INSTITUTE OF AERONAUTICAL ENGINEERING

(Autonomous)

Dundigal, Hyderabad -500 043

AERONAUTICAL ENGINEERING

COURSE LECTURE NOTES

Course Name	**HIGH SPEED AERODYNAMICS**
Course Code	AAE008
Programme	B.Tech
Semester	V
Course Coordinator	Mr. G. Staya Dileep, Assistant Professor, AE
Course Faculty	Mr. G. Staya Dileep, Assistant Professor, AE Ms D. Anitha, Assistant Professor, AE
Lecture Numbers	1-60
Topic Covered	All

COURSE OBJECTIVES (COs):

The course should enable the students to:	
I	Understand the effect of compressibility at high-speeds and the ability to make intelligent design decisions.
II	Explain the dynamics in subsonic, transonic and supersonic flow regimes in both internal and external geometries.
III	Analyze the airfoils at subsonic, transonic and supersonic flight conditions using the perturbed flow theory assumption.
IV	Formulate appropriate aerodynamic models to predict the forces and performance of realistic three dimensional configurations.

COURSE LEARNING OUTCOMES (CLOs):

Students, who complete the course, will have demonstrated the ability to do the following:

S. No	Description
AAE008.01	Demonstrate the concept of supersonic flow, how it is different from incompressible flow.
AAE008.02	Understand governing equations of supersonic flow in various form and thermodynamics properties.
AAE008.03	Describe the governing equations required for compressible flows.
AAE008.04	Illustrate the impact of supersonic flow in the presence of compression and expansion corner
AAE008.05	Demonstrate supersonic aircraft design and applications to aircrafts, supersonic wind tunnel, shock tubes.
AAE008.06	Understand the concepts of shock wave boundary layer interaction.

AAE008.07	Illustrate the concepts of quasi one dimensional flow for compressible flows
AAE008.08	Describe isentropic flow in nozzles, area Mach relations, choked flow, under and over expanded nozzles, slipstream line.
AAE008.09	Understand the impact of heat and Friction in duct flow and fanno flow
AAE008.10	Describe small perturbation equations for subsonic, transonic, supersonic and hypersonic flow
AAE008.11	Understand experimental characteristics of airfoils in compressible flow, supercritical airfoils and area rule.
AAE008.12	Explain supersonic nozzle design using method of characteristics.
AAE008.13	Illustrate working principle of subsonic wind tunnels, supersonic wind tunnels, shock tunnels
AAE008.14	Explain free-piston shock tunnel, detonation-driven shock tunnels, and expansion tubes and characteristic features, their operation and performance.
AAE008.15	Demonstrate flow visualization techniques for compressible flows.

SYLLABUS

Unit-I	INTRODUCTION TO COMPRESSIBLE FLOWS
Basic concepts: Introduction to compressible flow, brief review of thermodynamics and fluid mechanics, integral forms of conservation equations, differential conservation equations, continuum postulates, acoustic speed and Mach number, governing equations for compressible flows.	
Unit-II	**SHOCK AND EXPANSION WAVES**
Shocks and expansion waves: Development of governing equations for normal shock, stationery and moving normal shock waves, applications to aircrafts, supersonic wind tunnel, shock tubes, shock polars, supersonic pitot probes; oblique shocks, governing equations, reflection of shock, Prandtl-Meyer expansion flow, shock expansion method for flow over airfoil, introduction to shock wave boundary layer interaction.	
Unit-III	**DIMENSIONAL AND QUASI ONE DIMENSIONAL FLOW**
Quasi one dimensional flow: Isentropic flow in nozzles, area Mach relations, choked flow, under and over expanded nozzles, slip stream line. One dimensional flow: Flow in constant area duct with friction and heat transfer, Fanno flow and Rayleigh flow, flow tables and charts for Fanno flow and Rayleigh flow.	
Unit-IV	**APPLICATIONS OF COMPRESSIBLE FLOWS AND NUMERICAL TECHNIQUES**
Small perturbation equations for subsonic, transonic, supersonic and hypersonic flow; Experimental characteristics of airfoils in compressible flow, supercritical airfoils, area rule; Theory of characteristics, determination of the characteristic lines and compatibility equations, supersonic nozzle design using method of characteristics.	
Unit-V	**EXPERIMENTAL METHODS IN COMPRESSIBLE FLOWS**
Experimental methods: Subsonic wind tunnels, supersonic wind tunnels, shock tunnels, free-piston shock tunnel, detonation-driven shock tunnels, and expansion tubes and characteristic features, their operation and performance, flow visualization techniques for compressible flows	
Text Books:	
1. John D. Anderson, ―Modern Compressible flow with historical perspective‖, McGraw-Hill Education, 3rd Edition, 2002. 2. John D. Anderson, ―Fundamentals of Aerodynamics‖, McGraw-Hill Education, 6th Edition, 2016.	
Reference Books:	
1. Ascher H. Shapiro, ―The Dynamics and Thermodynamics of Compressible Fluid Flow‖ John Wiley & Sons; Volume 1 ed. Edition, 1977. 2. RadhakrishnanEthirajan, ―Gas Dynamics‖, John Wiley & Sons, 2nd edition 2010. 3. H W Liepmann and A Roshko, ―Elements of Gas Dynamics‖, John Wiley & Sons, 4th edition, 2003.	

UNIT - I

INTRODUCTION TO COMPRESSIBLE FLOW

1.1 Introduction

Compressible flow is often called as variable density flow. For the flow of all liquids and for the flow of gases under certain conditions, the density changes are so small that assumption of constant density remains valid.Let us consider a small element of fluid of volume. The pressure exerted on the element by the neighboring fluid is p. If the pressure is now increased by an amount dp, the volume of the element will correspondingly be reduced by the amount d .The compressibility of the fluid K is thus defined as

$$\boldsymbol{K = \frac{1}{\rho}.\frac{d\rho}{dp}}$$

However, when a gas is compressed, its temperature increases. Therefore, the above mentioned definition of compressibility is not complete unless temperature condition is specified. When the temperature is maintained at a constant level, the isothermal compressibility is defined as

$$K_\tau = -\frac{1}{\forall}\left(\frac{d\,\forall}{dP}\right)_\tau$$

Compressibility is a property of fluids. Liquids have very low value of compressibility (for ex. compressibility of water is 5×10^{-10} m^2/N at 1 atm under isothermal condition), while gases have very high compressibility (for ex. compressibility of air is 10^{-5} m^2/N at 1 atm under isothermal condition).

Categories of flow for external aerodynamics

Ma < 0.3: incompressible flow; change in density is negligible.

0.3< Ma < 0.8: subsonic flow; density changes are significant but shock waves do not appear.

0.8< Ma < 1.2: transonic flow; shock waves appear and divide the subsonic and supersonic regions of the flow. Transonic flow is characterized by mixed regions of locally subsonic and supersonic flow

1.2 < Ma < 3.0: supersonic flow; flow field everywhere is above acoustic speed. Shock waves appear and across the shock wave, the streamline changes direction discontinuously.

3.0<Ma: hypersonic flow; where the temperature, pressure and density of the flow increase almost explosively across the shock wave.

1.2 Brief Review of Thermodynamics and Fluid Mechanics

1.2.1 Thermodynamics Concepts:

1.2.1.1 System

A thermodynamic system is defined as a definite quantity of matter or a region in space upon which attention is focused in the analysis of a problem. We may want to study a quantity of matter contained with in a closed rigid walled chambers, or we may want to consider something such as gas pipeline through which the matter flows. The composition of the matter inside the system may be fixed or may change through chemical and nuclear reactions. A system may be arbitrarily defined. It becomes important when exchange of energy between the system and the everything else outside the system is considered. The judgment on the energetics of this exchange is very important.

1.2.1.2 Surroundings

Everything external to the system is surroundings. The system is distinguished from its surroundings by a specified boundary which may be at rest or in motion. The interactions between a system and its surroundings, which take place across the boundary, play an important role in thermodynamics. A system and its surroundings together comprise a universe.

1.2.1.3 Types of systems

Two types of systems can be distinguished. These are referred to, respectively, as closed systems and open systems or control volumes. A closed system or a control mass refers to a fixed quantity of matter, whereas a control volume is a region in space through which mass may flow. A special type of closed system that does not interact with its surroundings is called an Isolated system.

Two types of exchange can occur between the system and its surroundings:

1. energy exchange (heat or work)and
2. Exchange of matter (movement of molecules across the boundary of the system and surroundings).

Based on the types of exchange, one can define

- isolated systems: no exchange of matter andenergy
- closed systems: no exchange of matter but some exchange ofenergy
- open systems: exchange of both matter andenergy

If the boundary does not allow heat (energy) exchange to take place it is called adiabatic boundary otherwise it is diathermalboundary.

1.2.2 Laws of thermodynamics

1.2.2.1 Zeroth law of thermodynamics: This law states that ‘when system A is in thermal equilibrium with system B and system B is separately in thermal equilibrium with system C then system A and C are also in thermalequilibrium’.

This law portrays temperature as a property of the system and gives basis of temperature measurement.

1.2.2.2 First law of thermodynamics: It states the energy conservation principle, ‘energy can neither be created nor be destroyed but one form of the energy can be converted toother’.

Implementationof first law for a thermodynamic process defines ‘internal energy’ as a property of the system. According to this law for a closed system, when some amount of heat is supplied to the system, part of it is used to convert into work and rest is stored in the system in the form of internal energy. For the open system, heat supplied splits into enthalpy change and work done by the system.

First law of thermodynamics is thus associated with a corollary which states that there can be no machine (Perpetual Motion Machine of first kind) which can produce continuous work output without having any heat interaction with the surrounding.

1.2.2.3 Second law of thermodynamics: There are two statements of second law of thermodynamics.

1.2.2.4 Clausius statement: It is impossible to construct a system which will operate in a cycle, transfers heat from the low temperature reservoir (or object) to the high temperature reservoir (or object) without any external effect or work interaction with surrounding.

1.2.2.5 Kelvin Plank statement: It is impossible to construct a heat engine which produces work in a cycle while interacting with only one reservoir.

Kelvin Plank statement necessarily states that Perpetual Motion Machine of second kind is impossible. These statements introduce a new property termed as entropy which is the measure of disorder.

There are some corollaries of second law of thermodynamics

1. All the reversible heat engines working between same temperature limits have same efficiency.
2. Irreversible heat engine working in the same temperature limit as the reversible heat engine will have lower efficiency.

The second law of thermodynamics leads to an inequality called as Clausius inequality which is valid for general process happening in any system. This law grades the energies according to which work is high grade energy and heat is low grade energy. Therefore second law of thermodynamics directs that low grade energy can not be completely converted into high grade energy. This law also states the possibility of having a process in reality unlike the first law which can at most quantify the effects of the process if process becomes a reality. Hence according to second law only those processes are possible in which entropy of the universe increase or at least remain constant. This is called as entropy increase principle which also states that entropy of an isolated system always increases or remains constant.

1.2.2.6 Third law of thermodynamics: 'Entropy of pure substance in thermodynamic equilibrium is zero at absolute zero temperature'. Therefore according to this law, there exists zero Kelvin temperature on the temperature scale but it is difficult to achieve the same.

1.2.2.7 Isentropic relations:

Isentropic relations are the relations between thermodynamic properties if the system undergoes isentropic process.

Consider a closed system interacting dQ amount of energy with the surrounding. If dU is the change in internal energy of the system and pdV is work done by the system against pressure p due to volume change dV. According to First Law of Thermodynamics we know that,

$$dQ = dU + pdV$$

From Second law of Thermodynamics,

$$\frac{dQ}{T} = dS$$

$$dQ = TdS$$

Here dS is the entropy change due to reversible heat interaction dQ.

Therefore, combining First and Second Laws of Thermodynamics,

$$TdS = dU + pdV$$

However, we know that, if H is enthalpy of the system then,

$$H = U + pV$$

$$dH = dU + pdV + VdP$$

$$dH - Vdp = dU + pdV$$

Combining equations we get,

$$TdS = dH – Vdp$$

For system with unit mass of matter, above equation can be written as,

$$Tds = dh - vdp$$

For the special case, if system undergoes reversible adiabatic or isentropic process then, entropy change of the system (dS) is zero.

$$C_p dT = v dp$$

From ideal gas relation, v = RT/P, above equations becomes

$$C_P\,dT = \frac{RT}{P}\,dp$$

$$\frac{C_P\,dT}{RT} = \frac{dp}{P}$$

We know the relations between specific heats as

$$C_P - C_v = R$$

$$C_P - \frac{C_P}{\gamma} = R$$

$$C_P = \frac{\gamma R}{\gamma - 1}$$

$$C_v = \frac{R}{\gamma - 1}$$

Substituting above expression for Cp in equation (1.4), we get

$$\frac{\gamma}{\gamma - 1}\frac{dT}{T} = \frac{dp}{P}$$

Integrating above equation from state 1 to state 2 for the isentropic process of the system, we get,

$$\frac{\gamma}{\gamma - 1}\,ln\left(\frac{T_2}{T_1}\right) = \ln\left(\frac{P_2}{P_1}\right)$$

$$\left(\frac{P_2}{P_1}\right) = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$$

Since the system is undergoing adiabatic process from state 1 to state 2,

$$\frac{P}{\rho^{\gamma}} = const$$

$$\left(\frac{\rho_1}{\rho_2}\right) = \left(\frac{T_2}{T_1}\right)^{\frac{1}{\gamma-1}}$$

Here relations are called as isentropic equations.

1.3 Important Fluid Properties

1.3.1 Continuum:

Fluid matter is made up of molecules or atoms. However for most of the fluid dynamic calculations discrete presence of these elementary objects is neglected and fluid matter is assumed to be continuous for prediction of various fluid flow phenomenon or for measurement of fluid properties. If a pressure probe concentrates to a very small area, of molecular dimensions, in the fluid flow, then fluctuations can be observed in the measurements due to random presence of molecules. Therefore any measurement probe used for experiments or fluid property estimation should not have very small probe area in order to incur molecular fluctuations. If average number of molecules colliding the probing area over the time is very large then these fluctuations die out and we get a constant macroscopic value of the property. For these conditions we can comfortably make the assumptions of continuum of matter.

Continuous presence of the matter is called as continuum. This is the assumption which we will be using for most of the derivations of this course. This assumption helps us for calculation of gradient and flow variables smoothly. The governing non-dimensional parameter for prediction of continuum is Knudsen number which is defined as the ratio of mean free path to the characteristic length of the object. Here mean free path should be understood as the mean distance traveled by a molecule between two successive collisions with other molecules. Thus calculated Knudsen number should be close to zero or below 0.3 for use of most of the relations or governing equations. Mean free path of standard atmosphere is 5×10^{-8} m due to very high density of air near the earth’s surface. Therefore Knudsen number remains close to zero for general fluid dynamic situations. Validation of continuum assumption and in turn the usage of governing equations in their standard form remain intact till the altitude of around 90 km from earth surface where Knudsen number is below 0.3.

Above the first critical value of Knudsen number (0.3), usefulness of governing equations in their standard form is intact however the nature of boundary conditions changes due to existence of velocity and temperature slip on the wall. From 90 Km till 150 km from earth surface, density becomes low as a effect of which fluid velocity and temperature at the surface do not remain in equilibrium with the surface. Therefore for the Knudsen number range 0.3 to 1, which is also called as transitional regime, slip wall boundary conditions should be used along with the usual governing equations based on continuum assumption

Presence of free molecular flow can be assured If Knudsen number crosses 1. Kinetic theory of gases and related equations are generally prescribed if the Knudsen number crosses its second critical value. This conditions persists beyond,150 km from earth’s surface where density of air is very low.

1.3.2 Transport phenomenon:

Diffusion of mass, momentum and heat always take place from the region of higher concentration to the region of lower concentration.

Concentration gradient of mass leads to mass transfer. Fick's law gives the quantification of mass transport through the linear relation between massflux and gradient of concentration. The proportionality constant (mass diffusion coefficient) depends on gases involved in the diffusion and their thermodynamic state. According to this law, diffusion of nitrogen in oxygen is different from diffusion of nitrogen in methane. Diffusion of nitrogen in oxygen also depends on pressure and temperature conditions. Sainted stick is the example of concentration based mass diffusion.

Temperature variation leads to heat transfer. Fourier's law gives the relation between heatflux and gradient for temperature. The proportionality constant is the material property (thermal conductivity) and depends mainly on temperature of the material. Heat transfer taking place in boiler walls, fins etc are the examples of conduction heat transfer.

Newton's law of momentum diffusion states that momentum flux is proportional to the gradient of velocity. The proportionality constant of this relation is the fluid property (viscosity) which depends mainly on temperature of the fluid. Presence of boundary layer near the wall, in case of fluid flow over the same, is the example of momentum diffusion.

Transport equations have flux equated with gradient where only first derivatives appear in gradient terms. The reason behind this is that, diffusion is the microscopic phenomenon hence it does not account for curvature involved with higher derivatives. Along with this there is no higher power term in this equation since these equations are valid for small concentration gradients but it has been seen that these equations hold good for higher concentrations also.

1.3.3. Compressibility of fluid and flow:

If application of pressure changes volume or density of the fluid then fluid is said to be compressible.

$$compressibility(\tau) = -\frac{1}{\vartheta}\frac{\partial\vartheta}{\partial P} = \frac{1}{\rho}\frac{\partial\rho}{\partial P}$$

Compressibility is thus inverse of bulk modulus. Hence compressibility can be defined as the incurred volumetric strain for unit change in pressure. Negative sign in the above expression is the fact that volume decreases with increase in applied pressure. For example, air is more compressible than water. Since definition of compressibility involves change in volume due to change in pressure, hence compressibility can be isothermal, where volume change takes place at constant temperature or isentropic where volume change takes place at constant entropy.

$$Isothermal\ compressibility(\tau) = -\frac{1}{\vartheta}\left(\frac{\partial\vartheta}{\partial P}\right)_{\tau=constant} = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial P}\right)_{\tau=constant}$$

$$Isentropic\ compressibility(\tau) = -\frac{1}{\vartheta}\left(\frac{\partial\vartheta}{\partial P}\right)_{s=constant} = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial P}\right)_{s=constant}$$

Also, for isothermal compressibility we know,

$$\tau = -\frac{1}{\vartheta}\left(\frac{\partial\vartheta}{\partial P}\right)_{\tau=constant}$$

Since $P\vartheta = RT$ for ideal gas, we have,

$$\left(\frac{\partial\vartheta}{\partial P}\right)_{\tau=constant} = -\frac{\vartheta}{P}$$

Hence, isothermal compressibility is

$$\tau=1/\rho$$

For isentropic compressibility we know,

$$\tau = -\frac{1}{\vartheta}\left(\frac{\partial \vartheta}{\partial P}\right)_{s=constant}$$

Since, Pv^{γ} = constant for isentropic process for an ideal gas, we have,

$$\left(\frac{\partial \vartheta}{\partial P}\right)_{s=constant} = -\frac{\vartheta}{\gamma P}$$

Hence, isentropic compressibility is

$$\Gamma = 1/\Upsilon P$$

Comparing equations we can see that, isothermal compressibility is always higher than isentropic compressibility of gas since specific heat ratio is always greater than one. This inturn means that it is simpler to change the volume of a gas isothermally than isentropically. In otherwards, it means that we need lesser amount of pressure to bring a particular amount of change in volume during isothermal process than during isentropic process.

Fluid flow is said to be compressible if density of the fluid changes roughly 5% of its original density during its flow.

$$d\rho = \rho \times \tau \times dP$$

From this relation it is very clear that, percentage change in density of fluid flow will be higher if either compressibility of the fluid is higher or pressure difference is high. Hence, compressible fluids exposed to smaller pressure difference situations can exhibit incompressible flow and at the same time incompressible fluids exposed to high pressure difference situations can exhibit compressible flow.

1.3.4Important Properties of Compressible Flows

The simple definition of compressible flow is the variable density flows. In general, the density of gases can vary either by changes in pressure and temperature. In fact, all the high speed flows are associated with significant pressure changes. So, let us recall the following fluid properties important for compressible flows;

Bulk modulus (E_v): It is the property of that fluid that represents the variation of density(ρ) with pressure(P)at constant temperature(T). Mathematically, it is represented as,

$$E_v = \forall\left(\frac{\partial p}{\partial \forall}\right)_{\tau} = \rho\left(\frac{\partial \rho}{\partial T}\right)_{\tau}$$

In terms of finite changes, it is approximated as,

$$E_v = \frac{(\Delta\forall/\forall)}{\Delta T} = -\frac{(\Delta\rho/p)}{\Delta T}$$

Coefficient of volume expansion (β): It is the property of that fluid that represents the variation of density with temperature at constant pressure. Mathematically, it is represented as,

$$\beta = \forall\left(\frac{\partial p}{\partial \forall}\right)_{\tau} = \rho\left(\frac{\partial \rho}{\partial T}\right)_{\tau}$$

Compressibility (K): It is defined as the fractional change in the density of the fluid element per unit change in pressure. One can write the expression for Kas follows;

$$\boldsymbol{K = \frac{1}{\rho}.\frac{d\rho}{dp}} \quad \mathbf{d\rho = \rho\, K\, dp}$$

In order to be more precise, the compression process for a gas involves increase in temperature depending on the amount of heat added or taken away from the gas. If the temperature of the gas remains constant, the definition is refined as *isothermal compressibility (K_T)*. On the other hand, when no heat is added/taken away from the gases and in the absence of any dissipative mechanisms, the compression takes place isentropically. It is then, called as *isentropic compressibility(K_s)*.

$$K_T = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_T, K_s = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_s$$

1.3.5 Viscosity (μ):

- Viscosity is a fluid property whose effect is understood when the fluid is in motion.
- In a flow of fluid, when the fluid elements move with different velocities, each element will feel some resistance due to fluid friction within the elements.
- Therefore, shear stresses can be identified between the fluid elements with different velocities.
- The relationship between the shear stress and the velocity field was given by Sir Isaac Newton.
- Newton postulated that τ is proportional to the quantity Δu/ Δy where Δy is the distance of separation of the two layers and Δu is the difference in their velocities.
- In the limiting case of, Δu / Δy equals du/dy, the velocity gradient at a point in a direction perpendicular to the direction of the motion of the layer.

According to Newton τ and du/dy bears the relation

$$\tau = \mu\frac{du}{dy}$$

Where, the constant of proportionality μ is known as the coefficient of viscosity or simply viscosity which is a property of the fluid and depends on its state. Sign of τ depends upon the sign of du/dy. For the profile, du/dy is positive everywhere and hence,τ is positive. Both the velocity and stress are considered positive in the positive direction of the coordinate parallel to them.

Equation

$$\tau = \mu\frac{du}{dy}$$

defining the viscosity of a fluid, is known as Newton's law of viscosity. Common fluids, viz. water, air, mercury obey Newton's law of viscosity and are known as Newtonian fluids.

Other classes of fluids, viz. paints, different polymer solution, blood do not obey the typical linear relationship, of τ and du/dy and are known as non-Newtonian fluids.

1.4 Integral forms of conservation equations

1.4.1 Conservation of Mass or Continuity Equation (Integral Form)

Consider the fluid domain. Our aim is to derive the mass conservation equation using this fluid domain of arbitrary shape.

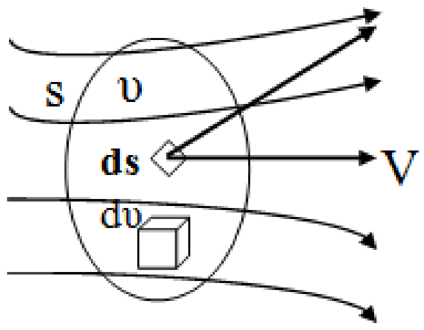


Fig.1.1 Finite Control Volume fixed in space

If there is no mass source in the control volume, we can equate the rate of change of mass inside the control volume with the difference in influx and outflux of mass. Consider ρ as the density of the flow which is function of space coordinates and time (ρ = ρ(x, y, z, t)). Let V be velocity vector of the flow which is also function of space coordinates and time and has u, v and was three components aligned to the coordinate axes x, y and z respectively. Consider elemental surface area (dS) of the control volume. This area along with the unit normal.

Total mass inside the control volume can be found by summing the mass of elemental volumes (dv) occupying the complete finite volume. We know that ρdυ is the mass of an elemental volume, hence total mass inside the control volume can be written as

Total mass in the control volume (CV) = $\oiiint_v \rho dv$

Hence rate of change of this mass is,

Rate of change of mass in CV= $\frac{\partial}{\partial t}\oiiint_v \rho dv$

Thus the time rate of decrease of mass inside the CV is $-\frac{\partial}{\partial t}\oiiint_v \rho dv$

The mass flow rate of any moving fluid across any fixed surface is equal to the product of density, area of surface and component of velocity normal to the surface. Therefore, the elemental mass flow across the area ds is expressed as

$$\rho V_n dS = \rho V.dS$$

Where V_n is the component of velocity normal to the surface. Thus the net mass flow out of the entire volume through the control surface S is summation of mass flow rates through all elemental areas dS of control surface S. Hence net flux through the control surface is

$$\oiint_{s} \rho V \, . ds$$

Here mass flux entering the CV is negative and mass flux leaving the CV is positive. Therefore equation gives the net mass leaving the CV which should be equal to rate of decrease of mass flux in the CV. Thus, equating equations, we get,

$$\oiint_{s} \rho V \, . ds = -\frac{\partial}{\partial t} \oiiint_{v} \rho dv$$

Or

$$\frac{\partial}{\partial t} \oiiint_{v} \rho dv + \oiint_{s} \rho V \, . ds = 0$$

Equation is called as integral form of mass conservation equation or continuity equation.

1.4.2 Conservation of Momentum (Integral Form)

Momentum equation is based on Newton's second law. This can be expressed as

$$\frac{d}{dt}(mV) = F$$

Consider the same CV shown in Figure for deriving the momentum conservation equation. Right hand side of equation is the summation of all forces like surface forces and body forces. Let Fb and P be the net body force per unit mass and pressure exerted on control surface respectively. The body force on the elemental volume dυ is therefore

$$\rho F_b dv$$

and the total body force exerted on the fluid in the control volume is

$$\oiiint_{v} \rho \, F_b dv$$

The elemental surface force due to pressure acting on the element of area ds is -Pds
where the negative sign indicates that the force is in the direction opposite of ds. Therefore, total pressure force is the summation of the elemental forces over the entire control surface expressed as

$$-\oiint_{s} Pds$$

Let $F_{viscous}$ be the total viscous force exerted on the control surface. Hence, the resultant force experienced by the fluid as it is sweeping through the fixed control volume is given by the sum of total pressure force, body force and viscous force i.e,

$$F = -\oiint_{s} Pds + \oiiint_{v} \rho \, F_b dv + F_{viscous}$$

Consider the left side of the Equation. This term gives the time rate of change of momentum following a fixed fluid element or substantial derivative of the momentum. Hence we can evaluate this using equation. Therefore we will have to evaluate sum of net flow of momentum leaving the control volume through the control surface s and time rate of change of momentum due to fluctuations of flow properties inside the control volume.

The mass flow across the elemental area ds is (ρV.ds).Therefore, the flow of momentum per second across ds is

$$(\rho V.ds)V$$

The net flow of momentum out of the control volume through s is the summation of the above elemental contributions, namely,

$$\oiint_s (\rho V.ds)V$$

The momentum of the fluid in the elemental volume dυ is (ρdu)V. The momentum contained at any instant inside the control volume is therefore

$$\oiiint_v \rho\, V dv$$

and its time rate of change due to unsteady flow fluctuation is

$$\frac{\partial}{\partial t}\oiiint_v \rho\, V dv$$

Combining Equations to obtain the left hand side of equation, we get

$$\frac{d}{dt}(mV) = \frac{\partial}{\partial t}\oiiint_v \rho V dv + \oiint_s (\rho V.ds)V$$

Thus, substituting Equations we have

$$\frac{\partial}{\partial t}\oiiint_v \rho V dv + \oiint_s (\rho V.ds)V = -\oiint_s P ds + \oiiint_v \rho F_b dv + F_{viscous}$$

This is the momentum equation in integral form. It is a general equation, applies to the unsteady, three-dimensional flow of any fluid, compressible or incompressible, viscous or inviscid.

1.4.3 Conservation of Energy (Integral Form)

The law of conservation of energy states that "energy can neither be created nor destroyed; it can only change its form". Consider the CV as the thermodynamic system. Let amount of heat δq be added to the system from the surrounding. Also let δw be the work done on the system by the surroundings. Both heat and work are the forms of energy. Addition of any form of the energy to the system, changes the amount of internal energy of the system. Let's denote this change of internal energy by *de*.

From the principle of energy conservation

$$\delta q + \delta w = de$$

Therefore in terms of rate of change the above equation changes to

$$\frac{\delta q}{\delta t}+\frac{\delta w}{\delta t}=\frac{\delta e}{\delta t}$$

If the system to be considered as a open system then the change will take place for all the forms of energies owed by the system, like internal energy and kinetic energy. Hence right hand side of the above equation is just the representation of change in energy content of the system.

Let's first concentrate on first term on left hand side of above equation and evaluate the same. There are various sources of heat addition in the system some of which are external heat addition from the surrounding, heat addition by chemical energy release, heat addition by radiation etc. Let's consider some of those sources of heat addition and derive the energy equation. If q is the amount of heat added per unit mass, then rate of heat addition for any elemental volume will be q(ρdv). Summing over the complete control volume gives us total external volumetric heat addition. Heat might get added by viscous effects like conduction. Hence net rate of heat addition can be,

$$\frac{\delta q}{\delta t}=\oiiint_{v} q\,\rho\,dv+\dot{Q}_{viscous}$$

Consider the second term on left hand side of equation and evaluate the same. There are various ways by which work transfer can be achieved to or from the system. Main source is the surface forces like pressure, body force etc.

Considering the elemental area ds of the control surface. The pressure force on this elemental area is -Pds and the rate of work done on the fluid passing through ds with velocity V is *(-Pds).V.* Hence, summing over the complete control surface, rate of work done due to pressure force is,

$$-\oiint_{s}(Pds).V$$

In addition, consider an elemental volume dυ inside the control volume. The rate of work done on the elemental volume due to body force is (ρF_bdu).V. Here F_b is the body force per unit mass. Summing over the complete control volume, we obtain, rate of work done on fluid inside υ due to body forces is

$$\oiiint_{v}(\rho F_b dv).V$$

If the flow is viscous, the shear stress on the control surface will also do work on the fluid as it passes across the surface. Let $\dot{W}_{viscous}$ denote the work done due to the shear stress. Therefore, the total work done on the fluid inside the control volume is the sum of terms and $\dot{W}_{viscous}$ that is

$$\frac{\delta w}{\delta t}=-\oiint_{s} PV.ds+\oiiint_{v}\rho(F_b.V)dv+\dot{W}_{viscous}$$

Now consider the right hand side of equation and evaluate the rate of internal energy change of the fluid. However since we are considering the open system we will have to consider the change in internal energy as well as the change in kinetic energy. Therefore right hand side of equation should deal with total energy (sum of internal and kinetic energies) of the system. Let, *e* be the internal energy per unit mass of the system and kinetic energy per unit mass due to local velocity V be *V2/2*. Hence the rate of change of total energy is

$$\frac{\partial}{\partial t}\oiiint_v \rho(e+\frac{V^2}{2})dv$$

Total energy in the control volume might also change due to influx and outflux of the fluid. The elemental mass flow across ds is (ρV.ds). Therefore the elemental flow of total energy across the ds is $(\rho V.ds)(e+V^2/2)$. Summing over the complete control surface, we obtain net rate of flow of total energy across control surface as,

$$\oiint_s (\rho V.ds)e+\frac{V^2}{2}$$

Hence the net energy change of the control volume is,

$$\frac{de}{dt}=\frac{\partial}{\partial t}\oiiint_v \rho(e+\frac{V^2}{2})dv+\oiint_s (\rho V.ds)\left(e+\frac{V^2}{2}\right)$$

Thus, substituting Equations, we have

$$\oiiint_v q\rho dv+\dot{Q}_{viscous}-\oiint_s PV.ds+\oiiint_v \rho(F_b.V)dv+\dot{W}_{viscous}=\frac{\partial}{\partial t}\oiiint_v \rho(e+\frac{V^2}{2})dv$$

$$+\oiint_s \rho\left(e+\frac{V^2}{2}\right)V.ds$$

This is the energy equation in the integral form. It is essentially the first law thermodynamics applied to fluid flow or open system.

1.5 Differential Conservation Equations

1.5.1 Conservation of Mass or Continuity Equation (Differential Form)

Using Gauss divergence theorem, we can express the right hand side term of Equation as

$$\oiint_s (\rho V).ds=\oiiint_v \nabla.(\rho V)dv$$

Substituting Equation, we obtain

$$\oiiint_v \frac{\partial \rho}{\partial t}dv+\oiiint_v \nabla.(\rho V)dv=0$$

or,

$$\oiiint_v \left[\frac{\partial \rho}{\partial t}+\nabla.(\rho V)\right]dv=0$$

For infinitely small elemental volumes we can always write equation as,

$$\frac{\partial \rho}{\partial t}+\nabla.(\rho V)=0$$

This is the continuity equation in the form of a partial differential equation. This is the conservation form of equation. For unsteady, compressible and three dimensional flows the Equation can be expressed as

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0$$

Equation can be re-written as

$$\frac{\partial \rho}{\partial t} + \rho \nabla . V + V . \nabla \rho = 0$$

or,

$$\frac{\partial \rho}{\partial t} + V . \nabla \rho + \rho \nabla . V = 0$$

However, the sum of the first two terms of the Equation is the substantial derivative of ρ. Thus, from Equation,

$$\frac{D\rho}{Dt} + \rho \nabla . V = 0$$

This is the form of continuity equation written in terms of the substantial derivative. This is also called as the non-conservative form of mass conservation or continuity equation.

1.5.2 Conservation of Momentum (Differential Form)

The right hand side of momentum conservation equation involves surface integral for pressure. This term can be re-written as,

$$-\oiint_s P ds = -\iiint_v \nabla P dv$$

Also, because the control volume is fixed, the time derivative in Equation can be placed inside the integral. Hence, Equation can be written as

$$\iiint_v \frac{\partial (\rho V)}{\partial t} dv + \oiint_s (\rho V . ds) V = -\iiint_v \nabla P dv + \iiint_v \rho F_b dv + F_{viscous}$$

This equation is a vector equation. It would be convenient to decompose this equation in terms of components. Since we know,

V = ui + vj + wk

Therefore, the x component of Equation is

$$\iiint_v \frac{\partial (\rho u)}{\partial t} dv + \oiint_s (\rho V . ds) u = -\iiint_v \frac{\partial P}{\partial x} dv + \iiint_v \rho F_{b,x} dv + F_{viscous,x}$$

Applying Gauss divergence theorem to the flux term of above equation, we get

$$\oiint_s (\rho V . ds) u = \oiint_s (\rho u V) . ds = \iiint_v \nabla . (\rho u V) dv$$

Substituting equation, we have

$$\iiint_v \frac{\partial (\rho u)}{\partial t} dv + \iiint_v \nabla . (\rho u V) dv = -\iiint_v \frac{\partial P}{\partial x} dv + \iiint_v \rho F_{b,x} dv + F_{viscous,x}$$

Or

$$\iiint_v \frac{\partial (\rho u)}{\partial t} dv + \iiint_v \nabla . (\rho u V) dv + \iiint_v \frac{\partial P}{\partial x} dv - \iiint_v \rho F_{b,x} dv - F_{viscous,x} = 0$$

Or

$$\iiint_v \left[\frac{\partial (\rho u)}{\partial t} + \nabla . (\rho u V) + \frac{\partial P}{\partial x} - \rho F_{b,x} - F'_{viscous,x} \right] dv = 0$$

where $F'_{viscous,x}$ is the representation of x component of the viscous shear stresses acting on the control volume. For the elemental volumes,

$$\frac{\partial(\rho u)}{\partial t} + \nabla.(\rho u V) + \frac{\partial P}{\partial x} - \rho F_{b,x} - F'_{viscous,x} = 0$$

Or

$$\frac{\partial(\rho u)}{\partial t} + \nabla.(\rho u V) = -\frac{\partial P}{\partial x} + \rho F_{b,x} + F'_{viscous,x}$$

This is the x component of the momentum equation in differential form. Similarly we can expressed the y and z component of the momentum equation respectively as

$$\frac{\partial(\rho v)}{\partial t} + \nabla.(\rho v V) = -\frac{\partial P}{\partial y} + \rho F_{b,y} + F'_{viscous,y}$$

or

$$\frac{\partial(\rho w)}{\partial t} + \nabla.(\rho w V) = -\frac{\partial P}{\partial z} + \rho F_{b,z} + F'_{viscous,z}$$

where the subscripts *y* and *z* on F_b and $F'_{viscous}$ denote the y and z component of the body and viscous forces respectively.

Consider the x component of momentum equation given in the form of Equation. The first term in the left hand side of this equation can be expanded as

$$\frac{\partial(\rho u)}{\partial t} = \rho\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t}$$

and second term of the equation can be expanded as,

$$\nabla.(\rho u V) = \nabla.[u(\rho V)] = u\nabla.(\rho V) + (\rho V).\nabla u$$

Substituting Equations, we obtain

$$\rho\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t} + u\nabla.(\rho V) + (\rho V).\nabla u = -\frac{\partial P}{\partial x} + \rho F_{b,x} + F'_{viscous,x}$$

Or

$$\rho\frac{\partial u}{\partial t} + u\left[\frac{\partial \rho}{\partial t} + \nabla.(\rho V)\right] + (\rho V).\nabla u = -\frac{\partial P}{\partial x} + \rho F_{b,x} + F'_{viscous,x}$$

Examine the two terms inside the square brackets in the above equation. They represent the continuity equation. Hence the sum inside the square bracket is zero. Therefore, equation becomes

$$\rho\frac{\partial u}{\partial t} + (\rho V).\nabla u = -\frac{\partial P}{\partial x} + \rho F_{b,x} + F'_{viscous,x}$$

Or

$$\rho\left(\frac{\partial u}{\partial t} + V.\nabla u\right) = -\frac{\partial P}{\partial x} + \rho F_{b,x} + F'_{viscous,x}$$

Examine the two terms inside the parentheses of above equation. They represent the substantial derivative of x-component of velocity, u. Hence, Equation becomes

$$\rho \frac{Du}{Dt} = -\frac{\partial \mathrm{P}}{\partial \mathrm{x}} + \rho F_{b,x} + F'_{viscous,x}$$

In a similar manner, y and z momentum equations are given respectively as below,

$$\rho \frac{Dv}{Dt} = -\frac{\partial \mathrm{P}}{\partial \mathrm{y}} + \rho F_{b,y} + F'_{viscous,y}$$

And

$$\rho \frac{Dw}{Dt} = -\frac{\partial \mathrm{P}}{\partial \mathrm{z}} + \rho F_{b,z} + F'_{viscous,z}$$

Equations are the x, y and z components of the momentum equation written in terms of the substantial derivative. The momentum equation in the vector form can be written as

$$\rho \frac{DV}{Dt} = -\nabla \mathrm{P} + \rho F_{b,z} + F'_{viscous}$$

1.5.3 Conservation of Energy (Differential Form)

As the control volume is fixed, the time derivative in this equation can be taken inside the integral. Hence, this equation can be written as

$$\oiiint_v q\rho dv + \dot{Q}_{viscous} - \oiint_s PV.ds + \oiiint_v \rho(F_b.V)dv + \dot{W}_{viscous} = \frac{\partial}{\partial t}\oiiint_v \rho(e + \frac{V^2}{2})dv + \oiint_s \rho\left(e + \frac{V^2}{2}\right)V.ds$$

$$\oiiint_v \frac{\partial}{\partial t}\left[\rho(e + \frac{V^2}{2}\right] dv + \oiint_s \rho\left(e + \frac{V^2}{2}\right)V.ds - \oiiint_v q\rho dv + \oiint_s PV.ds - \oiiint_v \rho(F_b.V)dv - \dot{Q}_{viscous} - \dot{W}_{viscous} = 0$$

Applying the divergence theorem to the surface integrals given in the above equation, we get

$$\oiint_s \rho\left(e + \frac{V^2}{2}\right)V.ds = \oiiint_v \nabla.\left[\rho(e + \frac{V^2}{2}\right] dv$$

And

$$\oiint_s PV.ds = \oiiint_v \nabla.(PV)dv$$

Substituting Equations we get,

$$\oiiint_v \frac{\partial}{\partial t}\left[\rho(e+\frac{V^2}{2})\right]dv + \oiiint_v \nabla.\left[\rho(e+\frac{V^2}{2})\right]dv$$

$$-\oiiint_v q\rho dv + \oiint_s PV.ds - \oiiint_v \rho(F_b.V)dv - \dot{Q}_{viscous} - \dot{W}_{viscous} = 0$$

or

$$\oiiint_v \left[\frac{\partial}{\partial t}\left\{\rho\left(e+\frac{V^2}{2}\right)\right\} + \nabla.\left\{\rho\left(e+\frac{V^2}{2}\right)V\right\} - q\rho + \nabla.(PV) - \rho(F_b.V) - \dot{Q}_{viscous} - \dot{W}_{viscous}\right]dv = 0$$

where $\dot{W}_{viscous}$ and $\dot{Q}_{viscous}$ give the representation of viscous work and heat transfer. We know that,

$$\frac{\partial}{\partial t}\left\{\rho\left(e+\frac{V^2}{2}\right)\right\} + \nabla.\left\{\rho\left(e+\frac{V^2}{2}\right)V\right\} - q\rho + \nabla.(PV) - \rho(F_b.V) - \dot{Q}_{viscous} - \dot{W}_{viscous} = 0$$

or

$$\frac{\partial}{\partial t}\left\{\rho\left(e+\frac{V^2}{2}\right)\right\} + \nabla.\left\{\rho\left(e+\frac{V^2}{2}\right)V\right\} = q\rho - \nabla.(PV) + \rho(F_b.V) + \dot{Q}_{viscous} + \dot{W}_{viscous}$$

This is the energy equation in differential form. This equation can also be written as,

$$\rho\frac{D}{Dt}\left(e+\frac{V^2}{2}\right) = q\rho - \nabla.(PV) + \rho(F_b.V) + \dot{Q}_{viscous} + \dot{W}_{viscous}$$

This is the form of energy equation written in terms of the substantial derivative.

1.6 CONTINUUM POSTULATES

The concept of continuum is a kind of idealization of the continuous description of matter where the properties of the matter are considered as continuous functions of space variables. Although any matter is composed of several molecules, the concept of continuum assumes a continuous distribution of mass within the matter or system with no empty space, instead of the actual conglomeration of separate molecules.

Describing a fluid flow quantitatively makes it necessary to assume that flow variables (pressure, velocity etc.) and fluid properties vary continuously from one point to another. Mathematical descriptions of flow on this basis have proved to be reliable and treatment of fluid medium as a continuum has firmly become established. For example density at a point is normally defined as

$$\rho = \lim_{\Delta\forall\to 0}\left(\frac{m}{\Delta\forall}\right)$$

Here $\Delta\forall$ is the volume of the fluid element and m is the mass

- If $\Delta\forall$ is very large ρ is affected by the in homogeneities in the fluid medium. Considering another extreme if $\Delta\forall$ is very small, random movement of atoms (or molecules) would change their number at different times. In the continuum approximation point density is defined at the smallest magnitude of $\Delta\forall$, before statistical fluctuations become significant. This is called continuum limit and is denoted by

$\Delta\forall_c$.

- $$\rho = \lim_{\Delta\forall \to \Delta\forall_c} \left(\frac{m}{\Delta\forall}\right)$$

One of the factors considered important in determining the validity of continuum model is molecular density. It is the distance between the molecules which is characterized by mean free path (λ). It is calculated by finding statistical average distance the molecules travel between two successive collisions. If the mean free path is very small as compared with some characteristic length in the flow domain (i.e., the molecular density is very high) then the gas can be treated as a continuous medium. If the mean free path is large in comparison to some characteristic length, the gas cannot be considered continuous and it should be analyzed by the molecular theory.

A dimensionless parameter known as Knudsen number, $K_n = \lambda / L$, where λ is the mean free path *and* L is the characteristic length. It describes the degree of departure from continuum.

Usually when $K_n > 0.01$, the concept of continuum does not hold good.

Beyond this critical range of Knudsen number, the flows are known as

- ✓ slip flow ($0.01 < K_n < 0.1$),
- ✓ transition flow ($0.1 < K_n < 10$) and
- ✓ free-molecule flow ($K_n > 10$).

However, for the flow regimes considered in this course, K_n is always less than 0.01 and it is usual to say that the fluid is a continuum.

Other factor which checks the validity of continuum is the elapsed time between collisions. The time should be small enough so that the random statistical description of molecular activity holds good.

In continuum approach, fluid properties such as density, viscosity, thermal conductivity, temperature, etc. can be expressed as continuous functions of space and time.

1.7 ACOUSTIC SPEED AND MACH NUMBER

Consider an acoustic wave moving in a stationary fluid with speed 'a'. Properties of fluid change due in the presence of the acoustic wave. These property variations can be predicted using 1D conservation equations. For simplicity we can assume the acoustic wave to be stationary and the fluid to be passing across the wave with velocity 'a'. Consider the control volume, for understanding; central hatched portion can be exaggered as the acoustic wave. Let P, ρ and a be pressure, density and velocity ahead the acoustic wave respectively. Acoustic wave being a small amplitude disturbance, induces small change properties while fluid passing across it. Hence the properties behind the acoustic wave are P+dP, in ρ+dρ and a+da pressure, density and velocity respectively. Application of mass conservation and momentum conservation equations between inlet and exit stations of control volume, we get,

$$\rho a = (\rho + d\rho)(a + da)$$
$$P + \rho u^2 = (P + dP) + (\rho + d\rho)(a + da)^2$$

From mass equation $\rho a = \rho a + \rho da + a d\rho + da d\rho$

We will neglect $da d\rho$ since both are small quantities. Hence their product will be even smaller.
Therefore $\rho da + a d\rho = 0$ and

$$\frac{d\rho}{da} = \frac{\rho}{a}$$

From momentum equations we get,

$$p + \rho a^2 = (p + dp) + (\rho + d\rho)(a^2 + 2ada + da^2)$$

neglecting da^2

$$p + \rho a^2 = (p + dp) + (\rho + d\rho)(a^2 + 2ada)$$

$$p + \rho a^2 = (p + dp) + (\rho a^2 + 2a\rho da + a^2 d\rho + 2adad\rho)$$

neglecting 2adads,

$$p + \rho a^2 = (p + dp) + (\rho a^2 + 2a\rho da + a^2 d\rho)$$

$$0 = dp + 2a\rho da + a^2 d\rho$$

$$\frac{dp}{d\rho} + 2a\rho\frac{da}{d\rho} + a^2 = 0$$

Incorporating Equation in above equation, we get,

$$\frac{dp}{d\rho} + 2a\rho - \frac{\rho}{a} + a^2 = 0$$

$$\frac{dp}{d\rho} - a^2 = 0$$

$$a^2 = \frac{dp}{d\rho} \; or \; a = \sqrt{\frac{dP}{d\rho}}$$

This is the general formula for acoustic speed or speed of sound.

We can express the same in terms of bulk modulus or compressibility using the definition of the compressibility (τ)

$$d\rho = \rho\tau dp$$

$$\frac{dp}{d\rho} = \frac{1}{\tau\rho}$$

$$a = \sqrt{\frac{1}{\tau\rho}}$$

Now this τ can be isothermal or adiabatic compressibility. However, changes in properties across sound wave are small and we have also not considered any dissipative effect like viscous effects, therefore we can treat the compressibility as the isentropic one. This proves that acoustic wave is isentropic (adiabatic reversible) in nature. Both the formulas derived for acoustic speed are valid for any state of matter. But if we consider gas then we can further simplify the expression as below.

$$a = \sqrt{\left(\frac{dp}{d\rho}\right)}_{s=constant}$$

1.7.1 Definition of Mach number

Mach number is defined as the ratio of the particle (local) speed to the (local) speed of sound.

$$mach\ number = \frac{speed\ of\ the\ fluid\ particle(sound)}{acoustic\ apeed\ of\ that\ atmosphere}$$

$$M = \frac{V}{a}$$

Here, 'V' represents the speed of the fluid particle at a particular instant at a particular position and it is related to kinetic energy which is direct form of energy. Kinetic energy here is termed as directed energy since it has capacity to do work. If energy is present in random form then there is no capacity to do work. As 'a' represents acoustic speed and is $\sqrt{\gamma RT}$ for gas, it clearly shows that, it is related to random velocity of molecule, obtained from kinetic theory of gases, $(\frac{8RT}{\pi})$. Hence Mach number can be thought of as the ratio of directed energy to random energy. Essentially, ratio of kinetic energy (KE) and internal energy (IE) of the flow depicts the ratio of directed energy and random energy and it is function of Mach number.

$$\frac{\text{K.E}}{\text{Internal energy(I.E)}} = \frac{V^2/2}{CvT} = \frac{V^2/2}{\left(R/\gamma-1\right)T} = \frac{\gamma V^2/2}{\left(\gamma R/\gamma-1\right)T} = \gamma \frac{V^2/2}{\gamma RT}(\gamma-1) = \frac{\gamma(\gamma-1)}{2} \frac{V^2/2}{a^2} = \frac{\gamma(\gamma-1)}{2} M^2$$

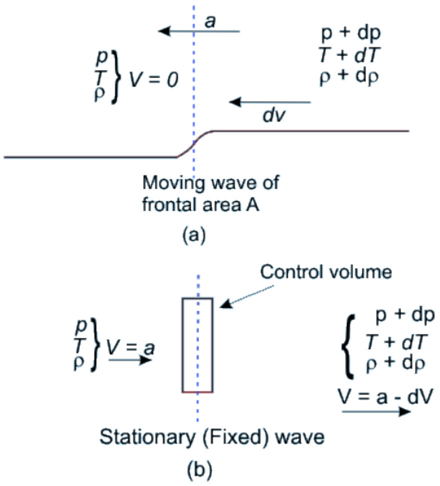
This clearly shows that, in order to increase the Mach number we will have to either decrease the internal energy or increase the kinetic energy.

- The so-called sound speed is the rate of propagation of a pressure pulse of infinitesimal strength through a still fluid. It is a thermodynamic property of a fluid.

- A pressure pulse in an incompressible flow behaves like that in a rigid body. A displaced particle displaces all the particles in the medium. In a compressible fluid, on the other hand, displaced mass compresses and increases the density of neighbouring mass which in turn increases density of the adjoining mass and so on. Thus, a disturbance in the form of an elastic wave or a pressure wave travels through the medium. If the amplitude and therefore the strength of the elastic wave are infinitesimal, it is termed as acoustic wave or sound wave.

- Figure shows an infinitesimal pressure pulse propagating at a speed "a" towards still fluid (V = 0) at the left. The fluid properties ahead of the wave are p,T and ρ, while the properties behind the wave are p+dp, T+dT and ρ+dρ. The fluid velocity dV is directed toward the left following wave but much slower.

In order to make the analysis steady, we superimpose a velocity "a" directed towards right, on the entire system. The wave is now stationary and the fluid appears to have velocity "a" on the left and (a - dV) on the right. The flow in figure is now steady and one dimensional across the wave. Consider an area A on the wave front. A mass balance gives



Propagation of a sound wave
(a) Wave Propagating into still Fluid (b) Stationary Wave

$$\rho A\alpha = (\rho + d\rho)A(a - dV)$$
$$dV = \propto \left[\frac{d\rho}{\rho + d\rho}\right]$$

This shows that

(a) dv>0 if dρ is positive.

(b) A compression wave leaves behind a fluid moving in the direction of the wave.

(c) Equation also signifies that the fluid velocity on the right is much smaller than the wave speed " a ". Within the framework of infinitesimal strength of the wave (sound wave), this "a" itself is very small.

Applying the momentum balance on the same control volume, it says that the net force in the x direction on the control volume equals the rate of outflow of x momentum minus the rate of inflow of x momentum. In symbolic form, this yields
$$\rho A - (\rho + d\rho)A = (A\rho a)(a - dV) - (A\rho a)(a)$$

In the above expression, Aρa is the mass flow rate. The first term on the right hand side represents the rate of outflow of x-momentum and the second term represents the rate of inflow of x momentum.

- Simplifying the momentum equation, we get
- If the wave strength is very small, the pressure change is small.
- Combining Equation, we get

$$a^2 = \frac{dp}{d\rho}\left(1 + \frac{d\rho}{\rho}\right)$$
$$a^2 = \frac{dp}{d\rho}$$
$$a = \sqrt{\left(\frac{dp}{d\rho}\right)_s}$$

For a perfect gas, by using of $\frac{P}{\rho^{\gamma}} = const$, and P= ρRT, we deduce the speed of sound as

$$a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma RT}$$

For air at sea-level and at a temperature of 15^0C, a=340 m/s

1.7.2 Pressure Field Due to a Moving Source

- Consider a point source emanating infinitesimal pressure disturbances in a still fluid, in which the speed of sound is "a". If the point disturbance, is stationary then the wave fronts are concentric spheres. As wave fronts are present at intervals of Δt.
- Now suppose that source moves to the left at speed U < a. Figure shows four locations of the source, 1 to 4, at equal intervals of time Δt, with point 4 being the current location of the source.
- At point 1, the source emanated a wave which has spherically expanded to a radius 3aΔt in an interval of time 3Δt . During this time the source has moved to the location 4 at a distance of 3uΔt from point 1. The figure also shows the locations of the wave fronts emitted while the source was at points 2 and 3, respectively.

When the source speed is supersonic U > a, the point source is ahead of the disturbance and an observer in the downstream location is unaware of the approaching source. The disturbance emitted at different points of time is enveloped by an imaginary conical surface known as "Mach Cone". The half angle of the cone α, is known as Mach angle and given by

$$\sin\alpha = \frac{a\Delta t}{U\Delta t} = \frac{1}{Ma}$$

$$\alpha = \sin^{-1}(1/Ma)$$

- Since the disturbances are confined to the cone, the area within the cone is known as zone of action and the area outside the cone is zone of silence.

  An observer does not feel the effects of the moving source till the Mach Cone covers his position.

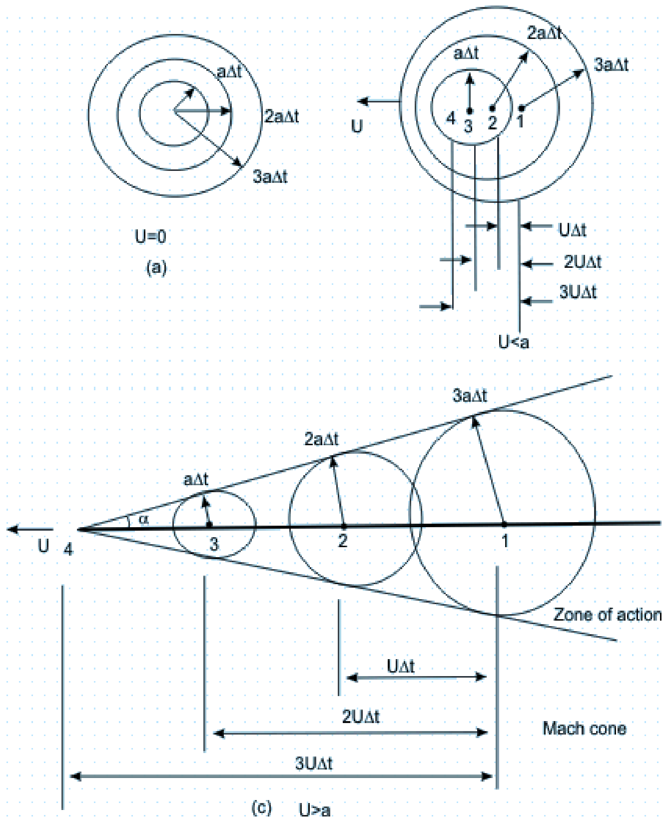


Fig 1.2 Wave fronts emitted from a point source in a still fluid when the source speed is
(a) U = 0 (still Source) (b) U < a (Subsonic) (c) U > a (Supersonic)

It may be seen that the speed of sound is the thermodynamic property that varies from point to point. When there is a large relative speed between a body and the compressible fluid surrounds it, then the compressibility of the fluid greatly influences the flow properties. Ratio of the local speed (V) of the gas to the speed of sound (a) is called as local Mach number (M).

$$M = \frac{V}{a} = \frac{V}{\sqrt{\gamma RT}}$$

There are few physical meanings for Mach number;

(a) It shows the compressibility effect for a fluid i.e. M<0.3 implies that fluid is incompressible.

(b) It can be shown that Mach number is proportional to the ratio of kinetic to internal energy.

$$\frac{\left(V^2/2\right)}{e} = \frac{V^2/2}{c_v T} = \frac{V^2/2}{RT/(\gamma-1)} = \frac{(\gamma/2)V^2}{a^2/(\gamma-1)} = \frac{\gamma(\gamma-1)}{2}M^2$$

(c) It is a measure of directed motion of a gas compared to the random thermal motion of the molecules.

$$M^2 = \frac{V^2}{a^2} = \frac{\text{directed kinetic energy}}{\text{random kinetic energy}}$$

1.7.3 Compressible Flow Regimes

In order to illustrate the flow regimes in a compressible medium, let us consider the flow over an aerodynamic body. The flow is uniform far away from the body with free stream velocity V_∞ while the speed of sound in the uniform stream is a_∞. Then, the free stream Mach number becomes $M_\infty = (V_\infty/a_\infty)$. The streamlines can be drawn as the flow passes over the body and the local Mach number can also vary along the streamlines. Let us consider the following distinct flow regimes commonly dealt with in compressible medium.

Subsonic flow : It is a case in which an airfoil is placed in a free stream flow and the local Mach number is less than unity everywhere in the flow field. The flow is characterized by smooth streamlines with continuous varying properties. Initially, the streamlines are straight in the free stream, but begin to deflect as they approach the body. The flow expands as it passed over the airfoil and the local Mach number on the top surface of the body is more than the free stream value. Moreover, the local Mach number (M) in the surface of the airfoil remains always less than 1, when the free stream Mach number (M_∞) is sufficiently less than 1. This regime is defined as subsonic flow which falls in the range of free stream Mach number less than 0.8.

Transonic flow: If the free stream Mach number increases but remains in the subsonic range close to 1, then the flow expansion over the air foil leads to supersonic region locally on its surface. Thus, the entire regions on the surface are considered as mixed flow in which the local Mach number is either less or more than 1 and thus called as *sonic pockets*. The phenomena of sonic pocket is initiated as soon as the local Mach number reaches 1 and subsequently terminates in the downstream with a shock wave across which there is discontinuous and sudden change in flow properties. If the free stream Mach number is slightly above unity, the shock pattern will move towards the trailing edge and a second shock wave appears in the leading edge which is called as *bow shock.* In front of this bow shock, the streamlines are straight and parallel with a uniform supersonic free stream Mach number. After passing through the bow shock, the flow becomes subsonic close to the free stream value. Eventually, it further expands over the airfoil surface to supersonic values and finally terminates with trailing edge shock in the downstream. The mixed flow patterns sketched, is defined as the *transonic regime.*

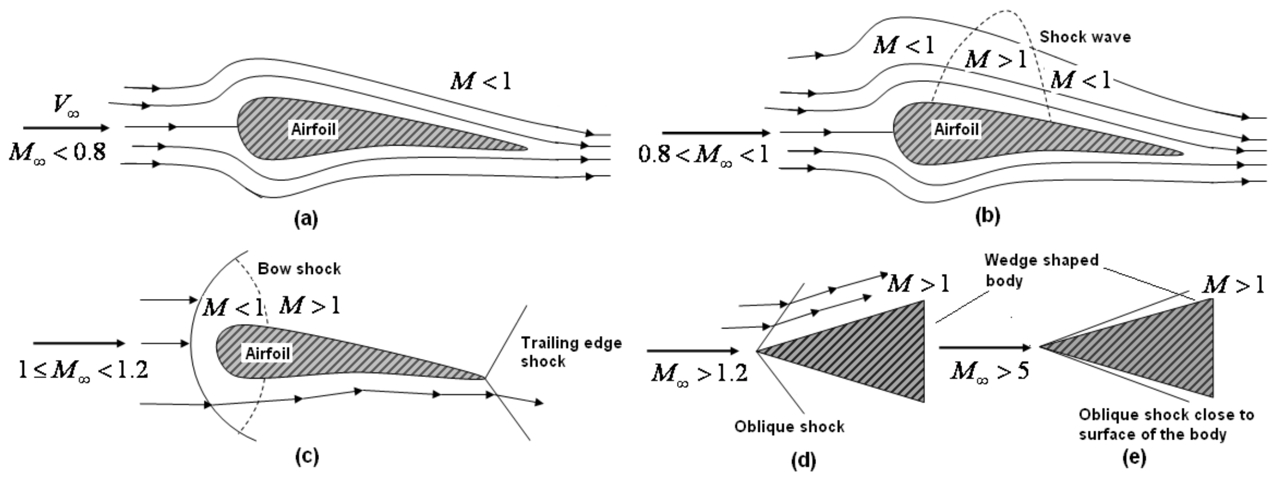


Fig. 1.3: Illustration of compressible flow regime: (a) subsonic flow; (b & c) transonic flow; (d) supersonic flow; (d) hypersonic flow.

Supersonic flow: In a flow field, if the Mach number is more than 1 everywhere in the domain, then it defined as supersonic flow. In order to minimize the drag, all aerodynamic bodies in a supersonic flow, are generally considered to be sharp edged tip. Here, the flow field is characterized by straight, oblique shock as shown in Fig. 1.3(d). The stream lines ahead of the shock the streamlines are straight, parallel and horizontal. Behind the oblique shock, the streamlines remain straight and parallel but take the direction of wedge surface. The flow is supersonic both upstream and downstream of the oblique shock. However, in some exceptional strong oblique shocks, the flow in the downstream may be subsonic.

Hypersonic flow: When the free stream Mach number is increased to higher supersonic speeds, the oblique shock moves closer to the body surface (Fig. 1.3-e). At the same time, the pressure, temperature and density across the shock increase explosively. So, the flow field between the shock and body becomes hot enough to ionize the gas. These effects of thin shock layer, hot and chemically reacting gases and many other complicated flow features are the characteristics of *hypersonic flow*. In reality, these special characteristics associated with hypersonic flows appear gradually as the free stream Mach numbers is increased beyond 5.

As a rule of thumb, the compressible flow regimes are classified as below;

$M < 0.3$ (incompressible flow)

$M < 1$ (subsonic flow)

$0.8 < M < 1.2$ (transonic flow)

$M > 1$ (supersonic flow)

$M > 5$ an above (hypersonic flow)

Rarefied and Free Molecular Flow: In general, a gas is composed of large number of discrete atoms and molecules and all move in a random fashion with frequent collisions. However, all the fundamental equations are based on overall macroscopic behavior where the continuum assumption is valid. If the mean distance between atoms/molecules between the collisions is large enough to be comparable in same order of magnitude as that of characteristics dimension of the flow, then it is said to be low density/rarefied flow. Under extreme situations, the mean free path is much larger than the characteristic dimension of the flow. Such flows are defined as free molecular flows. These are the special cases occurring in flight at very high altitudes (beyond 100 km) and some laboratory devices such as electron beams.

1.8 GOVERNING EQUATIONS FOR COMPRESSIBLE FLOWS

One dimensional form of Mass conservation or Continuity Equation

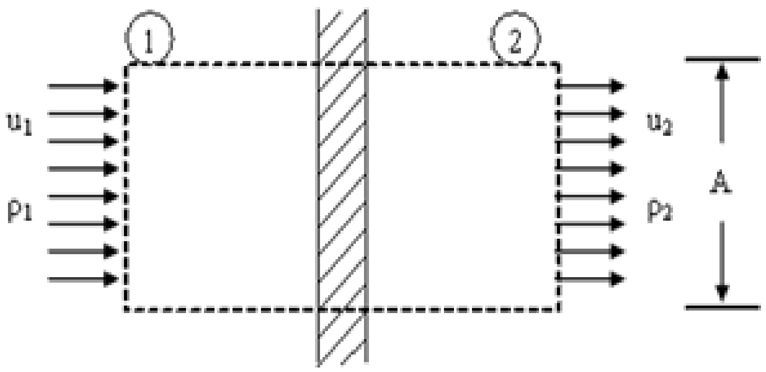


Fig. 1.4 One-dimensional flow

For steady flow Equation becomes

$$\oiint_s \rho V.ds = 0$$

Applying the surface integral over the control volume of Figure, this equation becomes

$$-P_1 u_1 A + P_2 u_2 A = 0$$

Or

$$\rho_1 u_1 = \rho_2 u_2$$

For flow over the varying cross-sectional area such as nozzle and diffuser, equation modifies to

$$-\rho_1 u_1 A_1 + \rho_2 u_2 A_2 = 0$$

One dimensional form of Momentum conservation Equation

For the steady and inviscid flow with no body forces, the Equation reduces to

$$\oiint_s (\rho V.ds)V = \oiint_s Pds$$

Above equation is a vector equation. However, since we are dealing with the one-dimensional flow, we need to consider only the scalar x component of equation.

$$\oiint_s (\rho V.ds)u = -\oiint_s (Pds)_x$$

Considering the control volume shown in Fig (1.4), above equation transforms to,

$$\rho_1(-u_1 A)u_1 + \rho_2(-u_2 A)u_2 = -(-P_1 A + P_2 A)$$

or

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2$$

This is the momentum equation for steady inviscid one-dimensional flow.

1.8.1 One dimensional form of Conservation of Energy

Consider the control shown in Figure for steady inviscid flow without body force, and then the equation reduces to,

$$\oiiint_v q\rho dv - \oiint_s PV.ds = \oiint_s \rho\left(e + \frac{V^2}{2}\right)V.ds$$

Let us denote the first term on left hand side of above equation by $\dot{Q}$ to represent the total external heat addition in the system. Thus, above equation becomes

$$\dot{Q} - \oiint_s PV.ds = \oiint_s \rho\left(e + \frac{V^2}{2}\right)V.ds$$

Evaluating the surface integrals over the control volume in Figure, we obtain

$$\dot{Q} - (-P_1 u_1 A + P_2 u_2 A) = -\rho_1 \left(e_1 + \frac{u_1^2}{2}\right) u_1 A + \rho_2 \left(e_2 + \frac{u_2^2}{2}\right) u_2 A$$

Or

$$\frac{\dot{Q}}{A} + P_1 u_1 + \rho_1 \left(e_1 + \frac{u_1^2}{2}\right) u_1 = P_2 u_2 + \rho_2 \left(e_2 + \frac{u_2^2}{2}\right) u_2$$

or

$$\frac{\dot{Q}}{\rho_1 u_1 A} + \frac{P_1}{\rho_1} + e_1 + \frac{u_1^2}{2} = \frac{P_2}{\rho_2} + e_2 + \frac{u_2^2}{2}$$

or

$$\frac{\dot{Q}}{\rho_1 u_1 A} + P_1 v_1 + e_1 + \frac{u_1^2}{2} = P_2 v_2 + e_2 + \frac{u_2^2}{2}$$

Here, $\dot{Q}/\rho_1 u_1 A$ is the external heat added per unit mass, q. Also, we know e + Pu = h.
Hence, above equation can be re-written as,

$$h_1 + \frac{u_1^2}{2} + q = h_2 + \frac{u_2^2}{2}$$

This is the energy equation for steady one-dimensional flow for inviscid flow.

1.8.2 Fundamental Equations for Compressible Flow

Consider a compressible flow passing through a rectangular control volume as shown in Figure. The flow is one-dimensional and the properties change as a function of x, from the region ‘1' to ‘2' and they are velocity (u), pressure (p), temperature (T), density (ρ) and internal energy (e). The following assumptions are made to derive the fundamental equations;

- Flow is uniform over left and right side of control volume.
- Both sides have equal area (A), perpendicular to the flow.
- Flow is inviscid, steady and nobody forces are present.
- No heat and work interaction takes place to/from the control volume.

Let us apply mass, momentum and energy equations for the one dimensional flow.

Conservation of Mass:

$$-P_1 u_1 A + P_2 u_2 A = 0$$

$$\Rightarrow \rho_1 u_1 + \rho_2 u_2$$

Conservation of Momentum:

$$\rho_1(-u_1 A)u_1 + \rho_2(-u_2 A)u_2 = -(-P_1 A + P_2 A)$$

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2$$

Steady Flow Energy Conservation:

$$\frac{P_1}{\rho_1} + e_1 + \frac{u_1^2}{2} = \frac{P_2}{\rho_2} + e_2 + \frac{u_2^2}{2}$$

$$h_1 + \frac{u_1^2}{2} + q = h_2 + \frac{u_2^2}{2}$$

Here, the enthalpy h(=e+P/ρ) is defined as another thermodynamic property of the gas.

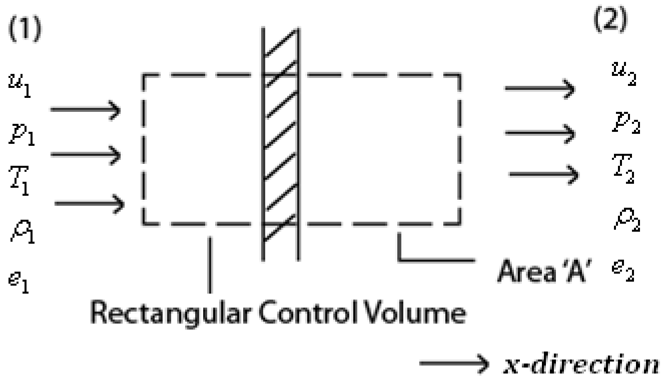


Fig.1.5 Schematic representation of one-dimensional flow

UNIT II

SHOCK AND EXPANSION WAVES

2.1 DEVELOPMENT OF GOVERNING EQUATIONS FOR NORMAL SHOCK

2.1.1 Shock Waves

Let us consider a subsonic and supersonic flow past a body. In both the cases, the body acts as an obstruction to the flow and thus there is a change in energy and momentum of the flow. The changes in flow properties are communicated through pressure waves moving at speed of sound everywhere in the flow field (i.e. both upstream and downstream). If the incoming stream is subsonic i.e. $M_\infty<1$; $V_\infty< a_\infty$, the sound waves propagate faster than the flow speed and warn the medium about the presence of the body. So, the streamlines approaching the body begin to adjust themselves far upstream and the flow properties change the pattern gradually in the vicinity of the body.

In contrast, when the flow is supersonic, i.e. $M_\infty>1$; $V_\infty> a_\infty$, the sound waves overtake the speed of the body and these weak pressure waves merge themselves ahead of the body leading to compression in the vicinity of the body. In other words, the flow medium gets compressed at a very short distance ahead of the body in a very thin region that may be comparable to the mean free path of the molecules in the medium. Since, these compression waves propagate upstream, so they tend to merge as *shock wave*. Ahead of the shock wave, the flow has no idea of presence of the body and immediately behind the shock; the flow is subsonic.

The thermodynamic definition of a shock wave may be written as “the instantaneous compression of the gas”. The energy for compressing the medium, through a shock wave is obtained from the kinetic energy of the flow upstream the shock wave. The reduction in kinetic energy is accounted as heating of the gas to a static temperature above that corresponding to the isentropic compression value. Consequently, in flowing through the shock wave, the gas experiences a decrease in its available energy and accordingly, an increase in entropy. So, the compression through a shock wave is considered as an irreversible process.

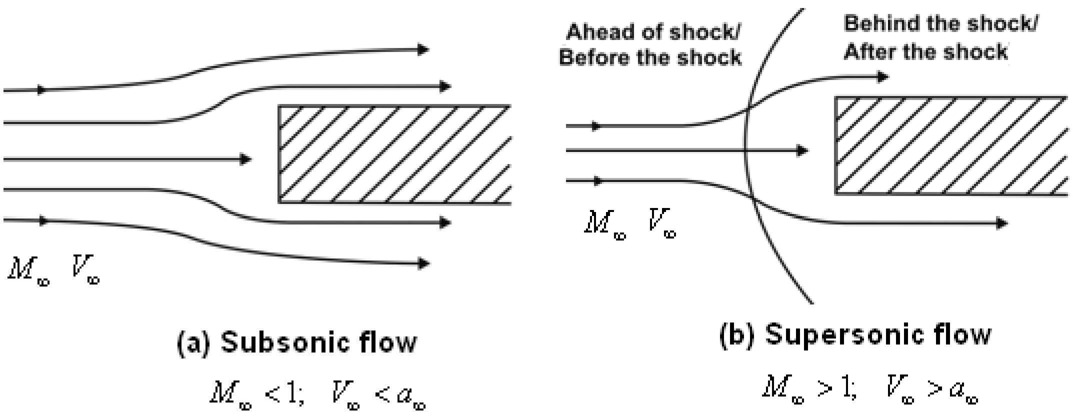


Fig.2.1 Illustration of shock wave phenomena

2.1.2 Normal Shock Waves

A normal shock wave is one of the situations where the flow properties change drastically in one direction. The shock wave stands perpendicular to the flow. The quantitative analysis of the changes across a normal shock wave involves the determination of flow properties. All conditions of are known ahead of the shock and the unknown flow properties are to be determined after the shock. There is no heat added or taken away as the flow traverses across the normal shock. Hence, the flow across the shock wave is adiabatic$(q \doteq 0)$.

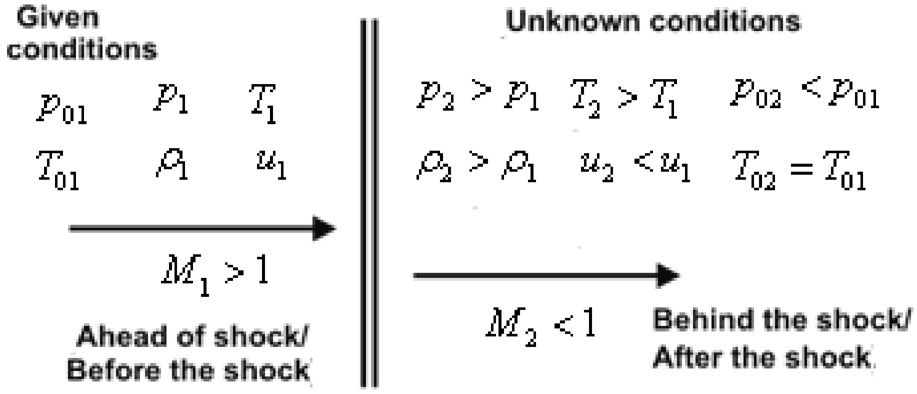


Fig.2.2 Schematic diagram of a standing normal shock wave

The basic one dimensional compressible flow equations can be written as below;

$$\rho_1 u_1 + \rho_2 u_2$$

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2$$

$$h_1 + \frac{u_1^2}{2} + q = h_2 + \frac{u_2^2}{2}$$

For a calorically perfect gas, thermodynamic relations can be used,

$$P = \rho RT; h = c_p T; a = \sqrt{\frac{\gamma p}{\rho}}$$

The continuity and momentum equations of Equation can be simplified to obtain,

$$\frac{a_1^2}{\gamma u_1} - \frac{a_2^2}{\gamma u_2} = u_2 - u_1$$

Since,$a^* = \sqrt{\gamma RT^*}$ and$M^* = \frac{V}{a^*}$, the energy equation is written as,

$$\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{\gamma + 1}{2(\gamma - 1)} a^{*2}$$

$$a^2 = \frac{\gamma + 1}{2} a^{*2} - \frac{\gamma - 1}{2} u^2$$

Both$a_1^2$and a_2^2 can now be expressed as,

$$a_1^2 = \frac{\gamma+1}{2}a^{*2} - \frac{\gamma-1}{2}u_1^2; a_2^2 = \frac{\gamma+1}{2}a^{*2} - \frac{\gamma-1}{2}u_2^2$$

solve fora^{*2}

$$a^{*2} = u_1 u_2 \quad \Rightarrow M_2^* = \frac{1}{M_1^*}$$

Recall the relation for M and M* and substitute in equation,

$$M^{*2} = \frac{(\gamma+1)M^2}{2+(\gamma+1)M^2}$$

Solve for M_2

$$M_2^2 = \frac{1+\left(\frac{\gamma-1}{2}\right)M_1^2}{\gamma M_1^2 - \left(\frac{\gamma-1}{2}\right)}$$

Using continuity equation and Prandtl relation, we can write,

$$\frac{\rho_2}{\rho_1} = \frac{u_2}{u_1} = \frac{u_1^2}{u_1 u_2} = \frac{u_1^2}{a^{*2}} = (M_1^*)^2$$

Solve for density and velocity ratio across the normal shock.

$$\frac{\rho_2}{\rho_1} = \frac{u_2}{u_1} = \frac{(\gamma+1)M_1^2}{2+(\gamma+1)M_1^2}$$

The pressure ratio can be obtained by the combination of momentum and continuity equations i.e.

$$P_2 - P_1 = \rho_1 u_1(u_1 - u_2) = \rho_1 u_1^2\left(1 - \frac{u_2}{u_1}\right)$$

$$\frac{P_2 - P_1}{P_1} = \gamma u_1^2\left(1 - \frac{u_2}{u_1}\right)$$

Simplifying for the pressure ratio across the normal shock, we get,

$$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1}(M_2^2 - 1)$$

For a calorically perfect gas, equation of state relation can be used to obtain the temperature ratio across the normal shock i.e.

$$\frac{h_2}{h_1} = \frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)\left(\frac{\rho_1}{\rho_2}\right) = \left[1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1)\right]\left[\frac{2+(\gamma-1)M_1^2}{(\gamma+1)M_1^2}\right]$$

Thus, the upstream Mach number is the powerful tool to dictating the shock wave properties. The "stagnation properties" across the normal shock can be computed as follows;

$$\frac{P_{02}}{P_{01}} = \frac{(P_{02}/P_2)}{(P_{01}/P_1)}\left(\frac{P_2}{P_1}\right)$$

Here, the ratios$\left(\frac{P_{01}}{P_1}\right)$ *and* $\left(\frac{P_{02}}{P_2}\right)$ can be obtained from the isentropic relation for the regions '1 and 2' respectively. Knowing the upstream Mach number M_1, gives the downstream Mach numberM_2. Further, Equation can be used to obtain the static pressure ratio (p_2/p_1). After substitution of these ratios, reduces

to,

$$\frac{P_{02}}{P_{01}} = \frac{\left(1+\frac{\gamma-1}{2}M_2^2\right)^{\frac{\gamma}{\gamma-1}}}{\left(1+\frac{\gamma-1}{2}M_1^2\right)^{\frac{\gamma}{\gamma-1}}}\left[1+\frac{2\gamma}{\gamma+1}(M_1^2-1)\right]$$

Many a times, another significant pressure ratio (p_{02}/p_1) is important for a normal shock which is normally called as *Rayleigh Pitot Tube* relation.

$$\frac{P_{02}}{P_{01}} = \left(\frac{P_{02}}{P_2}\right)\left(\frac{P_2}{P_1}\right)$$

$$\frac{P_{02}}{P_2} = \left(1+\frac{\gamma-1}{2}M_2^2\right)^{\frac{\gamma}{\gamma-1}}\left[1+\frac{2\gamma}{\gamma+1}(M_1^2-1)\right]$$

Recall the energy equation for a calorically perfect gas:

$$c_pT_1 + \frac{u_1^2}{2} = c_pT_2 + \frac{u_2^2}{2}$$

$$c_pT_{01} = c_pT_{02}$$

Thus, the stagnation temperatures do not change across a normal shock.

2.1.3Entropy across a normal shock

The compression through a shock wave is considered as irreversible process leading to an increase in entropy. The change in entropy can be written as a function of static pressure and static temperature ratios across the normal shock.

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right)$$

Mathematically, it can be seen that the entropy change across a normal shock is also a function of the upstream Mach number. The *second law of thermodynamics* puts the limit that 'entropy' must increase (s_2-$s_1$$\geq 0$) for a process to occur in a certain direction. Hence, the upstream Mach number (M_1) must be greater than 1 (i.e. supersonic). It leads to the fact that $M_2\leq1$;($p_2/p_1\geq1$;(ρ_2/ρ_1)≥1;(T_2/T_1)≥1 .

The entropy change across a normal shock can also be calculated from another simple way by expressing the thermodynamic relation in terms of total pressure. It is seen that the discontinuity occurs only in the thin region across the normal shock. If the fluid elements is brought to rest isentropically from its real state (for both upstream and downstream conditions), then they will reach an imaginary state '1a and 2a'. The expression for entropy change between the imaginary states can be written as,

$$s_{2a} - s_{1a} = c_p \ln\left(\frac{T_{2a}}{T_{1a}}\right) - R\ln\left(\frac{p_{2a}}{p_{1a}}\right)$$

Since, $s_{2a} = s_2; s_{1a} = s_1; T_{2a} = T_{1a} = T_0; \ p_{2a} = p_{02}$ and $p_{1a} = p_{01}$, the Equation reduces to,

$$s_2 - s_1 = -R \ln\left(\frac{p_{02}}{p_{01}}\right) \quad \Rightarrow \frac{p_{02}}{p_{01}} = e^{-(s_2 - s_1)/R}$$

Because of the fact $s_2 > s_1$, Equation implies that $p_{02} < p_{01}$. Hence, the stagnation pressure always decreases across a normal shock.

2.1.4 Normal shock relations

It had already been discussed that the subsonic flow is pre-warned and supersonic flow is not. The reason behind this fact is that, any small amplitude disturbance travels with acoustic speed, however speed of fluid particle is more than the speed of sound in case of supersonic flows. Therefore the message of presence of the obstacle cannot propagate upstream. Hence a messenger gets developed in front of the obstacle to warn the flow in order to avoid its direct collision with the obstacle. This messenger is called as shock. In the presence of normal shock, fluid velocity decreases to the extent where flow Mach number behind the shock attains value below one. Due to this subsonic speed attainment of the flow, it becomes aware about the presence of the obstacle well in advance in the narrow space between shock and obstacle. Herewith we will deal for computation of flow properties behind the normal shock.

In the presence of a general obstacle the shock pattern.

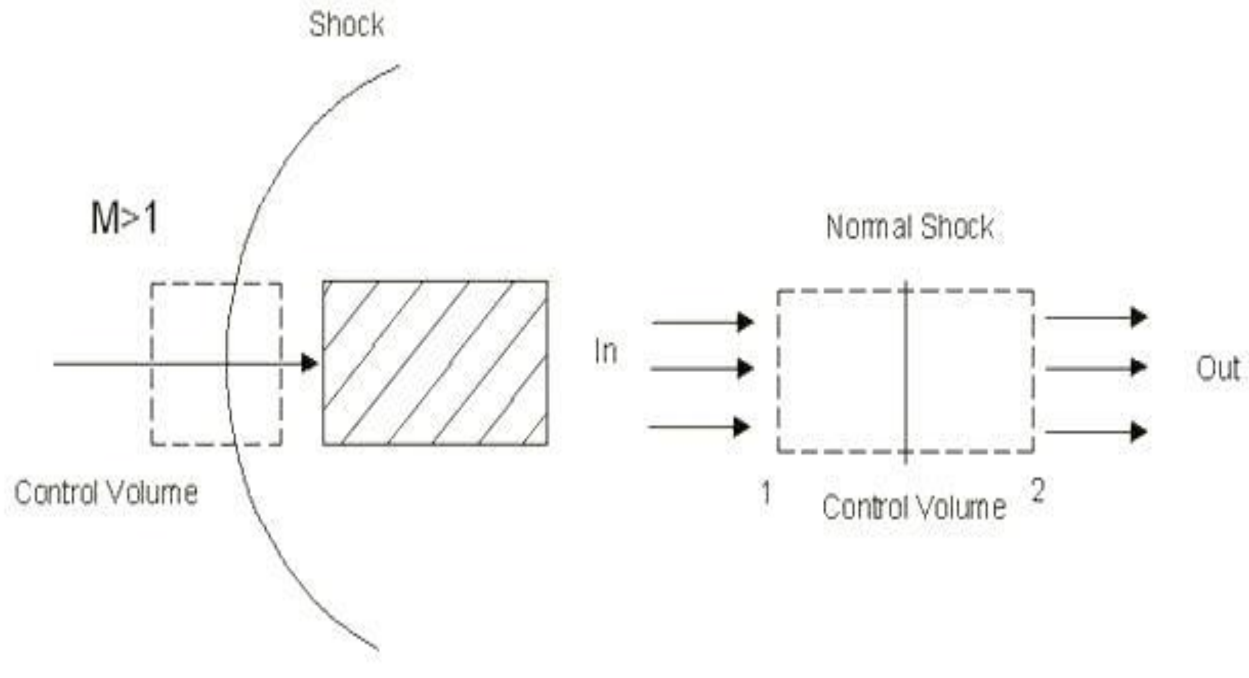


Fig. 2.3 Shock pattern for a blunt or bluff obstacle

The shock for the stagnation streamline can be considered as normal to it. Therefore we can use the earlier derived 1D flow relations along with the assumptions of flow steady, adiabatic and inviscid flow. Consider a small control volume around normal shock for application of these relations between two stations of the control volume, mainly, inlet and outlet.

$$\rho_1 u_1 = \rho_2 u_2$$
$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$$
$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

Lets us examine the reference star properties of the flow in the process to calculate the flow properties behind the normal shock from the known inlet conditions. We can take the advantage of using stared temperature since the flow is adiabatic in nature. Imagine that flow is adiabatically brought to Mach number one on either sides of the shock independantly. In this case, we should get same stared temperature on either sides of shock. We can also show that total temperature is also same on either sides. The explicit formulation using the star temperature and concerned acoustic speed before the normal shock is,

$$h_1 + \frac{u_1^2}{2} = h_1^* + \frac{u_1^*}{2}$$
$$h_1 + \frac{u_1^2}{2} = C_p T^* + \frac{a^*}{2}$$
$$h_1 + \frac{u_1^2}{2} = C_p T^* + \frac{\gamma R T^*}{2} = \frac{\gamma R T^*}{\gamma - 1} + \frac{\gamma R T^*}{2} \left[\because C_p = \frac{\gamma R}{\gamma - 1} \right]$$
$$h_1 + \frac{u_1^2}{2} = \gamma \left[\frac{1}{\gamma - 1} + \frac{1}{2} \right] R T^* = \frac{\gamma(\gamma+1)}{2(\gamma-1)} R T^* = \frac{(\gamma+1)}{2(\gamma-1)} a^{*2}$$

Applying same strategy at the outlet we get,

$$h_2 + \frac{u_2^2}{2} = \frac{(\gamma+1)}{2(\gamma-1)} a^{*2}$$

However, we can write static enthalpy in terms of acoustic speed as,

$$h = C_p T = \frac{\gamma R}{\gamma - 1} T = \frac{\gamma R T}{\gamma - 1} = \frac{a^2}{\gamma - 1}$$

Therefore, the energy equation at the inlet becomes,

$$a_1^2 = \frac{(\gamma+1)}{2} a^{*2} - \frac{(\gamma-1)}{2} u_1^2$$

Similarly for the outlet station we have

$$a_2^2 = \frac{(\gamma+1)}{2} a^{*2} - \frac{(\gamma-1)}{2} u_2^2$$

Let's obtain the expression for velocity using mass and momentum equations to replace the acoustic speed term from equations.

From 1D mass and momentum conservation equations we have

$$\rho_1 u_1 = \rho_2 u_2$$
$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$$

Therefore,

$$\frac{p_1 + \rho_1 u_1^2}{\rho_1 u_1} = \frac{p_2 + \rho_2 u_2^2}{\rho_2 u_2}$$

$$\frac{p_1}{\rho_1 u_1} + u_1 = \frac{p_2}{\rho_2 u_2} + u_2 \left[\because \rho_1 u_1 = \rho_2 u_2 \text{ mass conservation}\right]$$

$$\frac{a_1^2}{\gamma u_1} + u_1 = \frac{a_2^2}{\gamma u_2} + u_2$$

$$u_2 - u_1 = \frac{a_1^2}{\gamma u_1} - \frac{a_2^2}{\gamma u_2}$$

Using, above equation transforms to

$$u_2 - u_1 = \frac{1}{\gamma u_1}\left[\frac{(\gamma+1)}{2} a^{*2} - \frac{(\gamma-1)}{2} u_1^2\right] - \frac{1}{\gamma u_2}\left[\frac{(\gamma+1)}{2} a^{*2} - \frac{(\gamma-1)}{2} u_2^2\right]$$

Rearranging the terms of above equation, we get

$$u_2 - u_1 = \frac{1}{\gamma u_1}\frac{(\gamma+1)}{2} a^{*2} - \frac{1}{\gamma u_1}\frac{(\gamma-1)}{2} u_1^2 - \frac{1}{\gamma u_2}\frac{(\gamma+1)}{2} a^{*2} + \frac{1}{\gamma u_2}\frac{(\gamma-1)}{2} u_2^2$$

$$u_2 - u_1 = \frac{1}{\gamma u_1}\frac{(\gamma+1)}{2} a^{*2} - \frac{1}{\gamma u_2}\frac{(\gamma+1)}{2} a^{*2} + \frac{1}{\gamma u_2}\frac{(\gamma-1)}{2} u_2^2 - \frac{1}{\gamma u_1}\frac{(\gamma-1)}{2} u_1^2$$

A further rearrangement gives

$$u_2 - u_1 = \frac{(\gamma+1)}{2\gamma} a^{*2}\left[\frac{u_2 - u_1}{u_1 u_2}\right] + \frac{(\gamma-1)}{2\gamma}[u_2 - u_1]$$

$$u_2 - u_1 = \frac{(\gamma+1)}{2\gamma} a^{*2}\left[\frac{1}{u_1} - \frac{1}{u_2}\right] + \frac{(\gamma-1)}{2\gamma}[u_2 - u_1]$$

$$u_2 - u_1 = \frac{(\gamma+1)}{2\gamma}\left[\frac{a^{*2}}{u_1 u_2}\right] + \frac{(\gamma-1)}{2\gamma}$$

Necessary rearrangement for the above equation is as given,

$$\left[\frac{a^{*2}(\gamma+1)}{u_1 u_2}\right]\frac{1}{2\gamma} = \frac{2\gamma - \gamma + 1}{2\gamma}$$

$$a^{*2} = u_1 u_2$$

$$\frac{u_1}{a^{*2}} = \frac{u_2}{a^{*2}} \Rightarrow M_1^{*2} = \frac{1}{M_2^{*2}}$$

This expression shows that, M_1^{*2} and M_2^{*2} are reciprocal of each other for a normal shock. This equation is called as Prandtl's relation for normal shock which can be used to prove that Mach number becomes subsonic behind the normal shock

- Shock waves are highly localized irreversibilities in the flow.
- Within the distance of a mean free path, the flow passes from a supersonic to a subsonic state, the velocity decreases suddenly and the pressure rises sharply. A shock is said to have occurred if there is an abrupt reduction of velocity in the downstream in course of a supersonic flow in a passage or around a body.
- Normal shocks are substantially perpendicular to the flow and oblique shocks are inclined at any angle.
- Shock formation is possible for confined flows as well as for external flows.
- Normal shock and oblique shock may mutually interact to make another shock pattern.

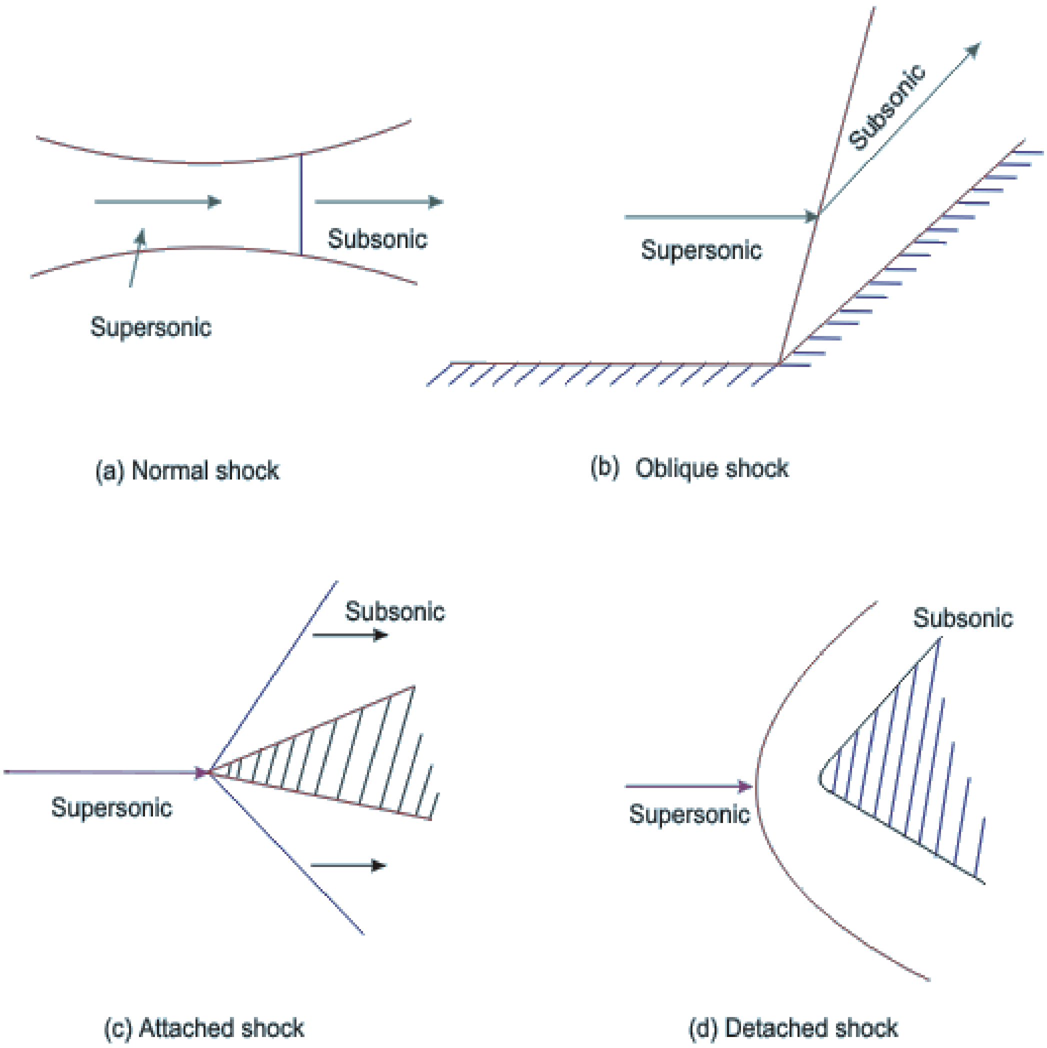


Fig.2.4 Different type of Shocks

Figure below shows a control surface that includes a normal shock.

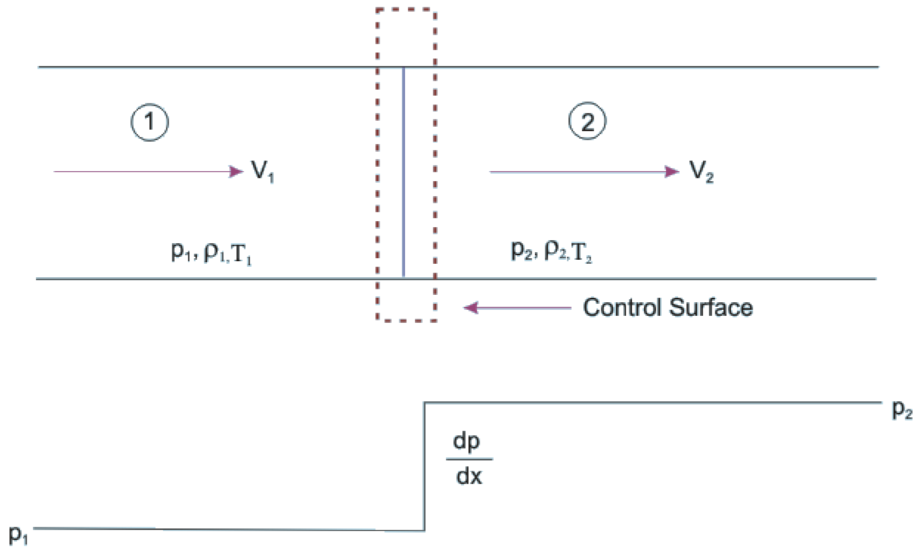


Fig 2.5 One Dimensional Normal Shock

The fluid is assumed to be in thermodynamic equilibrium upstream and downstream of the shock, the properties of which are designated by the subscripts 1 and 2, respectively.

Continuity equation can be written as

$$\frac{\dot{m}}{A} = \rho_1 V_1 = \rho_2 V_2 = G$$

where G is the mass velocity kg/ m^2 s, and $\dot{m}$ is mass flow rate

From momentum equation, we can write

$$p_1 - p_2 = \frac{\dot{m}}{A}(V_2 - V_1) = \rho_2 V_2^2 - \rho_1 V_1^2$$
$$\Rightarrow p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2$$
$$\Rightarrow \quad F_1 = F_2$$

where $p + \rho V^2$ is termed as Impulse Function .

The energy equation is written as

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} = h_{01} = h_{02} = h_0$$

where h_0 is stagnation enthalpy.

From the second law of thermodynamics, we know

$$s_2 - s_1 \geq 0$$

To calculate the entropy change, we have

$$T ds = dh - \forall dp$$

For an ideal gas

$$ds = c_p \frac{dT}{T} - R \frac{dp}{p}$$

For an ideal gas the equation of state can be written as

$$p = \rho RT$$

For constant specific heat, the above equation can be integrated to give

$$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$$

Above equations are the governing equations for the flow of an ideal gas through normal shock.
If all the properties at state 1 (upstream of the shock) are known, then we have six unknowns T_2, P_2, ρ_2, V_2, h_2, $s_2$in these five equations.
We know relationship between h and T for an ideal gas,dh = c_PdT.
For an ideal gas with constant specific heats,

$$\Delta h = h_2 - h_1 = c_p (T_2 - T_1)$$

Thus, we have the situation of six equations and six unknowns.
If all the conditions at state "1"(immediately upstream of the shock) are known, how many possible states 2 (immediate downstream of the shock) are there? The mathematical answer indicates that there is a unique state 2 for a given state 1.

2.1.5 Stationary and Moving Normal Shock Waves

The easiest way to analyze a normal shock is to consider a control surface around the wave. The continuity equation, the momentum equation and the energy equation have already been discussed earlier. The energy equation can be simplified for an ideal gas as

$$T_{01}=T_{02}$$

By making use of the equation for the speed of sound equation and the equation of state for ideal gas equation, the continuity equation can be rewritten to include the influence of Mach number as:

$$\frac{p_1}{RT_1} Ma_1 \sqrt{\gamma RT_1} = \frac{p_2}{RT_2} Ma_2 \sqrt{\gamma RT_2}$$

Introducing the Mach number in momentum equation, we have

$$p_1 + \frac{p_1}{RT_1} V_1^2 = p_2 + \frac{p_2}{RT_2} V_2^2$$

$$\rho_2 V_2^2 - \rho_1 V_1^2 = p_1 - p_2$$

Therefore,

$$p_1 \left(1 + \gamma Ma_1^2\right) = p_2 \left(1 + \gamma Ma_2^2\right)$$

Rearranging this equation for the static pressure ratio across the shock wave, we get

$$\frac{p_2}{p_1} = \frac{\left(1 + \gamma\, Ma_1^2\right)}{\left(1 + \gamma\, Ma_2^2\right)}$$

As already seen, the Mach number of a normal shock wave is always greater than unity in the upstream and less than unity in the downstream; the static pressure always increases across the shock wave.

The energy equation can be written in terms of the temperature and Mach number using the stagnation temperature relationship as

$$\frac{T_2}{T_1} = \frac{\{1+(\gamma-1)/2\}Ma_1^2}{\{1+(\gamma-1)/2\}Ma_2^2}$$

Substituting Equation yields the following relationship for the Mach numbers upstream and downstream of a normal shock wave:

$$\frac{Ma_1}{1+\gamma\, Ma_1^2}\left[1+\frac{\gamma-1}{2}Ma_1^2\right]^{\frac{1}{2}} = \frac{Ma_2}{1+\gamma\, Ma_2^2}\left[1+\frac{\gamma-1}{2}Ma_2^2\right]^{\frac{1}{2}}$$

Then, solving this equation for Ma_2 as a function of Ma_1 we obtain two solutions. One solution is trivial $Ma_2 = Ma_1$, which signifies no shock across the control volume. The other solution is

$$Ma_2^2 = \frac{(\gamma-1)Ma_1^2 + 2}{2\gamma Ma_1^2 - (\gamma-1)}$$

$Ma_1 = 1$ In Equation results in $Ma_2 = 1$

Above Equations also show that there would be no pressure or temperature increase across the shock. In fact, the shock wave corresponding to $Ma_1 = 1$ is the sound wave across which, by definition, pressure and temperature changes are infinitesimal. Therefore, it can be said that the sound wave represents a degenerated normal shock wave. The pressure, temperature and Mach number (Ma_2) behind a normal shock as a function of the Mach number Ma_1, in front of the shock for the perfect gas can be represented in a tabular form.

2.2 APPLICATION TO AIRCRAFTS

Air intakes for supersonic aircraft present an excellent example for the application of the theory of shock waves in steady flow. This chapter describes the operation of such devices given that it is only an introduction to the aerodynamics of air intakes whose design has many aspects that will not be mentioned here. We know that the propulsion of aircraft is provided by the so-called propulsion nacelle, or more simply the nacelle, which brings together the various elements contributing to the propulsion, namely:

- the air intake, capturing air to feed the engine;
- the engine that can be a turbojet, a ramjet, or even a scramjet engine for air breathing hypersonic vehicles;
- a possible reheat duct or afterburner, where kerosene is burnt downstream of the engine in order to provide extra thrust; and
- the exhaust nozzle whose aerodynamics.

The engines of current airliners additionally have a fan, arranged in front of the compressor and drawing an airflow that does not cross the compressor–turbine set. The operation of the fan is characterized by the bypass ratio, a ratio of the total drawn airflow and the airflow through the compressor. This bypass ratio is around 10 for modern engines. Engines for combat aircraft and missiles have no fan.

2.3 SUPERSONIC WIND TUNNEL

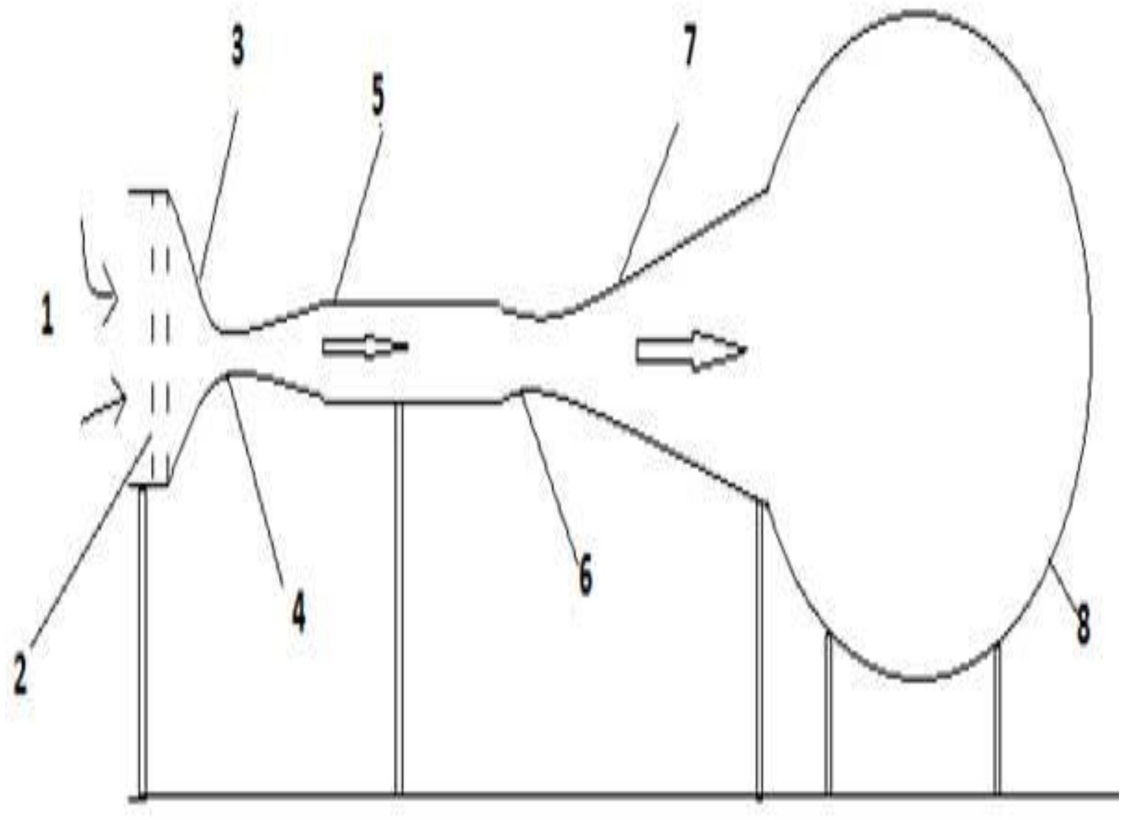


Fig 2.6 Schematic of indraft tunnel

Following are the components of this tunnel.

1. Air inlet
2.Settling chamber
3.Nozzle
4.Sonic throat
5.Test-section
6.Second throat
7.Diffuser
8.Vacuum chamber.

Some blow down tunnels, called as in draft tunnels, do not use a high pressure chamber, but use atmosphere as the reservoir. The in draft tunnel uses the low pressure (vacuum) chamber downstream of the test section to produce flow. During the operation of this tunnel, air from the atmosphere enters the nozzle and gets expanded to require Mach number defined by the area ratio of the nozzle. Further flow of the air from the test section, second throat decelerates the air and let it fill the vacuum chamber. The main advantage of this configuration is that the conditions in the inlet remain constant and there is no need for a pressure regulator. The disadvantage is that the pressure ratio across the test section is usually lower than a closed configuration and therefore the maximum Mach number (M>2) is not possible.

2.3.1 Closed Continuous Wind Tunnel

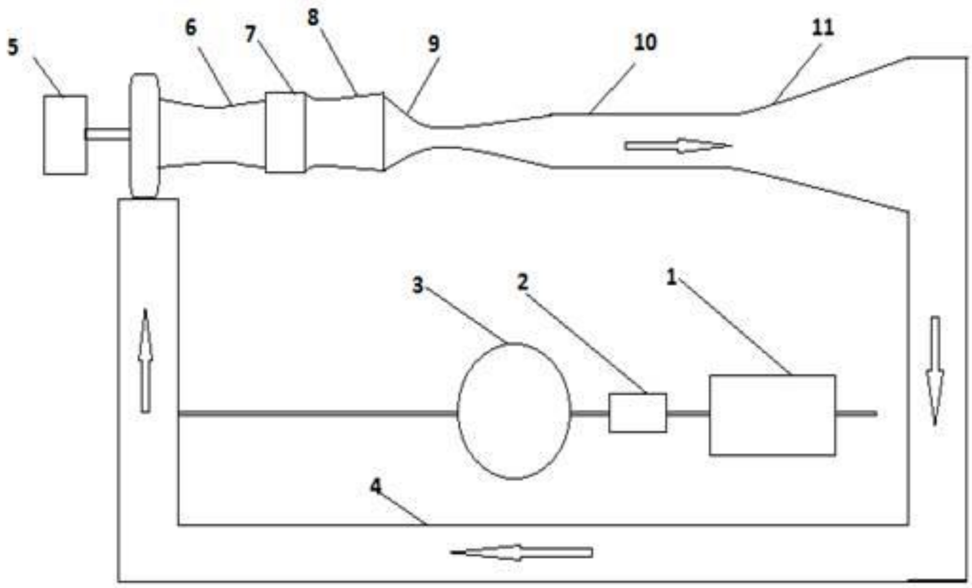


Fig 2.7 Schematic of closed circuit continuous wind tunnel

Following are the parts of this wind tunnel.

1. Air compressor
2.Dryer
3.Dry air storage
4.Return passage
5.Driver motor
6.Compressor
7.Cooler
8.Settling chamber
9.Nozzle
10.Test-section
11.Diffuser

In the closed return tunnel, air is passed from the exit of the test section back to the inlet by a series of turning vanes at the corner of the tunnel. Exiting the diffuser, the air is returned to the nozzle and back through the test section. Air is continuously circulated through the duct work of the closed return tunnel. To prevent condensation in the test section because of low pressure, the air entering the tunnel is often passed over a dryer bed. There is usually an additional throat placed in the tunnel downstream of the test section to decelerate the supersonic flow to subsonic. The continuous closed-circuit supersonic tunnel has some advantages and some disadvantages relative to the intermittent tunnel. Advantages of the closed return tunnels include longer run times relative to the blow down tunnel, superior flow quality in the test section, flow turning vanes in the corner and flow straighteners near the test section ensure relatively uniform flow in the test section, lesser noisy operation, the test section can be designed for high Mach numbers (M>4) and large-size models, low operating costs. Also once the air is circulating in the tunnel, the fan and motor only needs to overcome losses along the wall and through the turning vanes. The fan does not have to constantly accelerate the air.On the same grounds disadvantages of the closed return tunnels include higher construction cost because of the added vanes and ducting. This tunnel must be designed to purge exhaust products that accumulate in the tunnel.

2.3.2 Supersonic Diffuser

Supersonic diffuser with second throat should be designed critically because increasing diffuser efficiency will lower the power requirement considerably. The first throat is the one upstream of the test-section through which the flow accelerates. An ideal diffuser would be characterized by an isentropic compression to lower velocities. This situation is sketched in Figure 2.8. Because the flow is isentropic the stagnation pressures at both ends of the diffuser would be equal. But it is extremely difficult to slow a supersonic flow without generating shock waves in the process. Moreover, in practical situation, the flow is viscous. As a result there will be an entropy increase within the boundary layers on the walls of the diffuser.

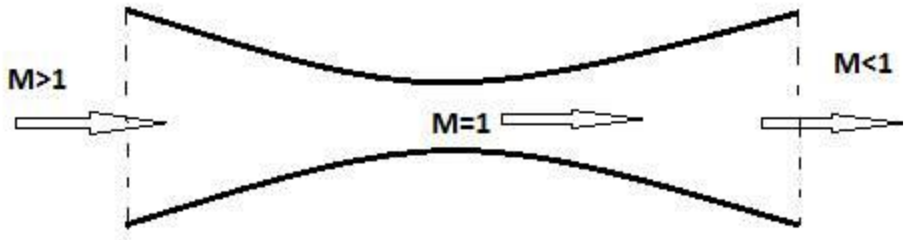
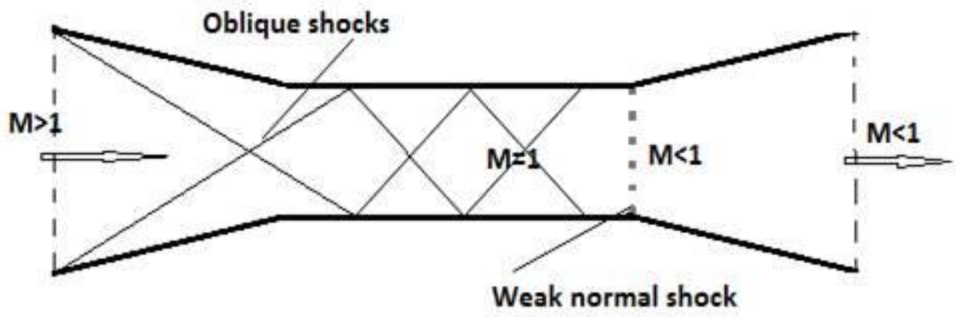


Fig 2.8 Schematic of ideal diffuser

1. An actual supersonic diffuser is sketched in fig 2.8. Here, the incoming flow is slowed by a series of reflected oblique shocks, first in a convergent section and then in a constant area throat (called as second throat). Due to the interaction of shock waves with the boundary layer near the wall, the reflected shock pattern eventually weakens and becomes diffused. Finally, the subsonic flow downstream of the constant area throat is further slowed by moving through a divergent section.



Schematic of ideal diffuser

2. It may be shown that a shock wave in the converging portion of the second throat would be unstable, and in practice the second-throat Mach number is chosen large enough for the breakdown shock system to be located well downstream of the throat, to ensure stability under all operating conditions. When the tunnel is started up, the second throat must be rather larger than the first throat in order to choke first, so that supersonic flow can be established in the test-section.

2.4 Shock Tube

The shock tube is a simple duct closed at both ends. A diaphragm divides this duct into two compartments called as driver and driven sections. Driver section of the shock tube is the high pressure section which is supplied by the high pressure gas from the reservoir. Driven section of the shock tube is the low pressure section which contains the low pressure driven gas. These two sections are separated by a metal diaphragm.

2.4.1 Working of Shock Tube

Typical shock tube experimental set up is as shown in Figure. A high pressure driver gas reservoir is connected with the driver section. However a vacuum pump is connected to the driven section to arrive at the accurate driven section pressure. Pressure sensors are generally mounted along the driven tube for pressure measurement. These pressure sensors are connected to the data acquisition system.

2.4.2 Operating steps for this set up are as follows.

1. Set the metal diaphragm between the driver and driven section.
2. Fill the driven gas in the driven section from the driven gas reservoir. This step is not required for case of air as the driven gas.
3. Set the required pressure in the driven section using vacuum pump.
4. Start filling the driver gas (say helium) in the driver section from the high pressure reservoir.

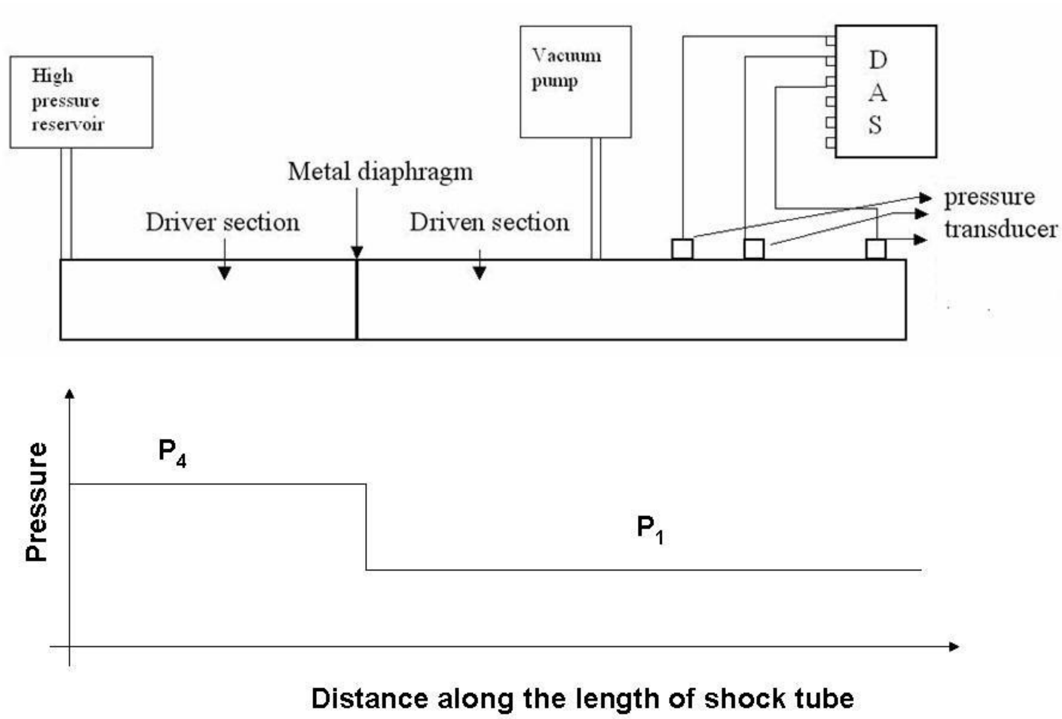
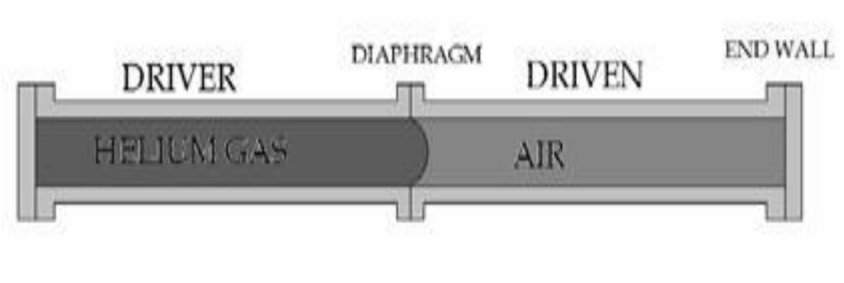
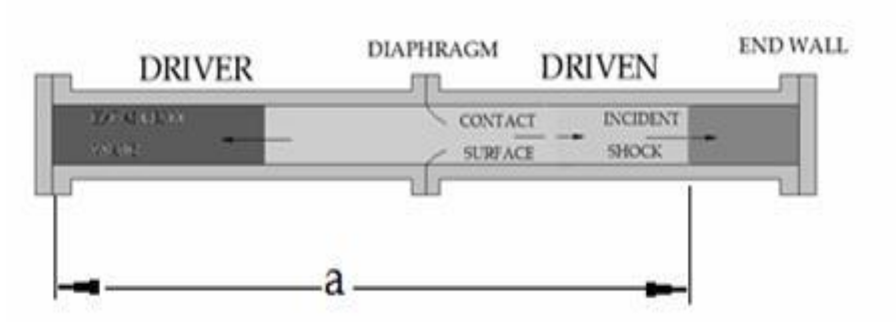


Fig 2.9 Schematic of the shock tube

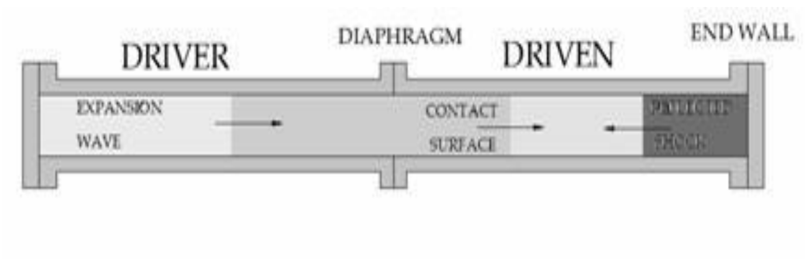
Metal diaphragm bursts at a particular driver and driven gas pressure difference. Dynamics of operation of shock tube after diaphragm burst is shown in steps in the following Fig.2.10.



(a) On set of diaphragm burst.



(b) Propagation of shock, expansion and contact surface in the shock tube



(c) Reflection of shock and expansion from the ends of the shock tube

Fig.2.10 Motion of various waves inside the shock tube

Thus shock tube gets operated solely by slowly increasing the driver gas pressure for a given driven pressure until the diaphragm gets ruptured. Burst of the diaphragm creates compression waves which propagate in the driven section and expansion waves traversing in the driver section. (Similar experience can be gained individually for compression and expansion in a tube with fixed and movable in presence of a piston. Sudden motion of the piston towards the closed end creates compression wave in the tube and sudden motion of the piston away from the closed end sets expansion of the inner fluid). However, in a short interval of time, compression waves coalesce to form the shock wave which propagate in the driven section. The stationary low pressure driven gas raises its pressure and temperature on arrival of the shock wave. Driver gas at the same time encounters the expansion in the presence of expansion waves. Moreover, the driver and driven gases do not mix due to the presence of contact surface which moves in the driven section. Thus contact surface separates the driver and driven gases and provides the position of diaphragm. The pressure and velocity are same across the contact surface. Both the expansion fan and shock reflect from the closed ends of the shock tube. Reflected shock cancels the motion of the driven fluid initiated by the primary shock. The strength of the shock wave and expansion fan thus produced depends on the initial pressure ratio across the diaphragm and the physical properties of the gases in the driver and driven sections. Higher the strength of the primary shock more is its speed in the shock tube which intern increases the pressure and temperature rise across it along with the pressure rise across the reflected shock.

Following are the notations of the regions between different waves in the shock tube

Region 1: This is the region infront of the primary shock.

Region 2: It is the region between contact surface and primary shock.

Region 3: This is the region between tail of the expansion fan and contact surface.

Region 4: It is the region ahead of the leading expansion wave.

Region 5: It is the region behind the reflected shock.

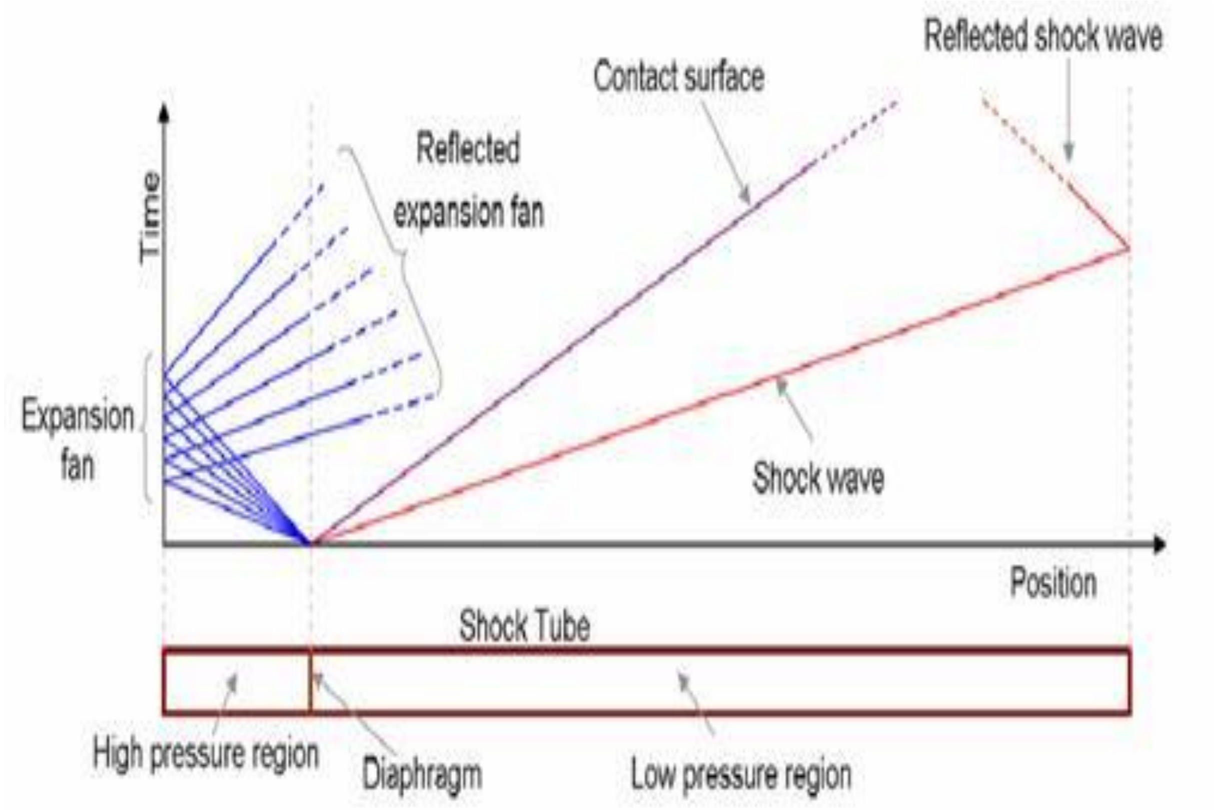


Fig 2.11 Space time diagram for a typical shock tube

2.4.3 Applications of shock tube:

Shock tube is very useful equipment and can be implemented for various applications in its various variants. Shock tube in its simplest format is used for the external flow aerodynamics studies in the higher subsonic, transonic and lower supersonic Mach numbers. Higher diameter tubes are advisable in these cases to avoid the domination of viscous effects. For the experimentation in supersonic or hypersonic flow regimes, driven tube end is connected to a convergent divergent nozzle along with the test section and dump tank to comprise a shock tunnel. This formation helps to expand the stagnant gas behind the reflected shock through the nozzle to achieve high speed flow in the test section. Shock tube can also be used for chemical kinetic studies for measurement of reaction rates. Use of shock tube for contact less drug delivery has also been thought for its application in medical sciences.

2.5 Shock Polar

In this section we are going to discuss about the graphical representation of pre and post shock velocities. Consider the supersonic flow of Mach number M_1 passing over a wedge of deflection angle θ.

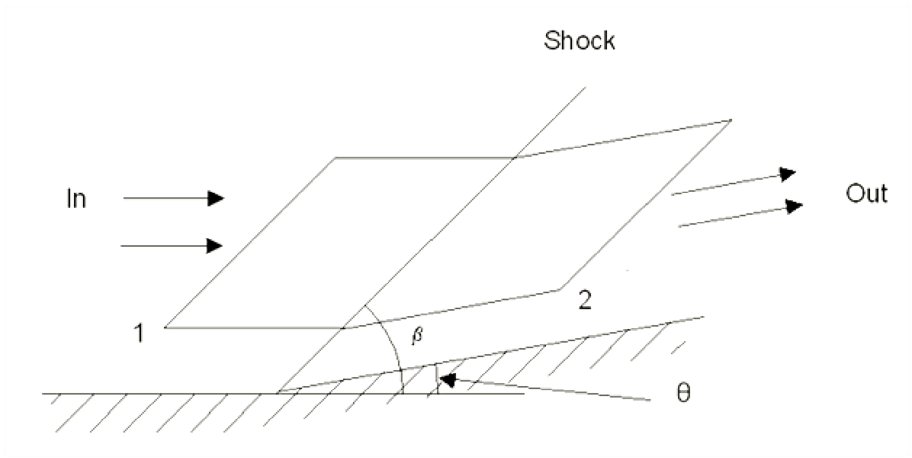


Fig 2.12 Supersonic flow over a wedge

Let V_x be the component of velocity in x-direction while V_y be the component of velocity in y direction. Hence V_{x1} is the x-direction velocity component for upstream velocity and V_{x2} is the x-direction velocity component for post shock velocity. Similarly V_{y1} and V_{y2} are the y-direction velocity component for upstream and post shock velocities respectively. If the upstream velocity is parallel to x axis then V_{y1} is equal to zero since V_{x1} is equal to V_1. Now let's plot velocities along their respective directions. Here the resultant velocity V_1 is parallel to x axis while V_2 makes an angle θ with x-axis, which is the flow deflection angle.

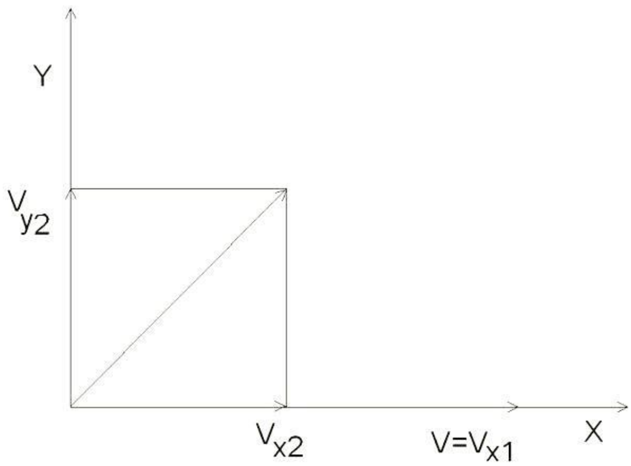


Fig 2.13 Velocity plot for shocked flow

Now if we increase the deflection angle then V_{x2} will decrease and V_{y2} will increase since in such cases resultant velocity V_2 will have to make an increased angle θ with the x-axis. Figure represents the same process. In the same way if we change the wedge angle for all possible attached shock solutions and join them together then such a plot is called as hodograph.

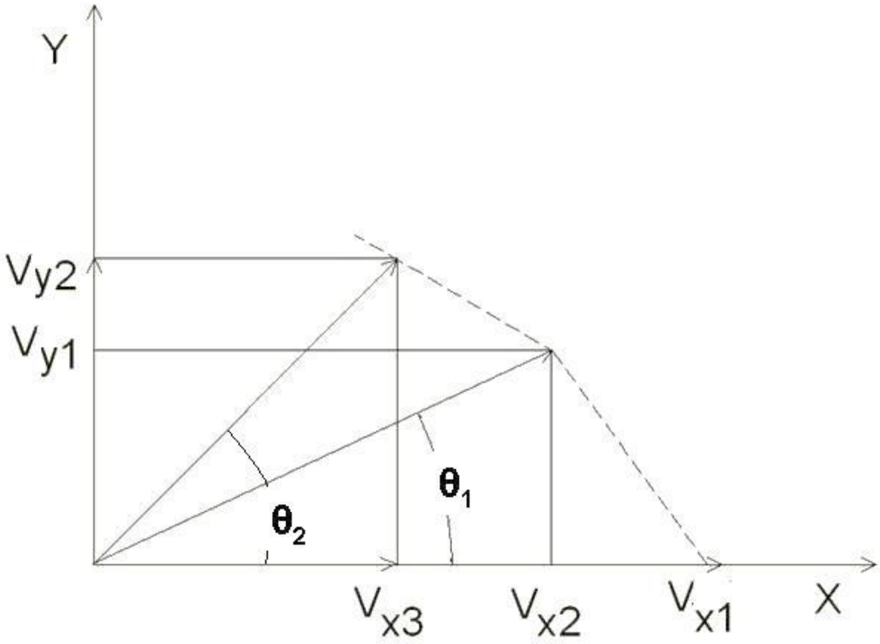


Fig 2.14 Velocity plot for various shocked flows

In the same way we can plot for various free stream Mach number for all the possible deflection angles. However, if we increase the freestream Mach number by increasing free stream velocity then it becomes impossible to plot such a graph when V_{x1} becomes ∞ for M_1 equal to ∞. To make such a plot possible, lets divide x axis and y axis by a^* which is the reference or stared quantity. This non-dimensionalisation makes it possible to represent V_{x1} since it will be represented by M_1^*. We can divide by a^* to post shock velocities as well, since a^* is constant in the flow field for an adiabatic flow. Hence V_2 gets transformed to M_2^* which will make an angle equal to the flow deflection angle with the x-axis. This plot is called as shock polar. Such a shock polar can be plotted for very high Mach numbers M_1 also due to the fact that, if M_1 is equal to ∞, M_1^* is equal to 2.54. Therefore this plot is equally helpful for representing the pre and post shock velocities, flow deflection angle and the shock angle.

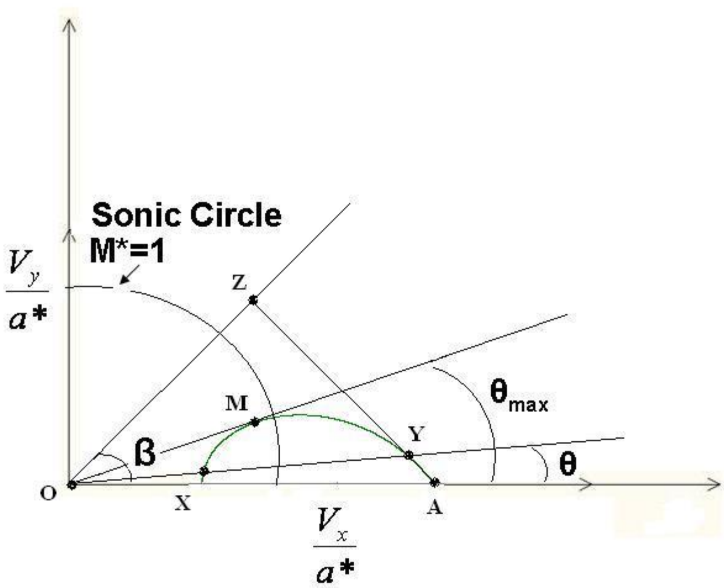


Fig 2.15 Typical Shock Polar

2.5.1 Analysis of Shock Polar

Shock polar can be used as a tool to evaluate the post shock properties from known pre shock conditions. We can calculate a^* and hence M_1^* which is the required pre shock input condition from the known freestream conditions. Hence using flow deflection angle θ and M_1^*, we can evaluate the post shock condition as, M_2^* using a given shock polar. Consider a shock polar given in Figure. Any line, say OXY, drawn from origin which intersects the given shock polar at two locations, X and Y. Point Y of line OXY represents higher value of M_2^* than the point X. Hence point Y represents the weak shock solution while point X on shock polar represents the strong shock solution. Angle made by line OXY gives the flow deflection angle. To find out the shock angle, we have to draw a line AY and extend the same same till it intersects a line (OZ) drawn from the origin at 90 degree. The angle made by line OZ with x-axis is the shock angle since this arrangement satisfies the constraint of equality of component of velocity parallel to the shock before and after the shock. If we draw any line OM, tangent to the shock polar, then the angle made by the line OM with x-axis represents maximum deflection angle possible for the given freestream Mach number. If we draw a circle of radius equal to a^* then such a circle cuts the shock polar at N which represents the Mach number equal 1 behind the shock.

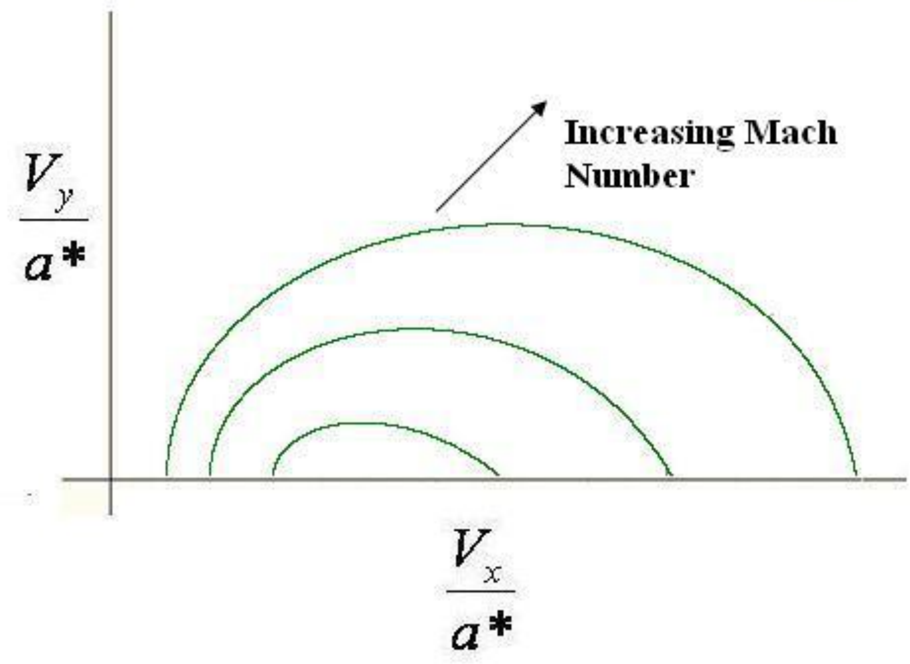


Fig 2.17 Shock polar for various Mach numbers

2.5.2 Applications of P-θ diagram

Pressure-deflection diagram is very useful for understanding various flow features like shock-shock interaction, shock reflection etc. Therefore the main objective of working with this diagram is to understand the deflection of flow which encounters various shocks. Consider a supersonic flow experiencing two left running shock waves..

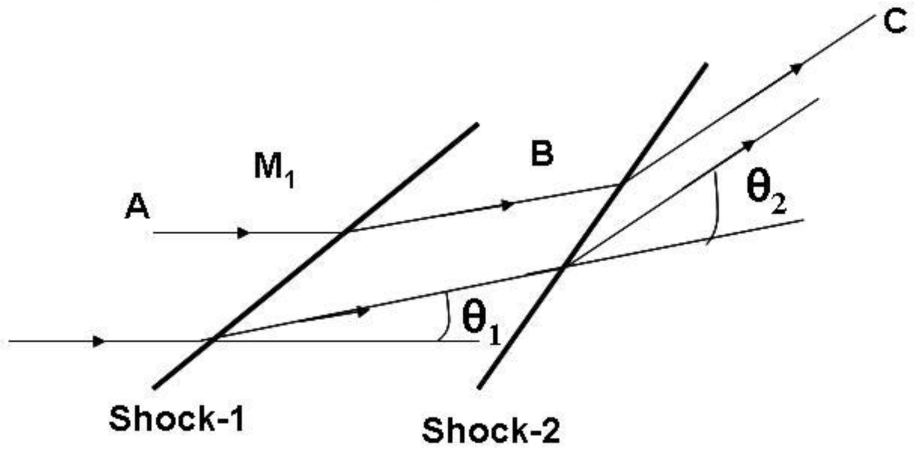


Fig 2.18 Supersonic flow passing through two left running shock waves

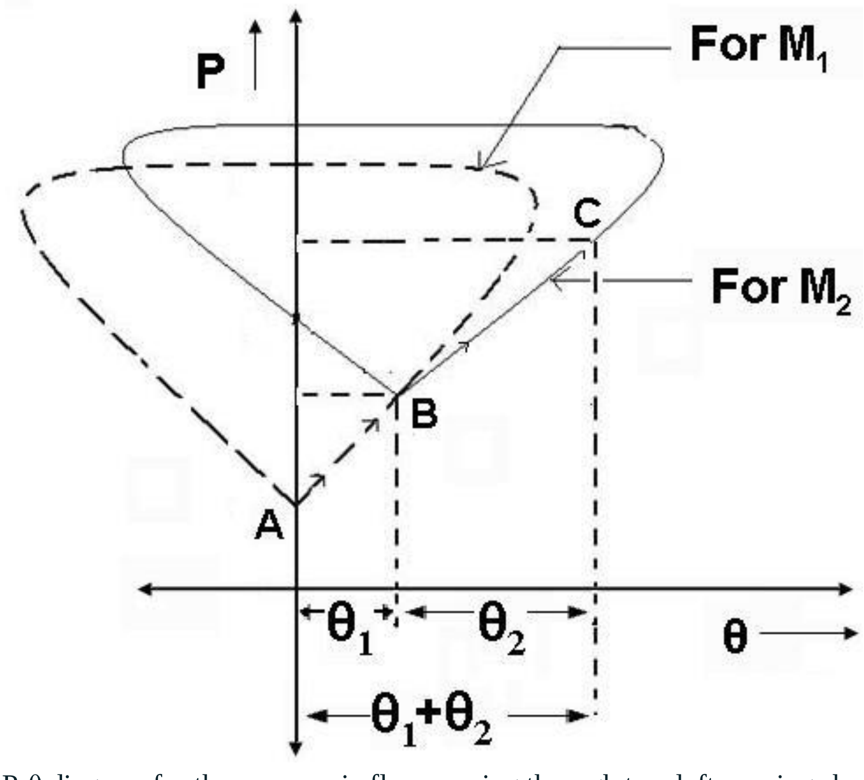


Fig 2.19 P-θ diagram for the supersonic flow passing through two left running shock waves

Here we can clearly understand the usefulness of this diagram. Consider a streamline ABC in this flow field. The initial or pre shock conditions are given point A on P-axis which corresponds to zero deflection. After passing through the first shock, the flow gets deflected by an angle θ1 which is represented by point B. We have to follow the P-θ diagram starting from point A, corresponding to its free stream conditions, to get the point B. The point B clearly portrays increase in pressure and positive deflection of the flow in the presence of left running shock wave. Further experience of next shock brings in the change in flow properties which are represented by point C. We have to draw a new P-θ diagram at point B, corresponding to its freestream conditions, to arrive at point C. For this fact, deflection of the flow at point B would be treated as zero and relative displacement between point B and

C (θ_2) should be plotted to get the point C. However state of the point C given in this diagram gives total flow deflection and final pressure.

Consider a supersonic flow experiencing a left running shock wave followed by a right running shock wave. Corresponding P-θ is shown in Figure. Consider a streamline XYZ in this flowfield.

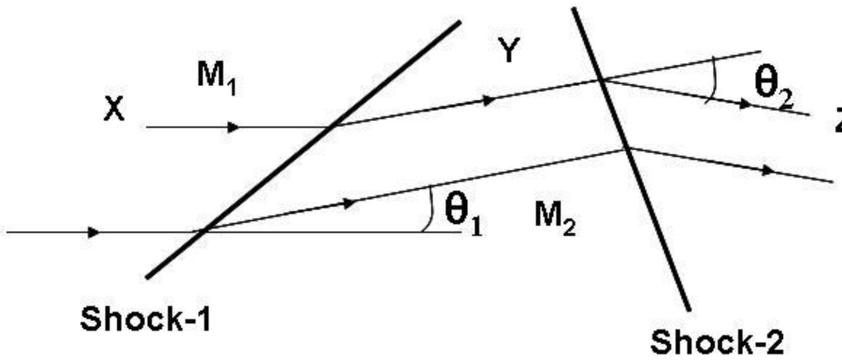


Fig 2.19 Supersonic flow passing through left and right running shock waves respectively

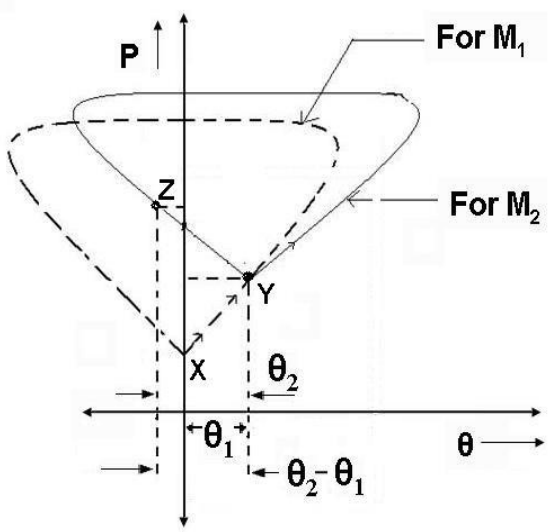


Fig 2.20 P-θ diagram for the supersonic flow passing through left and right running shock waves respectively

The initial or pre shock conditions are given point X on P-axis which corresponds to zero deflection. After passing through the first shock which is left running shock, the flow gets deflected by an angle θ1 which is represented by point Y. Here as well, we have to follow the P-θ diagram starting from point A, corresponding to its freestream conditions, to get the point Y. The point Y clearly portrays increase in pressure and positive deflection of the flow in the presence of left running shock wave. Further experience of right running shock wave brings in the change in flow properties which are represented by point Z. We have to draw a new P-θ diagram at point Y, corresponding to its freestream conditions, to arrive at point Z. For this fact, deflection of the flow at point Y would be treated as zero and relative displacement between point Y and Z (θ2) should be plotted to get the point Z. Since the flow is facing right running shock wave, the deflection θ2 is negative in reference with the origin at Y. However presence of the shock increases pressure of the flow. Therefore, state of the point Z given in this diagram gives total flow deflection and final pressure.

2.6 Supersonic Pitot Probes

2.6.1 Measurements of Flow Velocity

In most of the cases, the flow velocity is obtained through simultaneous measurement of static and stagnation pressures using a *Prandtl Pitot Static probe.* It has opening at the nose for stagnation pressure communications while several number of equal size holes are made around the circumference of the probe at the location downstream of the nose. The difference pressure gives the dynamic pressure. Further, Bernoulli equation can be applied to calculate the flow velocity.

$$\frac{dp}{\rho} + VdV + gdz = 0 \qquad \Rightarrow \int \frac{dp}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

Now, replace the integral of Equation with the isentropic relation for gases;

$$p = c\,\rho^{\gamma} \quad \Rightarrow \frac{dp}{\rho} = c^{\gamma} \rho^{\gamma - 2} d\rho$$

where, P, ρ and V are the pressure, density and velocity, respectively, z is the elevation difference, ϒ is the specific heat ratio and C is a constant. Combine Equation and simplify to obtain the Bernoulli equation for one-dimensional frictionless isentropic flow for compressible fluid.

$$\frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

Apply Equation along a stream line at the location of stagnation point and any desired location to obtain the flow velocity.

$$\frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{V_\infty^2}{2} = \frac{\gamma}{\gamma - 1}\frac{p_0}{\rho_0} \qquad \Rightarrow V_\infty = \left[\frac{2\gamma}{\gamma - 1}\left(\frac{p_0}{\rho_0} - \frac{p}{\rho}\right)\right]^{-\frac{1}{2}}$$

The subscripts, 0 and ∞ refer to stagnation and free stream conditions, respectively. Had the flow been incompressible, the density term in Equation becomes constant quantity and the stagnation and static pressure difference is expressed as follows:

$$p_0 - p = \frac{1}{2}\rho V_\infty^2 \quad \Rightarrow V_\infty = \sqrt{\frac{2(p_0 - p)}{\rho}}$$

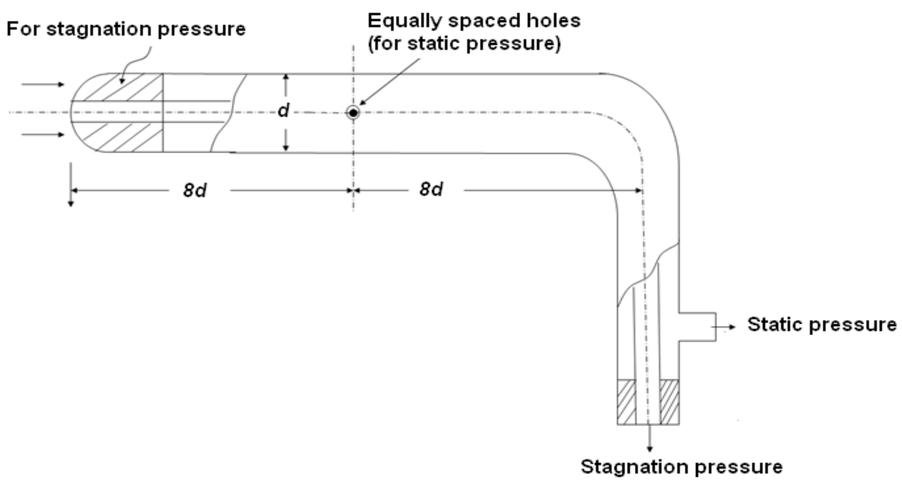


Fig2.21 Prandtl Pitot static probe for simultaneous measurement

2.6.2 Measurements for Subsonic and Supersonic Flows

The flow Mach number is one of the important parameter for subsonic and supersonic flows. All the flow parameters and their variations are the functions of local Mach number (M). The pressure measurements are one of the common practices to determine the Mach number. In subsonic flow, the simultaneous measurement of static(P) and stagnation pressures (p_0)using a *Prandtl Pitot Static tube* are made in a similar way as shown in Figure. Subsequently, the isentropic relation is used to determine the flow Mach number.

$$\frac{P_0}{P_\infty} = \left(1 + \frac{\gamma - 1}{2} M_\infty^2\right)^{\frac{\gamma}{\gamma-1}}$$

The characteristic feature of a supersonic flow is the formation of a shock wave. So, the introduction of a *Pitot probe* into the flow stream, leads to a detached bow shock. Due to this shock wave at certain distance from the measurement location, the stagnation pressure located indicated by the probe will be much higher than the stagnation pressure of the free stream. For the stagnation stream lines, the curved shock is normal to the free stream and the measured value represents the stagnation pressure downstream of the normal shock (P_{02}). While conducting experiment, the static pressure (P_∞) of the free stream (upstream of the shock) is also measured simultaneously by any of the methods. However, the static pressure measurement must be done far upstream of the shock so that its influence on the measurement will be minimized. The Mach number relation connecting the static and stagnation pressure measurements is expressed by *Rayleigh-Pitot* formula for supersonic flows.

$$\frac{P_{02}}{P_\infty} = \frac{\left(\frac{\gamma-1}{2} M_\infty^2\right)^{\frac{\gamma}{\gamma-1}}}{\left(\frac{2\gamma}{\gamma+1} M_\infty^2 - \frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{\gamma-1}}}$$

The *Rayleigh-Pitot formula* with air as free stream is presented graphically in figure. The dynamic pressure(P_d)obtained from static pressure and the Mach number is then given by the following expression.

$$\frac{P_d}{P_\infty} = \frac{\gamma M_\infty^2}{2}$$

Thus, the Mach number calculation through static and stagnation measurements gives complete information of a supersonic flow field.

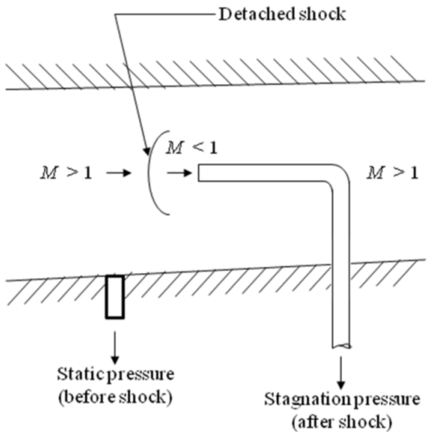


Fig 2.22 Detached shocks ahead of the measuring pressure probe in a supersonic flow.

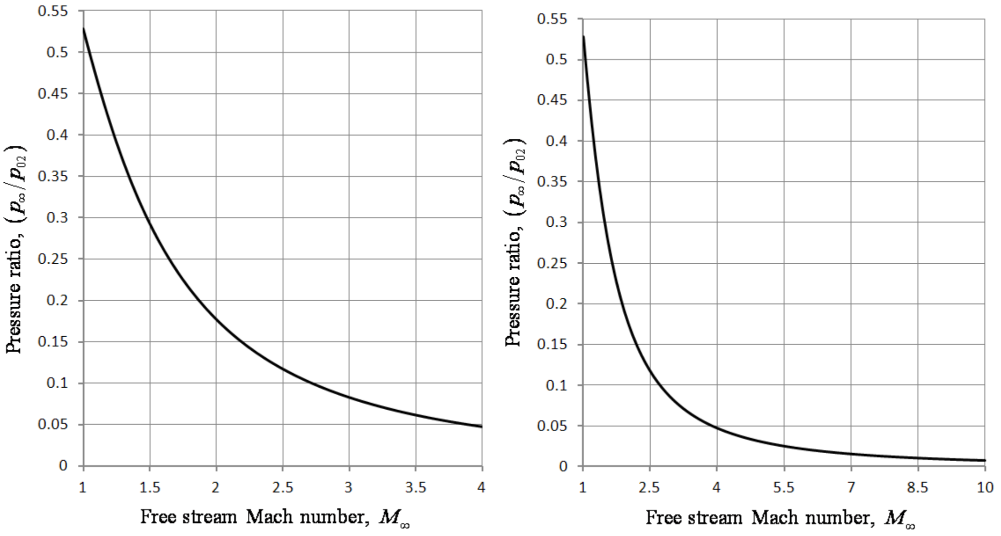


Fig 2.22 Mach number determination from Pitot tube measurement in a supersonic flow

2.6.3 Sonic Nozzle: It is an obstruction device often used to measure high flow rates for gases. When the flow rate is sufficiently high, the pressure differential is also expected to be large. Under this condition, a sonic flow condition is achieved at the minimum flow area and the flow is said to be *choked.* Such a device is known as *sonic nozzle.* In this case, the flow rate takes the maximum value for a given inlet condition. If this inlet refers to a reservoir pressure(p_0), temperature(T_0) and the flow is said to be choked at certain area(A*), then the pressure at this location(P*) can be obtained from isentropic relation,

$$\frac{p^*}{p_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

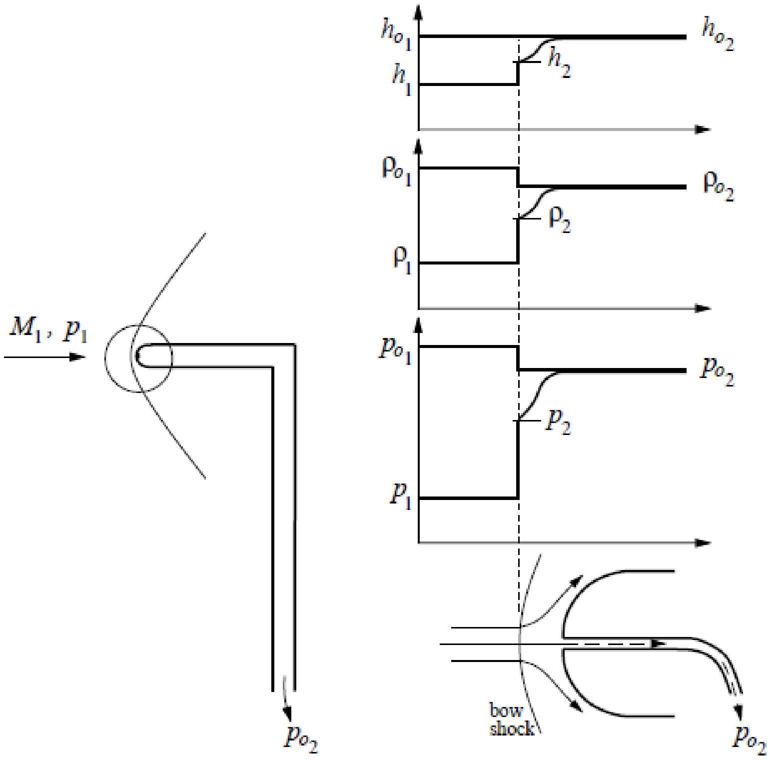
This relation is known as *critical pressure ratio for a choked nozzle.* The choked mass flow rate can be obtained by the following expression,

$$\dot{m}^* = \frac{p_0 A^*}{\sqrt{T_0}} \sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

By designing the geometric parameter of a sonic nozzle, it is possible to achieve the discharge coefficient up to 0.97 corresponding to theoretical expression of flow rate.

A pitot probe in a supersonic stream will have a *bow shock* ahead of it. This complicates theflowmeasurement,sincethebowshockwillcauseadropinthetotalpressure,fromp_{o1} to p_{o2} , the latter being sensed by the pitot port. It's useful to note that the shock will also cause a drop in ρ_o, but h_o will notchange.

The pressures and Mach number immediately behind the shock are related by



In addition, we also have M_2 and p_2/p_1 as functions of M_1 from the earlier normal-shock analysis. Combining these produces the relation between the p_{o_2}measured by the pitot probe,thestaticp_1,andtherequiredflowMachnumberM_1.Aftersomemanipulation,theresult is the *Rayleigh Pitot tube formula*.

The figure shows p_{o_2}/p_1 versus M_1, compared with the isentropic ratio p_{o1} $/p_1$.OnlythelatterisplottedforM_1<1,wherethereisnobowshock,andsoequationdoes notapply. Theeffectofthepitotbowshock'stotalpressureloss,indicatedbythedifference $p_{o1} - p_{o2}$, becomes substantial at larger Machnumbers.Ideally, we would like to have the pitot formula give M_1 as an explicit function of the pitot/static pressure ratio p_{o_2}/p_1. However, this is not possible due to its complexity, so a numerical solution is required.

2.7 OBLIQUE SHOCKS

2.7.1 Introduction

The shock encountered by such flow is the normal shock. Herewith we are going to deal the compressible flows in two dimensions, 2D. There are two entities which are going to tackle the expansion and compression of the 2D compressible flow. Such situations are two situations which the flow might encounter. If the flow turns in to itself then the compression of flow takes place through shock. Since this shock makes certain angle with the flow, it is called as the oblique shock. However if the flow turns away from itself then the expansion of the flow takes place through an expansion fan. Deriving various relations for angles or properties ratios for the two dimensional compressible flows in the presence of shock or expansion fan is the central objective here onwards.

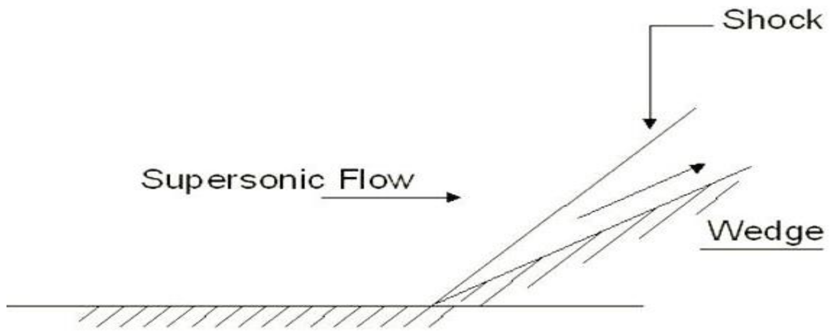


Fig 2.23 Supersonic flow turning into itself in the presence of shock

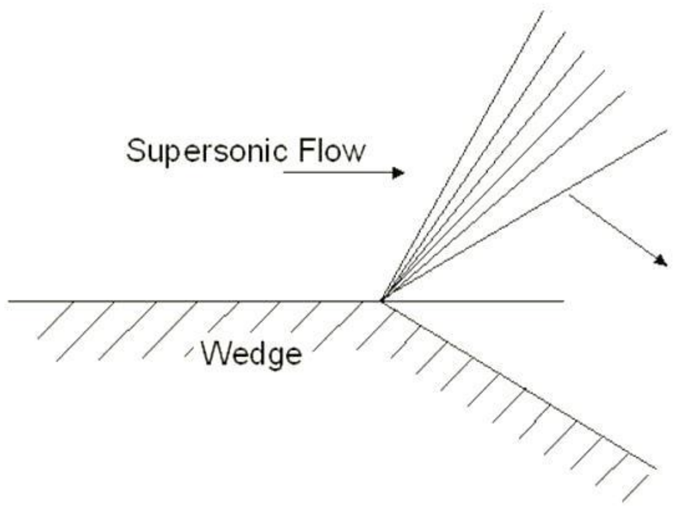


Fig 2.24 Supersonic flow turning away from itself in the presence of expansion fan

The main reason of having shock or expansion fan at the deflection corners is to deflect the flow accordingly so as to maintain its parallelism with the wall in order to avoid the direct collision. Therefore the angle of shock with the wall or freestream velocity will be dependant on flow deflection angle and freestream velocity.

2.7.2 Mach Wave

Let's consider a train traveling along its track from location A to location B. If it makes a beep sound while leaving station A and at intermediate locations C and D then the sound wave travels spherically out with the velocity (say) 'a' m/s. The distance traveled by the sound waves originated from A,C and D locations are given by the circles drawn, considering respective points as centers. Now if the speed of the train is less than acoustic velocity (a), say v m/s, then the train will always remain inside the circles. This statement also means that the distance travels by the train in some time interval is always lower than the distance traveled by the acoustic wave in the same time interval. Now if we consider the speed of train to be more than speed of sound then the train will always cross the respective circle of acoustic speed in that time interval. Now if we draw the tangent to all circles, representing position of acoustic wave, from the end location of train (B), that such a tangent makes an angle μ with the train track.

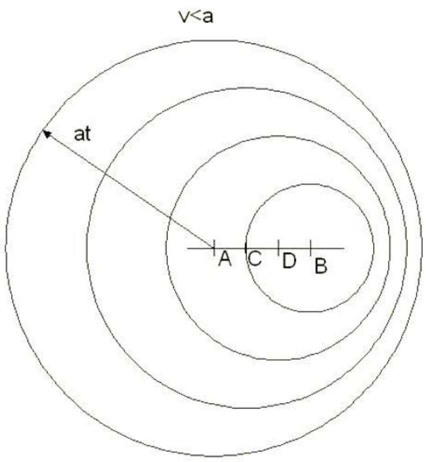


Fig 2.25 Train travelling from A to B at subsonic speed

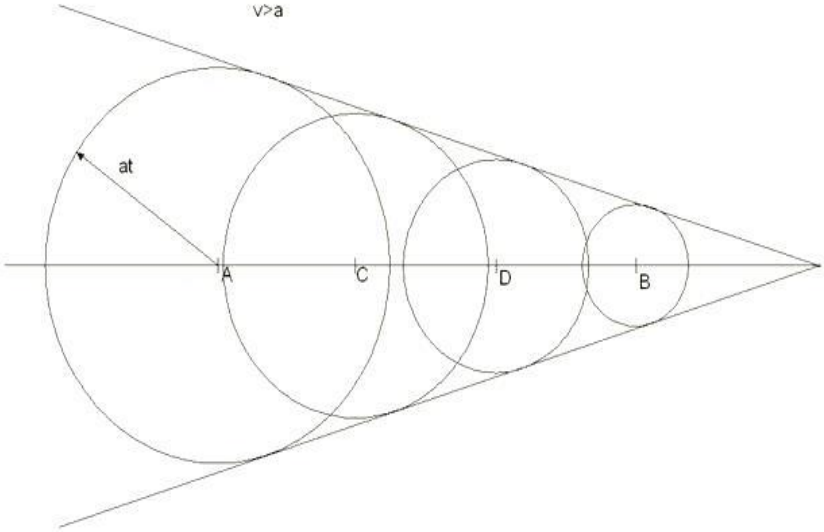


Fig 2.26 Train travelling from A to B at supersonic speed

The expression for this angle is as,

$$\sin(\mu) = \frac{at}{vt}$$

$$\sin(\mu) = \frac{1}{M}$$

$$\mu = \sin^{-1}\frac{1}{M}$$

This angle (μ) is called as Mach angle and it becomes the reference angle for our following discussions. Shock angle which we are going to understand is always more than the Mach angle

2.7.3 Governing equations for Oblique Shock

Consider the flow taking place along a wedge as shown in Figure. Let θ be the wedge angle and β be the shock angle with the wall which is parallel to the approaching freestream.

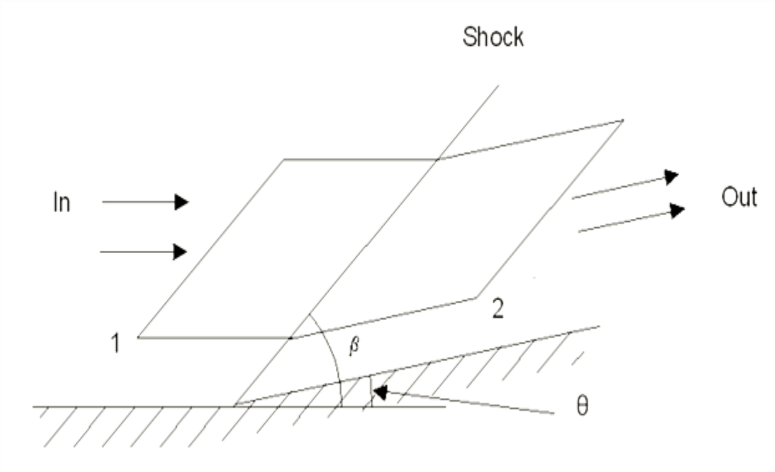


Fig 2.27 An oblique shock for a supersonic flow over the wedge

As we have already proved that shock exists only for supersonic flows, consider a supersonic flow of Mach number M_1 approaching the wedge. In the presence of the shock, flow deflects by an angle θ which is the wedge angle. Lets solve the mass, momentum and energy equations for this flow. Consider the control volume as shown in Figure. In this special control volume, inlet and outlet are parallel to the shock. Other two faces of the control volume are parallel to the streamline hence these faces will not contribute to the mass, momentum and energy fluxes. Let u be the velocity normal to the shock and w be the velocity parallel to the shock. Graphical demonstration of these velocities is given in Figure. Station 1 corresponds to inlet or preshock conditions while station 2 corresponds to outlet or post shock conditions.

$$\oiiint_v \frac{\partial \rho}{\partial t} d\upsilon + \oiint_s \rho V.ds = 0$$

Mass conservation in integral form

Lets assume the flow to be steady, hence,

$$\oiint_s \rho V.ds = 0$$

$$(\rho V.ds)_{post} - (\rho V.ds)_{pre} = 0$$

$$(\rho V.ds)_{post} = (\rho V.ds)_{pre}$$

But

$$(V.ds)_{post} = u_2 A \quad \text{and} \quad (V.ds)_{pre} = u_1 A$$

$$(\rho u)_2 = (\rho u)_1 \text{ or } \rho_1 u_1 = \rho_2 u_2$$

Hence,

This is the mass conservation equation for oblique shock conditions expressed in terms of velocities normal to the shock.

Now consider the momentum conservation equation for the same flow. Since momentum is the vector equation, we have to consider, two equations, viz, normal and parallel to the shock. Let's initially consider the momentum equation in integral form for inviscid flow.

$$\frac{\partial}{\partial t}\iiint^{v} \rho\vec{V}\,dv + \iint^{s}(\rho\vec{V}.\overrightarrow{ds})\vec{V} = -\iint^{s}(p\overrightarrow{ds})$$

For steady flow,this equation becomes,

$$\iint^{s}(\rho\vec{V}.\overrightarrow{ds})\vec{V} = -\iint^{s}(p\overrightarrow{ds})$$

Now consider the momentum equation in the direction parallel to the shock wave. Since there is no pressure difference in this direction, the right hand side will be zero. Hence,

$$\iint^{s}(\rho\vec{V}.\overrightarrow{ds})w = \mathbf{0}$$

$$\iint^{s}(\rho\vec{V}.\overrightarrow{ds})w = \mathbf{0}$$

but using mass conservation Equation we can re-write it as,

$$(w)_{pre} = (w)_{post} \text{ or } w_1 = w_2$$

This expression clearly suggests that velocity parallel to the shock remains conserved.

Now consider the momentum equation normal to the shock.

$$\iint^{s}(\rho\vec{V}.\overrightarrow{ds})u = -\iint^{s}(pds)$$

$$\iint^{s}(\rho uu + p)ds = \mathbf{0}$$

$$\iint^{s}(p + \rho u^2)ds = \mathbf{0}$$

$$(p + \rho u^2)_{post} = (p + \rho u^2)_{post}$$

Or

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$$

We can clearly see that the momentum equation looks exactly same as that for the normal shock relations. Here u is the velocity normal to the shock. Therefore only velocity component normal to the shock wave is responsible for the change in momentum since momentum and velocity tangential to shock are conserved.

2.7.4 θ-β-M relation

It has been already observed that the Mach number normal to the shock is responsible for all the property variations for given shock angle. However this shock angle can be easily calculated from the upstream or freestream Mach number for given wedge or deflection angle. Consider the same control volume shown in Figure. Relation between velocities and angles before and after the shocks are,

$$\tan\beta = \frac{u_1}{w_1}$$ Before the shock

$$\tan(\beta - \theta) = \frac{u_2}{w_2}$$ After the shock

But we know that,

$$w_1 = w_2$$

Hence,

$$\frac{\tan(\beta - \theta)}{\tan\beta} = \frac{u_2}{u_1}$$

But

$$\frac{u_2}{u_1} = \frac{\rho_1}{\rho_2}$$

$\rho_1 u_1 = \rho_2 u_2$ hence,

Therefore,

$$\frac{\tan(\beta - \theta)}{\tan(\beta)} = \frac{\rho_1}{\rho_2}$$

From density ratio Equation, we have

$$\frac{\tan(\beta-\theta)}{\tan\beta}=\frac{(\gamma+1)M_{n1}^{2}}{2+(\gamma-1)M_{n1}^{2}}$$

From the expression of upstream Mach number, $M_{n1}=M_1\sin\beta,$

$$\frac{\tan(\beta-\theta)}{\tan\beta}=\frac{2+(\gamma-1)M_1^2\sin^2\beta}{(\gamma+1)M_1^2\sin^2\beta}$$

$$\tan\theta=2\cot\beta\left[\frac{M_1^2\sin^2\beta-1}{M_1^2(\gamma+\cos 2\beta)+2}\right]$$

This is the expression between upstream Mach number, shock angle and wedge angle. In most general case, we need to know the shock angle for given Mach number and wedge angle. Following figure provides the information about the same. In this figure, each curve corresponds to various possible shock angles for a given Mach number and flow deflection angle.

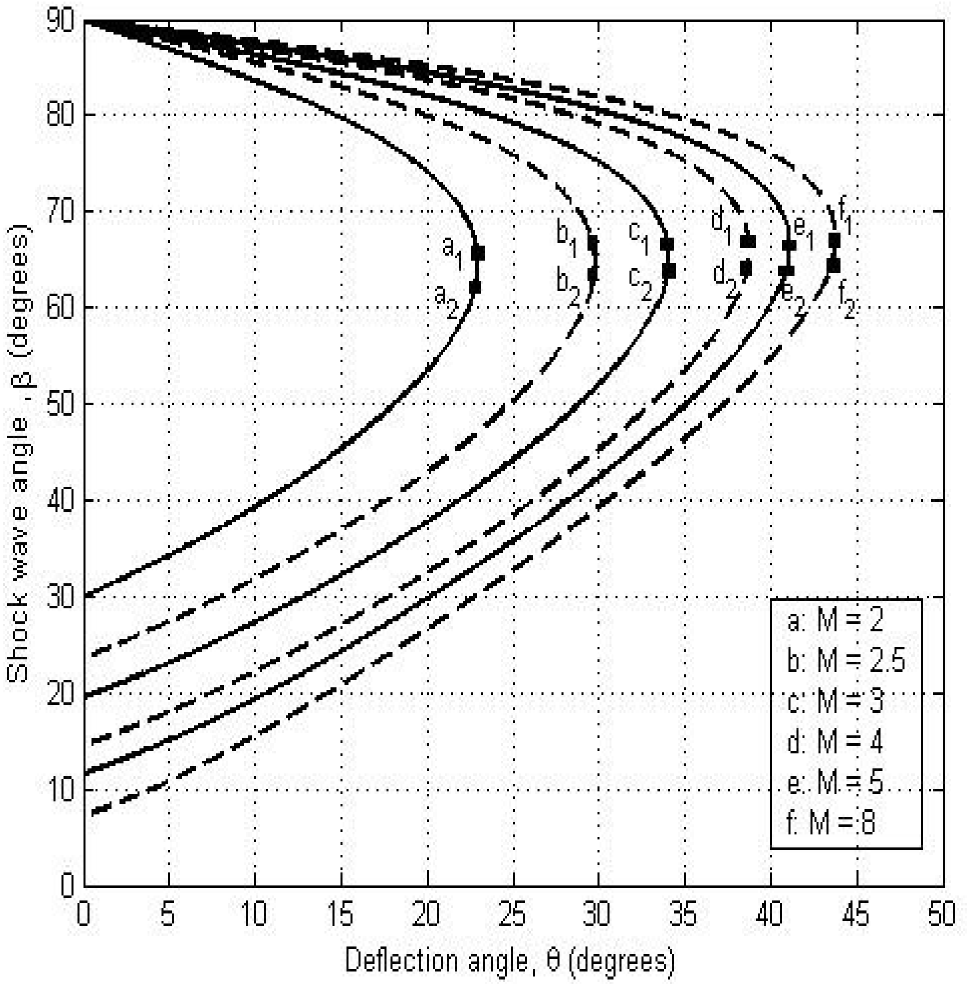


Fig 2.28 θ-β-M relation

2.7.5 Following points should be noted from this figure

For a given Mach number and wedge angle, there are two possible shock angles. The lower shock angle corresponds to a weak shock solution and higher shock angle corresponds to strong shock solution. Weak shock solution is the most familiar solution or situation we encounter in the nature. Higher shock angle values, obtained out of two possible solutions for the given Mach number and deflection angle, is referred as the strong shock solution. The strong shock solution may exist for a given Mach number and deflection angle if the pressure on the wedge can be increased independently. However this situation is most uncommon. Points having subscript 1 in Figure. are the reference points for the known Mach number above which shock is said to be strong shock and below which shock is said to be weak shock. Points having subscript 2 in Figure are the reference points for the known Mach number above which shock flow behind the shock is subsonic and below which flow behind the shock is supersonic.

If we consider the weak shock solution, then for a given Mach number, increase in deflection angle increases the shock angle. Thus Increased shock angle decreases the Mach number behind the shock. Hence shock strength increases for a given Mach number with increase in deflection angle. For a particular deflection angle, we get, maximum shock angle for a given Mach number above which there is no attached shock solution. Hence there exists no solution or deflection angle for that Mach number for which shock is attached.

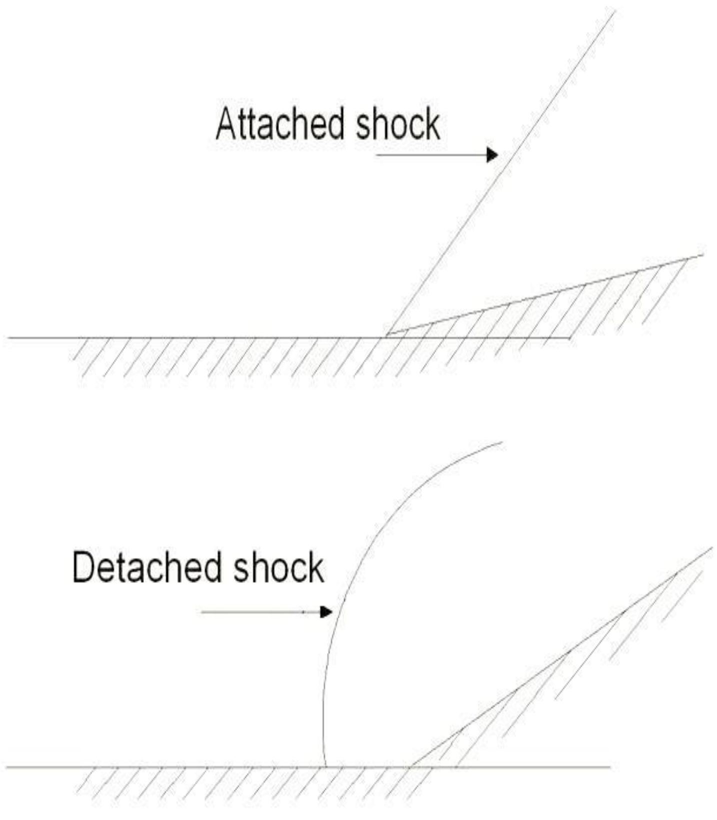


Fig 2.29 Attached and detached shock solutions.

If we consider the weak shock solution, then, for a given deflection angle, increase in Mach number decreases the shock angle and shock becomes weaker.

All the curves for various Mach numbers can be seen to meet at 90 degree shock angle for zero degree deflection angle. This situation corresponds to normal shock. However, the other solution for zero degree deflection angle corresponds to Mach angle, μ=sin-1(1/M). Therefore intersection of the curve with shock angle for weak solution for all the Mach number is different.

The discontinuities in supersonic flows do not always exist as normal to the flow direction. There are oblique shocks which are inclined with respect to the flow direction. Refer to the shock structure on an obstacle, as depicted qualitatively.

The segment of the shock immediately in front of the body behaves like a normal shock.

Oblique shock can be observed in following cases-

1. Oblique shock formed as a consequence of the bending of the shock in the free-stream direction
2. In a supersonic flow through a duct, viscous effects cause the shock to be oblique near the walls, the shock being normal only in the core region.
3. The shock is also oblique when a supersonic flow is made to change direction near a sharp corner

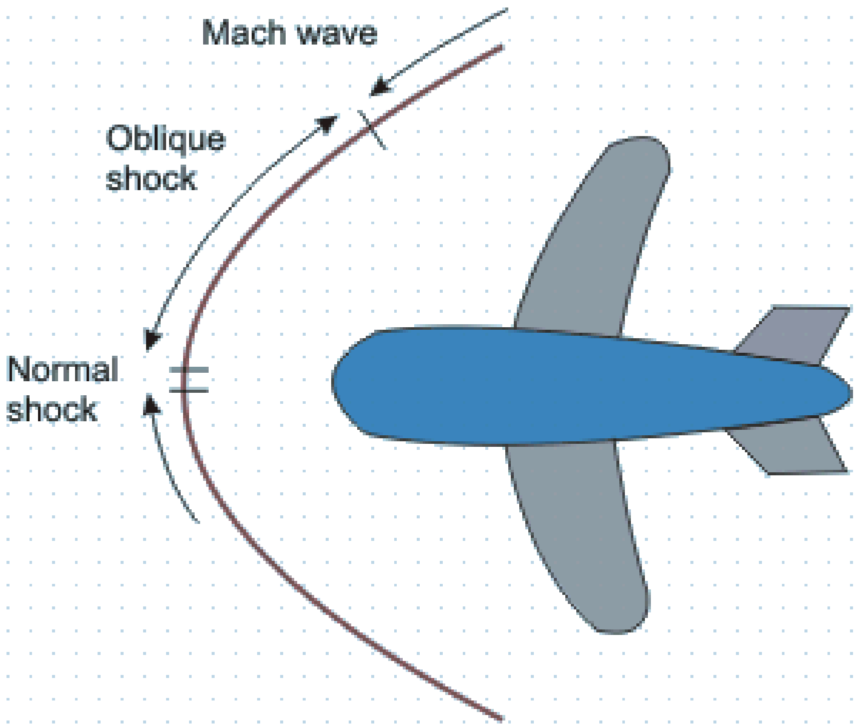


Fig 2.30 Normal and oblique Shock in front of an Obstacle

- The relationships derived earlier for the normal shock are valid for the velocity components normal to the oblique shock. The oblique shock continues to bend in the downstream direction until the Mach number of the velocity component normal to the wave is unity. At that instant, the oblique shock degenerates into a so called Mach wave across which changes in flow properties are infinitesimal.

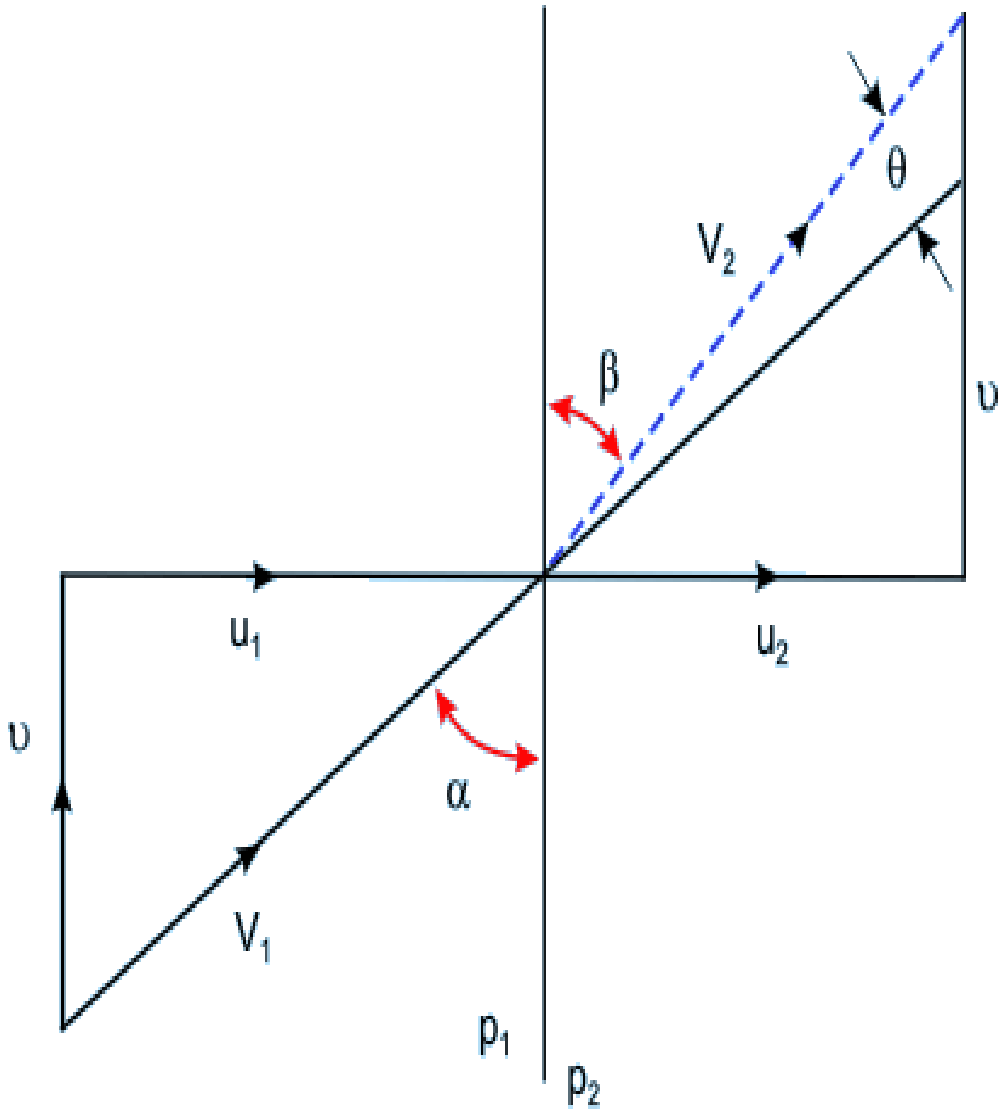


Fig 2.31 Two dimensional Oblique Shock

For analyzing flow through such a shock, it may be considered as a normal shock on which a velocity v (parallel to the shock) is superimposed. The change across shock front is determined in the same way as for the normal shock. The equations for mass, momentum and energy conservation, respectively, are

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1 (u_1 - u_2) = p_2 - p_1$$

$$h_{01} = h_{02}$$

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

$$\frac{\gamma}{\gamma-1} \cdot \frac{p_1}{\rho_1} + \frac{u_1^2}{2} = \frac{\gamma}{\gamma-1} \cdot \frac{p_2}{\rho_2} + \frac{u_2^2}{2}$$

These equations are analogous to corresponding equations for normal shock. In addition to these, we have

$$\frac{u_1}{a_1} = Ma_1 \sin\alpha$$

And

$$\frac{u_2}{a_2} = Ma_2 \sin\beta$$

Modifying normal shock relations by writing $Ma_1 \sin\alpha$ and $Ma_2 \sin\beta$ in place of Ma_1 and Ma_2 , we obtain

$$\frac{p_2}{p_1} = \frac{2\gamma Ma_1^2 \sin\alpha - \gamma + 1}{\gamma + 1}$$

$$\frac{u_2}{u_1} = \frac{\rho_1}{\rho_2} = \frac{\tan\alpha}{\tan\beta} = \frac{\gamma - 1}{\gamma + 1} + \frac{2}{(\gamma + 1) + Ma_1^2 \sin^2\alpha}$$

$$Ma_2^2 \sin^2\beta = \frac{2 + (\gamma - 1) Ma_1^2 \sin^2\alpha}{1 + \tan^2\alpha (\tan\beta / \tan\alpha)}$$

Note that although $Ma_2 \sin\beta < 1$, Ma_2 might be greater than 1. So the flow behind an oblique shock may be supersonic although the normal component of velocity is subsonic.

In order to obtain the angle of deflection of flow passing through an oblique shock, we use the relation

$$\tan\theta = \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \cdot \tan\beta}$$

$$= \frac{\tan\alpha - (\tan\beta / \tan\alpha) \tan\alpha}{1 + \tan^2\alpha (\tan\beta / \tan\alpha)}$$

Having substituted ($\tan\beta/\tan\alpha$) from Equation, we get the relation

$$\tan\theta = \frac{Ma_1^2 \sin 2\alpha - 2\cot\alpha}{Ma_1^2 (\gamma + \cos 2\alpha) + 2}$$

Sometimes, a design is done in such a way that an oblique shock is allowed instead of a normal shock. The losses for the case of oblique shock are much less than those of normal shock.This is the reason for making the nose angle of the fuselage of a supersonic aircraft small.

2.8 Shock Reflection

Reflection of shock wave from the wall is mainly of two types, regular reflection and irregular or Mach reflection.

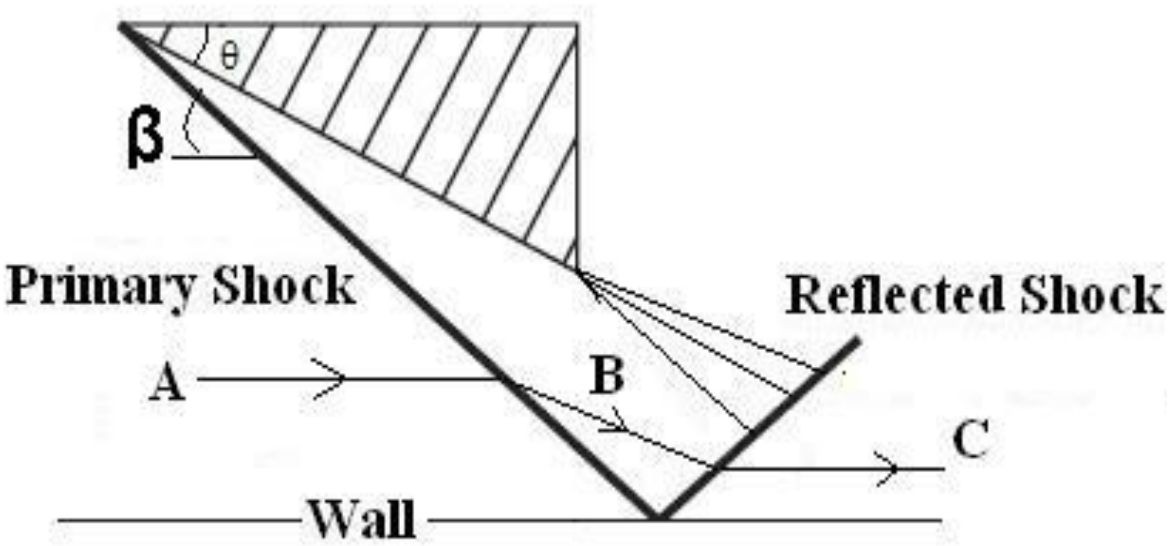


Fig 2.32 Typical regular reflection of shock wave

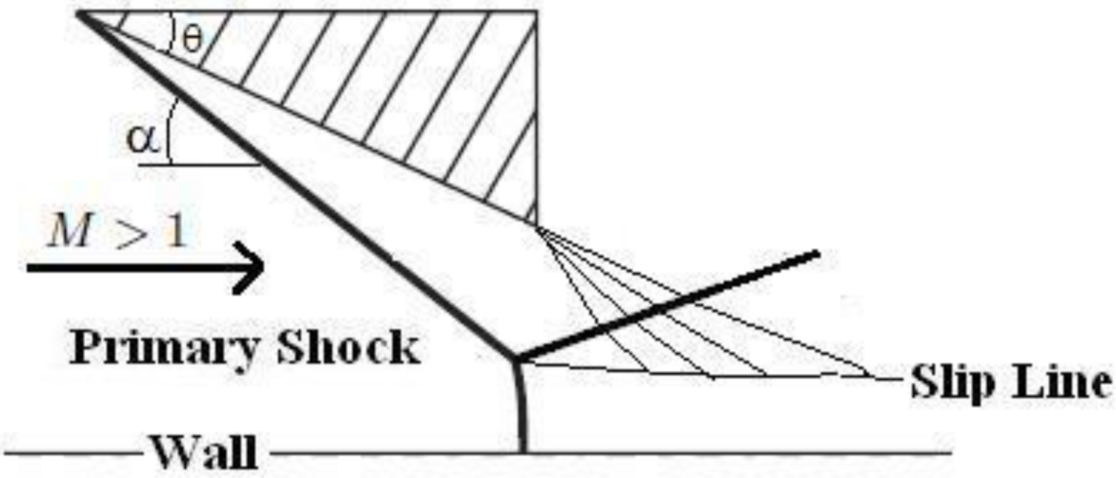


Fig 2.33 Typical Mach reflection of shock wave

The shock that originates from the corner of the compression wedge makes an angle β with the freestream velocity vector. Flow behind this shock gets deflected by angle θ and becomes parallel to the wedge. However, when the shock hits the top wall, the part of the flow which was parallel to the top wall, upstream of the shock, also gets deflected by angle θ. Hence the flow downstream of the shock does not remain parallel to the top wall. This contradicts the necessary condition of formation of shock, hence the primary shock gets reflected from the top wall to cancel out the deflection of the flow and to make it parallel to the wall. In this process shock reflection can be observed as the reflection of light ray, however the incident and reflection angle of shock are dependent on shock upstream Mach number. Therefore this reflection is called as the regular reflection. Typical pressure deflection diagram for the streamline ABC.

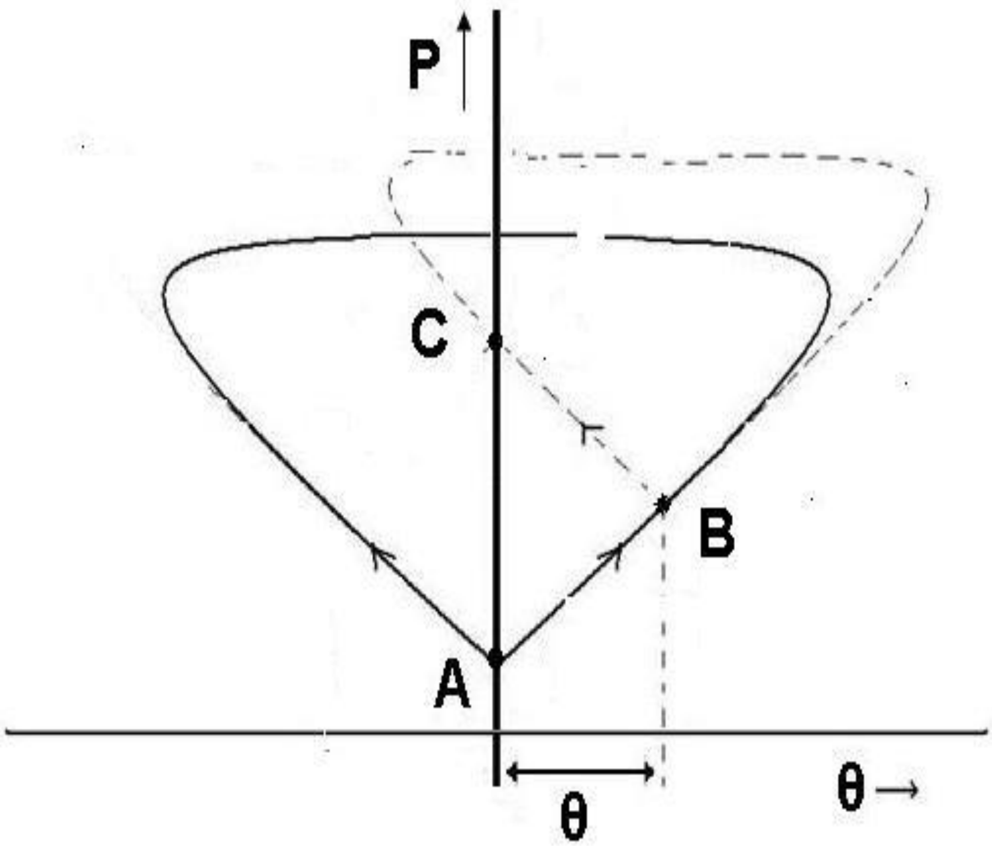


Fig 2.34 Pressure deflection diagram for the regular reflection case

Providing equal and opposite deflection to the flow as that provided by incident shock is the objective of reflected shock. However possibility of such a deflection behind the reflected shock depends on the Mach number behind the incident shock or Mach number upstream to the reflected shock. If the deflection required is more than the maximum possible deflection for that Mach number then the reflection of shock does not remain as regular reflection since the reflected shock gets detached from the wall.

2.8.1 Mach Reflection

Let's understand the reason for deviation from regular reflection to Mach reflection. We know from Figure that the presence of reflected shock is mainly to deflect the flow in region B and make it parallel with the wall. The amount of this flow deflection expected from the reflected shock is same as that from the primary or incident shock. However, regular reflection ensures that this deflection is less than the maximum possible defection corresponding to the Mach number in region B in the same figure. A situation has been shown in Figure which shows the possibility of where the expected deflection using reflected shock more than the maximum deflection possible corresponding to Mach number in region B.

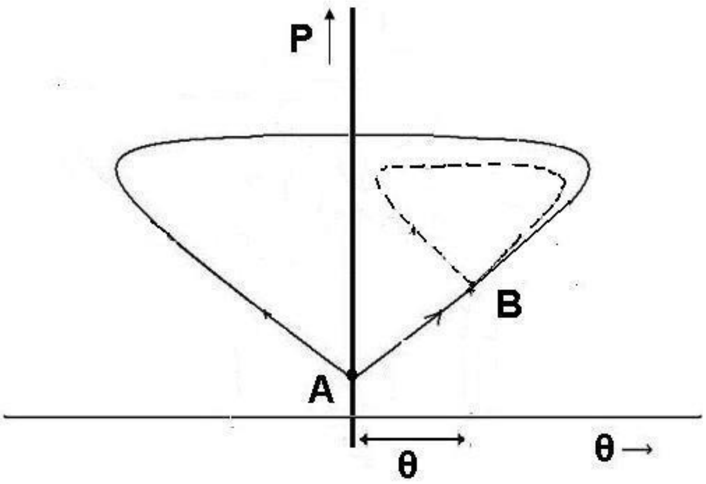


Fig 2.35 Reason for Mach reflection using pressure deflection diagram

Therefore, since reflected shock cannot provide the necessary deflection to the flow so as to make it parallel to the wall, the incident or primary shock does not reflect like the ray of light or regular reflection.

2.8.2 Supersonic expansion

Expansion Fan

We have already seen that compression of supersonic flow takes while passing through the shock. In other words, when the supersonic flow turns into itself then it undergoes the compression through a shock. Exactly opposite situation can be encountered when the supersonic flow turns out of itself where, expansion of the supersonic flow takes place. This expansion unlike compression takes place smoothly through infinite expansion waves hence called as expansion fan. This expansion fan is comprised of infinite number of expansion waves or Mach waves where every wave is responsible for infinitesimal amount of deflection. Where supersonic flow turns outward by an angle θ.

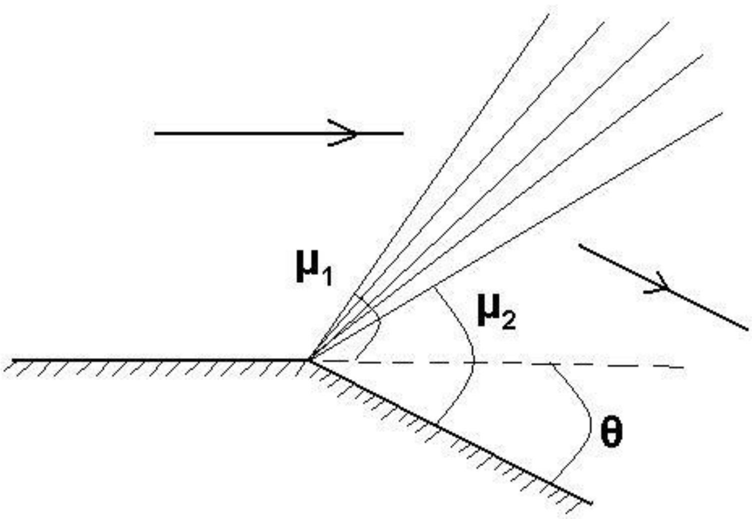


Fig 2.36 Schematic of expansion fan

For better understanding of expansion of supersonic flow, consider that p_1, T_1 and M_1 be the properties of flow before expansion or upstream of the expansion fan and p_2, T_2 and M_2 be the properties of the flow after expansion or downstream of the expansion fan due to outward deflection by an angle θ. For the know upstream flow properties and deflection angle it should be possible for us to calculate the downstream flow properties. Since the expansion is the continuous and smooth process carried out via infinite Mach waves, let's consider one such wave across upstream of which velocity is V and Mach number is M. Angle made by this Mach wave with the upstream velocity vector is μ. Let's consider dV be the change in velocity brought by the Mach wave by turning through an angle dθ. Hence V+dV is downstream velocity and M+dM is the downstream Mach number. If we construct the velocity triangle as shown in Figure,then we can use the sin law for triangle as,

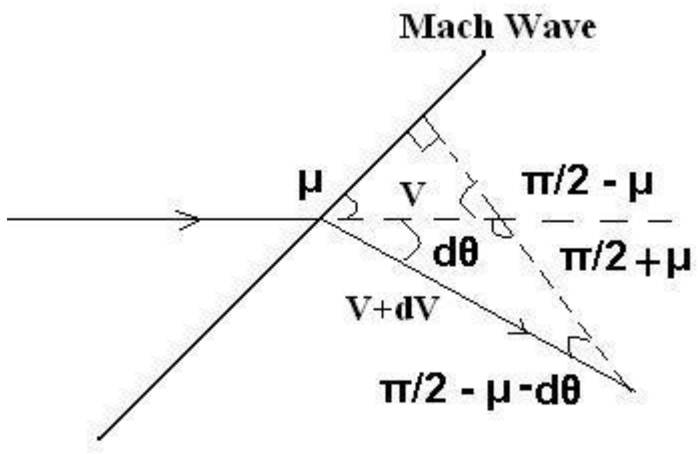


Fig 2.37 Velocity triangle across a typical Mach wave during supersonic expansion

$$\frac{V+dV}{V}=\frac{\sin\left(\frac{\pi}{2}+\mu\right)}{\sin(\frac{\pi}{2}-\mu-d\theta)}$$

But we know that

$$\sin\left(\frac{\pi}{2}+\mu\right)=\sin\left(\frac{\pi}{2}-\mu\right)=\cos\mu$$

and

$$\sin(\frac{\pi}{2}-\mu-d\theta)=\cos\left(\mu+d\theta\right)=\cos\mu\cos d\theta-\sin\mu\sin d\theta$$

Hence we can re-write Equation as,

$$1+\frac{dV}{V}=\frac{\cos\mu}{\cos\mu\cos d\theta-\sin\mu\sin d\theta}$$

We can approximate as

sin dθ ≈ dθ and cos dθ ≈ 1, Therefore Equation can be simplified as,

$$1+\frac{dV}{V}=\frac{\cos\mu}{\cos\mu-d\theta\sin\mu}=\frac{1}{1-d\theta\tan\mu}$$

since dθ tan μ < 1, lets recall the expansion for x<1,

$$\frac{1}{1-x}=1=x+x^2+x^3+....$$

Neglecting higher order term, we can express Equation as,

$$1+\frac{dV}{V}=1+d\theta\tan\mu+...$$

$$d\theta=\frac{dV/V}{\tan\mu}$$

But we know that

$$\mu=\sin^{-1}\frac{1}{M}$$

and hence

$$\tan\mu=\frac{1}{\sqrt{M^2-1}}$$

Hence above equation becomes,

$$d\theta=\sqrt{M^2-1}\,\frac{dV}{V}$$

2.9 PRANDTL-MEYER EXPANSION FLOW

Prandtl Meyer Function

We can see that since for positive value of dθ, we get positive dV which leads to expansion. This formula is also valid for small angles for compression where we get negative dV. If we integrate this formula for the total expansion angle then we can get the downstream Mach number.

$$\int_{\theta_1}^{\theta_2} d\theta=\int_{M_1}^{M_2}\sqrt{M^2-1}\,\frac{dV}{V}$$

Before integrating we can express the integrant in Mach number,

$$V = Ma$$

$$\ln V = \ln M + \ln a$$

$$\frac{dV}{V}=\frac{dM}{M}+\frac{da}{a}$$

We can express here the second term on right hand side in terms of Mach number using the isentropic relations as,

$$\left(\frac{a_o}{a}\right)^2=\frac{T_0}{T}=1+\frac{\gamma-1}{2}M^2$$

$$a=a_o\left(1+\frac{\gamma-1}{2}M^2\right)^{-1/2}$$

$$\frac{da}{a}=-\left(\frac{\gamma-1}{2}\right)M\left(1+\frac{\gamma-1}{2}M^2\right)^{-1}dM$$

$$\int_{\theta_1}^{\theta_2} d\theta = \theta_2 - 0 = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}$$

Integration of right hand side is as,

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)} - \tan^{-1}\sqrt{M^2-1}$$

Here, v is called as the Prandtl-Meyer function.

$$\theta = v(M_2) - v(M_1)$$

Therefore upstream Mach number (M_1) we can calculate the upstream Prandtl-Meyer function. Hence for known flow deflection angle and upstream Mach number we can get the downstream Prandtl-Meyer function and hence the downstream Mach number.

Process of expansion of supersonic flow is an isentropic process. However, while passing through the expansion fan, pressure, temperature and density of the flow decreases while Mach number and velocity increases for the supersonic flow. Moreover, all the total properties remain constant. We can calculate the total pressure, temperature and density upstream of the expansion using isentropic relations for the known flow Mach number. From the calculated downstream Mach number, we can calculate all the static flow properties from known stagnation or total properties.

2.10 Shock Expansion Method for Flow over Airfoil

It is possible to solve many problems in two-dimensional supersonic flow by patching together appropriate combinations of the oblique shock wave, and the Prandtl-Mayer expansion fan. For example, let us consider the flow over a simple two-dimensional airfoil section.

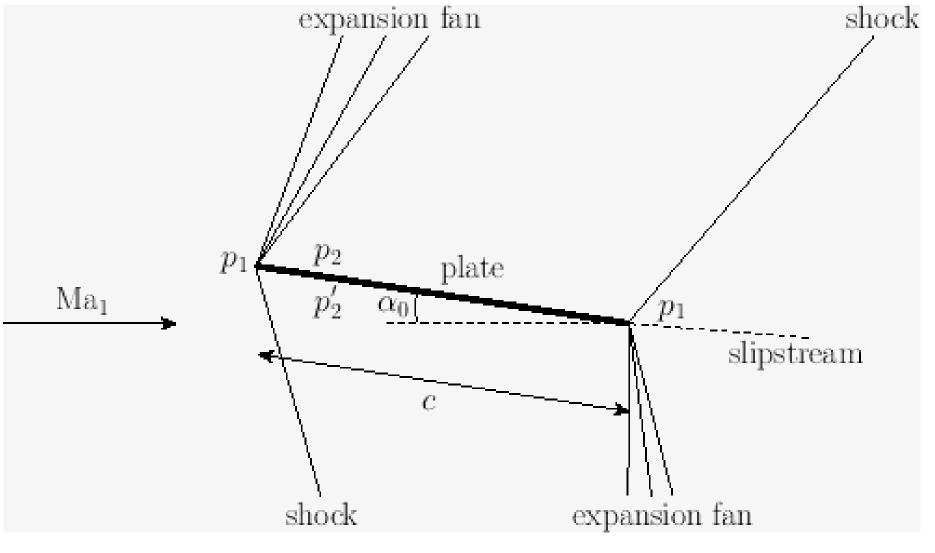


Fig2.37: A flat lifting plate. Ma_1 is the upstream Mach number, p_1 , p_2 , et cetera, denote pressures, and α_o is the angle of attack.

Figure shows a flat plate inclined at an *angle of attack*, α_o, to the oncoming supersonic flow. The streamline ahead of the leading edge is non-inclined, because there is no upstream influence. Moreover, the flow streams over the upper and lower surfaces are completely independent of one another. Thus, the flow on the upper surface is turned through an expansion angle $\alpha_o$by means of a Prandtl-Mayer expansion fan attached to the leading edge of the airfoil, whereas the flow on the lower side is turned through a compression angle α_o by means of an oblique shock. The flow on the upper surface is recompressed to the upstream pressure p_1 by means of an oblique shock wave attached to the trailing edge of the airfoil. Likewise, the flow on the lower surface is re-expanded to the upstream pressure by means of an expansion fan. The uniform pressures,p_2 andP_2' , respectively, on the upper and lower surfaces of the airfoil can easily be calculated by means of oblique shock theory and Prandtl-Mayer expansion theory. Given the pressures, the lift and drag per unit transverse length of the airfoil are simply

$$L = (P_2' - P_2)\, c \cos\alpha_0,$$

$$D = (P_2' - P_2)\, c \sin\alpha_0,$$

respectively, where c is the chord-length (i.e., width). The increase in entropy of the flow along the upper surface of the airfoil is not the same as that for the flow along the lower surface, because the upper and lower shock waves occur at different Mach numbers. Consequently, the streamline attached to the trailing edge of the airfoil is a *slipstream*--that is, it separates flows with the same pressures, but slightly different speeds, temperatures, and densities--inclined at a small angle relative to the free stream. (The angle of inclination is determined by the requirement that the pressures on both sides of the slipstream be equal to one another.)
Note that the drag that develops on the airfoil is of a completely different nature to the previously discussed drags that develop on subsonic airfoils, such as friction drag, form drag, and induced drag. This new type of drag is termed *supersonic wave drag*, and exists even in an idealized, inviscid fluid. It is ultimately due to the trailing shock waves attached to the airfoil.

Comparatively far from the airfoil, the attached shock waves and expansion fans intersect one another. The expansion fans then attenuate the oblique shocks, making them weak and curved. At very large distances, the shock waves asymptote to free-stream Mach lines. However, this phenomenon does not affect the previous calculation of the lift and drag on a flat-plate airfoil.

The Physical Picture of the Flow through a Normal Shock

- It is possible to obtain physical picture of the flow through a normal shock by employing some of the ideas of Fanno line and Rayleigh line Flows. Flow through a normal shock must satisfy Equations.

- Since all the condition of state "1" are known, there is no difficulty in locating state "1" on T-s diagram. In order to draw a Fanno line curve through state "1", we require a locus of mathematical states that satisfy Equations. The Fanno line curve does not satisfy Equation.

- While Rayleigh line curve through state "1" gives a locus of mathematical states that satisfy Equations. The Rayleigh line does not satisfy Equation. Both the curves on a same T-s diagram are shown in Figure.

- We know normal shock should satisfy all the six equations stated above. At the same time, for a given state" 1", the end state "2" of the normal shock must lie on both the Fanno line and Rayleigh line passing through state "1." Hence, the intersection of the two lines at state "2" represents the conditions downstream from the shock.

- In Figure, the flow through the shock is indicated as transition from state "1" to state "2". This is also consistent with directional principle indicated by the second law of thermodynamics, i.e. $s_2 > s_1$.

- From Figure, it is also evident that the flow through a normal shock signifies a change of speed from supersonic to subsonic. Normal shock is possible only in a flow which is initially supersonic.

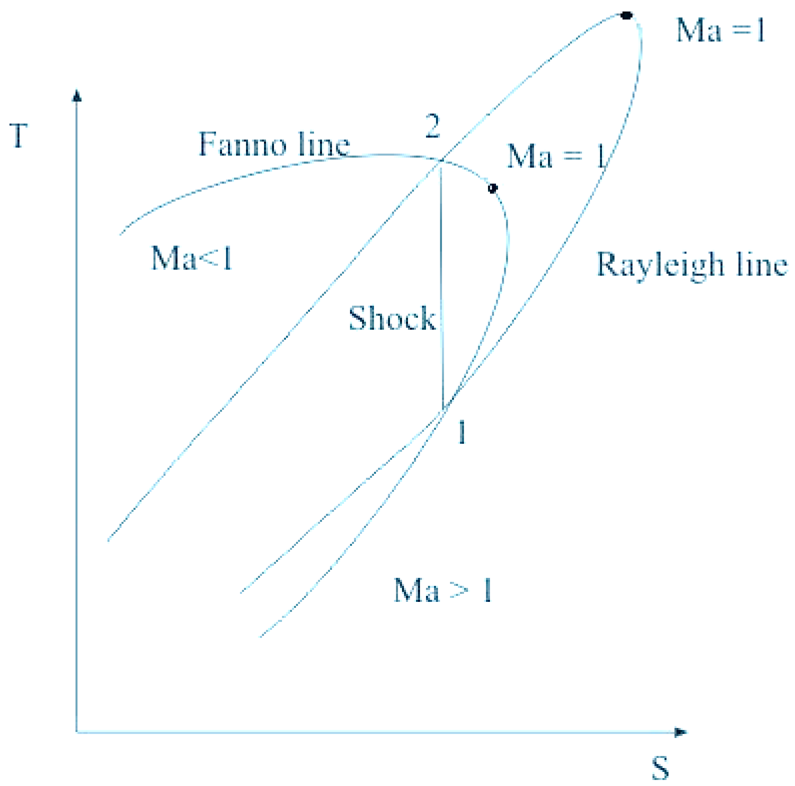


Fig 2.38 Intersection of Fanno line and Rayleigh line and the solution for normal shock condition

UNIT III

QUASI-ONE DIMENSIONAL FLOWS

3.1 Isentropic flow in nozzles

One and two dimensional compressible flows have already been seen in earlier sections. This section deals with the quasi-one dimensional flows. For Rayleigh and Fanno flows we had considered the flow to be one dimensional. The main reason for this assumption is the constancy of the cross sectional area of duct in the direction of flow. If area of the duct varies in the direction of flow then flow should be treated as three dimensional flows. However if such a variation is gradual then we can neglect the changes taking place in cross stream directions. Such flows are called as quasi-one dimensional flows.

Example of a strictly one dimensional and a quasi-one dimensional flow is given in Figure.

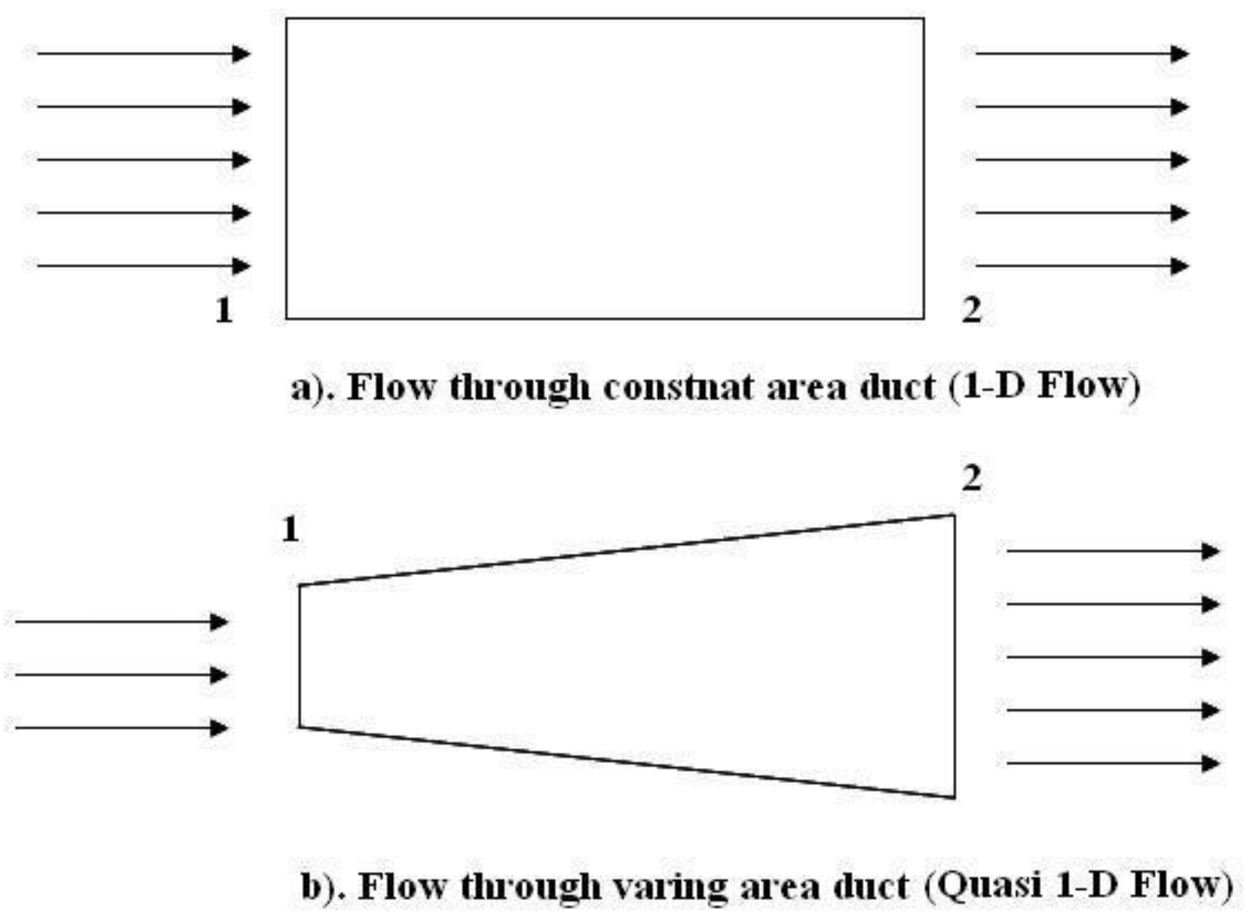


Fig 3.1 Control volume for 1D and Quasi 1D flows

Consider a typical control volume shown in Figure. (b) where inlet and outlet properties of the flow are given by subscripts 1 and 2 respectively.

3.1.1 Mass conservation equation

The integral form of mass conservation equation is,

$$\frac{\partial}{\partial t}\iiint\limits^{v} \rho\, dv + \iint\limits^{s} (\rho\vec{V}.\overrightarrow{ds}) = \mathbf{0}$$

For the steady flow, above equation reduces to,

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$$

$$\rho uA = const$$

Hence the differential form of mass conservation equation is

$$d(\rho uA) = 0$$

3.1.2 Momentum conservation equation

The integral form of momentum conservation equation for inviscid flow is,

$$\frac{\partial}{\partial t}\iiint\limits^{v} \rho\vec{V}\, dv + \iint\limits^{s} (\rho\vec{V}.\overrightarrow{ds})\vec{V} = -\iint\limits^{s} (p\overrightarrow{ds})$$

For the steady flow this equation transforms to,

$$p_1A_1 + \rho_1 u_1^2 A_1 + \int_{A_1}^{A_2} p\,dA = p_2A_2 + \rho_2 u_2^2 A_2$$

Here integration of the pressure is on the stream wise surfaces. This equation is not the usual algebraicequation. The differential form of this equation can be written as,

$$pA + \rho u^2 A + pdA = (p+dp)(A+dA) + (\rho+d\rho)(u+du)^2(A+dA)$$

Neglecting the higher order terms we get,

$$Adp + Au^2 d\rho + \rho u^2 dA + 2\rho uAdu = 0$$

We can expand the equation as

$$uAd\rho + u\rho dA + \rho Adu = 0$$

multiplying this equation by u we get,

$$\rho u^2 dA + \rho uAdu + Au^2 d\rho = 0$$

Subtracting the above equation from Eq. We get,

$$dp = -\rho\, udu$$

3.1.3 Energy conservation equation

The integral form of energy conservation equation for inviscid adiabatic flow is,

$$\frac{\partial}{\partial t}\iiint_{v} \rho\left[e+\frac{V^2}{2}\right]dv + \iint_{s}(\rho\vec{V}.\overrightarrow{ds})\left[e+\frac{V^2}{2}\right] = -\iint_{s}(p\vec{V}.\overrightarrow{ds})$$

For the steady flow above equation reduces to,

$$\left(\rho_1\left[e_1+\frac{u_1^2}{2}\right]\right)(-u_1A_1) + \left(\rho_2\left[e_2+\frac{u_2^2}{2}\right]\right)(u_2A_2) = -(-p_1u_1A_1 + p_2u_2A_2)$$

$$p_1u_1A_1 + \left(\rho_1u_1A_1\left[e_1+\frac{u_1^2}{2}\right]\right) = p_2u_2A_2 + \left(\rho_2u_2A_2\left[e_2+\frac{u_2^2}{2}\right]\right)$$

But we know that,

$$h = e + \frac{P}{\rho}$$

Hence,

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

Or

$$\mathrm{h} + \frac{\mathrm{u}^2}{2} = \mathrm{const\ and\ h_0 = const}$$

The differential form of energy equation is,

dh +udu = Constant

Area-velocity relation for Quasi-One Dimensional flow

Consider the differential mass conservation equation

$$d(\rho\, uA) = 0$$

Expanding this equation we get,

$$uA[d(\rho)] + \rho u[d(A) + \rho A[d(u)] = 0$$

dividing both the sides by puA=const , we get

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0$$

To obtain the relation between velocity and area we need to replace the term dρ/ρ of above equation. We know differential form or momentum equation as,

$$dp = -\rho\, udu$$

$$\frac{dp}{\rho} = \frac{dp}{d\rho}\frac{d\rho}{\rho} = -udu$$

But we know that,

$$\frac{dp}{d\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s = a^2$$

hence, momentum equation becomes

$$a^2 \frac{d\rho}{\rho} = -udu$$

$$\frac{d\rho}{\rho} = -\frac{udu}{a^2} = -\frac{u^2 du}{a^2 u} = -M^2 \frac{du}{u}$$

Using the above equation, can be written as,

$$\frac{dA}{A} = \left(M^2 - 1\right)\frac{du}{u}$$

3.1.4 Many conclusions can be drawn from this equation

1. For positive dA, du will be positive for M>1. Hence for supersonic flows, velocity of the flow increases with increase in area or divergent portion acts as the nozzle. Similarly, for convergent portion acts as the diffuser for supersonic flows.

2. For negative dA, du will be positive for M<1. Hence for subsonic flows, velocity of the flow increases with decrease in area or convergent portion acts as the nozzle. Similarly, for divergent portion acts as the diffuser for supersonic flows.

3. We can always achieve supersonic flow using a convergent-divergent duct having subsonic flow at the entry. In such a case, for M=1, we get dA=0, means Mach one will be achieved at the minimum cross section of the duct. Therefore the minimum cross-section where sonic conditions are achieved in the convergent divergent duct, is called as throat.

3.2 Isentropic flow through varying area duct

Let's consider the varying area duct as shown in Fig. 3.2. Areas at different stations are mentioned in the same figure. The minimum cross-sectional area of this duct is called as throat if local Mach number of the same cross-section is 1. We can find out the area of throat under this constraint for known inlet or outlet area of the duct. We know that mass flow rate at the throat is,

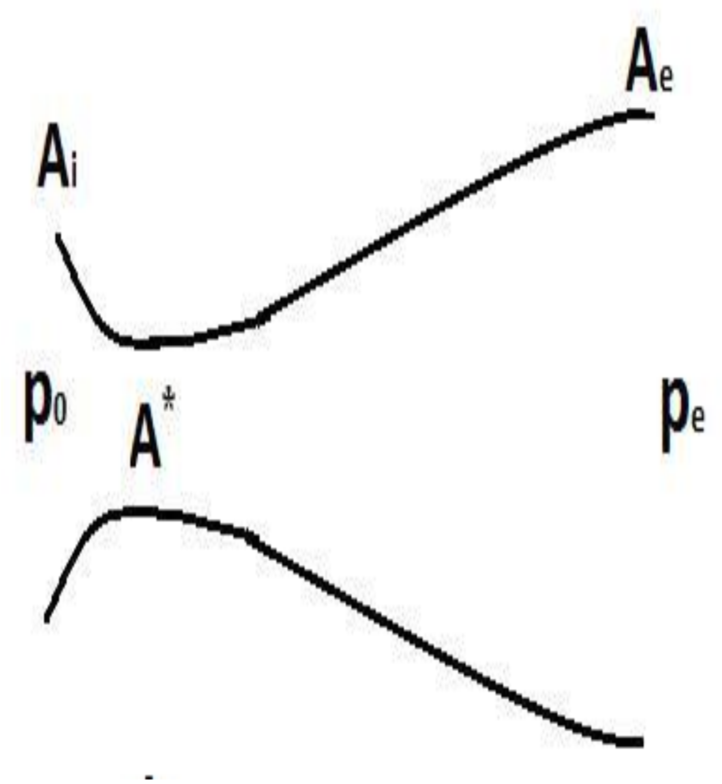


Fig. 3.2 Flow through convergent divergent duct.

$$\dot{m} = \rho^* u^* A^*$$

Where, ρ^*, A^* and u^* are geometric and flow properties at the throat.
For the steady flow, mass flow rate at any cross-section having geometric and flow properties as ρ, A and u will be equal to the mass flow rate of the throat. Hence,

$$\rho u A = \rho^* u^* A^*$$

$$\frac{A}{A^*} = \frac{\rho^* u^*}{\rho u} = \frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho}\frac{u^*}{u} = \frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho}\frac{a^*}{u}$$

$$\left(\frac{A}{A^*}\right)^2 = \left(\frac{\rho^*}{\rho_o}\right)^2 \left(\frac{\rho_o}{\rho}\right)^2 \left(\frac{a^*}{A}\right)^2$$

But we know that

$$\frac{\rho_0}{\rho} = \left[1 + \frac{(\gamma - 1)}{2} M^2\right]^{\frac{1}{(\gamma-1)}}$$

$$\frac{\rho_0}{\rho^*} = \left[\frac{(\gamma + 1)}{2}\right]^{\frac{1}{(\gamma-1)}}$$

$$\left[\frac{u}{a^*}\right]^2 = M^{*2} = \frac{\frac{(\gamma+1)}{2} M^2}{1 + \frac{(\gamma-1)}{2} M^2}$$

Hence the area relation can be written as,

$$\left[\frac{A}{A^*}\right]^2 = \frac{1}{M^2}\left[\frac{2}{(\gamma + 1)}\left(1 + \frac{(\gamma - 1)}{2} M^2\right)\right]^{\frac{(\gamma+1)}{(\gamma-1)}}$$

Hence, If we know Mach number M at any cross section and corresponding area A then we can calculate the area of the throat for the duct. From this expression it is also clear that the Mach number at any cross-section upstream or downstream of the throat is not dependent on the nature of variation of cross-sectional area of the duct in the stream wise direction.

3.2.1 Nozzle flow

Consider the convergent divergent duct. Left end of the duct corresponds to the stagnation or total conditions (T_o, P_o and ρ_o) due to its connection to the reservoir while right end of the duct is open to the atmosphericpressure (P_e). If initially exit pressure (P_e) is same as the reservoirpressure (P_o) then there will not be any flow through the duct. If we decrease the exit pressure by small amount then flow takes place through the duct. Here convergent portion acts as nozzle where pressure decreases and Mach number increases in the stream wise direction while divergent portion acts as diffuser which leads to increase in pressure and Mach number along the length of the nozzle. Variation of pressure and Mach number for this condition.

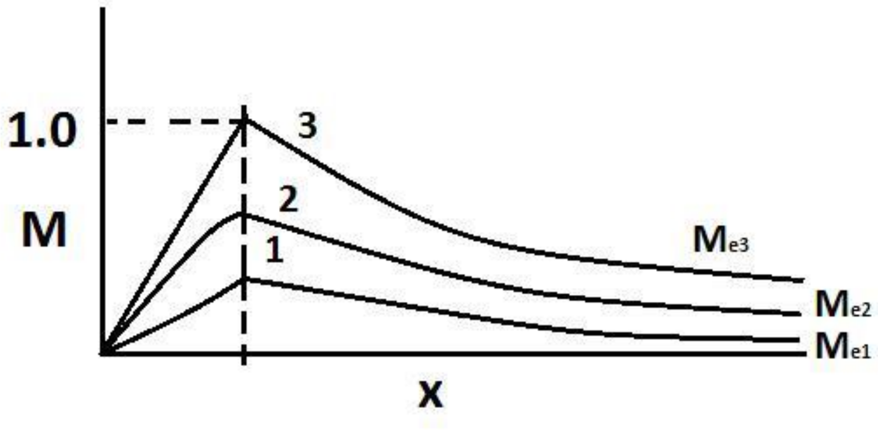


Fig 3.3 Mach number variation along the length of the duct for various exit pressure conditions

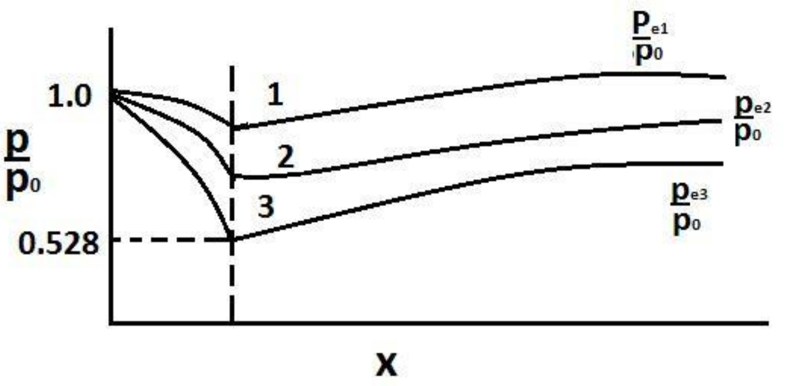


Fig 3.4 Pressure variation along the length of the duct for various exit pressure conditions

Further decrease in pressure at the exit of the duct shifts the pressure and Mach number curves as shown in Figure tagged by 2. Mass flow continues to increase with decreasing the exit pressure from conditions from 1 to 2. Condition 3 in this figure represents first critical condition or a particular value of exit pressure at which Mach number at the minimum cross-section of the duct becomes 1 or sonic. From Figure a it is clear that convergent portion continues to act as nozzle while divergent portion acts as diffuser. Pressure at the throat where Mach number has reached 1 attains the reference star value which is equal to 0.528 times the reservoir pressure for isentropic air flow. Further decrease in exit pressure beyond the first critical pressure (corresponding to situation 3), does not change the role of convergent portion as the nozzle. The pressure and Mach number in the convergent portion also remain unchanged with further decrease in exit pressure. Once the sonic state is achieved at the minimum cross section, mass flow rate through the duct attains saturation. Hence duct or the nozzle is said to be choked for any pressure value lower than the first critical condition. Typical mass flow rate variation for air flow with change in exit pressure is shown in Figure.

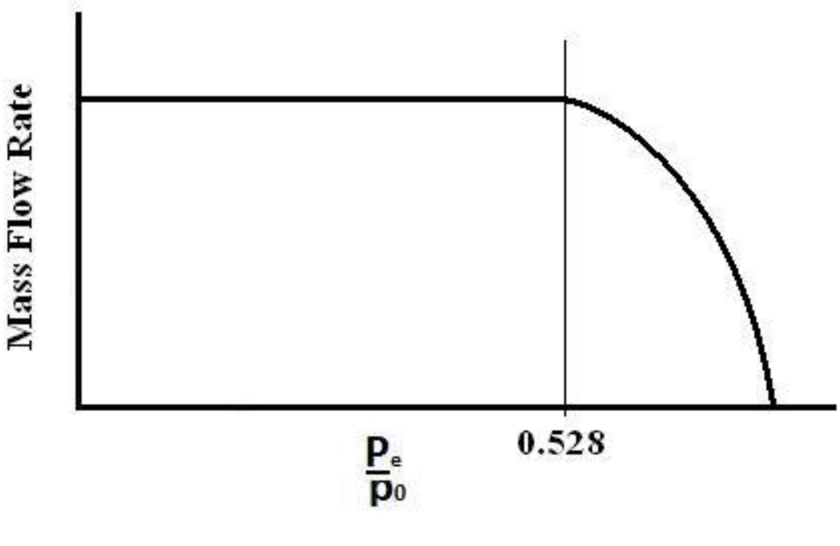


Fig 3.5 Variation of mass flow rate for air with change in exit pressure

Variation of pressure and Mach number for a typical exit pressure just below first critical conditions is shown in Figure a and b respectively. As discussed earlier, for this situation also pressure decreases and Mach number increases in the convergent portion of the duct. Thus Mach number attains value 1 at the end of convergent section or at the throat. Fluid continues to expand in the initial part of the divergent portion which corresponds to decrease in pressure and increase in Mach number in the supersonic regime in that part of the duct. However, if fluid continues to expand in the rest part of the duct then pressure of the fluid is expected to reach a value at the exit which is much lower than the exit pressure. Therefore, a normal shock gets created after initial expansion in the divergent portion to increase the pressure and decrease the Mach number to subsonic value. Hence rest of the portion of divergent duct acts as diffuser to increase the pressure in the direction of flow to reach the exit pressure value smoothly.

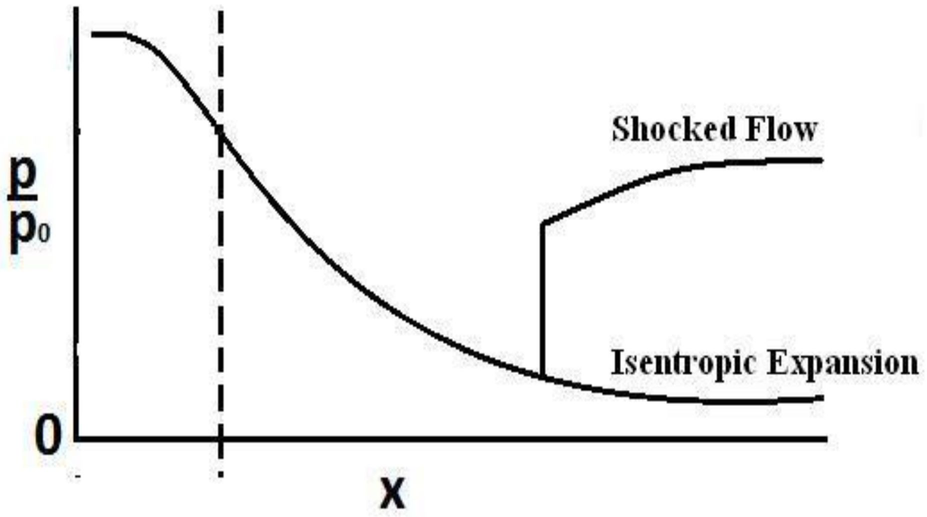


Fig 3.6 Pressure variation along the length of the nozzle

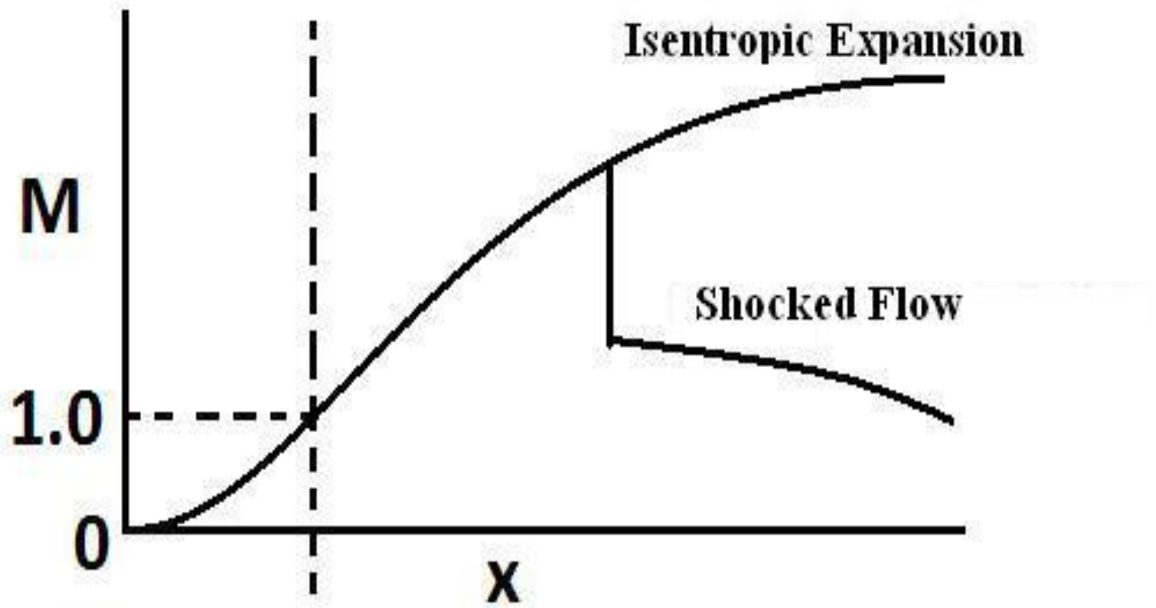


Fig 3.7 Mach number variation along the length of the nozzle

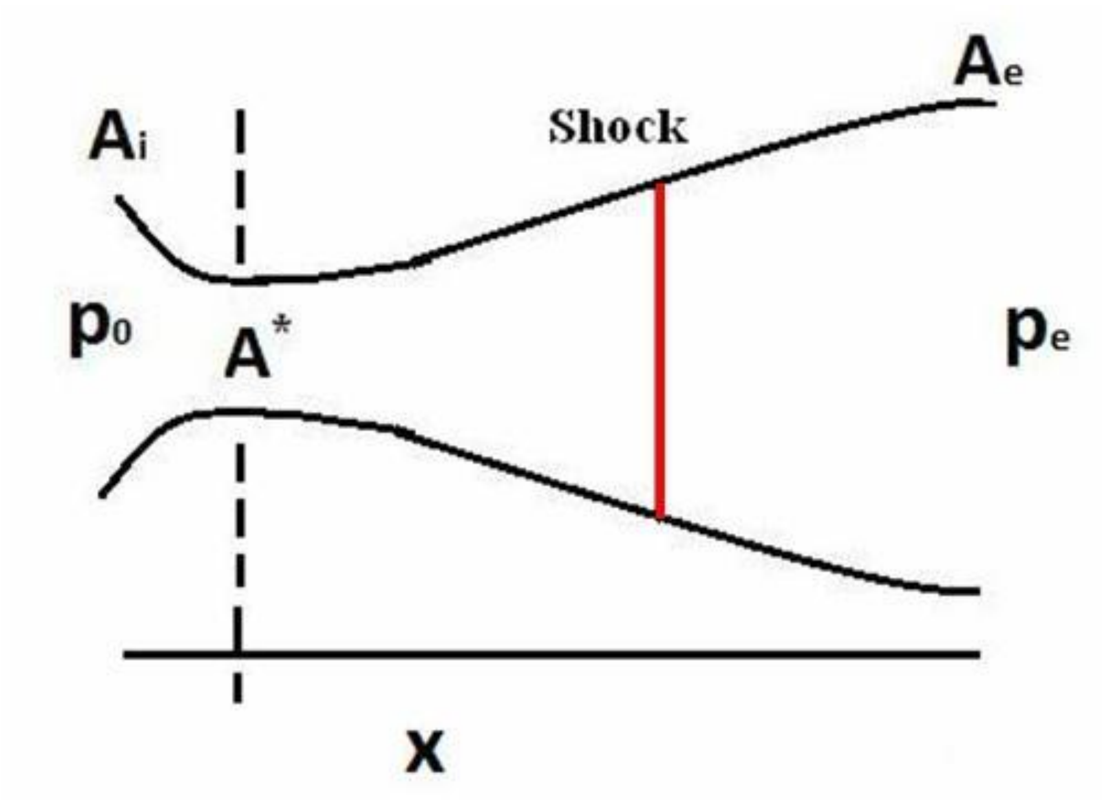


Fig 3.8 Presence of normal shock in the nozzle flow

For the further decrease in exit pressure for the same reservoir condition, the portion of divergent part acting as nozzle, intern the normal shock moves towards exit of the duct. For a particular value of exit pressure normal shock stands at the exit of the convergent divergent duct. Decrease in the exit pressure beyond this condition provides oblique shock pattern originating from the edge of the duct to rise pressure in order to attain the exit pressure conditions. Corresponding condition is shown in Figure.

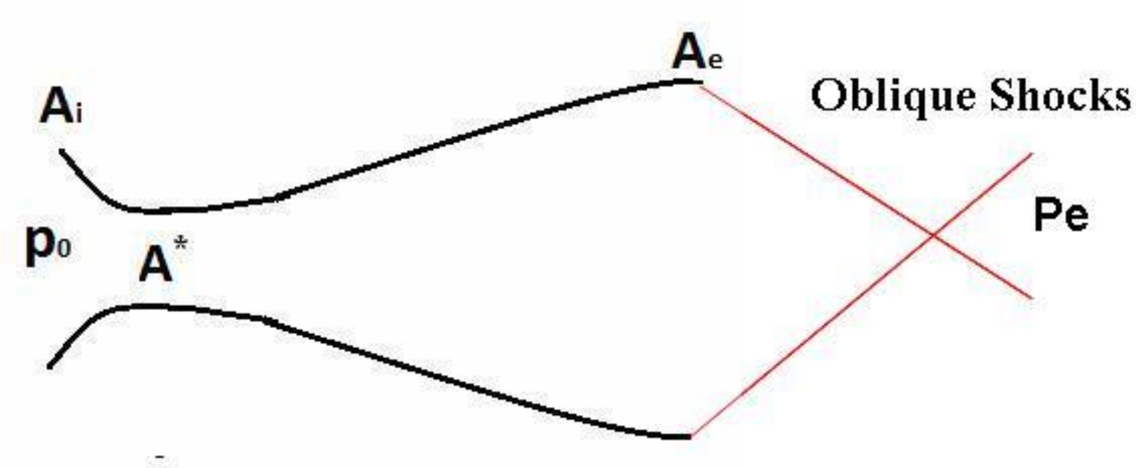


Fig 3.9 Oblique shock pattern for over expanded condition

For this exit pressure condition, the flow inside the duct is isentropic. However fluid attains the pressure at the exit of the duct which is lower than the exit pressure, hence it has to pass through the oblique shock and attain the pressure as that of exit pressure by the non-isentropic process. Hence such condition of the duct is called as over expanded nozzle' and it is shown in Figure. Decrease in exit pressure beyond this over expanded condition, decreases the strength of oblique shock and hence the amount of pressure rise. Hence at a particular value of exit pressure fluid pressure at the exit of the duct becomes exactly equal to the exit pressure and flow becomes completely isentropic for the duct. In this condition both convergent and divergent portions of the duct act as nozzle to expand the flow smoothly, hence duct is called as convergent divergent nozzle. Expansion of the flow in the convergent divergent nozzle is mentioned as 'isentropic expansion' in Figures a and b. Further decrease in exit pressure beyond the isentropic condition corresponds to more fluid pressure at the exit in comparison with the ambient pressure. Hence expansion fan gets originated from the edge of the nozzle to decrease the pressure smoothly to reach the ambient condition isentropically. Nozzle flow for such a situation is termed as 'under expanded nozzle flow'. Corresponding flow pattern is shown in Figure.

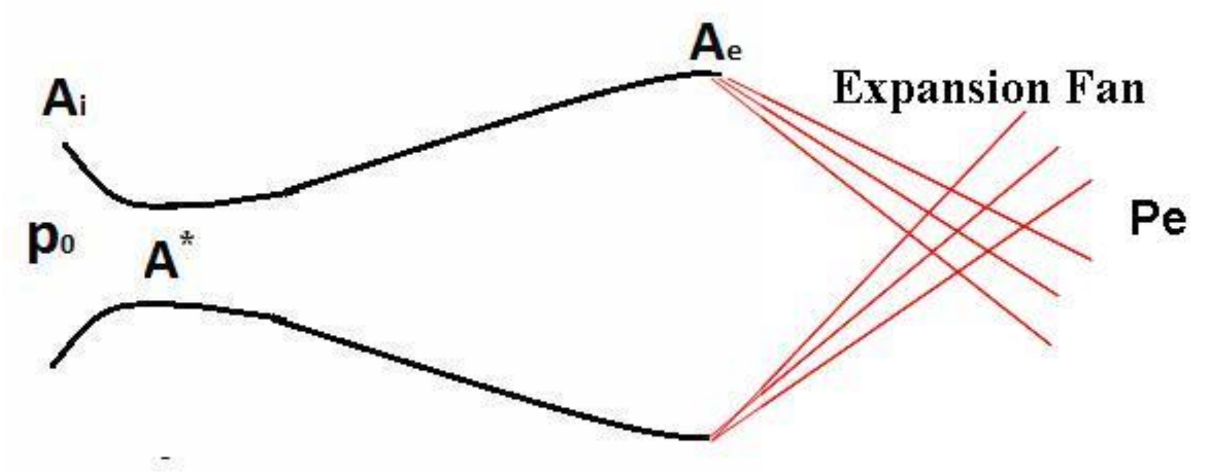


Fig 3.10 Expansion fan pattern for under expanded condition

3.4 Choked Flow

Choked flow is a compressible flow effect. The parameter that becomes "choked" or "limited" is the fluid velocity. Choked flow is a fluid dynamic condition associated with the Venturi effect. When a flowing fluid at a given pressure and temperature passes through a restriction (such as the throat of a convergent-divergent nozzle or a valve in a pipe) into a lower pressure environment the fluid velocity increases. At initially subsonic upstream conditions, the conservation of mass principle requires the fluid velocity to increase as it flows through the smaller cross-sectional area of the restriction. At the same time, the Venturi effect causes the static pressure, and therefore the density, to decrease downstream beyond the restriction. Choked flow is a limiting condition where the mass flow rate will not increase with a further decrease in the downstream pressure environment while upstream pressure is fixed.

For homogeneous fluids, the physical point at which the choking occurs for adiabatic conditions is when the exit plane velocity is at sonic conditions i.e. at a Mach number of 1. At choked flow, the mass flow rate can be increased only by increasing density upstream and at the choke point.

The choked flow of gases is useful in many engineering applications because the mass flow rate is independent of the downstream pressure, and depends only on the temperature and pressure andhencethedensityofthegasontheupstreamsideoftherestriction.Underchokedconditions, valves and calibrated orifice plates can be used to produce a desired mass flowrate.

3.4.1 Chocked mass flow rate of the nozzle

We have already seen in Figure, that mass flow rate of the nozzle remains unaltered after flow gets chocked. This chocked mass flow rate can be calculated as,

$$\dot{m} = \rho^* u^* A^*$$

But we know that,

$$\frac{\rho_0}{\rho^*} = \left[\frac{(\gamma+1)}{2}\right]^{\frac{1}{(\gamma-1)}}$$

$$u^* = a^* = \sqrt{\gamma R T^*}$$

Hence

$$\dot{m} = \rho_0 \left[\frac{(\gamma+1)}{2}\right]^{\frac{-1}{(\gamma-1)}} \sqrt{\gamma R T^*} A^*$$

$$\dot{m} = \rho_0 \left[\frac{(\gamma+1)}{2}\right]^{\frac{-1}{(\gamma-1)}} \sqrt{\gamma R \frac{T^*}{T_0} T_0 A^*}$$

However,

$$\rho_0 = \frac{p_0}{R T_0}$$

$$\frac{T^*}{T_0} = \left[1 + \frac{(\gamma-1)}{2}\right]^{-1}$$

Hence

$$\dot{m} = \frac{p_0}{R T_0} \left[\frac{(\gamma+1)}{2}\right]^{\frac{-1}{(\gamma-1)}} \sqrt{\gamma R \frac{T^*}{T_0} T_0 A^*}$$

$$\dot{m} = p_0 \left[\frac{(\gamma+1)}{2}\right]^{\frac{-(\gamma+1)}{2(\gamma-1)}} \sqrt{\frac{\gamma}{R T_0}} A^*$$

From this expression it is clear that for a convergent divergent nozzle, for given throat area, choked mass flow rate remains constant for the fixed reservoir (P_0 and T_0) conditions. Therefore choked mass flow rate can be increased by increasing the reservoir pressure P_0 or decreasing reservoir temperature T_0.

3.4.2 Over Expanded and Under Expanded flows innozzle

UnderexpandedNozzle

1. Discharges fluid at an exit pressure greater than the externalpressure
2. This owes to the exit area being too small for an optimum arearatio
3. The expansion of the fluid is incomplete
4. Further expansion happens outside of thenozzle
5. Nozzle exit pressure is greater than local atmosphericpressure

Overexpanded Nozzle

- Fluid exits at lower pressure than theatmosphere
- This owes to an exit area too large for optimum CurveAB:variation of axial pressure with optimum back pressure for given nozzle arearatio Curves AC thruAF:
  Variation of axial pressure for increasingly higher external pressure. Sudden rise in pressure represents flow separation. This resultin shock formation (sharp pressure rise) within thenozzle.
- This shock is pushed upstream toward the nozzle as ambient pressure increases Curve AG: Significant back pressure has caused the nozzle to "unstart" meaning the nozzle is no longer choked.
- The diverging section now decelerates the flow CurveAH.
- Further back pressure increase drives reduces the pressure and resulting exitvelocity

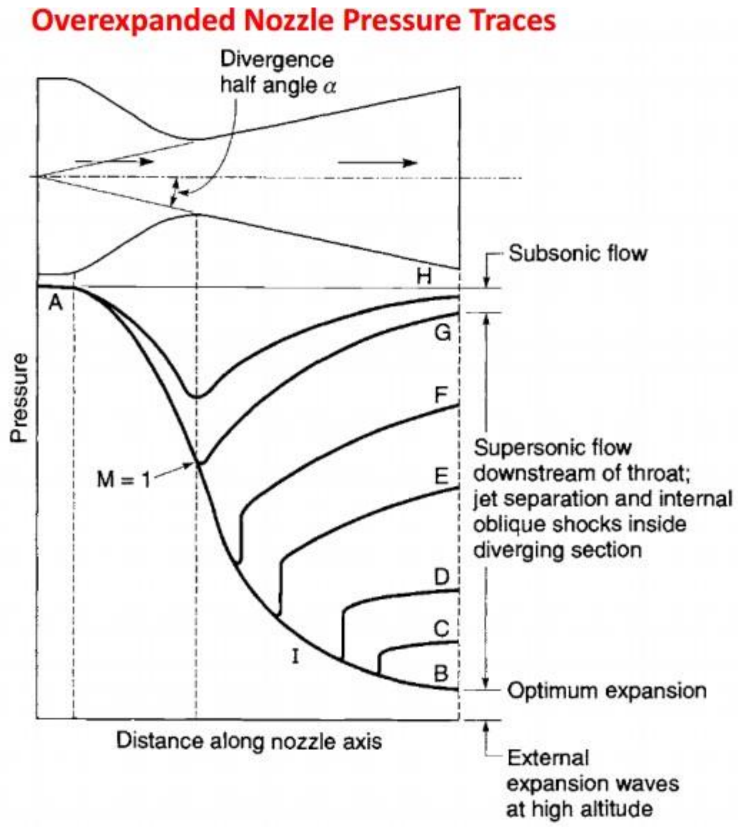


Fig 3.11 Overexpanded Nozzle Pressure Traces

3.5 FLOW IN CONSTANT AREA DUCT WITH FRICTION AND HEAT TRANSFER

3.5.1 Flow with friction

Consider a flow through constant area pipe as shown in Figure. A subsonic or supersonic flow enters in the pipe at section 1 and leaves at section 2. Thermodynamic properties along with the velocity of the flow change from their initial value at station 1 to the station 2 in the presence of friction force. This 1D flow with friction is called as Fanno flow. Analysis of this flow would lead to prediction of properties of the flow at the exit for known inletconditions and pipe configuration.

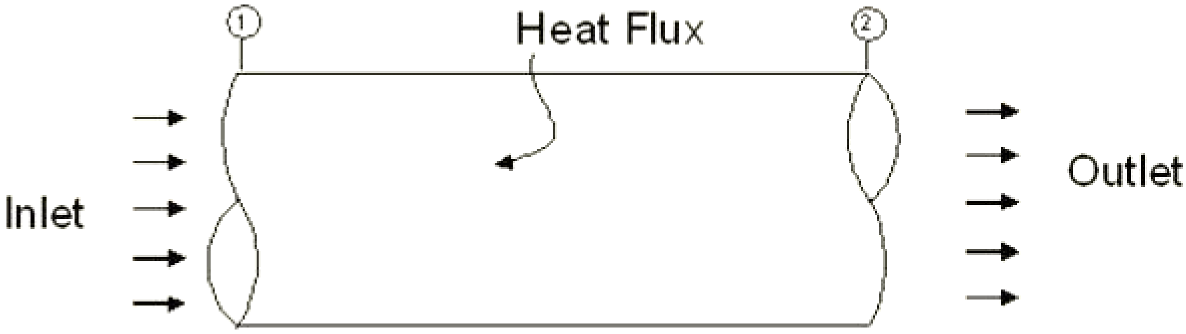


Fig 3.12 Flow through constant area pipe

Here we will be considering the effect of friction between pipe wall and fluid. However this assumption will be used only in momentum equation. Hence total temperature can be considered to be constant in the flow process. The 1D governing equations for this flow are,

$$\rho_1 u_1 = \rho_2 u_2 \text{(mass conservation)}$$

And

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \quad \text{(energy conservation)}$$

The main change takes place in momentum equation. Therefore consider the integral form of momentum equation for 1D flow.

$$\frac{\partial}{\partial t}\oint_v \rho du + \oint_s (\rho u ds)u = -\oint_s pds + \oint_s \tau ds$$

Let's assume the flow to be steady through the pipe. Hence,

$$\oint_s (\rho u ds)u = -\oint_s pds + \oint_s \tau ds$$

for integration of pressure and momentum terms, area is the cross sectional area while for the shear stress term areas the circumferential area of the pipe. Therefore the wetted area for shear stress includes diameter and length of the pipe. Negative sign should be associated with the shear stress term since shear acts in the direction opposite to the flow. Hence the momentum equation is,

$$p_1 + \rho u_1^2 = p_1 + \rho u_1^2 + F$$

where term F corresponds the frictional force and can be expressed in terms of pipe dimensions and friction coefficient.

Let's try to express the change in static and total properties of the flow from station 1 and 2 of the pipe due to consideration of wall shear or friction.

Since total temperature is constant, we can express the static temperature ratio as

$$\frac{T_2}{T_1} = \frac{T_0/T_1}{T_0/T_2} = \frac{\left(1+\frac{\gamma-1}{2}M_1^2\right)}{\left(1+\frac{\gamma-1}{2}M_2^2\right)}$$

Hence,

$$\therefore \frac{T_2}{T_1} = \frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}$$

From, mass conservation equation, we can get,

$$\rho_1 u_1 = \rho_2 u_2$$

But,

$$a^2 = \frac{\gamma p}{\rho}$$

$$\therefore \rho = \frac{\gamma p}{a^2}$$

Substituting this expression in mass conservation equation, we get,

$$\therefore \left(\frac{\gamma p_1}{a_1^2}\right)u_1 = \left(\frac{\gamma p_2}{a_2^2}\right)u_2$$

$$\therefore \frac{p_1 M_1}{a_1} = \frac{p_2 M_2}{a_2}$$

$$\frac{p_2}{p_1} = \frac{M_1}{M_2}\frac{a_2}{a_1} = \frac{M_1}{M_2}\sqrt{\frac{T_2}{T_1}}$$

$$\therefore \frac{p_2}{p_1} = \frac{M_1}{M_2}\left[\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}\right]^{0.5}$$

We can use ideal gas equation to calculate the density ratio from pressure and temperature ratio.

$$\frac{\rho_2}{\rho_1} = \frac{p_2}{p_1}\frac{T_1}{T_2}$$

$$\therefore \frac{\rho_2}{\rho_1} = \frac{M_1}{M_2}\sqrt{\frac{T_2}{T_1}}\frac{T_1}{T_2} = \frac{M_1}{M_2}\sqrt{\frac{T_1}{T_2}}$$

$$\frac{\rho_2}{\rho_1} = \frac{M_1}{M_2}\left[\frac{2+(\gamma-1)M_1^{\,2}}{2+(\gamma-1)M_2^{\,2}}\right]^{-0.5}$$

The earlier equation are for static property ratios. For total property ratios between two stations we have,

$$\frac{T_{0_2}}{T_{0_1}} = 1$$

$$\frac{p_{0_2}}{p_{0_1}} = \frac{\left(p_{0_2}/p_2\right)p_2}{\left(p_{0_1}/p_1\right)p_1} = \frac{\left(1+\frac{\gamma-1}{2}M_2^{\,2}\right)^{\gamma/\gamma-1}}{\left(1+\frac{\gamma-1}{2}M_1^{\,2}\right)^{\gamma/\gamma-1}}\frac{p_2}{p_1}$$

$$\frac{p_{0_2}}{p_{0_1}} = \left[\frac{2+(\gamma-1)M_2^{\,2}}{2+(\gamma-1)M_1^{\,2}}\right]^{\gamma+1/2(\gamma-1)}\frac{M_1}{M_2}$$

$$\frac{\rho_{0_2}}{\rho_{0_1}} = \frac{p_{0_2}}{p_{0_1}}\frac{T_{0_1}}{T_{0_2}} = \frac{p_{0_2}}{p_{0_1}}$$

These expressions provide the ratios of thermodynamic properties for the known Mach number at station 1 and 2.

3.5.2 Reference conditions for Fanno flow

If the inlet flow, either subsonic or supersonic, attains Mach number equal to 1 or sonic condition, at the station 2, then such a condition is taken as reference for calculations of Fanno flow. The corresponding length of the pipe is terms as critical length of the pipe. We can use the reference conditions for frictional pipe flow analysis. The expressions for property ratios are then given as,

$$\therefore \frac{T}{T^*} = \frac{\gamma+1}{2+(\gamma-1)M_\infty^{\ 2}}$$

$$\frac{p}{p^*} = \frac{1}{M_\infty}\left(\frac{\gamma+1}{2+(\gamma-1)M_\infty^{\ 2}}\right)^{1/2}$$

$$\frac{\rho}{\rho^*} = \frac{1}{M_\infty}\left(\frac{\gamma+1}{2+(\gamma-1)M_\infty^{\ 2}}\right)^{-1/2}$$

$$\frac{p_0}{p_0^*} = \frac{1}{M}\left[\frac{2+(\gamma-1)M_\infty^{\ 2}}{\gamma+1}\right]^{\gamma+1/2(\gamma-1)}$$

Since M_2 is equal to 1, the Mach number at station 1 (M_1) is the freestream Mach number (M_∞).

3.5.3 Differential relations and analysis of flow with friction,

We know the mass, momentum and energy equations for 1D flow with friction. Herewith we will try to derive the differential form for the same.

$\rho u = Constant$ mass conservation

$$d(\rho u) = 0$$

$\dfrac{d\rho}{\rho} + \dfrac{du}{u} = 0$ Differential form of conservation equation

Energy equation

$$h + \frac{u^2}{2} = constant$$

Differential form of energy equation

$$dh + udu = 0$$

For momentum equation, consider the control volume shown in Fig. 2.1 and initially consider the steady integral form of the momentum equation

$$\oint_s (\rho u ds)u = -\oint_s pds + \oint_s \tau ds$$

We know that, for integration of wall shear, the area of to be considered is the circumferential area. For integration of pressure and momentum term, the area of to be considered is cross-sectional area, $\frac{\pi}{4}D^2$, hence

$$\therefore \oint_s \left(p + (\rho u ds) u \right) ds = -\oint_s \tau ds = -\int_0^l \tau D dx \pi = -\pi D \int_0^l \tau dx$$

If we consider the distance between two stations to be infinitesimal (dx) Hence the differential form of the momentum equation is as,

$$\therefore (p + dp)A + A(\rho + d\rho)(u + du)^2 - (p + \rho u^2)A = -\tau A_5$$

$$(pA + dpA) + A(\rho + d\rho)(u^2 + 2udu + du^2) - (pA + A\rho u^2) = -\tau A_5$$

$$Adp + A\rho(u^2 + 2udu + du^2) + Adp(u^2 + 2udu + du^2) - A\rho u^2 = -\tau A_5$$

$$Adp + 2\rho udu + Ad\rho u^2 = -\tau A_5$$

We can use the mass conservation equation and simplify the above equation,

$$\frac{d\rho}{\rho} + \frac{du}{u} = 0$$

$$\frac{d\rho}{\rho} = -\frac{du}{u}$$

$$d\rho = -\frac{du}{u}\rho$$

Using above equation, the equation can be simplified as,

$$Adp + 2\rho uduA + Au(-\rho du) = -\tau A_5$$

$$Adp + \rho uduA = -\tau A_5$$

$$\frac{dp}{\rho u^2} + \frac{du}{u} = -\frac{\tau}{\rho u^2}\frac{A_5}{A}$$

As we know,

$$A = \frac{\pi}{4} D^2$$

$$A_5 = \pi D dx$$

$$\frac{A_5}{A} = \frac{4}{D} dx$$

Using this equation we can re-write the Equation as,

$$\frac{dp}{\rho u^2} + \frac{du}{u} = -\frac{\tau}{\rho u^2}\frac{4dx}{D}$$

We can use the definition of skin friction coefficient as,

$$C_f = \frac{\tau}{\rho u^2 / 2}$$

$$\frac{C_f}{2} = \frac{\tau}{\rho u^2}$$

Hence differential form of momentum equation becomes,

$$\frac{dp}{\rho u^2} + \frac{du}{u} = -\frac{C_f}{2}\frac{4dx}{D}$$

From further simplification, we can replace the dynamic pressure, as,

$$\rho u^2 = \rho u^2 \frac{\gamma p}{\gamma p} = u^2 \frac{\rho}{\gamma p}\gamma p = \frac{u^2}{a^2}\gamma p = M^2 \gamma p$$

Hence,

$$\frac{1}{M^2\gamma}\frac{dp}{p} + \frac{du}{u} = -\frac{C_f}{2}\frac{4dx}{D}$$

Above equation represents the differential form of momentum equation.

We can use the Equation for further analysis of Fanno flow. For this purpose lets replace dp/p and du/u of this equation. We can use ideal gas equation for replacing dp/p as,

$p = \rho RT$ (Differential form of ideal gas equation)

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

using mass differential form of mass conservation equation, we get,

$$\frac{dp}{p} = -\frac{du}{u} + \frac{dT}{T}$$ (Modified differential form of ideal gas equation)

For this purpose let’s re-express the terms du/u and dT/T.

As we know

$$dh + udu = 0$$

$$c_p dT + udu = 0$$

$$\frac{dT}{T} + \frac{udu}{c_p T} = 0$$

But

$$C_p = \frac{\gamma}{\gamma - 1} R$$

$$\frac{dT}{T} + \frac{udu}{\frac{\gamma}{\gamma - 1} RT} = 0$$

$$\frac{dT}{T} + \frac{(\gamma - 1)udu}{\gamma RT} = 0$$

$$\frac{dT}{T} + \frac{(\gamma - 1)udu}{a^2} = 0$$

$$\frac{dT}{T} + \frac{(\gamma - 1)u^2}{a^2} \frac{du}{u} = 0$$

$$\frac{dT}{T} + (\gamma - 1)M^2 \frac{du}{u} = 0$$

$$\frac{dT}{T} = -(\gamma - 1)M^2 \frac{du}{u}$$

$$\frac{du}{u} = \frac{-1}{(\gamma - 1)M^2} \frac{dT}{T}$$

We can use this equation for replacing the term du/u of Equation. However we can further simplify this equation by replacing dT/T. As,

$$M = \frac{u}{a} = \frac{u}{(\gamma RT)^{0.5}}$$

$$M(\gamma RT)^{0.5} = u$$

$$dM.(\gamma RT)^{0.5} + \frac{1}{2} M.(\gamma R)^{0.5} (T)^{-0.5} dT = du$$

Dividing by $M(\gamma RT)^{0.5}$ both the sides

$$\frac{dM}{M} + \frac{dT}{2T} = \frac{du}{u}$$

$$\frac{dM}{M} + \frac{dT}{2T} = \frac{du}{u}$$

We can use the Equation to replace the term dT/T

$$\frac{dM}{M} = \left(1 + \frac{(\gamma - 1)M^2}{2}\right) \frac{du}{u}$$

$$\frac{du}{u} = \left(1 + \frac{(\gamma - 1)M^2}{2}\right)^{-1} \frac{dM}{M}$$

This is the simplest form of du/u term expressed in terms of Mach number and specific heat ratio.

3.5.4 Analysis of Fanno Flow

We can use the Equation for critical analysis of Fanno flow. For this purpose let's replace dp/p and du/u of this equation.
Let again consider Equation again and replace the term du/u, we get,

$$\frac{dM}{M}+\frac{dT}{2T}=\frac{-1}{(\gamma-1)M^2}\frac{dT}{T}$$

$$\frac{dM}{M}=-\frac{dT}{T}\left(\frac{1}{(\gamma-1)M^2}+\frac{1}{2}\right)$$

$$\frac{dM}{M}=-\frac{dT}{T}\left(\frac{2+(\gamma-1)M^2}{2(\gamma-1)M^2}\right)$$

$$\frac{dT}{T}=-\left(\frac{2(\gamma-1)M^2}{2+(\gamma-1)M^2}\right)\frac{dM}{M}$$

Modified differential form of ideal gas equation can be re-written using equation as

$$\frac{dp}{p}=-\left(1+\frac{(\gamma-1)M^2}{2}\right)^{-1}\frac{dM}{M}-\left(\frac{(\gamma-1)M^2}{1+\left((\gamma-1)/2\right)M^2}\right)\frac{dM}{M}$$

$$\frac{dp}{p}=-\left(1+\frac{(\gamma-1)M^2}{2}\right)^{-1}\frac{dM}{M}\left(1+(\gamma-1)M^2\right)$$

Now let's consider the differential form of momentum Equation and replace the term du/u and dp/p using Equation, we get,

$$\frac{-1}{M^2\gamma}\left(1+\frac{(\gamma-1)M^2}{2}\right)^{-1}\frac{dM}{M}\left(1+(\gamma-1)M^2\right)+\left(1+\frac{(\gamma-1)M^2}{2}\right)^{-1}\frac{dM}{M}=-\frac{C_f}{2}\frac{4dx}{D}$$

$$\left(1+\frac{(\gamma-1)M^2}{2}\right)^{-1}\frac{dM}{M}\left(\frac{-1-\gamma M^2+M^2}{M^2\gamma}\right)+\left(1+\frac{(\gamma-1)M^2}{2}\right)^{-1}\frac{dM}{M}=-\frac{C_f}{2}\frac{4dx}{D}$$

$$\left(1+\frac{(\gamma-1)M^2}{2}\right)^{-1}\frac{dM}{M}\left(\frac{-1-M^2(\gamma-1)}{M^2\gamma}\right)+\left(1+\frac{(\gamma-1)M^2}{2}\right)^{-1}\frac{dM}{M}=-\frac{C_f}{2}\frac{4dx}{D}$$

$$\left(1+\frac{(\gamma-1)M^2}{2}\right)^{-1}\frac{dM}{M}\left(\frac{-1}{M^2\gamma}-\frac{(\gamma-1)}{\gamma}\right)+\left(1+\frac{(\gamma-1)M^2}{2}\right)^{-1}\frac{dM}{M}=-\frac{C_f}{2}\frac{4dx}{D}$$

$$\left(1+\frac{(\gamma-1)M^2}{2}\right)^{-1}\frac{dM}{M}\left(\frac{-1}{M^2\gamma}-\frac{(\gamma-1)}{\gamma}+1\right)=-\frac{C_f}{2}\frac{4dx}{D}$$

$$\left(1+\frac{(\gamma-1)M^2}{2}\right)^{-1}\frac{dM}{M}\left(\frac{M^2-1}{M^2\gamma}\right)=-\frac{C_f}{2}\frac{4dx}{D}$$

$$C_f\frac{4dx}{D}=2\left(1+\frac{(\gamma-1)M^2}{2}\right)^{-1}\frac{dM}{M}\left(\frac{1-M^2}{M^2\gamma}\right)$$

$$\frac{dM}{dx}=C_f\frac{2}{D}\left(1+\frac{(\gamma-1)M^2}{2}\right)\left(\frac{M^3\gamma}{1-M^2}\right)$$

We can clearly observe that $\frac{dM}{dx} > 0$ if M<1 and $\frac{dM}{dx} < 0$ for M>1.

Hence from Equation, we can clearly state that, for subsonic flow Mach number increases in the presence of friction, while Mach number for supersonic flow decreases in the presence of friction. From Equation, it becomes clear that, for increases in Mach number, velocity of the subsonic flow increases while that of supersonic flow decreases due to decrease in Mach number. From Equation, it becomes clear that temperature of the subsonic flow decreases due to increase in Mach number in the presence of friction while temperature increases for supersonic flow with friction. We can prove from Equation that pressure decreases for subsonic flow while pressure increases for supersonic flow with friction. For entropy change we can arrive at the expression as,

$$Tds = dh - vdp$$

$$ds = cp\frac{dT}{T} - R\frac{dp}{p}$$

$$ds = -c_p\left(\frac{(\gamma-1)M^2}{1+\left(\frac{\gamma-1}{2}\right)M^2}\right)\frac{dM}{M} + R\left(1+\frac{(\gamma-1)M^2}{2}\right)^{-1}\frac{dM}{M}\left(1+(\gamma-1)M^2\right)$$

$$ds = -\frac{\gamma RM^2}{1+\left(\frac{\gamma-1}{2}\right)M^2}\frac{dM}{M} + R\left(1+\frac{(\gamma-1)M^2}{2}\right)^{-1}\frac{dM}{M}\left(1+(\gamma-1)M^2\right)$$

$$ds = R\left(1+\frac{(\gamma-1)M^2}{2}\right)^{-1}\frac{dM}{M}\left(-\gamma M^2+1+(\gamma-1)M^2\right)$$

$$ds = R\left(1+\frac{(\gamma-1)M^2}{2}\right)^{-1}\frac{dM}{M}\left(-\gamma M^2+1+\gamma M^2-M^2\right)$$

$$ds = R\left(1+\frac{(\gamma-1)M^2}{2}\right)^{-1}\frac{dM}{M}\left(1-M^2\right)$$

For supersonic flow dM is negative however $1 - M^2$ is also negative, therefore ds is positive for supersonic flow. For subsonic flow dM is positive however $1 - M^2$ is also positive, therefore ds is positive for supersonic flow. This expression clearly proves that entropy increases for both subsonic and supersonic flows while sonic flow is isentropic.

3.6 Fanno line or curve

Typical Fanno line for a particular mass flow rate is shown in Fig. 3.13. Previously proved facts are clearly evident in this figure. For a subsonic flow through a frictional pipe, enthalpy decreases and entropy increases. With increase in length of pipe, more expansion of the flow takes place for the subsonic flow due to increase in Mach number, decrease in pressure and increase in velocity. For a particular length of pipe flow, inlet subsonic flow attains sonic state at the exit. Corresponding length of the pipe is called as critical length. Further increase in length of pipe doesn't change state at the exit however inlet conditions change and subsonic flow becomes lower subsonic. Hence the flow condition for the critical pipe length is called as choked flow. For supersonic flow at the entry to a constant area pipe, deceleration of flow takes place and flow attains lower supersonic conditions. With increase in length of the pipe, Mach number at the exit decreases due to deceleration and at a particular pipe length flow becomes sonic at the exit. Further increase length of the pipe doesn't change exit conditions while inlet conditions become subsonic. Therefore choked conditions can be said to be attained for critical length of pipe for which entry is supersonic. This curve shows that there is only one critical point for flow with friction. This critical point corresponds to maximum entropy and hence the sonic state.

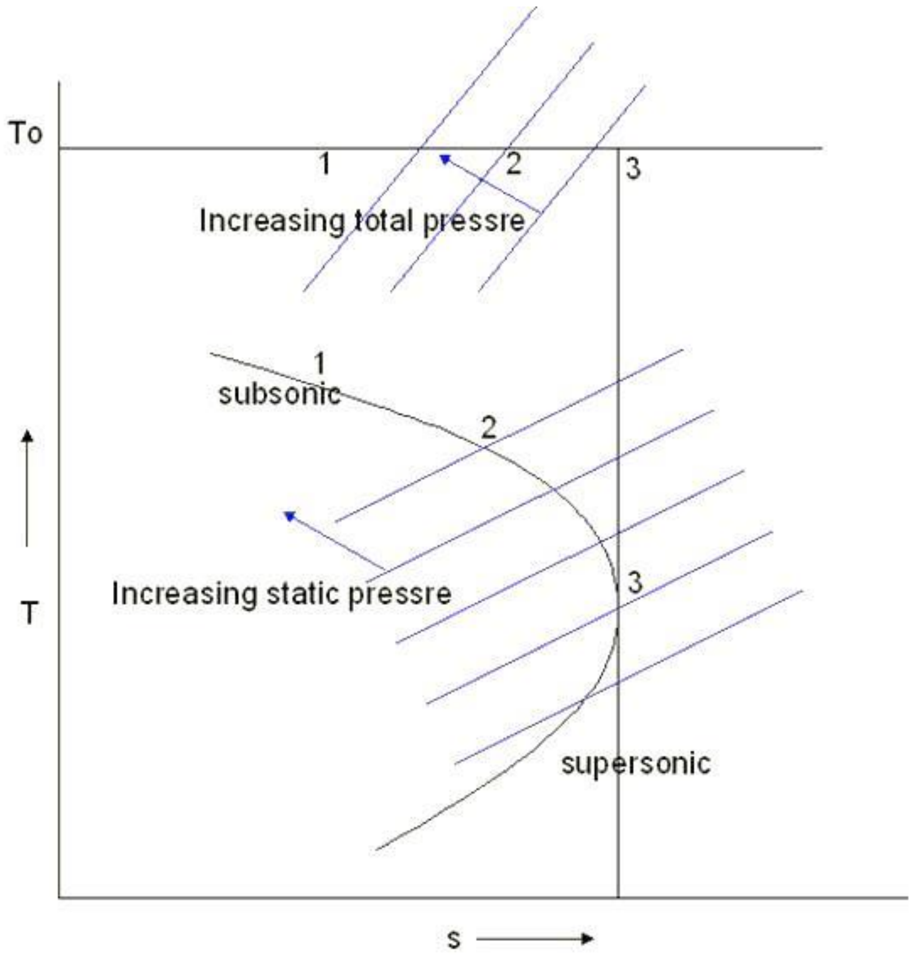


Fig. 3.13 h-s diagram for Fanno flow.

Like Rayleigh curve, h-s diagram for Fanno flow shown in Fig.3.13 corresponds to a particular value of mass flow rate. Fanno curve for various mass flow rate conditions is given in Fig.3.14. Explanation for Fanno curve for various mass flow rates can be obtained on similar line as it has been obtained for Rayliegh flow for various mass flow rates where instead of changing the applied heat flux we have to change the pipe length. Fanno curve for increased mass flow rate is also seen to be shrunk like Rayleigh curve.

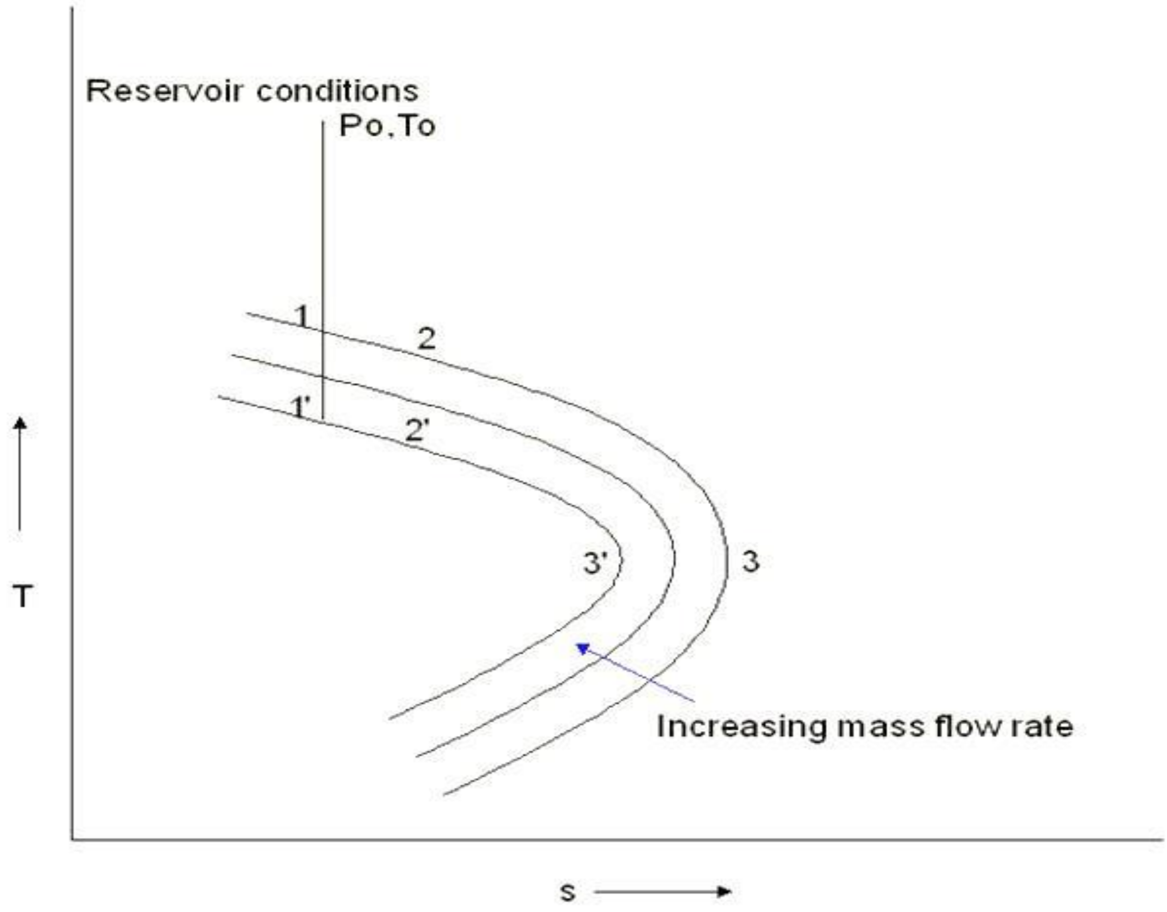


Fig. 3.14 h-s diagram for Fanno flow for various mass flow rates.

3.6.1 Fanno Line Flows

- If we consider a problem of frictional adiabatic flow through a duct, the governing are valid between any two points "1" and "2".
- Equation requires to be modified in order to take into account the frictional force, R_x, of the duct wall on the flow and we obtain

$$R_x + p_1 A - p_2 A = \dot{m} V_2 - \dot{m} V_1$$

So, for a frictional flow, we have the situation of six equations and seven unknowns.

- If all the conditions of "1" are known, the no. of possible states for "2" is 2. With an infinite number of possible states "2" for a given state "1", what do we observe if all possible states "2" are plotted on a T - s diagram, The locus of all possible states "2" reachable from state "1" is a continuous curve passing through state "1". The question is how to determine this curve? The simplest way is to assume different values of T_2. For an assumed value of T_2, the corresponding values of all other properties at " 2 " and R_x can be determined.

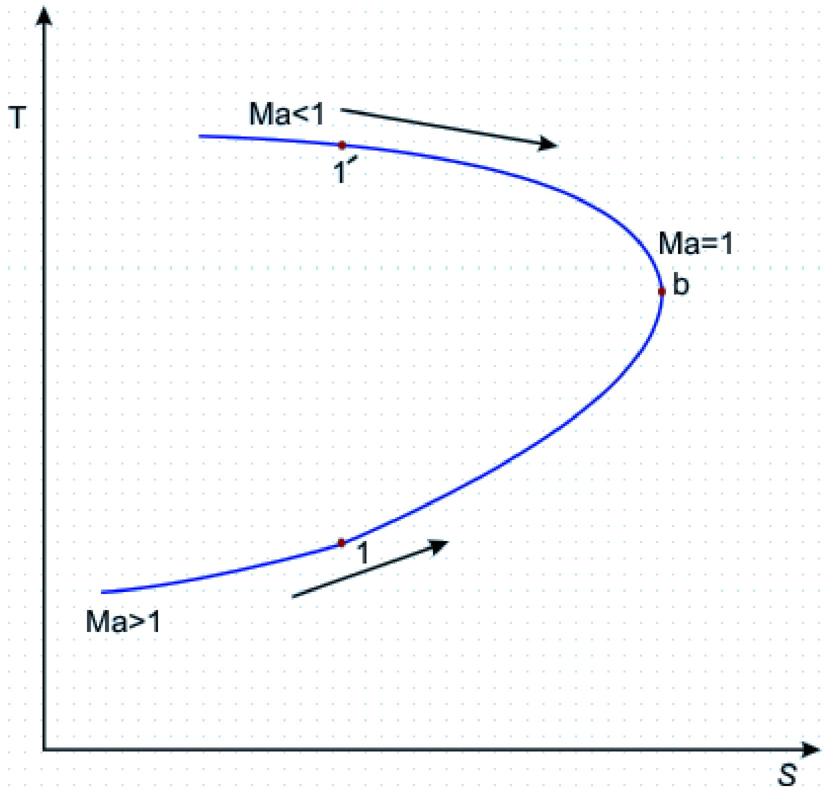


Fig 3.15 Fanno line representation of constant area adiabatic flow

- The locus of all possible downstream states is called Fanno line and is shown in Fig. 3.15. Point " b " corresponds to maximum entropy where the flow is sonic. This point splits the Fanno line into subsonic (upper)and supersonic (lower) portions.
- If the inlet flow is supersonic and corresponds to point 1 in Fig. 41.3, then friction causes the downstream flow to move closer to point "b" with a consequent decrease of Mach number towards unity.
- Note that each point on the curve between point 1 and "b" corresponds to a certain duct length L. As L is made larger, the conditions at the exit move closer to point "b". Finally, for a certain value of L, the flow becomes sonic. Any further increase in L is not possible without a drastic revision of the inlet conditions.
- Consider the alternative case where the inlet flow is subsonic, say, given the point 1' in Fig. 41.3. As L increases, the exit conditions move closer to point "b". If L is increased to a sufficiently large value, then point "b" is reached and the flow at the exit becomes sonic. The flow is again choked and any further increase in L is not possible without an adjustment of the inlet conditions.

3.6.2 One dimensional flow with heat addition

Consider the control volume as shown in Fig.3.16 for 1D flow with heat addition. The fluid flow of this kind is called as Raylaigh flow. Here station 1 is representative station before heat addition while station 2 is representative station after heat addition. This control volume is necessarily a constant cross-section pipe hence variation is the inviscid flow properties is expected in the direction of the flow due to addition of heat.

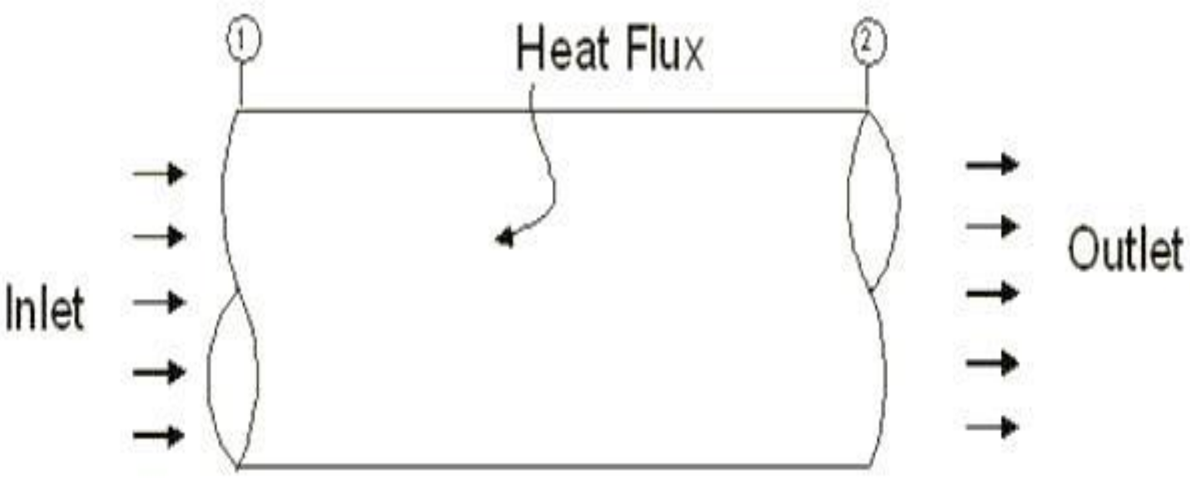


Fig.3.16: Typical Control volume for 1D flow with heat addition.

Assume the flow to be inviscid and steady between these two stations. Therefore the mass and momentum conservation equations remain unaltered from the normal shock case but energy equation will have a term corresponding to external heat addition in comparison with equation. Hence the 1D conservation equations for flow with heat addition are as follows.

$$\rho_1 u_1 = \rho_2 u_2$$
$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2$$
$$h_1 + \frac{u_1^2}{2} + q = h_2 + \frac{u_2^2}{2}$$

Here 'q' is amount of heat added per unit mass. Hence,

$$q = (h_1 - h_2) + \left(\frac{v_2^2}{2} - \frac{v_1^2}{2}\right)$$

However, we know that

$$h + \frac{v^2}{2} = ho$$

$$q = ho_2 - ho_1 = cp(To_2 - To_1)$$

This equation suggests that change in total temperature takes place due to heat addition between two stations.

Let's represent the ratios of static and total properties in terms of upstream (station 1) and downstream (station 2) Mach number and specific heat ratio. Let's consider the momentum equation,

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$$
$$p_1 + \gamma p_1 \frac{\rho_1}{\gamma p_1} u_1^2 = p_2 + \gamma p_2 \frac{\rho_2}{\gamma p_2} u_2^2$$
$$p_1 + \gamma p_1 \frac{u_1^2}{a_1^2} = p_2 + \gamma p_2 \frac{u_2^2}{a_2^2}$$
$$p_1 + \gamma p_1 M_1^2 = p_2 + \gamma p_2 M_2^2$$
$$p_1\left(1+\gamma M_1^2\right) = p_2\left(1+\gamma M_2^2\right)$$
$$\frac{p_2}{p_1} = \frac{\left(1+\gamma M_1^2\right)}{\left(1+\gamma M_2^2\right)}$$

Also from ideal gas assumption

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_2}{\rho_1}$$

But

$$\rho_1 u_1 = \rho_2 u_2$$
$$\frac{\rho_1}{\rho_2} = \frac{u_2}{u_1},$$

Therefore,

$$\frac{T_2}{T_1} = \frac{\rho_2}{\rho_1}\frac{u_2}{u_1}$$

$$\frac{T_2}{T_1} = \frac{\rho_2}{\rho_1}\frac{M_2}{M_1}\frac{a_2}{a_1}$$

$$\frac{T_2}{T_1} = \frac{\rho_2}{\rho_1}\frac{M_2}{M_1}\sqrt{\frac{T_2}{T_1}}$$

$$\sqrt{\frac{T_2}{T_1}} = \frac{\rho_2}{\rho_1}\frac{M_2}{M_1}$$

$$\frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1}\right)^2\left(\frac{M_2}{M_1}\right)^2$$

Hence from Eq. we get,

$$\frac{T_2}{T_1} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2\left(\frac{M_2}{M_1}\right)^2$$

Therefore,

$$\frac{\rho_2}{\rho_1} = \frac{p_2}{p_1}\frac{T_1}{T_2}$$

$$\frac{\rho_2}{\rho_1} = \frac{\left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)}{\left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2\left(\frac{M_2}{M_1}\right)^2}$$

$$\frac{\rho_2}{\rho_1} = \frac{1}{\left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)\left(\frac{M_2}{M_1}\right)^2}$$

$$\frac{\rho_2}{\rho_1} = \frac{1+\gamma M_2^2}{1+\gamma M_1^2}\left(\frac{M_1}{M_2}\right)^2$$

For ratio of total properties,

$$\frac{p_{0_2}}{p_{0_1}} = \left(\frac{p_{0_2}/p_2}{p_{0_1}/p_1}\right)\frac{p_2}{p_1}$$

$$\frac{p_{0_2}}{p_{0_1}} = \frac{\left(1+\frac{\gamma-1}{2}M_2^2\right)^{\gamma/\gamma-1}}{\left(1+\frac{\gamma-1}{2}M_1^2\right)^{\gamma/\gamma-1}}\frac{1+\gamma M_1^2}{1+\gamma M_2^2}$$

$$\frac{p_{0_2}}{p_{0_1}} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2}\left[\frac{\left(1+\frac{\gamma-1}{2}M_2^2\right)}{\left(1+\frac{\gamma-1}{2}M_1^2\right)}\right]^{\gamma/\gamma-1}$$

Similarly

$$\frac{T_{0_2}}{T_{0_1}} = \left(\frac{T_{0_2}/T_2}{T_{0_1}/T_1}\right)\frac{T_2}{T_1}$$

$$\frac{T_{0_2}}{T_{0_1}} = \left[\frac{\left(1+\frac{\gamma-1}{2}M_2^2\right)}{\left(1+\frac{\gamma-1}{2}M_1^2\right)}\right]\left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2\left(\frac{M_2}{M_1}\right)^2$$

From these two ratios we can find out $\frac{\rho_{0_2}}{\rho_{0_1}}$ as

$$\frac{\rho_{0_2}}{\rho_{0_1}} = \frac{p_{0_2}}{p_{0_1}}\frac{T_{0_1}}{T_{0_2}}$$

3.6.3 Reference conditions for flows with heat addition

We have represented all the ratios in terms of upstream and downstream Mach numbers. If we consider a particular case where heat addition leads to downstream Mach number equal to one or post heat addition Mach number is unity, then equations can be written as,

$$\frac{p_2}{p_1} = \frac{1+\gamma M_1^{\ 2}}{1+\gamma M_2^{\ 2}}$$

Since $M_2 = 1$ & $p_2 = p^*$ & $p_1 = p_\infty$ & $M_1 = M_\infty$. Here flow properties after heat addition are the stared quantities due to unity of the local Mach number. Hence these quantities are of very much of importance since can be used as reference quantities.

$$\frac{p^*}{p_\infty} = \frac{1+\gamma M_\infty^{\ 2}}{1+\gamma}$$

$$\frac{p_\infty}{p^*} = \frac{1+\gamma}{1+\gamma M_\infty^{\ 2}}$$

Similarly

$$\frac{T_\infty}{T^*} = M_\infty^{\ 2}\left(\frac{1+\gamma}{1+\gamma M_\infty^{\ 2}}\right)^2$$

$$\frac{\rho_\infty}{\rho^*} = \frac{1}{M_\infty^{\ 2}}\left(\frac{1+\gamma M_\infty^{\ 2}}{1+\gamma}\right)$$

$$\frac{p_0}{p_0^*} = \frac{1+\gamma}{1+\gamma M_\infty^{\ 2}}\left[\frac{2+(\gamma-1)M_\infty^{\ 2}}{1+\gamma}\right]^{\frac{\gamma}{\gamma-1}}$$

$$\frac{T_0}{T_0^*} = \frac{(1+\gamma)M_\infty^{\ 2}}{1+\gamma M_\infty^{\ 2}}\left[2+(\gamma-1)M_\infty^{\ 2}\right]$$

From all the ratios for 1D flow with heat addition, following conclusions can be drawn for supersonic and subsonic flows.

1. Addition of heat in supersonic flows

- Decreases Mach number
- Increases static pressure
- Increases static temperature
- Decreases total pressure
- Increases total temperature
- Decreases velocity

2. Addition of heat in subsonic flow

- Increases Mach number
- Decreases static pressure
- Increases static temperature if$M_\infty < \gamma^{-1/2}$ and decreases if$M_\infty > \gamma^{-1/2}$
- Decreases total pressure
- Increases total temperature
- Increases velocity

Hence supersonic flow decelerates towards sonic value while subsonic flow accelerates towards the same due to heat addition. In a way, having a subsonic flow to start with we can keep on adding heat to reach sonic and then remove heat to attain required supersonic flow conditions. Exactly reverse procedure needs to be followed if givenflow is supersonic and the target is to attain specified subsonic condition.

It had been shown that addition of heat in subsonic flow increases the static temperature till$M_\infty < \gamma^{-1/2}$ and decreases afterwards. Main reason for this phenomenon is explained herewith. Addition of heat to subsonic flow increases the velocity and temperature initially for small amount of heat addition. Therefore both Kinetic energy (KE) and Internal energy (IE) of the flow increase due to externally added heat in subsonic flow. However for a given mass flow rate, rate of increase of KE is more than IE for given amount of heat addition. Therefore after particular amount of addition of external heat, it becomes impossible to increase IE (hence temperature) and KE (hence velocity) both keeping mass flow rate same. As a result beyond particular amount of heat addition for a given mass flow rate condition, temperature (hence IE) decreases but velocity (hence KE) continues to increase. We can as well interpreted the same phenomenon as, after certain critical amount of heat addition in subsonic flow added external heat becomes insufficient to increase the velocity (hence KE) while keeping the mass flow rate same, hence required extra energy is supplied by the flow itself from its internal energy, by virtue of which temperature decrease though we add heat in subsonic flow.

From the above mentioned formulae for sonic conditions or 'star' properties, we can calculate total temperature of the sonic flow after heat addition from any given initial conditions and hence the amount of heat required to be added to reach sonic condition from any given initial conditions. The properties at this sonic conditions for a given mass flow rate remain independent of upstream or freestream Mach number. Therefore, we can use this concept or these properties as reference properties for handling 1D flows with heat addition.

If the amount of heat added in the flow is more than the critical heat required to reach sonic condition, then flow cannot accommodate this heat. The main reason for this fact is the anchoring of conditions after heat addition to sonic point. Hence to accommodate the added extra heat, upstream conditions of the flow change from supersonic to subsonic or from subsonic to lower subsonic for which the externally added heat is the heat required to reach sonic condition.

3.7 Rayleigh curve

As we know, Rayleigh flow is called as the flow with heat addition. The curve or plot or state chart dealing with heat addition is called Rayleigh curve. Let's derive the expression for this curve in p-v and h-s chart.

$\rho v = const = k$ Mass Conservation

$p + \rho v^2 = const$ or $p + \frac{k^2}{\rho} = const$ Momentum Conservation

For 1D, it can be expressed as $pA + \rho v^2 A = const$, called as Impulse function or Thrust function.

$h_1 + \frac{v_1^2}{2} + q = h_2 + \frac{v_2^2}{2}$ Energy Conservation

We can draw a p-v diagram for flow with heat addition or Rayleigh flow like Hugoniot curve, using momentum equation as,

$$p + \frac{k_1^2}{\rho} = const \quad \text{Or} \quad p + \vartheta k^2 = const$$

This equation is the equation for straight line on p-v diagram where k^2 corresponds to slope which eventually is the mass flow rate. Hence, slope of the line joining any point, corresponding to initial state and final state on Rayleigh curve, represents mass fluxes or mass flow rates. This fact is same as that observed for Hugoniot curve.

Slope of this Rayleigh line can be calculated as

$$p_1 + v_1 k^2 = p_2 + v_2 k^2$$

$$\frac{p_2 - p_1}{v_2 - v_1} = -k^2$$

Such curve represented by points 1, 2, 3 and 4 for flow with heat addition is as shown in Figure 3.16 along with the isentropic and isothermal line on p-v chart

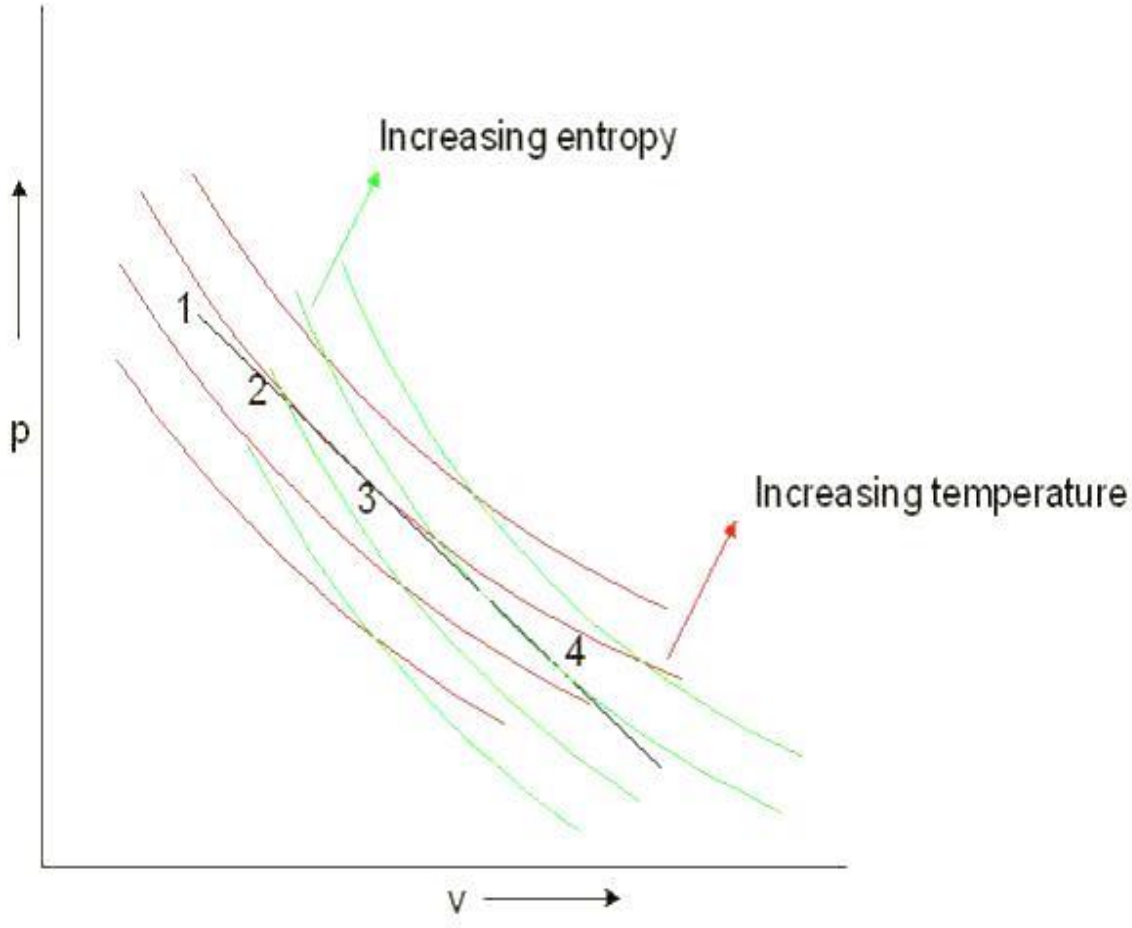


Fig. 3.16 P-V diagram for the heat addition process

For better understanding, consider the process of heat addition in subsonic flow. Suppose given conditions are described by point 1 in Fig.3.17. Change in thermodynamic states of the flow in the process of heat addition is shown in the figure by a straight line. Slope of this straight line is proportional to the mass flow rate and is given by Eq. 3.17. Here point 2 represents the conditions after certainly amount of heat addition. Point 2 essentially has lesser pressure and higher specific volume as that of point 1 since expansion of the subsonic flow takes place due to heat addition. Increase in temperature can also be observed here for the subsonic flow. Sufficient amount of heat addition would lead to reach point 3 from initial conditions 1. We can clearly see in this figure that the Rayleigh line is tangent to an isotherm at point 3, hence the temperature given by the corresponding isotherm is the maximum attainable temperature by adding heat in the given subsonic flow of initial conditions 1. Conditions represented by point 4 become possible by further addition of heat. It can also be seen here that Rayleigh line is tangent to an isentropic at point 4, hence point 4 represents the maximum entropy point or sonic point. Reduction is temperature in the process 3-4 is clearly evident in the presence of heat addition. Therefore there are two critical points in Rayleigh curve, one of which corresponds to maximum enthalpy or temperature and other corresponds to maximum entropy or total temperature or total enthalpy.

We can as well use h-s diagram to explain the heat addition process in the same subsonic flow as shown in Fig.3.17.

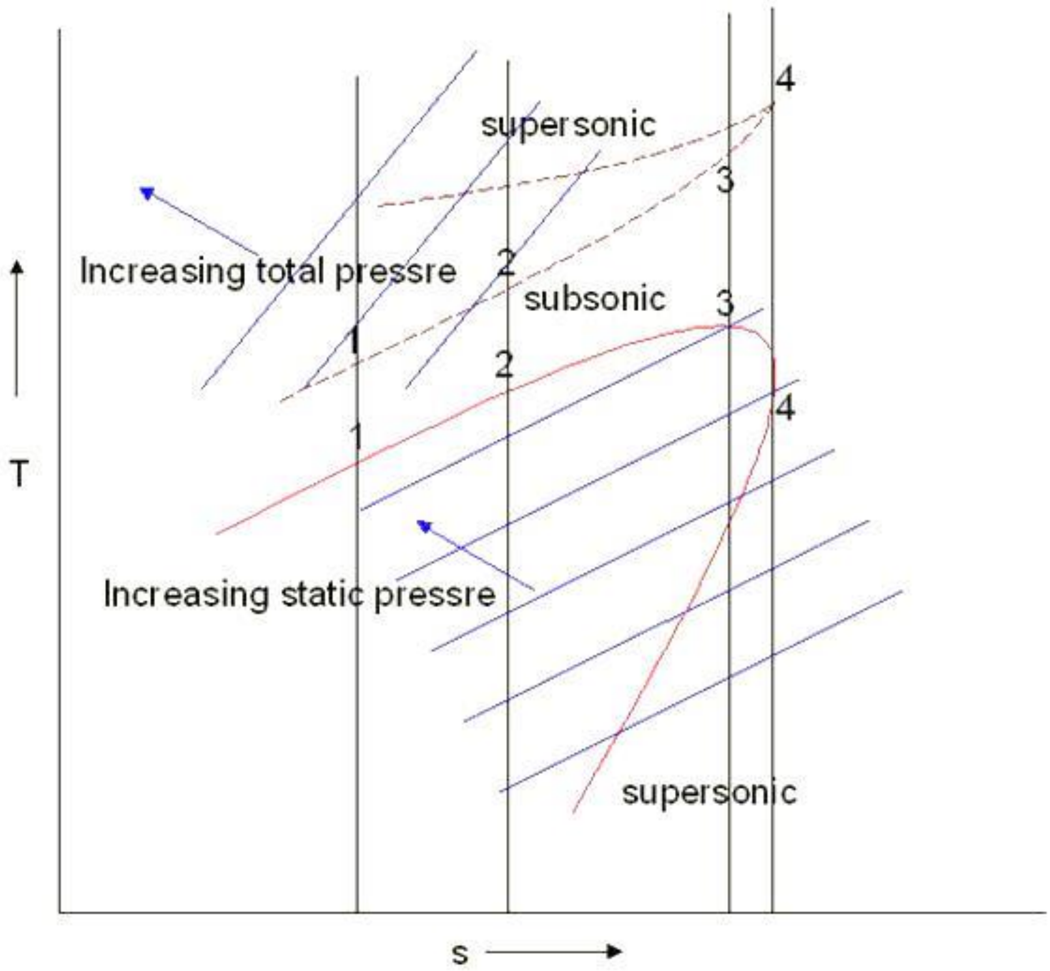


Fig.3.17 Rankine curve in h-s chart

3.7.1 Analysis of Critical Points of Rayleigh Curve

The same conservation equations for 1D heat addition can be re-written in differential form as,

For mass conservation equation

$$\rho u = const = k$$

$d(\rho u) = 0$ Mass Conservation

$$d\rho = -\frac{\rho}{u}\,du \quad \text{or} \quad du = -\frac{u}{\rho}\,d\rho$$

For momentum conservation equation

$$P + \rho u^2 = const$$

$$dp - \frac{k^2}{\rho^2}\,d\rho = 0$$

Momentum Conservation

For energy conservation equation

$$\left(h_2 - h_1\right) + \left(\frac{v_2^{\,2}}{2} - \frac{v_1^{\,2}}{2}\right) = q$$

dh + udu = dq   Energy Conservation

$$P + vk^2 = \text{const}$$

From laws of thermodynamics,

$$Tds = dh - vdp$$

However for point 3 (where Rankine line is tangent to an isotherm) dh=0 or dT=0, hence.

$$Tds = -\,vdp$$

From energy equation, for the same point we have,

$$Tds = udu$$

Hence,

$$-\,vdp = udu$$

Using differential form of mass conservation equation, we can replace du and we get,

$$-vdp = u\left(-\frac{u}{\rho}d\rho\right)$$

$$\frac{dp}{d\rho} = v^2$$

However, we know that point 3 also lies on an the isotherm, we can calculate the slope of the isotherm

to obtain another expression for dP/dρ as

$$p = \rho RT \qquad dp = d\rho RT \qquad \frac{dp}{d\rho} = RT$$

Hence we can put this dp/dρ in equation,

$$RT = u^2$$

$$\frac{\gamma RT}{\gamma} = u^2$$

$$\frac{a^2}{\gamma} = u^2$$

$$\frac{1}{\gamma} = \frac{u^2}{a^2}$$

$$M^2 = \gamma^{-1}$$

$$M = \gamma^{-1/2}$$

This procedure provides a formal proof for the Mach number at the point which corresponds to maximum temperature on Rayleigh curve. This expression also proves the fact that upper branch of Rankine curve in h-s diagram corresponds to subsonic flow.
Similarly we can also prove that the sonic point (M=1), represented by point 4, corresponds to maximumentropy.

We know the momentum conservation equation as

$$p + vk^2 = const$$

The differential form of the same can be written as

$$dp + k^2 dv = const$$

Hence slope of Rankine line on p-v diagram is

$$\frac{dp}{dv} = -k^2$$

We know that, at the point of maximum entropy, Rankine line is tangent to the isentropic. Hence slope of theisentropic at point 4 is

$$pv^\gamma = const$$

Differentiating this equation we get

$$v^\gamma dp + p\gamma v^{\gamma-1} dv = 0$$

$$\frac{dp}{dv} = -\frac{p\gamma}{v}$$

Equating the slopes we get,

$$\frac{dp}{dv} = -\frac{p\gamma}{v} = -k^2 = -(\rho u)^2$$

$$\frac{p\gamma}{v} = (\rho u)^2$$

$$\frac{p\gamma}{\rho} = u^2$$

$$a^2 = u^2$$

$$M = 1$$

This expression proves the fact that maximum entropy point on Rayleigh curve corresponds to Mach number 1 or sonic condition. We can extend our analysis about heat addition in subsonic flow and attainment of maximum entropy as follows.

We know that addition of heat increases the total temperature and hence total enthalpy of the flow. We know that,

$$h_0 = h + \frac{v^2}{2}$$

Hence the differential form of this equation is,

$$dh_0 = dh + udu$$

However added heat gives rise to this change in total enthalpy, hence

$$dq = dh_0$$

$$\text{But } dq = Tds.$$

Therefore

$$Tds = dh_0 = dh + udu$$

However at the maximum entropy point, ds = 0, therefore $dh_0 = 0$. Hence we can conclude that, maximum entropy point also corresponds to maximum total enthalpy and hence total temperature point. Therefore further addition of heat is not possible for the given mass flow rate. Hence point 4 also represents the chocking condition. Change in upstream conditions takes place if we add heat more that the heat requiredchocking the flow.

3.7.2 Choking of the flow with Heat Addition

It had been observed that the point of maximum entropy or maximum total temperature represents sonic condition. This condition also represents the maximum possible heat addition in a flow. However further increase in amount of heat addition in the flow decreases the mass flow rate through change in upstream or inlet conditions. Expected streamline pattern for un-choked and choked flow is shown in Figure.

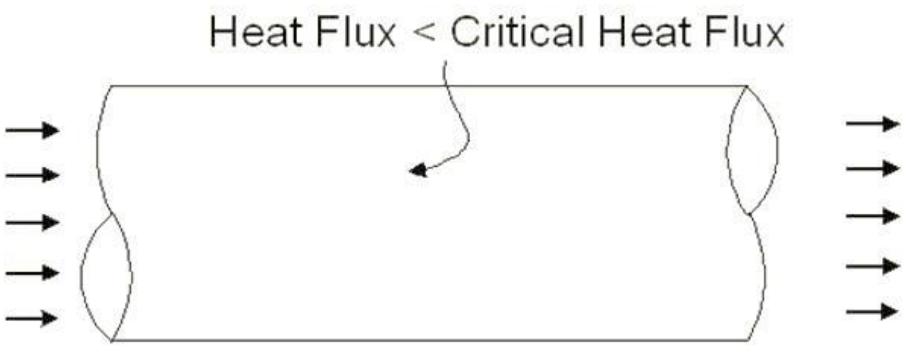


Fig.3.18 Streamline pattern for unchocked condition.

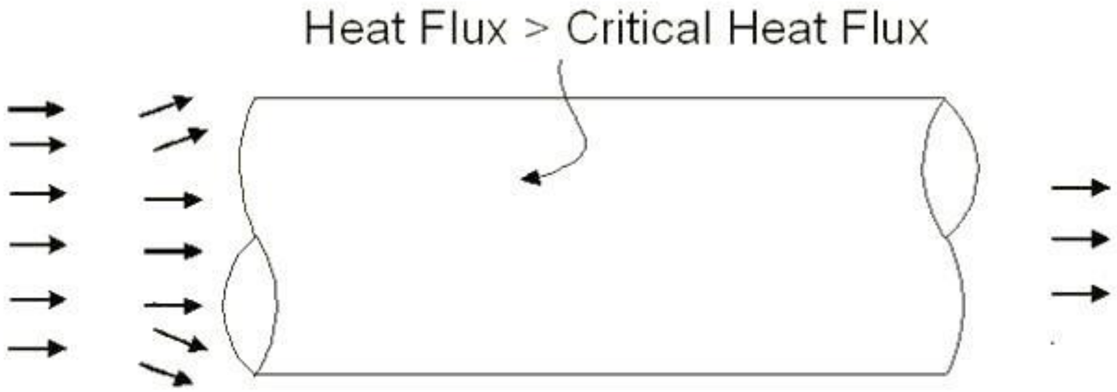


Fig.3.19 Streamline pattern for chocked condition.

3.7.3 Total Property Line of Rayleigh Curve

Total enthalpy and entrpoy for 1D flow with heat addition has also been plotted in Figure. Slope of this total enthalpy curve can be obtained from the above equation as,

$$\frac{dh_0}{ds} = T$$

Value of the local temperature on h_0-s diagram gives the slope of the curve. The Figure has been plotted for the single value of mass flow rate and we have already proved that upper branch of h-s diagram of Rayleigh line corresponds to subsonic flow. Hence bottom brach showing monotonic h-s curve for lower static enthalpy at the same mass flow rate corresponds to supersonic flow. Since subsonic flow has lesser kinetic energy, it has higher static enthalpy in turn higher static temperature in comparison with the supersonic flow for the same mass flow rate. Hence total enthalpy line for supersonic flow has lower slope than subsonic flow total enthalpy line. However both the lines should meet at highest total enthalpy point or highest entropy or sonic point for this mass flux condition. Hence, supersonic flow total enthalpy line should be at the top of subsonic total enthalpy line which is unlike the static enthalpy entropy lines. This fact can also be interpreted from the definition of total enthalpy where velocity appears in square. Since, mass flow rate is same and velocity of supersonic flow is higher than the velocity of subsonic flow, total enthalpy of supersonic flow will be higher than that of subsonic flow. Both static and total enthalpy lines cut the constant pressure lines on the corresponding charts which represent static and total pressures respectively. We can clearly see from Figure that static pressure increases due to heat addition in supersonic flow and the same decreases for subsonic flow. It is also clear from this figure that total pressure decreases for both supersonic and subsonic flows due to heat addition.

Understanding of process of heat addition in supersonic flow is much simpler than that for subsonic flow. It is also evident from the h-s diagram which represents monotonic curve for supersonic flow. It is clearly evident from this figure that static enthalpy, entropy and pressure increase due to heat addition for supersonic flow. From equation of Rayleigh curve for p-v chart, we can also know that increase in pressure leads to decrease in specific volume and hence increase in density for supersonic flow with heat addition. Highest amount of heat addition in the given supersonic flow leads to choking, maximum entropy and maximum temperature. Therefore point 4 represents the choking of supersonic flow as it represented for the subsonic flow of same mass flow rate. It can be clearly concluded that the process of heat addition can lead to supersonic flow to sonic state and further heat rejection to lower subsonic state. Moreover, heat addition in subsonic flow can fetch the sonic conditions and further heat rejection leads to supersonic condition. Hence possibility of reversible heat interaction eventually leads to trace the complete h-s plot of Rayleigh curve for a given mass flow rate.

3.7.3 Effect **of Change in Mass Flow Rate on T-S diagram for Rayleigh flow**

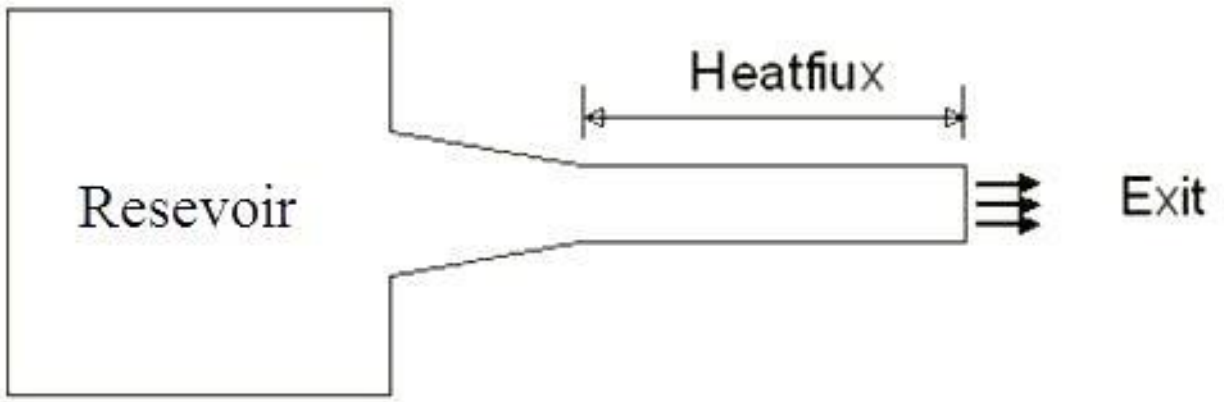


Fig 3.20 A proposed experimental set up for 1D heat addition studies

The h-s plot shown for heat addition process in Figure belongs to a particular value of mass flow rate. If we change the mass flow rate then the curve changes. Hence we need to find out the new curve in case of decreased or increased mass flow rate conditions. Consider the proposed experimental set up shown in Figure for the Rayleigh flow studies. Here we can change the mass flow rate by changing the reservoir conditions (pressure and temperature) and also the exit pressure. During the experiment, gas stored in the reservoir expands through the nozzle and achieves certain mass flow rate while flowing through the constant area duct where heat addition takes place. We are going to learn the topic "flow through nozzle" in detail in the coming lectures, however for now, we can assume that, mass flow rate for subsonic flow can be increased by decreasing the exit pressure or increasing the reservoir pressure. Similarly, we can increase the mass flow rate of supersonic flow by increase in reservoir pressure or by decreasing the reservoir temperature.

Consider the case of subsonic flow with increased mass flow rate. Initial isentropic expansion in the nozzle is given by the vertical line joining reservoir conditions and point 1 in Fig.3.21. If we add heat in the flow through constant area duct then the process of heat addition is represented by the dotted curve in the same figure. Suppose we increase the mass flow rate by decreasing exit pressure, then static enthalpy or static temperature at the entry to the heating duct decreases. Hence length of the initial vertical line increases and the nozzle exit conditions for subsonic flow are represented by 1'.

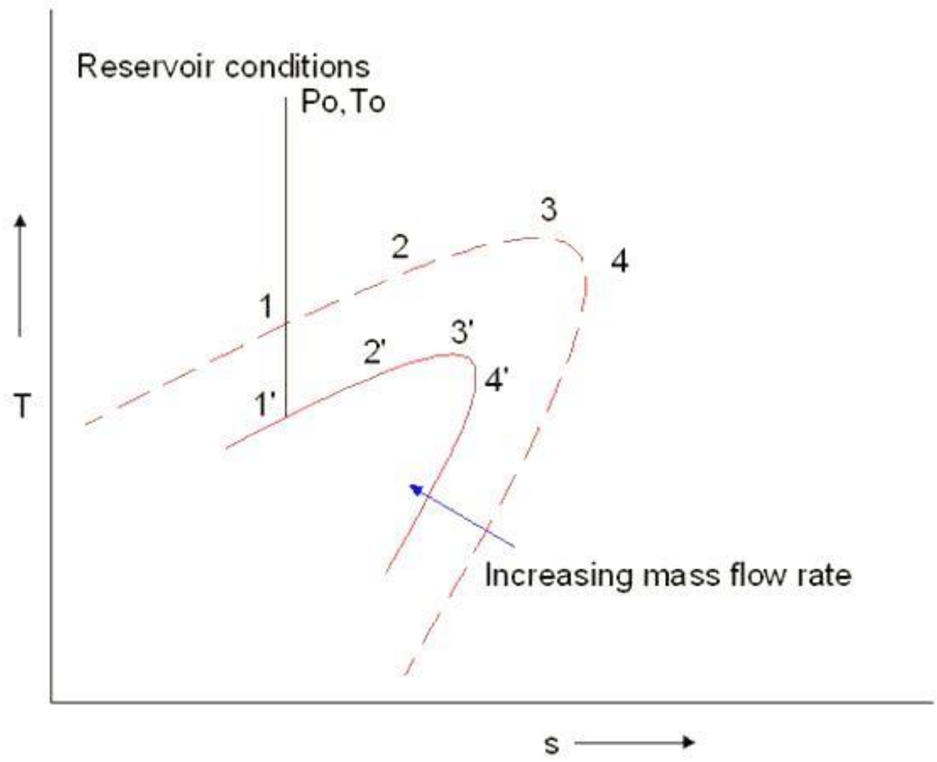


Fig 3.21 Rayleigh for increase in mass flow rate conditions

Further process of heat addition is shown by the thick line in the Fig. 3.21. Hence increase in mass flow rate of subsonic flow shifts the Rayleigh curve downwards. Now consider the process of heat addition in supersonic flow. We will have to replace the nozzle connecting the reservoir and heat addition section in Fig. 3.21 by corresponding supersonic nozzle to attain required supersonic Mach number at the entry to the constant cross-section heat addition section. Dotted Rayleigh curve given in Fig.3.21 represents the process of heat addition in the initial mass flow rate of supersonic flow. Increase in this mass flow rate can be achieved by increasing the reservoir pressure which is necessarily the total pressure of the flow. Since we have to increase the total pressure to increase the mass flow rate of supersonic flow, the static pressure, enthalpy and hence temperature increase at the entry to the heating duct. Further process of heat addition is given by the thick line in Fig.3.21. Hence increase in mass flow rate of supersonic flow shifts the Rayleigh curve upwards.

Two Rayleigh curves drawn for two mass flow rates cannot cut each other since, in such a case, the point of intersection of two curves will represent two different values of a thermodynamic property like density which is impossible. Therefore complete shrinking of the Rayleigh curve is necessary to represent the increased mass flow rate. This proves that maximum entropy attained by addition of heat decreases with increase in mass flow rate.

This understanding helps in assessing the situation where addition of heat is more than the required to reach sonic state for subsonic or supersonic flow. In both the situations, upstream mass flow rate at the heat addition station decreases. Hence entering subsonic flow at the heating station becomes lower subsonic or entering supersonic flow becomes subsonic in the presence of a shock to accommodate the extra heat.

3.7.4 Differential form of equations for 1D heat addition

We know the mass, momentum and energy equations for 1D heat addition process.

$$\rho u = const$$ (mass conservation)

$$\frac{d\rho}{\rho} = -\frac{du}{u}$$ (differential form of Mass conservation)

$$p + \rho u^2 = const$$ (momentum conservation)

$$dp + \rho u du = 0$$ (differential form of Momentum conservation)

$$dq = h_{0_2} - h_{0_1} = C_p T_{0_2} - C_p T_{0_1} = C_p dT_0$$ (energy conservation)

But we know that

$$h_0 = h + \frac{u^2}{2}$$

$$C_p T_0 = C_p T + \frac{u^2}{2}$$

Differential form of this equation is

$$dT_0 = dT + \frac{udu}{C_p}$$

Putting above equation in energy equation we get,

$$\frac{dq}{C_p} = dT + \frac{udu}{C_p}$$

Also,

$$\frac{dq}{C_p} = C_p \frac{dT}{du} + u$$ (differential form of energy equation)

We also know the ideal gas equation,

$$p = \rho RT$$ (ideal gas equation)

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}$$ (differential form of ideal gas equation)

We can use all the above mentioned differential forms for better understanding of the flow with heat addition.

We will initially consider the differential form of energy equation to replace the term dT/du. In this process, we can use differential form of mass conservation and ideal gas equation as

$$\frac{dp}{p} = -\frac{du}{u} + \frac{dT}{T}$$

Using differential form of momentum conservation we can replace dP of the above equation as,

$$-\frac{\rho udu}{p} = -\frac{du}{u} + \frac{dT}{T}$$

Hence,

$$-\frac{\rho u}{p}+\frac{1}{u}=\frac{dT}{du}\frac{1}{T}$$

$$\frac{dT}{du}=\frac{T}{u}-\frac{\rho u T}{p}=\frac{T}{u}-\frac{u}{R}$$

Now we can use above equation to replace dT/du from differential form of energy equation. Hence differential form of energy equation becomes,

$$\frac{dq}{du}=\left(\frac{T}{u}-\frac{u}{R}\right)C_p+u$$

$$\frac{dq}{du}=C_p\frac{T}{u}-u\left(\frac{C_p}{R}-1\right)$$

But we know that,

$$C_p=\frac{\gamma R}{\gamma-1} \text{ or } \frac{C_p}{R}=\frac{\gamma}{\gamma-1}$$

Therefore,

$$\frac{dq}{du}=C_p\frac{T}{u}-\left[\frac{\gamma}{\gamma-1}-1\right]u$$

$$\frac{dq}{du}=C_p\frac{T}{u}-\frac{u}{\gamma-1}$$

Therefore, there are three cases viz. dq/du to be zero, positive or negative. Consider first case to start with,

$$C_p\frac{T}{v}=\frac{\gamma}{\gamma-1}$$

$$u^2=C_p(\gamma-1)T=\gamma RT=a^2$$

M = 1

This particular case can be interpreted as further change in velocity is impossible with heat addition. Hence, dq/duto be zero represents maximum entropy or choking or sonic condition.

Now consider dq/du to be positive,

$$C_p\frac{T}{u}>\frac{u}{\gamma-1}$$

$$\therefore a^2>u^2$$

This particular case belongs to subsonic flow. Therefore for subsonic flow, dq/duis positive which implicitly means velocity of the subsonic flow increases (du > 0) if heat is added in it (dq > 0). At the same time, velocity of the subsonic flow decreases (du < 0) if heat is rejected from it (dq < 0).

Now consider dq/duto be negative,

$$C_p\frac{T}{u}<\frac{u}{\gamma-1}$$

$$\therefore a^2<u^2$$

This particular case belongs to supersonic flow. Therefore for supersonic flow, dq/duis negative which implicitly means velocity of the supersonic flow decreases (du < 0) if heat is added in it (dq > 0). At the same time, velocity of the supersonic flow increases (du > 0) if heat is rejected from it (dq < 0).

Now consider Equation

$$\frac{dT}{du} = \frac{T}{u} - \frac{\rho u T}{p} = \frac{T}{u} - \frac{u}{R}$$

we can clearly see the existence of three cases here also, in which dT/du can be either zero, positive, or negative. Consider dT/duto be zero so,

$$\frac{T}{u} = \frac{u}{R}$$

$$\therefore u^2 = RT = \frac{\gamma RT}{\gamma} = \frac{a^2}{\gamma}$$

$$M = \gamma^{-1/2}$$

Hence for this condition we will have dT/du zero means dT = 0. This condition corresponds to maximum temperature attained by heat addition. From$M = \gamma^{-1/2}$, it becomes clear that, this condition belongs to subsonic flow where further increase in temperature is not possible with heat addition.

Now consider the case where, dT/du is positive. Hence

$$\frac{T}{u} > \frac{u}{R}$$

$M^2 < \gamma^{-1/2}$This means that for subsonic flow till$M^2 < \gamma^{-1/2}$, dT/du is positive. We have already seen that for subsonic flow du is positive. This proves that, for subsonic flow, dT is positive till$M^2 < \gamma^{-1/2}$.

Now for the third case with dT/du is negative, we will get $> \gamma^{-1/2}$. However for subsonic flow du is always positive, hence for any subsonic Mach number which is $M > \gamma^{-1/2}$ till M = 1, dT is negative. Therefore in this range of subsonic Mach number added heat decreases temperature and increases the velocity. However from present expression, dT/du is negative for all$> \gamma^{-1/2}$. Hence for supersonic flows also dT/du is negative. But we have already seen that du is negative for supersonic flow with heat addition, hence dT is positive for supersonic flow. Hence externally added heat in supersonic flow decreases velocity and increases temperature.

We can modify the right hand side of Equation as

$$\frac{TR - u^2}{Ru} = \frac{(\gamma RT)/_{u^2}}{Ru} = \frac{a^2/_{\gamma} - u^2}{Ru}$$

Hence, Equation can be written as,

$$\frac{dT}{du} = \frac{a^2/_{\gamma} - u^2}{Ru} = \frac{1 - M^2\gamma}{Ru}$$

The proof about all the critical points can be achieved from this expression as well.

UNIT – IV

APPLICATIONS OF COMPRESSIBLE FLOWS AND NUMERICAL TECHNIQUES

4.1 Small perturbation equations for subsonic, transonic, supersonic and hypersonic flow

4.1.1 Small-Perturbation Theory

A great number of problems of interest in compressible fluid mechanics are concerned with the perturbation of a known flow pattern. The most common case is that of uniform, steady flow. Let U denote the uniform flow velocity, which is directed parallel to the x -axis. The density, pressure, and temperature are also assumed to be uniform, and are denoted ρ_∞, P_∞ and T_∞ respectively. The corresponding sound speed is c_∞ , and the Mach number is $Ma_\infty = U/c_\infty$

Finally,the velocity field of the unperturbed flow pattern is

$$u = U,$$
$$v = 0$$

Suppose that a solid body, such as an airfoil, is placed in the aforementioned flow pattern. The cross-section of the body is assumed to be independent of the Cartesian coordinate Z. The body disturbs the flow pattern, and changes its velocity field, which is now written

$$u = U + u',$$
$$v = v'$$

where u' and v' are known as induced velocity components. We are interested in situations in which

$$u'/U \ll 1 \text{ and } v'/V \ll 1$$

Equation can be combined with the previous two equations to give

$$C^2 = C_\infty^2 - \frac{1}{2}(\gamma - 1)(2Uu' + u'^2 + v'^2)$$

It then follows from Equation that

$$(1 - Ma_\infty^2)\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = Ma_\infty^2\left[(\gamma-1)\frac{u'}{U} + \left(\frac{\gamma+1}{2}\right)\left(\frac{u'}{U}\right)^2 + \left(\frac{\gamma-1}{2}\right)\left(\frac{v'}{U}\right)^2\right]\frac{\partial u'}{\partial x} + Ma_\infty^2\left[(\gamma-1)\frac{v'}{U} + \left(\frac{\gamma+1}{2}\right)\left(\frac{v'}{U}\right)^2 + \left(\frac{\gamma-1}{2}\right)\left(\frac{u'}{U}\right)^2\right]\frac{\partial u'}{\partial y} + Ma_\infty^2\frac{u'}{U}\left(1 - \frac{u'}{U}\right)\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right)$$

The previous equation is exact. However, if u'/U and v'/V are small then it becomes possible to neglect many of the terms on the right-hand side. For instance, neglecting terms that are third-order in small quantities, we obtain

$$(1 - Ma_\infty^2)\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \simeq Ma_\infty^2(\gamma+1)\frac{u'}{U}\frac{\partial u'}{\partial x} + Ma_\infty^2(\gamma-1)\frac{u'}{U}\frac{\partial v'}{\partial y} + Ma_\infty^2\frac{v'}{U}\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right)$$

Furthermore, if we neglect terms that are second-order in small quantities then we get the linear equation

$$1 - Ma_\infty^2\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) \simeq 0$$

Note, however, than in so-called *transonic* flow, where$Ma_\infty \simeq 1$, the coefficient of$\partial u'/_{\partial x}$ on the left-hand side of Equation becomes very small. In this situation, it is not possible to neglect the first term on the right-hand side. However, the condition$Ma_\infty \simeq 1$ does not affect the term $\partial v'/_{\partial y}$ on the left-hand side of Equation, and so the other terms on the right-hand side can still be neglected. Thus, transonic flow is governed by the non-linear equation

$$(1 - \mathrm{Ma}_\infty^2)\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \simeq \mathrm{Ma}_\infty^2(\gamma+1)\frac{u'}{U}\frac{\partial u'}{\partial x}.$$

On the other hand, subsonic (i.e.,$\mathrm{Ma}_\infty<1$) and supersonic flow (i.e., $\mathrm{Ma}_\infty>1$) are both governed by Equation . Another situation in which certain terms on the right-hand side of Equation must be retained is *hypersonic* flow (i.e., $\mathrm{Ma}_\infty>>1$). This follows because, although u'/U and v'/V are small, their products withMa_∞^2 can still be non-negligible. Roughly speaking, Equation is valid for $0 \leq \mathrm{Ma}_\infty \leq 0.8$ and $1.2 \leq \mathrm{Ma}_\infty \leq 5$ and . In other words, transonic flow corresponds to $0.8 \leq \mathrm{Ma}_\infty \leq 1.2$, and hypersonic flow to $\mathrm{Ma}_\infty>5$.
The pressure coefficient is defined

$$C_p = \frac{p - p_\infty}{(1/2)\,\rho_\infty U^2} = \frac{2}{\gamma\,\mathrm{Ma}_\infty^2}\frac{p - p_\infty}{p_\infty}.$$

Equation implies that

$$c^2 + \frac{1}{2}(\gamma-1)\,q^2 = c_\infty^2 + \frac{1}{2}(\gamma-1)\,U^2,$$

Where

$$q^2 = (U + u')^2 + v'^2$$

.

Moreover, Equation yields

$$\frac{p}{p_\infty} = \left[\frac{2 + (\gamma-1)\,\mathrm{Ma}_\infty^2}{2 + (\gamma-1)\,\mathrm{Ma}^2}\right]^{\gamma/(\gamma-1)},$$

Where Ma = q/c . The previous two equations can be combined to give

$$\frac{p}{p_\infty} = \left[1 + \frac{1}{2}(\gamma-1)\,\mathrm{Ma}_\infty^2\left(1 - \frac{q^2}{U^2}\right)\right]^{\gamma/(\gamma-1)}.$$

Hence, we obtain

$$C_p = \frac{2}{\gamma\,\mathrm{Ma}_\infty^2}\left(\left[1+\frac{1}{2}(\gamma-1)\,\mathrm{Ma}_\infty^2\left(1-\frac{q^2}{U^2}\right)\right]^{\gamma/(\gamma-1)}-1\right),$$

which reduces to

$$C_p = \frac{2}{\gamma\,\mathrm{Ma}_\infty^2}\left(\left[1-\frac{1}{2}(\gamma-1)\,\mathrm{Ma}_\infty^2\left(\frac{2\,u'}{U}+\frac{u'^2+v'^2}{U^2}\right)\right]^{\gamma/(\gamma-1)}-1\right).$$

Using the binomial expansion on the expression in square brackets, and neglecting terms that are third-order, or higher, in small quantities, we obtain

$$C_p \simeq -\left[\frac{2u'}{U}+(1-\mathrm{Ma}_\infty^2)\left(\frac{u'}{U}\right)^2+\left(\frac{v'}{U}\right)^2\right].$$

For two-dimensional flows, in the limit in which Equation is valid, it is consistent to retain only first-order terms in the previous equation, so that

$$C_p \simeq -\frac{2\,u'}{U}.$$

Let

f(x,y) = 0

be the equation of the surface of the solid body that perturbs the flow. At the surface, the velocity vector of the flow must be perpendicular to the local normal: that is, the flow must be tangential to the surface. In other words,

$$q.\nabla f = 0$$

which reduces to

$$(U+u')\,\frac{\partial f}{\partial x}+v'\,\frac{\partial f}{\partial y}=0.$$

Neglecting u' with respect to U , we obtain

$$\frac{v'}{U}\simeq-\frac{\partial f/\partial x}{\partial f/\partial y}=\frac{dy}{dx},$$

Where dy/dx is the slope of the surface, and v'/u the approximate slope of a streamline.

Now, the body has to be thin in order to satisfy our assumption that the induced velocities are relatively small. This implies that the coordinate y differs little from zero (say) on the surface of the body. Hence, we can write

$$v'(x,y)=v'(x,0)+\left(\frac{\partial v'}{\partial y}\right)_{y=0}y+\cdots$$

in the immediate vicinity of the surface. Within the framework of small-perturbation theory, it is consistent to neglect all terms on the right-hand side of the previous equation after the first. Hence, the boundary condition reduces to

$$v'(x,0) = U\left(\frac{dy}{dx}\right)_{\text{body}},$$

respectively.
Because a homenergic, homentropic flow pattern is necessarily irrotational, we can write

$$\mathbf{q} = U\,\mathbf{e}_x - \nabla\phi,$$

Where ϕ is the perturbed velocity potential.
It follows that

$$U^2 = -\frac{\partial\phi}{\partial x},$$
$$v^2 = -\frac{\partial\phi}{\partial y},$$

Hence, Equationsbecome

$$(1-\mathrm{Ma}_\infty^2)\,\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \simeq 0,$$

$$C_p \simeq \frac{2}{U}\,\frac{\partial\phi}{\partial x},$$

$$\frac{\partial\phi(x,0)}{\partial y} \simeq -U\left(\frac{dy}{dx}\right)_{\text{body}},$$

4.2 Experimental characteristics of airfoils in compressible flow

4.2.1 Linearization of Velocity Potential Equation

Consider the steady irrotational flow around the thin aerofoil as shown in **Fig. 4.1**

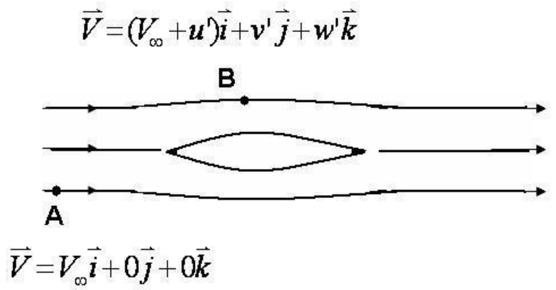


Fig. 4.1 Schematic of the perturbed velocity field.

At location A, velocity is only in x direction. However, presence of body perturbs the components of velocity at location B. Let's represent the general velocity field as,

$$\vec{V} = V_x\vec{i} + V_y\vec{j} + V_z\vec{k}$$

here,$V_x = V_\infty + u'$ and u', v', w' are the perturbed velocities in the x, y and z directions respectively such that.

$$\mathbf{V_y = v'}$$
$$\mathbf{V_w = w'}$$

Since the velocity field is irrotational, we can represent the velocity field using gradient of velocity potential as,

$$\nabla\Phi = \vec{V} = (V_\infty + u')\vec{i} + v'\vec{j} + w'\vec{k}$$

Let the perturbed velocity field be presented by perturbed velocity potential, ϕ. Hence,

$$\frac{\partial\phi}{\partial x} = u' \quad \frac{\partial\phi}{\partial y} = v' \quad \frac{\partial\phi}{\partial z} = w'$$

Therefore,

$\Phi(x, y, z) = V_\infty x + \phi(x, y, z)$

Such that,

$$V_x = \frac{\partial\Phi}{\partial x} = V_\infty x + u' = V_\infty x + \frac{\partial\phi}{\partial x}$$
$$V_y = \frac{\partial\Phi}{\partial y} = \frac{\partial\phi}{\partial y} = v'$$
$$V_z = \frac{\partial\Phi}{\partial z} = \frac{\partial\phi}{\partial z} = w'$$

and

$$\Phi_{xx} = \frac{\partial^2\phi}{\partial x^2} = \phi_{xx}, \quad \Phi_{yy} = \frac{\partial^2\phi}{\partial y^2} = \phi_{yy}, \quad \Phi_{zz} = \frac{\partial^2\phi}{\partial z^2} = \phi_{zz}$$

We can use these expressions in the known velocity potential equation.

$$\left(1 - \frac{\Phi_x^2}{a^2}\right)\Phi_{xx} + \left(1 - \frac{\Phi_y^2}{a^2}\right)\Phi_{yy} + \left(1 - \frac{\Phi_z^2}{a^2}\right)\Phi_{zz}$$
$$-2\frac{\Phi_x\Phi_y}{a^2}\Phi_{xy} - 2\frac{\Phi_x\Phi_z}{a^2}\Phi_{xz} - 2\frac{\Phi_y\Phi_z}{a^2}\Phi_{yz} = 0$$

This expression in the form of perturbed velocity potential can be written as,

$$\left[a^2-\left(V_\infty+\frac{\partial\phi}{\partial x}\right)^2\right]\frac{\partial^2\phi}{\partial x^2}+\left[a^2-\left(\frac{\partial\phi}{\partial y}\right)^2\right]\frac{\partial^2\phi}{\partial y^2}+\left[a^2-\left(\frac{\partial\phi}{\partial z}\right)^2\right]\frac{\partial^2\phi}{\partial z^2}$$

$$-2\left(V_\infty+\frac{\partial\phi}{\partial x}\right)\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y}-2\left(V_\infty+\frac{\partial\phi}{\partial x}\right)\frac{\partial\phi}{\partial z}\frac{\partial^2\phi}{\partial x\partial z}-2\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial z}\frac{\partial^2\phi}{\partial y\partial z}=0$$

or,

$$\left[a^2-(V_\infty+u')^2\right]\frac{\partial u'}{\partial x}+\left[a^2-v'^2\right]\frac{\partial v'}{\partial y}+\left[a^2-w'^2\right]\frac{\partial w'}{\partial z}$$

$$-2(V_\infty+u')v'\frac{\partial v'}{\partial y}-2(V_\infty+u')w'\frac{\partial u'}{\partial z}-2v'w'\frac{\partial v'}{\partial z}=0$$

But we know that, total enthalpy is constant in flow field. We can use this fact to represent the speed of sound encountered in the above equation as,

h_o = const.

$$h_0=h_\infty+\frac{V_\infty^2}{2}=h+\frac{V^2}{2}=h+\frac{(V_\infty+u')^2+v'^2+w'^2}{2}$$

$$\frac{a_\infty^2}{\gamma-1}+\frac{V_\infty^2}{2}=\frac{a^2}{\gamma-1}+\frac{(V_\infty+u')^2+v'^2+w'^2}{2}$$

$$a^2=a_\infty^2-\frac{\gamma-1}{2}\left(2u'V_\infty+u'^2+v'^2+w'^2\right)$$

Using this expression and further simplification, Eq. can be written as,

$$\left(1-M_\infty^2\right)\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}$$

$$=M_\infty^2\left[(\gamma+1)\frac{u'}{V_\infty}+\left(\frac{\gamma+1}{2}\right)\frac{u'^2}{V_\infty^2}+\left(\frac{\gamma-1}{2}\right)\left(\frac{v'^2+w'^2}{V_\infty^2}\right)\right]\frac{\partial u'}{\partial x}$$

$$+M_\infty^2\left[(\gamma-1)\frac{u'}{V_\infty}+\left(\frac{\gamma+1}{2}\right)\frac{v'^2}{V_\infty^2}+\left(\frac{\gamma-1}{2}\right)\left(\frac{w'^2+u'^2}{V_\infty^2}\right)\right]\frac{\partial v'}{\partial x}$$

$$+M_\infty^2\left[(\gamma-1)\frac{u'}{V_\infty}+\left(\frac{\gamma+1}{2}\right)\frac{w'^2}{V_\infty^2}+\left(\frac{\gamma-1}{2}\right)\left(\frac{u'^2+v'^2}{V_\infty^2}\right)\right]\frac{\partial w'}{\partial x}$$

$$+M_\infty^2\left[\frac{v'}{V_\infty}\left(1+\frac{u'}{V_\infty}\right)\left(\frac{\partial u'}{\partial y}+\frac{\partial v'}{\partial x}\right)+\frac{w'}{V_\infty}\left(1+\frac{u'}{V_\infty}\right)\left(\frac{\partial u'}{\partial z}+\frac{\partial w'}{\partial x}\right)+\frac{u'w'}{V_\infty}\left(\frac{\partial w'}{\partial y}+\frac{\partial v'}{\partial z}\right)\right]\frac{\partial v'}{\partial x}$$

The equation is the exact equation for steady irrotational flow around the thin configurations.

We can simplify this equation, since the perturbed velocities, u', v' and w'are small in comparison with the freestream velocity V_∞ .

Hence,

$$\frac{u'}{V_\infty},\ \frac{v'}{V_\infty} \text{ and } \frac{w'}{V_\infty} <<1$$

So,

$$\left(\frac{u'}{V_\infty}\right)^2 \left(\frac{v'}{V_\infty}\right)^2 \left(\frac{w'}{V_\infty}\right)^2 <<<1$$

This approximations leads to two facts,

1. Except for the Transonic flows (Flows having Mach number in the range 0.8 to 1.2)

$$M_\infty^2\left[(\gamma+1)\frac{u'}{V_\infty}+\cdots\right]\frac{\partial u'}{\partial x} << \left(1-M_\infty^2\right)\frac{\partial u'}{\partial x}$$

2. If Mach number of the flow is less than 5.0

$$M_\infty^2\left[(\gamma+1)\frac{u'}{V_\infty}+\cdots\right]\frac{\partial u'}{\partial x} << \left(1-M_\infty^2\right)\frac{\partial u'}{\partial x}$$

$$M_\infty^2\left[(\gamma-1)\frac{u'}{V_\infty}+\cdots\right]\frac{\partial v'}{\partial y} << \frac{\partial v'}{\partial y},\quad M_\infty^2\left[(\gamma-1)\frac{u'}{V_\infty}+\cdots\right]\frac{\partial w'}{\partial z} << \frac{\partial w'}{\partial z}$$

$$M_\infty^2\left[(\gamma-1)\frac{v'}{V_\infty}\left(1+\frac{u'}{V_\infty}\right)\left(\frac{\partial u'}{\partial y}+\frac{\partial v'}{\partial x}\right)+\cdots\right] <<1 \quad (\approx 0)$$

For these two facts we get,

$$\left(1-M_\infty^2\right)\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0$$

Or,

$$\left(1-M_\infty^2\right)\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}+\frac{\partial^2 \phi}{\partial z^2}=0$$

This is the linear equation and this approximation is valid for steady irrotational subsonic and supersonic flows under the assumption of small perturbation.

4.3 Supercritical airfoil

A Supercritical airfoil is an airfoil designed, primarily, to delay the onset of wave drag inthe transonic speed range. Supercritical airfoils are characterized by their flattened upper surface, highly cambered (curved) aft section, and smaller leading edge radius compared with traditional airfoil shapes. The supercritical airfoils were designed in the 1960s, by then NASA Engineer Richard Whitcomb, and were first tested on a modified North American T-2C Buckeye. After this first test, the airfoils were tested at higher speeds on the TF-8A Crusader While the design was initially developed as part of the supersonic transport (SST) project at NASA, it has since been mainly applied to increase the fuel efficiency of many high subsonic aircraft. The supercritical airfoil shape is incorporated into the design of a supercritical wing.

Supercritical airfoils feature four main benefits: they have a higher drag divergence Machnumber,they develop shock waves further aft than traditional airfoils,they greatly reduce shock-induced boundary layer separation, and their geometry allows for more efficient wing design (e.g., a thicker wing and/or reduced wing sweep, each of which may allow for a lighter wing). At a particular speed for a given airfoil section, the critical Mach number, flow over the upper surface of an airfoil can become locally supersonic, but slows down to match the pressure at the trailing edge of the lower surface without a shock. However, at a certain higher speed,the drag divergence Mach number, a shock is required to recover enough pressure to match the pressures at the trailing edge. This shock causes transonic wave drag, and can induce flow separation behind it; both have negative effects on the airfoil's performance.

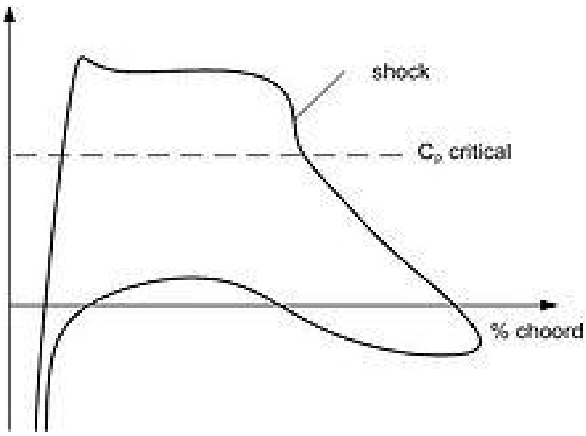


Fig 4.2 Supercritical airfoil Mach Number/pressure coefficient diagram The sudden increase in pressure coefficient at midchord is due to the shock. (y-axis: Mach number (or pressure coefficient, negative up); x-axis: position along chord, leading edge left)

At a certain point along the airfoil, a shock is generated, which increases the pressure coefficient to the critical value $C_{p\text{-}crit}$, where the local flow velocity will be Mach 1. The position of this shockwave is determined by the geometry of the airfoil; a supercritical foil is more efficient because the shockwave is minimized and is created as far aft as possible thus reducingdrag. Compared to a typical airfoil section, the supercritical airfoil creates more of its lift at the aft end, due to its more even pressure distribution over the uppersurface.

4.3.1 Swept Wing

A swept wing is a wing planform favored for high subsonic and supersonic speeds, and is found on almost all jet aircraft in one form or another, as well as some high speed propeller aircraft.
Compared with straight wings common to slower aircraft, they have a "swept" wing root to wingtip direction angled beyond (usually aftward) the spanwise axis. This has the effect of delaying the aerodynamic drag rise caused by fluid compressibility near the speed of sound, increasing performance.

As an aircraft enters the transonic speeds just below the speed of sound, the pressure waves associated with subsonic flight converge and begin to impinge on the aircraft. As the pressure waves converge the air in front of the aircraft begins to compress. This creates a force known as wave drag. This wave drag increases steeply until the whole aircraft is supersonic and then reduces.
However, shock waves can form on some parts of an aircraft at a speed where the aircraft is moving at less than M1.0 because the *local* speed of sound is determined by air density. A rapid drop in air density caused by reductions in aircraft contours will cause a drop in the local speed of sound and shock waves may form.
With objects where there is a sudden reduction in profile/thickness and the local air expands rapidly to fill the space taken by the solid object or where a rapid angular change is imparted tothe airflow causing a momentary increase of volume/decrease in density, an oblique shock

wave is generated. This is why shock waves are often associated with the part of a fighter aircraft cockpit canopy with the highest local curvature, appearing immediately behind this point.
At the point where the density drops, the local speed of sound correspondingly drops and a shock wave can form. This is why in conventional wings, shock waves form first *after* the maximum Thickness/Chord and why all airliners designed for cruising in the transonic range (above M0.8) have supercritical wings that are flatter on top resulting in minimized angular change of flow to upper surface air. The angular change to the air that is normally part of lift generation is decreased and this lift reduction is compensated for by deeper curved lower surfaces accompanied by a reflex curve at the trailing edge. This results in a much weaker standing shock wave towards the rear of the upper wing surface and a corresponding *increase*in critical Mach number.

4.3.2 Second order equation for transonic flows

The numerical solution of the compressible Euler and Navier-Stokes equations in primitive variables form requires the use of artificial viscosity or upwinding. Methods that are first-order- accurate are too dissipative and reduce the effective Reynolds number substantially unless a very fine grid is used. A first-order finite element method for the solution of the Euler and Navier- Stokes equations can be constructed by adding Laplacians of the primitive variables to the governing equations. Second-order schemes may require fourth-order dissipation and higher- order elements. A finite element approach is proposed in which the fourth-order dissipation is recast as the difference of two Laplacian operators, allowing the use of bilinear elements. The Laplacians of the primitive variables of the first-order scheme are thus balanced by additional terms obtained from the governing equations themselves, tensor identities or other forms of nodal averaging.

4.3 Area Rule

Whitcomb's Area Rule

The Whitcomb area rule, also called the transonic area rule, is a design technique used to reducean aircraft's drag transonic and supersonic speeds, particularly between Mach 0.75 and 1.2.This is one of the most important operating speed ranges for commercial and military fixed-wingaircraft today, with transonic acceleration being considered an important performance metric for combat aircraft and necessarily dependent upon transonic drag.

At high-subsonic flight speeds, the local speed of the airflow can reach the speed of sound where the flow accelerates around the aircraft body and wings. The speed at which this development occurs varies from aircraft to aircraft and is known as the critical Mach number. Theresulting shock waves formed at these points of sonic flow can greatly reduce power, which is experienced by the aircraft as a sudden and very powerful drag, called wave drag. To reduce the number and power of these shock waves, an aerodynamic shape should change in crosssectional area as smoothly as possible.

The area rule says that two airplanes with the same longitudinal cross-sectional area distribution have the same wave drag, independent of how the area is distributed laterally (i.e. in the fuselage or in the wing). Furthermore, to avoid the formation of strong shock waves, this total area distribution must be smooth. As a result, aircraft have to be carefully arranged so that at the location of the wing, the fuselage is narrowed or "waisted", so that the total area doesn't change much. Similar but less pronounced fuselage waisting is used at the location of a bubble canopy and perhaps the tail surfaces.

The area rule also holds true at speeds exceeding the speed of sound, but in this case the body arrangement is in respect to the Mach line for the design speed. For example, consider that at Mach 1.3 the angle of the Mach cone formed off the body of the aircraft will be at about μ = arcsin(1/M) = 50.3° (μ is the angle of the Mach cone, or simply Mach angle). In this case the "perfect shape" is biased rearward; therefore, aircraft designed for high speed cruise usually have wings towards the rear.A classic example of such a design is the Concorde. When applying the transonic area rule, the condition that the plane defining the cross-section meets the longitudinal axis at the Mach angle μ no longer prescribes a unique plane for μ other than the 90° given by M= 1. The correct procedure is to average over all possible orientations of the intersecting plane.

4.4 Theory of characteristics (method of characteristics), determination of the characteristic lines and compatibility equations,

Theory of characteristics

4.4.1 Introduction

The method of characteristics has been used for many years to compute supersonic irrotational flows. Although the method has a strong analytical basis, its practical implementation is, essentially, always numerical and it is then used to compute the values of the flow variables at a series of distinct points in the flow rather than continuously throughout the flow field.

Let's consider a general steady two-dimensional irrotational flowfield. We have already derived thevelocity potential (φ) equation for such flowfield.

$$\left(1-\frac{u^2}{a^2}\right)\frac{\partial^2\phi}{\partial x^2}+\left(1-\frac{v^2}{a^2}\right)\frac{\partial^2\phi}{\partial y^2}-\frac{2uv}{a^2}\frac{\partial^2\phi}{\partial x\partial y}=0$$

Consider the change in any flow variable, *f*, *df* which can be determined by small changes in the coordinates *dx* and *dy* as illustrated in Figure. The change in the variable, *df*,

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$

If $f=\partial\emptyset/\partial x$ then this will give:

$$d\left(\frac{\partial\phi}{\partial x}\right)=\frac{\partial^2\phi}{\partial x^2}dx+\frac{\partial^2\phi}{\partial x\partial y}dy$$

But $u=\partial\emptyset/\partial x$ so this equation gives:

$$du=\frac{\partial^2\phi}{\partial x^2}dx+\frac{\partial^2\phi}{\partial x\partial y}dy$$

Similarly if $f=\partial\emptyset/\partial y$ then this will give:

$$d\left(\frac{\partial\phi}{\partial y}\right)=\frac{\partial^2\phi}{\partial x\partial y}dx+\frac{\partial^2\phi}{\partial y^2}dy$$

But $v = {}^{\partial \emptyset}/_{\partial y}$ so this equation gives:

$$dv = \frac{\partial^2 \phi}{\partial x \partial y} dx + \frac{\partial^2 \phi}{\partial y^2} dy$$

Now consider eqs. which involve second derivatives of φ and can be solved using Cramer's Rule. For example

$$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{(1 - u^2/a^2) du dy + (1 - v^2/a^2) dv dx}{(1 - u^2/a^2)(dy)^2 + (2uv/a^2) dx dy + (1 - v^2/a^2)(dx)^2}$$

In general, this equation can be solved for any chosen values of dx and dy , i.e., for any chosen direction, to give $\partial^2\emptyset/\partial x\partial y$at a selected point in the flow. However, it is possible have$\partial^2\emptyset/\partial x\partial y$ indeterminate in certain directions. More over this differential is expected to be finite at that point. Hence in these directions, eq. must give an indeterminate value.

$$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{0}{0}$$

Let's consider the denominator of equation since along a particular direction$\partial^2\emptyset/\partial x\partial y$ is indeterminate. Hence the denominator should be zero, as:

$$\left(1 - \frac{u^2}{a^2}\right)(dy)^2 + \frac{2uv}{a^2} dx dy + \left(1 - \frac{v^2}{a^2}\right)(dx)^2 = 0$$

Dividing by$(dx)^2$ gives:

$$\left(1 - \frac{u^2}{a^2}\right)\left(\frac{dy}{dx}\right)^2 + \frac{2uv}{a^2}\frac{dy}{dx} + \left(1 - \frac{v^2}{a^2}\right) = 0$$

The subscript 'ch' on *dy/dx* indicates that the slope of the characteristic line or a specific direction is being considered along which differentials are indeterminate. Solving above equation we get:

$$\left(\frac{dy}{dx}\right)_{ch} = \frac{-\left(\frac{uv}{a^2}\right) \pm \sqrt{\frac{(u^2+v^2)}{a^2} - 1}}{\left(1 - \frac{u^2}{a^2}\right)}$$

Let'srepresent the component of velocities in terms of velocity vector and angle made by the streamline with co-ordinate axes as,

$V^2 = u^2+v^2$

u = V cosθ

v = V sinθ

Hence, we get the same equation for slope as,

$$\left(\frac{dy}{dx}\right)_{ch} = \frac{-M^2 \cos\theta \sin\theta \pm \sqrt{M^2 - 1}}{(1 - M^2 \cos^2\theta)}$$

where, M=V/a.

We can introduce the local Mach angle, *a*, where (M=1/sinα, $\sqrt{M^2 - 1}$=1/tan α) by replacing Mach number as,

$$\left(\frac{dy}{dx}\right)_{ch} = \frac{\cos\theta\sin\theta \pm \cos\alpha\sin\alpha}{\sin^2\alpha - \cos^2\alpha}$$

After much manipulation and rearrangement, it can be shown that this equation gives:

$$\left(\frac{dy}{dx}\right)_{ch} = \tan(\theta \pm \propto)$$

There are two characteristic lines. This clearly means that the characteristic lines or lines along which derivatives are inderminate makes Mach angle with the streamline. Hence the net angle made by the characteristic line with the x-axis is the summation of the angle made by the streamline with x-axis and angle made by the Mach wave with streamline. Hence Mach waves are the characteristic lines.

4.4.2 Governing Equation

We know about the direction of the characteristic line obtained from the indeterminacy of the equation for zero denominators. However finiteness of the differential compels the zero value of the numerator. This condition evolves the equation to be solved along the characteristic lines.

Hence for the numerator equation we have,

$$(1 - u^2/a^2)dudy + (1 - v^2/a^2)dvdx = 0$$

i.e.:

$$\frac{dv}{du} = \frac{(1 - u^2/v^2)}{(1 - v^2/a^2)(dy/dx)}$$

Substituting for slope (dy/dx) of the characteristic lines from equation that gives:

$$\frac{dv}{du} = \frac{uv/a^2 \pm \sqrt{(u^2+v^2)/a^2 - 1}}{(1 - v^2/a^2)}$$

Using the velocity relation we have,

$$\frac{dv}{du} = \frac{M^2\sin\theta\cos\theta \pm \sqrt{M^2 - 1}}{1 - M^2\sin^2\theta}$$

However,

$$\frac{dv}{du} = \frac{d(V\sin\theta)}{d(V\cos\theta)} = \frac{\sin\theta dV + V\cos\theta d\theta}{\cos\theta dV - V\sin\theta d\theta}$$

$$\Rightarrow \frac{dv}{du} = \frac{d(V\sin\theta)}{d(V\cos\theta)} = \frac{\sin\theta(dV/V) + \cos\theta d\theta}{\cos\theta(dV/V) - \sin\theta d\theta}$$

Substituting this expression in equation we get,

$$\frac{\sin\theta(dV/V)+\cos\theta d\theta}{\cos\theta(dV/V)-\sin\theta d\theta}=\frac{M^2\sin\theta\cos\theta\pm\sqrt{M^2-1}}{1-M^2\sin^2\theta}$$

i.e.

$$\sin\theta\left(\frac{dV}{V}\right)+\cos\theta d\theta-M^2\sin^3\theta\left(\frac{dV}{V}\right)-M^2\sin^2\theta\cos\theta d\theta$$
$$=M^2\sin\theta\cos^2\theta\left(\frac{dV}{V}\right)\pm\sqrt{M^2-1}\cos\theta\left(\frac{dV}{V}\right)$$
$$-M^2\sin^2\theta\cos\theta d\theta\pm\sqrt{M^2-1}\sin\theta d\theta$$

i.e.

$$[\cos\theta\pm\sqrt{M^2-1}sin\theta]$$
$$=[-\sin\theta+M^2\sin\theta(\sin^2\theta+\cos^2\theta)\pm\sqrt{M^2-1}\cos\theta]\left(\frac{dV}{V}\right)$$

i.e.

$$d\theta=-\left[\frac{\sin\theta(M^2-1)\pm\sqrt{M^2-1}\cos\theta}{\pm\sqrt{M^2-1}\sin\theta-\cos\theta}\right]\left(\frac{dV}{V}\right)$$

i.e.

$$d\theta=-\left[\frac{\sqrt{M^2-1}\sin\theta\pm\cos\theta}{\pm\sqrt{M^2-1}\sin\theta-\cos\theta}\right]\sqrt{M^2-1}\left(\frac{dV}{V}\right)$$

$$d\theta=\pm\sqrt{M^2-1}\,\frac{dV}{V} \quad \text{-----------------(33.2)}$$

This is the equation governing the changes in the variables along the characteristic lines. It can be noted that it is an ordinary differential equation whereas the original equation, velocity potential equation, was a partial differential equation.

We have already seen the solution for equation during consideration of expansion fan. On the similar line, the integration of equation leads to,

$$\int d\theta\pm\int\sqrt{M^2-1}\,\frac{dV}{V}=\text{constant}$$

and the $\int\sqrt{M^2-1}\,\frac{dV}{V}$ is the Prandtl Mayer function,

$$\nu(M)=\int_0^M\frac{\sqrt{M^2-1}}{[1+(\gamma-1)M^2/2]}\frac{dM}{M}$$

Hence,

$$\theta\pm\nu=constant$$

This is the simple algebraic equation which we will have to solve along the characteristic line.

Following figure represents the details of the characteristic lines.

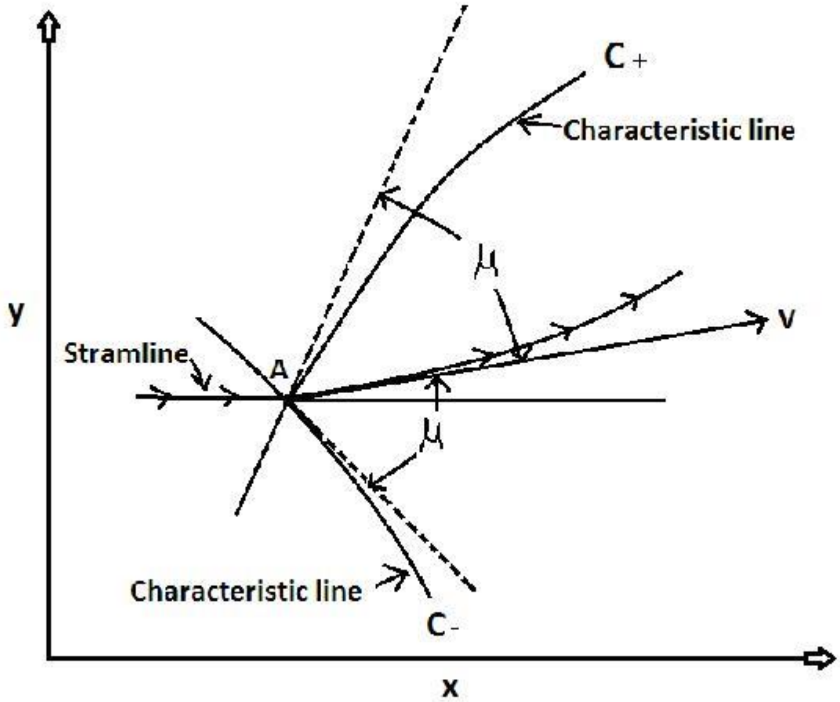


Fig.4.3 Schematic representation of the details of the characteristic line

We know that there are two characteristic lines from equation.
For C+ characteristics line of Figure

$$\left(\frac{dy}{dx}\right)_{ch} = \tan(\theta + \alpha)$$

$$\theta - \vartheta = K^+$$

For C- characteristics line of Figure

$$\left(\frac{dy}{dx}\right)_{ch} = \tan(\theta - \alpha)$$

$$\theta + \vartheta = K^-$$

K+ and K - are constants along the + and - characteristic lines.

4.4.3 Strategy to solve numerically along the characteristic line

Consider any point 3 in the flowfield at which properties are to be evaluated. Point 1 and 2 supposed to the points of known properties. Consider C+ characteristic line passing through point 2 and C- passing through 1 intersecting at point 3. Hence point 3 lies on both the characteristic lines. Therefore for point 1 and 3 we have,

$$\theta_1 + \vartheta_1 = (K^-)_1 = \theta_3 + \vartheta_3 = (K^-)_3$$

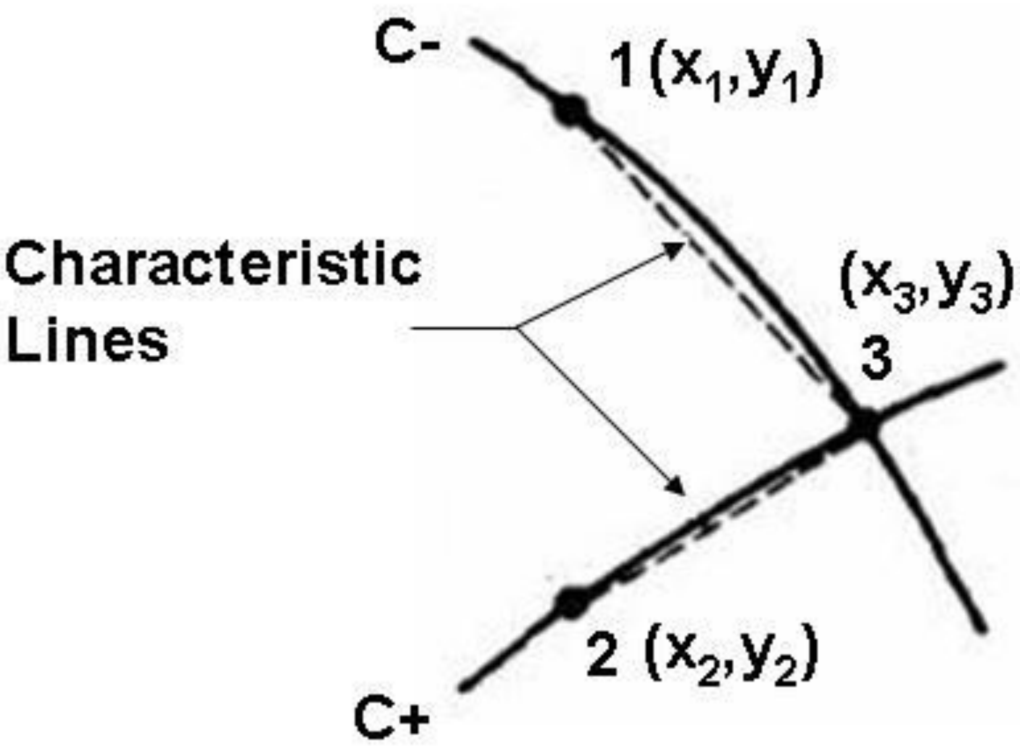


Fig.4.4 Solution strategy at a point in the flowfield.

Similarly for the point 2 and 3

$$\theta_2 - \vartheta_2 = (K^+)_2 = \theta_3 + \vartheta_3 = (K^+)_3$$

Hence,

$$\theta_3 + \vartheta_3 = (K^-)_1$$

$$\theta_3 - \vartheta_3 = (K^+)_2$$

First adding these two equations and then subtracting them gives:

$$\theta_3 = [(K^-)_1 + (K^+)_2]/2$$

$$\vartheta_3 = [(K^-)_1 - (K^+)_2]/2$$

which can be also written as:

$$\theta_3 = ((\theta_1 + \theta_2)/2) + ((\vartheta_1 - \vartheta_2)/2)$$

$$\vartheta_3 = ((\theta_1 - \theta_2)/2) + ((\vartheta_1 + \vartheta_2)/2)$$

This helps in finding θ_3 and v_3 from known properties. Since v_3 depends only on M, this allows $M_3$and hence a_3 can be evaluated. Since the stagnation pressure and temperature are constant throughout the flow field therefore using M_3 we can calculate P_3, T_3, a_3, and ρ_3 and then V_3. The characteristic lines are, in general, curved. Their local slope depends on the local values of v and θ. However, if points 1 and 3 and 2 and 3 are close together, the characteristic lines can be assumed to be straight with a slope equal to the average of the values at the end points.

$$[(\theta_1 - \alpha_1) + (\theta_3 - \alpha_3)]/2 = \tan^{-1}\left[\frac{y_3 - y_1}{x_3 - x_1}\right]$$

$$[(\theta_2 + \alpha_2) + (\theta_3 + \alpha_3)]/2 = \tan^{-1}\left[\frac{y_3 - y_2}{x_3 - x_2}\right]$$

Since θ_3 and a_3 are determined by solving above two equations we can determine x_3 and y_3.

The procedure discussed above was for an "internal" point, i.e., a point 3 in the flow field that did not lie on a boundary. If a point lies on the boundary, the flow direction at this point will be determined by the slope of the boundary, e.g., consider the point 5 shown in Fig. 33.3 which lies on a solid wall. The flow direction at this point θ_5 is equal to the slope of the wall as indicated. Consider the characteristic line between points 4 and 5 as shown in the same figure. Since (K -)4 = (K -)5 it follows that:

$$\theta_4 + \nu_4 = \theta_5 + \nu_5$$

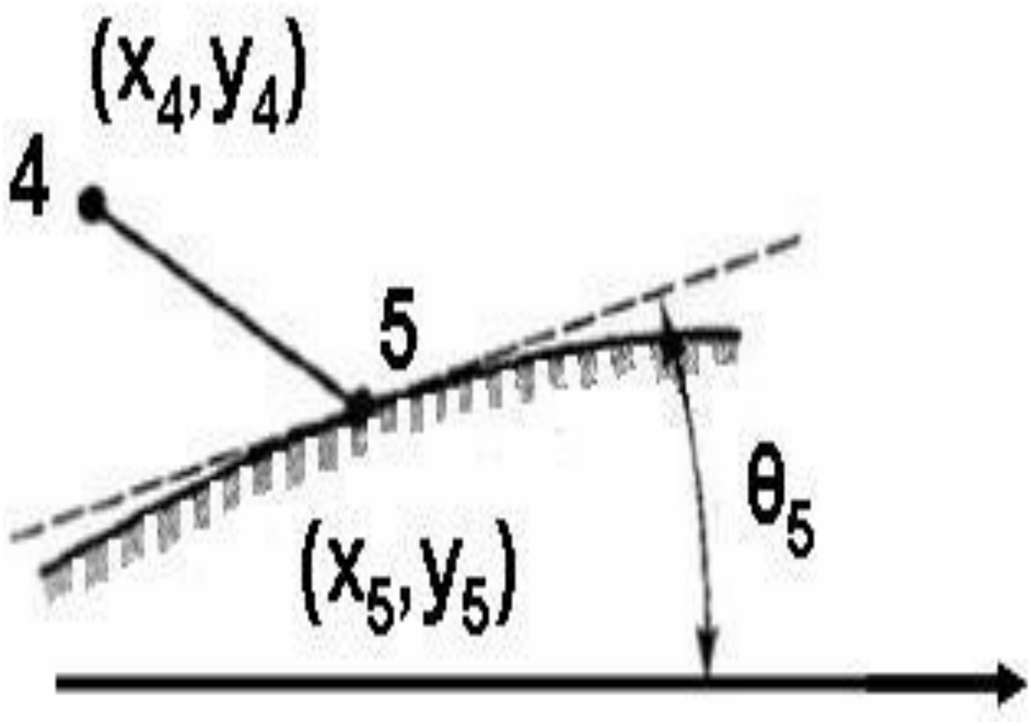


Fig. 4.5 Demonstration of wall boundary condition

However, θ_5 is known, hence ν_5 is given by:

$$\nu_5 = \theta_4 - \theta_5 + \nu_4$$

With ν_5 determined, the values of all the flow properties at 5 can be determined as discussed before. The characteristic line between 4 and 5 is, of course, assumed to be straight which determines the position of the point 5.

4.5 Flow through duct.

The procedure for using the method of characteristic lines is to numerically calculate the flow in a duct is as follows:

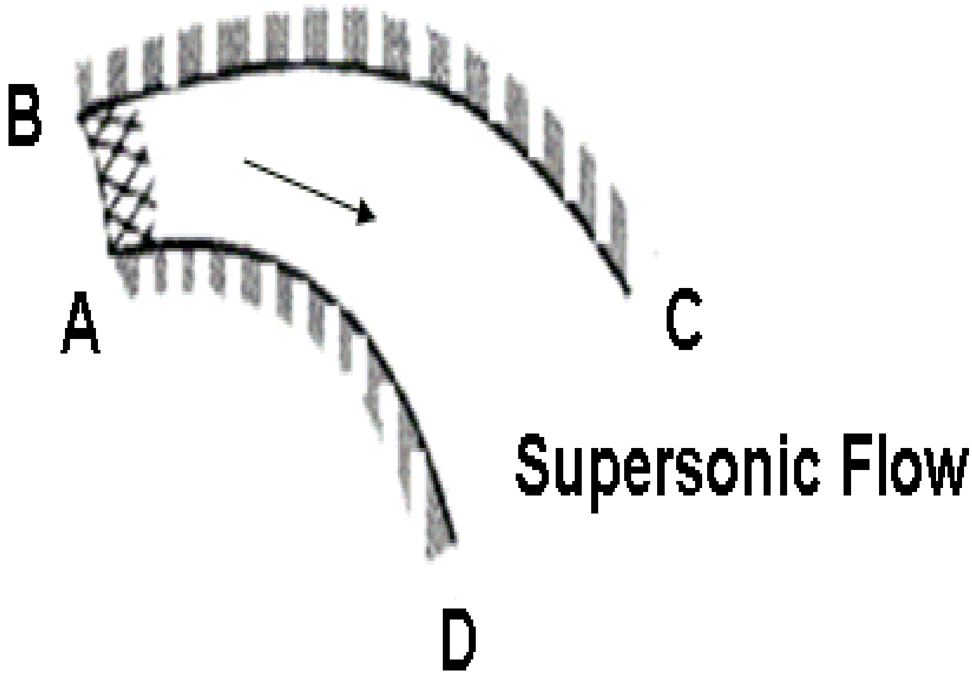


Fig. 4.6. Supersonic flow through duct

1. The conditions on some initial line must be specified, e.g., conditions on the line AB in Fig.4.6 must be specified.
2. The shape of the walls, e.g., AD and BC in Fig. 4.6, must be known.
3. Using the initial values of the variables on line A, determine the stagnation pressure, temperature, etc.
4. Starting with a series of chosen points on line AB, march the solution forward to the points defined by the intersection of characteristics with each other or with the wall as indicated.
5. At each point, use the calculated values of v and θ to get flow variables.
   A computer program based on this procedure can be easily developed.

4.5.1 Nozzle Design

Supersonic nozzles are used in a variety of engineering applications to expand a flow to desired supersonic conditions. Supersonic nozzles can be divided into two different types: gradual-expansion nozzles and minimum-length nozzles. Gradual-expansion nozzles are typically used in applications where maintaining a high-quality flow at the desired exit conditions is of importance (e.g., supersonic wind tunnels). For other types of applications (e.g., rocket nozzles), the large weight and length penalties associated with gradual-expansion nozzles make them unrealistic; therefore minimum-length nozzles, which utilize a sharp corner to provide the initial expansion, are commonly used.

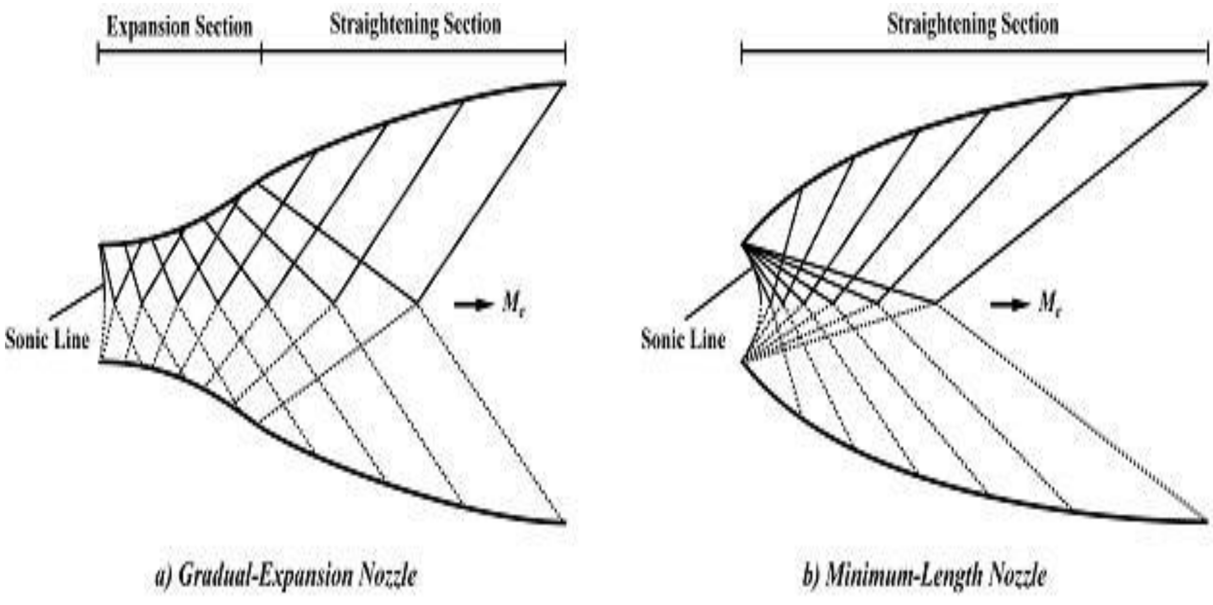


Fig. 4.7 Types of nozzles.

For both gradual-expansion and minimum-length nozzles, the flow can be divided into simple and non-simple regions. A non-simple region is characterized by Mach wave reflections and intersections. In order to meet the requirement of uniform conditions at the nozzle exit, it is desirable to minimize the non-simple region as much as possible. This can be performed by designing the nozzle surface such that Mach waves (e.g., characteristics) are not produced or reflected while the flow is straightened. The Method of Characteristics is therefore applied to allow the design of a supersonic nozzle which meets these requirements.

4.5.2 Design of Minimum-Length Nozzle (MLN)

It should be noted that for this two-dimensional nozzle configuration, flow symmetry implies that only half of the nozzle is physically required, assuming that the characteristic reflections in the non-simple region are maintained. Therefore, we can make the assumption of a half-symmetric minimum-length nozzle, in which a nozzle flap is extended from the symmetry plane such that it meets the length requirement for the last characteristic intersecting the nozzle surface.

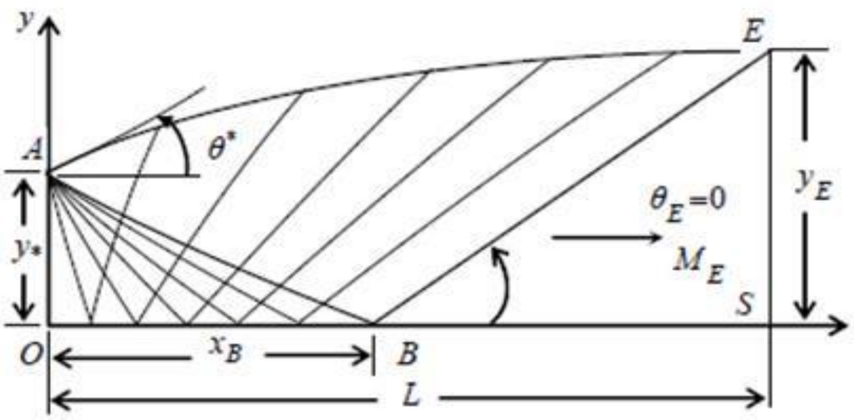


Fig.4.8. Schematic of characteristic lines for MLN.

4.5.3 Implementation of Method of Characteristics

The two-dimensional Method of Characteristics is a relatively simple analytical model for analyzing supersonic two-dimensional flow. This analysis is performed by considereing the characteristic lines in the flow. Points along each characteristic have five important properties: M (Mach number), θ (flow angle), ν (Prandtl-Meyer function), and x and y (position). For the assumption of steady, supersonic, we know that,

$$\theta \pm v = constant$$

The constant for summation can be said to be K+ and for subtraction to be K-. These constants are the Riemann invariants, which are constant along the characteristics C+ and C-.

4.5.4 Design of Gradual expansion nozzle

The steps involved in this calculation are precisely same as those used for the minimum length nozzle except for the fact that the expansion fan at the sharp corner is now replaced by a series of right running characteristic lines originating from the arc of the circle. One major assumption that has been made here is that a characteristic originating from any point on the expansion section is always reflected from the axis in such a way that it reaches the straightening portion of the nozzle. Multiple reflections of characteristic lines within the smooth expansion portion of the nozzle would make the problem much complicated without really improving the results much.

4.5.4 Governing equation in 2-D

For subsonic flows density has a known constant value, i.e. it is no longer an unknown.. Conservation of Mass: For constant density which means that the velocity field of an incompressible flow should be divergence free, this is known as the divergence free constraint in CFD literature.

We have $\nabla \cdot \vec{V} = 0$.

Note that there is no time derivative in the continuity equation even for unsteady flows, which is one of the reasons that make numerical solution of incompressible flows difficult.

Conservation of Linear Momentum:

$$\rho \left[\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} \right] = -\nabla p \ + \ \mu \nabla^2 \vec{V} + \rho \vec{f}$$

Dividing the equation by density we get the following form of the Navier-Stokes equation.

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = -\frac{1}{\rho}\nabla p \ + \ \nu \nabla^2 \vec{V} + \vec{f}$$

The term on the right hand side is known as the convective term. It is the term which makes the Navier-Stokes equation nonlinear. For diffusion dominated flows the convective term can be dropped and the simplified equation is called the Stokes equation, which is linear. Stokes equations can be used to model very low speed flows known as creeping flows or flows with very small length scales (micro or nano flows) where Reynolds number is small. Convectiondominated flows, which are typically characterized by high Reynolds numbers, are much more difficult to solve numerically compared to diffusion dominated flows. For most solid mechanics problems convection (flow of material) does not exist, which is the main reason of the differences seen in mathematical modeling (Eulerian vs. Lagrangian formulations) and in numerical solution techniques (e.g. need for up winding) used in the disciplines of fluid and solid mechanics. Conservation of Energy: Conservation of energy can be simplified by considering the fact that density is constant for incompressible. Note that for incompressible flows. Here it is important to note that for incompressible flows equation of state does not exist. In practice this means that the energy equation is decoupled from the other two equations. Therefore we can first solve continuity and Navier-Stokes equations to find the unknown velocity and pressure distribution without knowing the temperature (We assume that fluid properties are taken to be constant, i.e. not functions of temperature. If fluid properties change with temperature all equations becomes coupled as in the case ofcompressible flows). After finding the velocity field, energy equation can be solved by itself to find the temperature distribution. However for buoyancy driven flows (natural convection) where the density changes due to temperature variations are considered in the body force term of the momentum equation (Boussinesq approximation), all three conservation equations again become coupled. Heat transfer and therefore the energy equation is not always a primary concern in an incompressible flow. For isothermal (constant temperature) incompressible flows energy equation (and therefore temperature) can be dropped and only the mass and linear momentum equations are solved to obtain the velocity and pressure fields. Numerical solution of incompressible flows is usually considered to be more difficult compared to compressible flows. The main numerical difficulty of solving incompressible flows lies in the role of pressure. For incompressible flows pressure is no longer a thermodynamic quantity and it cannotbe related to density or temperature through an equation of state. It just establishes itself instantaneously in a flow field so that the velocity field always remains divergence free. In the continuity equation there is no pressure term and in the momentum equation there are only the derivatives of pressure, but not the pressureitself.This means that the actual value of pressure in an incompressible flow solution is not important only the changes of pressure in space are important. Additionally there is no time derivative of pressure, even for incompressible flows.

UNIT-V

EXPERIMENTAL METHODS IN COMPRESSIBLE FLOWS

With the advancement of aerospace vehicles, the human's dream is to fly faster and higher. Speed of manned and unmanned flight vehicles has increased by some orders of magnitude over the last few decades. As the speed of the vehicle is increased, the aerodynamic environment becomes increasingly hostile. At speeds around the local speed of sound (Transonic) and higher (Supersonic), the aerodynamic loads increase and their distributions change. When the speed of the vehicle becomes several times higher than the speed of sound (Hypersonic), additional problem of aerodynamic heating demands the change in design geometry and materials. At still higher speeds,the behavior of air begins to change significantly; both physically and chemically. There is a conventional 'rule-of-thumb' that defines the flow regimes based on the free stream Mach number i.e.Subsonic:$0<M_\infty<0.8$;Transonic:$0.8<M_\infty>1.2$ Supersonic:$1.2>M_\infty>5$;Hypersonic:$M_\infty>5$

The past four decades have seen major flights cruising from subsonic to hypersonic speeds. The most routine flights made possible when American Jet Transportstarteditsfirstflight(Boeing-707)onOctober26,1958cruisingatMach Traveling at speeds faster than sound speed and thus breaking 'Sound Barrier' became reality with the taste of supersonic travel from London to Bahrain in the aircraft 'Concorde' commenced by British airways on January 21, 1976. The first ever-fastest commercial passenger aircraft 'Concorde' with its cruising altitude 18 km at Mach 2, crossed Atlantic Ocean from London to New York little less than 3.5 hours as opposed to about eight hours for a subsonic flight. This mile stone was achieved on November 22, 1977. Now the age of 'hypersonic flight' is about to dawn with the evolution of orbital space crafts, hypersonic airliners and re-entry vehicles. The first such major landmark has been achieved after the launch of USSR satellite SPUTNIK- II on November 3, 1957. It was the first man-made earth satellite that carried living organism(adognamedLAIKA)intothespaceandremainedinorbittillApril13,1958. In the bumper year of 1961, Yuri Gagarin (USSR) became the first man in the history to fly in space with an orbital space craft VOSTAK-I that entered the earth atmosphere at Mach 25 on April 12. His safe return from the space has inspired the future objectives of hypersonic flight. In the same year i.e. on June 23, U.S. air force test pilot Major R. White accomplished the concept of 'miles per second' flight in an X-15 airplane by flying at Mach 5.3. White again extended this record with same X- 15 flight at Mach 6. Since 1961, major space programs carried out by U.S. space agency NASA and Indian Space Research Organization (ISRO) have achieved milestones in the development of satellites and aero-assisted space transfer vehicles.

The aerodynamic flow fields at very high Mach numbers experience a very high pressure, temperature and density. Even, the temperature rise can be so high that the gaseous medium gets decomposed and thereby the properties (specific heat, gas constant and specific heat ratio) can change as well. So, even at same free stream Mach number, different velocities can be obtained. Thus, the high Mach number flows where the behavior of the medium begins to change significantly, are normally classified in terms their velocities: suborbital velocities speed (4-7 km/s), super orbital speed (8-12 km/s) and escape velocities (>13 km/s). However, these flow conditions are normally achieved at different trajectories/altitudes of a flight vehicle in the earth atmosphere. So, the more realistic way is to simulate these conditions experimentally in the laboratory for entire range of operational speeds.

Some of these aerodynamic test facilities are broadly summarized and discussed in thismodule.

- Low speed wind tunnel (continuous type; up to 40m/s)

- High speed wind tunnel (intermittent/blow down type; Mach 3,600m/s)
- Shock tunnel (impulse type; Mach 7,2km/s)
- Free piston shock tunnel (impulse type, Mach 4-10,5km/s)
- Expansion tube (impulse type, Mach 10,10km/s)

5.1 SUBSONIC (Low Speed) Wind Tunnel

In general, the wind tunnels are the devices which provide an airstream flowing under controlled conditions for external/internal flow simulations in the laboratory. The most fundamental experiments undertaken in the wind tunnel are the force/heat transfer measurement and flow visualization on aerodynamic models. The flows generated in the test section of the tunnel may be laminar/turbulent, steady/unsteady etc. The other features may be study of boundary layer separation, vortex flow generation etc. The low speed wind tunnels limit their speed to 50-60 m/s and based on the need, the tunnel may be designed. Depending on the discharge of the air flow to atmosphere or recirculation of air, it is classified as open or closed circuit wind tunnels.

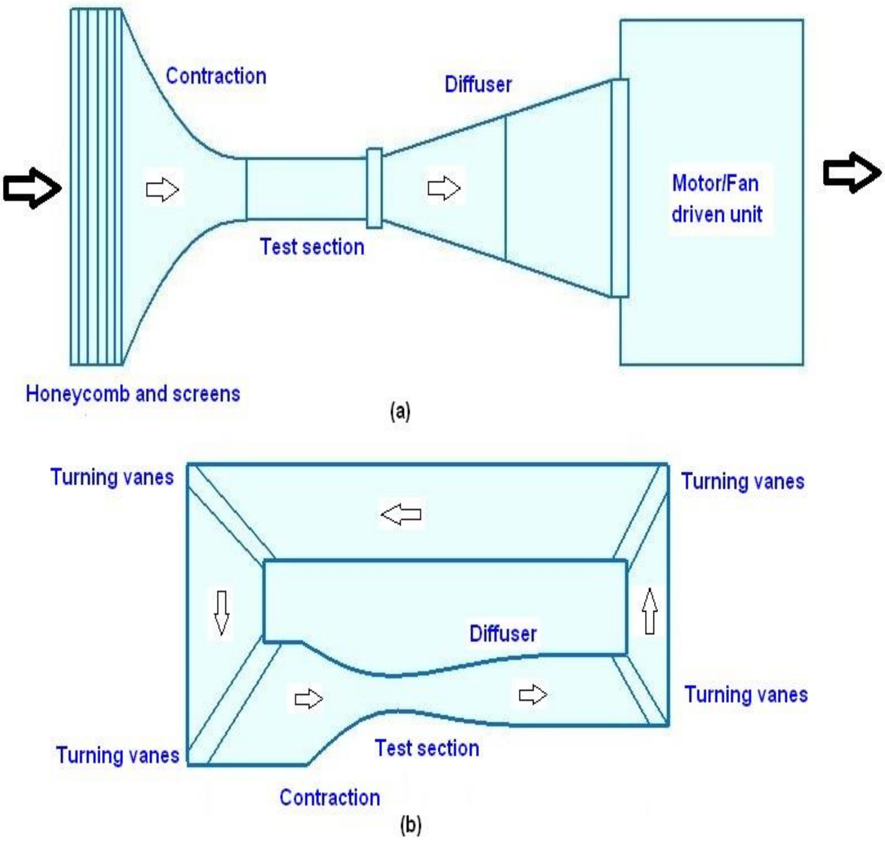


Fig. 5.1 Schematic representation of wind tunnel: (a) open circuit wind tunnel; (b) closed circuit wind tunnel.

Remarks for open/closed circuit wind tunnel

- In a closed circuit wind tunnel, the high quality flow can be assured in the test section and power requirement is less as compared to open circuit wind tunnel. However, it is not suitable for smoke flow visualization and incurs high capital/constructioncost.
- An open circuit wind tunnel is more adaptive for flow visualization experiments because due to its direct connection with the atmosphere. In order to assure the flow quality in the test

section, one needs to install special devices such as flow straightener explicitly aligning the flow axially. Of course, it requires more power as compared to closed circuit windtunnel.

5.1.1 Wind tunnel components

The important components of the wind tunnel are listed below;

Motor/Fan Driven unit: This is the air supply unit that drives the air flow in the wind tunnel. Typically, the fan is axial/centrifugal type and the axial fan is a better choice in the closed circuit tunnels since it produces a static pressure rise necessary to compensate for the total pressure loss in the rest of the circuit. The fans with higher ratio of tip speed to axial velocity generally produce the required pressure rise in a small blade area. The wind tunnels fitted with blower are generally driven by a centrifugal impeller of squirrel-cage type. While in operation, the fan draws air from the atmosphere through the honeycomb/screensection.

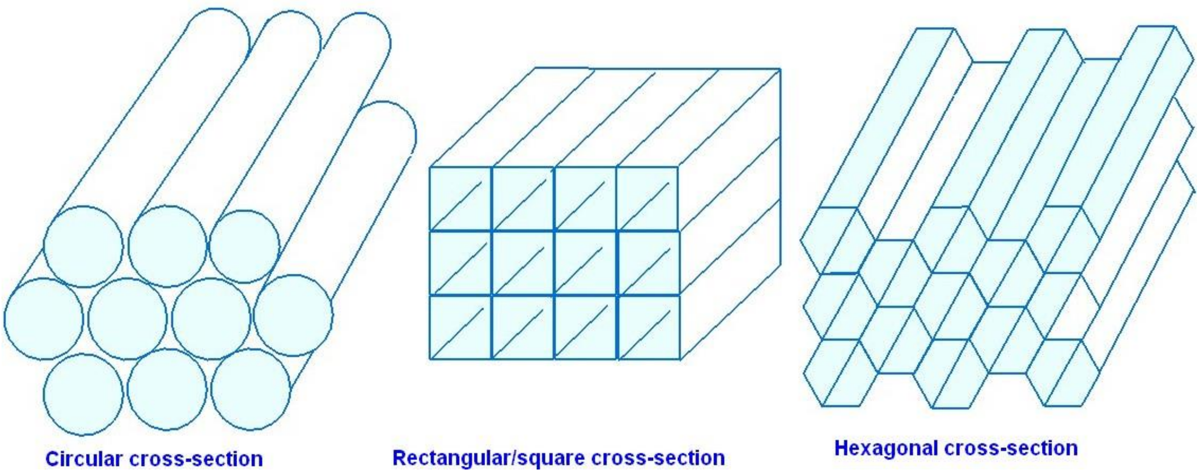


Fig.5.2: Honeycomb structures for low speed wind tunnels.

Settling chamber and flow straightener: It mainly comprises of honeycomb and screens as combination. The main function is to reduce the turbulence and straighten the flow only in the axial direction. In principle, the air can enter to the tunnel from any directions. But, only the axial flow is desired in the test section. The main purpose of the screen is to reduce the turbulent intensity in the flow and not to allow any unwanted objects to enter the tunnels. The honeycomb can be made with cells or various shapes as shown in Fig. 5.2. These cells are aligned in the stream wise direction in the settling chamber thereby straightens the flow. The honeycomb has a longer length that reduces the transverse velocity components of the flow with minimal pressure drop in the stream wise direction. The minimum length required for this honeycomb is six to eight times the cell size. The number of screens required in the settling chamber depends on the flow quality requirement in the test section. Moreover, the power requirement is more when the number of screens is increased. The preferable length of the settling chamber is about 0.5 times the diameter of its inlet.

Contraction: The prime objective is to accelerate incoming flow from the settling chamber and supplies it to the test section at desired velocity. This section essentially reduces the cross-sectional velocity variation and maintains the flow uniformity. In general, small radius of curvature is used at the entry to this section and curvature of large radius is considered at the exit of the contraction section. However, the boundary-layer separation should be avoided at both the ends of this section. The

contraction length is expected to be small so that large contraction area ratios are preferred.
Test Section: It is the basic element of wind tunnel on which all other designs are generally made. All the aerodynamic models are mounted in the test section when the tunnel is operated with desired flow velocity. Various shapes for the test section are considered for constructing the wind tunnel viz. hexagonal, octagonal, rectangle etc. The test section is generally designed on the basis of utility and aerodynamic considerations since cost of construction depends on the test section area. Length of the test section is mostly equal to major dimension of the cross-section of the same or twice of it. In addition, the test section should also be provided with facilities as per the testing requirement. The test section velocity is generally specified as percentage variationfromtheaverage ofthecross-section.The idealtestsectionhassteady uniform velocity at the inlet, no cross flow, less or no turbulence and less operating cost.

Diffuser: It is basically a duct with increase in area attached downstream of the test section. After the test section, it is desired that the air must pass smoothly out of the test section. So, this geometry is made to decrease the flow velocity and increase in pressure. In order to avoid flow reversal, the exit pressure should be higher than the atmospheric in case of open circuit wind tunnel. This is a very critical section in design since the incurred pressure rise reduces the power requirement for the wind tunnel which is proportional to the cube of velocity. Hence maximum pressure recovery to be achieved at least possible distance is the main objective of diffuser design. In general practice, the cone angle of the diffuse is 7^o or less so as to avoid boundary layerseparation.

Turning vanes: In a closed circuit wind tunnel, the air has to circulate in a controlled manner. Typically, the corners of the wind tunnel are of two bends aligned 90^0 each other. These corners are provided with turning vanes for smooth passage of the flow. Chambered aerofoils of bent planes are generally accepted as the turning vanes. These vanes are purposely made as adjustable for smooth operation thereby avoiding under/over tuning.

5.1.2 Special purpose low speed wind tunnels

Low turbulence tunnels: Low intensity turbulence is the prime requirement for many experimental situations. Hence, these tunnels are specifically designed with a wide- angle diffuser just ahead of the test section-settling chamber. The major task of reducing free stream turbulence is done by large settling chamber having honeycombs and the screens. Large contraction ratio of contraction section and several number of screens are some of the important components for this type of tunnel.
Two-dimensional tunnels: These tunnels are intended for testing objects like cylinder, aerofoils etc. These tunnels can be of open or closed types but the main design feature of this tunnel is its height to width ratio of the test section which is more than or equal to 2.

Smoke tunnels: Flow visualization is sole application of smoke tunnels. So, these tunnels are mostly three-dimensional open circuit type tunnels. The source of the smoke source is kept upstream of the test section so as to mix the smoke with free stream before passing over the object of interest. Very high contraction ratio and large number of screens are sometimes preferred for visualization of laminar flow.
Water tunnels: These are the closed circuit type tunnels that are mostly preferred for hydrodynamics application like water flow over immersed bodies. These are preferred for flow visualization studies with particle image velocimetry (PIV) method because it is very simple to implement this technique in water tunnels. These tunnels are comprised of pumps for through circulation of water unlike fans in windtunnels.

5.2 High Speed (supersonic) Wind Tunnel

Experimental facilities for supersonic and hypersonic flow regimes are different from those used at subsonic speeds. In supersonic flows, the interest lies in simulating flow Reynolds number and the Mach numbers in the test section of the tunnel. In addition to these parameters, the total energy content (i.e. enthalpy) of the flow also becomes important at hypersonic speeds. The wind tunnels used in the Mach number range 1 to 5arecalledassupersonictunnelswhilethetunnelsusedforhigherMachnumbers(>5) are called as hypersonic tunnels.

The high speed tunnels can be of open/closed circuit type. The open circuit wind tunnel takes the air from atmosphere and rejects them to a vacuum chamber. In contrast, the same air is re-circulated in a closed circuit wind tunnel. In the case of subsonic wind tunnels, experiments can be performed by running the tunnel continuously. But, when the velocity of air in the test section increases, the power requirement becomes very high because it is proportional to the cube of the velocity. Thus, in many cases, it is preferred to run high speed wind tunnels for a short duration and gather all the experimental data in this short time period (~1-5s). So, such types of tunnels are called as blow down tunnels. These tunnels operate intermittently using high pressure tanks and/or vacuumtanks.

5.2.1 Blow down wind Tunnel (Open circuit type)

The high pressure chamber, vacuum section, nozzle and test section are few important componentsofablowdownwindtunnel.TheschematicdiagramisshowninFig.5.3. Since, the air from the high pressure chamber flows towards the vacuum section, it is referred as pressure-vacuum tunnel. First, the air is taken from the atmosphere and after compression; it is stored in a tank. Simultaneously, the low pressure section is evacuated by a vacuum pump. Pressure regulator controls the air flow from reservoir to the settling chamber during the actual experiment and maintains the desired constant pressure. The nozzle expands the flow by increasing velocity and decreasing pressure and provides desired Mach number in the test section. The high speed air encounters deceleration while passing through the second throat.

These blow down tunnels are inherent intermittent tunnels but bear many advantages over continuous wind tunnels such as high Mach capability (up to 4), easy tunnel starting, large size test section, lower construction/operating costs, superior design for propulsion experiments and smoke visualization.

The limitations of the blow down tunnels are requirement of faster data acquisition system, noisy operation and necessity of pressure regulator valves. The blow down tunnel can be of different types based on the driving pressure difference is achieved. In one such tunnel, air expands from high pressure to the atmospheric pressure where low pressure chamber is excluded (intermittent blow down tunnel). In other cases, the atmospheric air can expand up to a very low vacuum (intermittent in-draft wind tunnel). Depending on the requirement of Mach number in the test section, the tunnel is chosen accordingly.

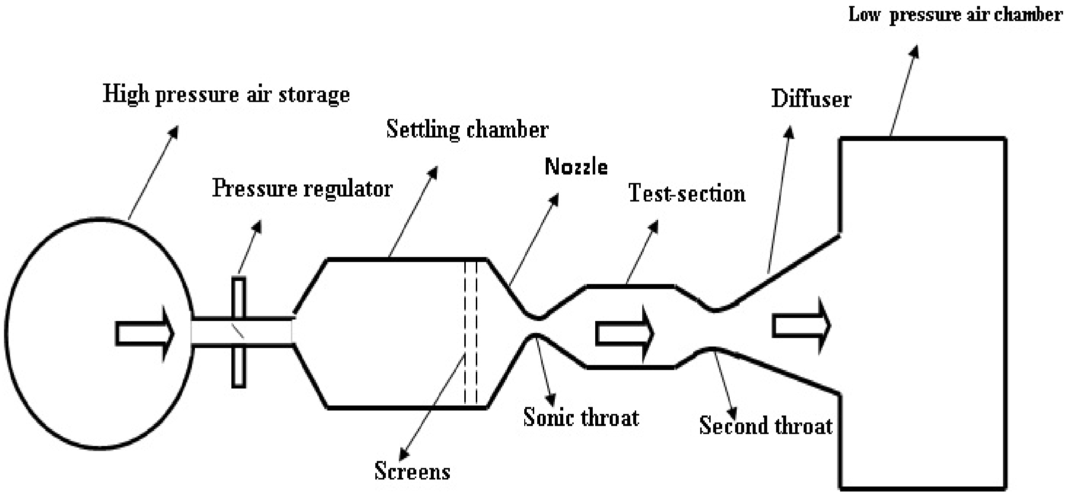


Fig. 5.3: Schematic diagram of a blow down wind tunnel (open circuit)

5.2.2 Continuous wind Tunnel (Closed circuit type)

The major disadvantage of an open circuit wind tunnel is the test flow duration available for the model. However, if the Mach number/velocity requirement is not too high, than continuous operation of the blow down tunnel can be done by re-circulating the air. So, the schematic representation of Fig. 5.3 is modified through a return passage for air circulation. One more additional drier unit is also desired in the return passage to avoid condensation of re-circulating air in the test section of the tunnel. In addition to the advantages of longer run times, this tunnel also provides good flow quality in the test section and the noise level is also less. However, the high construction cost is the major disadvantage of thistunnel.

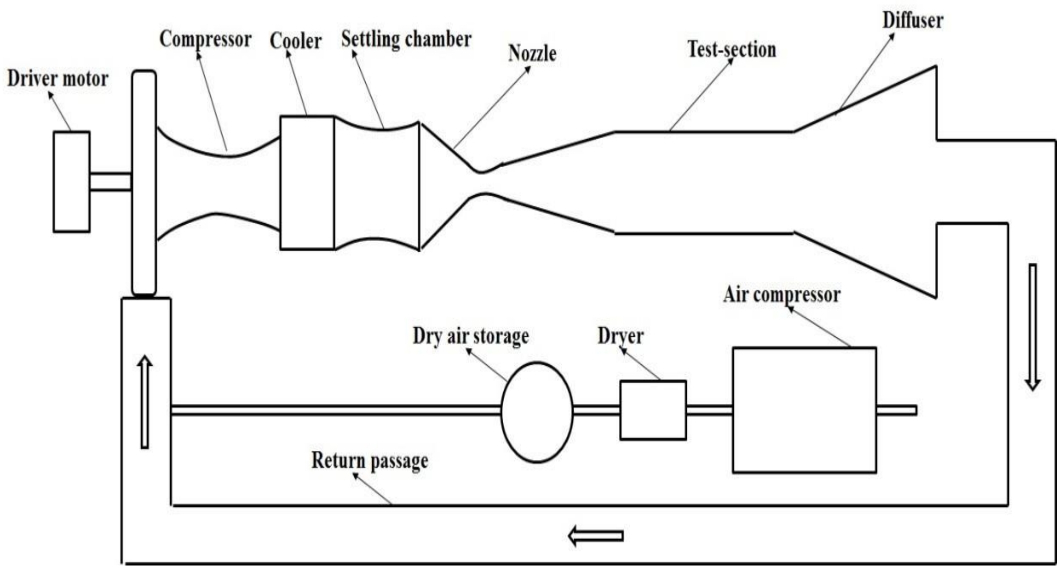


Fig. 5.4: Schematic diagram of a blow down wind tunnel (closed circuit).

5.3 Impulse-Type Tunnels

The conventional hypersonic wind tunnels face the challenges of simulating re-entry flight phenomena, long range ballistic missile test conditions that occurs at very altitudes. Since, the temperature can rise up to 10000K, there will be dissociations, chemical reactions as well. Hence, such investigations need real gas simulation conditions and it is almost impossible to achieve them in conventional hypersonic wind tunnels. So, the concept of impulse type aerodynamic facilities is introduced where it is desired to produce low density and high enthalpy flows for a very short duration (~ few milliseconds).

Expansion and shock tunnels are the typical aerodynamic testing facilities with a specific interest in high speeds and high temperature testing. Shock tunnels use steady flow nozzle expansion whereas expansion tunnels use unsteady expansion with higher enthalpy/thermal energy. In both cases the gases are compressed and heated until the gases are released, expanding rapidly down the expansion chamber. The tunnels reach speeds from Mach numbers ranging 3 to 30, thus creating testing conditions similar to that of very high altitudes.

Shock Tunnel: It consists of two major parts, the shock tube and the wind tunnel portion. The general schematic layout of the shock tunnel, consisting of shock tube and wind tunnel section, is shown in Fig. 5.5. The shock tube portion consists of a constant area tube separated by a diaphragm (generally, metal) into regions of high and low pressures. High and low pressure regions are called ‘driver tube’ and ‘driven tube’ respectively. The shock tube works on the principle of using a high-pressure gas in the driver tube to set up a shock wave, which propagates into the low pressure gas in the driven tube at the instant of diaphragm rupture. The propagating shock wave compresses and heats the low-pressure test gas in the driven tube to a high pressure and temperature, and also imparts the test gas a high kinetic energy, with which it starts moving at a supersonic Mach number behind the propagating shock wave. This shock wave ideally travels through the driven section at a constant velocity, and there exists a region of steady supersonic flow of high temperature and pressure between the moving driver/driven gas interface and the shockwave.

The wind tunnel portion of the shock tunnel consists of a hypersonic nozzle that is attached to the driven end of the shock tube, a test section at the exit of the nozzle where the measurements are carried out, and a dump tank portion to accommodate the gas. The shocked high pressure/temperature gas from the shock tube is then expanded through the hypersonic nozzle to the desired Mach number and velocity in the tunnel test section.

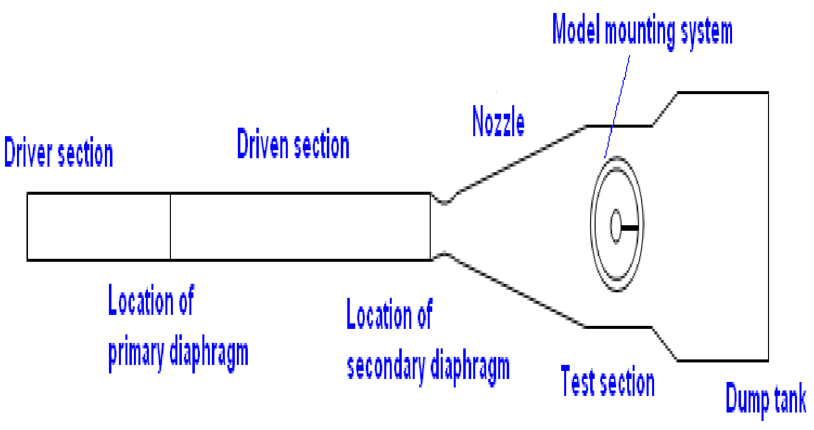


Fig.5.5: Schematic representation of a shock tunnel.

The ability of the shock tube portion of the hypersonic shock tunnel to heat the test gas to the conditions encountered in hypersonic flight makes it an attractive tool for hypersonic flow research. The test flow duration in a shock tunnel is in the range of1ms.

5.4 Free Piston Shock Tunnel

In a typical shock tunnel, the high pressure in the driver section is achieved by filling the tube from a high pressure gas cylinder. Thus, the driver gas is normally at room temperature. In the low density environment, it is likely that condensation may arise in the test section. So, many test facilities involve preheating the driver gas. One of them is to use a free piston driver to operate the shock tunnel. Here, the compression tube is filled with a driver gas (typically at atmospheric pressure) and is coupled to the driven section of low pressure region at one end and to a reservoir at very high pressure air (~100 bar). In this way, the piston separates the reservoir from the compression tube. When the piston is released for firing, it accelerates in the compression tube by acquiring the energy of expanding reservoir gas. The piston then approaches down to the compression tube and transfers all its energy to the driver gas. In this way, both temperature and pressure can be increased due to adiabatic compression mechanism. In a typical facility, the pressure up 900 bar and temperature of 4500K can be achieved. At this high pressure the primary diaphragm ruptures and shock tube flow is initiated in the similar manner as discussed earlier. In a well designed free piston shock tunnel, it is possible to achieve test flow duration up to2ms.

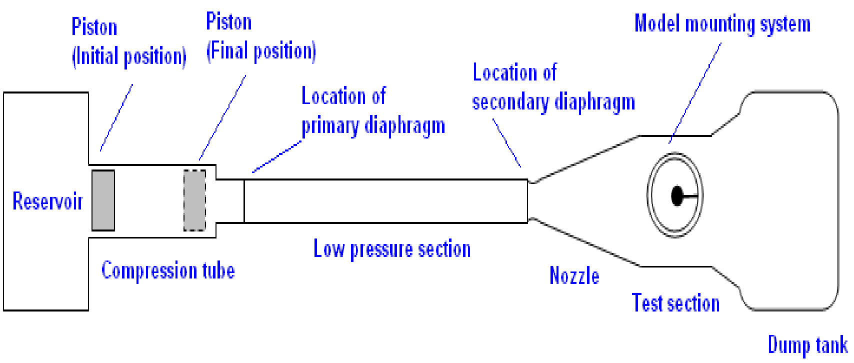


Fig. 5.6: Schematic diagram of a free piston shock tunnel.

5.5 Detonation-driven shock tunnels (tube)

The detonation-driven shock tube, first proposed by Bird,3 has been studied by several investigators. In a detonation-driven shock tube or shock tunnel, the conventional driver is replaced by a detonation section filled with a detonable gas mixture. Typically, an oxyhydrogen mixture is used as the driver gas with helium or argon dilution. Helium dilution raises the sonic speed in the driver gas and also somewhat reduces the danger associated with premature detonation.

A. Upstream Mode

Two modes of operation are possible. In the upstream-or backward propagation mode, the ignition source is just upstream of the primary diaphragm between the detonation and driven sections. In this case, the detonation wave propagates upstream. The corresponding wave diagram is shown in Fig. 5.7.

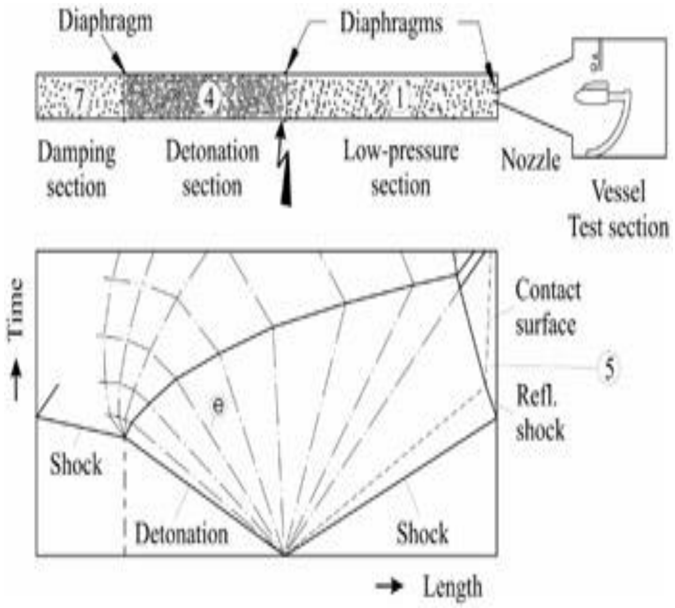


Fig 5.7: Principle of an upstream detonation-driven shock tunnel.

Direct initiation leads to a stable detonation wave where the products are at high temperature and pressure. The pressure rise following the detonation wave is constant, but the momentum imparted to the driver gas by the detonation wave is directed upstream, that is, opposite to the main flow. This leads to a reduction of the effective driver performance. Shortly after initiation, the main diaphragm opens and the burnt products exhaust into the low-pressure section, driving the incident shock that compresses and heats the test gas. A Taylor expansion immediately follows the detonation wave, decelerating the burnt gas to zero velocity along the characteristic labeled *e* in Fig. 5.7. This is common with the usual Taylor expansion as it develops behind a detonation wave starting in a closed- end tube to match the boundary conditions there. After the main diaphragm rupture, the shock tube expansion develops, which merges with the Taylor Expansion. It follows that for the upstream mode, the effective driver condition for the shock tube process is not achieved immediately behind the detonation wave but along the characteristic e where the flow velocity is zero.

The upstream running detonation was creates a tremendous pressure, of as much as 200 times the initial filling pressure, as it reflects from the end wall of the driver. Indeed, a shock tube was damaged because of this over pressure. To avoid or at least reduce the mechanical loading of the facility because of wave reflection, Yu proposed installing a damping section upstream of the detonation section. This damping section is filled with nitrogen at a relatively low pressure of approximately 3-7 kPa. Between the damping and detonation section is a thin plastic diaphragm. The detonation wave enters the damping section without significant reflection. There it extinguishes and propagates as a shock toward the end wall, where the pressure behind this shock is much smaller than that behind the detonation wave. With this technique, the mechanical loading on the facility is reduced acceptably.

The second advantage of the damping section lies in the increase of duration of constant driver properties. Without a damping section, the detonation wave reflects at the driver end wall and propagates downstream as a shock. If a damping tube is attached to the driver, a rarefaction wave is generated at the diaphragm station between the driver and damping section. The tail of this rarefaction wave travels towards slower than the reflected shock, the damping section yields a longer flow duration of constant driver conditions.

B. Downstream mode

In the downstream or forward propagation mode, the ignition source is located at the upstream end of the detonation section, producing a detonation wave that propagates downstream. The main processes are shown schematically in Fig 5.8. The detonation wave, depicted as a solid line from the lower left-hand corner in the diagram, propagates downstream into region 4. The burnt gas following the detonation wave also flows downstream. The Taylor rarefaction decelerates the gas so that the flow velocity is zero at the end wall of the driver section. The detonation wave is reflected at the diaphragm to yield an effective, unsteady condition given by 4".

Shortly after reflection of the detonation wave, the main diaphragm ruptures and a shock wave propagates along the driven section , driven by the high enthalpy detonation process products at state 4 ". As usual, a contact surface separates the shock-compressed gas in region 2 from the expanded driver in region 3.

The flow direction, that is, the momentum of the burnt gas, is in the downstream direction and would usually result in a stronger shock for the same detonation overpressure than for the upstream mode. However, measurements showed that the shock speeds achieved are considerably lower than expected, thus producing drastically lower pressure and temperature levels in state 2 of the driven gas. The primary reason was attributed to the Taylor rarefaction that catches up with the incident shock, resulting in significant shock attenuation. In fact, early investigators concluded that the downstream mode cannot meet requirements of high - enthalpy aerodynamic tests.

Several concepts were proposed to improve the performance of the forward detonation driver, including overcoming the problem of shock attenuation by Taylor rarefaction. One of them is lengthening the driver section. Experiments have shown that the pressure drop in the Taylor expansion from the C-J pressure in the detonation front to the end wall pressure occurs in approximately half the distance from the front to the initiation point, as shown schematically in figure. The longer this distance, the smaller the pressure drop per unit length because the total pressure drop is fixed at approximately 60% of the C-J pressure. This means that a longer detonation driver can provide a more uniform driving flow behind the detonation front. However the use of the method is limited by laboratory space.

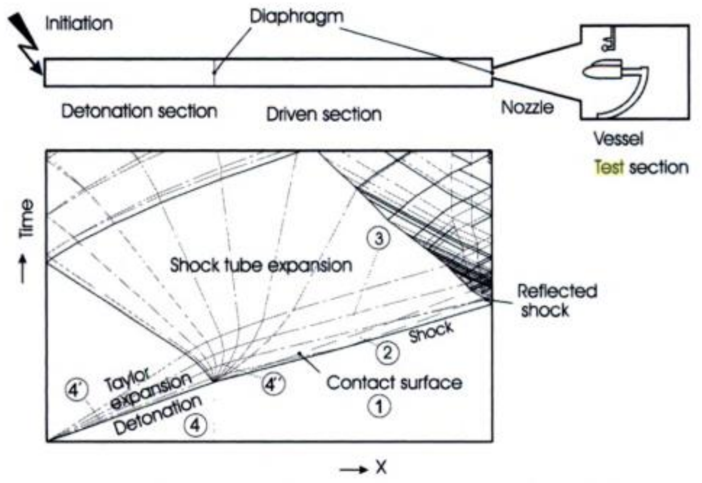


Fig 5.8: Principle of a downstream detonation-driven shock tunnel.

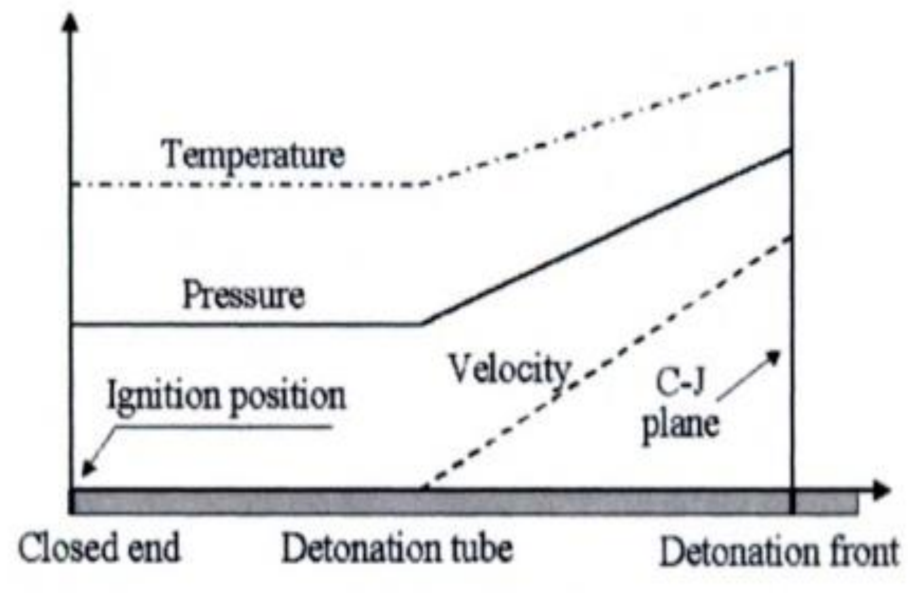


Fig 5.9 A simple model for the distribution of detonation wave along the length of a detonation tube.

5.6 Expansion Tunnel

It uses a dual-diaphragm system where the diaphragms act as rupture discs, or a pressure relief. The tunnel is separated into three sections: driver, driven, and acceleration as shown in Fig. 5.10. The driver section can be fired either by a high pressure gas or by a free piston driver. The driven section is filled with a lower pressure test gas. The acceleration section is filled with an even lower pressurized test gas. Each section is divided by a diaphragm, which is meant to be ruptured in sequence causing the first diaphragm to rupture, mixing and expanding the driver and the driven. When the shock wave hits the second diaphragm, it ruptures casing the two gases to mix with the acceleration and expand down the enclosed test section. In this way, the flow of the test flow can be increased for hypervelocity conditions. However, the operation time is approximately 250μs.

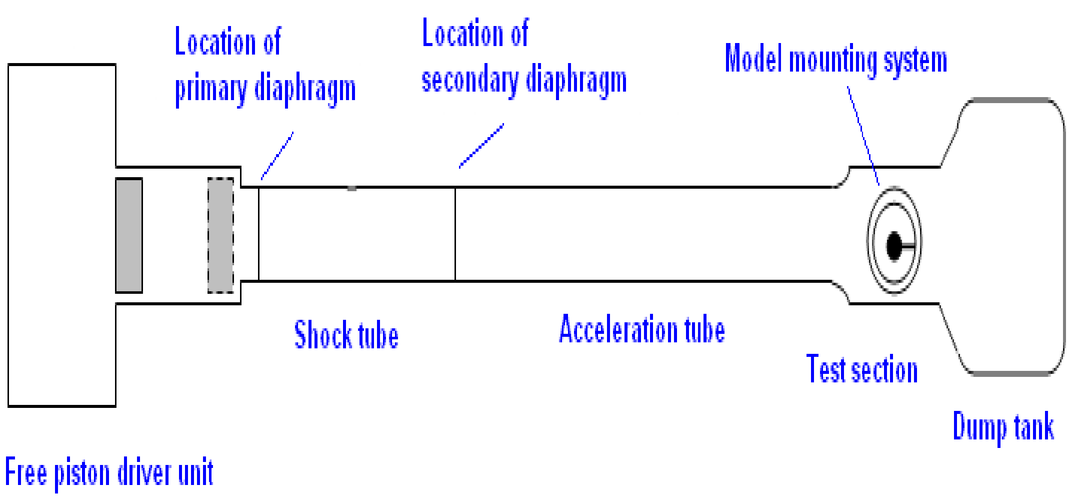


Fig. 51.0: Schematic diagram of an expansion tube.

5.7 Flow Visualization Techniques for Compressible Flows

5.7.1 Introduction

Flow visualization is carried out to understand the physics of the flow in detail. Understanding of the flow provides excellent description which in turn helps to calculate flow properties for many problems of practical interests in both compressible subsonic and supersonic flow regimes. Flow visualization can be of interest for various problems. Since air is transparent, thus their flow patterns are invisible to us without incorporating special techniques to visualize the same. Therefore the present section deals with some important flow visualization techniques.

5.7.2 Interferometer

The interferometer which is an optical method is specifically suited for qualitative determination of the density field of high speed flows. From the theory of light, we know that when light travels through a gas the velocity of propagation is affected by the physical properties of the gas. Principle of working of the interferometer is based on this fact.

Figure 5.11 shows the essential features of the Mach-Zhender Interferometer. Light emitted from the source is allowed to pass through the lens L1 which makes the light rays parallel with each other. These parallel light beams are then passed through a monochromatic filter. The path of the light wave is then made by two ways as M1-M2-M4 and M1-M3-M4 in the way to fall on the screen as shown in Fig.5.11. Here M represents mirror shown in same figure. These two paths are possible due to half silvered mirror M1 which divides a light beam into two beams. These two beams travel same distance in their respective paths before recombining at lens L2 and getting projected on the screen. Although the length of travel for both the beams is same, however the medium of travel is necessarily different since one beam which travels via M1-M3-M4 path actually encounters room air whereas the other beam travelling through M1-M2-M4 passes through the test section. For the situation without any flow through the test section, both the rays face similar medium and fall on the screen as a single ray after combining. Thus a screen gets illuminated by a uniform patch of light. However, the situation changes with in the presence of flow in the test section where density change is encountered by one ray in the presence of shock. This increased density retards the beam passing through the test section and hence creates phase difference between the two beams. If this phase difference is $\lambda/2$ where λ is wave length, then two rays mark contrast spots on the screen. Therefore, we can observe dark and white bands on the screen for appreciable difference in the density.

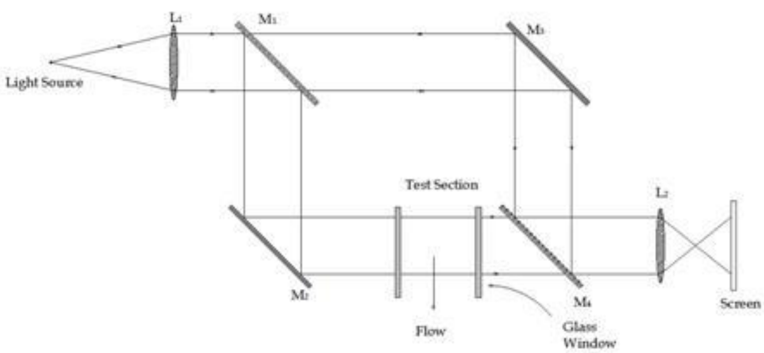


Fig 5.11 Mach-Zhender Interferometer.

5.7.3 Schlieren Technique

The Schlieren method is one more technique prevalently used for visualizing the compressible flow due to presence of large density gradients. Schematic diagram of a typical Schlieren arrangement used for supersonic flow visualization is shown in Fig. 5.12.

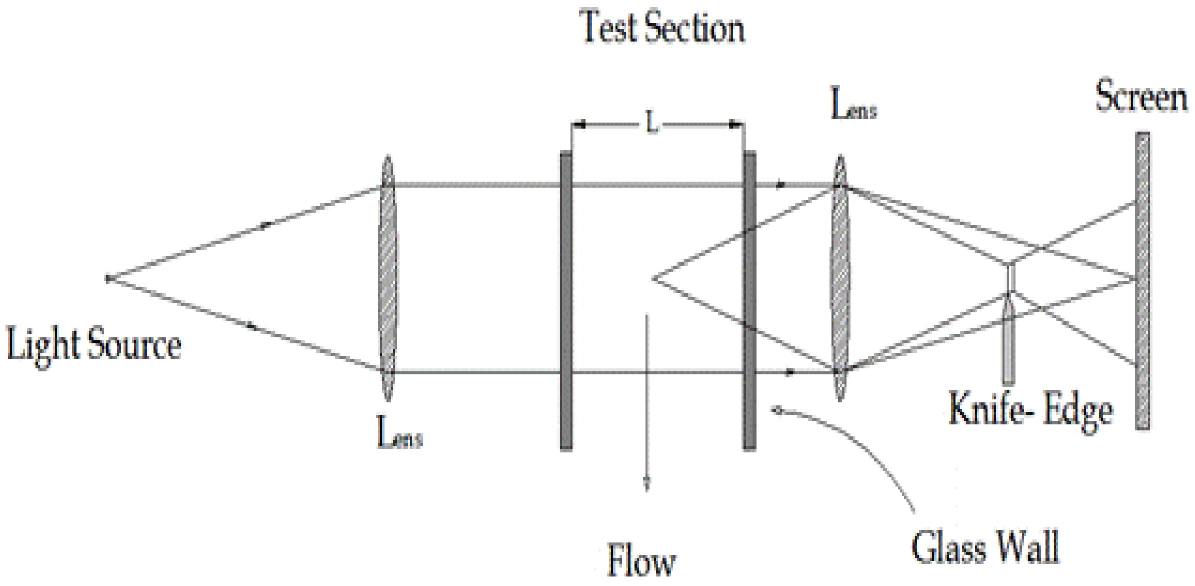


Fig. 5.12 Schematic of Schlieren arrangement

This techniques also uses a light source and lens where Light emitted from the source is collimated by the lens before passing through the test section. These light beams are then passed through one more lens before getting on the screen. A knife edge is placed at the focal point of the second lens where the images of the source is formed. Kinfe edge can be any opaque object which can be placed at the same location. This object or knife object is obstruct the light beam. If the beams of light escape the knife edge the screen gets uniformly illumination. This situation is seen for no flow case through the test section since both the beams pass though the medium of same density. However during the flow taking place in the test section with test model mounted in it, both the light beams encounter different densities therefore make different deflections one of the beams pass through uniform or freestream density region while other beam passes through the portion of shock placed ahead the test model. In this way one beam passes through a region of uniform density while other passes through differential density regions or sees the density gradient. It is similar to the light beam passing through a prism when that ray of light bends. This is the reason for orientation of the knife edge for the known density gradient since the ray of light which encounters the density gradient creates differential illumination on the screen presence of knife edge. In this way Schlieren technique makes it possible to visualize density gradient by differential illuminations on the screen. A photographic plate or camera are generally used for viewing instead of screen.

The major requirement for the Schlieren imaging is to have high optical quality lenses and with large diameter and long focal length. The need for large diameter is necessitate that the coverage of entire flow field. The demand for larger focal length is to acquire proper images on the screen. The quality of the lenses should also be higher in order to avoid chromatic and spherical aberrations and lessen the astigmatism.

It has been observed that is difficult to visualize images using this technique for very large cross section wind tunnels due to unavailability of high optical quality and diameter of lenses. The cost of such lenses is the major concern is such cases. Use of concave mirrors is the immediate remedy on such situations. This is due to the fact that these lenses are easy to fabricate and also to correct during the experimental setting. The minimum optical quality is implicit in these lenses. The Schlieren arrangement with such low cost lenses is as shown in Fig 5.13.

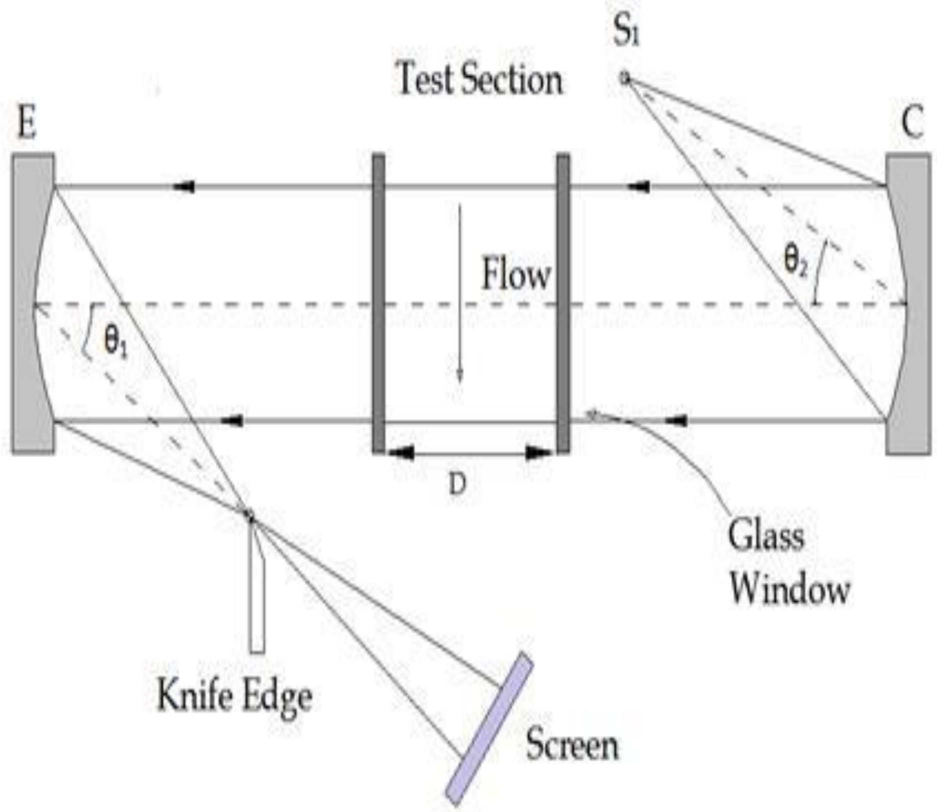


Fig. 5.13. Schematic of twin Schlieren arrangement

5.7.4 Planar Laser Induced Fluorescence (PLIF)

Optical techniques are considered for flow visualization, Planar Laser Induced Fluorescence (PLIF) is among such methods. This lased based technique is popular due to its abilities like remote controlling, precise prediction and non-intrusive. In this technique, a laser sheet is used to illuminate the flow and captures the fluorescence. Radicals which are most active species on the flow are chosen to track in this technique using the laser sheet to excite the species and hence to emitted the fluorescence which is captured on charge-coupled device (ICCD) camera.

A basic arrangement of PLIF is illustrated below in Fig 5.14. Consider a burner whose fluorescence is expected to capture. Therefore a laser beam passes through a cylindrical lens to convert it into a thin laser sheet. This laser sheet passes through the flame produced by a flat flame burner and excites the OH radical which inturn causes fluorescence. This fluorescence signal is then acquired in the camera in the form of the images. The images captured in the ICCD can be further considered for processing.

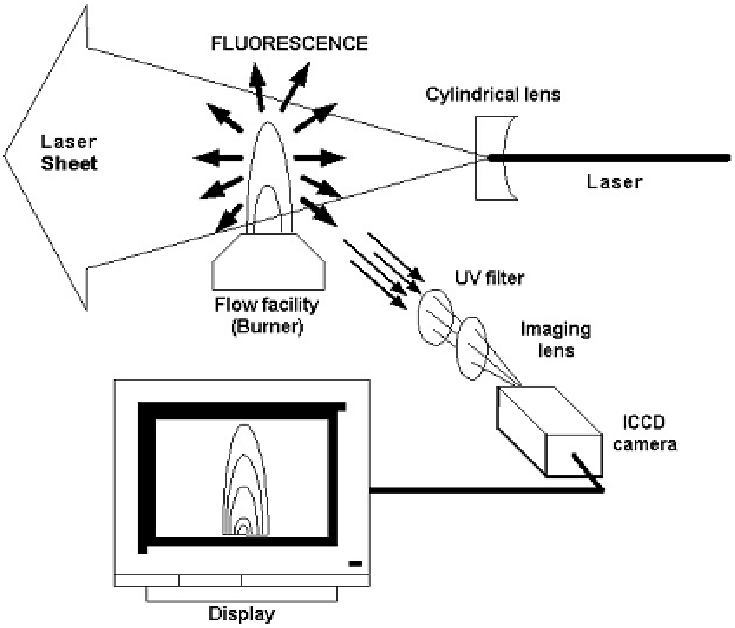


Fig 5.14 Basic Arrangement of PLIF

Disadvantages of PLIF: The major disadvantage is the quenching of the fluorescence at high pressure conditions due to high number of collisions of the molecules.