INSTITUTE OF AERONAUTICAL ENGINEERING
(Autonomous)
Dundigal, Hyderabad -500 043
MECHANICAL ENGINEERING

COURSE LECTURE NOTES

Course Name	**HEAT TRANSFER**
Course Code	AME016
Programme	B.Tech
Semester	VI
Course Coordinator	Dr. Ch. Sandeep, Associate Professor
Course Faculty	Dr. K.Ch. Apparao, Associate Professor
Lecture Numbers	1-355
Topic Covered	All

COURSE OBJECTIVES (COs):

The course should enable the students to:	
I	Understand the basic modes of heat transfer like conduction, convection and radiation with and without phase change in solid liquids and gases.
II	Design and analyze thermal fluidic components in engineering systems to energy mechanisms (in the form of heat transfer) for steady and unsteady state.
III	Conduct experiments in laboratories and analyze the results with theoretical ones to evolve research oriented projects in the field of heat transfer as well as propulsion.
IV	Apply the concepts of heat transfer with convective mode in internal and external flows involved in engineering components and work in real time problems in Industry.

COURSE LEARNING OUTCOMES (CLOs):

Students, who complete the course, will have demonstrated the ability to do the following:

AME016.01	Understand basic concepts of heat transfer modes, Fourier Law and First law of thermodynamics.
AME016.02	Remember the basic laws of energy involved in the heat transfer mechanisms.
AME016.03	Understand the physical system to convert into mathematical model depending upon the mode of Heat Transfer.
AME016.04	Understand the thermal response of engineering systems for application of Heat Transfer mechanism in both steady and unsteady state problems.
AME016.05	Understand heat transfer process and systems by applying conservation of mass and energy into a system.
AME016.06	Understand the steady state condition and mathematically correlate different forms of heat transfer
AME016.07	Analyse finned surfaces, and assess how fins can enhance heat transfer

AME016.08	Remember dimensionless numbers which are used for forced and free convection phenomena.
AME016.09	Understand the applications of Buckingham Pi Theorem in deriving various non dimensional numbers and their applications in heat transfer
AME016.10	Remember and use the methodology presented in tutorial to solve a convective heat transfer problems
AME016.11	Understand the various forms of free and forced convection and the application of the same in day to day problems
AME016.12	Calculate local and global convective heat fluxes using Nusselt's Theory.
AME013.13	Understand the method to evolve hydrodynamic and thermal boundary layers applied mathematically to vertical plates and Tubes
AME016.14	Understand the physical mechanisms of phase change involving pool, nucleate and film boiling processes
AME016.15	Understand Nusselt's theory of condensation for the application in film and dropwise condensation
AME016.16	Correlate the empirical relations in terms of vertical and horizontal cylinders during film condensation
AME016.17	Understand the concepts of black and gray body radiation heat transfer.
AME016.18	Understand the concept of shape factor and evolve a mechanism for conducive radiation shields
AME016.19	Understand the various classifications of heat exchangers based on arrangement and correlate the effects of fouling
AME016.20	Understand the LMTD and NTU methods and apply the same for solving real time problems in heat exchangers

SYLLABUS

UNIT-I	BASIC CONCEPTS	Classes: 10
Modes and mechanisms of heat transfer, basic laws of heat transfer, applications of heat transfer; conduction heat transfer: Fourier rate equation, general three dimensional heat conduction equations in cartesian, cylindrical and spherical coordinates; Simplification and forms of the field equation, steady and unsteady and periodic heat transfer, initial and boundary conditions.		
UNIT -II	**ONE DIMENSIONAL STEADY STATE AND TRANSIENT CONDUCTION HEAT TRANSFER**	**Classes: 14**
One dimensional steady state conduction heat transfer: Homogeneous slabs, hollow cylinders and spheres, overall heat transfer coefficient, electrical analogy, Critical radius of insulation; one dimensional steady state conduction; heat transfer: with variable thermal conductivity and systems with internal heat generation, extended surfaces (Fins) long, short and insulated tips; one dimensional transient heat conduction: Systems with negligible internal resistance, significance of Biot and Fourier numbers, chart solutions of transient conduction systems.		
UNIT -III	**CONVECTIVE HEAT TRANSFER**	**Classes: 11**
Classification of systems based on causation of flow, condition of flow, configuration of flow and medium of flow, dimensional analysis as a tool for experimental investigation, Buckingham Pi Theorem and method, application for developing semi, empirical non-dimensional correlation for convection heat transfer, significance of non dimension numbers, concepts of continuity, momentum and energy equations; Forced convection: external flows: Concepts of hydrodynamic and thermal boundary layer and use of empirical correlations for convective heat transfer, flat plates and cylinders; Internal flows, Concepts about Hydrodynamic and thermal entry lengths, division of internal flows based on this, use of empirical correlations for horizontal pipe flow and annulus flow; free convection: Development of hydrodynamic and thermal boundary layer along a vertical plate, use of empirical relations for vertical plates and pipes		

UNIT -IV	HEAT TRANSFER WITH PHASE CHANGE	Classes: 11
Boiling: Pool boiling- regimes Calculations on Nucleate boiling, Critical heat flux, Film boiling; Condensation: Film wise and drop wise condensation, Nusselt‘s theory of condensation on a vertical plate Film condensation on vertical and horizontal cylinders using empirical correlations; Radiation heat transfer: Emission characteristics, laws of black-body radiation, Irradiation, total and Monochromatic quantities, laws of Planck, Wien, Kirchhoff, Lambert, Stefan and Boltzmann, heat exchange between two black bodies, concepts of shape factor, emissivity, heat exchange between grey bodies, radiation shields, electrical analogy for radiation networks.		
UNIT -V	**HEAT EXCHANGERS**	**Classes: 10**
Classification of heat exchangers, overall heat transfer Coefficient and fouling factor, Concepts of LMTD and NTU methods, Problems using LMTD and NTU methods.		
Text Books:		
1. Yunus A. Cengel, “Heat Transfer A Practical Approach”, Tata McGraw hill Education (P) Ltd, New Delhi, India. 4th Edition, 2012. 2. R. C. Sachdeva, “Fundamentals of Engineering, Heat and Mass Transfer”, New Age, New Delhi, India, 3rd Edition, 2012.		
Reference Books:		
1. Holman, ―Heat Transfer, Tata McGraw-Hill education, 10th Edition, 2011. 2. P. S. Ghoshdastidar, ―Heat Transfer, Oxford University Press, 2nd Edition, 2012. 3. D. S. Kumar, ―Heat and Mass Transfer, S.K. Kataria & sons, 9th Edition 2015.		

UNIT-1
BASIC CONCEPTS

Introduction: - We recall from our knowledge of thermodynamics that heat is a form of energy transfer that takes place from a region of higher temperature to a region of lower temperature solely due to the temperature difference between the two regions. With the knowledge of thermodynamics we can determine the amount of heat transfer for any system undergoing any process from one equilibrium state to another. Thus the thermodynamics knowledge will tell us only how much heat must be transferred to achieve a specified change of state of the system. But in practice we are more interested in knowing the rate of heat transfer (i.e. heat transfer per unit time) rather than the amount. This knowledge of rate of heat transfer is necessary for a design engineer to design all types of heat transfer equipments like boilers, condensers, furnaces, cooling towers, dryers etc. The subject of heat transfer deals with the determination of the rate of heat transfer to or from a heat exchange equipment and also the temperature at any location in the device at any instant of time.

The basic requirement for heat transfer is the presence of a "temperature difference". The temperature difference is the driving force for heat transfer, just as the voltage difference for electric current flow and pressure difference for fluid flow. One of the parameters, on which the rate of heat transfer in a certain direction depends, is the magnitude of the temperature gradient in that direction. The larger the gradient higher will be the rate of heat transfer.

Heat Transfer Mechanisms: - There are three mechanisms by which heat transfer can take place. All the three modes require the existence of temperature difference. The three mechanisms are: (i) conduction, (ii) convection and (iii) radiation

Conduction: - It is the energy transfer that takes place at molecular levels. Conduction is the transfer of energy from the more energetic molecules of a substance to the adjacent less energetic molecules as a result of interaction between the molecules. In the case of liquids and gases conduction is due to collisions and diffusion of the molecules during their random motion. In solids, it is due to the vibrations of the molecules in a lattice and motion of free electrons.

Fourier's Law of Heat Conduction: - The empirical law of conduction based on experimental results is named after the French Physicist Joseph Fourier. The law states that the rate of heat flow by conduction in any medium in any direction **is proportional to the area normal to the direction of heat flow and also proportional to the temperature gradient in that direction**. For example the rate of heat transfer in x- direction can be written according to Fourier's law as

$$Q_x \; \alpha - A \, (dT / dx) \ldots\ldots\ldots\ldots\ldots\ldots(1.1)$$

or

$$Q_x = - k \, A \, (dT / dx) \;\; W \ldots\ldots\ldots\ldots(1.2)$$

In equation (1.2), Q_X is the rate of heat transfer in positive x-direction through area A of the medium normal to x-direction, (dT/dx) is the temperature gradient and k is the constant of proportionality and is a material property called *"thermal conductivity"*. Since heat transfer has to take place in the direction of decreasing temperature, (dT/dx) has to be negative in the direction of heat transfer. Therefore negative sign has to be introduced in equation (1.2) to make Q_X positive in the direction of decreasing temperature, thereby satisfying the second law of thermodynamics.

If equation (1.2) is divided throughout by A we have

$$q_x = (Q_x / A) = -k\,(dT / dx)\,W/m^2 \ldots\ldots\ldots\ldots\ldots(1.3)$$

q_x is called the *heat flux*.

Thermal Conductivity: - The constant of proportionality in the equation of Fourier‟s law of conduction is a material property called the thermal conductivity.The units of thermal conductivity can be obtained from equation (1.2) as follows:

Solving for k from Eq. (1.2) we have $k = -q_x / (dT/dx)$

Therefore units of k = (W/m^2) (m/ K) = W / (m – K) or W / (m – ^{0} C). Thermal conductivity is a measure of a material‟s ability to conduct heat. The thermal conductivities of materials vary over a wide range as shown in Fig. 1.1.

It can be seen from this figure that the thermal conductivities of gases such as air vary by a factor of 10 from those of pure metals such as copper. The kinetic theory of gases predicts and experiments confirm that the thermal conductivity of gases is proportional to the *square root of the absolute temperature*, and inversely proportional to the *square root of the molar mass M.* Hence, the thermal conductivity of gases increases with increase in temperature and decrease with increase in molar mass. It is for these reasons that the thermal conductivity of helium (M=4) is much higher than those of air (M=29) and argon (M=40).For wide range of pressures encountered in practice the thermal conductivity of gases is *independent of pressure*.

The mechanism of heat conduction in liquids is more complicated due to the fact that the molecules are more closely spaced, and they exert a stronger inter-molecular force field. The values of k for liquids usually lie between those for solids and gases. Unlike gases, the thermal conductivity for most liquids decreases with increase in temperature except for water. Like gases the thermal conductivity of liquids decreases with increase in molar mass.

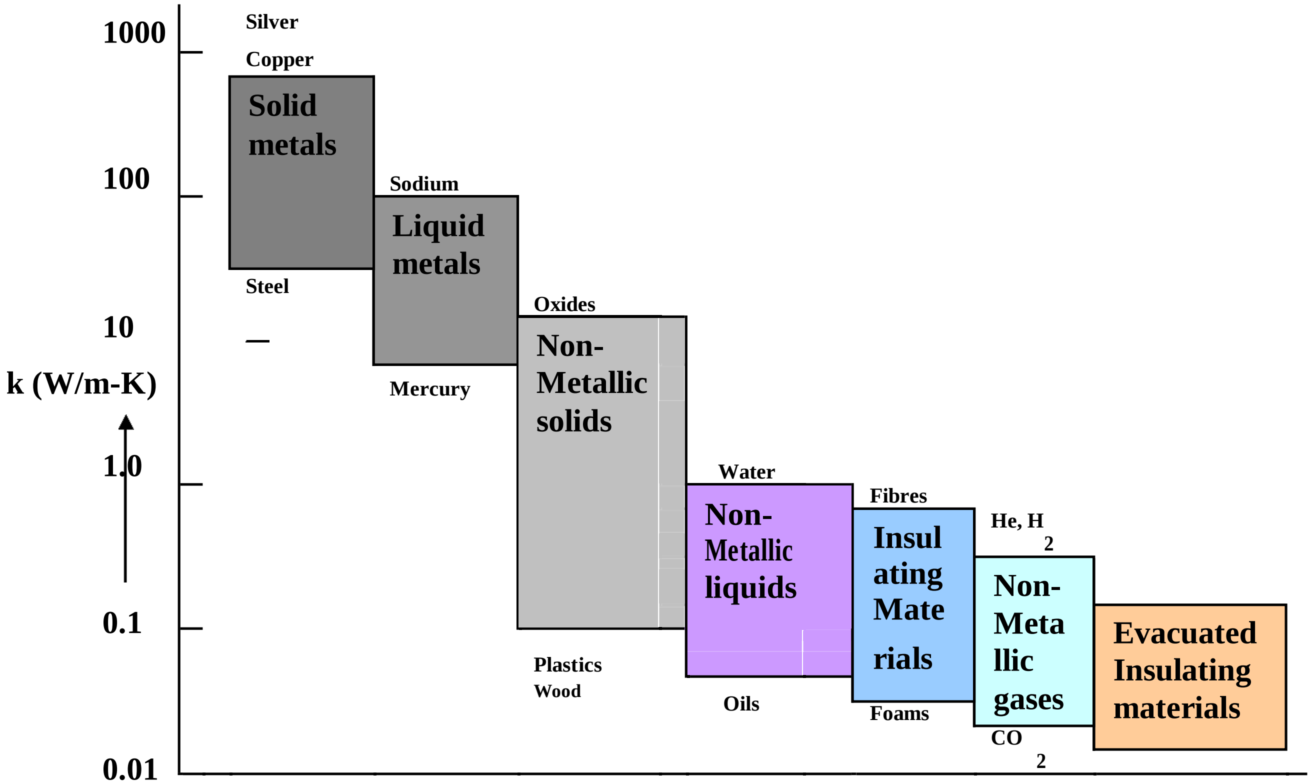


Fig. 1.1: Typical range of thermal conductivities of various materials

In the case of solids heat conduction is due to two effects: the vibration of lattice induced by the vibration of molecules positioned at relatively fixed positions , and energy transported due to the *motion of free electrons*. The relatively high thermal conductivities of pure metals are primarily due to the electronic component. The lattice component of thermal conductivity strongly depends on the way the molecules are arranged. For example, diamond, which is highly ordered crystalline solid, has the highest thermal conductivity at room temperature.

Unlike metals, which are good electrical and heat conductors, *crystalline solids* such as diamond and semiconductors such as silicon are good heat conductors but poor electrical conductors. Hence such materials find widespread use in electronic industry. Despite their high price, diamond heat sinks are used in the cooling of sensitive electronic components because of their excellent thermal conductivity. Silicon oils and gaskets are commonly used in the packaging of electronic components because they provide both good thermal contact and good electrical insulation.

One would expect that metal alloys will have high thermal conductivities, because pure metals have high thermal conductivities. For example one would expect that the value of the thermal conductivity k of a metal alloy made of two metals with thermal conductivities k_1 and k_2 would lie between k_1 and k_2.But this is not the case. In fact k of a metal alloy will be less than that of either metal.

The thermal conductivities of materials vary with temperature. But for some materials the variation is insignificant even for wide temperature range.At temperatures near absolute zero, the thermal conductivities of certain solids are extremely large. For example copper at 20 K will have a thermal conductivity of 20,000 W / (m-K), which is about 50 times the conductivity at room temperature. The temperature dependence of thermal conductivity makes the conduction heat transfer analysis more complex and involved. As a first approximation analysis for solids with variable conductivity is carried out assuming constant thermal conductivity which is an average value of the conductivity for the temperature range of interest.

Thermal Diffusivity:- This is a property which is very helpful in analyzing transient heat conduction problem and is normally denoted by the symbol α . It is defined as follows.

$$\alpha = \frac{\text{Heat conducted}}{\text{Heat Stored per unit volume}} = \frac{k}{\rho C_p} \quad (m^2/s) \;\ldots\ldots(1.4)$$

It can be seen from the definition of thermal diffusivity that the numerator represents the ability of the material to conduct heat across its layers and the denominator represents the ability of the material to store heat per unit volume. Hence we can conclude that larger the value of the thermal diffusivity, faster will be the propagation of heat into the medium. A small value of thermal diffusivity indicates that heat is mostly absorbed by the material and only a small quantity of heat will be conducted across the material.

Convection :- Convection heat transfer is composed of *two mechanisms.* Apart from energy transfer due to *random molecular motion,* energy is also transferred due to *macroscopic motion* of the fluid. Such motion in presence of the temperature gradient contributes to heat transfer. Thus in convection the total heat transfer is due to random motion of the fluid molecules together with the bulk motion of the fluid, the major contribution coming from the latter mechanism. Therefore bulk motion of the fluid is a necessary condition for convection heat transfer to take place in addition to the temperature gradient in the fluid. Depending on the force responsible for the bulk motion of the fluid, convective heat transfer is classified into *"forced convection"* and *"natural or free convection".* In the case of forced convection, the fluid flow is caused by an external agency like a pump or a blower where as in the case of natural or free convection the force responsible for the fluid flow (normally referred to as the buoyancy force) is generated within the fluid itself due to density differences which are caused due to temperature gradient within the flow field. Regardless of the particular nature of convection, the rate equation for convective heat transfer is given by

$$\mathbf{q = h\,\Delta T \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.5)}$$

where q is the heat flux, ΔT is the temperature difference between the bulk fluid and the surface which is in contact with the fluid, and „h" is called the *"convective heat transfer coefficient"* or *"surface film coefficient"*. Eq.(1.5) is generally referred to as the Newton"s law of cooling.If T_s is the surface temperature , T_f is the temperature of the bulk fluid and if $T_s > T_f$, then Eq. (1.5) in the direction of heat transfer can be written as

$$\mathbf{q = h\,[T_s - T_f]} \quad \textbf{(1.6a)}$$

and if $T_s < T_f$, the equation reduces to

$$\mathbf{q = h\,[T_f - T_s]} \quad \textbf{(1.6b)}$$

The heat transfer coefficient *h* depends on (i) the type of flow (i.e. whether the flow is laminar or turbulent), (ii) the geometry of the body and flow passage area, (iii) the thermo-physical properties of the fluid namely the density ρ, viscosity μ, specific heat at constant pressure C_p and the thermal conductivity of the fluid k and (iv) whether the mechanism of convection is forced convection or free convection. The heat transfer coefficient for free convection will be generally lower than that for forced convection as the fluid velocities in free convection are much lower than those in forced convection. The heat transfer coefficients for some typical applications are given in table 1.2.

Table 1.2: Typical values of the convective heat transfer coefficient h

Type of flow	h ,W / (m^2 – K)
Free convection	
Gases	2– 25
Liquids	50 – 1000
Forced Convection	
Gases	25 – 250
Liquids	50 – 20,000
Convection with change of phase	
Boiling or condensation	2500– 100,000

Thermal Radiation:- Thermal radiation is the energy emitted by matter (solid, liquid or gas) by virtue of its temperature. This energy is transported by electromagnetic waves (or alternatively, photons).While the transfer of energy by conduction and convection requires the presence of a material medium, radiation does not require.Infact radiation transfer occurs most effectively in vacuum.

Consider radiation transfer process for the surface shown in Fig.1.2a.Radiation that

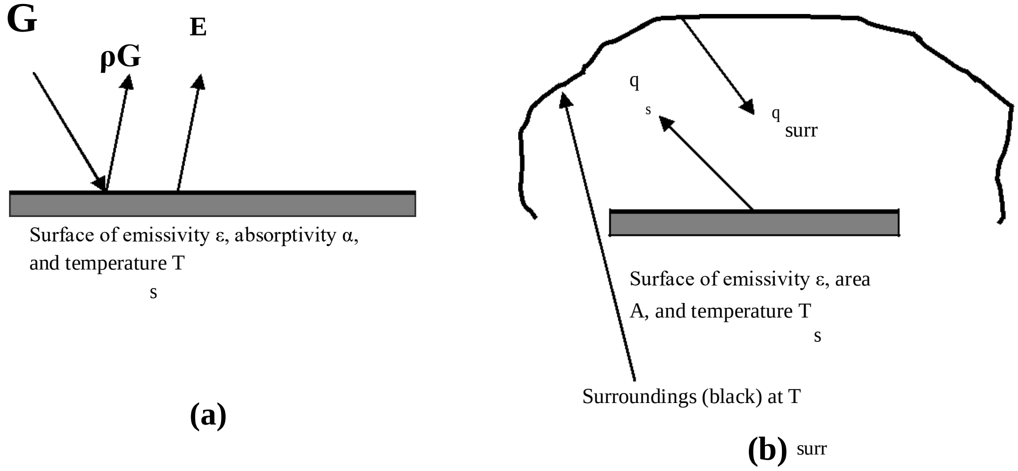


Fig.1.2: Radiation exchange: (a) at a surface and (b) between a surface and large surroundings

is emitted by the surface originates from the thermal energy of matter bounded by the surface, and the rate at which this energy is released per unit area is called as the surface *emissive power E.*An ideal surface is one which emits maximum emissive power and is called *an ideal radiator or a black body.*Stefan-Boltzman‟s law of radiation states that the emissive power of a black body is proportional to the fourth power of the absolute temperature of the body. Therefore if E_b is the emissive power of a black body at temperature T ^{0}K, then

$$E_b \; \alpha \; T^4$$

(or)

$$\mathbf{E_b = \zeta\, T^4} \text{..}$$

ζ is the *Stefan-Boltzman constant* ($\zeta = 5.67 \times 10^{-8}$ W / ($m^2 - K^4$)). For a non black surface the emissive power is given by

$$\mathbf{E = \varepsilon\, \zeta\, T^4} \text{..} \quad \mathbf{(1.8)}$$

where ε is called the emissivity of the surface ($0 \le \varepsilon \le 1$).The emissivity provides a measure of how efficiently a surface emits radiation relative to a black body. The emissivity strongly depends on the surface material and finish.

Radiation may also *incident* on a surface from its surroundings. The rate at which the radiation is incident on a surface per unit area of the surface is calle the *"irradiation"* of the surface and is denoted by G. The fraction of this energy absorbed by the surface is called *"absorptivity"* of the surface and is denoted by the symbol α. The fraction of the incident energy is reflected and is called the *"reflectivity"* of the surface denoted by ρ and the remaining fraction of the incident energy is transmitted through the surface and is called the *"transmissivity"* of the surface denoted by η. It follows from the definitions of α, ρ, and η that

$$\boldsymbol{\alpha + \rho + \eta = 1} \quad \textbf{(1.9)}$$

Therefore the energy absorbed by a surface due to any radiation falling on it is given by

$$\boldsymbol{G_{abs} = \alpha G} \quad \textbf{(1.10)}$$

The absorptivity α of a body is generally different from its emissivity. However in many practical applications, to simplify the analysis α is assumed to be equal to its emissivity ε.

Radiation Exchange:- When two bodies at different temperatures "see" each other, heat is exchanged between them by radiation. If the intervening medium is filled with a substance like air which is transparent to radiation, the radiation emitted from one body travels through the intervening medium without any attenuation and reaches the other body, and vice versa. Then the hot body experiences a net heat loss, and the cold body a net heat gain due to radiation heat exchange between the two. The analysis of radiation heat exchange among surfaces is quite complex which will be discussed in chapter 10. Here we shall consider two simple examples to illustrate the method of calculating the radiation heat exchange between surfaces.

As the first example" let us consider a small opaque plate (for an opaque surface η = 0) of area A, emissivity ε and maintained at a uniform temperature T_s. Let this plate is exposed to a large surroundings of area A_{su} (A_{su} >> A) which is at a uniform temperature T_{sur} as shown in Fig. 1.2b.The space between them contains air which is transparent to thermal radiation.

The radiation energy emitted by the plate is given by

$$Q_{em} = A\, \varepsilon\, \zeta\, T_s^4$$

The large surroundings can be approximated as a black body in relation to the small plate. Then the radiation flux emitted by the surroundings is $\zeta\, T_{sur}^4$ which is also the radiaton flux incident on the plate. Therefore the radiation energy absorbed by the plate due to emission from the surroundings is given by

$$Q_{ab} = A\, \alpha\, \zeta\, T_{sur}^4.$$

The net radiation loss from the plate to the surroundings is therefore given by

$$Q_{rad} = A\, \varepsilon\, \zeta\, T_s^4 - A\, \alpha\, \zeta\, T_{sur}^4.$$

Assuming $\alpha = \varepsilon$ for the plate the above expression for Q_{net} reduces to

$$\mathbf{Q_{rad} = A\, \varepsilon\, \zeta\, [T_s^4 - T_{sur}^4] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1.11)}$$

The above expression can be used to calculate the net radiation heat exchange between a small area and a large surroundings.
As the second example, consider two finite surfaces A_1 and A_2 as shown in Fig. 1.3.

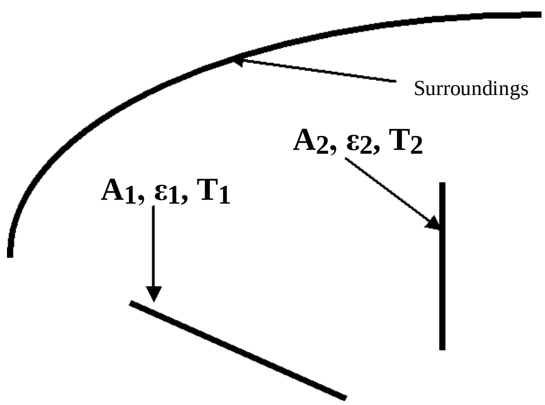


Fig.1.3: Radiation exchange between surfaces A_1 and A_2

The surfaces are maintained at absolute temperatures T_1 and T_2 respectively, and have emissivities ε_1 and ε_2. Part of the radiation leaving A_1 reaches A_2, while the remaining energy is lost to the surroundings. Similar considerations apply for the radiation leaving A_2.If it is assumed that the radiation from the surroundings is negligible when compared to the radiation from the surfaces A_1 and A_2 then we can write the expression for the radiation emitted by A_1 and reaching A_2 as

$$Q_{1\to 2} = F_{1-2}\, A_1 \varepsilon_1 \zeta\, T_1^4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1.12)$$

where F_{1-2} is defined as the fraction of radiation energy emitted by A_1 and reaching A_2. Similarly the radiation energy emitted by A_2 and reaching A_1 is given by

$$Q_{2\to 1} = F_{2-1}\, A_2\, \varepsilon_2\, \zeta\, T_2^4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1.13)$$

where F_{2-1} is the fraction of radiation energy leaving A_2 and reaching A_1. Hence the net radiation energy transfer from A_1 to A_2 is given by

$$Q_{1-2} = Q_{1\to 2} - Q_{2\to 1}$$

$$= [F_{1-2} A_1 \varepsilon_1 \zeta T_1^4] - [F_{2-1} A_2 \varepsilon_2 \zeta T_2^4]$$

F_{1-2} is called the view factor (or geometric shape factor or configuration factor) of A_2 with respect to A_1 and F_{2-1} is the view factor of A_1 with respect to A_2.It will be shown in chapter 10 that the view factor is purely a geometric property which depends on the relative orientations of A_1 and A_2 satisfying the reciprocity relation, $A_1 F_{1-2} = A_2 F_{2-1}$.

Therefore $$\mathbf{Q_{1-2} = A_1F_{1-2}\, \zeta\, [\varepsilon_1 T_1^4 - \varepsilon_2 T_2^4] \ldots\ldots\ldots (1.13)}$$

Radiation Heat Transfer Coefficient:- Under certain restrictive conditions it is possible to simplify the radiation heat transfer calculations by defining a radiation heat transfer coefficient h_r analogous to convective heat transfer coefficient as

$$Q_r = h_r A\, \Delta T$$

For the example of radiation exchange between a surface and the surroundings [Eq. (1. 11)] using the concept of radiation heat transfer coefficient we can write

$$Q_r = h_r A[T_s - T_{sur}] = A\, \varepsilon\, \zeta\, [T_s^4 - T_{sur}^4]$$

Or $$h_r = \frac{\varepsilon\, \zeta\, [T_s^4 - T_{sur}^4]}{[T_s - T_{sur}]} = \frac{\varepsilon\, \zeta\, [T_s^2 + T_{sur}^2][T_s + T_{sur}][T_s - T_{sur}]}{[T_s - T_{sur}]}$$

Or $$\mathbf{h_r = \varepsilon\, \zeta\, [T_s^2 + T_{sur}^2][T_s + T_{sur}] \ldots\ldots\ldots (1.14)}$$

First Law of Thermodynamics (Law of conservation of energy) as applied to Heat Transfer Problems :-

The first law of thermodynamics is an essential tool for solving many heat transfer problems. Hence it is necessary to know the general formulation of the first law of thermodynamics.

First law equation for a control volume:- A *control volume* is a region in space bounded by a *control surface* through which energy and matter may pass.There are two options of formulating the first law for a control volume. One option is formulating the law on a *rate basis*. That is, at any instant, there must be a balance between all *energy rates.* Alternatively, the first law must also be satisfied over any *time interval* Δt. For such an interval, there must be a balance between the *amounts* of all energy changes.

First Law on rate basis: - The rate at which thermal and mechanical energy enters a control volume, plus the rate at which thermal energy is generated within the control volume, minus the rate at which thermal and mechanical energy leaves the control volume must be equal to the rate of increase of stored energy within the control volume. Consider a control volume shown in Fig. 1.4 which shows that thermal and mechanical energy are entering the control volume at a rate denoted by $\dot{E}_{in}$, thermal and

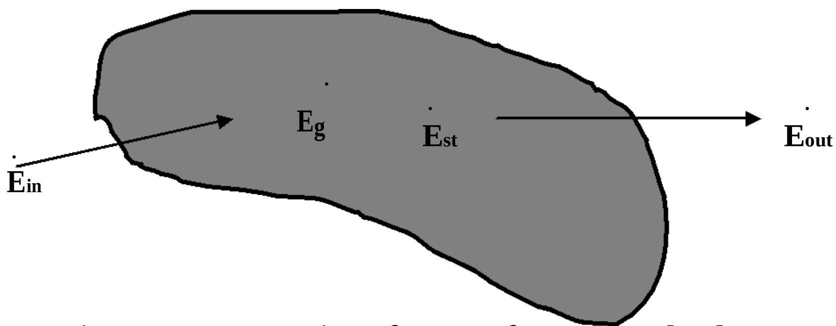


Fig. 1.4: Conservation of energy for a control volume on rate basis

mechanical energy are leaving the control volume at a rate denoted by $\dot{E}_{out}$. The rate at which energy is generated within the control volume is denoted by $\dot{E}_g$ and the rate at which energy is stored within the control volume is denoted by $\dot{E}_{st}$. The general form of the energy balance equation for the control volume can be written as follows:

$$\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st} \qquad \text{.......................................}(1.15)$$

$\dot{E}_{st}$ is nothing but the rate of increase of energy within the control volume and hence can be written as equal to dE_{st} / dt.

First Law over a Time Interval Δt:- Over a time interval Δt, the amount of thermal and mechanical energy that enters a control volume, plus the amount of thermal energy generated within the control volume minus the amount of thermal energy that leaves the control volume is equal to the increase in the amount of energy stored within the control volume.
The above statement can be written symbolically as

$$E_{in} + E_g - E_{out} = \Delta E_{st} \qquad \text{.......................................}(1.16)$$

The inflow and outflow energy terms are *surface phenomena.* That is they are associated exclusively with the processes occurring at the boundary surface and are proportional to the surface area.

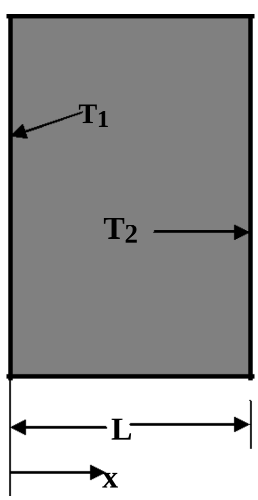
The energy generation term is associated with conversion from some other form (chemical, electrical, electromagnetic, or nuclear) to thermal energy. It is a *volumetric phenomenon.*That is, it occurs within the control volume and is proportional to the magnitude of this volume. For example, exothermic chemical reaction may be taking place within the control volume. This reaction converts chemical energy to thermal energy and we say that energy is generated within the control volume. Conversion of electrical energy to thermal energy due to resistance heating when electric current is passed through an electrical conductor is another example of thermal energy generation

Energy storage is also a volumetric phenomenon and energy change within the control volume is due to the changes in kinetic, potential and internal energy of matter within the control volume.

Illustrative Examples: **A. Conduction**

Example 1.1:- *Heat flux through a wood slab 50 mm thick, whose inner and outer surface temperatures are 40 ^{0} C and 20 ^{0} C respectively, has been determined to be 40 W/m^2. What is the thermal conductivity of the wood slab?*

Solution:



Given:- $T_1 = 40\ ^0$ C; $T_2 = 20\ ^0$ C; L = 0.05

m q = Q/A = 40 W / m^2.

To find*: k*

.

Assuming steady state conduction across the thickness of the slab and noting that the slab is not generating any thermal energy, the first law equation for the slab can be written as

Rate at which thermal energy (conduction) is entering the slab at the surface x = 0 is equal to the rate at which thermal energy is leaving the slab at the surface x = L That is

$$Q_x|_{x=0} = Q_x|_{x=L} = Q_x = \text{constant}$$

By Fourier‟s law we have $Q_x = -\, kA\,(dT / dx)$.

Separating the variables and integrating both sides w.r.t. „x‟ we have

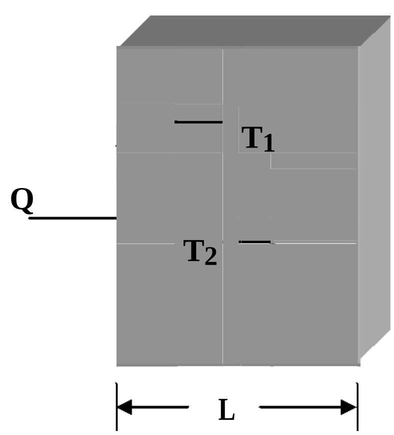
$$Q_x \int_0^L dx = -\, kA \int_{T1}^{T2} dT \text{ . Or } Q_x = kA\,(T_1 - T_2) / L$$

Heat flux = $q = Q_x / A = k(T_1 - T_2) / L$

Hence $k = q\,L / (T_1 - T_2) = 40 \times 0.05 / (40 - 20) = 0.1$ W / (m – K)

***Example 1.2:-** A concrete wall, which has a surface area of 20 m^2 and thickness 30 cm, separates conditioned room air from ambient air.The temperature of the inner surface of the wall is 25 ^{0} C and the thermal conductivity of the wall is 1.5 W / (m-K).Determine the heat loss through the wall for ambient temperature varying from – 15 ^{0} C to 38 ^{0} C which correspond to winter and summer conditions and display your results graphically.*

Solution:



Data:- $T_1 = 25\,^0$ C ; A = 20 m^2; L = 0.3 m

K = 1.5 W /(m-K) ;

By Fourier‟s law,

$Q = kA(T_1 - T_2) / L$

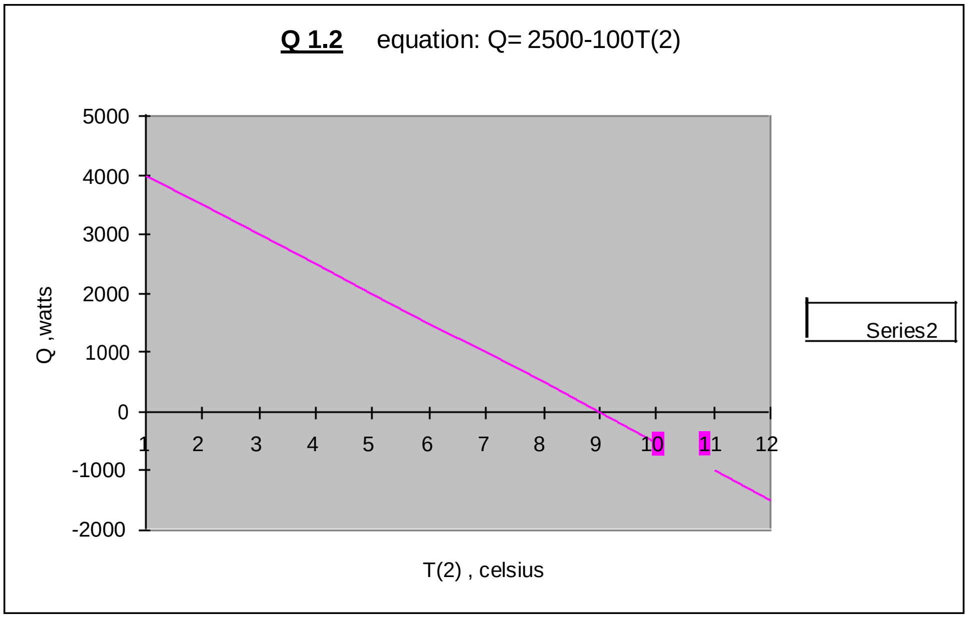
$$= \frac{1.5 \times 20 \times (25 - T_2)}{0.30}$$

Or $Q = 2500 - 100\,T_2$ ....(1)

Heat loss Q for different values of T_2 ranging from – 15 ^{0} C to + 38 ^{0} C are obtained from Eq. (1) and the results are plotted as shown

Scale x-axis : 1cm= 5 C

y-axis : 1cm =1000 W



Example 1.3:-*What is the thickness required of a masonry wall having a thermal conductivity of 0.75 W/(m-K), if the heat transfer rate is to be 80 % of the rate through another wall having thermal conductivity of 0.25 W/(m-K) and a thickness of 100 mm? Both walls are subjected to the same temperature difference.*

Solution:- Let subscript 1 refers to masonry wall and subscript 2 refers to the other wall.

By Fourier"s law, $Q_1 = k_1 A(T_1 - T_2) / L_1$

And $Q_2 = k_2 A(T_1 - T_2) / L_2$

Therefore

$$\frac{Q_1}{Q_2} = \frac{k_1 L_2}{k_2 L_1}$$

$$L_1 = \frac{Q_2 k_1}{Q_1 k_2} L_2$$

$$= (1 / 0.80) \times (0.75/0.25) \times 100 = 375 \text{ mm}$$

B. Convection:

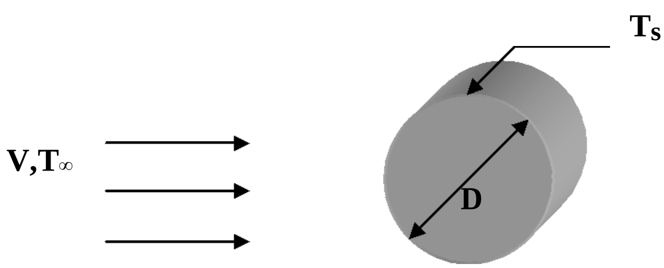
Example 1.4:- *Air at 40 ^{0}C flows over a long circular cylinder of 25 mm diameter with an embedded electrical heater. In a series of tests, measurements were made of power* *per unit length, P required to maintain the surface temperature of the cylinder at 300 ^{0} C for different stream velocities V of the air. The results are as follows:*

Air velocity, V (m/s) :	*1*	*2*	*4*	*8*	*12*
Power, P (W/m) :	*450*	*658*	*983*	*1507*	*1963*

(a) Determine the convective heat transfer coefficient for each velocity and display your results graphically. (h = P / 20.43)

(b)Assuming the dependence of the heat transfer coefficient on velocity to be of the form $h = CV^n$, determine the parameters C and n from the results of part (a).

Solution:-



Data:- D = 0.025 m : T_S = 300 ^{0} C ; T_∞ = 40 ^{0} C;

If h is the surface heat transfer coefficient then the power dissipated by the cylinder by convection is given by

$$P = hA_s (T_s - T_\infty)$$

Where A_s is the area of contact between the fluid and the surface of the cylinder. Therefore

$$P = h\, \pi DL (T_s - T_\infty)$$

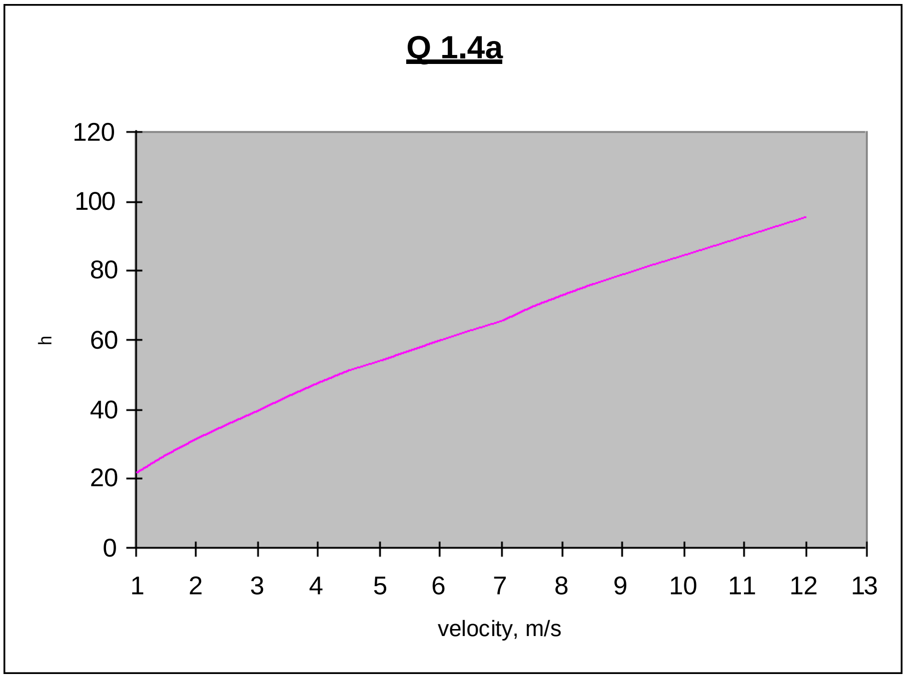
Or $\quad h = P / [\pi DL(T_s - T_\infty)] = P / [\pi \times 0.025 \times 1 \times (300 - 40)]$

Or $\quad h = P / 20.42$ W/m^2-k.. (1)

Values of h for different flow velocities are obtained and tabulated as follows:

Air Velocity, V (m/s) :	1	2	4	8	12
Power,P (W/m) :	450	658	983	1507	1963
h, $(W / (m^2 - K))$:	22.04	32.22	48.14	73.8	96.13

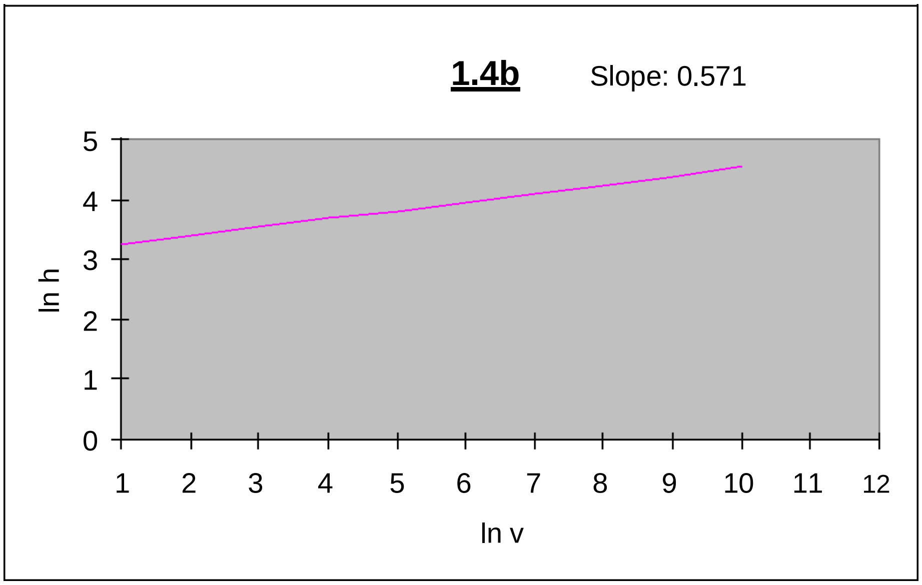
(a) A graph of h versus V can now be plotted as shown in Fig. P 1.4
(a). Scale: X axis 1cm= 1m/s
Y axis 1cm= 10 W/m^2k



(b) $$h = CV^n$$

Therefore $$\ln h = \ln C + n \ln V \qquad (2)$$

If ln h is plotted against ln V it will be straight line and the slope of which will give the value of n. Also the intercept of this line w.r.t the axis on which ln V is plotted will give the value of ln C from which C can be determined. The log –log plot is as shown in Fig. P 1.4(b). Scale X axis 1cm=0.25
Y axis 1cm=0.5



$$\ln C = 3.1 \text{ or } C = 22$$

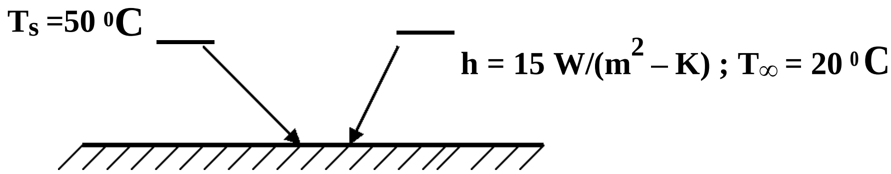
and

$$n = \frac{(\ln h - \ln C)}{\ln V} = \frac{(4.55 - 3.10)}{2.5}$$

$$= 0.571$$

Therefore $h = 22.2\, V^{0.571}$ is the empirical relation between h and V.

Example 1.5:- *A large surface at 50 ^{0} C is exposed to air at 20 ^{0} C. If the heat transfer coefficient between the surface and the air is 15 W/(m^2-K), determine the heat transferred from 5 m^2 of the surface area in 7 hours.*



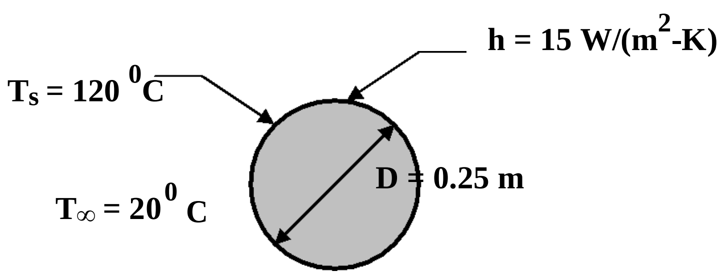
A = 5 m^2 : time = t = 7 h ;

$Q_{total} = Q\,t = hA(T_S - T_\infty)\,t = 15 \times 5 \times (50 - 20) \times 7 \times 3600$ J

$= 56.7 \times 10^6$ J = 56.7 MJ

Example 1.6:- *A 25 cm diameter sphere at 120 ^{0} C is suspended in air at 20 ^{0} C. If the convective heat transfer coefficient between the surface and air is 15 W/(m^2-K), determine the heat loss from the sphere.*

Solution:-



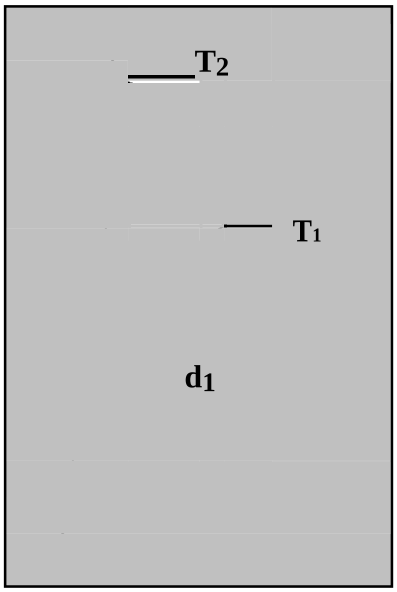
$Q = hA_S(T_S - T_\infty) = h\,4\pi R^2\,(T_S - T_\infty) = 15 \times 4\pi \times (0.25/2)^2 \times (120 -$

$20) = 294.52$ W

C. Radiation:

Example 1.7:- *A sphere 10 cm in diameter is suspended inside a large evacuated chamber whose walls are kept at 300 K. If the surface of the sphere is black and maintained at 500 K what would be the radiation heat loss from the sphere to the walls of the chamber?. What would be the heat loss if the surface of the sphere has an emissivity of 0.8?*

Solution:



$T_1 = 500$ K ; $T_2 = 300$ K ; $d_1 = 0.10$ m

Surface area of the sphere $= A_s = 4\pi R_1^2$
$= 4\pi x\ (0.1/2)^2$
$= 0.0314\ m^2$

If the surface of the sphere is black then

$Q_{black} = \zeta\ A_s\ (T_1^4 - T_2^4)$

$= 5.67 \times 10^{-8} \times 0.0314 \times (500^4 - 300^4)$

$= 96.85$ W

If the surface is having an emissivity of 0.8 then

$Q = 0.8\ Q_{black} = 0.8 \times 96.85 = 77.48$ W.

Example 1.8:- *A vacuum system as used in sputtering conducting thin films on micro circuits, consists of a base plate maintained at a temperature of 300 K by an electric heater and a shroud within the enclosure maintained at 77 K by circulating liquid nitrogen. The base plate insulated on the lower side is 0.3 m in diameter and has an emissivity of 0.25.*

(a) How much electrical power must be provided to the base plate heater?

(b) At what rate must liquid nitrogen be supplied to the shroud if its latent heat of vaporization is 125 kJ/kg?

Solution:- $T_1 = 300$ K ; $T_2 = 77$ K ; d = 0.3 m ; $\varepsilon_1 = 0.25$

Surface area of the top surface of the base plate $= A_s = (\pi / 4)d_1^2 = (\pi / 4) \times 0.3^2$

$$= 0.0707 \ m^2$$

(a) $Q_r = \varepsilon_1 \zeta \, A_s \, (T_1^4 - T_2^4)$

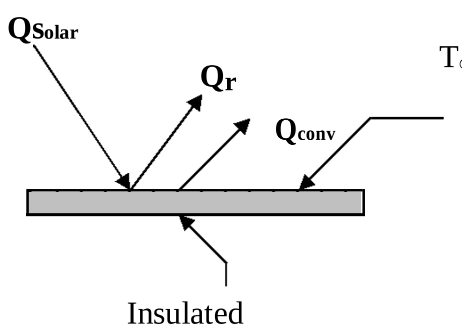
$$= 0.25 \times 5.67 \times 10^{-8} \times 0.0707 \times (300^4 - 77^4) = 8.08 \ W$$

(b) If $\dot{m}_{N2}$ = mass flow rate of nitrogen that is vapourised then

$$\dot{m}_{N2} = Q_r / h_{fg} = \frac{8.08}{125 \times 1000} = 6.464 \times 10^{-5} \text{ kg/s or } 0.233 \text{ kg/s}$$

Example 1.9:- *A flat plate has one surface insulated and the other surface exposed to the sun. The exposed surface absorbs the solar radiation at a rate of 800 W/m^2 and dissipates heat by both convection and radiation into the ambient at 300 K. If the emissivity of the surface is 0.9 and the surface heat transfer coefficient is 12 W/(m^2-K), determine the surface temperature of the plate.*

Solution:-



$T_\infty = 300$ K ; $q_{solar} = 800$ W / m^2

T_s ; $\varepsilon = 0.9$; h = 12 W / (m^2 – K)

Energy balance equation for the top surface of the plate is given by

$$Q_{solar} = Q_r + Q_{conv}$$

$$q_{solar} A_s = \varepsilon \, \zeta \, A_s \, (T_s^4 - T_\infty^4) + h \, A_s (T_s - T_\infty)$$

Therefore $\quad 800 = 0.9 \times 5.67 \times 10^{-8} \times (T_s^4 - 300^4) + 12 \times (T_s - 300)$

On simplifying the above equation we get

$$(T_s / 100)^4 + 2.35 \, T_s = 943 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (1)$$

Equation (1) has to be solved by trial and error.

Trial 1:- Assume T_S = 350 K. Then LHS of Eq. (1) = 972.6 which is more than RHS of Eq.(1). Hence T_S < 350 K.

Trial 2 :- Assume T_S = 340 K. Then LHS of Eq. (1) = 932.6 which is slightly less than

RHS. Therefore T_S should lie between 340 K and 350 K but closer to 340 K. ***Trial 3:-***

Assume T_S = 342.5 K. Then LHS of Eq.(1) = 942.5 = RHS of Eq. (1). Therefore Ts =

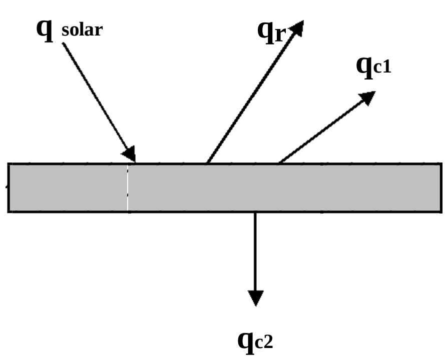
K

Example 1.10:- *The solar radiation incident on the outside surface of an aluminum shading device is 1000 W/m^2. Aluminum absorbs 12 % of the incident solar energy and dissipates it by convection from the back surface and by combined convection and radiation from the outer surface. The emissivity of aluminum is 0.10 and the convective heat transfer coefficient for both the surfaces is 15 W/(m^2 –K). The ambient temperature of air may be taken as 20 ^{0} C. Determine the temperature of the shading device.*

Solution:- q_{solar} = 1000 W / m^2 ; absorptivity of aluminum = α = 0.12 ; emissivity

of aluminum = ε = 0.10 ; h = 15 W /(m^2 – K) ; T_∞ = 20 + 273 = 293 K ;

Solar radiation flux absorbed by aluminum = $q_a = \alpha\, q_{solar}$ = 0.12 x 1000 = 120 W / m^2.



The energy absorbed by aluminum is dissipated by convection from the back surface and by combined convection and radiation from the outer surface. Hence the energy balance equation can be written as

$q_a = q_r + q_{c1} + q_{c2}$

Therefore, $\quad q_a = \varepsilon \zeta T_S^4 - \alpha \zeta T_\infty^4) + h_1(T_S - T_\infty) + h_2 (T_S - T_\infty)$

Or $\quad 120 = 5.67 \times 10^{-8} \times (0.10 T_S^4 - 0.12 \times 293^4) + (T_S - 293) \times (15 + 15)$

On simplifying we get, $(T_S / 100)^4 + 53\, T_S = 15873$ ....(1)

Eq.(1) has to be solved by trial and error.

Trail 1:- Assume T_S = 300 K. Then LHS = 15981 which is > RHS.

Trail 2 :- Assume T_S = 295 K. Then LHS = 15710.73 which is < RHS. Hence T_S should lie between 300K and 295 K.

Trial 3 :- Assume T_S = 297 K . Then LHS = 15819 which is almost equal to RHS (Within 0.34 %)
Therefore T_S = 297 K.
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UNIT-II

ONE DIMENSIONAL STEADY STATE AND TRANSIENT CONDUCTION HEAT TRANSFER

Introduction: In this chapter, the governing basic equations for conduction in Cartesian coordinate system is derived. The corresponding equations in cylindrical and spherical coordinate systems are also mentioned. Mathematical representations of different types of boundary conditions and the initial condition required to solve conduction problems are also discussed. After studying this chapter, the student will be able to write down the governing equation and the required boundary conditions and initial condition if required for any conduction problem.

One – Dimensional Conduction Equation : In order to derive the one-dimensional conduction equation, let us consider a volume element of the solid of thickness Δx along x – direction at a distance „x" from the origin as shown in Fig. 2.1.Q_X represents the rate

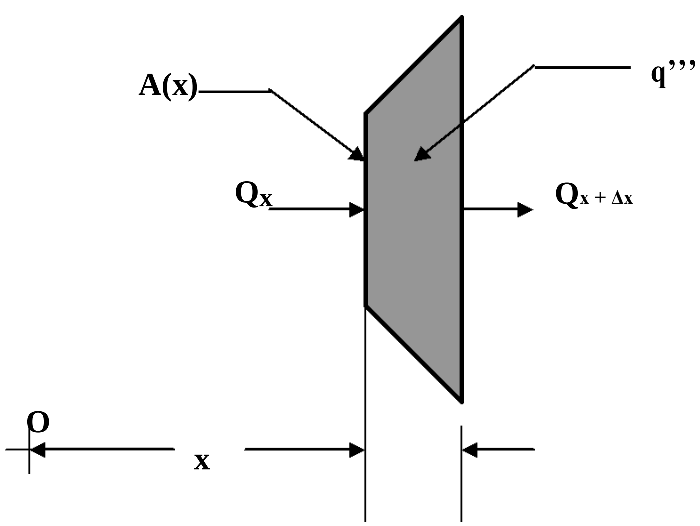

Fig. 2.1: Nomenclature for one dimensional conduction equation

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of heat transfer in x – direction entering into the volume element at x, A(x) area of heat flow at the section x ,q""" is the thermal energy generation within the element per unit volume and $Q_{X+\Delta X}$ is the rate of conduction out of the element at the section x + Δx. The energy balance equation per unit time for the element can be written as follows:

[Rate of heat conduction into the element at x + Rate of thermal energy generation within the element − Rate of heat conduction out of the element at x + Δx]

= Rate of increase of internal energy of the element.

i.e., $Q_x + Q_g – Q_{x+\Delta x} = \partial E / \partial t$

or $Q_x + q''' A(x)\, \Delta x – \{Q_x + (\partial Q_x / \partial x)\Delta x + (\partial^2 Q_x / \partial x^2)(\Delta x)^2 / 2! + \ldots\ldots\}$

$$= \partial / \partial t\, (\rho A(x)\Delta x C_p T)$$

Neglecting higher order terms and noting that ρ and Cp are constants the above equation simplifies to

$$Q_x + q''' A(x)\, \Delta x – \{Q_x + (\partial Q_x / \partial x)\Delta x = \rho A(x)\Delta x C_p (\partial T/ \partial t)$$

Or $$− (\partial Q_x / \partial x) + q''' A(x) = \rho A(x)\, C_p (\partial T/ \partial t)$$

Using Fourier's law of conduction , $Q_x = − k\, A(x)\, (\partial T / \partial x)$, the above equation simplifies to

$$− \partial/ \partial x\, \{− k\, A(x)\, (\partial T / \partial x)\} + q''' A(x) = \rho A(x)\, C_p (\partial T/ \partial t)$$

Or $$\boldsymbol{\{1/A(x)\}\, \partial/ \partial x\, \{k\, A(x)\, (\partial T / \partial x)\} + q''' = \rho\, C_p (\partial T/ \partial t)} \quad \textbf{....................(2.1)}$$

Eq. (2.1) is the most general form of conduction equation for one-dimensional unsteady state conduction.

Equation for one-dimensional conduction in plane walls :- For plane walls, the area of heat flow A(x) is a constant. Hence Eq. (2.1) reduces to the form

$$\boldsymbol{\partial/ \partial x\, \{k\, (\partial T / \partial x)\} + q''' = \rho\, C_p (\partial T/ \partial t)} \quad \textbf{...........................(2.2)}$$

(i) If the thermal conductivity of the solid is constant then the above equation reduces to

$$\boldsymbol{(\partial^2 T / \partial x^2) + (q''' / k) = (1/\alpha)(\partial T/ \partial t)} \quad \textbf{...................................(2.3)}$$

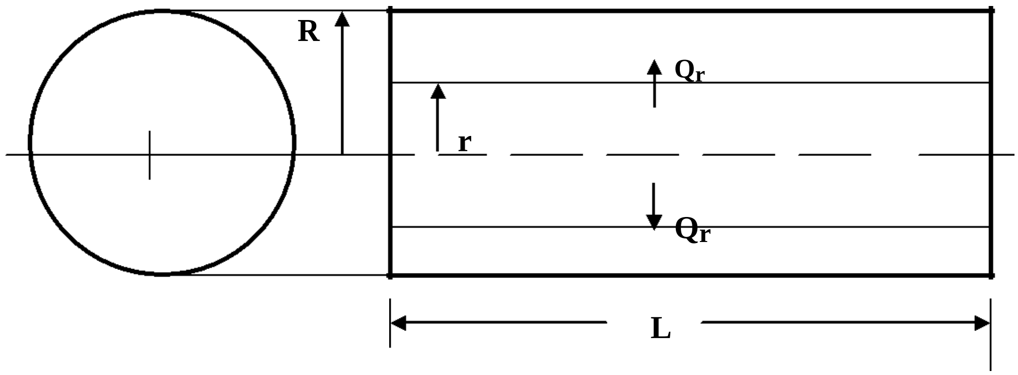
(ii) For steady state conduction problems in solids of constant thermal conductivity temperature within the solid will be independent of time (i.e.$(\partial T/ \partial t) = 0$) and hence Eq. (2.3) reduces to

$$\boldsymbol{(d^2 T / dx^2) + (q''' / k) = 0} \quad \textbf{.. (2.4)}$$

(iii) For a solid of constant thermal conductivity for which there is no thermal energy generation within the solid q"""""" = 0 and the governing for steady state conduction is obtained by putting q"""""" = 0 in Eq. (2.4) as

$$(d^2T / dx^2) = 0 \text{ .... } (2.4)$$

Equation for one-dimensional radial conduction in cylinders:-



For radial conduction in cylinders, by convention the radial coordinate is denoted by „r" instead of „x" and the area of heat flow through the cylinder of length L,at any radius r is given by A(x) = A(r) = 2πrL. Hence substituting this expression for A(x) and replacing x by r in Eq. (2.1) we have

$$\{1/(2\pi rL)\partial/\partial r\ \{k\ 2\pi rL\ (\partial T / \partial r)\} + q'''''' = \rho\ C_p\ (\partial T/\partial t)$$

Or

$$(1/r)\ \partial/\partial r\ \{k\ r\ (\partial T / \partial r)\} + q''' = \rho\ C_p\ (\partial T/\partial t) \text{ .... } (2.5)$$

(i) For cylinders of constant thermal conductivity the above equation reduces to

$$(1/r)\ \partial/\partial r\ \{ r\ (\partial T / \partial r)\} + q''' / k = (1 / \alpha)\ (\partial T/\partial t) \text{ .... } (2.6)$$

(ii) For steady state radial conduction (i.e. $(\partial T/\partial t) = 0$) in cylinders of constant k, the above equation

reduces to $$\mathbf{(1/r)\, d/dr \{ r (dT/\partial r)\} + q''' / k = 0 \text{ .... } (2.7)}$$

(iii) For steady state radial conduction in cylinders of constant k and having no thermal energy generation (i.e. q‘‘‘‘‘‘ = 0) the above equation reduces to

$$\mathbf{d/dr \{ r (dT/\partial r)\} = 0 \text{ .... } (2.8)}$$

Equation for one-dimensional radial conduction in spheres:- For one-imensional radial conduction in spheres, the area of heat flow at any radius r is given by $A(r) = 4\pi r^2$. Hence Eq.(2.1) for a sphere reduces to

$$\{1/(4\pi r^2)\}\partial/\partial r \{k\, 4\pi r^2 (\partial T/\partial r)\} + q''' = \rho C_p (\partial T/\partial t)$$

Or $$\mathbf{1/r^2\, \partial/\partial r \{k\, r^2 (\partial T/\partial r)\} + q''' = \rho\, C_p (\partial T/\partial t) \text{ .... } (2.9)}$$

(i) For spheres of constant thermal conductivity the above equation reduce to

$$\mathbf{1/r^2\, \partial/\partial r \{ r^2 (\partial T/\partial r)\} + q''' / k = (1/\alpha) (\partial T/\partial t) \text{ .... } (2.10)}$$

(ii) For steady state conduction in spheres of constant k the above equation further reduce to

$$\mathbf{1/r^2\, \partial/\partial r \{ r^2 (\partial T/\partial r)\} + q''' / k = 0 \text{ .... } (2.11)}$$

(iii) For steady state conduction in spheres of constant k and without any thermal energy generation the above equation further reduces to

$$\mathbf{1/r^2\, d/dr \{ r^2 (dT/dr)\} = 0 \text{ .... } (2.12)}$$

Equation in compact form:- The general form of one – dimensional conduction equations for plane walls, cylinders and spheres {equations (2..2), (2.5) and (2.9)} can be written in a compact form as follows:

$$\mathbf{1/r^n\, \partial/\partial r \{k\, r^n (\partial T/\partial r)\} + q''' = \rho\, C_p (\partial T/\partial t) \text{ .... } (2.13)}$$

Where n = 0 for plane walls,
n = 1 for radial conduction in cylinders
n = 2 for radial conduction in spheres,
and for plane walls it is customary to replace the „r‟ variable by „x‟ variable.

Three dimensional conduction equations: While deriving the one – dimensional conduction equation, we assumed that conduction heat transfer is taking place only along one direction. By allowing conduction along the remaining two directions and following the same procedure we obtain the governing equation for conduction in three dimenions.

Three dimensional conduction equation in Cartesian coordinate system: Let us consider a volume element of dimensions Δx, Δy and Δz in x y and z directions respectively. The conduction heat transfer across the six surfaces of the element is shown in Fig. 2.3.

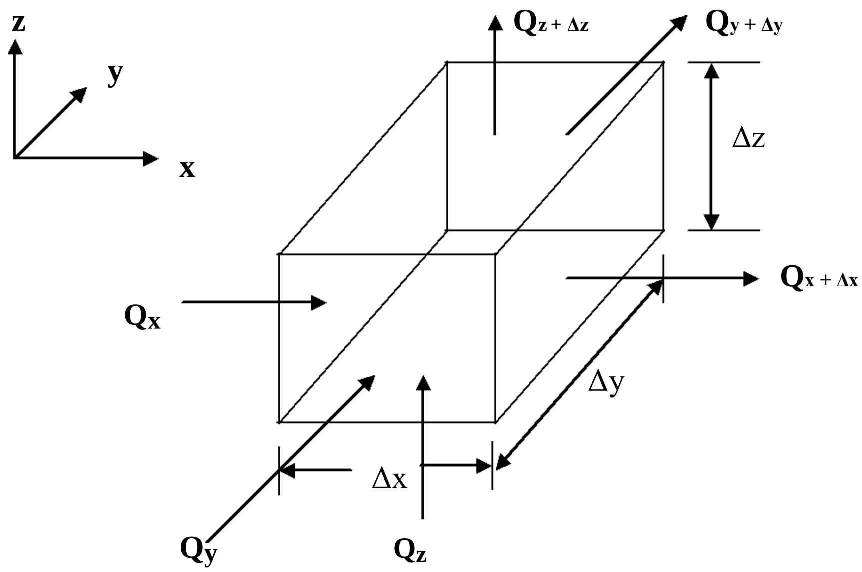


Fig. 2.3: Conduction heat transfer across the six faces of a volume element

Net Rate of conduction into the element in x-direction $= Q_x - Q_{x+\Delta x}$

$$= Q_x - [Q_x + (\partial Q_x/\partial x)\,\Delta x + (\partial^2 Q_x/\partial x^2)(\Delta x)^2 / 2! + \ldots.]$$

$$= -(\partial Q_x/\partial x)\,\Delta x \text{ by neglecting higher order terms.}$$

$$= -\partial/\partial x\,[-k_x\,\Delta y\,\Delta z(\partial T/\partial x)]\,\Delta x$$

$$= \partial/\partial x[k_x\,(\partial T/\partial x)]\,\Delta x\,\Delta y\,\Delta z$$

Similarly the net rate of conduction into the element in y – direction $= \partial/\partial y[k_y\,(\partial T/\partial y)]\,\Delta x\,\Delta y\,\Delta z$

and in z – direction $= \partial/\partial z[k_z\,(\partial T/\partial z)]\,\Delta x\,\Delta y\,\Delta z.$

Hence the net rate of conduction into the element from all the three directions

$$Q_{in} = \{\partial / \partial x[k_x (\partial T / \partial x)] + \partial / \partial y[k_y (\partial T / \partial y)] + \partial / \partial z[k_z (\partial T / \partial z)] \}\ \Delta x\ \Delta y\ \Delta z$$

Rate of heat thermal energy generation in the element = $Q_g = q''''''\ \Delta x\ \Delta y\ \Delta z$

Rate of increase of internal energy within the element = $\partial E / \partial t = \rho\ \Delta x\ \Delta y\ \Delta z\ C_p\ (\partial T\ /$

$\partial t)$ Applying I law of thermodynamics for the volume element we have

$$Q_{in} + Q_g = \partial E / \partial t$$

Substituting the expressions for Q_{in}, Q_g and $\partial E / \partial t$ and simplifying we get

$$\{\partial / \partial x[k_x (\partial T / \partial x)] + \partial / \partial y[k_y (\partial T / \partial y)] + \partial / \partial z[k_z (\partial T / \partial z)] \} + q''' = \rho\ C_p\ (\partial T / \partial t) \qquad \text{……………………(2.14)}$$

Equation (2.14) is the most general form of conduction equation in Cartesian coordinate system. This equation reduces to much simpler form for many special cases as indicated below.

Special cases:- (i) For isotropic solids, thermal conductivity is independent of direction; i.e., $k_x = k_y = k_z = k$. Hence Eq. (2.14) reduces to

$$\{\partial / \partial x[k (\partial T / \partial x)] + \partial / \partial y[k (\partial T / \partial y)] + \partial / \partial z[k (\partial T / \partial z)] \} + q''' = \rho\ C_p\ (\partial T / \partial t) \qquad \text{……………………..(2.15)}$$

(ii) For isotropic solids with constant thermal conductivity the above equation further reduces to

$$\partial^2 T / \partial x^2 + \partial^2 T / \partial y^2 + \partial^2 T / \partial z^2 + q''' / k = (1 / \alpha)\ (\partial T\ / \partial t)\ldots\ldots \text{(2.16)}$$

Eq.(2.16) is called as the *"Fourier – Biot equation"* and it reduces to the following forms under specified conditions as mentioned below:

(iii) Steady state conduction [i.e., $(\partial T / \partial t) = 0$]

$$\partial^2 T / \partial x^2 + \partial^2 T / \partial y^2 + \partial^2 T / \partial z^2 + q''' / k = 0 \ldots\ldots \text{(2.17)}$$

Eq. (2.17) is called the *"Poisson equation"*.

(iv) No thermal energy generation [i.e. $q'''''' = 0$]:

$$\partial^2 T / \partial x^2 + \partial^2 T / \partial y^2 + \partial^2 T / \partial z^2 = (1 / \alpha)\ (\partial T / \partial t)\ldots\ldots \text{(2.18)}$$

Eq. (2.18) is called the *"diffusion equation"*.

(v) Steady state conduction without heat generation [i.e., $(\partial T / \partial t) = 0$ and $q''' = 0$]:

$$\partial^2 T / \partial x^2 + \partial^2 T / \partial y^2 + \partial^2 T / \partial z^2 = 0 \qquad \text{......(2.19)}$$

Eq. (2.19) is called the *"Laplace equation"*.

Three dimensional conduction equation in cylindrical coordinate system:

It is convenient to express the governing conduction equation in cylindrical coordinate system when we want to analyse conduction in cylinders. Any point P in space can be located by using the cylindrical coordinate system r, θ and z and its relation to the Cartesian coordinate system (See Fig. 2.4) can be written as follows:

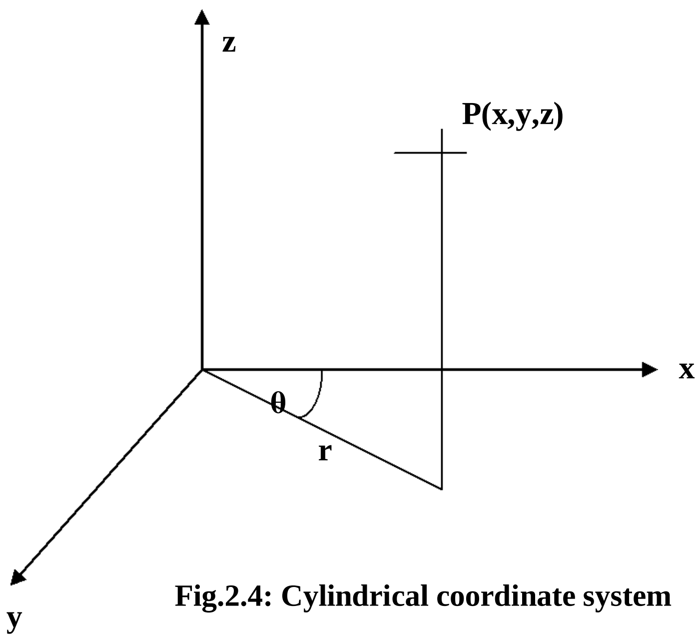


Fig.2.4: Cylindrical coordinate system

$x = r \cos\theta$; $y = r \sin\theta$; $z = z$. Using these transformations and after laborious simplifications Eq. (2.15) simplifies to

$$\frac{1}{r}\frac{\partial}{\partial r}\left[k\, r\, \frac{\partial T}{\partial r}\right] + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left[k \frac{\partial T}{\partial \theta}\right] + \frac{\partial}{\partial z}\left[k \frac{\partial T}{\partial z}\right] + q''' = \rho\, C_p \frac{\partial T}{\partial t} \qquad \text{..................(2.20)}$$

The above equation is valid for only for isotropic solids.

2.3.2. Three dimensional conduction equation in Spherical coordinate system:

For spherical solids,it is convenient to express the governing conduction equation in spherical coordinate system. Any point P on the surface of a sphere of radius r can be located by using the spherical coordinate system r, θ and θ and its relation to the Cartesian coordinate system (See Fig. 2.5) can be written as follows:

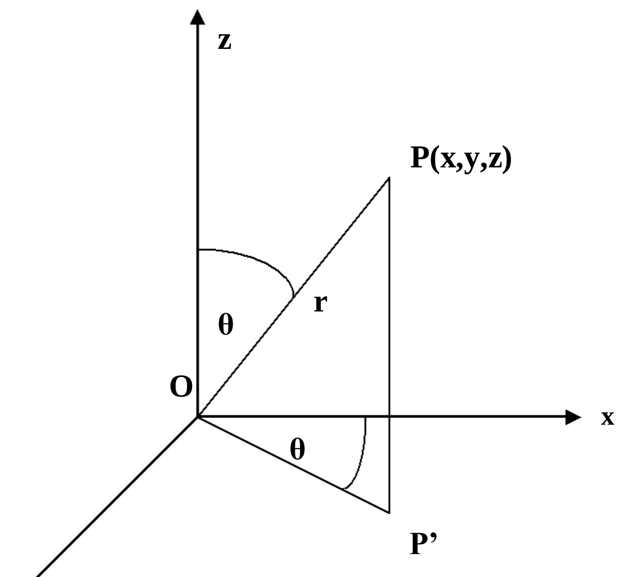


Fig: 2.5: Spherical coordinate system

OP'' = r sin θ.Hence

$x = r \sin\theta \cos\theta$;

$y = r \sin\theta \sin\theta$;

$z = r \cos\theta$

Using the relation between x, y ,z and r, θ and θ, the conduction equation (2.15) can be transformed into the equation in terms of r, θ and θ as follows.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[kr^2\frac{\partial T}{\partial r}\right]+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \theta}\left[k\frac{\partial T}{\partial \theta}\right]+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left[k\sin\theta\frac{\partial T}{\partial \theta}\right]+q'''=\rho C_p\frac{\partial T}{\partial t}\quad\text{.....................(2.21).}$$

Boundary and Initial Conditions:

The temperature distribution within any solid is obtained by integrating the above conduction equation with respect to the space variable and with respect to time.The solution thus obtained is called the *"general solution"* involving arbitrary constants of integration. The solution to a particular conduction problem is arrived by obtaining these constants which depends on the conditions at the bounding surfaces of the solid as well as the initial condition. The thermal conditions at the boundary surfaces are called the *"boundary conditions"* **.** Boundary conditions normally encountered in practice are:

*(i) Specified temperature (*also called as *boundary condition of the first kind),*

(ii) Specified heat flux (also known as *boundary condition of the second kind),*

(iii) Convective boundary condition (also known as *boundary condition of the third kind)* and

(iv) radiation boundary condition. The mathematical representations of these boundary conditions are illustrated by means of a few examples below.

Specified Temperatures at the Boundary:- Consider a plane wall of thickness Lwhose outer surfaces are maintained at temperatures T_0 and T_L as shown in Fig.2.6. For one-dimensional unsteady state conduction the boundary conditions can be written as

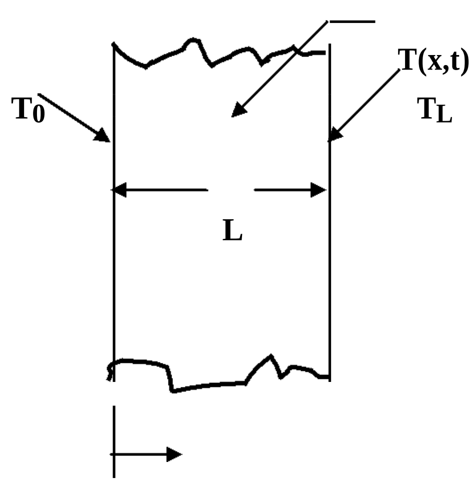


Fig. 2.6: Boundary condition of first kind for a plane wall

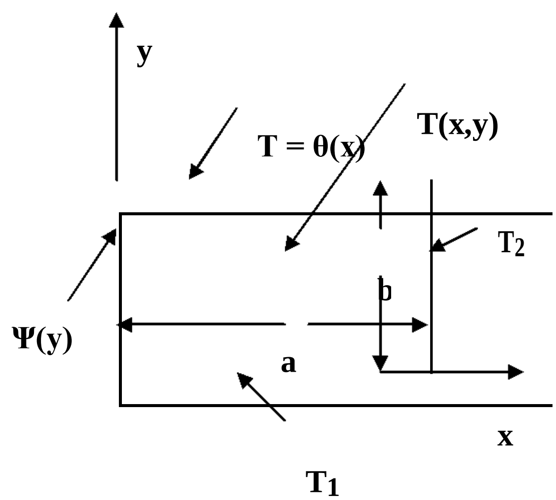


Fig.2.7: Boundary conditions of first kind for a rectangular plate

(i) at x = 0, $T(0,t) = T_0$; (ii) at x = L, $T(L,t) = T_L$.

Consider another example of a rectangular plate as shown in Fig. 2.7. The boundary conditions for the four surfaces to determine two-dimensional steady state temperature distribution T(x,y) can be written as follows.

(i) at x = 0, $T(0,y) = \Psi(y)$; (ii) at y = 0, $T(x,0) = T_1$ for all values of y

(iii) at x = a, $T(a,y) = T_2$ for all values of y; (iv) at y = b, $T(x,b) = \theta(x)$

Specified heat flux at the boundary:- Consider a rectangular plate as shown in Fig. 2.8 and whose boundaries are subjected to the prescribed heat flux conditions as shown in the figure. Then the boundary conditions can be mathematically expressed as follows.

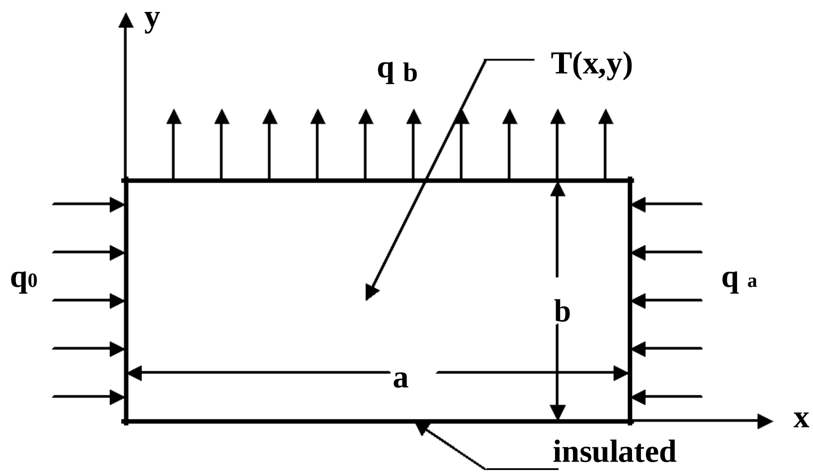


Fig.2.8: Prescribed heat flux boundary conditions

(i) at $x = 0, -k\,(\partial T / \partial x)|_{x = 0} = q_0$ for $0 \le y \le b$;

(ii) at $y = 0$, $(\partial T / \partial y)|_{y = 0} = 0$ for $0 \le x \le a$;

(iii) at $x = a$, $k\,(\partial T / \partial x)|_{x = a} = q_a$ for $0 \le y \le b$;

(iv) at $y = b, -k\,(\partial T / \partial y)|_{y = b} = 0$ for $0 \le x \le a$;

Boundary surface subjected to convective heat transfer:- Fig. 2.9 shows a plane wall whose outer surfaces are subjected to convective boundary conditions.The surface at $x = 0$ is in contact with a fluid which is at a uniform temperature T_i and the surface heat transfer coefficient is h_i. Similarly the other surface at $x = L$ is in contact with another fluid at a uniform temperature T_0 with a surface heat transfer coefficient h_0. This type of boundary condition is encountered in heat exchanger wherein heat is transferred from hot fluid to the cold fluid with a metallic wall separating the two fluids. This type of boundary condition is normally referred to as the boundary condition of third kind. The mathematical representation of the boundary conditions for the two surfaces of the plane wall can be written as follows.

(i) at $x = 0$, $q_{convection} = q_{conduction}$; i.e., $h_i[T_i - T|_{x = 0}] = -k(dT / dx)|_{x = 0}$

(ii) at $x = L$, $-k(dT / dx)|_{x = L} = h_0\,[T|_{x = L} - T_0]$

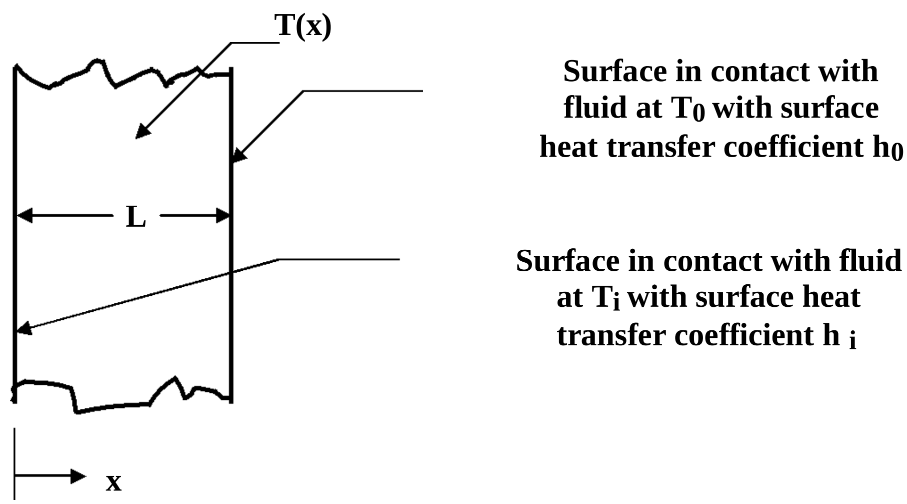


Fig. 2.9: Boundaries subjected to convective heat transfer for a plane wall

Radiation Boundary Condition: Fig. 2.10 shows a plane wall whose surface at x =L is having an emissivity „ε‟ and is radiating heat to the surroundings at a uniform temperature T_S. The mathematical expression for the boundary condition at x = L can be written as follows:

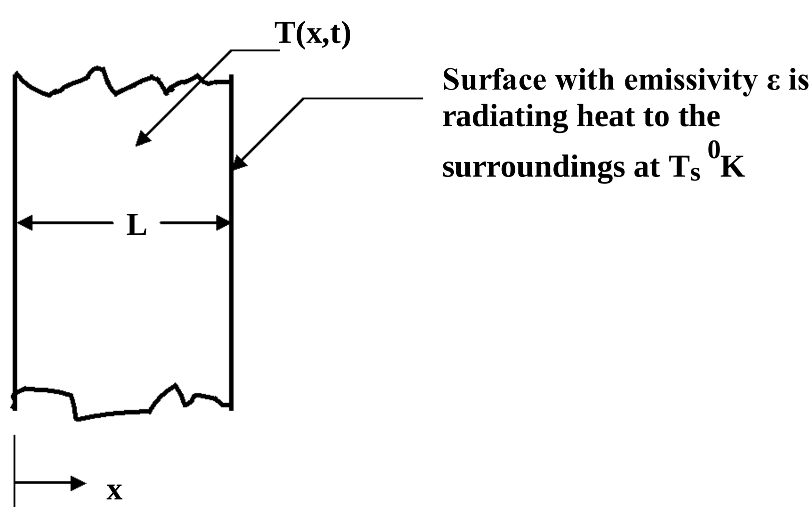


Fig. 2.10: Boundary surface at x = L subjected to radiation heat transfer

(i) at x = L, $q_{conduction} = q_{radiation}$; i.e., $-k\,(dT / dx)|_{x = L} = \zeta\,\varepsilon\,[(T|_{x = L})^4 - T_s^{\,4}]$

In the above equation both $T|_{x = L}$ and T_S should be expressed in degrees Kelvin.

General form of boundary condition (combined conduction, convection and radiation boundary condition):

There are situations where the boundary surface is subjected to combined conduction, convection and radiation conditions as illustrated in Fig. 2.11.It is a south wall of a house and the outer surface of the wall is exposed to solar radiation. The interior of the room is at a uniform temperature T_i. The outer air is at uniform temperature T_0 . The sky, the ground and the surfaces of the surrounding structures at this location is modeled as a surface at an effective temperature of T_{sky}.

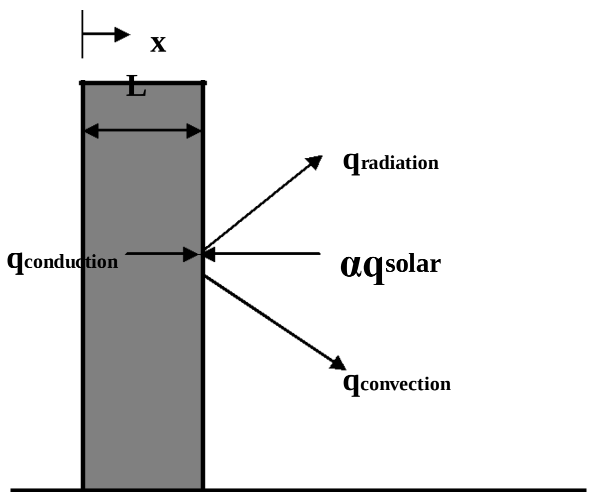


Fig. 2.11: Schematic for general form of boundary condition

Energy balance for the outer surface is given by the equation

$$q_{conduction} + \alpha\, q_{solar} = q_{radiation} + q_{convection}$$

$$-\,k\,(dT / dx)|_{x = L} + \alpha q_{solar} = \varepsilon\, \zeta\, [(T|_{x = L})^4 - T_{sky}^4] + h_0[T|_{x = L} - T_0]$$

<u>Illustrative Examples:</u>

A. Derivation of conduction Equations:

By writing an energy balance for a differential cylindrical volume element in the „r" variable (r is any radius), derive the one-dimensional time dependent heat conduction equation with internal heat generation and variable thermal conductivity in the cylindrical coordinate system.

By writing an energy balance for a differential spherical volume element in the „r" variable (r is any radius), derive the one-dimensional time dependent heat conduction equation with internal heat generation and variable thermal conductivity in the spherical coordinate system.

By simplifying the three-dimensional heat conduction equation, obtain one- dimensional steady-state conduction equation with heat generation and constant thermal conductivity for the following coordinate systems:

(a) Rectangular coordinate in the „x" variable.
(b) Cylindrical coordinate in the r variable.
(c) Spherical coordinates in the „r" variable

B.Mathematical Formulation of Boundary conditions:

A plane wall of thickness L is subjected to a heat supply at a rate of q_0 W/m^2 at one boundary surface and dissipates heat from the surface by convection to the ambient which is at a uniform temperature of $T\infty$ with a surface heat transfer coefficient of h_∞. Write the mathematical formulation of the boundary conditions for the plane wall.

Consider a solid cylinder of radius R and height Z. The outer curved surface of the cylinder is subjected to a uniform heating electrically at a rate of q_0 W / m^2.Both the circular surfaces of the cylinder are exposed to an environment at a uniform temperature T_∞ with a surface heat transfer coefficient h. Write the mathematical formulation of the boundary conditions for the solid cylinder.

A hollow cylinder of inner radius r_i, outer radius r_0 and height H is subjected to the following boundary conditions.

(d) The inner curved surface is heated uniformly with an electric heater at a constant rate of q_0 W/m^2,
(e) the outer curved surface dissipates heat by convection into an ambient at a uniform temperature, T_∞ with a convective heat transfer coefficient, h
(f) the lower flat surface of the cylinder is insulated, and
(g) the upper flat surface of the cylinder dissipates heat by convection into the ambient at T_∞ with surface heat transfer coefficient h. Write the mathematical formulation of the boundary conditions for the hollow cylinder.

B. Formulation of Heat Conduction Problems:

A plane wall of thickness L and with constant thermal properties is initially at a uniform temperature T_i. Suddenly one of the surfaces of the wall is subjected to heating by the flow of a hot gas at temperature T_∞ and the other surface is kept insulated. The heat transfer coefficient between the hot gas and the surface exposed to it is h. There is no heat generation in the wall. Write the mathematical formulation of the problem to determine the one-dimensional unsteady state temperature within the wall.

A copper bar of radius R is initially at a uniform temperature T_i. Suddenly the heating of the rod begins at time t=0 by the passage of electric current, which generates heat at a uniform rate of q''' W/m^2. The outer surface of the dissipates heat into an ambient at a uniform temperature T_∞ with a convective heat transfer coefficient h. Assuming that thermal conductivity of the bar to be constant, write the mathematical formulation of the heat conduction problem to determine the one-dimensional radial unsteady state temperature distribution in the rod.

Consider a solid cylinder of radius R and height H. Heat is generated in the solid at a uniform rate of q''' W/m^3. One of the circular faces of the cylinder is insulated and the other circular face dissipates heat by convection into a medium at a uniform temperature of T_∞ with a surface heat transfer coefficient of h. The outer curved surface of the cylinder is maintained at a uniform temperature of T_0. Write the mathematical formulation to determine the two-dimensional steady state temperature distribution T(r, z) in the cylinder.

Consider a rectangular plate as shown in Fig. P2.10.The plate is generating heat at a uniform rate of q''' W/m^3. Write the mathematical formulation to determine two-dimensional steady state temperature distribution in the plate.

Consider a medium in which the heat conduction equation is given in its simple form as

$$\partial^2 T / \partial x^2 = (1/\alpha)\,(\partial T / \partial t)$$

(a) Is heat transfer in this medium steady or transient?
(b) Is heat transfer one-, two- or three-dimensional?
(c) Is there heat generation in the medium?
(d) Is thermal conductivity of the medium constant or variable?

Consider a medium in which the heat conduction equation is given in its simple form as $(1/r)\, d / dr(r\, k\, dT/dr) + q''' = 0$.

(a) Is heat transfer steady or unsteady?
(b) Is heat transfer one-, two- or three-dimensional?
(c) Is there heat generation in the medium?
(d) Is the thermal conductivity of the medium constant or variable?

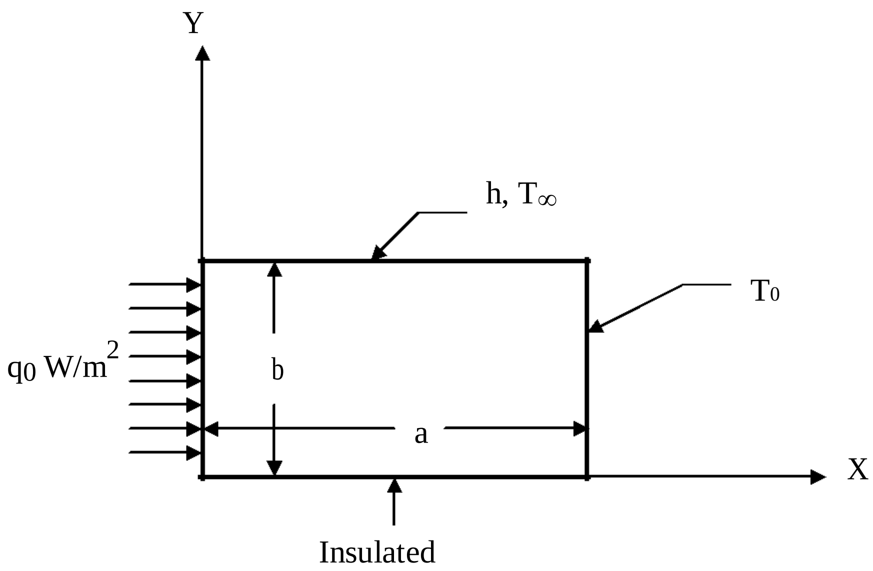


Fig. P 2.10: Schematic for problem 2.10

Consider a medium in which heat the heat conduction equation in its simplest form is given as

$$(1/r^2)\ \partial/\partial r\ (r^2\ \partial T/\partial r) = (1/\alpha)\ (\partial T/\partial t)$$

(a) Is heat transfer steady state or unsteady state?
(b) Is heat transfer one-, two- or three-dimension?
(c) Is there heat generation in the medium?
(d) Is the thermal conductivity constant or variable?

Consider a medium in which the heat conduction equation is given in its simplest form as

$$(1/r)\ \partial/\partial r\ (k\ r\ \partial T/\partial r) + \partial/\partial z\ (k\ \partial T/\partial z) + q'''''' = 0$$

(a) Is heat transfer steady state or unsteady state?
(b) Is heat transfer one-, two- or three-dimension?
(c) Is there heat generation in the medium?
(d) Is the thermal conductivity constant or variable?

Consider a medium in which heat the heat conduction equation in its simplest form is given as

$$(1/r^2)\ \partial/\partial r\ (r^2\ \partial T/\partial r) + \frac{1}{r^2 \sin^2 \varphi}\left[\frac{\partial^2 T}{\partial \theta^2}\right] = (1/\alpha)\ (\partial T/\partial t)$$

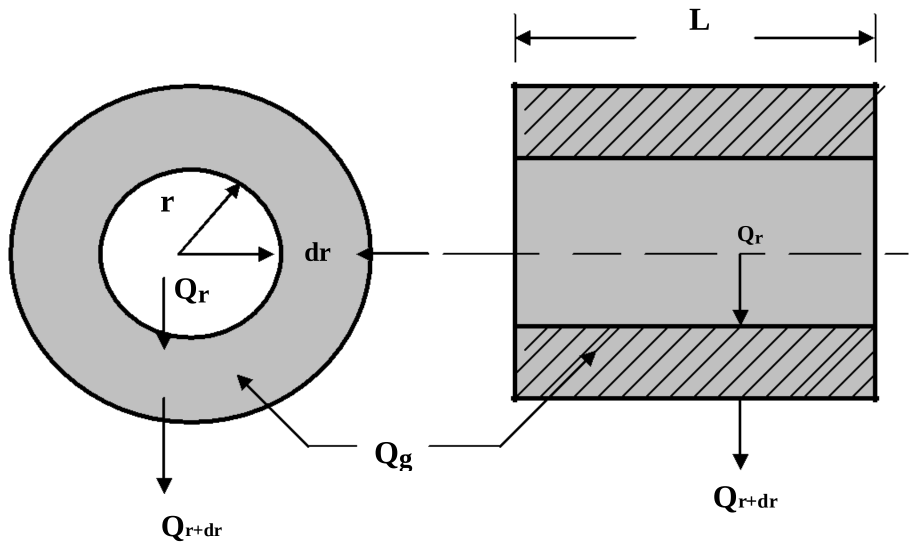
(a) Is heat transfer steady state or unsteady state?
(b) Is heat transfer one-, two- or three-dimension?
(c) Is there heat generation in the medium?
(d) Is the thermal conductivity constant or variable?

Consider the north wall of a house of thickness L. The outer surface of the wall exchanges heat by both convection and radiation. The interior of the house is maintained at a uniform temperature of T_i, while the exterior of the house is at a uniform temperature T_0. The sky, the ground, and the surfaces of the surrounding structures at this location can be modeled as a surface at an effective temperature of T_{sky} for radiation heat exchange on the outer surface. The radiation heat exchange between the inner surface of the wall and the surfaces of the other walls, floor and ceiling are negligible. The convective heat transfer coefficient for the inner and outer surfaces of the wall under consideration are h_i and h_0 respectively. The thermal conductivity of the wall material is K and the emissivity of the outer surface of the wall is „ε_0‟. Assuming the heat transfer through the wall is steady and one dimensional, express the mathematical formulation (differential equation and boundary conditions) of the heat conduction problem

Solutions to Tutorial Problems

A. Derivation of Conduction Equations:

Solution:-



A cylindrical element of thickness dr in the radial direction at a radius r is shown in the figure above. For unsteady state one dimensional radial conduction with heat generation is given by

$$Q_r + Q_g – Q_{r+dr} = (\partial E / \partial t)$$

Or $$Q_r + Q_g – [Q_r + (\partial Q_r \partial r)dr] = (\partial E / \partial t)$$

Or $$– (\partial Q_r \partial r)dr + Q_g = (\partial E / \partial t) \qquad (1)$$

where Qr is the rate of conduction into the element at radius r = – k $2\pi rL$ $(\partial T / \partial r)$

Qg is the rate of heat generation within the element = 2π rL dr q''''''

$(\partial E / \partial t)$ is the rate of increase of the energy of the element. = 2π rL dr $\rho C_p (\partial T / \partial t)$
where $dV=2\pi rLdr$ --------volume

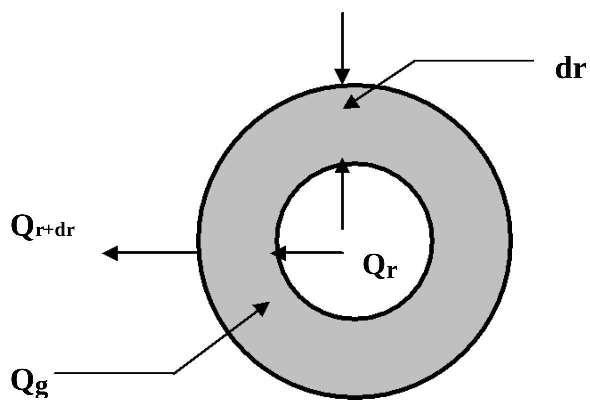
Substituting these expressions in Eq.(1) we get

$$[-\partial/\partial r(-2\pi rLk(\partial T/\partial r))]dr + 2\pi rL\, dr\, q'''''' = 2\pi rL\, dr\, \rho C_p (\partial T / \partial t)$$

Simplifying we get

$$(1/r)\, \partial/\partial r\,[kr(\partial T/\partial r)] + q'''''' = \rho C_p (\partial T / \partial t)$$

Solution:



Consider a spherical element of thickness dr at any radius r as shown in the figure above. The energy balance equation for one – dimensional radial unsteady state conduction with heat generation is given by

$$Q_r + Q_g - Q_{r+dr} = (\partial E / \partial t)$$

Or $$Q_r + Q_g - [Q_r + (\partial Q_r / \partial r)\, dr] = (\partial E / \partial t)$$

Or $$-(\partial Q_r / \partial r)\, dr + Q_g = (\partial E / \partial t) \qquad \text{...............................(1)}$$

Where Q_r = rate of heat conducted in to the element at radius r = $-k\, 4\pi r^2 (\partial T / \partial r)$,

Q_g = rate of heat generation within the element = $(4/3)\pi\, [(r + dr)^3 - r^3]\, q''''''$

$(\partial E / \partial t)$ = rate of increase of energy of the element = $\rho\,(4/3)\pi\,[\,(r + dr)^{3} - r^{3}\,](\partial T/\partial t)$ Now $(r + dr)^3 - r^3 = r^3 + 3r^2 dr + 3r(dr)^2 + (dr)^3 - r^3$

$$= 3r^2 dr + 3r(dr)^2 + (dr)^3$$

Neglecting higher order terms like $(dr)^3$ and $(dr)^2$ we have

$$(r + dr)^3 - r^3 = 3\,r^2\,dr.$$

Therefore $\quad Q_g = 4\,\pi\,r^2\,dr\,q'''$

And $\quad (\partial E / \partial t) = \rho\,4\,\pi\,r^2\,dr\,C_p(\partial T/\partial t).$

Substituting the expressions for Q_r, Q_g and $(\partial E / \partial t)$ in Eq. (1) we have

$$-\,[\partial /\partial r\{- k\,4\pi\,r^2\,(\partial T / \partial r)\}]dr + 4\,\pi\,r^2\,dr\,q''' = \rho\,4\,\pi\,r^2\,dr\,C_p(\partial T/\partial t)$$

Simplifying the above equation and noting that if k is given to be constant we have

$$\partial /\partial r\{\,r^2\,(\partial T / \partial r)\} + r^2\,(\,q'''/ k) = (\rho\,r^2\,C_p / k)(\partial T/\partial t)$$

Or $\quad (1 / r^2)\,\partial /\partial r\{\,r^2\,(\partial T / \partial r)\} + (\,q'''/ k) = (1 / \alpha)\,(\partial T/\partial t)$; where $\alpha = k / (\rho\,C_p)$

Solution:- (a) The general form of conduction equation for an isotropic solid in rectangular coordinate system is given by

$$\partial / \partial x\,(k\partial T / \partial x) + \partial / \partial y\,(k\partial T / \partial y) + \partial / \partial z\,(k\partial T / \partial z) + q''' = (\rho\,C_p)\,(\partial T / \partial t) \quad \ldots\ldots\ldots\ldots..(1)$$

For steady state conduction $(\partial T / \partial t) = 0$; For one dimensional conduction in x – direction we have

$$\partial T / \partial y = \partial T / \partial z = 0\ .\ \text{Therefore}\ \partial T / \partial x = dT / dx\ .$$

Therefore Eq. (1) reduces to

$$d / dx\,(k\,dT / dx) + q''' = 0.$$

For constant thermal conductivity the above equation reduces to

$$d^2\,T / dx^2 + q''' / k = 0.$$

(b) The general form of conduction equation in cylindrical coordinate system is given by $(1 / r)\,\partial / \partial r\,(kr\,\partial T / \partial r) + (1 / r^2)\,\partial / \partial\theta(k\,\partial T / \partial\theta) + \partial / \partial z\,(k\partial T / \partial z) + q''' = \rho C_p(\,\partial T / \partial t)$

For steady state conduction, ($\partial T / \partial t$) = 0 ; For one-dimensional radial conduction we have

$\partial T / \partial \theta = 0$ and $\partial T / \partial z = 0$. Therefore $\partial T / \partial r = dT / dr$. With these simplifications the general form of conduction equation reduces to

$$(1 / r)\, d / dr\, (kr\, dT/dr) + q'''''' = 0$$

For constant thermal conductivity the above equation reduces to

$$(1 / r)\, d / dr\, (r\, dT/dr) + q'''''' / k = 0.$$

© The general form of conduction equation in spherical coordinate system is given by $(1/r^2)\, \partial / \partial r(kr^2\, \partial T / \partial r) + \{1/(r^2 \sin^2 \theta)\}\partial / \partial \theta\, (k\, \partial T/\partial \theta)$

$$+ \{1/(r^2 \sin \theta)\}\, \partial / \partial \theta\, (k \sin \theta\, \partial T/\partial \theta) + q'''''' = \rho C_p\, (\partial T/ \partial t) \qquad \text{..........(1)}$$

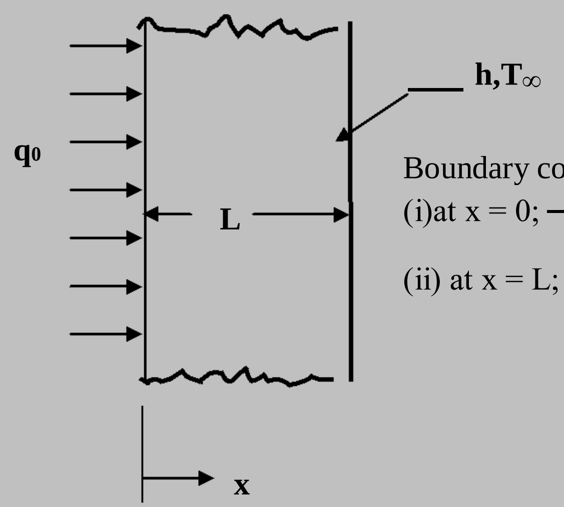
For steady state conduction ($\partial T\ \partial t$) = 0 ; For one dimensional radial conduction we have $\partial T/\partial \theta = 0$ and $\partial T/\partial \theta = 0$. Therefore $\partial T / \partial r = dT / dr$. Substituting these conditions in Eq. (1) we have $(1/r^2)\, d / dr\, (kr^2\, dT / dr) + q'''''' = 0$.

For constant thermal conductivity the above equation reduces to

$$(1/r^2)\, d / dr\, (r^2\, dT / dr) + q'''''' / k = 0.$$

B. Mathematical Formulation of the Boundary Conditions:

Solution:-

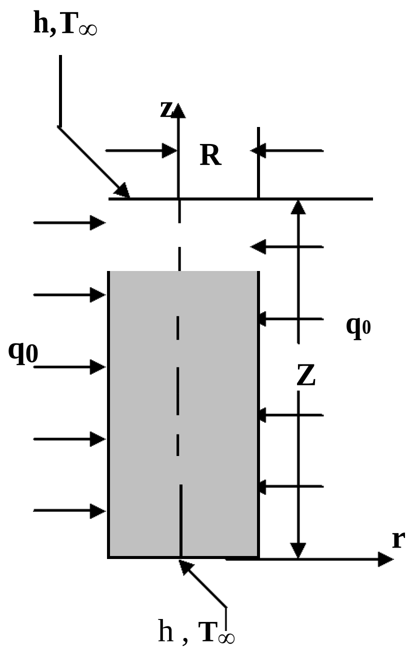


Boundary conditions are :

(i)at $x = 0; -k\,(dT / dx)_{x=0} = q_0$

(ii) at $x = L; -k(dT / dx)_{x=L} = h(T|_{x=L} - T_\infty)$

2.5. Solution:-



Boundary conditions are:

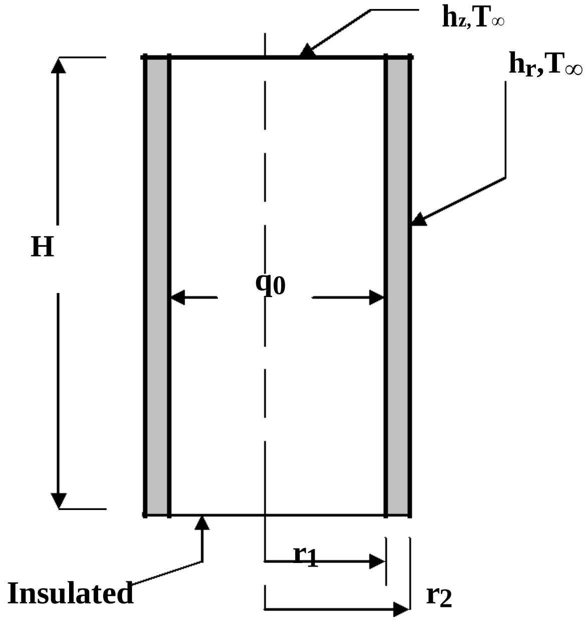
(i) at $r = 0$; $(\partial T/\partial r) = 0$ (axis of symmetry)

(ii) at $r = R$; $k(\partial T/\partial r) = q_0$

(iii) at $z = 0$; $h[T|_{z=0} - T_\infty] + k(\partial T/\partial z)_{z=0} = 0$

(iv) at $z = Z$; $-k(\partial T/\partial r) = h\,[\,T|_{z=z} - T_\infty]$

Solution:-



Boundary conditions are:

(i) at $r = r_1$, $-k(\partial T/\partial r) = q_0$ for all z;

(ii) at $r = r_2$, $-k(\partial T/\partial r) = h_r[T|_{r=r_2} - T_\infty]$ for all z

(iii) at $z = 0$, $(\partial T/\partial z) = 0$ for all r.

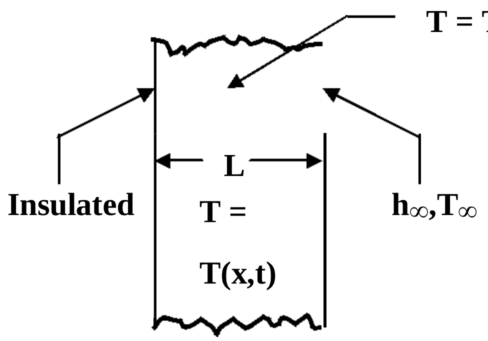
(iv) at $z = H$,

$-k(\partial T/\partial z)_{z=H} = h_z[T|_{z=H} - T_\infty]$ for all r

from the problem: $h_z=h_r=h$

C. Mathematical Formulation of Conduction Problems:

Solution:-



Governing differential equation to determine T(x,t) is given by

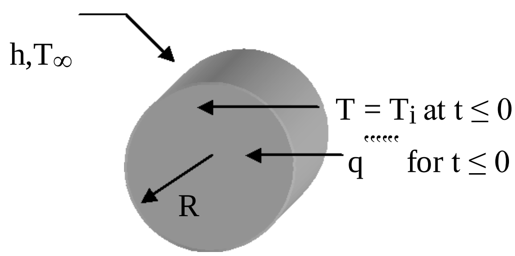
$(\partial^2 T / \partial x^2) = (1/\alpha)(\partial T / \partial t)$

where α is the thermal diffusivity of the wall. Initial condition is

at time t = 0 $T = T_i$ for all x.

The boundary conditions are : (i) at x = 0, $(\partial T / \partial x)_{x=0} = 0$. (Insulated) for all t >0 (ii) at x = L, $-k\,(\partial T / \partial x)_{x=L} = h_\infty\,[T|_{x=L} - T_\infty]$ for all t>0

Solution:-



The governing differential equation to determine T(r,t) is given by

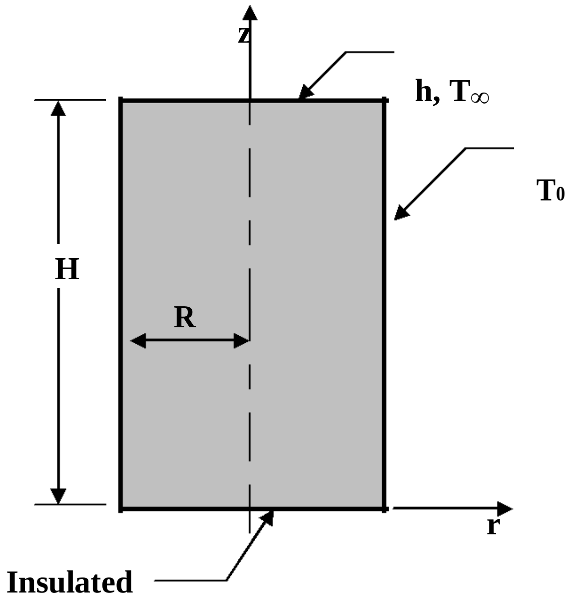
$(1/r)\,\partial / \partial r\,(r\,\partial T / \partial r) + q_0 / k = (1/\alpha)\,(\partial T / \partial t)$.

Boundary conditions are: (i) at r = 0, $(\partial T / \partial r) = 0$ (Axis of symmetry)

(ii) at r = R, $-k\,(\partial T / \partial r)|_{r=R} = h\,[T|_{r=R} - T_\infty]$

Initial condition is: At t = 0, $T = T_i$: for all r

Solution:-



The governing differential equation to determine T(r,z) is given by

$$(1/r)\partial/\partial r(r\partial T/\partial r) + \partial^2 T/\partial z^2 + q'''/k = 0$$

Boundary conditions are:

(i) at $r = 0$, $\partial T/\partial r|_{r=0} = 0$, for all z (axis of symmetry).

(ii) at $r = R$, $T = T_0$ for all z.

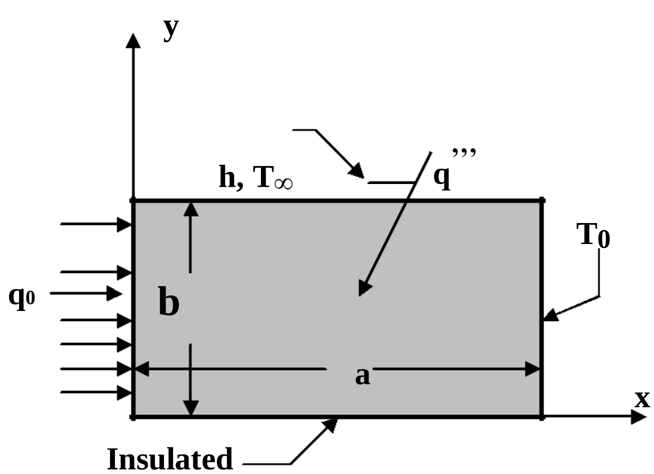
(iii) at $z = 0$, $\partial T/\partial z|_{z=0} = 0$ for all r.

(iv) at $z = H$,

$$-k(\partial T/\partial z)_{z=H} = h(T|_{z=H} - T_\infty)$$

Solution:-

for all x.



The governing differential equation to determine T(x,y) is given by

$$\partial^2 T/\partial x^2 + \partial^2 T/\partial y^2 + q'''/k = 0$$

Boundary conditions are:

(i) at x=0, $-k(\partial T / \partial x)|_{x=0} = q_0$ for all y ; (ii) at x = a, $T = T_0$ for all y

(iii) at y = 0, $\partial T / \partial y = 0$ for all x ; (iv) at y = b, $-k(\partial T / \partial y)|_{y=b} = h[T|_{y=b} - T_\infty]$.

Solution: The given differential equation is

$$\partial^2 T / \partial x^2 = (1/\alpha)(\partial T / \partial t)$$

It can be seen from this equation that T depends on one space variable x and the time variable t. Hence the problem is one dimensional transient conduction problem. No heat generation term appears in the equation indicating that the medium is not generating any heat. The thermal conductivity of the medium does not appear within the differential symbol indicating that the conductivity of the medium is constant.

Solution: The given differential equation is

$$(1/r)\, d / dr(r\, k\, dT/dr) + q''' = 0.$$

It can be seen from this equation that the temperature T depends only on one space variable „r‟ and it does not depend on time t. Also the heat generation term q‟‟‟ appears in the differential equation. Hence the problem is a one-dimensional steady state conduction problem with heat generation. Since the thermal conductivity appears within the differential symbol, it follows that the thermal conductivity of the medium is not a constant but varies with temperature.

Solution: The given differential equation is

$$(1/r)\, \partial/\partial r\, (k\, r\, \partial T /\partial r) + \partial/\partial z\, (k\, \partial T /\partial z) + q''' = 0$$

It can be seen from the above equation that the temperature T depends on two space variables r and z and does not depend on time. There is the heat generation term appearing in the equation and the thermal conductivity k appears within the differential symbol $\partial/\partial r$ and $\partial/\partial z$. Hence the problem is two-dimensional steady state conduction with heat generation in a medium of variable thermal conductivity.

Solution: The given differential equation is

$$(1/r^2)\, \partial/\partial r\, (r^2\, \partial T /\partial r) + \frac{1}{r^2 \sin^2 \varphi} \left[\frac{\partial^2 T}{\partial \theta^2} \right] = (1/\alpha)(\partial T/\partial t)$$

It can be seen from the given equation that the temperature T depends two space variables r and θ and it also depends on the time variable t. There is no heat generation term appearing in the given equation. Also the thermal conductivity k do not appear within the differential symbol. Hence the given equation represents two-dimensional, steady state conduction in a medium of constant thermal conductivity and the medium is not generating any heat.

2.15. Solution:

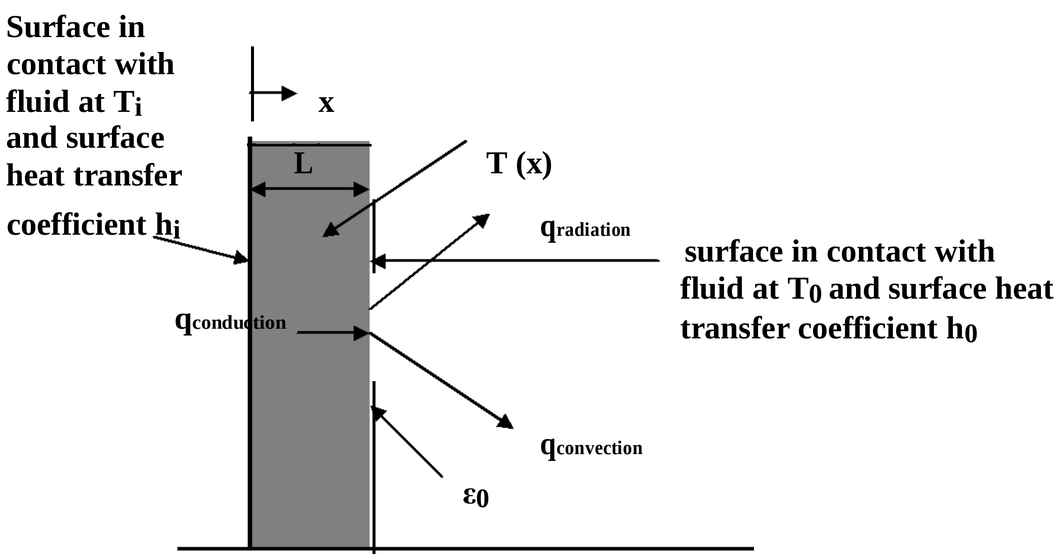


Fig. P.2.15: Schematic for problem 2.15.

The problem is one-dimensional steady state conduction without any heat generation and the wall is of constant thermal conductivity. Hence the governing differential equation is $d^2T / dx^2 = 0$.

The boundary conditions are:

(i) at x = 0, $h_i [T_i - T|_{x=0}] = -k\,(dT/dx)|_{x=0}$;

(ii) at x = L, $q_{conduction} = q_{convection} + q_{radiation}$

Or $\quad -k\,(dT/dx)|_{x=L} = h_0[T|_{x=L} - T_0] + \varepsilon_0\, \zeta\, [\{T|_{x=L}\}^4 - T_{sky}{}^4\}$

CHAPTER 3

CONVECTIVE HEAT TRANSFER

Introduction:- In this chapter the problems of one-dimensional steady state conduction without and with thermal energy generation in slabs, cylinders and spheres and subjected to different types of boundary conditions are analyzed to determine the temperature distribution and rate of heat flow. The concept of thermal resistance is introduced and the use of this concept, for solving conduction in composite layers is illustrated. The problem of critical thickness of insulation for cylinder and sphere are also analyzed. The effects of variable thermal conductivity on temperature distribution and rate of heat transfer are also studied. Finally the problems of one dimensional heat conduction in extended surfaces (fins) subjected to different types of boundary conditions are examined.

Conduction Without Heat Generation

The Plane Wall (The Slab):- The statement of the problem is to determine the temperature distribution and rate of heat transfer for one dimensional steady state conduction in a plane wall without heat generation subjected to specified boundary conditions.

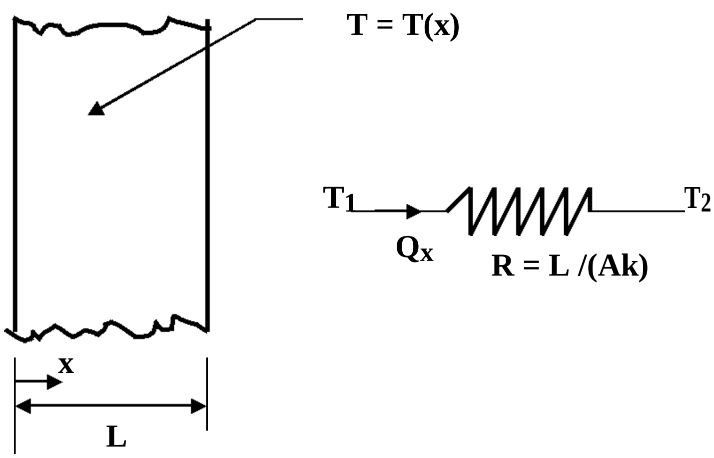


Fig. 3.1: One dimensional steady state conduction in a slab

The governing equation for one − dimensional steady state conduction without heat generation is given by

$$\frac{d^2T}{dx^2} = 0 \quad \text{……………………………………}(3.1)$$

Integrating Eq. (3.1) twice with respect to x we get

$$T = C_1x + C_2 \quad \text{………………………………}(3.2)$$

where C_1 and C_2 are constants which can be evaluated by knowing the boundary conditions.

Plane wall with specified boundary surface temperatures:- If the surface at x = 0 is maintained at a uniform temperature T_1 and the surface at x = L is maintained at another uniform temperature T_2, then the boundary conditions can be written as follows:
(i) at x = 0, $T(x) = T_1$; (ii) at x = L, $T(x) = T_2$.

Condition (i) in Eq.(3.2) gives $\quad T_1 = C_2.$

Condition (ii) in Eq. (3.2) gives $\quad T_2 = C_1L + T_1$

Or $\quad C_1 = \dfrac{T_2 - T_1}{L}.$

Substituting for C_1 and C_2 in Eq. (3.2), we get the temperature distribution in the plane wall as

$$T(x) = (T_2 - T_1)\frac{x}{L} + T_1$$

Or

$$\mathbf{\frac{T(x) - T_1}{(T_2 - T_1)} = \frac{x}{L} \quad \text{……………………………..}(3.3)}$$

Expression for Rate of Heat Transfer:

The rate of heat transfer at any section x is given by Fourier‟s law as

$$Q_x = - k\,A(x)\,(dT / dx)$$

For a plane wall A(x) = constant = A. From Eq. (3.3), $dT/dx = (T_2 - T_1) / L$.

Hence $\quad Q_x = - k\,A\,(T_2 - T_1) / L.$

$$\text{Or} \qquad Q_x = \frac{kA(T_1 - T_2)}{L} \qquad \text{.......(3.4)}$$

Concept of thermal resistance for heat flow:

It can be seen from the above equation that Q_x is independent of x and is a constant. Eq. (3.4) can be written as

$$Q_x = \frac{(T_1 - T_2)}{\{L/(kA)\}} = \frac{(T_1 - T_2)}{R} \qquad \text{.......(3.5)}$$

Where R = L / (A k).

Eq. (3.5) is analogous to Ohm‟s law for flow of electric current. In this equation $(T_1 - T_2)$ can be thought of as *"thermal potential"*, R can be thought of as *"thermal resistance"*, so that the plane wall can be represented by an equivalent *"thermal circuit"* as shown in Fig.3.1.The units of thermal resistance R are

K / W.

Plane wall whose boundary surfaces subjected to convective boundary conditions:

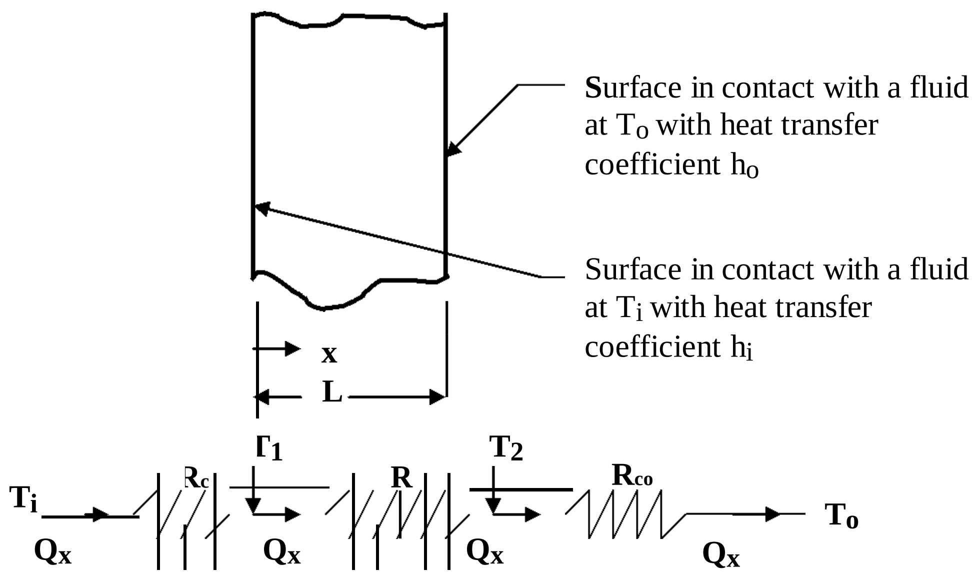


Fig.3.2: Thermal Circuit for a plane wall with convective boundary conditions

Let T_1 be the surface temperature at x = 0 and T_2 be the surface temperature at x = L. If we assume that $T_i > T_o$, then for steady state conduction heat will transfer by convection from the fluid at T_i to the surface at x = 0, then it is conducted across the plane wall and finally heat is transferred by convection from the surface at x = l to the fluid at T_o.

The expression for rate of heat transfer Q_x can be written as follows:

$$Q_x = h_i A [T_i - T_1]$$

or
$$Q_x = \frac{(T_i - T_1)}{1 / (h_i A)} = \frac{(T_i - T_1)}{R_{ci}} \quad \text{..................................... (3.6a)}$$

$R_{ci} = 1 / (h_i A)$ is called thermal resistance for convection at the surface at x = 0

Similarly
$$Q_x = \frac{(T_1 - T_2)}{R} \quad \text{...(3.6b)}$$

where R = L /(Ak) is the thermal resistance offered by the wall for conduction and

$$Q_x = \frac{(T_2 - T_o)}{R_{co}} \quad \text{.. (3.6c)}$$

Where $R_{co} = 1 / (h_o A)$ is the thermal resistance offered by the fluid at the surface at x = L for convection. It follows from Equations (3.6a), (3.6b) and (3.6c) that

$$Q_x = \frac{(T_i - T_1)}{R_{ci}} = \frac{(T_1 - T_2)}{R} = \frac{(T_2 - T_0)}{R_{co}}$$

Or
$$\mathbf{Q_x = \frac{(T_i - T_o)}{[R_{ci} + R + R_{co}]}} \quad \textbf{......................................(3.7)}$$

Radial Conduction in a Hollow Cylinder:

The governing differential equation for one-dimensional steady state radial conduction in a hollow cylinder of constant thermal conductivity and without thermal energy generation is given by Eq.(2.10b) with n = 1: i.e.,

$$\frac{d}{dr} [r (dT / dr)] = 0 \quad \text{.......................................(3.8)}$$

Integrating the above equation once with respect to „r‟ we get

$$r (dT / dr) = C_1$$

or
$$(dT / dr) = C_1 / r$$

Integrating once again with respect to „r‟ we get

$$T(r) = C_1 \ln r + C_2 \quad \text{...(3.9)}$$

where C_1 and C_2 are constants of integration which can be determined by knowing the boundary conditions of the problem.

Hollow cylinder with prescribed surface temperatures: Let the inner surface at $r = r_1$ be maintained at a uniform temperature T_1 and the outer surface at $r = r_2$ be maintained at another uniform temperature T_2 as shown in Fig. 3.3.

Substituting the condition at r_1 in Eq.(3.9) we get

$$T_1 = C_1 \ln r_1 + C_2 \quad \text{..(3.10a)}$$

and the condition at r_2 in Eq. (3.9) we get

$$T_2 = C_1 \ln r_2 + C_2 \quad \text{..(3.10b)}$$

Solving for C_1 and C_2 from the above two equations we get

$$C_1 = \frac{(T_1 - T_2)}{[\ln r_1 - \ln r_2]} = \frac{(T_1 - T_2)}{\ln (r_1 / r_2)}$$

and $$C_2 = T_1 - \frac{(T_1 - T_2)}{\ln (r_1 / r_2)} \ln r_1$$

Substituting these expressions for C_1 and C_2 in Eq. (3.9) we have

$$T(r) = \frac{(T_1 - T_2)}{\ln (r_1 / r_2)} \ln r + T_1 - \frac{(T_1 - T_2)}{\ln (r_1 / r_2)} \ln r_1$$

or $$\boldsymbol{\frac{[T(r) - T_1]}{[T_2 - T_1]} = \frac{\ln (r / r_1)}{\ln (r_2 / r_1)}} \quad \textbf{..(3.11)}$$

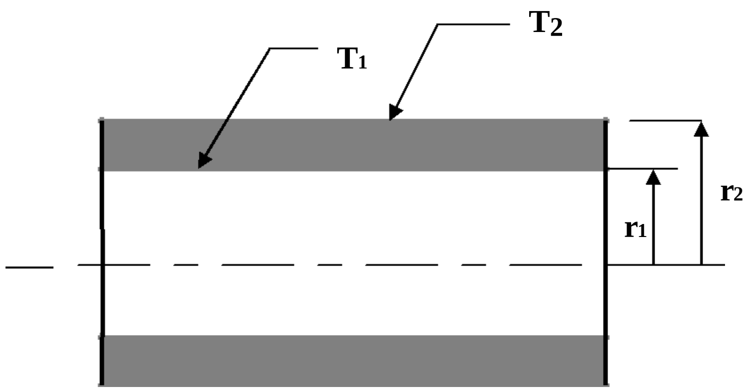


Fig.3.3: Hollow cylinder with prescribed surface temperatures

Eq. (3.11) gives the temperature distribution with respect to the radial direction in a hollow cylinder. The plot of Eq. (3.11) is shown in Fig. 3.4.

Expression for rate of heat transfer:- For radial steady state heat conduction in a hollow cylinder without heat generation energy balance equation gives

$$Q_r = Q_r|_{r = r1} = Q_r|_{r = r2}$$

Hence $$Q_r = - k\,[A(r)\,(dT / dr)]\,|_{r = r1} \quad \text{......(3.12)}$$

Now $A(r)\,|_{r = r1} = 2\,\pi\, r_1\, L$ .From Eq. (3.11) we have

$$(dT / dr) = \{[\,T_2 - T_1] / \ln (r_2 / r_1)\,\}(1/r)$$

Hence $$(dT / dr)|_{r = r1} = \{[\,T_2 - T_1] / \ln (r_2 / r_1)\,\}(1/\,r_1).$$

Substituting the expressions for $A(r)|_{r = r1}$ and $(dT / dr)|_{r = r1}$ in Eq. (3.12) we get the expression for rate of heat transfer as

$$Q_r = \frac{2\,\pi\, L\, k\,(T_1 - T_2)}{\ln (r_2 / r_1)} \quad \text{......(3.13)}$$

Thermal resistance for a hollow cylinder: Eq. 3.13 can be written as:

$$Q_r = (T_1 - T_2) / R \quad \text{......(3.14a)}$$

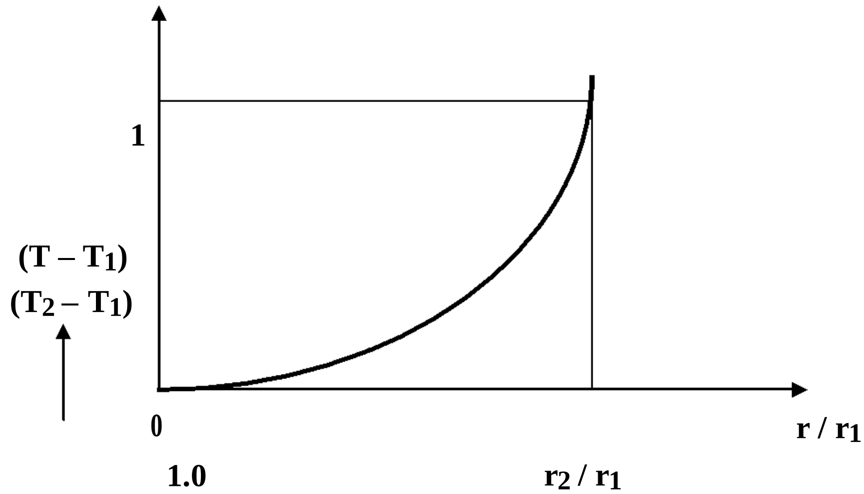


Fig. 3.4: Radial temperature distribution for a hollow cylinder

where $$R = \frac{\ln (r_2 / r_1)}{2 \pi L k} = \frac{1}{k A_m} \quad \text{.........} (3.14b)$$

Where $A_m = (A_2 - A_1) / \ln (A_2 / A_1)$, when $A_2 = 2\pi r_2 L$ = Area of the outer surface of the cylinder and $A_1 = 2\pi r_1 L$ = Area of the inner surface of the cylinder, and A_m is logarithmic mean area.

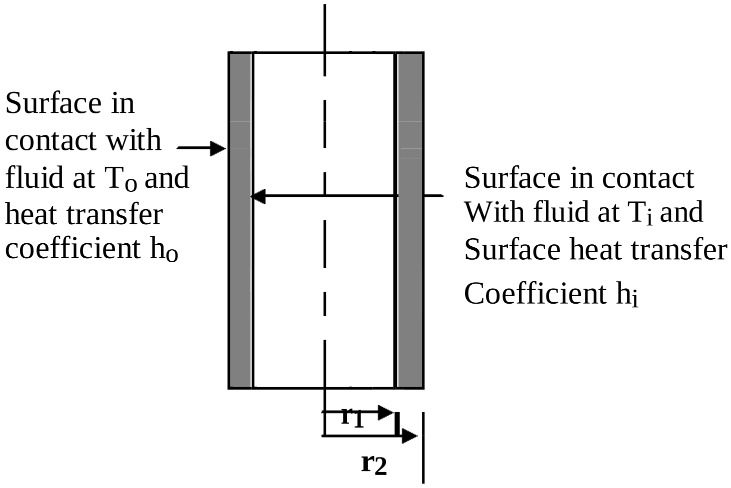
Hollow cylinder with convective boundary conditions at the surfaces:- Let for the hollow cylinder, the surface at $r = r_1$ is in contact with a fluid at temperature T_i with a surface heat transfer coefficient h_i and the surface at $r = r_2$ is in contact with another fluid at a temperature T_o as shown in Fig.3.5.By drawing the thermal circuit for this problem and using the concept of thermal resistance it is easy and straight forward to write down the expression for the rate of heat transfer as shown.

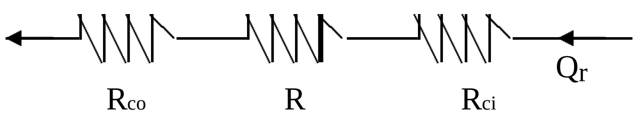
Now $$Q_r = h_i A_i (T_i - T_1) = 2\pi r_1 L h_i (T_i - T_1) = \frac{(T_i - T_o)}{R_{ci}} \quad \text{.........} (3.15a)$$

where $R_{ci} = 1 / (2\pi r_1 L h_i)$… .... (3.15b)

Also $$Q_r = \frac{(T_1 - T_2)}{R} \quad \text{.........} (3.15c)$$

where $R = \ln(r_2 / r_1) / (2\pi Lk)$ ....(3.15d)





And $$Q_r = \frac{(T_2 - T_o)}{R_{co}} \quad \text{......(3.15e)}$$

Where $$R_{co} = \frac{1}{(2\pi r_2 L h_o)} \quad \text{......(3.15f)..}$$

From Eqs.(3.15a), (3.15c) and (3.15e) we have

$$Q_r = \frac{(T_i - T_1)}{R_{ci}} = \frac{(T_1 - T_2)}{R} = \frac{(T_2 - T_o)}{R_{co}}$$

Or $$Q_r = \frac{\mathbf{(T_i - T_o)}}{\mathbf{R_{ci} + R + R_{co}}} \quad \textbf{......(3.16)}$$

where R_{ci}, R and R_{co} are given by Eqs.(3.15b), (3.15d) and (3.15f) respectively.

Radial Conduction in a Hollow Sphere:

The governing differential equation for one-dimensional steady state radial conduction in a hollow sphere without thermal energy generation is given by Eq.(2.10b) with n = 1: i.e.,

$$\frac{d}{dr}\left[r^2 (dT / dr)\right] = 0 \quad \text{.......................................} (3.17)$$

Integrating the above equation once with respect to „r" we get

$$r^2 (dT / dr) = C_1$$

or $$(dT / dr) = C_1 / r^2$$

Integrating once again with respect to „r" we get

$$T(r) = - C_1 / r + C_2 \quad \text{..} (3.18)$$

where C_1 and C_2 are constants of integration which can be determined by knowing the boundary conditions of the problem.

Hollow sphere with prescribed surface temperatures:

(i) Expression for temperature distribution:-Let the inner surface at $r = r_1$ be maintained at a uniform temperature T_1 and the outer surface at $r = r_2$ be maintained at another uniform temperature T_2 as shown in Fig. 3.6.

The boundary conditions for this problem can be written as follows:

(i) at $r = r_1$, $T(r) = T_1$ and (ii) at $r = r_2$, $T(r) = T_2$.

Condition (i) in Eq. (3.18) gives $T_1 = - C_1 / r_1 + C_2$ .... (3.19a)

Condition (ii) in Eq. (3.18) gives $T_2 = - C_1 / r_2 + C_2$ .... (3.19b)

Solving for C_1 and C_2 from Eqs. (3.19a) and (3.19b) we have

$$C_1 = \frac{(T_1 - T_2)}{[1 / r_2 - 1 / r_1]} \text{ and } C_2 = T_1 + \frac{(T_1 - T_2)}{r_1[1 / r_2 - 1 / r_1]}$$

Substituting these expressions for C_1 and C_2 in Eq. (3.18) we get

$$T(r) = - \frac{(T_1 - T_2) / r}{[1 / r_2 - 1 / r_1]} + T_1 + \frac{(T_1 - T_2) / r_1}{[1 / r_2 - 1 / r_1]}$$

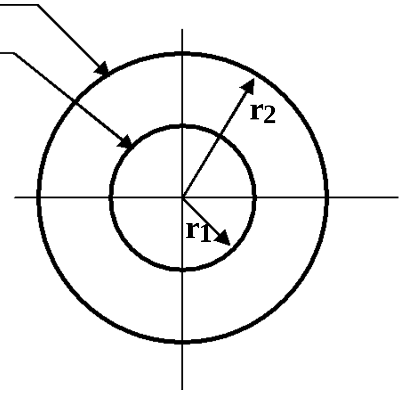


Fig. 3.6: Radial conduction in a hollow sphere with prescribed surface temperatures

Or
$$\frac{T(r) - T_1}{[T_1 - T_2]} = \frac{[1/r_2 - 1/r]}{[1/r_2 - 1/r_1]} \quad \textbf{.............(3.20)}$$

(ii) <u>*Expression for Rate of Heat Transfer:-*</u> The rate of heat transfer for the hollow sphere is given by

$$Q_r = -k\, A(r)(d\,T / dr) \quad \text{.............. (3.21)}$$

Now at any radius for a sphere $A(r) = 4\pi\, r^2$ and from Eq. (3.20)

$$dT / dr = [T_1 - T_2] \frac{1}{[1/r_2 - 1/r_1]} (1/r^2)$$

Substituting these expressions in Eq. (3.21) and simplifying we get

$$\mathbf{Q_r = \frac{4\,\pi\, k\, r_1\, r_2\, [T_1 - T_2]}{[r_2 - r_1]}} \quad \textbf{.............(3.22)}$$

Eq.(3.22) can be written as $\mathbf{Q_r = [T_1 - T_2] / R}$ **.............(3.23a)**

Where R is the thermal resistance for the hollow sphere and is given by

$$\mathbf{R = (r_2 - r_1) / \{4\,\pi\, k\, r_1\, r_2\}} \quad \textbf{.............(3.23b)}$$

Hollow sphere with convective conditions at the surfaces: - Fig. 3.7 shows a hollow sphere whose boundary surfaces at radii r_1 and r_2 are in contact with fluids at temperatures T_i and T_0 with surface heat transfer coefficients h_i and h_0 respectively.

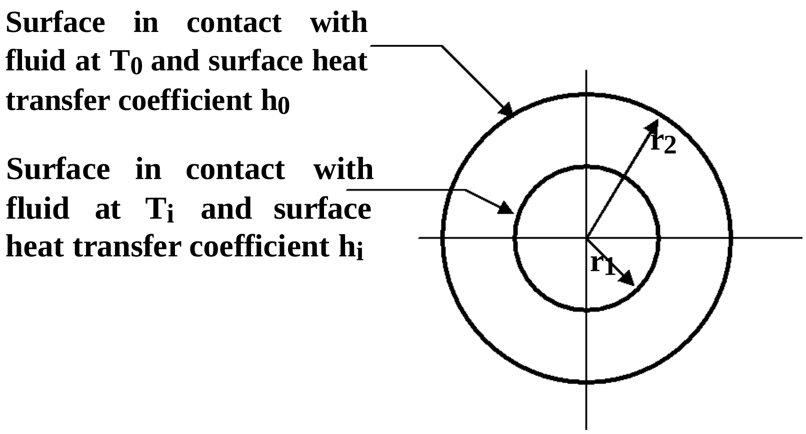


Fig. 3.7: Radial conduction in a hollow sphere with convective conditions at the two boundary surfaces

The thermal resistance network for the above problem is shown in Fig.3.8

$$Q_{ci} = Q_r = Q_{co} \quad \text{.............................} (3.24)$$

Where Q_{ci} = heat transfer by convection from the fluid at T_i to the inner surface of the hollow sphere and is given by

$$Q_{ci} = h_i A_i [T_i - T_1] = \frac{[T_i - T_1]}{R_{ci}} \quad \text{.....}(3.25)$$

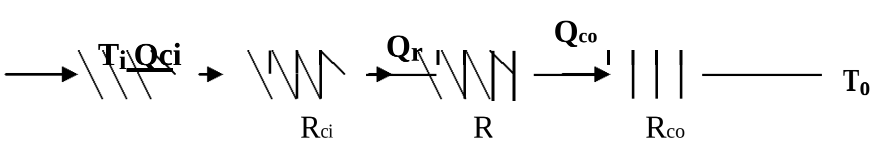


Fig. 3.8: Thermal circuit for a hollow sphere with convective boundary conditions 3.12

When T_1 = the inside surface temperature of the sphere and

$R_{ci} = 1 / (h_i A_i)$ = the thermal resistance for convection for the inside surface

Or $R_{ci} = 1 / (4 \pi r_1^2 h_i)$.. (3.25b)

Q_r = Rate of heat transfer by conduction through the hollow sphere

$= [T_1 - T_2] / R$ with $R = (r_2 - r_1) / \{4 \pi k r_1 r_2\}$

And Q_{co} = Rate of heat transfer by convection from the outer surface of the sphere to the outer fluid and is given by

$$Q_{co} = h_o A_o [T_2 - T_o] = \frac{[T_2 - T_0]}{R_{co}} \quad \text{....................(3.26a)}$$

Where T_2 = outside surface temperature of the sphere and

A_o = outside surface area of the sphere = $4 \pi r_2^2$ so that

$$R_{co} = 1 / \{4 \pi r_2^2 h_o\} \quad \text{..(3.26b)}$$

Now Eq.(3.24) can be written as

$$Q_r = h_i A_i [T_i - T_1] = \frac{[T_i - T_1]}{R_{ci}} = \frac{[T_1 - T_2]}{R} = \frac{[T_2 - T_1]}{R_{co}}$$

$$\mathbf{Q_r = \frac{[T_i - T_o]}{[R_{ci} + R + R_{co}]}} \quad \textbf{...(3.27)}$$

<u>Steady State conduction in composite medium:</u>

There are many engineering applications in which heat transfer takes place through a medium composed of several different layers, each having different thermal conductivity. These layers may be arranged in series or in parallel or they may be arranged with combined series-parallel arrangements. Such problems can be conveniently solved using electrical analogy as illustrated in the following sections.

Composite Plane wall:- **(i) Layers in series:** Consider a plane wall consisting of three layers in series with perfect thermal contact as shown in Fig. 3.10.The equivalent thermal resistance network is also shown. If Q is the rate of heat transfer through an area A of the composite wall then we can write the expression for Q as follows:

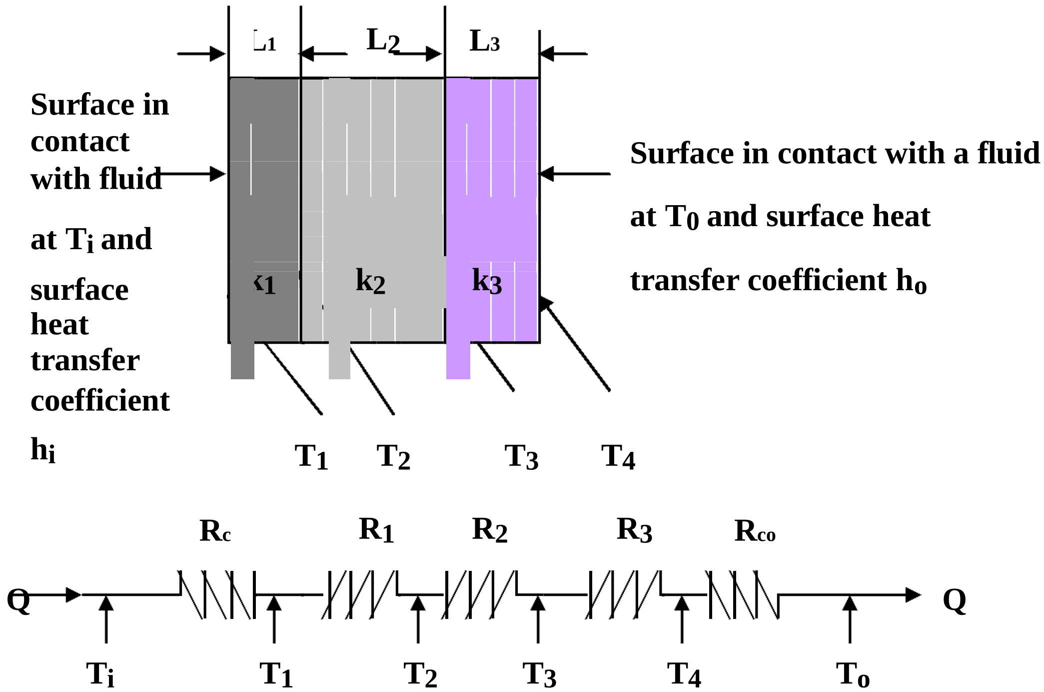


Fig. 3.10: A composite plane wall with three layers in series and the equivalent thermal resistance network

$$Q = \frac{(T_2 - T_3)}{R_{co}} = \frac{(T_1 - T_2)}{R_1} = \frac{(T_1 - T_2)}{R_2} = \frac{(T_2 - T_3)}{R_3} = \frac{(T_3 - T_{co})}{R_{co}}$$

$$\text{Or } Q = \frac{(T_i - T_0)}{R_{ci} + R_1 + R_2 + R_3 + R_{co}} = \frac{T_i - T_0)}{R_{total}} \quad \text{.. (3.28)}$$

Overall heat transfer coefficient for a composite wall: **-** It is sometimes convenient to express the rate of heat transfer through a medium in a manner which is analogous to the Newton‟s law of cooling as follows:

If U is the overall heat transfer coefficient for the composite wall shown in Fig. (3.10) then

$$Q = U\,A\,(T_i - T_o) \quad \text{..(3.29)}$$

Comparing Eq. (3.28) with Eq. (3.29) we have the expression for U as

$$U = \frac{1}{A\,R_{total}} \quad \text{..(3.30)}$$

$$\text{Or } U = \frac{1}{A\,[R_{ci} + R_1 + R_2 + R_3]} = \frac{1}{A[1/(h_iA) + L_1/(Ak_1) + L_2/(Ak_2) + L_3/(Ak_3)]}$$

$$\text{Or } \mathbf{U} = \frac{\mathbf{1}}{[\,1/h_i + L_1/k_1 + L_2/k_2 + L_3/k_3\,]} \quad \text{.....(3.31)}$$

(ii) Layers in Parallel:- Fig.3.11 shows a composite plane wall in which three layers are

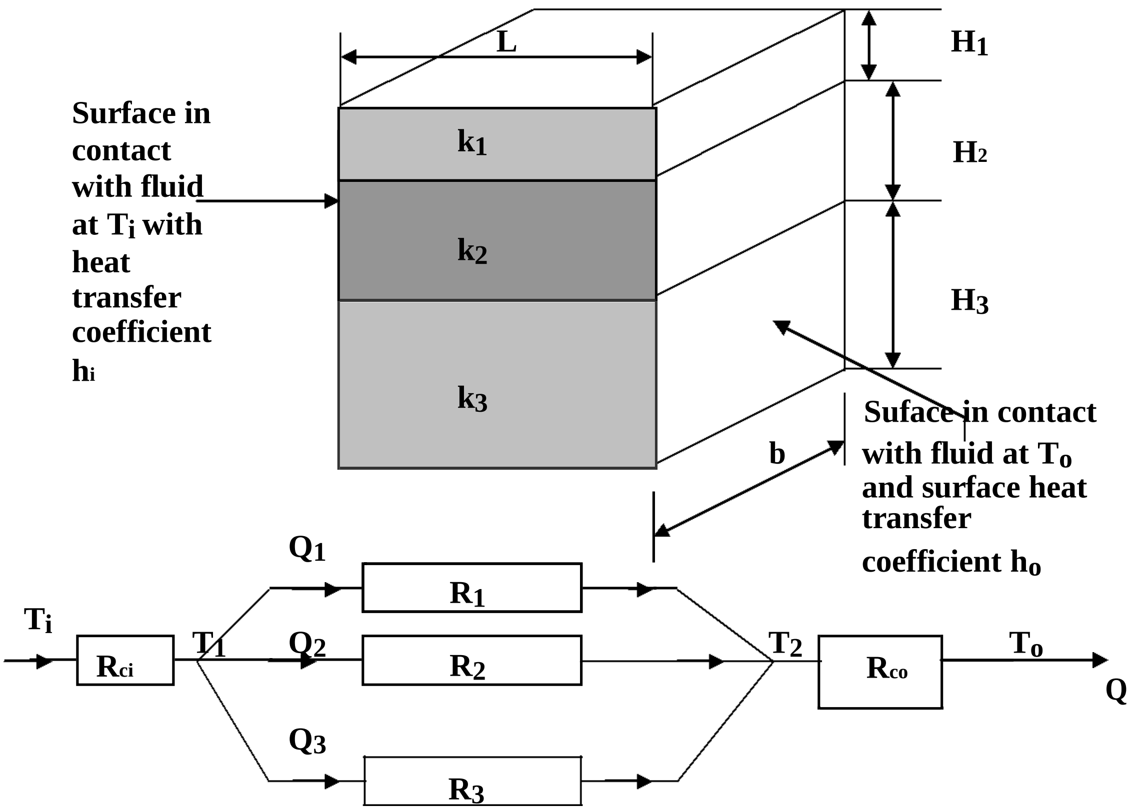


Fig. 3.11: Schematic and equivalent thermal circuit for a composite wall with layers in parallel

arranged in parallel. Let „b" be the dimension of these layers measured normal to the plane of the paper. Let one surface of the composite wall be in contact with a fluid at temperature T_i and surface heat transfer coefficient h_i and the other surface of the wall be in contact with another fluid at temperature T_o with surface heat transfer coefficient h_o. The equivalent thermal circuit for the composite wall is also shown in Fig. 3.11. The rate of heat transfer through the composite wall is given by

$$Q = Q_1 + Q_2 + Q_3 \quad \text{.....(3.32)}$$

where Q_1 = Rate of heat transfer through layer 1,

Q_2 = Rate of heat transfer through layer 2, and

Q_3 = Rate of heat transfer through layer 3.

$$\text{Now } Q_1 = \frac{(T_1 - T_2)}{R_1} \qquad \text{.......(3.33a)}$$

Where $R_1 = \{L / (H_1 b k_1)\}$

$$\text{Similarly } Q_2 = \frac{(T_1 - T_2)}{R_2} \qquad \text{.......(3.33b)}$$

Where $R_2 = \{L / (H_2 b k_2)\}$

$$\text{and } Q_3 = \frac{(T_1 - T_2)}{R_3} = \qquad \text{.......(3.33c)}$$

Where $R_3 = \{L / (H_3 b k_3)\}$

Substituting these expressions in Eq. (3.32) and simplifying we get

$$Q = \frac{(T_1 - T_2)}{R_1} + \frac{(T_1 - T_2)}{R_2} + \frac{(T_1 - T_2)}{R_3} = \frac{(T_1 - T_2)}{R_e} \qquad \text{.......(3.34)}$$

Where $1 / R_e = 1/R_1 + 1/R_2 + 1/R_3$

$$\text{Hence } Q = \frac{(T_i - T_1)}{R_{ci}} = \frac{(T_1 - T_2)}{R_e} = \frac{(T_2 - T_o)}{R_{co}} = \frac{(T_i - T_o)}{[R_{ci} + R_e + R_{co}]} \qquad \text{.......(3.35)}$$

Composite Coaxial Cylinders:- Fig. 3.12. shows a composite cylinder having two layers in series. The equivalent thermal circuit is also shown in the figure. The rate of heat transfer through the composite layer is given by

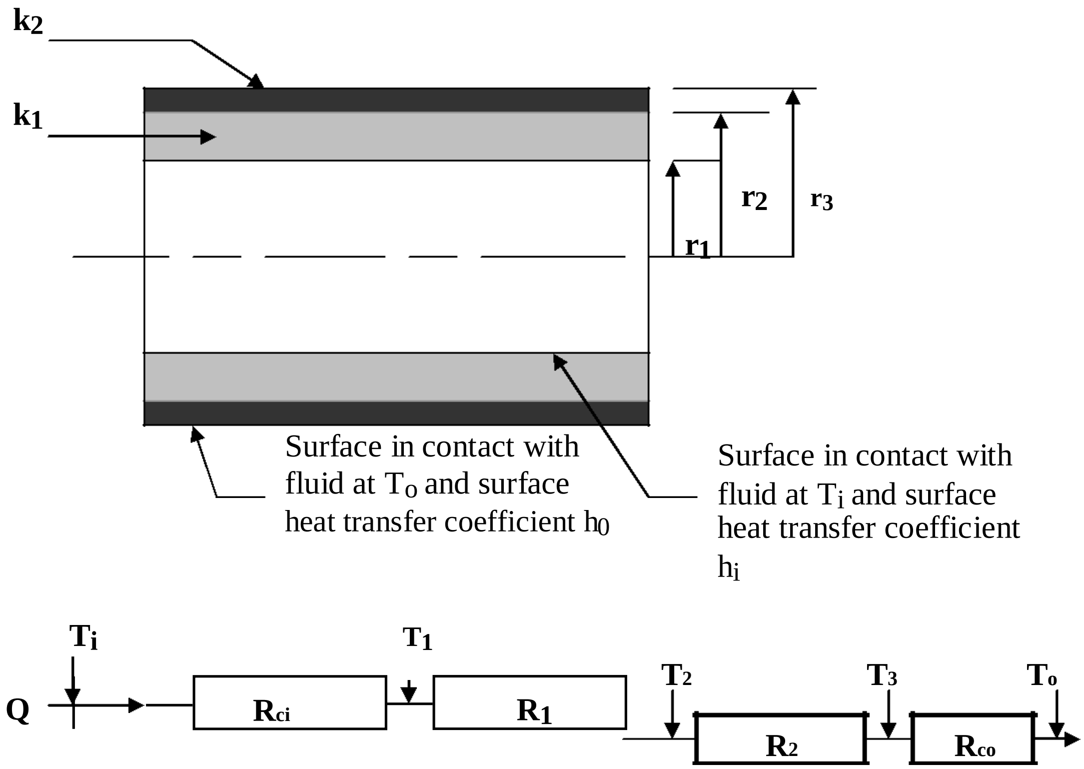


Fig. 3.12: Schematic and thermal circuit diagrams for a composite cylinder

$$\text{Now } Q = \frac{(T_i - T_1)}{R_{ci}} = \frac{(T_1 - T_2)}{R_1} = \frac{(T_2 - T_3)}{R_2} = \frac{(T_3 - T_o)}{R_{co}} = \frac{(T_i - T_0)}{[R_{ci} + R_1 + R_2 + R_{co}]} \quad \text{……………..(3.36)}$$

$$\text{Where } R_{ci} = 1 / [h_i A_i] = \frac{1}{2\,\pi\, r_1 L\, h_i}\,; \qquad R_1 = \frac{1}{2\,\pi\, L\, k_1} \ln(r_2 / r_1)$$

$$R_{co} = 1 / [h_o A_o] = \frac{1}{2\,\pi\, r_3 L\, h_o}\,; \qquad R_2 = \frac{1}{2\,\pi\, L\, k_2} \ln(r_3 / r_2)$$

The above expression for Q can be extended to any number of layers.

Overall Heat Transfer Coefficient for a Composite Cylinder:- For a cylinder the area of heat flow in radial direction depends on the radius r we can define the overall heat transfer coefficient either based on inside surface area or based on outside surface area of the composite cylinder. Thus if U_i is the overall heat transfer coefficient based on inside surface area A_i and U_o is the overall heat transfer coefficient based on outside surface area A_o then

$$Q = U_i A_i (T_i - T_o) \qquad \text{.......... (3.37)}$$

From equations (3.36) and (3.37) we have

$$\text{Now } U_i A_i (T_i - T_o) = \frac{(T_i - T_0)}{[R_{ci} + R_1 + R_2 + R_{co}]}$$

Substituting the expressions for A_i, R_{ci},R_1,R_2 and R_{co} in the above equation we have

$$2 \pi r_1 L\, U_i = \frac{1}{[1/(2\pi r_1 L h_i) + \{1/(2\pi L k_1)\}\ln (r_2 / r_1) + \{1/(2\pi L k_2)\}\ln (r_3 / r_2) + 1/(2\pi r_3 L h_o)]}$$

$$\textbf{Or} \quad \mathbf{U_i = \frac{1}{[\,1/h_i + (r_1 / k_1) \ln (r_2/r_1) + (r_1/k_2) \ln (r_3/r_2) + (r_1/r_3)(1/h_o)\,]}} \qquad \textbf{............ (3.38)}$$

Similarly it can be shown that

$$\mathbf{U_o = \frac{1}{[(r_3/ r_2)(1/h_i) + (r_3 / k_1) \ln (r_2/r_1) + (r_3/k_2) \ln (r_3/r_2) + (1/h_o)\,]}} \qquad \textbf{....... (3.39)}$$

Composite Concentric Spheres:- Fig.3.13 shows a composite sphere having two layers with the inner surface of the composite sphere in contact with fluid at a uniform temperature T_i and surface heat transfer coefficient h_i and the outer surface in contact with another fluid at a uniform temperature T_o and surface heat transfer coefficient h_o. The corresponding thermal circuit diagram is also shown in the figure.

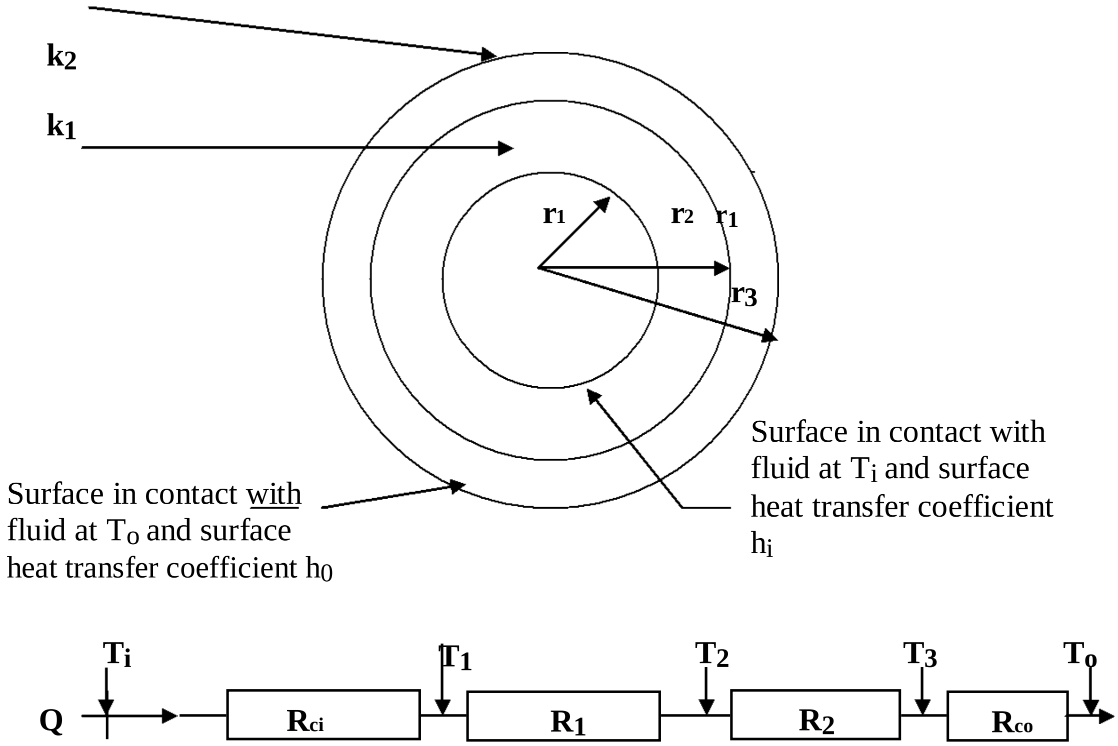


Fig. 3.13: Schematic and thermal circuit diagrams for a composite sphere

Eq. (3.36) is also applicable for the composite sphere of Fig. 3.13 except that the expression for individual resistance will be different. Thus

$$Q = \frac{(T_i - T_o)}{[R_{ci} + R_1 + R_2 + R_{co}]} \qquad \ldots (3.40)$$

where $R_{ci} = \dfrac{1}{h_i A_i} = \dfrac{1}{4 \pi r_1^2 h_i}$; $R_1 = \dfrac{(r_2 - r_1)}{4 \pi k_1 r_1 r_2}$;

$R_{co} = \dfrac{1}{h_o A_o} = \dfrac{1}{4 \pi r_3^2 h_o}$; $R_2 = \dfrac{(r_3 - r_2)}{4 \pi k_2 r_2 r_3}$;

Example 3.1:- Consider a plane wall 100 mm thick and of thermal conductivity 100 W/(m-k). Steady state conditions are known to exist with T_1 = 400 K and T_2 = 600 K. Determine the heat flux (magnitude and direction) and the temperature gradient dT/dx for the coordinate system shown in Fig. P3.1.(2000 ^{0}C/m; - 200,00 W/m^2;- 2000 ^{0}C/m,200,000 W/m^2; 2000 ^{0}C/m, - 200,000 W/m^2)

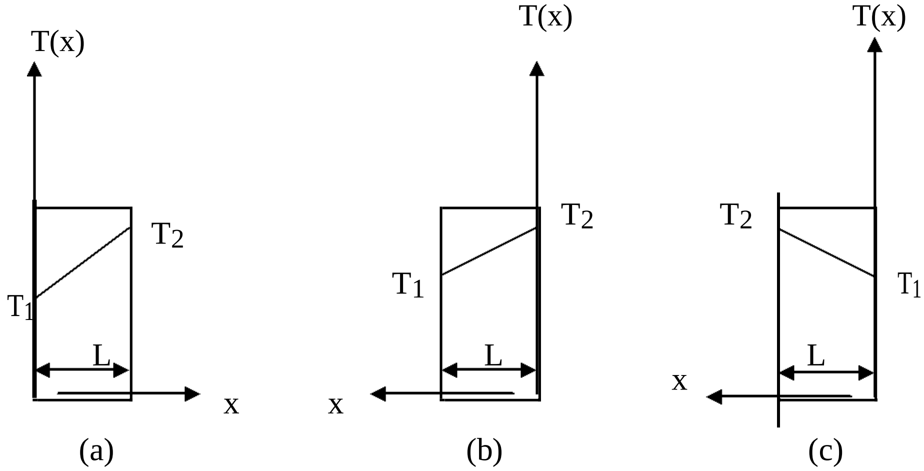
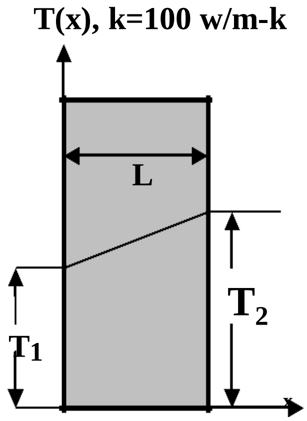


Fig.P3.1: Schematic for problem 3.1

Solution:-



(a) It can be seen from the figure that the temperature is increasing with increase in x: i.e., dT/dx is +ve.

Therefore $dT/dx = (T_2 – T_1) / L = (600 – 400)/0.10$

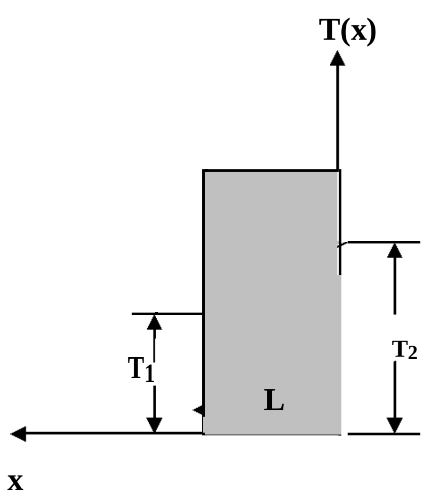
$= 2000\ ^0C / m.$

Heat flux $= q_x = - k\ dT/dx = - 100 \times 2000\ W/m^2$

$= - 2 \times 10^5\ W / m^2.$

The negative sign indicates that heat transfer takes place in the direction opposite to the +ve direction of x.

(b)



It can be seen from the figure shown that temperature is decreasing with increase in x or in other words dT/dx is – ve.

Therefore $dT/dx = (T_1 – T_2)/L$

$= (400 – 600) / 0.1$

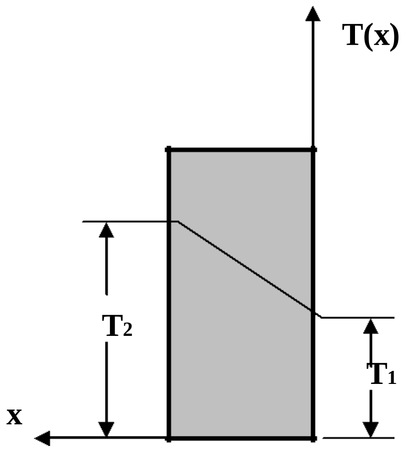
$= - 2000\ ^0C / m.$

$q_x = - k\ dT/dx = - 100 \times (- 2000)$

$= + 200{,}000\ W / m^2.$

+ ve sign for q_x indicates that heat transfer is taking place in the + ve direction of x.

(c)



It can be seen from the figure shown that the temperature increasing with increase in x: i.e., dT/dx is + ve.

Therefore dT/dx = (600 – 400) / 0.1

$$= 2000\ ^0C / m.$$

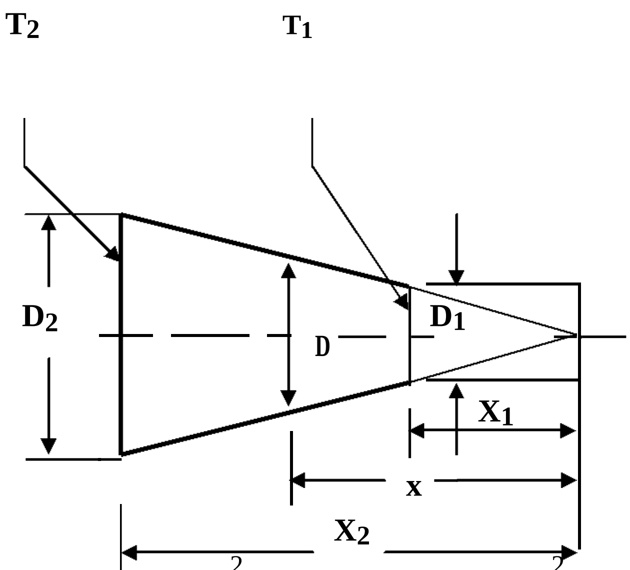
Heat flux = $q_x = - k\ dT/dx$

$$= -100 \times 2000 = - 200{,}000\ W/m^2.$$

Negative sign in q_x indicates that heat transfer takes place in a direction opposite to the + ve direction of x.

Example 3.2:-Fig. P3.2 shows a frustum of a cone (k = 3.46 W/m-K). It is of circular cross section with the diameter at any x is given by D = ax, where a = 0.25. The smaller cross section is at x_1 = 50 mm and the larger cross section is at x_2 = 250 mm. The corresponding surface temperatures are T_1 = 400 K and T_2 = 600 K. The lateral surface of the cone is completely insulated so that conduction can be assumed to take place in x-direction only.

(i) Derive an expression for steady state temperature distribution, T(x) in the solid and
(ii) calculate the rate of heat transfer through the solid.(T(x) = 400 + 12.5{20 – 1/x} ; Q_x = - 2.124 W)



By Fourier‟s law, the rate of heat transfer in x-direction across any plane at a distance x from the origin „o‟ is given by

$$Q_X = -k A_X (dT/dx).$$

For steady state conduction without heat generation Q_X will be a constant. Also at any x, D = ax.

Therefore, $Q_X = -k (\pi D^2/4) (dT/dx) = -k [\pi(ax)^2/4] (dT/dx)$.

Separating the variables we get, $dT = -(4/\pi a^2 k) Q_X (dx/x^2)$

Integrating the above equation we have

$$\int_{T_1}^{T} dT = -(4Q_x / \pi a^2 k) \int_{X_1}^{x} (dx / x^2)$$

Or $$T - T_1 = -(4Q_x / \pi a^2 k) [(1/x) - (1/X_1)]$$

Or $$T = T_1 - \frac{(4 Q_x)}{(\pi a^2 k)} ((1/x) - (1/X_1)) \quad \text{................(1)}$$

At $x = X_2$, $T = T_2$. Substituting this condition in Eq.(1) and solving for Q_x we get

$$Q_x = \frac{(\pi a^2 k)(T_2 - T_1)}{4 (1/X_2 - 1/X_1)} \quad \text{................................. (2)}$$

Substituting this expression for Q_X in Eq. (1) we get the temperature distribution in the cone as follows:

$$T(x) = T_1 + \frac{(T_2 - T_1)(1/x - 1/X_1)}{(1/X_2 - 1/X_1)} \quad .... (3)$$

Substituting the given numerical values for X_1, X_2, T_1 and T_2 in Eq.(3) we get the temperature distribution as follows:

$$T(x) = 400 + \frac{(600 - 400) [1/ x - 1/0.05]}{[1/0.25 - 1/0.05\}}$$

Or $\quad T(x) = 400 + 12.5 [20 - 1/x] \rightarrow$ Temperature distribution

And $\quad Q_x = \dfrac{\pi \times (0.25)^2 \times 3.46 \times [600 - 400]}{4 \times [1/0.25 - 1/0.05]} = - 2.123 \text{ W}$

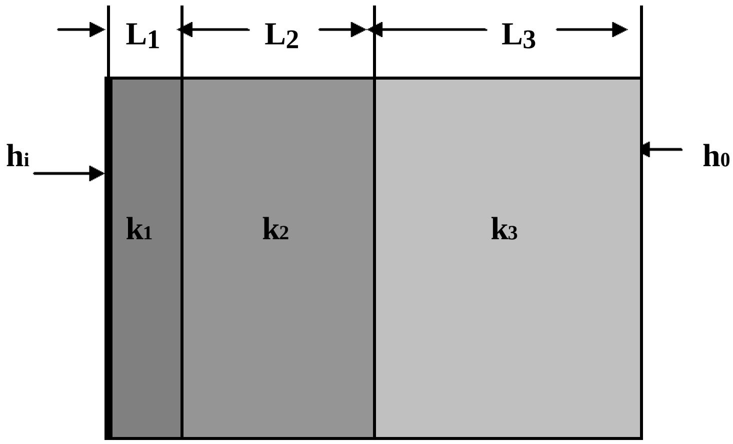
Example 3.3: -A plane composite wall consists of three different layers in perfect thermal contact. The first layer is 5 cm thick with k = 20 W/(m-K), the second layer is 10 cm thick with k = 50 W/(m-K) and the third layer is 15 cm thick with k = 100 W/(m-K). The outer surface of the first layer is in contact with a fluid at 400 ^{0}C with a surface heat transfer coefficient of 25 W/ (m ^{2} – K), while the outer surface of the third layer is exposed to an ambient at 30 ^{0}C with a surface heat transfer coefficient of 15 W/(m ^{2}-K).Draw the equivalent thermal circuit indicating the numerical values of all the thermal resistances and calculate the heat flux through the composite wall. Also calculate the overall heat transfer coefficient for the composite wall.

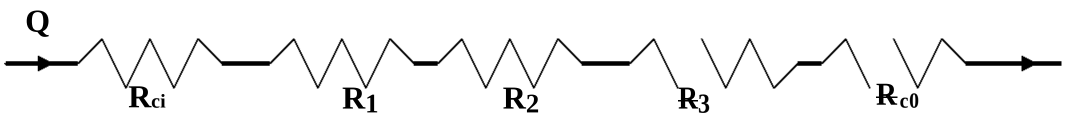
Solution: Data :- $L_1 = 0.05$ m ; $L_2 = 0.10$ m ; $L_3 = 0.15$ m ; $k_1 = 20$ W /(m-K) ;

$k_2 = 50$ W /(m-K) ; $k_3 = 100$ W/(m-K) ; $h_i = 25$ W /(m^2 – K) ; $h_0 = 15$ W/(m^2 – K)

; $T_i = 400\ ^0$ C ; $T_0 = 30\ ^0$ C.

$$R_{ci} = 1 / (h_i A_1) = \frac{1}{25 \times 1} = 0.04 \text{ m}^2 - \text{K / W} \quad (A_1 = A_2 = A_3 = A_4 = 1 \text{ m}^2)$$





$$R_1 = L_1/(k_1A_1) = \frac{0.05}{20 \times 1} = 0.0025\, m^2 - K/W.$$

$$R_2 = L_2/(k_2A_2) = \frac{0.10}{50 \times 1} = 0.002\ m^2 - K/W.$$

$$R_3 = L_3/(k_3A_3) = \frac{0.15}{100 \times 1} = 0.0015\ m^2 - K/W.$$

$$R_{co} = 1/(h_0A_4) = \frac{1}{15 \times 1} = 0.067\ m^2 - K/W.$$

$$\sum R = R_{ci} + R_1 + R_2 + R_3 + R_{co} = 0.04 + 0.0025 + 0.002 + 0.0015 + 0.067$$

Or $\sum R = 0.113\ m^2\text{-K/W}.$

$$\text{Heat Flux through the composite slab} = q = \frac{(T_i - T_0)}{\sum R} = \frac{(400 - 30)}{0.113}$$

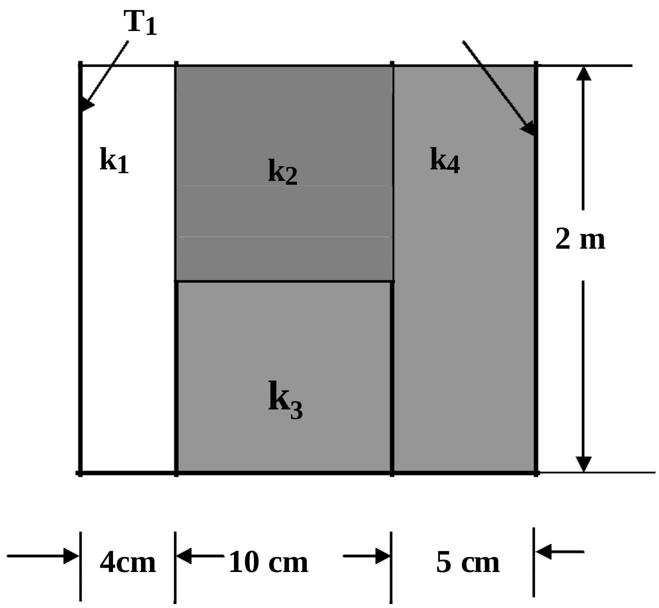
$$= 3274.34\ W/m^2.$$

If „U" is the overall heat transfer coefficient for the given system then

$$U = \frac{Q}{(T_i - T_0)} = \frac{1}{\sum R} = \frac{1}{0.113}$$

$$= 8.85 \text{ W / (m}^2 - \text{K)}.$$

Example 3.4:-A composite wall consisting of four different materials is shown in Fig P3.10. Using the thermal resistance concept determine the heat transfer rate per m^2 of the exposed surface for a temperature difference of 300 ^{0}C between the two outer surfaces. Also draw the thermal circuit for the composite wall.



k_1 = 100 W/(m-K) ; L_1 =0.04 m;

k_2 = 0.04 W/(m-K) ; L_2 = 0.1 m;

k_3 = 20 W/(m-K) ; L_3 = 0.1 m ;

k_4 = 70 W/(m-K) ; L_4 = 0.05 m;

W = Width of the wall perpendicular to the plane of paper = 1 m (assumed).

$T_1 - T_4 = 300\ ^0$C.
$A_1 = A_4 = 1 \times 2 = 2\ m^2$.
$A_2 = A_3 = 1 \times 1 = 1\ m^2$.

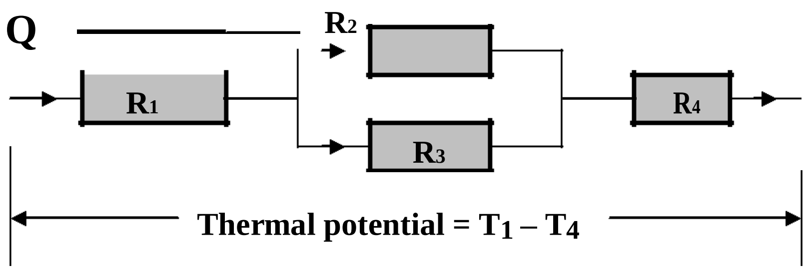
Solution:

$$R_1 = L_1 / (A_1 k_1) = \frac{0.04}{2 \times 100} = 0.0002\ ^0\text{C / W}.$$

$$R_2 = L_2 / (A_2 k_2) = \frac{0.10}{1 \times 70} = 0.00143\ ^0\text{C / W}.$$

$$R_3 = L_3 / (A_3 k_3) = \frac{0.10}{1 \times 20} = 0.005\ ^0\text{C / W}.$$

$$R_4 = L_4 / (A_4 k_4) = \frac{0.05}{2 \times 70} = 0.00036\ ^0C / W.$$



R_2 and R_3 are resistances in parallel and they can be replaced by a single equivalent resistance R_e, where

$$1 / R_e = 1 / R_2 + 1 / R_3 \text{ or } R_e = \frac{R_2 R_3}{(R_2 + R_3)} = \frac{0.00143 \times 0.005}{(0.00143 + 0.005)} = 0.0011\ ^0C/W$$

Now R_1, R_e and R_4 are resistances in series so that

$$Q = \frac{(T_1 - T_4)}{(R_1 + R_e + R_4)} = \frac{300}{[0.002 + 0.0011 + 0.00036]} = 86.705 \times 10^3\ W$$

Heat transfer per unit area of the exposed surface is given by

$$q = Q / A_1 = 86.705 / 2.0 = 43.35\ kW.$$

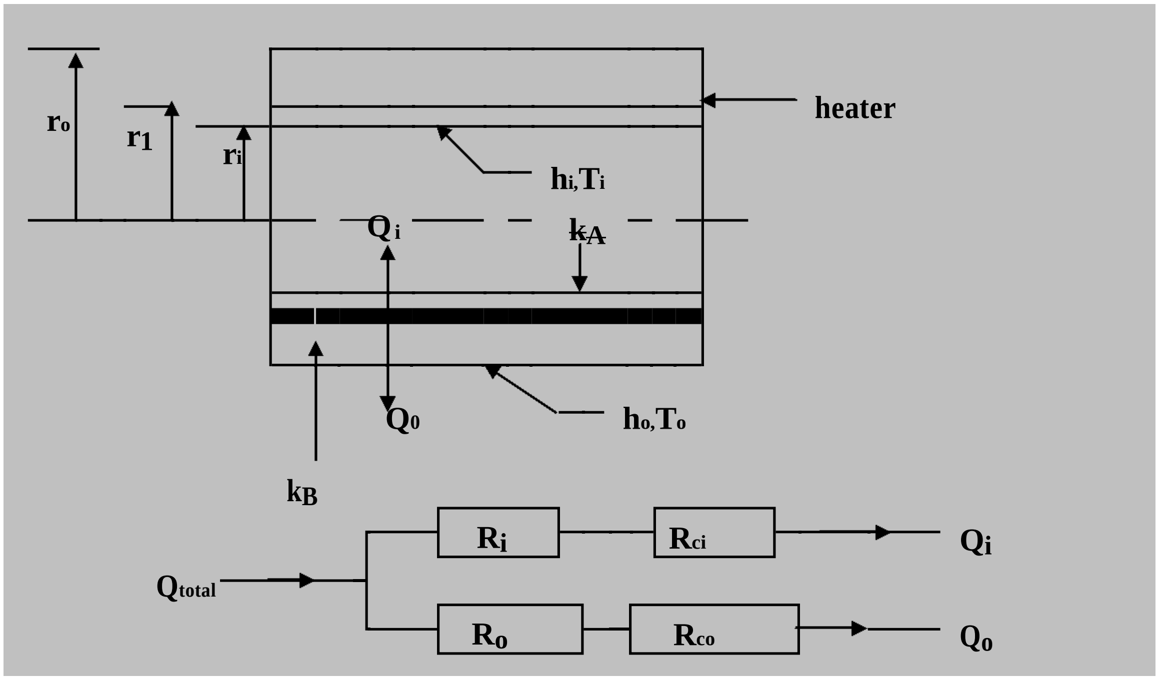
Example 3.5:-A composite cylindrical wall is composed of two materials of thermal conductivity k_A and k_B. A thin electric resistance heater for which the interfacial contact resistances are negligible separates the two materials. Liquid pumped through the inner tube is at temperature T_i with the inside surface heat transfer coefficient h_i. The outer surface of the Composite wall is exposed to an ambient at a uniform temperature of T_o with the outside surface heat transfer coefficient h_o. Under steady state conditions a uniform heat flux of q_h is dissipated by the heater.

(a) Sketch the equivalent thermal circuit for the composite wall and express all thermal resistances in terms of the relevant variables

(b)Obtain an expression that may be used to determine the temperature of the heater,T_h.

(b) Obtain an expression for the ratio of heat flows to the outer and inner fluid, q_o/q_i.

Solution:



$$R_i = \frac{1}{(2\pi L k_A)} \ln(r_1/r_i) \; ; \; R_{ci} = \frac{1}{h_i A_i} = \frac{1}{(2\pi r_i L h_i)}$$

$$R_o = \frac{1}{(2\pi L k_B)} \ln(r_o/r_1) \; ; \; R_{co} = \frac{1}{h_o A_o} = \frac{1}{(2\pi r_o L h_o)}$$

$$Q_i = (T_h - T_i) / [R_i + R_{ci}] \; ; \qquad Q_o = (T_h - T_o) / [R_o + R_{co}] \; ;$$

$$Q_{total} = Q_i + Q_o = (T_h - T_i) / (R_i + R_{ci}) + (T_h - T_o) / (R_o + R_{co})$$

$$2\pi r_1 L q_h = \frac{(T_h - T_i)}{\dfrac{\ln(r_1/r_i)}{2\pi L k_A} + \dfrac{1}{2\pi L r_i h_i}} + \frac{(T_h - T_o)}{\dfrac{\ln(r_o/r_1)}{2\pi L k_B} + \dfrac{1}{2\pi L r_o h_o}}$$

$$\text{Therefore } q_h = \frac{(T_h - T_i)}{[\,(r_1/k_A) \ln (r_1/r_i) + r_1/(r_i h_i)]} + \frac{(T_h - T_o)}{[(r_1/k_B) \ln (r_o/r_1) + r_1/(r_o h_o)}$$

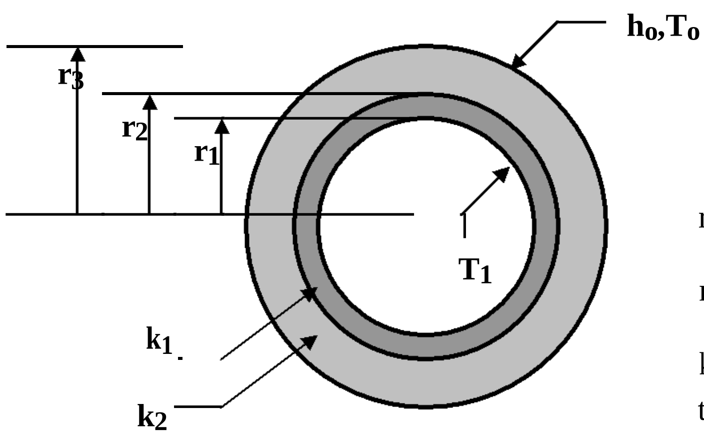
The temperature T_h of the heater can be obtained from the above equation.

$$\text{Now } \frac{Q_o}{Q_i} = \frac{(T_h - T_o) / (R_o + R_{co})}{(T_h - T_i) / (R_i + R_{ci})} = \frac{(T_h - T_o) \times (R_i + R_{ci})}{(T_h - T_i) \times (R_o + R_{co})}$$

$$= \frac{(T_h - T_o)}{(T_h - T_i)} \times \frac{[\,1 / (r_i h_i) + (1 / k_A) \ln (r_1/r_i)\,]}{[\,1 / (r_o h_o) + (1 / k_B) \ln (r_o/r_1)\,]}$$

Example 3.8:- A hollow aluminum sphere with an electrical heater in the centre is used to determine the thermal conductivity of insulating materials. The inner and outer radii of the sphere are 15 cm and 18 cm respectively and testing is done under steady state conditions with the inner surface of the aluminum maintained at 250^0 C. In a particular test, a spherical shell of insulation is cast on the outer surface of the aluminum sphere to a thikness of 12 cm. The system is in a room where the air temperature is $20\ ^0$ C and the convection coefficient is 30 W/(m^2 – K). If 80 W are dissipated by the heater under steady state conditions, what is the thermal conductivity of the insulating material?

Solution:



r_1 = 0.15 m ; r_2 = 0.18 m ;

r_3 = 0.18 + 0.12 = 0.3 m ;

k_1 = 204 W/(m-K) from tables; k_2 = 0.30 W/(m-K)

h_o = 30 W/(m^2-K);Q = 60 W

T_1 = 250 ^{0} C ; T_o = 20 ^{0} C.

$$R_1 = \frac{(r_2 - r_1)}{4\pi k_1 r_1 r_2} = \frac{(0.18 - 0.15)}{4\,\pi \times 204 \times 0.18 \times 0.15} = 4.335 \times 10^{-4}\ ^0C / W.$$

$$R_2 = \frac{(r_3 - r_2)}{4\pi\, k_2\, r_2\, r_3} = \frac{(0.30 - 0.18)}{4\,\pi \times k_2 \times 0.30 \times 0.18} = 0.177 / k_2 \;\; {}^0C / W.$$

$$R_{co} = 1/(h_0 A_0) = \frac{1}{4\pi\, r_3^2\, h_o} = \frac{1}{4\pi \times (0.3)^2 \times 30} = 0.0295 \;\; {}^0C / W.$$

$$Q = \frac{(T_1 - T_o)}{R_1 + R_2 + R_{co}} \quad \text{or } R_2 = (T_1 - T_o) / Q - (R_1 + R_{co})$$

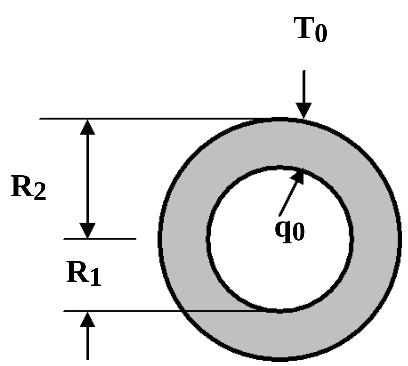
$$\text{Or } R_2 = (250 - 20) / 80 - (4.335 \times 10^{-4} + 0.0295) = 2.874$$

Therefore $\quad 0.177 / k_2 = 2.845$

Or $\quad k_2 = 0.177 / 2.845 = 0.062 \text{ W / (m-K)}$

Example 3.8:- In a hollow sphere of inner radius 10 cm and outer radius 20, the inner surface is subjected to a uniform heat flux of 1.6×10^5 W/m^2 and the outer surface is maintained at a uniform temperature of 0 ^{0}C.The thermal conductivity of the material of the sphere is 40 W /(m – K).Assuming one-dimensional radial steady state conduction determine the temperature of the inner surface of the hollow sphere.

Solution:-



The governing equation for one-dimensional steady-state radial conduction in a sphere without heat generation is given by

$$d/dr\,(r^2\, dT / dr) = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The boundary conditions are : (i) at $r = R_1$, $-k\,(dT/dr)|_{r=R1} = q_0$

(ii) at $r = R_2$ $\quad T(r) = 0$.

Integrating Eq. (1) w.r.t. r once, we get

$$r^2\,(dT/dr) = C_1$$

$$\text{or } dT / dr = C_1 / r^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Integrating once again w.r.t. r we get

$$T(r) = -\,C_1 / r + C_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

From (2) $\quad (dT/dr)_{r\,=\,R1} = C_1 / R_1^2$

Hence condition (i) gives

$$-\,kC_1 / R_1^2 = q_0$$

Or $\quad C_1 = -\,q_0 R_1^2 / k$

Condition (ii) in Eq.(2) gives $\quad 0 = -\,C_1 / R_2 + C_2$

Or $\quad C_2 = C_1 / R_2 = -\,(q_0 R_1^2) / (kR_2)$

Substituting the expressions for C_1 and C_2 in Eq. (2) we have

$$T(r) = \frac{q_0 R_1^2}{k\,r} - \frac{q_0 R_1^2}{kR_2}$$

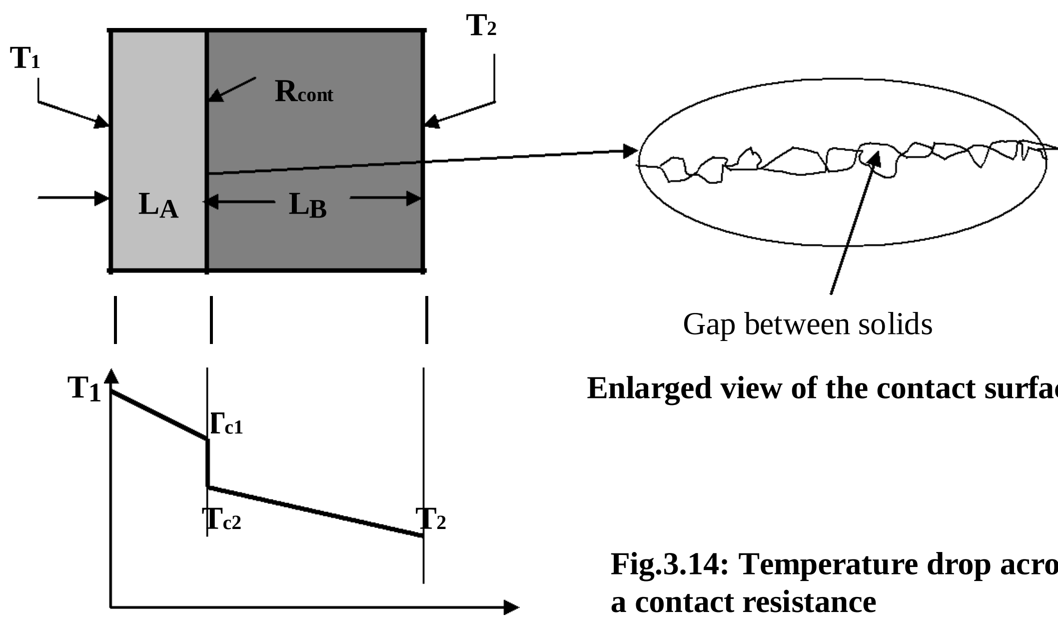
Substituting the numerical values for q_0, k, R_1 and R_2 we have

$$T(r) = \frac{1.6 \times 10^5 \times 0.1^2}{40} / r - \frac{1.6 \times 10^5 \times 0.1^2}{40 \times 0.2}$$

Or $\quad T(r) = (40 / r) - 200$

Therefore $T(r)|_{r = R1} = (40 / 0.1) - 200 = 200\ ^0C$.

Thermal Contact Resistance: In the analysis of heat transfer problems for composite medium it was assumed that there is "perfect thermal contact" at the interface of two layers. This assumption is valid only the two surfaces are smooth and they produce a perfect contact at each point.But in reality, even flat surfaces that appear smooth to the naked eye would be rather rough when examined under a microscope as shown in Fig. 3.14 with numerous peaks and valleys.



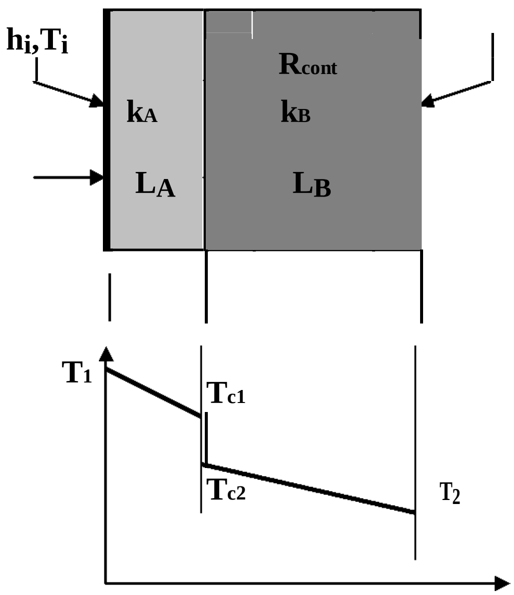
Enlarged view of the contact surface

Fig.3.14: Temperature drop across a contact resistance

The physical significance of thermal contact resistance is that the peaks will form good thermal contact, but the valleys will form voids filled with air.As a result the air gaps act as insulation because of poor thermal conductivity of air.Thus the interface offers some resistance to heat conduction and this resistance is called the *"thermal contact resistance,R_{cont}"*. The value of R_{cont} is determined experimentally and is taken into account while analyzing the heat conduction problems involving multi-layer medium.The procedure is illustrated by means of a few examples below.

Example 3.4:- A composite wall consists of two different materials A [k = 0.1 W/(m-k)] of thickness 2 cm and B[k = 0.05 W/(m-K)] of the thickness 4 cm. The outer surface of layer A is in contact with a fluid at 200^0C with a surface heat transfer coefficient of 15 W/(m^2-K) and the outer surface of layer B is in contact with another fluid at 50 ^{0} C with a surface heat transfer coefficient of 25 W/(m^2-K). The contact resistance between layer A and layer B is 0.33 (m^2-K) /W. Determine the heat transfer rate through the composite wall per unit area of the surface. Also calculate the interfacial temperatures and the inner and outer surface temperatures.

Solution:



$T_i = 200\ ^0C$; $T_0 = 50\ ^0C$;

$h_i = 15\ W/(m^2 - K)$; $h_0 = 25 W/(m^2 - K)$

$k_A = 0.1\ W/(m\text{-}K)$; $k_B = 0.05\ W/(m\text{-}K)$

$R_{cont} = 0.33\ (m^2 - K)/W$.

The equivalent thermal circuit is also shown in the figure.

$$R_{ci} = 1/(h_iA_A) = \frac{1}{(15 \times 1)} = 0.067\ m^2\text{-K/W}$$

$R_1 = L_A/(k_AA_A) = 0.02 / (0.1 \times 1)$
$= 0.2\ m^2\text{-K / W}.$

$R_2 = L_B / (k_BA_B) = 0.04 / (0.05 \times 1) = 0.8\ m^2 - K / W.$

$R_{co} = 1 / (h_oA_B) = 1 / (25 \times 1) = 0.04\ m^2 - K / W.$

$\sum R = R_{ci} + R_1 + R_{cont} + R_2 + R_{co} = 0.067 + 0.2 + 0.33 + 0.8 + 0.04 = 1.437\ m^2 - K / W.$

$$\text{Heat flux} = q = (T_i - T_o) / \sum R = \frac{(200 - 50)}{1.437} = 104.4\ W/m^2$$

Now $q = (T_i - T_A) / R_{ci}$ or $T_A = T_i - q R_{ci} = 200 - (104.4 \times 0.067) = 193\ ^0C.$

Similarly $T_{c1} = T_A - q R_1 = 193 - (104.4 \times 0.2) = 172.12\ ^0C.$

$T_{c2} = T_{c1} - q R_{cont} = 172.12 - (104.4 \times 0.33) = 137.67\ ^0C.$

$T_B = T_{c2} - q R_2 = 137.67 - (104.4 \times 0.8) = 54.15\ ^0C.$

Check : $T_o = T_B - q R_{co} = 54.15 - (104.4 \times 0.04) = 49.97\ ^0C$

Example 3.6:- A very thin electric heater is wrapped around the outer surface of a long cylindrical tube whose inner surface is maintained at 5 ^{0}C. The tube wall has inner and outer radii of 25 mm and 75 mm respectively and a thermal conductivity of 10 W/(m-K). The thermal contact resistance between the heater and the outer surface of the tube per unit length is 0.01 (m-K) / W. The outer surface of the heater is exposed to a fluid with a temperature of – 10 ^{0}C and a convection coefficient of 100 W/(m^2-K). Determine the heater power required per length of the tube to maintain a heater temperature of 25 ^{0}C.

Solution: Data: r_1 = 0.025 m ; r_2 = 0.075 m ; k = 10 W /(m-K) ; T_∞ = – 10 ^{0} C ; h = 100 W/(m^2 – K); T_1 = 5 ^{0} C ; R_{cont} = 0.01 m – K / W ; T_2 = 25 ^{0} C.

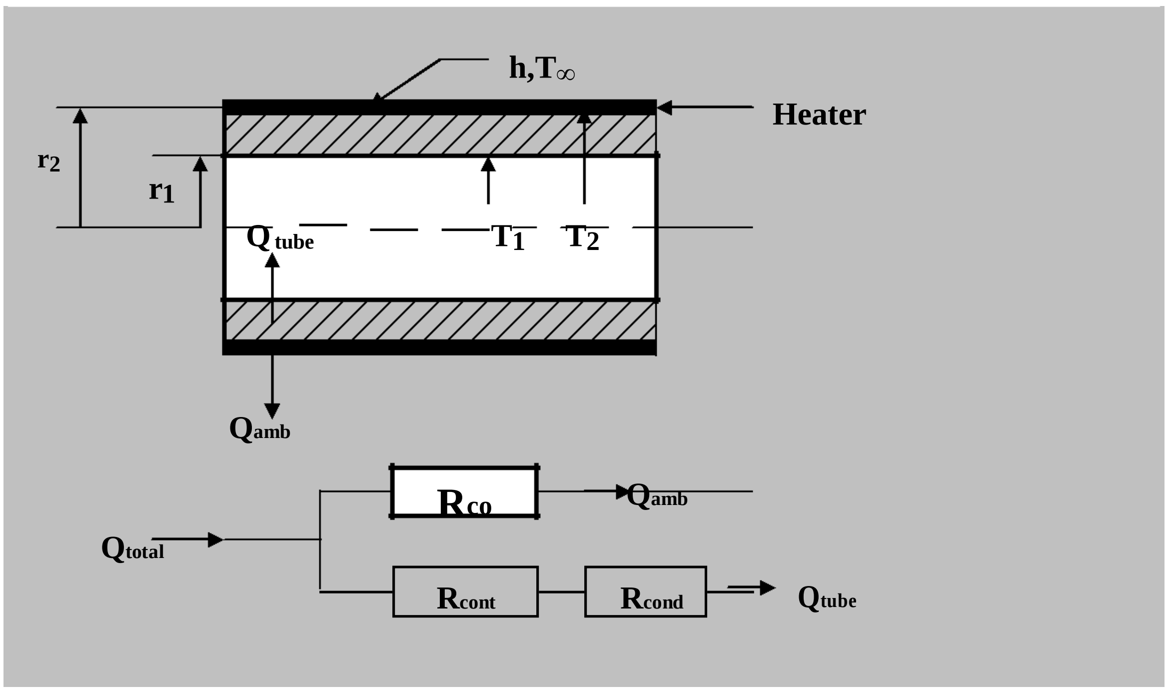


Fig. P3.6: Schematic and Thermal circuit for Example 3.6

$$R_{co} = 1 / (h\,A_o) = \frac{1}{100 \times 2\pi \times 0.075} = 0.0212 \text{ (m- K) / W.}$$

$$R_{cond} = \frac{1}{2\pi\,L\,k} \ln(r_2 / r_1) = \frac{1}{(2\times \pi \times 1 \times 10)} \ln(0.075 / 0.025) = 0.0175 \text{ (m-K) / W.}$$

$$Q_{amb} = (T_2 - T_\infty)/R_{co} = \frac{[25 - (-10)]}{0.0212} = 1651 \text{ W / m.}$$

$$Q_{tube} = \frac{(T_2 - T_1)}{R_{cont} + R_{cond}} = \frac{[25 - 5]}{(0.01 + 0.0175)} = 727.3 \text{ W / m}$$

Power required = $Q_{total} = Q_{amb} + Q_{tube}$ = 1650 + 727.3 = 2378.3 W/m

One Dimensional Steady State Conduction With Heat Generation:

The governing equation for one – dimensional steady state conduction in solids which are generating is given as follows.

(i) Plane wall :$(d^2T / dx^2) + q''' / k = 0$..(3.41)

(ii) Radial conduction in cylinder: $(1/r)\, d / dr \{r\, dT/dr\} + q''' / k = 0$… ....(3.42)

(iii) Radial conduction in spheres: $(1/r^2)\, d / dr \{r^2\, dT/dr\} + q''' / k = 0$… ....(3.43)

The following examples illustrate the method of analysis of steady state heat conduction In solids generating heat.

Example 3.7:-A plane wall of thickness L and thermal conductivity k has one of its surfaces insulated and the other surface is kept at a uniform temperature T_0. Heat is generated in the wall at a rate $q'''(x)$ where $q'''(x) = q_0 \cos\{(\pi x) / (2L)\}$ W / m^3 where q_0 is a constant.

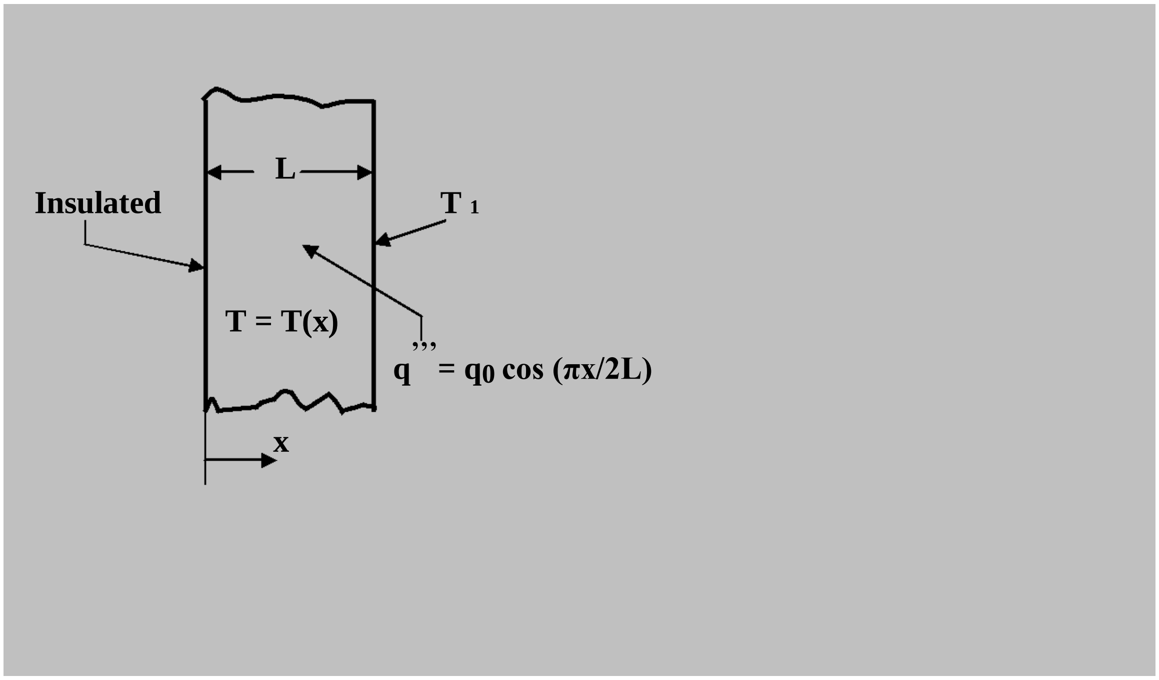
(a) Develop an expression for one-dimensional steady state temperature distribution in the solid and (b) develop an expression for the temperature of the insulated surface. [$T(x) = (\pi/2L)^2 (q_0 / k) \cos(\pi x / 2L) + T_2$]

Solution:- The governing equation to determine T(x) is given

by $d^2T / dx^2 + q''' / k = 0$

Substituting the given expression for q''' the above equation reduces to

$d^2T / dx^2 + (q_0 / k) \cos(\pi x / 2L) = 0$ .... (1)



The boundary conditions are ; (i) at x = 0 (dT/dx) = 0 (Insulated)

(ii) at x = L, $T = T_1$

Integrating Eq.(1) once w.r.t. x we get

$$dT/dx = -(\pi/2L)\ (q_0/k)\ \sin(\pi x/2L) + C_1 \ldots\ldots\ldots\ldots\ldots\ldots\ (2)$$

Integrating once again w.r.t. x we get

$$T(x) = -(\pi/2L)^2\ (q_0/k)\ \cos(\pi x/2L) + C_1x + C_2 \ldots\ldots\ldots\ldots\ldots\ldots\ (3)$$

Condition (i) in Eq. (2) gives $0 = 0 + C_1$ or $C_1 = 0$.

Condition (ii) in Eq. (3) gives $T_1 = 0 + 0 + C_2$ or $C_2 = T_1$.

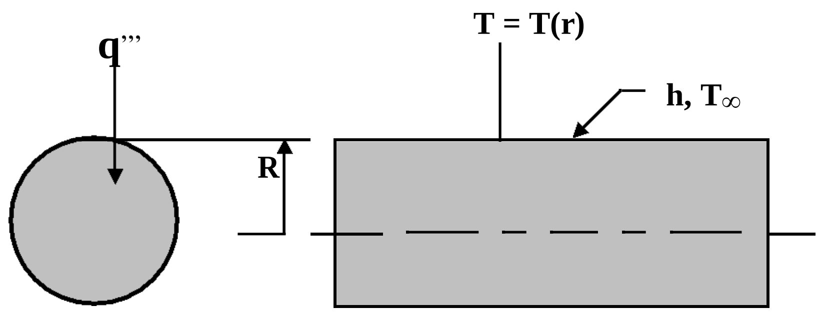
Substituting the values of C_1 and C_2 in Eq. (3) we get the temperature distribution as

$$T(x) = -(\pi/2L)^2\ (q_0/k)\ \cos(\pi x/2L) + T_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (4).$$

At the insulated surface (x = 0) the temperature therefore is given

$$\text{by } T(x)\,|_{x=0} = (\pi/2L)^2\ (q_0/k) + T_1.$$

Example 3.8:- A long cylindrical rod of radius 5 cm and k = 10 W/(m-K) contains radioactive material which generates heat uniformly within the cylinder at a rate of 3×10^5 W/ m^3. The rod is cooled by convection from its cylindrical surface by ambient air at 50 ^{0}C with a heat transfer coefficient of 60 W/(m^2-K). Assuming one-dimensional radial conduction determine the temperature at the centre of the rod as well as at the outer surface of the rod.(175 ^{0} C)



The governing differential equation to determine one-dimensional steady state radial conduction with heat generation is given by

$$(1/r)\ d / dr\ (r\ dT/dr) + q''' / k = 0 \ldots \qquad (1)$$

The boundary conditions are : (i) at r = 0, dT/dr = 0 (axis of symmetry)

(ii0 at r= R, $-k\ (dT/dr)|_{r=R} = h\ [T\ |_{r=} - T_\infty]$

Eq. (1) can be written as

$$d / dr\ (r\ dT/dr) + q'''\ r / k = 0.$$

Integrating once w.r.t. r we get

$$r\ dT/dr + (q'''\ r^2) / 2k = C_1$$

or

$$dT/dr + (q'''\ r) / 2k = C_1 / r \qquad (2)$$

Integrating once again w.r.t. r we have

$$T(r) = -\frac{q'''\ r^2}{4\ k} + C_1 \ln r + C_2 \qquad (3)$$

Condition (i) in Eq. (2) gives $0 + 0 = C_1 / 0$ or $C_1 = 0$.

From Eq. (3) we have

$$T(r)|_{r=R} = -\frac{q'''\ R^2}{4\ k} + C_2 \qquad (3)$$

and from Eq. (2) we have $(dT/dr)\ |_{r=R} = -(q'''\ R) / 2k$

Therefore condition (ii) gives

$$-k[-(q'''/R)/2k] = h[-(q''' R^2/4k) + C_2 - T_\infty]$$

Or $$C_2 = \frac{q''' R}{2h} + \frac{q''' R^2}{4k} + T_\infty$$

Substituting the expressions for C_1 and C_2 in Eq.(3) we get the temperature distribution in the cylinder as

$$T(r) = T_\infty + \frac{q''' R^2}{4k}[1 - (r/R)^2] + (q''' R)/2h \qquad \text{.........................(4)}$$

Now $$q''' R^2/(4k) = \frac{3 \times 10^5 \times (0.05)^2}{4 \times 10} = 18.75\ ^0C.$$

$$q''' R/(2h) = \frac{3 \times 10^5 \times 0.05}{2 \times 60} = 125\ ^0C$$

Therefore $T(r) = 50 + 18.75 [1 - (r/R)^2] + 125$

$= 175 + 18.75 [1 - (r/R)^2]$

At the centre $T(r)|_{r=0} = 175 + 18.75 = 193.75\ ^0C$.

At the surface $T(r)|_{r=R} = 50 + 18.75 [1 - 1] + 125 = 175\ ^0C$.

Example 3.9:- In a cylindrical fuel element for a gas-cooled nuclear reactor, the heat generation rate within the fuel element due to fission can be approximated by the equation

$$q''' = q_o [1 - (r/R)^2]\ W/m^3,$$

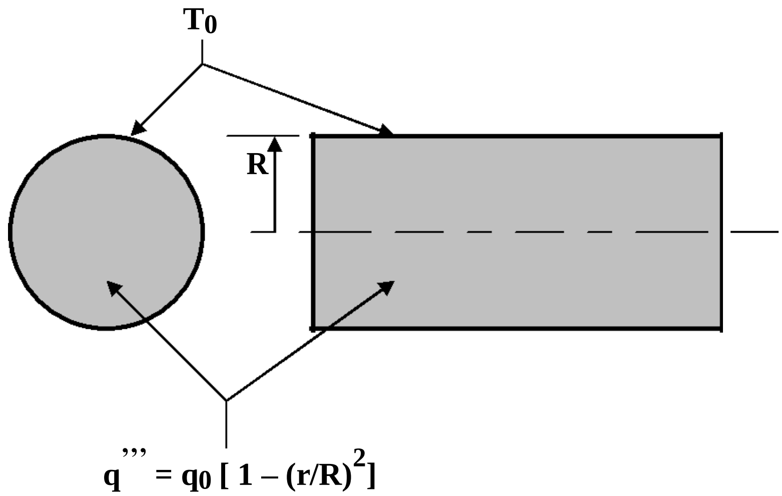
where R is the outer radius of the fuel element and q_o is a constant. The outer surface of the cylinder is maintained at a uniform temperature T_o. Assuming one-dimensional radial conduction obtain an expression for the temperature distribution in the element. If R = 2 cm, k = 10 W/(m-K) and $q_o = 1.16 \times 10^5\ W/m^3$, what would be the temperature difference between centre temperature and the outer surface temperature.(0.87 ^{0}C)

Solution: The governing equation to determine the one-dimensional steady-state radial temperature distribution in a cylinder with heat generation is given by

$$(1/r)\, d/dr(r\, dT/dr) + q'''/k = 0.$$

Multiplying by r and substituting for q''' the given expression we have

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{q_0\, r\,[1-(r/R)^2]}{k} = 0 \quad \text{.................................}(1)$$



Boundary conditions are: (i) at $r = 0$, $dT/dr = 0$ (axis of symmetry)

(ii) at $r = R$, $T = T_0$

Integrating Eq. (1) w.r.t. r once we get

$$\left(r\frac{dT}{dr}\right) + \frac{q_0}{K}\left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right] = C_1$$

Or

$$\frac{dT}{dr} + \frac{q_0}{K}\left[\frac{r}{2} - \frac{r^3}{4R^2}\right] = \frac{C_1}{r} \quad \text{………..}(2)$$

Integrating once again w.r.t. r we have

$$T(r) = -\frac{q_0}{k}\left[\frac{r^2}{4} - \frac{r^4}{16R^2}\right] + C_1 \ln r + C_2 \text{.....................} \qquad (3)$$

Condition(i) in Eq.(2) gives

$$0 + 0 = C_1 / 0 \text{ or } C_1 = 0.$$

Condition (ii) in Eq.(3) gives

$$T_0 = -\frac{q_0}{k}\left[\frac{R^2}{4} - \frac{R^4}{16 R^2}\right] + C_2$$

Or $$C_2 = T_0 + \frac{3 q_0 R^2}{16 k}$$

Substituting the expressions for C_1 and C_2 in Eq. (3) we have

$$T(r) = -\frac{q_0}{k}\left[\frac{r^2}{4} - \frac{r^4}{16R^2}\right] + T_0 + \frac{3 q_0 R^2}{16 k}$$

Or $$T(r) - T_0 = -\frac{q_0}{k}\left[\frac{r^2}{4} - \frac{r^4}{16R^2}\right] + \frac{3 q_0 R^2}{16 k}$$

Therefore $$T(r)|_{r=0} - T_0 = \frac{3 q_0 R^2}{16 k} = \frac{3 \times 1.16 \times 10^5 \times (0.02)^2}{16 \times 10} = 0.87\ {}^0C.$$

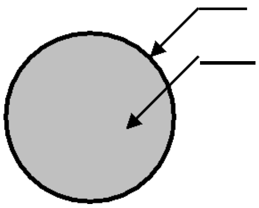
Example3.10:-Develop an expression for one-dimensional radial steady state temperature distribution in a solid sphere of radius R in which heat is generated at a rate given by

$$q''' = q_o [\,1 - (r/R)] \quad W/m^3.$$

Assume that the outer surface is maintained at a uniform temperature T_o.

$[T(r) = T_0 + (q_0 R^2 / 12k) \{1 - 2 (r/R)^2 + (r/R)^3\}]$

Solution:-



The governing differential equation to find the one-dimensional radial steady state temperature distribution in a sphere with heat generation is given by

$$(1/r^2)\, d/dr(r^2\, dT/dr) + q'''/k = 0.$$

Multiplying throughout by r^2 and substituting the given expression for q''' we have

$$d/dr\,(r^2\, dT/dr) + [q_0\, r^2\{1 - (r/R)\}]/k = 0 \quad \text{.......................(1).}$$

Boundary conditions are : (i) at $r = 0$, $dT/dr = 0$ (axis of symmetry)

(ii) at $r = R$, $T = T_0$.

Integrating Eq. (1) w.r.t. r once we get

$$(r^2\, dT/dr) + \frac{q_0}{k}\left[\frac{r^3}{3} - \frac{r^4}{4R}\right] = C_1$$

Or

$$dT/dr = -\frac{q_0}{k}\left[\frac{r}{3} - \frac{r^2}{4R}\right] + C_1/r^2 \quad \text{.................................(2)}$$

Integrating once again w.r.t. r we get

Or

$$T(r) = -\frac{q_0}{k}\left[\frac{r^2}{6} - \frac{r^3}{12R}\right] - C_1/r + C_2 \quad \text{........................(3)}$$

Condition (i) in Eq. (2) gives $0 = 0 + C_1/0$ or $C_1 = 0$.

Condition (ii) in Eq. (3) gives

$$T_0 = -\frac{q_0}{k}\left[\frac{R^2}{6} - \frac{R^2}{12}\right] + C_2$$

or

$$C_2 = T_0 + q_0 R^2 / 12k$$

Substituting the expressions for C_1 and C_2 in Eq. (3) we have

$$T(r) = -\frac{q_0}{k}\left[\frac{r^2}{6} - \frac{r^3}{12R}\right] + T_0 + \frac{q_0 R^2}{12\,k}$$

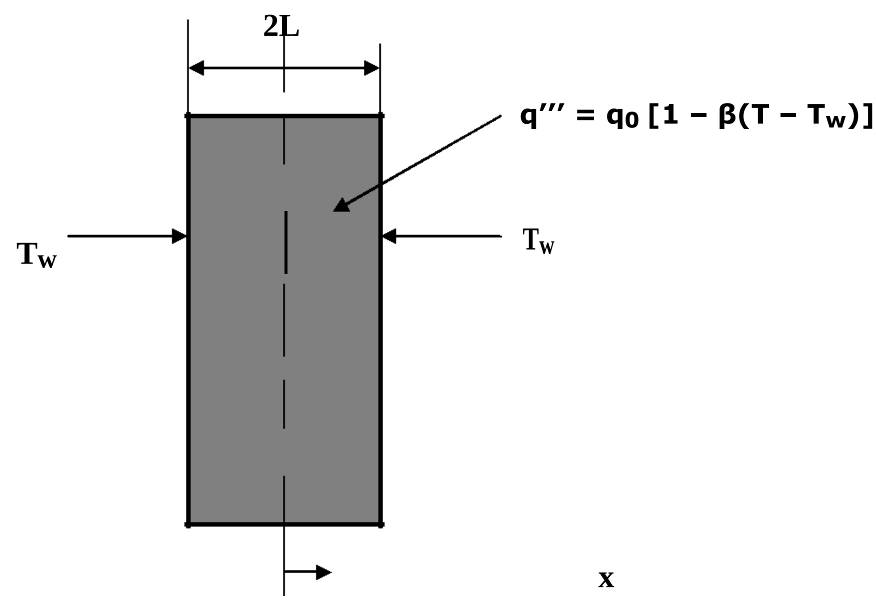
or $$T(r) = T_0 + \frac{q_0 R^2}{12k}\left[1 - 2\,(r/R)^2 + (r/R)^3\right]$$

Example 3.11:- A plane wall of thickness 2L is generating heat according to the law

$q''' = q_0\,[1 - \beta(T - T_w)]$

where q_o, β, and T_w are constants and T is the temperature at any section x from the mid-plane of the wall. The two outer surfaces of the wall are maintained at a uniform temperature T_w. Determine the one-dimensional steady state temperature distribution, T(x) for the wall.

Solution:



Governing differential equation for one-dimensional steady state conduction in a plane wall which generating heat is given by

$$d^2T / dx^2 + q'''''' / k = 0.$$

Substituting for q'''''' we have

$$d^2T / dx^2 + q_o [1 - \beta(T - T_w)] /k = 0$$

Defining a new variable $\theta = T - T_w$, the above equation can be written as

$$d^2\theta / dx^2 + q_o [1 - \beta\theta] /k = 0$$

or $$d^2\theta / dx^2 - q_o \beta\theta /k = q_o / k$$

or $$d^2\theta / dx^2 - m^2\theta = q_o / k \text{ ....(1a)}$$

where $$m^2 = q_0 \beta /k \text{ .... (1b)}$$

Eq.(1a) is a second order linear non-homogeneous differential equation whose solution is given by

$$\theta (x) = \theta_h(x) + \theta_p(x) \text{ .... (2)}$$

where $\theta_h(x)$ satisfies the differential equation

$$d^2\theta_h / dx^2 - m^2\theta_h = 0 \text{ .... (3)}$$

Solution to Eq.(3) is given by

$$\theta_h(x) = A_1 e^{mx} + A_2 e^{-mx} \text{ .... (4)}$$

$\theta_p(x)$ satisfies the differential equation

$$d^2\theta_p / dx^2 - m^2\theta_p = q_o / k \text{… .... (5)}$$

The term q_o/k makes the governing differential equation non-homogeneous. Since this is a constant $\theta_p(x)$ is also assumed to be constant. Thus let $\theta_p(x) = C$, where C is a constant. Substituting this solution in Eq. (5) we get

$$- m^2C = q_o / k$$

Or $$C = - q_o /(km^2)$$

Substituting for m^2 we get $$C = - 1 / \beta.$$

Hence $$\theta_p(x) = - 1 / \beta \text{ .... (6)}$$

The complete solution $\theta(x)$ is therefore given by

$$\theta(x) = A_1 e^{mx} + A_2 e^{-mx} - 1 / \beta \text{ ....(7)}$$

The constants A_1 and A_2 in Eq.(7) can be determined by using the two boundary conditions, which are:

(i) at $x = 0$, $dT / dx = 0$ (axis of symmetry) i.e., $d\theta / dx = 0$

(ii) at $x = L$, $T = T_w$; i.e., $\theta = 0$

From Eq.(7), $d\theta / dx = m[A_1 e^{mx} - A_2 e^{-mx}]$

Substituting condition (i) we get $m[A_1 - A_2] = 0$

Or $A_1 = A_2$.

Substituting condition (ii) in Eq.(7) we get $A_1[e^{mL} + e^{-mL}] = 1 / \beta$

Or
$$A_1 = \frac{(1/\beta)}{[e^{mL} + e^{-mL}]}$$

Substituting the expressions for A_1 and A_2 in Eq. (7) we get the temperature distribution in the plane wall as

$$\theta(x) = T(x) - T_w = \frac{(1/\beta)}{[e^{mL} + e^{-mL}]}[e^{mx} + e^{-mx}] - 1/\beta$$

Or
$$T(x) - T_w = \frac{1}{\beta}\left[\frac{e^{mx} + e^{-mx}}{e^{mL} + e^{-mL}} - 1\right]$$

or
$$T(x) - T_w = (1/\beta)\frac{[e^{m(L-x)} + e^{-m(L-x)}]}{[e^{mL} + e^{-mL}]} = (1/\beta)\frac{\cosh m(L-x)}{\cosh mL}$$

<u>Critical Radius of Insulation</u>:- For a plane wall adding more insulation will result in a decrease in heat transfer as the area of heat flow remains constant .But adding insulation to a cylindrical pipe or a conducting wire or a spherical shell will result in an increase in thermal resistance for conduction at the same will result in a decrease in the convection resistance of the outer surface because of increase in surface area for convection. Therefore the heat transfer may either increase or decrease depending on the relative magnitude of these two resistances.

Critical Radius of Insulation for Cylinder:- Let us consider a cylindrical pipe of outer radius r_s maintained at a constant temperature of T_s. Let the pipe now be insulated with a material of thermal conductivity k and outer radius r. Let the outer surface of the insulation be in contact with a fluid at a uniform temperature T_∞ with a surface heat transfer coefficient h. Then the thermal circuit for this arrangement will be as shown in Fig.3.15.

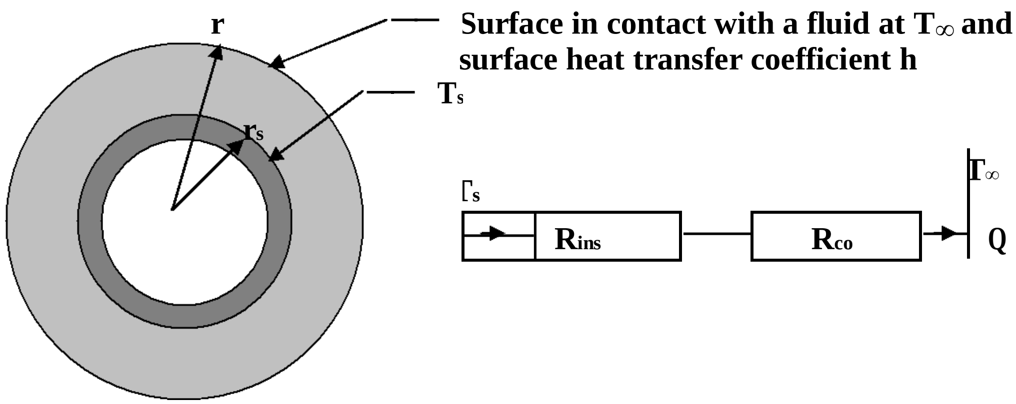


Fig.3.15: Schematic of a cylindrical pipe covered with an insulation and exposed to an ambient and the corresponding thermal circuit

The rate of heat transfer from the pipe to the ambient is given by

$$Q = \frac{(T_s - T_\infty)}{[R_{ins} + R_{co}]} = \frac{(T_s - T_o)}{\dfrac{\ln (r / r_s)}{2 \pi L k} + \dfrac{1}{2 \pi r L h}} \quad \text{.......................................(3.44)}$$

It can be seen from Eq. (3.44) that if T_S and h are assumed not to vary with „r" then Q depends only on r and the nature of variation of Q with r will be as shown in Fig.3.16. The value of r at which Q reaches a maximum can be determined as follows.

Eq. (3.44) can be written as $Q = \dfrac{(T_S - T_O)}{F(r)}$

where $F(r) = \dfrac{\ln (r / r_S)}{2 \pi L k} + \dfrac{1}{2 \pi r L h}$

Hence for Q to be maximum, F(r) has to be minimum: i.e., dF(r) / dr = 0

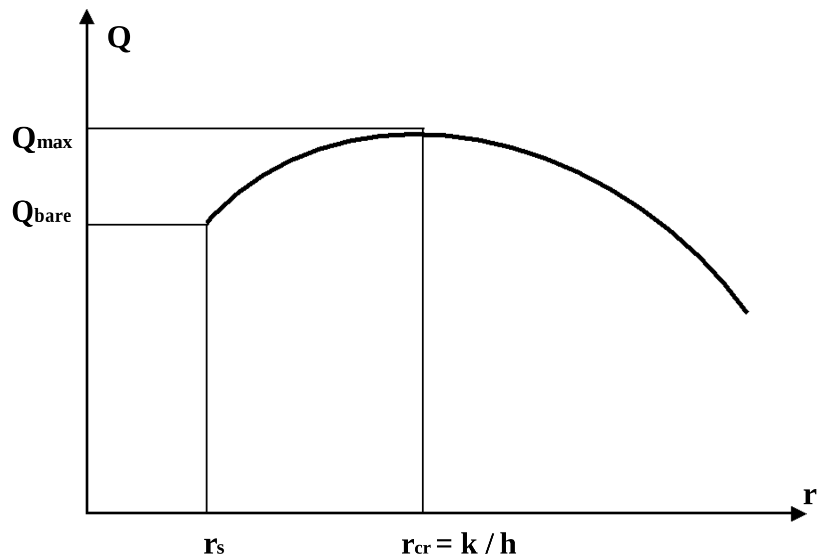


Fig.3.16: Variation of Q with outer radius of insulation

Now $dF / dr = (1 / 2\pi Lk)(1/r) - (1/2\pi Lh)(1/r_2) = 0$

Or $r = k / h$.

This value of r is called *"critical radius of insulation, r_{cr}"*.

Therefore **$r_{cr} = k / h$..(3.45)**

It can be seen from Fig.(3.16) that if the outer radius of the bare tube or bare wire is greater than the critical radius then, any addition of insulation on the tube surface decreases the heat loss to the ambient. But if the outer radius of the tube is less than the critical radius , the heat loss will increase continuously with the addition of insulation until the outer radius of insulation equals the critical radius. The heat loss becomes maximum at the critical radius and begins to decrease with addition of insulation beyond the critical radius.

The value of critical radius r_{cr} will be the largest when k is large and h is small. The lowest value of h encountered in practice is about 5 W/(m^2 – K) for free convection in a gaseous medium and the thermal conductivity of common insulating materials is about 0.05 W/(m – K). Hence the largest value of r_{cr} that we may likely to encounter is given by

$$r_{cr} = \frac{0.05}{5} = 0.01 \text{ m} = 1 \text{ cm}$$

The critical radius would be much less in forced convection (it may be as low as 1mm) because of large values of h associated with forced convection. Hence we can insulate hot water or steam pipes freely without worrying about the possibility of increasing the heat loss to the surroundings by insulating the pipes.

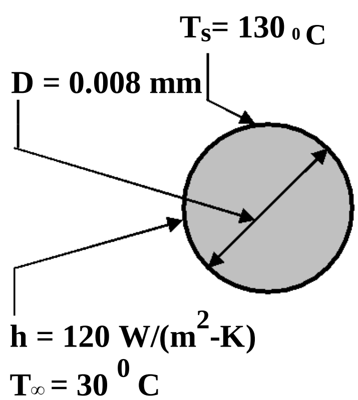
The radius of electric wires may be smaller than the critical radius. Therefore, the plastic electrical insulation may enhance the heat transfer from electric wires, there by keeping their steady operating temperatures at lower and safer levels.

Critical Radius Insulation for a Sphere:- The analysis described above for cylindrical pipes can be repeated for a sphere and it can be shown that for a sphere the critical radius of insulation is given by

$$r_{cr} = \frac{2k}{h} \qquad \text{.....(3.46)}$$

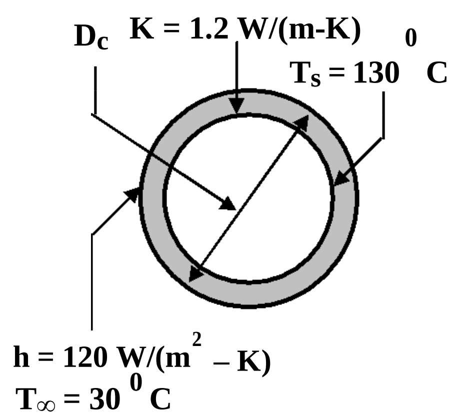
Example 3.12:-A conductor with 8 mm diameter carrying an electric current passes through an ambient at 30 ^{0} C with a convection coefficient of 120 W/(m^2 – K). The temperature of the conductor is to be maintained at 130 ^{0} C. Calculate the rate of heat loss per metre length of the conductor when (a) the conductor is bare and (b) conductor is covered with bakelite insulation [k = 1.2 W/(m-K)] with radius corresponding to the critical radius of insulation.

Solution:



(a) Conductor without

Insulation.



(b) Conductor with critical thickness of insulation

(a) When the conductor is bare the rate of heat loss to the ambient is given by

$$Q = h\,\pi D\,L\,(T_S - T_\infty) = 120 \times \pi \times 0.008 \times 1 \times (130 – 30) = 301.6 \text{ W/m.}$$

(b) When the conductor is covered with critical thickness of insulation,

$D_c = 2\, r_c = 2\,(k/h) = 2 \times (1.2 / 120) = 0.02$ m.

$$R_{insulation} = \frac{1}{2\pi L k} \ln (D_c / D) = \frac{1}{2\pi \times 1.0 \times 1.2} \ln (0.02 / 0.008)$$

$$= 0.1215\ (m - {}^0C) / W.$$

$$R_{co} = 1/(hA_c) = \frac{1}{\pi D_c L h} = \frac{1}{\pi \times 0.02 \times 1 \times 120} = 0.133\ (m - {}^0C)/W.$$

$$\sum R = R_{insulation} + R_{co} = 0.1215 + 0.133 = 0.2545\ (m - {}^0C) / W.$$

$$Q_{insulation} = \frac{(T_s - T_\infty)}{\sum R} = \frac{(130 - 30)}{0.145} = 392.93\ W / m.$$

Example 3.13: -An electrical current of 700 A flows through a stainless steel cable having a diameter of 5 mm and an electrical resistance of 6×10^{-4} ohms per metre length of the cable. The cable is in an environment at a uniform temperature of 30 ^{0}C and the surface heat transfer coefficient of 25 W/(m^2 – ^{0}C).

(a) What is the surface temperature of the cable when it is bare?

(b)What thickness of insulation of k = 0.5 W/(m – K) will yield the lowest value of the maximum insulation temperature? What is this temperature when the thickness is used?

Solution: <u>(a) When the cable is Bare: -</u> Electrical Resistance = $R_e = 6 \times 10^{-4}$ Ω / m

Current through the cable = I = 700 A; D = 0.005 m ; h = 25 W/(m^2-K) ; $T_\infty = 30\ {}^0$

C. Power dissipated = $Q = I^2 R_e = (700)^2 \times 6 \times 10^{-4} = 294$ W / m.

But $Q = hA(T_s - T_\infty)$ or $T_s = T_\infty + Q / [(\pi D L) \times h]$

Or $\quad T_s = 30 + 294 / [(\pi \times 0.005 \times 1) \times 25] = 779\ {}^0$C.

(b) <u>When the cable is covered with insulation:</u>

k = 0.5 W/(m-K) ; Hence critical radius = $r_c = k / h = 0.5 / 25 = 0.02$ (m-K) /

W. Thickness of insulation = $r_c - D/2 = 0.02 - 0.005 / 2 = 0.0175$ m

$$R_{insulation} = \frac{1}{(2\pi\ L\ k)} \ln (r_c\ / r_o) = \frac{1}{(2\pi \times 1 \times 0.5)} \ln (0.02 / 0.0025) = 0.662 \text{ (m-K)/W}$$

$$R_{co} = 1 / (hA_o) = \frac{1}{(2\pi\ r_c L\ h)} = \frac{1}{(2 \times \pi \times 0.02 \times 1 \times 25)} = 0.318 \text{ (m-K) / W.}$$

$$Q = \frac{(T_s - T_\infty)}{R_{insulation} + R_{co}} \quad \text{or} \quad T_s = T_\infty + Q\ (R_{insulation} + R_{co})$$

Or $$T_s = 30 + 294 \times (0.662 + 0.318)$$

$$= 318.12\ {}^0\text{C.}$$

Example 3.14:- A 2 mm-diameter and 10 m-long electric wire is tightly wrapped With a 1 mm-thick plastic cover whose thermal conductivity is 0.15 W / (m-K). Electrical measurements indicate that a current of 10 A passes through the wire and there is a voltage drop of 8 V along the wire. If the insulated wire is exposed to a medium at 30 ^{0}C with a heat transfer coefficient of 24 W / (m^2 – K), determine the temperature at the interface of the wire and the plastic cover in steady operation. Also determine if doubling the thickness of the plastic cover will increase or decrease this interface temperature.

Given: Outer radius of the bare wire = r_S = 1 mm = 0.001 m ; Length of the wire = L = 10 m ; outer radius of plastic insulation = r = 1 + 1 = 2 mm = 0.002 m ;
Current through the wire = I = 10 A ; Voltage drop in the wire = V = 8 V ; Ambient temperature = T_∞ = 30 ^{0}C ; Thermal conductivity of the plastic cover = k = 0.15 W /(m– K) ; Surface heat transfer coefficient = h = 24 W /(m^2 – K).

To find: (i) Interface temperature = T_S ; (ii) Whether T_S increases or decreases when the thickness of insulation is doubled.

Solution: (i) Q = VI = 8 x 10 = 80 W.

The thermal circuit for the problem is shown in Fig. P3.14.

$$R_{ins} = \frac{\ln (r / r_s)}{2\ \pi\ L\ k} = \frac{\ln (\ 0.002 / 0.001)}{2 \times \pi \times 10 \times 0.15} = 0.0735 \text{ K / W}$$

$$R_{co} = \frac{1}{h\ A_o} = \frac{1}{2\ \pi\ L\ r\ h} = \frac{1}{2 \times \pi \times 10 \times 0.002 \times 24} = 0.3316 \text{ K / W}$$

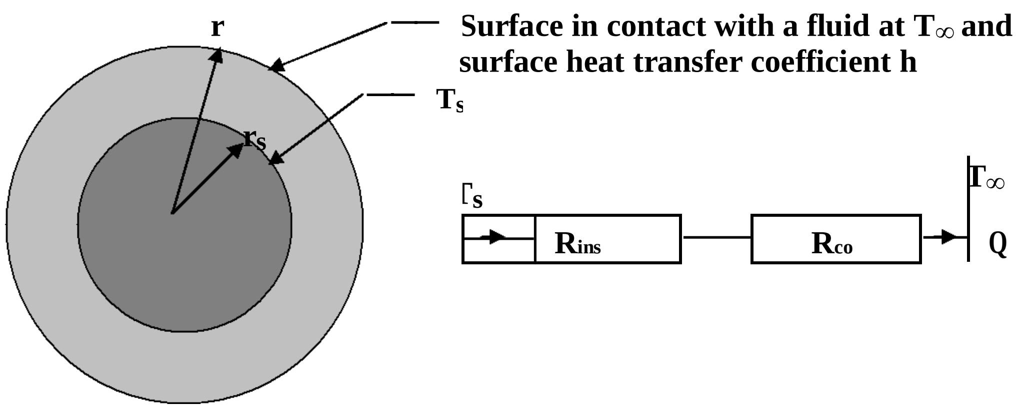


Fig.P3.14: Schematic of an electric wire covered with an insulation and exposed to an ambient and the corresponding thermal circuit

Hence $R_{total} = R_{ins} + R_{co} = 0.0735 + 0.3316 = 0.405$ K / W.

Now $Q = (T_S - T_\infty) / R_{total}$.

Hence $T_S = T_\infty + Q\,R_{total} = 30 + 80 \times 0.405 = 62.4\ ^0C$

(ii) Critical radius of insulation = $r_{cr} = k / h = 0.15 / 24 = 0.00625$.

Since $r_{cr} > r$, increasing the thickness of plastic insulation will increase the heat transfer rate if Ts is held constant or for a given heat transfer rate the interface temperature Ts will decrease till the critical radius is reached. Now when the thickness is doubled then r = 3 mm = 0.003 m . Therefore

$$R_{ins} = \frac{\ln(0.003 / 0.001)}{2 \times \pi \times 10 \times} = 0.1166 \text{ K / W}$$

$$R_{co} = \frac{1}{2 \times \pi \times 10 \times 0.003 \times 24} = 0.221 \text{ K / W}$$

Therefore $R_{total} = 0.1166 + 0.221 = 0.3376$ K / W.

and $T_S = 30 + 80 \times 0.3376 = 57\ ^0C$

Extended Surfaces (Fins):-

Solution to tutorial problems:

Example 3.15:- A steel rod of diameter 2 cm and thermal conductivity 50 W/(m – K) is exposed to ambient air at 20 ^{0}C with a heat transfer coefficient 64 W/(m^2 – K). One end of the rod is maintained at a uniform temperature of 120 ^{0}C. Determine the rate of heat from the rod to the ambient and the temperature of the tip of the rod exposed to ambient if (i) the rod is very long, (ii) rod is of length 10 cm with negligible heat loss from its tip, (ii) rod is of length 25 cm with heat loss from its tip.

Solution: (i) Given:- D = 0.02 m ; k = 50 W/(m-K); T_∞ = 20 ^{0}C; T_o = 120 ^{0}C; h = 64 W/(m^2-K); Very long fin ($x \to \infty$)

$m = \sqrt{[(hP) / (kA)]} = \sqrt{[(h\pi D / (\pi D^2/4)]} = \sqrt{[(4h) / (kD)]}$

$$= \sqrt{\left[\frac{4 \times 64}{50 \times 0.02}\right]} = 16$$

For a very long fin the rate of heat transfer is given by

$Q = kmA(T_0 - T_\infty) = 50 \times 16 \times (\pi / 4) \times 0.02^2 \times [120 – 20] = 25.13$ W

(ii) L = 0.10 m. Hence mL = 16 x 0.1 = 1.6

$Q = kmA(T_0 - T_\infty) \tanh mL = 50 \times 16 \times (\pi / 4) \times 0.02^2 \times (120 – 20) \times \tanh 1.6 = 23.16$ W

(iii) When the heat loss from the rod tip is not negligible, then we can use the same formula as in case (ii) with modified length L_e given by

$L_e = L + A /P = L + (\pi D^2/4) /(\pi D) = L + D / 4 = 0.1 + 0.02/4 = 0.105$ Hence $mL_e = 16 \times 0.105 = 1.68$ and $\tanh mL_e = \tanh 1.68 = 0.9329$ Hence $Q = 25.13 \times 0.933 = 23.44$ W

Example 3.16:-A thin rod of uniform cross section A, length L and thermal conductivity k is thermally attached from its ends to two walls which are maintained at temperatures T_1 and T_2. The rod is dissipating heat from its lateral surface to an ambient at temperature T_∞ with a surface heat transfer coefficient h.

(a) Obtain an expression for the temperature distribution along the length of the rod

(b) Also obtain an expression for the heat dissipation from the rod to the ambient

Solution: The general solution for the one-dimensional steady-state temperature distribution along the length of a rod dissipating heat by convection from its lateral surface is given by

$$\theta(x) = C_1 \cosh mx + C_2 \sinh mx \quad \text{.......................................} (1)$$

where $\theta(x) = T(x) - T_\infty$; $m = \sqrt{(hP) / (kA)}$:
P = perimeter of the rod = π D and A = Area of cross section of the rod = $\pi D^2 / 4$.

The boundary conditions are: (i) at x = 0, T = T_1 or $\theta = T_1 - T_\infty = \theta_o$ (say).

(ii) at x = L, T = T_2 or $\theta = T_2 - T_\infty = \theta_L$ (say).

Condition(i) in Eq. (1) gives $\theta_o = C_1$.

Condition (ii) in Eq. (1) gives $\theta_L = \theta_0 \cosh mL + C_2 \sinh mL$

$$C_2 = \frac{(\theta_L - \theta_o \cosh mL)}{\sinh mL}.$$

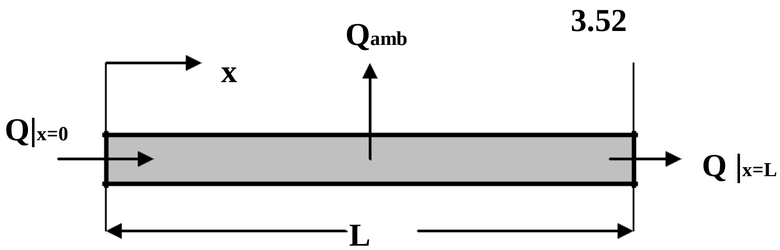
Substituting for C_1 and C_2 in Eq. (1) we have

$$\theta(x) = \theta_o \cosh mx + \frac{(\theta_L - \theta_o \cosh mL)}{\sinh mL} \sinh mx$$

$$\text{0r } \theta(x) = \frac{\theta_o \cosh mx \sinh mL + \theta_L \sinh mx - \theta_o \cosh mL \sinh mx}{- \sinh mL}$$

$$\text{0r } \theta(x) = \frac{\theta_L \sinh mx + \theta_o \sinh m(L - x)}{\sinh mL} \quad \text{..} (2)$$

Expression for the rate of heat dissipation from the rod:



Energy balance for the rod is given by

$$Q_{amb} = Q|_{x=0} - Q|_{x=L}$$

$$= - kA\,(d\theta / dx)|_{x=0} + kA\,(d\theta / dx)|_{x=L} \quad \text{..........(3)}$$

From Eq. (2) we have

$$(d\theta / dx) = \frac{- m\,[\theta_L \cosh mx + \theta_0 \cosh m(L - x)]}{\sinh mL}$$

Therefore

$$(d\theta / dx)|_{x=0} = \frac{- m\,[\theta_L + \theta_0 \cosh mL]}{\sinh mL}$$

and

$$(d\theta / dx)|_{x=L} = \frac{- m\,[\theta_L \cosh mL + \theta_0]}{\sinh mL}$$

Hence

$$Q_{amb} = \frac{kmA\,[\theta_L + \theta_0 \cosh mL - \theta_L \cosh mL - \theta_0]}{\sinh mL}$$

$$= \frac{kmA\,[(\theta_L - \theta_0) - (\theta_L - \theta_0)\cosh mL]}{\sinh mL}$$

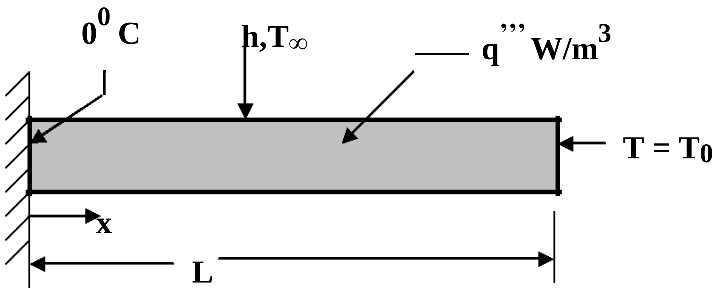
or

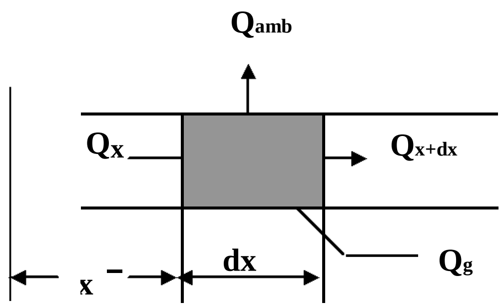
$$Q_{amb} = \frac{kmA(\theta_L - \theta_0)\,(1 - \cosh mL)}{\sinh mL}$$

Example 3.17:-Heat is generated at a constant rate of q''' W/m^3 in a thin circular rod of length L and diameter D by the passage of electric current. The two ends of the rod are maintained at uniform temperatures with one end at temperature T_0 and the other end at 0 ^{0}C, while heat is being dissipated from the lateral surface of the rod to an ambient at 0 ^{0}C with a surface heat transfer coefficient h.

(a) Derive the one-dimensional steady state energy equation to determine the temperature distribution along the length of the rod
(b) Solve the above equation and obtain the temperature distribution.

Solution: Since the rod is generating heat and dissipating heat to the ambient, the governing differential equation to determine the one-dimensional steady state temperature distribution has to be obtained from first principles as illustrated below.





Consider an elemental length „dx"of the rod as shown in the figure above. The various energies crossing the boundaries of the rod as well as the energy generated are also shown in the figure. For steady state condition the energy balance equation for the rod element can be written as

$$Q_x + Q_g = Q_{x+dx} + Q_{amb}$$

Or $$Q_x + Q_g = Q_x + (dQ_x/dx)\, dx + Q_{amb}$$

Or $$(dQ_x/dx)\, dx + Q_{amb} = Q_g$$

$$d/dx(- kA\, dT/dx)\, dx + hPdx\, (T - T_\infty) = Adx\, q''''''$$

Or $$(d^2T / dx^2) – (hP / kA)\, (T - T_\infty) = - (q'''''' / k)$$

Let $T - T_\infty = \theta$ and $(hP / kA) = m^2$. then the above equation reduces to

$$(d^2 \theta /d x^2) - m^2\, \theta = - (q'''''' / k) \text{ ....(1)}$$

Eq.(1) is a non-homogeneous linear second order ordinary differential equation whose solution can be written as

$$\theta(x) = \theta_h(x) + \theta_p(x) \text{ ------------------------------------ } (2)$$

where $\theta_h(x)$ satisfies the homogeneous part of the differential equation namely

$$(d^2 \theta_h / d x^2) - m^2 \theta_h = 0 \text{ ------------------------------------ } (3)$$

and $\theta_p(x)$ is the particular integral which satisfies Eq. (1). Solution to Eq.(3) is given by

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx} \text{ ----------------------------- } (4)$$

To find $\theta_p(x)$:- Since the RHS of Eq.(1) is a constant let us assume $\theta_p(x) = B$, where B is a constant. Substituting this solution in Eq.(1) we have

$$0 - m^2 B = -(q''' / k)$$

Or $$B = (q''' / km^2)$$

Therefore the complete solution for Eq. (1) can be written as

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx} + (q''' / km^2)$$

Or $T(x) = T_\infty + C_1 e^{mx} + C_2$ e Boundary $^{-mx} + (q''' / km^2)$..........................(5)

conditions are: (i) at $x = 0$, $T = 0$

(ii) at $x = L$, $T = T_0$

Condition (i) in Eq. (5) gives

$$0 = T_\infty + C_1 + C_2 + (q''' / km^2)$$

Or $$C_1 + C_2 = -T_\infty - (q'''/km^2) \text{ -------------------------------- } (a)$$

Condition(ii) in Eq.(5) gives $T_0 = T_\infty + C_1 e^{mL} + C_2 e^{-mL} + (q'''/km^2)$

Or $$C_1 e^{mL} + C_2 e^{-mL} = T_0 - T_\infty - (q'''/km^2) \text{ ----------------------------- } (b)$$

From Eq.(a) $C_2 = -C_1 - T_\infty - (q'''/km^2)$. Substituting this expression in Eq.(b) we have $C_1 e^{mL} - [C_1 + T_\infty + (q'''/km^2)] e^{-mL} = T_0 - T_\infty - (q'''/km^2)$

Solving for C_1 we get

$$C_1 = \frac{T_0 - \{T_\infty + (q'''/km^2)\}\{1 - e^{-mL}\}}{\{e^{mL} - e^{-mL}\}}$$

$$C_2 = -\{T_\infty + (q'''/km^2)\} - \frac{T_0 - \{T_\infty + (q'''/km^2)\}\{1 - e^{-mL}\}}{\{e^{mL} - e^{-mL}\}}$$

$$C_2 = \frac{-\{T_\infty + (q'''/km^2)\}\{e^{mL} - e^{-mL}\} - T_0 + \{T_\infty + (q'''/km^2)\}\{1 - e^{-mL}\}}{\{e^{mL} - e^{-mL}\}}$$

$$C_2 = \frac{\{T_\infty + (q'''/km^2)\}[-e^{mL} + e^{-mL} + 1 - e^{-mL}] - T_0}{\{e^{mL} - e^{-mL}\}}$$

$$C_2 = \frac{\{T_\infty + (q'''/km^2)\}[1 - e^{mL}] - T_0}{\{e^{mL} - e^{-mL}\}}$$

Substituting the expressions for C_1 and C_2 in Eq. (5) and simplifying we get

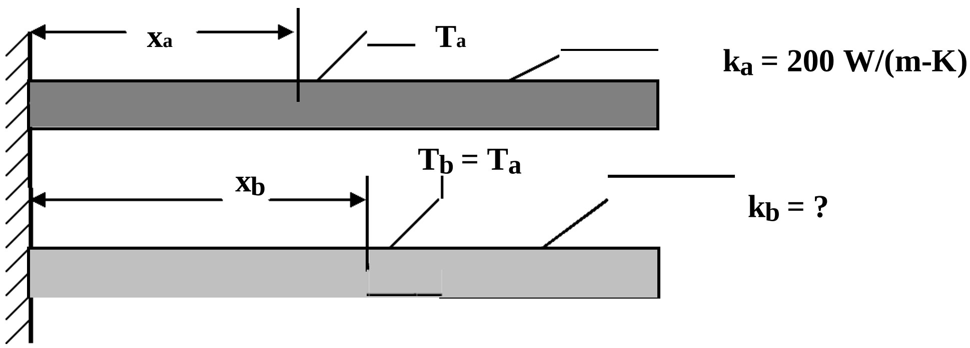
$$T(x) = T_\infty + (q'''/km^2) + \frac{[T_0 - \{T_\infty + (q'''/km^2)\}\{1 - e^{-mL}\}]\, e^{mx}}{\{e^{mL} - e^{-mL}\}} + \frac{[\{T_\infty + (q'''/km^2)\}[1 - e^{mL}] - T_0]\, e^{-mx}}{\{e^{mL} - e^{-mL}\}}$$

$$T(x) = T_\infty + (q'''/km^2) + \frac{T_0(e^{mx} - e^{-mx})}{\{e^{mL} - e^{-mL}\}} +$$

$$\frac{[-\{T_\infty + (q'''/km^2)\}\{1 - e^{-mL}\}]\, e^{mx} + \{T_\infty + (q'''/km^2)\}[1 - e^{mL}]\, e^{-mx}]}{\{e^{mL} - e^{-mL}\}}$$

Example 3.18:- Two very long slender rods of the same diameter are given. One rod is of aluminum (k = 200 W/(m-K)). The thermal conductivity of the other rod is not known. To determine this, one end of each rod is thermally attached to a metal surface maintained at a uniform temperature T_0. Both rods are losing heat to the ambient air at T_∞ by convection with a surface heat transfer coefficient h. The surface temperature of each rod is measured at various distances from hot base surface. The temperature of the aluminum rod at 40 cm from the base is same as that of the rod of unknown thermal conductivity at 20 cm from the base. Determine the unknown thermal conductivity.

Solution:



For very long slender rods the steady-state one-dimensional temperature distribution along the length of the rod is given by

$$\theta(x) = \theta_0 e^{-mx}$$

……………………………..(1) where $\theta(x) = T(x) - T_\infty$ and $\theta_0 = T_0 - T_\infty$.

For rod A Eq.(1) can be written as $\theta_a(x) = \theta_0 e^{-m_a x_a}$………………………………………(2)

And for rod B it can be written as $\theta_b(x) = \theta_0 e^{-m_b x_b}$………………………………………(3)

It is given that when $x_a = 0.4$ m and $x_b = 0.2$ m, $\theta_a(x_a) = \theta_b(x_b)$

Therefore we have $\theta_0 e^{-0.4 m_a} = \theta_0 e^{-0.2 m_b}$

Or $m_b = 2 m_a$

Or $\sqrt{[(hP_b) / (k_b A_b)]} = 2\sqrt{[(hP_a) / (k_a A_a)]}$

Since $P_a = P_b$ and $A_a = A_b$, we have $\sqrt{k_a} = 2\sqrt{k_b}$ or $k_a = 4 k_b$

Therefore $k_b = 200/4 = 50$ W/(m-K).

Example 3.19:- Show that for a finned surface the total heat transfer rate is given by

$$\mathbf{Q_{total} = [\eta\, \beta + (1 - \beta)]\, a\, h\, \theta_0 = \acute{\eta}\, a\, h\, \theta_0}$$

Where η = fin efficiency ; $\beta = a_f / a$: a_f = surface area of the fin, a = total heat transfer area (i.e. finned surface + unfinned surface) ; $\theta_0 = T_0 - T_\infty$, with T_0 = fin base temperature and T_∞ = ambient temperature, and $\acute{\eta}$ = area – weighted fin efficiency.

Solution:

$$Q_{total} = Q_{fin} + Q_{bare}$$

Where Q_{total} = Total heat transfer rate, Q_{fin} = Heat transfer rate from the finned surface and Q_{bare} = Heat transfer rate from the bare surface.

Therefore $$Q_{total} = \eta\, h\, a_f\, \theta_0 + h(a - a_f)\, \theta_0$$

$$= h\, a\, \theta_0 \left[(\eta\, a_f) / a + (1 - a_f/a)\right]$$

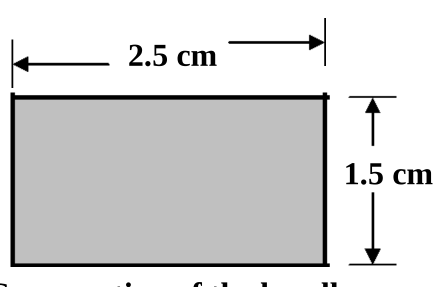
here, $\beta = a_f/a$

$$= ha\, \theta_0 [\eta\beta + (1 - \beta)]$$

$$= \acute{\eta}\, ha\, \theta_0 \text{ , where } \acute{\eta} = [\eta\beta + (1 - \beta)]$$

Example 3.20:- The handle of a ladle used for pouring molten lead at 327 ^{0}C is 30 cm long and is made of 2.5 cm x 1.5 cm mild steel bar stock (k = 43 W/(m-K)). In order to reduce the grip temperature, it is proposed to make a hollow handle of mild steel plate 1.5 mm thick to the same rectangular shape. If the surface heat transfer coefficient is 14.5 W/(m^2-K) and the ambient temperature is 27 ^{0}C, estimate the reduction in the temperature of the grip. Neglect the heat transfer from the inner surface of the hollow shape.

Solution: (a) When the handle is made of solid steel bar:



Cross section of the handle

h = 14.5 W/(m^2-K) ;

k = 43 W/(m-K)

$\theta_0 = 327 - 27 = 300\ ^0$C

Area of cross section of the bar = A = 2.5 x 1.5 x 10^{-4} m^2 = 3.75 x 10^{-4} m^2

Perimeter of the bar = P = 2 [2.5 + 1.5] x 10^{-2} m = 8 x 10^{-2} m

$$\text{Therefore } m = \frac{(hP)^{(1/2)}}{(kA)^{(1/2)}} = \frac{\sqrt{[14.5 \times 8 \times 10^{-2}]}}{\sqrt{[43 \times 3.75 \times 10^{-4}]}} = 8.48\ (1/m)$$

Therefore $mL = 8.48 \times 0.3 = 2.54$.

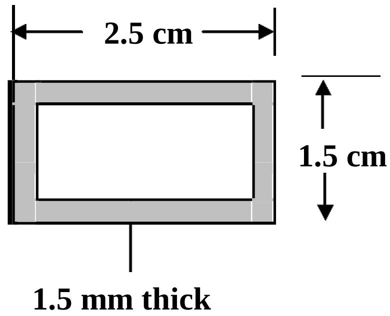
When the heat loss from the tip of the handle is neglected the temperature at any point along the length of the handle is given by

$$\theta(x) = \theta_0 \frac{\cosh m(L - x)}{\cosh mL}$$

Therefore $\theta(x)|_{x=L} = \theta_0 / \cosh mL = 300 / \cosh 2.54 = 47\ ^0C.$

Or $T(x)|_{x=L} = 47 + 27 = 74\ ^0C.$

(b) When the handle is hollow made out of a sheet:



Area of the cross section of the fin is

$A = [(2.5 \times 1.5) – (2.5 – 0.3) \times (1.5 – 0.3)]$

$= 1.11\ cm^2 = 1.11 \times 10^{-4}\ m^2$

$P = 2 \times [\ 2.5 + 1.5\ ] = 8\ cm = 8 \times 10^{-2}\ m$

$$m = \sqrt{(hP) / (kA)} = \frac{\sqrt{(14.5 \times 8 \times 10^{-2})}}{\sqrt{(43 \times 1.11 \times 10^{-4})}}$$

Or m = 15.59 1/m. Therefore mL = 15.59 x 0.3 = 4.68

$$\theta(x)|_{x=L} = \frac{\theta_0}{\cosh mL} = \frac{(327 – 27)}{\cosh 4.68} = 5.57\ ^0C.$$

Therefore $T(x)|_{x=L} = 5.57 + 27 = 32.57\ ^0C.$

Reduction in grip temperature = $74 – 32.57 = 41.43\ ^0C.$

Example 3.21:- Derive an expression for the overall heat transfer coefficient across a plane wall of thickness 'b' and thermal conductivity 'k' having rectangular fins on both sides. Given that over an overall area A of the wall, the bare area on both sides, not covered by the fins are A_{u1} and A_{u2}, the fin efficiencies are η_1 and η_2, and the heat transfer coefficients h_1 and h_2.

Solution:

Let T_i be the temperature of the fluid in contact with the surface 1, T_0 be the temperature of the fluid in contact with surface 2, T_1 be the temperature of surface 1 and T_2 be the temperature of surface 2.Let $T_i > T_0$. Then the rate of heat transfer from T_i to T_0 is given by

$$Q = Q_{bare} + Q_{fin}$$
$$= h_iA_{u1} (T_i – T_1) + h_i\, \eta_1 A_{f1}(T_i – T_1)$$

Or

$$Q = \frac{(T_i - T_1)}{(1/h_1A_{u1})} - \frac{(T_i - T_1)}{(1/h_1\eta_1A_{f1})}$$

$$Q = \frac{(T_i - T_1)}{[(1/h_1A_{u1}) + (1/h_1\eta_1A_{f1})]} \quad \text{.......................................(1)}$$

$$Q = \frac{(T_2 - T_0)}{[(1/h_2A_{u2}) + (1/h_2\eta_2A_{f2})]} \quad \text{.......................................(2)}$$

Rate of heat transfer is also given by

$$Q = \frac{(T_1 - T_2)}{(b/Ak)} \quad \text{..(3)}$$

Therefore as A/B = C/D = E/F = (A+C+E)/(B+D+F) →

$$Q = \frac{(T_i - T_1) + (T_1 - T_2) + (T_2 - T_0)}{[(1/h_1A_{u1}) + (1/h_1\eta_1A_{f1}) + (1/h_1A_{u1}) + (1/h_1\eta_1A_{f1}) + (b/Ak)]}$$

$$Q = \frac{(T_i - T_0)}{[(1/h_1A_{u1}) + (1/h_1\eta_1A_{f1}) + (1/h_1A_{u1}) + (1/h_1\eta_1A_{f1}) + (b/Ak)]} \quad \text{……(4)}$$

If U = overall heat transfer coefficient for the plane wall then

$$Q = UA(T_i - T_0)$$

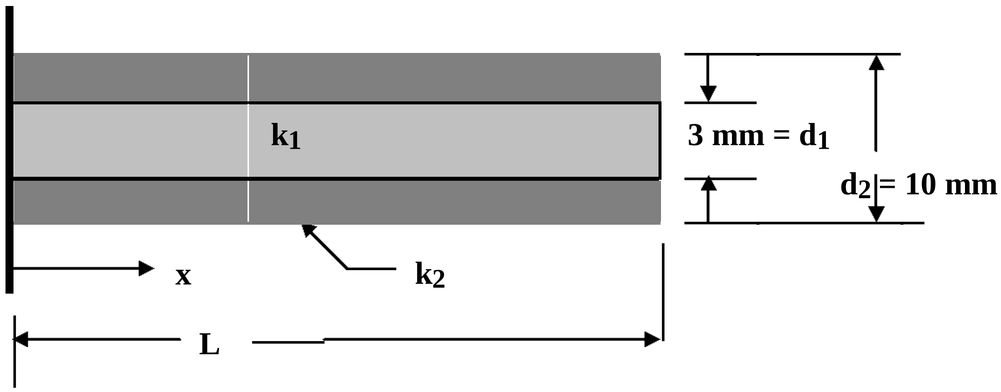
$$= \frac{(T_i - T_0)}{(1/UA)} \quad \text{…………………………………….(5)}$$

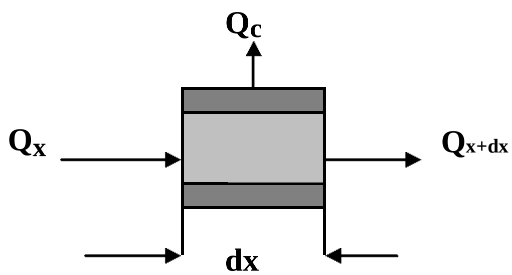
From Eqs. (4) and (5) we have

$$U = \frac{1}{A\,[(1/h_1A_{u1}) + (1/h_1\eta_1A_{f1}) + (1/h_1A_{u1}) + (1/h_1\eta_1A_{f1}) + (b/Ak)]}$$

Example 3.22:- Calculate the effectiveness of the composite pin fin shown in Fig.P3.22. Assume k_1 = 15 W/(m-K), k_2 = 50 W/(m-K) and h = 12 W/(m^2 – K).

Solution:





K_1=15 W/m-k, K_2=50W/m-k, K_3=12 W/m-k.

(b) Energy transfer across the surfaces of the fin element

Energy balance equation for the fin element is given by

$$Q_x = Q_{x+dx} + Q_c$$

$$= Q_x + (dQ_x/dx)\,dx + Q_c$$

Or $\qquad dQ_x / dx + Q_c = 0$ .... (1)

Q_x consists of two components namely the heat transfer Q_{x1} through the material of thermal conductivity k_1 and the rate of heat transfer Q_{x2} through the material of conductivity k_2.

Therefore $\qquad Q_x = Q_{x1} + Q_{x2} = -\,k_1A_1\,(dT / dx) - k_2A_2\,(dT / dx)$

$$= -\,(k_1A_1 + k_2A_2)\,(dT / dx)$$

And $\qquad Q_c = (hP_2\,dx)\,(T - T_\infty)$.

Substituting these expressions for Q_x and Q_c in equation (1) we get

$$(d_2T / dx_2) - \frac{hP_2}{(k_1A_1 + k_2A_2)} (T - T_\infty) = 0$$

Or $$(d_2\theta / dx_2) - m_2\,\theta = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Where $\theta = T - T\infty$ and $m = \sqrt{[\, hP_2 / (k_1A_1 + k_2A_2) \,]}$.

When the heat loss from the fin tip is negligible , the solution to equation (2) is given by

$$\theta(x) = \theta_0 \; \frac{\cosh [m(L - x)]}{\cosh mL} \ldots\ldots\ldots\ldots \quad (3)$$

The rate of heat transfer from the fin base is given by

$$Q_x|_{x=0} = -\,(k_1A_1 + k_2A_2)\,(d\theta / dx)|_{x=0}$$

$$= \frac{-\,(k_1A_1 + k_2A_2) \sinh [m(L - x)]_{x=0}\ (-\,m)\theta_0}{\cosh mL}$$

$$= \quad m\theta_0\,(k_1A_1 + k_2A_2) \tanh mL$$

Now $$\eta = Q_x|_{x=0} / Q_{max}$$

$$= \frac{m\theta_0\ (k_1A_1 + k_2A_2) \tanh mL}{hP_2L\ \theta_0}$$

Noting that $hP_2 / (k_1A_1 + k_2A_2) = m^2$, the above expression for η simplifiers to

$$\eta = \frac{\tanh mL}{mL} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

In the given problem $A_1 = (\pi / 4) \times (0.003)^2 = 7.1 \times 10^{-6}\ m^2$.

$$A_2 = (\pi / 4) \times [\, (0.01)^2 - (0.003)^2] = 7.15 \times 10^{-5}$$

$$P_2 = \pi \times 0.01\ 0.0314\ m.$$

$$m = \frac{\sqrt{[12 \times 0.0314]}}{\sqrt{[(15 \times 7.1 \times 10^{-6}) + (50 \times 7.15 \times 10^{-5})]}} = 10.12$$

Therefore $mL = 10.12 \times 0.1 = 1.012$

$$\eta = \frac{\tanh(1.012)}{1.012} = 0.757$$

Example 3.23:- Why is it necessary to derive a fresh differential equation for determining the one-dimensional steady state temperature distribution along the length of a fin?

Solution:- While deriving the conduction equation in differential form we will have considered a differential volume element within the solid so that the heat transfer across the boundary surfaces of the element is purely by conduction. But in the case of a fin the lateral surface is exposed to an ambient so that the heat transfer across the lateral surfaces is by convection. Therefore we have to derive the differential equation afresh taking into account the heat transfer by convection across the lateral surfaces of the fin.

<u>Solutions to Problems on Conduction in solids with variable thermal conductivity</u>

Example 3.24:- A plane wall 4 cm thick has one of its surfaces in contact with a fluid at 130 ^{0}C with a surface heat transfer coefficient of 250 W/(m^2 – K) and the other surface is in contact with another fluid at 30 ^{0}C with a surface heat transfer coefficient of 500 W/(m^2-K). The thermal conductivity of the wall varies with temperature according to the law

$$k = 20 [1 + 0.001 T]$$

where T is the temperature. Determine the rate of heat transfer through the wall and the surface temperatures of the wall.

Given:- L = 0.04 m; T_i = 130 ^{0}C; h_i = 250 W/(m^2-k); T_o = 30 ^{0}C; h_o = 500

W/(m^2-K); k = 20 [1 + 0.001 T].

To find:- (i) Q_x (ii) T_1 and T_2

Solution:

R_{ci} = Thermal resistance for convection at the surface at T_i = 1/(h_iA) = 1 / (250 x 1) = 0.004 m^2 – K /W

R_{co} = Thermal resistance for convection at the surface at T_o = 1/(h_oA) = 1/(500 x 1)

Or $R_{co} = 0.002\ m^2\text{-K/W}$

Now $Q = (T_i - T_1) / R_{ci}$, where T_1 = Surface temperature in contact with fluid at T_i.

Hence $T_1 = T_i - QR_{ci} = 130 - 0.004\ Q$ ....(1)

Similarly $Q = (T_2 - T_o) / R_{co}$

Or $T_2 = T_o + QR_{co} = 30 + 0.002Q$ .... (2)

From equations (1) and (2) we have

$T_1 - T_2 = 100 - 0.006Q$...(3)

And $T_m = (T_1 + T_2) / 2 = 80 - 0.001Q$..(4)

Hence $k_m = k_o [1 + \beta T_m] = 20 \times [1 + 0.001 \times \{80 - 0.001Q\}]$

$= 21.6 - 2 \times 10^{-5}Q$

Hence thermal resistance offered by the wall = $R = L/(Ak_m)$

Or $$R = \frac{0.04}{[21.6 - 2 \times 10^{-5}Q]}$$

$$Q = \frac{[T_1 - T_2]}{R} = \frac{[100 - 0.006Q]}{0.04} \times \frac{[21.6 - 2 \times 10^{-5}Q]}{}$$

Cross multiplying we have

$0.04Q = 2160 - 0.1316Q + 1.2 \times 10^{-7} Q^2$

Or $Q^2 - 1.41 \times 10^6 Q + 1.8 \times 10^{10} = 0$.Hence $Q = (1.41 \times 10^6 \pm 1.39 \times 10^6) / 2$

For physically meaningful solution T_1 should lie between T_i and T_o. This is possible only If

$Q = (1.41 \times 10^6 - 1.39 \times 10^6) / 2 = 10000$ W.

Now $T_1 = T_i - QR_{ci} = 130 - 10000 \times 0.004 = 90\ ^0C$

and $T_2 = T_0 + Q\ R_{co} = 30 + 10000 \times 0.002 = 50\ ^0C$.

Example 3.25:- The thermal conductivity of a plane wall varies with temperature according to the equation

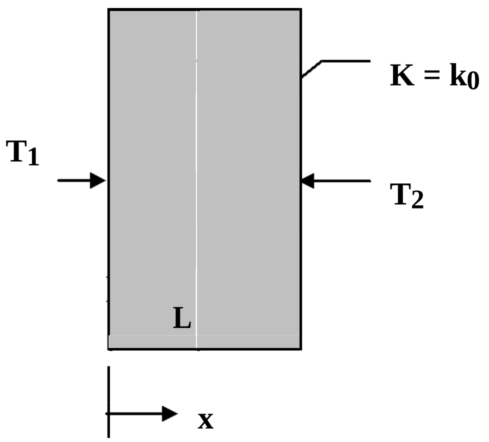
$$\mathbf{k(T) = k_0 [\ 1 + \beta\ T^2\ ]}$$

where k_0 and β are constants.

(a) Develop an expression for the heat transfer through the wall per unit area of the wall if the two surfaces are maintained at temperatures T_1 and T_2 and the thickness of the wall is L.

(b) Develop a relation for the thermal resistance of the wall if the heat transfer area is A.

Solution:



For steady state conduction we have

$$Q_x = -\,kA(dT / dx) = \text{constant.}$$

Or $Q_x = -\,k_0[1 + \beta T^2]A(dT/dx)$

$$Q_x dx = -\,k_0[1 + \beta T^2]A\ dT$$

Integrating the above equation between x = 0 and x = L we have

$$\int_0^L Q_x dx = -\,k_0 A \int_{T1}^{T2} [1 + \beta T^2] dT$$

Or $$Q_x L = -\,k_0 A\ [(T_2 - T_1) + (\beta/3)(T_2^3 - T_1^3)]$$

Or $$Q_x = (k_0 A / L)(T_1 - T_2)\ [1 + (\beta/3)(T_1^2 + T_1 T_2 + T_2^2)]$$

$$Q_x = \frac{(T_1 - T_2)}{\dfrac{1}{(k_0 A/L)\ [1 + (\beta/3)(T_1^2 + T_1 T_2 + T_2^2)]}}$$

Therefore thermal resistance of the wall is given by

$$R = \frac{1}{(k_0A/L)\,[1 + (\beta/3)(T_1^2 + T_1T_2 + T_2^2)]}$$

Example 3.26:- Find the steady -state heat flux through the composite slab made up of two materials as shown in Fig. P 3.26. Also find the interfacial temperature. The thermal conductivities of the two materials vary linearly with the temperature in the following manner:

k_1 = 0.05 [1 + 0.008 T] W/m-K and k_2 = 0.04 [1 + 0.0075 T] W/m-K

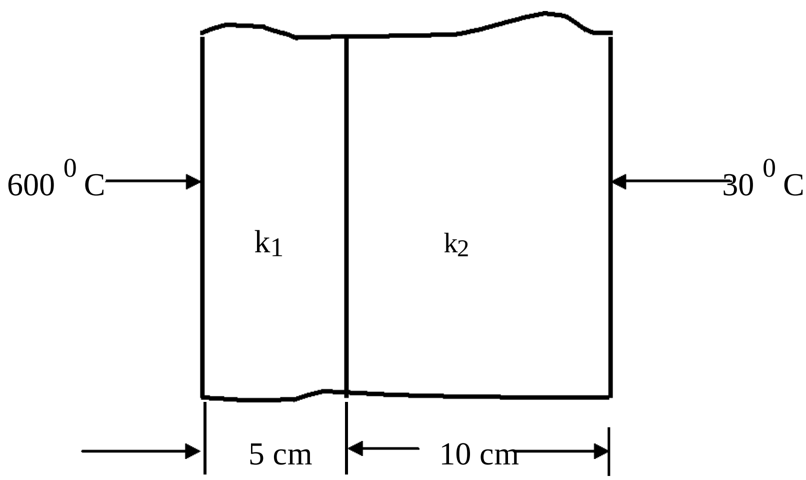
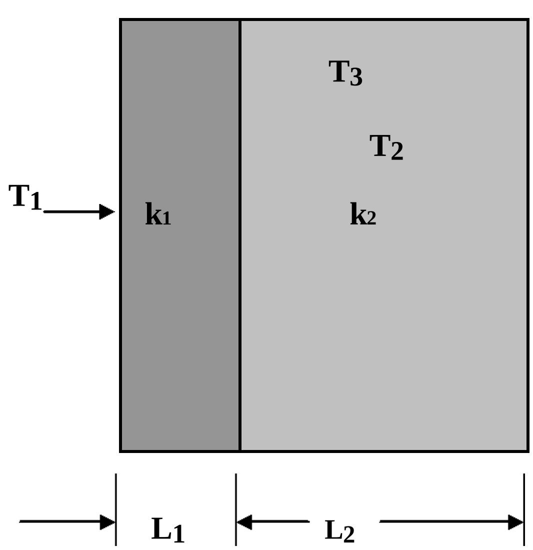


Fig. P 3.26: Schematic for problem 3.26



Data:- $T_1 = 600\,^0C$; $T_2 = 30\,^0C$;

$L_1 = 0.05$ m ; $L_2 = 0.10$ m ;

$k_1 = 0.05\,[\,1 + 0.008T\,]$;

$k_2 = 0.04\,[\,1 + 0.0075T\,]$

To find T_3 and q

Mean thermal conductivity for the first layer is given by

$$k_{m1} = 0.05 [1 + 0.008(T_1 + T_3) /2] = 0.05[1 + 0.004(600 +T_3)]$$

$$= 0.17 + 2 \times 10^{-4} T_3 \text{ W/(m-K)}$$

Similarly $$k_{m2} = 0.04 [1 + 0.0075(T_3 + T_2) /2] = 0.04[1 + 0.00375(T_3 + 30)]$$

$$= 0.0445 + 1.5 \times 10^{-4} T_3 \text{ W/(m-K)}$$

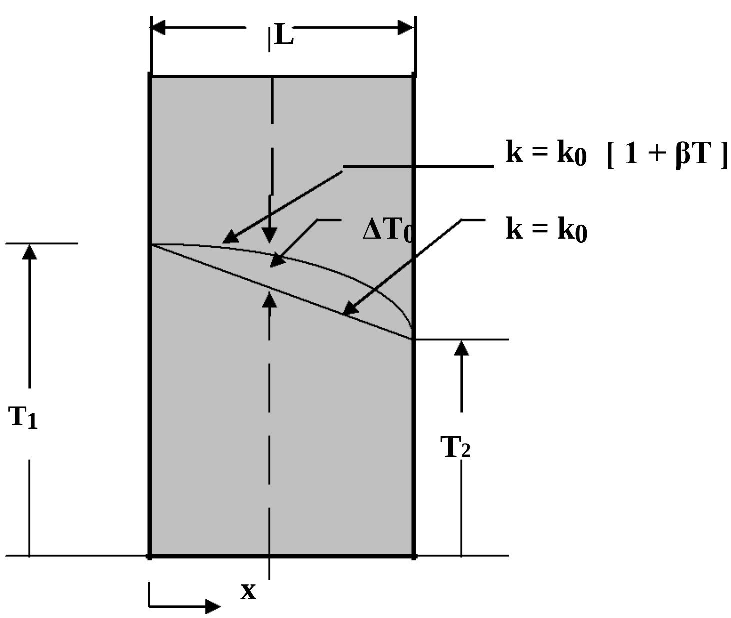
Example 3.27:- Consider a slab of thickness L in which heat is generated at a uniform rate of q''' W/m ^{3}. The two boundary surfaces are maintained at temperatures T_1 and T_2. The thermal conductivity of the slab varies with temperature according to the equation

$$k(T) = k_0 [1 + \beta T]$$

where k_0 and β are constants. Develop an expression for the heat flux q(x) in the slab.

Measurements show that steady -state conduction through a plane wall without heat generation produced a convex temperature distribution such that the mid-point temperature was ΔT_0 higher than expected for a linear temperature distribution. Assuming that the thermal conductivity has a linear dependence on temperature [k = $k_0(1 + \beta T)$], where k_0 and β are constants, develop a relationship to evaluate β in terms of ΔT_0, T_1 and T_2.

Solution:



For constant thermal conductivity k_0, the temperature distribution in the wall is linear and is given by

$$T(x) = T_1 – (T_1 – T_2)\, x / L$$

Therefore $\quad T(x)|_{x=L/2} = (T_1 + T_2) / 2$ ....(1)

When the thermal conductivity varies with temperature the temperature distribution in the wall is determined as follows.

$$Q_x = -\, kA(dT/dx) = -\, k_0(1 + \beta T)A\, (dT/dx)$$

Therefore $\quad Q_x\, dx = -\, k_0A(1 + \beta T)\, dT$ .... (2)

Integrating the above equation between x = 0 and any x at which the temperature is given by T*(x) we have

$$\int_0^x Q_x dx = \int_{T1}^{T^*} -\, k_0A(1 + \beta T)\, dT$$

Or $\quad Q_x\, x = -\, k_0A\, [(T^* - T_1) + (\beta/2)(T^{*2} – T_1^2)]$ .... (3)

If equation is integrated between x = 0 and x = L we get

$$Q_x\, L = -\, k_0A\, [(T_2 - T_1) + (\beta/2)(T_2^2 – T_1^2)]$$

Or $\quad Q_x = (k_0A)[1 + \beta(T_1 + T_2) / 2]\, (T_1 – T_2) / L$

Or $\quad Q_x = k_mA\, (T_1 – T_2) / L$..(4)

Where $\quad k_m = k_0\, [1 + \beta(T_1 + T_2) / 2]$.

Substituting this expression for Q_x in equation (3) we get

$$k_m\, (T_1 – T_2)\, (x / L) = -\, k_0[(T^* - T_1) + (\beta/2)(T^{*2} – T_1^2)]$$

The above equation simplifies to

$$T^{*2} + (2/\beta)T^* + [\, (2/\beta)\{(k_m/k_0)(T_1 – T_2)(x/L) – T_1\} – T_1^2\,] = 0.$$

Therefore

$$T^* = -(1/\beta) \pm \sqrt{(1/\beta^2) - (2k_m/\beta k_0)(x/L)(T_1 - T_2) + T_1^2 + (2/\beta)T_1}$$

$$T^*|_{x=L/2} = -(1/\beta) \pm \sqrt{(1/\beta^2) - (k_m/\beta k_0)(T_1 - T_2) + T_1^2 + (2/\beta)T_1}$$

$$= -(1/\beta) \pm \sqrt{(1/\beta^2) - [k_0\{1 + \beta(T_1 + T_2)/2\}/\beta k_0](T_1 - T_2) + T_1^2 + (2/\beta)T_1}$$

$$= -(1/\beta) \pm \sqrt{(1/\beta^2) - [\{1 + \beta(T_1 + T_2)/2\}/\beta](T_1 - T_2) + T_1^2 + (2/\beta)T_1}$$

But $T^*|_{x=L/2} = T(x)|_{x=L/2} + \Delta T_0 = (T_1 + T_2)/2 + \Delta T_0$.

Hence we have

$$(T_1 + T_2)/2 + \Delta T_0 = -(1/\beta) \pm \sqrt{(1/\beta^2) - [\{1 + \beta(T_1 + T_2)/2\}/\beta](T_1 - T_2) + T_1^2 + (2/\beta)T_1}$$

Example 3.28:- A slab of thickness „L" has its two surfaces at x=0 and x = L maintained at uniform temperatures of T_0 and T_L respectively. The thermal conductivity of the slab has spatial variation according to the law $k = k_0 [1 + \alpha x]$, where k_o and α are constants. Obtain expressions for (i) temperature distribution in the slab, and (ii) rate of heat transfer through the slab assuming one dimensional steady state conduction.

Solution:

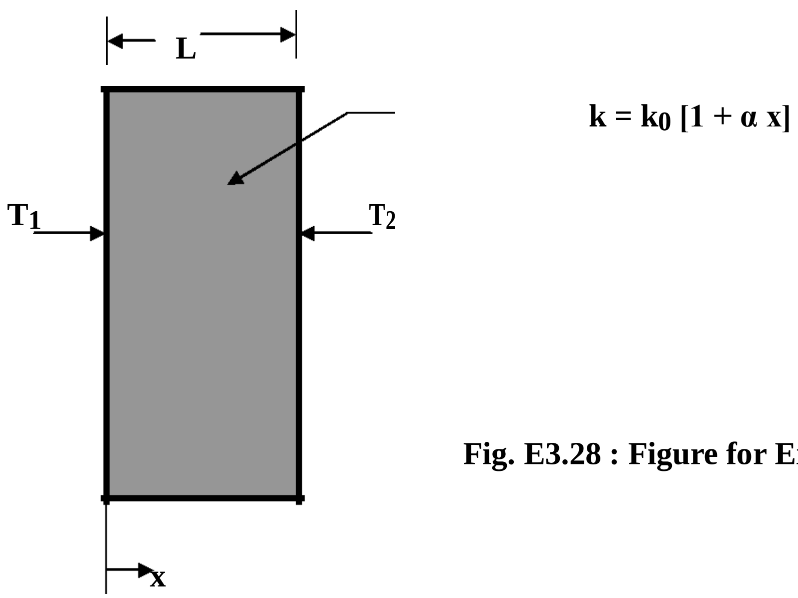


Fig. E3.28 : Figure for Example 3.28

The governing differential equation for one dimensional steady state conduction without heat generation is given by

$d / dx\, [k\, dT/dx] = 0$ substituting for k we have

$d / dx\, [k_0\, (1 + \alpha x)\, dT/dx] = 0$ .... (1)

Let $1 + \alpha x = y$; Then $\alpha\, dx = dy$ or $dy/dx = \alpha$. Then Eq.(1) can be written in terms of the variable „y‟ as follows.

$\{d / dy\, [k_0\, y\, (dT/dy)\, \alpha\,]\, \}\alpha = 0.$

Or $\{d / dy\, [k_0\, y\, (dT/dy)\, \alpha\,]\, \} = 0$

Integrating w.r.t „y‟ once we get

$[k_0\, y\, (dT/dy)\, \alpha\,] = C_1$

Or $dT = [C_1 /(\alpha\, k_0)]\, (dy / y)$

Integrating once we have,

$T = [C_1 /(\alpha\, k_0)] \ln y + C_2$

Substituting for „y‟ in terms of „x‟ we have

$T = [C_1 /(\alpha\, k_0)] \ln (1 + \alpha x) + C_2$ .... (2)

Eq.(2) is the general solution of Eq.(1). The values of C_1 and C_2 can be obtained from the two boundary conditions at x = 0 and at x = L as follows.

(i) at $x = 0$, $T = T_1$; and (ii) at $x = L$, $T = T_2$;

Condition (i) in Eq. (2) gives $T_1 = C_2$;

Condition (ii) in Eq. (20 gives $T_2 = [C_1 /(\alpha\, k_0)] \ln (1 + \alpha L) + T_1$

Or $C_1 = (T_2 – T_1)\, (\alpha\, k_0) / \ln (1 + \alpha L)$.

Substituing the values of $C_1$and C_2 in Eq.(2) we get the temperature distribution as:

$$\mathbf{T(x) = (T_2 – T_1) \ln ((1 + \alpha x) / \ln (1 + \alpha L) + T_1}$$

Expression for rate of heat transfer:

At any „x‟ $Q_x = - k\, A\, (dT/dx)$

Or $\quad Q_x = -[k_0 (1 + \alpha x)] A (dT/dx)$

$$= -[k_0 (1 + \alpha x)] A \, d/dx\{(T_2 - T_1) [\ln(1 + \alpha x) / \ln(1 + \alpha L)] + T_1\}$$

$$= -[k_0 (1 + \alpha x)] \frac{A (T_2 - T_1) \alpha}{\ln(1 + \alpha L)} \times \frac{1}{(1 + \alpha x)}$$

Or $$\mathbf{Q_x = \frac{k_0 \alpha A (T_1 - T_2)}{\ln(1 + \alpha L)}}$$

Example 3.29:- If in the above problem the thermal conductivity varies with distance according to the law

$$K = k_0 [1 + \alpha x^2]$$

Obtain expressions for (i) the temperature distribution T(x) and (ii) the rate of heat transfer.

Solution:

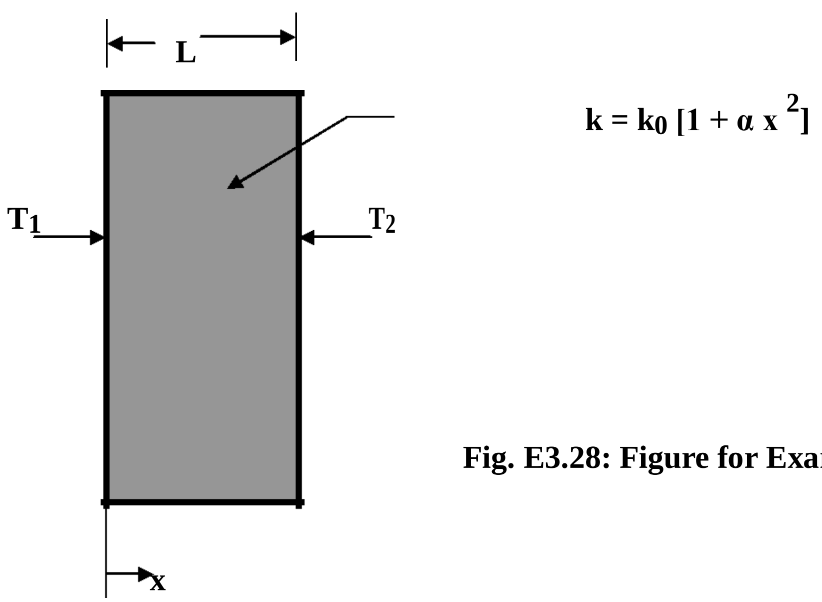


Fig. E3.28: Figure for Example 3.28

The governing differential equation for one dimensional steady state conduction without heat generation is given by

$$d / dx [k \, dT/dx] = 0 \text{ substituting for k we have}$$

$$d / dx [k_0 (1 + \alpha x^2) \, dT/dx] = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Eq. (2) can be solved by using the following substitution.

Let $\sqrt{\alpha}\, x = \tan y$. Differnatiating both sides w.r.t."x" we have

$\sqrt{\alpha} = \sec^2 y\,(dy/dx)$

or $(dy/dx) = \sqrt{\alpha} / \sec^2 y = \dfrac{\sqrt{\alpha}}{[1 + \alpha x^2]}$.

Or $[1 + \alpha x^2]\,(dy / dx) = \sqrt{\alpha}$.

Eq. (1) can be written in terms of variable „y" as

$$d / dy[k_0 (1 + \alpha x^2) (dy/dx) (dT/dy)] (dy/dx) =$$

0 Substituting for $(1 + \alpha x^2)(dy/dx)$ we get

$$d/ dy\, [k_0 \sqrt{\alpha}\, (dT/dy)]\, [\sqrt{\alpha} / \sec^2 y] = 0.$$

Or $$d/ dy\, [k_0 \sqrt{\alpha}\, (dT/dy)] = 0.$$

Integrating w.r.t."y" we have, $k_0 \sqrt{\alpha}\, (dT / dy) = C_1$

Or $$dT = [C_1 /(k_0 \sqrt{\alpha})]\, dy.$$

Integrating we get $T = [C_1/ (k_0 \sqrt{\alpha})]\, y + C_2$

Or $$T(x) = [C_1/(k_0 \sqrt{\alpha})] \tan^{-1}(\sqrt{\alpha}\, x) + C_2 \quad \text{.......(3)}$$

Boundary conditions are: (i) at $x = 0$ $T(x) = T_1$ and (ii) at $x = L$, $T = T_2$.

Condition (i) in Eq. (3) gives $C_2 = T_1$.

Condition (ii) in Eq. (3) gives $T_2 = [C_1/(k_0 \sqrt{\alpha})] \tan^{-1}(\sqrt{\alpha}\, L) + T_1$

Or $$C_1 = \frac{[T_2 - T_1] k_0 \sqrt{\alpha}}{\tan^{-1}(\sqrt{\alpha}\, L)}$$

Substituting the values of C_1 and C_2 we have

$$\mathbf{T(x) = [T_2 - T_1] \frac{\tan^{-1}(\sqrt{\alpha}\, x)}{\tan^{-1}(\sqrt{\alpha}\, L)}}$$

Expression for Rate of Heat Transfer:

$$Q_x = -\, k_0\, [1 + \alpha\, x^2]\, A\, (dT/dx)$$

$$= -\, k_0\, [1 + \alpha\, x^2]\, A\, [T_2 - T_1]\, \frac{\sqrt{\alpha}}{[1 + \alpha\, x^2] \tan^{-1} (\sqrt{\alpha}\, L)}$$

Or $$\mathbf{Q_x = \frac{k_0\, A\, \sqrt{\alpha}\, [T_1 - T_2]}{\tan^{-1} (\sqrt{\alpha}\, L)}}$$

Example 3.30:- A hollow cylinder has its internal surface at radius r_1 maintained at a uniform temperature T_1 and external surface at radius r_2 maintained at a uniform temperature T_2. The thermal conductivity of the material of the cylinder varies with radius according to the law $k = k_0\, [1 + \alpha\, r]$, where k_0 and α are constants. Derive expressions for (i) radial temperature distribution in the cylinder and (ii) rate of heat transfer through the cylinder. Assume one-dimensional radial steady state conduction in the cylinder.

Example 3.31:- A hollow cylinder has its internal surface at radius r_1 maintained at a uniform temperature T_1 and external surface at radius r_2 maintained at a uniform temperature T_2. The thermal conductivity of the material of the cylinder varies with radius according to the law $k = k_0\, [1 + \alpha\, r^2]$, where k_0 and α are constants. Derive expressions for (i) radial temperature distribution in the cylinder and (ii) rate of heat transfer through the cylinder. Assume one-dimensional radial steady state conduction in the cylinder.

Example 3.32:- A hollow sphere has its internal surface at radius r_1 maintained at a uniform temperature T_1 and external surface at radius r_2 maintained at a uniform temperature T_2. The thermal conductivity of the material of the cylinder varies with radius according to the law $k = k_0\, [1 + \alpha\, r]$, where k_0 and α are constants. Derive expressions for (i) radial temperature distribution in the cylinder and (ii) rate of heat transfer through the cylinder. Assume one-dimensional radial steady state conduction in the cylinder.

Example 3.33:- A hollow sphere has its internal surface at radius r_1 maintained at a uniform temperature T_1 and external surface at radius r_2 maintained at a uniform temperature T_2. The thermal conductivity of the material of the cylinder varies with radius according to the law $k = k_0\, [1 + \alpha\, r^2]$, where k_0 and α are constants. Derive expressions for (i) radial temperature distribution in the cylinder and (ii) rate of heat transfer through the cylinder. Assume one-dimensional radial steady state conduction in the cylinder.

Introduction:- In general, the temperature of a body varies with time as well as position. In chapter 3 we have discussed conduction in solids under steady state conditions for which the temperature at any location in the body do not vary with time. But there are many practical situations where in the surface temperature of the body is suddenly altered or the surface may be subjected to a prescribed heat flux all of a sudden. Under such circumstances the temperature at any location within the body varies with time until steady state conditions are reached. In this chapter, we take into account the variation of temperature with time as well as with position. However there are many practical applications where in the temperature variation with respect to the location in the body at any instant of time is negligible. The analysis of such heat transfer problems is called the *"lumped system analysis"*. Therefore in lumped system analysis we assume that the temperature of the body is a function of time only.

Lumped system analysis:- Consider a solid of volume V, surface area A, density ρ, Specific heat C_p and thermal conductivity k be initially at a uniform temperature T_i.Suddenly let the body be immersed in a fluid which is maintained at a uniform temperature T_∞, which is different from T_i.The problem is illustrated in Fig.4.1.Now if

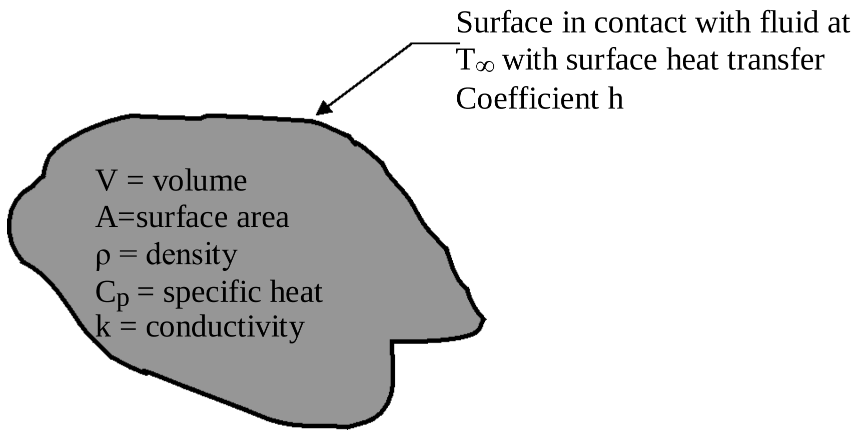


Fig.4.1: Nomenclature for lumped system analysis of transient Conduction heat transfer

T(t) is the temperature of the solid at any time t, then the energy balance equation for the solid at time t can be written as

Rate of increase of energy of the solid = Rate of heat transfer from the fluid to the solid

i.e., $$\rho V C_p (dT / dt) = hA[T_\infty - T(t)]$$

Or $$dT / dt = \frac{h\,A}{\rho\,V\,C_p}[T_\infty - T(t)]$$

For convenience, a new temperature $\theta(t) = T(t) - T_\infty$ is defined and denoting $m = (hA)/(\rho V C_p)$ the above equation can be written as

$$(d\theta / dt) = - m\,\theta \qquad (4.1)$$

Eq.(4.1) is a first order linear differential equation and can be solved by separating the variables. Thus

$$d\theta / \theta = - m\,dt$$

Integrating we get $\ln \theta = - mt + \ln C$, where $\ln C$ is a constant.

Or

$$\theta = C\,e^{-mt} \qquad (4.2)$$

At time $t = 0$, $T(t) = T_i$ or $\theta = T_i - T_\infty = \theta_i$ (say). Substituting this condition in Eq. (4.2) we get

$$C = \theta_i.$$

Substituting this value of C in eq. (4.2) we get the temperature $\theta(t)$ as follows.

$$\theta(t) = \theta_i\,e^{-mt}$$

or

$$\frac{\theta(t)}{\theta_i} = e^{-mt} \qquad (4.3)$$

Since LHS of Eq.(4.3) is dimensionless, it follows that 1/m has the dimension of time and is called the time constant.Fig. 4.2 shows the plot of Eq.(4.3) for different values of m. Two observations can be made from this figure and Eq. (4.3).

1. Eq. (4.3) can be used to determine the temperature T(t) of the solid at any time t or to determine the time required by the solid to reach a specified temperature.

2. The plot shows that as the value of m increases the solid approaches the surroundings temperature in a shorter time. That is any increase in m will cause the solid to respond more quickly to approach the surroundings temperature.

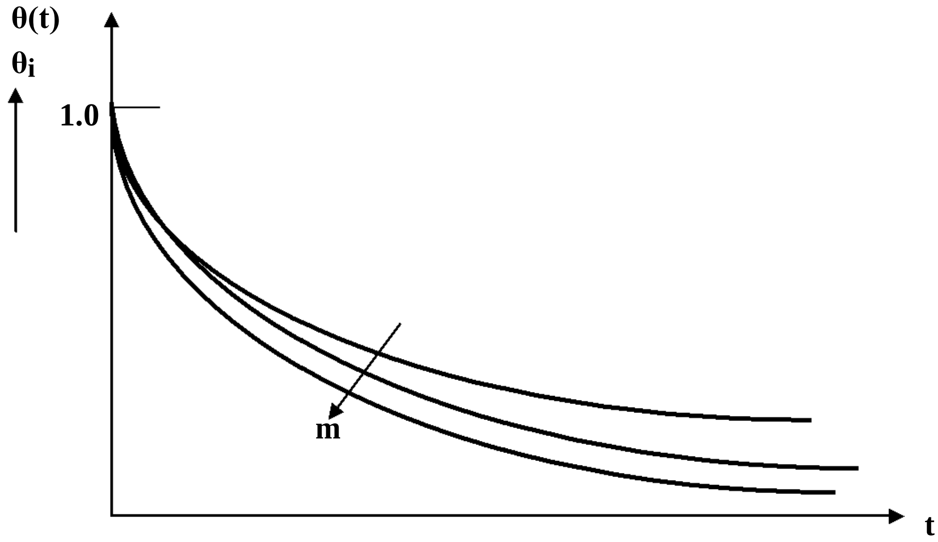


Fig.4.2: Dimensionless temperature as a function of time for a solid with negligible internal temperature gradients

The definition of m reveals that increasing the surface area for a given volume and the heat transfer coefficient will increase m. Increasing the density, specific heat or volume decreases m.

Criteria for Lumped System Analysis:- To establish a criterion to neglect internal temperature gradient of the solid so that lumped system analysis becomes applicable, a *Characteristic length L_S* is defined as

$$L_S = V / A \qquad \text{.......(4.4)}$$

and the Biot. number Bi as

$$Bi = \frac{h L_S}{k} \qquad \text{.......(4.5)}$$

For solids like slabs, infinite cylinder, and sphere, it has been found that the error by neglecting internal temperature gradients is less than 5 %, if

$$Bi < 0.1 \qquad \text{.......(4.6)}$$

The physical significance of Biot number can be understood better by writing the expression for Biot number as follows

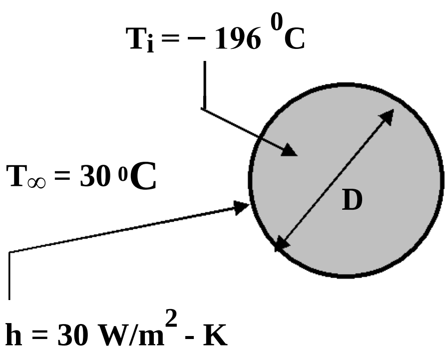
$$Bi = \frac{h L_s}{k} = \frac{(L_s / Ak)}{(1 / hA)} = \frac{\text{Thermal resistance for conduction}}{\text{Thermal resistance for convection}}$$

Hence a very low value of Biot number indicates that resistance for heat transfer by conduction within the solid is much less than that for heat transfer by convection and therefore a small temperature gradient within the body could be neglected.

<u>Illustrative examples on lumped system analysis</u>

Example 4.1: - A copper cylinder 10 cm diameter and 15 cm long is removed from a liquid nitrogen bath at $-196\ ^0C$ and exposed to room temperature at $30\ ^0C$. Neglecting internal temperature gradients find the time taken by the cylinder to attain a temperature of $0\ ^0C$, with the following assumptions:
Surface heat transfer coefficient = 30 W / m^2 – K.
Density of the copper cylinder = 8800 kg / m^3.
Specific heat of the cylinder = 0.38 kJ/(kg-K) Thermal conductivity of the cylinder = 350 W / (m-K).

Solution: :



Other data:- D = 10 cm or R = 0.05 m; L = 0.15 m

k = 350 W / (m-K) ; ρ = 8800 kg / m^3 ;

c_p = 0.38 kJ / (kg-K) ; T(t) = 0

Let $\theta(t) = T(t) - T_\infty$

Biot Number = hR / k = 30 x 0.05 / 350 = 0.0043 which is << 0.1. Hence internal temperature gradients can be neglected. In that case we have

$\theta(t) = T(t) - T_i = \theta_0\, e^{-(hA/\rho V c_p)\, t}$, where $\theta_0 = T_i - T_\infty$

$$(hA/\rho V c_p) = \frac{2\{\pi R^2 + \pi RL)h}{\pi R^2 L\, \rho c_p} = \frac{2\{R+L\}h}{\rho c_p RL} = \frac{2 \times \{0.05 + 0.15\} \times 30}{8800 \times 0.38 \times 1000 \times 0.05 \times 0.15}$$

$$= 4.785 \times 10^{-4} \quad 1 / s$$

Now

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-(hA/\rho V c_p)\, t}$$

$$\text{Hence} \qquad \frac{0 - 30}{-196 - 30} = \exp(-4.785 \times 10^{-4} \times t)$$

Solving for t we get t = 4226 s = 1 hr 10.43 mins.

Example 4.2:- A thin copper wire having a diameter D and length L (insulated at the ends) is initially at a uniform temperature of T_0. Suddenly it is exposed to a gas stream, the temperature of which changes with time according to the equation

$$\mathbf{T_g = T_f\,(1 - e^{-ct}) + T_0}$$

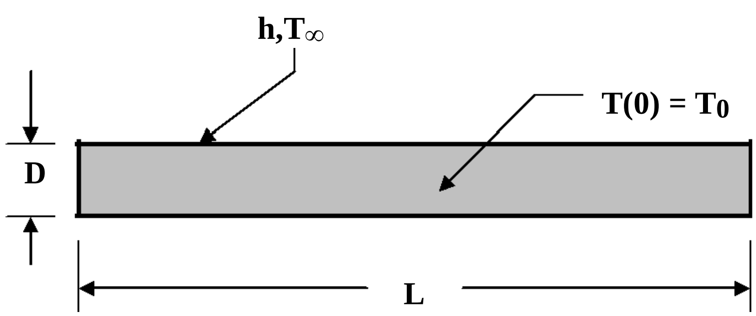
where T_f, T_0 and c are constants. The surface heat transfer coefficient is h. Obtain an expression for the temperature of the wire as a function of time t.

Solution:

Let T(t) be the temperature of the cylinder at any time t. Energy balance for the cylinder for a time interval dt is given by

$$hA\,[T_\infty - T(t)]\,dt = \rho V C_p\,dT$$

where dT is the increase in temperature of the cylinder in time dt.



$$\text{Or} \qquad dT/dt = (hA/\rho V C_p)\,[T_\infty - T(t)]$$

Putting $m = (hA/\rho V C_p)$, the above equation reduces to

$$dT/dt + m\,T(t) = m\,T_\infty$$

Substituting the given expression for T_∞ we have

$$dT/dt + m\,T(t) = m\,[T_0 + T_f\,(1 - e^{-ct})]$$

or $dT/dt + m\,[T(t) - T_0] = mT_f\,(1 - e^{-ct})$

Let $\theta(t) = T(t) - T_0$. Then the above equation reduces to

$$d\theta / dt + m\,\theta(t) = mT_f(1 - e^{-ct}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

This equation is of the form $dy / dx + Py = Q$, which is solved by multiplying throughout by an integrating factor and then integrating. For equation (1) the integrating factor is $e^{\int m dt}\, e^{mt}$. therefore multiplying equation (1) by e^{mt} we get

$$e^{mt}(d\theta / dt) + m\,e^{mt}\,\theta(t) = mT_f[\,e^{mt} - e^{(m-c)t}]$$

or $$d / dt\,(e^{mt}\theta) = mT_f[\,e^{mt} - e^{(m-c)t}]$$

Integrating with respect to t we have

$$e^{mt}\theta(t) = mT_f[\,(e^{mt} / m) - e^{(m-c)t} / (m-c)\,] + C_1$$

or $$\theta(t) = T_f - \frac{m}{(m-c)} T_f e^{-ct} + C_1 e^{-mt} \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

When $t = 0$, $T(0) = T_0$ i.e., $\theta(0) = 0$. Substituting this condition in equation (2) we get

or $$0 = T_f - \frac{m}{(m-c)} T_f + C_1$$

Or $$C_1 = [\,c / (m-c)]\,T_f.$$

Substituting this expression for C_1 in equation (2) we get the temperature of the cylinder as

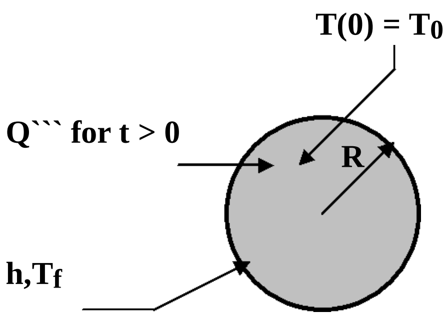
$$\theta(t) = T_f - \frac{m}{(m-c)} T_f e^{-ct} + \frac{c}{(m-c)} T_f e^{-mt}$$

Or $$T(t) - T_0 = T_f[\,1 - m / (m-c)\,e^{-ct} + c / (m-c)\,e^{-mt}]$$

Where $m = hA / (\rho V C_p) = \pi D L h / \{(\pi D^2/4) L \rho C_p\} = (4h) / (\rho D C_p)$.

Example 4.3:- A solid sphere of radius R is initially at a uniform temperature T_0. At a certain instant of time (t = 0), the sphere is suddenly exposed to the surroundings at a temperature T_f and the surface heat transfer coefficient, 'h'. In addition from the same instant of time, heat is generated within the sphere at a uniform rate of q''' units per unit volume. Neglecting internal temperature gradients, derive an expression for the temperature of sphere as a function of time

Solution:



Energy balance equation for the sphere at any time t can be written as

$$(4/3)\pi R^3 q''''''' + 4\pi R^2 h\,[T_f - T(t)]$$

$$=(4/3)\pi R^3 \rho C_p\,(dT/dt)$$

$$\text{Or } (dT/dt) + (3h/\rho RC_p)[T(t) - T_f] = (q'''/\rho C_p)$$

Let $\theta(t) = T(t) - T_f$. Then the above equation reduces to

$$(d\theta / dt) + m\theta = q_0 \text{ ....(1)}$$

Where $m = (3h/\rho RC_p)$ and $q_0 = (q'''/\rho C_p)$

Multiplying equation (1) by the integrating factor e^{mt} we

have $e^{mt}(d\theta / dt) + e^{mt} m\theta = q_0 e^{mt}$

or $\quad d / dt(\theta e^{mt}) = q_0 e^{mt}$

Integrating throughout w.r.t. t we get

$$\theta e^{mt} = (q_0 / m)\, e^{mt} + C_1$$

or $$\theta = (q_0 / m) + C_1 e^{-mt} \text{ ....(2)}$$

At t = 0 , $T = T_0$ or $\theta = T_0 - T_f = \theta_0$ (say). Substituting this condition in equation (2) we

get $C_1 = (T_0 - T_f) - (q_0 / m)$. Therefore the temperature in the sphere as a function of time is given by

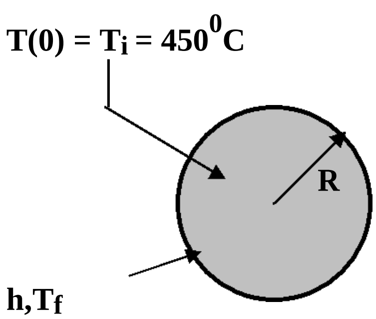
$$\theta(t) = [(T_0 - T_f) - (q_0 / m)]\, e^{-mt} + (q_0 / m)$$

or $$\theta(t) = (q_0 / m)[\,1 - e^{-mt}\,] + (T_0 - T_f)\, e^{-mt}$$

$$\text{where } (q_0 / m) = \frac{q'''}{(\rho C_p)} \times \frac{(\rho R C_p)}{3h} = (q''' R / 3h)$$

Example 4.4:- A solid steel ball (ρ =8000 kg/m^3 ; c_p = 0.42 kJ/kg-K) 5 cm in diameter is at a uniform temperature of 450 ^{0} C. It is quenched in a controlled environment which is initially at 90 ^{0}C and whose temperature increases linearly with time at the rate of 10 ^{0}C per minute. If the surface heat transfer coefficient is 58 W/(m^2-K), determine the variation of the temperature of the ball with time neglecting internal temperature gradients. Find the value of the minimum temperature to which the ball cools and the time taken to reach this minimum temperature.

Solution:



Other data:- $h = 58$ W / (m^2 – K) ;

$C_p = 0.42$ kJ / (kg – K) ; $\rho = 8000$ kg / m^3 ;

$T_f = a + bt$, where a and b are constants ;

at t = 0, $T_f = 90\ ^0C$;$(dT_f / dt) = 10\ ^0C$ / min

$= (1/6)\ ^0C$ / s

Therefore $a = 90\ ^0C$ and $b = ;(dT_f / dt) = (1/6)\ ^0C$ / s.

Or $\qquad T_f = 90 + t / 6$, t in seconds ....(1)

Energy balance equation for the sphere at any time t can be written as

$$\rho V C_p (dT / dt) = hA\,[T_f(t) – T(t)]$$

Or
$$(dT / dt) = (hA/ \rho V C_p)\,[T_f(t) – T(t)]$$

Letting $m = (hA/ \rho V C_p)$ the above equation can be written as

$$(dT / dt) + mT(t) = m\,T_f(t)$$

Substituting for $T_f(t)$ from equation (1) we have

$$(dT / dt) + mT(t) = m\,[90 + t / 6\,]$$

Multiplying the above equation with the integrating factor e^{mt} we get

$$e^{mt}(dT / dt) + mT(t)\, e^{mt} = m\,[90 + t / 6\,]\, e^{mt}$$

or $$d / dt\,(T\, e^{mt}) = m\,[90 + t / 6\,]\, e^{mt}$$

Integrating throughout w.r.t t we have

$$(T\, e^{mt}) = m \int [90 + t / 6\,]\, e^{mt}\, dt + C_1$$

Or $$T(t) = m\, e^{-mt} \int [90 + t / 6\,]\, e^{mt}\, dt + C_1\, e^{-mt}$$

$$= m\, e^{-mt}\,[(90e^{mt} / m) + (te^{mt}/6m) - (e^{mt}/6m^2)] + C_1\, e^{-mt}$$

Or $$T(t) = [\,90 + (t / 6\,) - (1/6m)] + C_1\, e^{-mt} \ldots\ldots\ldots\ldots (2)$$

When t = 0 , T(t) = T_i. Substituting this condition in the above equation and solving for C_1 we get

$$C_1 = [T_i - 90 + 1 / 6m]$$

Therefore the temperature of sphere as a function of time is given by

$$\mathbf{T(t) = [\,90 + (t / 6\,) - (1/6m)] + [T_i - 90 + 1 / 6m]\, e^{-mt}} \ldots\ldots\ldots (3)$$

For T(t) to be extremum (dT / dt) = 0.

Therefore we have $(dT / dt) = 1/6 + [T_i - 90 + 1 / 6m]\, e^{-mt}(-m) = 0$

Substituting $T_i = 450\,^0$C and simplifying we get

$$(360\, m + 1/6)\, e^{-mt} = 1/6$$

Or $$e^{mt} = (2160\, m + 1) \text{ ---------- } (4)$$

$$\text{Now } m = (hA / \rho VC_p) = \frac{4\pi R^2 h}{[(4/3)\pi R^3\, \rho C_p]} = (3h/ \rho C_p R) = \frac{3 \times 58}{(\,8000 \times 0.025 \times 0.42 \times 10^{3})}$$

$$= 2.07 \times 10^{-3}. \text{--------} \quad (5)$$

Using (4) & (5) in (3),

$$T(t)= 90+ (t/6)- (1/(6x2.07x10^{-3})) + [\,240-90+(1/(6x2.07x10^{-3}))]\ x\ exp\{-2.07x10^{-3}\ t\}$$

$T(t)=9.4857+(230.5152)x(0.9979)^t$

$T(t) > 0$

Hence value of t will be minimum.

Therefore $e^{mt} = [\, 2160 \times 2.07 \times 10^{-3} + 1 \,] = 5.47$

$mt = 1.7$

Or $t = 1.7 / m = 1.7 / (2.07 \times 10^{-3}) = 821 \text{ s} = 13.7 \text{ min}$

Substituting this value of t in equation (3) we get the minimum temperature

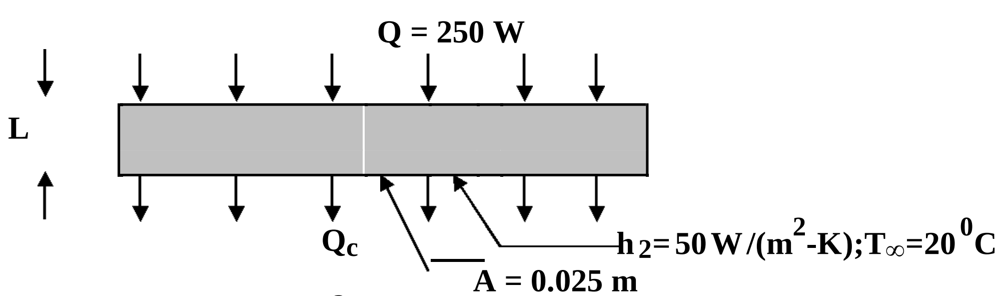
as $T_{minimum} = [90 + (821/7) - \{1 / (6 \times 2.07 \times 10^{-3}) \}\,]$

$+ [\, 450 - 90 + \{1 / (6 \times 2.07 \times 10^{-3}) \}\,]\, e^{-1.7} = 226.7\ ^0C.$

Example 4.5:- A house hold electric iron has a steel base [ρ =7840 kg/m^3 ; c_p = 450 J/(kg-K) ;k = 70 W/(m-K)] which weighs 1 kg. The base has an ironing surface area of 0.025 m^2 and is heated from the other surface with a 250 W heating element. Initially the iron is at a uniform temperature of 20 ^{0} C with a heat transfer coefficient of 50 W/(m^2-K).

(b) What would be the equilibrium temperature of the iron if the control of the iron box did not switch of the current?

Solution:



Other data:- ρ = 7840 kg / m^3 ; C_p = 450 J / (kg – K) ; k = 70 W /(m – K) ; m = 1 kg ; t = 5 min = 300 s.

$V = m / \rho = 1 / 7840 = 0.0001275 \text{ m}^3 = 1.275 \times 10^{-4} \text{ m}^3.$

$$L = V / A = \frac{1.275 \times 10^{-4}}{0.025} = 0.005 \text{ m}$$

$$Bi = (hL / k) = \frac{50 \times 0.005}{70} = 0.00364$$

Since Bi < 0.1, it can be assumed that temperature gradients within the plate are negligible. Hence the temperature of the plate depends only on time till steady state condition is reached.

Energy balance at any time t for the plate can be written as

$$Q - Q_c = \rho V C_p (dT/dt)$$

Or $$Q - hA(T - T_\infty) = \rho V C_p (dT/dt)$$

Or $$(dT/dt) + m(T - T_\infty) = (Q / \rho V C_p) \quad \ldots\ldots (1)$$

Where $m = (hA / \rho V C_p)$. Letting $\theta = T - T_\infty$, equation (1) can be written as

$$(d\theta / dt) + m\,\theta = (Q / \rho V C_p)$$

Multiplying the above equation by the integrating factor e^{mt},($e^{\int m dt} = e^{mt}$) we get

$$(d\theta / dt)\, e^{mt} + m\,\theta e^{mt} = (Q / \rho V C_p)\, e^{mt}$$

Or $$d/dt\,(\theta e^{mt}) = (Q / \rho V C_p)\, e^{mt}$$

Or $$(\theta e^{mt}) = (Q / \rho V C_p)\, e^{mt}\, (1/m) + C_1$$

Or $$\theta = (Q / \rho V C_p m) + C_1 e^{-mt} \quad \ldots\ldots (2)$$

When $t = 0$, $T = T_i$ or $\theta = T_i - T_\infty = 20 - 20 = 0\,^0C$.

Substituting this condition in equation (2) we get

$$0 = (Q / \rho V C_p m) + C_1 \text{ or } C_1 = -(Q / \rho V C_p m)$$

Therefore the temperature in the plate as a function of time is given by

$$\theta = (Q / \rho V C_p m)\, [\, 1 - e^{-mt}$$

] But $\rho V C_p m = hA$. Therefore

$$\theta = (Q / hA) [1 - e^{-mt}] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$Q / hA = \frac{250}{50 \times 0.025} = 200 \; ; \; m = \frac{50 \times 0.025}{1 \times 450} = 2.8 \times 10^{-3}$$

Therefore $\theta = 200 [1 - e^{-0.028t}]$

When t = 300 s, $\theta = T - T_\infty = 200 \times [1 - e^{-0.028 \times 300}] = 113.7$

Or $T = 113.7 + 20 = 133.7\ ^0 C.$

(b) When the control switch is not switched off and the iron is left in the ambient, steady state condition will be attained as t tends to ∞ so that the heat transferred to the baseplate will be convected to the ambient. i.e.,

$$Q = Q_c$$

Therefore $250 = 50 \times 0.025 \times [T - 20]$

Or $T = 220\ ^0 C.$

This answer can also be obtained by putting $t = \infty$ in equation (3) and solving for T.

One-dimensional Transient Conduction (Use of Heissler's Charts): There are many situations where we cannot neglect internal temperature gradients in a solid while analyzing transient conduction problems. Then we have to determine the temperature distribution within the solid as a function of position and time and the analysis becomes more complex. However the problem of one-dimensional transient conduction in solids without heat generation can be solved readily using the method of separation of variables.The analysis is illustrated for solids subjected to convective boundary conditions and the solutions were presented in the form of transient – temperature charts by **Heissler.** These charts are now familiarly known as "Heissler‟s charts".

One-dimensional transient conduction in a slab:- Let us consider a slab of thickness 2L, which is initially at a uniform temperature T_i. Suudenly let the solid be exposed to an environment which is maintained at a uniform temperature of T_∞ with a surface heat transfer coefficient of h for time t > 0.Fig.4.3 shows the geometry , the coordinates and the boundary conditions for the problem. Because of symmetry in the problem with respect to the centre of the slab the „x‟ coordinate is measured from the centre line of the slab as shown in the figure.

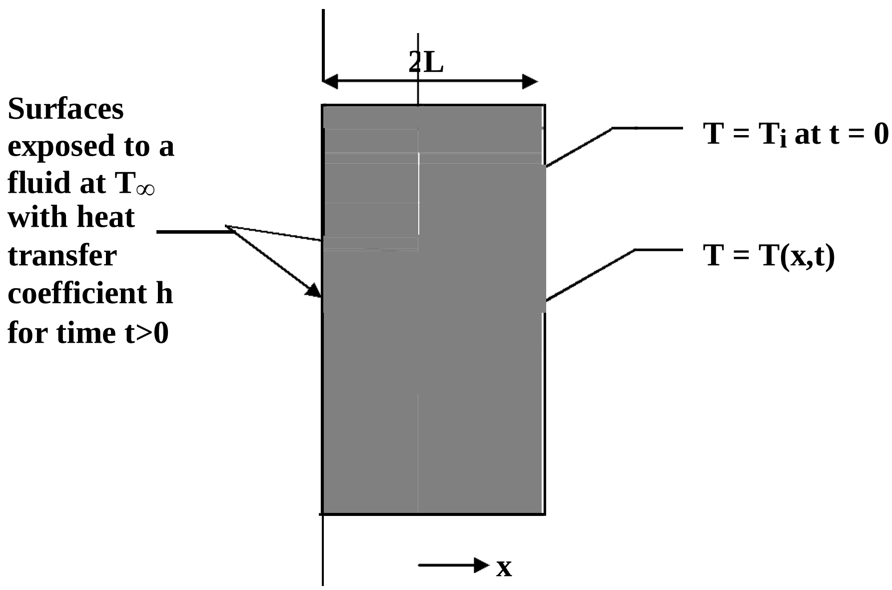


Fig.4.3: Geometry, coordinates and boundary conditions for transient conduction in a slab

The mathematical formulation of this transient conduction problem is given as follows:

Governing differential equation: $\partial^2 T / \partial x^2 = (1/\alpha)\, \partial T / \partial t$ ....(4.7a)

Initial condition : at $t = 0$, $T = T_i$ in $0 < x < L$ ....(4.7b)

Boundary conditions are :

(i) at $x = 0$, $\partial T / \partial x = 0$ (axis of symmetry) for all $t > 0$… .... (4.7c)

(ii) at $x = L$, $-k\,(\partial T / \partial x)|_{x = L} = h(T|_{x = L} - T_\infty)$ for all $t > 0$.................................... (4.7d)

It is more convenient to analyze the problem by using the variable $\theta(x,t)$, where

$\theta(x,t) = T(x,t) - T_\infty$. Then equations (4.7a) to (4.7d) reduce to the following forms:

$$\partial^2\theta / \partial x^2 = (1/\alpha)\, \partial\theta / \partial t \quad \text{.....................................(4.8a)}$$

Initial condition : at $t = 0$, $\theta = T_i - T_\infty = \theta_i$ in $0 < x < L$....................................... (4.8b)

Boundary conditions reduce to :

(i) at $x = 0$, $\partial\theta / \partial x = 0$ for all $t > 0$...(4.8c)

(ii) at $x = L$, $-k\,(\partial\theta / \partial x)|_{x = L} = h\theta|_{x = L}$ for all $t > 0$...(4.8d)

Eq.(4.8a) can be solved by the method of separation of variables as shown below.

Let $\theta(x,t) = X(x)\, Y(t)$ ....(4.9)

Substituting this in Eq. (4.8a) we get

$$Y\,(d^2X / dx^2) = (X/\alpha)\,(dY / dt)$$

Or

$$\frac{1}{X}\,(d^2X / dx^2) = \frac{1}{(Y\alpha)}\,(dY / dt) \quad \text{................................(4.10)}$$

LHS of Eq. (4.10) is a function of x only and the RHS of Eq. (4.10) is a function of t only.They can be equal only to a constant say $-\lambda^2$.(The reason to choose the negative sign is to get a physically meaningful solution as explained later in this section).Hence we have two equations namely

$(1 / X)\,(d^2X / dx^2) = -\lambda^2$ and $[1/(Y\alpha)]\,((dY / dt) = -\lambda^2$

Or $\quad (d^2X / dx^2) + \lambda^2 X = 0$ ....(4.11)

and $\quad (dY/dt) = -\alpha\lambda^2\, Y$ ....(4.12)

Solution to Eq. (4.11) is $X(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x)$ ....(4.13)

and solution to Eq. (4.12) is $\quad Y(t) = D \exp(-\alpha\lambda^2 t)$ ....(4.14)

with C_1, C_2 and D as constants of integration. Substituting these solutions in Eq.(4.9) we have

$$\theta(x,t) = D \exp(-\alpha\lambda^2 t)\,[C_1 \cos(\lambda x) + C_2 \sin(\lambda x)]$$

or $\quad \theta(x,t) = \exp(-\alpha\lambda^2 t)\,[A_1 \cos(\lambda x) + A_2 \sin(\lambda x)]$ .... (4.15)

Eq.(4.15) is the general solution involving the constants A_1, A_2 and λ which can be determined using the two boundary conditions and the initial condition as illustrated below.

Now from Eq. (4.15), $\partial\theta / \partial x = \lambda \exp(-\alpha\lambda^2 t)\,[-A_1 \sin(\lambda x) + A_2 \cos(\lambda x)]$

Substituting boundary condition (i) we have $0 = \lambda \exp(-\alpha\lambda^2 t)\,[0 + A_2]$ for all

t. Hence $A_2 = 0$. Therefore Eq. (4.15) reduce to

$$\theta(x,t) = A_1 \exp(-\alpha\lambda^2 t)\cos(\lambda x) \quad \ldots\ldots(4.16)$$

Now $\theta(L,t) = A_1 \exp(-\alpha\lambda^2 t)\cos(\lambda L)$

and $\partial\theta / \partial x = \lambda \exp(-\alpha\lambda^2 t)\,[-A_1 \sin(\lambda x)]$

Hence $[\partial\theta / \partial x]_{x=L} = -\lambda A_1 \exp(-\alpha\lambda^2 t)\sin(\lambda L)$

Therefore boundary condition (ii) can be written as

$$k\lambda A_1 \exp(-\alpha\lambda^2 t)\sin(\lambda L) = h A_1 \exp(-\alpha\lambda^2 t)\cos(\lambda L)$$

or $\tan(\lambda L) = h / (k\lambda)$

or $$\lambda L \tan(\lambda L) = Bi \quad \ldots\ldots(4.17)$$

where $Bi = hL / k$.

Equation (4.17) is called the *"characteristic equation"* and has infinite number of roots namely $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots$ Corresponding to each value of λ we have one solution and hence there are infinite number of solutions. Sum of all these solutions will also be a solution as the differential equation is linear. Therefore the solution $\theta(x,t)$ can be written as follows.

$$\boldsymbol{\theta(x,t) = \sum A_n \exp(-\alpha\lambda_n^2 t)\cos(\lambda_n x) \quad \ldots\ldots(4.18)}$$

To find A_n:- The constants A_n in Eq. (4.18) can be found using the orthogonal property of trigonometric functions as shown below. Substituting the initial condition we have

$$\theta_i = \sum A_n \cos(\lambda_n x)$$

Multiplying both sides of Eq.(4.18) by $\cos\lambda_m x$ and integrating w.r.t „x" between the limits 0 and L we have

$$\int_1^L \theta_i \cos(\lambda_m x)\,dx = \int_1^L \sum A_n \cos(\lambda_m x)\cos(\lambda_n x)\,dx$$

Using the orthogonal; property

$$\int A_n \cos(\lambda_m x)\cos(\lambda_n x)\,dx = 0 \text{ for } \lambda_n \neq \lambda_m$$

The above equation reduce to

$$\int_0^L \theta_i \cos(\lambda n x)\,dx = \int_0^L A_n \cos^2(\lambda_n x)\,dx$$

Or $$A_n = \frac{\theta_i \int_0^L \cos(\lambda_n x)\,dx}{\int_0^L \cos^2(\lambda_n x)\,dx}$$

It is very convenient to express Eq. (4.18) in dimension less form as follows:

$$\frac{\theta(x,t)}{\theta i} = \sum (A_n^* \exp(-\lambda_n^{*2} Fo) \cos(\lambda_n^* x / L) \text{..} (4.19)$$

where $A_n^* = A_n / \theta_i$; $\lambda_n^* = \lambda_n L$; Fo = Fourier Number = $\alpha t / L^2$;

Heissler's Charts for transient conduction:- For values of Fo > 0.2 the above series solution converges rapidly and the solution will be accurate within 5 % if only the first term in the series is used to determine the temperature. In that case the solution reduces to

$$\frac{\theta(x,t)}{\theta i} = A_1^* \exp(-\lambda_1^{*2} Fo) \cos(\lambda_1^* x / L) \text{...} (4.20)$$

From the above equation the dimensionless temperature at the centre of the slab (x =0) can be written as

$$\frac{\theta(0,t)}{\theta i} = A_1^* \exp(-\lambda_1^{*2} Fo) \text{..} (4.21)$$

The values of A_1^* and λ_1^* for different values of Bi are presented in the form of a table (See Table 4.1). These values are evaluated using one term approximation of the series solution.It can also be concluded from Eq.(4.20) at any time „t" the ratio θ(x,t) / θ(0,t) will be independent of temperature and is given by

$$\frac{\theta(x,t)}{\theta(0,t)} = \cos(\lambda_1^* x / L) \text{...} (4.22)$$

Heissler has represented Eq. (4.21) and (4.22) in the form of charts and these charts are normally referred to as Heissler"s charts. Eq. (4.21) is plotted as Fourier number Fo versus dimensionless centre temperature $\theta(0,t) / \theta_i$ using [Fig.4.4(b)].
reciprocal of Biot number1 / Bi as the parameter [Fig.4.4(a)], where as Eq. (4.22) is plotted as θ(x,t) / θ(0,t) versus reciprocal of Biot number using the dimensionless distance x / L as the parameter.In Fig.[4.4(a)], the curve for 1/Bi = 0 corresponds to the case
$h \rightarrow \infty$, or the outer surfaces of the slab are maintained at the ambient temperature $T\infty$. For large values of 1 / Bi, the Biot number is small, or the internal conductance is large in comparison with the surface heat transfer coefficient. This in turn, implies that the temperature distribution within the solid is sufficiently uniform and hence lumped system analysis becomes applicable.

Fig. (4.5) shows the dimensionless heat transferred Q / Q_0 as a function of dimensionless time for different values of the Biot number for a slab of thickness 2L. Here Q represents the total amount of thermal energy which is lost by the slab up to any time t during the transient conduction heat transfer. The quantity Q_0, defined as

$$Q_0 = \rho V C_p [T_i - T_\infty] \qquad (4.23)$$

represents the initial thermal energy of the slab relative to the ambient temperature.

<u>Transient-Temperature charts for Long cylinder and sphere:</u> The dimensionless transient-temperature distribution and the heat transfer results for infinite cylinder and sphere can also be represented in the form of charts as in the case of slab. For infinite cylinder and sphere the radius of the outer surface R is used as the characteristic length so that the Biot number is defined as Bi = hR / k and the dimensionless distance from the centre is r/R where r is any radius ($0 \le r \le R$).These charts are illustrated in Figs. (4.6) to (4.9).

Illustrative examples on the use of Transient Temperature Charts:-Use of the transient temperature charts for slabs, infinite cylinders and spheres is illustrated in the following examples.

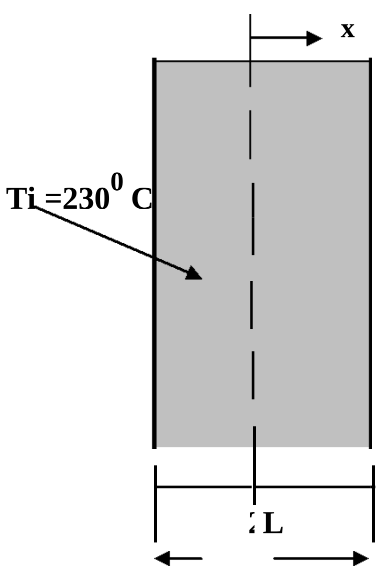
Example 4.6:- A brick wall ($\alpha = 0.5 \times 10^{-6}$ m^2/s, k = 0.69 W/(m-K) and ρ = 2300 kg/m^3) of 10 cm thick is initially at a uniform temperature of 230 ^{0}C. The wall is suddenly exposd to a convective environment at 30 ^{0}C with a surface heat transfer coefficient of 60 W/(m^2-K). Using the transient-temperature charts, determine

(a) the centre temperature at ½ hour and 2 hours after the exposure to the cooler ambient,

(c) energy removed from the wall per m^2 during ½ hour and during 2 hours.

(d) What would be the time taken for the surface of the wall to reach a temperature of 40 ^{0}C.

Solution:-



Data:- $\alpha = 0.5 \times 10^{-6}\ m^2/s$; k = 0.69 W / (m – K)

ρ = 2300 kg / m^3 ; 2L = 0.1 m ; Ti = 230 ^{0}C ;

T_∞ = 30 ^{0}C; h = 60 W / (m^2 – K) ;

(a) (i) t = 0.5 h = 0.5 x 3600 = 1800 s.

$$Bi = hL / k = \frac{60 \times 0.05}{0.69} = 4.35.$$

Since Bi > 0.1, internal temperature gradients cannot be neglected. i.e. T = T(x,t)

Hence Heissler"s transient temperature charts are to be used.

$$1/Bi = 1 / 4.35 = 0.23 \; ; \; Fo = (\alpha t / L^2) = \frac{(0.5 \times 10^{-6} \times 1800)}{0.05^2} = 0.36$$

From the Heissler"s chart for a slab of thickness 2L,

$$\theta_0 = \frac{(T_0 - T_\infty)}{(T_i - T_\infty)} = 0.8 \quad (T_0 = \text{Centre temperature of the slab})$$

Therefore $T_0 = T_\infty + 0.8 (T_i - T_\infty) = 30 + 0.8 \times (230 - 30)$

$= 190\,^0C.$

(ii) when t = 2 h = 7200 s we have $Fo = \frac{(0.5 \times 10^{-6} \times 7200)}{(0.05^2)} = 1.44.$

From chart, $\frac{(T_0 - T_\infty)}{(T_i - T_\infty)} = 0.125.$

Therefore $T_0 = T_\infty + 0.125 (T_0 - T_\infty) = 30 + 0.125 \times (230 - 30) = 55\,^0C.$

(b) (i) t = 1800 s. At the surface x / L = 1.0 ; k / hL = 0.23 ;

Hence from the chart $\frac{(T|_{x=L} - T_\infty)}{(T_0 - T_\infty)} = 0.275$

Or $T|_{x=L} = T_\infty + 0.275 (T_0 - T_\infty) = 30 + 0.275 \times (190 - 30)$

$= 74\,^0C.$

(ii) t = 7200 s. Hence $T|_{x=L} = T_\infty + 0.275 (T_0 - T_\infty) = 30 + 0.275 \times (55 - 30)$

$= 36.9\,^0C.$

(c) (i) $Bi^2 Fo = 4.35^2 \times 0.36 = 6.81$; From chart $Q / Q_{max} = 0.50$.

$$Q_{max} = \rho C_p V(T_i - T_\infty) = (k / \alpha)V(T_i - T_\infty) = \frac{0.69 \times 1 \times 0.1}{0.5 \times 10^{-6}} \times (230 - 30)$$

$= 27.6 \times 10^{6} = 27.6$ MJ / m

Therefore $Q = 0.5 \times 27.6 = 13.8$ MJ

(ii) when t = 7200 s , $Bi^2 Fo = 4.35^{2} \times 1.44 = 27.25$

From chart, $Q / Q_{max} = 0.9$; Therefore $Q = 0.9 \times 27.6 = 24.8$ MJ.

(d) It is given that $T(L,t) = 30\ ^0C$.

Hence $$\frac{\theta (L,t)}{\theta_i} = \frac{(40 - 30)}{(230 - 30)} = 0.05$$

Now for x / L = 1.0 and 1 / Bi = 0.23 from the chart the ratio of surface temperature difference to the centre tempaerature difference can be read as

$$\theta (L,t) / \theta (0,t) = 0.225$$

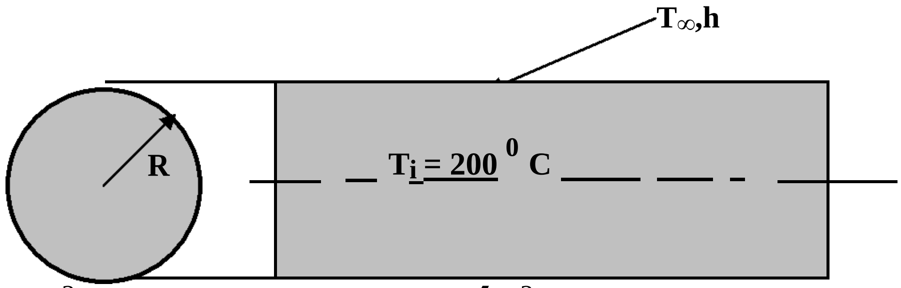
Hence $\theta (0,t) / \theta_i = 0.05 / 0.225 = 0.2222$

From the chart corresponding to this value of $\theta (0,t) / \theta_i$ and 1 / Bi = 0.23, the Fourier number can be read as

$Fo = (\alpha t) / L^2 = 1.0$. Therefore $t = L^2 / \alpha = 0.05^{2} / 0.5 \times 10^{-6} = 5000$ s = 1.39 h

Example 4.7:- A long solid cylinder [$\alpha = 0.05\ m^2/h$, k = 50 W/(m-K)] of 5 cm diameter is initially at 200 ^{0} C. Suddenly it is immersed in water at a temperature of 20 ^{0} C. Assuming the heat transfer coefficient to be 200 W/(m^2-K), determine (a) the centre and the surface temperatures after 10 minutes have elapsed, and (b) the energy removed from the cylinder during this 10 minute period.

Solution :-



Data:- $\alpha = 0.05\ m^2 / h = 0.05 / 3600 = 1.39 \times 10^{-5}\ m^2 / s$; $k = 50\ W/(m - K)$;

$T_i = 200\ ^0C$; $T_\infty = 20\ ^0C$; $R = 0.025\ m$; $h = 200\ W/(m^2 - K)$;$t = 10 \times 60 = 600$

s (a)(i) To find centre temperature T_0 :-$Bi = (hR / k) = (200 \times 0.025) / 50 = 0.1$.

Since Bi = 0.1, internal temperature gradients cannot be neglected.

$$1/Bi = 1 / 0.1 = 10 \ ; \ Fo = (\alpha t / R^2) = \frac{(1.39 \times 10^{-5} \times 600)}{0.025^2}$$

From chart for transient conduction in an infinite cylinder we have

$$\frac{(T_0 - T_\infty)}{(T_i - T_\infty)} = 0.08 \ ; \text{Hence } T_0 = T_\infty + 0.08\,(T_i - T_\infty)$$

Or $T_0 = 20 + 0.08 \times (200 - 20)$

$= 34.4\ ^0C.$

(ii) To find the surface temperature, $T|_{r=R}$:

$$r / R = 1.0 \ ; \ 1 / Bi = 0.1 \ ; \text{From chart } \frac{(T|_{r=R} - T_\infty)}{(T_0 - T_\infty)} = 0.13$$

Therefore $T|_{r=R} = T_\infty + 0.13\,(T_0 - T_\infty) = 20 + 0.13 \times (34.4 - 20)$

$= 21.9\ ^0C.$

(b) $Bi^2\ Fo = (0.1^2) \times 13.34 = 1.33 \times 10^{-1}$.

From energy chart for the infinite cylinder, $Q / Q_{max} = 0.875$

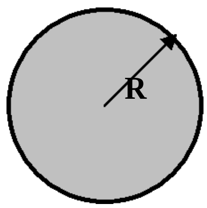
$Q_{max} = \rho V C_p\,(T_i - T_\infty) = (\pi \times 0.025^2 \times 1) \times (50 / 1.35 \times 10^{-6})\,(200 - 20)$

$= 1.27 \times 10^6\ kJ / m = 1.27\ MJ/m$

Therefore $Q = 0.875 \times 1.27 = 1.11\ MJ / m$.

Example 4.8:-An orange of 10 cm diameter is initially at a uniform temperature of 30 ^{0}C. Suddenly it is placed in a refrigerator in which the air temperature is 2 ^{0} C. If the surface heat transfer coefficient is 50 W/(m^2-K), determine the time required for the centre of the orange to reach 10 ^{0} C. Assume for the orange α = 1.4 x 10 $^{-7}$ m^2/s and k = 0.59 W/(m-K).

Solution: Orange is assumed to be in the shape of a sphere.



R = 0.05 m ; T_i = 30 ^{0} C ; T_∞ = 2 ^{0} C ; T_0 = 10 ^{0} C ;

h = 50 W/(m^2 –K);k = 0.59 W/(m – K);

α =1.4 x10^{-7} m^2/s

Now $$\frac{(T_0 - T_\infty)}{(T_i - T_\infty)} = \theta_0 = \frac{(10-2)}{(30-2)} = 0.286$$

1 / Bi = k / hR = 0.59 / (50 x 0.05) = 0.24 ; From chart for transient conduction in

Sphere we have Fo = (αt / R^2) = 0.3.

$$\text{Therefore } t = \frac{(0.3 \times 0.05^2)}{(1.4 \times 10^{-7})} = 5357 \text{ s} = 1.5 \text{ h}$$

Solution using Tables: - For the given problem we have

Bi = 4.24. Therefore from Table 4 – 1 , by interpolation λ_1 = 2.4831 and A_1 = 1.7362.

$$\text{Therefore} \quad \frac{(T_0 - T_\infty)}{(T_i - T_\infty)} = 0.286 = 1.7362 \exp[-\lambda_1^2 Fo]$$

Solving for λ_1^2Fo we get λ_1^2Fo = 1.804 or Fo = 1.804 / (2.4831^2) = 0.2925.

Therefore αt / (R^2) = 0.2925

$$\text{Or } t = 0.2925\, R^2 / \alpha = \frac{(0.2925 \times 0.05^2)}{1.4 \times 10^{-7}} = 5223 \text{ s} = 1 \text{ hr } 27 \text{ mins.}$$

Transient conduction in semi-infinite solids:- A semi-infinite solid is an idealized body that has a *single plane surface* and extends to infinity in all directions.The transient conduction problems in semi-infinite solids have numerous practical applications in engineering. Consider, for example, temperature transients in a slab of finite but large thickness, initiated by a sudden change in the thermal condition at the boundary surface. In the initial stages, the temperature transients near the boundary surface behave similar to those of semi-infinite medium, because some time is required for the heat to penetrate the slab before the other boundary condition begins to influence the transients.The earth for example, can be considered as a semi-infinite solid in determining the variation of its temperature near its surface

We come across basically three possibilities while analyzing the problem of one-dimensional transient conduction in semi-infinite solids.These three problems are as follows:

Problem 1:- The solid is initially at a uniform temperature T_i and suddenly at time t>0 The boundary-surface temperature of the solid is changed to and maintained at a uniform temperature T_0 which may be greater or less than the initial temperature T_i.

Problem 2:- The solid is initially at a uniform temperature T_i and suddenly at time t>0 the boundary surface of the solid is subjected to a uniform heat flux of q_0 W/m^2.

Problem 3:- The solid is initially at a uniform temperature T_i. Suddenly at time t>0 the boundary surface is exposed to an ambience at a uniform temperature T_∞ with the surface heat transfer coefficient h. T_∞ may be higher or lower than T_i.

Solution to Problem 1:- The schematic for problem 1 is shown in Fig. 4.10. The mathematical formulation of the problem to determine the unsteady temperature distribution in an infinite solid T(x,t) is as follows:

The governing differential equation is

$$\partial^2 T / \partial x^2 = (1/\alpha)\,(\partial T / \partial t) \quad \text{4.24(a)}$$

The initial condition is at time t = 0, $T(x,0) = T_i$ .... 4.24(b)

and the boundary condition is at x = 0, $T(0,t) = T_0$ .... 4.24(c)

It is convenient to solve the above problem in terms of the variable θ(x,t), where θ(x,t) is defined as

$$\theta(x,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty} \quad \text{4.25}$$

The governing differential equation in terms of θ(x,t) will be

$$\partial^2 \theta / \partial x^2 = (1/\alpha)\,(\partial \theta / \partial t) \quad \text{4.26(a)}$$

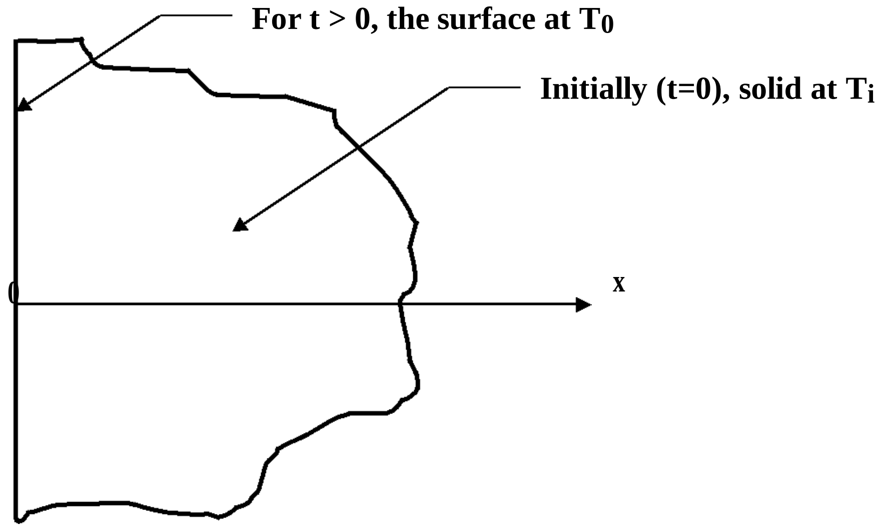


Fig. 4.10: Semi-infinite solid with specified surface temperature T_0 for t > 0

The initial condition will be at time t = 0, $\theta(x,0) = T_i - T_\infty$ .... 4.26(b)

And the boundary condition will be at x = 0, $\theta(0,t) = T_0 - T_\infty$................................ 4.26(c)

This problem has been solved analytically and the solution θ(x,t) is represented graphically as θ(x,t) as a function of the dimensionless variable $x / [2\sqrt{(\alpha t)}]$ as shown in Fig. 4.11.

In engineering applications, the heat flux at the boundary surface x = 0 is also of interest. The analytical expression for heat flux at the surface is given by

$$q_s(t) = \frac{k(T_0 - T_i)}{\sqrt{(\pi \alpha t)}} \quad \text{.................................4.27}$$

Solution to problem 2:- The schematic for this problem is shown in Fig. 4.12.

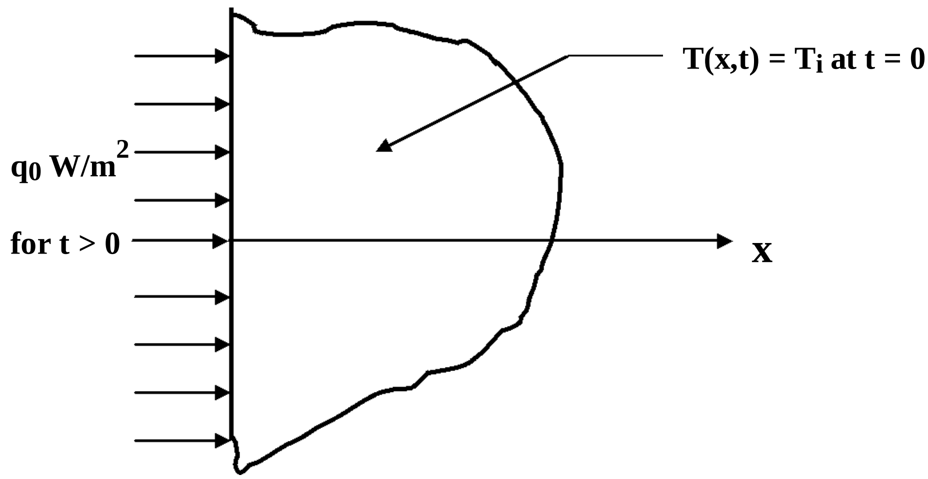


Fig. 4.12: An infinite solid subjected to a constant heat flux at x = 0 for t > 0

Governing differential equation in terms of T(x,t) and the initial condition are same that for problem 1[i.e. equations 4.26(a) and 4.26(b)].

The boundary condition is : at x = 0, $- k (\partial\theta / \partial x)|_{x = 0} = q_0$.

The temperature distribution within the solid T(x,t) is given by

$$T(x, t) = T_i + \frac{2q_0}{k} (\alpha t)^{1/2} [(1 / \sqrt{\pi}) \exp (- \xi^2) + \xi \,\mathrm{erf}(\xi) - \xi] \quad \text{.............................(4.28 a)}$$

$$\text{where } \xi = x / (2\sqrt{\alpha t}) \text{ and } \mathrm{erf}(\xi) = \frac{2}{\sqrt{\pi}} \int_0^{\xi} \exp(- y^2)\, dy \quad \text{...................................(4.28b)}$$

Here erf (ξ) is called the *"error function"* of argument ξ and its values for different values of ξ are tabulated.

Solution to Problem 3 :- The solid is initially at a uniform temperature T_i and suddenly for t >0 the surface at x = 0 is brought in contact with a fluid at a uniform temperature T_∞ with a surface heat transfer coefficient h. For this problem the solution is represented in the form of a plot where the dimensionless temperature $[1 - \theta(x,t)]$ is plottedagainst dimensionless distance $x / \sqrt{(\alpha t)}$, using $h\sqrt{(\alpha t)} / k$ as the parameter. It can be noted that the case $h \to \infty$ is equivalent to the boundary surface ay x = 0 maintained at a constant temperature T_∞.

Illustrative examples on Transient Conduction in Semi – Infinite solids

Example 4.9:-A thick stainless steel slab [$\alpha = 1.6 \times 10^{-5}$ m^2/s and k = 61 W/(m-K)] is initially at a uniform temperature of 150 ^{0} C. Its surface temperature is suddenly lowered to 20 ^{0} C. By treating this as a one-dimensional transient conduction problem in a semi-infinite medium, determine the temperature at a depth 2 cm from the surface and the heat flux 1 minute after the surface temperature is lowered

Solution:

$T_i = 150\,^0C$; $T_0 = T|_{x=0} = 20\,^0C$; $\alpha = 1.6 \times 10^{-5}\ m^2/s$; k = 61 W/(m – K) ; x = 0.02 m ;

T = 1 min = 60 s

$$\xi = \frac{x}{2\sqrt{(\alpha t)}} = \frac{0.02}{2 \times \sqrt{(1.6 \times 10^{-5} \times 60)}} = 0.323$$

$$\text{From chart, } \frac{T(x,t) - T_0}{T_i - T_0} = 0.35$$

Therefore $T(x,t) = T_0 + 0.35 (T_i - T_0) = 20 + 0.35 \times (150 - 20) = 65.5\,^0C$.

$$q_s(t) = \frac{k(T_0 - T_i)}{\sqrt{(\pi\alpha t)}} = \frac{61x(20-150}{\sqrt{(\pi \times 1.6 \times 10^{-5} \times 60)}} = -435.5\ W/m^2$$

Example 4.10:- A semi-infinite slab of copper ($\alpha = 1.1 \times 10^{-4}$ m^2/s and k = 380 W/(m-K) is initially at a uniform temperature of 10 ^{0} C. Suddenly the surface at x = 0 is raised to 100 ^{0}C. Calculate the heat flux at the surface 5 minutes after rising of the surface temperature . How long will it take for the temperature at a depth of 5 cm from the surface to reach 90 ^{0} C?

Solution:

$T_i = 10\,^0C$; $T_0 = 100\,^0C$; k = 380 W / (m – K) ; $\alpha = 1.1 \times 10^{-4}\ m^2/s$; t = 300 s ;

$$q_s(t) = \frac{k(T_0 - T_i)}{\sqrt{(\pi\alpha t)}} = \frac{380 \times (100 - 10)}{\sqrt{(\pi \times 1.1 \times 10^{-4} \times 300)}} = 11012\ W/m^2 = 11.012\ kW/m^2$$

$$\theta(x,t) = \frac{T(x,t) - T_0}{T_i - T_0} = \frac{90 - 100}{10 - 100} = 0.11 \text{ . From chart } \xi = 0.1$$

$$\xi = \frac{x}{2\sqrt{(\alpha t)}} \quad \text{or } t = \frac{x^2}{4\,\alpha\,\xi^2} = \frac{0.05^2}{4 \times 1.1 \times 10^{-4} \times (0.1)^2}$$

$$= 586 \text{ s} = 9.46 \text{ min}$$

Example 4.11:-A thick bronze [$\alpha = 0.86 \times 10^{-5}\ m^2/s$ and $k = 26$ W/(m-K)] is initially at 250 ^{0} C. Suddenly the surface is exposed to a coolant at 25 ^{0} C. If the surface heat transfer coefficient is 150 W/(m^2-K), determine the temperature 5 cm from the surface 10 minutes after the exposure.

Solution:

$T_i = 250\ ^0C$; $T_\infty = 25\ ^0C$; $h = 150\ W/(m^2 - K)$; $k = 26\ W/(m - K)$; $\alpha = 0.86 \times 10^{-5}\ m^2/s$

$t = 600$ s ; $x = 0.05$ m ;

$$\xi = \frac{x}{2\sqrt{(\alpha\, t)}} = \frac{0.05}{2 \times \sqrt{(0.86 \times 10^{-5} \times 600)}} = 0.35$$

$$\frac{h\sqrt{(\alpha\, t)}}{K} = \frac{150 \times \sqrt{[0.86 \times 10^{-5} \times 600]}}{26} = 0.414$$

Therefore from chart $$1 - \frac{[T(x,t) - T_\infty]}{(T_i - T_\infty)} = 0.15$$

Solving for T(x,t) we have $T(x,t) = T_\infty + (1 - 0.15)(T_i - T_\infty)$

$$= 25 + 0.85 \times (250 - 25) = 216.25\ ^0C.$$

Example 4.12:- A thick wood [$\alpha = 0.82 \times 10^{-7}\ m^2/s$ and $k = 0.15$ W/(m-K)] is initially at 20 ^{0} C. The wood may ignite at 400 ^{0} C. Suddenly the surface of the wood is exposed to gases at 500 ^{0} C. If the surface heat transfer coefficient is 45 W/(m^2-K), how long will it take for the surface of the wood to reach 400 ^{0} C ?

Solution:

$T_i = 20\ ^0C$; $T_\infty = 500\ ^0C$; $h = 45\ W/(m^2 - K)$; $k = 0.15\ W/(m - K)$

; $\alpha = 0.82 \times 10^{-7}\ m^2 / s$.

$$\frac{h\sqrt{(\alpha t)}}{k} = \frac{45 \times \sqrt{(0.82 \times 10^{-7} \times t)}}{0.15} = 0.086\sqrt{t}$$

$$x/2\sqrt{(\alpha t)} = 0. \text{ Also } 1 - \frac{T(x,t) - T_\infty}{T_i - T_\infty} = 1 - \frac{(400 - 500)}{(20 - 5000)} = 0.9799$$

Hence from chart $h\sqrt{(\alpha t)}/k = 2.75$.

Therefore $0.086\sqrt{t} = 2.75$ or $t = 1023$ s $= 17$ min.

UNIT-III

CONVECTIVE HEAT TRANSFER

Definition of Convective Heat Transfer:- When a fluid flows over a body or inside a channel and if the temperatures of the fluid and the solid surface are different, heat transfer will take place between the solid surface and the fluid due to the macroscopic motion of the fluid relative to the surface. This mechanism of heat transfer is called as "*convective heat transfer*". If the fluid motion is due to an external force (by using a pump or a compressor) the heat transfer is referred to as "*forced convection*". If the fluid motion is due to a force generated in the fluid due to buoyancy effects resulting from density difference (density difference may be caused due to temperature difference in the fluid) then the mechanism of heat transfer is called as *"natural or free convection"*. For example, a hot plate suspended vertically in quiescent air causes a motion of air layer adjacent to the plate surface because the temperature gradient in the air gives rise to a density gradient which in turn sets up the air motion.

Heat Transfer Coefficient:- In engineering application, to simplify the heat transfer calculations between a hot surface say at temperature T_W and a cold fluid flowing over it at a bulk temperature T_∞ as shown in Fig. 5.1 a term called "*heat transfer coefficient,h*" is defined by the equation

$$q = h(T_W - T_\infty) \qquad \text{.......... 5.1(a)}$$

where q is the heat flux (expressed in W / m^2) from the surface to the flowing fluid. Alternatively if the surface temperature is lower than the flowing fluid then the heat transfer takes place from the hot fluid to the cold surface and the heat flux is given by

$$q = h(T_\infty - T_W) \qquad \text{.......... 5.1(b)}$$

The heat flux in this case takes place from the fluid to the cold surface.If in equations 5.1(a) and 5.1(b) the heat flux is expressed in W / m^2, then the units of heat transfer coefficient will be W /(m^2 – K) or W / (m^2 – ^{0}C).

The heat transfer coefficient is found to vary with (i) the geometry of the body, (ii) the type of flow (laminar or turbulent), (iii) the transport properties of the fluid (density, viscosity and thermal conductivity),(iv) the average temperature, (v) the position along the surface of the body, and (vi) whether the heat transfer is by forced convection or free convection. For convection problems involving simple geometries like flow over a flat plate or flow inside a circular tube, the heat transfer coefficient can be determined analytically

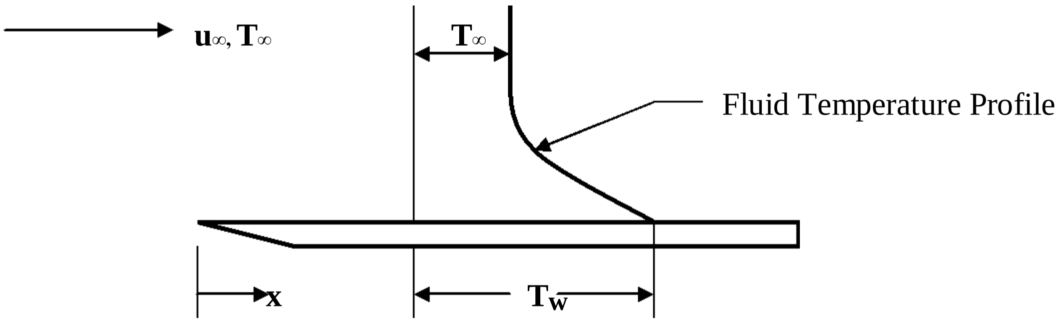


Fig. 5.1: Temperature distribution of the fluid at any x for $T_W > T_\infty$

But for flow over complex configurations, experimental / numerical approach is used to determine h. There is a wide difference in the range of values of h for various applications. Typical values of heat transfer coefficients encountered in some applications are given in Table 5.1.

Table 5.1: Typical Values of heat transfer coefficients

Type of flow		**h [W /(m^2 – K)]**
Free convection	air	5 – 15
-----do ---------	oil	25 – 60
-----do---------	water	400 –800
Forced Convection	air	15 –300
-------do------------	oil	50 –1700
-------do-----------	water	300 – 12000
Boiling	water	3000 – 55000
Condensing	steam	5500 – 120000

Basic concepts for flow over a body:- When a fluid flows over a body, the velocity and temperature distribution at the vicinity of the surface of the body strongly influence the heat transfer by convection. By introducing the concept of boundary layers (velocity boundary layer and thermal boundary layer) the analysis of convective heat transfer can be simplified.

Velocity Boundary Layer:- Consider the flow of a fluid over a flat plate as shown in Fig. 5.2. The fluid just before it approaches the leading edge of the plate has a velocity u_∞ which is parallel to the plate surface. As the fluid moves in x-direction along the plate,

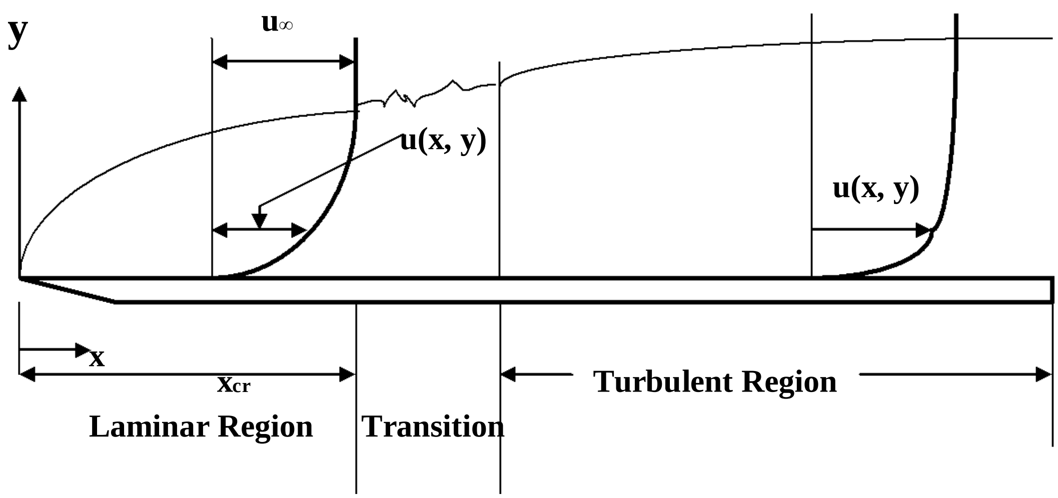


Fig. 5.2: Velocity boundary layer for flow over a flat plate

those fluid particles that makes contact with the plate surface will have the same velocity as that of the plate. Therefore if the plate is stationary, then the fluid layer sticking to the plate surface will have zero velocity.But far away from the plate ($y = \infty$) the fluid will have the velocity u_∞.Therefore starting from the plate surface ($y = 0$) there will be retardation of the fluid in x- direction component of velocity u(x,y).This retardation effect is reduced as we move away from the plate surface.At distances sufficiently long from the plate($y = \infty$) the retardation effect is completely reduced: i.e. $u \rightarrow u_\infty$ as $y \rightarrow \infty$. This means that there is a region surrounding the plate surface where the fluid velocity changes from zero at the surface to the velocity u_∞ at the outer edge of the region. This region is called ***the velocity boundary layer***. The variation of the x- component of velocity u(x,y) with respect to y at a particular location along the plate is shown in Fig. 5.2.The distance measured normal to the surface from the plate surface to the point at which the fluid attains 99% of u_∞ is called "***velocity boundary layer thickness***" and denoted by ***δ(x)*** Thus for flow over a flat plate, the flow field can be divided into two distinct regions, namely, (i) ***the boundary layer region*** in which the axial component of velocity u(x,y) varies rapidly with y with the result the velocity gradient ($\partial u / \partial y$) and hence the shear stress are very large and (ii) ***the potential flow region*** which is outside the boundary layer region, where the velocity gradients and shear stresses are negligible.

The flow in the boundary layer, starting from the leading edge of the plate will be initially laminar in which the fluid particles move along a stream line in an orderly manner. In the laminar region the retardation effect is due to the viscosity of the fluid and therefore the shear stress can be evaluated using Newton‟s law of viscosity. The laminar

flow continues along the plate until a critical distance „x_{cr}‟ is reached. After this the small disturbances in the flow begin to grow and fluid fluctuations begin to develop. This characterizes the end of the laminar flow region and the beginning of transition from laminar to ***turbulent boundary layer.*** A dimensionless parameter called ***Reynolds number*** is used to characterize the flow as laminar or turbulent. For flow over a flat plate the Reynolds number is defined as

$$Re_X = \frac{u_\infty x}{\nu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 5.2$$

where u_∞ = free-stream velocity of the fluid, x = distance from the leading edge of the plate and ν = kinematic viscosity of the fluid.

For flow over a flat plate it has been found that the transition from laminar flow to turbulent flow takes place when the Reynolds number is $\approx 5 \times 10^5$.This number is called as the critical Reynolds number Re_{cr} for flow over a flat plate. Therefore

$$Re_{cr} = \frac{u_\infty x_{cr}}{\nu} = 5 \times 10^5 \ldots\ldots\ldots\ldots\ldots\ldots\ 5.3$$

The critical Reynolds number is strongly dependent on the surface roughness and the turbulence level of the free stream fluid. For example, with very large disturbances in the free stream, the transition from laminar flow to turbulent flow may begin at Re_X as low as 1×10^5 and for flows which are free from disturbances and if the plate surface is smooth transition may not take place until a Reynolds number of 1×10^6 is reached. But it has been found that for flow over a flat plate the boundary layer is always turbulent for $Re_X \geq 4 \times 10^6$.In the turbulent boundary layer next to the wall there is a very thin layer called *"the viscous sub-layer"*, where the flow retains its viscous flow character. Next to the viscous sub-layer is a region called *"buffer layer"* in which the effect of fluid viscosity is of the same order of magnitude as that of turbulence and the mean velocity rapidly increases with the distance from the plate surface. Next to the buffer layer is *"the turbulent layer"* in which there is large scale turbulence and the velocity changes relatively little with distance.

Drag coefficient and Drag force:- If the velocity distribution u(x,y) in the boundary layer at any „x‟ is known then the viscous shear stress at the wall can be determined using Newton‟s law of viscosity. Thus if $\eta_W(x)$ is the wall-shear stress at any location x then

$$\eta_W(x) = \mu(\partial u / \partial y)_{y=0} \quad \text{................................ 5.4}$$

where μ is the absolute viscosity of the fluid. The drag coefficient is dimensionless wall shear stress. Therefore *the local drag coefficient, C_X* at any „x‟ is defined as

$$C_X = \frac{\eta_W(x)}{(1/2)\,\rho u_\infty^2} \quad \text{..................................... 5.5}$$

Substituting for $\eta_W(x)$ in the above equation from Eq. 5.4 and simplifying we get

$$C_X = \frac{2\nu\,(\partial u / \partial y)_{y=0}}{u_\infty^2} \quad \text{......................... 5.6}$$

Therefore if the velocity profile u(x,y) at any x is known then the local drag coefficient C_X at that location can be determined from Eq. 5.6.The average value of C_X for a total length L of the plate can be determined from the equation

$$C_{av} = (1/L)\int_0^L C_X\,dx \quad \text{.................................5.7}$$

Substituting for C_X from Eq. 5.5 we have

$$C_{av} = \frac{\int^L \eta_W(x)dx}{L\,(1/2)\,\rho u_\infty^2}$$

Or

$$C_{av} = \frac{\bar{\eta}_W}{(1/2)\,\rho u_\infty^2} \quad \text{……………….5.8}$$

Where $\bar{\eta}_W$ is the average wall-shear stress for total length L of the plate.

The total drag force experienced by the fluid due to the presence of the plate can be written as

$$F_D = A_S\,\bar{\eta}_W \quad \text{...5.9}$$

Where A_S is the total area of contact between the fluid and the plate. If „W‟ is the width of the plate then A_S = LW if the flow is taking place on one side of the plate and A_S = 2LW if the flow is on both sides of the plate.

Thermal boundary layer:- Similar to the velocity boundary layer one can visualize the development of a thermal boundary layer when a fluid flows over a flat plate with the temperature of the plate being different from that of the free stream fluid.Consider that a fluid at a uniform temperature T_∞ flows over a flat plate which is maintained at a uniform temperature T_W.Let T(x,y) is the temperature of the fluid at any location in the flow field.Let the dimensionless temperature of the fluid θ(x,y) be defined as

$$\theta(x,y) = \frac{T(x,y) - T_W}{T_\infty - T_W} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 5.10$$

The fluid layer sticking to the plate surface will have the same temperature as the plate surface $[T(x,y)_{y=0} = T_W]$ and therefore θ(x,y) = 0 at y = 0.Far away from the plate the fluid temperature is T_∞ and hence $\theta(x,y) \rightarrow 1$ as $y \rightarrow \infty$. Therefore at each location x along the plate one can visualize a location $y = \delta_t(x)$ in the flow field at which θ(x,y) = 0.99. $\delta_t(x)$ is called ***"the thermal boundary layer thickness"*** as shown in Fig. 5.3. The locus of such points at which θ(x,y) = 0.99 is called the edge of the thermal boundary layer. The relative thickness of the thermal boundary layer $\delta_t(x)$ and the velocity

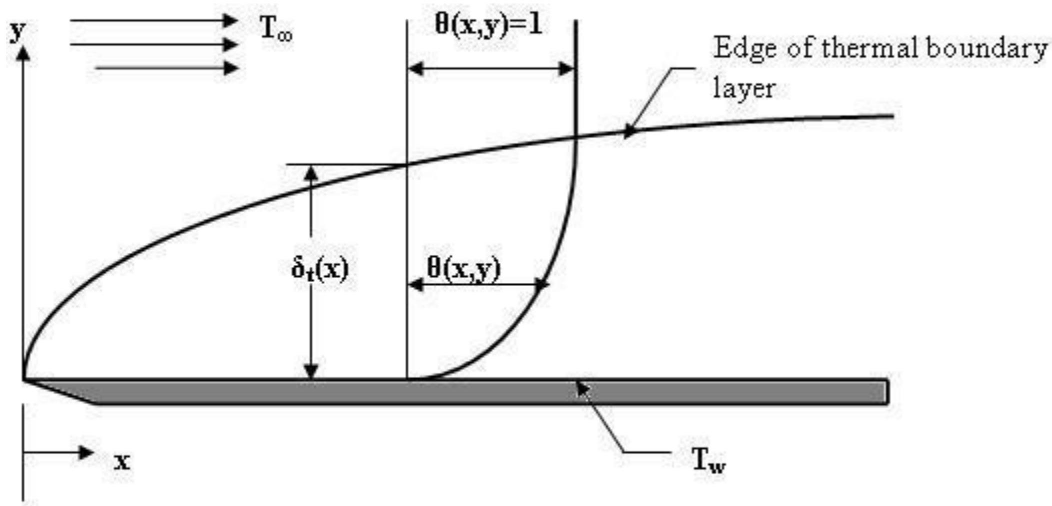


Fig. 5.4: Growth of thermal boundary layer for flow over a flat plate

boundary layer δ(x) depends on a dimensionless number called ***"Prandtl number"*** of the fluid.It is denoted by Pr and is defined as

$$Pr = \frac{\mu C_p}{k} = \frac{(\mu/\rho)}{(k/\rho C_p)} = \frac{\nu}{\alpha} \ldots\ldots\ldots\ldots\ldots 5.11$$

Where μ is the absolute viscosity of the fluid, C_p is the specific heat at constant pressure

and k is the thermal conductivity of the fluid. The Prandtl number for fluids range from 0.01 for liquid metals to more than 100,000 for heavy oils. For fluids with Pr = 1 such as gases $\delta_t(x) = \delta(x)$, for fluids with Pr << 1such as liquid metals $\delta_t(x) >> \delta(x)$ and for fluids with Pr >> 1, like oils $\delta_t(x) << \delta(x)$.

General expression for heat transfer coefficient:- Let us assume that $T_w > T_\infty$. Then heat is transferred from the plate to the fluid flowing over the plate.Therefore at any „x" the heat flux is given by

$$q = -k\,(\partial T/\partial y)_{y=0} \quad\text{...}\;5.12(a)$$

In terms of the local heat transfer coefficient h_x, the heat flux can also be written as

$$q = h_x\,(T_w - T_\infty) \quad\text{.......................................}\;5.12(b)$$

From equations 5.12(a) and 5.12(b) it follows that

$$h_x = \frac{-k\,(\partial T/\partial y)_{y=0}}{(T_w - T_\infty)} \quad\text{...........................}\;5.13$$

From equation 5.10 we have $(\partial T/\partial y)_{y=0} = [T_\infty - T_w]\,(\partial\theta/\partial y)_{y=0}$. Substituting this expression in Eq.5.13 and simplifying we get the general expression for h_x as

$$h_x = k\,(\partial\theta/\partial y)_{y=0} \quad\text{...}\;5.14$$

The same expression for h_x could be obtained even when $T_w < T_\infty$. Equation 5.14 can be used to determine the local heat transfer coefficient for flow over a flat plate if the dimensionless temperature profile $\theta(x,y)$ is known.

Average heat transfer coefficient:- For a total length L of the plate the average heat transfer coefficient is given by

$$h_{av} = (1/L)\int_0^L h_x\,dx \quad\text{......................................}\;5.15$$

Substituting for h_x from Eq. 5.14 we get

$$h_{av} = (1/L)\int_0^L k\,(\partial\theta/\partial y)_{y=0}\,dx \quad\text{...............................}5.15$$

Since $(\partial\theta/\partial y)_{y=0}$ at any x depends on whether the flow at that section is laminar or turbulent the expression for h_{av} can be written as

$$h_{av} = (1/L)\left\{\int^{x_{cr}} k\,[(\partial\theta/\partial y)_{y=0}]_{laminar}\,dx + \int^{L} k\,[(\partial\theta/\partial y)_{y=0}]_{turbulent}\,dx\right\}\ldots\ldots 5.16$$

Illustrative examples on flow over a flat plate:

Example 5.1:- *Assuming the transition from laminar to turbulent flow takes place at a Reynolds number of 5 x 10 ^{5}, determine the distance from the leading edge of a flat plate at which transition occurs for the flow of each of the following fluids with a velocity of 2 crm/s at 40 ^{0} C.(i) Air at atmospheric pressure;(ii)Hydrogen at atmospheric pressure;(iii) water;(iv) Engine oil;(v) mercury. Comment on the type of flow for the 5 fluids if the total length of the plate is 1 m.*

Solution: Data:- $Re_{cr} = 5 \times 10^{5}$; $u_\infty = 2$ m/s ; $T_\infty = 40^{0}$ C

(i)Air at atmospheric pressure :- At 40 ^{0} C, $\nu = 17 \times 10^{-6}$ m^2/s.

$$Re_{cr} = \frac{u_\infty x_{cr}}{\nu} \quad \text{or } x_{cr} = \frac{Re_{cr}\,\nu}{u_\infty} = \frac{5 \times 10^{5} \times 17 \times 10^{-6}}{2} = 4.25 \text{ m.}$$

(ii) Hydrogen :- For hydrogen at 40 ^{0} C, $\nu = 117.9 \times 10^{-6}$ m^2/s.

Therefore $$x_{cr} = \frac{5 \times 10^{5} \times 117.9 \times 10^{-6}}{2} = 29.5 \text{ m}$$

(iii) Water :- For water at 40 ^{0} C, $\nu = 0.658 \times 10^{-6}$ m^2/s.

Therefore $$x_{cr} = \frac{5 \times 10^{5} \times 0.658 \times 10^{-6}}{2} = 0.1645 \text{ m}$$

(iv) Engine oil :- For engine oil at 40 ^{0} C, $\nu = 0.24 \times 10^{-3}$ m^2/s.

Therefore $$x_{cr} = \frac{5 \times 10^{5} \times 0.24 \times 10^{-3}}{2} = 60 \text{ m}$$

(v) Mercury :- For mercury at 40 ^{0} C, $\nu = 0.107 \times 10^{-6}$ m^2/s.

Therefore $$x_{cr} = \frac{5 \times 10^{5} \times 0.107 \times 10^{-6}}{2} = 0.027 \text{ m}$$

Comments on the type of flow

Sl.No	Type of fluid	x_{cr}	x_{cr} vs L	Type of Flow
1	Air	4.25	$x_{cr} > L$	Flow is Laminar for entire length
2	Hydrogen	29.5	$x_{cr} >> L$	Flow is laminar for entire length
3	Water	0.1645	$x_{cr} < L$	Flow is partly Laminar & Partly Turbulent
4	Engine oil	60	$x_{cr} >> L$	Flow is laminar for entire length
5	Mercury	0.027	$x_{cr} << L$	Flow is turbulent for almost entire length

Example 5.2:- *An approximate expression for the velocity profile u(x,y) for laminar boundary layer flow along a flat plate is given by*

$$u(x, y)/ u_\infty = 2[y / \delta(x)] - 2[y / \delta(x)]^3 + [y / \delta(x)]^4$$

where δ(x) is the velocity boundary layer thickness given by the expression

$$\delta(x) / x = 5.83 / (Re_x)^{1/2}$$

(a) *Develop an expression for the local drag coefficient.*
(b) *Develop an expression for the average drag coefficient for a length L of the plate.*
(c) *Determine the drag force acting on the plate 2 m x 2 m for flow of air with a free stream velocity of 4 m /s and a temperature of 80 ^{0}C.*

Solution:- (a) The velocity profile u(x,y) is given as

$$u(x, y) = u_\infty \{2[y / \delta(x)] - 2[y / \delta(x)]^3 + [y / \delta(x)]^4\}$$

Therefore $\quad (\partial u / \partial y)_{y=0} = 2u_\infty / \delta(x)$

$$\eta_w(x) = \mu\, (\partial u / \partial y)_{y=0} = (2\, \mu u_\infty) / \delta(x)$$

$$= \frac{(2\, \mu u_\infty)\, Re_x}{5.83\, x} = \frac{(2\, \mu u_\infty)\, [(u_\infty x) / \nu]^{1/2}}{5.83\, x} = 0.343\, (\mu u_\infty)\, [u_\infty /(x\, \nu)]^{1/2} \quad \ldots(1)$$

The local drag coefficient at any x is given by

$$C_x = \frac{\eta_w(x)}{(1/2)\,\rho u_\infty^2} = \frac{0.343\,(\mu u_\infty)\,[u_\infty/(x\,\nu)]^{1/2}}{(1/2)\,\rho u_\infty^2}$$

$$= \frac{0.686}{\{(u_\infty x)/\nu\}^{1/2}} = \frac{0.686}{(Re_x)^{½}}$$

(b) The average drag coefficient is given by

$$C_{av} = (1/L)\int_0^L C_x\,dx = (1/L)\int_0^L 0.686\,(Re_x)^{-½}\,dx$$

$$= \frac{\{0.686\,(u_\infty/\nu)^{-½}\}}{L}\int_0^L x^{-½}\,dx$$

Or

$$C_{av} = \frac{2 \times 0.686}{(u_\infty L/\nu)^{½}} = \frac{1.372}{(Re_L)^{½}}$$

(c) At 80 ^{0}C for air $\nu = 20.76 \times 10^{-6}\ m^2/s$; $\rho = 1.00\ kg/m^3$

$$Re_L = \frac{u_\infty L}{\nu} = \frac{4 \times 2}{21.09 \times 10^{-6}} = 3.793 \times 10^5$$

$$\text{Average drag coefficient} = C_{av} = \frac{1.372}{Re_L^{0.5}} = \frac{1.372}{(3.793 \times 10^5)^{0.5}} = 2.228 \times 10^{-3}$$

Drag force assuming that the flow takes place on one side of the plate is given by

$$F_D = \eta_w\,LW = (1/2)\rho\,u_\infty^2\,C_{av}\,LW \text{ for flow over one side of the plate}$$

$$= (1/2) \times 1.00 \times 4^2 \times 2.2228 \times 10^{-3} \times 2 \times 2 = 0.071\ N$$

Example 5.3:- *An approximate expression for temperature profile θ(x,y) in the thermal boundary layer region is given by*

$$\theta(x,y) = 2y / \delta_t - [y / \delta_t]^2$$

where the thermal boundary layer thickness δ_t *is given by*

$$\delta_t / x = \frac{5.5}{Re_x^{0.5} Pr^{1/3}}$$

; Re_x is the Reynolds number based on „x" and Pr is the Prandtl number of the fluid. Develop an expression for (i) the local heat transfer coefficient h_x and (ii) the average heat transfer coefficient for total length L of the plate.

Solution: (i) The local heat transfer coefficient h_x is given by

$$h_x = k\,(\partial\theta / \partial y)|_{y=0}.$$

Now $$\theta(x,y) = 2y / \delta_t - [y / \delta_t]^2$$

Hence $$(\partial\theta / \partial y)|_{y=0.} = 2 / \delta_t = \frac{2\,Re_x^{0.5} Pr^{1/3}}{5.5.x}$$

Or $$h_x = \frac{2\,k\,Re_x^{0.5} Pr^{1/3}}{5.5.x} = 0.364\,(k / x)\,Re_x^{0.5}\,Pr^{1/3}$$

Or $$\frac{h_x\,x}{K} = 0.364\,Re_x^{0.5}\,Pr^{1/3}$$

$\frac{h_x\,x}{k}$ is a dimensionless number involving local heat transfer coefficient and is called "*local Nusselt number*".

(ii) The average heat transfer coefficient for a total length L of the plate is given by

$$h_{av} = (1 / L)\int_0^L h_x\,dx = (1 / L)\int 0.364\,(k / x)\;Re_x^{0.5}\,Pr^{1/3}\,dx$$

Or $$= (1 / L)\,0.364\,Pr^{1/3}\,k\,(U_\infty / \nu)^{0.5}\int_0^L x^{-0.5}\,dx$$

$$= (1 / L)\,\frac{L^{0.5}}{0.5}\,0.364\,Pr^{1/3}\,k\,(U_\infty / \nu)^{0.5}$$

$$= 0.728\,(k / L)\,(U_\infty L / \nu)^{0.5}\,Pr^{1/3}$$

Or $$h_{av} L / k = 0.728\, Re_L^{0.5}\, Pr^{1/3}$$

$h_{av} L / k$ is a dimensionless number involving the average heat transfer coefficient and is called the "average Nusselt number".

Example 5.4:- *The heat transfer rate per unit width from a longitudinal section $x_2 - x_1$ of a flat plate can be expressed as $q_{12} = h_{12} (x_2 - x_1)(T_s - T_\infty)$, where h_{12} is the average heat transfer coefficient for the section length of $(x_2 - x_1)$. Consider laminar flow over a flat plate with a uniform temperature T_s. The spatial variation of the local heat transfer coefficient is of the form $h_x = C\, x^{-0.5}$, where C is a constant.*

(a) Derive an expression for h_{12} in terms of C, x_1 and x_2.

(b) Derive an expression for h_{12} in terms of x_1, x_2, and the average coefficients h_1 and h_2 corresponding to lengths x_1 and x_2 respectively.

Solution:

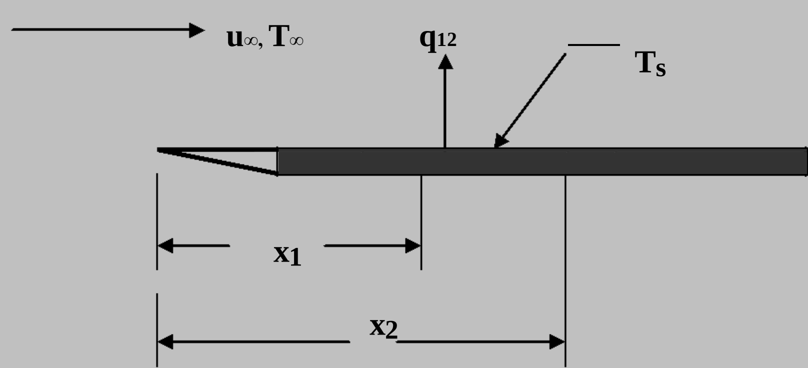


Fig. P5.5: Schematic for problem 5.5

(a) $$h_x = C\, x^{-0.5}$$

Therefore $$\bar{h}_{12} = \frac{1}{(x_2 - x_1)} \int_{x_1}^{x_2} h_x\, dx$$

$$= \frac{1}{(x_2 - x_1)} \int_{0}^{x_2} C\, x^{-0.5}\, dx$$

$$= \frac{2C}{(x_2 - x_1)} [x_2^{0.5} - x_1^{0.5}]$$

(b) $$\bar{h}_1 = (1/x_1) \int_0^{x_1} C\, x^{-0.5}\, dx$$

$$= 2C / \sqrt{x_1}$$

Similarly $$\bar{h}_2 = 2C / \sqrt{x_2}$$

Therefore Since $\int_0^{x_1} h_x dx = x_1 \bar{h}_1$, $$\bar{h}_{12} = \frac{1}{(x_2 - x_1)} \left[\int_0^{x_2} h_x dx - \int_0^{x_1} h_x dx \right]$$

$$\bar{h}_{12} = \frac{\bar{h}_2 x_2 - \bar{h}_1 x_1}{x_2 - x_1}$$

<u>Basic Concepts For Flow Through Ducts</u> :- The basic concepts developed on the development of velocity and thermal boundary layers for flow over surfaces are also applicable to flows at the entrance region of the ducts.

Velocity Boundary Layer:- Consider the flow inside a circular tube as shown in Fig.5.4.

Lat u_o be the uniform velocity with which the fluid approaches the tube. As the fluid enters the tube, a *"velocity boundary layer"* starts to develop along the wall-surface. The velocity of the fluid layer sticking to the tube-surface will have zero velocity and the fluid layer slightly away from the wall is retarded. As a result the velocity in the central portion of the tube increases to satisfy the continuity equation (law of conservation of mass).The thickness of the velocity boundary layer $\delta(z)$ continuously grows along the tube-surface until it fills the entire tube. The region from the tube inlet up to little beyond the hypothetical location where the boundary layer reaches the tube centre is called *"hydrodynamic entrance region or hydrodynamically developing region"* and the corresponding length is called *"hydrodynamic entrance length L_h".* In the hydrodynamically developing region the shape of the velocity profile changes both in axial and radial direction, i.e., u = u(r,z). The region beyond the hydrodynamic entry length is called *"Hydrodynamically developed region"*, because in this region the velocity profile is invariant with distance along the tube,i.e., u = u(r).

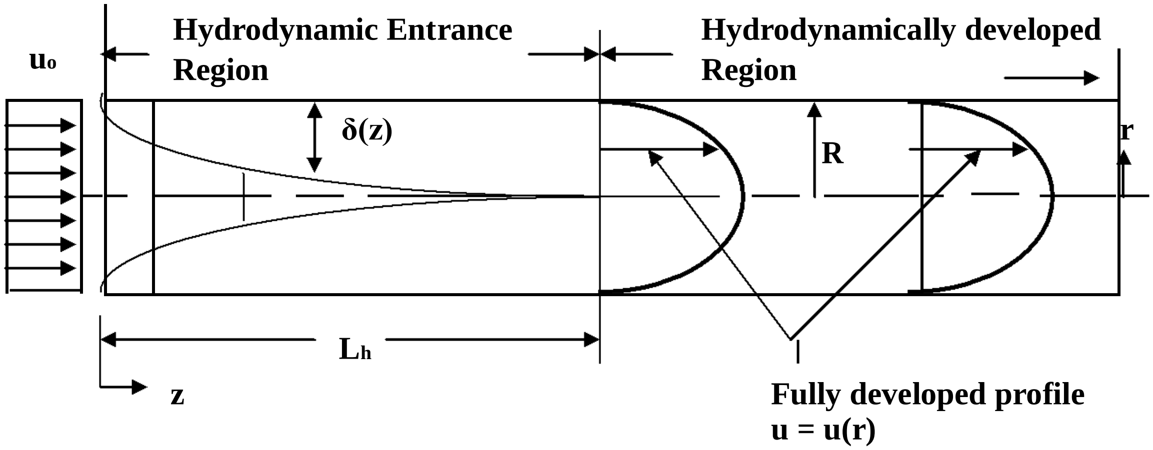


Fig. 5.4: development of velocity boundary layer at entrance region of a tube

If the boundary layer remains laminar until it fills the tube, then laminar flow will prevail in the developed region. However if the boundary layer changes to turbulent before its thickness reaches the tube centre, fully developed turbulent flow will prevail in the hydrodynamically developed region. The velocity profile in the turbulent region is flatter than the parabolic profile of laminar flow. The Reynolds number, defined as

$$Re_d = (u_m\, D_h) / \nu \quad \text{...(5.17)}$$

is used as a criterion for change from laminar flow to turbulent flow. In this definition, u_m is the average velocity of the fluid in the tube, D_h is the hydraulic diameter of the tube and ν is the kinematic viscosity of the fluid. The hydraulic diameter is defined as

$$D_h = \frac{4 \text{ x Area of flow}}{\text{Wetted Perimeter}} \quad \text{................................(5.18)}$$

For flows through ducts it has been observed that turbulent flow prevails for

$$Re_d \geq 2300 \text{..(5.19)}$$

But this critical value is strongly dependent on the surface roughness, the inlet conditions and the fluctuations in the flow. In general, transition may occur in the range $2000 < Re_d < 4000$. It is a common practice to assume a value of 2300 fro transition from laminar flow to turbulent flow.

Friction Factor and Pressure Drop Relations For Hydrodynamically Developed Laminar Flow

In engineering applications, the pressure gradient (dp / dz) associated with the flow is a quantity of interest, because this decides the pumping power required to overcome thr frictional losses in the pipe of a given length.

Consider a differential length dz of the tube at a distance z from the entrance and let this length be in the fully developed region. The various forces acting on the fluid element in the direction of flow are shown in Fig.5.5.

Resultant force in the direction of motion = F = $(pA)_z - (pA)_{z+dz} - \eta_w \, Sdz$

where S is the perimeter of the duct.
Using Taylor‟s series expansion and neglecting higher order terms we can write

$$(pA)_{z+dz} = (pA)_z + d/dz(pA)\, dz$$

Therefore F = $d/dz(pA)\, dz - \eta_w \, Sdz$

Rate of change of momentum in the direction of flow = 0 because the velocity u does not vary with respect to z in the fully developed region.

Hence $$d/dz(pA)\, dz - \eta_w \, Sdz = 0$$

For duct of uniform cross section A is constant. Therefore the above equation reduces to

$$\mathbf{dp/dz = -\eta_w S/A} \text{.. } \mathbf{(5.20)}$$

For laminar flow $\eta_w = -\mu\, (du / dr)|_{wall}$. Hence Eq. (5.20) reduces to

$$\frac{dp}{dz} = \frac{\mu S}{A} (du/dr)|_{wall} \text{ .... } (5.21)$$

Eq.(5.21) is not practical for the determination of (dp/dz), because it requires the evaluation f the velocity gradient at the wall. Hence for engineering applications a parameter called *"friction factor, f "* is defined as follows:

$$f = -\frac{(dp/dz)\, D_h}{\frac{1}{2}(\rho u_m^2)} \text{ .... } (5.22a)$$

Substituting for (dp/dz) from Eq. (5.21) we have

$$f = -\frac{(\mu S/A)\,(du/dr)|_{wall}\, D_h}{\frac{1}{2}(\rho u_m^2)} \text{ …………………..} (5.22b)$$

For a circular tube $S = \pi D_i$, and $A = \pi D_i^2 / 4$. Hence $D_h = D_i$

Hence for a circular tube Eq. (5.22b) reduces to

$$f = -\frac{8\mu}{(\rho u_m^2)} (du/dr)|_{wall} \quad \text{.......(5.22c)}$$

Also from Eq. (5.22a) we have

$$dp = -\frac{(\frac{1}{2})(\rho u_m^2) f}{D_h} dz$$

Integrating the above equation over a total length L of the tube we have

$$\int_{p_1}^{p_2} dp = -\frac{(\frac{1}{2})(\rho u_m^2) f}{D_h} \int_0^L dz$$

or pressure drop = $\mathbf{\Delta p = (p_1 - p_2) = (\frac{1}{2})(L/D_h) f \rho u_m^2}$ .... **(5.23)**

Pumping power is given by $P = \dot{V} \Delta p$.......(5.24)

where $\dot{V}$ = volume flow rate of the fluid.

<u>Thermal Boundary Layer:</u> In the case of temperature distribution in flow inside a tube, it is more difficult to visualize the development of thermal boundary layer and the existence of thermally developed region. However under certain heating or cooling conditions such as *constant wall-heat flux* or *constant wall-temperature* it is possible to have thermally developed region.

Consider a laminar flow inside a circular tube subjected to uniform heat flux at the wall. Let „r‟ and „z‟ be the radial and axial coordinates respectively and T(r,z) be the local fluid temperature. A dimensionless temperature θ(r,z) is defined as

$$\theta(r,z) = \frac{T(r,z) - T_w(z)}{T_m(z) - T_w(z)} \quad \text{.......(5.25a)}$$

where $T_w(r,z)$ = Tube wall-temperature and $T_m(z)$ = Bulk mean temperature of the fluid. The bulk mean temperature at any cross section „z‟ is defined as follows:

$$T_m(z) = \frac{\int \rho(2\pi r dr)\, u(r,z)\, C_p\, T(r,z)}{\int \rho(2\pi r dr)\, u(r,z)\, C_p} = \frac{\int r dr\, u(r,z) T(r,z)}{\int r dr)\, u(r,z)} \quad \text{.......(5.25b)}$$

At the tube wall it is clear that θ(r,z) = 0 and attains some finite value at the centre of the tube. Thus we can visualize the development of thermal boundary layer along the tube surface as shown in Fig. 5.5.The thickness of the thermal boundary layer δ_t continuously grows along the tube surface until it fills the entire tube. The region from the tube inlet to the hypothetical location where the thermal boundary layer thickness reaches the tube centre is called the *"thermal entry section"*. In this region the shape of the dimensionless temperature profile θ(r,z) changes both in axial and in radial directions. The region beyond the thermal entry section is called as the *"thermally developed region"*, because in this region the dimensionless temperature profile θ remains invariant with respect to z. That is in this region θ = θ(r).It is difficult to explain qualitatively why θ should be independent of z even though the temperature of the fluid T depends both on r and z. However it can be shown mathematically that, for both constant wall-heat flux and constant wall-temperature conditions, θ depends only on r for large values of z.For constant wall-heat flux condition the wall-temperature $T_W(z)$ increases with z.

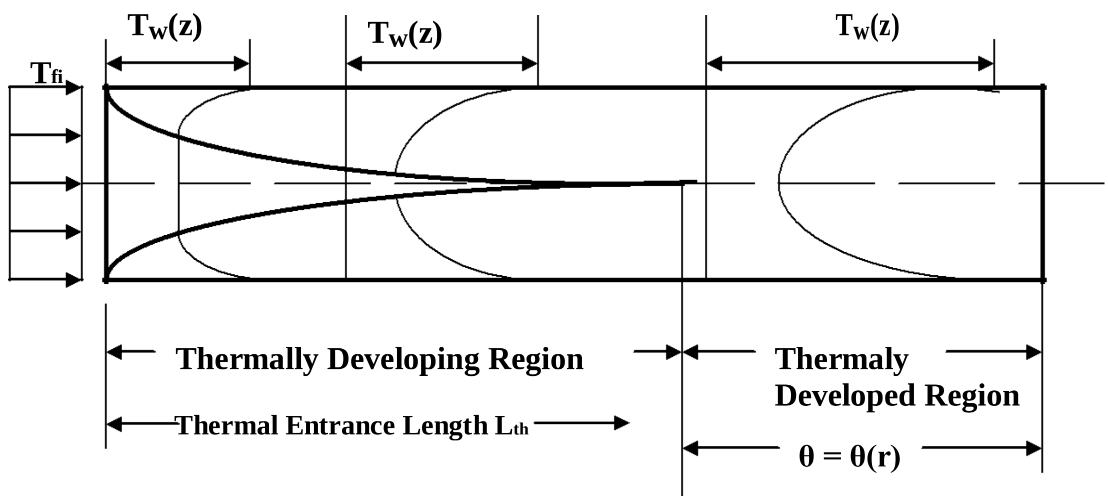


Fig. 5.5: Development of Thermal Boundary Layer In a Flow Through A Tube Subjected to Constant Wall-Heat Flux Condition

The variation of wall-temperature and the bulk fluid temperature as we proceed along the length of the tube for constant wall-heat flux conditions is shown in Fig. 5.6.

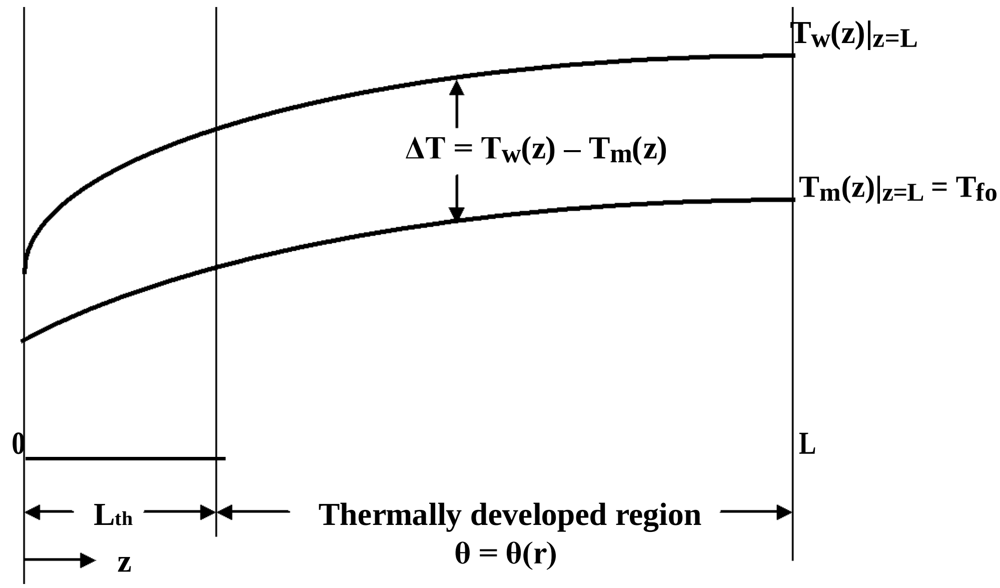


Fig. 5.6: Variation of tube wall-temperature and bulk fluid temperature along the length of the tube

It can be shown that for constant wall-heat flux condition the temperature difference ΔT between the tube wall and the bulk fluid remains constant along the length of the tube.

The growth of the thermal boundary layer for constant wall-temperature conditions is similar to that for constant wall-heat flux condition except that the wall temperature does not vary with respect to z. Therefore the temperature profile T(r,z) becomes flatter and flatter as shown in Fig. 5.7 as we proceed along the length of the tube and eventually the fluid temperature becomes equal to the wall temperature. Since the

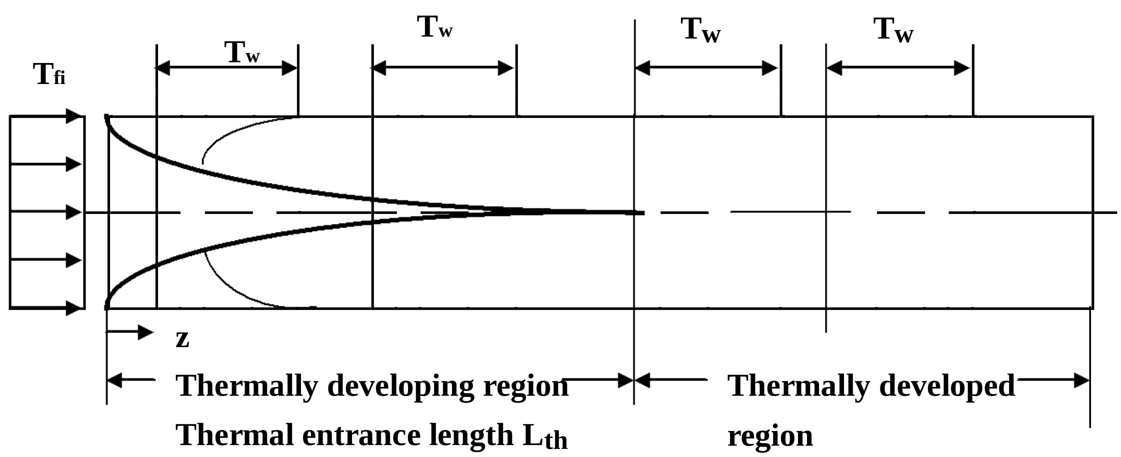


Fig.5.7: Growth of thermal boundary layer for flow through a tube with constant wall-temperature

wall-temperature remains constant and the bulk fluid temperature varies along the length

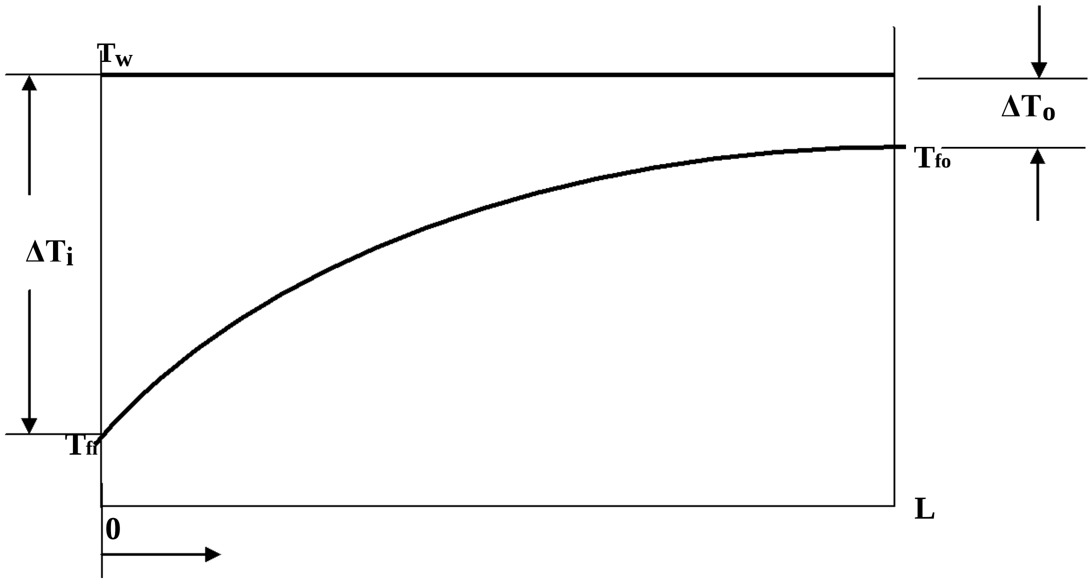

Tw
ΔTo
Tfo
ΔTi
Tfi
L
0

Fig. 5.8: Variation of bulk fluid temperature along the length of the tube for tube with constant wall-temperature

the temperature difference between the tube wall and the bulk fluid varies along the length of the tube as shown in Fig. 5.8.

5.4.4. Mean Temperature Difference, ΔT_m: If Q is the total heat transfer rate between the fluid and the tube surface , A_s is the area of contact between the fluid and the surface, h_m is the average heat transfer coefficient for the total length of the tube then we can write

$$Q = h_m A_s \Delta T_m \qquad (5.26)$$

Where ΔT_m = mean temperature difference between the tube wall and the bulk fluid. For a tube with constant wall-heat flux condition, since the temperature difference between the fluid and the tube surface remains constant along the length of the tube it follows that

$$\Delta T_m = [T_w(z)|_{z=0} - T_{fi}] = [T_w(z)|_{z=L} - T_{fo}] \qquad (5.27a)$$

For a tube with constant wall-temperature condition the mean temperature difference is given by

$$\Delta T_m = \frac{\Delta T_i - \Delta T_o}{\ln (\Delta T_i / \Delta T_o)} \qquad (5.27b)$$

General expression for heat transfer coefficient **:** Let the fluid be heated as it flows through the tube. Then at any z the heat flux from the tube surface to the fluid is given by Fourier‟s law as

$$q_w(z) = k (\partial T / \partial r)|_{wall} \qquad (5.28)$$

[Note that when the fluid is heated $T_w > T_m$ so that $(\partial T / \partial r)|_{wall}$ will be positive). If h_z is the heat transfer coefficient then

$$q_w(z) = h_z [T_w(z) – T_m(z) \qquad (5.29)$$

Therefore from Eq. (5.28) and (5.29) we have

$$h_z = \frac{k (\partial T / \partial r)|_{wall}}{[T_w(z) – T_m(z)]} \qquad (5.30)$$

$[T(r,z) – T_w(z)]$

Now $\theta(r,z) = \dfrac{}{[T_m(z) - T_w(z)]}$

Therefore $(\partial T / \partial r)|_{wall} = [T_m(z) - T_w(z)]\ (\partial\theta / \partial r)|_{wall}$

Substituting this expression in Eq. (5.30) and simplifying we get

$$h_z = -\ k\ (\partial\theta / \partial r)|_{wall} \qquad (5.31)$$

In the thermally developed region θ depends only on r. Hence

$$h_z = -\ k\ (d\theta / dr)|_{wall} \qquad (5.32)$$

Since $(d\theta / dr)|_{wall}$ is independent of z it follows that the heat transfer coefficient h_z is independent of z. This is true both for constant wall-temperature and constant wall-heat flux conditions.

Illustrative Examples on Flow Through Ducts:

Example 5.5:- The velocity profile for hydrodynamically developed laminar flow inside a circular tube of radius R is given by

$$u(r) = 2u_m[\ 1 - (r/R)^2]$$

where u_m is the average velocity of the fluid in the tube.Develop an expression for the friction factor f and express it in terms of the Reynolds number Re_d where Re_d is defined as $Re_d = (u_m D) / v$.

Solution:

$$u(r) = 2u_m[\ 1 - (r/R)^2]$$

Therefore $(du / dr) = 2u_m\ [\ 0 - 2r / R^2]$

Or $(du/dr)|_{wall} = (du/dr)|_{r=R} = -4u_m/R = -8u_m/D$

Therefore $(dp/dz) = -(S/A)\eta_w = -[(\pi D)/(\pi D^2/4)]\{-\mu(du/dr)|_{wall}\}$

$$= (4\mu/D)(-8u_m/D) = -32\,\mu u_m/D^2$$

$$\text{Friction factor} = f = \frac{-(dp/dz)\,D}{\tfrac{1}{2}(\rho u_m^2)} = \frac{32\,\mu u_m/D}{\tfrac{1}{2}(\rho u_m^2)} = \frac{64}{(\rho u_m D/\mu)}$$

Or $f = 64/Re_d$

Example 5.6:- The velocity profile u(y) for hydro dynamically developed laminar flow between two parallel plates a distance 2L apart is given by $u(y)/u_m = (3/2)[1-(y/L)^2]$ where u_m is the mean flow velocity and the coordinate axis y is as shown in Fig. P5.6.

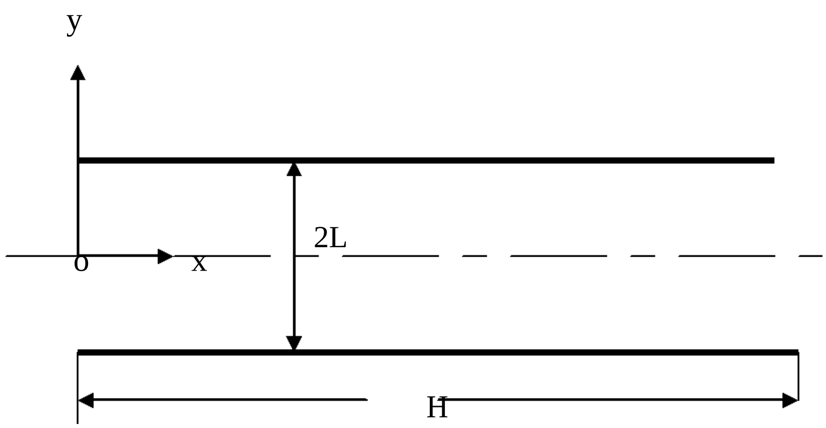


Fig. P 5.6 : Schematic for problem 5.6

(a) Develop an expression for the friction factor f.
(b) Write the expression for calculating the pressure drop Δp over a length H of the channel.

Solution:

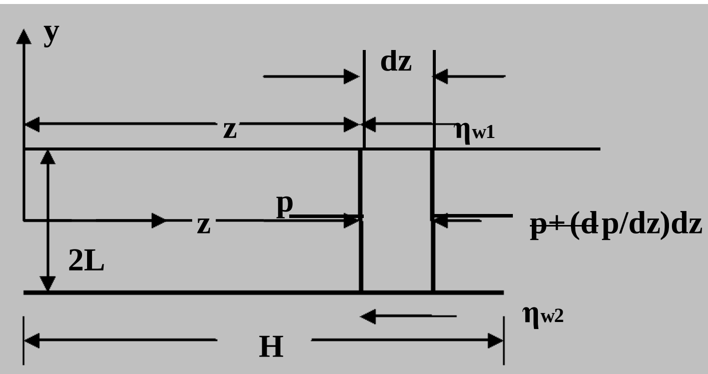


Fig. P5.6 : Schematic for problem 5.6

Consider a fluid element of length dz at a distance z from the origin as shown in the figure. Resultant force acting on the fluid element is given by

$$F = p\ (2L \times 1) - [p + (dp/dz)\ dz]\ (2L \times 1) - \eta_{w1}\ (dz \times 1) - \eta_{w2}\ (dz \times 1)$$

$$= -\ 2L\ (dp/dz)dz - (\eta_{w1} + \eta_{w2})\ dz$$

For fully developed flow there is no change in the momentum of the fluid in z-direction. Hence by Newton‟s second law F = 0.

Therefore we have $-\ 2L\ (dp/dz)dz - (\eta_{w1} + \eta_{w2})\ dz = 0$

Or $$(dp/dz) = -\ (\eta_{w1} + \eta_{w2}) / 2L \text{.....................................}(1).$$

It is given that,

$$u = (3/2)\ u_m\ [\ 1 - (y/L)^2\ ]$$

Therefore $$(du / dy) = -\ (3/2)(2y / L^2)$$

By Newton‟s law of viscosity $\eta_{w1} = -\ \mu(du/dy)|_{y=L} = -\ \mu\ [\ -\ 3u_m / L]$

$$= (3\mu u_m)/L$$

Similarly $$\eta_{w2} = +\ \mu(du/dy)|_{y=-L} = +\ \mu\ [\ +\ 3u_m / L]$$

$$= (3\mu u_m)/L$$

Substituting these expressions for η_{w1} and η_{w2} we have

$$(dp/dz) = -\,[(3\mu u_m)/L + (3\mu u_m)/L\,]\,/\,2L$$

$$= -\,[(3\mu u_m)/\,L^2]$$

(a) The friction factor f is given by

$$f = \frac{-\,(dp/dz)\,d_h}{(1/2)\,\rho u_m^2} = \frac{[(3\mu u_m)/\,L^2]\,d_h}{(1/2)\,\rho u_m^2}$$

as (dp/dz) = (1/2) ρ u_m^2 f (1/d_h), d_h = hyd. Diameter = $\frac{4 \times 2L}{2}$ = 4L

$$= \frac{12 \times 2}{(\rho u_m L)/\mu} = \frac{24}{Re_L}$$

b) The total pressure drop for length H of the plate is given by

$$\Delta p = p_1 - p_2 = -\int_{p1}^{p2} dp$$

$$= -\int_0^H -[(3\mu u_m)\,/\,L^2]\,dz = 3\,(H/L)(\mu u_m\,/\,L)$$

$$= \frac{3\,(H/L)\,(\rho u_m^2)}{(\rho u_m L/\mu)} = 3\,(H/L)\,(\rho u_m^2)\,/\,Re_L$$

Example 5.7:-The friction factor for hydro dynamically developed laminar flow through a circular tube is given by

$$f = 64\,/\,Re_d\;;\;Re_d = (u_m d)\,/\,v.$$

Water at a mean temperature of 60 ^{0}C and a mean velocity of 10 cm/s flows inside a tube of 1 cm ID. Calculate the pressure drop for a length of 10 m of the tube and also the corresponding pumping power required.

Solution:

Properties of water at 60 ^{0}C are : $\rho = 985\ kg/m^3$; $\mu = 0.78 \times 10^{-3}$ kg / (m – s);

Mean velocity of water = u_m = 0.1 m/s ; D_i = 0.01 m ; L = 10 m.

$$Re_d = (\rho u_m D_i)/\mu = \frac{985 \times 0.1 \times 0.01}{0.78 \times 10^{-3}} = 2060.7 \text{ or } 2061$$

Friction factor = f = $64/Re_d$ = 64 / 2061 = 0.031

$$\text{Pressure drop} = \Delta p = f\,(L/D_i)\,(1/2)\rho u_m^2 = \frac{0.031 \times 10 \times 985 \times (0.1)^2}{2 \times 0.01}$$

$$= 152.68 \text{ N / m}^2.$$

$$\text{Volume flow rate} = \dot{V} = (\pi D_i^2/4)\,u_m = \frac{\pi \times (0.01)^2 \times 0.1}{4} = 7.85 \times 10^{-6} \text{ m}^3/\text{s}$$

$$\text{Pumping power} = \Delta p\,\dot{V} = 152.68 \times 7.85 \times 10^{-6} = 1198.5 \times 10^{-6} \text{ J / s.}$$

Example 5.8:- Engine oil [ν = 0.8 x 10 $^{-4}$ m^2/s and k = 0.14 W /(m-K)] is in laminar flow between two parallel plates a distance 3 cm apart and subjected to a constant heat flux of 2500 W/m^2. The average heat transfer coefficient for the hydro dynamically and thermally developed flow is given by

$$(h_m\, 4L)/k = 8.235,$$

where 2L is the distance between the plates. Calculate the temperature difference between the plate surface and the mean fluid temperature.

Solution:

2L = 0.03 m ; ν = 0.8 x 10 $^{-4}$ m^2 / s ; k = 0.14 W /(m-K) ; q = 2500 W / m^2; $(h_m 4L)$ / k = 8.235

Therefore h_m = (8.235 k) / (2 x2L)

= (8.235 x 0.14)/(2 x 0.03) = 19.215 W / (m^2 – K) Temperature difference = ΔT = q / h_m = 2500 / 19.215 = 130.11 ^{0} C.

Example 5.9:-Consider a special case of parallel flow of an incompressible fluid between two parallel plates where one plate is stationary and the other plate is moving with an uniform velocity U.A distance L separates the two plates (refer Fig.

P5.9). The stationary plate is maintained at temperature T_o and the moving plate at temperature T_L. This type of flow is referred to as COUETTE flow and occurs, for example, in journal bearing. The continuity, momentum and energy equations for such a flow are given as follows:

Continuity equation: $\partial u / \partial x = 0$

Momentum equation: $\partial^2 u / \partial y^2 = 0$

Energy equation: $k\,(\partial^2 T / \partial y^2) + \mu\,(\partial u / \partial y)^2 = 0$

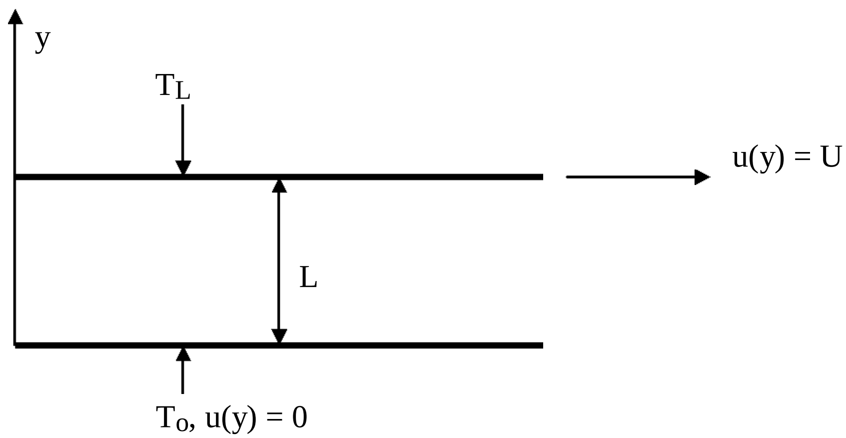


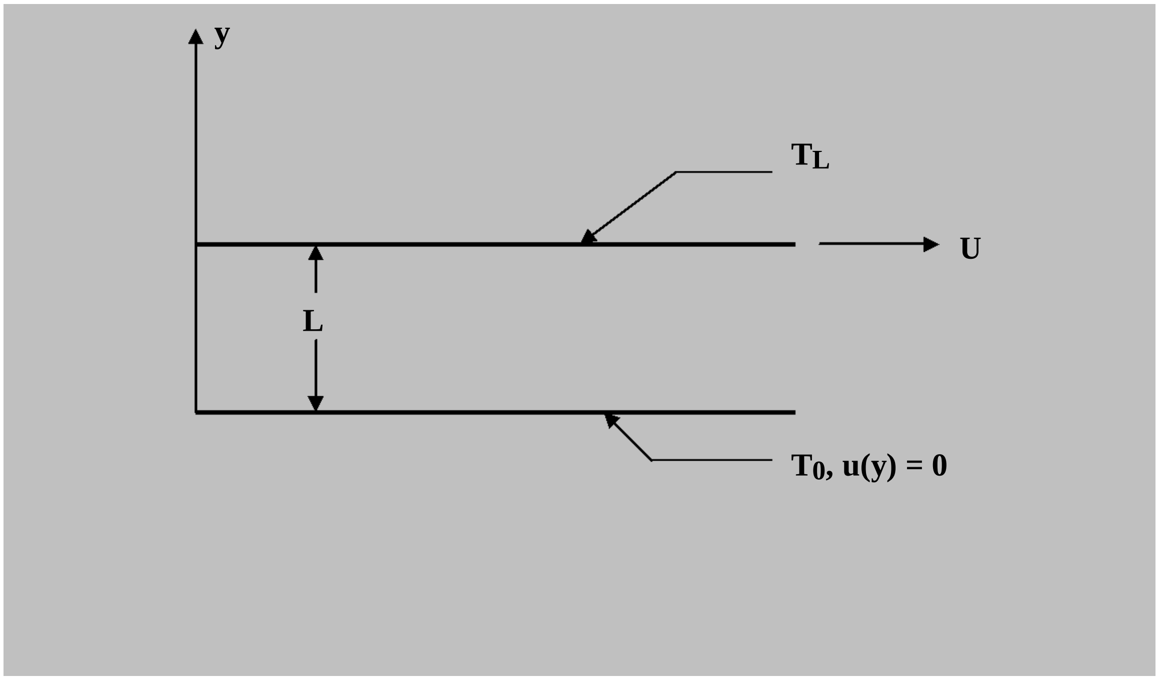
Fig. P5.9: Schematic for problem 5.9

Show that the temperature distribution is given by

$$T(y) = T_o + (\mu/2k)U^2\,[\,(y/L) - (y/L)^2] + (T_L - T_o)y/L$$

Also show that the heat flux at any y is given by

$$q(y) = -k\,[(\mu/2k)U^2\,(1/L - 2y/L^2) + (T_L - T_o)/L$$



It is given that the momentum equation is

$$(\partial^2 u / \partial y^2) = 0 \text{ ....} (1)$$

With the boundary conditions: (i) at y = 0, u(y) = 0

and (ii) at y = L, u(y) = U.

Integrating equation (1) twice, we have

$$u(y) = C_1 y + C_2 \text{ ....} (2).$$

Condition (i) in equation (2) gives $C_2 = 0$.

Condition (ii) in equation (2) gives $U = C_1 L$ or $C_1 = U / L$.

Substituting these values of C_1 and C_2 in equation (2) we get the velocity distribution as

$$u(y) = (Uy) / L \text{ ....} (3)$$

The energy equation is given by

$$k(\partial^2 T / \partial y^2) = -\mu\,(\partial u / \partial y)^2 \text{ ....} (4)$$

From equation (3) $(\partial u / \partial y) = U / L$ substituting this in equation (4)

We have $k(\partial^2 T / \partial y^2) = -\mu(U/L)^2$

Or $(\partial^2 T / \partial y^2) = -(\mu/k)(U/L)^2$

Integrating twice with respect to y we get

$$T(y) = -(\mu/k)(U/L)^2 (y^2/2) + C_1 y + C_2 \text{.........................}(5)$$

The boundary conditions are: (i) at $y = 0$ $T = T_0$ and (ii) at $y = L$, $T = T_L$.

Condition (i) in equation (5) gives $C_2 = T_0$.

Condition (ii) in equation (5) gives $T_L = -(1/2)(\mu U^2/k) + C_1 L + T_0$

Or $C_1 = (1/L)[(T_L - T_0) + (1/2)(\mu U^2/k)]$

Substituting the expressions for C_1 and C_2 in Equation (5) we get the temperature as

$$T(y) = -(\mu/k)(U/L)^2 (y^2/2) + (y/L)[(T_L - T_0) + (1/2)(\mu U^2/k)] + T_0$$

Or
$$\frac{T(y) - T_0}{T_L - T_0} = \frac{y}{L} + \frac{1}{(T_L - T_0)\,2k}\mu U^2 [(y/L) - (y/L)^2]$$

Heat flux at any y is given by

$$q(y) = -k(\partial T / \partial y)$$

$$= -k[(\mu U^2/2k)\{(1/L) - (2y/L^2) + (T_L - T_0)/L]$$

$$= k(\mu U^2/2)\{(2y/L^2) - (1/L)\} - k(T_L - T_0)/L$$

5.10. Consider Couette flow with heat transfer for which the lower plate moves with a velocity of U = 15 m/s and is perfectly insulated (see Fig. P5.10). The upper plate is stationary and is made of material with k_{up} = 1.5 W/(m-K) and thickness L_{up} = 3 mm. Its outer surface is maintained at T_{up} = 40 ^{0} C. The plates are separated by a distance of L_0 = 5 mm which is filled with an engine

oil of viscosity μ = 0.8 N-s / m^2 and thermal conductivity k_0 = 0.145 W /(m-K).

(a) On T(y) − y coordinates, sketch the temperature distribution in the oil film and in the moving plate.

(b) Obtain an expression for the temperature at the lower surface of the film T_0 in terms of the plate speed U, the stationary plate parameters T_{up},k_{up},L_{up} and the oil parameters μ, k_0, L_0. Calculate this temperature for the prescribed conditions.

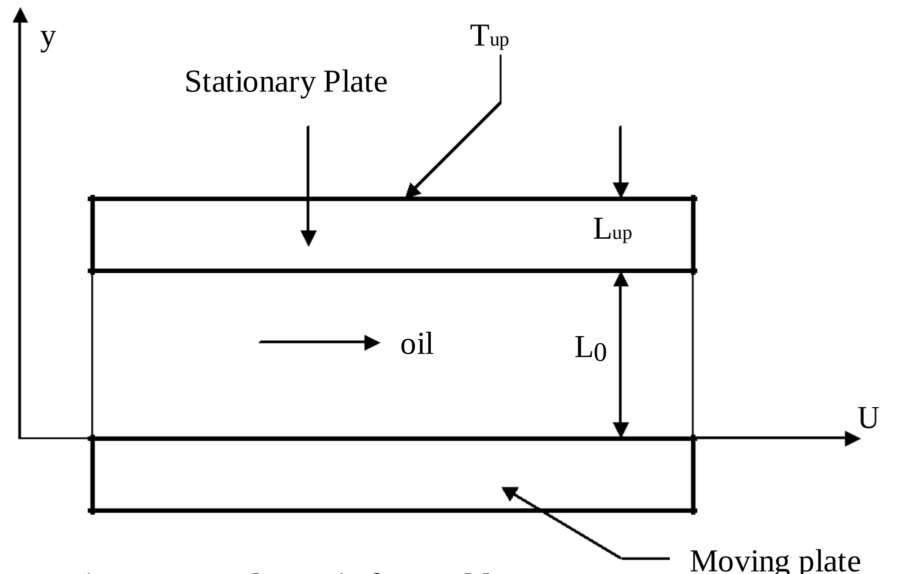


Fig. P5.10: Schematic for problem 5.10

Solution:

For Couette flow the momentum equation is :

$$d^2u/dy^2 = 0.$$

Integrating twice with respect to y we have

$$u(y) = C_1 y + C_2 \quad \text{.......(1)}$$

The boundary conditions are (i) at y = 0, u(y) = U ;

And (ii) at $y = L_0$, u(y) = 0.

Condition (i) in equation (1) gives $C_2 = U$ and condition (ii) in equation (1) gives $C_1 = -U/L_0$.

Substituting the expressions for C_1 and C_2 in equation (1) we get the velocity distribution as

$$u(y) = U\,[1 - (y/L_0)] \quad \text{.......(2)}$$

The energy equation for Couette flow is

$$k(d^2T / dy^2) = -\mu\, (du/dy)^2 \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

From equation (2) we have $(du/dy) = -\,(U / L_0)$.

Therefore equation (3) reduces to

$$k_0(d^2T / dy^2) = -\mu\, (U / L_0)^2$$

or

$$(d^2T / dy^2) = -\mu\, (U / L_0)^2 / k_0$$

Integrating once with respect to y we have

$$dT / dy = -(\mu/k_0)\, (U / L_0)^2\, y + C_1 \ldots\ldots\ldots\ldots(4)$$

Integrating once again with respect to y we have

$$T = -(\mu/k_0)\, (U / L_0)^2\, (y^2/2) + C_1 y + C_2 \ldots\ldots\ldots\ldots(5)$$

At y = 0 the surface is insulated i.e. $(dT/dy) = 0$. Substituting this condition in equation (4) we have

$$C_1 = 0.$$

At $y = L_0$, the condition is

$$-\,k_0\, (dT/dy)|_{y=L0} = k_{up}[\, T|_{y = L0} - T_{up}] / L_{up} \ldots\ldots\ldots\ldots(6)$$

From equation (4) we have $(dT / dy)|_{y = L0} = -(\mu/k_0)\, (U / L_0)^2\, L_0 = -\,\mu U^2/(L_0 k_0)$.

From equation (5) we have $T|_{y = L0} = -(\mu/k_0)(U^2/2) + C_2$.

Therefore equation (6) reduces to $\mu U^2/L_0 = k_{up}\, [-(\mu/k_0)(U^2/2) + C_2 - T_{up}] / L_{up}$

Or $$C_2 = \mu\, U^2 / k_0\, [\, (½) + (\, k_0\, L_{up}/ k_{up}\, L_0)\,] + T_{up}$$

Therefore $$T|_{y = L0} = T_{up} + \{\, \mu\, U^2 / k_0\, [\, (½) + (\, k_0\, L_{up}/ k_{up}\, L_0)\,]\, \} - (\mu/k_0)(U^2/2)$$

$$= T_{up} + (\, k_0\, L_{up}/ k_{up}\, L_0)$$

Or Temperature distribution is given by :

$$T = -(\mu/k_0)\, (\, U/L_0)^2\, (y^2/2) + \{\, \mu\, U^2 / k_0\, [\, (½) + (\, k_0\, L_{up}/ k_{up}\, L_0)\,]\, \} + T_{up}$$

At lower surface y = 0

Therefore $$T|_{y=0} = T_{up} + \{\, \mu\, U^2 / k_0\, [\, (½) + (\, k_0\, L_{up}/ k_{up}\, L_0)\,]\, \}$$

Forced Convection Heat Transfer

A. Hydro dynamically and thermally developed flow through tubes:

Determine the friction factor, the pressure drop and pumping power for fully developed laminar flow of water at 21 ^{0} C [μ = 9.8 kg/(m-s) ; ρ = 997.4 kg/m^3]through a 2.5 cm diameter, 100 m long tube for a mass flow rate of
0.015 kg/s. What are the mean and maximum velocities of flow?

Determine the friction factor, the pressure drop and pumping power required for the flow of water at 0.5 kg/s and 40 ^{0} C through a tube of square cross section of 2 cm x 2 cm and 12 m long. What would be the corresponding values if the pipe is of equilateral-triangular cross section of side 2 cm and length 5 m ?

Water at 30 ^{0} C with a mass flow rate of 2 kg/s enters a 2.5 cm-ID tube whose wall is maintained at a uniform temperature of 90 ^{0} C. Calculate the length of the tube required to heat the water to 70 ^{0} C.

Water at 20 ^{0} C with a mass flow rate of 5 kg/s enters a circular tube of 5 cm- ID and 10 m long. If the tube surface is maintained at 80 ^{0} C, determine the exit temperature of water.

Air at 27 ^{0} C with a flow rate of 0.01 kg/s enters a rectangular tube 0.6 cm x 1.0 cm in cross section and 2 m long. The duct wall is subjected to a uniform heat flux of 5 kW/m^2. Determine the outlet temperature of air and the duct surface temperature at the exit assuming that the flow is hydro dynamically and thermally developed.

Three kg/min of liquid sodium is heated from a bulk mean temperature of 400 ^{0} C to 500 ^{0} C, as it flows through a stainless steel tube of 5 cm-ID and 2 mm thick. The sodium is heated by a constant wall-heat flux, which maintains the tube-wall temperature at 30 ^{0} C above the bulk mean temperature of sodium all along the length of the tube. Calculate the length of the tube required. Assume the following properties for liquid sodium.

ρ = 846.7 kg/m^3 ; k = 68.34 W/(m-K) ; C_p = 1.274 kJ/(kg-K) ; Pr = 0.00468 ;

ν =0.2937 x 10 $^{-6}$ m^2 /s.

Consider hydro dynamically and thermally developed turbulent flow of water with a mass flow rate of M kg/s inside a circular tube of inside diameter 'D'. The Dittus-Boelter equation can be used to determine the heat transfer coefficient. If the tube's inside diameter is changed from D to D/2 while the mass flow remains same, determine the resulting change in the heat transfer coefficient.

Mercury at a temperature of 100 ^{0} C and with a velocity of 1 m/s enters a1.25 cm ID tube,

which is maintained at a uniform temperature of 250 ^{0} C. Determine the length of the tube required.

B. Hydrodynamic and thermal entry lengths:

Determine the hydro dynamic entry lengths for flow at 60 ^{0} C and at a rate of 0.015 kg/s of water, ethylene glycol and engine oil through a circular tube of 2.5 cm ID.

Determine the hydro dynamic entry length, thermal entry length and the heat transfer coefficient for fully developed flow for engine oil at 60 ^{0} C flowing at a rate of 0.01 kg/s through a square duct 1 cm x 1cm cross section and subjected to a uniform wall-temperature. Assume the following physical properties for the engine oil:

ρ = 864 kg/m^3 ; C_p = 2047 J/(kg-K) ; k = 0.14 W/(m-K) ; μ =0.0725 kg/(m-s) ; Pr = 1050

C. Flow over a flat plate:

Atmospheric air at 25 ^{0} C flows over both the surfaces of a flat plate 1 m long with a velocity of 5 m/s. The plate is maintained at a uniform temperature of 75^{0}C.

(a) Determine the velocity boundary layer thickness, the surface shear stress and the heat flux at the trailing edge of the plate.

(b) Determine the drag force on the plate and the total heat transfer from the plate to air.

Air at 30 ^{0} C flows with a velocity of 10 m/s along a flat plate 4 m long. The plate is maintained at a uniform temperature of 130 ^{0} C. Assuming a critical Reynolds number of 2 x 10 ^{5} and width of plate to be 1 m determine (a) the heat flux at the trailing edge of the plate, (b) the heat transfer from the laminar portion of the plate, (c) the total heat transfer from the plate and (d) the heat transfer from the turbulent portion of the plate.

A highly conducting thin wall 2 m long separates the hot and cold air streams flowing on both sides parallel to the plate surface. The hot stream at 250 ^{0} C is flowing with a velocity of 50 m/s while the cold stream at 50 ^{0} C is flowing with a velocity of 15 m/s. Calculate (a) the average heat transfer coefficients for both the air streams and the heat transfer between the two streams per metre width of the plate and (b) the local heat flux at the mid point of the plate. Assume that the wall is at the arithmetic mean of the temperature of the two streams for the purposes of calculating the physical properties of the two streams and the critical Reynolds number to be 2 x 10 ^{5}.

A flat plate of width 1 metre is maintained at a uniform temperature of 150 ^{0} C by using independently controlled heat generating rectangular modules of thickness 10 mm and length 50 mm. Each module is insulated from its neighbours, as well as its back side.(see Fig. P 6.14). Atmospheric air at 25 ^{0} C flows over the plate at a velocity of 30 m/s. The thermo-physical properties of the module are : k = 5.2 W/(m-K) ; C_p = 320 kJ/(kg-K) ; ρ = 2300 kg / m^3.

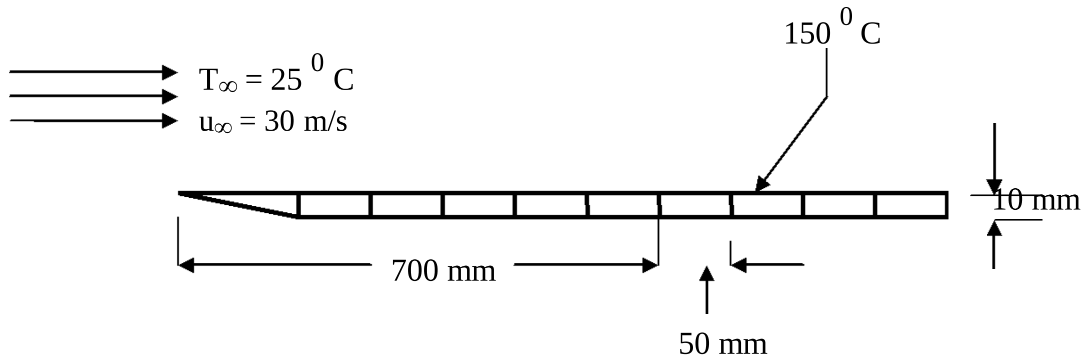


Fig. P 6.14 : Schematic for problem 6.14

(a) Find the required power generation in W/m^2 in a module positioned at a distance of 700 mm from the leading edge of the plate.
(b) Find the maximum temperature in the heat generating module.

D. Flow across a cylinder:

A circular pipe of 25 mm OD is placed in an air stream at 25 ^{0} C and 1 atm pressure. The air moves in cross flow over the pipe at 15 /s, while the outer surface of the pipe is maintained at 115 ^{0} C. What is the drag force exerted on the pipe per unit length of the pipe? What is the rate of heat transfer per unit length of the pipe?

A long cylindrical heating element [k = 240 W/(m-K), ρ = 2700 kg/m^3 and C_p = 900 kJ/(kg-K)] of diameter 10 mm is installed in a duct in which air moves in cross flow over the heating element at a temperature of 27 ^{0} C with a velocity of 10 m/s.

(c) Estimate the steady state surface temperature of the heater when electrical energy is being generated at a rate of 1000 W per metre length of the cylinder.
(d) If the heater is activated from an initial temperature of 27 ^{0} C, estimate the time required for the surface temperature to come to within 10 ^{0} C of its steady state value.

Air at 40 ^{0} C flows over a long 25 mm diameter cylinder with an embedded electrical heater. Measurements of the effect of the free stream velocity V on the power per unit length P, required to maintain the cylinder surface temperature at 300 ^{0} C yielded the following results:

V (m/s) :	1	2	4	8	12
P (W/m) :	450	658	983	1507	1963

(a) Determine the convection coefficient for each of the above test conditions. Display your results graphically.
(b) For the corresponding Reynolds number range, determine the suitable constants C and m for use with an empirical correlation of the form $Nu_m = C\ Re_d^m\ Pr^{1/3}$.

A thermocouple is inserted into a hot air duct to measure the air temperature. The thermocouple (T_1) is soldered to the tip of a steel *thermocouple well* of length 15 cm and inner and outer diameters of 5 mm and 10 mm respectively. A second thermocouple (T_2) is used to measure the duct wall temperature (see Fig. P 6.18).

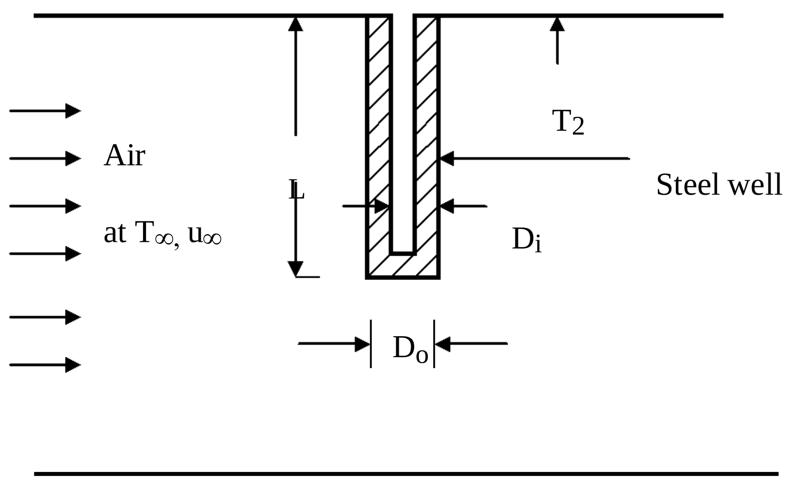


Fig. P 6.18 : Schematic for problem 6.18

Consider the conditions for which the air velocity in the duct u_∞ = 3 m/s and the two thermocouples register temperatures of T_1 = 450 K and T_2 = 375 K.

Neglecting radiation determine the air temperature T_∞. Assume that for steel k = 35 W/(m-K), and for air ρ = 0.774 kg / m^3, μ = 251 x 10^{-7} N-s / m^2, k = 0.0373 W/(m-K), and Pr = 0.686

E. Flow across tube bundles:

Air at atmospheric pressure and 30 ^{0} C flows over a bank of tubes consisting 1 cm OD tubes, 10 rows deep. The velocity of air before it enters the bundle is 1 m/s.

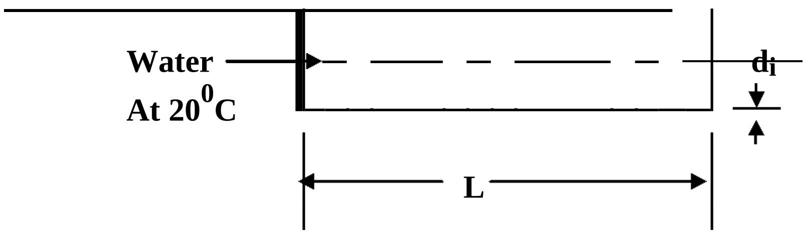
(a) Determine the friction factor and the pressure drop and (b) the average heat transfer coefficient for the following cases:

(i) Tubes are in equilateral-triangular arrangement with S_T / D = S_D / D = 1.25 (ii) Tubes are in square arrangement with S_T / D = S_L / D = 1.25

6.21. Hot flue gases at 375 ^{0} C flow across a tube bank consisting of 1.25 cm OD tubes, which are maintained at a uniform surface temperature of 30 ^{0} C by flowing water through the tubes. The tube bundle is 10 rows deep in the direction of flow and contains 40 tubes in each row. The tubes are 1 m long and have an *in-line* arrangement with S_L / D = S_T / D = 2. he velocity of the flow gases entering the tube matrix is 7 m/s. Determine the average heat transfer coefficient and the total heat transfer rate. Assume that thermo- physical properties of the flow gases to be same as that of air at any temperature.

A. Hydro-dynamically and Thermally developed flow through ducts

6.1. Solution:-



Mass flow rate = m = 0.015 kg / s ; d_i = 0.025 m ; L = 100 m ;

Properties of water at 20 ^{0}C are: ρ = 1000 kg / m^3 ; ν = 1.006 x 10^{-6} m^2 / s;

Reynolds Number = $Re_d = u_{av}\, d_h / \nu$, where u_{av} = average velocity of the fluid in the pipe and d_h = hydraulic diameter for the pipe.

Now $m = \rho(\pi d_i^2/4)u_{av}$.

Or $$u_{av} = (4m) / \rho(\pi d_i^2) = \frac{4 \times 0.015}{1000 \times \pi \times 0.025^2} = 0.0305 \text{ m / s}$$

$d_h = d_i$ for a circular pipe.

Therefore $$Re_d = \frac{0.0305 \times 0.025}{1.006 \times 10^{-6}} = 757.95$$

Since Re_d < 2300, flow is laminar. For hydro-dynamically developed laminar flow we have friction factor as

$$f = 64 / Re_d = 64 / 745.5 = 0.084.$$

Pressure drop for a total length L is given by $\Delta p = (1/2) f (L/d_h) \rho u_{av}^2$

$$= ½ \times 0.086 \times (100 / 0.025) \times 1000.52 \times 0.0305^2 = 156.28 \text{ N/m}^2.$$

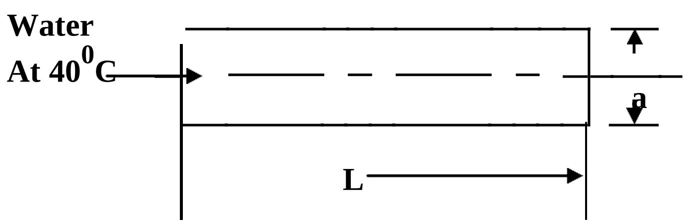
Pumping power = $\Delta p (m/\rho) = 156.28 \times 0.015 / 1000 = 2.34 \times 10^{-3}$ W

The velocity of the fluid at any radius for fully developed laminar flow through a circular tube is given by

$$u(r) = 2u_{av} [1 – (r/R)^2]$$

Therefore $u_{max} = u(r)_{r=0} = 2 u_{av} = 2 \times 0.0305 = 0.061$ m/s

6.2. Solution



Mass flow rate = m = 0.5 kg / s ; a = b = 0.02 m ; L = 12 m ;

Properties of water at 40 ^{0}C are: $\rho = 994.59$ kg / m^3 ; $\nu = 0.658 \times 10^{-6}$ m^2 / s;

$$u_{av} = m / (\rho ab) = \frac{0.5}{994.59 \times 0.02^2} = 1.26 \text{ m / s.}$$

Hydraulic diameter = $d_h = 4ab / 2(a + b) = 2ab / (a + b) = 2a^2/2a = a = 0.02$ m.

$$\text{Reynolds number } Re_d = u_{av} d_h / \nu = \frac{1.26 \times 0.02}{0.658 \times 10^{-6}} = 38299$$

Since $Re_d > 2300$ flow is turbulent.

For fully developed turbulent flow through a pipe of square cross section the friction factor f is given by (Moody chart, smooth pipe)

$$f = 0.02175.$$

Pressure drop = $\Delta p = (1/2) f (L/d_h) \rho u_{av}^2 = 0.5 \times 0.02175 \times (12 / 0.02) \times 994.59 \times$

$$1.26^2 = 10303 \text{ N / m}^2$$

Pumping power = Δp x m / ρ = 10303 x 0.5 / 994.59 = 5.18 W

For a tube of equilateral triangular cross section, $d_h = 4\,\{\sqrt{3} \times a^2/4\} / 3a$, where a is the side of the triangle.

Hence $d_h = a/\sqrt{3} = 0.02/\sqrt{3} = 0.0115$ m

$$\text{Average velocity} = u_{av} = \frac{0.5}{994.59 \times (\sqrt{3}/4) \times 0.02^2} = 2.9 \text{ m/s}$$

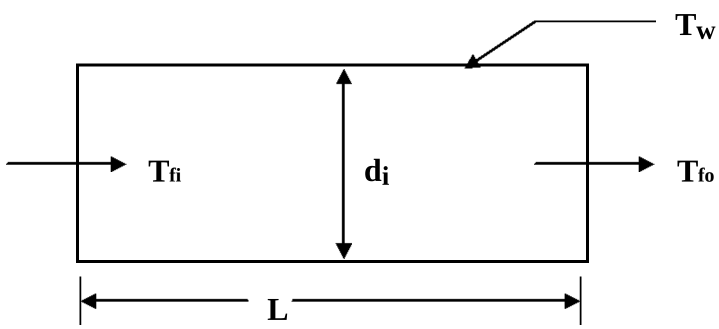
$$\text{Reynolds number} = Re_d = \frac{2.9 \times 0.0115}{0.658 \times 10^{-6}} = 50684$$

Since $Re_d > 2300$, flow is turbulent. Hence from Moody chart we

have f = 0.02

Pressure drop = $\Delta p = 0.5 \times 994.59 \times 2.9^2 \times 0.02 \times (5 / 0.0115) = 36367.4$ N / m^2 Pumping power = 36367.4 x 0.5 / 994.59 = 18.28 W.

6.3.
Solution:



Data :- $T_{fi} = 30\,^0C$; $T_{fo} = 70\,^0C$; $T_w = 90\,^0C$; m = 2 kg / s ; d_i = 2.5 cm = 0.025 m.

To find L , assuming flow is hydrodynamically and thermally developed.

For pipe of circular cross section hydraulic diameter = $d_h = d_i = 0.025$.

Bulk mean temperature of water = $T_m = \frac{1}{2}(T_{fi} + T_{fo}) = 0.5 \times (30 + 70) = 50\,^0$ C.

Properties of water at 50 ^{0}C are : ρ = 990 kg/m^3 ; c_p = 4181 J/kg-K

; k =0.644 W / (m-K) ; μ = 0.547 x 10^{-3} kg / (m-s) ; Pr = 3.55

Since nothing has been specified in the problem regarding the type of flow, it is assumed that the flow is hydro dynamically and thermally developed.

$$\text{Average velocity} = u_{av} = m / (\rho x\, \pi d_i^2/4) = \frac{4 \times 2}{990 \times \pi \times 0.025^2} = 4.11 \text{ m/s.}$$

$$\text{Reynolds number} = Re_d = \rho\, u_{av}\, d_h / \mu = \frac{990 \times 4.11 \times 0.025}{0.547 \times 10^{-3}} = 1.86 \times 10^5$$

Since Re_d >2300, flow is turbulent. For fully developed turbulent flow the Nusselt number is given by

$$Nu_d = 0.023\, Re_d^{0.8}\, Pr^n \text{ with } n = 0.4 \text{ for } T_w > T_f.$$

Therefore $$Nu_d = 0.023 \times (1.86 \times 10^5)^{0.8} \times (3.55)^{0.4} = 628$$

Hence the heat transfer coefficient, h = Nu_d k / d_h

$$= 628 \times 0.644 / 0.025 = 16177 \text{ W/(m}^2\text{-K)}$$

To find the length of the tube L, we write the energy balance equation for the entire length of the tube as

Heat supplied to fluid from the tube wall = Increase of energy of the fluid

Therefore $$h\, (\pi d_i L)\, \Delta T_m = mc_p(T_{fo} - T_{fi})$$

$$L = mc_p(T_{fo} - T_{fi}) / h\pi d_i\, \Delta T_m \quad \text{.................................... (1)}$$

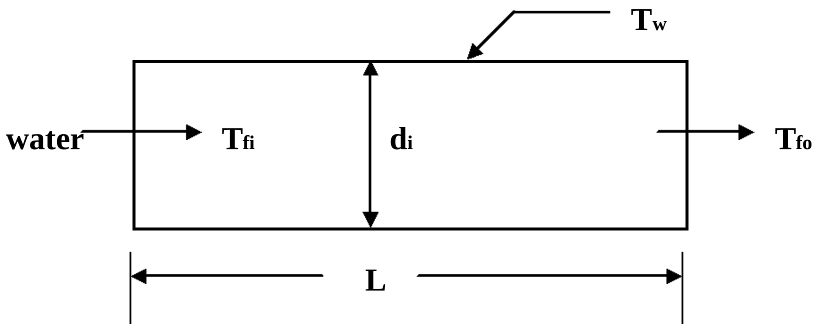
Where ΔT_m = logarithmic mean temperature difference = $[\Delta T_i - \Delta T_o] / \ln(\Delta T_i / \Delta T_o)$

$\Delta T_i = T_w - T_{fi}$ = 90 – 30 = 60 ^{0}C and $\Delta T_o = T_w - T_{fo}$ = 90 – 70 = 20 ^{0}C.

Therefore ΔT_m = [60 – 20] / ln(60 / 20) = 36.41 ^{0}C.

$$\text{Hence} \qquad L = \frac{[2 \times 4181 \times (70 - 30)]}{16171 \times \pi \times 0.025 \times 36.41} = 7.23 \text{ m.}$$

6.4. Solution:



Data :- $T_{fi} = 20\ ^0C$; m = 5 kg / s ; $d_i = 0.05$ m ; L = 10 m ; $T_w = 80\ ^0C$.

To find T_{fo}.

Since T_{fo} is not known we cannot determine the bulk fluid mean temperature to know the properties of the fluid. Hence this problem has to be solved by trial and error method as shown below.

Trial No. 1:- Assume suitable value for T_{fo} noting that $T_{fo} < T_w$.

Let $T_{fo} = T_w = 60\ ^0C$. Hence bulk mean temperature $= T_m = ½(T_{fi} + T_{fo})$

$$= 0.5 \times (20 + 60) = 40\ ^0C.$$

Properties of water at $40\ ^0C$ are : $\rho = 994.59$ kg/m^3 ; $c_p = 4178.4$ J/kg-K ; Pr = 4.34 ;

$\nu = 0.658 \times 10^{-6}$ m^2 / s ; k = 0.628 W /(m-K).

$$\text{Average velocity of water} = u_{av} = 4m /(\pi d_i^2 \rho) = \frac{4 \times 5}{\pi \times (0.05)^2 \times 994.59}$$

$$= 2.56 \text{ m/s.}$$

For a circular tube $d_h = d_i = 0.05$ m.

$$\text{Reynolds number} = Re_d = u_{av} d_h / \nu = \frac{2.56 \times 0.05}{0.658 \times 10^{-6}} = 1.945 \times 10^{5}$$

Since $Re_d > 2300$, flow is turbulent.

Assuming the flow to be thermally and hydrodynamically developed,

$$Nu_d = 0.023\, Re_d^{0.8}\, Pr^{n} \text{ with } n = 0.4 \text{ (as the fluid is heated)}$$

$$= 0.023 \times (1.945 \times 10^{5})^{0.8} (4.34)^{0.4}$$

$$= 704.5$$

$$\text{Heat transfer coefficient} = h = Nu_d\, k / d_h = \frac{704.5 \times 0.628}{0.05} = 8848.5 \text{ W}/(\text{m}^2 - \text{K}).$$

Heat balance equation for the total length of the tube can be written as

$$h\, \pi d_i L\, \Delta T_m = mc_p[T_{fo} - T_{fi}]$$

or

$$\frac{h\, \pi d_i L\, [\Delta T_i - \Delta T_o]}{\ln[\Delta T_i/\Delta T_o]} = mc_p\, [\Delta T_i - \Delta T_o]$$

or

$$\Delta T_o = \Delta T_i / \exp\{(h\, \pi d_i L)/(mc_p)\}$$

$$= \frac{[80 - 20]}{\exp\{(8848.5 \times \pi \times 0.05 \times 10) / (5 \times 4178.4)\}}$$

$$= 30.85\ ^0\text{C}.$$

Therefore $T_{fo} = 80 - 30.85 = 49.15\ ^0\text{C}$.

Trial 2:- Assume $T_{fo} = 49\ ^0$C. Therefore $T_m = (49+20)/2 = 34.5\ ^0$C.

Properties of water at $34.5\ ^0$C are : $\rho = 996.22$ kg/m^3; $c_p = 4179.3$ J/kg-K; Pr = 5.077; k = 0.6195 W/(m-K); $\nu = 0.7537 \times 10^{-6}$ m^2/s.

$$u_{av} = \frac{4 \times 5}{\pi \times 0.05^{2} \times 996.22} = 2.556 \text{ m/s}; \quad Re_d = \frac{2.556 \times 0.05}{0.7537 \times 10^{-6}} = 1.696 \times 10^{5}$$

Therefore

$$Nu_d = 0.023 \times (1.696 \times 10^{5})^{0.8} \times (5.077)^{0.4}$$

$$= 672.2$$

Hence

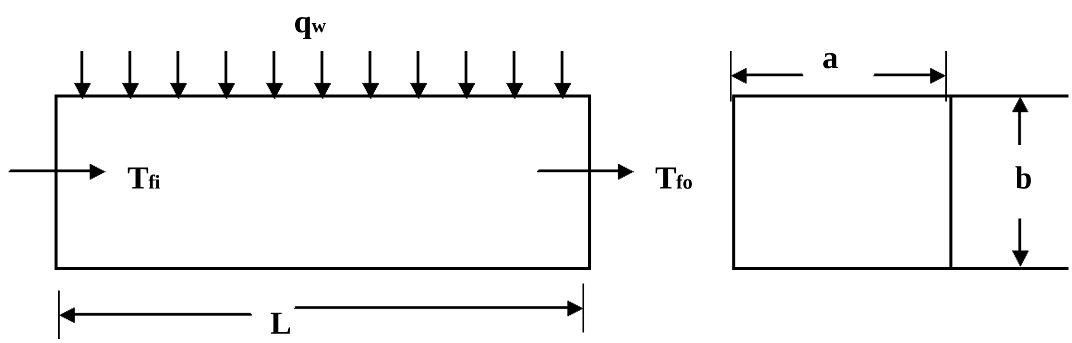
$$h = 672.2 \times 0.6195 / 0.05 = 8328.6 \text{ W}/(\text{m}^2 - \text{K})$$

Therefore $$T_{fo} = 80 - \frac{[80 - 20]}{\exp\{(8328.6 \times \pi \times 0.05 \times 10) / (5 \times 4179.3)\}}$$

$= 48\ ^0C$ which is very close to the assumed value of T_{fo}. Hence the iteration can be stopped.

Therefore $T_{fo} = (49 + 48) / 2 = 48.5\ ^0C$.

Solution:



Data :- Fluid is air ; $T_{fi} = 27\ ^0C$; m = 0.01 kg/s; a = 0.01 m; b = 0.006 m; L = 2 m; $q_w = 5000\ W / m^2$.

find (i) T_{fo} ; (ii) $T_w|_{z = L}$

Energy balance equation for total length of the tube can be written as

$$mc_p(T_{fo} - T_{fi}) = q_w\, 2(a+b)L$$

Or $$T_{fo} = T_{fi} + [q_w\, 2(a+b)L] / (mc_p)$$

Since T_{fo} is not known the property c_p is read at T_{fi}.

Therefore c_p = 1005.7 J/kg-K.

Therefore $$T_{fo} = 27 + \frac{2 \times (0.01 + 0.006) \times 2 \times 5000}{0.01 \times 1005.7}$$

$$= 58.8\ ^0C.$$

Therefore $T_m = ½(T_{fi} + T_{fo}) = 0.5 \times (27 + 58.8) = 42.9\ ^0C$.

Properties of air at 42.9 ^{0}C are : ρ = 1.12 kg / m^3; 1006.8 J/kg-K; ν = 17.30 x 10^{-6} m^2/s;

Pr = 0.7045 ; k = 0.02745 W/(m-K).(It should be noted that the variation of c_p with temperature between T_{fi} and T_m is very negligible and hence this problem does not require trial and error solution)

$$\text{Hydraulic diameter} = d_h = \frac{4ab}{2[a+b]} = \frac{2ab}{[a+b]} = \frac{2 \times 0.01 \times 0.006}{[0.01 + 0.006]}$$

$$= 0.0075 \text{ m.}$$

$$\text{Average velocity} = u_{av} = m / [\rho ab] = \frac{0.01}{1.12 \times 0.01 \times 0.006} = 149 \text{ m/s.}$$

$$\text{Reynolds number} = Re_d = u_{av} d_h / \nu = \frac{149 \times 0.0075}{17.3 \times 10^{-6}} = 64595$$

Since Re_d >2300 flow is turbulent. Assuming that the flow is hydrodynamically and thermally developed we have

$Nu_d = 0.023\, Re_d^{\,0.8} Pr^{\,n}$, with n = 0.4 as air is being heated.

Therefore $Nu_d = 0.023 \times [64595]^{0.8} \times [0.7045]^{0.4}$

$= 140.9$

$$\text{Heat transfer coefficient} = h = Nu_d\, k / d_h = \frac{140.9 \times 0.02745}{0.0075} = 515.7 \text{ W/(m}^2 - \text{K).}$$

At the exit of the tube we have $q_w = h\, [T_w|_{z=l} - T_{fo}]$

Therefore $T_w|_{z=l} = q_w / h + T_{fo} = 5000 / 515.7 + 58.8$

$= 68.5\ ^0$C

Solution:

Data: Fluid is liquid sodium; m = 3/60 =0.05 kg/s; T_{fi} = 400 ^{0}C; T_{fo} = 500 ^{0}C;d_i = 0.05 m $\Delta T_i = \Delta T_m = \Delta T_o$= 30 ^{0}C; ρ = 846.7 kg/m^3; k = 68.34 W/(m-K); Pr = 0.00468 c_p = 1274 J/kg-K; ν = 0.2937 x 10^{-6} m^2/s.

$$\text{Average velocity} = u_{av} = 4m / (\rho\pi d_i^2) = \frac{4 \times 0.05}{846.7 \times \pi \times 0.05^2} = 0.03 \text{ m/s.}$$

$$\text{Reynolds number} = Re_d = u_{av}d_h / \nu = \frac{0.03 \times 0.05}{0.2937 \times 10^{-6}} = 5107$$

Since $Re_d > 2300$, flow is turbulent.Assuming the flow to be hydrodynamically and thermally developed and since $Pr << 1$ (Liquid metal), the Nusselt number for constant wall heat flux condition is given by

$$Nu_d = 4.82 + 0.0185 (Re_d\, Pr)^{0.827}$$

$$= 4.82 + 0.0185 \times [5107 \times 0.00468]^{0.827}$$

$$= 5.075$$

$$\text{Heat transfer coefficient} = h = Nu_d\, k / d_h = \frac{5.075 \times 68.34}{0.05} = 6936.5 \text{ W/(m}^2 - \text{K).}$$

Energy balance equation for the total length of the tube can be written as

$$h (\pi d_i L)\Delta T_m = mc_p(T_{fo} - T_{fi})$$

or $$L = \frac{mc_p(T_{fo} - T_{fi})}{h (\pi d_i)\Delta T_m} = \frac{0.05 \times 1274 \times (500 - 400)}{6936.5 \times \pi \times 0.05 \times 30}$$

$$= 0.195 \text{ m.}$$

Solution:

The Dittus-Boetler correlation for hydrodynamically and thermally developed flow is given by

$$Nu_d = hd_h/k = 0.023\, Re_d^{0.8}\, Pr^{n} \qquad (1)$$

$$\text{For a circular tube of diameter D, } Re_d = u_{av}D / \nu = \frac{4M}{\rho\pi D\nu}$$

Hence Eq.(1) can be written as $$\frac{h_1 D}{k} = 0.023\, [4M/(\rho\pi D\nu)]^{0.8}\, Pr^{n}$$

Or $$h_1 = 0.023k\,[4M/(\rho\pi\nu)]^{0.8}\,Pr^{n} D^{-1.8} \qquad (2)$$

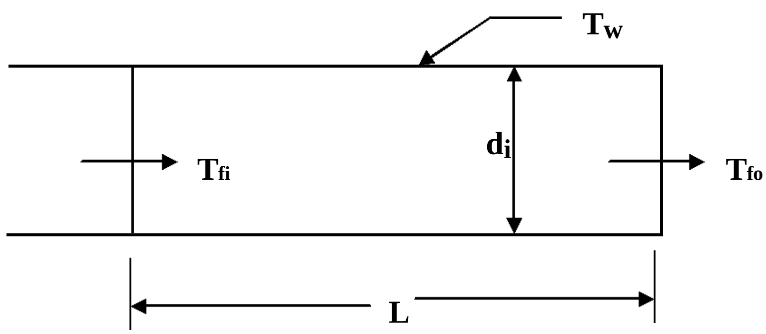
Similarly when the diameter of the tube is reduced to D/2, for the same mass flow rate the heat transfer coefficient is given by

$$h_2 = 0.023k\,[4M/(\rho\pi\nu)]^{0.8}\,Pr^{n}(D/2)^{-1.8} \qquad (3)$$

Dividing Eq.(3) by Eq.(2) we get

$$h_2/h_1 = 2^{1.8} = 3.5 \text{ or } h_2 = 3.5\,h_1.$$

6.8.
Solution:



Data:- Fluid is mercury(Liquid metal) ; $T_{fi} = 100^0C$; $T_{fo} = 200^0C$; $d_i = d_h = 0.0125m$; $u_{av} = 1$ m/s; $T_w = 250^0C$;

To find L

Bulk mean temperature of mercury = $T_m = ½(T_{fi} + T_{fo}) = 0.5 \times (100 + 200) = 150\ ^0C$. Properties of mercury at 150 ^0C are:- $\rho = 13230$ kg/m^3; $c_p = 137.3$ J/kg-K;

$\nu = 0.0865 \times 10^{-6}$ m^2 / s; k = 9.65 W/(m-K) ; Pr = 0.0162

$$\text{Reynolds number} = Re_d = u_{av} d_h / \nu = \frac{1.0 \times 0.0125}{0.0865 \times 10^{-6}} = 1.445 \times 10^5$$

Therefore Peclet number = $Pe = Re_d\,Pr = 1.445 \times 10^5 \times 0.0162 = 2341$

Since $Re_d > 2300$, flow is turbulent. Therefore for liquid metal flow subjected to uniform wall-temperature Nusselt number is given by

$$Nu_d = 5.0 + 0.025\ Pe^{0.8}$$

$$= 5.0 + 0.025 \times 2341^{0.8} = 17.4$$

Heat transfer coefficient, $h = Nu_d k/d_h = \dfrac{15.97 \times 11.425}{0.0125} = 13433\ W/(m^2\text{-}K)$

$\Delta T_i = T_w - T_{fi} = 250 - 100 = 150\ ^0C$; $\Delta T_o = T_w - T_{fo} = 250 - 200 = 50\ ^0C$.

Mean temperature difference = $\Delta T_m = [\Delta T_i - \Delta T_o] / \ln(\Delta T_i / \Delta T_o)$

$$= [150 - 50] / \ln(150 / 50) = 91\ ^0C$$

Mass flow rate of mercury = $m = \rho(\pi d_i^2/4)u_{av} = 13230 \times (\pi \times 0.0125^2/4) \times$

$1.0 = 1.624$ kg/s

Energy balance equation for the total length of the pipe is given by

$$h\ \pi d_i L\ \Delta T_m = mc_p(T_{fo} - T_{fi})$$

Therefore $L = \dfrac{mc_p(T_{fo} - T_{fi})}{h\ \pi d_i\ \Delta T_m} = \dfrac{1.624 \times 137.3 \times (200 - 100)}{13433 \times \pi \times 0.0125 \times 91}$

$= 0.465$

B Hydrodynamic and Thermal Entry Lengths

Solution:

Data:- $T_{fi} = 60\ ^0C$; $m = 0.015$ kg / s; $d_i = d_h = 0.025$ m.

(i) Fluid is water. Hence at $60\ ^0C$, $\rho = 985.46$ kg / m^3 ; $\nu = 0.478 \times 10^{-6}$ m^2 / s.

Average velocity = $u_{av} = 4m / \rho\pi d_i^2 = \dfrac{4 \times 0.015}{985.46 \times \pi \times 0.025^2} = 0.032$ m/s.

Reynolds number = $Re_d = u_{av} d_h / \nu = \dfrac{0.031 \times 0.025}{0.478 \times 10^{-6}} = 1674$

Since $Re_d < 2300$, flow is laminar. Hence the hydrodynamic entrance length L_h for a circular pipe is given by

$$\frac{L_h / d_h}{Re_d} = 0.056$$

Therefore $L_h = 0.056\ Re_d\ d_h = 0.056 \times 1674 \times 0.025$

$= 2.34$ m

(ii) Fluid is ethylene glycol: $\rho = 1087.66\ kg/m^3$; $\nu = 4.75 \times 10^{-6}\ m^2/s$.

$$\text{Average velocity} = u_{av} = 4m / \rho\pi d_i^2 = \frac{4 \times 0.015}{1087.66 \times \pi \times 0.025^2} = 0.0281 \text{ m/s.}$$

$$\text{Reynolds number} = Re_d = \frac{0.0281 \times 0.025}{4.75 \times 10^{-6}} = 147.9$$

Since $Re_d < 2300$, flow is laminar.

Therefore $L_h = 0.056\ Re_d\ d_h = 0.056 \times 147.9 \times 0.025$

$= 0.21$ m.

(iii) Fluid is engine oil; $\rho = 864.04\ kg/m^3$; $\nu = 0.839 \times 10^{-4}\ m^2/s$.

$$\text{Average velocity} = u_{av} = 4m / \rho\pi d_i^2 = \frac{4 \times 0.015}{864.04 \times \pi \times 0.025^2} = 0.0354 \text{ m/s.}$$

$$\text{Reynolds number} = Re_d = \frac{0.0354 \times 0.025}{0.839 \times 10^{-4}} = 10.55$$

Therefore $L_h = 0.056\ Re_d\ d_h = 0.056 \times 10.55 \times 0.025 = 0.013$ m.

Solution: Data: Fluid is engine oil ; $T_{fi} = 60\ ^0C$; $m = 0.01$ kg/s; square duct with $a = 0.01$m ; $\rho = 864\ kg/m^3$; $c_p = 2047$ J/kg-K; $k = 0.14$ W/(m-K); $\mu = 0.0725$ kg/(m-s); $Pr = 1050$.

To find (i) L_h ;(ii)L_t ; (iii) h for fully developed flow.

(i) Hydraulic diameter $= d_h = 4a^2/(4a) = a = 0.01$ m.

$$\text{Average velocity} = u_{av} = m / (\rho a^2) = \frac{0.01}{864 \times (0.01)^2} = 0.1157 \text{ m/s.}$$

$$\text{Reynolds number} = Re_d = (\rho u_{av} d_h) / \mu = \frac{864 \times 0.1157 \times 0.01}{0.0725} = 13.8$$

Since $Re_d < 2300$, flow is laminar. Hence

$$\frac{(L_h / d_h)}{Re_d} = 0.09 \text{ for a tube of square section.}$$

Therefore $L_h = 0.09\, Re_d\, d_h = 0.09 \times 13.8 \times 0.01 = 0.0124$ m.

(ii) For constant wall temperature condition we have

$$\frac{(L_t / d_h)}{Pe} = 0.041$$

Hence $L_t = 0.041\, Pe\, d_h = 0.041 \times (13.8 \times 1050) \times 0.01$

$= 5.94$ m

(iii)For fully developed laminar flow through a square tube Nusselt number is given by

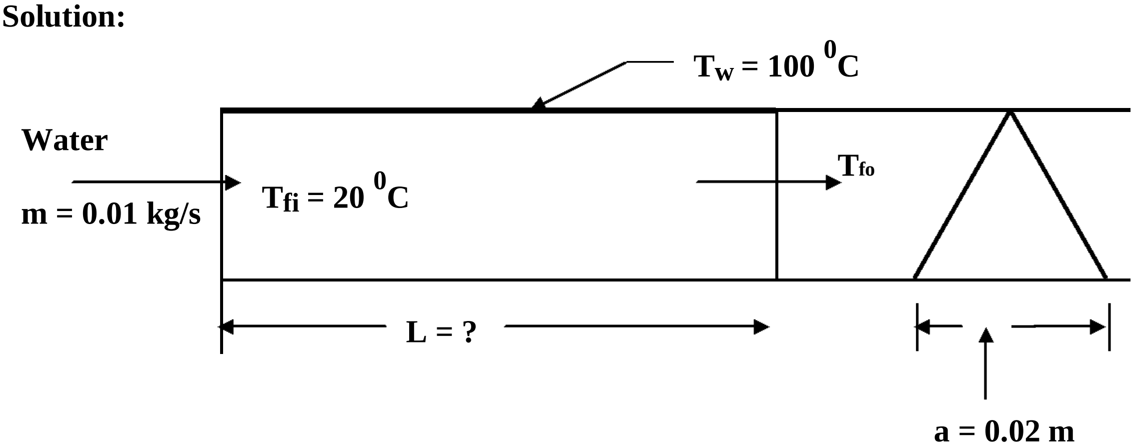
$$Nu_T = hd_h / k = 2.976$$

Hence $h = 2.976\, k / d_h = 2.976 \times 0.14 / 0.01$

$= 41.66$ W/(m^2-K)

Consider the flow of water at a rate of 0.01 kg/s through an equilateral triangular duct of sides 2 cm and whose walls are kept at a uniform temperature of 100 ^{0}C. Assuming the flow to be hydrodynamically and thermally developed, determine the length of the tube required to heat the water from 20^{0}C to 70 ^{0}C.

Solution:



Bulk mean temperature of water = ½(T_w + T_∞) = 0.5x (20 + 70) = 45 ^{0}C.

Properties of water at bulk mean temperature are: ρ = 992.3075 kg/m^3; Pr = 4.01
ν = 0.598 x 10^{-6} m^2/s; k = 0.63375 W/(m-K); c_p = 4179.9J/kg-K

For an equilateral triangular tube, area of flow = A = (√3 / 4)a^2 = (√3 / 4) x 0.02^2

$$= 1.732 \times 10^{-4} \text{ m}^2$$

$$\text{Hydraulic diameter} = d_h = \frac{4\,[(\sqrt{3}/4)a^2]}{3a} = \frac{a}{\sqrt{3}} = \frac{0.02}{\sqrt{3}} = 0.01155 \text{ m}$$

$$\text{Average velocity of water} = u_{av} = \frac{m}{\rho A} = \frac{0.01}{992.3075 \times 1.732 \times 10^{-4}}$$

$$= 0.0582 \text{ m/s}$$

$$\text{Reynolds number} = Re_d = u_{av} d_h / \nu = \frac{0.0582 \times 0.01155}{0.598 \times 10^{-6}} = 1124$$

Since Re_d <2300 flow is laminar.For thermally developed laminar flow with constant wall-temperature the Nusselt number is given by

$$Nu_d = h_{av}\, d_h / k = 2.47$$

Therefore

$$h_{av} = \frac{2.47 \times 0.63375}{0.01155} = 135.53 \text{ W/(m}^2\text{-K)}$$

Mean temperature difference between the surface and the bulk fluid is given by

$$\Delta T_m = [\Delta T_i - \Delta T_o] / \ln[\Delta T_i / \Delta T_o]$$

Now $\Delta T_i = T_w - T_{fi}$ = 100 – 20 = 80 ^{0}C; $\Delta T_o = T_w - T_{fo}$ = 100 – 70 = 30

Hence ΔT_m = [80 – 30] / ln{80/30} = 50.1 ^{0}C. Rate of heat transfer to water

= Q = $mc_p(T_{fo} - T_{fi})$ = 0.01 x 4179.9 x (70 – 20)

= 2090 W

But $Q = h_{av} P L \, \Delta T_m$, where P is the perimeter of the duct = 3a

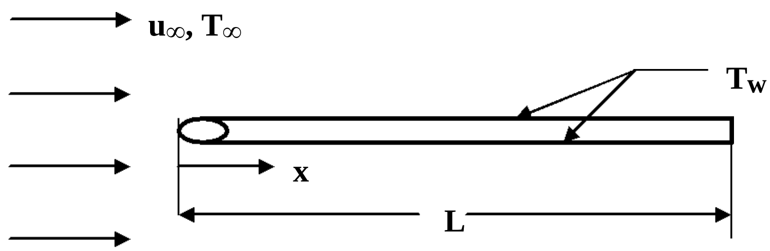
Hence $Q = h_{av} \, 3aL \, \Delta T_m$

Or
$$L = \frac{Q}{h_{av} \, 3a \, \Delta T_m} = \frac{2090}{135.53 \times 3 \times 0.02 \times 50.1}$$

$= 5.13$ m

C Flow over a flat plate:

Solution:



Data:- Fluid is air; $u_\infty = 5$ m/s; $T_\infty = 25\,^0$C; L = 1 m; $T_W = 75\,^0$C.

Mean temperature = $½(T_W + T_\infty) = 0.5 \times (75 + 25) = 50\,^0$C.

Properties of air at $50\,^0$C are : $\rho = 1.093$ kg/m^3; $\nu = 18.02 \times 10^{-6}$ m^2/s; Pr =0.703

k = 0.028 W/(m-K).

a) 1) Reynolds number at the trailing edge = $Re_L = (u_\infty L) / \nu = 5 \times 1 / (18.02 \times 10^{-6})$

$= 2.775 \times 10^5$

Assuming the critical Reynolds number to be 5×10^5, the flow is laminar at the trailing edge. Therefore from heat transfer data hand book we have

$$\delta(x)|_{x=L} = 5 L \, Re_L^{-0.5}$$

$= 5 \times 1 \times (2.775 \times 10^5)^{-0.5}$

$= 0.0088$ m.

2) Local drag coefficient at the trailing edge is given by

$$C_x|_{x=L} = 0.664\, Re_L^{-0.5} = 0.664 \times (2.775 \times 10^5)^{-0.5}$$

$$= 1.26 \times 10^{-3}$$

Therefore $\eta_w(x)|_{x=L} = ½(\rho u_\infty^2)\, C_x|_{x=L} = ½ \times 1.095 \times 5^2 \times 1.26 \times 10^{-3}$

Or $\eta_w(x)|_{x=L} = 0.0173$ N/m ^{2}.

3) Local Nusselt number at the trailing edge is given by

$$Nu_x|_{x=L} = 0.332\, Re_L^{0.5}\, Pr^{0.333}$$

$$= 0.332 \times (2.775 \times 10^5)^{0.5} \times (0.703)^{0.333}$$

$$= 155.5$$

Hence local heat transfer coefficient at the trailing edge is given by

$$h_x|_{x=L} = (Nu_x|_{x=L}\, k) / L = 155.5 \times 0.028 / 1$$

$$= 4.354 \text{ W/(m}^2\text{-K)}$$

Heat flux at the trailing edge = $q_w|_{x=L} = [h_x|_{x=L}]\,(T_w - T_\infty)$

$$= 4.354 \times (75 - 25) = 217.7 \text{ W/(m}^2 - \text{K)}$$

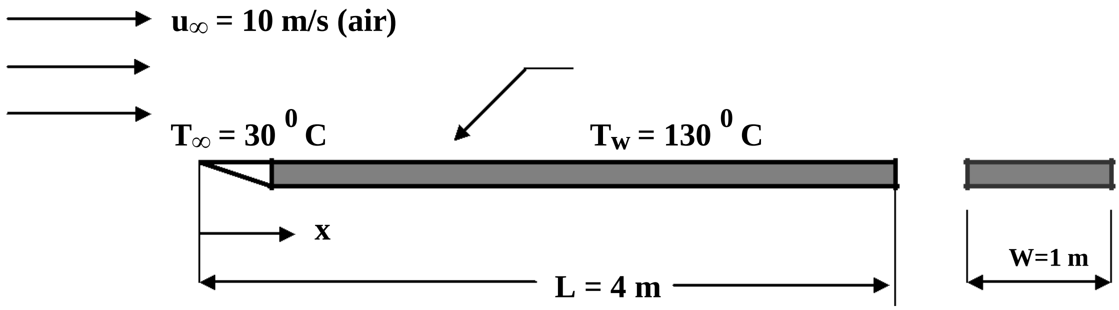
(4) Drag force: $F_d = \eta_{wavg} .A = 2 \times \eta_{w(x)|x=L} \times 1 = 2 \times 0.0173 = 0.0346$ N

Therefore average heat transfer coefficient is $h_{avg} = 2\, h_x|_{x=L} = 4.354 \times 2$

$$= 8.708 \text{ W/(m}^2 - \text{K) and}$$

Average heat flus = $q_w = [h_{avg}]\,(T_w - T_\infty) = 8.708 \times 50 = 435.4$ W/(m ^{2} – K)

Solution:-



To find :- (a) $q_w(x)|_{x=L}$; (b) $Q_{laminar}$; (c) $Q_{turbulent}$ assuming $Re_{cr} = 2 \times 10^5$ Mean film temperature = ½($T_w + T_\infty$) = 0.5 x (130 + 30) = 80 ^{0}C.

Properties of air at mean film temperature are: ν =21.48 x 10^{-6} m^2/s; Pr = 0.692 k = 0.03047 W /(m-K).

$$Re_L = u_\infty L / \nu = \frac{10 \times 4}{21.48 \times 10^{-6}} = 1.86 \times 10^{-6}$$

Since $Re_L > Re_{cr}$ flow is turbulent at the trailing edge.

For turbulent flow of air over a flat plate the local Nusselt number is given by

$$Nu_x = 0.0296\, Re_x^{0.8}\, Pr^{1/3}.$$

Hence $\quad Nu_x|_{x=L} = 0.0296 \times [1.86 \times 10^6]^{0.8} (0.692)_{1/3}$

$= 2721$

Therefore $[h_x|_{x=L}]\, L / k = 2803$ or $\quad h_x|_{x=L} = \dfrac{2721 \times 0.03047}{4} = 21$ W /(m^2-K)

Heat flux at the trailing edge = $q_w(x)|_{x=L} = [h_x|_{x=L}]\,(T_w - T_\infty)$

$$= 21 \times (130 - 30) = 2100 \text{ W / m}^2.$$

(b) $Re_{cr} = u_\infty x_{cr} / \nu.$

Or
$$x_{cr} = \frac{Re_{cr}\,\nu}{u_\infty} = \frac{2 \times 10^5 \times 20.76 \times 10^{-6}}{10} = 0.415 \text{ m}$$

Hence flow is laminar up to x_{cr}.

Average Nusselt number for the laminar region is given by

$$[Nu_{av}]_{laminar} = 0.664\, Re_{cr}^{0.5}\, Pr^{1/3}$$

$$= 0.664 \times [2 \times 10^5]^{0.5} (0.697)^{1/3} = 263.3$$

Hence average heat transfer coefficient for the laminar region is

$$[h_{av}]_{laminar} = [Nu_{av}]_{laminar}\, k / x_{cr} = \frac{263.3 \times 0.03003}{0.415}$$

$$= 19.00 \text{ W /(m}^2 \text{–K)}$$

Heat transfer rate from laminar portion = $Q_{laminar} = [h_{av}]_{laminar} [x_{cr} W] (T_W - T_\infty)$

$$= 19.0 \times [\,0.415 \times 1] \times (130 - 30)$$

$$= 789.0 \text{ W}$$

(c)Average Nusselt number for the entire length of the plate is given

by $Nu_{av} = Pr^{1/3} [\,0.037 Re_L^{0.8} - A]$

Where $A = 0.037 Re_{cr}^{0.8} - 0.664 Re_{cr}^{0.5}.$

For this problem $A = 0.037 \times [2 \times 10^5]^{0.8} - 0.664 \times [2 \times 10^5]^{0.5} = 356$

Therefore $Nu_{av} = (0.697)^{1/3} [0.037 \times \{1.93 \times 10^6\}^{0.8} - 356]$

$= 3180$

Hence
$$h_{av} = Nu_{av}\, k / L = \frac{3180 \times 0.03047}{4}$$

$$= 24.2 \text{ W/(m}^2 - \text{K)}$$

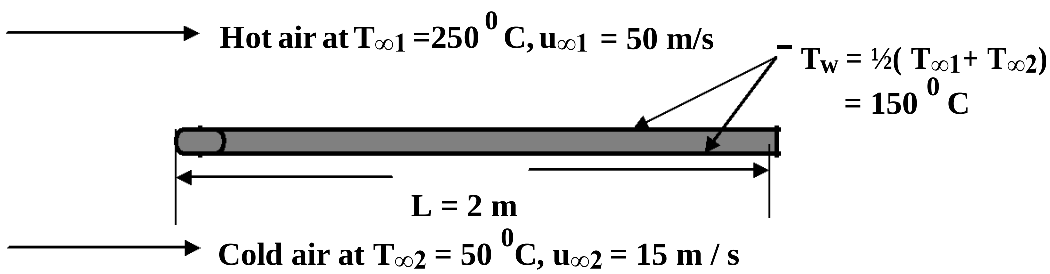
$Q_{total} = h_{av}(LW)(T_w - T_\infty) = 24.2 \times (4 \times 1) \times (130 - 30)$

$= 9680$ W

$Q_{turbulent} = Q_{total} - Q_{laminar} = 9680 - 789$

$= 8891$ W

Solution:



Additional data:- $Re_{cr} = 2 \times 10^5$

To find (i) h_{av} for the hot surface ; (ii) h_{av} for the cold surface ; (iii) $q_w(x)|_{x=L/2}$

(i) Mean film temperature for the hot fluid = $\frac{1}{2}(T_w + T_{\infty 1}) = 0.5 \times (150 + 250) = 200\ ^0C$.

Properties of air at 200 ^0C are: $\nu = 34.85 \times 10^{-6}\ m^2/s$; $k = 0.03931$ W/(m-K); Pr = 0.68 Reynolds number at the trailing edge of the plate = $Re_L = u_{\infty 1}L/\nu$

Or $$Re_L = \frac{50 \times 2}{34.85 \times 10^{-6}}$$

$$= 2.865 \times 10^6$$

Since $Re_L > Re_{cr}$, flow is partly laminar and partly turbulent. Therefore the average Nusselt number is given by

$$Nu_{av} = Pr^{1/3}[0.037\, Re_L^{0.8} - A]$$

Where $$A = 0.037\, Re_{cr}^{0.8} - 0.664\, Re_{cr}^{0.5}.$$

Or $$A = 0.037 \times [2 \times 10^5]^{0.8} - 0.664 \times [2 \times 10^5]^{0.5} = 347.25$$

Hence $Nu_{av} = (0.68)^{1/3} [0.037 \times (3.028 \times 10^6)^{0.8} - 347.25]$

$= 4465$

Hence for the hot surface $[h_{av}]_{hot} = Nu_{av}\, k / L = \dfrac{4465 \times 0.0391}{2}$

$= 987.76\ W/(m^2\text{-}K)$

(ii) Mean film temperature for the cold surface = ½(150 + 50) = 100 ^{0}C.

Properties of air at the mean film temperature are: $\nu = 23.33 \times 10^{-6}\ m^2/s$; Pr = 0.693
k = 0.03184 W/(m-K)

$$Re_L = u_{\infty 2} L / \nu = \frac{15 \times 2}{23.33 \times 10^{-6}} = 1.286 \times 10^6$$

Therefore $Nu_{av} = (0.693)^{1/3}[0.037 \times (1.286 \times 10^6)^{0.8} - 347]$

$= 2219$

Hence for the cold surface $[h_{av}]_{cold} = \dfrac{2219 \times 0.03184}{2} = 35.33\ W/(m^2\text{-}K)$.

(iii) The rate of heat transfer from the hot air stream to cold air stream is given by

$$Q = (T_{\infty 1} - T_{\infty 2}) / [R_{c1} + R + R_{c2}]$$

Where R_{c1} = Thermal resistance offered by hot surface for convection,

R_{c2} = Thermal resistance offered by cold surface for convection,

and R = Thermal resistance offered by the plate for conduction.

Now $R_{c1} = 1 / [h_{av}]_{hot} A = \dfrac{1}{90.3 \times (2 \times 1)} = 0.00554\ m^2 - K / W.$

Similarly $R_{c2} = 1 / [h_{av}]_{cold} A = \dfrac{1}{35.33 \times (2 \times 1)} = 0.01415\ m^2 - K / W.$

R = L / Ak . Since k is not given it is assumed that k is very large i.e R = 0.

Therefore $Q = [250 – 50] / [0.00554 + 0.01415]$

$= 10,157$ W.

(iv) At mid point of the plate $x = L / 2$.

Therefore for the hot fluid $Re|_{x=L/2} = ½ Re_L = 0.5 \times 2.865 \times 10^6$

$= 1.4325 \times 10^6$ which is $> Re_{cr}$.

Therefore flow is turbulent at mid point of the plate.

Hence $[Nu|_{x=L/2}]_{hot} = 0.037 [Re|_{x=L/2}]^{0.8} Pr^{1/3}$

$= 0.037 \times [1.5014 \times 10^6]^{0.8} \times (0.6815)^{1/3}$

$= 2739$

Hence $[h_x|_{x=L/2}]_{hot} = 2739 \times 0.03931 / 1 = 107.7$ W/(m^2-K).

Similarly $[Nu|_{x=L/2}]_{cold} = 0.037 \times [0.5 \times 1.286 \times 10^6]^{0.8} \times (0.693)^{1/3}$

$= 1451$

Hence $[h_x|_{x=L/2}]_{cold} = 1451 \times 0.03184 / 1 = 46.2$

Heat flux at the mid point of the plate is given by

$q_w(x)|_{x=L/2} = [h_x|_{x=L/2}]_{hot} [T_{\infty 1} – T_w] = [h_x|_{x=L/2}]_{cold} [T_w - T_{\infty 2}]$

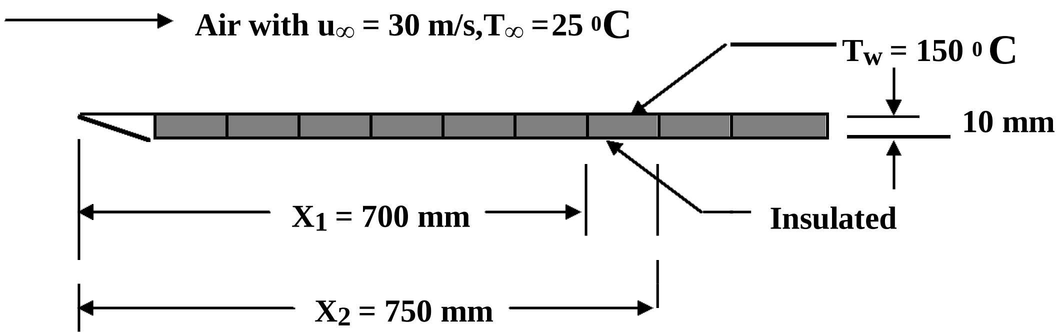
Therefore $T_W = \dfrac{CT_{\infty 1} + T_{\infty 2}}{1 + C}$; $C = \dfrac{[h_x|_{x=L/2}]_{hot}}{[h_x|_{x=L/2}]_{cold}} = \dfrac{107.7}{46.2} = 2.33$

Hence $T_W = \dfrac{(2.33 \times 250) + 50}{1 + 2.375} = 187.41\ ^0C$

Therefore $q_W(x)|_{x=L/2} = 107.7 \times [250 – 187.41] = 6743$ W/m^2

Also check for $q_W(x)|_{x=L/2}$; $q_W(x)|_{x=L/2} = 46.2 \times (187.41 – 50) = 6348.34$ W/m^2

Solution:



Properties of the module: $k_m = 5.2$ W/(m-K); $c_{pm} = 320$ kJ/kg-K; $\rho_m = 2300$ kg/m^3.

Mean film temperature = ½($T_w + T_\infty$) = 0.5 x(25 + 150) = 87.5 ^{0}C.
Properties of air at 87.5 ^{0}C are : $\nu = 21.79 \times 10^{-6}$ m^2/s; k = 0.03075 W/(m-K); Pr = 0.695. Assume $Re_{cr} = 5 \times 10^5$

Reynolds number at $x_1 = Re_{x1} = u_\infty x_1/\nu = 30 \times 0.7 / (21.79 \times 10^{-6}) = 9.637 \times 10^5$.

Similarly $\quad Re_{x2} = u_\infty x_2/\nu = 30 \times 0.75 / (21.79 \times 10^{-6}) = 10.325 \times 10^5$

Since $Re_{x1} > Re_{cr}$, the flow is turbulent at x_1 and the flow will be turbulent over the module under consideration. Therefore the average heat transfer coefficient for the module can be written as

$$[h_{av}]_m = \{1 / (x_2 - x_1)\} \int_{x_1}^{x_2} h_x dx$$

$$= \{1 / (x_2 - x_1)\} \int (Nu_x\, k/x)dx$$

$$= \{1 / (x_2 - x_1)\} \int_{x_1}^{x_2} \{k[\,0.037\,(Re_x)^{0.8} - 871]Pr^{1/3}/x\}dx$$

$$= \frac{k\,Pr^{1/3}}{(x_2 - x_1)} \left\{ \int_{x_1}^{x_2} [0.037\,(u_\infty/\nu)^{0.8}\, x^{-0.2} - 871/x]dx \right\}$$

$$= \frac{k\,Pr^{1/3}}{(x_2 - x_1)} \left\{0.037\,(u_\infty/\nu)^{0.8} \int x^{-0.2}\, dx - 871\int (dx/x) \right\}$$

$$= \frac{k\,Pr^{1/3}}{(x_2 - x_1)} \{\,0.04625[Re_{x2}^{0.8} - Re_{x1}^{0.8}] - 871 \ln (x_2 / x_1)\}\,(x_2 - x_1)$$

$$\frac{0.03075 \times (0.695)^{1/3}\{0.04625[(10.327 \times 10^5)^{0.8} - (9.325 \times 10^5)^{0.8}] - 871 \ln (0.75/0.7)\}}{(0.75 - 0.7)} =$$

$= 95.16$ W / (m^2-K)

For the module, power generation = $q_W = [h_{av}]_m \{T_W - T_\infty\} = 95.16 \times (150 - 25)$

$= 11895$ W/m^2 $= 11.895$ kW/ m^2

(b) Since the bottom surface of the module is insulated, all the heat generated in the module is transferred to air from the top surface of the module. Hence if q'''''' is the heat generated per unit volume then

$q'''''' (x_2 - x_1)\delta W = q_W(x_2 - x_1)W$, where δ is the thickness of the module.

Therefore $q'''''' = q_W / \delta = 11895 / 0.001$

$= 11.895 \times 10^6$ W/m^3.

For the module the governing conduction equation is

$$d^2T/dy^2 + q'''''' / k = 0 \qquad \text{... (a)}$$

where y is the coordinate measured in the direction of the thickness of the module.

The boundary conditions are (i) at y = 0, the surface is insulated i.e. dT/dy = 0

and at $y = \delta$, T = Tw. The solution of Eq.(a) subject to the boundary conditions is given by

$$T(y) + q''''''y^2 / 2k = T_W + q''''''\delta^2 / 2k$$

Since the bottom surface is insulated, the maximum temperature of the module will be at the bottom surface (y = 0) and is therefore given by

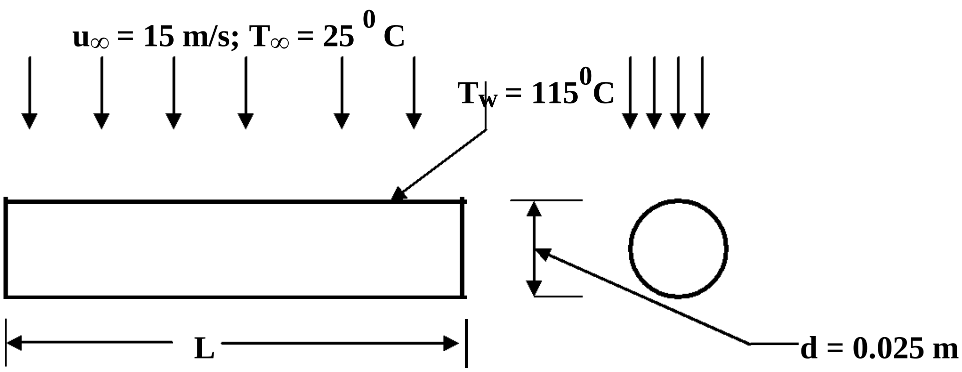
$$T_{max} = T_W + q''''''\delta^2 / 2k$$

$$= 150 + \frac{11.895 \times 10^6 \times (0.001)^2}{2 \times 5.2}$$

$$= 151.1\ ^0C$$

D) Flow across a cylinder

Solution:



Mean film temperature = ½ $(T_w + T_\infty)$ = 0.5 x (115 + 25) = 70 ^{0}C.

Properties of air at 70 ^{0}C are: ρ = 1.0231 kg/m^3; ν = 20.05 x 10 $^{-6}$m^2/s; Pr = 0.699; k = 0.0295 W/(m-K); To find (i) Drag force F_D ; (ii) Q

(i) Reynolds number = $Re_d = u_\infty d / \nu$ = 15 x 0.025 / 20.05 x 10 $^{-6}$

= 18703

From the chart the drag coefficient ,C_D = 1.2

Therefore drag force = $F_D = ½(\rho u_\infty{}^{2)}$ LD C_D = 0.5 x 1.0231 x 15^2 x 1 x 0.025 x 1.2

= 3.453 N

(ii) From Heat transfer data hand book for gases the average Nusselt number is given by ($\mu_w = \mu_\infty$ can be considered for air)

$$Nu_{av} = h_{av}D/k = (0.4Re_d^{0.5} + 0.06\, Re_d^{2/3})\, Pr^{0.4}$$

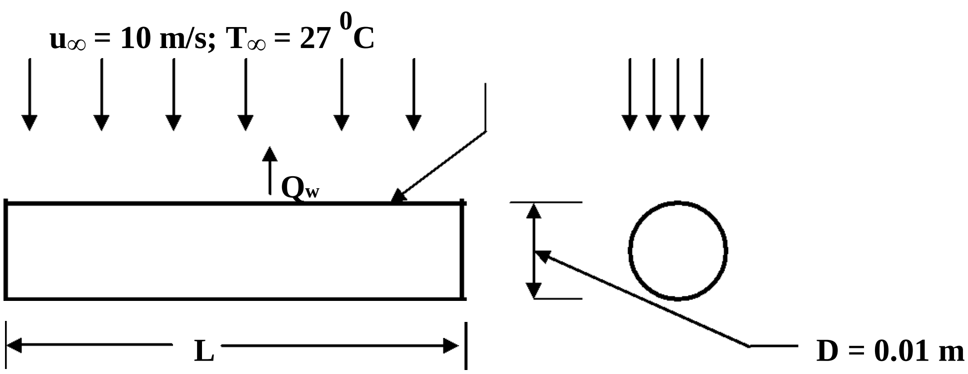
$$= (0.4 \text{ x } 18703^{\,0.5} + 0.06 \text{ x } 18703^{\,2/3}) \text{ x } (0.699)^{\,0.4}$$

= 84.04

Therefore $$h_{av} = Nu_{av}\, k/d = \frac{84.04 \text{ x } 0.0295}{0.025} = 99.17 \text{ W/(m}^2\text{-K)}$$

Heat transfer = Q = $h_{av}\pi dL(T_w - T_\infty)$ = 99.17 x π x 0.025 x 1 x (115 – 25) = 701 W

6.16 Solution:



Given:- Q_W = 1000 W/m; k_s = 240 W/(m-K); ρ_s = 2700 kg/m^3; C_{ps} = 900 kJ/kg-K To find (i) T_W under steady state condition

(ii) time „t" required for the surface to reach a temperature of $(T_W - 10)\ ^0C$

Since T_W is not known it is not possible to read the properties at the mean film temperature. Hence the problem has to be solved by trail and error procedure.

Trial 1:- Calculations are started using the properties of air at T_∞.
Properties of air at 27^{0}C are:

ρ = 1.1774 kg/m^3; c_p = 1.0057 kJ/kg-K; $\nu = 15.68 \times 10^{-6}$ m^2/s; k = 0.02624 W/(m-K);

Pr = 0.708

$$Re_d = u_\infty D / \nu = \frac{10 \times 0.01}{15.68 \times 10^{-6}} = 6377.5$$

($\mu_W = \mu_\infty$ can be considered for air)

Therefore $Nu_{av} = h_{av}D/k = (0.4Re_d^{0.5} + 0.06\ Re_d^{2/3})\ Pr^{0.4}$

$$= (0.4 \times 6377.5^{0.5} + 0.06 \times 6377.5^{2/3}) \times (0.708)^{0.4} = 45.79$$

Therefore $$h_{av} = Nu_{av}\ k/D = \frac{45.79 \times 0.02624}{0.01} = 120.15 \text{ W/(m}^2\text{- K)}$$

Now $Q_W = h_{av}\ \pi DL\ (T_W - T_\infty)$

Or $T_w = T_\infty + Q_w / h_{av} \pi DL = 27 + 1000 / (120.15 \times \pi \times 0.01 \times 1) = 291.93^0$ C

Trial 2:- Assume $T_w = 291.93\ ^0$C .Mean film temperature = ½(291.93 +27) = 159.5 ^{0}C.

Properties of air at 159.5^{0}C are: $\nu = 30.09 \times 10^{-6}$ m^2/s; k = 0.03640 W/(m-K); Pr = 0.682.

$$Re_d = u_\infty D / \nu = \frac{10 \times 0.01}{30.09 \times 10^{-6}} = 3323.4$$

$Nu_{av} = (0.4 \times 3323.4^{0.5} + 0.06 \times 3323.4^{2/3}) \times (0.682)^{0.4} = 31.25$

Hence $h_{av} = 31.25 \times 0.03640 / 0.01 = 113.75$ W/(m^2-K)

Therefore $T_w = 27 + 1000 / (113.75 \times \pi \times 0.01 \times 1) = 306.8\ ^0$C. Since this value of T_w is considerably different from the value got in the first trial, one more iteration is required.

Trial 3:- Assume $T_w = 303\ ^0$ C. Mean film temperature = 0.5 (303 + 27) = 165 ^{0} C.

Properties of air at 165 ^{0} C are: $\nu = 30.88 \times 10^{-6}$ m^2/s; k = 0.0369 W/(m-K);

Pr = 0.682

$$Re_d = u_\infty D / \nu = \frac{10 \times 0.01}{30.88 \times 10^{-6}} = 3238.34$$

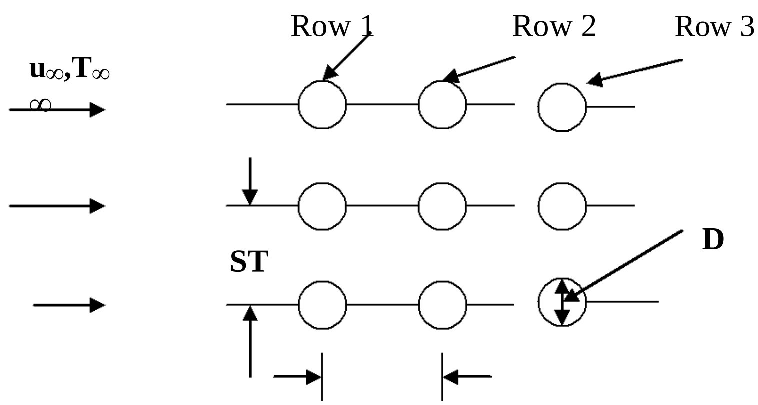
$Nu_{av} = (0.4 \times 3238.34^{0.5} + 0.06 \times 3238.34^{2/3}) \times (0.682)^{0.4} =$

30.8 Hence $h_{av} = 30.8 \times 0.0369 / 0.01 = 113.65$W/(m^2-K)

Therefore $T_w = 27 + 1000 / (113.65 \times \pi \times 0.01 \times 1) = 307\ ^0$C which is very close to the assumed value of 303 ^{0} C and hence the iteration can be stopped.
Therefore $T_w = 307\ ^0$C.

E. Flow across tube bundles

Solution: *Case (i) Square Arrangement*



SL

Data:- $T_\infty = 30\ ^0C$; $D = 0.01$ m; $N = 10$; $u_\infty = 1$ m/s; $S_T / D = S_L / D = 1.25$

To find (i) friction factor,f; (ii) pressure drop Δp ; (iii) h_{av}

Since the surface temperature of the tubes is not known properties of air are evaluated at T_∞. Hence properties of air at 30 ^{0}C are:

$\rho = 1.1774$ kg/m^3; $\mu = 1.983 \times 10^{-5}$ kg/(m-s); $c_p = 1005.7$ J/kg-K; $k = 0.02624$ W/(m-K); $Pr = 0.708$

(i) For square arrangement the maximum velocity is given by

$$U_{max} = u_\infty \frac{S_T / D}{[S_T / D - 1]} = 1 \times \frac{1.25}{[1.25 - 1]}$$

$$= 5 \text{ m/s.}$$

$$G_{max} = \rho U_{max} = 1.1774 \times 5 = 5.887 \text{ kg/(m}^2\text{-s)}$$

$$\text{Reynolds number} = Re = DG_{max} / \mu = \frac{0.01 \times 5.887}{1.983 \times 10^{-5}} = 2969$$

From the graph friction factor f = 5.5 and Z = 1 for square arrangement as $S_T = S_L$.

Now

$$\Delta p = f \frac{N (G_{max})^2}{\rho} Z = 5.5 \times \frac{10 \times (5.887)^2}{1.1774} \times 1.0$$

$$= 1619 \text{ N / m}^2$$

(ii) For $N \geq 20$ the average Nusselt number is given by

$$Nu_{av} = c_2 Re^m Pr^{0.36} (Pr/Pr_w)^n$$

Here $c_2 = 0.27$; $m = 0.63$; $n = 0$.

Therefore $$Nu_{av} = 0.27 \times (2969)^{0.63} (0.708)^{0.36} = 36.74$$

Since N <20, the above value of Nu_{av} has to be multiplied by a correction factor.

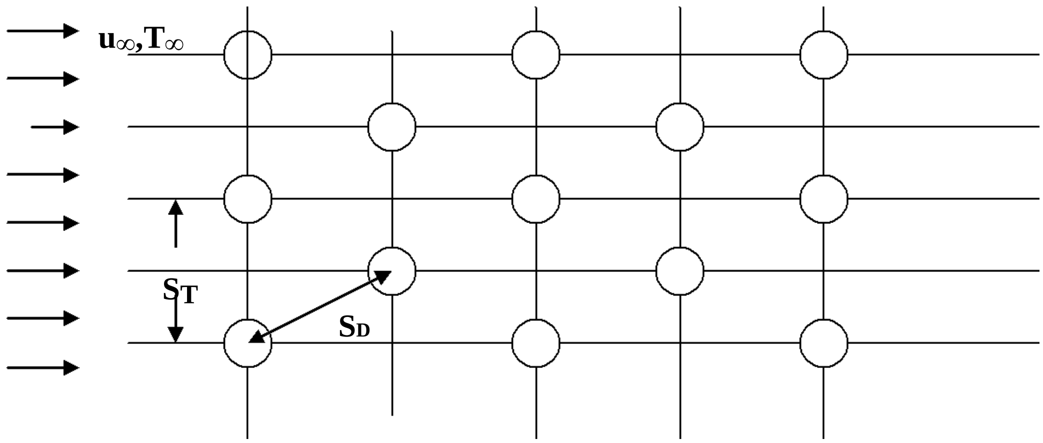
Hence $[Nu_{av}]_{N=10} = c_3[Nu_{av}]_{N\geq 20}$

For Re = 2969 and N = 10 from the graph $c_3 = 0.98$

Therefore $[Nu_{av}]_{N=10} = 0.98 \times 36.74 = 36$

Therefore $h_{av} = 36 \times 0.02624 / 0.01 = 94.5\ W/(m^2\text{-}K)$

Case (ii); Equilateral Triangular Arrangement:



Given: $S_T / D = S_D / D = 1.25$

Maximum velocity $$U_{max} = u_\infty \frac{S_T / D}{[S_T / D - 1]} = 5 \text{ m/s as calculated above}$$

Or $$U_{max} = (½)u_\infty \frac{S_T / D}{[S_D / D - 1]} = 0.5 \times 1 \times \frac{1.25}{(1.25 - 1)} = 2.5 \text{ m/s}$$

We have to choose the higher of the two maximum velocities to calculate the Reynolds number.Hence U_{max} and Reynolds number will be same as the above case.

From chart for staggered tube arrangement, f = 7.0 and Z = 1

Therefore $$\Delta p = 7 \times \frac{10 \times 5.887^2}{1.1774} 1 = 2061\ N/m^2$$

From data hand book the average Nusselt number is given by

$$[Nu_{av}]_{N\geq 20} = c_2 \, Re^m \, Pr^{0.36} (Pr / Pr_w)^n$$

For staggered arrangement $c_2 = 0.35 \times (S_T / S_L)^{0.2}$

Now $S_L^2 = S_D^2 - (S_T/2)^2$ or $S_L / D = [(S_D/D)^2 - (S_T/2D)^2]^{½}$

$$= (\sqrt{3}/2)(S_T/D)$$

Or $\quad S_T/S_L = 2 / \sqrt{3} = 1.155$

Hence $\quad c_2 = 0.35 \times [1.155]^{0.2} = 0.36$;

For staggered arrangement $\quad m = 0.6$; for air $n = 0$.

Hence $\quad [Nu_{av}]_{N\geq 20} = 0.36 \times (2969)^{0.6} (0.708)^{0.36}$

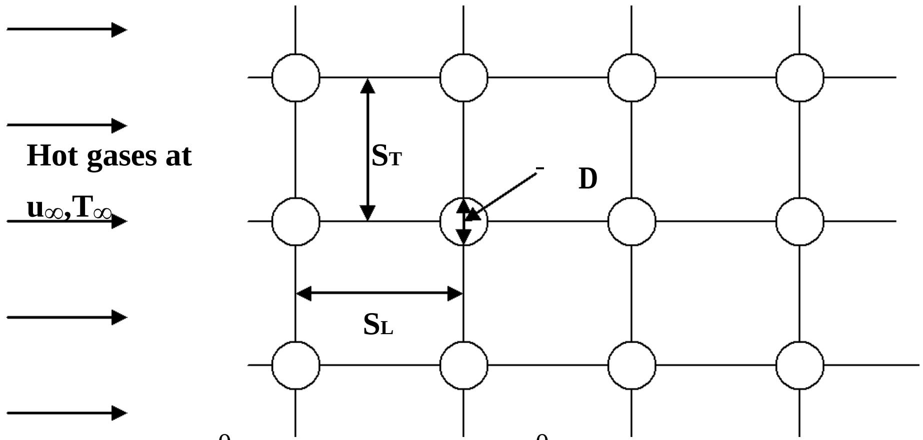
$$= 38.54$$

$$[Nu_{av}]_{N=10} = c_3 [Nu_{av}]_{N\geq 20}$$

From the graph $c_3 = 0.98$. Hence $[Nu_{av}]_{N=10} = 0.98 \times 38.54 = 37.8$

Therefore $\quad h_{av} = 37.8 \times 0.02624 / 0.01 = 99.2$ W/(m^2-K).

Solution:



Data:- $T_\infty = 375\ ^0C$; $u_\infty = 7$ m/s; $T_W = 30\ ^0C$; Number of rows = N = 10 ;
D = 0.0125 m; number of tubes in each row = m = 40; L = 1 m ;
In-line arrangement; $S_L/D = S_T/D = 2$.
To find:- (i) h_{av} ; (ii) Q

Mean film temperature = 0.5 x (375 + 30) = 202.5 ^0C.

Properties of air at 202.5 ^0C are: $\rho = 0.9403$ kg/m^3; $c_p = 1.0115$ kJ/(kg-K);

$\mu = 2.1805 \times 10^{-5}$ m^2/s; k = 0.03184 W/(m-K); Pr = 0.693

For inline arrangement $U_{max} = u_\infty (S_T/D) / [(S_T/D) - 1] = 7 \times 2 /(2 - 1) = 14$ m/s. Mass velocity = $G_{max} = \rho\, U_{max} = 0.9403 \times 14 = 13.164$ kg/(m^2-s)

$$\text{Reynolds number} = Re = DG_{max} / \mu = \frac{0.0125 \times 13.164}{2.1805 \times 10^{-5}} = 7546$$

The average Nusselt number for N ≥ 20 is given by $[Nu_{av}]_{N\geq20} = c_2\, Re^m Pr^{0.36}$
For in-line arrangement from data hand book, $c_2 = 0.27$ and m = 0.63

Therefore $[Nu_{av}]_{N\geq20} = 0.27 \times (7546)^{0.63} \times (0.693)^{0.36} = 170$

$[Nu_{av}]_{N=10} = c_3 [Nu_{av}]_{N\geq 20}$ with $c_3 = 0.96$

Hence $[Nu_{av}]_{N=10} = 0.96 \times 170 = 163.2$

$$\text{Average heat transfer coefficient} = h_{av} = [Nu_{av}]_{N=10}\, k / D = \frac{163.2 \times 0.03184}{0.0125}$$

$= 416\ W/(m^2\text{-}K)$.

Energy balance between the tubes surfaces and hot gases can be written as

Heat transfer from hot gases to the tube surfaces = $Q = (\pi DLNm)h_{av}(T_\infty - T_w)$

Or $Q = \pi \times 0.0125 \times 1 \times 10 \times 40 \times 416 \times (375 - 30) = 2254 \times 10^3\ W = 2254\ kW$

Free Convective Heat Transfer

A. Free convection from/to plane surfaces:

A vertical plate 30 cm high and 1 m wide and maintained at a uniform temperature of 120 ^{0} C is exposed to quiescent air at 30 ^{0} C.Calculate the average heat transfer coefficient and the total heat transfer rate from the plate to air.

An electrically heated vertical plate of size 25 cm x 25 cm is insulated on one side and dissipates heat from the other surface at a constant rate of 600 W/m^2 by free convection into quiescent atmospheric air at 30 ^{0} C. Determine the surface temperature of the plate.

Determine the heat transfer by free convection from a plate 30 cm x 30 cm whose surfaces are maintained at 100 ^{0} C and exposed to quiescent air at 20 ^{0} C for the following conditions: (a) the plate is vertical. (b) Plate is horizontal

A circular plate of 25 cm diameter with both surfaces maintained at a uniform temperature of 100 ^{0} C is suspended in horizontal position in atmospheric air at 20 ^{0} C. Determine the heat transfer from the plate.

Consider an electrically heated plate 25 cm x 25 cm in which one surface is thermallt insulated and the other surface is dissipating heat by free convection into atmospheric air at 30 ^{0} C. The heat flux over the surface is uniform and results in a mean surface temperature of 50 ^{0} C. The plate is inclined making an angle of 50 ^{0} from the vertical. Determine the heat loss from the plate for (i) heated surface facing up and (ii) heated surface facing down.

A thin electric strip heater of width 20 cm is placed with its width oriented vertically. It dissipates heat by free convection from both the surfaces into atmospheric air at 20 ^{0} C. If the surface temperature of the heater is not to exceed 225 ^{0} C, determine the length of the heater required in order to dissipate 1 kW of energy into the atmospheric air.

A plate 75 cm x 75 cm is thermally insulated on the one side and subjected to a solar radiation flux of 720 W/m^2 on the other surface. The plate makes an angle of 60 ^{0} with the vertical such that the hot surface is facing upwards. If the surface is exposed to quiescent air at 25 ^{0} C and if the heat transfer is by pure free convection determine the equilibrium temperature of the plate.

B. Free convection from/to Cylinders:

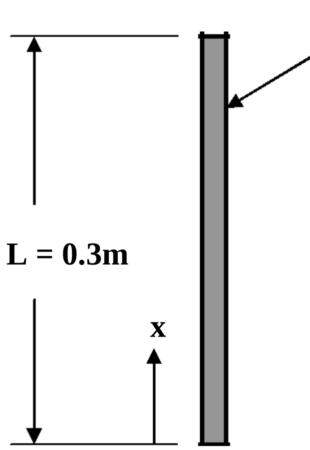
A 5 cm diameter, 1.5 m long vertical tube at a uniform temperature of 100 ^{0}C is exposed to quiescent air at 20 ^{0} C. calculate the rate of heat transfer from the surface to air. What would be the heat transfer rate if the tube were kept horizontally?

A horizontal electrical cable of 25 mm diameter has a heat dissipation rate of 30 W/m. If the ambient air temperature is 27 ^{0} C, estimate the surface temperature of the cable.

An electric immersion heater, 10 mm in diameter and 300 mm long is rated at 550 W. If the heater is horizontally positioned in a large tank of water at 20 ^{0}C, estimate its surface temperature. What would be its surface temperature if the heater is accidentally operated in air.

A.Free Convection to or from plane surfaces

Solution:



$T_W = 120\ ^0$ C

- **$T_\infty = 30\ ^0$C**

Mean film temperature of air = 0.5 x (120 +30) = 75^{0}C

Properties of air at 75^{0}C are:

β = 1/ (273 + 75) = 2.874 x 10^{-3} 1/K; Pr = 0.693

k = 0.03 W/(m-K) ; ν = 20.555 x 10^{-6} m^2/s ;

\

First we have to establish whether the flow become turbulent within the given length of the plate by evaluating the Rayleigh number at x = L.

$$Gr_L = (g\beta\Delta TL^3) / \nu^2 = \frac{9.81 \times 2.874 \times 10^{-3} \times (120 - 30) \times 0.3^3}{20.555 \times 10^{-6}}$$

$$= 1.62 \times 10^8$$

Rayleigh number = $Ra_L = Gr_L Pr$ = 1.62 x 10^8 x 0.693 = 1.12 x 10^8.

Since $Ra_L < 10^9$ flow is laminar for the entire height of the plate. Hence the average Nusselt number is given by (from data hand book)

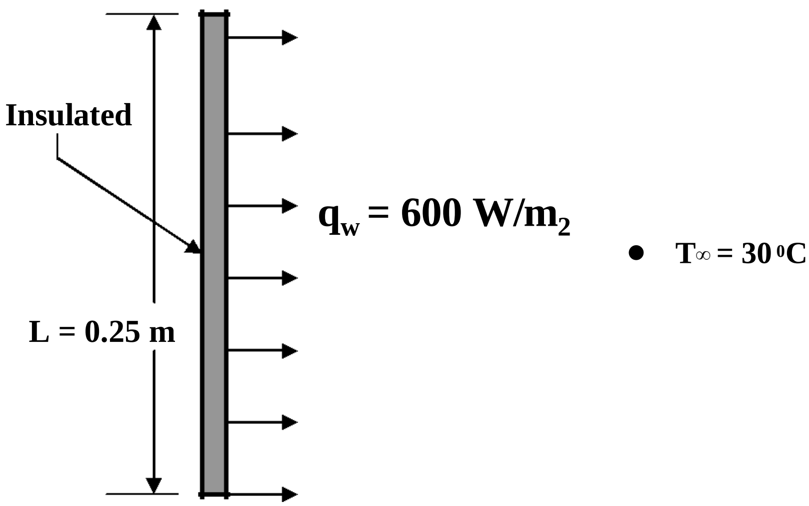
$$Nu_{av} = 0.59 \times (Ra_L)^{0.25} = 0.59 \times (1.12 \times 10^8)^{0.25} = 60.695$$

$$\text{Therefore} \qquad h_{av} = Nu_{av}\, k / L = \frac{60.6 \times 0.03}{0.3} = 6.069 \text{ W}/(\text{m}^2 - \text{K}).$$

Total heat transfer fro both sides of the plate per unit width of the plate is given by

$$Q_{total} = h_{av}(2LW)\, T = 6.06 \times (2 \times 0.3 \times 1) \text{ X } (120 - 30) = 327.726 \text{ W/m}.$$

Solution:



Since T_W is not known, it is not possible to determine the mean film temperature at which fluid properties have to be evaluated. Hence this problem requires a trial and error solution either by assuming T_W and then calculate T_W by using the heat balance equation and check for the assumed value or assume a value for h_{av} ,calculate T_W and then calculate h_{av} and check for the assumed value of h_{av}.Since it is difficult to guess a reasonable value for T_W to reduce the number of iterations, it is preferable to guess a reasonable value for h_{av} for air as we know that for air h_{av} varies anywhere between 5 and 15 W/(m^2-K).

Trial 1:- Assume h_{av} = 10 W/(m^2-K).

Now $q_W = h_{av}[T_W - T_\infty]$ or $T_W = T_\infty + q_W / h_{av} = 30 + 600 / 10 = 90^0$C.

Hence mean film temperature = 0.5 x [90 + 30] = 60 ^{0}C.

Properties of air at 60 ^{0}C are: $\beta = 1 / (60 + 273) = 3.003 \times 10^{-3}$ 1/K; Pr = 0.696;

k = 0.02896 W/(m-K); $\nu = 18.97 \times 10^{-6}$ m^2/s.

$$Ra_L^* = Gr_L^* Pr = [(g\beta q_w x^4)/(k\nu^2)]Pr = \frac{9.81 \times 3.003 \times 10^{-3} \times 600 \times (0.25)^4}{0.02896 \times (18.97 \times 10^{-6})^2} \times 0.696$$

Or $$Ra_L^* = 4.61 \times 10^9 .$$

Since $Ra_L^* > 10^9$ flow is turbulent for the entire length of the plate

Hence $$Nu_{av} = 1.25\, Nu_x|_{x=L} = 1.25 \times 0.17 \times (4.61 \times 10^9)^{0.2} = 55.37$$

Therefore $$h_{av} = 55.37 \times 0.02896 / 0.25 = 6.41\ W/(m^2 - K)$$

Since the calculated value of h_{av} deviates from the assumed value by about 34 %, one more iteration is required.

Trial 2:- Assume h_{av} = 6.41 W/(m^2-K)

Hence $T_w = 30 + 600 / 6.41 = 123.6\ ^0C \cong 120\ ^0C$

Mean film temperature = 0.5 x (120 + 30) = 75 ^{0}C

Properties of air at 75 ^{0}C are:- $\beta = 1/(75 + 273) = 2.873 \times 10^{-3}$ 1/K. Pr =

0.686 k = 0.03338 W /(m-K); $\nu = 25.45 \times 10^{-6}\ m^2/s$.

$$Ra_L^* = \frac{9.81 \times 2.873 \times 10^{-3} \times 600 \times 0.25^4}{0.03338 \times (25.45 \times 10^{-6})^2} \times 0.686 = 2.06 \times 10^9$$

Flow is turbulent for the entire length of the plate.

Hence $$Nu_{av} = 1.25\, Nu_x|_{x=L} = 1.25 \times 0.17 \times (2.06 \times 10^9)^{0.25} = 45.27$$

Therefore $$h_{av} = 45.27 \times 0.03338 / 0.25 = 6.04\ W/(m^2 - K).$$

Since the calculated value of h_{av} is very close to the assumed value, the iteration is stopped. The surface temperature of the plate is therefore given by

$$T_w = 30 + 600 / 6.04 = 129.3\ ^0C.$$

Solution:- *Case(i) When the plate is vertical*

Data:- Characteristic length = L = height of the plate = 0.3 m; $T_w = 100\ ^0C$; $T_\infty = 20\ ^0C$; Mean film temperature = 0.5 x (100 + 20) = 60 ^{0}C.

Properties of air at 60 ^{0}C are: $\beta = 1 / (60 + 273) = 3.003 \times 10^{-3}$ 1/K; Pr = 0.696;

k = 0.02896 W/(m-K); $\nu = 18.97 \times 10^{-6}$ m^2/s.

$$Ra_L = Gr_L Pr = (g\beta\Delta T L^3 / \nu^2)\, Pr$$

$$= \frac{9.81 \times 3.003 \times 10^{-3} \times (100 - 20) \times (0.3)^3}{(18.97 \times 10^{-6})^2} \, 0.696$$

$$= 1.23 \times 10^8$$

From data hand book corresponding to this value of Ra_L have

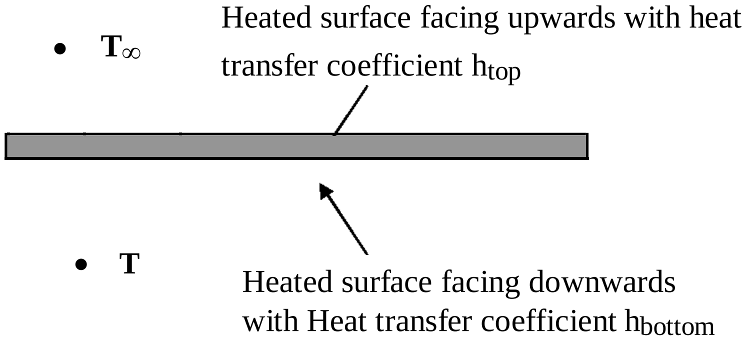
$$Nu_{av} = 0.59 \times (1.23 \times 10^8)^{0.25} = 62.13$$

Therefore $h_{av} = 62.13 \times 0.02896 / 0.3 = 5.99$ W/(m^2-K).

Rate of heat transfer = $Q = h_{av}(2LW)(\Delta T) = 5.99 \times (2 \times 0.3 \times 0.3) \times (100 - 20)$

$= 86.256$ W

Case (ii) When the plate is horizontal



Data:- $T_\infty = 20$ ^{0}C; Temperature of both the surfaces = $T_W = 100$ ^{0}C;
Mean film temperature = 0.5 x (100 + 20) = 60 ^{0}C; L = W = 0.3 m
Properties of air at 60 ^{0}C are: $\beta = 1 /(60 + 273) = 3.003 \times 10^{-3}$ 1/K; Pr = 0.696
$\nu = 18.97 \times 10^{-6}$ m^2/s; k = 0.02896 W/(m-K)

(a) To find h_{top}:- Characteristic length = L = A/P = LW {2(L+W)}

$$= L^2 / 4L = L/4 = 0.3 / 4 = 0.075 \text{ m}$$

$$Ra_L = \frac{g\beta\Delta T L^3}{\nu^2} Pr = \frac{9.81 \times (3.003 \times 10^{-3}) \times (100 - 20) \times (0.075)^3}{(18.97 \times 10^{-6})^2} \, 0.696$$

Or $Ra_L = 1.923 \times 10^6$.

From data hand book for heated surface facing upwards with constant surface temperature the average Nusselt number is given by

$$Nu_{top} = h_{top}L/k = 0.54 \times (Ra_L)^{0.25} = 0.54 \times (1.923 \times 10^6)^{0.25}$$

$$= 20.11$$

Hence $h_{top} = 20.11 \times 0.02896 / 0.075 = 7.76 \text{ W/(m}^2\text{-K)}$

(b)To find h_{bottom}:- From data hand book for heated surface facing downwards with constant surface temperature, the average Nusselt number is given by

$$Nu_{bottom} = h_{bottom}L/k = 0.27 \times (1.923 \times 10^6)^{0.25} = 10.05$$

$$H_{bottom} = 10.05 \times 0.02896 / 0.075 = 3.88 \text{ W/(m}^2\text{-K)}$$

Total heat loss to air = $Q_{total} = Q_{top} + Q_{bottom} = (LW)h_{top}\,\Delta T + (LW)h_{bottom}\Delta T$

$$= (0.3 \times 0.3) \times (100 - 20) \times (7.76 + 3.88) = 83.808 \text{ W}$$

Solution: Data:- Horizontal circular plate with D = 0.25 m; $T_W = 100\ ^0C$; $T_\infty = 20\ ^0C$

This problem is similar to the previous problem except for the characteristic length. For a horizontal circular plate of diameter D the characteristic length is given by

$$L = A/P = (\pi D^2/4) / (\pi D) = D/4$$

$$= 0.25 / 4 = 0.0625 \text{ m}$$

Mean film temperature = 0.5 x (100 + 20) = 60 ^{0}C; L = W = 0.25 m

Properties of air at 60 ^{0}C are: $\beta = 1/(60 + 273) = 3.003 \times 10^{-3}$ 1/K; Pr = 0.696

$\nu = 18.97 \times 10^{-6}$ m^2/s; k = 0.02896 W/(m-K).

$$Ra_L = \frac{g\beta\Delta TL^3}{\nu^2} Pr = \frac{9.81 \times (3.003 \times 10^{-3}) \times (100 - 20) \times (0.0625)^3}{(18.97 \times 10^{-6})^2} \times 0.696$$

Or $Ra_L = 1.112 \times 10^6$.

From data hand book for heated surface facing upwards with constant surface temperature the average Nusselt number is given by

$Nu_{top} = h_{top}L/k = 0.54 \times (Ra_L)^{0.25} = 0.54 \times (1.112 \times 10^6)^{0.2} = 17.53$

Hence $h_{top} = 17.53 \times 0.02896 / 0.0625 = 8.12\,W/(m^2\text{-}K)$

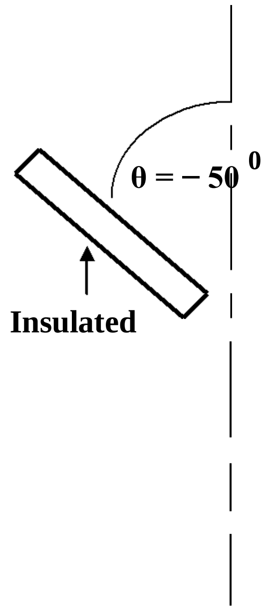
(b)To find h_{bottom}:- From data hand book for heated surface facing downwards with constant surface temperature, the average Nusselt number is given by

$Nu_{bottom} = h_{bottom}L/k = 0.27 \times (1.112 \times 10^6)^{0.25} = 8.76$ $h_{bottom} = 8.76 \times 0.02896 / 0.0625 = 4.059\ W/(m^2\text{-}K)$ Total heat loss to air $= Q_{total} = Q_{top} + Q_{bottom} = (\pi DL)h_{top}\,\Delta T + (\pi DL)h_{bottom}\Delta T$

$= (\pi \times 0.25 \times 1) \times (100 - 20) \times (8.76 + 4.059) = 805.44\ W$

Soloution: Data:- L = W = 0.25 m; $T_\infty = 30\ ^0C$; $T_W = 50\ ^0C$;
Mean film temperature = 50 − 0.25 x (50 - 30) = 45 ^{0}C; Properties of air at 45 ^{0}C are:
Pr = 0.6835; k = 0.02791 W/(m-K); $\nu = 17.455 \times 10^{-6}\ m^2/s$.
$\beta = 1/[\{30 + 0.25x(50\text{-}30)\} + 273] = 3.25 \times 10^{-3}$

(i) Inclined plate with heated surface facing upwards:



Characteristic length = L = 0.25 m

$$Gr_L = \frac{g\beta\Delta T L^3}{\nu^2}$$

$$= \frac{9.81 \times (50 - 30) \times 3.25 \times 10^{-3} \times (0.25)^3}{(17.455 \times 10^{-6})^2}$$

$$= 3.27 \times 10^7$$

Hence $Ra_L = 3.27 \times 10^7 \times 0.6835 = 2.23 \times 10^7$

From data hand book, for inclined plate with heated surface facing upwards the Nusselt number is given by

$$Nu_{av} = 0.145\,[Ra_L^{1/3} - (Gr_c Pr)^{1/3}] + 0.56(Gr_c Pr\cos\theta)^{1/4}$$

The above correlation is valid only if $Gr_L > Gr_c$.

From data hand book for $\theta = -50^0$, $Gr_c = 4 \times 10^8$ which is more than Gr_L. Hence the above correlation cannot be used. Instead the following correlation has to be used.

$$Nu_{av} = 0.59\,(Gr_L \cos\theta\, Pr)^{1/4} = 0.59 \times (2.23 \times 10^7 \times \cos 50^0)^{1/4} = 36.3$$

Hence $\quad h_{av} = 36.3 \times 0.02791 / 0.25 = 4.05\ W/(m^2 - K)$.

Therefore $\quad Q = 4.05 \times (0.25 \times 0.25) \times (50 - 30) = 5.062\ W$

(ii) Inclined plate with heated surface facing downwards:

The correlation for Nusselt number when the heated surface is facing downwards is given by

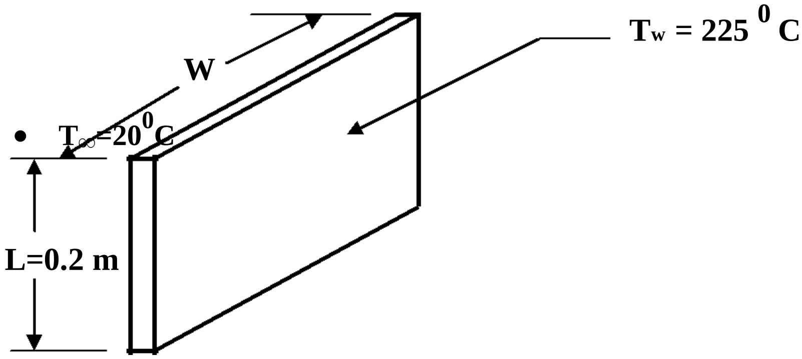
$$Nu_{av} = 0.56\,(Gr_L \cos\theta\, Pr)^{1/4} = 0.56 \times (2.23 \times 10^7 \times \cos 50^0)^{1/4}$$

$$= 34.45$$

$$h_{av} = 34.45 \times 0.02791 / 0.25 = 3.84\ W/(m^2\text{-}K).$$

Hence $\quad Q = 3.84 \times (0.25 \times 0.25) \times (50 - 30) = 4.8\ W$

Solution:



Mean film temperature = 0.5 x (225 + 20) = 122.5 ^{0}C. Properties of air at 122.5^{0}C are: β = 1/(122.5 + 273) = 2.5 x 10^{-3} 1/K; ν = 25.90 x 10^{-6} m^2/s; Pr = 0.6865; k = 0.03365 W/(m-K).

$$Gr_L = \frac{9.81 \times 2.5 \times 10^{-3} \times (225 - 20) \times 0.2^3}{(25.9 \times 10^{-6})^2} = 5.99 \times 10^7$$

Ra_L = 5.99 x 10^7 x 0.6865 = 4.11 x 10^7

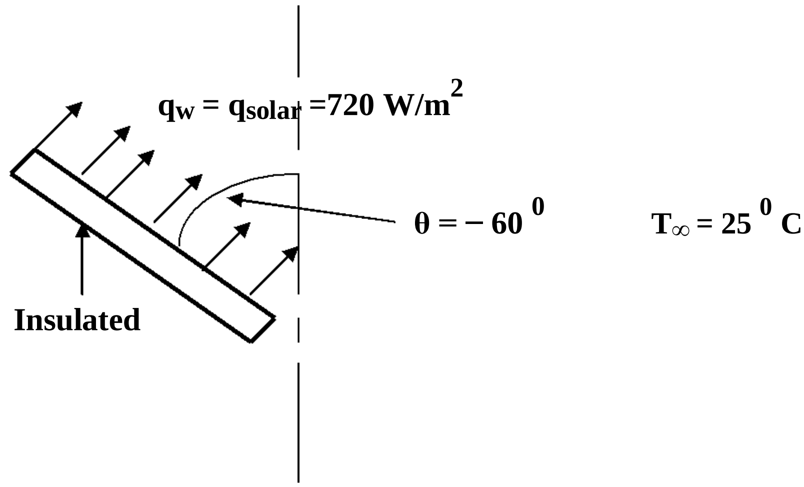
Hence Nu_{av} = 0.59 x (4.11 x 10^7)$^{0.25}$ = 47.24

Therefore h_{av} = 47.24 x 0.03365/0.2 = 7.94 W/(m^2-K)

Now Q = h_{av}LWΔT or
$$W = Q / (h_{av}L\Delta T) = \frac{1000}{7.94 \times 0.2 \times (225 - 20)}$$

= 3.0718 m

Solution:



Since T_W is not known, mean film temperature to evaluate the fluid properties cannot be determined. Hence the problem requires a trial- and – error method by suitably assuming a value for h_{av} and then check for this assumpotion.

Trial 1:- Assume h_{av} = 5 W/(m^2-K):

$T_W = T_\infty + q_W / h_{av} = 25 + 720 / 5 = 169\ ^0C$.

$T_m = T_W - 0.25[T_W - T_\infty] = 169 - 0.25 \times [169 - 25] = 133\ ^0C$.

Properties of air at 133 ^{0}C are: $\nu = 26.62 \times 10^{-6}\ m^2/s$; k = 0.03413 W/(m-K); Pr = 0.685 Mean temperature to evaluate β is given by $T_\beta = T_\infty + 0.25[T_W - T_\infty]$

$= 25 + 0.25 \times (169 - 25) = 61\ ^0C$

Therefore $\beta = 1/[61 + 273] = 2.994 \times 10^{-3}$ 1/K.

$$Gr_L = \frac{9.81 \times 2.994 \times 10^{-3} \times [169 - 25]\ 0.75^3}{(26.62 \times 10^{-6})^2} = 2.51 \times 10^9$$

For $\theta = -60^0$, $Gr_C = 10^8$. Since $Gr_L > Gr_C$ the average Nusselt number is given by

$$Nu_{av} = 0.145\ [(Gr_L Pr)^{1/3} - (Gr_C Pr)^{1/3}] + 0.56[Gr_C Pr]^{1/4}$$

$= 0.145 \times [(2.51 \times 10^9 \times 0.685)^{1/3} - (10^8 \times 0.685)^{1/3}] + 0.56 \times (10^8 \times 0.685)^{1/4}$

= 164.45

Therefore $h_{av} = 164.45 \times 0.03413 / 0.75 = 7.48\ W/(m^2\text{-}K)$.

Since the calculated value of h_{av} is quite different from the assumed value one more iteration is required.

Trial 2: Assume $h_{av} = 7.48\ W/(m^2\text{-}K)$.

With this assumption, following the steps shown in trial 1 we get

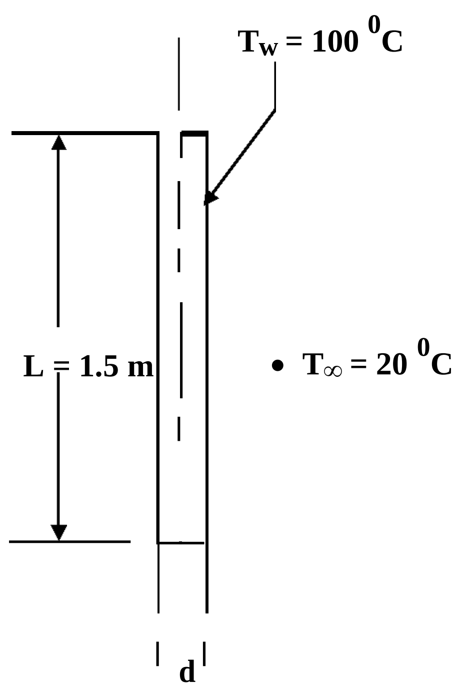
$T_w = 121.25\ ^0C$;$T_m = 97.18\ ^0C \cong 100\ ^0C$;$T_\beta = 49.06\ ^0C$; $Gr_L = 2.311 \times 10^9$;$Nu_{av} = 159.96$

Hence $h_{av} = 6.88\ W/(m^2\text{-}K)$. This value agrees with the assumed value within 8 %. Hence the iteration is stopped and the equilibrium temperature of the plate surface is calculated as

$$T_w = 25 + 720 /\{0.5(7.48 + 6.88)\} = 125.27\ ^0C.$$

B Free convection from/to cylinders

Solution: *(i) When the tube is vertical:*



Mean film temperature = 0.5x(100+20)

= $60\ ^0C$.

Properties of air at $60\ ^0C$ are:Pr = 0.696

k = 0.0290 W/(m-K); $\nu = 18.97\times10^{-6}$ m^2/s;$\beta = 1/(60 + 273) = 3.003\times10^{-3}$ 1/K.

$$Gr_L = \frac{g\beta\Delta T L^3}{\nu^2} = \frac{9.81 \times 3.003 \times 10^{-3} \times (100 - 20) \times 1.5^3}{(18.97\times10^{-6})^2} = 2.21 \times 10^{10}$$

$$\text{Now } \frac{(L/d)}{Gr_L^{1/4}} = \frac{[1.5 / 0.05]}{[2.21 \times 10^{10}]^{1/4}} = 0.0078$$

$$\text{Since } \frac{(L/d)}{Gr_L^{1/4}} < 0.025, \text{ the vertical tube/cylinder can be treated as a vertical flat surface}$$

Hence $Nu_{av} = 0.1 \times (1.538 \times 10^{10})^{1/3} = 248.7$ Therefore $h_{av} = 248.7 \times 0.029 / 1.5 =$

$4.81\ W/(m^2 - K)$. Rate of heat transfer $= Q = \pi dLh_{av}\Delta T = \pi \times 0.05 \times 1.5 \times (100 - 20)$

$\times\ 4.81 = 90.67\ W$

(ii) When the pipe is horizontal:- When the pipe is horizontal, the characteristic length is the diameter. Hence

$$Gr_d = \frac{g\beta\Delta Td^3}{\nu^2} = \frac{9.81 \times 3.003 \times 10^{-3} \times (100 - 20) \times 0.05^3}{(18.97\times10^{-6})^2} = 8.185 \times 10^5$$

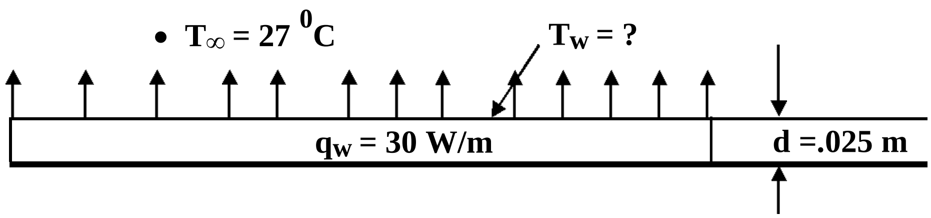
$$[Nu_{av}]^{1/2} = 0.60 + 0.387 \left\{ \frac{Ra_d}{[1 + (0.559/Pr)^{9/16}]^{8/27}} \right\}^{1/6}$$

$$[Nu_{av}]^{1/2} = 0.60 + 0.387 \left\{ \frac{(5.697\times 10^5)}{[1 + (0.559/0.696)^{9/16}]^{8/27}} \right\}^{1/6} = 4.01 \text{ or } Nu_{av} = 16.08$$

Therefore $h_{av} = 16.08 \times 0.029 / 0.05 = 9.32 W/(m^2 - K)$.

$Q = \pi \times 0.05 \times 1.5 \times (100 - 20) \times 9.32 = 175.67\ W$

Solution:



Since T_W is not known, it is not possible to evaluate the fluid properties at the mean film temperature. Hence the problem has to be solved by trial and error solution by assuming a suitable value for h_{av} and check for the assumed value.

Trial 1:- Assume h_{av} = 10 W/(m^2-K)

$q_W = \pi d h_{av}[T_W - T_\infty]$ or $T_W = T_\infty + q_W/(\pi d h_{av}) = 27 + 30/(\pi \times 0.025 \times 10) = 65\ ^0C$.

Mean film temperature = 0.5 x (27 + 65) = 46 ^{0}C.

Properties of air at 46 ^{0}C are: $\beta = 1/(46 + 273) = 3.135 \times 10^{-3}$ 1/K; k = 0.0280 W/(m-K)

Pr = 0.684; $\nu = 17.45 \times 10^{-6}$ m^2/s.

$$Gr_d = \frac{9.81 \times 3.135 \times 10^{-3} \times (65 - 27) \times 0.025^3}{[17.45 \times 10^{-6}]^2} = 6.0 \times 10^4.$$

$$\text{Hence } [Nu_{av}]^{½} = 0.60 + 0.387 \left\{ \frac{(6.0 \times 10^4 \times 0.684)}{[1 + (0.559/0.6985)^{9/16}]^{8/27}} \right\}^{1/6} = 2.8$$

Hence Nu_{av} = 7.84 or h_{av} = 7.84 x 0.028 / 0.025 = 8.78 W/(m^2 – K).

Since the calculated value of h_{av} deviates very much from the assumed value one more iteration is required.

(ii) Trial 2: Assume h_{av} = 8.78 W/(m^2-K).

Proceeding in the same way as in trial 1 we have T_W = 70.5 ^{0}C. Hence T_m = 48.75 ^{0} ≈ 50^{0}C Properties of air at 50 ^{0}C are: $\beta = 1/(50+273) = 3.05 \times 10^{-3}$ 1/K; Pr = 0.698; k = 0.02826 W/(m-K); $\nu = 17.95 \times 10^{-6}$ m^2/s.

$$Gr_d = \frac{9.81 \times 3.05 \times 10^{-3} \times (70.5 - 27) \times 0.025^3}{[17.95 \times 10^{-6}]^2} = 6.4 \times 10^4.$$

$$[Nu_{av}]^{1/2} = 0.60 + 0.387 \left\{ \frac{(6.4 \times 10^4 \times 0.698)}{[1 + (0.559/0.698)^{9/16}]^{8/27}} \right\}^{1/6} = 2.89 \text{ or } Nu_{av} = 8.35$$

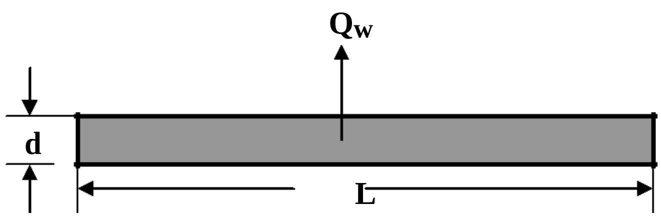
Hence $h_{av} = 8.35 \times 0.02865 / 0.025 = 9.5691$ W/(m^2-K).

The calculated value of h_{av} agrees with the assumed value within 5 % iteration is stopped.

The equilibrium temperature of the surface = $T_w = 27 + 30 /(\pi \times 0.025 \times 9.5691)$

$= 67\ ^0$C.

Solution:



Data:- L = 0.3 m; d = 0.01 m; Q_w = 550 W; $T_\infty = 20\ ^0$C;

$$\text{Wall heat flux} = q_w = Q_w/(\pi dL) = \frac{550}{(\pi \times 0.01 \times 0.3)} = 58357 \text{ W/m}^2$$

Since T_w is not known, fluid properties cannot be evaluated at the mean temperature and hence the problem has to be solved by trial and error procedure by assuming a suitable value for h_{av} and then check for the assumed value.

<u>Case(i):- When the heater is immersed in water</u>

For free convection in liquids the order of heat transfer coefficient is around 10 to 1000 W/(m^2-K).Let us assume h_{av} = 1000 W/(m^2-K).

Hence $T_w = T_\infty + q_w / h_{av} = 20 + 58357 / 1000 = 78.4\ ^0$C.

Mean film temperature = 0.5 x (20 + 78.4) = 49.2 ^{0}C.

Properties of water at 49.2 ^{0}C are: $\beta = 3.103 \times 10^{-3}$ 1 / K;

Pr = 3.68; k = 0.639 W/(m-K) ; $\nu = 0.5675 \times 10^{-6}$ m^2/s;

$$Gr_d = \frac{9.81 \times 3.103 \times 10^{-3} \times (78.4 - 20) \times (0.01)^3}{(0.5675 \times 10^{-6})^2} = 5.519 \times 10^6$$

$Ra_d = 5.519 \times 10^6 \times 3.68 = 2.03 \times 10^7$.

For horizontal cylinders Nusselt number is given by

$Nu_d = C\, Ra_d^{\,n}$ with C = 0.125 and n = 1/3 for this value of Ra_d.

Hence $Nu_d = 0.125 \times (2.03 \times 10^7)^{1/3} = 33.908$

Hence $h_{av} = 33.908 \times 0.639 / 0.01 = 2166.21$ W/(m^2 –K)

Trial 2:- Assume h_{av} = 2166.21 W/(m^2-K)

$T_w = 20 + 58357 / 2166.21 = 46.94^0$C.

Hence mean film temperature =0.5x(20 + 46.94) = 33.47 ^{0}C.

Properties of water at 33 ^{0}C ≈ 30^{0}C are: k = 0.6129 W/(m-K); Pr = 5.68; $\beta = 3.3 \times 10^{-3}$; $\nu = 0.831 \times 10^{-6}$ m^2 /s.

$$Gr_d = \frac{9.81 \times 3.3 \times 10^{-3} \times (46.94 - 20) \times (0.01)^3}{(0.831 \times 10^{-6})^2} = 2.2 \times 10^6.$$

$Ra_d = 2.2 \times 10^6 \times 5.68 = 1.24 \times 10^7$

Hence, $Nu_{av} = 0.125 \times (1.24 \times 10^7)^{0.333} = 28.77$

Therefore, $h_{av} = 28.77 \times 0.6129 / 0.01 = 1763.3$ W/(m^2 – K)

Trial 3:- Assume h_{av} =1763 W/(m^2-K)

$T_w = 20 + 58357 / 1763 = 53.1$ ^{0}C. Mean film temperature = 0.5 x (20 + 53.1) = 36.55 ^{0}C.

Properties of water at 40^{0}C are: k = 0.6280 W/(m-K); Pr = 4.34; $\beta = 3.19 \times 10^{-3}$ 1/K; $\nu = 0.657 \times 10^{-6}$ m^2/s.

$$Ra_d = \frac{9.81 \times 3.19 \times 10^{-3} \times (53 - 20) \times (0.01)^3}{(0.657 \times 10^{-6})^2} \times 4.34 = 1.03 \times 10^7$$

Hence, $Nu_{av} = 0.125 \times (1.03 \times 10^7)^{0.333} = 27.05$

Therefore, $h_{av} = 27.05 \times 0.6280 \times / 0.01 = 1698.74$ W/(m^2-K)

Since the calculated value of h_{av} agrees with the assumed value within 4%, iteration is stopped and the equilibrium temperature of the heater is calculated as

$$T_w = 20 + 58357 / 1698.74 = 54.35\ ^0C$$

Case (ii):- When the heater is exposed to air

When a heated surface is exposed to air the order of heat transfer coefficient varies between 5 and 20 $W/(m^2\text{-}K)$.

Trial 1:- Assume $h_{av} = 20\ W/(m^2\text{-}K)$

$T_w = 20 + 58357 / 20 = 2938\ ^0C$. Mean film temperature = 0.5 x (20 + 2938) = 1479 ^0C Properties of air at 1479 ^0C are : $\beta = 1/(1479 + 273) = 5.71 \times 10^{-4}$ 1/K; Pr = 0.7045; k = 0.108 W/(m-K); $\nu = 294.3 \times 10^{-6}\ m^2/s$.

$$Ra_d = \frac{9.81 \times 5.71 \times 10^{-4} \times (2938 - 20) \times (0.01)^3}{(294.3 \times 10^{-6})^2} \times 0.7045 = 133$$

Hence $h_{av} = 2.13 \times 0.108 / 0.01 = 23\ W/(m^2\text{-}K)$ This is 13% away from the assumed value and hence one more iteration is required.

$T_w = 20 + 58357 / 23 = 2557\ ^0C$; Mean film temperature = 0.5 x (20 + 2557) = 1289 ^0C Properties of air at 1289 ^0C are: $\beta = 6.4 \times 10^{-4}$ 1/K; $\nu = 244.34 \times 10^{-6}\ m^2/s$;Pr = 0.705; k = 0.0978 W/(m-K)

$$Ra_d = \frac{9.81 \times 6.4 \times 10^{-4} \times (2557 - 20) \times (0.01)^3}{(244.34 \times 10^{-6})^2} \times 0.705 = 188$$

$Nu_{av} = 0.850 \times (188)^{0.188} = 2.275$.Hence $h_{av} = 2.275 \times 0.0978 / 0.01 = 22.25\ W/(m^2 - K)$.

This value of h_{av} agrees with the assumed value within 4% and hence the iteration is stopped. The equilibrium temperature of the heater is therefore given by

$$T_w = 20 + 58357 / 22.25 = 2643\ ^0C.$$

UNIT-IV
HEAT TRANSFER WITH PHASE CHANGE

Introduction: Knowledge of heat transfer occurring during change of phase i.e. during condensation and boiling is very useful in a number of ways. For example in all power and refrigeration cycles, it is necessary to convert a liquid into a vapour and vice-versa. This is accomplished in boilers or evaporators and condensers.

Heat transfer coefficients in both condensation and boiling are generally much higher than those encountered in single phase processes. Values greater than 1000 W/(m^2-K) are almost always obtained. This fact has been used in several recent applications where it is desired to transfer high heat fluxes with modest temperature differences. An example is the "heat pipe" which is a device capable of transferring a large quantity of heat with very small temperature differences.

Film-wise and Drop-wise condensation:- Condensation occurs whenever a vapour comes into contact with a surface at a temperature lower than the saturation temperature of the vapour corresponding to its vapour pressure. The nature of condensation depends on whether the liquid thus formed wets the solid surface or does not wet the surface. If the liquid wets the surface, the condensate flows on the surface in the form of a film and the process is called "*film-wise condensation"*. If on the other hand, the liquid does not wet the surface, the condensate collects in the form of droplets, which either grow in size or coalesce with neighboring droplets and eventually roll of the surface under the influence of gravity. This type of condensation is called "*drop-wise condensation"*.

The rate of heat transfer during the two types of condensation processes is quite different. For the same temperature difference between the vapour and the surface, the heat transfer rates in drop-wise condensation are significantly higher than those in film- wise condensation. Therefore it is preferable to have drop-wise condensation from the designer"s point of view if the thermal resistance on the condensing side is a significant part of the total thermal resistance. However it is generally observed that, although drop-wise condensation may be obtained on new surfaces, it is difficult to maintain drop-wise condensation continuously and prolonged condensation results in a change to film-wise condensation. Therefore it is still the practice to design condensers under the conservative assumption that the condensation is of film type.

Nusselt's theory for laminar film-wise condensation on a plane vertical surface:-The problem of laminar film-wise condensation on a plane vertical surface was first analytically solved by Nusselt in 1916.He made the following simplifying assumptions in his analysis.

(i) The fluid properties are constant.

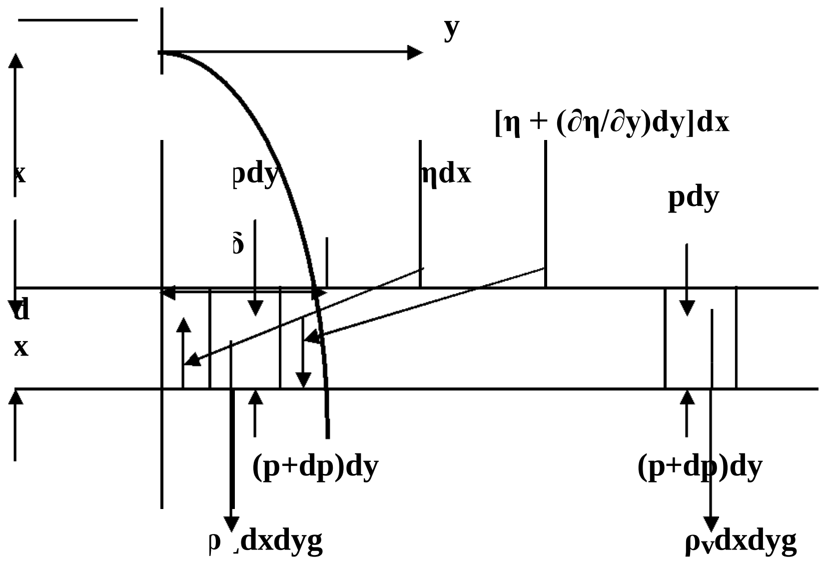
(ii) The plane surface is maintained at a uniform temperature, T_W which is less than the saturation temperature T_V of the vapour.

(iii) The vapour is stationary or has a very low velocity and so it does not exert any drag on the motion of the condensate: i.e., the shear stress at the liquid-vapour interface is zero.

(iv) The flow velocity of the condensate layer is so low that the acceleration of the condensate is negligible.

(v) The downward flow of the condensate under the action of gravity is laminar.

(vi) Heat transfer across the condensate layer is purely by conduction; hence the liquid temperature distribution is linear.



(a) Force balance on a condensate element

(b) Force balance on a vapour element at the same distance x from top

Fig. 8.1: Laminar film condensation on a vertical plate

Consider the film-wise condensation on a vertical plate as illustrated in Fig.8.1. Here „x‟ is the coordinate measured downwards along the plate, and „y‟ is the coordinate measured normal to the plate from the plate surface. The condensate thickness at any x is represented by δ [$\delta = \delta(x)$]. The velocity distribution u(y) at any location x can be determined by making a force balance on a condensate element of dimensions dx and dy in x and y directions as shown in Fig. 8.1(a). Since it is assumed that there is no acceleration of the liquid in x direction, Newton‟s second law in x direction gives

$$\rho_L dxdyg + pdy + [\eta + (\partial\eta/\partial y)dy]dx - \eta dx - (p + dp)dy = 0$$

or $$(\partial\eta/\partial y) = (dp/dx) - \rho_L g \quad \text{..(8.1)}$$

Expression for (dp/dx) in terms of vapour density ρ_V can be obtained by making a force

balance for a vapour element as shown in Fig. 8.1(b). The force balance gives

$$\rho_V dxdyg + pdy = (p + dp)\, dy$$

or $(dp/dx) = \rho_V g$ Substituting this expression for dp/dx in

Eq. (8.1) we have

$$(\partial\eta/\partial y) = (\rho_V - \rho_L)g$$

Since the flow is assumed to be laminar, $\eta = \mu_L(\partial u/\partial y)$

Therefore $\partial/\partial y\{\mu_L (\partial u/\partial y)\} = (\rho_v - \rho_L)g$

Integrating with respect to y we have $\mu_L (\partial u/\partial y) = (\rho_v - \rho_L)g\, y + C_1$

Or
$$(\partial u/\partial y) = \frac{(\rho_v - \rho_L)g\, y}{\mu_L} + \frac{C_1}{\mu_L} \quad \ldots\ldots(8.2)$$

Integrating once again with respect to y we get

$$u(y) = \frac{(\rho_v - \rho_L)g\, y^2}{2\,\mu_L} + \frac{C_1\, y}{\mu_L} + C_2 \quad \ldots\ldots(8.3)$$

The boundary conditions for the condensate layer are: (i) at y = 0, u = 0;

(ii) at $y = \delta$, $(\partial u/\partial y) = 0$.

Condition (i) in Eq. (8.3) gives $C_2 = 0$ and condition (ii) in Eq. (8.2) gives

$$0 = \frac{(\rho_v - \rho_L)g\, \delta}{2\,\mu_L} + \frac{C_1}{\mu_L}$$

Therefore
$$C_1 = -\frac{(\rho_v - \rho_L)g\, \delta}{2}$$

Substituting for C_1 and C_2 in Eq.(8.3) we get the velocity distribution in the condensate layer as

$$u(y) = \frac{g(\rho_L - \rho_v)}{\mu_L}[\,\delta y - (y^2/2)] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8.4)$$

If „m‟ is the mass flow rate of the condensate at any x then

$$m = \int_0^{\delta} \rho_L u\, dy$$

$$m = \int_0^{\delta} \rho_L\{\, g(\rho_L - \rho_v) / \mu_L\}[\,\delta y - (y^2/2)]dy$$

$$= \frac{g\,\rho_L(\rho_L - \rho_v)\,\delta^3}{3\,\mu_L} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8.5)$$

Hence $$dm = \frac{g\,\rho_L(\rho_L - \rho_v)\,\delta^2\,d\delta}{\mu_L}$$

Amount of heat transfer across the condensate element = dq = dm h_{fg}

Or $$dq = \frac{g\,\rho_L(\rho_L - \rho_v)\,\delta^2\,d\delta\,h_{fg}}{\mu_L} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8.6)$$

Energy balance for the condensate element shown in the figure can be written as

$$dq = k_L(T_v - T_w)dx / \delta$$

Or $$\frac{g\,\rho_L(\rho_L - \rho_v)\,\delta^2\,d\delta\,h_{fg}}{\mu_L} = k_L(T_v - T_w)dx / \delta \quad \ldots\ldots\ldots\ldots (8.6)$$

or $$\delta^3 d\delta = \frac{k_L\,\mu_L\,(T_v - T_w)dx}{g\,\rho_L(\rho_L - \rho_v)\,h_{fg}}$$

Integrating we get

$$\frac{\delta^4}{4} = \frac{k_L\,\mu_L\,(T_v - T_w)x}{g\,\rho_L(\rho_L - \rho_v)\,h_{fg}} + C_3$$

At x = 0, δ = 0. Hence C_3 = 0.

Therefore $$\frac{\delta^4}{4} = \frac{k_L\,\mu_L\,(T_v - T_w)x}{g\,\rho_L(\rho_L - \rho_v)\,h_{fg}}$$

or $$\boldsymbol{\delta = \left[\frac{4\,k_L\,\mu_L\,(T_v - T_w)x}{g\,\rho_L(\rho_L - \rho_v)\,h_{fg}}\right]^{1/4}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8.7)$$

Now

$$\frac{k_L (T_V - T_W)dx}{\delta} = h_x\, dx\, [T_V - T_W]$$

Therefore

$$h_x = \frac{k_L}{\delta} = \left[\frac{g\, \rho_L (\rho_L - \rho_V)\, h_{fg}\, k_L^3}{4\, \mu_L (T_V - T_W)x}\right]^{1/4}$$

Or

$$\mathbf{h_x = 0.707\left[\frac{g\, \rho_L (\rho_L - \rho_V)\, h_{fg}\, k_L^3}{\mu_L (T_V - T_W)x}\right]^{1/4}} \quad \ldots\ldots\ldots\ldots\ldots \quad (8.8)$$

The local Nusselt number Nu_X can therefore be written as

$$\mathbf{Nu_X = \frac{h_x x}{k_L} = 0.707\left[\frac{g\, \rho_L (\rho_L - \rho_V)\, h_{fg}\, x^3}{\mu_L (T_V - T_W)k_L}\right]^{1/4}} \quad \ldots\ldots\ldots\ldots\ldots \quad (8.8)$$

The average heat transfer coefficient for a length L of the plate is given by

$$h_{av} = (1/L) \int_0^L h_x dx \quad \ldots\ldots\ldots\ldots\ldots \quad (8.9)$$

It can be seen from Eq. (8.8) that $h_X = C\, x^{-1/4}$, where C is a constant given by

Or

$$C = 0.707\left[\frac{g\, \rho_L (\rho_L - \rho_V)\, h_{fg}\, k_L^3}{\mu_L (T_V - T_W)}\right]^{1/4} \quad \ldots\ldots\ldots\ldots\ldots (8.10)$$

Hence

$$h_{av} = (1/L)\, C \int_0 x^{-1/4}\, dx = (C / L)\, (4/3)\, L^{-1/4} = (4/3)C\, L^{-1/4}$$

Substituting for C from Eq. (8.10) we have

$$\mathbf{h_{av} = 0.943\left[\frac{g\, \rho_L (\rho_L - \rho_V)\, h_{fg}\, k_L^3}{\mu_L (T_V - T_W)L}\right]^{1/4} = (4/3)h_x|_{x = L}} \quad \ldots\ldots\ldots\ldots\ldots \mathbf{(8.11)}$$

Condensation on Inclined Surfaces : Nusselt,s analysis given above can readily be extended to inclined plane surfaces making an angle θ with the horizontal plane as shown in Fig. 8.2.

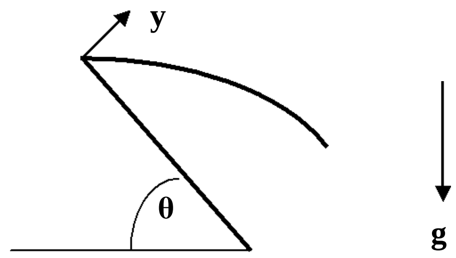


Fig. 8.2 : Condensation on an inclined plane surface

The component of the gravitational force along the length of the pate is g sin θ.The expressions for local and average heat transfer coefficients can therefore be written as

$$h_x = 0.707\left[\frac{g \sin\theta \rho_L (\rho_L - \rho_v) h_{fg} k_L^3}{\mu_L (T_v - T_w) x}\right]^{1/4} \quad \text{.......................(8.12)}$$

and

$$h_{av} = 0.943\left[\frac{g \sin\theta \rho_L (\rho_L - \rho_v) h_{fg} k_L^3}{\mu_L (T_v - T_w) L}\right]^{1/4} = (4/3) h_x|_{x=L} \quad \text{..............................(8.13)}$$

Condensation on a horizontal tube: The analysis of heat transfer for condensation on the outside surface of a horizontal tube is more complicated than that for a vertical surface. Nusselt,s analysis for laminar film-wise condensation on the surface of a horizontal tube gives the average heat transfer coefficient as

$$h_{av} = 0.725\left[\frac{g \rho_L (\rho_L - \rho_v) h_{fg} k_L^3}{\mu_L (T_v - T_w) D}\right]^{1/4} \quad \text{...............(8.14)}$$

where D is the outside diameter of the tube. A comparison of equations (8.11) and (8.14) for condensation on a vertical tube of length L and a horizontal tube of diameter D gives

$$\frac{[h_{av}]_{vertical}}{[h_{av}]_{horizontal}} = \frac{0.943}{0.725}(D/L)^{1/4} = 1.3\,(D/L)^{1/4} \quad \text{..................................... (8.15)}$$

This result implies that for a given value of ($T_V - T_W$), the average heat transfer coefficient for a vertical tube of length L and a horizontal tube of diameter D becomes equal when L = 2.856 D.For example when L = 100 D, theoretically $[h_{av}]_{horizontal}$ would be 2.44 times $[h_{av}]_{vertical}$. Therefore horizontal tube arrangements are generally preferred to vertical tube arrangements in condenser design.

Condensation on horizontal tube banks: Condenser design generally involves horizontal tubes arranged in vertical tiers as shown in Fig. 8.3 in such a way that the

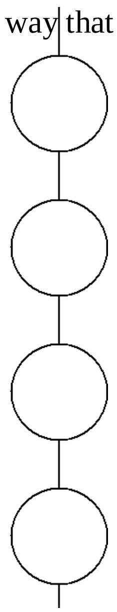


Fig. 8.3 : Film-wise condensation on horizontal tubes arranged in a vertical tier.

condensate from one tube drains on to tube just below. If it is assumed that the drainage from one tube flows smoothly on to the tube below, then for a vertical tier of N tubes each of diameter D, the average heat transfer coefficient for N tubes is given by

$$[h_{av}]_{N\ tubes} = 0.725 \left[\frac{g\rho_L(\rho_L - \rho_V)h_{fg}\, k_L^3}{\mu_L(T_V - T_W)\, N\, D} \right]^{1/4} = \frac{1}{N^{1/4}} [h_{av}]_{1\ tube} \quad \ldots\ldots (8.16)$$

This relation generally gives a conservative value for the heat transfer coefficient. Since some turbulence and some disturbance of condensate are unavoidable during drainage, the heat transfer coefficient would be more than that given by the above equation.

Reynolds number for condensate flow: Although the flow hardly changes to turbulent flow during condensation on a single horizontal tube, turbulence may start at the lower portions of a vertical tube. When the turbulence occurs in the condensate film, the average heat transfer coefficient begins to increase with the length of the tube in contrast to its decrease with the length for laminar film condensation. To establish a criterion for transition from laminar to turbulent flow, a *"Reynolds number for condensate flow"* is defined as follows.

$$Re = \frac{\rho_L \, u_{av} \, D_h}{\mu_L} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8.17)$$

where u_{av} is the average velocity of the condensate film and D_h is the hydraulic diameter for the condensate flow given by

$$D_h = \frac{4 \text{ x (Cross sectional area for condensate flow)}}{\text{Wetted Perimeter}} = \frac{4A}{P}$$

Therefore

$$Re = \frac{4A \, \rho_L \, u_{av}}{P \, \mu_L} = \frac{4M}{P \, \mu_L} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8.18)$$

where M is mass flow rate of condensate at the lowest part of the condensing surface in kg/s. The wetted perimeter depends on the geometry of the condensing surface and is given as follows.

πD …..For vertical tube of outside diameter D.................(8.19 a)

P = 2L …...For horizontal tube of length L ....(8.19 b)

W ….. For vertical or inclined plate of width W.............. (8.19 c)

Experiments have shown that the transition from laminar to turbulent condensation takes place at a Reynolds number of 1800. The expression for average heat transfer coefficient for a vertical surface [Eq.(8.11)] can be expressed as follows.

$$h_{av} = 0.943 \left[\frac{g\,\rho_L(\rho_L - \rho_v)\,k_L^3\,h_{fg}}{\mu_L(T_v - T_w)} \right]^{1/4}$$

Generally $\rho_L >> \rho_v$. Therefore

$$h_{av} = 0.943 \left[\frac{g\,\rho_L^2\,k_L^3\,h_{fg}}{\mu_L(T_v - T_w)} \right]^{1/4} \quad \text{………………..(8.20)}$$

The above equation can be arranged in the form

$$h_{av}\,[\nu_L^2 / (g k_L^3)]^{1/3} = 1.47\,Re_L^{-1/3} \quad \text{…………………………………(8.21)}$$

The above equation is valid for $Re_L < 1800$.
It has been observed experimentally that when the value of the film Reynolds number is greater than 30, there are ripples on the film surface which increase the value of the heat transfer coefficient. Kutateladze has proposed that the value of the local heat transfer coefficient be multiplied by $0.8(RE / 4)^{0.11}$ to account for the ripples effect.

Using this correction it can be shown that

$$(h_{av}/k_L)(\nu_L^2/g)^{1/3} = \frac{Re_L}{[1.08\,Re_L^{1.22} - 5.2]} \quad \text{………………(8.22)}$$

Turbulent film condensation: For turbulent condensation on a vertical surface, Kirkbride has proposed the following empirical correlation based on experimental data.

$$h_{av}\,[\nu_L^2 / (g k_L^3)]^{1/3} = 0.0077\,(Re_L)^{0.4} \quad \text{……………………………(8.23)}$$

In the above correlation the physical properties of the condensate should be evaluated at the arithmetic mean temperature of T_v and T_w.

Film condensation inside horizontal tubes: In all the correlations mentioned above, it is assumed that the vapour is either stationary or has a negligible velocity. In practical applications such as condensers in refrigeration and air conditioning systems, vapour condenses on the inside surface of the tubes and so has a significant velocity. In such situations the condensation phenomenon is very complicated and a simple analytical treatment is not possible. Consider, for example, the film condensation on the inside surface of a long vertical tube.

The upward flow of vapour retards the condensate flow and causes thickening of the condensate layer, which in turn decreases the condensation heat transfer coefficient. Conversely the down ward flow of vapour decreases the thickness of the condensate film and hence increases the heat transfer coefficient.

Chato recommends the following correlation for condensation at low vapour velocities inside horizontal tubes:

$$h_{av} = 0.555 \left[\frac{g\, \rho_L(\rho_L - \rho_v)\, k^3\, h^*_{fg}}{\mu_L(T_v - T_w)\, D} \right]^{1/4} \qquad \text{.........................(8.24 –a)}$$

where $$h^*_{fg} = h_{fg} + (3/8)c_{p,L}(T_v - T_w) \qquad \text{.....................................(8.24 –b)}$$

This result has been developed for the condensation of refrigerants at low Reynolds number [$Re_v = (\rho_v u_v D) / \mu_v < 35{,}000$; Re_v should be evaluated at the inlet conditions.]

For higher flow rates, Akers, Deans and Crosser propose the following correlation for the average condensation heat transfer coefficient on the inside surface of a horizontal tube of diameter D:

$$\frac{h_{av} D}{k} = 0.026\, Pr^{1/3} \left[Re_L + Re_v(\rho_L / \rho_v)^{1/2}\right]^{0.8} \qquad \text{................(8.25)}$$

where $Re_L = (4M_L) / (\pi D \mu_L)$: $Re_v = (4M_v)/ (\pi D \mu_v)$ ....(8.26)

The above equation correlates the experimental data within 50 % for $Re_L > 5000$

and $Re_v > 20{,}000$.

Illustrative examples on film wise condensation:

Example 8.1: *Saturated steam at 1.43 bar condenses on a 1.9 cm OD vertical tube which is 20 cm long. The tube wall is at a uniform temperature of 109 ^0C . Calculate the average heat transfer coefficient and the thickness of the condensate film at the bottom of the tube.*

Solution: Data:- T_v = Saturation temperature at 1.43 bar = 110 ^{0}C (from steam tables) T_w = 109 ^{0}C ; Characteristic length = L = 0.2 m ; D = 0.019 m ;

To find : (i) h_{av} ; (ii) $\delta(x)|_{x=L}$;

Mean film temperature of the condensate (water) = 0.5 x (110 + 109) = 109.5 ^{0}C.

Properties of water at 109.5 ^{0}C are: ρ_L = 951.0 kg/m^3; μ_L = 258.9 x 10^{-6} N-s / m^2; k = 0.685 W/(m-K); ν = 0.2714 x 10^{-6} m^2/s; h_{fg} = 2230 kJ/kg. Also $\rho_L >>> \rho_v$.

Let us assume that the condensate flow is laminar and later check for this assumption.

$$h_{av} = 0.943 \left[\frac{g \rho_L^2 k_L^3 h_{fg}}{\mu_L (T_v - T_w) L} \right]^{1/4}$$

$$\text{Hence } h_{av} = 0.943 \times \left[\frac{9.81 \times (951)^2 \times (0.685)^3 \times 2230 \times 10^3}{258.9 \times 10^{-6} \times (110 - 109) \times 0.2} \right]^{1/4}$$

$$= 17{,}653 \text{ W / (m}^2\text{-K)}$$

(ii) $h_{av} = (4 / 3)h_x|_{x=L}$ or $h_x|_{x=L} = ¾ \times h_{av} = 0.75 \times 17{,}653 = 13{,}240$ W/(m^2-K).

Therefore $\delta(x)|_{x=L} = k_L / h_x|_{x=L} = 0.685 / 13240 = 5.174 \times 10^{-5}$ m = 0.0517 mm.

Check for Laminar flow assumption:- The relation between h_{av} and Reynolds number at the bottom of the tube is given by

$h_{av} [\nu_L^2 / (g k_L^3)]^{1/3} = 1.47 \, Re_L^{-1/3}$ or $Re_L = (1.47 / h_{av})^3 (g k_L^3 / \nu_L^2)$

Hence $Re_L = (1.47 / 17{,}653)^3 [9.81 \times 0.685^3 / \{0.2714 \times 10^{-6}\}^2]$

= 24.72

Since $Re_L < 1800$, our assumption that condensate flow is laminar is correct.

Example 8.2:- *Saturated steam at 80 ^{0}C condenses as a film on a vertical plate 1 m high. The plate is maintained at a uniform temperature of 70 ^{0}C. Calculate the average heat transfer coefficient and the rate of condensation. What would be the corresponding values if the effect of ripples is taken into consideration.*

Solution: Data:- $T_v = 80\ ^0$C; $T_w = 70\ ^0$C; Mean film temperature =0.5 x (80 + 70) = 75 ^{0}C.

Properties of condensate (liquid water) at 75 ^{0}C are: $\rho_L = 974.8$ kg/m^3;

$k_L = 0.672$ W /(m-K) ; $\mu_L = 381 \times 10^{-6}$ N-s/m^2; h_{fg} at 80 ^{0}C = 2309 kJ/kg-K;

$\nu_L = 0.391 \times 10^{-6}$ m^2/s.Charecteristic length = L = 1.0 m.

Assuming laminar film condensation the average heat transfer coefficient is given by

$$h_{av} = 0.943 \left[\frac{g \rho_L^2 k_L^3 h_{fg}}{\mu_L (T_v - T_w)} \right]^{1/4}$$

$$= 0.943 \times \left[\frac{9.81 \times (974.8)^2 \times (0.672)^3 \times 2309 \times 10^3}{381.6 \times 10^{-6} \times (80 - 70) \times 1.0} \right]^{1/4} = 6066.6 \text{ W}/(\text{m}^2 - \text{K}).$$

$$\text{Condensate rate} = M = \frac{h_{av} L (T_v - T_w)}{h_{fg}} = \frac{6066.6 \times 1.0 \times (80 - 70)}{2309 \times 10^3} \quad 0.0263 \text{ kg/s.}$$

Check for laminar flow assumption :- $Re_L = \dfrac{4M}{\mu_L P}$, where P = width of the plate for vertical flat plate. Hence $Re_L = \dfrac{4 \times 0.0263}{381 \times 10^{-6}} = 276$

Since $Re_L < 1800$, the condensate flow is laminar.

Since $Re_L > 30$, it is clear that the effects of ripples have to be considered.

$$\text{Now } Re_L = \frac{4M}{\mu_L P} = \frac{4 h_{av} L (T_v - T_w)}{\mu_L P h_{fg}}$$

$$\text{Hence } h_{av} = \frac{Re_L \mu_L P h_{fg}}{4L(T_v - T_w)} \quad \ldots\ldots(1)$$

When the effects of ripples are considered the relation between Re_L and h_{av} is given by Eq.(8.22) as follows:

$$1.08 Re_L^{1.22} - 5.2 = \frac{Re_L}{(h_{av}/k_L)(\nu_L^2/g)^{1/3}}$$ Substituting for h_{av} from Eq.(1) we have

$$1.08 Re_L^{1.22} - 5.2 = \frac{4L (T_v - T_w) k_L (g/\nu_L^2)^{1/3}}{\mu_L P h_{fg}}$$

$$1.08 Re_L^{1.22} - 5.2 = \frac{4 \times 1 \times (80 - 70) \times 0.672 \times \{9.81/(0.391 \times 10^{-6})^2\}^{1/3}}{381.6 \times 10^{-6} \times 1.0 \times 2309 \times 10^3}$$

$1.08\ Re_L^{1.22} - 5.2 = 1221.3$. Or $Re_L = 319.4$

Hence from Eq.(1) we have $$h_{av} = \frac{319.4 \times 381.6 \times 10^{-6} \times 1.0 \times 2309 \times 10^3}{4 \times 1.0 \times (80-70)}$$

$= 7036\ W/(m^2 - K)$.

Hence $$M = \frac{h_{av} L (T_v - T_w)}{h_{fg}} = \frac{7036 \times 1.0 \times (80-70)}{2309 \times 10^3} = 0.03047\ kg/s.$$

[It can be seen that the ripples on the surface increase the heat transfer coefficient by about 15 %].

Example 8.3:- *Air free saturated steam at 65 ^{0}C condenses on the surface of a vertical tube of OD 2.5 cm. The tube surface is maintained at a uniform temperature of 35 ^{0}C. Calculate the length of the tube required to have a condensate flow rate of 6 x 10 $^{-3}$kg/s.*

Solution: Data:- $T_v = 65\ ^0C$; $T_w = 35\ ^0C$; $D_0 = 0.025$ m; $M = 6 \times 10^{-3}$ kg/s. To

find length of the tube, L.

Mean film temperature = 0.5 x (65 + 35) = 50 ^{0}C.Properties of condensate

(liquid water) at 50 ^{0}C are: $k_L = 0.640$ W/(m-K); $\mu_L = 0.562 \times 10^{-3}$ N-s/m^2; $\rho_L = 990$

kg/m^3; At 65 ^{0}C, $h_{fg} = 2346 \times 10^3$ J/(kg-K).

$$\text{Reynolds number} = Re = \frac{4M}{\mu_L \pi D_o} = \frac{4 \times 6 \times 10^{-3}}{0.562 \times 10^{-3} \times \pi \times 0.025} = 544$$

Since Re < 1800 flow is laminar. It is more convenient to use Eq.(8.21)

$$h_{av} [\nu_L^2 / (g k_L^3)]^{1/3} = 1.47\ Re_L^{-1/3}$$

or $$h_{av} = 1.47 Re_L^{-1/3} \left[\frac{(g k_L^3)}{\nu_L^2}\right]^{1/3} = \frac{1.47 \times (544)^{-1/3} \times [9.81 \times 0.64^3]^{1/3}}{(0.562 \times 10^{-3} / 990)^2}$$

$= 3599\ W/(m^2 - K)$

Heat balance equation gives $M\, h_{fg} = h_{av}\, \pi D_o L\, [T_v - T_w]$

Therefore
$$L = \frac{M\, h_{fg}}{h_{av}\, \pi D_o\, [T_v - T_w]} = \frac{6 \times 10^{-3} \times 2346 \times 10^{3}}{3599 \times \pi \times 0.025 \times (65 - 35)}$$

$$= 1.66 \text{ m}$$

Example 8.4:- *Air free saturated steam at 85 ^0C condenses on the outer surfaces of 225 horizontal tubes of 1.27 cm OD, arranged in a 15 x 15 array. Tube surfaces are maintained at a uniform temperature of 75 ^0C. Calculate the total condensate rate per one metre length of the tube.*

Solution: Data:- $T_v = 85\ ^0C$; $T_w = 75\ ^0C$; $D_o = 0.0127$ m; L = 1 m; Number of tubes in vertical tier = N = 15 ; Total number of tubes = n = 225;

Mean film temperature = 0.5 x (85 + 75) = 80 ^0C. Properties of the condensate (liquid water) are: $k_L = 0.668$ W/(m-K); $\mu_L = 0.355 \times 10^{-3}$ N-s/m^2; $\rho_L = 974$ kg/m^3;

At 85 ^0C, $h_{fg} = 2296 \times 10^{3}$ J/(kg-K).

For N horizontal tubes arranged in a vertical tier, h_{av} is given by

$$h_{av} = 0.725 \left[\frac{g\, \rho_L^{2}\, h_{fg}\, k_L^{3}}{\mu_L (T_v - T_w) N D_o} \right]^{1/4}$$

$$h_{av} = \frac{0.725 \times [9.81 \times (974)^2 \times (0.668)^3]^{1/4}}{[0.355 \times 10^{-3} \times (85 - 75) \times 15 \times 0.0127]^{1/4}} = 7142 \text{ W/(m}^2 - \text{K)}$$

$$Q = h_{av} A_{total} (T_v - T_w) = h_{av}\, n\, \pi\, D_o L\, (T_v - T_w)$$

$$= 7142 \times 225 \times \pi \times 0.0127 \times 1 \times (85 - 75) = 641.14 \times 10^{3} \text{ W}$$

Mass flow rate of condensate = $M = Q / h_{fg} = 641.14 \times 10^{3} / 2296 \times 10^{3} = 0.28$ kg/ (s-m)

Example 8.5:- *Superheated steam at 1.43 bar and 200 ^0C condenses on a 1.9 cm OD vertical tube which is 20 cm long. The tube wall is maintained at a uniform temperature of 109 ^0C. Calculate the average heat transfer coefficient and the thickness of the condensate at the bottom of the tube. Assume c_p for super heated steam as 2.01 kJ/(kg-K).*

Solution: With a superheated vapour, condensation occurs only when the surface temperature is less than the saturation temperature corresponding to the vapour pressure. Therefore for a superheated vapour, the amount of heat to be removed per unit mass to condense it is given by

$$Q / M = h_{fg} + c_{pv}(T_v – T_{sat})$$

Where c_p is the specific heat of superheated steam and T_{sat} is the saturation temperature corresponding to the vapour pressure. If it is assumed that the liquid – vapour interphase is at the saturation temperature, then Eq.(8.20) still holds good with h_{fg} replaced by $h_{fg} + c_{pv}(T_v – T_{sat})$.

Hence

$$h_{av} = 0.943 \left[\frac{g\,\rho_L^2\,k_L^3\,\{h_{fg} + c_{pv}(T_v – T_{sat})\}}{\mu_L(T_{sat} – T_w)L} \right]^{1/4}$$

At 1.43 bar, T_{sat} = 110 ^{0}C.Mean film temperature = 0.5 x (110 + 109) = 109.5 ^{0}C.

Properties of the condensate at 109.5 ^{0}C are: k_L = 0.685W/(m-K); μ_L = 0.259 x 10^{-3} N-s/m^2; ρ_L= 951 kg/m^3;At 1.43 bar, h_{fg} = 2230 x10^3 J/(kg-K).

$$h_{av} = 0.943 \times \left[\frac{9.81 \times (951)^2 \times (0.685)^3 \times \{2230 \times 10^3 + 2010 \times (200 – 110)\}}{0.259 \times 10 \quad \times (110 – 109) \times 0.2} \right]^{1/4}$$

= 18,000 W /(m^2 – K).

Hence $h_x|_{x=L}$ = (¾) x 18000 = 13,500 W / (m^2 – K).

$\delta(x)|_{x=L} = k_L / h_x|_{x=L}$ = 0.685 / 13,500 = 5.07 x 10^{-5}m

Example 8.6:- *Air free saturated steam at 70 ^{0}C condenses on the outer surface of a 2.5 cm OD vertical tube whose outer surface is maintained at a uniform temperature of 50 ^{0}C. What length of the tube would produce turbulent film condensation?*

Solution: Data:- T_v = 70 ^{0}C; T_w = 50 ^{0}C; D_o = 0.025 m; Vertical tube.

To find L such that Re = 1800.

Mean film temperature = 0.5 x (70 + 50) = 60 ^{0}C. Properties of the condensate (liquid water) are : k_L = 0.659W/(m-K); μ_L = 0.4698 x 10^{-3} N-s/m^2; ρ_L= 983.2 kg/m^3;

At 70 ^{0}C h_{fg} = 2358 x 10 ^{3} J/(kg-K).

$$Re = 4M / (\mu_L \pi D_o) \text{ or } M = \frac{Re\,(\mu_L \pi D_o)}{4} = \frac{1800 \times 0.4698 \times 10^{-3} \times \pi \times 0.025}{4}$$

$$= 0.0166 \text{ kg / s.}$$

For turbulent flow $\quad h_{av}\,[\nu_L^{\,2} / (g k_L^{\,3})\,]^{\,1/3} = 0.0077\,(Re_L)^{\,0.4}$

Or $\quad h_{av} = 0.0077\,(Re_L)^{\,0.4}[\nu_L^{\,2} / (g k_L^{\,3})\,]^{\,-1/3}$

Hence $h_{av} = 0.0077 \times (1800)^{0.4} \times [\,(0.4698 \times 10^{-3}/983.2)^2 / (9.81 \times 0.659^3)\,]^{-1/3} = 3563.4$ W / (m^2 – K).

Heat balance equation is $M\,h_{fg} = h_{av}\,\pi\,D_o L\,(T_v - T_w)$

Hence
$$L = \frac{M\,h_{fg}}{h_{av}\,\pi\,D_o\,(T_v - T_w)} = \frac{0.0166 \times 2358 \times 10^3}{3563.4 \times \pi \times 0.025 \times (70 - 50)}$$

$$= 7 \text{ m}$$

Example 8.7:- *Saturated steam at 100 ^{0}C condenses on the outer surface of a 2 m long vertical plate. What is the temperature of the plate below which the condensing film at the bottom of the plate will become turbulent?*

Solution: Data:- T_v = 100 ^{0}C; L = 2 m. Since T_w is not known, properties of the condensate at the mean film temperature cannot be determined and therefore the problem has to be solved by trial and error procedure as follows:

Trial 1:- The properties of the condensate are read at T_v = 100 ^{0}C. The properties are k_L = 0.683 W/(m-K); μ_L = 0.2824 x 10 $^{-3}$ N-s/m^2; ρ_L= 958.4 kg/m^3; At 100 ^{0}C,

h_{fg} = 2257 x 10 ^{3} J/kg-K.

Since the flow has to become turbulent at the bottom of the plate we have

$$h_{av} = 0.0077\,(Re_L)^{\,0.4}[\nu_L^{\,2} / (g k_L^{\,3})\,]^{\,-1/3} \text{ with } Re_L = 1800$$

Hence $$h_{av} = 0.0077 \times (1800)^{0.4} \times \left[\frac{9.81 \times 0.683^3}{(0.2824 \times 10^{-3} / 958.4)^2} \right]^{1/3}$$

$$= 5098 \text{ W} / (\text{m}^2 - \text{K})$$

Now $M h_{fg} = h_{av} L W (T_v - T_w)$

Or $T_w = T_v - (M/W)h_{fg} / (h_{av} L)$. But $Re_L = 4M / (\mu_L W)$ or $M/W = Re_L \mu_L / 4$.

Therefore $$T_w = T_v - \frac{Re_L \mu_L h_{fg}}{4 h_{av} L} = 100 - \frac{1800 \times 0.2824 \times 10^{-3} \times 2257 \times 10^3}{4 \times 5098 \times 2}$$

$$= 72\ {}^0C$$

Trial 2:- Assume $T_w = 72\ {}^0C$. Mean film temperature = 0.5 x (100 + 72) = 86 ^0C. Properties of the condensate at 86 ^0C ; $k_L = 0.677$ W/(m-K); $\mu_L = 0.3349 \times 10^{-3}$ N-s/m^2;

ρ_L= 968.5kg/m^3; At 100 ^0C, $h_{fg} = 2257 \times 10^3$ J/(kg-K).

Hence $$h_{av} = 0.0077 \times (1800)^{0.4} \times \left[\frac{9.81 \times 0.677^3}{(0.3349 \times 10^{-3} / 968.5)^2} \right]^{1/3}$$

$$= 4541 \text{ W} /(\text{m}^2 - \text{K}).$$

Therefore $$T_w = 100 - \frac{1800 \times 0.3349 \times 10^{-3} \times 2257 \times 10^3}{4 \times 4541 \times 2} = 60\ {}^0C$$

Since the calculated value of T_w is quite different from the assumed value, one more iteration is required.

Trial 3:- Assuming $T_w = 60\ {}^0C$ and proceeding on the same lines as shown in trial 2 we get $h_{av} = 4365$ W /(m^2 – K) and hence $T_w = 59^0C$. This value is very close to the value assumed (difference is within 2 %). The iteration is stopped. Hence $T_w = 59\ {}^0C$.

Example 8.8:- *Air free saturated steam at 90 ^0C condenses on the outer surface of a 2.5 cm OD, 6 m long vertical tube, whose outer surface is maintained at a uniform temperature of 60 ^0C. Calculate the total rate of condensation of steam at the tube surface.*

Solution: Data:- $T_v = 90\ ^0C$; $T_w = 30\ ^0C$; $D_o = 0.025$ m; L = 6 m. Vertical tube.

Mean film temperature = 0.5 x (90 + 60) = 75 ^0C. Properties of the condensate at 75 ^0C are: k_L = 0.671 W/(m-K); $\mu_L = 0.3805 \times 10^{-3}$ N-s/m^2; ρ_L= 974.8 kg/m^3; At 90 ^0C, h_{fg} = 2283 x 10^3 J/(kg-K).

We do not know whether the condensate flow is laminar or turbulent. We start the calculations assuming laminar flow and then check for laminar flow condition. For laminar flow

$$h_{av} = 0.943 \left[\frac{g \rho_L^2 k_L^3 h_{fg}}{\mu_L (T_v - T_w) L} \right]^{1/4}$$

$$\text{Hence } h_{av} = 0.943 \times \left[\frac{9.81 \times (974.8)^2 \times (0.671)^3 \times 2283 \times 10^3}{0.3805 \times 10^{-3} \times (90 - 60) \times 6} \right]^{1/4}$$

$= 2935.3$ W /(m^2 – K).

For laminar flow $\qquad h_{av} [\nu_L^2 / (g k_L^3)]^{1/3} = 1.47\, Re_L^{-1/3}$

Or $\qquad Re_L = (1.47 / h_{av})^3 (g k_L^3 / \nu_L^2)$

$= (1.47 / 2935.3)^3 \times [9.81 \times 0.671^3 \times 974.8^2 / (0.3805 \times 10^{-3})^2]$

= 2443

Since $Re_L > 1800$, flow is turbulent.

For turbulent flow $\qquad h_{av} [\nu_L^2 / (g k_L^3)]^{1/3} = 0.0077\, (Re_L)^{0.4}$

Or $\qquad h_{av} = 0.0077\, (Re_L)^{0.4} / [\nu_L^2 / (g k_L^3)]^{1/3}$ .... (1)

$Re_L = 4M / (\mu_L \pi D_o)$. But $M h_{fg} = h_{av} \pi D_o L (T_v - T_w)$ or $M / (\pi D_o) = h_{av} L (T_v - T_w) / h_{fg}$

$$\text{Therefore} \qquad Re_L = \frac{4\, h_{av} L (T_v - T_w)}{h_{fg}\, \mu_L}$$

Substituting this expression for Re_L in equation (1) we have

$$h_{av} = 0.0077 \left[\frac{4\, h_{av} L\,(T_v - T_W)}{h_{fg}\,\mu_L}\right]^{0.4} [\nu_L^2/(g k_L^3)]^{-1/3}$$

$$(h_{av})^{0.6} = 0.0077 \left[\frac{4\, L\,(T_v - T_W)}{h_{fg}\,\mu_L}\right]^{0.4} [\nu_L^2/(g k_L^3)]^{-1/3}$$

$$= 0.0077 \times \left[\frac{4 \times 6 \times (90 - 60)}{2283 \times 10^3 \times 0.3805 \times 10^{-3}}\right]^{0.4} \times \left[\frac{(0.3805 \times 10^{-3})^2}{974.8^2 \times 9.81 \times (0.671)^3}\right]^{-1/3}$$

$= 192$. Hence $h_{av} = [192]^{1/0.6} = 6390\ \mathrm{W/(m^2-K)}$.

$$\text{Therefore } M = \frac{h_{av} \pi D_o L\,(T_v - T_W)}{h_{fg}} = \frac{6390 \times \pi \times 0.025 \times 6 \times (90 - 60)}{2283 \times 10^3} = 0.0396\ \mathrm{kg/s}$$

Dropwise Condensation: Experimental investigations on condensation have indicated that, if traces of oil are present in steam and the condensing surface is highly polished, the condensate film breaks into droplets. This type of condensation is called "*drop wise condensation*". The droplets grow, coalesce and run off the surface, leaving agreater portion of the condensing surface exposed to the incoming steam. Since the entire condensing surface is not covered with a continuous layer of liquid film, the heat transfer rate for ideal drop wise condensation is much higher than that for film wise condensation.

The heat transfer coefficient may be 2 to 3 times greater for drop wise condensation than for film wise condensation. Hence considerable research has been done with the objective of producing long lasting drop wise condensation. Various types of chemicals have been tried to promote drop wise condensation. Continuous drop wise condensation, obtainable with different promoters varies between 100 to 300 hours with pure steam and are shorter with industrial steam. Failure occurs because of fouling or oxidation of the surface, or by the flow of the condensate or by a combination of these effects.

It is unlikely that long lasting drop wise condensation can be produced under practical conditions by a single treatment of any of the promoters currently available. Therefore in the analysis of a heat exchanger involving condensation of steam, it is recommended that film wise condensation be assumed for the condensing surface.

Boiling Types: When evaporation occurs at a solid-liquid interface, it is called as *"boiling"*. The boiling process occurs when the temperature of the surface T_W exceeds the saturation temperature T_{sat} corresponding to the liquid pressure. Heat is transferred from the solid surface to the liquid, and the appropriate form of Newton"s law of cooling is

$$q_W = h\,[T_W - T_{sat}] = h\,\Delta T_e \quad \text{...(8.27)}$$

Where $\Delta T_e = [T_W - T_{sat}]$ and is termed as the *"excess temperature"*. The boiling process is characterized by the formation of vapour bubbles which grow and subsequently detach from the surface. Vapour bubble growth and dynamics depend, in a complicated manner, on the excess temperature ΔT_e, the nature of the surface, and the thermo-physical properties of the fluid, such as its surface tension. In turn the dynamics of vapour bubble growth affect fluid motion near the surface and therefore strongly influence the heat transfer coefficient.

Boiling may occur under varying conditions. For example if the liquid is quiescent and if its motion near the surface is due to free convection and due to mixing induced by bubble growth and detachment, then such a boiling process is called *"pool boiling"*. In contrast in *"forced convection boiling"*, the fluid motion is induced by an external means as well as by free convection and bubble induced mixing. Boiling may also be classified as *"sub-cooled boiling"* and *"saturated boiling"*. In sub-cooled boiling, the temperature of the liquid is below the saturation temperature and the bubbles formed at the surface may condense in the liquid. In contrast, in saturated boiling, the temperature of the liquid slightly exceeds the saturation temeperature, Bubbles formed at the surface are then propelled through the liquid by buoyancy forces, eventually escaping from a free surface.

Pool Boiling Regimes: The first investigator who established experimentally the different regimes of pool boiling was Nukiyama. He immersed an electric resistance wire into a body of saturated water and initiated boiling on the surface of the wire by passing electric current through it. He determined the heat flux as well as the temperature from the measurements of current and voltage. Since the work of Nukiyama, a number of investigations on pool boiling have been reported. Fig. 8.4 illustrates the characteristics of pool boiling for water at atmospheric pressure. This boiling curve illustrates the variation of heat flux or the heat transfer coefficient as a function of excess temperature ΔT_e. This curve pertains to water at 1 atm pressure.From Eq. (8.27) it can seen that q_W depends on the heat transfer coefficient h and the excess temperature ΔT_e.

Free Convection Regime(up to point A):- Free convection is said to exist if $\Delta T_e \leq 5\,^0$C. In this regime there is insufficient vapour in contact with the liquid phase to cause boiling at the saturation temperature. As the excess temperature is increased, the bubble inception will eventually occur, but below point A (referred to as onset of nucleate boiling,ONB), fluid motion is primarily due to free convection effects.Therefore,according to whether the flow is laminar or turbulent, the heat transfer coefficient h varies as $\Delta T_e^{1/4}$ or as $\Delta T_e^{1/3}$ respectively so that q_W varies as $\Delta T_e^{5/4}$ or as $\Delta T_e^{4/3}$.

Nucleate Boiling Regime(Between points A and C):- Nucleate boiling exists in the range $5\,^0C \leq \Delta T_e \leq 30\,^0$C. In this range, two different flow regimes may be distinguished. In the region A – B, isolated bubbles form at nucleation sites and separate from the surface, substantially increasing h and q_W. In this regime most of the heat exchange is through direct transfer from the surface to liquid in motion at the surface, and not through vapour bubbles rising from the surface. As ΔT_e is increased beyond $10\,^0$C (Region B-C), the nucleation sites will be numerous and the bubble generation rate is so high that continuous columns of vapour appear. As a result very high heat fluxes are obtainable in this region. In practical applications, the nucleate boiling regime is most desirable, because large heat fluxes are obtainable with small temperature differences. In the nucleate boiling regime, the heat increases rapidly with increasing excess temperature

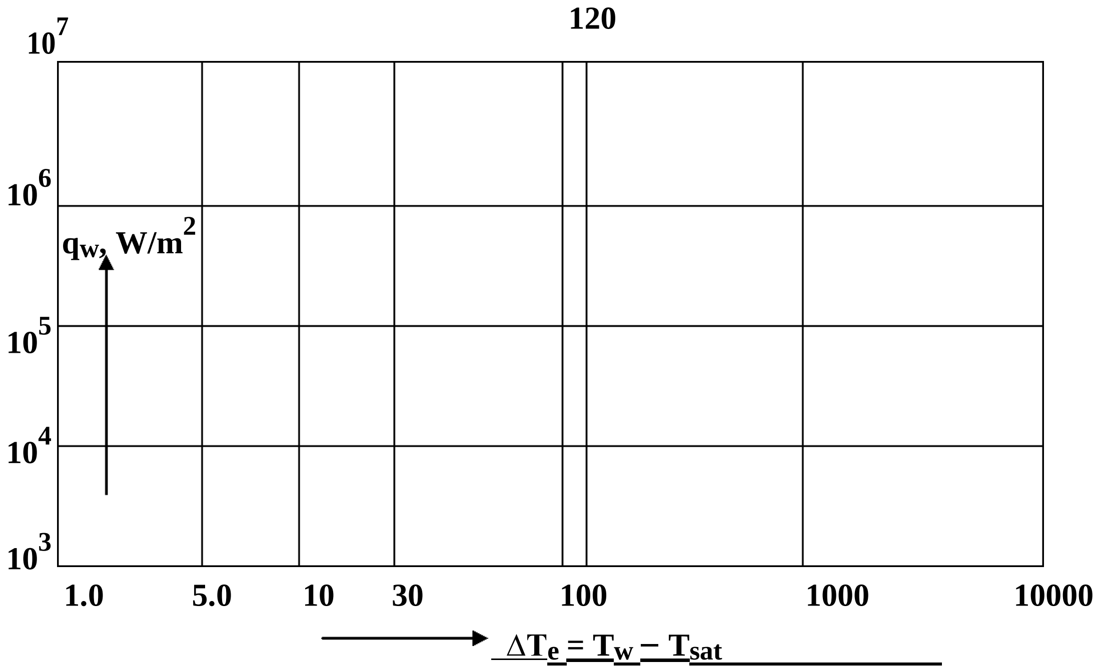


Fig. 8.4: Typical boiling curve for water at 1 atm; surface heat flux q_W as function of excess temperature ΔT_e

ΔT_e until the peak heat flux is reached. The location of this peak heat flux is called the *burnout point*, or *departure from nucleate boiling* (DNB), or the *critical heat flux* (CHF). The reason for calling the critical heat flux the burnout point is apparent from the Fig.

8.4. Such high values of ΔT_e may cause the burning up or melting away of the heating element.

Film Boiling Regime:- It can be seen from Fig. 8.4 that after the peak heat flux is reached, any further increase in ΔT_e results in a reduction in heat flux. The reason for this curious phenomenon is the blanketing of the heating surface with a vapour film which restricts liquid flow to the surface and has a low thermal conductivity. This regime is called the *film boiling regime.* The film boiling regime can be separated into three distinct regions namely (i) the *unstable film boiling region,* (ii) the *stable film boiling region* and

(iii) *radiation dominating region.* In the unstable film boiling region, the vapour film is unstable, collapsing and reforming under the influence of convective currents and the

iv) surface tension. Here the heat flux decreases as the surface temperature increases, because the average wetted area of the heater surface decreases. In the stable film boiling region, the heat flux drops to a minimum, because a continuous vapour film covers the heater surface.In the radiation dominating region, the heat flux begins to increase as the excess temperature increases, because the temperature at the heater surface is sufficiently high for thermal radiation effects to augment heat transfer through the vapour film.

Pool Boiling Correlations:

Correlation for The Nucleate Boiling Regime:- The heat transfer in the nucleate boiling regime is affected by the nucleation process, the distribution of active nucleation sites on the surface, and the growth and departure of bubbles.Numerous experimental investigations have been reported and a number of attempts have been made to correlate the experimental data corresponding to nucleate boiling regime.The most successful and widely used correlation was developed by Rohsenow. By analyzing the significance of various parameters in relation to forced - convection effects. He proposed the following empirical relation to correlate the heat flux in the entire nucleate boiling regime:

$$\frac{C_{pl}\,\Delta T_e}{h_{fg}\,Pr_l^{\,n}} = C_{sf}\left[\frac{q_w}{(\mu_l\,h_{fg})}\sqrt{\zeta^* / \{g\,(\rho_l - \rho_v)\}}\,\right]^{0.33} \;\ldots\ldots\ldots\ldots\ldots\ldots\;(8.28)$$

where C_{pl} = specific heat of saturated liquid, J /(kg -^{0}C)

C_{sf} = constant to be determined from experimental data depending upon Heating surface – fluid combination

h_{fg} = latent heat of vapourization, J / kg

g = acceleration due to gravity, m / s^2

Pr_l = Prandtl number of saturated liquid

q_w = boiling heat flux, W / m^2

ΔT_e = excess temperature as defined in Eq. (8.27)

μ_l = viscosity of saturated liquid, kg / (m – s)

ρ_l, ρ_v = density of liquid and saturated vapour respectively, kg /

m^3 ζ^* = surface tension of liquid – vapour interface, N / m.

In Eq. (8.28) the exponent n and the coefficient C_{sf} are the two provisions to adjust the correlation for the liquid – surface combination. Table 8-1 gives the experimentally determined values C_{sf} for a variety of liquid – surface combinations. The value of n *should be taken as 1 for water and 1.7 for all other liquids* shown in Table 8 – 1.

Table 8 – 1: Values of C_{sf} of Eq. (8.28) for various liquid – surface combinations

Liquid – surface combination	C_{sf}
Water – Copper	0.0130
Water – scored copper	0.0068
Water – chemically etched stainless steel	0.0130
Water – mechanically polished stainless steel	0.0130
Water – ground and polished stainless steel	0.0060
Water – brass	0.0060
Water – nickel	0.0060
Water – platinum	0.0130
n-Pentane – polished copper	0.0154
n-Pentane – lapped copper	0.0049
Benzene – chromium	0.1010
Ethyl alcohol – chromium	0.0027

Correlations for Peak Heat Flux:- The correlation given by Eq. (8.28) provides information for the heat flux in nucleate boiling, but it cannot predict the *peak heat flux.*Based on stability considerations, *Kutateladze and Zuber* veloped the following correlation to calculate the peak heat flux in pool boiling from an infinite horizontal plate facing up.

$$q_{max} = \frac{\Pi}{24} \rho_v h_{fg} \left[\frac{\zeta^* g (\rho_l - \rho_v)}{\rho_v^2}\right]^{1/4} \left[1 + \rho_v / \rho_l\right]^{1/2} \quad \text{..................} (8.29)$$

where ζ^* = surface tension of liquid – vapour interface, N / m

g = acceleration due to gravity, m / s^2

ρ_l, ρ_v = density of liquid and vapour respectively, kg / m^3 h_{fg} = latent heat of vapourization, J / kg

q_{max} = peak heat flux, W / m^2

It is apparent from this equation that large values of h_{fg}, ρ_v, g and ζ^* are desirable for a large value of the peak heat flux. For example, water has a large value of h_{fg}; hence the peak heat flux obtainable with boiling water is high. This equation also shows that a reduced gravitational field decreases the peak heat flux. For most situations, the quantity $[1 + \rho_v / \rho_l]^{1/2}$ is approximately equal to unity. Hence Eq. (8.29) can be written as

$$q_{max} = \frac{\pi}{24} \rho_v^{1/2} h_{fg} \left[\zeta^* g (\rho_l - \rho_v)\right]^{1/4} \quad \text{...} (8.30)$$

Lienhard and Dhir improved the analysis and considered the effect of the size of the horizontal plate. They showed that

$$q_{max} = 0.149\, h_{fg}\, \rho_v^{1/2} \left[\zeta^* g (\rho_l - \rho_v)\right]^{1/4} \quad \text{...} (8.31)$$

The above expression was shown to be valid as long as the plate is large and the dimensionless quantity $\mathbf{L\ [g\ (\rho_l - \rho_v) / \zeta^*\ ]^{1/2} \geq 2.7}$, where L is the characteristic dimension of the plate. For circular plate L is taken as the diameter, while for a square plate it is taken as the side of the plate.

For the case of horizontal cylinders of radius R, Lienhard & Sun recommended the following modified form of Eq. (8.30).

$$q_{max} = \frac{\pi}{24} \rho_v^{1/2} h_{fg} [\zeta^* g (\rho_l - \rho_v)]^{1/4} f(L') \ldots\ldots (8.32)$$

where $f(L'') = 0.89 + 2.27 \exp\{-3.44 \sqrt{L''}\}$, and $L'' = R\,[g\,(\rho_l - \rho_v) / \zeta^*]^{1/2}$.

Eq. (8.32) is valid for situations in which $L'' \geq 0.15$. This equation cal also be used for large spheres in which case $f(L'') = 0.84$ and $L'' = 4.26$. For small spheres
$f(L'') = 1.734 / \sqrt{L''}$, where $0.15 \leq L'' \leq 4.26$.

Correlations for Film boiling Regime:- No satisfactory correlation exists for the unstable film boiling region. For the stable film boiling region, Bromley has derived the following correlation for horizontal cylinders:

$$h_o = 0.62 \left[\frac{k_v^3 \rho_v (\rho_l - \rho_v) g h^*_{fg}}{D \mu_v \Delta T_e} \right]^{1/4} \ldots\ldots (8.33)$$

Where h_o = average boiling heat transfer coefficient in the absence of radiation W/(m^2 – K),
D = outside diameter of the tube,
h_{fg}^* = Difference between the enthalpy of the vapour at the mean film temperature, T_m ($T_m = [T_w + T_{sat}] / 2$) and the enthalpy of the liquid at the saturation temperature

$\approx h_{fg} + 0.8\, C_{pv}\, \Delta T_e$.

In the above correlation, all the vapour properties are evaluated at the mean film temperature while the liquid density is evaluated at the saturation temperature.

Since radiation is significant in film boiling, the radiation component has to be added in order to obtain the total heat transfer. Brmoley has shown that the total heat transfer coefficient h is given by the relation

$$h = h_o [h_o / h]^{1/3} + h_r \ldots\ldots (8.34)$$

where h_r is the radiation heat transfer coefficient which is calculated from the formula for the radiation heat exchange between parallel planes;

$$h_r = \frac{1}{[1/\varepsilon + 1/\alpha - 1]} \frac{\zeta\,[T_w^4 - T_{sat}^4]}{[T_w - T_{sat}]} \ldots\ldots (8.35)$$

where α = absorptivity of liquid and ε = emissivity of the hot tube .

Eq.(8.34) is difficult to use because a trial and error approach is needed to determine h. When h_r is smaller than h_o, Eq. (8.34) can be replaced by

$$\mathbf{h = h_o + \tfrac{3}{4}\, h_r} \qquad \textbf{(8.36)}$$

<u>Illustrative examples on pool boiling</u>

Example 8.9:- *Saturated water at 100 ^{0}C is boiled with a copper heating element having a heating surface area of 0.04 m^2 which is maintained at a uniform temperature of 115 ^{0}C. Calculate the surface heat flux and the rate of evaporation of water. Also calculate the critical flux.*

Solution: Given:- T_{sat} = 100 ^{0}C; T_w = 115 ^{0}C; Surface area = A = 0.04 m^2;

Properties of liquid water at 100 ^{0}C are: C_{pl} = 4216 J/(kg – K); h_{fg} = 2257 x 10^3 J/kg;

ρ_l = 960.6 kg/m^3; ρ_v = 0.5977 kg/m^3; Pr_l = 1.75; μ_l = 282.4 x 10^{-6} kg/(m – s);

ζ^* = 58.8 x 10^{-3} N/m ; ΔT_e = 115 – 100 = 15 ^{0}C.

Since ΔT_e lies between 5 ^{0}C and 30 ^{0}C, the boiling is in the nucleate regime.

C_{sf} = 0.0130 for water – copper combination and for water n = 1.For nucleate boiling region we can use Eq. (8.28) which is as follows:

$$\frac{C_{pl}\,\Delta T_e}{h_{fg}\,Pr_l^{\,n}} = C_{sf}\left[\frac{q_w}{(\mu_l\,h_{fg})}\,\sqrt{\zeta^*/\{g(\rho_l-\rho_v)\}}\,\right]^{0.33} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (8.28)$$

Substituting the given values we have

$$\frac{4216 \times 15}{(2257 \times 10^3) \times 1.75} = 0.013 \times \left[\frac{q_w}{282.4 \times 10^{-6} \times 2257 \times 10^3} \times \sqrt{\left\{\frac{58.8 \times 10^{-3}}{9.81 \times (960.6 - 0.5977)}\right\}}\,\right]^{0.33}$$

Solving for q_w we get q_w = 4.84 x 10^5 W / m^2.

Hence total heat transfer = Q = Aq_w = 0.04 x 4.84 x 10^5 = 19.36 x 10^3 W = 19.36 kW.

$$\text{Rate of evaporation} = M = Q / h_{fg} = \frac{19.36 \times 10^3}{2257 \times 10^3} = 8.58 \times 10^{-3}\ \text{kg / s.}$$

Critical heat flux can be calculated from Eq. (8.31) namely

$$q_{max} = 0.149\ h_{fg}\ \rho_v^{1/2}\ [\zeta^*\ g\ (\rho_l - \rho_v)]^{1/4} \ldots\ldots\ldots (8.31)$$

Hence $\quad q_{max} = 0.149 \times 2257 \times 10^{3} \times (0.5977)^{1/2}$

$$\times [\ 58.8 \times 10^{-3} \times 9.81 \times (960.6 - 0.5977)\ ]^{1/4}$$

$$= 1.261 \times 10^{6}\ W/m^2 = 1.261\ MW/m^2.$$

Example 8.10:- *A metal clad heating element of 6 mm diameter and emissivity equal to unity is horizontally immersed in a water bath. The surface temperature of the metal is 255 ^{0}C under steady state boiling conditions. If the water is at atmospheric pressure estimate the power dissipation per unit length of the heater.*

Solution: *Given:*- T_w = 255 ^{0}C ; T_{sat} = saturation temperature of water at 1 atm = 100 ^{0}C;

ΔT_e = 255 – 100 = 155 ^{0}C. Since ΔT_e > 120 ^{0}C, film boiling conditions will prevail.
The heat transfer in this regime is given by Eq.(8.33) namely

$$h_o = 0.62 \left[\frac{k_v^3\ \rho_v\ (\rho_l - \rho_v)\ g\ h^*_{fg}}{D\ \mu_v\ \Delta T_e} \right]^{1/4}$$

Properties of water at 100 ^{0}C are: ρ_l = 957.9 kg/m^3; h_{fg} = 2257 x 10 ^{3} J/kg;

ρ_v = 4.808 kg/m^3; C_{pv} = 2.56 x 10 ^{3} J/(kg-K); k_v = 0.0331 W / (m-^{0}C);

μ_v = 14.85 x 10 $^{-6}$ kg / (m-s).

Substituting these values in the expression for h_o we have

$$h_o = 0.62 \times \left[\frac{(0.0331)^3 \times 4.808 \times (957.9 - 4.808) \times 9.81 \times \{2257 \times 10^3 + 0.8 \times 2.56 \times 10^3 \times 155\}}{14.85 \times 10^{-6} \times 0.006 \times 155} \right]^{1/4}$$

$$= 460\ W/(m^2 - K)$$

$$h_r = \frac{1}{[1/\varepsilon + 1/\alpha - 1\ ]} \times \frac{\zeta\ \{T_w^4 - T_{sat}^4\}}{\{T_w - T_{sat}\}}$$

$$= \frac{1}{[\ 1/1 + 1/1 - 1\ ]} \times \frac{5.67 \times 10^{-8} \times \{\ 528^4 - 373^4\}}{\{\ 528 - 373\}}$$

$= 21.3$ W / (m^2-K).

Now $h \approx h_o + ¾\, h_r = 460 + ¾ \times 21.3 = 476$ W /(m^2 – K).

Hence $Q = h\, A\, \Delta T_e = 476 \times (\pi \times 0.006 \times 1) \times 155 = 1.36 \times 10^{3}$ W / m.

Example 8.11:- *A vessel with a flat bottom and 0.1 m^2 in area is used for boiling water at atmospheric pressure. Find the temperature at which the vessel must be maintained if a boiling rate of 80 kg/h is desired. Assume that the vessel is made of copper and the boiling is nucleate boiling. Take $\rho_v = 0.60$ kg/m^3.*

Solution: *Given:-* A = 0.1 m^2; $T_{sat} = 100\ ^0C$; M = 80 kg/h = 0.022 kg/s; $Pr_l = 1.75$

$h_{fg} = 2257 \times 10^{3}$ J /kg; $C_{pl} = 4216$ J/(kg-K); $\rho_l = 960.6$ kg/m^3; $\zeta^* = 58.8 \times 10^{-3}$ N/m;

$\mu_l = 282.4 \times 10^{-6}$ kg / (m-s); n = 1; For water-copper combination $C_{sf} = 0.0130$;

$$q_w = Q / A = \frac{M\, h_{fg}}{A} = \frac{0.022 \times 2257 \times 10^{3}}{0.1} = 4.965 \times 10^{3}\ W/m^2$$

For nucleate boiling Eq.(8.28) is used to calculate the excess temperature .ΔT_e

$$\frac{C_{pl}\, \Delta T_e}{h_{fg}\, Pr_l^{\,n}} = C_{sf} \left[\frac{q_w}{(\mu_l\, h_{fg})} \sqrt{\zeta^* / \{g\, (\rho_l - \rho_v)\}} \right]^{0.33}$$

$$\frac{4216 \times \Delta T_e}{2257 \times 10^3 \times 1.75} = 0.013 \times \{4.965 \times 10^{5} / (282.4 \times 10^{-6} \times 2257 \times 10^{3}) \times \sqrt{58.8 \times 10^{-3} / [9.81 \times (960.6 - 0.6)]}\ \}^{0.33}$$

Or $\Delta T_e = 15.2\ ^0C$

Hence $T_w = 100 + 15.2 = 115.2\ ^0C$.

Example 8.12:- *Calculate the heat transfer coefficient during stable film boiling of water from a 0.9 cm diameter horizontal carbon tube. The water is saturated and at 100 ^{0}C and the tube surface is at 1000 ^{0}C. Take the emissivity of the carbon surface to be 0.8 and assume that at the average film temperature, the steam has the following properties.*
k_v = 0.0616 W/(m-K); ρ_v = 0.266 kg/m^3; μ_v = 28.7 x 10^{-6} kg/(m-s); C_{pv} = 2168 J/(kg-K); ρ_l = 958.4 kg/m^3

Solution: *Given:-* D = 0.009 m; $\Delta T_e = T_w - T_{sat}$ = 1000 – 100 = 900 ^{0}C; ε = 0.8; α = 1.0 $h_{fg}^* = h_{fg} + 0.8\, C_{pv}\, \Delta T_e$ = 2257 x 10^3 + 0.8 x 2168 x 900 = 3818 x 10^3 J/kg.

For stable film boiling the convection coefficient is given by Eq.(8.33)

$$h_o = 0.62 \left[\frac{k_v^3\, \rho_v (\rho_l - \rho_v)\, g\, h_{fg}^*}{D\, \mu_v\, \Delta T_e} \right]^{1/4}$$

$$h_o = 0.62 \times \left[\frac{(0.0616)^3 \times 0.266 \times (958.4 - 0.266) \times 9.81 \times 3818 \times 10^3}{0.009 \times (28.7 \times 10^{-6}) \times 900} \right]^{1/4}$$

$$= 194 \text{ W/(m}^2 - \text{K)}$$

Radiation heat transfer coefficient is given by

$$h_r = \frac{1}{[1/\varepsilon + 1/\alpha - 1]} \; \frac{\zeta (T_w^4 - T_{sat}^4)}{(T_w - T_{sat})}$$

$$h_r = \frac{1}{[1/0.8 + 1/1 - 1]} \times \frac{5.67 \times 10^{-8} (1273^4 - 373^4)}{(1273 - 373)}$$

$$= 131.4 \text{ W/(m}^2 - \text{K)}.$$

Hence $h = h_o + \frac{3}{4} h_r$ = 194 + ¾ x 131.4 = 292.5

RADIATION HEAT TRANSFER

INTRODUCTORY CONCEPTS AND DEFINITIONS *THERMAL RADIATION*

When a body is placed in an enclosure whose walls are at temperatures below that of the body, the temperature of the body will decrease even if the enclosure is evacuated. This process by which heat is transferred from a body by virtue of its temperature, without the aid of any intervening medium is called “THERMAL RADIATION”. The actual mechanism of radiation is not yet completely understood. There are at present two theories by means of which radiation propagation is explained. According to Maxwell‟s electromagnetic theory, Radiation is treated as electromagnetic waves, while Max Planck‟s theory treats radiation as “Photons” or “Quanta of energy”. Neither theory completely describes all observed phenomena. It is however known that radiation travels with the speed of light, c (c = 3×10^{8} m/s) in a vacuum. This speed is equal to the product of the frequency of the radiation and the wavelength of this radiation,

OR **$c = \lambda\nu$ .... (10.1)**

Where $\boldsymbol{\lambda}$ = wavelength of radiation (m) and $\boldsymbol{\nu}$ = frequency (1/s).

Usually, it is more convenient to specify wavelength in micrometer, which is equal to 10^{-6} m.

From the viewpoint of electromagnetic wave theory, the waves travel at the speed of light, while from the quantum theory point of view, energy is transported by photons which travel at the speed of light. Although all the photons have the same velocity, there is always a distribution of energy among them. The energy associated with a photon, $\mathbf{e_p = h\nu}$ where h is the Planck‟s constant equal to 6.6256×10^{-34} Js. The entire energy spectrum can also be described in terms of the wavelength of radiation.

Radiation phenomena are usually classified by their characteristic wavelength, λ. At temperatures encountered in most engineering applications, the bulk of the thermal energy emitted by a body lies in the wavelengths between **λ**= 0.1 and 100 **µm**. For this reason, the portion of the wavelength spectrum between **λ**= 0.1 and 100 **µ**m is generally referred to as “THERMAL RADIATION”. The wavelength spectrum in the range **λ**= 0.4 and 0.7 **µ**m is visible to the naked eye, and this is called „light rays‟. The wavelength spectrum of radiation is illustrated in Fig 10.1

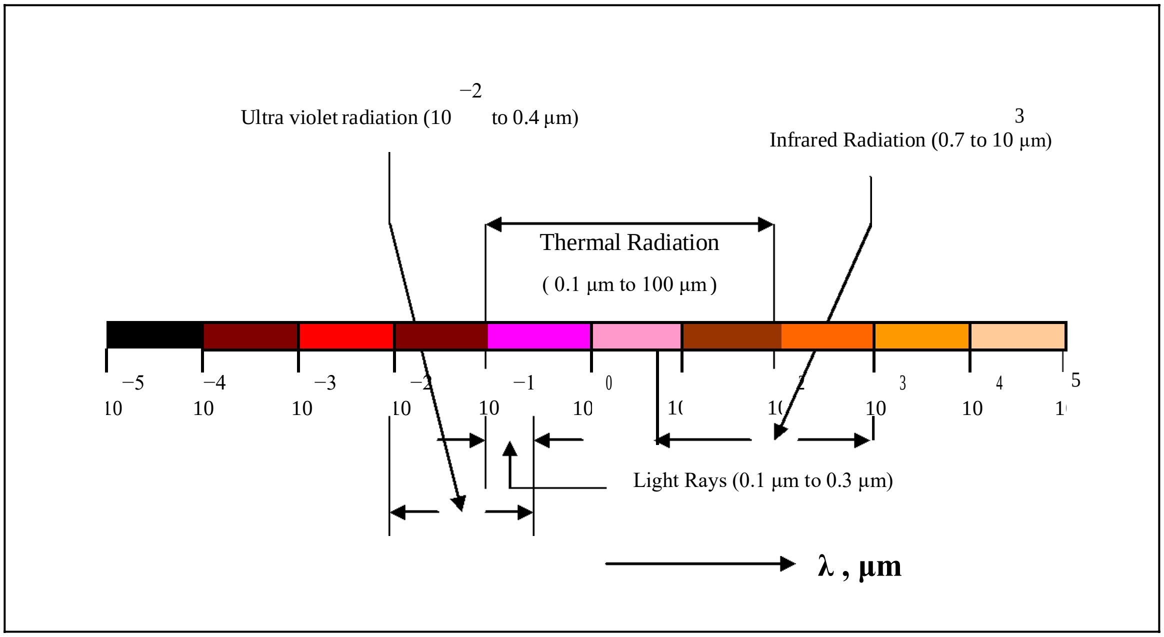


Fig. 10.1 Typical Spectrum of electromagnetic radiation

In the study of radiation transfer, a distinction should be made between bodies which are "semi-transparent" to radiation and those which are "opaque". If the material is semitransparent to radiation, such as glass, salt crystals, and gases at elevated temperatures, then the radiation leaving the body from its outer surfaces results from emissions at all depths within the material. The emission of radiation for such cases is a "BULK" or a "VOLUMETRIC PHENOMENON". If the material is opaque to thermal radiation, such as metals, wood, rock etc. then the radiation emitted by the interior regions of the body cannot reach the surface. In such cases, the radiation emitted by the body originates from the material at the immediate vicinity of the surface (i.e. within about 1**µ**m) and the emission is regarded as a "SURFACE PHENOMENON". It should also be noted that a material may behave as a semi transparent medium for certain temperature ranges, and as opaque for other temperatures. Glass is a typical example for such behaviour. It is semi transparent to thermal radiation at elevated temperatures and opaque at intermediate and low temperatures.

DEFINITIONS OF TERMS USED IN THERMAL RADIATION

- ***Monochromatic Emissive Power (E_λ):*** The monochromatic emissive power of a surface at any temperature **T** and wavelength **λ** is defined as the quantity which when multiplied by **dλ** gives the radiant flux in the wavelength range - **λ** to **λ+dλ**.

- ***Emissive Power (E):*** The emissive power of a surface is the energy emitted by a surface at a given temperature per unit time per unit area for the entire wavelength range, from **λ = 0** to **λ** = ∞.

$$E = \int_{0}^{\infty} E_\lambda d\lambda \quad \text{……………………………………………}(10.2)$$

- ***<u>Absorptivity, Reflectivity and Transmissibility of a body:</u>***

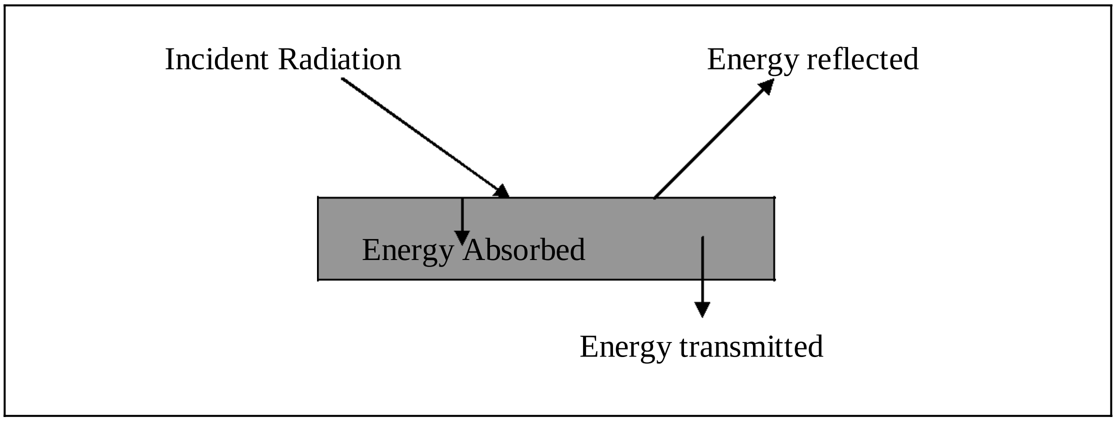


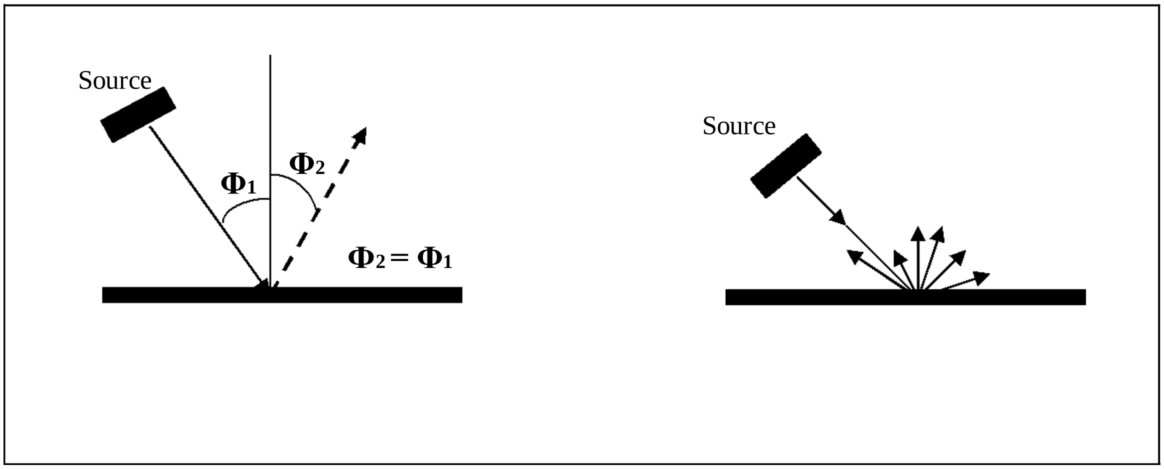
Fig.10.2: Effects of radiation incident on a surface

When a radiant energy strikes a material surface, part of the radiation is reflected, part is absorbed, and part is transmitted, as shown in Fig. 10.2. Reflectivity (ρ) is defined as the fraction of energy which is reflected, Absorptivity (α) as the fraction absorbed, and Transmissivity (η) as the fraction transmitted. Thus, $\boldsymbol{\rho + \alpha + \eta = 1}$.

Most solid bodies do not transmit thermal radiation, so that for many applied problems, the transmissivity may be taken as zero. Then

$$\rho + \alpha = 1 \text{...} \quad (10.3)$$

- ***<u>Specular Radiation and Diffuse Radiation</u>*:**



(a) Specular Radiation **(b) Diffuse Radiation**

Fig.10.3: Specular and Diffuse Radiation

When radiation strikes a surface, two types of reflection phenomena may be observed. If the angle of incidence is equal to the angle of reflection, the radiation is called Specular. On the other hand, when an incident beam is distributed uniformly in all directions after reflection, the radiation is called Diffuse Radiation. The two types of radiation are depicted in Fig. 10.3. Ordinarily, no real surface is either specular or diffuse. An ordinary mirror is specular for visible light, but would not necessarily be specular over the entire wavelength range. A rough surface exhibits diffuse behaviour better than a highly polished surface. Similarly, a highly polished surface is more specular than a rough surface.

- ***Black Body:***

A body which absorbs all incident radiation falling on it is called a blackbody. For a blackbody, $\alpha = 1$, $\rho = \eta = 0$. For a given temperature and wavelength, no other body at the same temperature and wavelength, can emit more radiation than a blackbody. Blackbody radiation at any temperature T is the maximum possible emission at that temperature. A blackbody or ideal radiator is a theoretical concept which sets an upper limit to the emission f radiation. It is a standard with which the radiation characteristics of other media are compared.

- ***Emissivity of a Surface (ε):***

The emissivity of a surface is the ratio of the emissive power of the surface to the emissive power of a black surface at the same temperature. It is denoted by the symbol **ε**.

$$\text{i.e. } \boldsymbol{\varepsilon} = [E/E_b]_T.$$

- ***Monochromatic Emissivity of a Surface (ε λ):***

The monochromatic emissivity of a surface is the ratio of the monochromatic emissive power of the surface to the monochromatic emissive power of a black surface at the same temperature and same wavelength.

$$\varepsilon_\lambda = [E_\lambda / E_{b\lambda}]_{\lambda,T}.$$

- ***Gray Body:***

A gray body is a body having the same value of monochromatic emissivity at all wavelengths. i.e.

$$\varepsilon = \varepsilon_\lambda, \text{ for a gray body.}$$

- ***Radiosity of a Surface (J):***

This is defined as the total energy leaving a surface per unit time per unit area of the surface. This definition includes the energy reflected by the surface due to some radiation falling on it.

- ***Irradiation of a surface(G):***

This is defined as the radiant energy falling on a surface per unit time, per unit area of the surface.

Therefore if E is the emissive power, J is the radiosity, ε is the irradiation and ρ the reflectivity of a surface, then,

$$\mathbf{J = E + \rho G}$$

For an opaque surface, $\boldsymbol{\rho + \alpha = 1}$ or $\boldsymbol{\rho = (1 - \alpha)}$

$$\mathbf{J = E + (1-\alpha)G} \quad \textbf{(10.4)}$$

LAWS OF RADIATION

STEFAN – BOLTZMANN LAW:

This law states that the emissive power of a blackbody is directly proportional to the fourth power of the absolute temperature of the body.

i.e., $\mathbf{E_b \ \alpha \ T^4}$

Or $\mathbf{E_b = \zeta T^4}$ ---------------------------------- **(10.5)**

where ζ is called the Stefan – Boltzmann constant.

In SI units $\zeta = 5.669 \times 10^{-8}$ W/(m^2-K^4).

PLANCK'S LAW:

This law states that the monochromatic power of a blackbody is given by

$$E_{b\lambda} = \frac{C_1}{\lambda^5 \left[e^{(C_2 / \lambda T)} - 1 \right]} \quad (10.6)$$

where C_1 and C_2 are constants whose values are found from experimental data; $C_1 = 3.7415 \times 10^{-16}$ Wm^2 and $C_2 = 1.4388 \times 10^{-2}$ m-K.

λ is the wavelength and T is the absolute temperature in K.

WEIN'S DISPLACEMENT LAW:

It can be seen from Eq. 10.6 that at a given temperature, $E_{b\lambda}$ depends only on $\boldsymbol{\lambda}$**.** Therefore the value of $\boldsymbol{\lambda}$ which gives maximum value of $E_{b\lambda}$ can be obtained by differentiating Eq(10.6) w.r.t $\boldsymbol{\lambda}$ and equating it to zero.

Let $C_2/\lambda T = y$. Then Eq. (10.6) reduces to

$$E_{b\lambda} = \frac{C_1}{[C_2/(yT)]^5[e^y - 1]}$$

Then $$\frac{dE_{b\lambda}}{dy} = \frac{C_1\, d/dy\, \{[C_2/(yT)]^5 [e^y - 1]\}}{\{[C_2/(yT)]^5 [e^y - 1]\}^2}$$

or $$d/dy\, \{[C_2/(yT)]^5 (e^y - 1)\} = 0$$

Or $$e^y(5 - y) = 5$$

By trial and error, y = 4.965

Therefore, if λ_m denotes the value of λ which gives maximum $E_{b\lambda}$, then

$$C_2/\lambda_m T = 4.965$$

Or $$\lambda_m T = C_2/4.965 = 1.4388\times10^{-2}/4.965$$

$$\boldsymbol{\lambda_m T = 0.002898 \text{ m-K}} \quad \textbf{(10.7)}$$

Equation (10.7) is called the Wein"s displacement law. From this equation it can be seen that the wavelength at which the monochromatic emissive power is a maximum decreases with increasing temperature. This is also illustrated in Fig 10.4(a). Fig 10.4(b) gives a comparison of monochromatic emissive powers for different surfaces at a particular temperature for different wavelengths.

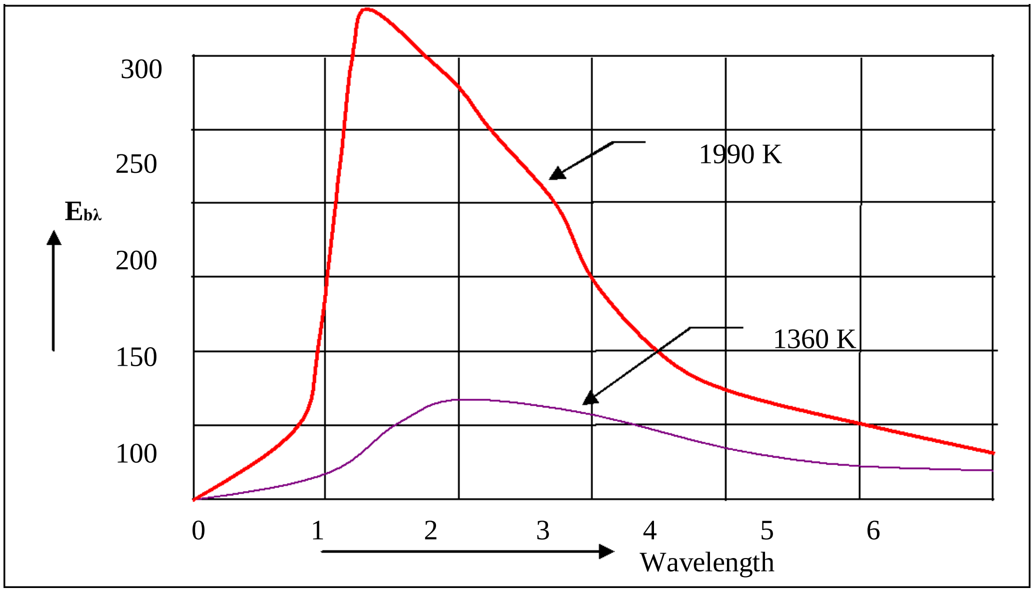


Fig. 10.4 (a) Black body emissive power as a function of wave length and Temperature

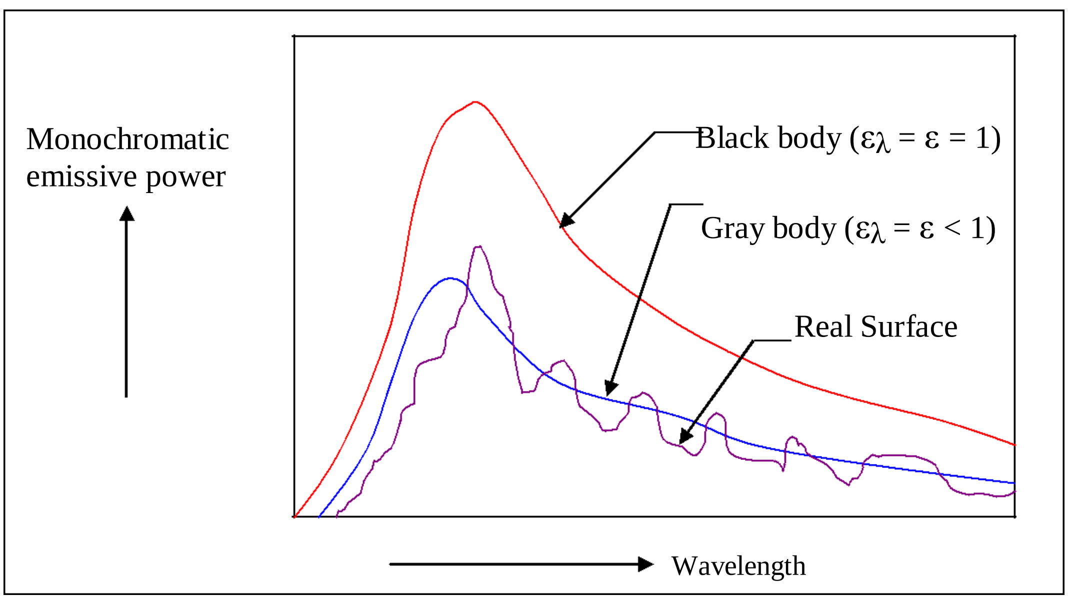


Fig. 10.4 (b) Comparison of emissive powers of different types o surfaces as a function of wavelength at a given temperature

KIRCHOFF'S LAW:

This law states that the emissivity of a surface is equal to its absorptivity when the surface is in thermal equilibrium with the surroundings.

Proof: Consider a perfect black enclosure i.e. the one which absorbs all the incident radiation falling on it (see Fig 10.5). Now let the radiant flux from this enclosure per unit area arriving at some area be $\mathbf{q_i}$ W/m^2.

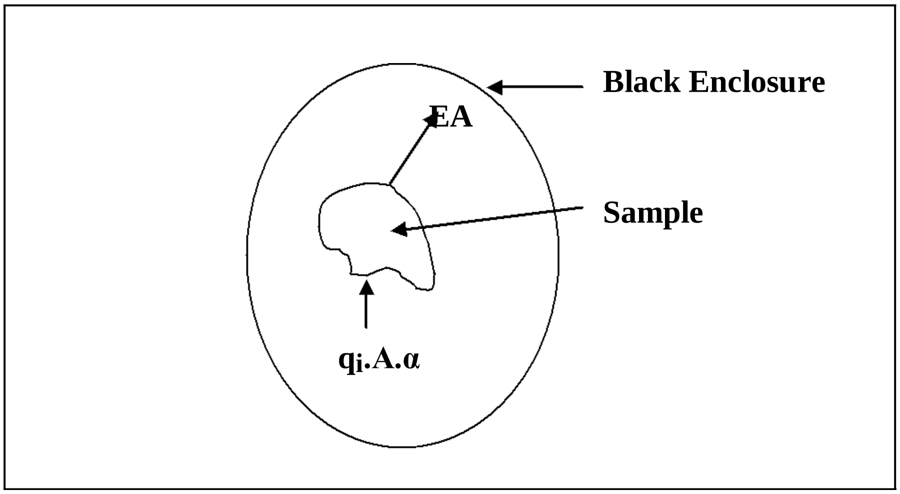


Fig. 10.5 : Model used for deriving Kirchoff law

Now suppose that a body is placed inside the enclosure and allowed to come to thermal equilibrium with it. At equilibrium, the energy absorbed by the body must be equal to the energy emitted; otherwise there would be an energy flow into or out of the body, which would raise or lower its temperature. At thermal equilibrium we may write

$$\mathbf{EA = q_i\ A\ \alpha} \quad \text{..(10.8)}$$

If we now replace the body in the enclosure with a black body of the same size and shape and allow it to come to thermal equilibrium with the enclosure,

$$\mathbf{E_bA = q_i\ A} \quad \text{.. (10.9)}$$

Since $\alpha = 1$ for a blackbody.

If Eq. 10.8 is divided by Eq. 10.9 we get

$$\mathbf{E/E_b = \alpha}$$

But by definition $E/E_b = \varepsilon$, the emissivity of the body, so that $\boldsymbol{\varepsilon = \alpha}$ **............. (10.10)**

Equation 10.10 is called Kirchoff‟s law and is valid only when the body is in thermal equilibrium with the surroundings. However, while analyzing radiation problems in practice we assume that Kirchoff‟s law holds good even if the body is not in thermal equilibrium with the surroundings, as the error involved is not very significant.

ILLUSTRATIVE EXAMPLES ON BASIC CONCEPTS

Example 10.1: *The emission of radiation from a surface can be approximated as blackbody radiation at 1000K.*

(a) What fraction of the total energy emitted is below λ = 5μm
(b) What is the wavelength below which the emission is 10.5% of the total emission at 1000K.
(c) What is the wavelength at which the maximum spectral emission occurs at 1000K.

Solution: The radiation flux emitted by the blackbody over the wavelength interval 0 – λ is given by

$$[E_b]_{0-\lambda} = \int_0^{\lambda} E_{b\lambda} d\lambda$$

The integration required in the above equation has been done numerically and the results are presented in the form of a table. The table gives the value of D $_{0-\lambda}$ where

$$D_{0-\lambda} = \frac{\int_0^{\lambda} E_{b\lambda} d\lambda}{\int_0^{\infty} E_{b\lambda} d\lambda} = \frac{1}{\zeta T^4} \int_0^{\lambda} E_{b\lambda}\, d\lambda$$

(a) From Table of Radiation properties, for λT = 5 x 1000 = 5000, $D_{0-\lambda}$ = 0.6337. This means that 63.37 % of the total emission occurs below λ = 5 μm.

(b) From the same table, for $D_{0-\lambda}$ = 0.105, λT = 2222. Hence λ = 2222/1000 = 2.222 μm.

(c) From Wein‟s displacement law, $\lambda_m T$ = 0.002898.
Hence for T = 1000 K, λ_m =0.002898 / 1000 = 2.898 x 10^{-6} m = 2.898 μm.

Example 10.2: *The monochromatic emissivity of a surface varies with the wavelength in the following manner:*

$$\varepsilon_\lambda = 0 \quad \text{for } \lambda < 0.3\mu m$$
$$= 0.9 \text{ for } 0.3\mu m < \lambda < 1\mu m$$
$$= 0 \text{ for } \lambda > 1\mu m$$

Calculate the heat flux emitted by the surface if it is at a temperature of 1500 K

Solution:

$$E_\lambda = \varepsilon_\lambda E_{b\lambda}$$

Therefore $$E = \int_0^\infty \varepsilon_\lambda E_{b\lambda} = \int_0^{0.3\ \mu m} 0.0\ E_{b\lambda}\, d\lambda + \int_{0.3\ \mu m}^{1\ \mu m} 0.9\ E_{b\lambda}\, d\lambda + \int_{1\ \mu m}^{\infty} 0.0\ E_{b\lambda}\, d\lambda$$

$$= 0.9 \int_{0.3\ \mu m}^{1\ \mu m} E_{b\lambda}\, d\lambda = 0.9 \left[\int_0^{1\ \mu m} E_{b\lambda}\, d\lambda - \int_0^{0.3\ \mu m} E_{b\lambda}\, d\lambda \right]$$

$$= 0.9\ \zeta\ T^4 \left[D_{0-1} - D_{0-0.3} \right]$$

For $\lambda = 1\mu m$, $\lambda T = 1500\ \mu m$-K, therefore $D_{0-1} = ½\ (0.01972 + 0.00779) =$

0.93755 For $\lambda = 0.3\mu m$, $\lambda T = 450\ \mu m$-K, therefore $D_{0-3} = 0$

Thus $E = 0.9 x 5.67 x 10^{-8} x 1500^4\ [0.013755 - 0] = 3553\ W/m^2$

Example 10.3: *Calculate the heat flux emitted due to thermal radiation from a black surface at 6000^0 C. At what wavelength is the monochromatic emissive power maximum and what is the maximum value?*

Solution: Temp of the black surface = 6273K

Heat Flux emitted = $E_b = \zeta T^4 = 5.67 x 10^{-8} x 6273^4 = 87798\ KW/m^2$

Wavelength corresponding to max monochromatic emissive power is given

by $\lambda_m T = 0.002898$ m-K

$\lambda_m = 0.002898/6273 = 4.62 x 10^{-7}$ m

The maximum monochromatic emissive power is given by

$$(E_{b\lambda})_{max} = \frac{2\ \pi\ C_1}{\lambda_{max}\ [\ \exp\{C_2 / (\lambda_{max} T)\} - 1]}$$

$$= \frac{2 \times \pi \times 0.596 \times 10^{-16}}{(4.62 \times 10^{-7})^5 \times [\ \exp\{\ 0.014387 / 0.002898\} - 1]}$$

$$= 1.251 \times 10^{14}\ W / m^2$$

Example 10.4: *The spectral hemispherical emissivity (monochromatic emissivity) of fire brick at 750K as a function of wavelength is as follows:*

$\varepsilon_1 = 0.1$ *for* $0 \leq \lambda \leq 2\mu m$
$\varepsilon_2 = 0.6$ *for* $2\mu m \leq \lambda \leq 14\mu m$
$\varepsilon_3 = 0.8$ *for* $14 \leq \lambda \leq \infty$
Calculate the hemispherical emissivity, ε for all wavelengths.

Solution:

$$\varepsilon = \frac{E}{E_b} = \frac{\int_0^\infty \varepsilon_\lambda E_{b\lambda}\, d\lambda}{\zeta T^4} = \frac{1}{\zeta T^4} \left[\varepsilon_1 \int_0^{\lambda 1} E_{b\lambda}\, d\lambda + \varepsilon_2 \int_{\lambda 1}^{\lambda 2} E_{b\lambda}\, d\lambda + \varepsilon_3 \int_{\lambda 2}^{\lambda 3} E_{b\lambda}\, d\lambda \right]$$

Where $\lambda_1 = 2\mu m$, $\lambda_2 = 14\mu m$, $\lambda_3 = \infty$

Thus $= \varepsilon_1 D_{0-\lambda 1} + \varepsilon_2 [D_{0-\lambda 2} - D_{0-\lambda 1}] + \varepsilon_3 [D_{0-\infty} - D_{0-\lambda 2}]$

Now, $\lambda_1 T = 2x750 = 1500$; $D_{0-\lambda 1} = 0.013$

$\lambda_2 T = 14x750 = 10500$; $D_{0-\lambda 2} = 0.924$ $\lambda_3 T = \infty$; $D_{0-\lambda 3} = 1$

Hence $\varepsilon = 0.1 \text{ x } 0.013 + 0.6 \text{ x } [0.924 - 0.013] + 0.8 \text{ x } [1 - 0.924] = 0.609$

Example 10.5: *the filament of a light bulb is assumed to emit radiation as a black body at 2400K. if the bulb glass has a transmissivity of 0.90 for radiation in the visible range, calculate the percentage of the total energy emitted by the filament that reaches the ambient as visible light.*

Solution: The wavelength range corresponding to the visible range is taken as $\lambda_1 = 0.38\mu m$ to $\lambda_2 = 0.76\mu m$. Therefore the fraction F of the total energy emitted in this range is given by

$$F = \eta \left[\frac{\int_{\lambda 1}^{\lambda 2} E_{b\lambda}\, d\lambda}{E_b (T)} \right] = \eta \left[\int_0^{\lambda 2} E_{b\lambda}\, d\lambda - \int_0^{\lambda 1} E_{b\lambda}\, d\lambda \right] / E_b$$

$$= \eta [D_{0-\lambda 2} - D_{0-\lambda 1}].$$

Now $\lambda_1 T = 0.38 \text{ x } 2400 = 912$. Hence $D_{0-\lambda 1} = 0.0002$

and $\lambda_2 T = 0.76 \text{ x } 2400 = 1824$. Hence $D_{0-\lambda 2} = 0.0436$

Therefore $F = 0.9 \text{ x } [0.0436 - 0.0002] = 0.039$.

Only 3.9 % of the total energy enters the ambient as light. The remaining energy produces heating.

RADIATION HEAT EXCHANGE BETWEEN INFINITE PARALLEL SURFACES IN THE PRESENCE OF NON PARTICIPATING MEDIUM

Assumptions:

(i) The medium does not participate in radiation heat exchange between the two surfaces.

(ii) The surfaces are flat and are at specified uniform temperatures.

: RADIATION EXCHANGE BETWEEN TWO PARALLEL BLACK SURFACES

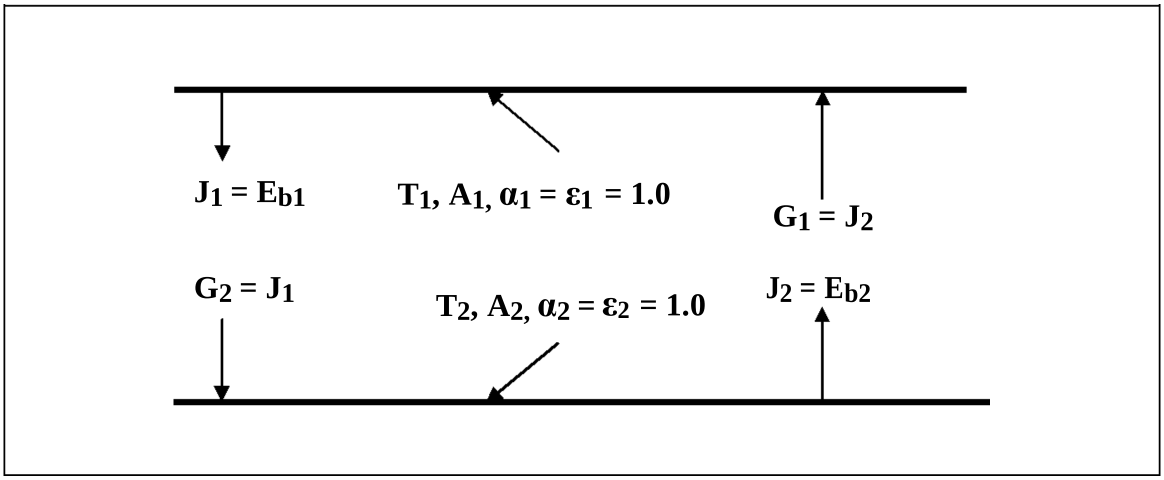


Fig: 10.6 Radiation heat exchange between two parallel black surfaces.

Since both surfaces are parallel, flat and infinite, radiosity of surface 1 = irradiation of surface 2 and vice versa. i.e. $J_1 = G_2$ and $J_2 = G_1$. Since both the surfaces are black, $\mathbf{J_1 = E_{b1} = \zeta T_1^4}$ and $\mathbf{J_2 = E_{b2} = \zeta T_2^4}$

Net radiation leaving $A_1 = Q_{r1} = A_1(J_1 - G_1)$ All this energy will reach A_2.

Net radiation leaving A_1 and reaching A_2 is given by

$Q_{1\text{-}2} = Q_{r1} = A_1(J_1 - G_1) = A_1[J_1 - J_2]$

Or $Q_{1\text{-}2} = A_1[E_{b1} - E_{b2}]$

Or $$\mathbf{Q_{1\text{-}2} = \zeta A_1[T_1^4 - T_2^4]} \qquad \textbf{(10.11)}$$

RADIATION HEAT EXCHANGE BETWEEN TWO PARALLEL INFINITE GRAYSURFACES:

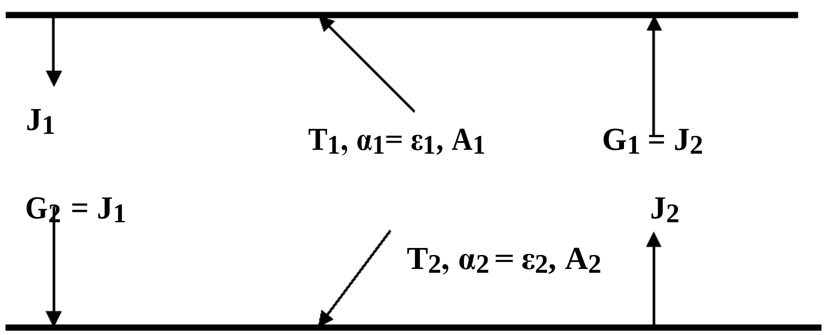


Fig: 10.7 Radiation Heat Exchange Between 2 Parallel Infinite Gray Surfaces.

Since the net radiation leaving A_1 will reach

A_2, $Q_{1\text{-}2} = Q_{r1} = A_1[J_1 – G_1]$ $J_1 = E_1 + (1-$ (10.12a)

$\alpha_1)G_1$ (10.12b)

$J_2 = E_2 + (1-\alpha_2)G_2$ (10.12c)

$J_1 = G_2$ (10.12d)

$J_2 = G_1$ (10.12e)

Equation (10.12b) can be written as

$$J_1 – (1 – \alpha_1)G_1 = E_1 \quad \text{..(4.12f)}$$

Equation (4.12c) with the help of Eqns. (10.12d) and Eqns. (10.12e) can be rewritten as

$$– (1 – \alpha_2)J_1 + G_1 = E_2 \qquad (10.12g)$$

Solving for J_1 and G_1 from Eq. (10.12f) and (10.12g) we get

$$J_1 = \frac{E_1 + (1 - \alpha_1) E_2}{1 - (1 - \alpha_1)(1 - \alpha_2)}$$

$$\text{Or } J_1 = \frac{\varepsilon_1 E_{b1} + (1 - \alpha_1)\varepsilon_2 E_{b2}}{1 - (1 - \alpha_1)(1 - \alpha_2)} \qquad \text{……………………………..(10.13a)}$$

$$\text{and } G_1 = \frac{\varepsilon_2 E_{b2} + (1 - \alpha_2)\varepsilon_1 E_{b1}}{1 - (1 - \alpha_1)(1 - \alpha_2)} \qquad \text{……………………………..(10.13b)}$$

Substituting these expressions for J_1 and G_1 in Eq.(10.12a) we get

$$Q_{1\text{-}2} = \frac{A_1}{[1 - (1 - \alpha_1)(1 - \alpha_2)]}[\varepsilon_1 E_{b1} + (1 - \alpha_1)\varepsilon_2 E_{b2} - \varepsilon_2 E_{b2} - (1 - \alpha_2)\varepsilon_1 E_{b1}]$$

$$\text{Or } Q_{1\text{-}2} = \frac{A_1 [\alpha_2 \varepsilon_1 E_{b1} - \alpha_1 \varepsilon_2 E_{b2}]}{[1 - (1 - \alpha_1)(1 - \alpha_2)]}$$

Substituting for E_{b1} and E_{b2} in terms of temperatures we get

$$\text{Or } Q_{1\text{-}2} = \frac{\zeta A_1 [\alpha_2 \varepsilon_1 T_1^4 - \alpha_1 \varepsilon_2 T_2^4]}{[1 - (1 - \alpha_1)(1 - \alpha_2)]} \qquad \text{…………………………………….(10.14)}$$

If Kirchoff"s law holds good then $\alpha_1 = \varepsilon_1$ and $\alpha_2 = \varepsilon_2$.

$$\text{Hence } Q_{1\text{-}2} = \frac{\zeta A_1 [\varepsilon_1 \varepsilon_2 T_1^4 - \varepsilon_1 \varepsilon_2 T_2^4]}{[1 - (1 - \varepsilon_1)(1 - \varepsilon_2)]}$$

$$\text{Or} \qquad \mathbf{Q_{1\text{-}2} = \frac{\zeta A_1 (T_1^4 - T_2^4)}{[1/\varepsilon_1 + 1/\varepsilon_2 - 1]}} \qquad \text{…………………………(10.15)}$$

PLANE RADIATION SHIELDS: It is possible to reduce the net radiation heat exchange between two infinite parallel gray surfaces by introducing a third surface in between them. If the third surface, known as the radiation shield is assumed to be very thin, then both sides of this surface can be assumed to be at the same temperature.

Fig.10.8 shows a scheme for radiation heat exchange between two parallel infinite gray surfaces at two different temperatures T_1 and T_2 in presence of a radiation shield at a uniform temperature, T_3.

Now
$$\frac{Q_{1-3}}{A_1} = \frac{\zeta\,(T_1^4 - T_3^4)}{[\,1/\varepsilon_1 + 1/\varepsilon_{13} - 1\,]} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10.16a)$$

And
$$\frac{Q_{3-2}}{A_1} = \frac{\zeta\,(T_3^4 - T_2^4)}{[\,1/\varepsilon_{32} + 1/\varepsilon_2 - 1\,]} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10.16b)$$

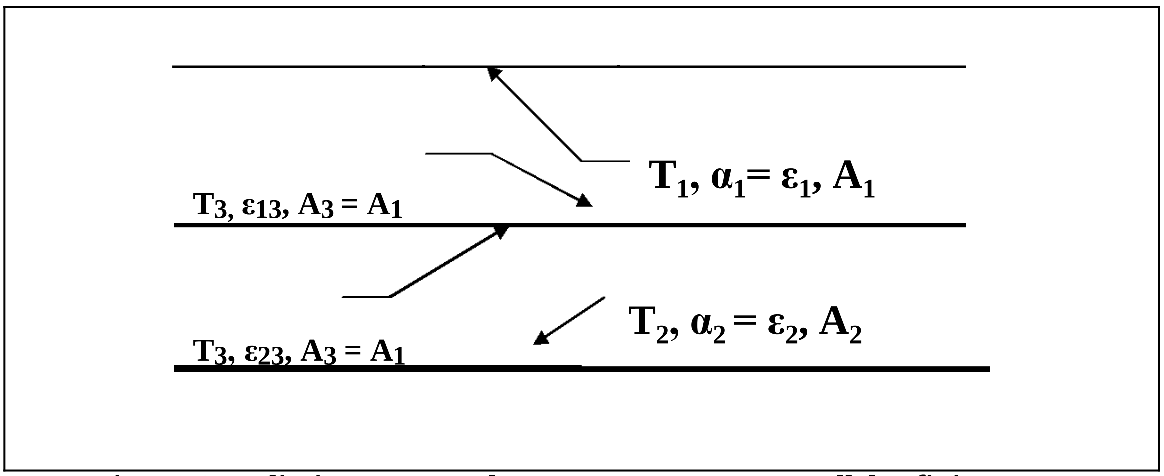


Fig: 10.8 Radiation Heat Exchange Between Two Parallel Infinite Gray surfaces in presence of a radiation shield

For steady state conditions, these two must be equal..Therefore we have

$$\frac{(T_1^4 - T_3^4)}{[\,1/\varepsilon_1 + 1/\varepsilon_{13} - 1\,]} = \frac{(T_3^4 - T_2^4)}{[\,1/\varepsilon_{32} + 1/\varepsilon_2 - 1\,]}$$

Let $X = [\,1/\varepsilon_1 + 1/\varepsilon_{13} - 1\,]$

and $Y = [\,1/\varepsilon_{32} + 1/\varepsilon_2 - 1\,]$

Then,

$$\frac{(T_1^4 - T_3^4)}{X} = \frac{(T_3^4 - T_2^4)}{Y}$$

Solving for T_3 we get

$$T_3 = \left[\frac{T_1^4 + (X/Y)T_2^4}{(1 + X/Y)}\right]^{1/4} \qquad \text{.................................}(10.16c)$$

Substituting this value of T_3 in Eq. (10.16a) we get

$$Q_{1-3} / A_1 = Q_{3-2} / A_1 = (Q_{1-2} / A)_{1\ \text{Rad.Shield}} = \zeta \{ T_2^{\ 4} - [\{T_1^4 + (X/Y)T_2^4\}/(1 + X/Y)] \} / X$$

……………………….(10.17a)

<u>Special case:</u>

When $\varepsilon_1 = \varepsilon_2 = \varepsilon_{13} = \varepsilon_{32} = \varepsilon$, then $X = Y = (2/\varepsilon) - 1$

Hence, $T_3 = [(T_1^4 + T_2^4) / 2]^{1/4}$ .... (10.18a)

and $[Q_{1-2} / A]_{1\ \text{rad shield}} = \dfrac{\zeta\{ T_1^4 - [(T_1^4 + T_2^4) / 2]}{[2/\varepsilon - 1]}$

$$= \frac{\zeta [T_1^4 - T_2^4]}{2 [2/\varepsilon - 1]} \qquad \text{…………………………………}(10.18b)$$

It can be seen from the above equation that when the emissivities of all surfaces are equal, the net radiation heat exchange between the surfaces in the presence of single radiation shield is 50% of the radiation heat exchange between the same two surfaces without the presence of a radiation shield. This statement can be generalised for N radiation shields as follows:

$$[Q_{1-2} / A]_{N\ \text{shields}} = \frac{1}{(N + 1)} [Q_{1-2} / A]_{\text{without shield}} \qquad \text{.......................}(10.18c)$$

4.5: ILLUSTRATIVE EXAMPLES ON PLANE RADIATION SHIELDS

Example 10.6: Two parallel infinite grey surfaces of emissivities 0.5 are at temperatures of 400K and 300K. Determine the net radiation heat flux between the two surfaces. Also determine the reduction in radiation flux when a plane radiation shield having emissivity of 0.5 on both its surfaces is placed between the two grey surfaces. Also determine the steady state temperature of the shield.

Solution: The radiation flux between two gray surfaces is given by

$$q = Q_{1-2} / A = \frac{\zeta\ (T_1^{\ 4} - T_2^{\ 4})}{[\ 1 / \varepsilon_1 + 1 / \varepsilon_2 - 1]}.$$

Since $\varepsilon_1 = \varepsilon_2 = \varepsilon = 0.5$, we have $\quad q = \dfrac{5.67 \times 10^{-8} \times (400^{\ 4} - 300^{\ 4})}{(2 / 0.5 - 1)}$

Or $\quad q = 330.75\ W / m^2.$

When a radiation shield of same emissivity is placed between two grey surfaces, the temperature of the shield T_3 is given by

$$T_3 = [(T_1^{\ 4} + T_2^{\ 4}) / 2\ ]^{1/4}$$

Hence $\quad T_3 = [(400^{\ 4} + 300^{\ 4}) / 2]^{1/4}$

Or, $\quad T_3 = 360.3\ K$

Also, since the emissivities of the plates and shield are equal we have

$$(q)_{1\ shield} = q / 2 = 330.75 / 2 = 163.375\ W / m^2$$

Example 10.7: Two parallel plates are at temperatures T_1 and T_2 and have emissivities $\varepsilon_1 = 0.8$ and $\varepsilon_2 = 0.5$. A radiation shield having the same emissivity ε_3 on both sides is placed between the plates. Calculate the emissivity ε_3 of the shield in order to reduce the radiation heat loss from the system to one tenth of that without shield.

Solution: Radiation flux between the two plates without the presence of a radiation shield is given by

$$q_{1-2} = \frac{\zeta\,[T_1^4 - T_2^4]}{[1/\varepsilon_1 + 1/\varepsilon_2 - 1]} = \frac{\zeta\,[T_1^4 - T_2^4]}{[1/0.8 + 1/0.5 - 1]}$$

or

$$q_{1-2} = \frac{\zeta\,[T_1^4 - T_2^4]}{2.25} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

When a shield is placed between the plates, the radiation flux is given by

$$(q)_{1radshield} = \frac{\sigma\left(T_1^4 - \dfrac{T_1^4 + (x/y)T_2^4}{1 + x/y}\right)}{x}$$

$$x = \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_3} - 1 = \frac{1}{0.8} + \frac{1}{\varepsilon_3} - 1 = 0.25 + \frac{1}{\varepsilon_3}$$

$$y = \frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_2} - 1 = \frac{1}{\varepsilon_3} + \frac{1}{0.5} - 1 = 1 + \frac{1}{\varepsilon_3}$$

$$\therefore (q)_{1radshield} = \frac{\sigma\left[(1 + x/y)T_1^4 - T_1^4 - (x/y)T_2^4\right]}{x(1 + x/y)}$$

$$(q)_{1radshield} = \frac{\sigma\left(T_1^4 - T_2^4\right)}{x + y}$$

But $(q)_{1radshield} = 0.1 \times q_{1-2}$

$$\therefore \frac{\sigma\left(T_1^4 - T_2^4\right)}{x + y} = \frac{1}{10}\,\frac{\sigma\left(T_1^4 - T_2^4\right)}{2.25} \Rightarrow x + y = 22.5 \Rightarrow 0.25 + \frac{1}{\varepsilon_3} + 1 + \frac{1}{\varepsilon_3} = 22.5$$

$$\Rightarrow \frac{2}{\varepsilon_3} = 21.25 \Rightarrow \varepsilon_3 = \frac{2}{21.25} = 0.094$$

Example 10.8: Two large parallel plates are at 800K and 600K have emissivities of 0.5 and 0.8 respectively. A radiation shield having emissivity of 0.1 on the surface facing 800K plate and 0.05 on the surface facing 600K plate is placed between the plates. Calculate the heat transfer rate per m^2 with and without the shield. Also calculate the temperature of the shield.

Solution:

The radiation flux without the radiation shield is given by

$$q = \frac{\zeta\,(T_1^4 - T_2^4)}{[1/\varepsilon_1 + 1/\varepsilon_2 - 1]} = \frac{5.67 \times 10^{-8} \times (800^4 - 600^4)}{[1/0.5 + 1/0.8 - 1]} = 7056\ W/m^2$$

When a radiation field is placed between thick plates the radiation flux is given by

$$(q)_{1\ \text{Rad. shield}} = \frac{\zeta\,(T_1^4 - T_3^4)}{x} \qquad \text{where } x = [\,1/\varepsilon_1 + 1/\varepsilon_{13} - 1]$$

$$\text{and } T_3 = \left[\frac{T_1^4 + (x/y)\,T_2^4}{1 + (x/y)}\right]^{1/4} \qquad \text{where } y = [\,1/\varepsilon_{32} + 1/\varepsilon_2 - 1]$$

Now x = [1 / 0.5 + 1 / 0.1 – 1] = 11 and y = [1 / 0.05 + 1 / 0.8 – 1] = 20.25

$$\text{Therefore} \quad T_3 = \left[\frac{800^4 + (11/20.25) \times 600^4}{1 + (11/20.25)}\right]^{1/4}$$

$$= 746.8\ \text{K}$$

$$\text{Hence } (q)_{1\ \text{Rad. shield}} = \frac{5.67 \times 10^{-8} \times (800^4 - 746.8^4)}{11} = 508\ \text{W/m}^2$$

Example 10.9: Find an expression for the net radiant flux between two infinite parallel diffuse grey surfaces at temperatures T_1 and T_2 degrees Kelvin when an infinite opaque plate of thickness b and thermal conductivity K is placed between them. Assume that all surfaces have the same emissivity.

Solution: The schematic for the problem is shown in Fig. E10.9. For steady state heat transfer we have

$$q = Q_{1\text{-}2}/A = Q_{1\text{-}3}/A = Q_{3\text{-}4}/A = Q_{4\text{-}2}/A$$

$$\text{Now } Q_{1\text{-}3}/A = \frac{\zeta\,(T_1^4 - T_3^4)}{(2/\varepsilon - 1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$Q_{3\text{-}4}/A = k\,(T_3 - T_4)/b \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

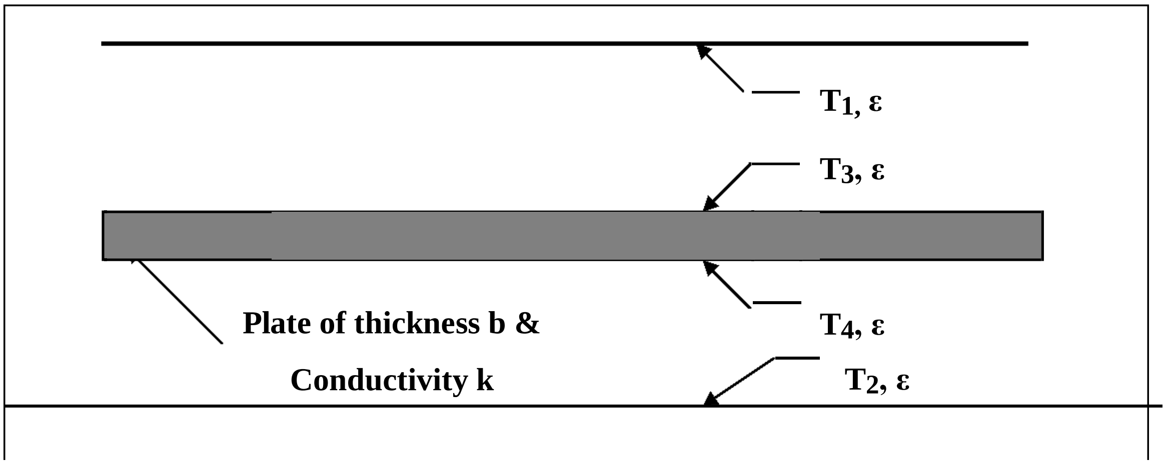


Fig. E10.9: Schematic for example 10.9

$$Q_{4-2} / A = \frac{\zeta (T_4^{\,4} - T_2^{\,4})}{(2/\varepsilon - 1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

From Eq. (1), $T_3 = [\, T_1^{\,4} - (q /\zeta)(2/\varepsilon - 1)]^{1/4}$.

Similarly from Eq. (3) we get $T_4 = [\, T_2^{\,4} - (q /\zeta)(2/\varepsilon - 1)]^{1/4}$.

Substituting these expressions for T_3 and T_4 in Eq. (2) we get

$$q = (k/b)[\{T_1^{\,4} - (q/\zeta)(2/\varepsilon - 1)\}^{1/4} - \{T_2^{\,4} + (q/\zeta)(2/\varepsilon - 1)\}^{1/4}]$$

$$\text{Or } (qb) / k = [\{T_1^{\,4} - (q/\zeta)(2/\varepsilon - 1)\}^{1/4} - \{T_2^{\,4} + (q/\zeta)(2/\varepsilon - 1)\}^{1/4}]$$

Example 10.10: Calculate the steady heat flow through the composite slab of Fig P10.10 consisting of two large plane walls with an evacuated space in between. The thicknesses of the walls are 20 and 30cm, they have thermal conductivities of 1.0 and 0.5 W/m-k and the emissivities of the surfaces facing each other are 0.5 and 0.4 respectively.

Solution :

If q is the heat flux through the composite slab then,

$$q = \frac{Q}{A} = \frac{(T_i - T_1)}{R_{ci} + R_1} = \frac{(473 - T_1)}{(1 / 20 + 0.2 /1)} = \frac{(473 - T_1)}{0.25} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(a)}$$

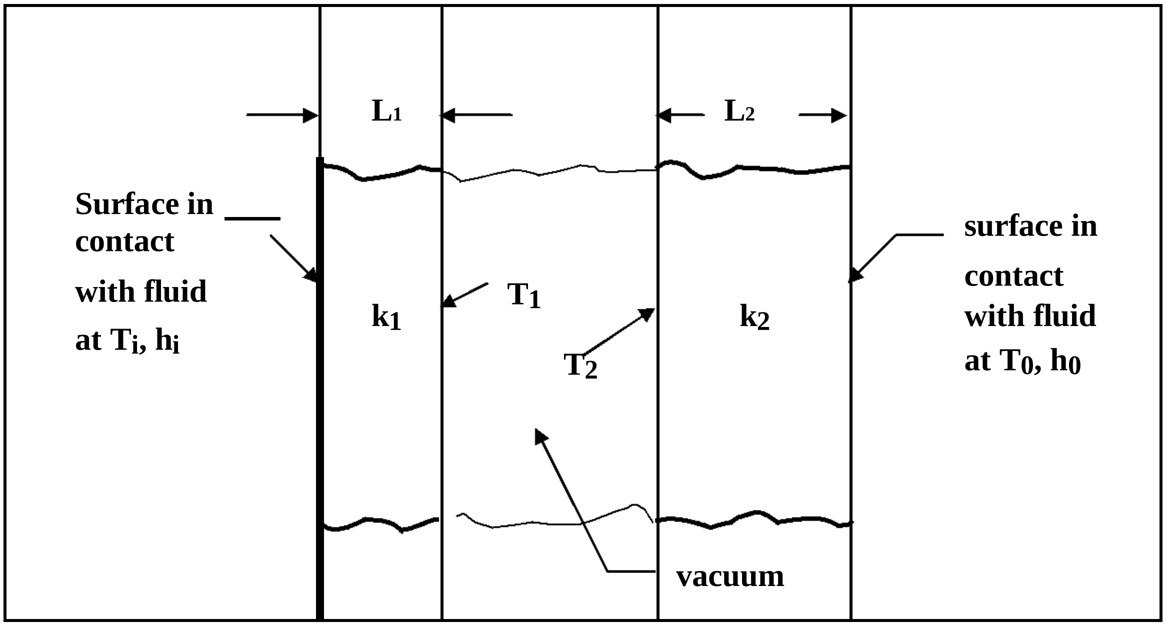


Fig. E10.10: Schematic for example 10.10

Also $q = \dfrac{\zeta (T_1^4 - T_2^4)}{(1/\varepsilon_1 + 1/\varepsilon_2 - 1)} = \dfrac{5.67 \times 10^{-8} \times [T_1^4 - T_2^4]}{[1/0.5 + 1/0.4 - 1]}$

Or $q = \dfrac{5.67 \times 10^{-8} \times [T_1^4 - T_2^4]}{3.5}$ .... (b)

and $q = \dfrac{(T_2 - T_0)}{R_2 + R_{co}} = \dfrac{(T_2 - 313)}{(0.3/0.5) + (1/10)} = \dfrac{(T_2 - 313)}{0.7}$ .... (c)

From Eq. (a) $T_1 = 473 - 0.25\,q$, and from Eq. (c) $T_2 = 313 + 0.7q$.

Substituing these expressions for T_1 and T_2 in Eq. (b) we get

$$q = \frac{5.67 \times 10^{-8} [(473 - 0.25q)^4 - (313 + 0.7q)^4]}{3.5}$$

Solving the above equatin by trial and error method we get $q = 139\ W/m^2$

VIEW FACTOR OR CONFIGURATION FACTOR:

In engineering applications, we come across problems involving radiation heat exchange between two or more finite surfaces. When the surfaces are separated from each other by a non participating medium that does not absorb, emit or reflect radiation, then the radiation heat exchange is not affected by the medium. A vacuum is a perfect non participating medium. However, air and many gases closely approximate this condition. For any two surfaces, the orientation of them with respect to each other affects the fraction of the radiation energy leaving one surface and striking the other directly. The concept of "VIEW FACTOR" (also called as CONFIGURATION FACTOR/SHAPE FACTOR) has been utilised to formalise the effects of orientation in the radiation heat exchange between surfaces. Before the concept of view factor is introduced, two more terms have to be defined.

SOLID ANGLE AND INTENSITY OF RADIATION:

Solid Angle: The solid angle dw subtended by an elemental area dA surrounding point P with respect to any other point O in space is defined as the component of the area dA in the direction OP divided by the square of the distance between O and P. This is illustrated in Fig. 10.9. Solid angle is measured in Steradian (Sr).

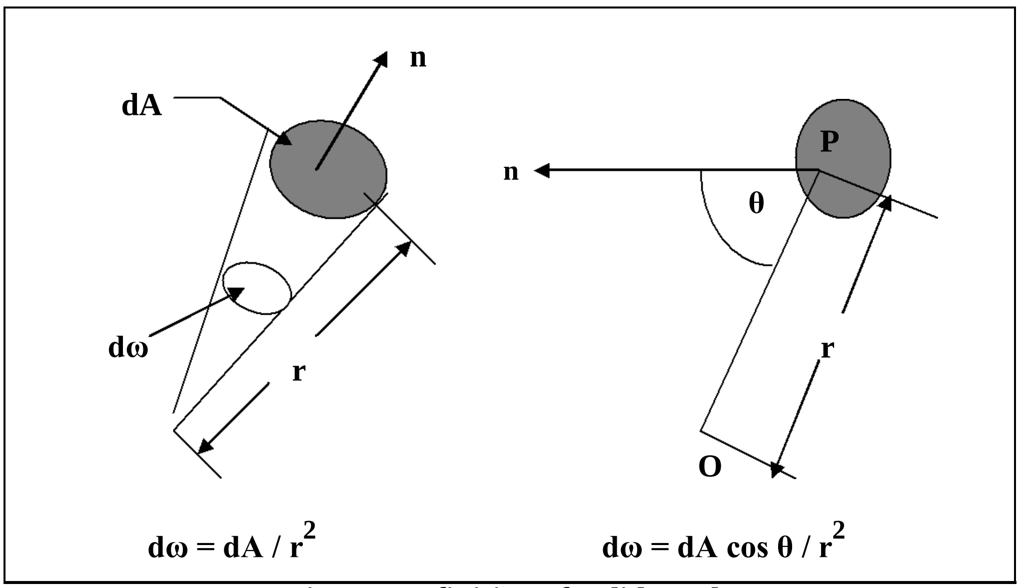


Fig 4.9: Definition of Solid Angle

Based on this definition, it can be readily inferred that the solid angle subtended by a hemispherical surface from its centre is 2π ($dw = 2\pi x r^2/r^2$) and by a full glass sphere from its centre is 4π.

Intensity of Radiation: The total intensity of radiation emitted by the surface in a given direction is equal to the radiant flux passing in that direction per unit solid angle. If I is the intensity of radiation and E is the total emissive power, then by definition

$$I = dE/dw \qquad \textbf{(10.19a)}$$

$$E = \int I dw \qquad \textbf{(10.19b)}$$

where the integration is carried out over all directions encompassed by a hemisphere.

Consider an elemental area dA_1 whose total emissive power is E_1. This total radiant energy emitted by dA_1 can be intercepted by a hemisphere as shown in Fig 10.10.

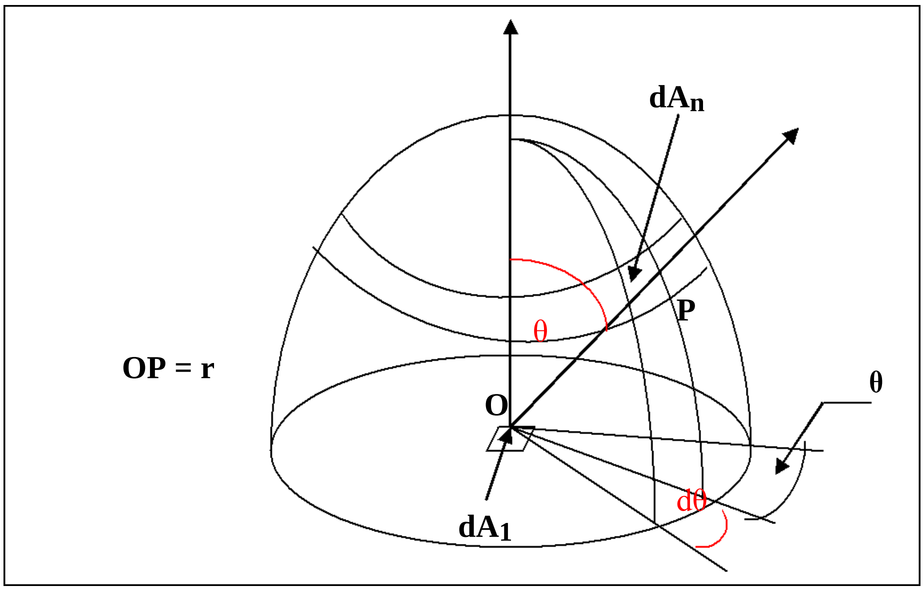


Fig 10.10: Radiation from a differential area dA_1 into surrounding hemisphere centered at dA_1.

If I is the intensity of radiation at any point P on the surface of the hemisphere due to emission by an elemental area dA_1 at O, then

$$E_1 = \int I_1 \cos\theta \, dw = \int I_1 \cos\theta \left(\frac{dA_n}{r^2} \right)$$

$$E_1 = \int_{\theta=0}^{\frac{\pi}{2}} \int_{\phi=0}^{2\pi} \left[\frac{I_1 \cos\theta \times r \sin\theta \, d\theta \times r \, d\phi}{r^2} \right]$$

Assuming that I_1 is same in all directions (Lambert‟s Law)

$$E_1 = I_1 \int_{\theta=0}^{\frac{\pi}{2}} \int_{\phi=0}^{2\pi} \cos\theta \sin\theta \, d\theta \, d\phi \qquad \textbf{(10.20)}$$

$$E_1 = \pi I_1$$

If the surface is a black surface then $\mathbf{E_b = \pi I_b}$ **(10.21)**

4.6.2: VIEW FACTOR BETWEEN TWO ELEMENTAL SURFACES.

Consider 2 elemental surfaces of area dA_1 and dA_2 as shown in Fig. 10.11. Let their normals n_1 and n_2 make angles θ_1 and θ_2 with the line joining the centroid of the two elemental areas. Let dw_2 be the solid angle subtended by dA_2 at dA_1 and dw_1 be the solid angle subtended by dA_1 at dA_2.

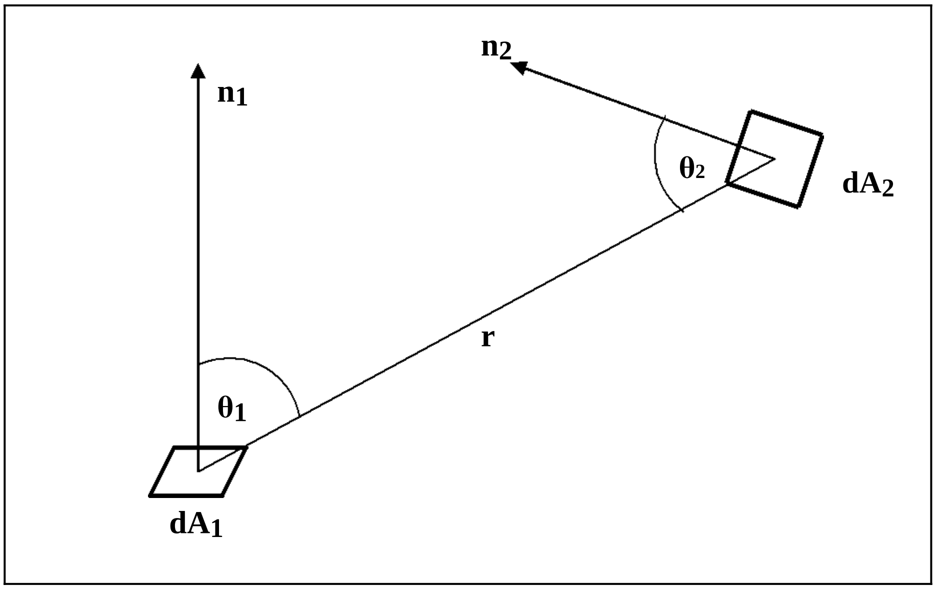


Fig. 10.11: View factor between two elemental areas

Let I_1 be the intensity of radiation from dA_1 striking dA_2 and I_2 be the intensity of radiation from dA_2 striking dA_1. If $dq_{1\text{->}2}$ is the radiant energy leaving dA_1 and striking dA_2 then

$$dq_{1\rightarrow 2} = I_1\, dA_1 \cos\phi_1\, dw_2$$

$$\text{Or}\; dq_{1\rightarrow 2} = I_1\, dA_1 \cos\phi_1 \frac{dA_2 \cos\phi_2}{r^2}$$

Radiation energy leaving $dA_1 = dq_{r1} = E_1 dA_1$

Fraction of energy leaving dA_1 and striking dA_2 is defined as the view factor of dA_2 with respect to dA_1 and is denoted by dF_{1-2}

$$\text{Therefore} \qquad dF_{1-2} = \frac{dq_{1-2}}{dq_{r1}} = \frac{I_1\, dA_1\, dA_2 \cos\theta_1 \cos\theta_2}{r^2 E_1\, dA_1}$$

Using the relation $E_1 = \pi\, I_1$ we have

$$dF_{1-2} = \frac{dA_2 \cos\theta_1 \cos\theta_2}{\pi r^2} \quad \textbf{(10.22)}$$

Similarly, the view factor of dA_1 with respect to dA_2 is denoted by dF_{2-1} and given by

$$dF_{2-1} = \frac{dA_1 \cos\theta_1 \cos\theta_2}{\pi r^2} \quad \textbf{(10.23)}$$

It follows from Eq. 10.22 and 10.23 that

$$dA_1 dF_{1-2} = dA_2 dF_{2-1} \quad \textbf{(10.24)}$$

10.6.3: VIEW FACTOR FOR FINITE SURFACES:

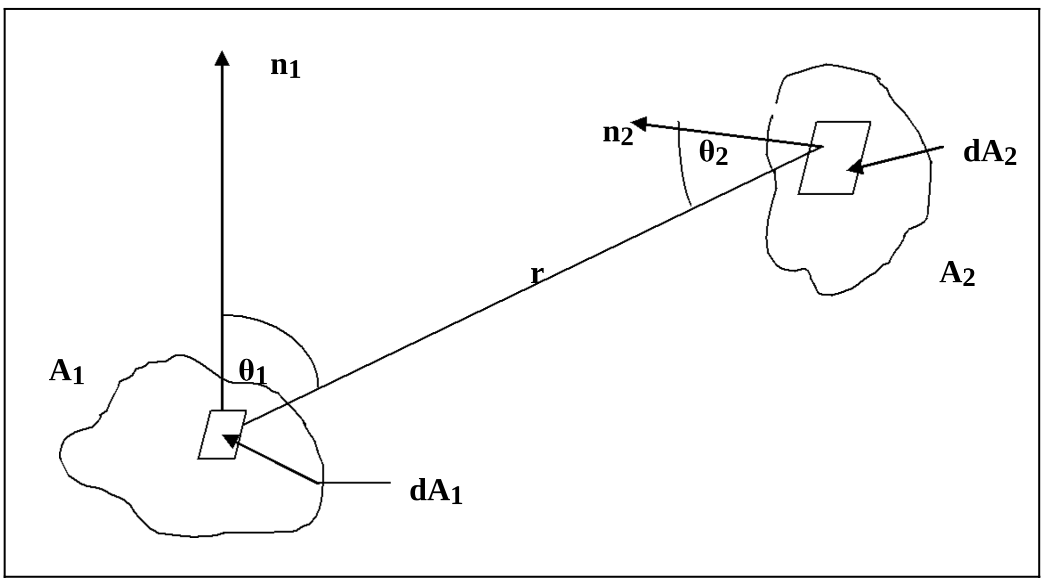


Fig.10.12: View Factor between two finite areas

Consider two finite surfaces of areas A_1 and A_2 as shown in Fig 10.12. If n_1 and n_2 are the Normals for elemental areas dA_1 and dA_2 then energy leaving dA_1 and reaching dA_2 is given by

$dq_{1\to2} = I_1\, dA_1 \cos\theta_1\, dA_2\ \cos\theta_2\ /\ r^2$

Hence $Q_{1\to2} = \int_{A1}\int_{A2}[\{ I_1\, dA_1\, dA_2 \cos\theta_2\} / r^2]$.

Total radiation emitted by $A_1 = Q_{r1} = E_1A_1 = \pi\, I_1\, A_1$

Fraction of enrgy which leaves A_1 and reaches A_2 is given by

$$F_{1-2} = \frac{Q_{1\rightarrow 2}}{Q_{r1}} = \frac{\int\int[\{ I_1\, dA_1\, dA_2 \cos\theta_2\} / r^2].}{\pi\, I_1\, A_1}$$

Or $$\mathbf{F_{1-2} = \frac{1}{A_1} [\int\int\{ I_1\, dA_1\, dA_2 \cos\theta_2\}/ (\pi r^2)] \text{..........................} (10.25a)}$$

Similarly $$\mathbf{F_{2-1} = \frac{1}{A_2} [\int\int\{ I_1\, dA_1\, dA_2 \cos\theta_2\}/ (\pi r^2)] \text{..........................} (10.25b)}$$

It follows from Equations (10.25a) and (10.25b) that

$$\mathbf{A_1F_{1-2} = A_2F_{2-1} \text{..} (10.26)}$$

<u>Properties of view factor:</u> Consider an enclosure consisting of N zones, each of surface area A_i ($i = 1,2,3 \ldots N$). The surface of each zone may be plane, convex or concave. For the enclosure, the following relations hold good.

1. $A_i F_{i-j} = A_j F_{j-i}$,$i= 1,2,3 \ldots N$,
$j = 1,2,3 \ldots N$

2. $F_{i-i} = 0$ if A_i is plane or convex (i.e. A_i cannot see itself)
$\neq 0$ if A_i is concave.

3. $F_{1-1} + F_{1-2} + \ldots + F_{1-N} = 1$
$F_{2-1} + F_{2-2} + \ldots + F_{2-N} = 1$
|
|
$F_{N-1} + F_{N-2} + \ldots + F_{N-N} = 1$

In short, $\sum_{j=1}^{N} F_{i,j} = 1$, $i = 1,2,3 \ldots N$

4. When there are two surfaces, one surface say A_1 is completely enclosed by A_2 and if A_1 cannot see itself then, $F_{1-2} = 1$ and $F_{2-1} = A_1/A_2$

5. The view factor F_{1-2} between surfaces A_1 and A_2 (Fig. 10.13) is equal to the sum of the view factors F_{1-3} and F_{1-4} if the two areas A_3 and A_4 together make up the area A_2.

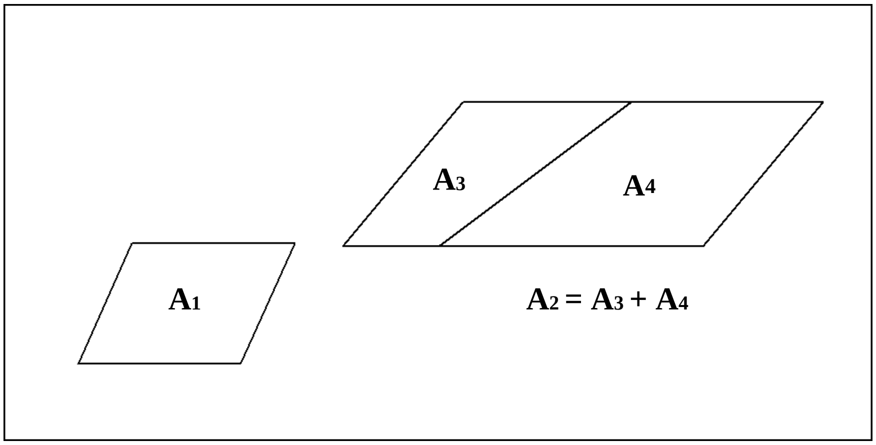


Fig. 10.13 : Additive relation between view factors

i.e., $F_{1-2} = F_{1-3} + F_{1-4}$. It should however be noted that $F_{2-1} \neq F_{3-1} + F_{3-4}$

View factors for standard configurations: The determination of view factors has been the object of considerable research. In cases where the integrals in Eq. 10.25 and Eq. cannot be solved analytically, numerical methods have been used. Some of these results are represented graphically for certain standard configurations like

(i) Shape factors between parallel rectangles of equal size.
(ii) Shape factors between rectangles perpendicular to each other and having a common edge
(iii) Shape factor from an elemental area dA_1 to a rectangular area A_2
(iv) Shape factor between two coaxial parallel discs
(v) Shape factors for concentric cylinders of finite length etc.

With the help of those charts and View Factor algebra, shape factors between surfaces not covered above can be determined.

ILLUSTRATIVE EXAMPLES ON VIEW FACTORS:

Example 10.11: Determine the view factor between an elemental area A_1 and a circular disc A_2 of radius R. The two areas are parallel to each other and positioned at a distance L form each other such that the perpendicular to A_1 passes through the centre of A_2.

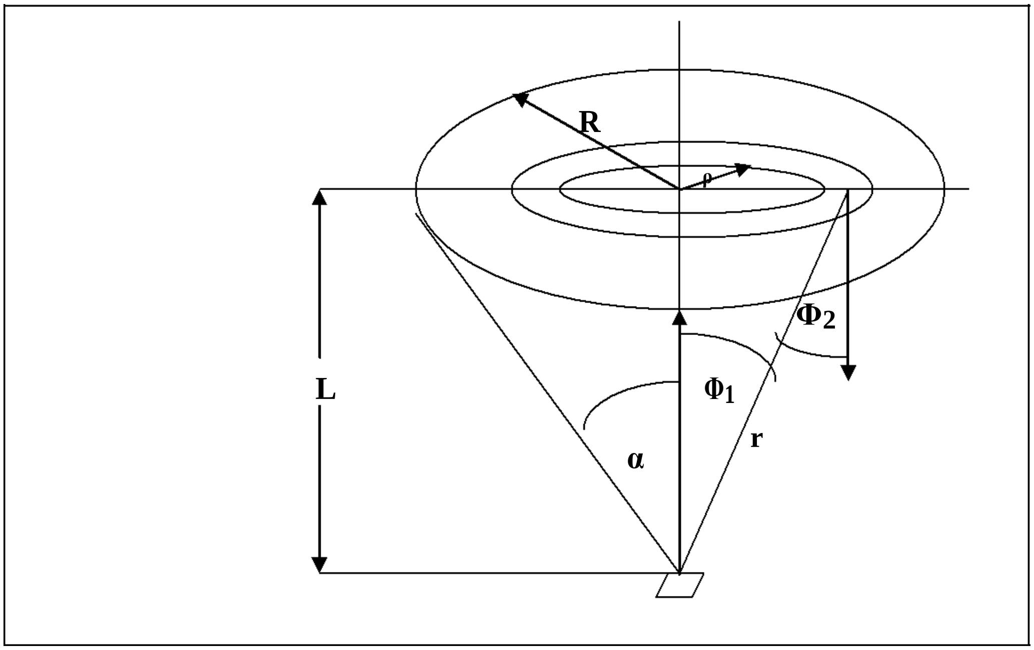


Fig.E10.11: Figure for example 10.11

$$A_1 F_{1-2} = \int_{A_1}\int_{A_2} \frac{dA_1\, dA_2 \cos\phi_1 \cos\phi_2}{\pi r^2}$$

Since $A_1 << A_2$ and $\phi_1 = \phi_2 = \phi$, the above exp can be written as

$$A_1 F_{1-2} = A_1 \int_{A_2} \frac{dA_2 \cos^2\phi}{\pi r^2} \quad \Rightarrow F_{1-2} = \int_0^R \frac{2\pi\rho\, d\rho \cos^2\phi}{\pi r^2}$$

Now $\cos\phi_1 = \cos\phi = \dfrac{L}{\sqrt{L^2+\rho^2}}$ and $r = \sqrt{L^2+\rho^2}$

$$\therefore F_{1-2} = \int_0^R \frac{2\pi\rho\, d\rho L^2}{(L^2+\rho^2)\pi(L^2+\rho^2)} = L^2 \int_0^R \frac{2\rho\, d\rho}{(L^2+\rho^2)^2}$$

$$\therefore F_{1-2} = \frac{R^2}{L^2+R^2} = \sin^2\alpha$$

Example 10.12: Obtain an expression for the shape factor for a conical cavity with respect to itself. The height of the cavity is H and the semi vertex angle of the cavity is α (See Fig. E10.12a)

Solution:

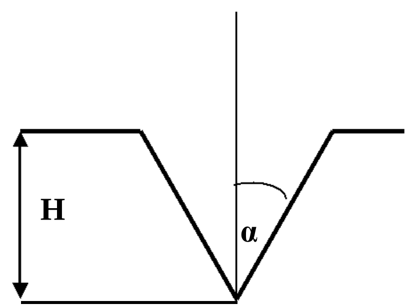


Fig. E10.12(a) : Figure for Example 10.12

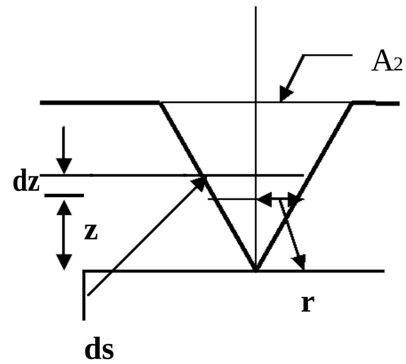


Fig. E10.12 (b):

Refering to Fig. 10.12(b) we have A_1 and A_2 form an enclosure.

Hence $F_{1\text{-}1} + F_{1\text{-}2} = 1$, and $F_{2\text{-}1} + F_{2\text{-}2} = 1$.

Since A_2 cannot see itself, $F_{2\text{-}2} = 0$. Hence $F_{2\text{-}1} = 1.0$

But $A_1F_{1\text{-}2} = A_2F_{2\text{-}1}$. Therefore $F_{1\text{-}2} = A_2 / A_1$.

From Fig. E10.12(b), $dA_1 = 2\pi\, r\, ds = 2\,\pi \sqrt{(dr^2 + dz^2)}$

Or $$dA_1 = 2\,\pi\, z \tan\alpha\, dz \sqrt{[(dr / dz)^2 + 1]}$$

$$= 2\,\pi\, z \tan\alpha\, dz \sqrt{[\tan^2\alpha + 1]}$$

$$= 2\,\pi\, z\, dz \tan\alpha \sec\alpha$$

Therefore $$A_1 = \int_0^H 2\,\pi\, z\, dz \tan\alpha \sec\alpha = \pi H^2 \tan\alpha \sec\alpha$$

Hence $$F_{1\text{-}1} = 1 - F_{1\text{-}2} = 1 - \frac{\pi H^2 \tan^2\alpha}{\pi H^2 \tan\alpha \sec\alpha} = 1 - \sin\alpha$$

<u>Example 10.13:</u> Consider 3 small surfaces each of area $dA_1 = dA_2 = dA_3 = 2\ cm^2$ as shown in fig E10.13. (a) Calculate the solid angle subtended by dA_2 with respect to a point on dA_1 (b) The solid angle subtended by dA_3 with respect to a point on dA_1 and (c) The elemental Diffuse factors $dF_{dA1\text{-}dA2}$ and $dF_{dA1\text{-}dA3}$.

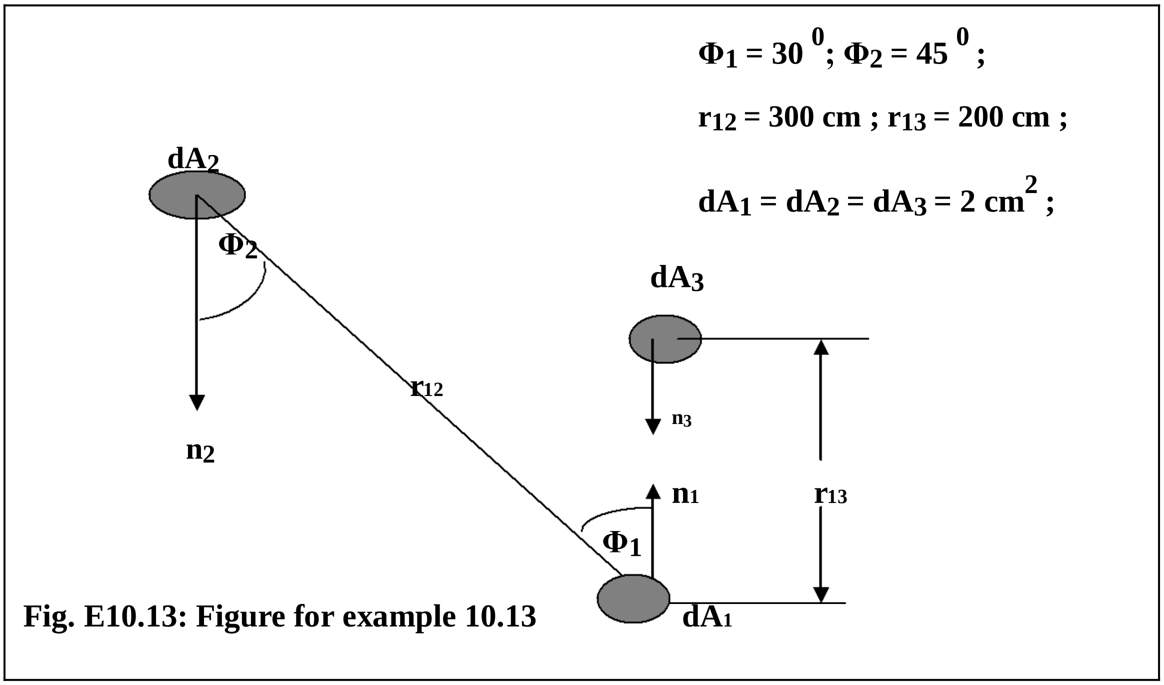


Fig. E10.13: Figure for example 10.13

If dw_{2-1} is the solid angle subtended by dA_2 w.r.t a point on dA_1 then

$$dw_{2-1} = \frac{dA_2 \cos\phi_2}{r_{12}^2} = \frac{2\cos 45^0}{300^2} = 1.57\times10^{-5} \quad sr$$

Similarly $dw_{3-1} = \dfrac{dA_3 \cos\phi_3}{r_{13}^2}$ (but $\phi_3 = 0^0$)

$$\therefore dw_{3-1} = \frac{dA_3}{r_{13}^2} = \frac{2}{200^2} = 5\times10^{-5} \quad sr$$

$$dF_{dA_1-dA_2} = \frac{dA_2 \cos\phi_{12} \cos\phi_2}{\pi(r_{12}^2)} = \frac{2\times\cos 30^0\times\cos 45^0}{\pi\times300^2} = 4.33\times10^{-6}$$

Similarly $dF_{dA_1-dA_3} = \dfrac{dA_3 \cos\phi_{13} \cos\phi_3}{\pi r_{13}^2} = \dfrac{2\times\cos^2 0_0}{\pi\times200^2} = 1.59\times10^{-5}$

Example 10.14: Determine the view factor F_{1-2} between an elemental surface dA_1 and the finite rectangular surface A_2 for the geometric arrangements shown in Fig E10.14

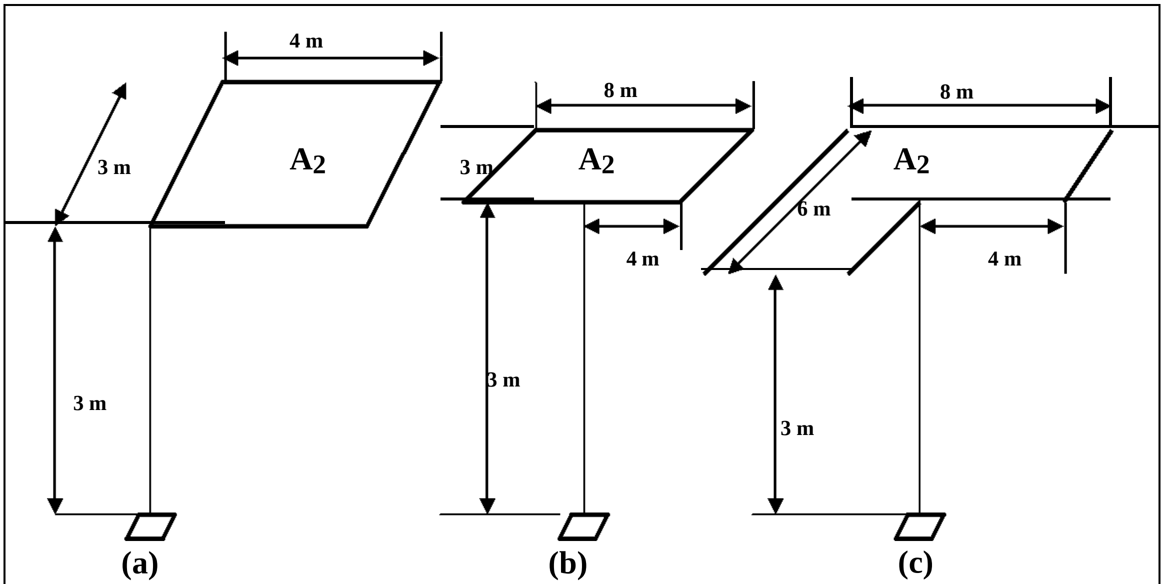


Fig. E10.14: Figures for example 10.14

Soluiton : (a) The configuration in Fig. E10.14(a) is a standard configuration for which the analytical expression for F_{1-2} is given by

$$F_{1-2} = \frac{1}{2\pi}\left[\frac{x}{\sqrt{(1+x^2)}}\tan^{-1}\left\{\frac{y}{\sqrt{(1+x^2)}}\right\} + \frac{y}{\sqrt{(1+y^2)}}\tan^{-1}\left\{\frac{x}{\sqrt{(1+y^2)}}\right\}\right]$$

Where $x = L_1 / D = 3 / 3 = 1$; and $y = L_2 / D = 4 / 3 = 1.33$

Substituting these values in the expression for F_{1-2} we get

$$F_{1-2} = \frac{1}{2\pi}\left[\frac{1}{\sqrt{(1+1^2)}}\tan^{-1}\left\{\frac{1.33}{\sqrt{(1+1^2)}}\right\} + \frac{1.33}{\sqrt{(1+1.33^2)}}\tan^{-1}\left\{\frac{1}{\sqrt{(1+1.33^2)}}\right\}\right]$$

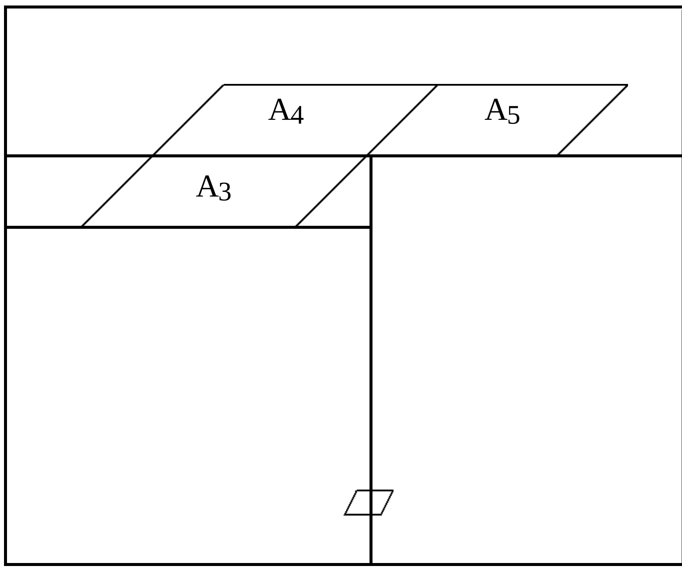
Or $\quad F_{1-2} = 0.154$

(a) The area A_2 is divided into two equal areas A_3 and A_4 both having the common edge of width $L_1 = 3$ m.Then

$$F_{1-2} = F_{1-3} + F_{1-4} = 2\,F_{1-3} \text{ (Because } F_{1-3} = F_{1-4})$$

$$= 2 \times 0.15\text{ v}4 = 0.308$$

(c) In this case A_2 is divided into three equal areas A_3, A_4, and A_5 as shown below



Now $F_{1-2} = F_{1-3} + F_{1-4} + F_{1-5}$

$= 3\ F_{1-3}$

$= 3 \times 0.154$

$= 0.462$

Example 10.15: Determine the Shape factor F_{1-2} for the configuration shown in Fig E10.15

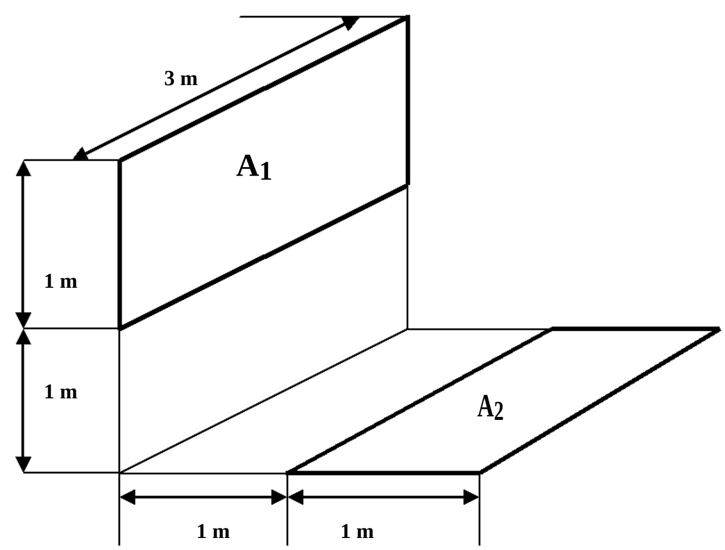


Fig. E10.15: Figure for example 10.15

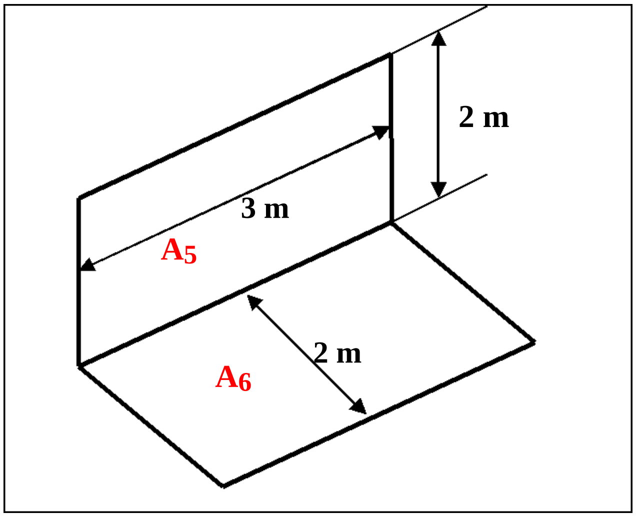
Solution:

$$F_{1-2} = F_{1-6} - F_{1-4} = \frac{A_6 F_{6-1}}{A_1} - \frac{A_4 F_{4-1}}{A_1}$$

$$= \frac{A_6}{A_1} [F_{6-5} - F_{6-3}] - \frac{A_4}{A_1} [F_{4-5} - F_{4-3}]$$

Values of F_{6-5}, F_{6-3}, F_{4-5} and F_{4-3} can be obtained from chart as follows.

To find F_{6-5}:



$A_5 = A_6 = 3 \times 2 = 6 \text{ m}^2$;

$L_1 / W = L_2/W = 2/3 = 0.667$;

From chart $F_{6-5} = 0.22$. Similarly we Get

$F_{6-3} = 0.16$; $F_{4-5} = 0.32$; $F_{4-3} = 0.27$

$$\text{Hence } F_{1-2} = \frac{6}{1 \times 3}[0.22 - 0.16] - \frac{3 \times 1}{3 \times 1} [0.32 - 0.27]$$

Thus $F_{1-2} = 0.07$

<u>Example 10.16:</u> Find F_{1-2} for the configuration shown in Fig. E10.16

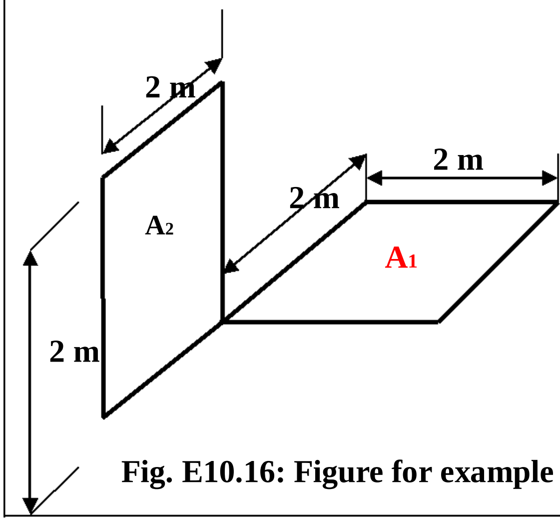


Fig. E10.16: Figure for example 10.16

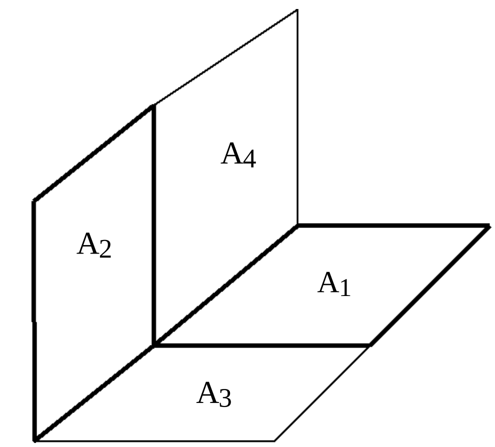


Fig. E10.16(a)

Refering to Fig. E10.16(a),let

$A_6 = A_2 + A_4 = 4 \text{ x } 2 = 6 \text{ m}^2$ and

$A_5 = A_1 + A_3\ 4 \text{ x } 2 = 6 \text{ m}^2$.

$A_1 = A_2 = A_3 = A_4 = 2 \text{ x } 2 = 4 \text{ m}^2$.

$F_{1\text{-}2} = F_{1\text{-}6} - F_{1\text{-}4}$

$= (A_6F_{6\text{-}1}) / A_1 - F_{1\text{-}4}$

$= (A_6/A_1)[F_{6\text{-}5} - F_{6\text{-}3}] - F_{1\text{-}4}$

$= (A_6/A_1)F_{6\text{-}5} - \{(A_3F_{3\text{-}6})/A_1\} - F_{1\text{-}4}$

Or $F_{1\text{-}2} = (A_6/A_1)F_{6\text{-}5} - (A_3/A_1)\ [\ F_{3\text{-}2} + F_{3\text{-}4}] - F_{1\text{-}4}$

But $F_{3\text{-}2} = F_{1\text{-}4}$ and $F_{3\text{-}4} = F_{1\text{-}2}$.

Hence $F_{1\text{-}2} = (A_6/A_1)F_{6\text{-}5} - (A_3/A_1)F_{1\text{-}4} - (A_3/A_1)\ F_{1\text{-}2} - F_{1\text{-}4}$

Or $$F_{1\text{-}2} = \frac{A_6F_{6\text{-}5} - A_3F_{1\text{-}4} - A_1F_{1\text{-}4}}{(A_1 + A_3)} = \frac{2A_1F_{6\text{-}5} - 2A_1F_{1\text{-}4}}{2A_1} = F_{6\text{-}5} - F_{1\text{-}4}$$

To find $F_{6\text{-}5}$:- $L_2 = L_1 = 2$ m; W = 4 m.

$L_1 / W = L_2 / W = 2 / 4 = 0.5$.

From chart $F_{5\text{-}6} = 0.25 = F_{6\text{-}5}$ since $A_5 = A_6$. Similarly $F_{1\text{-}4} = 0.2$.

Hence $F_{1\text{-}2} = 0.25 – 0.20 = 0.05$

Example 10.17: Find the Shape Factor $F_{2\text{-}1}$ for the configuration shown in fig E10.17

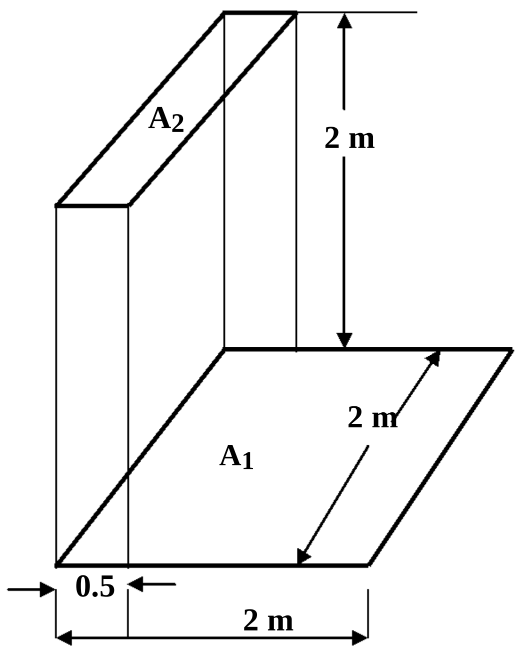


Fig. E10.17: Figure for example 10.17

Solution: Refer to Fig. 10.17 (a)

$F_{2\text{-}1} = F_{2\text{-}3} + F_{2\text{-}5} = F_{2\text{-}3} + (A_5 F_{5\text{-}2}) / A_2$

$= F_{2\text{-}3} + (A_5 / A_2) [F_{5\text{-}6} – F_{5\text{-}4}]$

$= F_{2\text{-}3} + (A_6F_{6\text{-}5}) / A_2 - (A_5F_{5\text{-}4}) / A_2$

$= F_{2\text{-}3} + (A_6 / A_2) [F_{6\text{-}1} - F_{6\text{-}3}] - (A_5F_{5\text{-}4}) / A_2$

$= F_{2\text{-}3} + (A_6 / A_2)F_{6\text{-}1} - (A_3F_{3\text{-}6}) / A_2 - (A_5F_{5\text{-}4}) / A_2$

$= F_{2\text{-}3} + (A_6 / A_2)F_{6\text{-}1} - (A_3F_{2\text{-}1}) / A_2 - (A_5F_{5\text{-}4}) / A_2$

Or $[(1 + (A_3 / A_2)] F_{2\text{-}1} = F_{2\text{-}3} + (A_6 / A_2)F_{6\text{-}1} - (A_5F_{5\text{-}4}) / A_2$

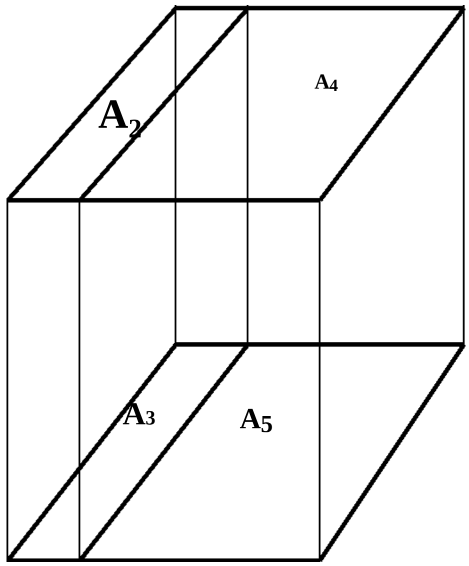


Fig. E10.17(a)

$A_1 = A_3 + A_5$; $A_6 = A_2 + A_4$

$A_2 = A_3 = 0.5 \times 2 = 1\ m^2$

$A_4 = A_5 = 1.5 \times 2 = 3\ m^2$

$A_1 = A_6 = 2 \times 2 = 4\ m^2$

$$\text{Or } F_{2-1} = \frac{F_{2-3} + (A_6 / A_2)F_{6-1} - (A_5F_{5-4}) / A_2}{[(1 + (A_3 / A_2)]}$$

From chart : $F_{2-3} = 0.06$; $F_{5-4} = 0.17$; $F_{6-1} = 0.2$

$$\text{Hence } F_{2-1} = \frac{0.06 + 4 \times 0.2 - 3 \times 0.17}{(1 + 1)} = 0.175$$

Example 10.18: Find the Shape factor F_{1-2} for the configuration shown in Fig E10.18

Solution: Refer Fig. E10.18(a).

$F_{1-2} = F_{1-8} + F_{1-4} + F_{1-6}$. But $F_{1-4} = F_{1-6}$.

Hence $F_{1-2} = F_{1-8} + 2F_{1-4}$

$= F_{1-8} + 2\,(A_4F_{4-1}) / A_1$

From example 10.16 we have $F_{4-1} = F_{7-10} - F_{4-3}$

Hence $F_{1-2} = F_{1-8} + 2\,(A_4 /A_1)\,[F_{7-10} - F_{4-3}]$

From chart $F_{1-8} = 0.15$; $F_{7-10} = 0.23$; $F_{4-3} = 0.2$

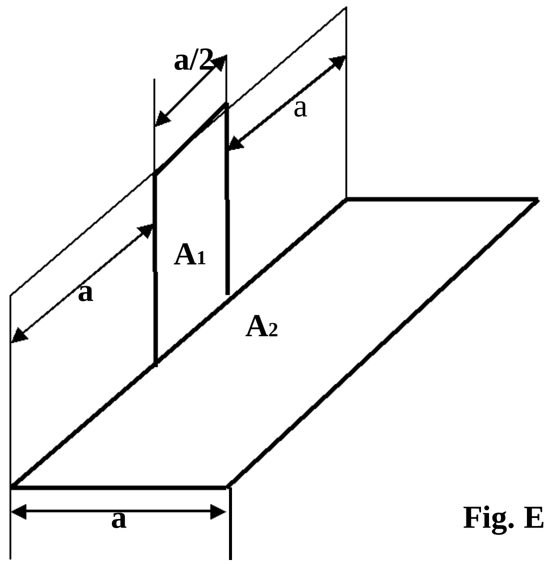
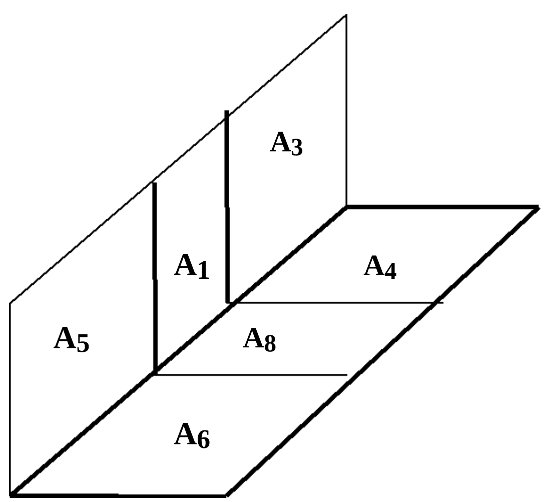


Fig. E10.18: Figure for example 10.18



$A_7 = A_1 + A_3$;

$A_{10} = A_4 + A_8$

Fig. E10.18 (a)

$$\text{Hence } F_{1\text{-}2} = 0.15 + 2 \times \frac{a^2}{[(a/2) \times a]} \times [0.23 - 0.20] = 0.27$$

Example 10.19 A1 and A2 are two rectangular flat surfaces having a common edge and inclined at an arbitrary angle α to each other. They are very long along the common edge and have lengths of ab and ac respectively in the other direction. Show that

$$F_{1-2} = \frac{(ab + ac) - bc}{2ab}$$

Solution: Refer Fig. E10.19

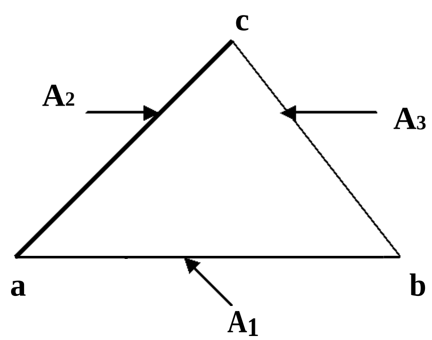


Fig. E10.19

A_1, A_2 and A_3 form an enclosure.

Hence $F_{1-1} + F_{1-2} + F_{1-3} = 1$.
But $F_{1-1} = 0$. Hence
$F_{1-2} + F_{1-3} = 1$ .... (a)
similarly
$F_{2-1} + F_{2-3} = 1$ .... (b)

$F_{3-1} + F_{3-2} = 1$ .... (c)

From Eq. (a) we get $F_{1-2} = 1 - F_{1-3}$ .... (d)

From (c) we get $F_{3-1} = 1 - F_{3-2}$

Or $\quad (A_1F_{1-3}) / A_3 = 1 - (A_2F_{2-3})/A_3$

Or $\quad F_{1-3} = (A_3 / A_1) - (A_2F_{2-3}) / A_1$

$= (A_3 / A_1) - (A_2 / A_1)\,[1 - F_{2-1}]$

$= (A_3 / A_1) - (A_2 / A_1) + F_{1-2}$

Substituting this expression in Eq. (d) we get

$$F_{1-2} = 1 - [(A_3 / A_1) - (A_2 / A_1) + F_{1-2}]$$

Solving for F_{1-2} we get

$$F_{1-2} = \frac{(A_1 + A_2) - A_3}{2A_1} = \frac{(ab \times 1) + (ac \times 1) - (bc \times 1)}{2 \times (ab \times 1)}$$

$$= \frac{(ab + ac - bc)}{2\,ab}$$

<u>Example 10.20: (Hottel's cross string formula)</u> Obtain an expression for the view factor between two flat surfaces, which extend to infinity in one direction.

Solution: Refer Fig. E10.20

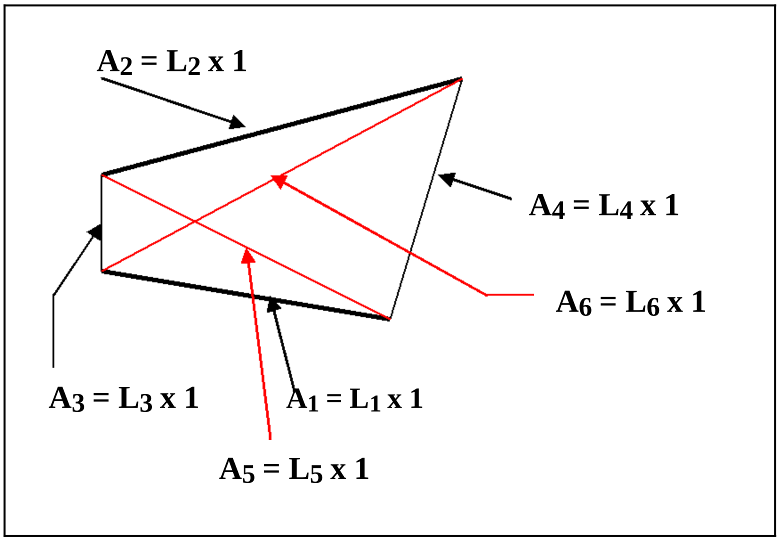


Fig. E10.20 : Figure for example 10.20

Consider unit width perpendicular to the plane of the paper. A_1, A_2, A_3 and A_4 form an enclosure. Hence we have

$$F_{1-1} + F_{1-2} + F_{1-3} + F_{1-4} = 1 \text{ and } F_{1-1} = 0.$$

Therefore $$F_{1-2} + F_{1-3} + F_{1-4} = 1 \qquad \text{(a)}$$

Similarly A_1, A_3 and A_5 form an enclosure. Therefore we have

$$F_{1-3} + F_{1-5} = 1 \qquad \text{(b)}$$

$$F_{3-1} + F_{3-5} = 1 \qquad \text{(c)}$$

Also $$F_{5-1} + F_{5-3} = 1 \qquad \text{(d)}$$

and $$F_{1-3} = 1 - F_{1-5}$$

From Eq. (b) we have $$= 1 - (A_5F_{5-1}) / A_1$$

$$= 1 - (A_5 / A_1) [1 - F_{5-3}]$$

$$= 1 - (A_5 / A_1) + (A_3F_{3-5}) / A_1$$

Or $F_{1-3} = 1 - (A_5 / A_1) + (A_3 / A_1) [1 - F_{3-1}]$

$$= 1 - (A_5 / A_1) + (A_3 / A_1) - F_{1-3}$$

Hence $F_{1-3} = \dfrac{1 - (A_5 / A_1) + (A_3 / A_1)}{2} = \dfrac{A_1 - A_5 + A_3}{2A_1}$

Similarly $F_{1-4} = \dfrac{A_1 - A_6 + A_4}{2A_1}$

Now from Eq. (a) we have $F_{1-2} = 1 - [F_{1-3} + F_{1-4}]$

Substituting the expressions obtained for F_{1-3} and F_{1-4} we get

$$F_{1-2} = 1 - \frac{A_1 - A_5 + A_3}{2A_1} - \frac{A_1 - A_6 + A_4}{2A_1}$$

$$= \frac{(A_5 + A_6) - (A_3 + A_4)}{2A_1} = \frac{(L_5 + L_6) - (L_3 + L_4)}{2L_1}$$

Example 10.21: A truncated cone has top and bottom diameters of 10cm and 20cm and a height of 10cm. Calculate the shape factor between the top surface and the side and the side and itself.

Solution: To find (i) F_{2-3} and (ii) F_{3-3} . Refer to Fig. E4.21.

(i) $F_{2-1} = (A_1F_{1-2}) / A_2$. F_{1-2} can be directly obtained fom chart as

follows: $L_1 / r = 10 / 10 = 1$; and $L_2 / r = 5 / 10 = 0.5$.

Hence fromchart $F_{1-2} = 0.12$.

Therefore $F_{2-1} = \dfrac{\pi \times (10)^2}{\pi \times (5)^2} \times 0.12 = 0.48$

$$F_{2-3} = 1 - F_{2-1} = 1 - 0.48 = 0.52$$

(ii) $F_{2-1} + F_{2-2} + F_{2-3} = 1$ and $F_{2-2} = 0$.

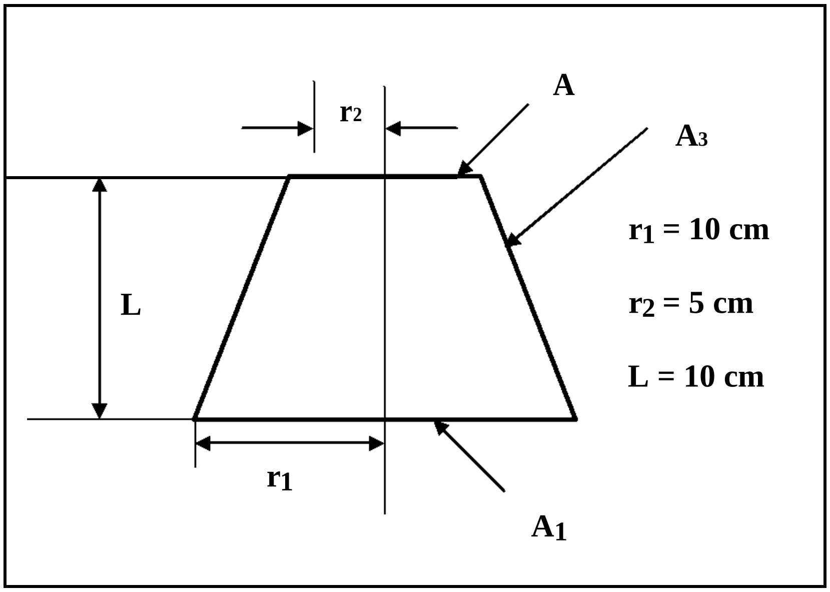


Fig. E10.21: Figure for example 10.21

Hence $F_{2-3} = 1 - F_{2-1} = 1 - 0.48 = 0.52$

Therefore $F_{3-2} = (A_2F_{2-3}) / A_3$.

Now $A_3 = \pi (r_1 + r_2) [(r_1 - r_2)^2 + L^2] = \pi \times (5 + 10) \times [5^2 + 10^2] = 526.9\ cm^2$

Therefore $F_{3-2} = \dfrac{\pi \times (5^2)}{526.9} \times 0.52 = 0.0775$

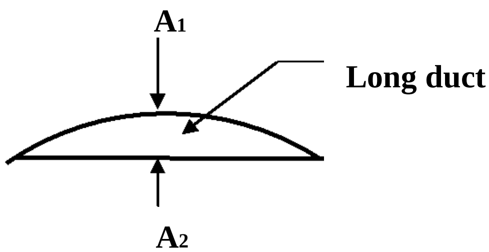
$F_{1-2} + F_{1-3} = 1$

Or $F_{1-3} = 1 - F_{1-2} = 1 - 0.12 = 0.88$

Hence $F_{3-1} = (A_1F_{1-3}) / A_3 = \dfrac{\pi \times (10^2)}{526.9} \times 0.88 = 0.525$

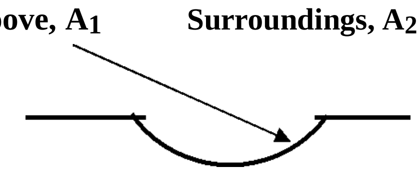
Therefore $F_{3-3} = 1 - [F_{3-1} + F_{3-2}] = 1 - [0.525 + 0.0775] = 0.397$

Example 10.22: Determine the shape factors for the geometries shown in Fig. E10.23(a) to E10.23(i)



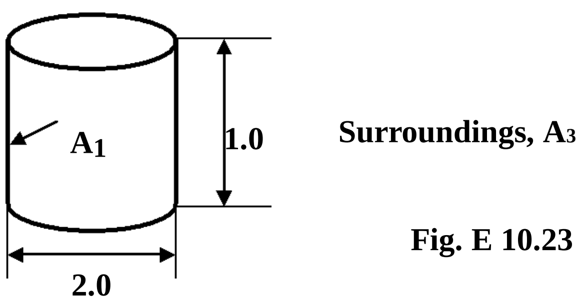
Find F_{1-2}, F_{2-1} and F_{1-1}

Fig. E 10.23 (a)



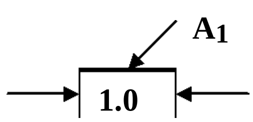
Find F_{1-2}, F_{2-1} and F_{1-1}

Fig. E 10.23 (b)



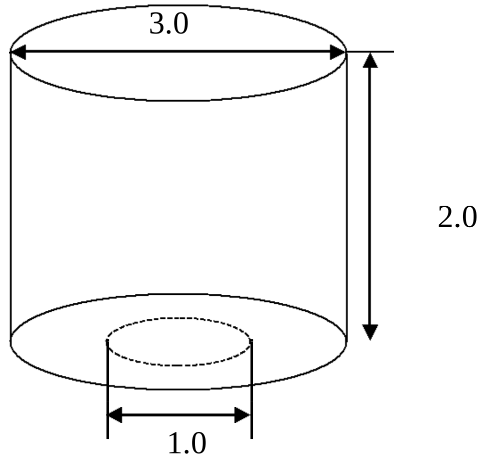
Find F_{1-1} and F_{1-3}

Fig. E 10.23 (c)

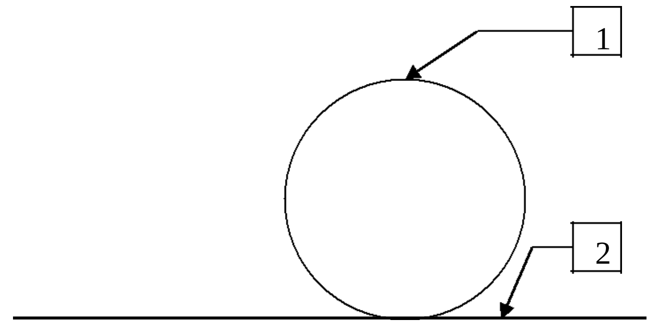


A_1 is covered by a hemispherical surface A_2 of radius 1.5. Find F_{1-2}, F_{1-3}, F_{2-1} and F_{2-2}

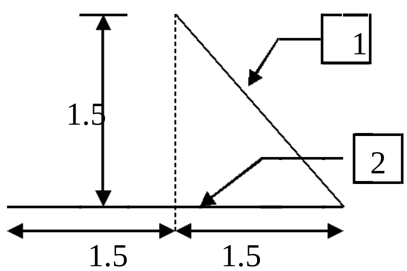
Fig. E 10.23 (d)



(e) $F_{1\text{-}3}$ $F_{2\text{-}3}$ $F_{2\text{-}2}$ $F_{2\text{-}1}$ $F_{1\text{-}2}$ for disc surrounded by a short cylinder

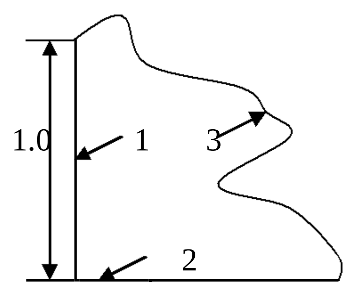


(f) $F_{1\text{-}2}$ $F_{2\text{-}1}$ fro sphere on infinite plane

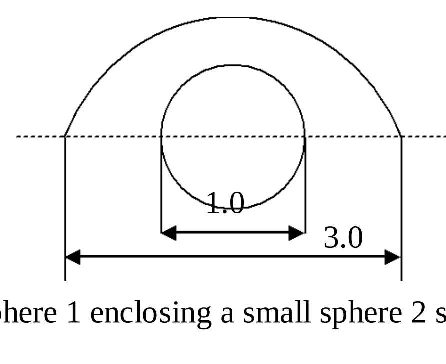


Surroundings 3

(g) $F_{1\text{-}2}$ $F_{2\text{-}1}$ $F_{2\text{-}3}$



(h) $F_{1\text{-}2}$ $F_{2\text{-}1}$ $F_{1\text{-}3}$ $F_{2\text{-}3}$



(i) $F_{1\text{-}2}$ $F_{2\text{-}1}$ $F_{1\text{-}1}$ $F_{1\text{-}3}$ hemisphere 1 enclosing a small sphere 2 surrounded by a large enclosure 3

Solution:

(1) and (2) form an enclosure

$\therefore F_{1-1} + F_{1-2} = 1$

$F_{2-1} + F_{2-2} = 1$

since (2) is a flat surface, $F_{2-2} = 0$

$\therefore F_{2-1} = 1$ but $A_1 F_{1-2} = A_2 F_{2-1}$

$$\therefore F_{1-2} = \frac{A_2}{A_1} F_{2-1} = \frac{A_2}{A_1} = \frac{2R\times 1}{2\pi R\times 1} = \frac{1}{\pi}$$

$$\therefore F_{1-1} = 1 - F_{1-2} = 1 - \frac{1}{\pi} = \frac{\pi - 1}{\pi}$$

All the radiation from (1) which goes to the surroundings can be intercepted by an imaginary surface 2^1 as shown in Fig. Now 1 and 2^1 form an enclosure. Therefore from the above example

$$F_{2'-1} = F_{2-1} = 1;\ F_{1-2'} = F_{1-2} = \frac{1}{\pi};\ F_{1-1} = \frac{\pi - 1}{\pi}$$

All the radiation from (1) which goes to the surroundings can be intercepted by an imaginary surface 2 and 4 as shown in Fig. Now, 1, 2 and 4 form an enclosure.

$\therefore F_{1-1} + F_{1-2} + F_{1-4} = 1$

but $F_{1-4} = F_{1-2}$

$\therefore F_{1-1} = 1 - 2F_{1-2}$ ----------- (a)

also $F_{2-1} + F_{2-2} + F_{2-4} = 1$ and $F_{2-2} = 0$

$\therefore F_{2-1} + F_{2-4} = 1 \Rightarrow F_{2-1} = 1 - F_{2-4}$

from chart for two parallel coaxial discs (2) and (4)

$F_{2-4} = 0.383$

$F_{2-1} = 1 - 0.383 = 0.617$

$$\therefore F_{1-2} = \frac{A_2 F_{2-1}}{A_1} = \frac{\pi R^2}{2\pi R L} \times 0.617 = \frac{1.0}{2\times 1.0} \times 0.617 = 0.309$$

$\therefore F_{1-1} = 1 - 2\times 0.309 = 0.383$

$F_{1-3} = 1 - F_{1-1} = 1 - 0.383 = 0.617$

$F_{1-1} + F_{1-2} + F_{1-3} = 1$ and $F_{1-1} = 0$; $F_{1-3} = 0$

$\therefore F_{1-2} = 1$ ______________________________ (a)

$F_{2-1} + F_{2-2} + F_{2-3} = 1$ ____________ (b)

$F_{3-1} + F_{3-2} + F_{3-3} = 1$ and $F_{3-3} = F_{3-1} = 0$

$\therefore F_{3-2} = 1$ ______________________________ (c)

$A_3 F_{3-2} = A_2 F_{2-3}$

$$\therefore F_{2-3} = \frac{A_3}{A_2} F_{3-2} = \frac{\pi\left[1.5^2 - 0.5^2\right]}{2\pi\left(1.5^2\right)} \times 1 = 0.444$$

$A_2 F_{2-1} = A_1 F_{1-2}$

$$\therefore F_{2-1} = \frac{A_1}{A_2} F_{1-2} = \frac{\pi\left(0.5^2\right)}{2\pi\left(1.5^2\right)} \times 1 = 0.055$$

From eq (b), $F_{2-2} = 1 - \left(F_{2-1} + F_{2-3}\right)$

$= 1 - \left(0.055 + 0.444\right)$

$\therefore F_{2-2} = 0.5$

Surroundings (3) can be replaced by imaginary surfaces (5) and (4) as shown in Fig. Now, 1, 2, 4 and 5 form an enclosure

$F_{1-1} + F_{1-2} + F_{1-4} + F_{1-5} = 1$ and $F_{1-1} \pm F_{1-4} = 0$

$\therefore F_{1-2} + F_{1-5} = 1$ ________________________ (a)

$F_{2-1} + F_{2-2} + F_{2-4} + F_{2-5} = 1$ and $F_{2-5} = F_{2-1} = F_{2-4}$

$\therefore F_{2-2} + 2F_{2-5} = 1$ ________________________ (b)

To find $F_{5-(1+4)}$:

$$R = \frac{d}{2L} = \frac{\square 3}{2 \times 2} = 0.75$$

$$X = \frac{2R^2 + 1}{R^2} = \frac{2 \times 0.75^2 + 1}{0.75^2} = 3.78$$

$$\therefore F_{(1+4)-5} = F_{5-(1+4)} = \frac{X - \sqrt{X^2 - 4}}{2} = \frac{3.78 - \sqrt{3.78^2 - 4}}{2}$$

or $F_{5-(1+4)} = 0.286$

$\Rightarrow$ From Eqn (d) $F_{5-2} = 1 - 0.286 = 0.714$

$$\therefore F_{2-5} = \frac{A_5 F_{5-2}}{A_2} = \frac{\pi(1.5^2)}{2\pi \times 1.5 \times 2} \times 0.714 = 0.268$$

$$\therefore \text{From (b) } F_{2-2} = 1 - 2 \times 0.268 = 0.465$$

$$\text{Now } F_{5-4} = F_{5-(1+4)} - F_{5-1} = F_{5-(1+4)} - \frac{A_1 F_{1-5}}{A_5}$$

$$= 0.286 - \frac{\pi(0.5^2)}{\pi(1.5^2)} \times 0.35 = 0.247$$

$$\therefore F_{4-5} = \frac{A_5 F_{5-4}}{A_4} = \frac{\pi(1.5^2)}{\pi(1.5^2 - 0.5^2)} \times 0.247 = 0.278$$

$$\text{From (c) } F_{4-2} = 1 - 0.278 = 0.722$$

$$\therefore F_{2-4} = \frac{A_4 F_{4-2}}{A_2} = \frac{\pi(1.5^2 - 0.5^2)}{2\pi \times 1.5 \times 2} \times 0.722 = 0.24$$

$$F_{2-1} = F_{2-5} - F_{2-4} = 0.268 - 0.240$$

$$\therefore F_{2-1} = 0.028$$

$$F_{2-3} = F_{2-5} + F_{2-(1+4)} = 2F_{2-5} = 2 \times 0.268$$

$$\therefore F_{2-3} = 0.536$$

RADIATION HEAT EXCHANGE BETWEEN FINITE SURFACES

RADIATION HEAT EXCHANGE BETWEEN FINITE BLACK SURFACES

Consider two black surfaces of area A_1 and A_2 and at temperatures T_1 and T_2 as shown in Fig 10.14. Then radiation leaving A_1 and reaching A_2 can be written as

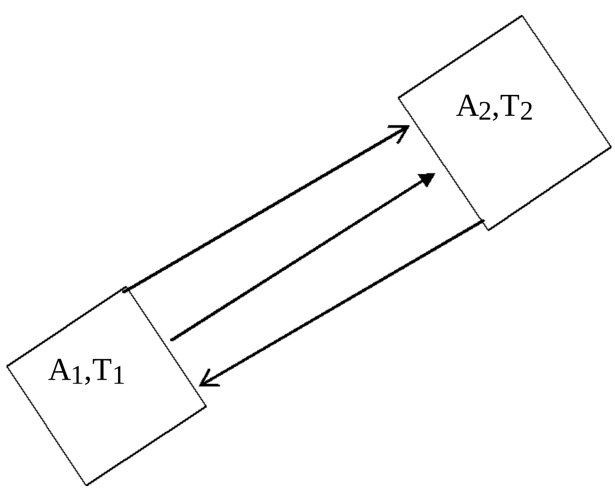


Fig 10.14: Radiation between two Finite Block Surfaces.

$Q_{1\to2} = A_1 J_1 F_{1-2}$ Since A_1 is a black surface, $J_1 = E_{b1}$ thus,

$Q_{1\to2} = A_1 E_{b1} F_{1-2}$

Similarly radiation leaving A_2 and reaching A_1 is given by

$Q_{2\to1} = A_2 E_{b2} F_{2-1}$

Net heat exchange between A_1 and A_2 can be written as

$$Q_{12} = Q_{1\to2} - Q_{2\to1}$$
$$= A_1 F_{1-2} E_{b1} - A_2 F_{2-1} E_{b2}$$

But, $A_1 F_{1-2} = A_2 F_{2-1}$

Thus, $Q_{1-2} = A_1F_{1-2} (E_{b1} - E_{b2})$

$$Q_{1-2} = \zeta A_1F_{1-2}(T_1^4 - T_2^4) \text{ ----------------- } \mathbf{(10.27)}$$

RADIATION HEAT EXCHANGE BETWEEN FINITE GREY SURFACES (NET WORK METHOD)

The calculation of radiation heat transfer between black surfaces is relatively easy because all the radiant energy which strikes a surface is absorbed by it. When non black bodies are involved, the situation is much more complex because all the energy striking a surface will not be absorbed: part will be reflected back to another heat transfer surface, and part may be reflected out of the system entirely. The problem can become complicated because the radiant energy can be reflected back and forth between heat transfer surfaces several times.

While deriving the expression for radiation exchange between any two finite grey surfaces the following assumptions are made

i. All the surfaces are diffuse and uniform in temperature.
ii. The reflection and emissive properties are constant over all the surface
iii. Radiosity and irradiation are uniform over each surface. This assumption is not strictly correct even for ideal grey diffuse surfaces, but the problems become exceedingly complex when this restriction is not imposed.
iv. The surfaces are opaque (i.e. Transmissivity is zero)

Now the net radiation from a surface is given by

$$Q_r = A\left(J - G\right)$$

$$\text{but } J = E + (1-\alpha)G$$

$$\text{assuming } \alpha = \varepsilon,\ J = \varepsilon E_b + (1-\varepsilon)G$$

$$\text{or } G = \frac{J - \varepsilon E_b}{1-\varepsilon}$$

$$\therefore Q_r = A\left[J - \frac{J - \varepsilon E_b}{1-\varepsilon}\right] = A\left[\frac{J(1-\varepsilon) - J + \varepsilon E_b}{1-\varepsilon}\right]$$

$$Q_r = \frac{\varepsilon A(E_b - J)}{1-\varepsilon} = \frac{E_b - J}{\frac{(1-\varepsilon)}{\varepsilon A}} = \frac{E_b - J}{R} \quad \text{-----------------------} (4.28)$$

Eq. 4.28 can be interpreted as follows… (E_b-J) can be thought of as thermal potential, R = (1-ε)/εA can be thought of as thermal resistance offered by the surface for radiation, as Q_r is the radiation heat flow rate. Therefore a radiating surface can be replaced by an element as shown in Fig 10.15

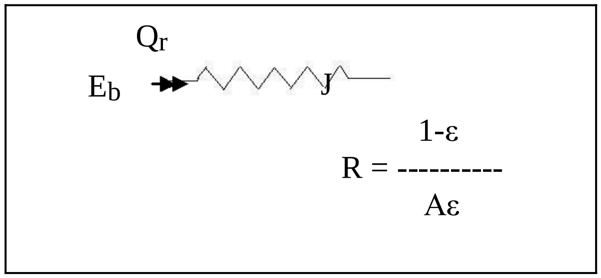


Fig 10.15: Element representing surface resistance in the Radiation network method

Now let us consider the radiation heat exchange between two surfaces A_1 and A_2. Radiation which leaves A_1 and strikes A_2 is given by

$$Q_{1\to 2} = A_1 J_1 F_{1-2}$$

Similarly Radiation which leaves A_2 and strikes A_1 is given by

$$Q_{2\to 1} = A_2 J_2 F_{2-1}$$

Therefore net radiation heat transfer from A_1 to A_2 is given by

$$Q_{12} = Q_{1\to 2} - Q_{2\to 1}$$

$$= A_1 J_1 F_{1-2} - A_2 J_2 F_{2-1}$$

$$\text{But } A_1 F_{1-2} = A_2 F_{2-1}$$

$$\therefore Q_{12} = A_1 F_{1-2}(J_1 - J_2)$$

$$\text{Or } Q_{12} = \frac{J_1 - J_2}{\left(\frac{1}{A_1 F_{1-2}}\right)} = \frac{\text{Thermal Potential}}{\text{Thermal Resistance}, R_{12}} \quad \text{-----------------} (4.29)$$

Eq. (4.29) can be represented by an element as shown in Fig 10.16

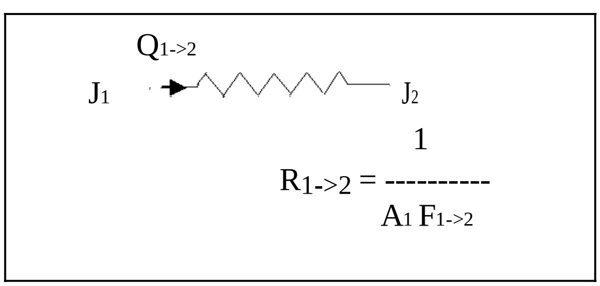


Fig 10.16: Element representing surface resistance in the Radiation network method

The two network elements shown in Fig 4.15 and 4.16 represent the essentials of radiation network method. To construct a network for a particular radiation heat transfer problem we need only to connect a "surface resistance" $(1-\varepsilon)/\varepsilon A$ to each surface and a "space resistance" $1/(A_iF_{i-j})$ between the radiosity potential potentials. This is illustrated below.

NETWORK METHOD FOR RADIATION HEAT EXCHANGE BETWEEN TWO PARALLEL INFINITE GREY SURFACES

The radiation network for the above problem will be as shown in Fig 10.17

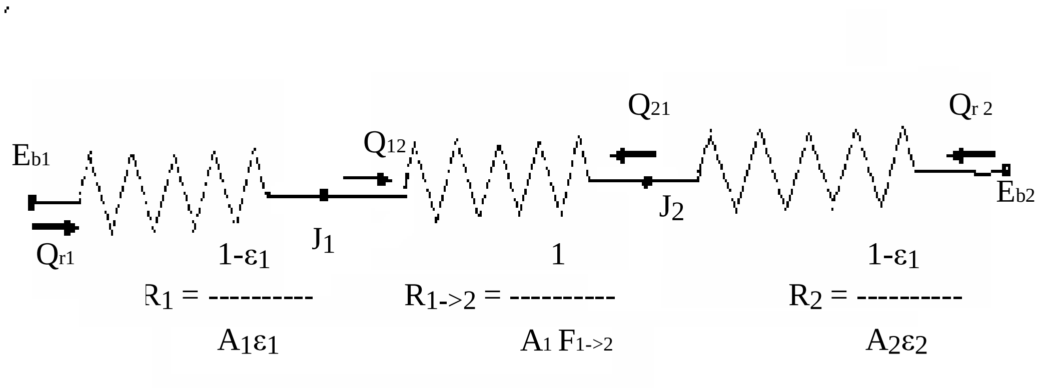


Fig 10.17 Radiation network for 2 parallel infinite grey surfaces

From the above Fig. we can write

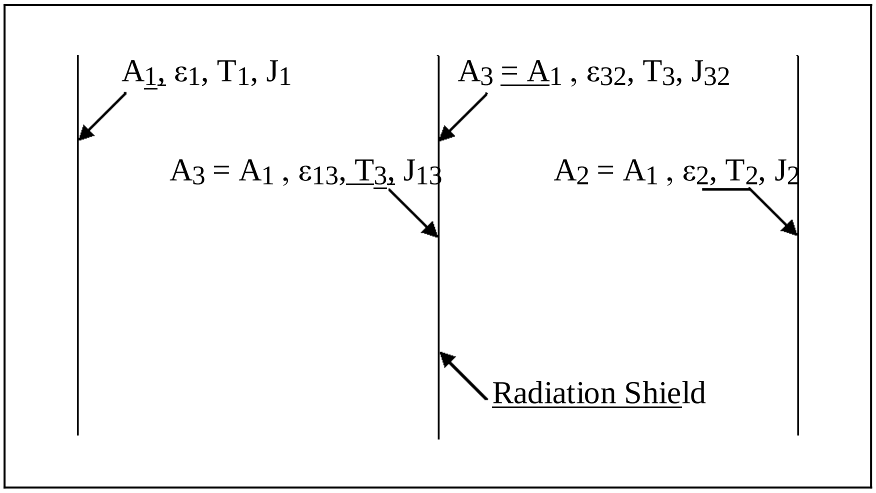
$$Q_{r_1} = Q_{12} = \frac{E_{b_1} - E_{b_2}}{R_1 + R_{12} + R_2} = \frac{\sigma\left(T_1^4 - T_2^4\right)}{\frac{(1-\varepsilon_1)}{A_1\varepsilon_1} + \frac{1}{A_1F_{1-2}} + \frac{(1-\varepsilon_2)}{A_2\varepsilon_2}}$$

For two parallel infinite grey surfaces, $A_2 = A_1$, $F_{1-2} = 1$

$$\therefore Q_{12} = \frac{\sigma A_1\left[T_1^4 - T_2^4\right]}{\frac{1-\varepsilon_1}{\varepsilon_1} + 1 + \frac{1-\varepsilon_2}{\varepsilon_2}} = \frac{\sigma A_1\left[T_1^4 - T_2^4\right]}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} + 1}$$

An expression which we have derived already [Eqn 4.15]

: NETWORK FOR RADIATION HEAT EXCHANGE BETWEEN TWO PARALLEL INFINITE GRAY SURFACES IN PRESENCE OF A RADIATION SHIELD



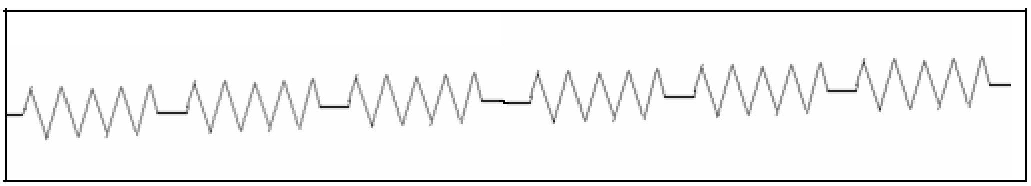


Fig 10.18: Radiation network for the parallel infinite gray surfaces in presence of a radiation shield

From Fig 10.18 the net radiation heat transfer from A_1 to A_2 is given by

$$Q_{12} = \frac{E_{b1} - E_{b2}}{R_1 + R_{13} + R_3 + R_{3'} + R_{32} + R_2}$$

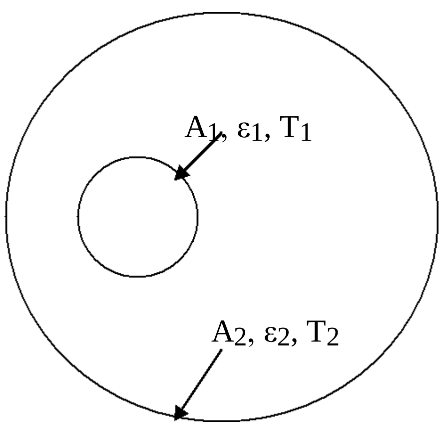
$$= \frac{\sigma\left[T_1^4 - T_2^4\right]}{\frac{1-\varepsilon_1}{A_1\varepsilon_1} + \frac{1}{A_1F_{13}} + \frac{1-\varepsilon_{13}}{A_3\varepsilon_{13}} + \frac{1-\varepsilon_{32}}{A_3\varepsilon_{32}} + \frac{1}{A_3F_{32}} + \frac{1-\varepsilon_2}{A_2\varepsilon_2}}$$

But $F_{13} = F_{32} = 1;\ A_1 = A_2 = A_3$

$$\therefore Q_{12} = \frac{\sigma A_1\left[T_1^4 - T_2^4\right]}{\frac{1-\varepsilon_1}{\varepsilon_1} + 1 + \frac{1-\varepsilon_{13}}{\varepsilon_{13}} + \frac{1-\varepsilon_{32}}{\varepsilon_{32}} + 1 + \frac{1-\varepsilon_2}{\varepsilon_2}}$$

$$Q_{12} = \frac{\sigma A_1\left[T_1^4 - T_2^4\right]}{\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1\right) + \left(\frac{1}{\varepsilon_{13}} + \frac{1}{\varepsilon_{32}} - 1\right)} \text{-----------------} (4.30)$$

<u>: RADIATION NETWORK FOR HEAT EXCHANGE BETWEEN TWO SURFACES, ONE SURFACE COMPLETELY ENCLOSING THE OTHER (The enclosed surface cannot see itself)</u>



$$F_{1-1} + F_{1-2} = 1 \text{ and } F_{1-1} = 0$$

$$\therefore F_{1-2} = 1$$

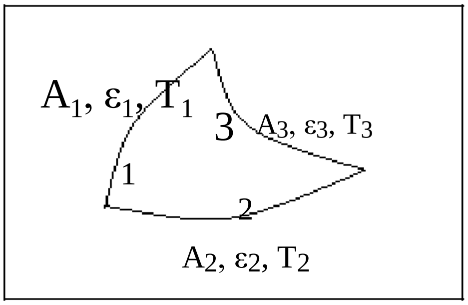
$$A_1F_{1-2} = A_2F_{2-1} \Rightarrow F_{2-1} = \frac{A_1}{A_2}$$

$$Q_{r1} = Q_{12} = \frac{E_{b1} - E_{b2}}{R_1 + R_{12} + R_2} = \frac{\sigma\left[T_1^4 - T_2^4\right]}{\frac{1-\varepsilon_1}{A_1\varepsilon_1} + \frac{1}{A_1F_{1-2}} + \frac{1-\varepsilon_2}{A_2\varepsilon_2}}$$

$$\therefore Q_{12} = \frac{\sigma A_1\left[T_1^4 - T_2^4\right]}{\frac{1}{\varepsilon_1} + \frac{A_1}{A_2}\left(\frac{1}{\varepsilon_2} - 1\right)} \quad (4.31)$$

: NETWORK METHOD FOR THREE ZONE ENCLOSURE

The network method described above can be readily generalised to enclosures involving three or more zones. However when there are more than three zones, the analysis becomes more involved and it is preferable to use the more direct "Radiosity Matrix" method. The radiation network for a three zone enclosure shown in Fig 4.20(a) is shown in Fig 4.20(b)



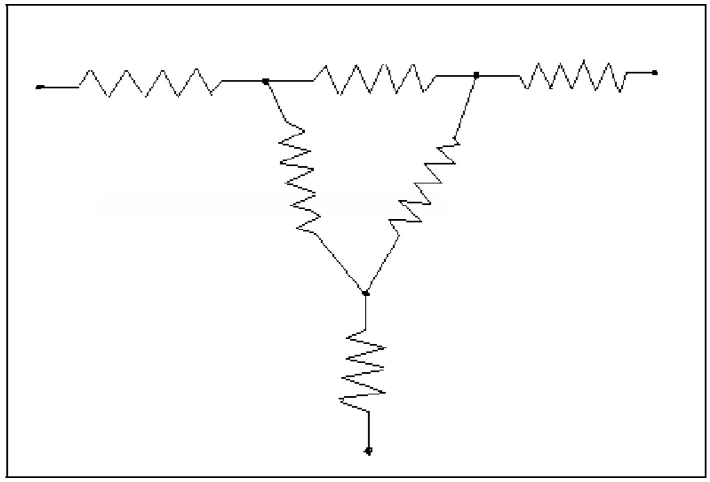
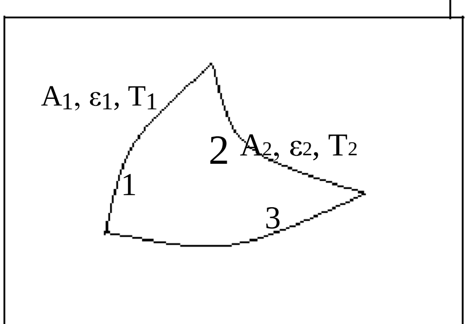


Fig 10.20: Radiation network for a three zone enclosure.

<u>Reradiating Surface:</u> In many practical situations one of the zones may be thermally insulated. In such a case, the net radiation heat flux in that particular zone is zero, because that surface emits as much energy as it receives by radiation from the surrounding zones. Such a zone is called a "RERADIATION ZONE" or an "ADIABATIC ZONE". Fig 10.21(a) represents a three zone enclosure with surface (3) being the reradiating surface and Fig 10.21(b) the corresponding radiation network.



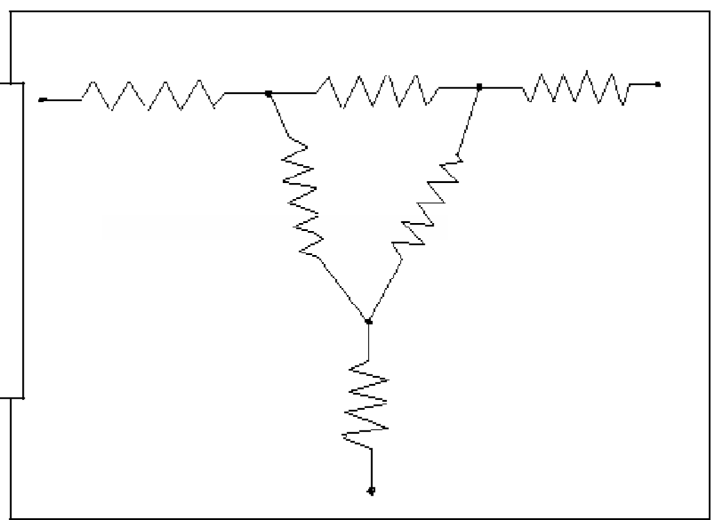


Fig 10.21: Radiation Heat Exchange in a 3 zone enclosure with one reradiating surface

For a three zone enclosure under steady state conditions, by I law of thermodynamics.

$$Q_{r1} + Q_{r2} + Q_{r3} = 0 \qquad (4.3$$

2) A_3 is reradiating, $Q_{r3} = 0$

$$\therefore Q_{r_1} = -Q_{r_2} = \frac{E_{b_1} - E_{b_2}}{R_{eq}} \qquad (4.33a)$$

$$\text{where } R_{eq} = R_1 + \left[\frac{1}{R_{12}} + \frac{1}{R_{13} + R_{23}}\right]^{-1} + R_2$$

$$\text{or } R_{eq} = \frac{1-\varepsilon_1}{A_1\varepsilon_1} + \left[A_1F_{1-2} + \frac{1}{\left(\frac{1}{A_1F_{13}} + \frac{1}{A_2F_{2-3}}\right)}\right]^{-1} + \frac{1-\varepsilon_2}{A_2\varepsilon_2} \qquad (4.33b)$$

ILLUSTRATIVE EXAMPLES ON NETWORK METHOD:

Example 4.24: Two square plates 1m x 1m are parallel to and directly opposite to each other at a distance of 1m. The hot plate is at 800K and has an emissivity of 0.8. The clod plate is at 600K and also has an emissivity of 0.8. The radiation heat exchange takes place between the plates as well as the ambient at 300K through the opening between the plates. Calculate the net radiation at each plate and the ambient.

Solution:

Data:-

T_1 = 800K, ϵ_1 = 0.8

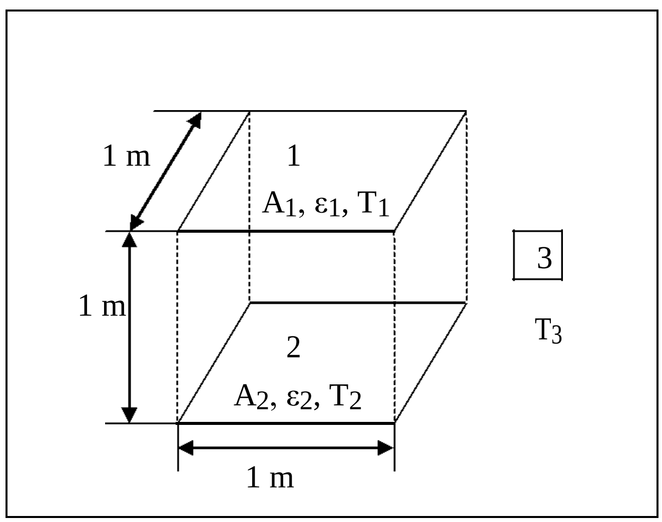
T_2 = 600K, ϵ_2 = 0.8

T_3 = 300K

To find:-

i) Q_{r1}, Q_{r2},

ii) Q_{r3}



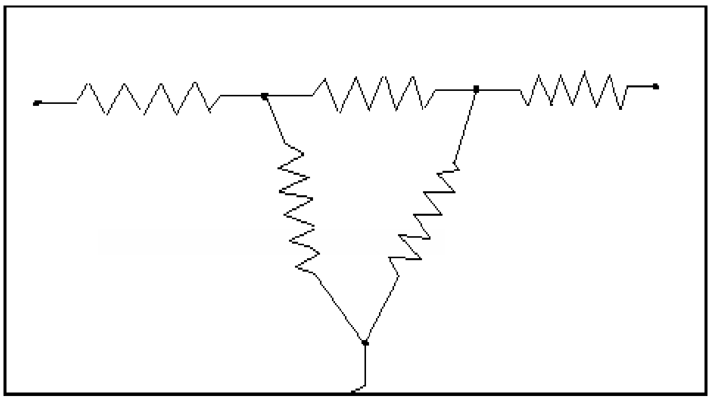
$$R_1 = \frac{1-\varepsilon_1}{A_1\varepsilon_1} = \frac{1-0.8}{1\times1\times0.8} = 0.25$$

$$R_2 = \frac{1-\varepsilon_2}{A_2\varepsilon_2} = \frac{1-0.8}{1\times1\times0.8} = 0.25$$

$$R_3 = \frac{1-\varepsilon_3}{A_3\varepsilon_3} : A_3 \text{ is the area of the surroundings } \therefore A_3 \; 1$$

$$\therefore R_3 = 0$$

$\therefore$ The radiation network for this problem will be as shown below



From chart $F_{1-2} = F_{2-1} = 0.20$

But $F_{1-1} + F_{1-2} + F_{1-3} = 1$ and $F_{1-1} = 0$

$\therefore F_{1-3} = 1 - F_{1-2} = 1 - 0.2 = 0.8 = F_{2-3}$

$$R_{12} = \frac{1}{A_1F_{1-2}} = \frac{1}{(1\times1)\,0.2} = 5$$

$$R_{13} = \frac{1}{A_1F_{1-3}} = \frac{1}{(1\times1)\,0.8} = 1.25$$

$$R_{23} = \frac{1}{A_1F_{2-3}} = \frac{1}{(1\times1)\,0.8} = 1.25$$

$$E_{b_1} = \sigma T_1^4 = 5.67\times10^{-8}\times800^4 = 23224\,W/m^2 = 23.224\,KW/m^2$$

$$E_{b_2} = \sigma T_2^4 = 5.67\times10^{-8}\times600^4 = 7348\,W/m^2 = 7.348\,KW/m^2$$

$$E_{b_3} = \sigma T_3^4 = 5.67\times10^{-8}\times300^4 = 459\,W/m^2 = 0.459\,KW/m^2$$

For steady state radiation, radiation energy cannot accumulate at nodes J_1, J_2 and J_3

$$\therefore Q_{r1} = Q_{12} + Q_{13}$$

$$\text{or } Q_{r1} - Q_{12} - Q_{13} = 0$$

$$\therefore \frac{E_{b_1} - J_1}{R_1} - \frac{(J_1 - J_2)}{R_{12}} - \frac{J_1 - J_3}{R_{13}} = 0$$

$$\text{or } \frac{23.224 - J_1}{0.25} - \frac{(J_1 - J_2)}{5} - \frac{(J_1 - 0.459)}{1.25} = 0 \_\_\_\_\_\_\_\_\_\_ (a)$$

$$\text{similarly } Q_{r_2} = Q_{21} + Q_{23} \Rightarrow Q_{r_2} - Q_{21} - Q_{23} = 0$$

$$\frac{E_{b_2} - J_2}{R_2} - \frac{(J_2 - J_1)}{R_{12}} - \frac{J_2 - E_{b_3}}{R_{23}} = 0$$

$$\frac{7.348 - J_2}{0.25} - \frac{(J_2 - J_1)}{5} - \frac{(J_2 - 0.459)}{1.25} = 0 \_\_\_\_\_\_\_\_\_\_ (b)$$

Solving Eqn (a) and (b) simultaneously we get

$$J_1 = 18.921\, KW/m^2;\ J_2 = 6.709\, KW/m^2$$

$$\therefore Q_{r_1} = \frac{E_{b_1} - J_1}{R_1} = \frac{23.224 - 18.921}{0.25} = 17.212\, KW$$

$$Q_{r_2} = \frac{E_{b_2} - J_2}{R_2} = \frac{7.348 - 6.709}{0.25} = 2.557\, KW$$

$$\text{But } Q_{r1} + Q_{r2} + Q_{r3} = 0 \Rightarrow Q_{r_3} = -\left[Q_{r_1} + Q_{r_2}\right] = -[17.212 + 2.557]$$

$$= -19.769\, KW$$

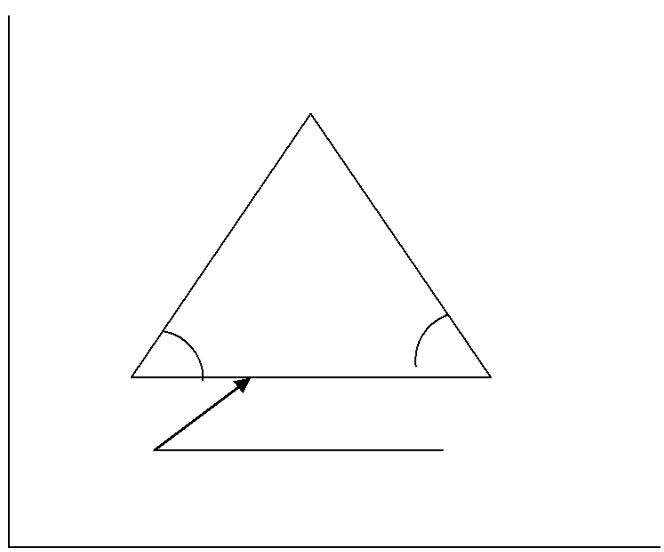
Example 4.25 The configuration of a furnace can be approximated as an equilateral triangular duct which is sufficiently long that the end efforts are negligible. The hot wall is at 900K with an emissivity of 0.8 and the cold wall is at 400K with emissivity of 0.8. The third wall is a reradiating wall. Determine the net radiation flux leaving the hot wall.

Solution:

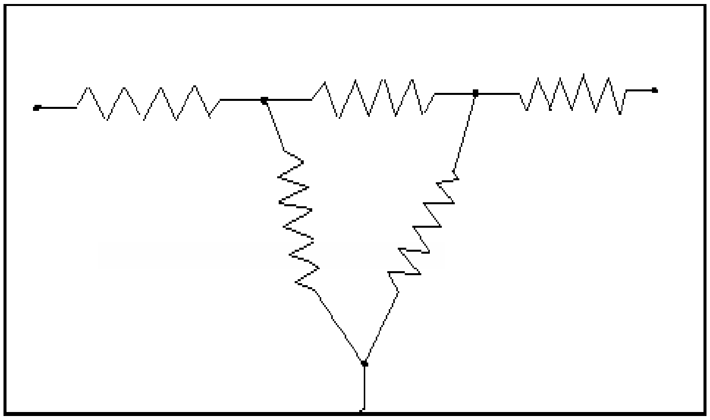
$A_1 = A_2 = A_3 = 1m^2$ (assumed)

$T_1 = 900K$, $\epsilon_1 = 0.8$

$T_2 = 400K$, $\epsilon_2 = 0.8$



The radiation network for the above problem will be as shown below



$$R_1 = \frac{1-\varepsilon_1}{A_1\varepsilon_1} = \frac{1-0.8}{1\times 0.8} = 0.25$$

$$R_2 = \frac{1-\varepsilon_2}{A_2\varepsilon_2} = \frac{1-0.8}{1\times 0.8} = 0.25$$

Using Hottel's crosss tring formula, we have

$$F_{1-2} = \frac{(A_1 + A_2) - A_3}{2A_1} = \frac{(1+1)-1}{2\times 1} = 0.5 = F_{1-3} = F_{2-3}$$

$$\therefore R_{12} = \frac{1}{A_1 F_{1-2}} = \frac{1}{1\times 0.5} = 2 = R_{23} = R_{13}$$

$$R_{eq} = R_1 + \left[\frac{1}{R_{12}} + \frac{1}{R_{13} + R_{23}}\right]^{-1} + R_2$$

$$= 0.25 + \left[\frac{1}{2} + \frac{1}{2+2}\right]^{-1} + 0.25 = 1.833$$

$$Q_{r_1} = \frac{E_{b_1} - E_{b_2}}{R_{eq}} = \frac{\sigma\left(T_1^4 - T_2^4\right)}{R_{eq}} = \frac{5.67\times 10^{-8}\times\left[900^4 - 400^4\right]}{1.833}$$

$$= 19503\ W/m^2$$

$$Q_{r_1} + Q_{r_2} + Q_{r_3} = 0 \text{ and } Q_{r_3} = 0 \Rightarrow Q_{r_2} = -Q_{r_1} = -19503\ W/m^2$$

Example 4.26 A short cylindrical enclosure is maintained at the temperatures as shown in Fig P4.26. Assuming $\epsilon_2 = \epsilon_3 = 1$; $\epsilon_1 = 0.8$ determine Q_{r1} and Q_{r2}

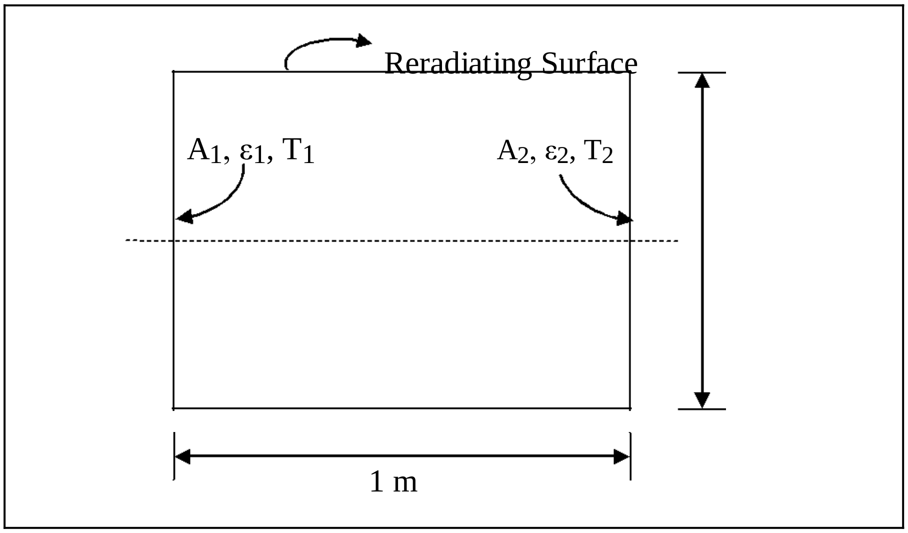
Solution:

From chart, $F_{1-2} = 0.175 = F_{2-1}$ $(A_2 = A_1)$

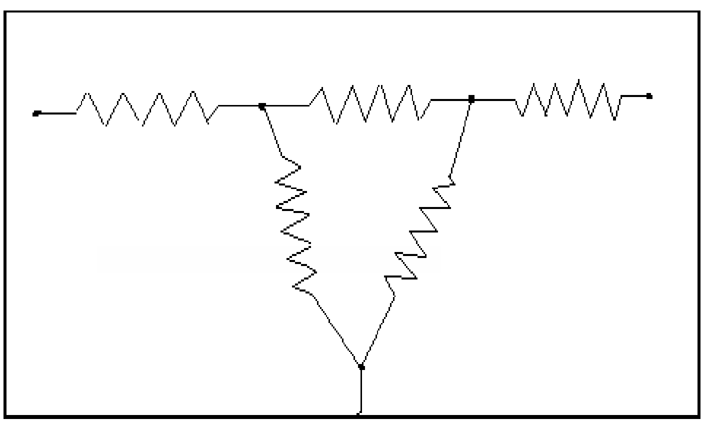
Also, $F_{1-1} + F_{1-2} + F_{1-3} = 1$ and $F_{1-1} = 0$

So, $F_{1-3} = 1 - F_{1-2} = 1 - 0.175$ or

$F_{1-3} = 0.825 = F_{2-3}$



The radiation network for the above problem will be as shown below



$$R_1 = \frac{1-\varepsilon_1}{A_1\varepsilon_1} = \frac{1-0.8}{\pi\left(0.5^2\right)0.8} = 0.318$$

$$R_2 = \frac{1-\varepsilon_2}{A_2\varepsilon_2} = \frac{1-1}{A_2\times 1} = 0$$

$$R_{12} = \frac{1}{A_1F_{1-2}} = \frac{1}{\pi\left(0.5^2\right)0.175} = 7.3$$

$$R_{13} = \frac{1}{A_1F_{1-2}} = \frac{1}{\pi\left(0.5^2\right)0.825} = 1.54$$

$$R_{23} = \frac{1}{A_2F_{2-3}} = \frac{1}{\pi\left(0.5^2\right)0.825} = 1.54$$

$$R_{eq} = R_1 + \left[\frac{1}{R_{12}} + \frac{1}{R_{13}+R_{23}}\right]^{-1} + R_2$$

$$= 0.318 + \left[\frac{1}{7.3} + \frac{1}{1.54+1.54}\right]^{-1} + 0 = 2.484$$

$$\therefore Q_{r_1} = \frac{E_{b1}-E_{b2}}{R_{eq}} = \frac{\sigma\left(T_1^4 - T_2^4\right)}{R_{eq}} = \frac{5.67\times10^{-8}\times\left[2000^4 - 1000^4\right]}{2.484}$$

$$= 329.14\times10^3\, W = 329.14\, KW$$

$$Q_{r1} + Q_{r2} + Q_{r3} = 0 \text{ and } Q_{r3} = 0$$

$$\therefore Q_{r2} = -Q_{r1} = -329.14\, KW$$

Example 4.27 A spherical tank with diameter 40cm fixed with a cryogenic fluid at 100K is placed inside a spherical container of diameter 60cm and is maintained at 300K. The emissivities of the inner and outer tanks are 0.15 and 0.2 respectively. A spherical radiation shield of diameter 50cm and having an emissivity of 0.05 on both sides is placed between the spheres. Calculate the rate of heat loss from the system by radiation and find also the rate of evaporation of the cryogenic liquid if the latent heat of vaporization of the fluid is $2.1\text{x}10^5$ W-s/Kg

Solution: The schematic and the corresponding network for the problem will be as shown in Fig P.10.27

$T_1 = 100$ K
$D_1 = 40$ cm
$\varepsilon_1 = 0.15$

$T_2 = 300$ K
$D_2 = 60$cm
$\varepsilon_2 = 0.2$

$D_3 = 50$cm
$\varepsilon_3 = 0.05$

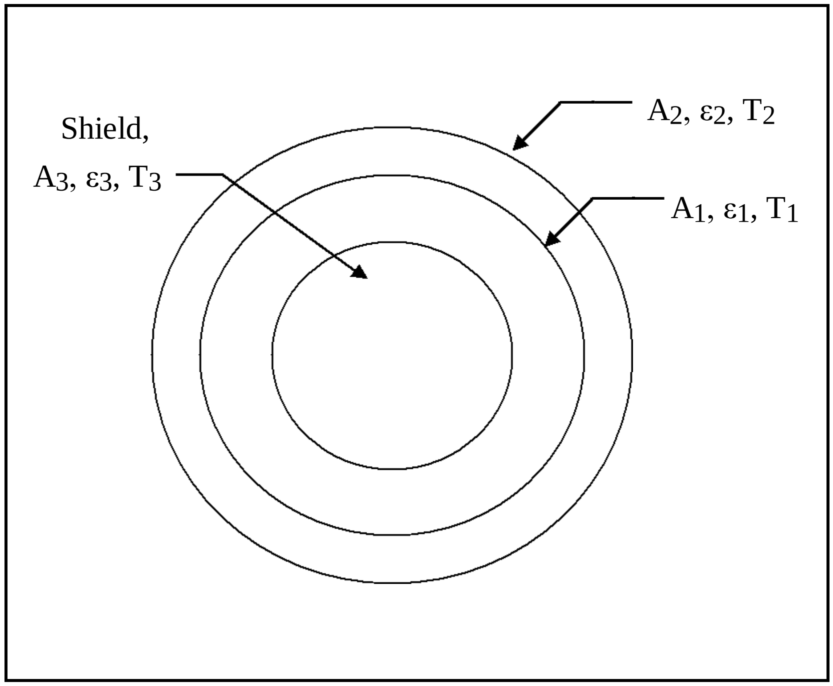


Fig P 10.27

$$Q_{r1} = Q_{12} = \frac{E_{b1} - E_{b2}}{R_1 + R_{13} + 2R_3 + R_{32} + R_2}$$

$$= \frac{\sigma\left(T_1^4 - T_2^4\right)}{\frac{1-\varepsilon_1}{A_1\varepsilon_1} + \frac{1}{A_1F_{1-3}} + 2\frac{1-\varepsilon_3}{A_3\varepsilon_3} + \frac{1}{A_3F_{3-2}} + \frac{1-\varepsilon_2}{A_2\varepsilon_2}}$$

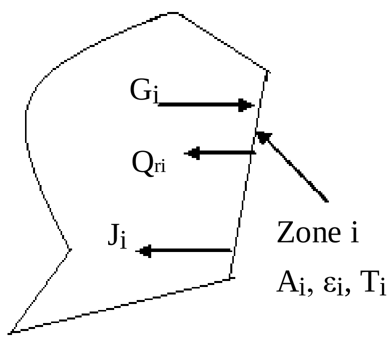
$$= \frac{\sigma A_1\left(T_1^4 - T_2^4\right)}{\frac{1}{\varepsilon_1} + \frac{A_1}{A_3}\left(\frac{2}{\varepsilon_3} - 1\right) + \frac{A_1}{A_2}\left(\frac{1}{\varepsilon_2} - 1\right)}$$

$$= \frac{5.67\times10^{-8}\times4\pi\times0.2^2\times\left[100^4 - 300^4\right]}{\frac{1}{0.15} + \left(\frac{40}{50}\right)^2\left(\frac{2}{0.05} - 1\right) + \left(\frac{40}{60}\right)^2\left[\frac{1}{0.2} - 1\right]} = -6.83\,W$$

$$\text{Evaporation Ratio} = \frac{Q_{r1}}{h_{fg}} = \frac{6.83}{2.1\times10^5} = 3.25\times10^{-5}\,Kg/s$$

RADIOSITY- MATRIX METHOD FOR RADIATION IN ENCLOSURES

The network method is quite easy to apply for determining the radiation exchange in enclosures having not more than 3 zones. As the number of zones forming the enclosure increases, the manipulations involved in the network method becomes enormous and the method is not so practical. Whereas the radiosity-matrix method is very straight forward and the method transforms the problem to the solution of an algebraic matrix equation for the unknown radiosities J_i [i=1,2,3,…….N] once these equations are solved for J_i then the net radiation flux or the zone temperature at any zone (i) can be immediately determined. This is illustrated below.



Consider an enclosure made up of N zones.
Let G_i be the irradiation at zone i
Let J_i be the Radiosity at zone i

Net radiation heat flux from zone i is given by

$$q_i = \frac{Q_{ri}}{A_i} = J_i - G_i \qquad (4.34)$$

Radiation leaving A_j and striking A_i is given by

$$Q_{j-i} = A_j F_{j-i} J_j = A_i F_{i-j} J_j$$

Radiation leaving all zones and striking A_i therefore will be

$$\sum_{j=1}^{N} Q_{j-i} = A_i \sum_{j=1}^{N} F_{i-j} J_j$$

$$\therefore G_i = \frac{\sum_{j=1}^{N} Q_{j-i}}{A_i} = \sum_{j=1}^{N} F_{i-j} J_j \qquad (4.35)$$

$$q_i = J_i - \sum_{j=1}^{N} F_{i-j} J_j \qquad (i = 1,2,3 \ldots N) \qquad (4.36)$$

$$\text{Also } q_i = \frac{E_{bi} - J_i}{\left(\frac{1-\varepsilon_i}{\varepsilon_i}\right)} = J_i - G_i \qquad (4.36a)$$

$$\therefore E_{bi} = \left(\frac{1-\varepsilon_i}{\varepsilon_i}\right) J_i + J_i - \left(\frac{1-\varepsilon_i}{\varepsilon_i}\right) G_i$$

$$\text{Or } E_{bi} = \frac{J_i - (1-\varepsilon_i)\sum_{j=1}^{N} F_{i-j} J_j}{\varepsilon_i} \qquad (i = 1,2,3 \ldots N) \qquad (4.37)$$

Equations (4.36) and (4.37) provide the fundamental relations for obtaining a system of N algebraic equations to determine the N unknown radiosities. Once the radiosities are known the net radiation heat flux q_i at any zone (i) can be computed using either Eq. 4.36 or 4.36a

The solution depends on the prescribed conditions for each of the zones. Two situations are of practical interest

1. Temperatures are prescribed for each of the N zones
2. Temperatures are prescribed for some of the zones and the net radiation flux are prescribed for the remaining zones

i. *Temperatures prescribed for all the zones*

Consider Eq. 4.37

$$E_{bi} = \frac{J_i - (1-\varepsilon_i)\sum_{j=1}^{N} F_{i-j} J_j}{\varepsilon_i} \qquad (i = 1,2,3 \ldots N)$$

And $E_{bi} = \zeta T_i^4$ is known because T "s are prescribed. Therefore the above set of equations can be solved for unknown radiosities $J_i (i = 1, 2, 3 \ldots N)$ and knowing J_i the net radiation flux q_i can be determined from Eq. 4.36a

Eq 4.37 can be rewritten as

$$\frac{J_i}{\varepsilon_i} - \left(\frac{1-\varepsilon_i}{\varepsilon_i}\right)\sum_{j=1}^{N} F_{i-j} J_j = \sigma T_i^4 \text{ ———————— } (4.38)$$

Equation (4.38) is of the form

$$[M]\{J\} = \{T\} \quad (4.39)$$

$$\text{Where } [M] = \begin{pmatrix} m_{11} & & m_{1N} \\ & & \\ m_{N1} & & m_{NN} \end{pmatrix} \quad (4.39a)$$

$$\{J\} = \begin{Bmatrix} J_1 \\ J_2 \\ \vdots \\ J_N \end{Bmatrix} \quad (4.39c)$$

$$\{T\} = \begin{Bmatrix} \sigma T_1^4 \\ \sigma T_2^4 \\ \sigma T_3^4 \\ \vdots \\ \sigma T_N^4 \end{Bmatrix} \quad (4.39d)$$

The elements of m_{ij} of matrix [M] can be determined from

$$m_{ij} = \frac{\delta_{ij} - (1-\varepsilon_i) F_{ij}}{\varepsilon_i} \text{ ______ } (4.39c)$$

where $\delta_{ij} = 1$ for $i = j$

$= 0$ for $i \neq j$

δ_{ij} is known as KRONECKER delta

ii. Temperature prescribed for some zones and net heat flux prescribed for others

In many practical situations, temperatures are prescribed for some of the zones and net heat fluxes for the remaining zones of an enclosure. In such problems we have to determine the net heat fluxes for the zones for which temperatures are specified and temperatures for the zones for which the net heat fluxes are prescribed. This can be done using the same equations (4.36) and (4.37) and illustrated below.

Let us assume that temperatures T_i are prescribed for zones i=1, 2, 3 ….k and the net heat fluxes q_i are prescribed for the remaining zones i=k+1, k+2….n

For zones 1 to k since temperatures are prescribed we can use eq. 4.38 that is

$$\frac{J_i}{\varepsilon_i} - \left(\frac{1-\varepsilon_i}{\varepsilon_i}\right) \sum_{j=1}^{N} F_{i-j} J_j = \sigma T_i^4 \quad \text{\_\_\_\_\_\_\_\_} (4.39)$$

For zones i = k+1, k+2, … N with prescribed heat fluxes we can use Eq (4.36) namely

$$q_i = J_i - \sum_{j=1}^{N} F_{i-j} J_j \quad (i = K+1, K+2, \ldots N) \quad (4.40)$$

It is more convenient to express Eq. 4.39 and 4.40 in matrix form as

$$[M]\{J\} = \{S\} \quad (4.41a)$$

$$\text{Where } [M] = \begin{pmatrix} m_{11} & \cdots & m_{1N} \\ \vdots & & \vdots \\ m_{N1} & \cdots & m_{NN} \end{pmatrix} \quad (4.41b)$$

$$m_{ij} = \frac{\delta_{ij} - (1-\varepsilon_i) F_{i-j}}{\varepsilon_i} \quad \text{for } i = 1, 2, 3 \ldots K \text{\_\_\_\_\_\_\_\_\_\_} (4.41c)$$

$$m_{ij} = \delta_{ij} - F_{i-j} \quad \text{for } i = k+1, k+2, \ldots N \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (4.41d)$$

$$\text{where } \delta_{ij} = 1 \text{ for } i = j$$

$$= 0 \text{ for } i \neq j \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (4.41e)$$

$$\{J\} = \begin{Bmatrix} J_1 \\ J_2 \\ \vdots \\ J_N \end{Bmatrix} \quad (4.41f)$$

$$[S] = \begin{Bmatrix} \sigma T_1^4 \\ \sigma T_2^4 \\ \sigma T_3^4 \\ \vdots \\ \sigma T_K^4 \\ q_{k+1} \\ q_{k+2} \\ \vdots \\ q_N \end{Bmatrix} \quad (4.39d)$$

Ones these equations are solved for unknown radiosities J_i, then the unknown radiation fluxes can be determined from the equation

$$q_i = \frac{\varepsilon_i}{1-\varepsilon_i}\left[E_{bi} - J_i\right],\ i = 1, 2, 3K \text{\_\_\_\_\_\_\_\_\_} (4.41h)$$

The unknown temperatures can be determined using

$$\sigma T_i^4 = J_i + \frac{1-\varepsilon_i}{\varepsilon_i} q_i \quad i = 1, 2, 3K \text{\_\_\_\_\_\_\_\_\_} (4.41i)$$

Noting that for a reradiating surface $q_i = 0$

$$\sigma T_i^4 = J_i \text{ for a reradiating surface\_\_\_\_\_\_\_\_\_\_\_} (4.41j)$$

4.10: ILLUSTRATIVE EXAMPLES ON RADIOSITY MATRIX METHOD

Example 4.28:- Solve Example 4.24 using radiosity matrix method.

Solution:

Data:- $T_1 = 800K$, $\epsilon_1 = 0.8$
$T_2 = 600K$, $\epsilon_2 = 0.8$
$T_3 = 300K$, $J_3 = E_{b3} = \zeta T_3^4$
$F_{1-2} = F_{2-1} = 0.2$
$F_{1-1} = F_{2-2} = 0$
$F_{1-3} = F_{2-3} = 0.8$

Since J_3 is already known, we have to solve only for J_1 and J_2. Thus the matrix form of equation for radiosities J_1 and J_2 can be written as

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} J_1 \\ J_2 \end{Bmatrix} = \begin{Bmatrix} \sigma T_1^4 \\ \sigma T_2^4 \end{Bmatrix}$$

$$m_{ij} = \frac{\delta_{ij} - (1-\varepsilon_i) F_{ij}}{\varepsilon_i}$$

where $\delta_{ij} = 1$ for $i = j$

$= 0$ for $i \neq j$

$$\therefore m_{11} = \frac{1-(1-0.8)\times 0}{0.8} = 1.25;\ \sigma T_1^4 = 5.67\times 10^{-8}\times 800^4 = 23224\, W/m^2$$

$$m_{12} = \frac{0-(1-0.8)\,0.2}{0.8} = -0.05;\ \sigma T_2^4 = 5.67\times 10^{-8}\times 600^4 = 7348\, W/m^2$$

$$m_{21} = \frac{0-(1-0.8)\,0.2}{0.8} = -0.05$$

$$m_{22} = \frac{1-(1-0.8)\times 0}{0.8} = 1.25$$

$$\therefore \begin{bmatrix} 1.25 & -0.05 \\ -0.05 & 1.25 \end{bmatrix} \begin{Bmatrix} J_1 \\ J_2 \end{Bmatrix} = \begin{Bmatrix} 23224 \\ 7348 \end{Bmatrix}$$

i.e,

$$1.25 J_1 - 0.05 J_2 = 23224$$

$$-0.05 J_1 + 1.25 J_2 = 7348$$

Solving for J_1 and J_2 we get $J_2 = 6632\, W/m^2$ and $J_1 = 18845\, W/m^2$

$$\therefore q_{r1} = \frac{E_{b1} - J_1}{\left(\frac{1-\varepsilon_1}{\varepsilon_1}\right)} = \frac{23224 - 18845}{\left(\frac{1-0.8}{0.8}\right)} = 17516\, W/m^2$$

$$q_{r2} = \frac{E_{b2} - J_2}{\left(\frac{1-\varepsilon_2}{\varepsilon_2}\right)} = \frac{7348 - 6632}{\left(\frac{1-0.8}{0.8}\right)} = 2864\, W/m^2$$

$$q_{r1} + q_{r2} + q_{r3} = 0 \Rightarrow q_{r3} = -[17516 + 2864] = -20380\, W/m^2$$

Example 4.29:- Solve example 4.25 by radiosity-matrix method.

Solution: Given $A_1 = A_2 = A_3 = 1\ m^2$; $T_1 = 900$ K ; $T_2 = 400$ K ; $Q_{r3} = 0$; $\varepsilon_1 = 0.8$; $\varepsilon_2 = 0.8$;

This is a case wherein temperature is specified for 2 zones and radiation flux is specified for the remaining zone.

$$m_{ij} = \frac{\delta_{ij} - (1-\varepsilon_i) F_{ij}}{\varepsilon_i} \quad \text{for } i = 1,2 \text{ and } m_{ij} = \delta_{ij} - F_{ij} \text{ for } i = 3$$

Therefore $m_{11} = (1-0)/0.8 = 1.25$; $m_{12} = \dfrac{0 - (1-0.8) \times 0.5}{0.8} = -0.125 = m_{21}$

$$m_{22} = \frac{(1-0)}{0.8} = 1.25 \text{ ; } m_{13} = \frac{0 - (1-0.8) \times 0.5}{0.8} = -0.125 = m_{23}$$

$m_{31} = 0 - F_{31} = -0.5$; $m_{32} = 0 - F_{32} = -0.5$; $m_{33} = 1 - F_{33} = 1.0$

The radiosity matrix equation can now be written as follows:

$[m_{ij}]\{J_i\} = \{S\}$ where $\{S\} = \begin{Bmatrix} \zeta T_1^4 \\ \zeta T_2^4 \\ 0 \end{Bmatrix}$

In expanded form the above equation can be written as:

$m_{11} J_1 + m_{12} J_2 + m_{13} J_3 = \zeta T_1^4$

$m_{21} J_1 + m_{22} J_2 + m_{23} J_3 = \zeta T_2^4$

$m_{31} J_1 + m_{32} J_2 + m_{33} J_3 = 0$

Substituting the numerical values fo m_{ij} , T_1, T_2 and ζ the above three equations can be solved for J_1, J_2 and J_3.

UNIT-V
HEAT EXCHANGERS

A. Overall heat transfer coefficient:

Water at 25 ^{0} C and a velocity of 1.5 m/s enters a brass condenser tube 6 m long, 1.34 cm ID, 1.58 cm OD and k = 110 W/(m-K). Steam is condensing on the outer surface of the tube with a heat transfer coefficient of 12,000 W/(m^2 – K). The fouling factors for the inner and outer surfaces are both equal to 0.00018 (m^2 – K) / W. Calculate the overall heat transfer coefficient based on (i) the inside surface area and (ii) the outside surface area.

A stainless steel tube [k = 45 W/(m-K)] of inner and outer diameters of 22 mm and 27 mm respectively, is used in a cross flow heat exchanger (see Fig. P 9.2). The fouling factors for the inner and outer surfaces are estimated to be 0.0004 and 0.0002 (m^2–K) /W respectively

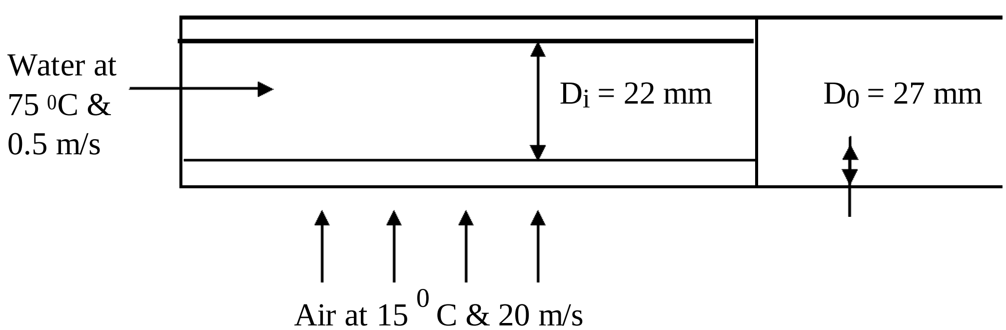


Fig. P 9.2: Schematic for problem 9.2

Determine the overall heat transfer coefficient based on the outside surface area of the tube. Compare the thermal resistances due to convection, tube-wall conduction and fouling and make comments.

B. Mean Temperature Difference Method:

In a heat exchanger hot fluid enters at 60 ^{0} C and leaves at 48 ^{0} C, where as the cold fluid enters at 35 ^{0} C and leaves at 44 ^{0} C. Calculate the mean temperature difference for (a) parallel flow, (b) counter flow, (c) single pass cross flow (both fluids unmixed),

(d) single pass cross flow (hot side fluid mixed, cold side fluid unmixed), and (e) single pass cross flow (hot side fluid unmixed, cold side fluid mixed).

A simple heat exchanger consisting of two concentric flow passages is used for heating 1110 kg/h of oil (C_p = 2.1 kJ/kg-K) from a temperature of 27 ^{0}C to 49 ^{0} C.The oil flows through the inner pipe made of copper (k = 350 W/m-K) with2.54 cm ID and 2.86 cm OD. The surface heat transfer coefficient on the oil side is 635 W/m^2 –K. The oil is heated by water supplied at a rate of 390 kg/h and at an inlet temperature of 93 ^{0} C. The water side heat transfer coefficient is 1270 W/m^2-K. the fouling factors o the oil side and water side are 0.0001 and 0.0004 m^2-K/W respectively. What is the length of the heat exchanger required for (i) parallel flow, and (ii) counter flow arrangements?

A one-shell pass, two-tube pass exchanger is to be designed to heat 0.5 kg/s of water entering the shell side at 10 ^{0} C. The hot fluid oil enters the tube at 80 ^{0} C with a mass flow rate of 0.3 kg/s and leaves the exchanger at 30 ^{0} C. The overall heat transfer coefficient is 250 W/m^2-K. Assuming the specific heat of oil to be 2 kJ/kg-K, calculate the surface area of the heat exchanger required.

A single pass cross flow heat exchanger uses hot gases (mixed) to heat water (unmixed) from 30 ^{0} C to 80 ^{0} C at a rate of 3 kg/s.The exhaust gases, having thermo-physical properties similar to air enter and leave the exchanger at 225 and 100 ^{0} C respectively. If the overall heat transfer coefficient is 200 W/m ^{2}-K, determine the required surface area of the exchanger.

A two-shell pass, four-tube pass heat exchanger is used to heat water with oil. Water enters the tubes at a flow rate of 2 kg/s and at 20 ^{0} C and leaves at 80 ^{0} C. Oil enters the shell side at 140 ^{0} C and leaves at 90 ^{0} C. If the overall heat transfer coefficient is 300 W/m^2-K, calculate the heat transfer area required.

A shell and tube heat exchanger is to be designed for heating water from 25 ^{0} C to 50 ^{0} C with the help of steam condensing at atmospheric pressure. The water flows through the tubes (2.5 cm ID, 2.9 cm OD and 2 m long) and the steam condenses on the outside of the tubes. Calculate the number of tubes required if the water flow rate is 500 kg/min and the individual heat transfer coefficients on the steam and water side are 8000 and 3000 W/ m^2-K respectively. Neglect all other resistances.

C. Effectiveness – NTU method:

Show that for counter flow heat exchanger with capacity ratio C = 1, the effectiveness is given by

$$\varepsilon = NTU / (1 + NTU)$$

The following data refer to a heat exchanger. Mass flow rate of the hot fluid = 4 kg/min. Mass flow rate of the cold fluid = 8 kg/min.

Specific heat of hot fluid = 4.20 kJ/kg-K.
Specific heat of the cold fluid = 2.52 kJ/kg-K.
Inlet temperature of hot fluid = 100 ^{0} C.
Inlet temperature of cold fluid = 20 ^{0} C.

What is the maximum possible effectiveness if the arrangement is (i) parallel flow and (ii) counter flow?

Calculate the exit temperature of the hot fluid and inlet temperature of the cold fluid for a counter flow heat exchanger having the following specifications.
Mass flow rate of hot fluid = 3 kg/s. Mass flow rate of cold fluid = 0.75 kg/s.
Cp for hot fluid = 1.05 kJ/kg-K.
Cp for cold fluid = 4.2 kJ/kg-K.

In a gas turbine power plant, heat is transferred in an exchanger from the hot gases leaving the turbine to the air leaving the compressor. The air flow rate is 5000 kg/h and the fuel-air ratio is 0.015 kg/kg. The inlet temperatures on the air side and the gas side are 170 ^{0} C and 450 ^{0} C respectively. The overall heat transfer coefficient for the exchanger is 52 W/m^2-K and the surface area of the exchanger is 50 m ^{2}. If the arrangement is cross flow with both fluids unmixed determine the exit temperatures of both the fluids and the rate of heat transfer. Take the specific heats of both the fluids as 1.05 kJ/kg-K.

A concentric-tube heat exchanger operates on the counter flow mode. The fluid flowing in the annular space enters the exchanger at 20 ^{0} and leaves at 70 ^{0} C. The fluid flowing through the inner tube enters at 110 ^{0} C and leaves at 65 ^{0} C. The length of the exchanger is 30 m. It is desired to increase the outlet temperature of the cold fluid to 80 ^{0} C by increasing only the length while maintaining the same mass flow rates, inlet temperatures and tube diameters. Make any justifiable assumption and calculate the new length.

It is proposed to cool 1000 kg/h of oil from150 ^{0} C to 50 ^{0} C in a heat exchanger using 1667 kg/h of water at an inlet temperature of 30 ^{0} C. Calculate the surface required assuming a single pass cross flow arrangement in which the oil is mixed and the water unmixed. Assume C_p for oil to be 2.087 kJ/kg-K and the overall heat transfer coefficient to be 550 W/m^2-K.
Solve the problem by the mean temperature difference method as well as by the ε – NTU method.

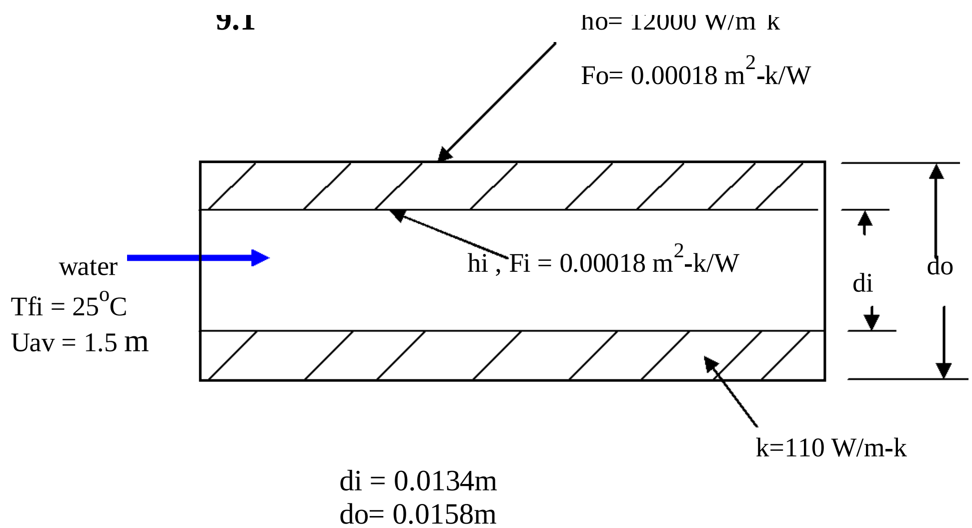
Two identical counter flow heat exchangers are available. Water at the rate of 5000 kg/h and at 30 ^{0} C is to be heated by cooling an oil (C_p = 2.1 kJ/kg-K) at 90 ^{0} C. The oil flow rate is 2000 kg/h. The heat transfer area is 3 m ^{2}. From the point of view of maximizing the heat transfer rate, which of the following is the best arrangement?

(a) Both the fluids flow in series.
(b) The oil flow is split up equally between the two exchangers, while the water flows in series.
(c) Both oil and water flows are split up equally in both the exchangers.

A counter flow double pipe heat exchanger is used to heat 1.25 kg/s of water from 35 ^{0} C to 80 ^{0} C by cooling an oil (C_p = 2.0 kJ/kg-K) from 150 ^{0} C to 85 ^{0} C. The overall heat transfer coefficient is 850 W/m^2-K. A similar arrangement is to be built at another location, but it is desired to compare the performance of the single counter flow heat exchanger with two smaller counter flow heat exchangers connected in series on the water side and in parallel on the oil side as shown in Fig. P 9.16. The oil flow is split equally between the two exchangers and it may be assumed that the overall heat transfer coefficient for the smaller exchangers is the same as for the large exchanger. If the smaller exchanger costs 20 % more per unit surface area, which would be the most economical arrangement – the one large exchanger or the two equal- sized small exchangers?

A.Overall heat transfer coefficient:

Solution:



To find : i. U_i ; ii. U_o

i. Overall heat transfer coefficient based on inside surface area is given by

$$U_i = \frac{1}{(1/h_i)+F_i+[d_i/2k]\log_e(d_o/d_i)+(d_i/d_o)F_o+(d_i/d_o)(1/h_o)}$$

To find h_i: Properties of water at 25^0C are:

k = 0.6805 W/(m-K) ; $\nu = 0.945\text{x}\ 10^{-6}\ m^2/s$; Pr =

6.22 $Re_d = U_{av}d_i/\nu = 1.5\text{x}0.0134/0.945\text{x}10^{-6} = 21270$

Since $Re_d > 2300$, flow is turbulent.

$Nu_d = h_id_i/k = 0.023(Re_d)^{0.8}\ \text{x}\ (Pr)^{0.4} = 138.5$

$h_i = 138.5\ \text{x}\ 0.6085 / 0.0134 = 6289.5\ W/(m^2\text{-K})$

$$U_i = \frac{1}{(1/6289.5)+0.00018+[0.0134/2\times110]\log_e(0.0158/0.0134)+(0.0134/0.0158)\times0.00018}$$ U_i

$= 1747\ W/(m^2\text{-}K)$

ii. $U_i \times A_i = U_o \times A_o$

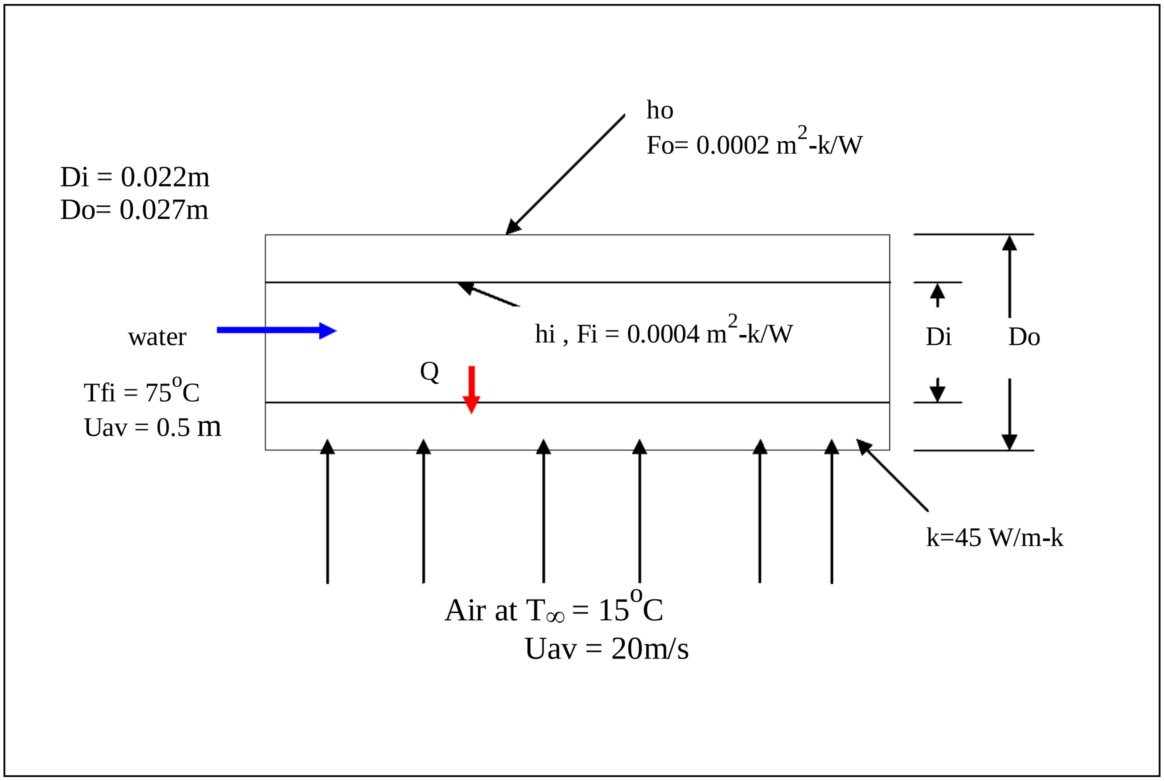
Hence $U_i \times \pi\, d_i\, L = U_o \times \pi\, d_o\, L$

Or $U_o = U_i\,(d_i/d_o)$

$= 1747 \times (0.0134/0.0158)$

$= 1481.6\ W/(m^2\text{-}K)$

Solution:



To find h_i: Properties of water at 75^0C are:

Since $Re_d > 2300$, flow is turbulent.

$Nu_d = h_i d_i/k = 0.023(Re_d)^{0.8} \times (Pr)^{0.4} = 118.2$

$h_i = 118.2 \times 0.6715/ 0.022 = 3608$ W/(m^2-K)

$Rc_i = 1/(h_i A_i) = \quad 1/ h_i \pi d_i L \quad = 1/ (\pi \times 0.022 \times 1 \times 3608)$

$= 4.01 \times 10^{-3}$ k/W

To find h_o:

Air is flowing across the cylinder.

Properties of air at 15^{0}C are :

$k = 0.0255$ W/(m-K) ; $\nu = 14.61 \times 10^{-6}$ m^2/s ; $Pr = 0.704$

$Re_d = U_\infty do/ \nu = \quad 20 \times 0.027/14.61 \times 10^{-6} = 36961$

From data hand book,

$$Nu_d = h_o d_o/k = [0.4\, Re_d^{0.5} + 0.06\, Re_d^{\ 0.6667}] \times Pr^{0.4} \times (\mu_\infty/\mu_w)^{0.25}$$

For gases, $(\mu_\infty/\mu_w) = 1$

Hence $\quad h_o d_o/k = [0.4 \times (36961)^{0.5} + 0.06\, (39961)^{0.6667}] \times 0.704^{0.4} = 130.4$

$h_o = 130.4 \times 0.0255/0.027 = 123$ W/(m^2-K)

$Rc_o = 1/(h_0 A_0) = \quad 1/ h_0 \pi d_0 L \quad = 1/ (\pi \times 0.027 \times 1 \times 123)$

$= 0.096$ k/W

$R_{fi} = F_i A_i = \pi d_i L F_i = \pi \times 0.022 \times 1 \times 0.004 = 2.765 \times 10^{-5}$ k/W

$R = (1/2\pi L k) \times \log_e (d_o/d_i) = (1/2 \times \pi \times 1 \times 45) \log_e (0.027/0.022)$

$= 7.24 \times 10^{-3}$ k/W

$R_{fo} = F_oA_o = \pi d_o LF_o = \pi \times 0.027 \times 1 \times 0.0002 = 1.697 \times 10^{-5}$ k/W

Total thermal resistance $= \sum R = 4.01 \times 10^{-3} + 2.765 \times 10^{-5} + 7.24 \times 10^{-3} + 1.697 \times 10^{-5} + 0.096$

$= 0.1073$ k/W

If U_o is the overall heat transfer coefficient based on outside area

then, $U_o A_o = 1/\sum R$

$U_o = 1 / A_o \sum R$

$= 1 / (\pi \times 0.027 \times 0.1073)$

$= 110$ W/(m^2-K)

Comparison between various resistances:

Rc_i	Thermal resistance for convection at the inside surface.	4×10^{-3}
R_{fi}	Resistance due to fouling at the inside surface	0.028×10^{-3}
R	Resistance of the tube wall for conduction	7.24×10^{-3}
R_{fo}	Resistance due to fouling at the outside surface	0.017×10^{-3}
Rc_o	Thermal resistance for convection at the outside surface	96×10^{-3}

The comparison shows that the thermal resistance for convection heat transfer ifrom the outer surface of the tube due to air is very large compared to the other resistances i.e,

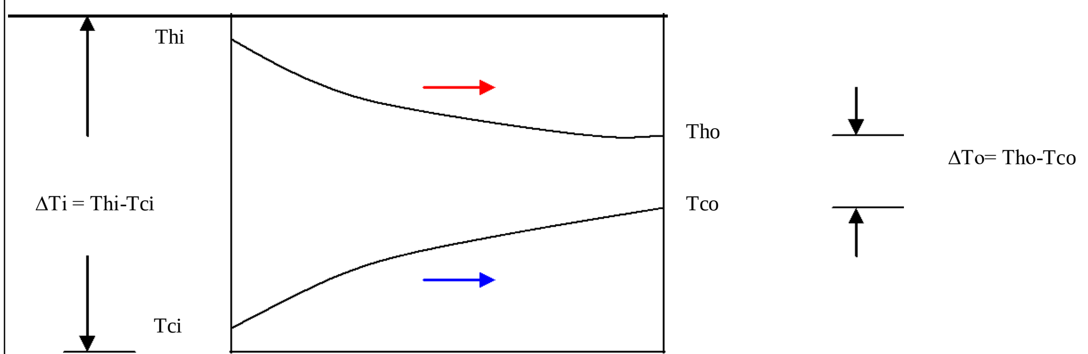
$\sum R \approx Rc_o$

B.Mean temperature difference method:

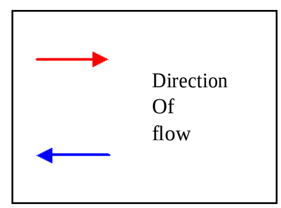
Solution:

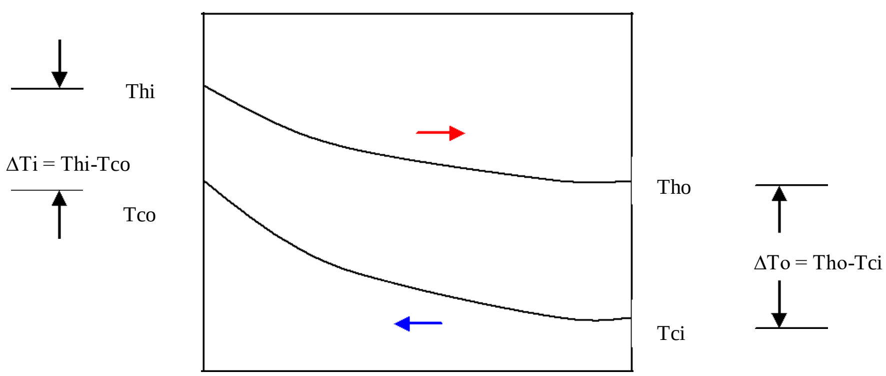
9.3

a) MTD for parallel flow:



b) MTD for counter flow:





$Th_i = 60^o$ C ; $Th_o = 48^o$ C ; $Tc_i = 35^o$ C ; $Tc_o = 44^o$ C

(a). MTD for parallel flow:

$\Delta T_i = Th_i - Tc_i = 60\text{-}35 = 25$

$\Delta T_o = Th_o - Tc_o = 48\text{-}44 = 04$

hence MTD = $\Delta T_i - \Delta T_o$ / $\log_e (25/4) = 11.5\ ^o$ C (b). MTD for counter flow:

$\Delta T_i = 60 - 44 = 16^o C$

$\Delta T_o = 48 - 35 = 13^o C$

hence MTD = $\Delta T_i - \Delta T_o / \log_e (16/13) \qquad = 14.45\ ^o C$

(c). Single pass cross flow (both fluids unmixed):

$T_1 = Th_i = 60^o C$; $T_2 = Th_o = 48^o C$; $t_1 = Tc_i = 35^o C$; $t_2 = Tc_o = 44^o C$

$R = T_1 - T_2 / t_2 - t_1 = 60 - 48 / 44 - 35 \qquad = 1.33$

$P = t_2 - t_1 / T_1 - t_1 = 44 - 35 / 60 - 35 \qquad = 0.36$

From chart , F = 0.94

Hence MTD = F x $(MTD)_{c.f} = 0.94 \times 14.45 = 13.583^o C$

(d). Single pass cross flow (hot fluid mixed ; cold fluid unmixed):

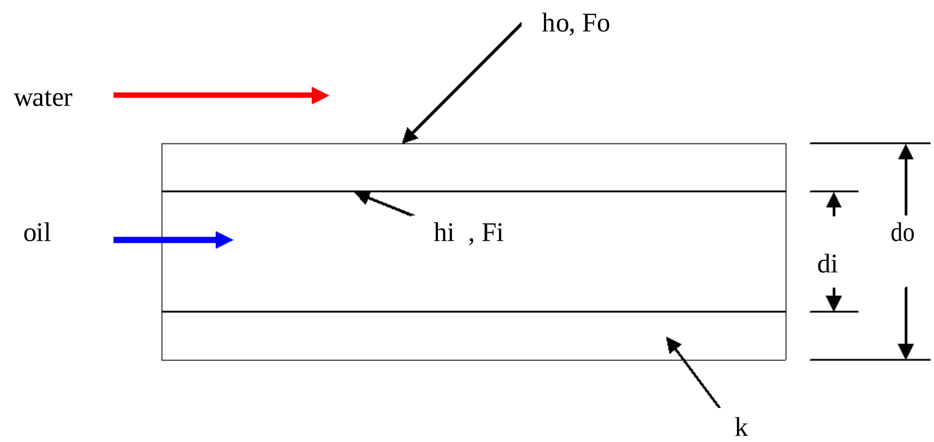
$R = T_1 - T_2 / t_2 - t_1 = 60 - 48 / 44 - 35 \qquad = 1.33$

$P = t_2 - t_1 / T_1 - T_2 = 44 - 35 / 60 - 35 \qquad = 0.36$

From chart , F = 0.98

Hence MTD = F x $(MTD)_{c.f} = 0.98 \times 14.45 = 14.16^o C$

Solution:



Cold fluid : Oil :

m_c = 0.3033 kg/s ; Cp_c = 2100 J/kg-K ;Tc_i = 27^oC ; Tc_o = 49^oC ; h_i = 635 W/ (m^2-K)

Fi = 0.0001 (m^2-K) / W

Hot fluid : Water :

m_h = 0.1083 kg/s ; Cp_h = 4200 J/kg-K ;Th_i = 93^oC ; h_o = 1270 W/ (m^2-K)

Fo = 0.0004 (m^2-K) / W

To find L for (i) parallel flow (ii) counter flow:

Overall heat transfer coefficient based on outside area of the inner tube is given by

$$U_o = \frac{1}{(d_o/d_i)(1/hi) + (d_o/d_i)\,Fi + (d_o/2k)\log_e(d_o/d_i) + F_o + (1/h_o)}$$

$$= \frac{1}{(0.0286/0.0254)(1/635)+(0.0286/0.0254)0.0001+(0.0286/700)\log_e(0.0286/0.0254)+0.0004+(1/1270)}$$

$$= 325\ \text{W/(m}^2\text{-K)}$$

Heat balance equation can be written as

$Q = m_c \times Cp_c\,(Tc_o – Tc_i) = 0.3083 \times 2100\,(49 – 27)$

$= 14243.5$ W

Also $Q = m_h\, Cp_h\,(Th_i – Th_o)$

Or $Th_o = Th_i – Q/m_h\, Cp_h = 93 – [14243.5/(0.1083 \times 4200)]$

$= 61.7^oC$

(i) Parallel flow arrangement :

$\Delta T_i = Th_i – Tc_i = 93 – 27 = 66^oC$

$\Delta T_o = Th_o – Tc_o = 61.7 – 49 = 12.7^oC$

hence MTD $= \Delta T_m = \Delta T_i - \Delta T_o / \log_e (66/12.7) = 32.34\ ^oC$

$Q = U_o\, \pi\, d_o\, L\, \pi\, \Delta T_m$

Or $L = Q / (U_o\, \pi\, d_o \Delta T_m) = 14243.5/(325 \times \pi \times 0.0286 \times 32.34)$

$= 15.1$ m.

(ii) Counter flow arrangement :

$\Delta T_m = (93 – 49) – (61.7 – 27) / \log_e [(93 – 49) / (61.7 – 27)]$

$= 39.2^oC$

Hence $L = 14243.5 / (325 \times \pi \times 0.0286 \times 39.2)$

$= 12.5$ m

Solution:

Shell side fluid : water :

m_c = 0.5 kg/s ; Cp_c = 4200 J/kg-K ;Tc_i = 10^oC ;

Tube side fluid : Oil :

m_h = 0.3 kg/s ; Cp_h = 2000 J/kg-K ;Th_i = 80^oC ; Th_o = 30^oC

U = 250 W/(m^2-K)

Heat balance equation is

$Q = m_h Cp_h (Th_i - Th_o) = 0.3 \times 2000 \times (80 - 30)$

$= 30000$ W

Also $Q = m_c \times Cp_c (Tc_o - Tc_i)$

Or $Tc_o = Tc_i + Q/ m_c Cp_c = 10 + 30000/(0.5 \times 4200)$

$= 24.3^oC$

$(\Delta T_m)_{c.f} = (80 - 24.3) - (30 - 10) / \log_e [(80 - 24.3) / (30 - 10)]$

$= 34.35^oC$

Single shell pass - two tube pass HE :

$T_1 = 10\ ^oC$; $T_2 = 24.3\ ^oC$; $t_1 = 80\ ^oC$; $t_2 = 30\ ^oC$

$R = T_1 - T_2 / t_2 - t_1 = 10 - 24.3 / 30 - 80 = 0.29$

$P = t_2 - t_1 / T_1 - t_1 = 30 - 80 / 10 - 80 = 0.714$

From chart , F = 0.875

Hence $(\Delta T_m) = F \times (MTD)_{c.f} = 0.875 \times 34.85 = 30.5^oC$

$Q = U A (\Delta T_m)$

Or $A = Q / U (\Delta T_m) = 30000/(250 \times 30.5) = 3.93\ m^2$

Solution:

Cross flow HE :

Cold water (Unmixed) : m_c = 3 kg/s ; Cp_c = 4200 J/kg-K ;Tc_i = 30^oC ; Tc_o = 80^oC ;

Hot gases (Mixed) : Cp_h = 1005 J/kg-K ;Th_i = 225^oC ; Th_o = 100^oC

U = 200 W/(m^2/K)

$Q = m_c \times Cp_c (Tc_o – Tc_i) = 3 \times 4200 (80 – 30)$

$= 630000$ W

Also

$P = t_2 – t_1 / T_1 – t_1 = 80 – 30 / 225 – 30 = 0.256$

$R = T_1 – T_2 / t_2 – t_1 = 225 – 100 / 80 – 30 = 2.5$

From chart, F = 0.93

$(\Delta T_m)_{c.f} = (225 – 80) – (100 – 30) / \log_e [(225 – 80) / (100 – 30)]$

$= 54.62^oC$

$(\Delta T_m) = 0.93 \times 54.62 = 50.8^oC$

$Q = U A (\Delta T_m)$ or $A = Q / A (\Delta T_m) = 630000 / (200 \times 54.62)$

$= 57.7 m^2$.

Solution:

Water : m_c = 2 kg/s; Cp_c = 4200 J/kg-K ; Tc_i = 20^oC ;Tc_o = 80^oC ;

Oil : Th_i = 140^oC ; Th_o = 90^oC

U = 300 W / (m^2 – K)

$Q = m_c \times Cp_c (Tc_o – Tc_i) = 2 \times 4200 (80 – 20)$

$= 504000$ W

Tube side fluid is water. Hence $t_1 = 20^oC$; $t_2 = 80\ ^oC$

Shell side fluid is oil. Hence $T_1 = 140\ ^oC$; $T_2 = 90\ ^oC$

$P = t_2 – t_1 / T_1 – t_1 = 80 – 20 / 140 – 20 = 0.5$

$R = T_1 – T_2 / t_2 – t_1 = 140 – 90 / 80 – 20 = 0.83$

From chart F = 0.97

$(\Delta T_m)_{c.f} = (140 – 80) – (90 – 20) / \log_e [(140 – 80) / (90 – 20)] =$

64.87^oC

$(\Delta T_m) = 0.97 \times 64.87 = 60.9^oC$

$Q = U A (\Delta T_m)$ or $A = Q / A (\Delta T_m)$ = 504000 / (300 x 62.9)

= 26.7 m^2.

9.8 Solution:

Shell and Tube HE :

Cold fluid : Water : $m_c = 8.33$ kg/s ;$Tc_i = 25^oC$; $Tc_o = 50^oC$;

Hot fluid : Steam condensing at atmospheric pressure.

Hence $Th_i = Th_o = T_{sat}$ at atmospheric pressure

= 99.6^o C (from steam tables)

h_{fg} = 2257×10^3 J / kg – K

Tube side fluid is water. Hence $h_i = 3000$ W / m^2 – K

Shell side fluid is oil. Hence $h_o = 8000$ W / m^2 – K

Inside dia of tube $= d_i = 0.025$ m

Outside dia of tube = $d_o = 0.029$ m

Length of the tube = L = 2 m

Overall heat transfer coefficient based on outside surface area is given by

$$U_o = \frac{1}{(d_o/d_i)(1/h_i) + (d_o/2k)\log_e(d_o/d_i) + (1/h_o)}$$

As $(d_o/2k)\log_e(d_o/d_i) = 0$

$$= \frac{1}{(0.029 / 0.025)(1/3000) + (1 / 8000)}$$

$= 1954\ W/(m^2\text{-}K)$

Since $Th_i = Th_o$, both parallel flow and counter flow arrangement will give the same value of (ΔT_m).

Hence $(\Delta T_m) = (99.6 - 25) - (99.6 - 50) / \log_e [(99.6 - 25) / (99.6 - 50)]$

$= 61.2\ ^oC$

$Q = m_c \times Cp_c (Tc_o - Tc_i) = 8.33 \times 4200 (50 - 25)$

$= 874650\ W$

$Q = U_o A_o (\Delta T_m)$ or $A_o = Q / U_o (\Delta T_m) = 874650 / (1954 \times 61.2)$

$= 7.314\ m^2$.

Surface area of each tube $= a_o = \pi d_o L = \pi \times 0.029 \times 2$

$= 0.1822\ m^2$

Hence number of tubes $= n = A_o / a_o = 7.314 / 0.1822 = 40.14 \approx 41$

B. Effectiveness - NTU method:

Solution:

For a counter flow HE effectiveness is given by

$$\varepsilon = \frac{1 - e^{[-(1-c)NTU]}}{1 - ce^{[-(1-c)NTU]}}$$

when c = 1 , the above expression gives

$$\varepsilon = \frac{1 - e^{0}}{1 - 1 \times e^{0}} = 0 / 0 = \text{indeterminate.}$$

Hence we have to find ε using L''hospital''s rule.

$$\varepsilon = Lt_{c \to 1} \frac{d/dt\,[1 - e^{[-(1-c)NTU]}]}{d/dt\,[1 - ce^{[-(1-c)NTU]}]}$$

$$= Lt_{c \to 1} \frac{0 - NTU\, e^{-(1-c)NTU}}{0 - \{e^{-(1-c)NTU} \times 1 + c\,(NTU)e^{-(1-c)NTU}\}}$$

$$\varepsilon = \frac{0 - NTU}{0 - \{1 + NTU\}} = \frac{NTU}{1 + NTU}$$

Solution:

$m_h = 0.067$ kg/s ; $Cp_h = 4200$ J/kg-K ;$Th_i = 100^oC$;

$m_c = 0.133$ kg/s; $Cp_c = 2520$ J/kg-K ; $Tc_i = 20^oC$;

$m_h\,Cp_h = 0.067 \times 4200 = 281.4$ J / s – K $m_c\,Cp_c =$

$0.133 \times 2520 = 335.16$ J / s – K

Since $m_h\,Cp_h < m_c\,Cp_c$, hence $c = \dfrac{m_h\,Cp_h}{m_c\,Cp_c} = \dfrac{281.4}{335.16} = 0.84$

For a parallel flow HE ,

$$\varepsilon = \frac{1 - e^{[-(1+c)\,NTU]}}{1 + c} \quad \text{-->}(1)$$

For a given value of c , ε will be max if [d ε/d(NTU)] = 0

From (1),

$$[d\,\varepsilon/d(NTU)] = \frac{1 \quad x\,[\,0 + (1+c)e^{-(1+c)NTU}]}{1+c} = 0$$

Therefore, $e^{-(1+c)NTU} = 0$ OR NTU = ∞

Substituting this condition in eqn (1) we have

$$\varepsilon_{max} = \frac{1 - e^{[-(1+c)\,\infty}}{1 + c} = \frac{1 - 0}{1 + c} = \frac{1}{1 + c}$$

$$= \frac{1}{1.84} = 0.5435$$

For a counter flow HE,

$$\varepsilon = \frac{1 - e^{[-(1-c)\,NTU]}}{1 - ce^{[-(1-c)\,NTU]}} \quad \text{------------------------>}\ (2)$$

$$\frac{d\,\varepsilon}{d(NTU)} = \frac{[1 - ce^{[-(1-c)NTU]}]\,[(1-c)e^{-(1-c)NTU}] - [1 - e^{[-(1-c)NTU]}][ce^{-(1-c)NTU}(1-c)]}{[1 - ce^{[-(1-c)\,NTU]}]^2}$$

Evaluating, we get 1 – c = 0 or c = 1 ; substituting this value of c, we have

ε = 1

[from (2)]

Solution:

Counter flow HE :

$m_h = 3$ kg/s ; $Cp_h = 1050$ J/kg-K ;$Th_i = 500^oC$;

$m_c = 0.75$ kg/s; $Cp_c = 4200$ J/kg-K ; $Tc_o = 85^oC$;

$U = 450$ W / (m^2- K)

$m_h\, Cp_h = 3 \times 1050 = 3150$ J / s – K

$m_c\, Cp_c = 0.75 \times 4200 = 3150$ J / s – K

$m_h\ Cp_h = m_c\ Cp_c$ Hence $Th_i - Th_o = Tc_o - Tc_i$ and c = 1

$$\text{Also} \quad NTU = \frac{UA}{m_c\ Cp_c} \quad \text{or} \quad \frac{UA}{m_h\ Cp_h}$$

$$= \frac{450 \times 1}{3150} = 0.1428$$

$$\varepsilon = \frac{1 - e^{[-(1-c)NTU]}}{1 - c e^{[-(1-c)NTU]}}$$

Since c = 1, $\varepsilon = \frac{NTU}{1 + NTU} = \frac{0.1428}{1.1428} = 0.1250$

But $\varepsilon = \frac{Th_i - Th_o}{Th_i - Tc_i}$ or $\frac{Tc_o - Tc_i}{Th_i - Tc_i}$ when c = 1

Hence $0.1250 = \frac{Tc_i}{500 - Tc_i}$

Or $62.5 - 0.125\, Tc_i = 85 - Tc_i$

Or $Tc_i = \frac{85 - 62.5}{(1 - 0.125)} = 25.7^oC$

Also $Th_i - Th_o = Tc_o - Tc_i$ when c = 1

Hence $500 - Th_o = 85 - 25.7$

$Th_o = 440.7^oC$

9.12 Solution:

Hot gases : $m_h = m_c (1 + 0.015) = 1.015\, m_c$

$= 1.015 \times (5000/3600) = 1.41$ kg/s

$Th_i = 450^oC$;

Cold fluid : Air : $m_c = 1.39$ kg/s; $Cp_c = Cp_h = 1050$ J/kg-K ; $Tc_i = 170^oC$;

$U = 52\ W / (m^2 - K)$; $A = 50\ m^2$

$m_h\ Cp_h = 1.39 \times 1050 = 1459.5$ J / s – K

$m_c\ Cp_c = 1.41 \times 1050 = 1480.5$ J / s – K

Since $m_h\ Cp_h > m_c\ Cp_c$, hence $c = \dfrac{m_c\ Cp_c}{m_h\ Cp_h} = \dfrac{1459.5}{1480.5} = 0.986$

$$NTU = \frac{UA}{m_c\ Cp_c} = \frac{52 \times 50}{1459.5} = 1.78$$

From chart for cross flow with both fluids unmixed,

$\varepsilon = 0.6$

when $m_h\ Cp_h > m_c\ Cp_c$,

$$\varepsilon = \frac{Tc_o - Tc_i}{Th_i - Tc_i}$$

Or $Tc_o = Tc_i + \varepsilon\ (Th_i - Tc_i)$

$= 170 + 0.6 \times [450 – 170]$

$= 338\ ^oC$

Also $m_c \times Cp_c\ (Tc_o – Tc_i) = m_h \times Cp_h\ (Th_i – Th_o)$

Hence $Th_o = Th_i - \dfrac{m_c\ Cp_c}{m_h\ Cp_h} \times [Tc_o – Tc_i]$

$= 450 - 0.986\ [338 – 170]$

$= 284.35\ ^oC$

Solution:

Counter flow HE :

case (i) $Th_i = 110^o$ C ; $Th_o = 65^o$ C ; $Tc_i = 20^o$ C ; $Tc_o = 77^oC$; L = 30 m ; U_1

case (ii) $Tc_i = 20^o$ C ; $Tc_o = 80^oC$; $Th_i = 110^o$ C ; $U_2 = U_1$ (assumed)

For case (i) $Q_1 = m_c \times Cp_c (Tc_o - Tc_i) = m_c \times Cp_c (70 - 20)$

For case (ii) $Q_2 = m_c \times Cp_c (80 - 20)$

Hence, $$\frac{Q_1}{Q_2} = \frac{m_c \times Cp_c (70 - 20)}{m_c \times Cp_c (80 - 20)} = \frac{5}{6}$$

Also for case (i) $m_c \times Cp_c (70 - 20) = m_h \times Cp_h (110 - 65)$

Also for case (ii) $m_c \times Cp_c (80 - 20) = m_h \times Cp_h [(110 - (Th_o)_2]$

Hence $$\frac{110 - 65}{110 - (Th_o)_2} = \frac{70 - 20}{80 - 20}$$

hence $(Th_o)_2 = 110 - (6/5)[110 - 65] = 56\ ^oC$

Hence for case (i) $$(\Delta T_m)_1 = \frac{(110 - 70) - (65 - 20)}{\log_e[(110 - 70)/(65 - 20)]} = 42.45^oC$$

Hence for case (ii) $$(\Delta T_m)_2 = \frac{(110 - 80) - (56 - 20)}{\log_e[(110 - 80)/(56 - 20)]} = 32.9\ ^oC$$

hence, $$\frac{Q_1}{Q_2} = \frac{U_1 \pi d L_1 (\Delta T_m)_1}{U_2 \pi d L_2 (\Delta T_m)_2} = \frac{5}{6}$$

Hence $$\frac{L_1 (\Delta T_m)_1}{L_2 (\Delta T_m)_2} = \frac{5}{6}$$

Or $L_2 = (6/5) \times [(\Delta T_m)_1 / (\Delta T_m)_2] \times L_1$

$= (6/5) \times (42.45/32.90) \times 30 = 46.45$ m

Solution:

Solution by MTD method:

Cross flow HE with oil mixed and water unmixed;

Hot fluid : oil : $m_h = 0.278$ kg/s ; $Cp_h = 2087$ J/kg-K ; $Th_i = 150^oC$; $Th_o = 50^oC$

Cold fluid : water : $m_c = 0.463$ kg/s ; $Cp_c = 4200$ J/kg-K ; $Tc_i = 30^oC$;

$U = 550 \text{ W} / (m^2 - K)$

$Q = m_c \times Cp_c (Tc_o - Tc_i) = 0.278 \times 2087 [150 - 50] = 58019$ W

Also $Q = m_h \times Cp_h (Th_i - Th_o)$

Or $Tc_o = Tc_i + (Q / m_c Cp_c)$

$= 30 + [58019 / (0.463 \times 4200)]$

$= 59.8\ ^oC$

$(\Delta T_m)_{c.f} = (150 - 59.8) - (50 - 30) / \log_e [(150 - 59.8) / (50 - 30)]$

$= 46.6\ ^oC$

$R = (150 - 50) / (59.8 - 30) = 3.35$

$P = (59.8 - 30) / (150 - 30) = 0.25$

From chart F = 0.80

$(\Delta T_m) = 0.8 \times 46.6 = 37.3\ ^oC$

$$A = \frac{Q}{U (\Delta T_m)} = \frac{58019}{550 \times 37.3} = 2.83 \text{ m}^2$$

Solution by ε – NTU method :

$m_h\,Cp_h$ = 0.278 x 2087 = 580.2 J / s – K

$m_c\,Cp_c$ = 0.463 x 4200 = 1944.6 J / s – K

Since $m_h\,Cp_h < m_c\,Cp_c$, hence $c = \dfrac{m_h\,Cp_h}{m_c\,Cp_c} = \dfrac{580.2}{1944.6} = 0.29$

when $m_h\,Cp_h < m_c\,Cp_c$,

$$\varepsilon = \frac{Th_i - Th_o}{Th_i - Tc_i} = \frac{150 - 50}{150 - 30} = 0.83$$

From chart, NTU = 2.8 = UA / $(mCp)_{min}$ = UA / $m_h\,Cp_h$

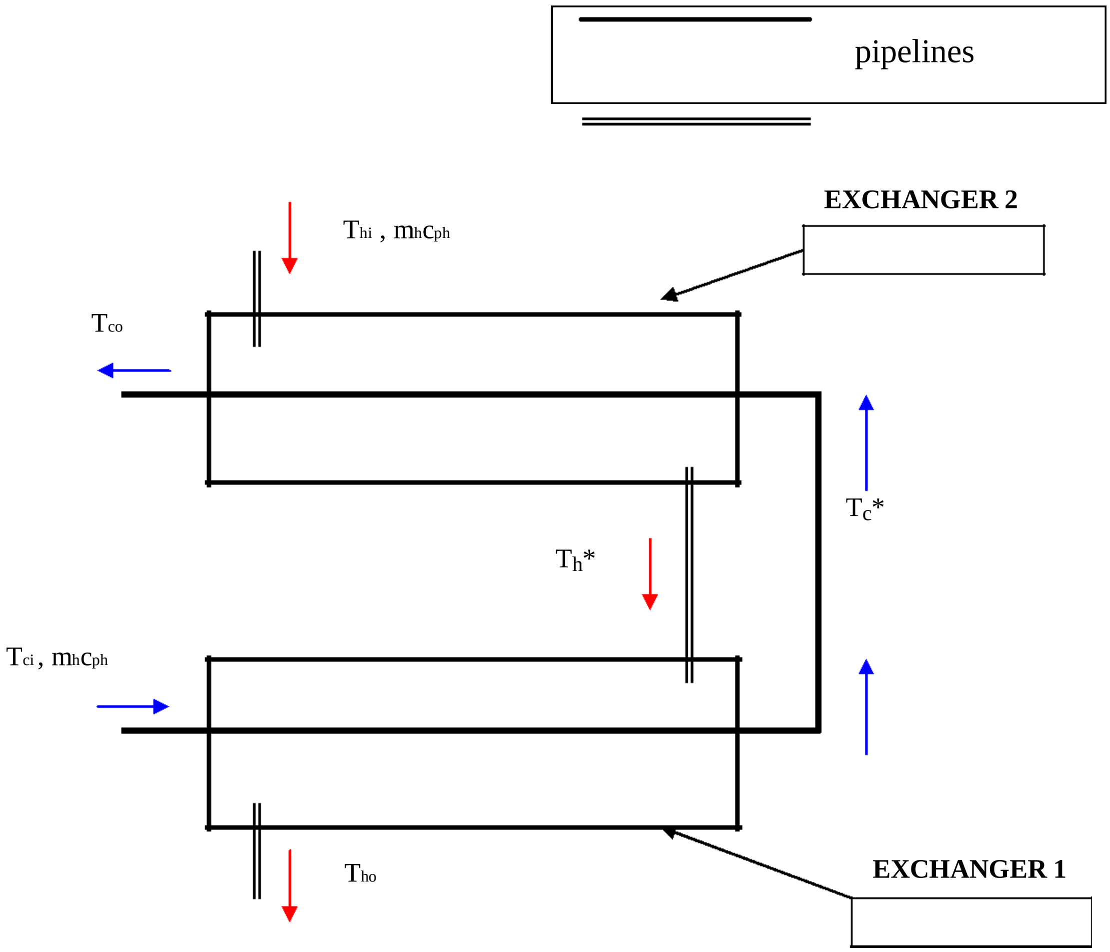
Hence A = 2.8 x 530.2 / 550 = 2.95 m^2

Cold: water: m_c= 5000/3600 = 1.39 kg/s ; T_{ci}= 30 ^oC ;C_{pc}= 4200 J/(kg-K) assumed

Hot : oil : m_h = 2000/3600 = 0.555 kg/s ;C_{ph} = 2100 J/(kg-K) ;T_{hi}=

90^oC A= $3m^2$; U = 465 W/(m^2-K)

Case 1) Both the fluids flow in series.
The arrangement for this case will be as shown with figure below:



As m_cC_{pc} = (5000/3600) x 4200 = 5833.3 J/(s-K)

And m_hC_ph= (2000/3600) x 2100 = 1166.7 J/(s-

K) therefore $m_cC_pc < m_hC_{ph}$

Therefore C= 1166.7/5833.3 = 0.20

$$NTU = \frac{UA}{m_h C_{ph}} = \frac{465*3}{1166.7} = 1.196$$

$$\varepsilon = \frac{1-\exp[-(1-C)NTU]}{1-C\exp[-(1-C)NTU]}$$

$$= \frac{1-\exp[-(1-0.2)1.196]}{1-0.2\exp[-(1-0.2)1.196]} = \frac{1-0.3841}{1-0.2*0.3841}$$

$$= 0.667$$

2) <u>When oil is split up equally between the two heat exchangers:</u>

In this case m_h= 1000/3600 = 0.278 kg/s

Therefore m_h C_{ph} = 0.278x2100= 583.8 J/(s-K)

Therefore C= 583.8/5833.3 = 0.10

$$NTU = \frac{UA}{m_h C_{ph}} = \frac{465*3}{583.8} = 2.4$$

$$\varepsilon = \frac{1-\exp[-(1-C)NTU]}{1-C\exp[-(1-C)NTU]}$$

$$= \frac{1-\exp[-(1-0.1)2.4]}{1-0.1\exp[-(1-0.1)2.4]} = \frac{1-0.1153}{1-0.1*0.1153}$$

$$= 0.895$$

3) <u>Both the oil and water flows are split up equally</u>:

In this case $m_c C_{pc}$ = (2500/3600) x 4200 = 2917 J/(s-K)
$m_h C_{ph}$= 0.278 x 2100 = 583.8 J/(s-K)

Therefore C= 583.8/2917 = 0.20

$$NTU = \frac{UA}{m_h C_{ph}} = \frac{465*3}{583.8} = 2.4$$

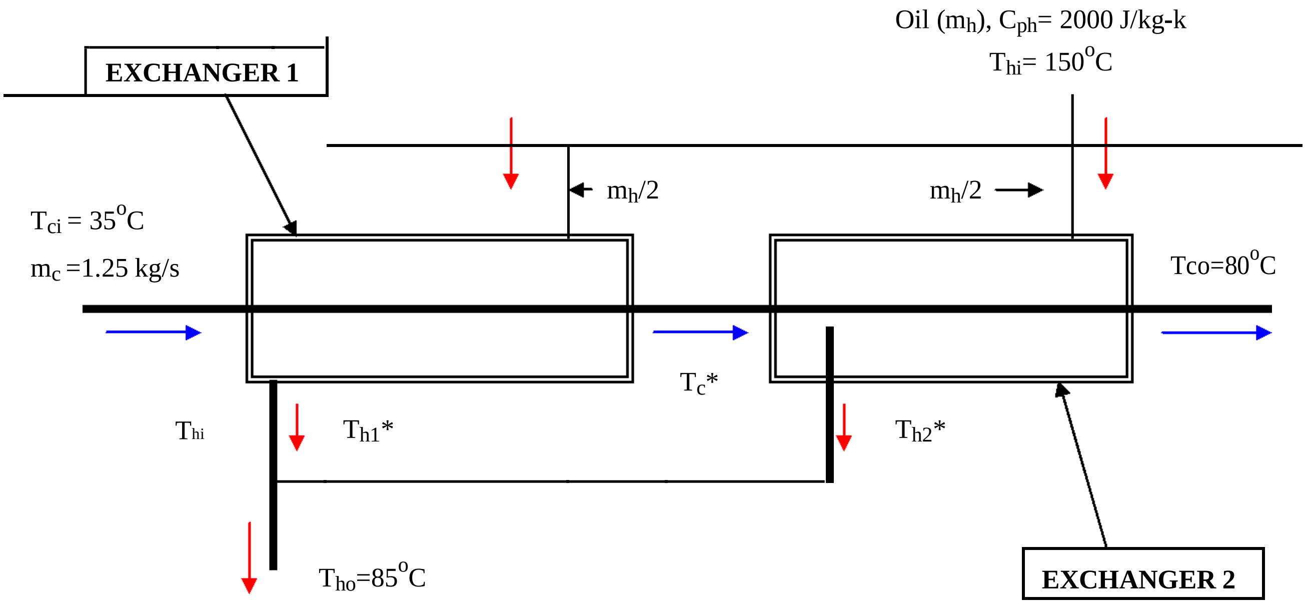
$$\varepsilon = \frac{1-\exp[-(1-C)NTU]}{1-C\exp[-(1-C)NTU]}$$

$$= \frac{1-\exp[-(1-0.2)2.4]}{1-0.2\exp[-(1-0.2)2.4]} = \frac{1-0.1467}{1-0.0293}$$

$$= 0.879$$

Since ε is the highest in case (2), this arrangement is the best for " <u>maximizing heat transfer</u>" point of view.

9.16 Solution:



$U_1 = U_2 = U = 850\ W / (m^2 - K)$

Case (i) single counter flow HE :

$Q = m_c \times Cp_c (Tc_o - Tc_i) = 1.25 \times 4200\ [80 - 35] = 236250$ W Q =

$m_h \times Cp_h (Th_i - Th_o)$

$$\text{Hence}\quad m_h = \frac{Q}{Cp\,[Th_i - Th_o]} = \frac{236250}{2000 \times [150 - 85]} = 1.817 \text{ kg/s}$$

$m_h\, Cp_h = 1.817 \times 2000 = 3634.6$ J / s – K

$m_c\, Cp_c = 1.25 \times 4200 = 5250$ J / s – K

$$\text{Since } m_h\, Cp_h < m_c\, Cp_c \text{, hence } c = \frac{m_h\, Cp_h}{m_c\, Cp_c} = \frac{3634.6}{5250} = 0.7$$

$$\varepsilon = \frac{Th_i - Th_o}{Th_i - Tc_i} = \frac{150 - 85}{150 - 35} = 0.565$$

From chart, NTU = 1.0 = UA / $(mCp)_{min}$ = UA / $m_h\ Cp_h$

Hence A = 1 x 3634.6 / 850 = 4.276 m^2

Case (ii) When two smaller heat exchangers are used:

For this case m_h = 1.317 / 2 = 0.9085 kg/s

$m_h\ Cp_h$ = 0.9085 x 2000 = 1817 J / s – K

$m_c\ Cp_c$ = 1.25 x 4200 = 5250 J / s – K

Hence c = 1817 / 5250 = 0.35

To find „ε", we should know the exit temperatures of hot and cold fluids for at least one HE. Since UA and $(mCp)_{min}$ is the same for both the exchangers NTU should be same for both the exchangers.

$$\text{Thus } \varepsilon_1 = \frac{Th_i - Th_1^*}{Th_i - Tc_i} = \varepsilon_2 = \frac{Th_i - Th_2^*}{Th_i - Tc^*}$$

Therefore $$\frac{150 - T_{hi}}{150 - 35} = \frac{150 - T_{h2}^*}{150 - T_c^*} = \varepsilon_1 = \varepsilon_2 \quad \text{.............................} (1)$$

Since the oil flow is the same in each exchanger and the average exit oil temperature must be 85°C …..it follows that

$$\frac{T_{h1}^* + T_{h2}^*}{2} = 85^oC \ldots\ldots\ldots\ldots\ldots \ldots\ldots\ldots\ldots\ldots (2)$$

Energy balance on the second heat exchanger gives

$$5250(80 - T_c^*) = 1817(150 - T_{h2}^*) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Equations 1,2 ,3 may be solved for the three unknowns T_{h1}^*, T_{h2}^*, T_c^*. The solutions are as follows:

Eqn 1can be rearranged after cross multiplying as:

$$150T_{h1}^* - 115T_{h2}^* + 150T_c^* - T_{h1}^* T_c^* = 5250 \ldots\ldots\ldots\ldots (4)$$

Eqn 2……………… $T_{h1}^* + T_{h2}^* = 170 \ldots\ldots\ldots\ldots\ldots (5)$

Eqn 3……………….. $T_{h2}^* - 2.9T_c^* = -82 \ldots\ldots\ldots\ldots\ldots (6)$

From eqn 6………………

$$T_c^* = \frac{T_{h2}^* + 82}{2.9} = 0.345T_{h2}^* + 28.3$$

From eqn 5………………. $T_{h1}^* = 170 - T_{h2}^*$

Substituting these expressions in Eqn 4 we have:

$$150(170 - T_{h2}^*) - 115T_{h2}^* + 150(0.345T_{h2}^* + 28.3) - (170 - T_{h2}^*)(0.345T_{h2}^* + 28.3) = 5250$$

Or $T_{h2}^{*2} - 706T_{h2}^* + 57055 = 0$

Therefore $T_{h2}^* = \dfrac{706 \pm \sqrt{706^2 - 4*57055}}{2} = 93 \, or \, 613$

T_{h2}^* cannot be 613 °C.

Therefore………. $T_{h2}^* = 93^oC$

Therefore……….. $T_{h1}^* = 170 - 93 = 77^oC$

And…………….. $T_c^* = 0.345*93 + 28.3 = 60.4^oC$

Therefore……………. $\varepsilon_1 = \varepsilon_2 = \dfrac{150 - 77}{150 - 33} = 0.635$

Therefore from chart, $NTU = 1.16$

Therefore…………….. $A_1 = \frac{1817 * 1.16}{850} = 2.48\, m^2 = A_2$

Therefore total area required to meet the heat load = 2.482*2= 4.92m^2

This is more than the 4.276 m^2 required in the one larger heat exchanger. In addition the cost per unit area is greater so that the most economical choice from the heat transfer point of view would be the single large heat exchanger.