LECTURE NOTES
ON
ENGINEERING MECHANICS

B. Tech III Semester (R-18)

Prepared By

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MECHANICAL ENGINEERING

INSTITUTE OF AERONAUTICAL ENGINEERING
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ENGINEERING MECHANICS

III Semester: ME

<table>
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<tr>
<th>Course Code</th>
<th>Category</th>
<th>Hours / Week</th>
<th>Credits</th>
<th>Maximum Marks</th>
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<td>AMEB03</td>
<td>Core</td>
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Contact Classes: 45  Tutorial Classes: Nil  Practical Classes: Nil  Total Classes: 45

COURSE OBJECTIVES:
The course should enable the students to:
I. Students should develop the ability to work comfortably with basic engineering mechanics concepts required for analyzing static structures.
II. Identify an appropriate structural system to studying a given problem and isolate it from its environment, model the problem using good free-body diagrams and accurate equilibrium equations.
III. Understand the meaning of centre of gravity (mass)/centroid and moment of Inertia using integration methods and method of moments
IV. To solve the problem of equilibrium by using the principle of work and energy, impulse momentum and vibrations for preparing the students for higher level courses such as Mechanics of Solids, Mechanics of Fluids, Mechanical Design and Structural Analysis etc...

MODULE-I  INTRODUCTION TO ENGINEERING MECHANICS  Classes: 10
Force Systems Basic concepts, Particle equilibrium in 2-D & 3-D; Rigid Body equilibrium; System of Forces, Coplanar Concurrent Forces, Components in Space – Resultant- Moment of Forces and its Application; Couples and Resultant of Force System, Equilibrium of System of Forces, Free body diagrams, Equations of Equilibrium of Coplanar Systems and Spatial Systems; Static Indeterminacy

MODULE-II  FRICTION AND BASICS STRUCTURAL ANALYSIS  Classes: 09
Types of friction, Limiting friction, Laws of Friction, Static and Dynamic Friction; Motion of Bodies, wedge friction, screw jack & differential screw jack; Equilibrium in three dimensions; Method of Sections; Method of Joints; How to determine if a member is in tension or compression; Simple Trusses; Zero force members; Beams &types of beams; Frames &Machines;

MODULE-III  CENTROID AND CENTRE OF GRAVITY AND VIRTUAL WORK AND ENERGY METHOD  Classes: 10
Centroid of simple figures from first principle, centroid of composite sections; Centre of Gravity and its implications; Area moment of inertia- Definition, Moment of inertia of plane sections from first principles, Theorems of moment of inertia, Moment of inertia of standard sections and composite sections; Mass moment inertia of circular plate, Cylinder, Cone, Sphere, Hook.
Virtual displacements, principle of virtual work for particle and ideal system of rigid bodies, degrees of freedom. Active force diagram, systems with friction, mechanical efficiency. Conservative forces and potential energy (elastic and gravitational), energy equation for equilibrium. Applications of energy method for equilibrium. Stability of equilibrium.

MODULE-IV  PARTICLE DYNAMICS AND INTRODUCTION TO KINETICS  Classes: 08
Particle dynamics- Rectilinear motion; Plane curvilinear motion (rectangular, path, and polar coordinates). 3-D curvilinear motion; Relative and constrained motion; Newton’s 2nd law (rectangular, path, and polar coordinates). Work-kinetic energy, power, potential energy. Impulse-momentum (linear, angular); Impact (Direct and oblique). Introduction to Kinetics of Rigid Bodies covering, Basic terms, general principles in dynamics; Types of motion, Instantaneous centre of rotation in plane motion and simple problems.
<table>
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<th>MODULE-V</th>
<th>MECHANICAL VIBRATIONS</th>
<th>Classes: 08</th>
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<td>Basic terminology, free and forced vibrations, resonance and its effects; Degree of freedom; Derivation for frequency and amplitude of free vibrations without damping and single degree of freedom system, simple problems, types of pendulum, use of simple, compound and torsion pendulums.</td>
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**Text Books:**


**Reference Books:**


**Web References:**

1. https://books.google.co.in/books/about/engineering_mechanics_Reference_Guide.html?id=6x1smAf_PAeC

**E-Text Books:**

1. https://books.google.co.in/books?id=6wFuw6wufTMC&printsec=frontcover#v=onepage&q&f=false
MODULE I
INTRODUCTION TO ENGINEERING MECHANICS

Mechanics

It is defined as that branch of science, which describes and predicts the conditions of rest or motion of bodies under the action of forces. Engineering mechanics applies the principle of mechanics to design, taking into account the effects of forces.

Statics

Statics deal with the condition of equilibrium of bodies acted upon by forces.

Rigid body

A rigid body is defined as a definite quantity of matter, the parts of which are fixed in position relative to each other. Physical bodies are never absolutely but deform slightly under the action of loads. If the deformation is negligible as compared to its size, the body is termed as rigid.

Force

Force may be defined as any action that tends to change the state of rest or motion of a body to which it is applied.

The three quantities required to completely define force are called its specification or characteristics. So the characteristics of a force are:

1. Magnitude
2. Point of application
3. Direction of application
Concentrated force/point load

Distributed force

Line of action of force

The direction of a force is the direction, along a straight line through its point of application in which the force tends to move a body when it is applied. This line is called line of action of force.

Representation of force

Graphically a force may be represented by the segment of a straight line.

Composition of two forces

The reduction of a given system of forces to the simplest system that will be its equivalent is called the problem of composition of forces.

Parallelogram law

If two forces represented by vectors AB and AC acting under an angle $\alpha$ are applied to a body at point A. Their action is equivalent to the action of one force, represented by vector AD, obtained as the diagonal of the parallelogram constructed on the vectors AB and AC directed as shown in the figure.
Force AD is called the resultant of AB and AC and the forces are called its components.

\[ R = \sqrt{(P^2 + Q^2 + 2PQ \cos \alpha)} \]

Now applying triangle law

\[ \frac{P}{\sin \gamma} = \frac{Q}{\sin \beta} = \frac{R}{\sin(\pi - \alpha)} \]

**Special cases**

**Case-I:** If \( \alpha = 0^\circ \)

\[ R = \sqrt{(P^2 + Q^2 + 2PQ \cos 0)} = \sqrt{(P + Q)^2} = (P + Q) \]

Case- II: If \( \alpha = 180^\circ \)

\[ R = \sqrt{(P^2 + Q^2 + 2PQ \cos 180)} = \sqrt{(P^2 + Q^2 - 2PQ)} = \sqrt{(P - Q)^2} = (P - Q) \]
Case-III: If $\alpha = 90^\circ$

\[ R = \sqrt{P^2 + Q^2 + 2PQ \times \cos 90^\circ} = \sqrt{P^2 + Q^2} \]

\[ \alpha = \tan^{-1} \left( \frac{Q}{P} \right) \]

**Resolution of a force**

The replacement of a single force by a several components which will be equivalent in action to the given force is called resolution of a force.

**Action and reaction**

Often bodies in equilibrium are constrained to investigate the conditions.

**Free body diagram**

Free body diagram is necessary to investigate the condition of equilibrium of a body or system. While drawing the free body diagram all the supports of the body are removed and replaced with the reaction forces acting on it.
1. Draw the free body diagrams of the following figures.

2. Draw the free body diagram of the body, the string CD and the ring.
3. Draw the free body diagram of the following figures.

**Equilibrium of colinear forces:**

**Equilibrium law:** Two forces can be in equilibrium only if they are equal in magnitude, opposite in direction and collinear in action.

(tension)

(compression)
Superposition and transmissibility

**Problem 1:** A man of weight \( W = 712 \) N holds one end of a rope that passes over a pulley vertically above his head and to the other end of which is attached a weight \( Q = 534 \) N. Find the force with which the man’s feet press against the floor.

\[
W = 712 \text{ N}, \quad Q = 534 \text{ N},
\]

Now applying parallelogram law.

\[
R = \sqrt{(W - s)^2 + 2 \times 890 \times 0.5}
\]

\[
R = 712 - 534 = 178 \text{ N} \quad (\downarrow)
\]

Reaction on the man’s feet: 178 N \( \uparrow \)

**Problem 2:** A boat is moved uniformly along a canal by two horses pulling with forces \( P = 890 \) N and \( Q = 1068 \) N acting under an angle \( \alpha = 60^\circ \). Determine the magnitude of the resultant pull on the boat and the angles \( \beta \) and \( \nu \).

\[
P = 890 \text{ N}, \quad \alpha = 60^\circ
\]

\[
Q = 1068 \text{ N}
\]

**Resultant Force:**

\[
R = \sqrt{(P^2 + Q^2 + 2PQ\cos\alpha)}
\]

\[
= \sqrt{(890^2 + 1068^2 + 2 \times 890 \times 1068 \times 0.5)}
\]

\[
= 1698.01 \text{ N}
\]
Resolution of a force

Equilibrium of collinear forces:
**Law of superposition**

The action of a given system of forces on a rigid body will no way be changed if we add to or subtract from them another system of forces in equilibrium.

**Problem 3:** Two spheres of weight P and Q rest inside a hollow cylinder which is resting on a horizontal force. Draw the free body diagram of both the spheres, together and separately.

**Problem 4:** Draw the free body diagram of the figure shown below.
**Problem 5:** Determine the angles $\alpha$ and $\beta$ shown in the figure.

$$\alpha = \tan^{-1}\left(\frac{762}{915}\right) = 39^\circ 47'$$

$$\beta = \tan^{-1}\left(\frac{762}{610}\right) = 51^\circ 19'$$
Problem 6: Find the reactions $R_1$ and $R_2$.

Problem 7: Two rollers of weight $P$ and $Q$ are supported by an inclined plane and vertical walls as shown in the figure. Draw the free body diagram of both the rollers separately.
**Problem 8:** Find $\theta_n$ and $\theta_t$ in the following figure.

![Diagram](image1.png)

**Problem 9:** For the particular position shown in the figure, the connecting rod BA of an engine exert a force of $P = 2225 \text{ N}$ on the crank pin at A. Resolve this force into two rectangular components $P_h$ and $P_v$ horizontally and vertically respectively at A.

$P_h = 2081.4 \text{ N}$

$P_v = 786.5 \text{ N}$

**Equilibrium of concurrent forces in a plane**

- If a body known to be in equilibrium is acted upon by several concurrent, coplanar forces, then these forces or rather their free vectors, when geometrically added must form a closed polygon.

- This system represents the condition of equilibrium for any system of concurrent forces in a plane.
\[ R = w \tan \alpha \]
\[ S = w \sec \alpha \]

**Lami’s theorem**

If three concurrent forces are acting on a body kept in equilibrium, then each force is proportional to the sine of angle between the other two forces and the constant of proportionality is same.

\[
\frac{P}{\sin \alpha} = \frac{Q}{\sin \beta} = \frac{R}{\sin \nu}
\]

\[
\frac{S}{\sin 90} = \frac{R_a}{\sin(180 - \alpha)} = \frac{W}{\sin(90 + \alpha)}
\]
**Problem:** A ball of weight $Q = 53.4\text{N}$ rests in a right angled trough as shown in figure. Determine the forces exerted on the sides of the trough at D and E if all the surfaces are perfectly smooth.

![Diagram of a ball in a trough](image1)

**Problem:** An electric light fixture of weight $Q = 178\text{N}$ is supported as shown in figure. Determine the tensile forces $S_1$ and $S_2$ in the wires BA and BC, if their angles of inclination are given.

![Diagram of an electric light fixture](image2)

\[
\frac{S_1}{\sin 135} = \frac{S_2}{\sin 75} = \frac{178}{\sin 150}
\]

\[
S_1 \cos \alpha = P
\]

\[
S = P \sec \alpha
\]
The point of application of a force may be transmitted along its line of action without changing the effect of force on any rigid body to which it may be applied.
\[\sum Y = 0\]
\[S_1 \cos 30 + 20 \sin 60 = S_2 \sin 30\]
\[
\frac{\sqrt{3}}{2} S_1 + 20 \frac{\sqrt{3}}{2} = \frac{S_2}{2}
\]
\[S_2 = \frac{\sqrt{3}}{2} S_1 + 10 \frac{2}{\sqrt{3}}\]
\[S_2 = \sqrt{3} S_1 + 20 \sqrt{3}\]  

(1)

\[\sum Y = 0\]
\[S_1 \sin 30 + S_2 \cos 30 = S_2 \cos 60 + 20\]
\[
\frac{S_1}{2} + \frac{\sqrt{3}}{2} S_2 = \frac{20}{2} + 20
\]
\[S_1 = \frac{\sqrt{3}}{2} S_2 = 30
\]
\[S_1 + \sqrt{3} S_2 = 60\]  

(2)

Substituting the value of \(S_2\) in Eq. 2, we get
\[S_1 + \sqrt{3} \left(\frac{3}{2} S_1 + 20 \frac{2}{\sqrt{3}}\right) = 60\]
\[S_1 + 3S_1 + 60 = 60\]
\[4S_1 = 0\]
\[S_1 = 0 \text{KN}\]
\[S_2 = 20 \sqrt{3} = 34.64 \text{KN}\]
**Problem:** A ball of weight $W$ is suspended from a string of length $l$ and is pulled by a horizontal force $Q$. The weight is displaced by a distance $d$ from the vertical position as shown in Figure. Determine the angle $\alpha$, forces $Q$ and tension in the string $S$ in the displaced position.

\[
\cos \alpha = \frac{d}{l} \\
\sin^2 \alpha + \cos^2 \alpha = 1
\]

\[
\Rightarrow \sin \alpha = \sqrt{1 - \cos^2 \alpha} = \sqrt{1 - \frac{d^2}{l^2}} = \frac{1}{l} \sqrt{l^2 - d^2}
\]

Applying Lami’s theorem,

\[
\frac{S}{\sin 90} = \frac{Q}{\sin(90 + \alpha)} = \frac{W}{\sin(180 - \alpha)}
\]
\[ Q \sin(90 + \alpha) = \frac{W}{\sin(180 - \alpha)} \]
\[ W \cos \alpha \quad \frac{W \left( d^2 \right)}{l} \]
\[ \Rightarrow Q = \frac{W d}{\sqrt{l^2 - d^2}} \]
\[ \Rightarrow Q = \frac{W d}{\sqrt{l^2 - d^2}} \]

\[ S = \frac{W}{\sin \alpha} = \frac{W}{1 \sqrt{l^2 - d^2}} \]
\[ = \frac{W l}{\sqrt{l^2 - d^2}} \]

**Problem:** Two smooth circular cylinders each of weight \( W = 445 \text{ N} \) and radius \( r = 152 \text{ mm} \) are connected at their centres by a string AB of length \( l = 406 \text{ mm} \) and rest upon a horizontal plane, supporting above them a third cylinder of weight \( Q = 890 \text{ N} \) and radius \( r = 152 \text{ mm} \). Find the forces in the string and the pressures produced on the floor at the point of contact.

\[ \cos \alpha = \frac{203}{304} \]
\[ \Rightarrow \alpha = 48.1^\circ \]

\[ \frac{R_g}{\sin 138.1} = \frac{R_e}{\sin 138.1} = \frac{Q}{83.8} \]
\[ \Rightarrow R_g = R_e = 597.86 \text{ N} \]
Resolving horizontally
\[ \sum X = 0 \]
\[ S = R_c \cos 48.1 \]
\[ = 597.86 \cos 48.1 \]
\[ = 399.27N \]

Resolving vertically
\[ \sum Y = 0 \]
\[ R_d = W + R_c \sin 48.1 \]
\[ = 445 + 597.86 \sin 48.1 \]
\[ = 890N \]
\[ R_e = 890N \]
\[ S = 399.27N \]

**Problem:** Two identical rollers each of weight \( Q = 445 \) N are supported by an inclined plane and a vertical wall as shown in the figure. Assuming smooth surfaces, find the reactions induced at the points of support A, B and C.

\[ \frac{R_a}{\sin 120} = \frac{S}{\sin 150} = \frac{445}{\sin 90} \]
\[ \Rightarrow R_a = 385.38N \]
\[ \Rightarrow S = 222.5N \]
Resolving vertically
\[ \sum Y = 0 \]
\[ R_b \cos 60 = 445 + S \sin 30 \]
\[ \Rightarrow R_b \frac{\sqrt{3}}{2} = 445 + \frac{222.5}{2} \]
\[ \Rightarrow R_b = 642.302N \]

Resolving horizontally
\[ \sum X = 0 \]
\[ R_c = R_b \sin 30 + S \cos 30 \]
\[ \Rightarrow 642.302 \sin 30 + 222.5 \cos 30 \]
\[ \Rightarrow R_c = 513.84N \]

**Problem:**

A weight Q is suspended from a small ring C supported by two cords AC and BC. The cord AC is fastened at A while cord BC passes over a frictionless pulley at B and carries a weight P. If P = Q and \( \alpha = 50^\circ \), find the value of \( \beta \).

Resolving horizontally
\[ \sum X = 0 \]
\[ S \sin 50 = Q \sin \beta \]

Resolving vertically
\[ \sum Y = 0 \]
\[ S \cos 50 + Q \sin \beta = Q \]
\[ \Rightarrow S \cos 50 = Q (1 - \cos \beta) \]

Putting the value of S from Eq. 1, we get
\[ S \cos 50 + Q \sin \beta = Q \]
\[ \Rightarrow S \cos 50 = Q(1 - \cos \beta) \]
\[ \Rightarrow \frac{S \sin \beta}{\sin 50} \cos 50 = Q(1 - \cos \beta) \]
\[ \Rightarrow \cot 50 = \frac{1 - \cos \beta}{\sin \beta} \]
\[ \Rightarrow 0.839 \sin \beta = 1 - \cos \beta \]

Squaring both sides,
\[ 0.703 \sin^2 \beta = 1 + \cos^2 \beta - 2 \cos \beta \]
\[ 0.703(1 - \cos^2 \beta) = 1 + \cos^2 \beta - 2 \cos \beta \]
\[ 0.703 - 0.703 \cos^2 \beta = 1 + \cos^2 \beta - 2 \cos \beta \]
\[ \Rightarrow 1.703 \cos^2 \beta - 2 \cos \beta - 0.297 = 0 \]
\[ \Rightarrow \cos^2 \beta - 1.174 \cos \beta + 0.297 = 0 \]
\[ \Rightarrow \beta = 63.13^\circ \]
Method of moments

Moment of a force with respect to a point:

- Considering wrench subjected to two forces P and Q of equal magnitude. It is evident that force P will be more effective compared to Q, though they are of equal magnitude.
- The effectiveness of the force as regards it is the tendency to produce rotation of a body about a fixed point is called the moment of the force with respect to that point.
- Moment = Magnitude of the force × Perpendicular distance of the line of action of force.
- Point O is called moment centre and the perpendicular distance (i.e. OD) is called moment arm.
- Unit is N m

Theorem of Varignon:

The moment of the resultant of two concurrent forces with respect to a centre in their plane is equal to the algebraic sum of the moments of the components with respect to some centre.

Problem 1:

A prismatic clear of AB of length l is hinged at A and supported at B. Neglecting friction, determine the reaction R_b produced at B owing to the weight Q of the bar.

Taking moment about point A,
\[ R_b \times l = Q \cos \alpha \frac{l}{2} \]
\[ \Rightarrow R_b = \frac{Q}{\cos \alpha 2} \]
Problem 2:
A bar AB of weight Q and length 2l rests on a very small friction less roller at D and against a smooth vertical wall at A. Find the angle $\alpha$ that the bar must make with the horizontal in equilibrium.

Resolving vertically,
$$R_d \cos \alpha = Q$$

Now taking moment about A,
$$\frac{R_d \cdot a}{\cos \alpha} - Q \cdot l \cos \alpha = 0$$

$$\Rightarrow \frac{Q \cdot a}{\cos^2 \alpha} - Q \cdot l \cos \alpha = 0$$

$$\Rightarrow Q \cdot a - Q \cdot l \cdot \cos^3 \alpha = 0$$

$$\Rightarrow \cos^3 \alpha = \frac{Q \cdot a}{Q \cdot l}$$

$$\Rightarrow \alpha = \cos^{-1} \left( \frac{a}{l} \right)$$

Problem 3:
If the piston of the engine has a diameter of 101.6 mm and the gas pressure in the cylinder is 0.69 MPa. Calculate the turning moment M exerted on the crankshaft for the particular configuration.
Area of cylinder
\[ A = \pi (0.1016)^2 = 8.107 \times 10^{-3} \, m^2 \]

Force exerted on connecting rod,
\[ F = \text{Pressure} \times \text{Area} \]
\[ = 0.69 \times 10^6 \times 8.107 \times 10^{-3} \]
\[ = 5593.83 \, N \]

Now \( \alpha = \sin^{-1} \left( \frac{178}{380} \right) = 27.93^\circ \)

\[ S \cos \alpha = F \]
\[ \Rightarrow S = \frac{F}{\cos \alpha} = 6331.29 \, N \]

Now moment entered on crankshaft,
\[ S \cos \alpha \times 0.178 = 995.7 \, N = 1 \, KN \]

**Problem 4:**

A rigid bar AB is supported in a vertical plane and carrying a load Q at its free end. Neglecting the weight of the bar, find the magnitude of tensile force S in the horizontal string CD.

Taking moment about A,
\[ \sum M_A = 0 \]
\[ S \frac{l}{\cos \alpha} = Q \frac{l}{\sin \alpha} \]
\[ \Rightarrow S = \frac{Q \sin \alpha}{l \cos \alpha} \]
\[ \Rightarrow S = 2Q \tan \alpha \]
Friction

- The force which opposes the movement or the tendency of movement is called **Frictional force or simply friction**. It is due to the resistance to motion offered by minutely projecting particles at the contact surfaces. However, there is a limit beyond which the magnitude of this force cannot increase.
- If the applied force is more than this limit, there will be movement of one body over the other. This limiting value of frictional force when the motion is impending, it is known as **Limiting Friction**.
- When the applied force is less than the limiting friction, the body remains at rest and such frictional force is called **Static Friction**, which will be having any value between zero and the limiting friction.
- If the value of applied force exceeds the limiting friction, the body starts moving over the other body and the frictional resistance experienced by the body while moving is known as **Dynamic Friction**. Dynamic friction is less than limiting friction.
- Dynamic friction is classified into following two types:
  a) Sliding friction
  b) Rolling friction
- Sliding friction is the friction experienced by a body when it slides over the other body.
- Rolling friction is the friction experienced by a body when it rolls over a surface.
- It is experimentally found that the magnitude of limiting friction bears a constant ratio to the normal reaction between two surfaces and this ratio is called **Coefficient of Friction**.

![Friction Diagram]

\[
\text{Coefficient of friction} = \frac{F}{N}
\]

where \( F \) is limiting friction and \( N \) is normal reaction between the contact surfaces.

Coefficient of friction is denoted by \( \mu \).

Thus, \( \mu = \frac{F}{N} \)
Laws of friction

1. The force of friction always acts in a direction opposite to that in which body tends to move.
2. Till the limiting value is reached, the magnitude of friction is exactly equal to the force which tends to move the body.
3. The magnitude of the limiting friction bears a constant ratio to the normal reaction between the two surfaces of contact and this ratio is called coefficient of friction.
4. The force of friction depends upon the roughness/smoothness of the surfaces.
5. The force of friction is independent of the area of contact between the two surfaces.
6. After the body starts moving, the dynamic friction comes into play, the magnitude of which is less than that of limiting friction and it bears a constant ratio with normal force. This ratio is called coefficient of dynamic friction.

Angle of friction

Consider the block shown in figure resting on a horizontal surface and subjected to horizontal pull P. Let F be the frictional force developed and N the normal reaction. Thus, at contact surface the reactions are F and N. They can be graphically combined to get the reaction R which acts at angle \( \theta \) to normal reaction. This angle \( \theta \) called the angle of friction is given by

\[
\tan \theta = \frac{F}{N}
\]

As P increases, F increases and hence \( \theta \) also increases. \( \theta \) can reach the maximum value \( \alpha \) when F reaches limiting value. At this stage,

\[
\tan \alpha = \frac{F}{N} = \mu
\]

This value of \( \alpha \) is called Angle of Limiting Friction. Hence, the angle of limiting friction may be defined as the angle between the resultant reaction and the normal to the plane on which the motion of the body is impending.

Angle of repose

![Diagram of Angle of Repose](image)
Consider the block of weight \( W \) resting on an inclined plane which makes an angle \( \theta \) with the horizontal. When \( \theta \) is small, the block will rest on the plane. If \( \theta \) is gradually increased, a stage is reached at which the block start sliding down the plane. The angle \( \theta \) for which the motion is impending, is called the angle of repose. Thus, the maximum inclination of the plane on which a body, free from external forces, can repose is called \textbf{Angle of Repose}.

Resolving vertically,
\[ N = W \cdot \cos \theta \]

Resolving horizontally,
\[ F = W \cdot \sin \theta \]

Thus, \( \tan \theta = \frac{F}{N} \)

If \( \phi \) is the value of \( \theta \) when the motion is impending, the frictional force will be limiting friction and hence,
\[ \tan \phi = \frac{F}{N} = \mu = \tan \alpha \]
\[ \Rightarrow \phi = \alpha \]

Thus, the value of angle of repose is same as the value of limiting angle of repose.

\textbf{Cone of friction}

- When a body is having impending motion in the direction of force \( P \), the frictional force will be limiting friction and the resultant reaction \( R \) will make limiting angle \( \alpha \) with the normal.
- If the body is having impending motion in some other direction, the resultant reaction makes limiting frictional angle \( \alpha \) with the normal to that direction. Thus, when the direction of force \( P \) is gradually changed through 360°, the resultant \( R \) generates a right circular cone with semi-central angle equal to \( \alpha \).
Problem 1: Block A weighing 1000N rests over block B which weighs 2000N as shown in figure. Block A is tied to wall with a horizontal string. If the coefficient of friction between blocks A and B is 0.25 and between B and floor is 1/3, what should be the value of P to move the block (B), if

(a) P is horizontal.
(b) P acts at 30° upwards to horizontal.

Solution: (a)

Considering block A,

\[ \sum V = 0 \]
\[ N_1 = 1000N \]

Since \(F_1\) is limiting friction,

\[ \frac{F_1}{N_1} = \mu = 0.25 \]
\[ F_1 = 0.25N_1 = 0.25 \times 1000 = 250N \]

\[ \sum H = 0 \]
\[ F_1 - T = 0 \]
\[ T = F_1 = 250N \]

Considering equilibrium of block B,

\[ \sum V = 0 \]
\[ N_2 - 2000 - N_1 = 0 \]
\[ N_2 = 2000 + N_1 = 2000 + 1000 = 3000N \]

\[ \frac{F_2}{N_2} = \mu = \frac{1}{3} \]
\[ F_2 = 0.3N_2 = 0.3 \times 1000 = 1000N \]
\[ \sum H = 0 \]
\[ P = F_1 + F_2 = 250 + 1000 = 1250N \]

(b) When P is inclined:

\[ \sum V = 0 \]
\[ N_2 - 2000 - N_1 + P \cdot \sin 30 = 0 \]
\[ \Rightarrow N_2 + 0.5P = 2000 + 1000 \]
\[ \Rightarrow N_2 = 3000 - 0.5P \]

From law of friction,

\[ F = \frac{1}{3} N_2 = \frac{1}{3} \left(3000 - 0.5P\right) = 1000 - \frac{0.5}{3}P \]

\[ \sum H = 0 \]
\[ P \cos 30 = F_1 + F_2 \]
\[ \Rightarrow P \cos 30 = 250 + \left(1000 - \frac{0.5}{3}P\right) \]
\[ \Rightarrow P \left(\cos 30 + 0.5 \cdot \frac{P}{3}\right) = 1250 \]
\[ \Rightarrow P = 1210.43N \]

**Problem 2:** A block weighing 500N just starts moving down a rough inclined plane when supported by a force of 200N acting parallel to the plane in upward direction. The same block is on the verge of moving up the plane when pulled by a force of 300N acting parallel to the plane. Find the inclination of the plane and coefficient of friction between the inclined plane and the block.

\[ \sum V = 0 \]
\[ N = 500 \cdot \cos \theta \]
\[ F_1 = \mu N = \mu \cdot 500 \cdot \cos \theta \]
\[ \sum H = 0 \]
\[ 200 + F_1 = 500 \sin \theta \]
\[ \Rightarrow 200 + \mu \cdot 500 \cos \theta = 500 \sin \theta \]

\[ \sum V = 0 \]
\[ N = 500 \cos \theta \]
\[ F_2 = \mu N = \mu \cdot 500 \cos \theta \]

\[ \sum H = 0 \]
\[ 500 \sin \theta + F_2 = 300 \]
\[ \Rightarrow 500 \sin \theta + \mu \cdot 500 \cos \theta = 300 \]

Adding Eqs. (1) and (2), we get

\[ 500 = 1000 \sin \theta \]
\[ \sin \theta = 0.5 \]
\[ \theta = 30^\circ \]

Substituting the value of \( \theta \) in Eq. 2,
\[ 500 \sin 30 + \mu \cdot 500 \cos 30 = 300 \]
\[ \mu = \frac{50}{500 \cos 30} = 0.11547 \]
**Parallel forces on a plane**

**Like parallel forces:** Coplanar parallel forces when act in the same direction. **Unlike parallel forces:** Coplanar parallel forces when act in different direction. **Resultant of like parallel forces:**

Let $P$ and $Q$ are two like parallel forces act at points $A$ and $B$. $R = P + Q$

**Resultant of unlike parallel forces:**

$R = P - Q$

$R$ is in the direction of the force having greater magnitude.

**Couple:**

Two unlike equal parallel forces form a couple.

The rotational effect of a couple is measured by its moment.

$\text{Moment} = P \times l$

**Sign convention:** Anticlockwise couple (Positive)

Clockwise couple (Negative)
**Problem 1:** A rigid bar CABD supported as shown in figure is acted upon by two equal horizontal forces $P$ applied at $C$ and $D$. Calculate the reactions that will be induced at the points of support. Assume $l = 1.2 \text{ m}$, $a = 0.9 \text{ m}$, $b = 0.6 \text{ m}$.

\[ \sum V = 0 \]
\[ R_a = R_b \]

Taking moment about $A$,
\[ R_a \times l + P \times b = P \times a \]
\[ \Rightarrow R_a = \frac{P(0.9 - 0.6)}{1.2} \]
\[ \Rightarrow R_a = 0.25P(\uparrow) \]
\[ \Rightarrow R_a = 0.25P(\downarrow) \]

**Problem 2:** Owing to weight $W$ of the locomotive shown in figure, the reactions at the two points of support $A$ and $B$ will each be equal to $W/2$. When the locomotive is pulling the train and the drawbar pull $P$ is just equal to the total friction at the points of contact $A$ and $B$, determine the magnitudes of the vertical reactions $R_a$ and $R_b$.

\[ \sum V = 0 \]
\[ R_a + R_b = W \]

Taking moment about $B$, 
\[ \sum M_B = 0 \]
\[ R_a \times 2a + P \times b = W \times a \]
\[ \Rightarrow R_a = \frac{W \cdot a - P \cdot b}{2a} \]
\[ \therefore R_a = W - R_b \]
\[ \Rightarrow R = W - \left( \frac{W \cdot a - P \cdot b}{2a} \right) \]
\[ \Rightarrow R_b = \frac{W \cdot a + P \cdot b}{2a} \]

**Problem 3:** The four wheels of a locomotive produce vertical forces on the horizontal girder AB. Determine the reactions \( R_a \) and \( R_b \) at the supports if the loads \( P = 90 \text{ KN} \) each and \( Q = 72 \text{ KN} \) (All dimensions are in m).

\[ \sum V = 0 \]
\[ R_a + R_b = 3P + Q \]
\[ \Rightarrow R_a + R_b = 3 \times 90 + 72 \]
\[ \Rightarrow R_a + R_b = 342 \text{ KN} \]
\[ \sum M_A = 0 \]
\[ R_b \times 9.6 = 90 \times 1.8 + 90 \times 3.6 + 90 \times 5.4 + 72 \times 8.4 \]
\[ \Rightarrow R_b = 164.25 \text{ KN} \]
\[ \therefore R_a = 177.75 \text{ KN} \]

**Problem 4:** The beam AB in figure is hinged at A and supported at B by a vertical cord which passes over a frictionless pulley at C and carries at its end a load \( P \). Determine the distance \( x \) from A at which a load \( Q \) must be placed on the beam if it is to remain in equilibrium in a horizontal position. Neglect the weight of the beam.
Problem 5: A prismatic bar AB of weight $Q = 44.5$ N is supported by two vertical wires at its ends and carries at D a load $P = 89$ N as shown in figure. Determine the forces $S_a$ and $S_b$ in the two wires.

Resolving vertically,

$\sum F = 0$

$S_a + S_b = P + Q$

$\Rightarrow S_a + S_b = 89 + 44.5$

$\Rightarrow S_a + S_b = 133.5N$
\[ \sum M_f = 0 \]
\[ S \times l = P \times \frac{l}{4} + Q \times \frac{l}{2} \]
\[ \Rightarrow S_b = \frac{P \times Q}{4} \times \frac{1}{2} \]
\[ \Rightarrow S_b = \frac{89 + 44.5}{4} \times \frac{1}{2} \]
\[ \Rightarrow S_b = 44.5 \]
\[ \therefore S_a = 133.5 - 44.5 \]
\[ \Rightarrow S_a = 89N \]
Centre of gravity

Centre of gravity: It is that point through which the resultant of the distributed gravity force passes regardless of the orientation of the body in space.

As the point through which resultant of force of gravity (weight) of the body acts.

Centroid: Centroid of an area lies on the axis of symmetry if it exits.

Centre of gravity is applied to bodies with mass and weight and centroid is applied to plane areas.

\[
x_c = \sum A_i x_i y_c = \sum A_i y_i
\]

\[
x_c = \frac{A_1 x_1 + A_2 x_2}{A_1 + A_2}
\]

\[
y_c = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2}
\]

\[
x = \frac{\text{Moment of area}}{\text{Total area}}
\]

\[
x_c = \frac{\int x \, dA}{A}
\]

\[
y_c = \frac{\int y \, dA}{A}
\]
Problem 1: Consider the triangle ABC of base ‘b’ and height ‘h’. Determine the distance of centroid from the base.

Let us consider an elemental strip of width ‘b₁’ and thickness ‘dy’.

\[ \Delta AEF \sim \Delta ABC \]

\[ \therefore \ \frac{b_1}{b} = \frac{h - y}{h} \]

\[ \Rightarrow b = b \left( \frac{h - y}{h} \right) \]

\[ \Rightarrow b = b \left( 1 - \frac{y}{h} \right) \]

Area of element EF (dA) = \( b₁ \times dy \)

\[ = b \left( 1 - \frac{y}{h} \right) dy \]

\[ y = \frac{\int y \cdot dA}{A} \]

\[ = \frac{\int y \cdot b \left( 1 - \frac{y}{h} \right) dy}{\frac{1}{2} b \cdot h} \]

\[ = \frac{b \left[ \frac{y^2}{2} - \frac{y^3}{3h} \right]}{\frac{1}{2} b \cdot h} \]

\[ = \frac{2 \left[ \frac{h^2}{6} - \frac{h^3}{3} \right]}{h} \]

\[ = \frac{h^2}{6} \]

Therefore, \( y_c \) is at a distance of \( h/3 \) from base.
Problem 2: Consider a semi-circle of radius $R$. Determine its distance from diametral axis.

Due to symmetry, centroid $y_c$ must lie on Y-axis.

Consider an element at a distance ‘$r$’ from centre ‘$o$’ of the semicircle with radial width $dr$.

Area of element = $(r \, d\theta) \times dr$

Moment of area about $x = \int y \, dA$

\[
\begin{align*}
\pi R & = \int_{0}^{\pi} r^2 \sin \theta \, dr \, d\theta \\
\pi R & = \int_{0}^{\pi} \left( r^2 \right) \sin \theta \, dr \, d\theta \\
\pi R & = \int_{0}^{\pi} \left( \frac{r^3}{3} \right) \sin \theta \, d\theta \\
\pi R & = \frac{R^3}{3} \sin \theta \, d\theta \\
\pi R & = \frac{R^3}{3} \left[ -\cos \theta \right]_0^\pi \\
\pi R & = \frac{2}{3} R^3
\end{align*}
\]

\[
y_c = \frac{\text{Moment of area}}{\text{Total area}}
\]
\[ \frac{2}{3} R^3 \]

\[ \frac{4R}{3\pi} \]

**Centroids of different figures**

<table>
<thead>
<tr>
<th>Shape</th>
<th>Figure</th>
<th>(\bar{x})</th>
<th>(\bar{y})</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangle</td>
<td></td>
<td>(\frac{b}{2})</td>
<td>(\frac{d}{2})</td>
<td>(bd)</td>
</tr>
<tr>
<td>Triangle</td>
<td></td>
<td>0</td>
<td>(\frac{h}{3})</td>
<td>(\frac{bh}{2})</td>
</tr>
<tr>
<td>Semicircle</td>
<td></td>
<td>0</td>
<td>(\frac{4R}{3\pi})</td>
<td>(\frac{\pi r^2}{2})</td>
</tr>
<tr>
<td>Quarter circle</td>
<td></td>
<td>(\frac{4R}{3\pi})</td>
<td>(\frac{4R}{3\pi})</td>
<td>(\frac{\pi r^2}{4})</td>
</tr>
</tbody>
</table>

**Problem 3:** Find the centroid of the T-section as shown in figure from the bottom.
Due to symmetry, the centroid lies on Y-axis and it is at distance of 80 mm from the bottom.

**Problem 4:** Locate the centroid of the I-section.

As the figure is symmetric, centroid lies on y-axis. Therefore, \( x = 0 \)

<table>
<thead>
<tr>
<th>Area ((A_i))</th>
<th>(x_i)</th>
<th>(y_i)</th>
<th>(A_i \cdot x_i)</th>
<th>(A_i \cdot y_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>0</td>
<td>110</td>
<td>10,000</td>
<td>22,000</td>
</tr>
<tr>
<td>2000</td>
<td>0</td>
<td>50</td>
<td>10,000</td>
<td>10,000</td>
</tr>
<tr>
<td>4000</td>
<td></td>
<td></td>
<td>20,000</td>
<td>32,000</td>
</tr>
</tbody>
</table>

\[
y = \frac{\sum A_i y_i}{A} = \frac{A_1 y_1 + A_2 y_2 + A_3 y_3}{A_1 + A_2 + A_3} = 80 \text{ mm}
\]

Thus, the centroid is on the symmetric axis at a distance 80 mm from the bottom.

**Problem 5:** Determine the centroid of the composite figure about x-y coordinate. Take \( x = 40 \) mm.

\[
y = \frac{\sum A_i y_i}{A} = \frac{A_1 y_1 + A_2 y_2 + A_3 y_3}{A_1 + A_2 + A_3} = 59.71 \text{ mm}
\]

Thus, the centroid is on the symmetric axis at a distance 59.71 mm from the bottom.

\[A_1 = \text{Area of rectangle} = 12x \times 14x = 168x^2\]

\[A_2 = \text{Area of rectangle to be subtracted} = 4x \times 4x = 16x^2\]
\[ A_3 = \text{Area of semicircle to be subtracted} = \frac{\pi R^2}{2} = 25.13 x^2 \]
\[ A_4 = \text{Area of quartercircle to be subtracted} = \frac{\pi R^2}{4} = 12.56 x^2 \]
\[ A_5 = \text{Area of triangle} = \frac{1}{2} \times 6x \times 4x = 12 x^2 \]

<table>
<thead>
<tr>
<th>Area (A_i)</th>
<th>x_i</th>
<th>y_i</th>
<th>A_i x_i</th>
<th>A_i y_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>A_1 = 268800</td>
<td>7x = 280</td>
<td>6x = 240</td>
<td>75264000</td>
<td>64512000</td>
</tr>
<tr>
<td>A_2 = 25600</td>
<td>2x = 80</td>
<td>10x = 400</td>
<td>2048000</td>
<td>10240000</td>
</tr>
<tr>
<td>A_3 = 40208</td>
<td>6x = 240</td>
<td>4x = 67.906</td>
<td>96499200</td>
<td>2730364448</td>
</tr>
<tr>
<td>A_4 = 20096</td>
<td>10x + \left( \frac{4x}{3} \right)</td>
<td>8x + \left( \frac{4x}{3} \right)</td>
<td>9889040.64</td>
<td>8281420.926</td>
</tr>
<tr>
<td>A_5 = 19200</td>
<td>14x + \frac{6x}{3} = 16x</td>
<td>\frac{4x}{3} = 53.33</td>
<td>12288000</td>
<td>1023936</td>
</tr>
</tbody>
</table>

\[ x = \frac{A_1 x_1 - A_2 x_2 - A_3 x_3 - A_4 x_4 + A_5 x_5}{A_1 - A_2 - A_3 + A_4 + A_5} = 326.404 \text{mm} \]

\[ y = \frac{A_1 y_1 - A_2 y_2 - A_3 y_3 - A_4 y_4 + A_5 y_5}{A_1 - A_2 - A_3 + A_4 + A_5} = 219.124 \text{mm} \]

**Problem 6:** Determine the centroid of the following figure.

\[ A_1 = \text{Area of triangle} = \frac{1}{2} \times 80 \times 80 = 3200 m^2 \]
\[ A_2 = \text{Area of semicircle} = \frac{\pi R^2}{8} = 2513.274 m^2 \]
\[ A_3 = \text{Area of semicircle} = \frac{\pi D^2}{2} = 1256.64 m^2 \]
Problem 7: Determine the centroid of the following figure.

\[ x_c = \frac{A_1 x_1 + A_2 x_2 - A_3 x_3}{A_1 + A_2 + A_3} = 49.57 \text{ mm} \]

\[ y_c = \frac{A_1 y_1 + A_2 y_2 - A_3 y_3}{A_1 + A_2 + A_3} = 9.58 \text{ mm} \]

A_1 = \text{Area of the rectangle}
A_2 = \text{Area of triangle}
A_3 = \text{Area of circle}
Numerical Problems (Assignment)

1. An isosceles triangle ADE is to cut from a square ABCD of dimension ‘a’. Find the altitude ‘y’ of the triangle so that vertex E will be centroid of remaining shaded area.

![Triangle ADE diagram]

2. Find the centroid of the following figure.

![Complex figure diagram]

3. Locate the centroid C of the shaded area obtained by cutting a semi-circle of diameter ‘a’ from the quadrant of a circle of radius ‘a’.

![Semi-circle quadrant diagram]

4. Locate the centroid of the composite figure.

![Composite figure diagram]
**Truss/ Frame:** A pin jointed frame is a structure made of slender (cross-sectional dimensions quite small compared to length) members pin connected at ends and capable of taking load at joints.

Such frames are used as roof trusses to support sloping roofs and as bridge trusses to support deck.

**Plane frame:** A frame in which all members lie in a single plane is called plane frame. They are designed to resist the forces acting in the plane of frame. Roof trusses and bridge trusses are the example of plane frames.

**Space frame:** If all the members of frame do not lie in a single plane, they are called as space frame. Tripod, transmission towers are the examples of spaceframes.

**Perfect frame:** A pin jointed frame which has got just sufficient number of members to resist the loads without undergoing appreciable deformation in shape is called a perfect frame. Triangular frame is the simplest perfect frame and it has 03 joints and 03 members.

It may be observed that to increase one joint in a perfect frame, two more members are required. Hence, the following expression may be written as the relationship between number of joint $j$, and the number of members $m$ in a perfect frame.

$$m = 2j - 3$$

- (a) When $LHS = RHS$, Perfect frame.
- (b) When $LHS < RHS$, Deficient frame.
- (c) When $LHS > RHS$, Redundant frame.

**Assumptions**

The following assumptions are made in the analysis of pin jointed trusses:

1. The ends of the members are pin jointed (hinged).
2. The loads act only at the joints.
3. Self weight of the members is negligible.

**Methods of analysis**

1. Method of joint
2. Method of section
Problems on method of joints

Problem 1: Find the forces in all the members of the truss shown in figure.

Joint C

\[ S_1 = S_2 \cos 45 \]

\[ \Rightarrow S_1 = 40 \text{KN (Compression)} \]

\[ S_1 \sin 45 = 40 \]

\[ \Rightarrow S_2 = 56.56 \text{KN (Tension)} \]

Joint D

\[ S_3 = 40 \text{KN (Tension)} \]

\[ S_4 = S_5 = 40 \text{KN (Compression)} \]

Joint B

Resolving vertically,

\[ \sum V = 0 \]

\[ S_5 \sin 45 = S_3 + S_2 \sin 45 \]
\[ S_5 = 113.137 \text{KN (Compression)} \]

Resolving horizontally,
\[ \sum H = 0 \]
\[ S_6 = S_5 \cos 45 + S_2 \cos 45 \]
\[ \Rightarrow S_6 = 113.137 \cos 45 + 56.56 \cos 45 \]
\[ \Rightarrow S_6 = 120 \text{KN (Tension)} \]

**Problem 2:** Determine the forces in all the members of the truss shown in figure and indicate the magnitude and nature of the forces on the diagram of the truss. All inclined members are at 60° to horizontal and length of each member is 2m.

Taking moment at point A,
\[ \sum M_A = 0 \]
\[ R_d \times 4 = 40 \times 1 + 60 \times 2 + 50 \times 3 \]
\[ \Rightarrow R_d = 77.5 \text{KN} \]

Now resolving all the forces in vertical direction,
\[ \sum V = 0 \]
\[ R_a + R_d = 40 + 60 + 50 \]
\[ \Rightarrow R_a = 72.5 \text{KN} \]

**Joint A**
\[ \sum V = 0 \]
\[ \Rightarrow R_a = S_1 \sin 60 \]
\[ \Rightarrow S_1 = 83.72 \text{KN (Compression)} \]

\[ \sum H = 0 \]
\[ \Rightarrow S_2 = S_1 \cos 60 \]
\[ S_1 = 41.86 \text{KN (Tension)} \]

**Joint D**

\[ \sum V = 0 \]
\[ S_7 \sin 60 = 77.5 \]
\[ \Rightarrow S_7 = 89.5 \text{KN (Compression)} \]

\[ \sum H = 0 \]
\[ S_6 = S_7 \cos 60 \]
\[ \Rightarrow S_6 = 44.75 \text{KN (Tension)} \]

**Joint B**

\[ \sum V = 0 \]
\[ S_1 \sin 60 = S_3 \cos 60 + 40 \]
\[ \Rightarrow S_3 = 37.532 \text{KN (Tension)} \]

\[ \sum H = 0 \]
\[ S_4 = S_1 \cos 60 + S_3 \cos 60 \]
\[ \Rightarrow S_1 = 37.532 \cos 60 + 83.72 \cos 60 \]
\[ \Rightarrow S_1 = 60.626 \text{KN (Compression)} \]

**Joint C**

\[ \sum V = 0 \]
\[ S_5 \sin 60 + 50 = S_7 \sin 60 \]
\[ \Rightarrow S_5 = 31.76 \text{KN (Tension)} \]
Plane Truss (Method of Section)

In cases of analyzing a plane truss, using method of section, after determining the support reactions, a section line is drawn passing through not more than three members in which forces are unknown, such that the entire frame is cut into two separate parts.

Each part should be in equilibrium under the action of loads, reactions, and the forces in the members.

Method of section is preferred for the following cases:

(i) Analyzing a large truss in which forces in only few members are required.

(ii) If method of joint fails to start or proceed. With analysis for not getting a joint with only two unknown forces.

Example 1:

\[ \sum F_y = 0 \]
\[ F_{kx} \times 9 \sin 60^\circ + 10 \times 1.2 = 10 \times 2 + 10 \times \sin 60^\circ \]
\[ F_{kx} = \frac{20 + 10 + 100}{2 + \sin 60^\circ} = 420 \]
\[ F_{kx} = -69.28 \text{ kN} \]
Negative sign indicates that direction should be opposite i.e. it is compressive in nature.

Now resolving all the forces vertically $\Sigma Y = 0$

\[ 10 + 10 + 10 + F \sin 60 = 35 \]
\[ \Rightarrow F \sin 60 = 35 - 30 \]
\[ \Rightarrow F = \frac{5.78 \text{ kN}}{\sin 60} \] (compressive)

Resolving all the forces horizontally $\Sigma X = 0$

\[ F \cos 60 + F_1 \cos 60 = F_1 \]
\[ \Rightarrow F_1 = 69.28 + 5.78 \cos 60 = 72.17 \text{ kN} \] (tension)

Using the method of sections determine the axial forces in bars 1, 2 and 3.

Taking moment about joint D $\Sigma M_D = 0$

\[ s_1 x a = F x h \Rightarrow s_1 = \frac{F h}{a} \] (tension) (1)

Similarly taking E as the moment centre $\Sigma M_E = 0$

\[ s_2 x a + F x 2h = 0 \]
\[ \Rightarrow s_2 = \frac{-2F h}{a} \] (negative sign indicates direction of force will be opposite and it will be compressive in nature)

Resolving all the forces horizontally $\Sigma X = 0$

\[ s_2 \cos \alpha = F \]
\[ \Rightarrow s_2 = \frac{F}{\cos \alpha} = \frac{4 + \sqrt{4 + h^2}}{a} \] (Ans.)

\[ \cos \alpha = \frac{a}{\sqrt{a^2 + h^2}} \]
\[ Bc = \tan 30' \]
\[ Bc = a \tan 30' = 0.578a \]

\[ \sum M_B = 0 \]
\[ s_2 \times 0.578a + Px = 0 \]
\[ \therefore s_2 = \frac{P}{0.578a} = -1.73P \]
(\textit{negative sign indicates direction is opposite and it is compressive in nature})

Resolving vertically, \( \sum y = 0 \)
\[ s_1 \sin 30 = 2P + s_2 \sin 30 \]
\[ \therefore s_1 = \frac{2P + s_2 / 2}{\sin 30} = (4P + s_2 / 2) - (2) \]

Now resolving horizontally, \( \sum x = 0 \)
\[ s_1 \cos 30 + s_2 \cos 30 = -1.73P \]
\[ \therefore (4P + s_2) \times \frac{\sqrt{3}}{2} + s_2 \frac{\sqrt{3}}{2} = -1.73P \]
\[ \therefore 2 \sqrt{3}P + \frac{\sqrt{3}}{2} s_2 + \frac{\sqrt{3}}{2} s_2 = -1.73P \]
\[ \therefore \frac{\sqrt{3}}{2} s_2 = -1.73P - 2 \sqrt{3}P \]
\[ \therefore s_2 = \frac{-1.73P - 2 \sqrt{3}P}{\sqrt{3}} = -1.73P \]
(\textit{negative sign indicates the direction is opposite and it is compressive})

Now \( s_1 = 4P + P = 5P \)

\( s_1 = 3P \) (\textit{tension})
Using method of sections:

5KN, find axial forces in each bar 1, 2 and 3 of the plane truss.

We have \( \tan \theta = \frac{2.5}{1.5} \) so \( \theta = 28.56^\circ \)

Considering section 1

Resolving vertically, \( \Sigma y = 0 \)

\[ \begin{align*}
\Sigma y &= 0 \\
5 &= 5 \text{ kN}
\end{align*} \]

Now taking moment about C

\[ \Sigma M_C = 0 \]

\[ s_2 \times 1.5 = 5 \times 3 = 0 \]

\[ s_2 = -10 \text{ kN} \]

We sign indicates direction should have been opposite

\[ s_2 = 10 \text{ kN} \] (Compression)

Considering section 4:2

Taking moment about F

\[ \Sigma M_F = 0 \]

\[ 5 \times 3 = 0 \]

\[ \Sigma s_2 = 0 \]

Assignment

Using method of joints and method of section find the axial forces in the bar 2.

Method of Joints

Considering the whole structure and taking moment about A: \( \Sigma M_A = 0 \).

\[ R_B \times 3 = P \times 1.5 \]

\[ \Rightarrow R_B = \frac{\sqrt{3}}{4} P \]
0.1 (6.3) Calculate the relation between active forces \( P \) and \( Q \) for equilibrium of system of bars. The bars are so arranged that they form identical rhombuses.

Let \( \theta \) = length of each side of bar.

\[ \theta = \text{angle made by each side of the rhombus} \]

Distance \( d \) from fixed point \( A \):

\[ d = 2l \cos \theta \]

Let the virtual displacement of \( P \) is \( B-B' \).

\[ B-B' = \frac{d}{\sin \theta} \]

Similarly, the virtual displacement of \( Q \) is \( C-C' \).

\[ C = 2l \sin \theta \]

Applying principle of virtual work, \( \Sigma W = 0 \):

\[ P \cdot d \sin \theta = Q \cdot 2l \sin \theta \]

\[ p = \frac{B}{3} \]

0.2 A prismatic bar \( AB \) of length \( l \) and \( BC \) stands in a vertical plane and is supported by smooth surfaces at \( A \) and \( B \). Using principle of virtual work find the magnitude of horizontal force \( P \) applied at \( A \) if the bar is in equilibrium.
Let the horizontal distance of \( P \) from \( D \) is \( x \)

\[ x = r \cos \theta \]

\[ A \theta = dx = r \sin \theta \, d\theta \]

Vertical distance of \( Q \) from \( D \) is \( y \)

\[ y = \frac{r}{2} \sin \theta \]

\[ C \theta = dy = \frac{r}{2} \cos \theta \, d\theta \]

Normal reactions \( R_a \) and \( R_b \) have no work along the plane.

Applying principle of virtual work \( \sum W_{C} \)

\[ P \, dx = R \, dy \]

\[ Pe \sin \theta \, d\theta = \frac{r^2}{2} \cos \theta \, d\theta \]

\[ \int P = \frac{b}{2} \csc \theta \]

\[ 0.3 \] (6.14)

Find axial forces in the bars of the simple truss by using method of virtual work.
Let $S$ be the compressive force in bar $EB$.

Consider the part $EBDF$ of the truss under the action of force $R_b$, $P$ and $S$.

Keeping $E$ fixed and giving $EB$ an angular displacement $d\alpha$:

$S W = 0$.

$R_b \times \frac{2}{3} = P \times a \\
BB' = \frac{l}{2} \, d\alpha \\
F_1 = h \, d\alpha \\
R_b \times \frac{l}{2} \, d\alpha = S \times a \, d\alpha \\
\Rightarrow S = \frac{R_b l}{2h}$ -- (1)

Now considering whole frame of equilibrium body $EYZD$:

$Ra + R_b = P$.

$R_b \times \frac{l}{2} = P \times \frac{l}{2} \Rightarrow R_b = \frac{P}{2} \quad -- (2)$

Substituting the value of $R_b$ in eq. (1):

$S = \frac{PL}{4h}$ -- (03)

8.4 (6.15)

Using principle of virtual work, find reactions $R_a$ for the truss.

Let the truss is virtual displaced by an amount $dy$.

$S W = 0$.

$R_a \times 4a = P \times dD$.

where $4a = dD = dy$.

$\Rightarrow R_a = P$
The moment of inertia of any plane figure with respect to x and y axes in its plane are expressed as:

\[ I_x = \int y^2 \, dA \quad I_y = \int x^2 \, dA \]

\( I_x \) and \( I_y \) are also known as second moment of inertia area about the axes as it is distance squared from corresponding axis.

Unit:

Unit of moment of inertia of area is expressed as m^4 or mm^4.

**Moment of Inertia of Plane Figures:**

1) Rectangle

Considering a rectangle of width \( b \) and depth \( d \).

\[ \text{Moment of Inertia about centroidal axis } \frac{b}{x} \text{ parallel to the short side } \frac{b}{2} \]

Now considering an elementary strip of width \( dy \)

\[ \text{Moment of Inertia of the elemental strip about centroidal axis } \frac{b}{x} \text{ is} \]

\[ I_{xx} = y^2 \, dA \]

\[ = y^2 \cdot bdy \]

So moment of inertia of entire area

\[ I_{xx} = \int y^2 \cdot bdy = \frac{b}{2} \left[ \frac{y^3}{3} \right]_{d/2}^{d/2} = b \left[ \frac{d^3}{24} + \frac{d^3}{24} \right] \]

\[ I_{xx} = \frac{bd^2}{12} \]

Similarly, moment of inertia about y-axis

\[ I_{yy} = \frac{d^3}{12} \]
C(i) Triangle 

(Moment of inertia of a triangle about its base)

Consider a small elementary strip of thickness dy, at a distance y from the base of thickness dy. Let da be the area of the strip.

\[ da = h \cdot dy \]

And \( b_1 = \frac{(h-y)}{h} \cdot b \).

Moment of inertia of strip about base AB

\[ y^2 da = y^2 \cdot b_1 \cdot dy = y^2 \cdot \left( \frac{h-y}{h} \right) \cdot b_1 \cdot dy \]

Moment of inertia of the triangle about AB

\[ I_{AB} = \int_0^h y^2 \left( \frac{h-y}{h} \right) \cdot b_1 \cdot dy = \int_0^h \left( y^2 - \frac{y^3}{h} \right) \cdot b_1 \cdot dy \]

\[ = b_1 \left[ \frac{y^3}{2} - \frac{y^4}{4h} \right]_0^h = b_1 \left[ \frac{h^3}{2} - \frac{h^4}{4h} \right] = b_1 \left[ \frac{h^2}{2} - \frac{h^3}{4} \right] = \frac{bh^3}{12} \]

\[ I_{AB} = \frac{bh^3}{12} \]

(C(ii)) Moment of inertia of a circle about its centroidal axis

Considering an elementary strip of thickness dy, the side of strip \( d\theta \).

Moment of inertia of strip about \( xx \)

\[ = y^2 \cdot da = (r \cdot \sin \theta) = r \cdot \sin \theta \cdot \cdot d\theta \cdot dy \]

Moment of inertia of circle about \( xx \) axis

\[ I_{xx} = \int \int r^2 \cdot \sin^2 \theta \cdot d\theta \cdot dy \]

\[ \int_{r=0}^R \int_{\theta=0}^{2\pi} r^2 \cdot \sin^2 \theta \cdot d\theta \cdot dr \]

\[ = \int_{r=0}^R \int_{\theta=0}^{2\pi} r^2 \cdot \sin^2 \theta \cdot d\theta \cdot dr \]

\[ = \frac{1}{2} \int_{r=0}^R \int_{\theta=0}^{2\pi} r^2 \cdot \sin^2 \theta \cdot d\theta \cdot dr \]

\[ = \frac{1}{2} \int_{r=0}^R \int_{\theta=0}^{2\pi} r^2 \cdot \sin^2 \theta \cdot d\theta \cdot dr \]
\[
\int_0^R \frac{r^3}{2} \left[ \theta - \frac{3\pi^2}{2} \right] \, dr
\]
\[
= \int_0^R \frac{r^3}{2} \left( 2\pi - \frac{3\pi^2}{2} \right) \, dr
\]
\[
= \left[ \frac{r^4}{8} \right]_0^R \left( 2\pi - 0 \right)
\]
\[
= \frac{R^4}{8} \cdot 2\pi = \frac{\pi R^4}{4}
\]

\[
\Rightarrow I_{xx} = \frac{\pi R^4}{4} = \frac{\pi D^4}{64}
\]

(Polar moment of inertia)

Moment of inertia about an axis perpendicular to the plane of area is called polar moment of inertia. It may be denoted as \( I \) or \( I_{xx} \)

\[
I_{xx} = \int \rho^2 \, dA
\]

Radius of gyration:

Radius of gyration may be defined by a relation

\[
K = \sqrt{\frac{I}{A}}
\]

where \( K \) = radius of gyration
\( I \) = moment of inertia
\( A \) = cross-sectional area

So, we can have the following relations:

\[
K_{xx} = \sqrt{\frac{I_{xx}}{A}}
\]

\[
K_{yy} = \sqrt{\frac{I_{yy}}{A}}
\]

\[
K_{xy} = \sqrt{\frac{I_{xy}}{A}}
\]
There are two theorems of moment of inertia.

(a) Perpendicular axis theorem.

(b) Parallel axis theorem.

Perpendicular axis theorem:

Moment of inertia of an area about an axis in its plane at any point \( O \) is equal to the sum of moments of inertia about any two mutually perpendicular axes through the same point \( O \) and lying in the plane of area.

\[
I_{xx} = I_{yy} + I_{xy}
\]

\[
I_{xx} = \sum x^2 dA
\]

\[
= \sum (x_0 y^2) dA
\]

\[
= \sum x^2 dA + \sum y^2 dA
\]

\[
\Rightarrow I_{xx} = I_{yy} + I_{xy}
\]

Parallel axis theorem:

Moment of inertia about an axis in the plane of an area is equal to the sum of moment of inertia about a parallel centroidal axis and the product of area and square of the distance between the two parallel axes.

\[
I_{xx} = I_{xx}^C + A \cdot h^2
\]
(i) Moment of inertia of a rectangle about its centroidal axes \( xx \)

\[ I_{xx} = \frac{bd^3}{12} \]

Similarly, moment of inertia about its centroidal axis \( yy \)

\[ I_{yy} = \frac{db^3}{12} \]

New moment of inertia of rectangle about its base \( AB \) can be obtained by applying parallel axis theorem:

\[ I_{AB} = I_{xx} + A h^2 \]

\[ = \frac{bd^3}{12} + \left( bd \right)^2 \left( \frac{d}{2} \right)^2 \]

\[ = \frac{bd^3}{12} + \frac{bd^2}{4} \]

\[ = \frac{3bd^3 + bd^2}{12} = \frac{bd^3}{2} \]

\[ I_{AB} = \frac{bd^3}{3} \]

(ii) Moment of inertia of a hollow rectangular section

Moment of inertia of hollow rectangular section

\[ I_{xx} = \frac{BD^3}{12} - \frac{bd^3}{12} = \frac{1}{12} \left( BD^3 - bd^3 \right) \]
(iii) Moment of Inertia of Triangle about its Base

Moment of Inertia of triangle about its base

\[ = \text{moment of inertia about its centroid} \]

\[ + Ah^2 \] (using parallel axis theorem)

\[ \Rightarrow I_{AB} = I_{xx} + Ah^2 \]

\[ \Rightarrow \frac{bh^3}{12} = I_{xx} + \frac{1}{2} bh \times (\frac{h}{3})^2 \]

\[ = I_{xx} + \frac{bh^3}{12} - \frac{bh^3}{18} \]

\[ \Rightarrow I_{xx} = \frac{bh^3}{12} - \frac{2}{18} \]

\[ = \frac{3bh^3 - 2bh^3}{36} \]

\[ \Rightarrow I_{xx} = \frac{bh^3}{36} \]

(iv) Moment of Inertia of Semi-circle

(a) About diametral axis

Moment of Inertia of semi-circle

About AB = \( \frac{1}{2} \pi d^4 \)

\[ = \frac{\pi d^4}{128} \]

(b) About centroidal axis xx

\[ P_{yy} = \frac{2d^2}{3\pi} \]

Area \( A = \frac{d^2}{2} \) \( \pi d^2 = \frac{d^2}{2} \)

Using parallel axis theorem

\[ I_{AB} = I_{xx} + Ah^2 \]

\[ \Rightarrow \frac{\pi d^4}{128} = I_{xx} + \frac{\pi d^2}{8} \times (\frac{2d^2}{\pi})^2 \]

\[ \Rightarrow \frac{\pi d^4}{128} = \frac{2d^4}{3\pi} \]
\[
\begin{align*}
\frac{{150}}{2} & = \frac{{\frac{{4^2}}{9} \times \pi d^4}}{2} \\
\frac{d^4}{12} & = \frac{4}{18} \\
\therefore \quad L_{xx} & = \left( \frac{\pi d^4}{12} - \frac{d^4}{18} \right)
\end{align*}
\]

Moment of inertia of composite figure:

0.1 Determine the moment of inertia of the composite section about an axis passing through the centroidal axis. Also determine its moments of inertia of symmetry and radius of gyration.

Divide the composite area into \( A_1 \) and \( A_2 \):

- \( A_1 = 150 \times 10 = 1500 \text{ mm}^2 \)
- \( A_2 = 140 \times 10 = 1400 \text{ mm}^2 \)

Distance of centroid from base of the composite figure:

\[
\bar{y} = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2} = \frac{150 \times 145 + 140 \times 70}{2900} = 108.79 \text{ mm}
\]

Moment of inertia of the area about \( xx \) axis:

\[
I_{xx} = \frac{150 \times 10^3}{12} + 1500 \times (145 - 108.79)^2 + 1400 \times (108.79 - 70)^2
\]

\[
= (12500 + 196674.615) + (286666.667 + 2106529.74)
\]

\[
= 6372448.557 \text{ mm}^4
\]

Similarly:

\[
L_{yy} = \frac{10 \times 150^3}{12} + \frac{14 \times 10^8}{12} = 2812500 + 1166666667
\]

\[
= 2824166667 \text{ mm}^4
\]
Radius of gyration $k = \sqrt{\frac{1}{A}}$

So $k_{xx} = \sqrt{\frac{E_{xx}}{A}} = \sqrt{\frac{6372442.5}{2900}} = 46.87 \text{ mm}$

Similarly $k_{yy} = \sqrt{\frac{E_{yy}}{A}} = \sqrt{\frac{2854166.67}{2900}} = 31.206 \text{ mm}$ (Ans)

Determine the ME of the section about its centroidal axis parallel to the leg. Also find the polar moment of inertia.

We have $A_1 = 125 \times 10 = 1250 \text{ mm}^2$
$A_2 = 75 \times 10 = 750 \text{ mm}^2$
Total area $A_1 + A_2 = 2000 \text{ mm}^2$

Distance of centroid from 1-1 axis:

$\bar{y} = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2}$

$= \frac{1250 \times 62.5 + 750 \times 5}{2000} = 40.9375 \text{ mm}$

Distance of centroidal axis $y' y$ from 2-2 axis:

$\bar{z} = \frac{A_1 z_1 + A_2 z_2}{A_1 + A_2}$

$= \frac{1250 \times 5 + 750 \times (\frac{75}{2} + 10)}{2000}$

$= \frac{1250 \times 5 + 750 \times 92.5}{2000} = 20.93 \text{ mm}$

Moment of inertia about xx axis:

$I_{xx} = \sum \frac{10 \times 125}{12} + 1250 \times (62.5 - 40.9375)^2 + \sum \frac{75 \times 10^2 + 750 \times (75 - 90.9375 - 5)^2}{12}$

$= (162760.9, 167 + 581176.7578) + (6250 + 968627.9277)$

$= 3183658.854 \text{ mm}^4$
Similarly, \( M_I \) about \( yy \) centroidal axis:

\[
I_{yy} = \frac{125 \times 10^3}{12} + 1250 \times (20.93 - 5)^2
+ \frac{10 \times 75^3}{12} + 750 \times (47.5 - 20.93)^2
\]

\[
= (10416.6667 + 317206.125) + (351562.5 + 529972.625)
= 1208658.967 \text{ mm}^4
\]

Polar moment of inertia \( I_{xx} = I_{xx} + I_{yy} \)

\[
= 4392317.821 \text{ mm}^4 \quad \text{(Ans)}
\]

\textbf{Q. 3}

Determine the \( M_I \) of the symmetrical I section about its centroidal axes \( xx \) and \( yy \). Also determine the polar moment of inertia of the section.

We have from the figure:

- \( A_1 = 200 \times 9 = 1800 \text{ mm}^2 \)
- \( A_2 = 232 \times 6.7 = 1554.9 \text{ mm}^2 \)
- \( A_3 = 200 \times 9 = 1800 \text{ mm}^2 \)

Position of centroidal axis \( xx \) from base:

\[
y = \frac{A_1 y_1 + A_2 y_2 + A_3 y_3}{A_1 + A_2 + A_3}
\]

\[
= \frac{1800 \times (9.5 + 23.2 + 9) + 1554.9 \times (28.92 + 9) + 1800 \times 9.5}{1800 + 1554.9 + 1800}
\]

\[
= 1800 \times 245.5 + 1554.9 \times 125 + 1800 \times 9.5
\]

\[
= 1800 \times 180 + 1554.9 \times 96.65 + 1800 \times 180
\]

\[
= 98.98 \text{ mm}
\]

\[
M_I = \frac{A_1 y_1^2 + A_2 y_2^2 + A_3 y_3^2}{A_1 + A_2 + A_3}
\]

\[
= \frac{1800 \times 100 + 1554.9 \times 96.65 + 1800 \times 180}{1800 + 1554.9 + 1800}
\]
\[ E_{xx} = \frac{200 \times 9^2}{12} + 1800 \times (125 - 4.5)^2 \frac{3}{12} + \frac{6.7 \times 282^2}{12} + 1554 \times (125 - 4.5)^2 \frac{3}{12} \]
\[ = (12150 + 26136450) + (6972002.132 + 0) \]
\[ + (12150 + 26136450) \]
\[ = 26148600 + 6972002.132 + 26148600 \]
\[ = 59269302.13 \text{ mm}^4 \]

**Moment about yy axis**
\[ E_{yy} = \frac{9 \times 9^2}{12} + \frac{282 \times 6.7^2}{12} + \frac{9 \times 282^2}{12} \]
\[ = 600000 + 58797.51 + 600000 \]
\[ = 12003814.75 \text{ mm}^4 \]

**Polar moment of inertia**
\[ I_{xx} = E_{xx} + E_{yy} \]
\[ = 71275016.88 \text{ mm}^4 \]

---

**Calculate the moment of inertia of the shaded area about x axis.**

- **ME of the shaded section about**
  - \[ E_{xx} = ME \text{ of triangle ABC about } x \]
  - \[ ME \text{ of semicircle ACS about } x \]
  - \[ ME \text{ of circle} \]
  \[
  = \frac{100 \times 100^3}{12} + \frac{\pi \times 100^4}{128} - \frac{\pi \times 50^4}{64} 
  \]
  \[ = 83833333.33 + 2459369.261 - 306796.1576 
  \]
  \[ = 11480906.49 \text{ mm}^4 \]
  \[ = 1.148 \times 10^7 \text{ mm}^4 \]
- **Rectilinear Translation**

In statics, it was considered that the rigid bodies are at rest. In dynamics, it is considered that they are in motion. Dynamics is commonly divided into two branches:

- **Kinematics**
  - Kinematics we are concerned with space-time relationship of the motion of a body and not at all with the forces that cause the motion.

- **Kinetics**
  - Kinetics we are concerned with finding the kind of motion that a given body or system of bodies will have under the action of given forces or with what forces must be applied to produce a desired motion.

**Displacement**

General displacement-time equation

\[ x = f(t) \]  

where \( f(t) \) = function of time,

**Example**

\[ x = c + wt \]

In the above equation, \( C \), represents the initial displacement at \( t = 0 \), while the constant \( w \) = the rate at which displacement increases. It is called uniform rectilinear motion.
Example: The rectilinear motion of a particle is defined by the displacement-time equation \( x = x_0 - u_0 t + \frac{1}{2} a t^2 \). Construct displacement-time and velocity diagrams for this motion and find the displacement and velocity at time \( t = 2 \) s. \( x_0 = 750 \text{ mm}, \ u_0 = 500 \text{ mm/s}, \ a = 0.125 \text{ m/s}^2 \). The equation of motion is

\[
x = x_0 - u_0 t + \frac{1}{2} a t^2 - (c) \]

\[
u = \frac{dx}{dt} = -u_0 + at - (c_2)
\]

Substituting \( x_0, u_0, \) and \( a \) in equation (1).

\[
x = 750 - 500 = -250 \text{ mm}
\]
A bullet leaves the muzzle of a gun with velocity $u = 750 \text{ m/s}$. Assuming constant acceleration (from breach to muzzle), find the time $t$ occupied by the bullet in travelling through gun barrel which is 750 mm long.

**Initial velocity of bullet $u = 0$**

**Final velocity of bullet $v = 750 \text{ m/s}$**, total distance $s = 0.75 \text{ m}$.

$t = 2$

We have $v^2 - u^2 = 2as$,

$\Rightarrow v^2 = 2as$  \(\Rightarrow\) $a = \frac{v^2}{2s} = \frac{750^2}{2 \times 0.75}$

Again $v = u + at$

$\Rightarrow 750 = 75000 \times t$

$\Rightarrow t = \frac{750}{75000} = 0.01 \text{ sec}$

A stone is dropped into a well and falls vertically with constant acceleration $a = 9.81 \text{ m/s}^2$. The sound of impact of the stone on the bottom of well is heard after 6.5 sec. If velocity of sound is 336 m/s, how deep is the well?

$V = 336 \text{ m/s}$

Let $s$ = depth of well

$t_1$ = time taken by the stone into the well

$t_2$ = time taken by the sound to be heard.

Total time $t = (t_1 + t_2) = 6.5 \text{ sec}$,

Now $s = ut + \frac{1}{2}at^2$

$\Rightarrow s = 0 + \frac{1}{2} \times 9.81 \times t^2$

$\Rightarrow t_1 = \sqrt{\frac{2s}{9.81}}$

When the sound travels with uniform velocity

$s = ut_2$  \(\Rightarrow\) $t_2 = \frac{s}{V}$
A rope AB is attached at B to a small block of negligible dimensions and passes over a pulley C so that its free end A hangs 1.5 m above the ground when the block rests on the floor. The end of the rope is moved horizontally in a straight line by a man walking with a uniform velocity \( v = 3 \text{ m/s} \). Plot the velocity-time diagram. 

At time \( t = 17.31 \text{ s} \) m.

A particle starts from rest and moves along a straight line with constant acceleration \( a \). After 1 s it acquires a velocity \( v = 3 \text{ m/s} \). After having travelled a distance \( s = 7.5 \text{ m} \), find the magnitude of acceleration.
Principles of Dynamics:

Newton's law of motion:
First law: Everybody continues in its state of rest or of uniform motion in a straight line, except in so far as it may be compelled by force to change that state.

Second Law:

The acceleration of a given particle is proportional to the force applied to it and takes place in the direction of the straight line in which the force acts.

Third law: To every action there is always an equal and contrary reaction or the mutual actions of any two bodies are always equal and oppositely directed.

General Equation of Motion of a Particle:

\[ F = ma \]

Differential equation of Rectilinear motion:

Differential form of equation for rectilinear motion can be expressed as

\[ \frac{d^2x}{dt^2} = a \]

where \( a \) = acceleration

\( x \) = Resultant acting force.

Example:

For the engine shown, if \( W = 950 \text{ N} \), crank radius \( r = 250 \text{ mm} \) and uniform speed of rotation \( n = 120 \text{ rpm} \), determine the magnitude of resultant force acting in piston \( C \) at extreme position and at the middle position.
piston has a simple harmonic motion represented by the displacement-time equation

\[ x = r \cos \omega t \quad (c.1) \]

\[ \omega = \frac{\text{displacement}}{\text{time}} = \frac{\text{amplitude}}{\text{period}} \]

\[ \dot{x} = -r \omega \sin \omega t \]

\[ \ddot{x} = -r \omega^2 \cos \omega t \quad (c.2) \]

Differential equation of motion

\[ \frac{W}{8} \ddot{x} = x \]

\[ \dot{y} = -\frac{W}{8} r \omega^2 \cos \omega t = x \]

\[ y = x = -\frac{450}{9.81} \times 0.25 \times (4\pi)^2 \cos (4\pi t) \]

For extreme position

\[ \cos \omega t = -1 \]

\[ x = 1810 + \ell \]

For middle position \( \cos \omega t = 0 \),

so resultant force \( = 0 \).

A balloon of mass \( m \) is falling vertically down with constant acceleration \( a \). What amount of ballast \( m_b \) must be thrown out in order to give the balloon an equal upward acceleration \( \ddot{y} \) = buoyant force.

\[ P = \text{buoyant force} \]

(i) Considering 1st case when balloon is falling,

\[ \frac{W}{8} a = W - P - c.1 \]

(ii) \[ \frac{W - m_b}{8} \]

Equation (i) + Equation (ii):

\[ \frac{g}{8} \cdot \ddot{a} = \frac{m_b}{8} \]

\[ \ddot{m} = \frac{2W}{m_b} \]

\[ \ddot{a} = \frac{2W + \dddot{m}}{m_b} \]

\[ \dddot{m} = 2W \]
\[
\frac{W - (W - R)}{8} = -T(W - R)
\]

\[
\frac{Wa + (W - R) a}{8} = \frac{W - \frac{a}{2} + \frac{a}{2} - (W - R)}{8}
\]

\[
Wa + Wa - Ra = \frac{8}{8}
\]

\[
2 Wa = 8g + Ra
\]

\[
\alpha = \frac{2W/a}{(8g + Ra)}
\]

---

A 60 kg person is supported in a vertical plane by a string and pulleys arranged as shown in Fig. If the tree and a boy at the string is pulled vertically downward with constant acceleration \(a = 18 \text{ m/s}^2\), find tension \(T\) in the string.

Differential equation of motion for the system is

\[
2s - W = \frac{W}{8} \times \frac{a}{2}
\]

\[
2 s = \frac{W}{25} + \frac{Wa}{25}
\]

\[
2 = \frac{W}{25} \left( 2 + \frac{a}{25} \right)
\]

\[
2 = W \left( 1 + \frac{a}{25} \right)
\]

\[
W = 2 \left( 1 + \frac{a}{25} \right)
\]

\[
2 = 4750 \left( 1 + \frac{18}{2 \times 9.81} \right) = 4265.28 \text{ N}.
\]
\[
\frac{W_a}{g} = (W - P) \\
\frac{(W - P)a}{g} = (W - P) \\
W_a + (W - P) \left( \frac{a}{g} \right) = W - P + a - (2W - P) = \Delta \\
\frac{W_a + W_a - ra}{g} = \Delta \\
2W_a = k \Delta + ra \\
\Delta = \frac{2W_a}{1 + k} \\
\text{A weight } W = 450 \text{ N is supported in a vertical plane by string and pulleys arranged as shown in Fig. If the free end of \( \Delta \) \( \text{the string is pulled vertically downward with constant acceleration} \)} \quad a = 18 \text{ m/s}^2 \text{ find tension } S \text{ in the string.} \\
\text{Differential equation of motion for the system is} \\
2s - W = \frac{W}{g} \times \frac{a}{2} \\
2s = W + \frac{W_a}{25} \\
2 = \frac{W}{25} \left( 2 + \frac{a}{25} \right) \\
= W \left( 1 + \frac{a}{25} \right) \\
2 = \frac{W}{25} \left( 1 + k \right) \\
2 = \frac{450}{25} \left( 1 + \frac{18}{2 \times 9.81} \right) = 4266.28 \text{ N.}
An elevator of mass \( m = 4450 \text{ N} \) starts to move upward direction with a constant acceleration and acquired a velocity \( v = 18 \text{ m/s} \). After travelling a distance \( x = 1.8 \text{ m} \), find the tension force \( F \) in the cable during its motion. \( v = 18 \text{ m/s} \).

\[
\begin{align*}
W &= 4450 \text{ N} \\
v &= 18 \text{ m/s} \\
\text{initial velocity} \ u &= 0 \\
\text{distance travelled} \ x &= 1.8 \text{ m}.
\end{align*}
\]

\[
F - W = \frac{W}{5} \cdot a
\]

\[
\Rightarrow a = \frac{W}{5} = \frac{4450}{5} \text{ m/s}^2 = 900 \text{ m/s}^2
\]

Now applying the equation of kinematic

\[
v^2 - u^2 = 2ax
\]

\[
\Rightarrow 18^2 - 0^2 = 2 \times 900 \times 1.8
\]

\[
a = \frac{18^2}{2 \times 900} = \frac{324}{1800} = 0.18 \text{ m/s}^2
\]

Substituting the value of \( a \) in \( \text{eq. (1)} \)

\[
F = 4450 \left( 1 + \frac{90}{9.81} \right) = 45275.7 \text{ N}
\]

A train weighing 1870 N without the locomotive starts to move with a constant acceleration along a straight track and in first 60 s acquires a velocity of 56 Km/h. Determine the tension \( F \) in the chord bar both locomotive and train if the air resistance is 0.005 times the wt. of the train.

\[
\begin{align*}
F &= 0.005W \\
W &= 1870 \text{ N} \\
V &= 56 \text{ Km/h} = 15.56 \text{ m/s}.
\end{align*}
\]
\[ s - f = \frac{W}{g} \]
\[ \Rightarrow s = 0.005W + \frac{Wq}{g} \]

From Eq. of Kinematics,
\[ v = u + at \]
\[ \Rightarrow a = \frac{15.56 - 0}{60} = 0.26 \text{ m/sec}^2 \]

Substituting the value of \( a \) in Eq. (1)
\[ s = W \left( 0.005 + \frac{a}{g} \right) \]
\[ = 1.57 \left( 0.005 + \frac{0.26}{9.81} \right) = 15.89 \text{ kN}. \]

A whre, \( W \) is attached to the end of a small flexible rope of dia, \( d = 6.25 \text{ mm} \), and is raised vertically by winding the rope on a reel. If the reel is turned uniformly at a rate of \( 2 \text{ r.p.m.} \), what will be the tension in rope?

Dia of rope \( d = 6.25 \text{ mm} = 0.00625 \text{ m} \),
No. of revolutions \( N = 2 \text{ r.p.m.} \).
1st \( r = \) initial radius of reel,
\( t = \) time taken for \( N \) revolutions,
Net radius after \( t \) sec:
\[ R = [r + (Nt + d)] \]

Now assume velocity \( v = R \omega \)
\[ w = \frac{2\pi}{N} \]
\[ v = (r + Nt + d) \frac{2\pi}{N} \]

Acceleration of rope \( a = \frac{dv}{dt} \)
\[ a = \frac{d}{dt} \left[ \frac{2\pi}{N}n + \alpha n N^2 t + d \right] = \frac{\pi}{N} N^2 d \]
\[ s - w = \frac{W}{g} \cdot q \]
\[ \Rightarrow s = W + \frac{Wq}{g} = W \left( 1 + \frac{q}{g} \right) \]
A mine case of \( \mathbf{W} = 8.9 \text{ KN} \) starts from rest and moves downward with constant acceleration travelling a distance \( s = 20 \text{ m} \) in 10 sec. Find the tensile force in the cable.

\begin{align*}
\text{Work done } \mathbf{W} &= 8.9 \text{ KN}.
\text{Initial velocity } \mathbf{v} &= 0.
\text{Distance travelled } s &= 30 \text{ m}.
\text{Time } t &= 10 \text{ sec}.
\end{align*}

\begin{align*}
s &= u t + \frac{1}{2} a t^2 \\
30 &= \frac{1}{2} a \times 10^2 \\
t &= \frac{60}{10^2} \approx 0.6 \text{ m/s}^2
\end{align*}

Differential equation of rectilinear motion:

\[ W - s = \frac{H}{s} \]

\[ \Rightarrow s = W - \frac{W}{s} a = W \left( 1 - \frac{a}{s} \right) \]

\[ = 8.9 \left( 1 - \frac{0.6}{9.87} \right) \]

\[ \Rightarrow s = 8.35 \text{ KN}. \]
D'Alembert's Principle

Differential equation of motion (rectilinear) can be written as

\[ x - m \ddot{x} = 0 \quad (1) \]

Where \( x \) = Resultant of all applied force in the direction of motion

\( m \) = mass of the particle

The above equation may be treated as equation of dynamic equilibrium. To reproduce this equation, in addition to the real force acting on the particle a fictitious force \( m \ddot{x} \) is required to be considered. This force is equal to the product of mass of the particle and its acceleration and directed opposite direction, and is called the inertia force of the particle.

\[ - \sum m_i \ddot{x} = - \ddot{x} \sum m_i = - \frac{W \ddot{x}}{g} \]

Where \( W \) = total weight of the body

So the equation of dynamic equilibrium can be expressed as:

\[ \sum x_i + \left( - \frac{W \ddot{x}}{g} \right) = 0 \quad (2) \]

**Example 1**

For the example shown considering the motion of pulley as shown by the arrow mark, we have upward acceleration \( \ddot{x} \) for \( W_2 \) and downward acceleration \( \ddot{y} \) for \( W_1 \), corresponding inertia forces and their direction are indicated by dotted line.

By adding inertia forces to the real forces (such as \( W_1, W_2 \) and tension in strings) we obtain, for each particle, a system of forces in equilibrium.

The equilibrium equation for the entire system is not \( S \)

\[ W_2 + m_2 \ddot{x} = W_1 - m_2 \ddot{x} \]

\[ \Rightarrow (m_1 + m_3) \ddot{x} = (W_1 - W_2) \Rightarrow \ddot{x} = \frac{W_1 - W_2}{g} \]
Example 2

A body is moving in upward direction by
a rope.

The equation of dynamic equilibrium considering
the real and inertia force,

\[ s = \frac{W}{s} \alpha = 0, \text{ so tensile force in rope} \]

\[ s = \frac{W (1 + \frac{\alpha}{s})}{s} \]

Find tensions in string during motion of the system
case if \( W_1 = 900 \text{ N}, W_2 = 450 \text{ N} \). Then \( \mu \) both the inclined plane
and block \( W_1 = 0.2 \)

When \( W_1 \) moves down, and in the inclined plane with an acceleration \( \alpha \), then acceleration of
\( W_2 = \frac{\alpha}{2} \).

Considering dynamic equilibrium of \( W_1 \), from D'Alembert's principle

\[ \left( W_1 \sin 45^\circ - \mu W_1 \cos 45^\circ - s \right) = \frac{W_1}{s} \alpha = 0 \]

\[ \Rightarrow \frac{W_1}{s} \alpha = W_1 \sin 45^\circ - \mu W_1 \cos 45^\circ - s \]

\[ \Rightarrow \alpha = \left( 900 \times \frac{1}{19.22} - 0.2 \times 900 \times \frac{1}{19.22} - s \right) \frac{9.81}{900} \]

\[ = \left( 636.4 - 127.28 - s \right) 9.81 \]

\[ \Rightarrow \alpha = \frac{636.4 - 127.28 - 0.0108 s}{c_1} \]

Similarly from for weight \( W_2 \)

\[ 2s - W_2 - \frac{W_2}{s} \alpha = 0 \]

\[ \Rightarrow \frac{W_2}{2s} = \frac{W_2}{s} \left( 1 + \frac{\alpha}{2s} \right) = 2s \]

\[ \Rightarrow \alpha = \frac{4500 \left( 1 + \frac{1}{962} \right)}{2} = 225 + 11.46 \text{ N} \]

Substituting the values \( \sin \) in eq. \( c_1 \)

\[ \alpha = 5.49 \]
\[ a = 6.93676 - 1.387352 - 0.0109 \left( \frac{225}{11} - 4.6 \right) \]
\[ = 6.935 - 5.49408 - 2.4525 - 0.129149 \]
\[ = 3.09608 - 0.129149 \]
\[ \Rightarrow a = 2.75 \text{ m/s}^2 \]

8.2 Two weights \( P \) and \( Q \) are connected by the arrangement shown in the figure. Neglecting friction and inertia of pulleys and cord, find the acceleration \( a \) of \( P \) and \( Q \).

Assume \( T = 178 \text{ N}, \quad \beta = 133.5 \text{ N}. \)

Applying d’Alembert’s principle for \( Q \):
\[ Q - \frac{\beta}{g} - \frac{\beta a}{g} = 0 \]
\[ \Rightarrow \beta = Q \left( 1 - \frac{a}{g} \right) \quad \text{(1)} \]

Applying d’Alembert’s principle to \( P \):
\[ 2T + \frac{\beta a}{g} = 0 \]
\[ \Rightarrow 2T = \frac{\beta a}{g} \left( 1 + \frac{a}{g} \right) \]
\[ \Rightarrow \beta = \frac{2T}{g} \left( 1 + \frac{a}{g} \right) \quad \text{(2)} \]

\[ 133.5 \left( 1 - \frac{a}{9.81} \right) = 89 \left( 1 + \frac{a}{19.62} \right) \]
\[ \Rightarrow 133.5 - 13.69a = 89 + 4.536a \]
\[ \Rightarrow 18.144a = 44.5 \]
\[ \Rightarrow a = 2.95 \text{ m/s}^2 \quad \text{(4ms)} \]

Assuming the car in the figure to have a velocity of 6 m/s, find the shortest distance in which it can come to rest with constant deceleration. Without disturbing the block, data: \( c = 0.6 \text{ m}, \quad h = 0.9 \text{ m}, \quad \mu = 0.5 \).
Two blocks of weights $W_1 = 150 \text{ N}$ and $W_2 = 500 \text{ N}$ are connected by an inextensible string. Find the accelerations of the blocks and tension in the string if $\mu = 0.1$, $\mu_0 = 0.2$, and $\alpha = 15^\circ$.

For Block 1:

$S - \mu N_1 = 0$

$\Rightarrow S = 0.1 \times 150 = 15 \text{ N}$.

For Block 2:

$W_1 = 890 \text{ N}$, $W_2 = 445 \text{ N}$.

$\mu = 0.2$, $\alpha = 94^\circ$.

Find $s$.

Considering equilibrium by $W_1$ and applying the first member principle:

$W_1 \sin 45^\circ - \mu N_1 - S = \frac{W_1}{s} \cdot a = 0$

$\Rightarrow S = \frac{W_1 \sin 45^\circ - \mu N_1}{\frac{W_1}{s} \cdot a}$

$= \frac{890}{\sqrt{2}} - 0.2 \times 890 \times \frac{1}{\sqrt{2}} = \frac{890}{9.4}$

$= 629.32 - 125.865 = 90.729$

$s = 503.955 - 90.729 \quad \text{(c1)}$

Applying the first member principle for $W_2$:

$2S - W_2 = \frac{W_2 \cdot a}{s} = 0$

$\Rightarrow 2S = W_2 \left(1 + \frac{a}{2S}\right)$

$\Rightarrow s = \frac{W_2}{2} \left(1 + \frac{a}{2S}\right) = \frac{445}{2} \left(1 + \frac{1}{11.349}\right) = 223.5 + 11.349$
Equations (1) and (2)

\[ 503.455 - 90.72 = 223.5 + 11.34a \]
\[ 102.5604a = 280.955 \]
\[ a = 2.75 \text{ m/s}^2 \]
\[ \cos \theta = \frac{223.5 + 11.34 \times 2.75}{2} = \frac{253.71}{N} \]

\[ W_A = 44.5 \text{ N} \quad W_B = 89 \text{ N} \]
\[ \alpha = 30^\circ \quad \mu_a = 0.15 \]
\[ \mu_B = 0.3 \]

Find pressure \( P \) between blocks.

\[ W_A \sin 30^\circ - P = 10g \mu_a R_A - \frac{W_A a}{b} = 0 \]
\[ P = W_A \sin 30^\circ - 10 \mu_a R_A - \frac{W_A a}{b} \approx 0 \]
\[ = 44.5 \times \frac{1}{2} - 0.15 \times 44.5 \times \frac{1}{2} \times 30 \]
\[ = \frac{44.5}{9.81} \]
\[ P = 22.25 - 5.78 - 4.53a \quad (a) \]
\[ = 16.47 - 4.53a \quad (a) \]

\[ P + W_B \sin 30^\circ - 14 \mu_B R_B - \frac{W_B a}{b} = 0 \]
\[ P = \frac{-W_B}{2} + 0.3 \times 89 \cos 30^\circ + \frac{89}{9.81} \]
\[ = \frac{-89}{2} + 23.122 + 9.07a \]
\[ = -21.378 + 9.07a \quad (a) \]

\[ 16.47 - 4.53a = -21.378 + 9.07a \]
\[ 13.69 = 37.848 \]
\[ a = 2.78 \text{ m/s}^2 \]
\[ P = 3.87 \text{ N} \]
We have the differential equation of rectilinear motion of a particle
\[ \frac{W}{S} \ddot{x} = x \]

Above equation may be written as
\[ \frac{W}{S} \frac{dx}{dt} = x \]

or
\[ \frac{d}{dt} \left( \frac{W}{S} \dot{x} \right) = x \dot{x} + \frac{W}{S} \dot{x} \]

or
\[ \frac{d}{dt} \left( \frac{W}{S} \dot{x} \right) = x \dot{x} + \frac{W}{S} \dot{x} \] — (1)

In the above equation, we still assume force \( x \) as a function of time represented by a force time diagram.

The right-hand side of eq. (1) is then represented by the area of shaded elemental strip of height \( x \) and width \( \dot{x} dt \). This quantity i.e.
\[ (x \dot{x} + \frac{W}{S} \dot{x}) \] is called impulse of the force \( x \) in time \( \dot{x} dt \). The expression on the left-hand side of the expression
\[ \frac{d}{dt} \left( \frac{W}{S} \dot{x} \right) \] is called momentum of the particle.

So the eq. (1) represents the differential change in momentum of a particle in time \( \dot{x} dt \).

Integrating eqn (1), we have
\[ \frac{W}{S} \dot{x} + C = \int_0^x x \dot{x} \, dt \] — (2)

where \( C \) is a constant of integration.

Now assuming an initial moment, \( x = 0 \), the particle has an initial velocity \( \dot{x}_0 \).

So
\[ C = -\frac{W}{S} \dot{x}_0 \] — (3)

So, equation (2) becomes
\[ \frac{W}{S} \dot{x} - \frac{W}{S} \dot{x}_0 = \int_0^x x \dot{x} \, dt \] — (4)
From equation (24) it is clear that the total change in momentum of a particle during a finite interval of time is equal to the impulse of acting force.

In other words

\[ F \cdot dt = \Delta (mV) \]

where \( m \times V \) = momentum

Required given

A man of wt 712 N stands in a boat so that he is 4.5 m from a pier on the shore. He walks 8.4 m in the boat towards the pier and then stops. How far from the pier will he be at the end of time? Wt of boat is 890 N.

Wt of man \( W_1 = 712 \) N

Wt of boat \( W_2 = 890 \) N

Let \( v_0 \) is the initial velocity of man and \( t \) is time

then \( v_0 t = x \)

\( \Rightarrow \) \( v_0 t = 8.4 \) m

\( \Rightarrow \) \( v_0 = \left( \frac{8.4}{t} \right) \) m/s.

Let \( V \) = velocity of boat towards right

According to conservation of momentum

\[ W_1 v_0 = (W_1 + W_2) V \]

\( \Rightarrow \) \( V = \frac{W_1 v_0}{W_1 + W_2} \)

Distance covered by boat

\[ s = v_0 t = \frac{W_1 v_0}{W_1 + W_2} + \frac{W_1 v_0}{W_1 + W_2} \]

\( \Rightarrow \) \( s = \frac{712 \times 8.4}{712 + 890} \)

\( \Rightarrow \) \( s = 1.067 \) m
A locomotive of 53.4 kN has a velocity of 16 kmph and hooks into a freight car of 86 kN that is at rest on a track. After coupling, at what velocity \( v \) the entire system continues to move. Neglect friction.

Conservation of momentum:

\[
W_1u_1 + W_2u_2 = (W_1 + W_2)v
\]

\[
\Rightarrow v = \frac{534 \times 4.45}{534 + 86} = 3.82 \text{ m/s}
\]

A 667.5 kg man sitting in a 333.75 kg canoe fires a rifle bullet horizontally. The final velocity \( u \) of the canoe is 660 m/s after the shot. The rifle has a muzzle velocity \( 660 \text{ m/s} \) and a ball is 0.28 N.

\[
W_1 \text{ of man } W_1 = 667.5 \text{ N},
\]

\[
W_2 \text{ of canoe } W_2 = 333.75 \text{ N},
\]

\[
W_3 \text{ of bullet } W_3 = 0.28 \text{ N},
\]

Velocity of muzzle \( u = 660 \text{ m/s} \).

According to conservation of momentum:

\[
W_1u_1 = (W_1 + W_2)v
\]

\[
\Rightarrow v = \frac{0.28 \times 660}{667.5 + 333.75} = 0.182 \text{ m/s}
\]
A wood block wt 22.25 N rests on a smooth horizontal surface. A revolver bullet weighing 0.14 N is shot horizontally into the side of block. If the block attains velocity of 2 m/s what is muzzle velocity.

\[ W_1 \text{ of wood block } W_1 = 22.25 \text{ N}, \]
\[ W_2 \text{ of bullet } W_2 = 0.14 \text{ N}. \]

velocity of bullet \( v = 2 \text{ m/s}. \)

velocity at muzzle \( u \)

According to conservation of momentum

\[ \frac{1}{2} u^2 = (W_1 + W_2) \cdot v \]
\[ \Rightarrow u = \sqrt{\frac{(22.25 + 0.14)}{0.14}} \]
\[ = 479.98 \text{ m/s}. \]

Conservation of momentum

When the sum of impulse due to external forces is zero, the momentum of the system remains conserved.

When \[ \sum \int F \cdot dt = 0 \]

\[ \sum \left( \frac{W}{s} \right) \cdot dx = \sum \left( \frac{W}{s} \right) \cdot \not{x}. \]

\[ \text{final momentum = initial momentum}. \]
When a moving particle describes a curved path, it is said to have curvilinear motion.

Consider a particle \( P \) in a plane on a curved path. To define the particle, we need two coordinates \( x \) and \( y \) as the particle moves, these coordinates move.

Change with time and the displacement time equations are

\[ x = f_1(t) \quad y = f_2(t) \]

The motion of the particle can also be expressed as

\[ y = f(x) \quad s = f(t) \]

where \( y = f(x) \) represents the equation of path OA, and \( s = f(t) \) gives displacement \( s \) measured along the path as a function of time.

Considering an infinitesimal time difference from \( t \) to \( t + dt \) during which the particle moves from \( P \) to \( P' \) along its path, then velocity of particle may be expressed as

\[ \overrightarrow{v}_{av} = \frac{\Delta s}{\Delta t} \]

\[ \overrightarrow{v}_{av,x} = \frac{\Delta x}{\Delta t} \]

\[ \overrightarrow{v}_{av,y} = \frac{\Delta y}{\Delta t} \]

(Average velocity along \( x \) and \( y \) coordinates)
It can also be represented as
\[ v_x = \frac{dx}{dt} = \dot{x} \]
\[ v_y = \frac{dy}{dt} = \dot{y} \]
so the total velocity may be represented by
\[ \mathbf{v} = \sqrt{\dot{x}^2 + \dot{y}^2} \]
and \( \cos(\mathbf{v}, \mathbf{x}) = \frac{\dot{x}}{v} \) and \( \cos(\mathbf{v}, \mathbf{y}) = \frac{\dot{y}}{v} \)
where \( (\mathbf{v}, \mathbf{x}) \) and \( (\mathbf{v}, \mathbf{y}) \) denote the angles between the direction of velocity vector \( \mathbf{v} \) and the coordinate axes.

**Acceleration:**

The acceleration particles may be described as
\[ a_x = \frac{d\dot{x}}{dt} = \ddot{x} \]
\[ a_y = \frac{d\dot{y}}{dt} = \ddot{y} \]
It is also known as instantaneous acceleration.

**Total acceleration**
\[ a = \sqrt{\ddot{x}^2 + \ddot{y}^2} \]

Considering particular path for above case:
\[ r = r \cos(\theta) + y = r \sin(\theta) \]
\[ \dot{x}^2 + \dot{y}^2 = r^2 \]
\[ \dot{x} = -rw \sin(\theta) \]
\[ \dot{y} = rw \cos(\theta) \]
\[ a = \sqrt{\ddot{x}^2 + \ddot{y}^2} = \sqrt{(-rw \cos(\theta))^2 + (rw \sin(\theta))^2} \]
\[ a = \sqrt{r^2 \cos^2(\theta) + r^2 \sin^2(\theta)} \]

**D’Alembert’s Principle in Curvilinear Motion**

**Acceleration during circular motion**

\[ V_B = \text{tangential velocity at } A \]
\[ V_A = \text{tangential velocity at } B \]
\[ V_B = V \]

Now \[ dv = v \frac{ds}{r} = v \frac{ds}{r} = \frac{ds}{r} \]

\[ \text{acceleration} = \frac{dv}{dt} = \frac{v^2}{r} \]

So when a body moves with uniform velocity \( v \) along a curved path of radius \( r \), it has a radial inward acceleration of magnitude \( \frac{v^2}{r} \).

Applying D’Alembert’s principle to set equilibrium condition an inertial force of magnitude \( \frac{W}{g} \) \( \frac{v^2}{r} \) must be applied in outward direction, it is known as centripetal force.

**Motion on a level road**

Consider a body is moving with uniform velocity on a curvilinear curve of radius \( r \). Let the road is flat.

Let \( W = \text{wt of the body} \)

And inertia force is given by

\[ \frac{W}{g} = \frac{W}{g} \frac{v^2}{r} \]
Condition for skidding:

Let \( W \) = weight of vehicle
\( R_1, R_2 \) = reactions at wheel
\( F \) = frictional force
\( \frac{W}{g} \) = inertial force

Skidding takes place when the frictional force reaches its limiting value i.e.

\[ F = \mu W \]

Then maximum permissible speed to avoid skidding

\[ v = \sqrt{\frac{\mu g}{2}} \frac{R}{h} \]

The distance between inner and outer wheel is equal to the gauge of railway track and expressed as \( G \),

so

\[ v = \sqrt{\frac{\mu g}{2}} \frac{G}{h} \]

**Desired speed and angle of braking**

\[ \tan \alpha = \frac{v^2}{\mu g} \]

Relation between the angle of braking and designed speed

\[ \tan \alpha = \frac{v^2}{\mu g} \]
(a) Condition for skidding

\[ v = \sqrt{\tan(\alpha + \phi) \times gR} \]

where:
- \( \alpha \) = angle of inclination
- \( \tan \phi = \mu \)
- \( g \) = gravitational acceleration
- \( R \) = radius of curve

Then the vehicle will skid if the velocity is more than this value.

(b) Condition for overturning

Limiting speed from consideration of overturning

\[ v = \sqrt{\frac{8 \times 9.8 + (2be/6)}{2h-e}} \]

A circular ring has a mean radius \( r = 500 \text{ mm} \) and is made of steel for which \( w = 77.12 \text{ kN/m}^2 \) and for which ultimate strength in tension is 413.25 MPa. Find the uniform speed of rotation about its geometrical axis perpendicular to the plane of the ring at which it will burst.
Mean radius $r = 50.3\text{ mm} = 0.503\text{ m}$.

Density of the wheel $\rho = 7.12\text{ kN/m}^3$.

Ultimate strength $T_u = 413.85\times 10^6\text{ kN/m}^2$.

Now, considering an infinitesimal small elementary ring extruded at an angle $\theta$.

Centrifugal force acting

\[ F_c = \frac{dW}{\rho} \cdot \frac{r^2}{\theta} \]

Let $p = \text{ tension on the ring}$

$A = \text{ cross-sectional area of ring}$

$dw = \text{ dt of the element}$

$W = \rho \times \text{ volume}$

$W = \rho \times A \times dr$

$W = \rho A \times r \times d\theta$

Now, centrifugal force

\[ \frac{W}{s} \left( A d\theta \right) \times \frac{r^2}{\theta} = \frac{W}{s} \times A/2 d\theta \times \frac{r^2}{\theta} = \frac{2 \rho A d\theta r^2}{s} \]

Balancing force along the radius $F_c$ and $F_t$

\[ F_c = \frac{2 \rho A d\theta r^2}{s} \]

As $d\theta$ is very small, $A d\theta = d\theta$

Eq. (1) may be written as

\[ 2F_c = \frac{2 \rho A d\theta r^2}{s} \]

\[ F_c = \frac{\rho A r^2 d\theta}{s} \] (2)

Tensile stress on the ring $T_t = \frac{F_c}{A}$

Now, substituting the values

$A = 113.85 \times 10^{-6} = 77.12 \times 10^3 \times 10^{-6}$

$\Rightarrow \theta = 229.45 \text{ m/rad}$

Now

$\theta = \frac{\pi DN}{60}$

$N = 60 \times 229.45 \times 15 \div 11 \times 1\text{ = 4362, 3 rpm}$. 

D1 Alemhert's Principle in Curvilinear Motion

Equation of motion of a particle may be written as

\[ X - m \ddot{x} = 0 \]
\[ Y - m \ddot{y} = 0 \]  \( - (C1) \)

Find the proper super elevation \( \epsilon \) for a 7.2 m highway curve of radius \( r = 600 \text{m} \) in order that a car travelling with a speed of 80 kmph will have no tendency to skid sideways.

\[ b = 7.2 \text{m} \quad r = 600 \text{m} \quad v = 80 \text{kmph} = 22.23 \text{m/s} \]

Resolving along the inclined plane,

\[ W \sin \theta = \frac{W}{s} \cdot \frac{v^2}{r^2} \cdot \cos \alpha \]

\[ \Rightarrow \tan \theta = \frac{v^2}{rg} \]

From the geometry, \( \sin \theta = \frac{r}{b} \), since \( \alpha \) is very small.

Let \( \sin \theta = \frac{r}{b} \)

\[ \frac{v^2}{rg} = \frac{b}{r} \]

\[ \Rightarrow r = \frac{bv^2}{rg} = \frac{7.2 \times 22.23^2}{600 \times 9.81} \]

\[ = 0.604 \text{m} \text{ (m/s)} \]
A racing car travels around a circular track of 300 m radius with a speed of 324 km/h. What angle $\theta$ should the floor of the track make with horizontal in order to safeguard against skidding.

Velocity $v = 324$ km/h $= 106.67$ m/s.

We have angle of braking force $\frac{v^2}{g}$

$\theta = \tan^{-1} \left( \frac{106.67^2}{300 \times 9.81} \right) \approx 75.5^\circ$ (Ans)

---

Two balls of wt $W_a = 44.571$ and $W_b = 56.751$ are connected by an elastic string and supported on a table. When the turntable is at rest, the tension in the string is $T = 222.5$ N and the balls exert this same force on each of the steps A and B. What forces will they exert on the steps when the turntable is rotating uniformly about the vertical axis CD at 60 rpm.?

![Diagram of two balls connected by a string.]

We have,

$W_a = 44.571$ N, $W_b = 56.751$ N

$r = 222.5$ N

$\omega = 60$ rpm

Radius of rotation $r = 1.42025$ m

$\omega = \frac{2 \pi \times 60}{60}$ rad/s
Considering the left-hand side ball:

\[ R_0 + \frac{W_a}{s} \times r_1 \times w^2 = S \]

\[ R_0 = 329.5 - \frac{44.5 \times 0.25 \times (2\pi)^2}{9.81} \]

\[ = 177.72 \text{ N.} \]

Considering the ball on right-hand side:

\[ R_b + \frac{W_b}{s} \times r_2 \times w^2 = S \]

\[ R_b = 329.5 - \frac{66.75 \times 0.25 \times (2\pi)^2}{9.81} \]

\[ = 155.39 \text{ N.} \]
Angular motion:

The rate of change of angular displacement with time is called angular velocity and denoted by \( \omega \).

\[
\omega = \frac{d\theta}{dt} \quad -(1)
\]

Angular acceleration may also be expressed as:

\[
\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \quad -(2)
\]

The rate of change of angular velocity with time is called angular acceleration and denoted by \( \alpha \).

\[
\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \quad -(2)
\]

Relationship between angular motion and linear motion from figure 2:

\[
s = r\theta \quad -(4)
\]

tangential velocity (linear) of the particle:

\[
v = \frac{dr}{dt} = \frac{d\theta}{dt} \quad -(4)
\]

Linear acceleration \( a_l = \frac{dv}{dt} = \frac{d^2\theta}{dt^2} \quad -(5) \)

\[
1D \quad \frac{v^2}{r} = \text{radial acceleration}
\]

Then \( a_n = \frac{v^2}{r} = r\omega^2 \quad -(6) \) where \( a_n \) = radial acceleration.

Uniform angular velocity \( \omega \):

\[
\omega = \frac{2\pi N}{60} \quad \text{rad/sec} \quad -(7)
\]
The step pulley starts from rest and accelerates at 2 rad/s². How much time is required for block A to move 2 m. Find also the velocity of A and B at that time.

When A moves by 2 m, the angular displacement of pulley B is given by

\[ \theta = \omega t \]

\[ \theta = 2 \text{ rad} \]

\[ \omega = 20 \text{ rad/s} \]

\[ a = 2 \text{ rad/s}^2 \quad \text{and} \quad \omega_0 = 0 \]

From kinematic relation

\[ \theta = \omega_0 t + \frac{1}{2} \alpha t^2 \]

\[ \Rightarrow 2 = 0 + \frac{1}{2} \times 2 \times t^2 \]

\[ \Rightarrow t = 4.472 \text{ sec} \]

Velocity of pulley at this time

\[ \omega = \omega_0 + \alpha t \]

\[ = 0 + 2 \times 4.472 \]

\[ = 8.944 \text{ rad/s} \]

Velocity of block A \( v_A = 1 \times 8.944 \]

\[ = 8.944 \text{ m/s} \]

Velocity of block B \( v_B = 0.75 \times 8.944 \]

\[ = 6.708 \text{ m/s} \]

**Kinematics of rigid body for rotation:**

Consider a wheel rotating about its axis in clockwise direction with an acceleration \( \alpha \). Let \( \delta m \) be mass of an element at a distance \( r \) from the axis of rotation. dp is the
resulting force on this element
\[ \delta p = \delta m \times a \quad (a = \text{tangential acceleration}) \]
\[ \text{but} \quad a = r \times \alpha \quad (\alpha = \text{angular acceleration}) \]
\[ \Rightarrow \delta p = \delta m \times \alpha \]

Rotational moment \( \delta M_t = \delta p \times r \)
\[ = \delta m \times r^2 \alpha \]
\[ M_t = \sum \delta M_t = \sum \delta m \times r^2 \alpha \]
\[ = \alpha \sum \delta m r^2 \]
\[ = \alpha L \]
\[ \Rightarrow M_t = \alpha L \quad (L = \text{mass moment of inertia}) \]

Product of mass moment of inertia and angular velocity of rotating body is called angular momentum

So, Angular momentum \( = I \omega \)

Kinetic energy of rotating bodies
\[ K.E = \frac{1}{2} I \omega^2 \]

0.2

A flywheel weighing 50 kg and having radius of gyration 1 m loses its speed from 400 rpm to 280 rpm in 2 min. Calculate
(a) retarding torque, (b) change in KE during the period, (c) change in angular momentum,

we have \( \omega_0 = 400 \text{ rpm} = \frac{400 \times 2\pi}{60} = 41.89 \text{ rad/s} \)
\( \omega = 280 \text{ rpm} = \frac{280 \times 2\pi}{60} = 29.32 \text{ rad/s} \)
\( t = 2 \text{ min} = 120 \text{ sec} \)
\[ \omega = \omega_0 + \alpha t \]
\[ \Rightarrow \alpha = \frac{\omega - \omega_0}{t} = \frac{29.32 - 41.89}{120} \text{ rad/s}^2 \]
\[ = -1.047 \text{ rad/s}^2 \]
\[ W = \text{hy} \Rightarrow 500 \text{N} \]

\[ m = \frac{5000}{9.8} = 509.684 \text{ kg} \]

\[ \text{Radius of gyration } k = 1 \text{ m}, \]

\[ L = mk^2 = 509.684 \times 1 = 509.684 \text{ Nm} \]

(a) Rotating torque

\[ L_a = 509.684 \times 0.1047 = 53.264 \text{ Nm} \]

(b) Change in KE

\[ = \text{initial KE} - \text{final KE} \]

\[ = \frac{1}{2} I w_0^2 - \frac{1}{2} I w^2 \]

\[ = \frac{1}{2} \times 509.684 \left( 41.89^2 - 29.32^2 \right) \]

\[ = 22864.4 \text{ Nm} \]

\[ = 228115.462 \text{ Nm} \]

(c) Change in angular momentum

\[ Iw_0 - lw \]

\[ = 509.684 \left( 41.89 - 29.32 \right) \]

\[ = 64067.298 \text{ Nm} \]

0.3

A cylinder weighing 500 N is welded to a 1 m long uniform bar of 200 N. Determine the acceleration with which the assembly will rotate about point \( \text{point A} \) if released from rest in horizontal position. Determine the reactions at \( \text{point A} \) at this instant.

\[ \begin{align*}
A & \quad B \quad C \\
\text{1 m} & \quad 3.4 \text{m} \\
\end{align*} \]
Let \( \alpha \) = angular acceleration of the assembly

\[ \text{mass moment of inertia of the assembly} \]

\[ I = I_b + M d^2 \]  

\( I \) = mass ML about \( A \):

\[ \frac{1}{2} \times \frac{500}{9.81} \times 0.2^2 + \frac{500}{9.81} \times 1.2^2 = 74.4 \]

\( M \) = mass of cylinder about \( A \):

\[ \frac{1}{2} \times \frac{500}{9.81} \times 0.1^2 + \frac{500}{9.81} \times 1.2^2 \]

\[ = 6.7968 \]

\( M \) + \( I \) = mass ML of system:

\[ = 6.7968 + 74.4 = 81.2097 \]

Rotational moment about \( A \):

\[ M_t = 200 \times 0.5 + 500 \times 1.2 = 700 \text{ N m} \]

\[ M_t = L \alpha \]

\[ \Rightarrow \alpha = \frac{700}{81.2097} \text{ rad/s}^2 \]

Instantaneous acceleration of rod \( AB \) is vertical and \( \alpha = 0.5 \times 8.6197 = 4.31 \text{ m/s}^2 \).

Similarly, instantaneous acceleration of cylinder \( \alpha = 1.2 \times 8.6197 = 10.34 \text{ m/s}^2 \).

Applying D'Alembert's dynamic equilibrium:

\[ R_A = 200 + 500 - \frac{200}{9.81} \times 4.31 - \frac{500}{9.81} \times 10.34 \]

\[ \Rightarrow R_A = 86.92 \text{ N} \]
Definitions and Concepts

Amplitude: Maximum displacement from equilibrium position; the distance from the midpoint of a wave to its crest or trough.

Equilibrium position: The position about which an object in harmonic motion oscillates; the center of vibration.

Frequency: The number of vibrations per unit of time.

Hooke’s law: Law that states that the restoring force applied by a spring is proportional to the displacement of the spring and opposite in direction.

Ideal spring: Any spring that obeys Hooke’s law and does not dissipate energy within the spring.

Mechanical resonance: Condition in which natural oscillation frequency equals frequency of a driving force.

Period: The time for one complete cycle of oscillation.

Periodic motion: Motion that repeats itself at regular intervals of time.

Restoring force: The force acting on an oscillating object which is proportional to the displacement and always points toward the equilibrium position.

Simple harmonic motion: Regular, repeated, friction-free motion in which the restoring force has the mathematical form $F = -kx$.

Simple Harmonic Motion

A pendulum, a mass on a spring, and many other kinds of oscillators exhibit a special kind of oscillatory motion called Simple Harmonic Motion (SHM).

SHM occurs whenever:

1. There is a restoring force proportional to the displacement from equilibrium: $F \propto -x$

2. The potential energy is proportional to the square of the displacement: $PE \propto x^2$

3. The period $T$ or frequency $f = 1 / T$ is independent of the amplitude of the motion.

4. The position $x$, the velocity $v$, and the acceleration $a$ are all sinusoidal in time.
As we will see, any one of these four properties guarantees the other three. If one of these 4 things is true, then the oscillator is a simple harmonic oscillator and all 4 things must be true.

Not every kind of oscillation is SHM. For instance, a perfectly elastic ball bouncing up and down on a floor: the ball's position (height) is oscillating up and down, but none of the 4 conditions above is satisfied, so this is not an example of SHM.

A mass on a spring is the simplest kind of Simple Harmonic Oscillator.

Hooke's Law: \( F_{\text{spring}} = -k x \)

\((-\) sign because direction of \( F_{\text{spring}} \) is opposite to the direction of displacement vector \( x \) (\textbf{bold} font indicates vector)

\( k = \) spring constant = stiffness,

units \([k] = \text{N} / \text{m}\)

Big \( k = \) stiff spring

Definition: \textit{amplitude} \( A = |x_{\text{max}}| = |x_{\text{min}}| \).

Mass oscillates between extreme positions \( x = +A \) and \( x = -A \)

Notice that Hooke's Law \((F = -kx)\) is condition i: restoring force proportional to the displacement from equilibrium. We showed previously (Work and Energy Chapter) that for a spring obeying Hooke's Law, the potential energy is \( U = (1/2)kx^2 \), which is condition ii. Also, in the chapter on Conservation of Energy, we showed that \( F = -dU/dx \), from which it follows that condition ii implies condition i. Thus, Hooke's Law and quadratic PE \((U \propto x^2)\) are equivalent.

We now show that Hooke's Law guarantees conditions iii (period independent of amplitude) and iv (sinusoidal motion).

We begin by deriving the \textit{differential equation} for SHM. A differential equation is simply an equation containing a derivative. Since the motion is 1D, we can drop the vector arrows and use sign to indicate direction.

\[
F_{\text{net}} = ma \quad \text{and} \quad F_{\text{net}} = -k x \quad \Rightarrow \quad ma = -k x
\]

\[
a = \frac{d^2x}{dt^2} \quad \Rightarrow \quad \frac{d^2x}{dt^2} = -\frac{k}{m} x
\]

The constants \( k \) and \( m \) and both positive, so the \( k/m \) is always positive, always. For notational convenience, we write \( k / m = \omega^2 \). (The square on the \( \omega \) reminds us that \( \omega^2 \) is always positive.) The differential equation becomes
This is the *differential equation* for SHM. We seek a solution \( x = x(t) \) to this equation, a function \( x = x(t) \) whose second time derivative is the function \( x(t) \) multiplied by a negative constant \((-\omega^2 = -k/m)\). The way you solve differential equations is the same way you solve integrals: you guess the solution and then check that the solution works.

Based on observation, sinusoidal solution: 

\[
x(t) = A \cos(\omega t + \varphi),
\]

where \( A, \varphi \) are any constants and (as we'll show) \( \omega = \sqrt{\frac{k}{m}} \).

\( A = \text{amplitude: } x \text{ oscillates between } +A \text{ and } -A \)

\( \varphi = \text{phase constant (more on this later)} \)

Danger: \( \omega t \) and \( \varphi \) have units of radians (not degrees). So set your calculators to radians when using this formula.

Just as with circular motion, the angular frequency \( \omega \) for SHM is related to the period by

\[
\omega = 2 \pi f = \frac{2 \pi}{T}, \quad T = \text{period}.
\]

(What does SHM have to do with circular motion? We'll see later.)

Let's check that \( x(t) = A \cos(\omega t + \varphi) \) is a solution of the SHM equation.

Taking the first derivative \( \frac{dx}{dt} \), we get

\[
v(t) = \frac{dx}{dt} = -A \sin(\omega t + \varphi).
\]

Here, we've used the Chain Rule:

\[
\frac{d}{dt} \cos(\omega t + \varphi) = \frac{d}{d(\theta)} \cos(\theta) \frac{d\theta}{dt}, \quad (\theta = \omega t + \varphi)
\]

\[
= -\sin \theta \cdot \omega = -\omega \sin(\omega t + \varphi)
\]

Taking a second derivative, we get

\[
a(t) = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{d}{dt}( -A \sin(\omega t + \varphi)) = -A \omega^2 \cos(\omega t + \varphi)
\]

\[
\frac{d^2x}{dt^2} = -\omega^2 [A \cos(\omega t + \varphi)]
\]

\[
\frac{d^2x}{dt^2} = -\omega^2 x
\]

This is the SHM equation, with \( \omega^2 = \frac{k}{m} \), \( \omega = \sqrt{\frac{k}{m}} \)

We have shown that our assumed solution is indeed a solution of the SHM equation. (I leave to the mathematicians to show that this solution is unique. Physicists seldom worry about that kind of thing, since we know that nature usually provides only one solution for physical systems, such as masses on springs.)

We have also shown condition iv: \( x, v, \) and \( a \) are all sinusoidal functions of time:
\[ x(t) = A \cos(\omega t + \phi) \]
\[ v(t) = -A \omega \sin(\omega t + \phi) \]
\[ a(t) = -A \omega^2 \cos(\omega t + \phi) \]

The period \( T \) is given by \( \omega = \frac{\sqrt{k}}{m} = \frac{2\pi}{T} \Rightarrow T = 2\pi \sqrt{\frac{m}{k}} \). We see that \( T \) does not depend on the amplitude \( A \) (condition iii).

Let's first try to make sense of \( \omega = \sqrt{k/m} \): big \( \omega \) means small \( T \) which means rapid oscillations. According to the formula, we get a big \( \omega \) when \( k \) is big and \( m \) is small. This makes sense: a big \( k \) (stiff spring) and a small mass \( m \) will indeed produce very rapid oscillations and a big \( \omega \).

**A closer look at** \( x(t) = A \cos(\omega t + \phi) \)

Let's review the sine and cosine functions and their relation to the *unit* circle. We often define the sine and cosine functions this way:

\[
\cos \theta = \frac{\text{adj}}{\text{hyp}} \quad \sin \theta = \frac{\text{opp}}{\text{hyp}}
\]

This way of defining sine and cosine is correct but incomplete. It is hard to see from this definition how to get the sine or cosine of an angle greater than 90°.

A more complete way of defining sine and cosine, a way that gives the value of the sine and cosine for *any* angle, is this: Draw a unit circle (a circle of radius \( r = 1 \)) centered on the origin of the x-y axes as shown:

Define sine and cosine as

\[
\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{1} = x \\
\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{1} = y
\]

This way of defining sin and cos allows us to compute the sin or cos of *any* angle at all.

For instance, suppose the angle is \( \theta = 210^\circ \). like this:

The point on the unit circle is in the third quadrant, where both \( x \) and \( y \) are negative. So both \( \cos \theta = x \) and \( \sin \theta = y \) are negative.

For any angle \( \theta \), even angles bigger than 360° (more than once around the circle), we can always compute sin and cos. When we plot sin and cos vs angle \( \theta \), we get functions that oscillate between +1 and −1 like so:
We will almost always measure angle \( \theta \) in radians. Once around the circle is \( 2\pi \) radians, so sine and cosine functions are periodic and repeat every time \( \theta \) increases by \( 2\pi \) rad. The sine and cosine functions have exactly the same shape, except that sin is shifted to the right compared to cos by \( \Delta \theta = \pi/2 \). Both these functions are called \textit{sinusoidal} functions.

The function \( \cos(\theta + \varphi) \) can be made to be anything in between \( \cos(\theta) \) and \( \sin(\theta) \) by adjusting the size of the \textit{phase} \( \varphi \) between 0 and \( -2\pi \).

\[
\cos \theta, \ (\varphi = 0) \rightarrow \sin \theta = \cos \left( \theta - \frac{\pi}{2} \right), \quad (\varphi = -\pi/2)
\]

The function \( \cos(\omega t + \varphi) \) oscillates between +1 and −1, so the function \( A\cos(\omega t + \varphi) \) oscillates between +A and −A.

Why \( \omega = \frac{2\pi}{T} \)? The function \( f(\theta) = \cos \theta \) is periodic with period \( \Delta \theta = 2\pi \). Since \( \theta = \omega t + \varphi \), and \( \varphi \) is some constant, we have \( \Delta \theta = \omega \Delta t \). One complete cycle of the cosine function corresponds to \( \Delta \theta = 2\pi \) and \( \Delta t = T \), (T is the period). So we have \( 2\pi = \omega T \) or \( \omega = \frac{2\pi}{T} \). Here is another way to see it: \( \cos(\omega t) = \cos \left( 2\pi \frac{t}{T} \right) \) is periodic with period \( \Delta t = T \). To see this, notice that when \( t \) increases by \( T \), the fraction \( t/T \) increases by 1 and the fraction \( 2\pi t/T \) increases by \( 2\pi \).
Now back to simple harmonic motion. Instead of a circle of radius 1, we have a circle of radius $A$ (where $A$ is the amplitude of the Simple Harmonic Motion).

**SHM and Conservation of Energy:**

Recall $PE_{\text{elastic}} = (1/2) k x^2 =$ work done to compress or stretch a spring by distance $x$.

If there is no friction, then the total energy $E_{\text{tot}} = KE + PE =$ constant during oscillation. The value of $E_{\text{tot}}$ depends on initial conditions – where the mass is and how fast it is moving initially. But once the mass is set in motion, $E_{\text{tot}}$ stays constant (assuming no dissipation.)

At any position $x$, speed $v$ is such that $\frac{1}{2} m v^2 + \frac{1}{2} k x^2 = E_{\text{tot}}$.

When $|x| = A$, then $v = 0$, and all the energy is $PE$: $KE_0 + \frac{PE}{(1/2)kA^2} = E_{\text{tot}}$

So total energy $E_{\text{tot}} = \frac{1}{2} k A^2$

When $x = 0$, $v = v_{\text{max}}$, and all the energy is KE: $\frac{KE}{(1/2)mv_{\text{max}}^2} + \frac{PE}{0} = E_{\text{tot}}$

So, total energy $E_{\text{tot}} = \frac{1}{2} m v_{\text{max}}^2$.

So, we can relate $v_{\text{max}}$ to amplitude $A$:\n$PE_{\text{max}} = KE_{\text{max}} = E_{\text{tot}} \Rightarrow \frac{1}{2} k A^2 = \frac{1}{2} m v_{\text{max}}^2 \Rightarrow$

$v_{\text{max}} = \sqrt{\frac{k}{m}} A$
**Example Problem:** A mass $m$ on a spring with spring constant $k$ is oscillating with amplitude $A$. Derive a general formula for the speed $v$ of the mass when its position is $x$.

Answer: $v(x) = A \sqrt{\frac{k}{m} \sqrt{1 - \left(\frac{x}{A}\right)^2}}$

Be sure you understand these things:

- **range of motion**
- $|x| = A$
- $v = 0$
- $PE = \text{max}$
- $KE = \text{min}$
- $|F| = \text{max}$
- $|a| = \text{max}$
- $x = 0$
- $|v| = \text{max}$
- $PE = \text{min}$
- $KE = \text{max}$
- $|F| = 0$
- $|a| = 0$
Pendulum Motion

A simple pendulum consists of a small mass \( m \) suspended at the end of a massless string of length \( L \). A pendulum executes SHM, if the amplitude is not too large.

\[ \theta = \frac{x}{L} \text{ (rads)} \]

Forces on mass:

\[ F_T = \text{tension} \]

\[ mg \sin \theta \]

\[ mg \cos \theta \]

The restoring force is the component of the force along the direction of motion:

restoring force \( = -mg \sin \theta \approx -mg \theta = -mg \frac{x}{L} \)

Claim: \( \sin \theta \approx \theta \) (rads) when \( \theta \) is small.

\[ \sin \theta = \frac{h}{L} \]

If \( \theta \) small, then \( h \approx s \), and \( L \approx R \), so \( \sin \theta = \theta \).

Try it on your calculator: \( \theta = 5^\circ = 0.087266.. \text{rad}, \sin \theta = 0.087156.. \)

\[ F_{\text{restore}} = -\left( \frac{mg}{L} \right)x \] is exactly like Hooke's Law \( F_{\text{restore}} = -kx \), except we have replaced the constant \( k \) with another constant (\( mg / L \)). The math is exactly the same as with a mass on a spring; all results are the same, except we replace \( k \) with (\( mg / L \)).

\[ T_{\text{spring}} = 2\pi \sqrt{\frac{m}{k}} \Rightarrow T_{\text{pend}} = 2\pi \sqrt{\frac{m}{mg/L}} = 2\pi \sqrt{\frac{L}{g}} \]

Notice that the period is independent of the amplitude; the period depends only on length \( L \) and acceleration of gravity. (But this is true only if \( \theta \) is not too large.)
SHM and circular motion

There is an exact analogy between SHM and circular motion. Consider a particle moving with constant speed $v$ around the rim of a circle of radius $A$.

The x-component of the position of the particle has exactly the same mathematical form as the motion of a mass on a spring executing SHM with amplitude $A$.

Angular velocity $\omega = \frac{d\theta}{dt} = \text{const} \Rightarrow \theta = \omega t$ so

This same formula also describes the sinusoidal motion of a mass on a spring.

That the same formula applies for two different situations (mass on a spring & circular motion) is no accident. The two situations have the same solution because they both obey the same equation. As Feynman said, "The same equations have the same solutions". The equation of SHM is $\frac{d^2x}{dt^2} = -\omega^2 x$. We now show that a particle in circular motion obeys this same SHM equation.

Recall that for circular motion with angular speed $\omega$, the acceleration of a the particle is toward the center and has magnitude $|\vec{a}| = \frac{v^2}{R}$. Since $v = \omega R$, we can rewrite this as $|\vec{a}| = \frac{(\omega R)^2}{R} = \omega^2 R$

Let's set the origin at the position vector $\vec{R}$ is that the acceleration direction opposite the component of this $a_x = -\omega^2 R_x$. If we $\frac{d^2x}{dt^2} = -\omega^2 x$, equation Done.

Example

A mass of 0.5 kg oscillates on the end of a spring on a horizontal surface with negligible friction according to the equation $x = A \cos(\omega t)$. The graph of $F$ vs. $x$ for this motion is shown below.
The last data point corresponds to the maximum displacement of the mass. Determine the
(a) angular frequency \( \omega \) of the oscillation,
(b) frequency \( f \) of oscillation,
(c) amplitude of oscillation,
(d) displacement from equilibrium position \( (x = 0) \) at a time of 2 s.

Solution:
(a) We know that the spring constant \( k = 50 \text{ N/m} \) from when we looked at this graph earlier. So,
\[ \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{50 \text{ N/m}}{0.5 \text{ kg}}} = 10 \frac{\text{rad}}{\text{s}} \]
(b) \[ f = \frac{\omega}{2\pi} = \frac{10 \text{ rad/s}}{2\pi} = 1.6 \text{ Hz} \]
(c) The amplitude corresponds to the last displacement on the graph, \( A = 1.2 \text{ m} \).
(d) \[ x = A \cos(\omega t) = (1.2 \text{ m}) \cos\left[\left(10 \frac{\text{rad}}{\text{s}}\right)(2 \text{ s})\right] = 0.5 \text{ m} \]

Example
A spring of constant \( k = 100 \text{ N/m} \) hangs at its natural length from a fixed stand. A mass of 3 kg is hung on the end of the spring, and slowly let down until the spring and mass hang at their new equilibrium position.
(a) Find the value of the quantity $x$ in the figure above. The spring is now pulled down an additional distance $x$ and released from rest.
(b) What is the potential energy in the spring at this distance?
(c) What is the speed of the mass as it passes the equilibrium position?
(d) How high above the point of release will the mass rise?
(e) What is the period of oscillation for the mass?

**Solution:**
(a) As it hangs in equilibrium, the upward spring force must be equal and opposite to the downward weight of the block.

\[
F_s = mg \\
kx = mg \\
x = \frac{mg}{k} = \frac{(3\text{kg})(10\text{ m/s}^2)}{100\text{ N/m}} = 0.3
\]

(b) The potential energy in the spring is related to the displacement from equilibrium position by the equation

\[
U = \frac{1}{2}kx^2 = \frac{1}{2}(100\text{ N/m})(0.3\text{ m})^2 = 4.5\text{ J}
\]
(c) Since energy is conserved during the oscillation of the mass, the kinetic energy of the mass as it passes through the equilibrium position is equal to the potential energy at the amplitude. Thus,

\[
K = U = \frac{1}{2}mv^2 \\
v = \sqrt{\frac{2U}{m}} = \sqrt{\frac{2(4.5\text{ J})}{3\text{ kg}}} = 1.7\text{ m/s}
\]
(d) Since the amplitude of the oscillation is 0.3 m, it will rise to 0.3 m above the equilibrium position.
(e) \[T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{3\text{ kg}}{100\text{ N/m}}} = 1.1\text{ s}\]

**Example**
A pendulum of mass 0.4 kg and length 0.6 m is pulled back and released from and angle of 10° to the vertical.

(a) What is the potential energy of the mass at the instant it is released. Choose potential energy to be zero at the bottom of the swing.
(b) What is the speed of the mass as it passes its lowest point?
(c) What is the acceleration due to gravity on this planet?

**Solution**
(a) First we must find the height above the lowest point in the swing at the instant the pendulum is released.

Recall from chapter 1 of this study guide that \[h = L - L\cos\theta\].
Then \[U = mg(L - L\cos\theta)\]
\[U = (0.4\text{ kg})(10\text{ m/s}^2)(0.6\text{ m} - 0.6\text{ m}\cos10^\circ) = 0.6\text{ m}\]
(b) Conservation of energy:
\[ U_{\text{max}} = K_{\text{max}} = \frac{1}{2}mv^2 \]

\[ v = \sqrt{\frac{2U}{m}} = \sqrt{\frac{2(0.4J)}{0.4kg}} = 1.4 \, m/s \]

\[ T = 2\pi \sqrt{\frac{L}{g}} \]

\[ g = \frac{4\pi^2L}{T^2} = \frac{4\pi^2(0.6m)}{(1.0s)^2} = 23.7 \, \frac{m}{s^2} \]

**COMPOUND PENDULUM**

**AIM:**
The aim of this experiment is to measure g using a compound pendulum.

**YOU WILL NEED:**

**WHAT TO DO:**
First put the knife edge through the hole in the metre rule nearest end A, and record the time for 10 oscillations. Hence work out the time for one oscillation (T).
Repeat this for each hole in the ruler for a series of different distances (d) from end A.

**ANALYSIS AND CALCULATIONS:**
Plot a graph of T against d.
From the graph record a series of values of the simple equivalent pendulum (L).
Calculate the value of g from the graph or from the formula:
\[ T^2 = 4\pi^2L/g \]

**Torsion Pendulum:**

1. **Introduction**
Torsion is a type of stress, which is easier to explain for a uniform wire or a rod when one end of the wire is fixed, and the other end is twisted about the axis of the wire by an external force. The external force causes deformation of the wire and appearance of counterforce in the material. If this end is released, the internal torsion force acts to restore the initial shape and size of the wire. This behavior is similar to the one of the released end of a linear spring with a mass attached.

Attaching a mass to the twisting end of the wire, one can produce a torsion pendulum with circular oscillation of the mass in the plane perpendicular to the axis of the wire.

To derive equations of rotational motion of the torsion pendulum, it would be useful to recall a resemblance of quantities in linear and rotational motion. We know that if initially a mass is motionless, its linear motion is caused by force \( F \); correspondingly, if an extended body does not rotate initially, its rotation is caused by torque \( r \). The measure of inertia in linear motion is mass, \( m \), while the measure of inertia in rotational motion is the moment of inertia about an axis of rotation, \( I \). For linear and angular displacement in a one-dimensional problem, we use either \( x \) or \( \theta \). Thus, the two equations of motion are:
\[ F_x = ma, \text{and} \quad \tau = I\alpha \quad (1) \]

where \( a\) and \( \alpha \) are the linear and the angular acceleration.

If the linear motion is caused by elastic, or spring, force, the Hooke's law gives \( F_x = -kx \), where \( k \) is the spring constant. If the rotation is caused by torsion, the Hooke's law must result in
\[ \tau = -\kappa \theta \]

where \( \kappa \) is the torsion constant, or torsional stiffness, that depends on properties of the wire. It is essentially a measure of the amount of torque required to rotate the free end of the wire 1 radian.

Your answer to the Preparatory Question 2 gives the following relationship between the moment of inertia \( I \) of an oscillating object and the period of oscillation \( T \) as:

This relationship is true for oscillation where damping is negligible and can be ignored. Otherwise the relationship between \( I \) and \( \kappa \) is given by
\[ I = \frac{\kappa}{\omega_0^2} \quad (3^*) \]

where \( \omega_0 \) can be found from
\[ \omega = \sqrt{\omega_0^2 - \left(\frac{c}{2I}\right)^2} \quad (3^{**}) \]

\[ \omega = \frac{2\pi}{T} = 2\pi f; f \text{ is the frequency of damped oscillation; and } c \text{ is the damping coefficient.} \]

The relationship between the torsion constant \( \kappa \) and the diameter of the wire \( d \) is given in [3] (check your answer to the Preparatory Question 1) as
\[ \kappa = \frac{\pi G d^4}{32l} \quad (4) \]

where \( l \) is the length of the wire and \( G \) is the shear modulus for the material of the wire.

As any mechanical motion, the torsional oscillation is damped by resistive force originating from excitation of thermal modes of oscillation of atoms inside the crystal lattice of the wire and air resistance to the motion of the oscillating object. We can estimate the torque of the resistive force as being directly proportional to the angular speed of the twisting wire, i.e. the torque \( \tau_R = -c \theta / dt \) (recall the drag force on mass on spring in viscous medium as \( R = -bv \)). Combining Eq.(1), (2) and the expression for \( \tau_R \), we obtain the equation of motion of a torsional pendulum as follows:
\[ I \frac{d^2 \theta}{dt^2} + c \frac{d\theta}{dt} + \kappa \theta = 0 \quad (5) \]

The solution of Eq.(5) is similar to the solution of the equation for damped oscillation of a mass on spring and is given by:
\[ \theta = Ae^{-\alpha t} \cos(\omega t + \phi) \quad (6) \]

where \( \alpha = c/2I \)

and \( \alpha = \beta^1 \) with \( \beta \) being the time constant of the damped oscillation; \( c \) is the damping coefficient; \( \omega \) is the angular frequency of torsional oscillation measured in the experiment; and \( \phi \) can be made zero by releasing the object on the wire at a position of the greatest deviation from equilibrium.
Equation (6) can be used to calculate $c$ (damping coefficient) and $\beta$ (time constant = amount of time to decay $e$ times) with DataStudio interface and software.

Another important formula is $\alpha = \omega_0/2Q$, where $Q$ is the quality factor and $\omega_0^2 = \kappa / I$ (see Eq.3'). The ratio $\zeta = \alpha/\omega_0 = (2Q)^{-1}$

is called the damping ratio.

**Free vibration of One Degree of Freedom Systems**

Free vibration of a system is vibration due to its own internal forces (free of external impressive forces). It is initiated by an initial deviation (an energy input) of the system from its static equilibrium position. Once the initial deviation (a displacement or a velocity or both) is suddenly withdrawn, the strain energy stored in the system forces the system to return to its original, static equilibrium configuration. Due to the inertia of the system, the system will not return to the equilibrium configuration in a straightforward way. Instead it will oscillate about this position — free vibration.

A system experiencing free vibration oscillates at one or more of its natural frequencies, which are properties of its mass and stiffness distribution. If there is no damping (an undamped system), the system vibrates at the (undamped) frequency (frequencies) forever. Otherwise, it vibrates at the (damped) frequency (frequencies) and dies out gradually. When damping is not large, as in most cases in engineering, undamped and damped frequencies are very close. Therefore usually no distinction is made between the two types of frequencies.

The number of natural frequencies of a system equals to the number of its degrees-of-freedom. Normally, the low frequencies are more important.

Damping always exists in materials. This damping is called material damping, which is always positive (dissipating energy). However, air flow, friction and others may ‘present’ negative damping.

**Undamped Free Vibration**

Equation of motion based on the free-body diagram

$$m\ddot{x} + kx = 0$$

\[ \dot{x} + \omega_n^2 x = 0 \]

$$\omega_n = \sqrt{\frac{k}{m}}$$

natural frequency

$$\tau = 2\pi \sqrt{\frac{m}{k}}$$

period

$$x(t) = A\sin\omega_n t + B\cos\omega_n t$$

$A$ and $B$ are determined by the initial conditions.
Vibration of a pendulum
How to establish the equation of motion?
What is its natural frequency?

\[ ml^2 \ddot{\theta} = -mg l \sin \theta \rightarrow l \ddot{\theta} + g \sin \theta = 0 \]

\[ l \ddot{\theta} + g \theta = 0 \quad \rightarrow \quad \omega_n = \sqrt{\frac{g}{l}} \]

Systems with Rotational Degrees-of-Freedom

Equation of Motion

\[ J_0 \ddot{\theta} + K \theta = 0 \]

natural frequency \[ \omega_n = \sqrt{\frac{K}{J_0}} \]

Systems involving rotational degrees-of-freedom are always more difficult to deal with, in particular when translational degrees-of-freedom are also present. Gear care is needed to identify both degrees-of-freedom and construct suitable equations of motion.
Damped Free Vibration (first hurdle in studying vibration)

\[ m\ddot{x} = -kx - cx \quad m\ddot{x} + c\dot{x} + kx = 0 \]

standard equation \[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = 0 \]

damping factor \[ \zeta = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{km}} \]

1. oscillatory motion (under-damped \( \zeta < 1 \))

\[ x(t) = \exp(-\zeta \omega_n t)[C_1 \exp(\sqrt{\zeta^2 - 1}\omega_n t) + C_2 \exp(-\sqrt{\zeta^2 - 1}\omega_n t)] \]

\[ x(t) = \exp(-\zeta \omega_n t)(A\sin\omega_d t + B\cos\omega_d t) = X \exp(-\zeta \omega_n t)\sin(\omega_d t + \phi) \]

\[ x(t) = \exp(-\zeta \omega_n t)[\frac{\dot{x}(0) + \zeta \omega_n x(0)}{\omega_d} \sin\omega_d t + x(0)\cos\omega_d t] \quad \omega_d = \omega_n \sqrt{1 - \zeta^2} \]

2. nonoscillatory motion (over-damped \( \zeta > 1 \))

\[ x(t) = \exp(-\zeta \omega_n t)[A\exp(\sqrt{\zeta^2 - 1}\omega_n t) + B\exp(-\sqrt{\zeta^2 - 1}\omega_n t)] \]
3. critically damped motion \((\zeta = 1)\)

\[ x(t) = (A + Bt)e^{-\omega_n t} \]

4. negative damping of \(\zeta < 0\) as a special case of \(\zeta < 1\):

\[ x(t) = \exp(-\zeta \omega_n t)[C_1 \exp(\sqrt{\zeta^2 - 1} \omega_n t) + C_2 \exp(-\sqrt{\zeta^2 - 1} \omega_n t)] \]

\textbf{Divergent} oscillatory motion (flutter) due to negative damping

\textbf{Determination of Damping}
\[ x(t) = X \exp(-\zeta \omega_n t) \sin(\omega_d t + \varphi) \]

\[
\begin{align*}
2 \exp(-0.05\pi t) \sin(0.9988 \pi t + \varphi) \\
\text{two consecutive peaks:} \\
x_1 &= X \exp(-\zeta \omega_n t_1) \sin(\omega_d t_1 + \varphi) \\
x_2 &= X \exp(-\zeta \omega_n t_2) \sin(\omega_d t_2 + \varphi) = X \exp(-\zeta \omega_n t_2) \sin(\omega_d t_1 + \varphi)
\end{align*}
\]

logarithm decrement
\[ \delta = \ln \frac{x_1}{x_2} = \zeta \omega_n \tau_d \Rightarrow \zeta = \frac{\delta}{\omega_n \tau_d} \]

Example:
The 2\textsuperscript{nd} and 4\textsuperscript{th} peaks of a damped free vibration measured are respectively 0.021 and 0.013. What is damping factor?

Solution:
\[
\frac{x(t_2)}{x(t_4)} = \exp(\zeta \omega_n 2\tau_d) \quad \Rightarrow \quad 2\zeta \omega_n \tau_d = \ln \left( \frac{x(t_2)}{x(t_4)} \right)
\]

\[
2\zeta \omega_n \tau_d = 2\zeta \omega_n \frac{2\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{4\pi \zeta}{\sqrt{1-\zeta^2}} = \ln \left( \frac{x(t_2)}{x(t_4)} \right)
\]

If a small damping is assumed, \(2\zeta \omega_n \tau_d = 4\pi \zeta = \ln \left( \frac{x(t_2)}{x(t_4)} \right)\). This leads to

\[ \zeta = \frac{1}{4\pi} \ln \left( \frac{x(t_2)}{x(t_4)} \right) = 0.0382 = 3.82\%. \]
If such an assumption is not made, then

$$\frac{\zeta}{\sqrt{1 - \zeta^2}} = \frac{1}{4\pi} \ln \left( \frac{x(t_2)}{x(t_4)} \right)$$

and hence

$$\frac{\zeta^2}{1 - \zeta^2} = \left[ \frac{1}{4\pi} \ln \left( \frac{x(t_2)}{x(t_4)} \right) \right]^2.$$  

This leads to

$$\zeta = \frac{1}{\sqrt{1 + \left[ \frac{1}{4\pi} \ln \left( \frac{x(t_2)}{x(t_4)} \right) \right]^2}} = 0.0381 = 3.81\%.$$  

So virtually the same value.

General differential equations

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_1 \frac{dx}{dt} + a_0 = 0$$

first solve the characteristic equation

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_1 \lambda + a_0 = 0$$

If all roots $\lambda_j$ are distinct, then the general solution is

$$x(t) = \sum_{j=1}^{n} b_j \exp(\lambda_j t)$$

where $b_j$ are constants to be determined.

If there are repeated roots, $t^m$ (integer $m > 1$) appears in a solution. These are not interesting cases for mechanical vibration.

$\lambda$ in response to the change of a parameter reveal stability properties.