots\cdots=\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{\Gamma(n+r+1)r!}=J_n(x)$$

Similarly, if we collect the coefficients of t^{-n} in the product, we get $J_{-n}(x)$.

Thus, $$e^{\frac{x}{2}(t-1/t)}=\sum_{n=-\infty}^{\infty}t^n J_n(x)$$

Result: $$e^{\frac{x}{2}(t-1/t)}=J_0(x)+\sum_{n=1}^{\infty}\left[t^n+(-1)^n t^{-n}\right]J_n(x)$$

Proof :

$$e^{\frac{x}{2}(t-1/t)}=\sum_{n=-\infty}^{\infty}t^n J_n(x)=\sum_{n=-\infty}^{-1}t^n J_n(x)+\sum_{n=0}^{\infty}t^n J_n(x)$$

$$=\sum_{n=1}^{\infty}t^{-n}J_{-n}(x)+J_0(x)+\sum_{n=1}^{\infty}t^n J_n(x)=J_0(x)+\sum_{n=1}^{\infty}t^{-n}(-1)^n J_n(x)+\sum_{n=1}^{\infty}t^n J_n(x)\quad \{\because J_{-n}(x)=(-1)^n J_n(x)\}$$

Thus,

$$e^{\frac{x}{2}(t-1/t)}=J_0(x)+\sum_{n=1}^{\infty}\left[t^n+(-1)^n t^{-n}\right]J_n(x)$$

Problem 6: Show that

(a) $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)d\theta$, *n being an integer*

(b) $J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\theta)d\theta$

(c) $J_0^2 + 2J_1^2 + 2J_2^2 + J_3^2 + \cdots\cdots = 1$

Solution :

We know that $e^{\frac{x}{2}(t-1/t)} = J_0(x) + \sum_{n=1}^{\infty}\left[t^n + (-1)^n t^{-n}\right]J_n(x)$

$= J_0(x) + tJ_1(x) + t^2 J_2(x) + t^3 J_3(x) + \cdots\cdots + t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + t^{-3}J_{-3}(x) + \cdots\cdots$

Since $J_{-n}(x) = (-1)^n J_n(x)$, we have

$e^{\frac{x}{2}(t-1/t)} = J_0(x) + J_1(x)(t - 1/t) + J_2(x)(t^2 + 1/t^2) + J_3(x)(t^3 - 1/t^3) + \cdots\cdots$ ----- (1)

Let t = cosθ + i sinθ so that t^p = cospθ + i sinpθ and $1/t^p$ = cospθ - i sinpθ.

From this we get, $t^p + 1/t^p$ = 2cospθ and $t^p – 1/t^p$ = 2i sinpθ

Using these results in (1), we get

$$e^{\frac{x}{2}(2i\sin\theta)} = e^{ix\sin\theta} = J_0(x)+2\left[J_2(x)\cos 2\theta+J_4(x)\cos 4\theta+\cdots\right]+2i\left[J_1(x)\sin\theta+J_3(x)\sin 3\theta+\cdots\right]$$
-----(2)

Since $e^{ixsin\theta}$ = $cos(xsin\theta) + i\ sin(xsin\theta)$, equating real and imaginary parts in (2) we get,

$$\cos(x\sin\theta)=J_0(x)+2\left[J_2(x)\cos 2\theta+J_4(x)\cos 4\theta+\cdots\right]$$
----- (3)

$$\sin(x\sin\theta)=2\left[J_1(x)\sin\theta+J_3(x)\sin 3\theta+\cdots\right]$$
----- (4)

These series are known as ***Jacobi Series***.

Now multiplying both sides of (3) by $cos\ n\theta$ and both sides of (4) by $sin\ n\theta$ and integrating each of the resulting expression between 0 and π, we obtain

$$\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\cos n\theta\, d\theta = \begin{cases} J_n(x), & \text{n is even or zero} \\ 0, & \text{n is odd}\end{cases}$$

and $$\frac{1}{\pi}\int_0^{\pi}\sin(x\sin\theta)\sin n\theta\, d\theta = \begin{cases} 0, & \text{n is even} \\ J_n(x), & \text{n is odd}\end{cases}$$

Here we used the standard result $$\int_0^{\pi}\cos p\theta\cos q\theta\, d\theta = \int_0^{\pi}\sin p\theta\sin q\theta\, d\theta = \begin{cases} \frac{\pi}{2}, & \text{if } p=q \\ 0, & \text{if } p\neq q\end{cases}$$

From the above two expression, in general, if n is a positive integer, we get

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi}[\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta]d\theta = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - x\sin\theta)d\theta$$

(b) Changing θ to (π/2) θ in (3), we get

$$\cos(x\cos\theta) = J_0(x) + 2[J_2(x)\cos(\pi - 2\theta) + J_4(x)\cos(\pi - 4\theta) + \cdots]$$

$$\cos(x\cos\theta) = J_0(x) - 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta - \cdots$$

Integrating the above equation w.r.t θ from 0 to π, we get

$$\int_0^{\pi}\cos(x\cos\theta)d\theta = \int_0^{\pi}[J_0(x) - 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta - \cdots]$$

$$\int_0^{\pi}\cos(x\cos\theta)d\theta = \left|J_0(x)\cdot\theta - 2J_2(x)\frac{\sin 2\theta}{2} + 2J_4(x)\frac{\sin 4\theta}{4} - \cdots\right|_0^{\pi} = J_0(x)\cdot\pi$$

Thus, $J_0(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\cos\theta)d\theta$

(c) Squaring (3) and (4) and integrating w.r.t. θ *from 0 to* π and noting that *m* and *n* being integers

$$\int_0^\pi \cos^2(x\sin\theta)d\theta = [J_0(x)]^2\cdot\pi + 4[J_2(x)]^2\frac{\pi}{2} + 4[J_4(x)]^2\frac{\pi}{2} + \cdots$$

$$\int_0^\pi \sin^2(x\sin\theta)d\theta = 4[J_1(x)]^2\frac{\pi}{2} + 4[J_3(x)]^2\frac{\pi}{2} + \cdots$$

Adding, $\int_0^\pi d\theta = \pi = \pi\left[J_0^2(x) + 2J_1^2(x) + 2J_2^2(x) + J_3^2(x) + \cdots\cdots\right]$

Hence, $J_0^2 + 2J_1^2 + 2J_2^2 + J_3^2 + \cdots\cdots = 1$

Orthogonality of Bessel Functions

If α *and* β *are the two distinct roots of* $\mathbf{J_n(x) = 0}$, *then*

$$\int_0^{\pi} xJ_n(\alpha x)J_n(\beta x)dx = \begin{cases} 0, & \text{if } \alpha \neq \beta \\ \frac{1}{2}\left[J_n^{/}(\alpha)\right]^2 = \frac{1}{2}\left[J_{n+1}(\alpha)\right]^2, & \text{if } \alpha = \beta \end{cases}$$

Proof:

We know that the solution of the equation

$$x^2u^{//} + xu^{/} + (\alpha^2x^2 - n^2)u = 0 \quad \text{-------- (1)}$$

$$x^2v^{//} + xv^{/} + (\beta^2x^2 - n^2)v = 0 \quad \text{-------- (2)}$$

are $u = J_n(\alpha x)$ *and* $v = J_n(\beta x)$ respectively.

Multiplying (1) by $\boldsymbol{v/x}$ and (2) by $\boldsymbol{u/x}$ and subtracting, we get

$x(u'' v - u v'') + (u' v - uv') + (\beta^2 - \alpha^2) xuv = 0$

or $\frac{d}{dx}\left\{x\left(u'v - uv'\right)\right\} = \left(\beta^2 - \alpha^2\right)xuv$

Now integrating both sides from 0 to 1, we get

$$\left(\beta^2 - \alpha^2\right)\int_0^1 xuvdx = \left[x\left(u'v - uv'\right)\right]_0^1 = \left(u'v - uv'\right)_{x=1} \quad \text{------- (3)}$$

Since $u = J_n(\alpha x)$, $u' = \frac{d}{dx}\left[J_n(\alpha x)\right] = \frac{d}{d(\alpha x)}\left[J_n(\alpha x)\right] \cdot \frac{d(\alpha x)}{dx} = \alpha J_n'(\alpha x)$

Similarly $v = J_n(\beta x)$ gives $v' = \frac{d}{dx}\left[J_n(\beta x)\right] = \beta J_n'(\beta x)$. Substituting these values in (3), we get

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \frac{\alpha J_n'(\alpha)J_n(\beta) - \beta J_n(\alpha)J_n'(\beta)}{\beta^2 - \alpha^2} \quad \text{------- (4)}$$

If α and β are the two distinct roots of **$J_n(x) = 0$**, then $J_n(\alpha) = 0$ and $J_n(\beta) = 0$, and hence (4) reduces to $\int_0^{\pi} xJ_n(\alpha x)J_n(\beta x)dx = 0$.

This is known as Orthogonality relation of Bessel functions.

When $\beta = \alpha$, the RHS of (4) takes 0/0 form. Its value can be found by considering α as a root of $J_n(x) = 0$ and β as a variable approaching to α. Then (4) gives

$$\underset{\beta\to\alpha}{Lt}\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \underset{\beta\to\alpha}{Lt}\frac{\alpha J_n^{/}(\alpha)J_n(\beta)}{\beta^2-\alpha^2}$$

Applying L'Hospital rule, we get

$$\underset{\beta\to\alpha}{Lt}\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \underset{\beta\to\alpha}{Lt}\frac{\alpha J_n^{/}(\alpha)J_n^{/}(\beta)}{2\beta} = \frac{1}{2}\left\{J_n^{/}(\alpha)\right\}^2 \quad \text{--------(5)}$$

We have the recurrence relation $J_n^{/}(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$.

$$\therefore \quad J_n^{/}(\alpha) = \frac{n}{\alpha}J_n(\alpha) - J_{n+1}(\alpha). \text{Since} \quad J_n(\alpha) = 0, \text{we have } J_n^{/}(\alpha) = -J_{n+1}(\alpha)$$

Thus, (5) becomes $\underset{\beta\to\alpha}{Lt}\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \frac{1}{2}\left\{J_n^{/}(\alpha)\right\}^2 = \frac{1}{2}\left\{J_{n+1}(\alpha)\right\}^2$


Thank you