INSTITUTE OF AERONAUTICAL ENGINEERING

(Autonomous)

Dundigal, Hyderabad -500 043

ELECTRONICS AND COMMUNICATION ENGINEERING

LECTURE NOTES

Course Title	**SIGNALS AND SYSTEMS**				
Course Code	**AECB14**				
Programme	B.Tech				
Semester	IV	ECE			
Course Type	Core				
Regulation	IARE - R18				
Course Structure	**Theory**			**Practical**	
	Lectures	**Tutorials**	**Credits**	**Laboratory**	**Credits**
	3	-	3	-	-
Chief Coordinator	Dr. V Padmanabha Reddy, Professor, ECE				
Course Faculty	Mrs. V. Bindusree, Assistant Professor, ECE Mr. P Sandeep Kumar, Assistant Professor, ECE				

COURSE OUTCOMES (COs):

CO 1	Apply the knowledge of linear algebra to represent any arbitrary signals in terms of complete sets of orthogonal functions and classify the signals and systems based on their properties.
CO 2	Analyze the spectral characteristics of continuous-time periodic and a periodic signals using Fourier analysis.
CO 3	Understand the properties of linear time invariant system, ideal filter characteristics through distortion less transmission and its bandwidth, causality with convolution and correlation.
CO 4	Apply the Laplace transform and Z- transform and their Region of convergence (ROC) properties for analysis of continuous-time and discrete-time signals and systems respectively.
CO 5	Understand the process of sampling to convert an analog signal into discrete signal and the effects of under sampling and study correlation, spectral densities.

COURSE LEARNING OUTCOMES (CLOs):

CLO Code	At the end of the course, the student will have the ability to:
AECB14.01	Apply the knowledge of vectors to find an analogy with signals.
AECB14.02	Understand Orthogonal signal space and orthogonal functions.
AECB14.03	Introduce the basic classification of signals in both continuous and discrete domain, exponential and sinusoidal signals, standard test signals
AECB14.04	Introduce the basic classification of systems in both continuous and discrete domain
AECB14.05	Representation of Fourier series for a periodic signal.

CLO Code	At the end of the course, the student will have the ability to:
AECB14.06	Deduce Fourier Transform from Fourier series
AECB14.07	Compute Fourier Transform of Periodic Signal
AECB14.08	Introduce the special transform-Hilbert transform
AECB14.09	Analyze time variance for linear systems.
AECB14.10	Understand the concept of distortion less transmission through a system
AECB14.11	Analyze Causality and Paley-Wiener criterion for physical realization.
AECB14.12	Understand the concept of convolution through graphical representation
AECB14.13	Introduce the concepts of Laplace transform for conversion to S-domain.
AECB14.14	Represent Region of Convergence for Laplace transforms and properties of Laplace Transforms.
AECB14.15	Understand the Z-Transform for discrete signals with issues of Region of Convergence
AECB14.16	Analyze the properties of Z-Transforms.
AECB14.17	Categorical analysis of sampling into different types.
AECB14.18	Understand how to reconstruct signals after sampling
AECB14.19	Understand cross correlation and auto correlation concepts.
AECB14.20	Analyze Power Spectral and Energy Spectral Characteristics

SYLLABUS

MODULE – I	SIGNAL ANALYSIS	Classes: 08

Analogy between Vectors and Signals, Orthogonal Signal Space, Signal approximation using Orthogonal functions, Mean Square Error, Closed or complete set of Orthogonal functions, Orthogonality in Complex functions, Classification of Signals and systems, Exponential and Sinusoidal signals, Concepts of Impulse function, Unit Step function, Signum function.

MODULE - II	FOURIER SERIES	Classes: 10

Representation of Fourier series, Continuous time periodic signals, Properties of Fourier Series, Dirichlet"s conditions, Trigonometric Fourier Series and Exponential Fourier Series, Complex Fourier spectrum.

Fourier Transforms:

Deriving Fourier Transform from Fourier series, Fourier Transform of arbitrary signal, Fourier Transform of standard signals, Fourier Transform of Periodic Signals, Properties of Fourier Transform, Fourier Transforms involving Impulse function and Signum function, Introduction to Hilbert Transforms.

MODULE - III	SIGNAL TRANSMISSION THROUGH LINEAR SYSTEMS	Classes: 10

Linear System, Impulse response, Response of a Linear System, Linear Time Invariant(LTI) System, Linear Time Variant (LTV) System, Transfer function of a LTI System, Filter characteristic of Linear System, Distortion less transmission through a system, Signal bandwidth, System Bandwidth, Ideal LPF, HPF, and BPF characteristics.

Causality and Paley-Wiener criterion for physical realization, Relationship between Bandwidth and rise time, Convolution and Correlation of Signals, Concept of convolution in Time domain and Frequency domain, Graphical representation of Convolution.

MODULE - IV	LAPLACE TRANSFORM AND Z-TRANSFORM	Classes: 08

Laplace Transforms

Laplace Transforms (L.T), Inverse Laplace Transform, Concept of Region of Convergence (ROC) for Laplace Transforms, Properties of L.T, Relation between L.T and F.T of a signal, Laplace Transform of certain signals using waveform synthesis. **Z–Transforms** Concept of Z- Transform of a Discrete Sequence, Distinction between Laplace, Fourier and Z Transforms, Region of Convergence in Z-Transform, Constraints on ROC for various classes of signals, Inverse Z-transform, Properties of Z- transforms.

MODULE - V	SAMPLING THEOREM	Classes: 09

Graphical and analytical proof for Band Limited Signals, Impulse Sampling, Natural and Flat top Sampling, Reconstruction of signal from its samples, Effect of under sampling – Aliasing, Introduction to Band Pass Sampling. **Correlation:** Cross Correlation and Auto Correlation of Functions, Properties of Correlation

Functions, Energy Density Spectrum, Parseval's Theorem, Power Density Spectrum, Relation between Autocorrelation Function and Energy/Power Spectral Density Function, Relation between Convolution and Correlation, Detection of Periodic Signals in the presence of Noise by Correlation, Extraction of Signal from Noise by filtering.

Text Books:

1. B.P. Lathi, "Signals, Systems & Communications", BSP, 2013.
2. Signals and Systems - A.V. Oppenheim, A.S. Willsky and S.H. Nawabi, 2nd Edition 2010.

Reference Books:

1. Simon Haykin and Van Veen, "Signals and Systems", Wiley Publications, 2nd Edition, 2010.
2. Fundamentals of Signals and Systems - Michel J. Robert, 2008, MGH International Edition.

MODULE – I

SIGNAL ANALYSIS

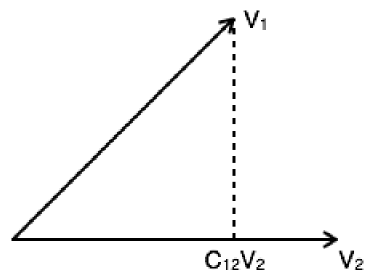
Analogy between Vectors and Signals, Orthogonal Signal Space, Signal approximation using Orthogonal functions, Mean Square Error, Closed or complete set of Orthogonal functions, Orthogonality in Complex functions, Classification of Signals and systems, Exponential and Sinusoidal signals, Concepts of Impulse function, Unit Step function, Signum function.

Analogy Between Vectors and Signals:

There is a perfect analogy between vectors and signals.

Vector
A vector contains magnitude and direction. The name of the vector is denoted by bold face type and their magnitude is denoted by light face type.

Example: V is a vector with magnitude V. Consider two vectors V1 and V2 as shown in the following diagram. Let the component of V1 along with V2 is given by C12V2. The component of a vector V1 along with the vector V2 can obtained by taking a perpendicular from the end of V1 to the vector V2 as shown in diagram:



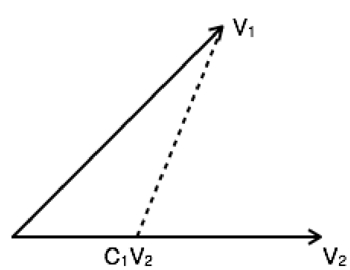
The vector V1 can be expressed in terms of vector V2

V1= C12V2 + Ve

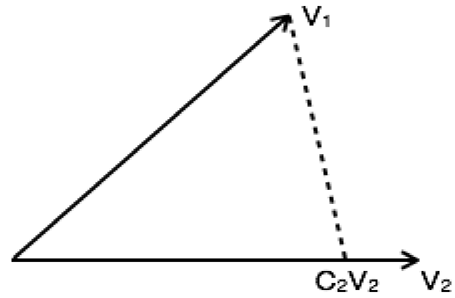
Where Ve is the error vector.

But this is not the only way of expressing vector V1 in terms of V2. The alternate possibilities are:

V1=C1V2+Ve1



V2=C2V2+Ve2



The error signal is minimum for large component value. If C12=0, then two signals are said to be orthogonal.

Dot Product of Two Vectors V1 . V2 = V1.V2 cosθ
θ = Angle between V1 and V2 V1. V2 =V2.V1
From the diagram, components of V1 a long V2 = C 12 V2

$$\frac{V_1 \, . \, V_2}{V_2 \, = C_1 2 \, V_2}$$

$$\Rightarrow C_{12} = \frac{V_1 \, . \, V_2}{V_2}$$

The concept of orthogonality can be applied to signals. Let us consider two signals f1(t) and f2(t).
Similar to vectors, you can approximate f1(t) in terms of f2(t) as f1(t) = C12 f2(t) + fe(t) for (t1 < t < t2)
⇒ fe(t) = f1(t) – C12 f2(t)

One possible way of minimizing the error is integrating over the interval t1 to t2.

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_e(t)] dt$$

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_1(t) - C_{12} f_2(t)] dt$$

However, this step also does not reduce the error to appreciable extent. This can be corrected by taking the square of error function.

$$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_e(t)]^2 dt$$

$$\Rightarrow \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_e(t) - C_{12} f_2]^2 dt$$

Where ε is the mean square value of error signal. The value of C12 which minimizes the error, you need to calculate *dε/dC*12=0

$$\Rightarrow \frac{d}{dC_{12}} [\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_1(t) - C_{12} f_2(t)]^2 dt] = 0$$

$$\Rightarrow \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [\frac{d}{dC_{12}} f_1^2(t) - \frac{d}{dC_{12}} 2 f_1(t) C_{12} f_2(t) + \frac{d}{dC_{12}} f_2^2(t) C_{12}^2] dt = 0$$

Derivative of the terms which do not have C12 term are zero.

$$\Rightarrow \int_{t_1}^{t_2} -2f_1(t)f_2(t)dt + 2C_{12}\int_{t_1}^{t_2}[f_2^2(t)]dt = 0$$

If $C_{12} = \frac{\int_{t_1}^{t_2} f_1(t)f_2(t)dt}{\int_{t_1}^{t_2} f_2^2(t)dt}$ component is zero, then two signals are said to be orthogonal.

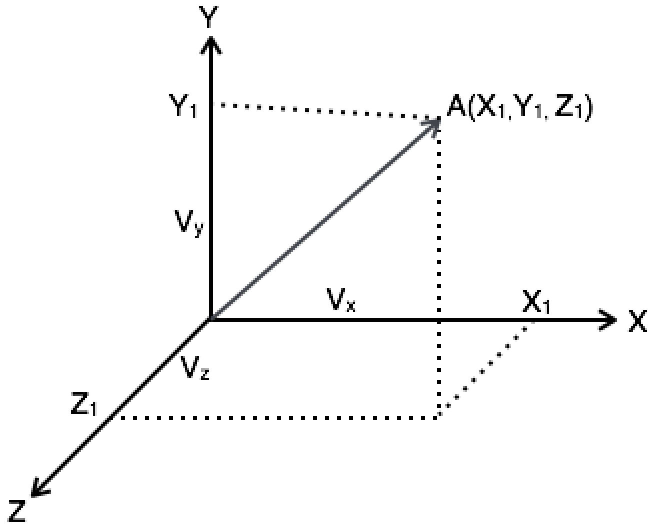
Put C12 = 0 to get condition for orthogonality.

$$0 = \frac{\int_{t_1}^{t_2} f_1(t)f_2(t)dt}{\int_{t_1}^{t_2} f_2^2(t)dt}$$

$$\int_{t_1}^{t_2} f_1(t)f_2(t)dt = 0$$

Orthogonal Vector Space

A complete set of orthogonal vectors is referred to as orthogonal vector space. Consider a three dimensional vector space as shown below:



Consider a vector A at a point (X1, Y1, Z1). Consider three unit vectors (VX, VY, VZ) in the direction of X, Y, Z axis respectively. Since these unit vectors are mutually orthogonal, it satisfies that

$$V_X . V_X = V_Y . V_Y = V_Z . V_Z = 1$$
$$V_X . V_Y = V_Y . V_Z = V_Z . V_X = 0$$

We can write above conditions as

The vector A can be represented in terms of its components and unit vectors as

$$V_a \,.\, V_b = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$$

$$A = X_1 V_X + Y_1 V_Y + Z_1 V_Z \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Any vectors in this three dimensional space can be represented in terms of these three unit vectors only.

If you consider n dimensional space, then any vector A in that space can be represented as

$$A = X_1 V_X + Y_1 V_Y + Z_1 V_Z + \ldots + N_1 V_N \ldots\ldots (2)$$

As the magnitude of unit vectors is unity for any vector A The component of A along x axis = A.VX
The component of A along Y axis = A.VY The component of A along Z axis = A.VZ

Similarly, for n dimensional space, the component of A along some G axis

=*A.VG* (3)
Substitute equation 2 in equation 3.

$$\Rightarrow CG = (X_1 V_X + Y_1 V_Y + Z_1 V_Z + \ldots + G_1 V_G \ldots + N_1 V_N) V_G$$

$$= X_1 V_X V_G + Y_1 V_Y V_G + Z_1 V_Z V_G + \ldots + G_1 V_G V_G \ldots + N_1 V_N V_G$$

$$= G_1 \quad \text{since } V_G V_G = 1$$

$$If V_G V_G \neq 1 \text{ i.e.} V_G V_G = k$$

$$AV_G = G_1 V_G V_G = G_1 K$$

$$G_1 = \frac{(AV_G)}{K}$$

Orthogonal Signal Space

Let us consider a set of n mutually orthogonal functions x1(t), x2(t)... xn(t) over the interval t1 to t2. As these functions are orthogonal to each other, any two signals xj(t), xk(t) have to satisfy the orthogonality condition. i.e.

$$\int_{t_1}^{t_2} x_j(t) x_k(t) dt = 0 \ \text{ where } j \neq k$$

$$\text{Let} \int_{t_1}^{t_2} x_k^2(t) dt = k_k$$

Let a function f(t), it can be approximated with this orthogonal signal space by adding the components along mutually orthogonal signals i.e.

$$f(t) = C_1 x_1(t) + C_2 x_2(t) + \ldots + C_n x_n(t) + f_e(t)$$
$$= \Sigma_{r=1}^{n} C_r x_r(t)$$
$$f(t) = f(t) - \Sigma_{r=1}^{n} C_r x_r(t)$$

Mean sqaure error $\varepsilon = \frac{1}{t_2 - t_2} \int_{t_1}^{t_2} [f_e(t)]^2 dt$

$$= \frac{1}{t_2 - t_2} \int_{t_1}^{t_2} [f[t] - \sum_{r=1}^{n} C_r x_r(t)]^2 dt$$

The component which minimizes the mean square error can be found by

$$\frac{d\varepsilon}{dC_1} = \frac{d\varepsilon}{dC_2} = \ldots = \frac{d\varepsilon}{dC_k} = 0$$

Let us consider $\frac{d\varepsilon}{dC_k} = 0$

$$\frac{d}{dC_k}[\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f(t) - \Sigma_{r=1}^{n} C_r x_r(t)]^2 dt] = 0$$

All terms that do not contain Ck is zero. i.e. in summation, r=k term remains and all other terms are zero.

$$\int_{t_1}^{t_2} -2f(t)x_k(t)dt + 2C_k \int_{t_1}^{t_2} [x_k^2(t)]dt = 0$$

$$\Rightarrow C_k = \frac{\int_{t_1}^{t_2} f(t)x_k(t)dt}{int_{t_1}^{t_2} x_k^2(t)dt}$$

$$\Rightarrow \int_{t_1}^{t_2} f(t)x_k(t)dt = C_k K_k$$

<u>Mean Square Error:</u>

The average of square of error function fe(t) is called as mean square error. It is denoted by ε (epsilon).

$$\varepsilon = \frac{1}{t_2-t_1} \int_{t_1}^{t_2} [f_e(t)]^2 dt$$

$$= \frac{1}{t_2-t_1} \int_{t_1}^{t_2} [f_e(t) - \Sigma_{r=1}^{n} C_r x_r(t)]^2 dt$$

$$= \frac{1}{t_2-t_1} [\int_{t_1}^{t_2} [f_e^2(t)]dt + \Sigma_{r=1}^{n} C_r^2 \int_{t_1}^{t_2} x_r^2(t)dt - 2\Sigma_{r=1}^{n} C_r \int_{t_1}^{t_2} x_r(t)f(t)dt$$

You know that $C_r^2 \int_{t_1}^{t_2} x_r^2(t)dt = C_r \int_{t_1}^{t_2} x_r(t)f(d)dt = C_r^2 K_r$

$$\varepsilon = \frac{1}{t_2-t_1} [\int_{t_1}^{t_2} [f^2(t)]dt + \Sigma_{r=1}^{n} C_r^2 K_r - 2\Sigma_{r=1}^{n} C_r^2 K_r]$$

$$= \frac{1}{t_2-t_1} [\int_{t_1}^{t_2} [f^2(t)]dt - \Sigma_{r=1}^{n} C_r^2 K_r]$$

$$\therefore \varepsilon = \frac{1}{t_2-t_1} [\int_{t_1}^{t_2} [f^2(t)]dt + (C_1^2 K_1 + C_2^2 K_2 + \ldots + C_n^2 K_n)]$$

The above equation is used to evaluate the mean square error.

Closed and Complete Set of Orthogonal Functions:

Let us consider a set of n mutually orthogonal functions x1(t), x2(t)...xn(t) over the interval t1 to t2. This is called as closed and complete set when there exist no function f(t) satisfying the condition

$$\int_{t_1}^{t_2} f(t)x_k(t)dt = 0$$

If this function is satisfying the equation

$$\int_{t_1}^{t_2} f(t)x_k(t)dt = 0$$

For k=1,2,.. then f(t) is said to be orthogonal to each and every function of orthogonal set.

This set is incomplete without f(t). It becomes closed and complete set when f(t) is included.

f(t) can be approximated with this orthogonal set by adding the components along mutually orthogonal signals i.e.

$$f(t) = C_1 x_1(t) + C_2 x_2(t) + \ldots + C_n x_n(t) + f_e(t)$$

If the infinite series $C_1 x_1(t) + C_2 x_2(t) + \ldots + C_n x_n(t)$ converges to ft then mean square error is zero.

Orthogonality in Complex Functions:

If f1(t) and f2(t) are two complex functions, then f1(t) can be expressed in terms of f2(t) as

$f1(t)=C12f2(t)$.. with negligible error

$$\text{Where } C_{12} = \frac{\int_{t_1}^{t_2} f_1(t)f_2^*(t)dt}{\int_{t_1}^{t_2} |f_2(t)|^2 dt}$$

Where f2*(t) is the complex conjugate of f2(t) If f1(t) and f2(t) are orthogonal then C12 = 0

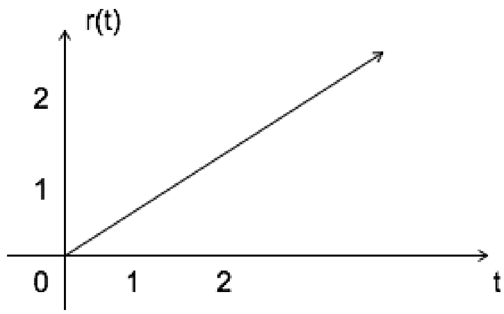
$$\frac{\int_{t_1}^{t_2} f_1(t) f_2^*(t) dt}{\int_{t_1}^{t_2} |f_2(t)|^2 dt} = 0$$

$$\Rightarrow \int_{t_1}^{t_2} f_1(t) f_2^*(dt) = 0$$

The above equation represents orthogonality condition in complex functions.

Ramp Signal

Ramp signal is denoted by r(t), and it is defined as r(t) = $\begin{cases} t & t \geqslant 0 \\ 0 & t < 0 \end{cases}$



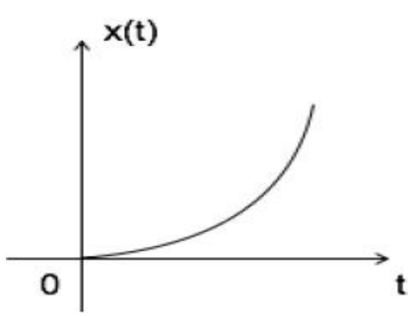
$$\int u(t) = \int 1 = t = r(t)$$

$$u(t) = \frac{dr(t)}{dt}$$

Area under unit ramp is unity.

Parabolic Signal

Parabolic signal can be defined as x(t) = $\begin{cases} t^2/2 & t \geqslant 0 \\ 0 & t < 0 \end{cases}$



$$\iint u(t)dt = \int r(t)dt = \int tdt = \frac{t^2}{2} = parabolicsignal$$

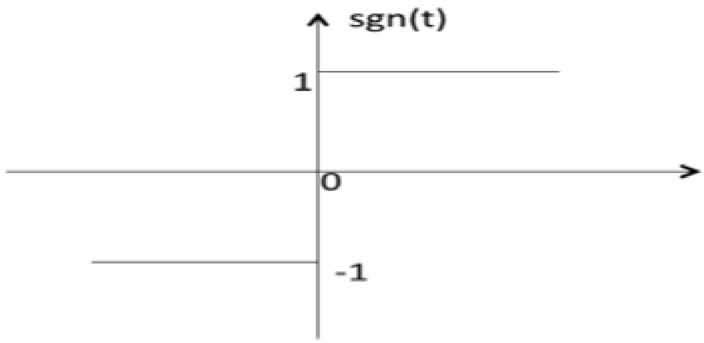
$$\Rightarrow u(t) = \frac{d^2x(t)}{dt^2}$$

$$\Rightarrow r(t) = \frac{dx(t)}{dt}$$

Signum Function

$$\begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

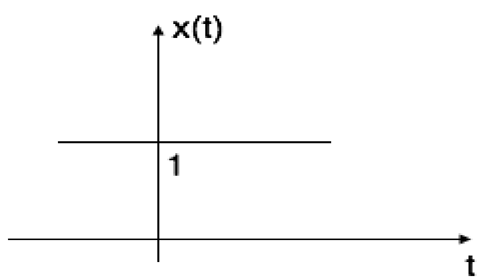
Signum function is denoted as sgn(t). It is defined as sgn(t) =



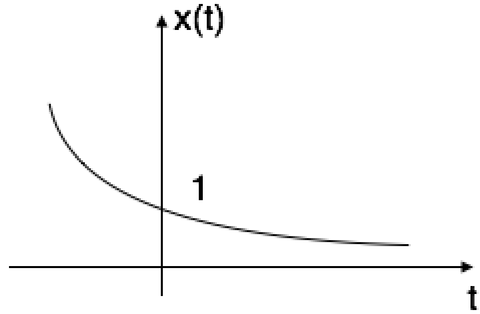
sgn(t) = 2u(t) – 1

Exponential Signal

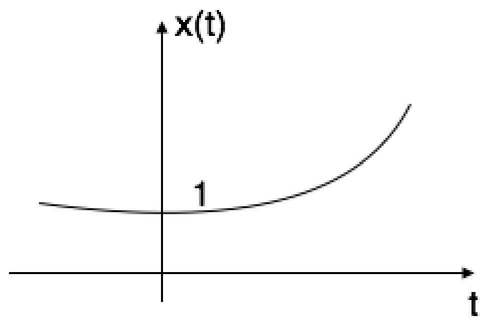
Exponential signal is in the form of x(t) = *eαt*
.The shape of exponential can be defined by *α*
Case i: if *α* = 0 → x(t) = *e*0= 1



Case ii: if $\alpha < 0$ i.e. -ve then x(t) = $e^{-\alpha t}$
. The shape is called decaying exponential.



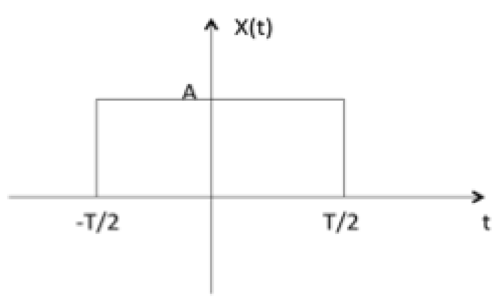
Case iii: if $\alpha > 0$ i.e. +ve then x(t) = $e^{\alpha t}$
. The shape is called raising exponential.

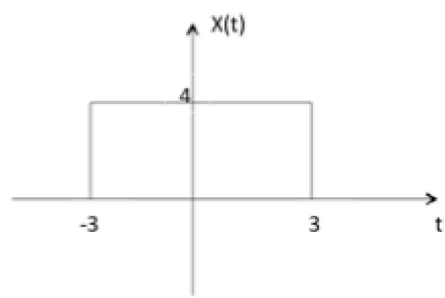


Rectangular Signal

Let it be denoted as x(t) and it is defined as

$$x(t) = A\, rect\left[\frac{r}{T}\right] \qquad \text{ex: } 4\, rect\left[\frac{r}{6}\right]$$

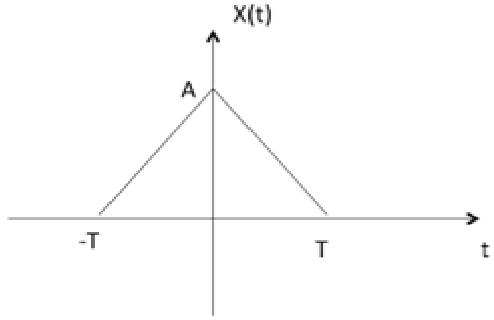


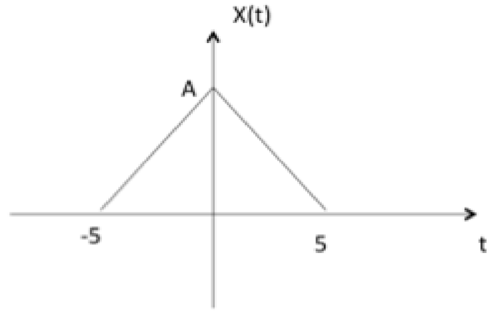


Triangular Signal

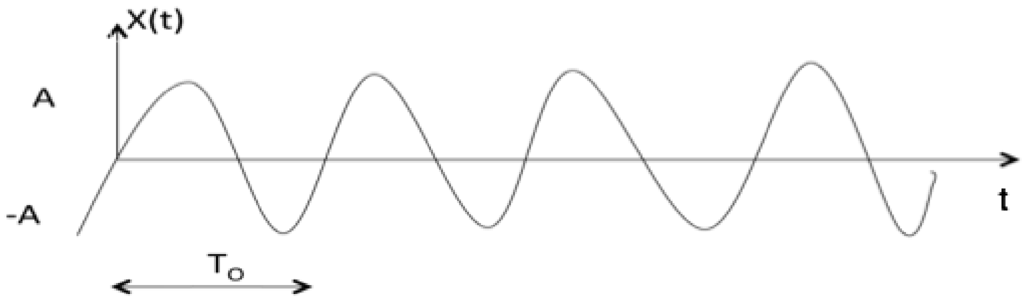
Let it be denoted as x(t)

$$x(t) = A\left[1 - \frac{|t|}{T}\right] \qquad \text{ex:}\ x(t) = A\left[1 - \frac{|t|}{5}\right]$$





Sinusoidal Signal
Sinusoidal signal is in the form of x(t) = A cos($w0\pm\phi$) or A sin($w0\pm\phi$)



Where T0 = $2\pi/w0$

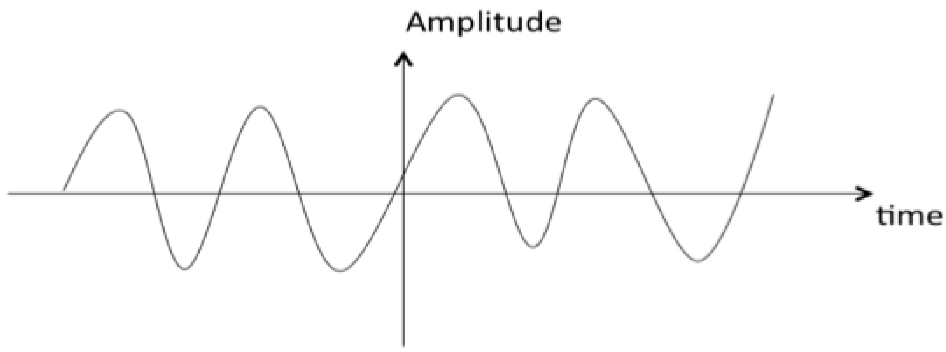
Classification of Signals:

Signals are classified into the following categories:

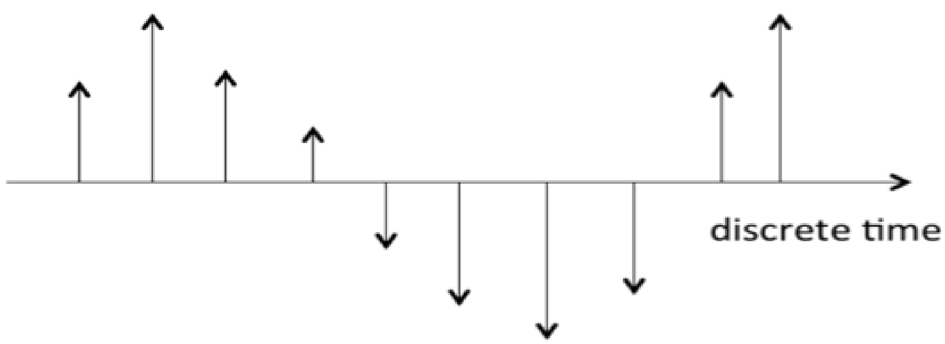
- Continuous Time and Discrete Time Signals
- Deterministic and Non-deterministic Signals
- Even and Odd Signals
- Periodic and Aperiodic Signals
- Energy and Power Signals
- Real and Imaginary Signals

Continuous Time and Discrete Time Signals

A signal is said to be continuous when it is defined for all instants of time.

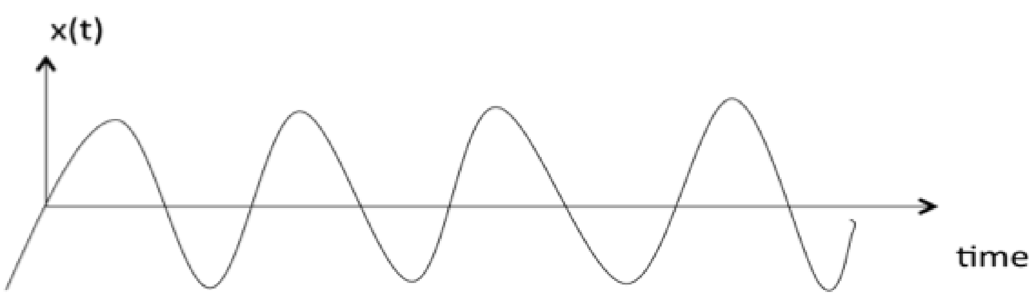


A signal is said to be discrete when it is defined at only discrete instants of time/

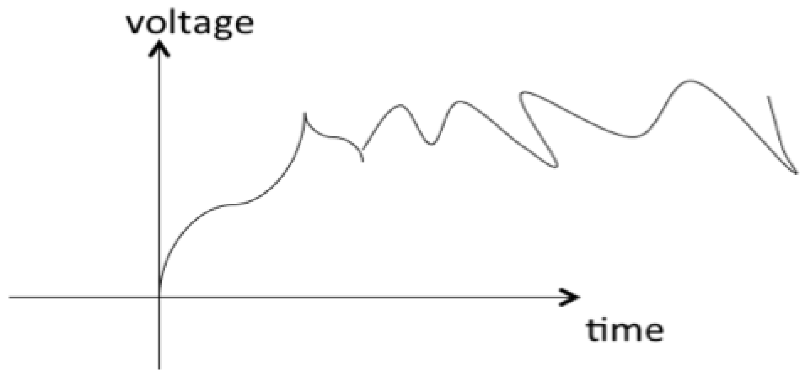


Deterministic and Non-deterministic Signals

A signal is said to be deterministic if there is no uncertainty with respect to its value at any instant of time. Or, signals which can be defined exactly by a mathematical formula are known as deterministic signals.



A signal is said to be non-deterministic if there is uncertainty with respect to its value at some instant of time. Non-deterministic signals are random in nature hence they are called random signals. Random signals cannot be described by a mathematical equation. They are modelled in probabilistic terms.



Even and Odd Signals

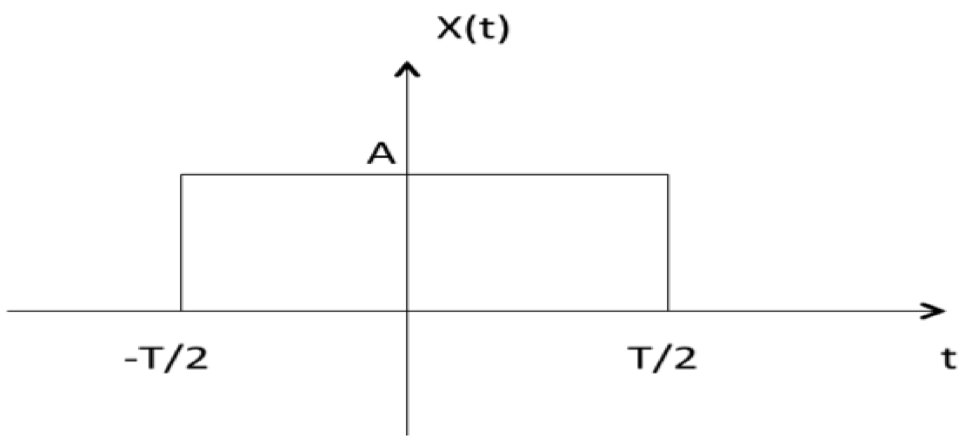
A signal is said to be even when it satisfies the condition x(t) = x(-t)

Example 1: t2, t4… cost etc.

Let x(t) = t2

x(-t) = (-t)2 = t2 = x(t)

∴ t2 is even function
Example 2: As shown in the following diagram, rectangle function x(t) = x(-t) so it is also even function.



A signal is said to be odd when it satisfies the condition x(t) = -x(-t)

Example: t, t3 ... And sin t Let x(t) = sin t
x(-t) = sin(-t) = -sin t = -x(t)

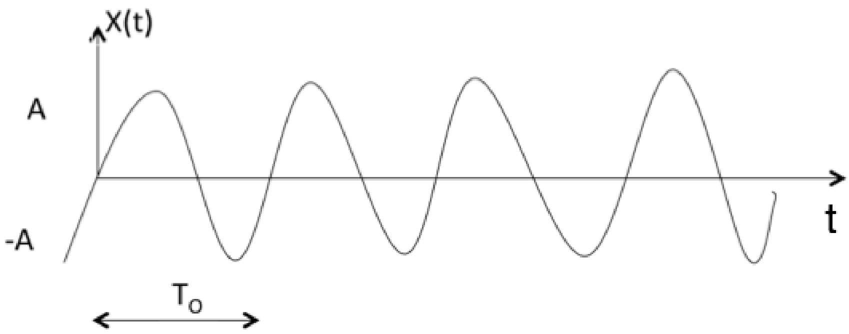
∴ sin t is odd function.

Any function *ƒ*(t) can be expressed as the sum of its even function *ƒ*e(t) and odd function *ƒ*o(t). *ƒ*(*t*) = *ƒ*e(*t*) + *ƒ*0(*t*)
where

$fe(t) = ½[f(t) + f(-t)]$

Periodic and Aperiodic Signals

A signal is said to be periodic if it satisfies the condition x(t) = x(t + T) or x(n) = x(n + N). Where T = fundamental time period, 1/T = f = fundamental frequency.



The above signal will repeat for every time interval T0 hence it is periodic with period T0.

Energy and Power Signals

A signal is said to be energy signal when it has finite energy.

$$\text{Energy } E = \int_{-\infty}^{\infty} x^2\,(t)dt$$

A signal is said to be power signal when it has finite power.

$$\text{Power } P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)dt$$

NOTE:A signal cannot be both, energy and power simultaneously. Also, a signal may be neither energy nor power signal.

Power of energy signal = 0 Energy of power signal = ∞

Real and Imaginary Signals

A signal is said to be real when it satisfies the condition x(t) = x*(t) A signal is said to be odd when it satisfies the condition x(t) = -x*(t) Example:
If x(t)= 3 then x*(t)=3*=3 here x(t) is a real signal.
If x(t)= 3j then x*(t)=3j* = -3j = -x(t) hence x(t) is a odd signal.
Note: For a real signal, imaginary part should be zero. Similarly for an imaginary signal, real part should be zero.

Basic operations on Signals:

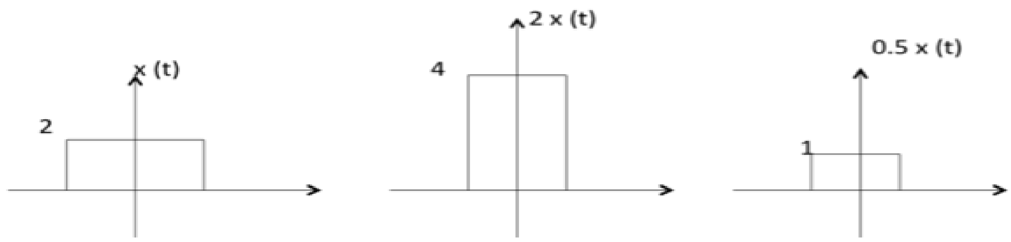
There are two variable parameters in general:

1. Amplitude
2. Time

(1) The following operation can be performed with amplitude:

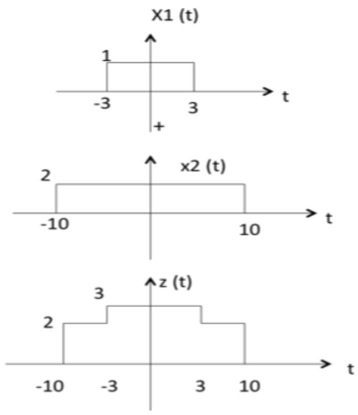
Amplitude Scaling

C x(t) is a amplitude scaled version of x(t) whose amplitude is scaled by a factor C.



Addition

Addition of two signals is nothing but addition of their corresponding amplitudes. This can be best explained by using the following example:



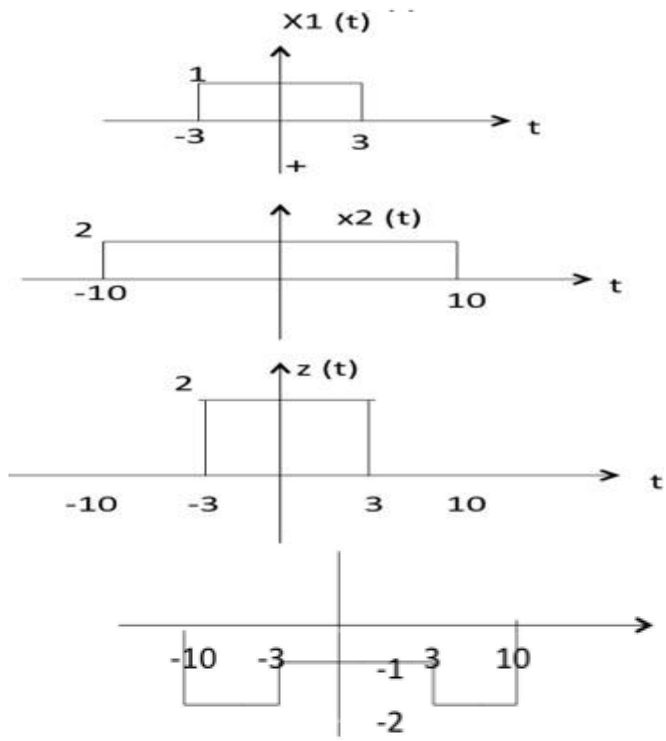
As seen from the previous diagram,

-10 < t < -3 amplitude of z(t) = x1(t) + x2(t) = 0 + 2 = 2

-3 < t < 3 amplitude of z(t) = x1(t) + x2(t) = 1 + 2 = 3 3 < t < 10 amplitude of z(t) = x1(t) + x2(t) = 0 + 2 = 2

Subtraction

subtraction of two signals is nothing but subtraction of their corresponding amplitudes.
This can be best explained by the following example:



As seen from the diagram above,

-10 < t < -3 amplitude of z (t) = x1(t) - x2(t) = 0 - 2 = -2

-3 < t < 3 amplitude of z (t) = x1(t) - x2(t) = 1 - 2 = -1 3 < t < 10 amplitude of z (t) = x1(t) - x2(t) = 0 - 2 = -2

Multiplication

Multiplication of two signals is nothing but multiplication of their corresponding amplitudes. This can be best explained by the following example:

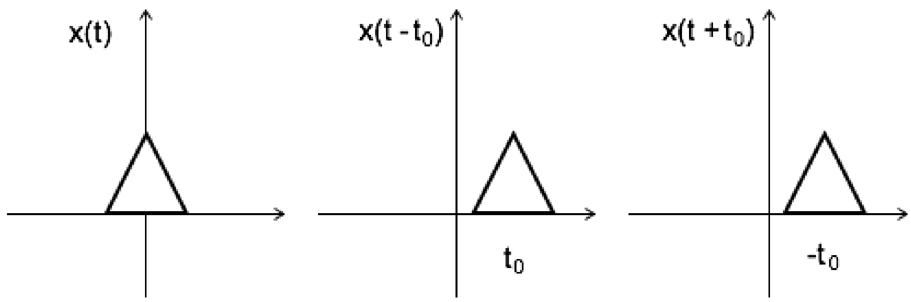
As seen from the diagram above,

-10 < t < -3 amplitude of z (t) = x1(t) ×x2(t) = 0 ×2 = 0
-3 < t < 3 amplitude of z (t) = x1(t) - x2(t) = 1 ×2 = 2 3 < t < 10 amplitude of z (t) = x1(t) - x2(t) = 0 × 2 = 0

(2) The following operations can be performed with time:

Time Shifting

x(t ±t0) is time shifted version of the signal x(t). x (t + t0) →negative shift
x (t - t0) →positive shift

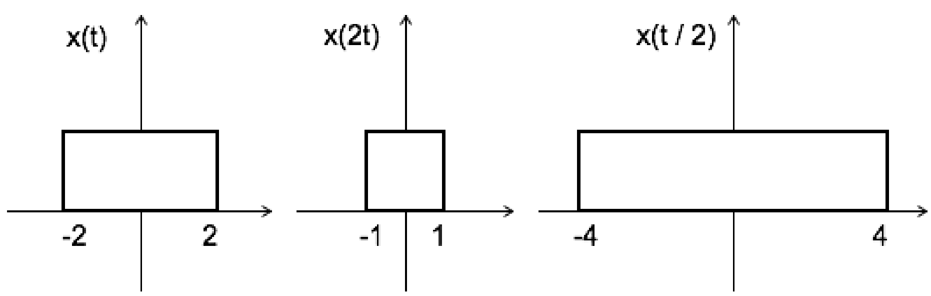


Time Scaling

x(At) is time scaled version of the signal x(t). where A is always positive.

|A| > 1 → Compression of the signal

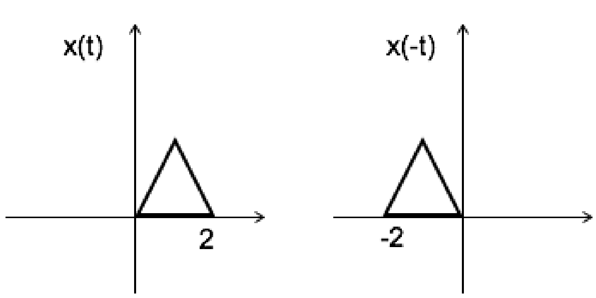
|A| < 1 → Expansion of the signal



Note: u(at) = u(t) time scaling is not applicable for unit step function.

Time Reversal

x(-t) is the time reversal of the signal x(t).



Classification of Systems:

Systems are classified into the following categories:

- Liner and Non-liner Systems
- Time Variant and Time Invariant Systems
- Liner Time variant and Liner Time invariant systems
- Static and Dynamic Systems
- Causal and Non-causal Systems
- Invertible and Non-Invertible Systems
- Stable and Unstable Systems

Linear and Non-linear Systems

A system is said to be linear when it satisfies superposition and homogenate principles. Consider two systems with inputs as x1(t), x2(t), and outputs as y1(t), y2(t) respectively. Then, according to the superposition and homogenate principles,

$T [a1\ x1(t) + a2\ x2(t)] = a1\ T[x1(t)] + a2\ T[x2(t)]$

$\therefore T [a1\ x1(t) + a2\ x2(t)] = a1\ y1(t) + a2\ y2(t)$
From the above expression, is clear that response of overall system is equal to response of individual system.

Example:

y(t) = x2(t) Solution:
y1 (t) = T[x1(t)] = x12(t)

y2 (t) = T[x2(t)] = x22(t)

$T [a1\ x1(t) + a2\ x2(t)] = [\ a1\ x1(t) + a2\ x2(t)]^2$

Which is not equal to a1 y1(t) + a2 y2(t). Hence the system is said to be non linear.

Time Variant and Time Invariant Systems

A system is said to be time variant if its input and output characteristics vary with time.
Otherwise, the system is considered as time invariant. The condition for time invariant system is:
y (n , t) = y(n-t)

The condition for time variant system is:
y (n , t) ≠ y(n-t)

Where y (n , t) = T[x(n-t)] = input change

y (n-t) = output change

Example:

$y(n) = x(-n)$

$y(n, t) = T[x(n-t)] = x(-n-t)$

$y(n-t) = x(-(n-t)) = x(-n + t)$

$\therefore y(n, t) \neq y(n-t)$. Hence, the system is time variant.

Liner Time variant (LTV) and Liner Time Invariant (LTI) Systems

If a system is both liner and time variant, then it is called liner time variant (LTV) system.

If a system is both liner and time Invariant then that system is called liner time invariant (LTI) system.

Static and Dynamic Systems

Static system is memory-less whereas dynamic system is a memory system.

Example 1: $y(t) = 2\ x(t)$

For present value t=0, the system output is $y(0) = 2x(0)$. Here, the output is only dependent upon present input. Hence the system is memory less or static.

Example 2: $y(t) = 2\ x(t) + 3\ x(t-3)$

For present value t=0, the system output is $y(0) = 2x(0) + 3x(-3)$.

Here $x(-3)$ is past value for the present input for which the system requires memory to get this output. Hence, the system is a dynamic system.

Causal and Non-Causal Systems

A system is said to be causal if its output depends upon present and past inputs, and does not depend upon future input.

For non causal system, the output depends upon future inputs also.

Example 1: $y(n) = 2\ x(t) + 3\ x(t-3)$

For present value t=1, the system output is $y(1) = 2x(1) + 3x(-2)$.

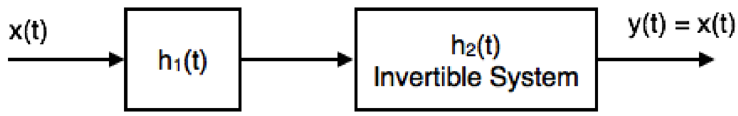
Here, the system output only depends upon present and past inputs. Hence, the system is causal.

Example 2: $y(n) = 2\ x(t) + 3\ x(t-3) + 6x(t + 3)$

For present value t=1, the system output is $y(1) = 2x(1) + 3x(-2) + 6x(4)$ Here, the system output depends upon future input. Hence the system is non-causal system.

Invertible and Non-Invertible systems

A system is said to invertible if the input of the system appears at the output.



Y(S) = X(S) H1(S) H2(S)

= X(S) H1(S) · 1(*H*1(*S*))

Since H2(S) = 1/(H1(S))

∴ Y(S) = X(S)

→ y(t) = x(t)
Hence, the system is invertible.

If y(t) ≠ x(t), then the system is said to be non-invertible.
Stable and Unstable Systems

The system is said to be stable only when the output is bounded for bounded input. For a bounded input, if the output is unbounded in the system then it is said to be unstable.

Note: For a bounded signal, amplitude is finite.

Example 1: y (t) = x2(t)

Let the input is u(t) (unit step bounded input) then the output y(t) = u2(t) = u(t) = bounded output.

Hence, the system is stable.

Example 2: y (t) = $\int x(t)dt$
Let the input is u (t) (unit step bounded input) then the output y(t) = $\int u(t)dt$ = ramp signal (unbounded because amplitude of ramp is not finite it goes to infinite when t → infinite).
Hence, the system is unstable.

1.1 Continuous-time and discrete-time Signals

1.1.1 Examples and Mathematical representation

Signals are represented mathematically as functions of one or more independent variables. Here we focus attention on signals involving a single independent variable. For convenience, this will generally refer to the independent variable as ***time***.

There are two types of signals: *continuous-time signals* and *discrete-time* signals.

Continuous-time signal: the variable of time is continuous. A speech signal as a function of time is a continuous-time signal.

Discrete -time signal: the variable of time is discrete. The weekly Dow Jones stock market index is an example of discrete-time signal.

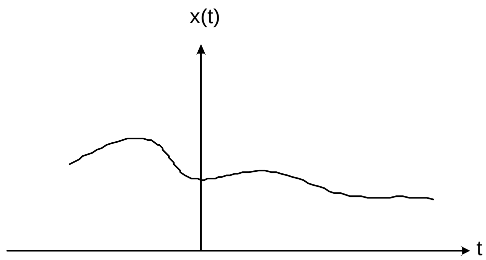


Fig. 1.1 Graphical representation of continuous-time signal.

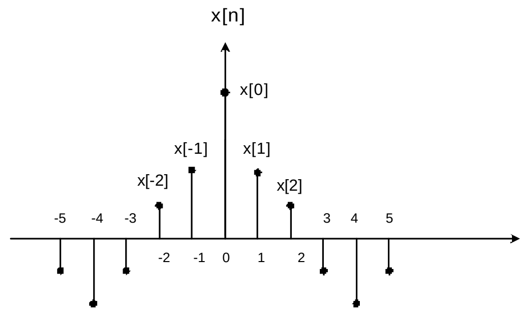


Fig. 1.2 Graphical representation of discrete-time signal.

To distinguish between continuous-time and discrete-time signals we use symbol t to denote the continuous variable and n to denote the discrete-time variable. And for continuous-time signals we will enclose the independent variable in parentheses $(\bullet)$, for discrete-time signals we will enclose the independent variable in bracket $[\bullet]$. ‘

A discrete-time signal $x[n]$ may represent a phenomenon for which the independent variable is inherently discrete. A discrete-time signal $x[n]$ may represent successive samples of an underlying phenomenon for which the independent variable is continuous. For example, the processing of speech on a digital computer requires the use of a discrete time sequence representing the values of the continuous-time speech signal at discrete points of time.

1.1.2 Signal Energy and Power

If $v(t)$ and $i(t)$ are respectively the voltage and current across a resistor with resistance R , then the instantaneous power is

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t) . \tag{1.1}$$

The total energy expended over the time interval $t_1 \le t \le t_2$ is

$$\int_{t_1}^{t_2} p(t)dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt , \tag{1.2}$$

and the average power over this time interval is

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt . \tag{1.3}$$

For any continuous-time signal $x(t)$ or any discrete-time signal $x[n]$, the total energy over the time interval $t_1 \le t \le t_2$ in a continuous-time signal $x(t)$ is defined as

$$\int_{t_1}^{t_2} |x(t)|^2 dt , \tag{1.4}$$

where $|x|$ denotes the magnitude of the (possibly complex) number x . The time-averaged power is $\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} |x(t)|^2 dt$. Similarly the total energy in a discrete-time signal $x[n]$ over the time interval $n_1 \le n \le n_2$ is defined as

$$\sum_{n_1}^{n_2} |x[n]|^2 \tag{1.5}$$

The average power is $\frac{1}{n_2 - n_1 + 1}\sum_{n_1}^{n_2} |x[n]|^2$

In many systems, we will be interested in examining the power and energy in signals over an infinite time interval, that is, for $-\infty \le t \le +\infty$ or $-\infty \le n \le +\infty$. The total energy in continuous time is then defined

$$E_\infty = \lim_{T\to\infty}\int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt , \tag{1.6}$$

and in discrete time

$$E_\infty = \lim_{N\to\infty}\sum_{-N}^{+N}|x[n]|^2 = \sum_{-\infty}^{+\infty}|x[n]|^2 . \qquad (1.7)$$

For some signals, the integral in Eq. (1.6) or sum in Eq. (1.7) might not converge, that is, if $x(t)$ or $x[n]$ equals a nonzero constant value for all time. Such signals have infinite energy, while signals with $E_\infty < \infty$ have finite energy.

The time-averaged power over an infinite interval

$$P_\infty = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2 dt \qquad (1.8)$$

$$P_\infty = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{-N}^{+N}|x[n]|^2 \qquad (1.9)$$

Three classes of signals:

- Class 1: signals with finite total energy, $E_\infty < \infty$ and zero average power, (Energy Signal)

$$P_\infty = \lim_{T\to\infty}\frac{E_\infty}{2T} = 0 \qquad (1.10)$$

- Class 2: with finite average power P_∞. If $P_\infty > 0$, then $E_\infty = \infty$. An example is the signal $x[n] = 4$, it has infinite energy, but has an average power of P_∞ =16. (Power Signal)

Class 3: signals for which neither P_∞ and E_∞ are finite. An example of this signal is $x(t) = t$.

1.2 Transformations of the independent variable

In many situations, it is important to consider signals related by a modification of the independent variable. These modifications will usually lead to reflection, scaling, and shift.

1.2.1 Examples of Transformations of the Independent Variable

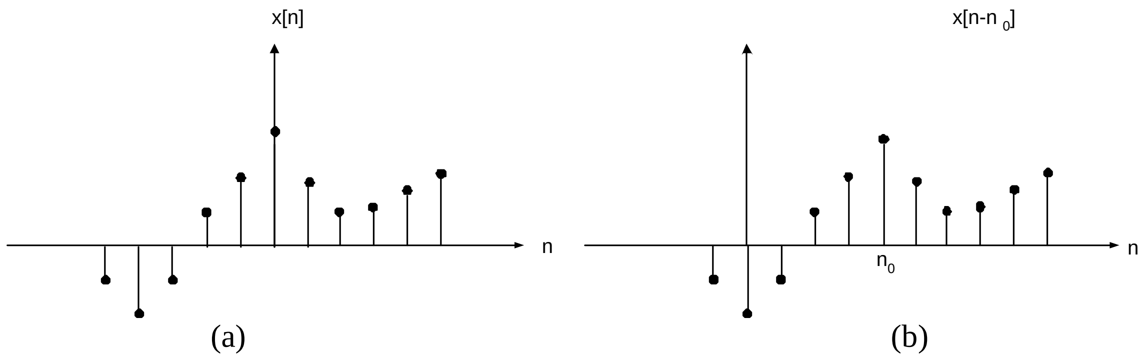


Fig.1.3 Discrete-time signals related by a time shift.

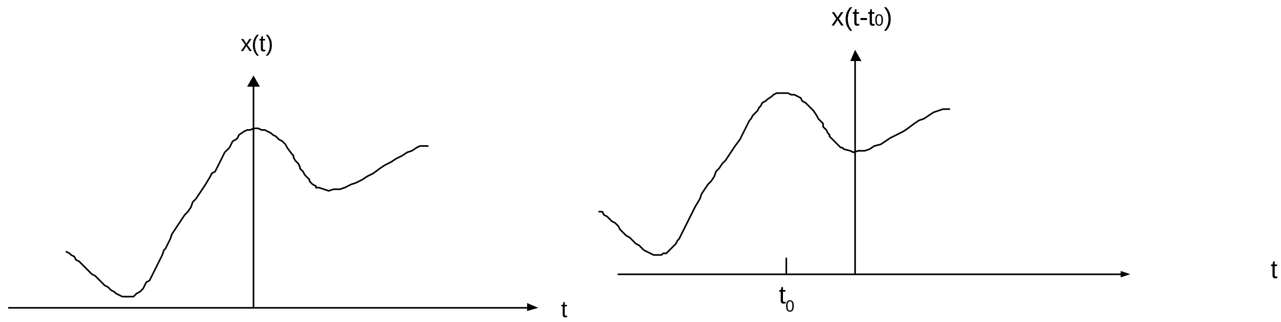


Fig. 1.4 Continuous-time signals related by a time shift.

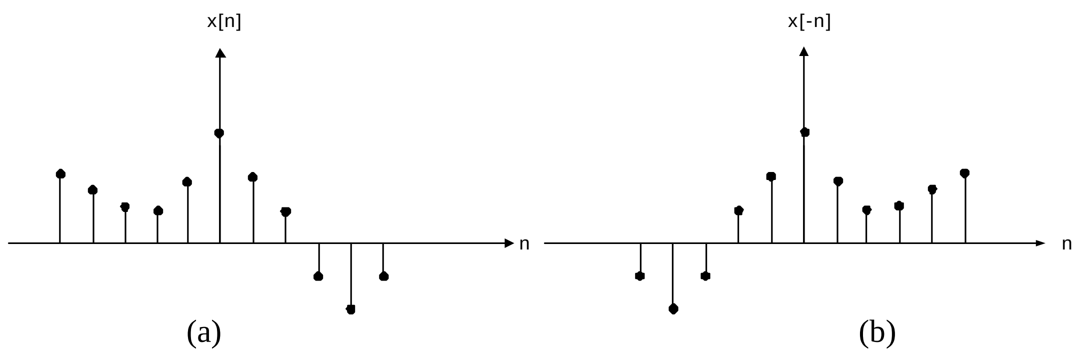


Fig. 1.5 (a) A discrete-time signal $x[n]$; (b) its reflection, $x[-n]$ about $n = 0$.

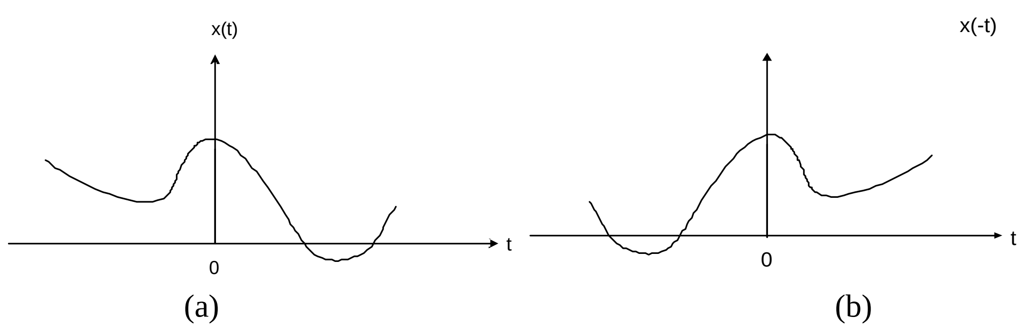


Fig. 1.6 (a) A continuous-time signal $x(t)$; (b) its reflection, $x(-t)$ about $t = 0$.

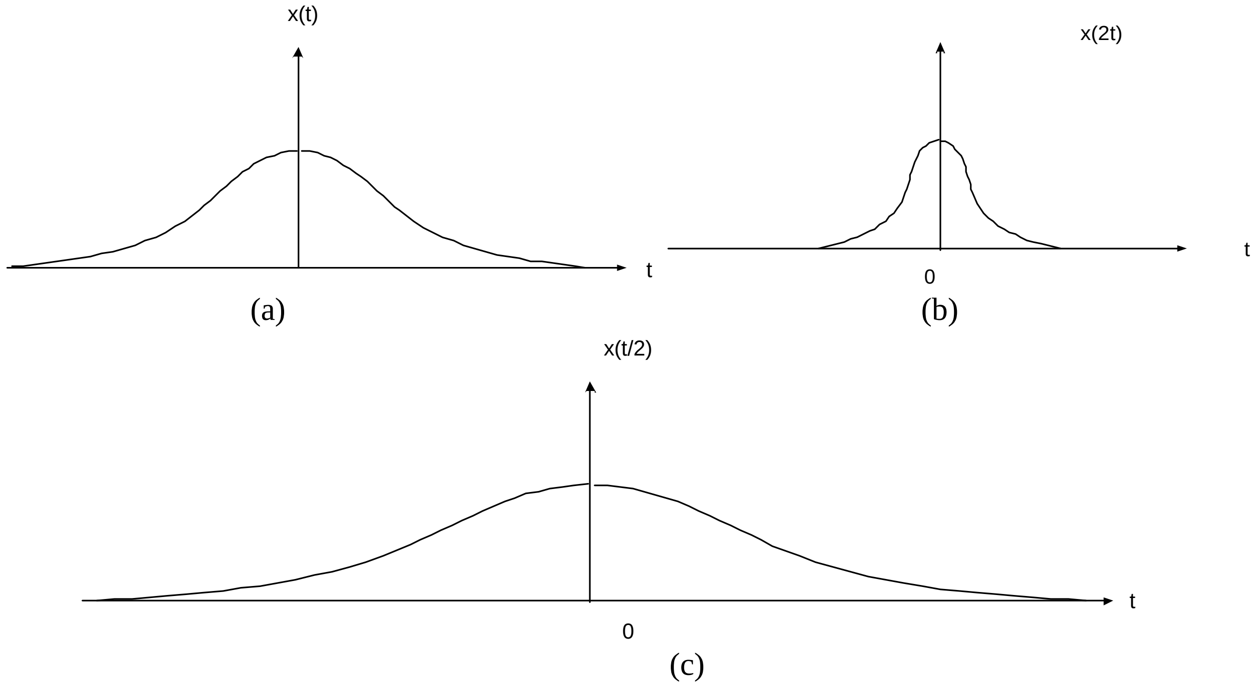


Fig. 1.7 Continuous-time signals related by time scaling.

1.2.2 Periodic Signals

A periodic continuous-time signal $x(t)$ has the property that there is a positive value of T for which

$$x(t) = x(t + T) \text{ for all } t \tag{1.11}$$

From Eq. (1.11), we can deduce that if $x(t)$ is periodic with period T, then $x(t) = x(t + mT)$ for all t and for all integers m. Thus, $x(t)$ is also periodic with period $2T$, $3T$, …. The fundamental period T_0 of $x(t)$ is the smallest positive value of T for which Eq. (1.11) holds.

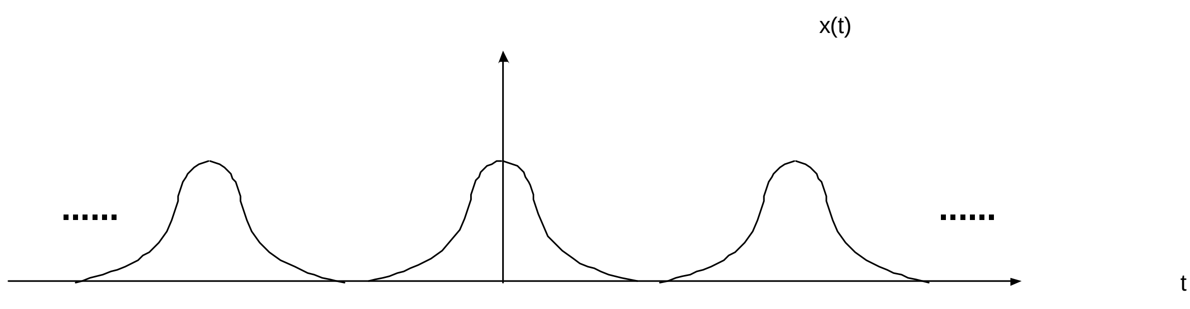


Fig. 1.8 Continuous-time periodic signal.

A discrete-time signal $x[n]$ is periodic with period N, where N is an integer, if it is unchanged by a time shift of N,

$$x[n] = x[n+N] \tag{1.12}$$

for all values of n. If Eq. (1.12) holds, then $x[n]$ is also periodic with period $2N$, $3N$, …. The fundamental period N_0 is the smallest positive value of N for which Eq. (1.12) holds.

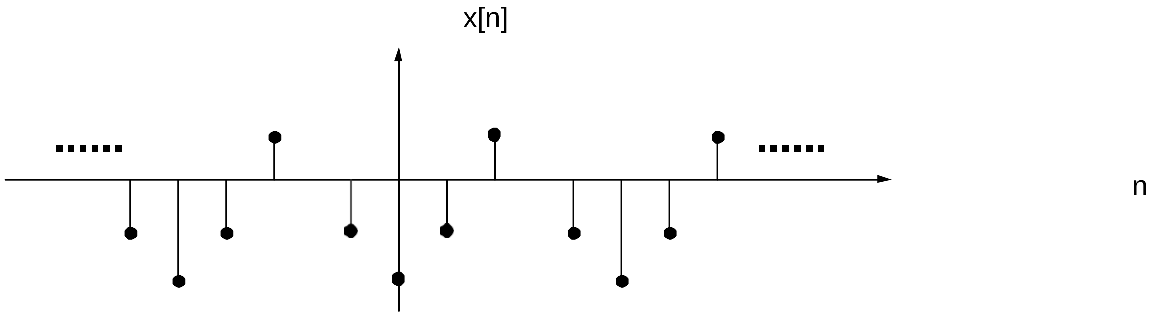


Fig. 1.9 Discrete-time periodic signal.

1.2.3 Even and Odd Signals

In addition to their use in representing physical phenomena such as the time shift in a radar signal and the reversal of an audio tape, transformations of the independent variable are extremely useful in examining some of the important properties that signal may possess.

Signal with these properties can be even or odd signal, periodic signal:

An important fact is that any signal can be decomposed into a sum of two signals, one of which is even and one of which is odd.

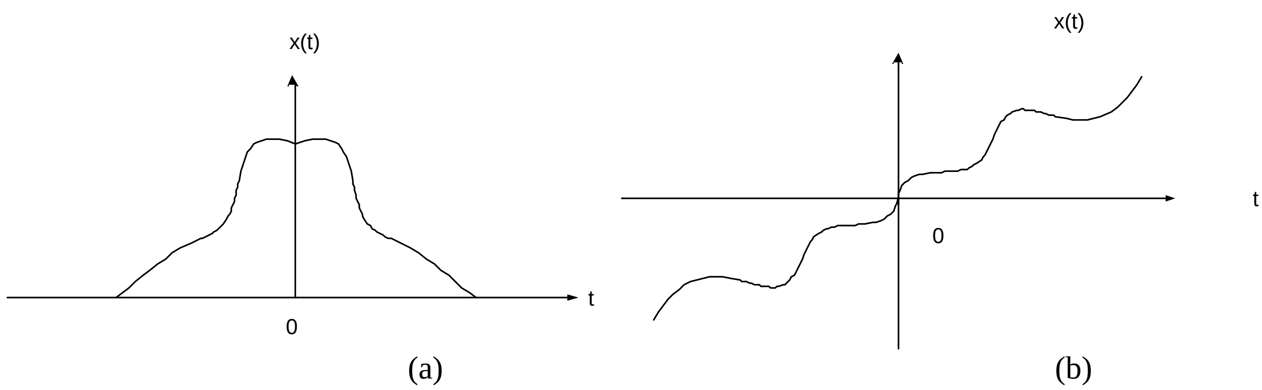


Fig. 1.10 An even continuous-time signal; (b) an odd continuous-time signal.

$$EV\{x(t)\} = \frac{1}{2}[x(t) + x(-t)] \quad (1.13)$$

which is referred to as the even part of $x(t)$. Similarly, the odd part of $x(t)$ is given by

$$OD\{x(t)\} = \frac{1}{2}[x(t) - x(-t)] \quad (1.14)$$

Exactly analogous definitions hold in the discrete-time case.

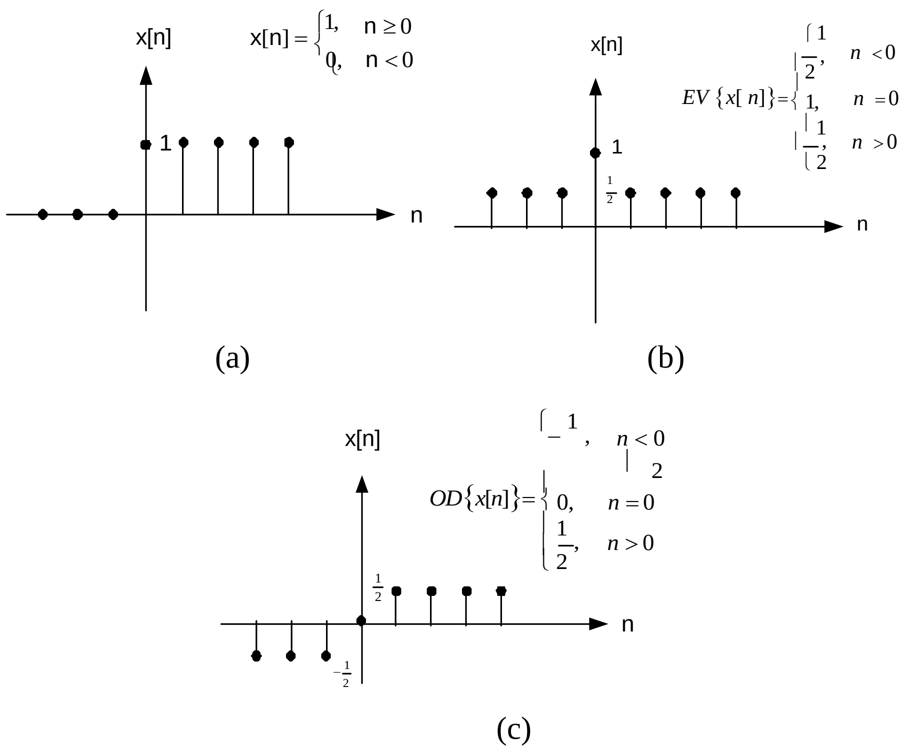


Fig.1.11 The even-odd decomposition of a discrete-time signal.

1.3 Exponential and sinusoidal signals

1.3.1 Continuous-time complex exponential and sinusoidal signals

The continuous-time complex exponential signal

$$x(t) = Ce^{at} \quad (1.15)$$

where C and a are in general complex numbers.

Real exponential signals

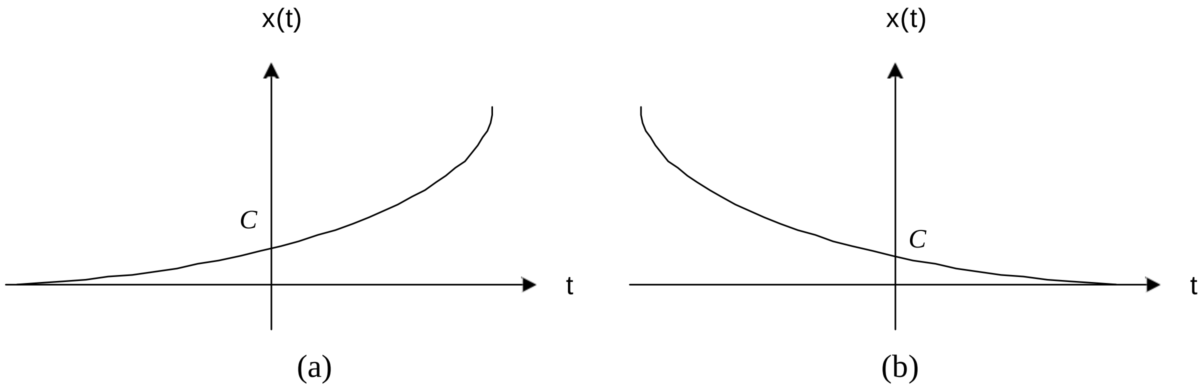


Fig. 1.12 The continuous-time complex exponential signal $x(t) = Ce^{at}$, (a) $a > 0$; (b) $a < 0$.

Periodic complex exponential and sinusoidal signals

If a is purely imaginary, we have

$$x(t) = e^{j\omega_0 t} \qquad (1.16)$$

An important property of this signal is that it is periodic. We know $x(t)$ is periodic with period T if

$$e^{j\omega_0 t} = e^{j\omega_0 (t+T)} = e^{j\omega_0 t} e^{j\omega_0 T} \qquad (1.17)$$

For periodicity, we must have

$$e^{j\omega_0 T} = 1 \qquad (1.18)$$

For $\xi_0 \neq 0$, the fundamental period T_0 is

$$T_0 = \frac{2\nu}{|\xi_0|} \qquad (1.19)$$

Thus, the signals $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ have the same fundamental period.

A signal closely related to the periodic complex exponential is the *sinusoidal signal*

$$x(t) = A\cos(\omega_0 t + \phi) \qquad (1.20)$$

With seconds as the unit of t, the units of $\exists$ and ξ_0 are radians and radians per second. It is also known $\xi_0 = 2\nu f_0$, where f_0 has the unit of circles per second or Hz.

The sinusoidal signal is also a periodic signal with a fundamental period of T_0.

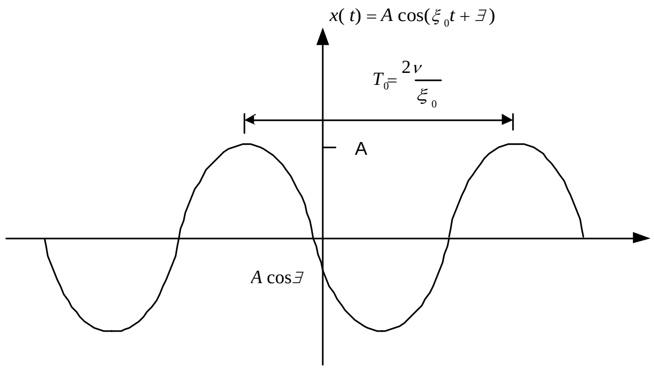


Fig. 1.13 Continuous-time sinusoidal signal.

Using Euler’s relation, a complex exponential can be expressed in terms of sinusoidal signals with the same fundamental period:

$$e^{j\xi_0 t} = \cos\xi_0 t + j\sin\xi_0 t \tag{1.21}$$

Similarly, a sinusoidal signal can also be expressed in terms of periodic complex exponentials with the same fundamental period:

$$A\cos(\xi_0 t + \exists) = \frac{A}{2} e^{j\exists} e^{j\xi_0 t} + \frac{A}{2} e^{-j\exists} e^{-j\xi_0 t} \tag{1.22}$$

A sinusoid can also be expresses as

$$A\cos(\xi_0 t + \exists) = A\mathrm{Re}\left\{ e^{j(\xi_0 t + \exists)} \right\} \tag{1.23}$$

and

$$A\sin(\xi_0 t + \exists) = A\,\mathrm{Im}\left\{ e^{j(\xi_0 t + \exists)} \right\} \tag{1.24}$$

Periodic signals, such as the sinusoidal signals provide important examples of signal with infinite total energy, but finite average power. For example:

$$E_{period} = \int_0^{T_0} \left| e^{j\xi_0 t} \right| dt = \int_0^{T_0} 1 dt = T_0 \tag{1.25}$$

$$P_{period} = \frac{1}{T_0} \int_0^{T_0} \left| e^{j\xi_0 t} \right| dt = \int_0^{T_0} 1 dt = 1 \tag{1.26}$$

Since there are an infinite number of periods as t ranges from $-\infty$ to $+\infty$, the total energy integrated over all time is infinite. The average power is finite since

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left| e^{j\xi_0 t} \right|^2 dt = 1 \tag{1.27}$$

Harmonically related complex exponentials:

$$\exists_k(t) = e^{jk\xi_0 t}, \quad k = 0, \pm 1, \pm 2, ... \tag{1.28}$$

ξ_0 is the fundamental frequency.

Example:

Signal $x(t) = e^{j2t} + e^{j3t}$ can be expressed as $x(t) = e^{j2.5t}(e^{-j0.5t} + e^{j0.5t}) = 2e^{j2.5t}\cos(0.5t)$, the magnitude of $x(t)$ is $|x(t)| = 2|\cos(0.5t)|$, which is commonly referred to as a full-wave rectified sinusoid, shown in Fig. 1.14.

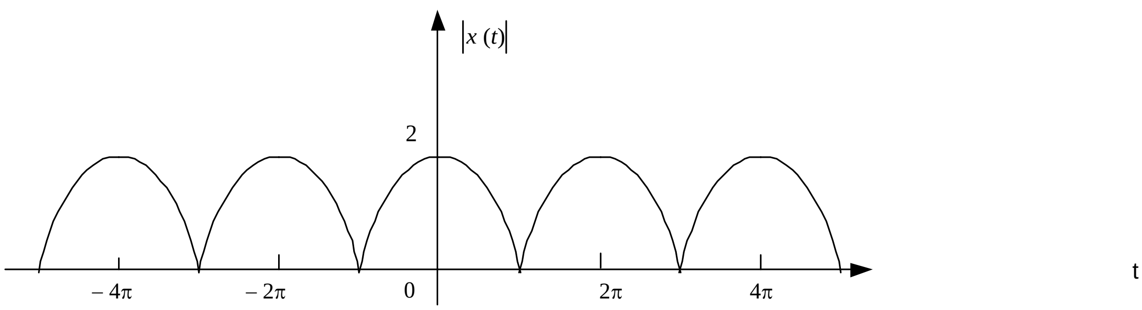


Fig. 1.14 Full-wave rectified sinusoid.

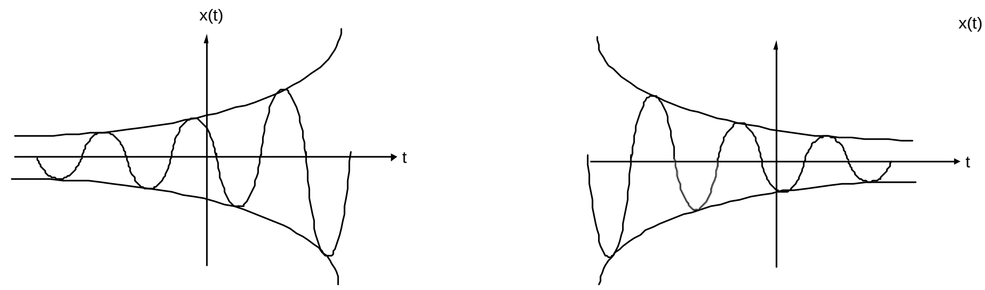
General complex Exponential signals

Consider a complex exponential Ce^{at}, where $C = |C|e^{j\theta}$ is expressed in polar and $a = r + j\xi_0$ is expressed in rectangular form. Then

$$Ce^{at} = |C|e^{j\theta}e^{(r+j\xi_0)t} = |C|e^{rt}e^{j(\xi_0 t+\theta)} = |C|e^{rt}\cos(\xi_0 t + \theta) + j|C|e^{rt}\sin(\xi_0 t + \theta). \tag{1.29}$$

Thus, for $r = 0$, the real and imaginary parts of a complex exponential are sinusoidal.
For $r > 0$, sinusoidal signals multiplied by a growing exponential. For $r < 0$, sinusoidal signals multiplied by a decaying exponential.

Damped signal – Sinusoidal signals multiplied by decaying exponentials are commonly refereed to as damped signal.



(a) (b)

Fig. 1.15 (a) Growing sinusoidal signal; (b) decaying sinusoidal signal.

1.3.2 Discrete-time complex exponential and sinusoidal signals

A discrete complex exponential or sequence is defined by

$$x[n] = C\alpha^n, \qquad (1.30)$$

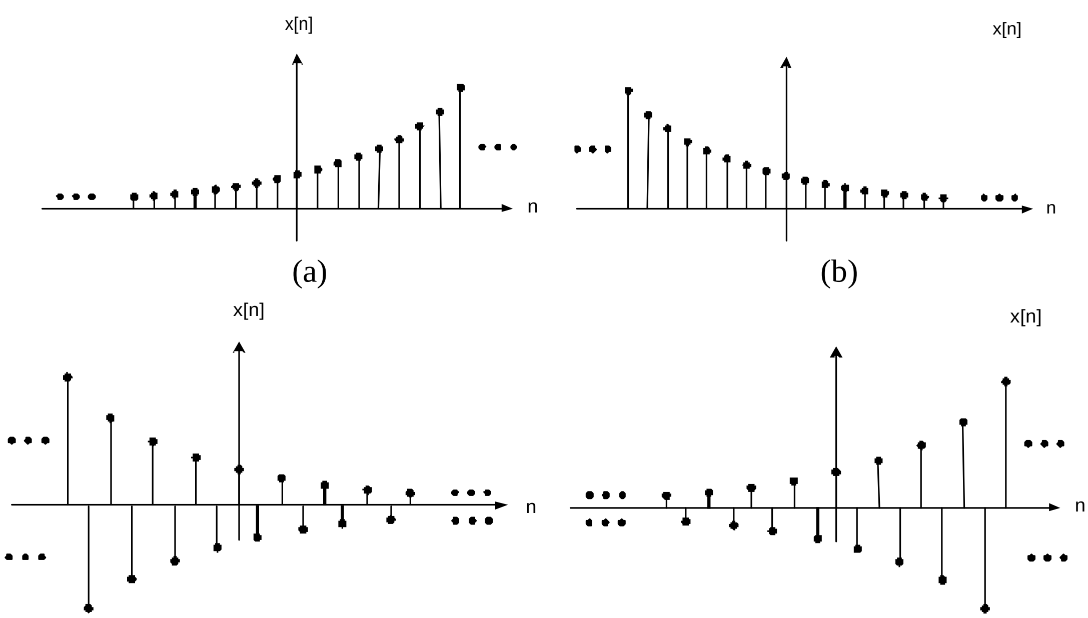
where C and α are in general complex numbers. This can be alternatively expressed

$$x[n] = Ce^{\beta n}, \qquad (1.31)$$

where $\alpha = e^{\beta}$.

Real Exponential Signals

If C and α are real, we have the real exponential signals.



(c) (d)

Fig. 1.16 Real Exponential Signal $x[n] = C\alpha^n$: (a) α >1; (b) 0<α<1; (c) –1<α<0; (d) α<-1.

Sinusoidal Signals

$$x[n] = e^{j\omega_0 n} \tag{1.32}$$

$$e^{j\omega_0 n} = \cos\omega_0 n + j\sin\omega_0 n \tag{1.33}$$

Similarly, a sinusoidal signal can also be expresses in terms of periodic complex exponentials with the same fundamental period:

$$A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n} \tag{1.34}$$

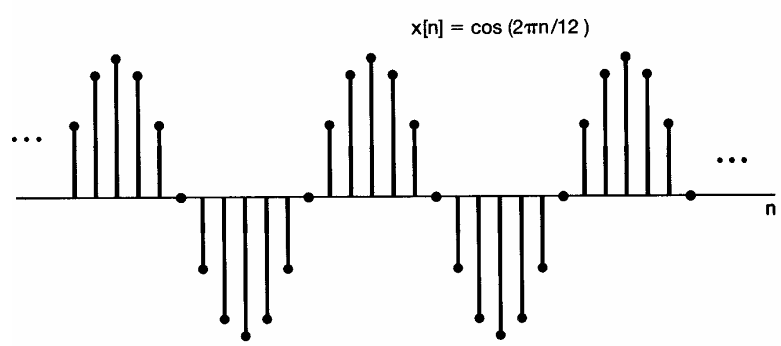
A sinusoid can also be expresses as

$$A\cos(\omega_0 n + \phi) = A\mathrm{Re}\left\{e^{j(\omega_0 n+\phi)}\right\} \tag{1.35}$$

and

$$A\sin(\omega_0 n + \phi) = A\mathrm{Im}\left\{e^{j(\omega_0 n+\phi)}\right\} \tag{1.36}$$

The above signals are examples of discrete signals with infinite total energy, but finite average power. For example: every sample of $x[n] = e^{j\omega_0 n}$ contributes 1 to the signal's energy. Thus the total energy $-\infty < n < +\infty$ is infinite, while the average power is equal to 1.



(a)

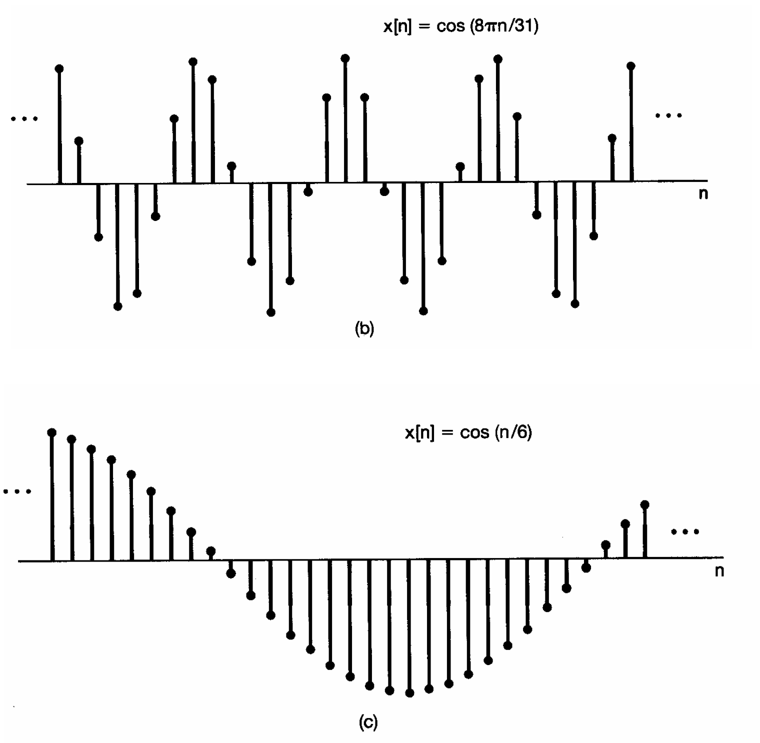


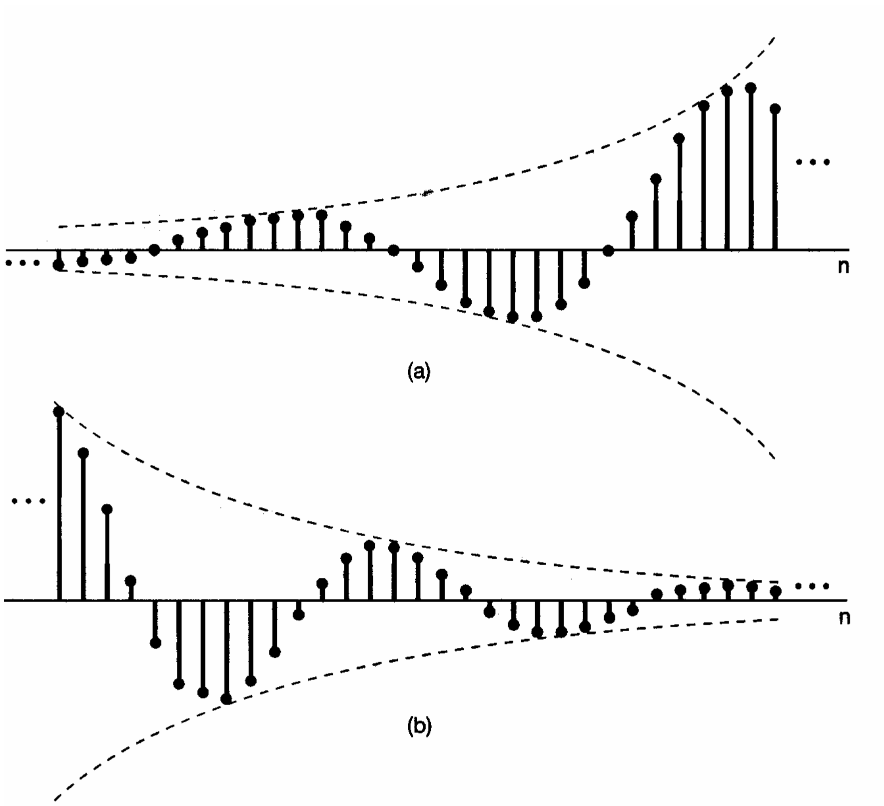
Fig.1.17 Discrete-time sinusoidal signal.

General Complex Exponential Signals

Consider a complex exponential $C\alpha^n$, where $C = |C|e^{j\theta}$ and $\alpha = |\alpha|e^{j\omega_0}$, then

$$C\alpha^n = |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin j(\omega_0 n + \theta). \tag{1.37}$$

Thus, for $|\alpha| = 1$, the real and imaginary parts of a complex exponential are sinusoidal.
For $|\alpha| < 1$, sinusoidal signals multiplied by a decaying exponential.
For $|\alpha| > 1$, sinusoidal signals multiplied by a growing exponential.



(a) (b)

Fig. 1.18 (a) Growing sinusoidal signal; (b) decaying sinusoidal signal.

1.3.3 Periodicity Properties of Discrete-Time Complex Exponentials

There are a number of important differences between continuous-time and discrete-time sinusoidal signals. The continuous-time signals $e^{j\omega_0 t}$ are distinct for distinct values of ξ_0. For discrete-time signals, however, these values are not distinct because the signal with ξ_0 is identical to the signals with frequencies $\xi_0 \pm 2\nu$, $\xi_0 \pm 4\nu$, and so on,

$$e^{j(\xi_0 \pm 2\nu)n} = e^{j(\xi_0 \pm 4\nu)n} = e^{j\xi_0 n}. \tag{1.38}$$

In considering discrete-time exponentials, we need only consider a frequency interval of 2ν. In most occasions, we will use the interval $0 \le \xi_0 < 2\nu$ or $-\nu \le \xi_0 < \nu$.

The discrete-time signal $x[n] = e^{j\omega_0 n}$ does not have a continuously increasing rate of oscillation as ξ_0 is increased in magnitude, but as ξ_0 is increased from 0, the signal oscillates more and more rapidly until ξ_0 reaches ν, and when ξ_0 is continuously increased, the rate of oscillation decreases until ξ_0 reaches 2ν. We conclude that the low-frequency discrete-time exponentials have values of ξ_0 near 0, 2ν, and any other even multiple of ν, while the high-frequencies are located near $\xi_0 = \pm\nu$ and other odd multiples of ν.

In order for the signal $x[n] = e^{j\omega_0 n}$ to be periodic with period $N > 0$, we must have

$$e^{j\xi_0 (n+N)} = e^{j\xi_0 n}, \tag{1.39}$$

or equivalently

$$e^{j\xi_0 N} = 1. \tag{1.40}$$

For Eq. (1.40) to hold, $\xi_0 N$ must be a multiple of 2ν. That is, there must be an integer *m* such that

$$\xi_0 N = 2\nu m, \tag{1.41}$$

or equivalently

$$\frac{\xi_0}{2\nu} = \frac{m}{N}. \tag{1.42}$$

From Eq. (1.40), $x[n] = e^{j\omega_0 n}$ is a periodic if $\xi_0 / 2\nu$ is a *rational number* and is not periodic otherwise.

The fundamental frequency of the discrete-time signal $x[n] = e^{j\omega_0 n}$ is

$$\frac{2\nu}{N} = \frac{\xi_0}{m}, \tag{1.43}$$

and the fundamental period of the signal can be

$$N = m\left(\frac{2\nu}{\xi_0}\right). \tag{1.44}$$

The comparison of the continuous-time and discrete-time signals are summarized in the table below:

Table 1 Comparison of the signals $e^{j\xi_0 t}$ and $e^{j\xi_0 n}$.

$e^{j\xi_0 t}$	$e^{j\xi_0 n}$
Distinct signals for distinct values of ξ_0	Identical signals for values of ξ_0 separated by multiples of 2ν
Periodic for any choice of ξ_0	Periodic only if $\xi_0 = 2\nu m / N$ for some integers $N > 0$ and m .
Fundamental frequency ξ_0	Fundamental frequency ξ_0 / m
Fundamental period $\xi_0 = 0$: undefined $\xi_0 \neq 0$: $\dfrac{2\nu}{\xi_0}$	Fundamental period $\xi_0 = 0$: undefined $\xi_0 \neq 0$: $m\left(\dfrac{2\nu}{\xi_0}\right)$

Example : Suppose that we wish to determine the fundamental period of the discrete-time signal

$$x[n] = e^{j(2\nu/3)n} + e^{j(3\nu/4)n} \qquad (1.45)$$

Solution:

The first exponential on the right hand side has a fundamental period of 3. The second exponential has a fundamental period of 8.

For the entire signal to repeat, each of the terms in Eq. (1.45) must go through an integer number of its own fundamental period. The smallest increment of *n* the accomplished this is 24. That is, over an interval of 24 points, the first term will have gone through 8 of its fundamental periods, and the second term through three of its fundamental periods, and the overall signal through exactly one of its fundamental periods.

Harmonically related periodic exponentials

$$\exists_k[n] = e^{jk(2\nu/N)n},\ k = 0, \pm 1, \ldots\ldots \qquad (1.46)$$

In the continuous-time case, all of the harmonically related complex exponentials $e^{jk(2\nu/N)t}$, $k = 0, \pm 1, \ldots\ldots..$, are distinct. But this is not the case for discrete-time signals:

$$\exists_{k+N}[n] = e^{j(k+N)(2\nu/N)n} = e^{j(k2\nu/N)n} e^{j2\nu n} = \exists_k[n] \qquad (1.47)$$

There are only *N* distinct period exponentials in the set given in Eq. (1.46).

1.4 The Unit Impulse and Unit Step Functions

The unit impulse and unit step functions in continuous and discrete time are considerably important in signal and system analysis.

1.4.1 The discrete-Time Unit Impulse and Unit Step Sequences

Discrete-time unit impulse is defined as

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}, \qquad (1.48)$$

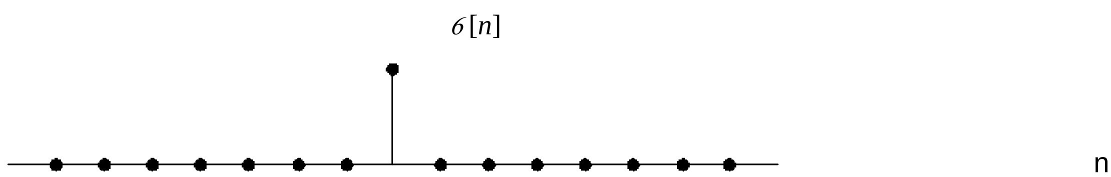


Fig. 1.19 Discrete-time unit impulse.

Discrete-time unit step is defined as

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}, \qquad (1.49)$$

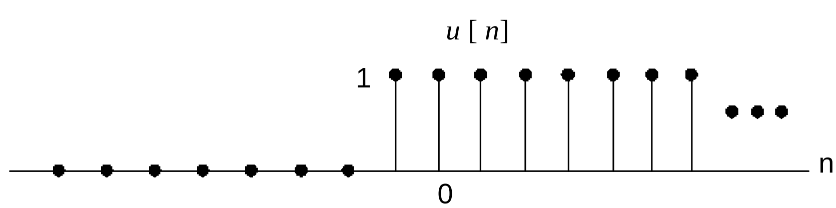


Fig. 1.20 Discrete-time unit step sequence.

The discrete-time impulse unit is the *first difference* of the discrete-time step

$$\delta[n] = u[n] - u[n-1], \qquad (1.50)$$

The discrete-time unit step is the *running sum* of the unit sample:

$$u[n] = \sum_{m=-\infty}^{n} \delta[m] \,, \tag{1.51}$$

It can be seen that for $n < 0$, the running sum is zero, and for $n \geq 0$, the running sum is 1.

If we change the variable of summation from m to $k = n - m$ we have, $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$.

The unit impulse sequence can be used to ***sample*** the value of a signal at $n = 0$. Since $\delta[n]$ is nonzero only for $n = 0$, it follows that

$$x[n]\delta[n] = x[0]\delta[n] \,. \tag{1.52}$$

More generally, a unit impulse $\delta[n - n_0]$, then

$$x[n]\delta[n - n_0] = x[n_0]\delta[n - n_0] \tag{1.53}$$

This ***sampling*** property is very important in signal analysis.

1.4.2 The Continuous-Time Unit Step and Unit Impulse Functions

Continuous-time unit step is defined as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \,, \tag{1.54}$$

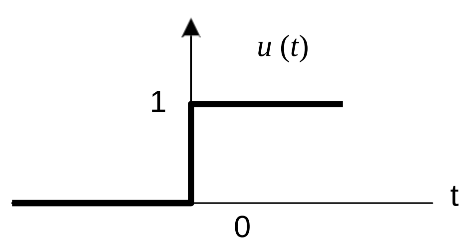


Fig. 1.21 Continuous-time unit step function. The continuous-time unit step

is the running integral of the unit impulse

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau \,. \tag{1.55}$$

The continuous-time unit impulse can also be considered as the first derivative of the continuous-time unit step,

$$\delta(t) = \frac{du(t)}{dt}. \tag{1.56}$$

Since $u(t)$ is discontinuous at $t = 0$ and consequently is formally not differentiable. This can be interpreted, however, by considering an approximation to the unit step $u_{\Delta}(t)$, as illustrated in the figure below, which rises from the value of 0 to the value 1 in a short time interval of length Δ.

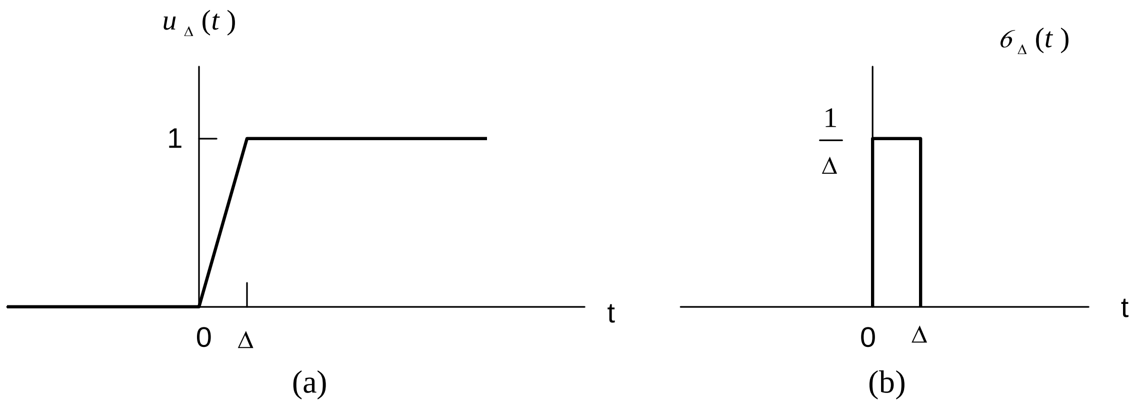


Fig. 1.22 (a) Continuous approximation to the unit step $u_{\Delta}(t)$; (b) Derivative of $u_{\Delta}(t)$.

The derivative is

$$\delta_{\Delta}(t) = \frac{du_{\Delta}(t)}{dt}, \tag{1.57}$$

$$\delta_{\Delta}(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & otherwise \end{cases}, \tag{1.58}$$

as shown in Fig. 1.22.

Note that $\delta_{\Delta}(t)$ is a short pulse, of duration Δ and with unit area for any value of Δ. As $\Delta \to 0$, $\delta_{\Delta}(t)$ becomes narrower and higher, maintaining its unit area. At the limit,

$$\delta(t) = \lim_{\Delta \to 0} \delta_{\Delta}(t), \tag{1.59}$$

$$u(t) = \lim_{\Delta \to 0} u_{\Delta}(t), \tag{1.60}$$

and

$$\delta(t) = \frac{du(t)}{dt} \,. \tag{1.61}$$

Graphically, $\delta(t)$ is represented by an arrow pointing to infinity at $t = 0$, "1" next to the arrow represents the area of the impulse.

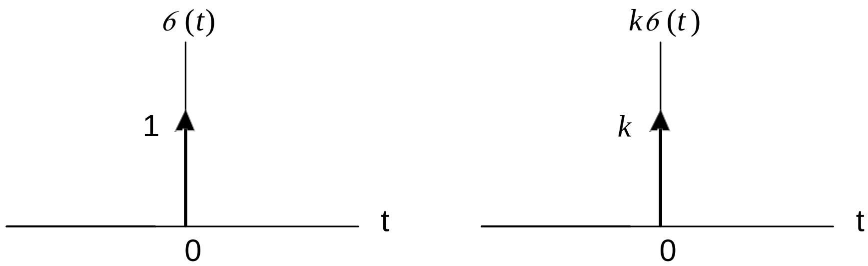


Fig. 1.23 Continuous-time unit impulse.

Sampling property of the continuous-time unit impulse:

$$x(t)\delta(t) = x(0)\delta(t) \,, \tag{1.62}$$

Or more generally,

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0) \tag{1.63}$$

Example:

Consider the discontinuous signal $x(t)$

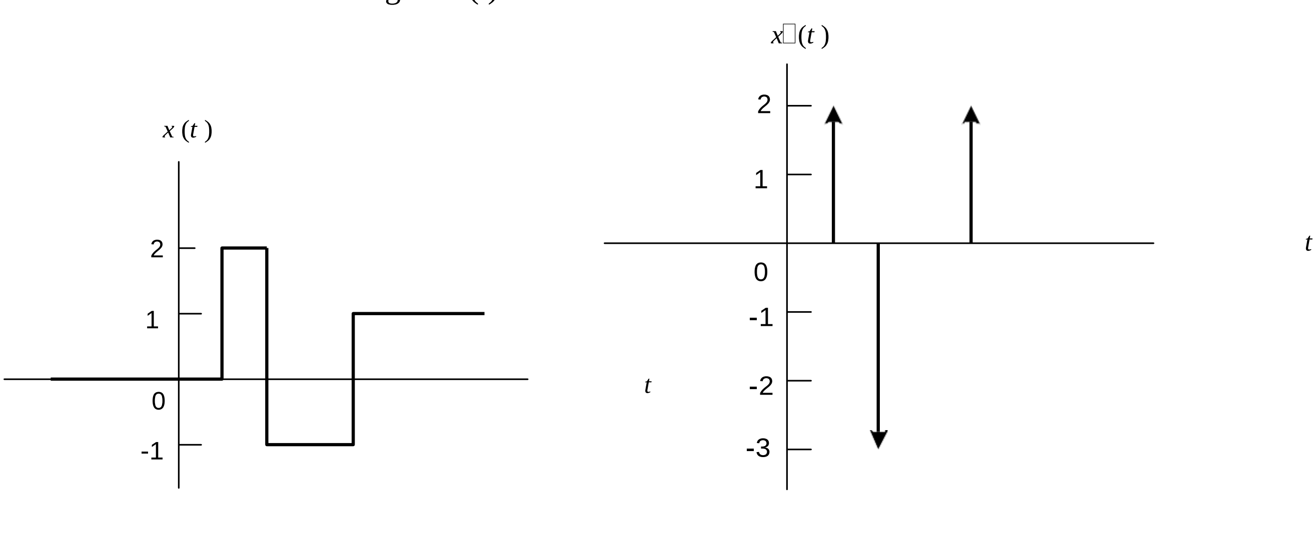


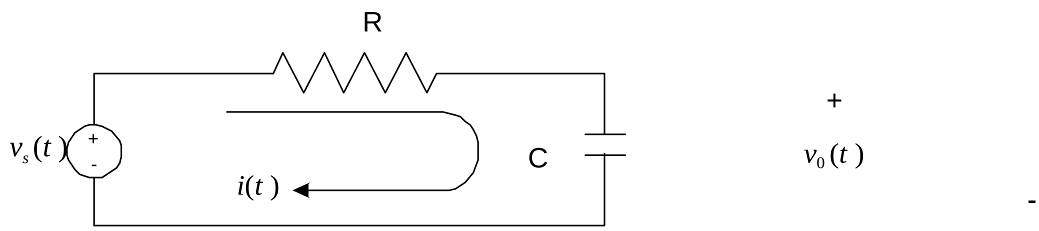
Fig. 1.24 The discontinuous signal and its derivative.

Note that the derivative of a unit step with a discontinuity of size of k gives rise to an impulse of area k at the point of discontinuity.

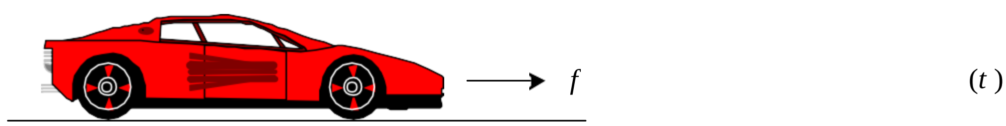
1.5 Continuous-Time and Discrete-Time Systems

A ***system*** can be viewed as a process in which input signals are transformed by the system or cause the system to respond in some way, resulting in other signals as outputs.

Examples



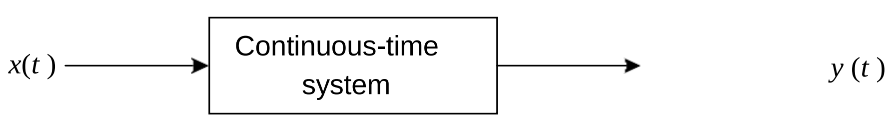
(a)



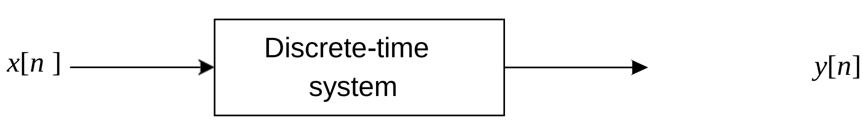
(a)

Fig. 1. 25 Examples of systems. (a) A system with input voltage $v_s(t)$ and output voltage $v_0(t)$.
(b) A system with input equal to the force $f(t)$ and output equal to the velocity $v(t)$.

A ***continuous-time system*** is a system in which continuous-time input signals are applied and results in continuous-time output signals.



A ***discrete-time system*** is a system in which discrete-time input signals are applied and results in discrete-time output signals.



1.5.2 Simple Examples of Systems

Example 1: Consider the RC circuit in Fig. 25 (a).

The current $i(t)$ is proportional to the voltage drop across the resistor:

$$i(t) = \frac{v_s(t) - v_C(t)}{R}. \tag{1.64}$$

The current through the capacitor is

$$i(t) = C\frac{dv_C(t)}{dt}. \tag{1.65}$$

Equating the right-hand sides of Eqs. 1.64 and 1.65, we obtain a differential equation describing the relationship between the input and output:

$$\frac{dv_C(t)}{dt} + \frac{1}{RC}v_C(t) = \frac{1}{RC}v_s(t), \tag{1.66}$$

Example 2: Consider the system in Fig. 25 (b), where the force $f(t)$ as the input and the velocity $v(t)$ as the output. If we let m denote the mass of the car and θv the resistance due to friction. Equating the acceleration with the net force divided by mass, we obtain

$$\frac{dv(t)}{dt} = \frac{1}{m}\left[f(t) - \theta v(t)\right] \quad \Rightarrow \quad \frac{dv(t)}{dt} + \frac{\theta}{m}v(t) = \frac{1}{m}f(t). \tag{1.67}$$

Eqs.1.66 and 1.77 are two examples of ***first-order linear differential equations*** of the form:

$$\frac{dy(t)}{dt} + ay(t) = bx(t). \tag{1.66}$$

Example 3: Consider a simple model for the balance in a bank account from month to month. Let $y[n]$ denote the balance at the end of nth month, and suppose that $y[n]$ evolves from month to month according the equation:

$$y[n] = 1.01y[n-1] + x[n], \tag{1.67}$$

or

$$y[n] - 1.01y[n-1] = x[n], \tag{1.68}$$

where $x[n]$ is the net deposit (deposits minus withdraws) during the nth month $1.01y[n-1]$ models the fact that we accrue 1% interest each month.

Example 4: Consider a simple digital simulation of the differential equation in Eq. (1.67), in which we resolve time into discrete intervals of length Δ and approximate $dv(t)/d(t)$ at $t = n\Delta$ by the first backward difference, i.e.,

$$\frac{v(n\Delta) - v((n-1)\Delta)}{\Delta}$$

Let $v[n] = v(n\Delta)$ and $f[n] = f(n\Delta)$, we obtain the following discrete-time model relating the sampled signals $v[n]$ and $f[n]$,

$$v[n] - \frac{m}{(m + \theta\Delta)} v[n-1] = \frac{\Delta}{(m + \theta\Delta)} f[n]. \qquad (1.69)$$

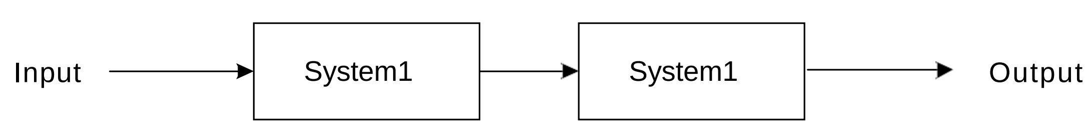
Comparing Eqs. 1.68 and 1.69, we see that they are two examples of the ***first-order linear difference equation***, that is,

$$y[n] + ay[n-1] = bx[n]. \qquad (1.70)$$

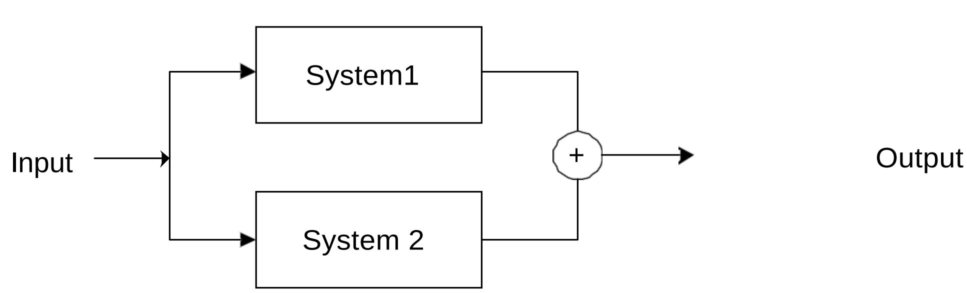
Some conclusions:

- Mathematical descriptions of systems have great deal in common;
- A particular class of systems is referred to ***as linear, time-invariant*** systems.
- Any model used in describing and analyzing a physical system represents an ***idealization*** of the system.

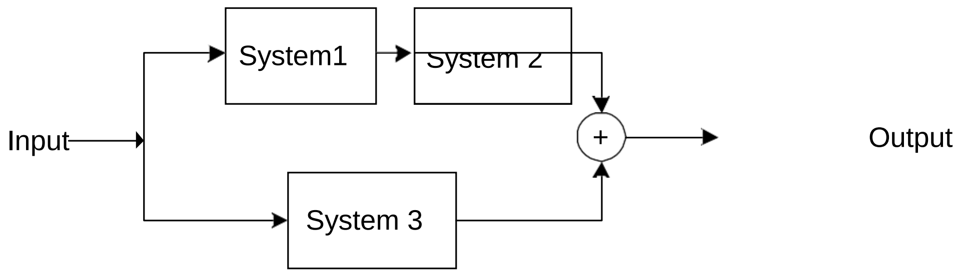
1.5.3 Interconnects of Systems



(a)



(b)



(c)

Fig. 1.26 Interconnection of systems. (a) A series or cascade interconnection of two systems; (b) A parallel interconnection of two systems; (c) Combination of both series and parallel systems.

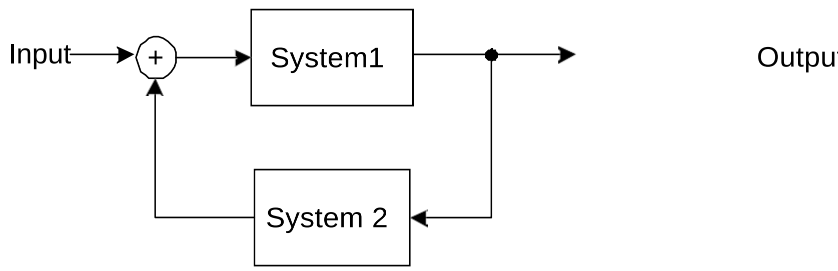
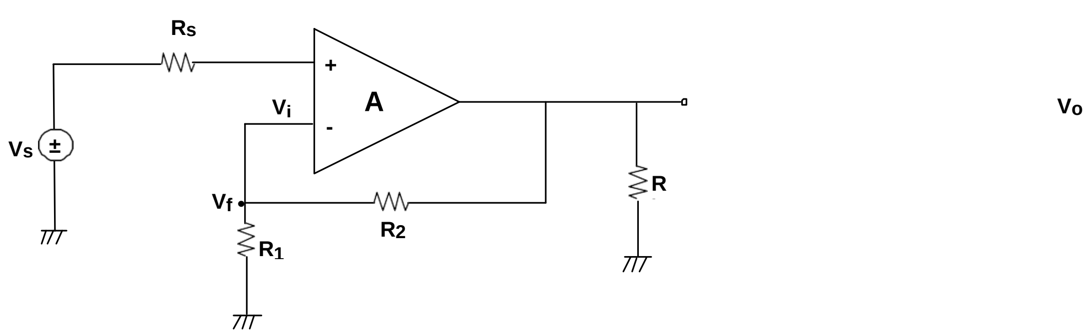
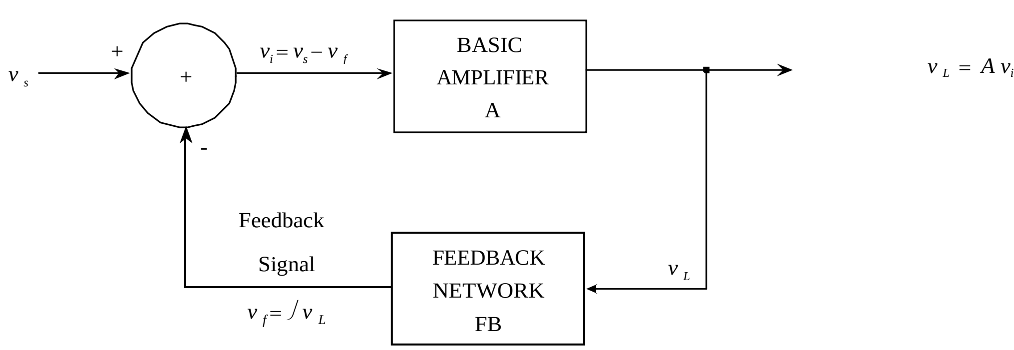


Fig. 1.27 Feedback interconnection.



(a)



(b)

Fig. 1.28 A feedback electrical amplifier.

1.6 Basic System Properties

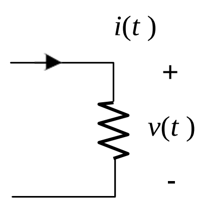
1.6.1 Systems with and without Memory

A system is ***memoryless*** if its output for each value of the independent variable as a given time is dependent only on the input at the same time. For example:

$$y[n] = (2x[n] - x^2[n])^2 , \tag{1.71}$$

is memoryless.

A resistor is a memoryless system, since the input current and output voltage has the relationship:



$$v(t) = Ri(t) , \tag{1.72}$$

where R is the resistance.

One particularly simple memoryless system is the ***identity system***, whose output is identical to its input, that is

$y(t) = x(t)$,   or $y[n] = x[n]$

An example of a discrete-time system with memory is an ***accumulator*** or ***summer***.

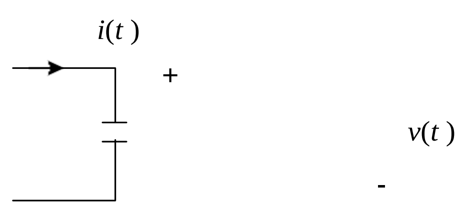
$$y[n] = \sum_{k=-\infty}^{n} x[k] = \sum_{k=-\infty}^{n-1} x[k] + x[n] = y[n-1] + x[n] \text{, or} \tag{1.73}$$

$$y[n] - y[n-1] = x[n] . \tag{1.74}$$

Another example is a ***delay***

$$y[n] = x[n-1] . \tag{1.75}$$

A capacitor is an example of a continuous-time system with memory,



$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau , \tag{1.76}$$

where C is the capacitance.

1.6.2 Invertibility and Inverse System

A system is said to be ***invertible*** if distinct inputs leads to distinct outputs.

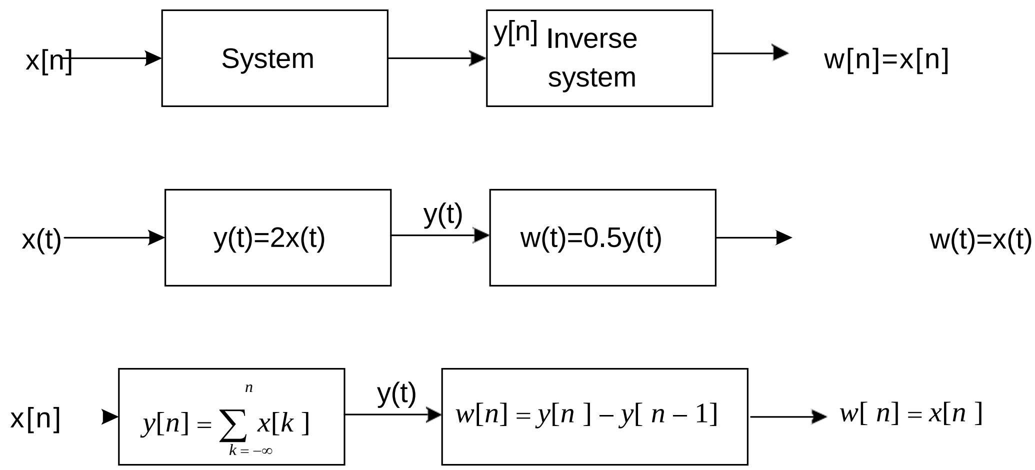


Fig. 1.29 Concept of an inverse system.

Examples of ***non-invertible systems***:

$y[n] = 0$,

the system produces zero output sequence for any input sequence.

$y(t) = x^2(t)$,

in which case, one cannot determine the sign of the input from the knowledge of the output.

Encoder in communication systems is an example of invertible system, that is, the input to the encoder must be exactly recoverable from the output.

1.6.3 Causality

A system is ***causal*** if the output at any time depends only on the values of the input at present time and in the past. Such a system is often referred to as being ***nonanticipative***, as the system output does not anticipate future values of the input.

The RC circuit in Fig. 25 (a) is causal, since the capacitor voltage responds only to the present and past values of the source voltage. The motion of a car is causal, since it does not anticipate future actions of the driver.

The following expressions describing systems that are not causal:

$$y[n] = x[n] - x[n+1], \tag{1.77}$$

and

$$y(t) = x(t+1). \tag{1.78}$$

All memoryless systems are causal, since the output responds only to the current value of input.

Example : Determine the Causality of the two systems:

(1) $y[n] = x[-n]$
(2) $y(t) = x(t)\cos(t+1)$

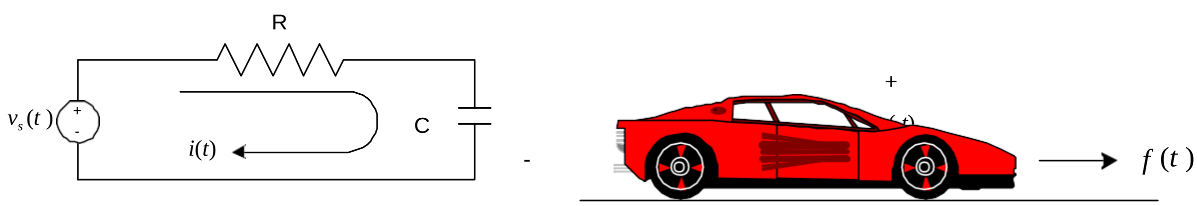
Solution: System (1) is not causal, since when $n < 0$, e.g. $n = -4$, we see that $y[-4] = x[4]$, so that the output at this time depends on a future value of input.

System (2) is causal. The output at any time equals the ***input at the same time*** multiplied by a number that varies with time.

1.6.4 Stability

A ***stable system*** is one in which small inp uts leads to responses that ***do not diverge***. More formally, if the input to a stable system is bounded, then the output must be also bounded and therefore cannot diverge.

Examples of stable systems and unstable systems:



(a) (b)

The above two systems are stable system.

The ***accumulator*** $y[n] = \sum_{k=-\infty}^{n} x[k]$ is not stable, since the sum grows continuously even if $x[n]$ is bounded.

Check the stability of the two systems:

- S1; $y(t) = tx(t)$;
- S2: $y(t) = e^{x(t)}$

- S1 is not stable, since a constant input $x(t) = 1$, yields $y(t) = t$, which is not bounded – no matter what finite constant we pick, $|y(t)|$ will exceed the constant for some *t*.

- S2 is stable. Assume the input is bounded $|x(t)| < B$, or $-B < x(t) < B$ for all *t*. We then see that $y(t)$ is bounded $e^{-B} < y(t) < e^{B}$.

1.6.5 Time Invariance

A system is ***time invariant*** if a time shift in the input signal results in an identical time shift in the output signal. Mathematically, if the system output is $y(t)$ when the input is $x(t)$, a time-invariant system will have an output of $y(t - t_0)$ when input is $x(t - t_0)$.

Examples:

- The system $y(t) = \sin[x(t)]$ is time invariant.

- The system $y[n] = nx[n]$ is not time invariant. This can be demonstrated by using ***counterexample***. Consider the input signal $x_1[n] = \delta[n]$, which yields $y_1[n] = 0$. However, the input $x_2[n] = \delta[n-1]$ yields the output $y_2[n] = n\delta[n-1] = \delta[n-1]$. Thus, while $x_2[n]$ is the shifted version of $x_1[n]$, $y_2[n]$ is not the shifted version of $y_1[n]$.

- The system $y(t) = x(2t)$ is not time invariant. To check using counterexample. Consider $x_1(t)$ shown in Fig. 1.30 (a), the resulting output $y_1(t)$ is depicted in Fig. 1.30 (b). If the input is shifted by 2, that is, consider $x_2(t) = x_1(t-2)$, as shown in Fig. 1.30 (c), we obtain the resulting output $y_2(t) = x_2(2t)$ shown in Fig. 1.30 (d). It is clearly seen that $y_2(t) \neq y_1(t-2)$, so the system is not time invariant.

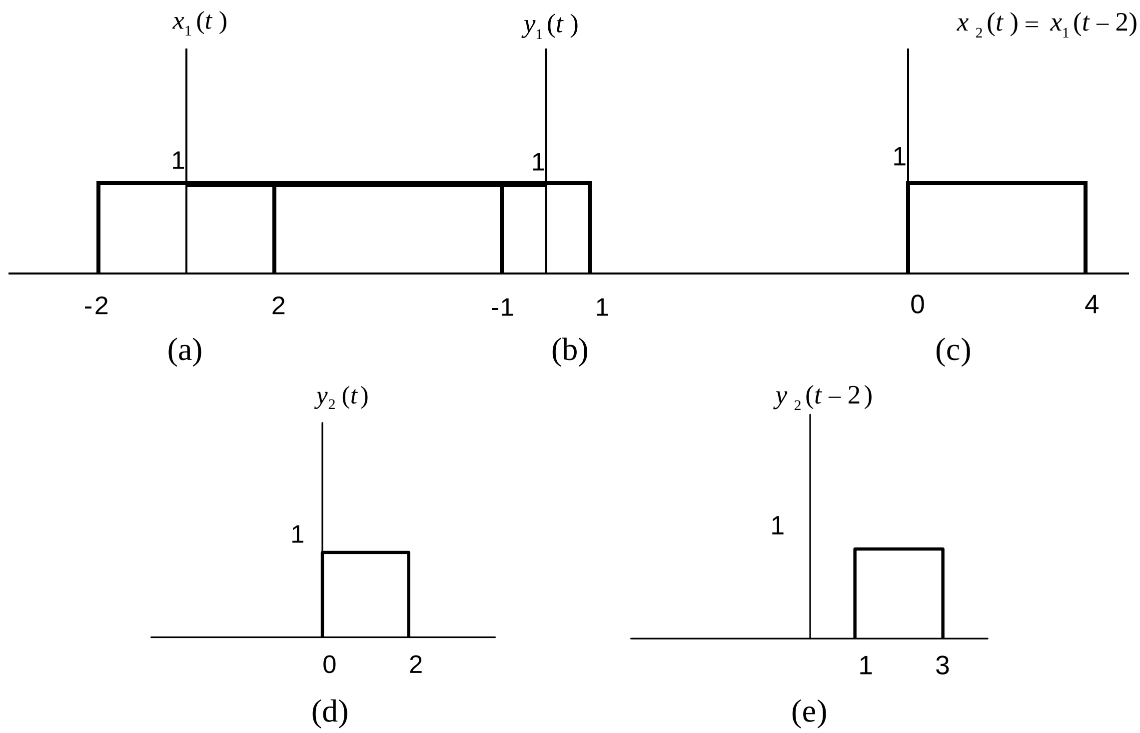


Fig. 1.30 Inputs and outputs of the system $y(t) = x(2t)$.

1.6.6 Linearity

The system is linear if

- The response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$ - ***additivity*** property
- The response to $ax_1(t)$ is $ay_1(t)$ - ***scaling*** or ***homogeneity*** property.

The two properties defining a linear system can be combined into a single statement:

- Continuous time: $ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t)$,
- Discrete time: $ax_1[n] + bx_2[n] \to ay_1[n] + by_2[n]$.

Here a and b are any complex constants.

Superposition property: If $x_k[n],\ k = 1, 2, 3, \ldots$ are a set of inputs with corresponding outputs $y_k[n],\ k = 1, 2, 3, \ldots$, then the response to a linear combination of these inputs given by

$$x[n] = \sum_k a_k x_k[n] = a_1 x_1[n] + a_2 x_2[n] + a_3 x_3[n] + \ldots , \qquad (1.79)$$

is

$$y[n]=\sum_k a_k y_k[n]=a_1 y_1[n]+a_2 y_2[n]+a_3 y_3[n]+\ldots, \qquad (1.80)$$

which holds for linear systems in both continuous and discrete time. For a linear system, ***zero input leads to zero output***.

Examples:

- The system $y(t)=tx(t)$ is a linear system.
- The system $y(t)=x^2(t)$ is not a liner system.
- The system $y[n]=\text{Re}\{x[n]\}$, is additive, but does not satisfy the homogeneity, so it is not a linear system.
- The system $y[n]=2x[n]+3$ is not linear. $y[n]=3$ if $x[n]=0$, the system violates the "zero-in/zero-out" property. However, the system can be represented as the sum of the output of a linear system and another signal equal to the zero-input response of the system. For system $y[n] = 2x[n] + 3$, the linear system is

  $x[n]\rightarrow 2x[n]$,

  and the zero-input response is

  $y_0[n]=3$

  as shown in Fig. 1.31.

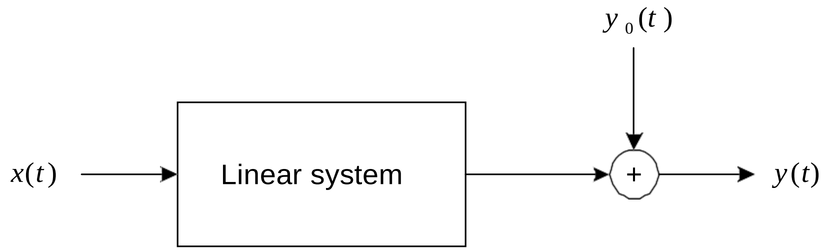


Fig. 1.31 Structure of an incrementally linear system. $y_0(t)$ is the zero-input response of the system.

The system represented in Fig. 1.31 is called incrementally linear system. The system responds linearly to the changes in the input.

The overall system output consists of the superposition of the response of a linear system with a zero-input response.

MODULE – II
FOURIER SERIES

Representation of Fourier series, Continuous time periodic signals, Properties of Fourier Series, Dirichlet‟s conditions, Trigonometric Fourier Series and Exponential Fourier Series, Complex Fourier spectrum.
Fourier Transforms: Deriving Fourier Transform from Fourier series, Fourier Transform of arbitrary signal, Fourier Transform of standard signals, Fourier Transform of Periodic Signals, Properties of Fourier Transform, Fourier Transforms involving Impulse function and Signum function, Introduction to Hilbert Transforms.

3.0 Introduction

- Signals can be represented using complex exponentials – ***continuous-time and discrete-time Fourier series and transform***.
- If the input to an LTI system is expressed as a linear comb ination of periodic complex exponentials or sinusoids, the output can also be expressed in this form.

3.1 A Historical Perspective

By 1807, Fourier had completed a work that series of harmonically related sinusoids were useful in representing temperature distribution of a body. He claimed that any periodic signal could be represented by such series – **Fourier Series**. He also obtained a representation for aperidic signals as weighted integrals of sinusoids **– Fourier Transform**.

Jean Baptiste Joseph Fourier

3.2 The Response of LTI Systems to Complex Exponentials

It is advantageous in the study of LTI systems to represent signals as linear combinations of basic signals that possess the following two properties:

- The set of basic signals can be used to construct a broad and useful class of signals.

- The response of an LTI system to each signal should be simple enough in structure to provide us with a convenient representation for the response of the system to any signal constructed as a linear combination of the basic signal.

Both of these properties are provided by Fourier analysis.

The importance of complex exponentials in the study of LTI system is that the response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude; that is

Continuous time: $e^{st} \to H(s)e^{st}$, (3.1)

Discrete-time: $z^{n} \to H(z)z^{n}$, (3.2)

where the complex amplitude factor $H(s)$ or $H(z)$ will be in general be a function of the complex variable s or z.

A signal for which the system output is a (possible complex) constant times the input is referred to as an ***eigenfunction*** of the system, and the amplitude factor is referred to as the system's ***eigenvalue***. Complex exponentials are eigenfunctions.

For an input $x(t)$ applied to an LTI system with impulse response of $h(t)$, the output is

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau \\ = \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau, \quad (3.3)$$

where we assume that the integral $\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$ converges and is expressed as

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau, \quad (3.4)$$

the response to e^{st} is of the form

$$y(t) = H(s)e^{st}, \quad (3.5)$$

It is shown the ***complex exponentials are eigenfunctions*** of LTI systems and $H(s)$ for a specific value of s is then the eigenvalues associated with the eigenfunctions.

Complex exponential sequences are eigenfunctions of discrete-time LTI systems. That is, suppose that an LTI system with impulse response $h[n]$ has as its input sequence

$$x[n] = z^n , \quad (3.6)$$

where z is a complex number. Then the output of the system can be determined from the convolution sum as

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k} . \quad (3.7)$$

Assuming that the summation on the right-hand side of Eq. (3.7) converges, the output is the same complex exponential multiplied by a consta nt that depends on the value of z . That is,

$$y[n] = H(z)z^n , \quad (3.8)$$

where $H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$. (3.9)

It is shown the ***complex exponentials are eigenfunctions*** of LTI systems and $H(z)$ for a specific value of z is then the eigenvalues associated with the eigenfunctions z^n .

The example here shows the usefulness of decomposing general signals in terms of eigenfunctions for LTI system analysis:

Let $x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$, (3.10)

from the eigenfunction property, the response to each separately is

$$a_1 e^{s_1 t} \rightarrow a_1 H_1(s_1) e^{s_1 t}$$

$$a_2 e^{s_2 t} \rightarrow a_2 H_2(s_2) e^{s_2 t}$$

$$a_3 e^{s_3 t} \rightarrow a_3 H_3(s_3) e^{s_3 t}$$

and from the superposition property the response to the sum is the sum of the responses,

$$y(t) = a_1 H_1(s_1) e^{s_1 t} + a_2 H_2(s_2) e^{s_2 t} + a_3 H_3(s_3) e^{s_3 t} , \quad (3.11)$$

Generally, if the input is a linear combination of complex exponentials,

$$x(t) = \sum_k a_k e^{s_k t} , \quad (3.12)$$

the output will be

$$y(t) = \sum_k a_k H(s_k) e^{s_k t} \ , \tag{3.13}$$

Similarly for discrete-time LTI systems, if the input is

$$x[n] = \sum_k a_k z_k^n \ , \tag{3.14}$$

the output is

$$y[n] = \sum_k a_k H(z_k) z_k^n \ , \tag{3.15}$$

3.3 Fourier Series representation of Continuous-Time Periodic Signals

3.31 Linear Combinations of harmonically Related Complex Exponentials

A periodic signal with period of T ,

$$x(t) = x(t+T) \text{ for all } t \ , \tag{3.16}$$

We introduced two basic periodic signals in Chapter 1, the sinusoidal signal

$$x(t) = \cos \xi_0 t \ , \tag{3.17}$$

and the periodic complex exponential

$$x(t) = e^{j\xi_0 t} \ , \tag{3.18}$$

Both these signals are periodic with fundamental frequency ξ_0 and fundamental period $T = 2\nu / \xi_0$. Associated with the signal in Eq. (3.18) is the set of *harmonically related* complex exponentials

$$\exists_k(t) = e^{jk\xi_0 t} = e^{jk(2\nu/T)t} \ , \qquad k = 0, \pm 1, \pm 2, \ldots\ldots \tag{3.19}$$

Each of these signals is periodic with period of T (although for $|k| \geq 2$, the fundamental period of $\exists_k(t)$ is a fraction of T). Thus, a linear combination of harmonically related complex exponentials of the form

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\xi_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\nu/T)t}, \tag{3.20}$$

is also periodic with period of T.

- $k = 0$, $x(t)$ is a constant.
- $k = +1$ and $k = -1$, both have fundamental frequency equal to ξ_0 and are collectively referred to as ***the fundamental components*** or ***the first harmonic components***.
- $k = +2$ and $k = -2$, the components are referred to as ***the second harmonic components***.
- $k = +N$ and $k = -N$, the components are referred to as ***the Nth harmonic components***.

Eq. (3.20) can also be expressed as

$$x(t) = x^*(t) = \sum_{k=-\infty}^{+\infty} a^*_k \; e^{-jk\xi_0 t}, \tag{3.21}$$

where we assume that $x(t)$ is real, that is, $x(t) = x^*(t)$.

Replacing k by $-k$ in the summation, we have

$$x(t) = \sum_{k=-\infty}^{+\infty} a^*_{-k} \; e^{jk\xi_0 t}, \tag{3.22}$$

which, by comparison with Eq. (3.20), requires that $a_k = a^*_{-k}$, or equivalently

$$a^*_k = a_{-k}. \tag{3.23}$$

To derive the alternative forms of the Fourier series, we rewrite the summation in Eq. (2.20) as

$$_0 \quad \sum_{k=1} a_k e \quad + a_{-k} e \quad]. \tag{3.24}$$

Substituting a^*_k for a_{-k}, we have

$$_0 \quad \sum_{k=1} a_k e \quad + a^*_k \, e \quad]. \tag{3.25}$$

Since the two terms inside the summation are complex conjugate of each other, this can be expressed as

$$x(t) = a_0 + \sum_{k=1}^{+\infty} 2\,\mathrm{Re}\left\{ a_k e^{jk\xi_0 t} \right\}. \tag{3.26}$$

If a_k is expressed in polar from as

$a_k = A_k e^{j\theta_k}$,

then Eq. (3.26) becomes

$$x(t) = a_0 + \sum_{k=1}^{+\infty} 2\mathrm{Re}\left\{A_k e^{j(k\xi_0 t + \theta_k)}\right\}.$$

That is

$$x(t) = a_0 + 2\sum_{k=1}^{+\infty} A_k \cos(k\xi_0 t + \theta_k). \qquad (3.27)$$

It is one commonly encountered form for the Fourier series of real periodic signals in continuous time.

Another form is obtained by writing a_k in rectangular form as

$a_k = B_k + jC_k$

then Eq. (3.26) becomes

$$x(t) = a_0 + 2\sum_{k=1}^{+\infty}\left[B_k \cos k\xi_0 t - C_k \sin k\xi_0 t\right]. \qquad (3.28)$$

For real periodic functions, the Fourier series in terms of complex exponential has the following *three* equivalent forms:

$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\xi_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\nu/T)t}$
$x(t) = a_0 + 2\sum_{k=1}^{+\infty} A_k \cos(k\xi_0 t + \theta_k)$
$x(t) = a_0 + 2\sum_{k=1}^{+\infty}\left[B_k \cos k\xi_0 t - C_k \sin k\xi_0 t\right]$

3.3.2 Determination of the Fourier Series Representation of a Continuous-Time Periodic Signal

Multiply both side of $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\xi_0 t}$ by $e^{-jn\xi_0 t}$, we obtain

$$x(t)e^{-jn\xi_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\xi_0 t} e^{-jn\xi_0 t} \quad , \tag{3.29}$$

Integrating both sides from 0 to $T = 2\nu / \xi_0$, we have

$$\int_0^T x(t)e^{-jn\xi_0 t} dt = \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{jk\xi_0 t} e^{-jn\xi_0 t} dt\right] = \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{j(k-n)\xi_0 t} dt\right], \tag{3.30}$$

Note that

$$\int_0^T e^{j(k-n)\xi_0 t} dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$$

So Eq. (3.30) becomes

$$a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\xi_0 t} dt , \tag{3.31}$$

The Fourier series of a periodic continuous-time signal

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\xi_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\nu/T)t} \tag{3.32}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\xi_0 t} dt = \frac{1}{T}\int_T x(t)e^{-jk(2\nu/T)t} dt \tag{3.33}$$

Eq. (3.32) is referred to as the ***Synthesis equation***, and Eq. (3.33) is referred to as ***analysis equation***. The set of coefficient $\{a_k\}$ are often called the Fourier series coefficients of the spectral coefficients of $x(t)$.

The coefficient a_0 is the ***dc*** or ***constant component*** and is given with $k = 0$, that is

$$a_0 = \frac{1}{T}\int_T x(t)dt \,, \tag{3.34}$$

Example: consider the signal $x(t) = \sin \xi_0 t$.

$$\sin \xi_0 t = \frac{1}{2j} e^{j\xi_0 t} - \frac{1}{2j} e^{-j\xi_0 t} .$$

Comparing the right-hand sides of this equation and Eq. (3.32), we have

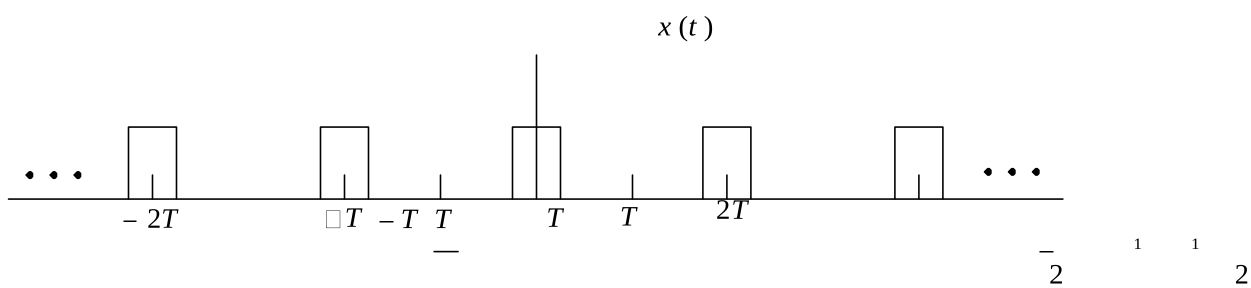
$a_1 = \frac{1}{2j},$ $\qquad\qquad a_{-1} = -\frac{1}{2j}$

$a_k = 0$, $\qquad\qquad k \neq +1 \; or -1$

Example : The periodic square wave, sketched in the figure below and define over one period is

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}, \tag{3.35}$$

The signal has a fundamental period T and fundamental frequency $\xi_0 = 2\nu / T$.



To determine the Fourier series coefficients for $x(t)$, we use Eq. (3.33). Because of the symmetry of $x(t)$ about $t = 0$, we choose $-T/2 \leq t \leq T/2$ as the interval over which the integration is performed, although any other interval of length T is valid the thus lead to the same result.

For $k = 0$,

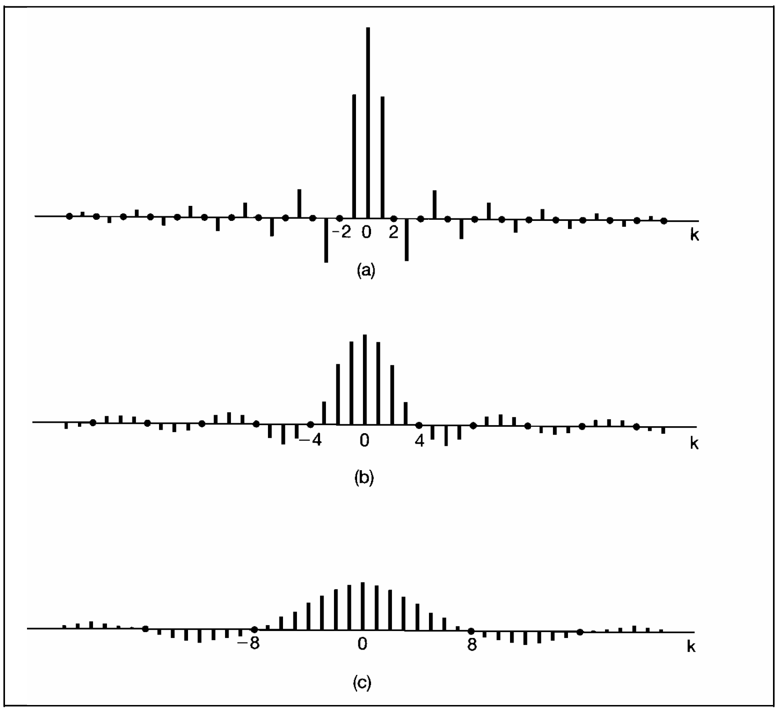
$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} x(t)dt = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T} , \tag{3.36}$$

For $k \neq 0$, we obtain

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\xi_0 t}dt = -\frac{1}{jk\xi_0 T} e^{-jk\xi_0 t}\Big|_{-T_1}^{T_1}$$

$$= \frac{2}{k\xi_0 T}\left[\frac{e^{jk\xi_0 T_1} - e^{-jk\xi_0 T_1}}{2j}\right] \qquad (3.37)$$

$$= \frac{2\sin(k\xi_0 T_1)}{k\xi_0 T} = \frac{\sin(k\xi_0 T_1)}{k\nu}$$



The above figure is a bar graph of the Fourier series coefficients for a fixed T_1 and several values of T. For this example, the coefficients are real, so they can be depicted with a single graph. For complex coefficients, two graphs corresponding to the real and imaginary parts or amplitude and phase of each coefficient, would be required.

3.4 Convergence of the Fourier Series

If a periodic signal $x(t)$ is approximated by a linear combination of finite number of harmonically related complex exponentials

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\xi_0 t}. \qquad (3.38)$$

Let $e_N(t)$ denote the approximation error,

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{N} a_k e^{jk\xi_0 t} . \tag{3.39}$$

The criterion used to measure quantitatively the approximation error is the energy in the error over one period:

$$E_N = \int_T |e_N(t)|^2 dt . \tag{3.40}$$

It is shown (problem 3.66) that the particular choice for the coefficients that minimize the energy in the error is

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\xi_0 t} dt . \tag{3.41}$$

It can be seen that Eq. (3.41) is identical to the expression used to determine the Fourier series coefficients. Thus, if $x(t)$ has a Fourier series representation, the best approximation using only a finite number of harmonically related complex exponentials is obtained by truncating the Fourier series to the desired number of terms.

The limit of E_N as $N \to \infty$ is zero.

One class of periodic signals that are representable through Fourier series is those signals which have finite energy over a period,

$$\int_T |x(t)|^2 dt < \infty , \tag{3.42}$$

When this condition is satisfied, we can guarantee that the coefficients obtained from Eq. (3.33) are finite. We define

$$e(t) = x(t) - \sum_{k=-\infty}^{\infty} a_k e^{jk\xi_0 t} , \tag{3.43}$$

then

$$\int_T |e(t)|^2 dt = 0 , \tag{3.44}$$

The convergence guaranteed when $x(t)$ has finite energy over a period is very useful. In this case, we may say that $x(t)$ and its Fourier series representation are ***indistinguishable***.

Alternative set of conditions developed by Dirichlet that guarantees the equivalence of the signal and its Fourier series representation:

Condition 1: Over any period, $x(t)$ must be absolutely integrable, that is

$$\int_T |x(t)| dt < \infty , \tag{3.45}$$

This guarantees each coefficient a_k will be finite, since

$$|a_k| = \frac{1}{T}\int_T |x(t)e^{-jk\xi_0 t}| dt = \frac{1}{T}\int_T |x(t)| dt < \infty . \tag{3.46}$$

A periodic function that violates the first Dirichlet condition is

$$x(t) = \frac{1}{t}, \qquad 0 < t < 1.$$

Condition 2: In any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during a single period of the signal. An example of a function that meets Condition1 but not Condition 2:

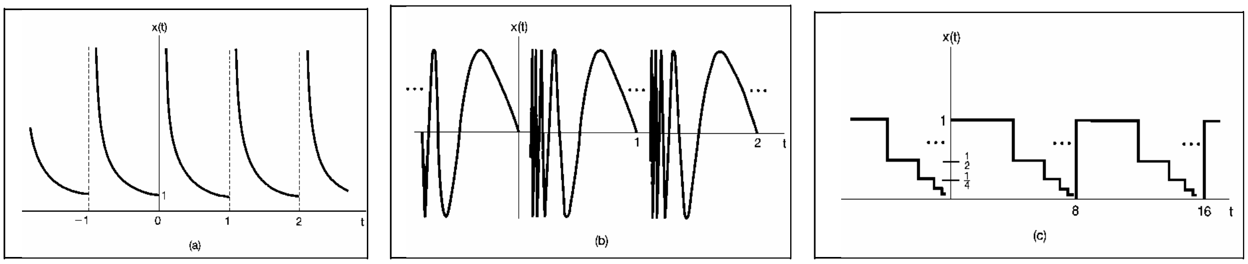
$$x(t) = \sin\left(\frac{2\nu}{t}\right), \qquad 0 < t \le 1, \tag{3.47}$$

Condition 3: In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

An example that violates this condition is a function defined as

$x(t) = 1$, $0 \le t < 4$, $x(t) = 1/2$, $4 \le t < 6$, $x(t) = 1/4$, $6 \le t < 7$, $x(t) = 1/8$, $7 \le t < 7.5$, etc.

The above three examples are shown in the figure below.



(a) (b) (c)

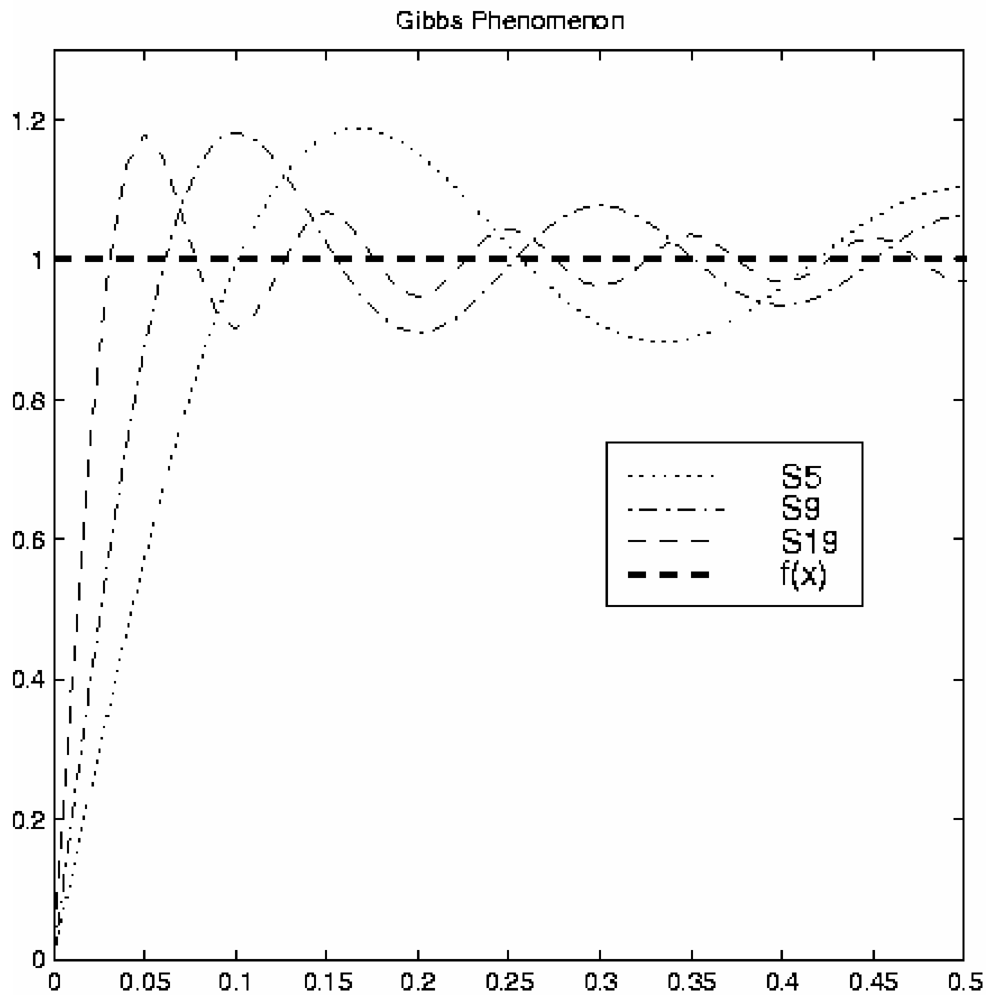
The above are generally pathological in nature and consequently do not typically arise in practical contexts.

Summary:

- For a periodic signal that has no discontinuities, the Fourier series representation converges and equals to the original signal at all the values of t .
- For a periodic signal with a finite number of discontinuities in each period, the Fourier series representation equals to the original signal at all the values of t except the isolated points of discontinuity.

Gibbs Phenomenon:

Near a point, where $x(t)$ has a jump discontinuity, the partial sums $x_N(t)$ of a Fourier series exhibit a substantial overshoot near these endpoints, and an increase in N will not diminish the amplitude of the overshoot, although with increasing N the overshoot occurs over smaller and smaller intervals. This phenomenon is called Gibbs phenomenon.



A large enough value of N should be chosen so as to guarantee that the total energy in these ripples is insignificant.

3.5 Properties of the Continuous-Time Fourier Series

Notation: suppose $x(t)$ is a periodic signal with period T and fundamental frequency ξ_0. Then if the Fourier series coefficients of $x(t)$ are denoted by a_k, we use the notation

$$x(t) \xleftrightarrow{FS} a_k,$$

to signify the pairing of a periodic signal with its Fourier series coefficients.

3.5.1 Linearity

Let $x(t)$ and $y(t)$ denote two periodic signals with period T and which have Fourier series coefficients denoted by a_k and b_k, that is

$$x(t) \xleftrightarrow{FS} a_k \text{ and } y(t) \xleftrightarrow{FS} b_k,$$

then we have

$$z(t) = Ax(t) + By(t) \xleftrightarrow{FS} c_k = Aa_k + Bb_k . \qquad (3.48)$$

3.5.2 Time Shifting

When a time shift to a periodic signal $x(t)$, the period T of the signal is preserved.

If $x(t) \xleftrightarrow{FS} a_k$, then we have

$$x(t - t_0) \xleftrightarrow{FS} e^{-jk\xi_0 t} a_k . \qquad (3.49)$$

The magnitudes of its Fourier series coefficients remain unchanged.

3.4.3 Time Reversal

If $x(t) \xleftrightarrow{FS} a_k$, then

$$x(-t) \xleftrightarrow{FS} a_{-k} . \qquad (3.50)$$

Time reversal applied to a continuous-time signal results in a time reversal of the corresponding sequence of Fourier series coefficients.

If $x(t)$ is even, that is $x(t) = x(-t)$, the Fourier series coefficients are also even, $a_{-k} = a_k$. Similarly, if $x(t)$ is odd, that is $x(-t) = -x(t)$, the Fourier series coefficients are also odd, $a_{-k} = -a_k$.

3.5.4 Time Scaling

If $x(t)$ has the Fourier series representation $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\xi_0 t}$, then the Fourier series representation of the time -scaled signal $x(\alpha t)$ is

$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\alpha \xi_0)t}. \quad (3.51)$$

The Fourier series coefficients have not changes, the Fourier series representation has changed because of the change in the fundamental frequency.

3.5.5 Multiplication

Suppose $x(t)$ and $y(t)$ are two periodic signals with period T and that

$$x(t) \overset{FS}{\longleftrightarrow} a_k,$$

$$y(t) \overset{FS}{\longleftrightarrow} b_k.$$

Since the product $x(t)\,y(t)$ is also periodic with period T, its Fourier series coefficients h_k is

$$x(t)y(t) \overset{FS}{\longleftrightarrow} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}. \quad (3.52)$$

The sum on the right-hand side of Eq. (3.52) may be interpreted as the discrete-time convolution of the sequence representing the Fourier coefficients of $x(t)$ and the sequence representing the Fourier coefficients of $y(t)$.

3.5.6 Conjugate and Conjugate Symmetry

Taking the complex conjugate of a periodic signal $x(t)$ has the effect of complex conjugation and time reversal on the corresponding Fourier series coefficients. That is, if

$x(t) \overset{FS}{\longleftrightarrow} a_k$, then

$$x^*(t) \overset{FS}{\longleftrightarrow} a^*_{-k}. \quad (3.53)$$

If $x(t)$ is real, that is, $x(t) = x^*(t)$, the Fourier series coefficients will be ***conjugate symmetric***, that is

$$a_{-k} = a^*_k. \quad (3.54)$$

From this expression, we may get various symmetry properties for the magnitude, phase, real parts and imaginary parts of the Fourier series coefficients of real signals. For example:

- From Eq. (3.54), we see that if $x(t)$ is real, a_0 is real and $|a_{-k}| = |a_k|$.
- If $x(t)$ is real and even, we have $a_k = a_{-k}$, from Eq. (3.54) $a_{-k} = a^*_k$, so $a_k = a^*_k \Rightarrow$ the Fourier series coefficients are real and even.
- If $x(t)$ is real and odd, the Fourier series coefficients are real and odd.

3.5.7 Parseval's Relation for Continuous-Time periodic Signals

Parseval's Relation for Continuous-Time periodic Signals is

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2 , \qquad (3.55)$$

Since

$$\frac{1}{T}\int_T \left|a_k e^{jk\xi_0 t}\right|^2 dt = \frac{1}{T}\int_T |a_k|^2 dt = |a_k|^2 ,$$

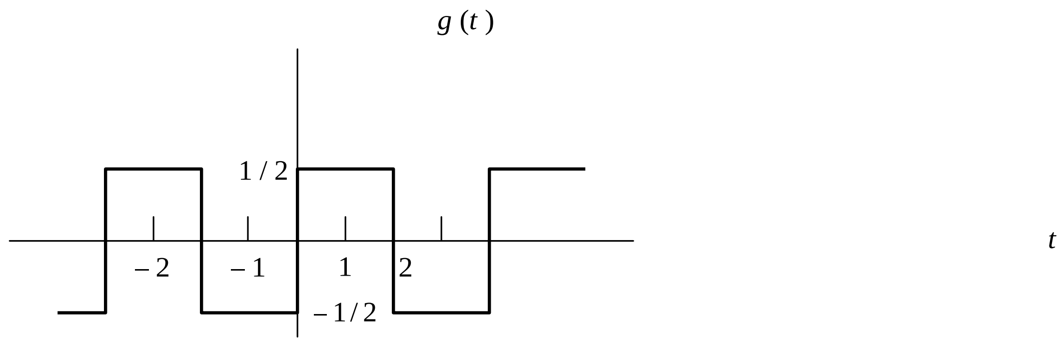
so that $|a_k|^2$ is the average power in the kth harmonic component.

Thus, Parseval's Relation states that the total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

3.5.8 Summary of Properties of the Continuous-Time Fourier Series

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{matrix} x(t) \\ y(t) \end{matrix}\right\}$ Periodic with period T and fundamental frequency $\xi_0 = 2\nu/T$	a_k b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	$x(t - t_0)$	$e^{-jk\xi_0 t}a_k$
Frequency shifting	$e^{jM\xi_0 t}x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a^*_{-k}
Time Reversal	$x(-t)$	a_{-k}
Time Scaling	$x(\alpha t)$, $\alpha > 0$ (Periodic with period T/α)	a_k
Periodic Convolution	$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	$x(t)\,y(t)$	$\sum_{l=-\infty}^{\infty} a_l b_{k-l}$
Differentiation	$\frac{dx(t)}{dt}$	$jk\xi_0 a_k = jk\frac{2\nu}{T}a_k$
Integration	$\int_{-\infty}^{t} x(t)dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\xi_0}\right)a_k = \left(\frac{1}{jk(2\nu/T)}\right)a_k$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathrm{Re}\{a_k\} = \mathrm{Re}\{a_{-k}\} \\ \mathrm{Im}\{a_k\} = -\mathrm{Im}\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \angle a_k = -\angle a_{-k} \end{cases}$
Real and Even Signals Real and Odd Signals Even-Odd Decomposition of Real Signals	$x(t)$ real and even $x(t)$ real and odd $\begin{cases} x_e(t) = Ev\{x(t)\} & [x(t)\ real] \\ x_e(t) = Od\{x(t)\} & [x(t)\ real] \end{cases}$	a_k real and even a_k purely imaginary and odd $\mathrm{Re}\{a_k\}$ $j\,\mathrm{Im}\{a_k\}$
	Parseval's Relation for Periodic Signals $\frac{1}{T}\int_T \lvert x(t)\rvert^2 dt = \sum_{k=-\infty}^{\infty} \lvert a_k\rvert^2$	

Example : Consider the signal $g(t)$ with a fundamental period of 4.



The Fourier series representation can be obtained directly using the analysis equation (3.33). We may also use the relation of $g(t)$ to the symmetric periodic square wave $x(t)$ discussed on page 8. Referring to that example, $T = 4$ and $T_1 = 1$,

$$g(t) = x(t-1) - 1/2\,. \tag{3.56}$$

The time-shift property indicates that if the Fourier series coefficients of $x(t)$ are denoted by a_k the Fourier series coefficients of $x(t-1)$ can be expressed as

$$b_k = a_k e^{-jk\nu/2}\,. \tag{3.57}$$

The Fourier coefficients of the *dc* offset in $g(t)$, that is the term –1/2 on the right-hand side of Eq. (3.56) are given by

$$c_k = \begin{cases} 0, & for\ \ k \neq 0 \\ -\dfrac{1}{2}, & for\ k = 0 \end{cases}\,. \tag{3.58}$$

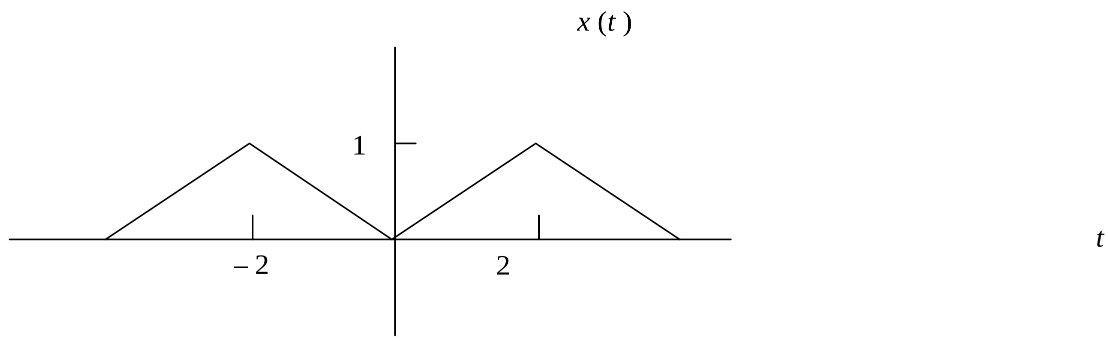
Applying the linearity property, we conclude that the coefficients for $g(t)$ can be expressed as

$$d_k = \begin{cases} a_k\, e^{-jk\nu/2}, & for\ k \neq 0 \\ a_0 - \dfrac{1}{2}, & for\ k = 0 \end{cases}, \tag{3.59}$$

replacing $a_k = \dfrac{\sin(\nu k/2)}{k\nu} e^{jk\nu/2}$, then we have

$$d_k = \begin{cases} \dfrac{\sin(\nu k/2)}{\nu k} e^{-jk\nu/2}, & for\ k \neq 0 \\ 0, & for\ k = 0 \end{cases}. \tag{3.60}$$

Example : The triangular wave signal $x(t)$ with period $T = 4$, and fundamental frequency $\xi_0 = \nu / 2$ is shown in the figure below.



The derivative of this function is the signal $g(t)$ in the previous preceding example. Denoting the Fourier series coefficients of $g(t)$ by d_k, and those of $x(t)$ by e_k, based on the differentiation property, we have

$$d_k = jk(\nu /2)e_k \ . \tag{3.61}$$

This equation can be expressed in terms of e_k except when $k = 0$. From Eq. (3.60),

$$e_k = \frac{2d_k}{jk\nu} = \frac{2 \sin(\nu k /2)}{j(k\nu)^2} e^{-jk\nu /2} \ . \tag{3.62}$$

For $k = 0$, e_0 can be simply calculated by calculating the area of the signal under one period and divide by the length of the period, that is

$$e_0 = 1/2 \ . \tag{3.63}$$

Example: The properties of the Fourier series representation of periodic train of impulse,

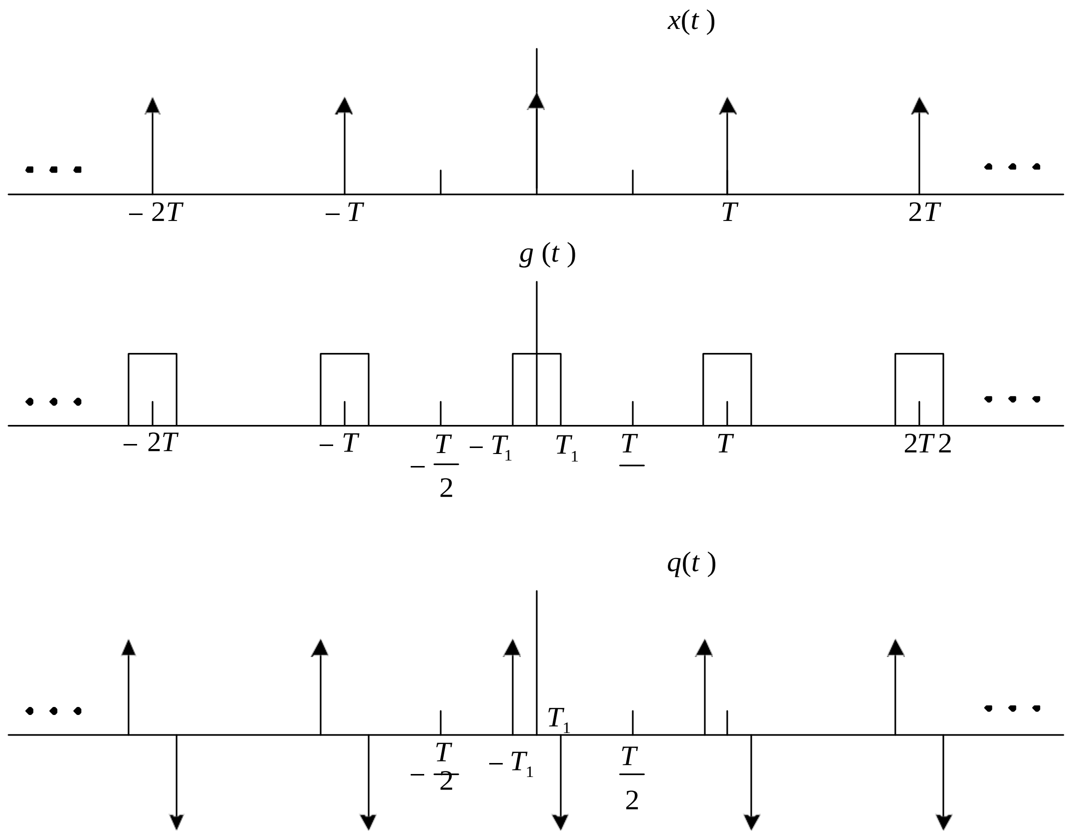
$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \ . \tag{3.64}$$

We use Eq. (3.33) and select the integration interval to be $-T/2 \le t \le T/2$, avoiding the placement of impulses at the integration limits.

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk(2\nu/T)t} dt = \frac{1}{T} \ . \tag{3.65}$$

All the Fourier series coefficients of this periodic train of impulse are identical, real and even.

The periodic train of impulse has a straightforward relation to square-wave signals such as $g(t)$ on page 8. The derivative of $g(t)$ is the signal $q(t)$ shown in the figure below,



which can also interpreted as the difference of two shifted versions of the impulse train $x(t)$. That is,

$$q(t) = x(t + T_1) - x(t - T_1)\,. \qquad (3.66)$$

Based on the time -shifting and linearity properties, we may express the Fourier coefficients b_k of $q(t)$ in terms of the Fourier series coefficient of a_k; that is

$$b_k = e^{jk\xi_0 T_1} a_k - e^{-jk\xi_0 T_1} a_k = \frac{1}{T}\left[e^{jk\xi_0 T_1} - e^{-jk\xi_0 T_1}\right], \qquad (3.67)$$

Finally we use the differentiation property to get

$$b_k = jk\xi_0 c_k\,, \qquad (3.68)$$

where c_k is the Fourier series coefficients of $g(t)$. Thus

$$c_k = \frac{b_k}{jk\xi_0} = \frac{2j\sin(k\xi_0 T_1)}{jk\xi_0 T} = \frac{2\sin(k\xi_0 T_1)}{k\xi_0 T}, \quad k \neq 0, \tag{3.69}$$

c_0 can be solve by inspection from the figure:

$$c_0 = \frac{2T_1}{T}. \tag{3.70}$$

Example: Suppose we are given the following facts about a signal $x(t)$

1. $x(t)$ is a real signal.
2. $x(t)$ is periodic with period $T = 4$, and it has Fourier series coefficients a_k.
3. $a_k = 0$ for $k > 1$.
4. The signal with Fourier coefficients $b_k = e^{-j\nu k/2} a_{-k}$ is odd.
5. $\frac{1}{4}\int_4 |x(t)|^2 dt = \frac{1}{2}$

Show that the information is sufficient to determine the signal $x(t)$ to within a sign factor.

- According to Fact 3, $x(t)$ has at most three nonzero Fourier series coefficients a_k: a_{-1}, a_0 and a_1. Since the fundamental frequency $\xi_0 = 2\nu / T = 2\nu / 4 = \nu / 2$, it follows that

$$x(t) = a_0 + a_1 e^{j\nu t/2} + a_{-1} e^{-j\nu t/2}. \tag{3.71}$$

- Since $x(t)$ is real (Fact 1), based on the symmetry property a_0 is real and $a_1 = a^*_{-1}$. Consequently,

$$x(t) = a_0 + a_1 e^{j\nu t/2} + \left(a_1 e^{j\nu t/2}\right)^* = a_0 + 2\,\mathrm{Re}\left\{a_1 e^{j\nu t/2}\right\}. \tag{3.72}$$

- Based on the Fact 4 and considering the time-reversal property, we note that a_{-k} corresponds to $x(-t)$. Also the multiplication property indicates that multiplication of kth Fourier series by $e^{-jk\nu/2}$ corresponds to the signal being shifted by 1 to the right. We conclude that the coefficients b_k correspond to the signal $x(-(t-1)) = x(-t+1)$, which according to Fact 4 must be odd. Since $x(t)$ is real, $x(-t+1)$ must also be real. So based the property, the Fourier series coefficients must be purely imaginary and odd. Thus, $b_0 = 0$, $b_{-1} = -b_1$.
- Since time reversal and time shift cannot change the average power per period, Fact 5 holds even if $x(t)$ is replaced by $x(-t+1)$. That is

$$\frac{1}{4}\int_4 |x(-t+1)|^2 dt = \frac{1}{2}. \tag{3.73}$$

Using Parseval's relation,

$$|b_1|^2 + |b_{-1}|^2 = 1/2 . \qquad (3.74)$$

Since $b_{-1} = -b_1$, we obtain $|b_1| = 1/2$. Since b_1 is known to be purely imaginary, it must be either $b_1 = j/2$ or $b_1 = -j/2$.

- Finally we translate the conditions on b_0 and b_1 into the equivalent statement on a_0 and a_1. First, since $b_0 = 0$, Fact 4 implies that $a_0 = 0$. With $k = 1$, this condition implies that $a_1 = e^{-j\nu/2} b_{-1} = -jb_{-1} = jb_1$. Thus, if we take $b_1 = j/2$, $a_1 = -1/2$, from Eq. (3.72), $x(t) = -\cos(\nu t/2)$. Alternatively, if we take $b_1 = -j/2$, the $a_1 = 1/2$, and therefore, $x(t) = \cos(\nu t/2)$.

3.6 Fourier Series Representation of Discrete-Time Periodic Signals

The Fourier series representation of a discrete-time periodic signal is ***finite***, as opposed to the ***infinite*** series representation required for continuous-time periodic signals

3.6.1 Linear Combination of Harmonically Related Complex Exponentials

A discrete-time signal $x[n]$ is periodic with period N if

$$x[n] = x[n+N] . \qquad (3.75)$$

The fundamental period is the smallest positive N for which Eq. (3.75) holds, and the fundamental frequency is $\xi_0 = 2\nu/N$.

The set of all discrete-time complex exponential signals that are periodic with period N is given by

$$\exists_k[n] = e^{jk\xi_0 n} = e^{jk(2\nu/N)n}, \qquad k = 0, \pm 1, \pm 2, ..., \qquad (3.76)$$

All of these signals have fundamental frequencies that are multiples of $2\nu/N$ and thus are harmonically related.

There are only N distinct signals in the set given by Eq. (3.76); this is because the discrete-time complex exponentials which differ in frequency by a multiple of 2ν are identical, that is,

$$\exists_k[n] = \exists_{k+rN}[n] . \qquad (3.77)$$

The representation of periodic sequences in terms of linear combinations of the sequences $\exists_k[n]$ is

$$x[n] = \sum_k a_k \exists_k[n] = \sum_k a_k e^{jk\xi_0 n} = \sum_k a_k e^{jk(2\nu/N)n} . \qquad (3.78)$$

Since the sequences $\exists_k[n]$ are distinct over a range of N successive values of k, the summation in Eq. (3.78) need include terms over this range. We indicate this by expressing the limits of the summation as $k = \langle N \rangle$. That is,

$$x[n] = \sum_{k=\langle N\rangle} a_k \exists_k[n] = \sum_{k=\langle N\rangle} a_k e^{jk\xi_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\nu/N)n} . \qquad (3.79)$$

Eq. (3.79) is referred to as the discrete-time Fourier series and the coefficients a_k as the Fourier series coefficients.

6.2 Determination of the Fourier Series Representation of a Periodic Signal

The discrete-time Fourier series pair:

$$_{k=\langle N\rangle} \; {}_{k\,k} \qquad \sum_{k=\langle N\rangle} {}_k \qquad \sum_{k=\langle N\rangle} {}_k \qquad , \qquad (3.80)$$

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk\xi_0 n} = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk(2\nu/N)n} \cdot \qquad (3.81)$$

Eq. (3.80) is called ***synthesis equation*** and Eq. (3.81) is called ***analysis equation***.

Example: Consider the signal $x[n] = \sin \xi_0 n$, (3.82)

$x[n]$ is periodic only if $2\nu / \xi_0$ is ***an integer***, or ***a ratio of integer***. For the case the when $2\nu / \xi_0$ is an integer N, that is, when

$$\xi_0 = \frac{2\nu}{N}, \qquad (3.83)$$

$x[n]$ is periodic with the fundamental period N. Expanding the signal as a sum of two complex exponentials, we get

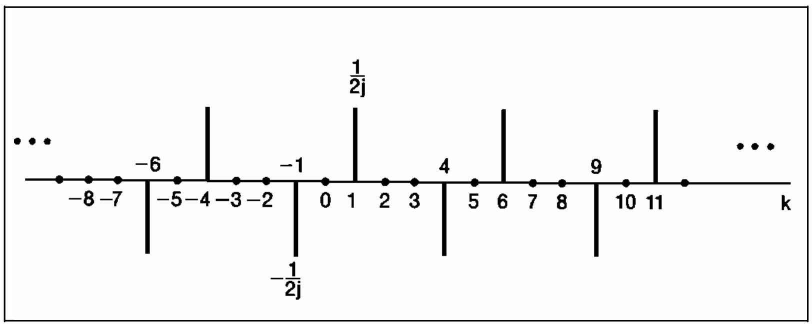
$$x[n] = \frac{1}{2j} e^{j(2\nu/N)n} - \frac{1}{2j} e^{-j(2\nu/N)n}, \tag{3.84}$$

From Eq. (3.84), we have

$$a_1 = \frac{1}{2j}, \; a_{-1} = -\frac{1}{2j}, \tag{3.85}$$

and the remaining coefficients over the interval of summation are zero. As discussed previously, these coefficients repeat with period *N*.

The Fourier series coefficients for this example with $N = 5$ are illustrated in the figure below.



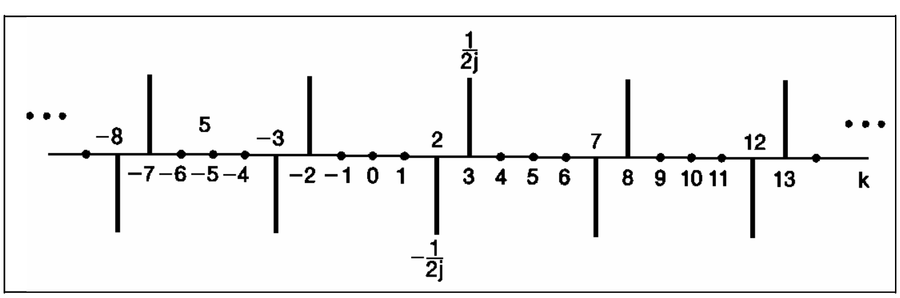
When $2\nu/\xi_0$ is a ratio of integer, that is, when

$$\xi_0 = \frac{2\nu M}{N}, \tag{3.86}$$

Assuming the *M* and *N* do not have any commo n factors, $x[n]$ has a fundamental period of *N*. Again expanding $x[n]$ as a sum of two complex exponentials, we have

$$x[n] = \frac{1}{2j} e^{jM(2\nu/N)n} - \frac{1}{2j} e^{-jM(2\nu/N)n}, \tag{3.87}$$

From which we determine by inspection that $a_M = (1/2j)$, $a_{-M} = -(1/2j)$, and the remaining coefficients over one period of length *N* are zero. The Fourier coefficients for this example with $M = 3$ and $N = 5$ are depicted in the figure below.



Example : Consider the signal

$$x[n] = 1 + \sin\left(\frac{2\pi}{N}\right)n + 3\cos\left(\frac{2\pi}{N}\right)n + \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right).$$

Expanding this signal in terms of complex exponential, we have

$$x[n] = 1 + \left(\frac{3}{2} + \frac{1}{2j}\right)e^{j(2\pi/N)n} + \left(\frac{3}{2} - \frac{1}{2j}\right)e^{-j(2\pi/N)n} + \left(\frac{1}{2}e^{j\pi/2}\right)e^{j2(2\pi/N)n} + \left(\frac{1}{2}e^{-j\pi/2}\right)e^{-j2(2\pi/N)n}.$$

Thus the Fourier series coefficients for this signal are

$a_0 = 1$,

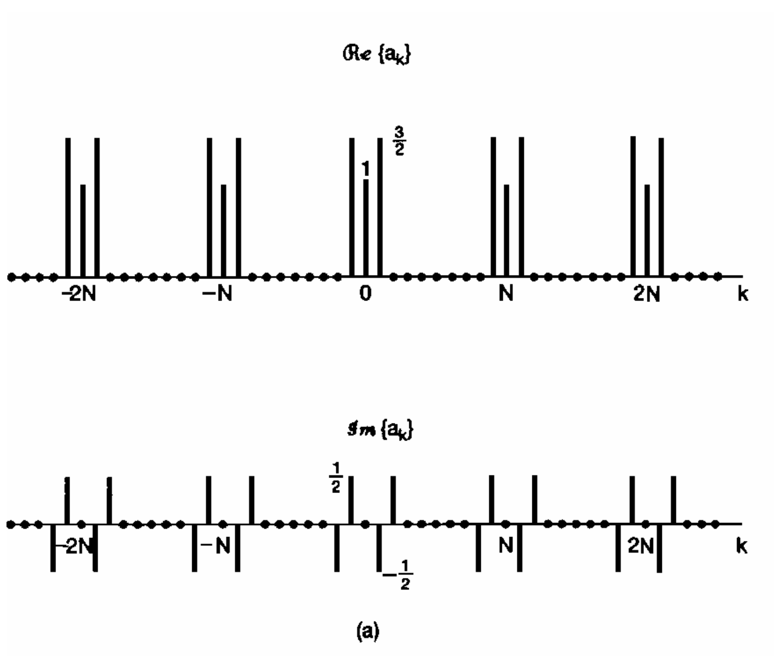
$a_1 = \frac{3}{2} + \frac{1}{2j} = \frac{3}{2} - \frac{1}{2}j$,

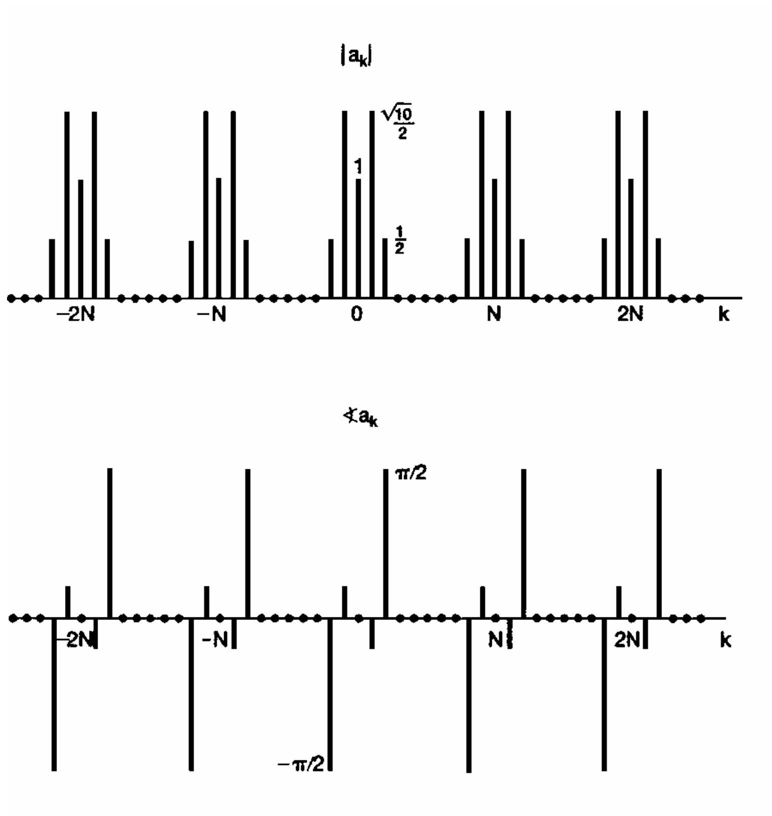
$a_{-1} = \frac{3}{2} - \frac{1}{2j} = \frac{3}{2} + \frac{1}{2}j$,

$a_2 = \frac{1}{2}j$,

$a_{-2} = -\frac{1}{2}j$.

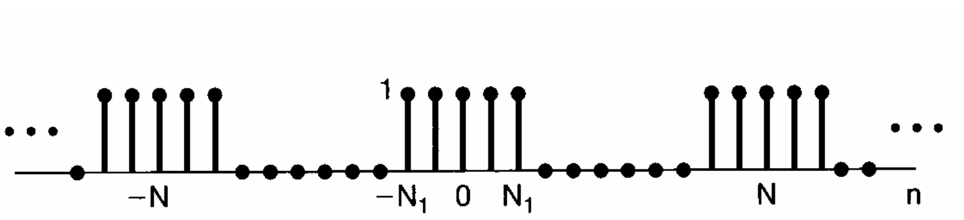
with $a_k = 0$ for other values of k in the interval of summation in the synthesis equation. The real and imaginary parts of these coefficients for $N = 10$, and the magnitude and phase of the coefficients are depicted in the figure below.





(b)

Example : Consider the square wave shown in the figure below.



Because $x[n] = 1$ for $-N_1 \le n \le N_1$, we choose the length-N interval of summation to include the range $-N_1 \le n \le N_1$. The coefficients are given

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk(2\nu/N)n}, \tag{3.88}$$

Let $m = n + N_1$, we observe that Eq. (3.88) becomes

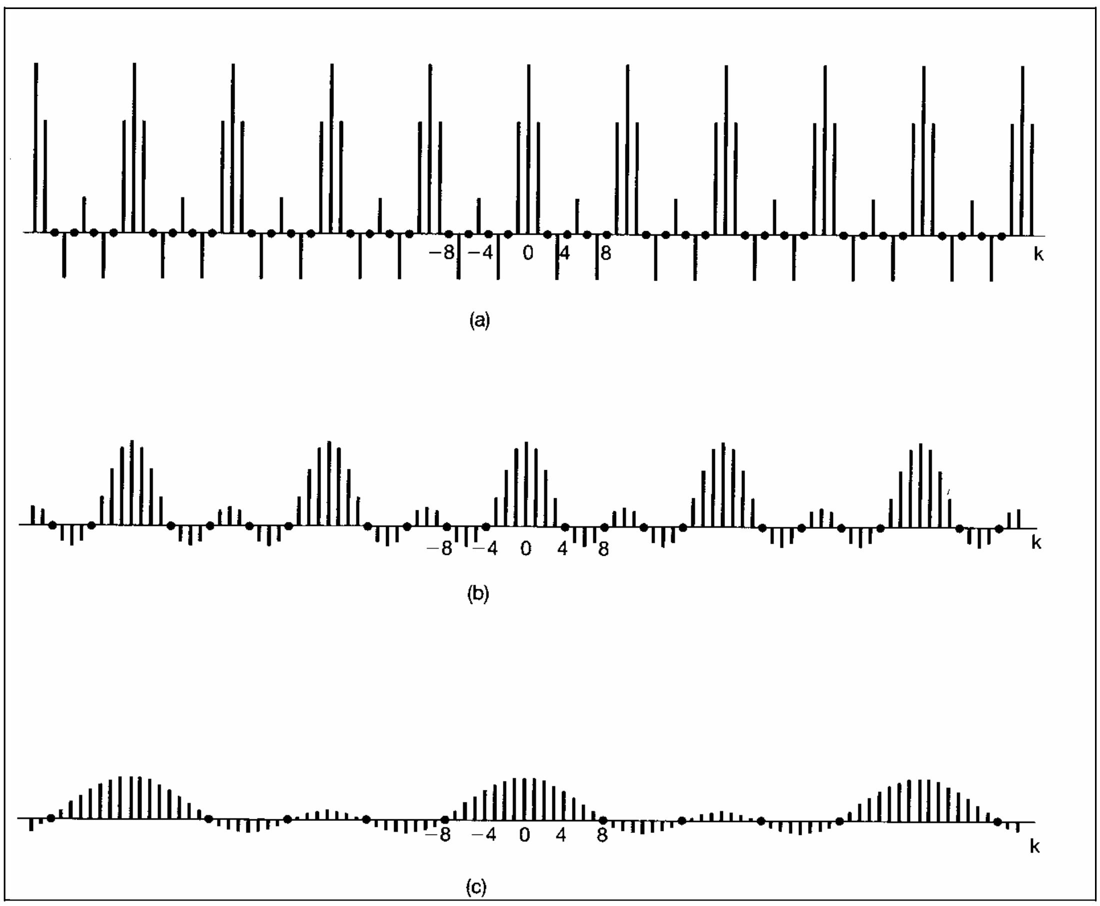
$$a_k = \frac{1}{N}\sum_{n=0}^{2N_1} e^{-jk(2\nu/N)(m-N_1)} = \frac{1}{N} e^{jk(2\nu/N)N_1} \sum_{n=0}^{2N_1} e^{-jk(2\nu/N)m}, \qquad (3.89)$$

$$a_k = \frac{1}{N} e^{jk(2\nu/N)N_1} \left(\frac{1-e^{jk2\nu(2N_1+1)/N}}{1-e^{-jk(2\nu/N)}} \right) = \frac{1}{N} \frac{\sin[2\nu k(N_1+1/2)/N]}{\sin(\nu k/N)}, \; k \neq 0, \pm N, \pm 2N, ... \qquad (3.90)$$

and

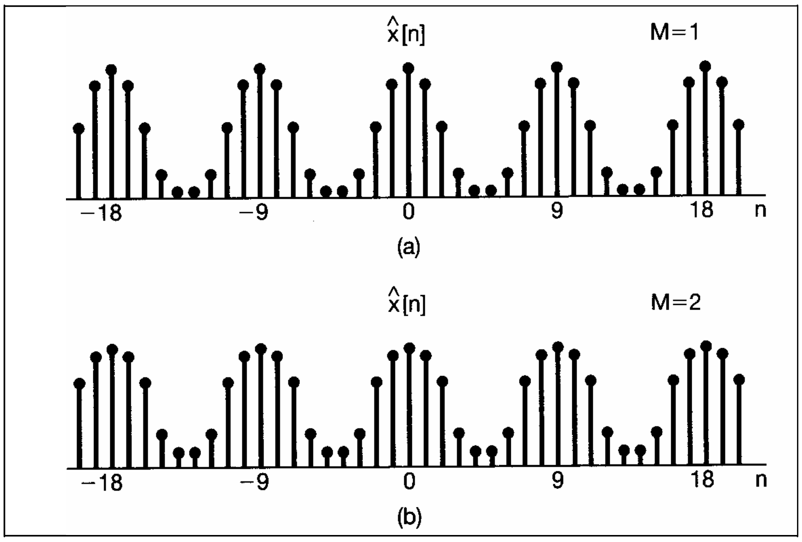
$$a_k = \frac{2N_1+1}{N}, k = 0, \pm N, \pm 2N, ... \qquad (3.91)$$

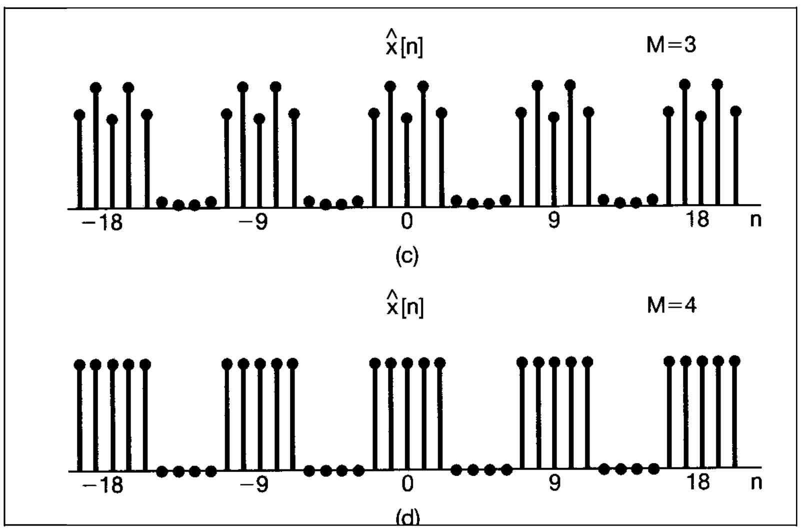
The coefficients a_k for $2N_1 + 1 = 5$ are sketched for $N = 10, 20,$ and 40 in the figure below.



The partial sums for the discrete-time square wave for $M = 1, 2, 3,$ and 4 are depicted in the figure below, where $N = 9$, $2N_1 + 1 = 5$.

We see for $M = 4$, the partial sum exactly equals to $x[n]$. In contrast to the continuous-time case, **there *are no convergence issues and there is no Gibbs phenomenon***.





3.7 Properties of Discrete-Time Fourier Series

Property	**Periodic Signal**	**Fourier Series Coefficients**
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamenta l frequency $\xi_0 = 2\nu$	$\begin{array}{l} a_k \\ b_k \end{array}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$e^{-jk(2\nu/N)t} q_k$
Frequency shifting	$e^{jM(2\nu/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}

Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & if\ n\ is\ a\ multiple of\ n \\ 0, & if\ n\ is\ a\ multiple of\ n \end{cases}$ (Periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix} viewed\ as\ periodic \\ with\ period\ mN \end{pmatrix}$
Periodic Convolution	$\sum_{r=[N]} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=<N>} a_l b_{k-l}$
Differentiation	$x[n] - x[n-1]$	$\left(1 - e^{-jk(2\pi/N)}\right)a_k$
Integration	$\sum_{k=-\infty}^{n} x[k]$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{1 - e^{-jk(2\pi/N)}}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathrm{Re}\{a_k\} = \mathrm{Re}\{a_{-k}\} \\ \mathrm{Im}\{a_k\} = -\mathrm{Im}\{a_{-k}\} \\ \|a_k\| = \|a_{-k}\| \\ \angle a_k = -\angle a_{-k} \end{cases}$
Real and Even Signals Real and Odd Signals Even-Odd Decomposition of Real Signals	$x[n]$ real and even $x[n]$ real and odd $\begin{cases} x_e[n] = Ev\{x[n]\} & [x[n]\ real] \\ x_e[n] = Od\{x[n]\} & [x[n]\ real] \end{cases}$	a_k real and even a_k purely imaginary and odd $\mathrm{Re}\{a_k\}$ $j\,\mathrm{Im}\{a_k\}$
	Parseval's Relation for Periodic Signals $\frac{1}{T}\sum_{n=<N>} \|x[n]\|^2 = \sum_{n=<N>} \|a_k\|^2$	

3.7.1 Multiplication

$$x[n]y[n] \xleftrightarrow{FS} \sum_{l=<N>} a_l b_{k-l} \quad (3.92)$$

Eq. (3.92) is analogous to the convolution, except that the summation variable is now restricted to in interval of N consecutive samples. This type of operation is referred to as a ***Periodic Convolution*** between the two periodic sequences of Fourier coefficients.

The usual form of the convolution sum, where the summation variable ranges from $-\infty$ to $+\infty$, is sometimes referred to as ***Aperiodic Convolution***.

3.7.2 First Difference

$$x[n]-x[n-1] \overset{FS}{\longleftrightarrow} \left(1-e^{-jk(2\nu/N)}\right)a_k . \qquad (3.93)$$

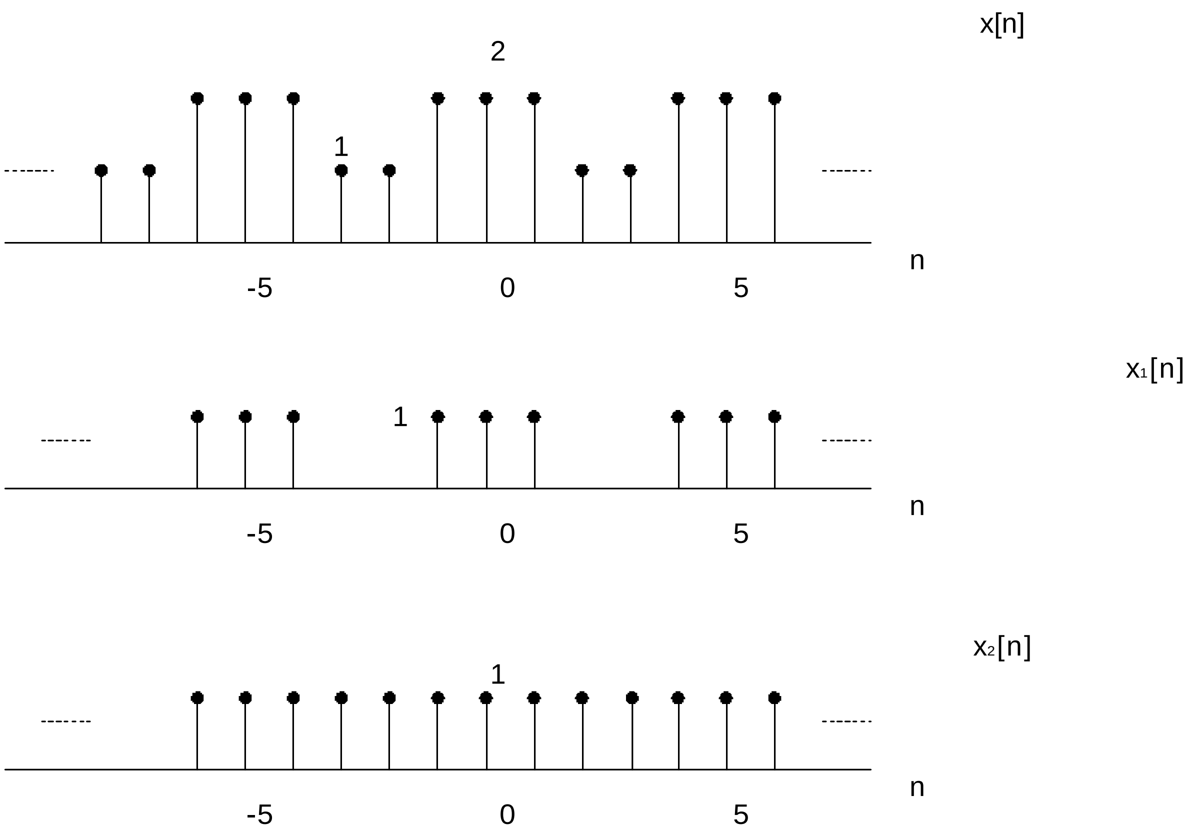
3.7.3 Parseval's Relation

$$\frac{1}{T}\sum_{n=<N>}|x[n]|^2 = \sum_{k=<N>}|a_k|^2 .$$

(3.94)

3.7.4 Examples

Example : Consider the signal shown in the figure below.



The signal $x[n]$ may be viewed as the sum of the square wave $x_1[n]$ with Fourier series coefficients b_k and $x_2[n]$ with Fourier series coefficients c_k.

$$a_k = b_k + c_k, \tag{3.95}$$

The Fourier series coefficients for $x_1[n]$ is

$$b_k = \begin{cases} \frac{1}{5}\frac{\sin(3\nu k/5)}{\sin(\nu k/5)}, & for\ k \neq 0, \pm 5, \pm 10, ... \\ \frac{3}{5}, & for\ k = 0, \pm 5, \pm 10, ... \end{cases}. \tag{3.96}$$

The sequence $x_2[n]$ has only a dc value, which is captured by its zeroth Fourier series coefficient:

$$c_0 = \frac{1}{5}\sum_{n=0}^{4} x_2[n] = 1, \tag{3.97}$$

Since the discrete-time Fourier series coefficients are periodic, it follows that $c_k = 1$ whenever k is an integer multiple of 5.

$$a_k = \begin{cases} \frac{1}{5}\frac{\sin(3\nu k/5)}{\sin(\nu k/5)}, & for\ k \neq 0, \pm 5, \pm 10, ... \\ \frac{8}{5}, & for\ k = 0, \pm 5, \pm 10, ... \end{cases} \tag{3.98}$$

Example : Suppose we are given the following facts about a sequence $x[n]$:

1. $x[n]$ is periodic with period $N = 6$.
2. $\sum_{n=0}^{5} x[n] = 2$.
3. $\sum_{n=2}^{7} (-1)^n x[n] = 1$.
4. $x[n]$ has minimum power per period among the set of signals satisfying the preceding three conditions.

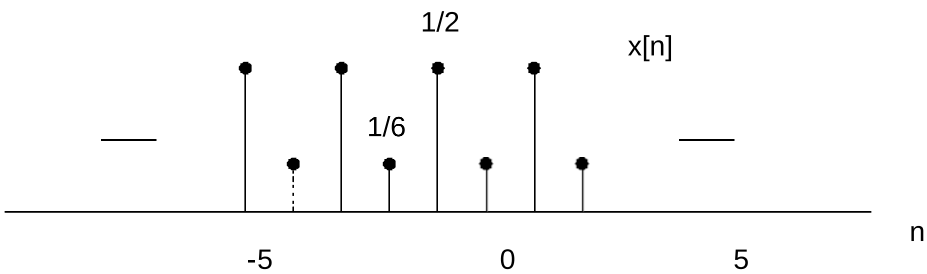
- From Fact 2, we have $a_0 = \frac{1}{6}\sum_{n=0}^{5} x[n] = \frac{1}{3}$.
- Note that $(-1)^n = e^{-j\nu n} = e^{-j(2\nu/6)3n}$, we see from Fact 3 that $a_3 = \frac{1}{6}\sum_{2}^{7} x[n] e^{-j3(2\nu/N)n} = \frac{1}{6}$.
- From Parseval's relation, the average power in $x[n]$ is

$$P = \sum_{k=0}^{5} \left| a_k^{\,2} \right| .$$

Since each nonzero coefficient contributes a positive amount to P, and since the values of a_0 and a_3 are specified, the value of P is minimized by choosing $a_1 = a_2 = a_4 = a_5 = 0$. It follows that

$$x[n] = a_0 + a_3 e^{j\nu n} = \frac{1}{3} + \frac{1}{6}(-1)^n ,$$

which is shown in the figure below.



3.8 Fourier Series and LTI Systems

We have seen that the response of a continuous-time LTI system with impulse response $h(t)$ to a complex exponential signal e^{st} is the same complex exponential multiplied by a complex gain:

$y(t) = H(s)e^{st}$, where

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau , \qquad (3.99)$$

In particular, for $s = j\xi$, the output is $y(t) = H(j\xi)e^{j\xi t}$. The complex functions $H(s)$ and $H(j\xi)$ are called the system function (or transfer function) and the frequency response, respectively.

By superposition, the output of an LTI system to a periodic signal represented by a Fourier series

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\xi_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\nu/T)t}$$ is given by

$$y(t) = \sum_{k=-\infty}^{+\infty} a_k H(jk\xi_0) e^{jk\xi_0 t} \quad . \tag{3.99}$$

That is, the Fourier series coefficients b_k of the periodic output $y(t)$ are given by

$$b_k = a_k H(jk\xi_0), \tag{3.100}$$

Similarly, for discrete-time signals and systems, response $h[n]$ to a complex exponential signal $e^{j\xi n}$ is the same complex exponential multiplied by a complex gain:

$$y[n] = H(jk\xi_0) e^{jk\xi_0 n}, \tag{3.101}$$

where

$$H(e^{j\xi}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\xi n} \quad . \tag{3.102}$$

Example: Suppose that the periodic signal $x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\nu t}$ with $a_0 = 1$, $a_1 = a_{-1} = \frac{1}{4}$, $a_2 = a_{-2} = \frac{1}{2}$, and $a_3 = a_{-3} = \frac{1}{3}$ is the input signal to an LTI system with impulse response

$$h(t) = e^{-t} u(t)$$

To calculate the Fourier series coefficients of the output $y(t)$, we first compute the frequency response:

$$H(j\xi) = \int_0^{\infty} e^{-\tau} e^{-j\xi\tau} d\tau = \frac{1}{1 + j\xi} e^{-\tau} e^{-\xi\tau} \Big|_0^{\infty} = \frac{1}{1 + j\xi}, \tag{3.103}$$

The output is

$$y(t) = \sum_{k=-3}^{+3} b_k e^{jk2\nu t}, \tag{3.104}$$

where $b_k = a_k H(jk\xi_0) = a_k H(jk2\nu)$, so that

$b_0 = 0$, $\quad b_1 = \frac{1}{4}\left(\frac{1}{1 + j2\nu}\right)$, $\quad b_{-1} = \frac{1}{4}\left(\frac{1}{1 - j2\nu}\right)$,

$$b_2 = \frac{1}{4}\left(\frac{1}{1+j4\nu}\right), \qquad b_{-2} = \frac{1}{4}\left(\frac{1}{1-j4\nu}\right),$$

$$b_3 = \frac{1}{4}\left(\frac{1}{1+j6\nu}\right), \qquad b_{-3} = \frac{1}{4}\left(\frac{1}{1-j6\nu}\right).$$

Example: Consider an LTI system with impulse response $h[n] = \alpha^n u[n]$, $-1 < \alpha < 1$, and with the input

$$x[n] = \cos\left(\frac{2\nu n}{N}\right). \tag{3.105}$$

Write the signal $x[n]$ in Fourier series form as

$$x[n] = \frac{1}{2} e^{j(2\nu/N)n} + \frac{1}{2} e^{-j(2\nu/N)n}.$$

Also the transfer function is

$$H(e^{j\xi}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\xi n} = \sum_{n=0}^{\infty} \left(\alpha\, e^{-j\xi}\right)^n = \frac{1}{1-\alpha\, e^{-j\xi}}. \tag{3.106}$$

The Fourier series for the output

$$\begin{aligned} y[n] &= \frac{1}{2} H\left(e^{j2\nu/N}\right) e^{j(2\nu/N)n} + \frac{1}{2} H\left(e^{-j2\nu/N}\right) e^{-j(2\nu/N)n} \\ &= \frac{1}{2}\left(\frac{1}{1-\alpha e^{-j\xi}}\right) e^{j(2\nu/N)n} + \frac{1}{2}\left(\frac{1}{1-\alpha e^{-j\xi}}\right) e^{-j(2\nu/N)n} \end{aligned} \tag{3.107}$$

3.9 Filtering

Filtering – to change the relative amplitude of the frequency components in a signal or eliminate some frequency components entirely.

Filtering can be conveniently accomplished through the use of LTI systems with an appropriately chosen frequency response.
LTI systems that change the shape of the spectrum of the input signal are referred to as *frequency-shaping filters*.

LTI systems that are designed to pass some frequencies essentially undistorted and significantly attenuate or eliminate others are referred to as *frequency-selective filters*.

Example : A first-order low-pass filter with impulse response $h(t) = e^{-t}u(t)$ cuts off the high frequencies in a periodic input signal, while low frequency harmonics are mostly left intact. The frequency response of this filter

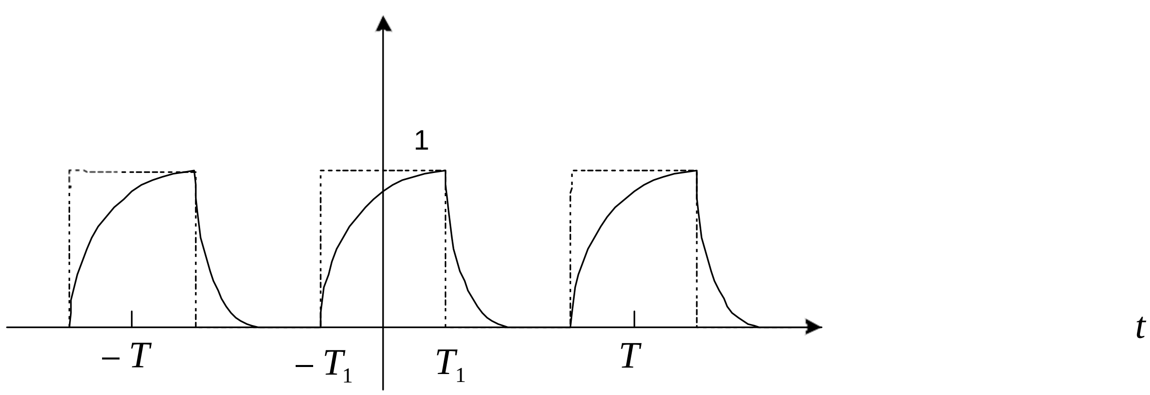
$$H(j\xi) = \int_0^{+\infty} e^{-\tau} e^{-j\xi\tau} d\tau = \frac{1}{1+j\xi}. \qquad (3.107)$$

We can see that as the frequency ξ increase, the magnitude of the frequency response of the filter $|H(j\xi)|$ decreases. If the periodic input signal is a rectangular wave, then the output signal will have its Fourier series coefficients b_k given by

$$b_k = a_k \; H(jk\xi_0) = \frac{\sin(k\xi_0 T_1)}{k\nu(1+jk\xi_0)}, \qquad k \neq 0 \qquad (3.108)$$

$$b_0 = a_0 \; H(0) = \frac{2T_1}{T}. \qquad (3.109)$$

The reduced power at high frequencies produced an output signal that is smother than the input signal.

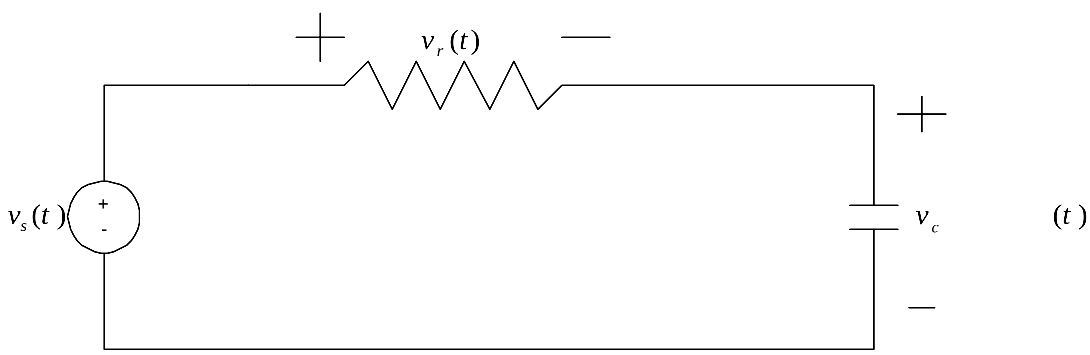


3.10 Examples of continuous-Time Filters Described By Differential Equations

In many applications, frequency-selective filtering is accomplished through the use of LTI systems described by linear constant-coefficient differential or difference equations. In fact, many physical systems that can be interpreted as performing filtering operations are characterized by differential or difference equation.

3.10.1 A simple RC Lowpass Filter

The first-order RC circuit is one of the electrical circuits used to perform continuous-time filtering. The circuit can perform either Lowpass or highpass filtering depending on what we take as the output signal.



If we take the voltage cross the capacitor as the output, then the output voltage is related to the input through the linear constant-coefficient differential equation:

$$RC\frac{dv_c(t)}{dt}+v_c(t)=v_s(t)\,. \qquad (3.111)$$

Assuming initial rest, the system described by Eq. (3.111) is LTI. If the input is $v_s(t)=e^{j\xi t}$, we must have voltage output $v_c(t)=H(j\xi)e^{j\xi t}$. Substituting these expressions into Eq. (3.111), we have

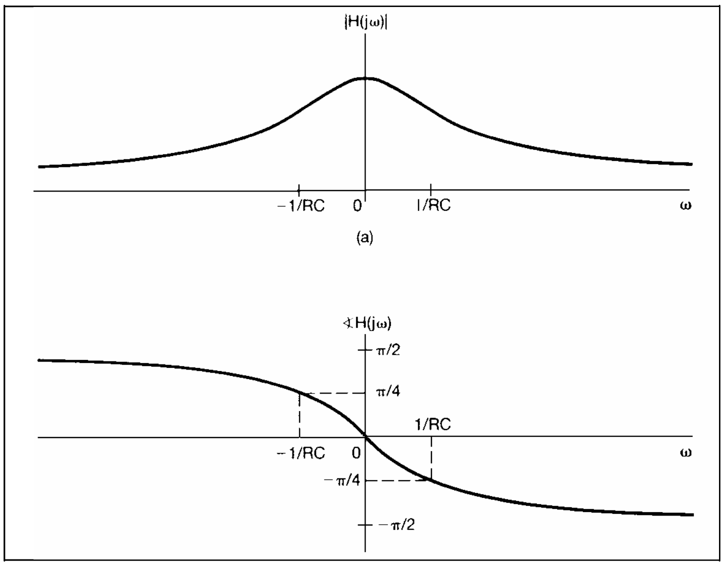
$$RC\frac{d}{dt}\left[H(j\xi)e^{j\xi t}\right]+H(j\xi)e^{j\xi t}=e^{j\xi t}\,, \qquad (3.112)$$

or

$$RCj\xi H(j\xi)e^{j\xi t}+H(j\xi)e^{j\xi t}=e^{j\xi t}\,, \qquad (3.113)$$

Then we have $H(j\xi) = \dfrac{1}{1 + RCj\xi}$. (3.114)

Te amplitude and frequency response $H(j\xi)$ is shown in the figure below.



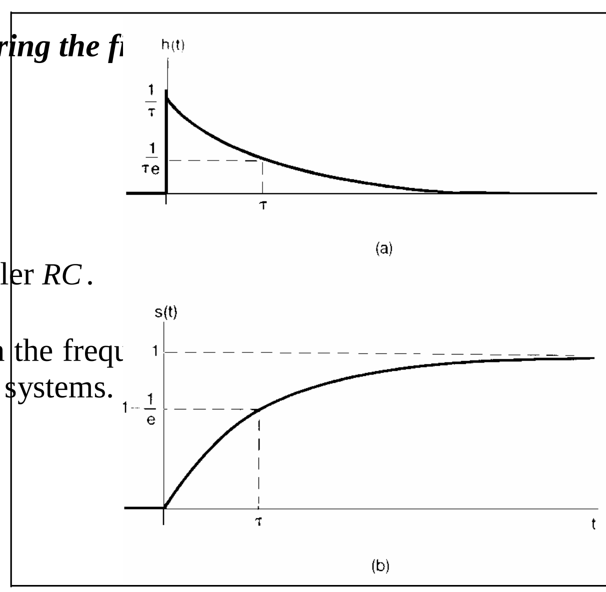
We can also get the impulse response

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t) , \quad (3.115)$$

and the step response is

$$h(t) = (1 - e^{-t/RC}) u(t) , \quad (3.116)$$

The fundamental trade-off can be found by comparing the fi



- To pass only very low frequencies, $1/RC$ should be small, or RC should be large.
- To have fast step response, we need a smaller RC.
- The type of trade-off between behaviors in the frequ cal of the issues arising in the design analysis of LTI systems.

3.10.2 A Simple RC Highpass Filter

If we choose the output from the resistor, then we get an RC highpass filter.

3.11 Examples of Discrete-Time Filter Described by Difference Equations

A discrete-time LTI system described by the first-order difference equation

$$y[n] - ay[n-1] = x[n] \ . \tag{3.116}$$

Form the eigenfunction property of complex exponential signals, if $x[n] = e^{j\xi n}$, then $y[n] = H(e^{j\xi})e^{j\xi n}$, where $H(e^{j\xi})$ is the frequency response of the system.

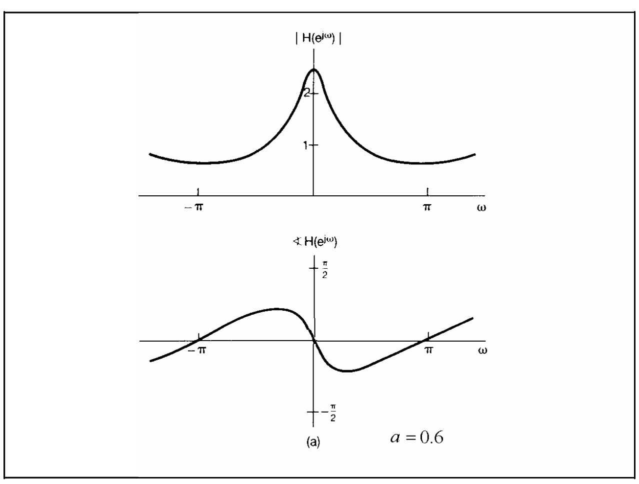
$$H(e^{j\xi}) = \frac{1}{1 - ae^{-j\xi}} \ . \tag{3.117}$$

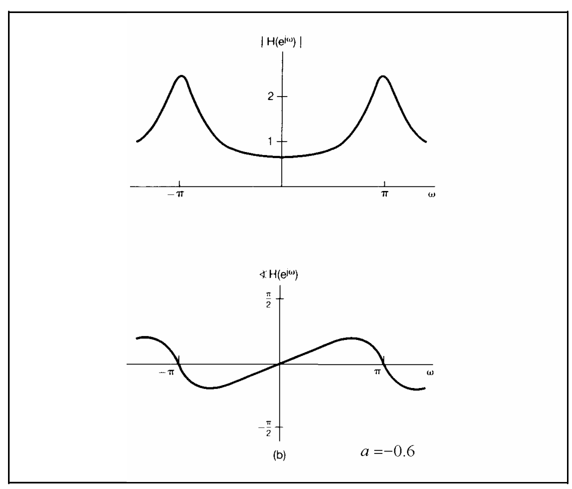
The impulse response of the system is

$$x[n] = a^n u[n]. \tag{3.118}$$

The step response is

$$s[n] = \frac{1 - a^{n+1}}{1 - a} u[n]. \tag{3.119}$$





From the above plots we can see that for $a = 0.6$ the system acts as a Lowpass filter and $a = -0.6$, the system is a highpass filter. In fact, for any positive value of $a < 1$, the system approximates a highpass filter, and for any negative value of $a > -1$, the system approximates a highpass filter, where a $|\;|$ controls the size of bandpass, with broader pass bands as $|a|$ in decreased.

The trade-off between time domain and frequency domain characteristics, as discussed in continuous time, also exists in the discrete-time systems.

3.11.2.2 Nonrecursive Discrete-Time Filters

The general form of an FIR norecursive difference equation is

$$y[n] = \sum_{k=-N}^{M} b_k x[n-k]. \tag{3.120}$$

It is a weighted average of the $(\mathrm{N}+\mathrm{M}+1)$ values of $x[n]$, with the weights given by the coefficients b_k.

One frequently used example is a *moving-average filter*, where the output of $y[n]$ is an average of values of $x[n]$ in the vicinity of n_0 - the result corresponding a smooth operation or lowpass filtering.

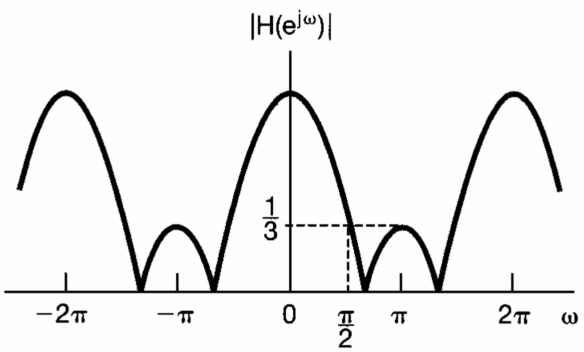
An example: $y[n] = \frac{1}{3}\left(x[n-1] + x[n] + x[n+1]\right).$ (3.121)

The impulse response is

$$h[n] = \frac{1}{3}\left(\delta[n-1] + \delta[n] + \delta[n+1]\right), \tag{3.122}$$

and the frequency response

$$H(e^{j\xi}) = \frac{1}{3}\left(e^{j\xi} + 1 + e^{-j\xi}\right). \tag{3.123}$$



Magnitude of the frequency response of a three-point moving-average lowpass filter.

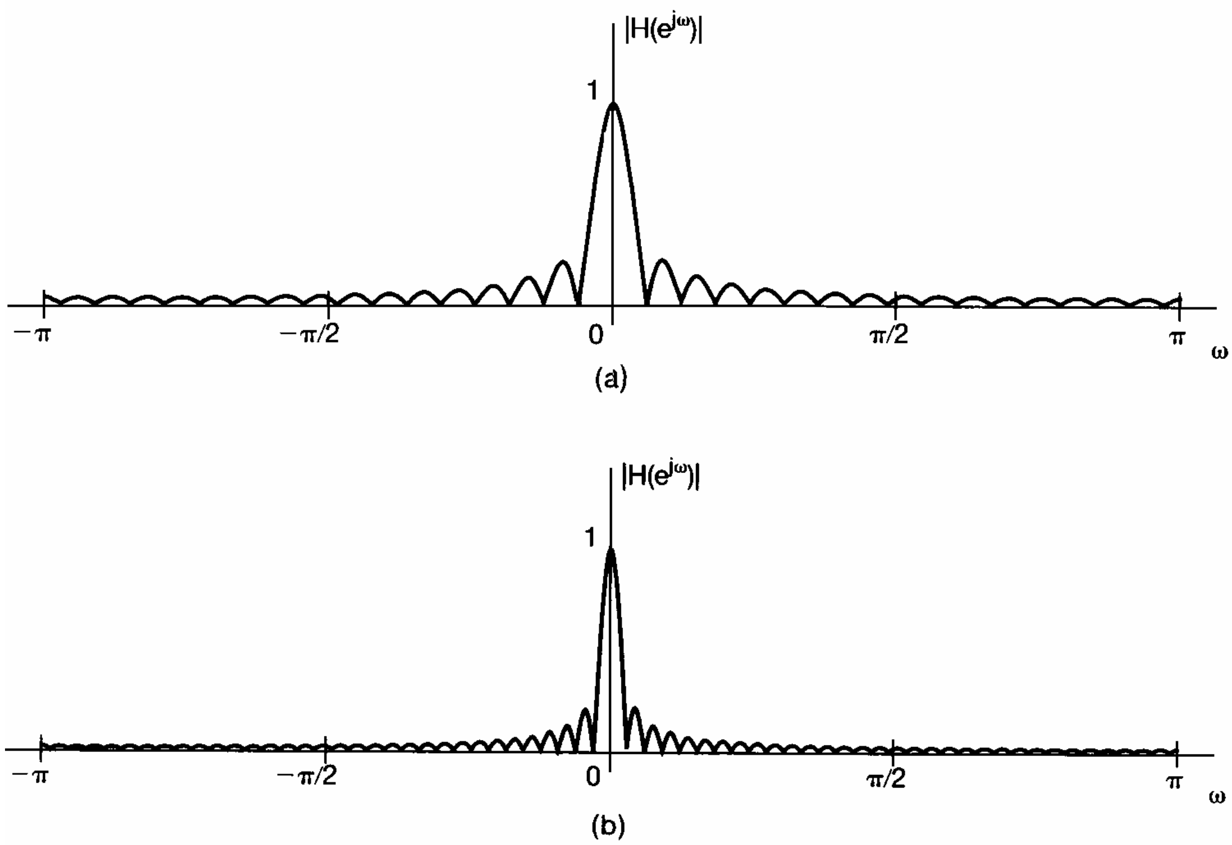
A generalized moving average filter can be expressed as

$$y[n] = \frac{1}{N+M+1}\sum_{k=-N}^{M} b_k x[n-k]\,. \qquad (3.124)$$

The frequency response is

$$H(e^{j\xi}) = \frac{1}{M+N+1}\sum_{k=-N}^{M} e^{-j\xi k} = \frac{1}{M+N+1} e^{j\xi[(N-M)/2]} \frac{\sin\left[\xi(M+N+1)/2\right]}{\sin(\xi/2)}\,. \qquad (3.125)$$

The frequency responses with different average window lengths are plotted in the figure below.



Magnitude of the frequency response for the lowpass moving-average filter of eq. (3.162): (a) $M = N = 16$; (b) $M = N = 32$.

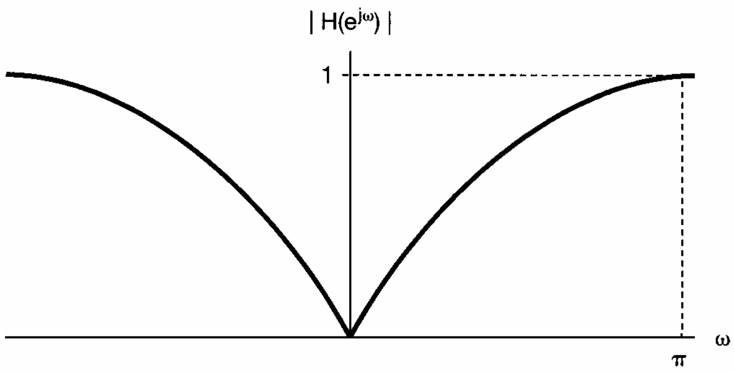
FIR norecursive highpass filter

An example of FIR norecursive highpass filter is

$$y[n] = \frac{x[n] - x[n-1]}{2}. \tag{3.126}$$

The frequency response is

$$H(e^{j\xi}) = \frac{1}{2}\left(1 - e^{-j\xi}\right) = je^{j\xi/2}\sin(\xi/2). \tag{3.127}$$



Frequency response of a simple highpass filter.

Continuous-Time Fourier Transform

4.0 Introduction

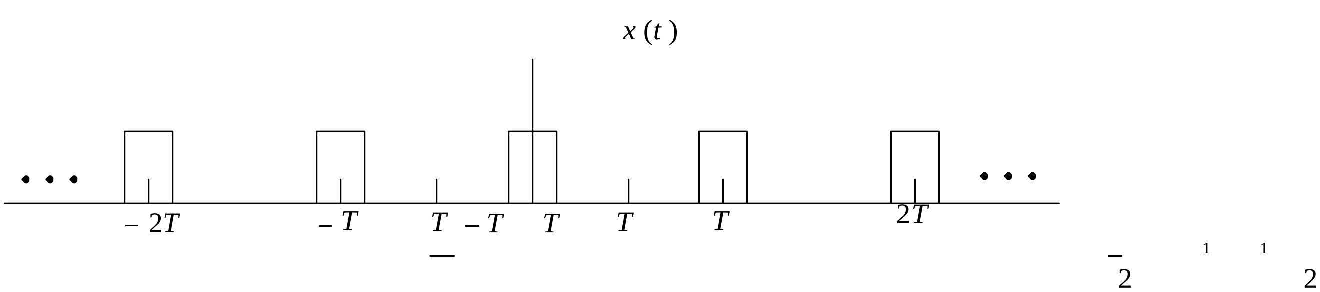
- A periodic signal can be represented as linear combination of complex exponentials which are harmonically related.
- An aperiodic signal can be represented as linear combination of complex exponentials, which are infinitesimally close in frequency. So the representation take the form of an integral rather than a sum
- In the Fourier series representation, as the period increases the fundamental frequency decreases and the harmonically related components become closer in frequency. As the period becomes infinite, the frequency components form a continuum and the Fourier series becomes an integral.

4.1 Representation of Aperiodic Signals: The Continuous-Time Fourier Transform

4.1.1 Development of the Fourier Transform Representation of an Aperiodic Signal

Starting from the Fourier series representation for the continuous-time periodic square wave:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}, \tag{4.1}$$



The Fourier coefficients a_k for this square wave are

$$a_k = \frac{2\sin(k\xi_0 T_1)}{k\xi_0 T}. \tag{4.2}$$

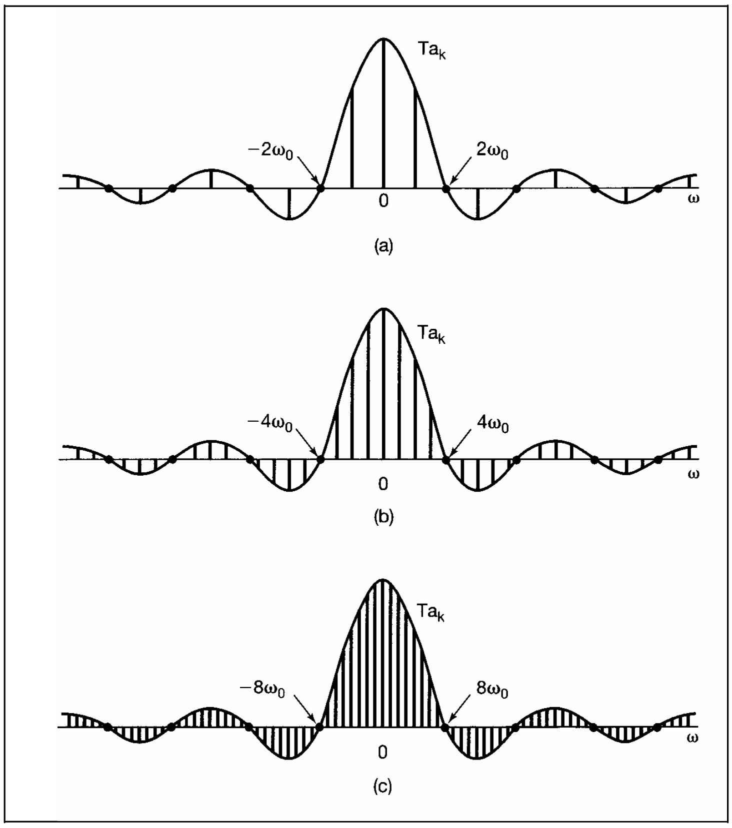
or alternatively

$$Ta_k = \left.\frac{2\sin(\xi T_1)}{\xi}\right|_{\xi = k\xi_0}, \tag{4.3}$$

where $2\sin(\xi T_1)/\xi$ represent the envelope of Ta_k

- When T increases or the fundamental frequency $\xi_0 = 2\nu / T$ decreases, the envelope is sampled with a closer and closer spacing. As T becomes arbitrarily large, the original periodic square wave approaches a rectangular pulse.

- Ta_k becomes more and more closely spaced samples of the envelope, as $T \to \infty$, the Fourier series coefficients approaches the envelope function.

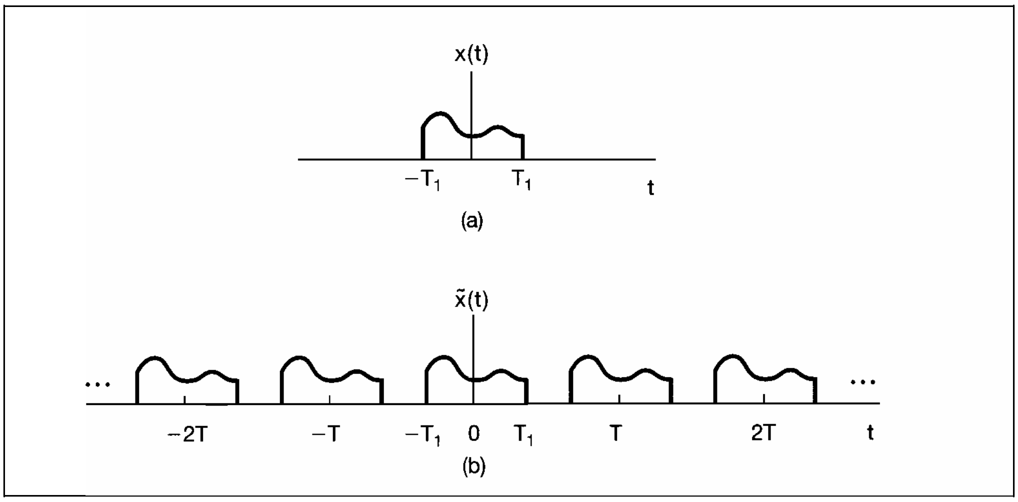


This example illustrates the basic idea behind Fourier's development of a representation for aperiodic signals.

Based on this idea, we can derive the Fourier transform for aperiodic signals.

Suppose a signal $x(t)$ with a finite duration, that is, $x(t) = 0$ for $|t| > T_1$, as illustrated in the figure below.

- From this aperiodic signal, we construct a periodic signal $\tilde{x}(t)$, shown in the figure below.



- As $T \to \infty$, $\tilde{x}(t) = x(t)$, for any infinite value of t.
- The Fourier series representation of $\tilde{x}(t)$ is

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\xi_0 t}, \qquad (4.4)$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\xi_0 t} dt. \qquad (4.5)$$

- Since $\tilde{x}(t) = x(t)$ for $|t| < T/2$, and also, since $x(t) = 0$ outside this interval, so we have

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\xi_0 t} dt = \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\xi_0 t} dt.$$

- Define the envelope $X(j\xi)$ of Ta_k as

$$X(j\xi) = \int_{-\infty}^{\infty} x(t) e^{-j\xi t} dt. \qquad (4.6)$$

we have for the coefficients a_k,

$$a_k = \frac{1}{T} X(jk\xi_0)$$

Then $\tilde{x}(t)$ can be expressed in terms of $X(j\xi)$, that is

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jk\xi_0) e^{jk\xi_0 t} = \frac{1}{2\nu} \sum_{k=-\infty}^{+\infty} X(jk\xi_0) e^{jk\xi_0 t} \xi_0. \qquad (4.7)$$

- As $T \to \infty$, $\tilde{x}(t) = x(t)$ and consequently, Eq. (4.7) becomes a representation of $x(t)$.
- In addition, $\xi_0 \to 0$ as $T \to \infty$, and the right-hand side of Eq. (4.7) becomes an integral.

We have the following Fourier transform:

$$x(t) = \frac{1}{2\nu}\int_{-\infty}^{\infty} X(j\xi)e^{j\xi t}d\xi \quad \textit{Inverse Fourier Transform} \tag{4.8}$$

and

$$X(j\xi) = \int_{-\infty}^{\infty} x(t)e^{-j\xi t}dt \quad \textit{Fourier Transform} \tag{4.9}$$

4.1.2 Convergence of Fourier Transform

If the signal $x(t)$ has finite energy, that is, it is square integrable,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty , \tag{4.10}$$

Then we guaranteed that $X(j\xi)$ is finite or Eq. (4.9) converges. If $e(t) = \tilde{x}(t) - x(t)$, we have

$$\int_{-\infty}^{\infty} |e(t)|^2 dt = 0 . \tag{4.11}$$

An alterative set of conditions that are sufficient to ensure the convergence:

Contition1: Over any period, $x(t)$ must be absolutely integrable, that is

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty , \tag{4.12}$$

Condition 2: In any finite interval of time, $x(t)$ have a finite number of maxima and mi nima.

Condition 3: In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

4.1.3 Examples of Continuous-Time Fourier Transform

Example : consider signal $x(t) = e^{-at}u(t)$, $a > 0$.

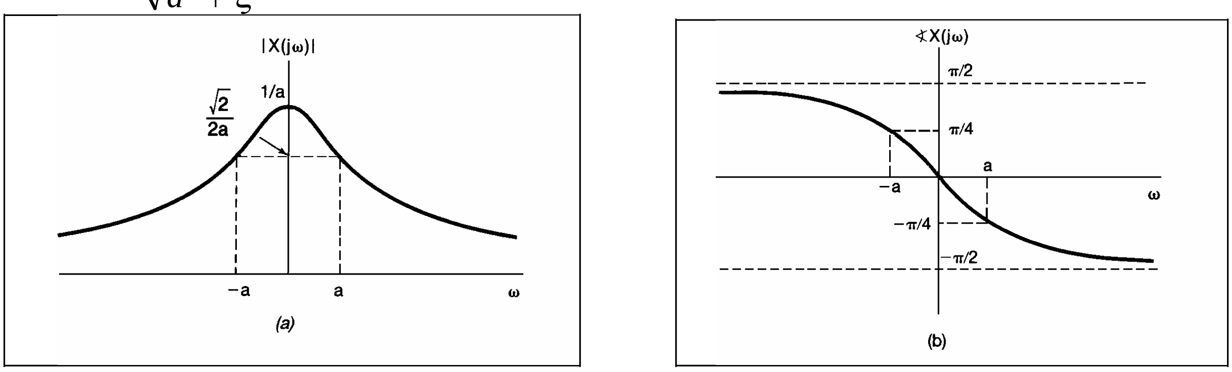
From Eq. (4.9),

$$X(j\xi) = \int_0^\infty e^{-at} e^{-j\xi t} dt = -\frac{1}{a + j\xi} e^{-(a+j\xi)t} \Big|_0^\infty = \frac{1}{a + j\xi}, \qquad a > 0 \tag{4.12}$$

If a is complex rather then real, we get the same result if $\mathrm{Re}\{a\} > 0$

The Fourier transform can be plotted in terms of the magnitude and phase, as shown in the figure below.

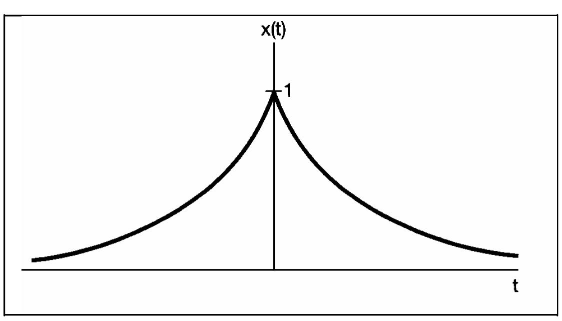
$$|X(j\xi)| = \frac{1}{\sqrt{a^2 + \xi^2}}, \qquad \angle X(j\xi) = -\tan^{-1}\left(\frac{\xi}{a}\right). \tag{4.13}$$

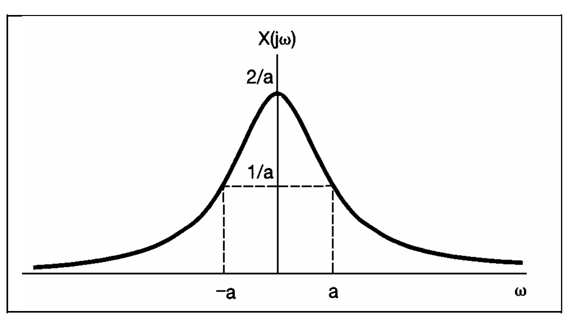


Example : Let $x(t) = e^{-a|t|}$, $\quad a > 0$

$$X(j\xi) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\xi t} dt = \int_{-\infty}^{0} e^{at} e^{-j\xi t} dt + \int_{0}^{\infty} e^{-at} e^{-j\xi t} dt = \frac{1}{a - j\xi} + \frac{1}{a + j\xi} = \frac{2a}{a^2 + \xi^2}$$

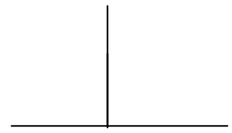
The signal and the Fourier transform are sketched in the figure below.

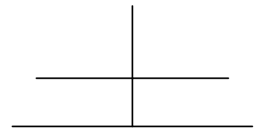




Example: $x(t) = \delta(t)$. (4.14)

$$X(j\xi) = \int_{-\infty}^{\infty} \delta(t) e^{-j\xi t}\, dt = 1. \quad (4.15)$$

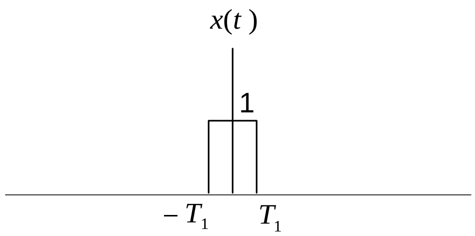


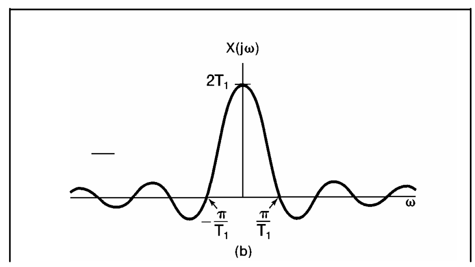


That is, the impulse has a Fourier transform consisting of equal contributions at all frequencies.

Example : Calculate the Fourier transform of the rectangular pulse signal

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}. \quad (4.16)$$





$$X(j\xi) = \int_{-\infty}^{\infty} x(t) e^{-j\xi t} dt = \int_{-T_1}^{T_1} 1 e^{-j\xi t} dt = 2\frac{\sin \xi T_1}{\xi}. \quad (4.17)$$

The Inverse Fourier transform is

$$\hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2 \frac{\sin \xi T_1}{\xi} e^{j\xi t} d\xi , \quad (4.18)$$

Since the signal $x(t)$ is square integrable,

$$e(t) = \int_{-\infty}^{\infty} |x(t) - \hat{x}(t)|^2 dt = 0 . \quad (4.19)$$

$\hat{x}(t)$ converges to $x(t)$ everywhere except at the discontinuity, $t = \pm T_1$, where $\hat{x}(t)$ converges to ½, which is the average value of $x(t)$ on both sides of the discontinuity.

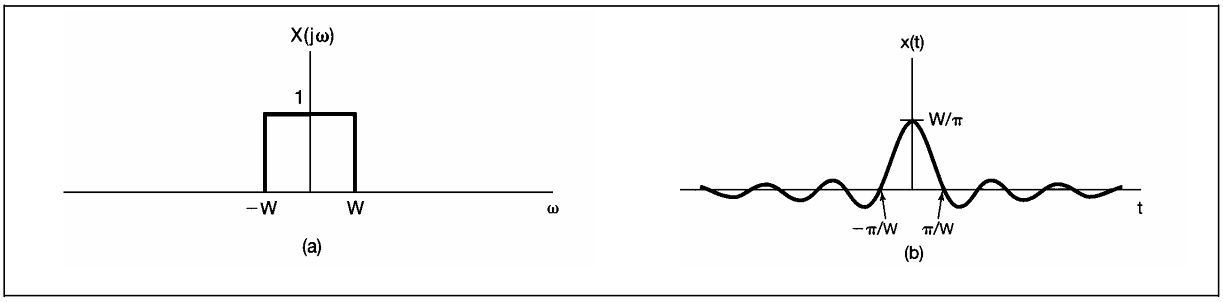
In addition, the convergence of $\hat{x}(t)$ to $x(t)$ also exhibits *Gibbs phenomenon*. Specifically, the integral over a finite-length interval of frequencies

$$\frac{1}{2\pi} \int_{-W}^{W} 2 \frac{\sin \xi T_1}{\xi} e^{j\xi t} d\xi$$

As $W \to \infty$, this signal converges to $x(t)$ everywhere, except at the discontinuities. More over, the signal exhibits ripples near the discontinuities. The peak values of these ripples do not decrease as W increases, although the ripples do become compressed toward the discontinuity, and the energy in the ripples converges to zero.

Example : Consider the signal whose Fourier transform is

$$X(j\xi) = \begin{cases} 1, & |\xi| < W \\ 0, & |\xi| > W \end{cases}.$$



The Inverse Fourier transform is

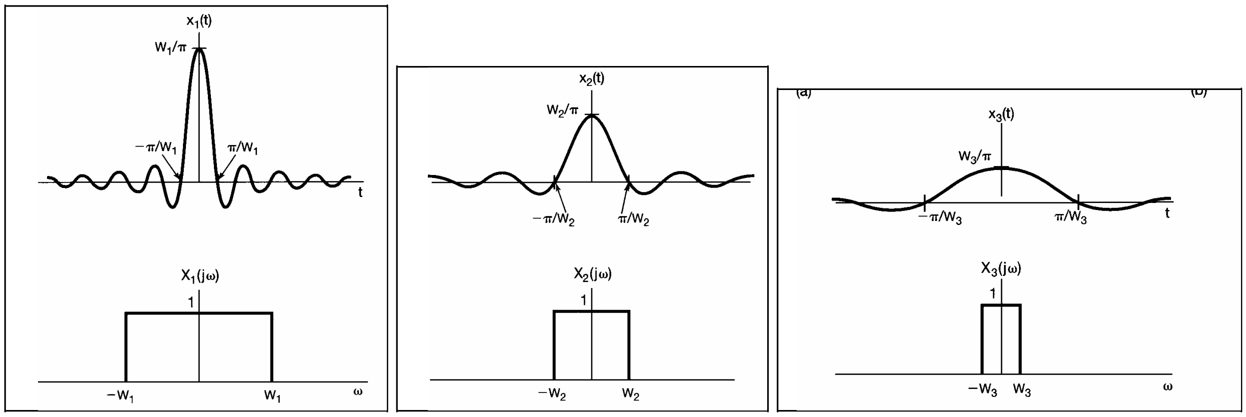
$$x(t) = \frac{1}{2\nu}\int_{-W}^{W} e^{j\xi t}\, d\xi = \frac{\sin Wt}{\nu t}.$$

Comparing the results in the preceding example and this example, we have

$$\textit{Square wave} \;\underset{FT^{-1}}{\overset{FT}{\rightleftarrows}}\; \textit{Sinc function}$$

This means a square wave in the time domain, its Fourier transform is a *sinc* function. However, if the signal in the time domain is a *sinc* function, then its Fourier transform is a square wave. This property is referred to as ***Duality Property***.

We also note that when the width of $X(j\xi)$ increases, its inverse Fourier transform $x(t)$ will be compressed. When $W \to \infty$, $X(j\xi)$ converges to an impulse. The transform pair with several different values of W is shown in the figure below.



4.2 The Fourier Transform for Periodic Signals

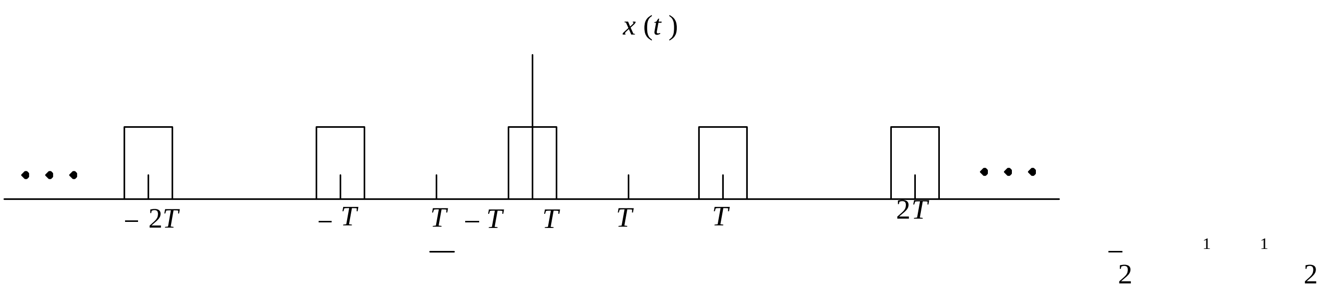
The Fourier series representation of the signal $x(t)$ is

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\xi_0 t} . \qquad (4.20)$$

It's Fourier transform is

$$X(j\xi) = \sum_{k=-\infty}^{\infty} 2\nu a_k 6(\xi - k\xi_0) . \qquad (4.21)$$

Example : If the Fourier series coefficients for the square wave below are given



$$a_k = \frac{\sin k\xi_0 T_1}{\nu k}, \qquad (4.22)$$

The Fourier transform of this signal is

$$X(j\xi) = \sum_{k=-\infty}^{\infty} \frac{2 \sin k\xi_0 T_1}{k} 6(\xi - k\xi_0) . \qquad (4.23)$$

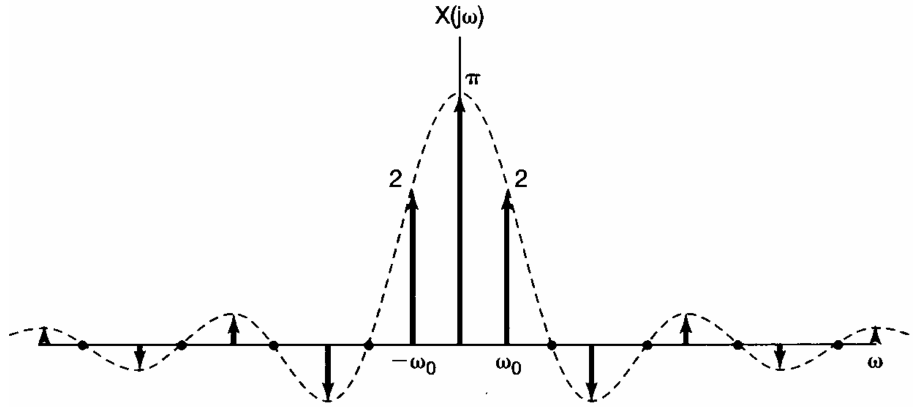
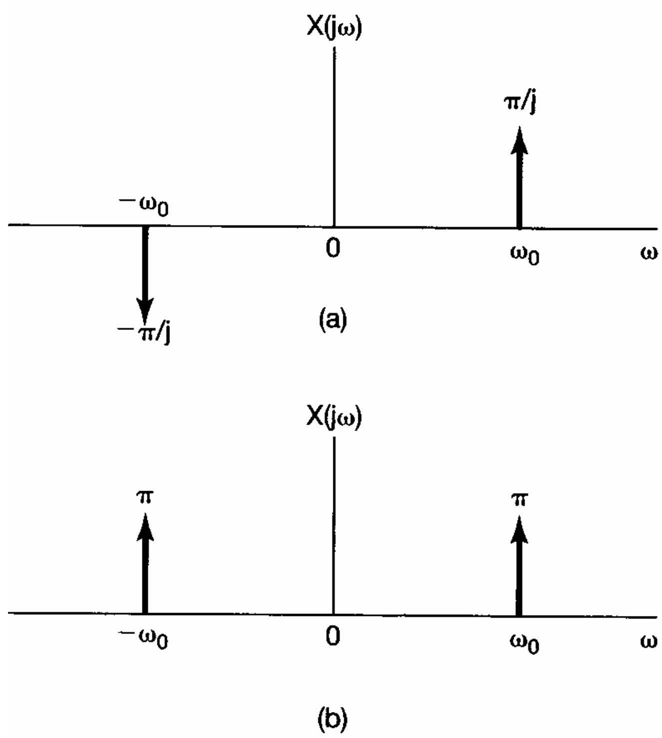


Figure 4.12 Fourier transform of a symmetric periodic square wave.

Example: The Fourier transforms for $x(t)=\sin\xi_0 t$ and $x(t)=\cos\xi_0 t$ are shown in the figure below.



Fourier transforms of (a) $x(t) = \sin \omega_0 t$; (b) $x(t) = \cos \omega_0 t$.

Example: Calculate the Fourier transform for signal $x(t) = \sum_{k=-\infty}^{\infty} 6(t - kT)$.

The Fourier series of this signal is

$$a_k = \frac{1}{T}\int_{-T/2}^{+T/2} 6(t)e^{-j\xi_0 t} = \frac{1}{T} .$$

The Fourier transform is

$$X(j\xi) = \frac{2\nu}{T}\sum_{k=-\infty}^{\infty} 6(\xi - \frac{2\nu k}{T_0}) .$$

The Fourier transform of a periodic impulse train in the time domain with period T is a periodic impulse train in the frequency domain with period $2\nu / T$, as sketched din the figure below.

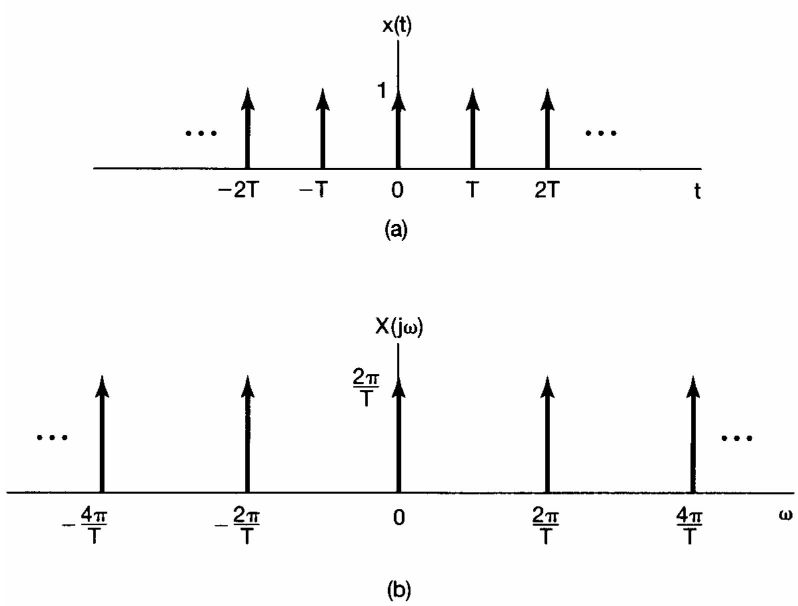


Figure 4.14 (a) Periodic impulse train; (b) its Fourier transform.

4.3 Properties of The Continuous-Time Fourier Transform

4.3.1 Linearity

If $x(t) \xleftrightarrow{F} X(j\xi)$ and $y(t) \xleftrightarrow{F} Y(j\xi)$

Then

$$ax(t)+by(t)\overset{F}{\longleftrightarrow}aX(j\xi)+bY(j\xi) \quad (4.20)$$

4.3.2 Time Shifting

If $x(t)\overset{F}{\longleftrightarrow}X(j\xi)$

Then

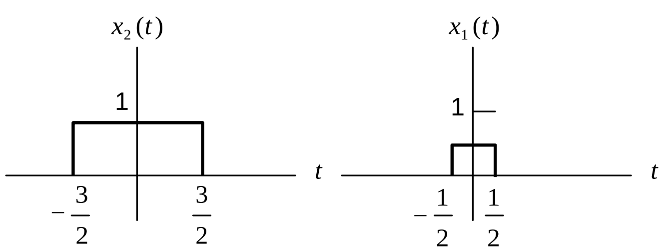
$$x(t-t_0)\overset{F}{\longleftrightarrow}e^{-j\xi t_0}X(j\xi) \quad (4.20)$$

Or

$$F\{x(t-t_0)\}=e^{-j\xi t_0}X(j\xi)=|X(j\xi)|e^{j[\angle X(j\xi)-\xi t_0]} \quad (4.20)$$

Thus, the effect of a time shift on a signal is to introduce into its transform a phase shift, namely, $-\xi_0 t$.

Example: To evaluate the Fourier transform of the signal $x(t)$ shown in the figure below.



The signal $x(t)$ can be expressed as the linear combination

$$x(t)=\frac{1}{2}x_1(t-2.5)+x_2(t-2.5). \quad (4.20)$$

$x_1(t)$ and $x_2(t)$ are rectangular pulse signals and their Fourier transforms are

$$X_1(j\xi) = \frac{2\sin(\xi/2)}{\xi} \quad \text{and} \quad X_2(j\xi) = \frac{2\sin(3\xi/2)}{\xi}$$

Using the linearity and time -shifting properties of the Fourier transform yields

$$X(j\xi) = e^{-j5\xi/2}\left\{\frac{\sin(\xi/2) + 2\sin(3\xi/2)}{\xi}\right\}$$

4.3.3 Conjugation and Conjugate Symmetry

If $x(t) \xleftrightarrow{F} X(j\xi)$

Then

$$x^*(t) \xleftrightarrow{F} X^*(-j\xi). \qquad (4.\ 20)$$

Since $X^*(j\xi) = \left[\int_{-\infty}^{+\infty} x(t)e^{-j\xi t}dt\right]^* = \int_{-\infty}^{+\infty} x^*(t)e^{j\xi t}dt$,

Replacing ξ by $-\xi$, we see that

$$X^*(-j\xi) = \int_{-\infty}^{+\infty} x^*(t)e^{-j\xi t}dt, \qquad (4.\ 20)$$

The right-hand side is the Fourier transform of $x^*(t)$.

If $x(t)$ is ***real***, from Eq. (4.20) we can get

$$X(-j\xi) = X^*(j\xi). \qquad (4.\ 20)$$

We can also prove that if $x(t)$ is both real and even, then $X(j\xi)$ will also be real and even. Similarly, if $x(t)$ is both real and odd, then $X(j\xi)$ will also be purely imaginary and odd.

A real function $x(t)$ can be expressed in terms of the sum of an even function $x_e(t) = Ev\{x(t)\}$ and an odd function $x_o(t) = Od\{x(t)\}$. That is

$$x(t) = x_e(t) + x_o(t)$$

Form the Linearity property,

$$F\{x(t)\} = F\{x_e(t)\} + F\{x_o(t)\},$$

From the preceding discussion, $F\{x_e(t)\}$ is real function and $F\{x_o(t)\}$ is purely imaginary. Thus we conclude with $x(t)$ real,

$$x(t) \overset{F}{\longleftrightarrow} X(j\xi)$$

$$Ev\{x(t)\} \overset{F}{\longleftrightarrow} \mathrm{Re}\{X(j\xi)\}$$

$$Od\{x(t)\} \overset{F}{\longleftrightarrow} j\,\mathrm{Im}\{X(j\xi)\}$$

Example: Using the symmetry properties of the Fourier transform and the result $e^{-at}u(t) \overset{F}{\longleftrightarrow} \dfrac{1}{a+j\xi}$ to evaluate the Fourier transform of the signal $x(t) = e^{-a|t|}$, where $a > 0$.

Since $x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t) = 2\left[\dfrac{e^{-at}u(t) + e^{at}u(-t)}{2}\right] = 2Ev\{e^{-at}u(t)\}$,

So $X(j\xi) = 2\,\mathrm{Re}\left(\dfrac{1}{a+j\xi}\right) = \dfrac{2a}{a^2+\xi^2}$

4.3.4 Differentiation and Integration

If $x(t) \overset{F}{\longleftrightarrow} X(j\xi)$

Then

$$\frac{dx(t)}{dt} \overset{F}{\longleftrightarrow} j\xi X(j\xi). \qquad (4.\ 20)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{F}{\longleftrightarrow} \frac{1}{j\xi}X(j\xi) + \pi X(0)\delta(\xi). \qquad (4.\ 20)$$

Example : Consider the Fourier transform of the unit step $x(t) = u(t)$.

It is know that

$g(t) = 6(t) \overset{F}{\longleftrightarrow} 1$

Also note that

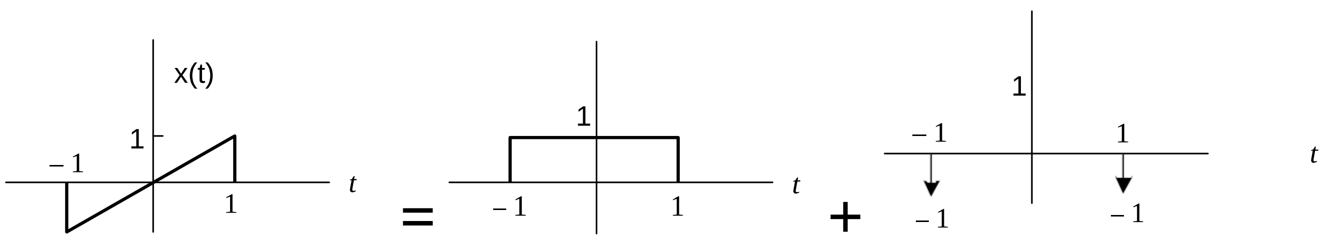
$$x(t) = \int_{-\infty}^{t} g(\lambda)d\lambda$$

The Fourier transform of this function is

$$X(j\xi) = \frac{1}{j\xi} + \nu G(0)6(\xi) = \frac{1}{j\xi} + \nu 6(\xi) \; .$$

where $G(0) = 1$.

Example: Consider the Fourier transform of the function $x(t)$ shown in the figure below.



$$g(t) = \frac{dx(t)}{dt}$$

From the above figure we can see that $g(t)$ is the sum of a rectangular pulse and two impulses.

$$G(j\xi) = \left(\frac{2\sin\xi}{\xi}\right) - e^{j\xi} - e^{-j\xi}$$

Note that $G(0) = 0$, using the integration property, we obtain

$$X(j\xi) = \frac{G(j\xi)}{j\xi} + \nu G(0)6(\xi) = \frac{2\sin\xi}{j\xi^2} - \frac{2\cos\xi}{j\xi} \; .$$

It can be found $X(j\xi)$ is purely imaginary and odd, which is consistent with the fact that $x(t)$ is real and odd.

4.3.5 Time and Frequency Scaling

$$x(t) \overset{F}{\longleftrightarrow} X(j\xi),$$

Then

$$x(at) \overset{F}{\longleftrightarrow} \frac{1}{|a|} X(\frac{j\xi}{a}). \qquad (4.\ 20)$$

From the equation we see that the signal is compressed in the time domain, the spectrum will be extended in the frequency domain. Conversely, if the signal is extended, the corresponding spectrum will be compressed.

If $a = -1$, we get from the above equation,

$$x(-t) \overset{F}{\longleftrightarrow} X(-j\xi). \qquad (4.\ 20)$$

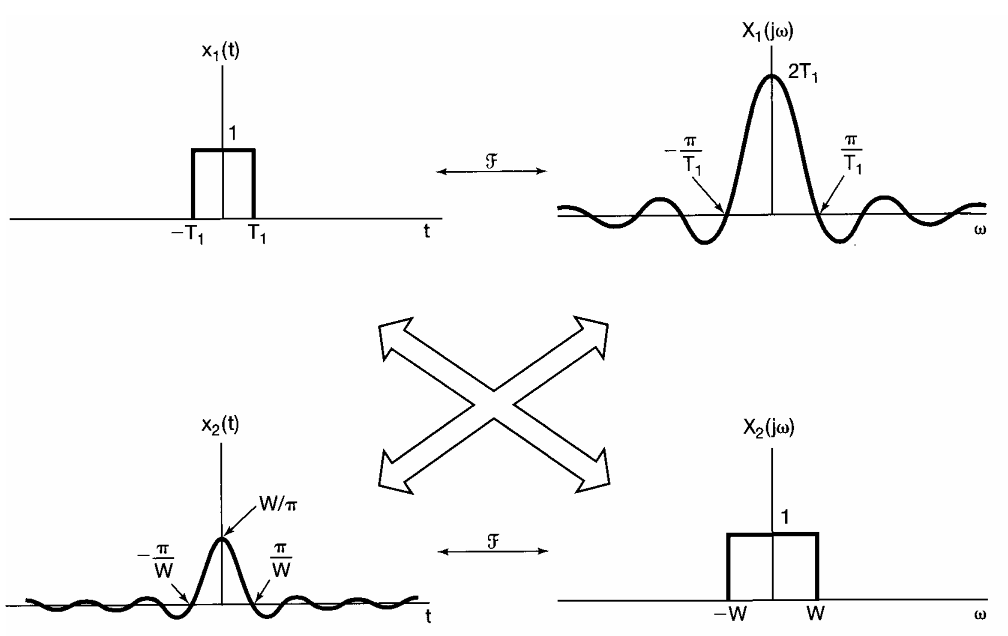
That is, reversing a signal in time also reverses its Fourier transform.

4.3.6 Duality

The *duality* of the Fourier transform can be demonstrated using the following example.

$$x_1(t) = \begin{cases} 1, & t < T_1 \\ 0, & t > T_1 \end{cases} \overset{F}{\longleftrightarrow} X_1(j\xi) = \frac{2\sin \xi T_1}{\xi}$$

$$x_2(t) = \frac{\sin WT_1}{\nu t} \overset{F}{\longleftrightarrow} X_2(j\xi) = \begin{cases} 1, & |\xi| < W \\ 0, & |\xi| > W \end{cases}$$



The symmetry exhibited by these two examples extends to Fourier transform in general. For any transform pair, there is a dual pair with the time and frequency variables interchanged.

Example : Consider using duality and the result $e^{-|t|} \overset{F}{\longleftrightarrow} X(j\xi) = \dfrac{2}{1+\xi^2}$ to find the Fourier transform $G(j\xi)$ of the signal

$$g(t) = \frac{2}{1+t^2}.$$

Since $e^{-|t|} \overset{F}{\longleftrightarrow} X(j\xi) = \dfrac{2}{1+\xi^2}$, that is,

$$e^{-|t|} = \frac{1}{2\nu}\int_{-\infty}^{\infty}\left(\frac{2}{1+\xi^2}\right)e^{j\xi t}d\xi,$$

Multiplying this equation by 2ν and replacing t by $-t$, we have

$$2\nu e^{-|t|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+\xi^2}\right)e^{-j\xi t}d\xi$$

Interchanging the names of the variables t and ξ, we find that

$$2\nu e^{-|\xi|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+t^2}\right)e^{-j\xi t}d\xi \;\Rightarrow\; F^{-1}\left(\frac{2}{1+t^2}\right) = 2\nu e^{-|\xi|}.$$

Based on the duality property we can get some other properties of Fourier transform:

$$-jtx(t) \overset{F}{\longleftrightarrow} \frac{dX(j\xi)}{d\xi}$$

$$e^{j\xi_0 t}x(t) \overset{F}{\longleftrightarrow} X(j(\xi - \xi_0))$$

$$-\frac{1}{jt}x(t) + \nu x(0)\delta(t) \overset{F}{\longleftrightarrow} \int_{-\infty}^{\xi} x(\psi)d\psi$$

4.3.7 Parseval's Relation

If $x(t) \stackrel{F}{\longleftrightarrow} X(j\xi)$,

We have

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\nu} \int_{-\infty}^{\infty} |X(j\xi)|^2 d\xi$$

Parseval's relation states that the total energy may be determined either by computing the energy per unit time $|x(t)|^2$ and integrating over all time or by computing the energy per unit frequency $|X(j\xi)|^2 / 2\nu$ and integrating over all frequencies. For this reason, $|X(j\xi)|^2$ is often referred to as the ***energy-density spectrum***.

4.4 The convolution properties

$$y(t) = h(t) * x(t) \stackrel{F}{\longleftrightarrow} Y(j\xi) = H(j\xi)X(j\xi)$$

The equation shows that the Fourier transform maps the convolution of two signals into product of their Fourier transforms.

$H(j\xi)$, the transform of the impulse response, is the frequency response of the LTI system, which also completely characterizes an LTI system.

Example : The frequency response of a differentiator.

$$y(t) = \frac{dx(t)}{dt}.$$

From the differentiation property,

$$Y(j\xi) = j\xi X(j\xi),$$

The frequency response of the differentiator is

$$H(j\xi) = \frac{Y(j\xi)}{X(j\xi)} = j\xi.$$

Example : Consider an integrator specified by the equation:

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau .$$

The impulse response of an integrator is the unit step, and therefore the frequency response of the system:

$$H(j\xi) = \frac{1}{j\xi} + \pi\delta(\xi) .$$

So we have

$$Y(j\xi) = H(j\xi)X(j\xi) = \frac{1}{j\xi}X(j\xi) + \pi X(0)\delta(\xi) ,$$

which is consistent with the integration property.

Example : Consider the response of an LTI system with impulse response

$$h(t) = e^{-at}u(t) , \qquad a > 0$$

to the input signal

$$x(t) = e^{-bt}u(t), \qquad b > 0$$

To calculate the Fourier transforms of the two functions:

$$X(j\xi) = \frac{1}{b + j\xi}, \text{ and}$$
$$H(j\xi) = \frac{1}{a + j\xi}.$$

Therefore,

$$Y(j\xi) = \frac{1}{(a + j\xi)(b + j\xi)} ,$$

using partial fraction expansion(assuming $a \neq b$), we have

$$Y(j\xi) = \frac{1}{b - a}\left[\frac{1}{a + j\xi} - \frac{1}{b + j\xi}\right]$$

The inverse transform for each of the two terms can be written directly. Using the linearity property, we have

$$y(t) = \frac{1}{b-a}\left[e^{-at}u(t) - e^{-bt}u(t)\right].$$

We should note that when $a = b$, the above partial fraction expansion is not valid. However, with $a = b$, we have

$$Y(j\xi) = \frac{1}{(a+j\xi)^2},$$

Considering $\frac{1}{(a+j\xi)^2} = j\frac{d}{d\xi}\left[\frac{1}{a+j\xi}\right]$, and

$$e^{-at}u(t) \xleftrightarrow{F} \frac{1}{a+j\xi}, \text{ and}$$

$$te^{-at}u(t) \xleftrightarrow{F} j\frac{d}{d\xi}\left[\frac{1}{a+j\xi}\right],$$

so we have

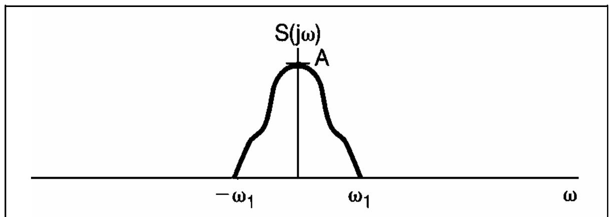
$$Y(t) = te^{-at}u(t).$$

4.5 The Multiplication Property

$$r(t) = s(t)p(t) \longleftrightarrow R(j\xi) = \frac{1}{2\nu}\int_{-\infty}^{+\infty} S(j\theta)P(j(\xi-\theta))d\theta$$

Multiplication of one signal by another can be thought of as one signal to scale or modulate the amplitude of the other, and consequently, the multiplication of two signals is often referred to as ***amplitude modulation***.

Example : Let $s(t)$ be a signal whose spectrum $S(j\xi)$ is depicted in the figure below.



(a)

Also consider the signal

$p(t) = \cos\xi_0 t$, then

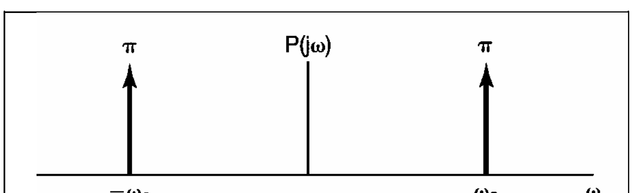
$P(j\xi) = \nu 6(\xi - \xi_0) + \nu 6(\xi + \xi_0)$.

The spectrum of $r(t) = s(t)\,p(t)$ is obtained by using the multiplication property,

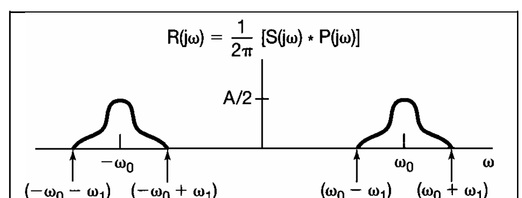
$$R(j\xi) = \frac{1}{2\nu}\int_{-\infty}^{+\infty} S(j\xi) P(j(\xi - 0))d0 ,$$

$$= \frac{1}{2}S(j\xi - \xi_0) + \frac{1}{2}S(j\xi + \xi_0)$$

which is sketched in the figure below.



(b)



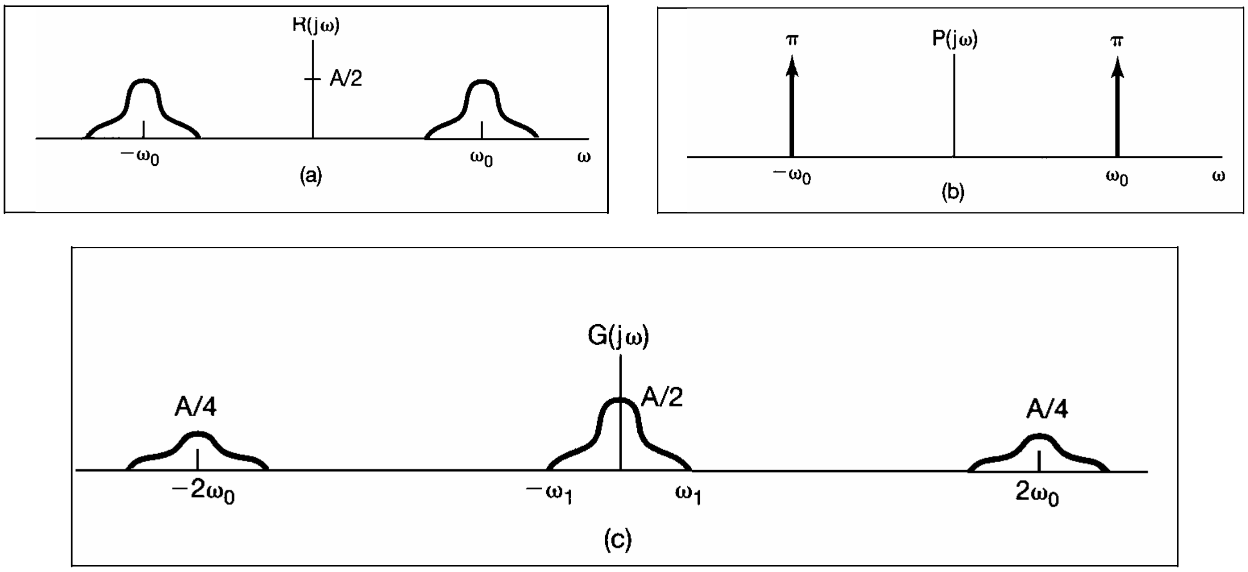
(c)

From the figure we can see that the signal is preserved although the information has been shifted to higher frequencies. This forms the basic for ***sinusoidal amplitude modulation*** systems for communications.

Example: If we perform the following multiplication using the signal $r(t)$ obtained in the preceding example and $p(t) = \cos\xi_0 t$, that is,

$g(t) = r(t)\,p(t)$

The spectrum of $P(j\xi)$, $R(j\xi)$ and $G(j\xi)$ are plotted in the figure below.



If we use a lowpass filter with frequency response $H(j\xi)$ that is constant at low frequencies and zero at high frequencies, then the output will be a scaled replica of $S(j\xi)$. Then the output will be scaled version of $s(t)$ - the modulated signal is recovered.

4.6 Summary of Fourier Transform Properties and Basic Fourier Transform Pairs

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t)\begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin\omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \ \mathcal{R}e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \ \mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

4.7 System Characterized by Linear Constant-Coefficient Differential Equations

An LTI system described by the following differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}, \tag{4. 67}$$

which is commonly referred to as an Nth-order differential equation. The frequency response of this LTI system

$$H(j\xi) = \frac{Y(j\xi)}{X(j\xi)}, \tag{4. 68}$$

where $X(j\xi)$, $Y(j\xi)$ and $H(j\xi)$ are the Fourier transforms of the input $x(t)$, output $y(t)$ and the impulse response $h(t)$, respectively.

Applying Fourier transform to both sides, we have

$$F\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = F\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}, \tag{4. 69}$$

From the linearity property, the expression can be written as

$$\sum_{k=0}^{N} a_k F\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k F\left\{\frac{d^k x(t)}{dt^k}\right\}. \tag{4. 70}$$

From the differentiation property,

$$\sum_{k=0}^{N} a_k (j\xi)^k Y(j\xi) = \sum_{k=0}^{M} b_k (j\xi)^k X(j\xi) \quad \Rightarrow \quad H(j\xi) = \frac{Y(j\xi)}{X(j\xi)} = \frac{\sum_{k=0}^{M} b_k (j\xi)^k}{\sum_{k=0}^{N} a_k (j\xi)^k} \tag{4. 71}$$

$H(j\xi)$ is a rational function, that is, it is a ratio of polynomials in $(j\xi)$.

Example : Consider a stable LTI system characterized by the differential equation

$\frac{dy(t)}{dt} + ay(t) = x(t)$, with $a > 0$.

The frequency response is

$$H(j\xi) = \frac{1}{j\xi + a}.$$

Te impulse response of this system is then recognized as

$$h(t) = e^{-at}u(t).$$

Example : Consider a stable LTI system that is characterized by the differential equation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t).$$

The frequency response of this system is

$$H(j\xi) = \frac{(j\xi) + 2}{(j\xi)^2 + 4(j\xi) + 3} = \frac{j\xi + 2}{(j\xi + 1)(j\xi + 3)}.$$

Then, using the method of partial-fraction expansion, we find that

$$H(j\xi) = \frac{1/2}{j\xi + 1} + \frac{1/2}{j\xi + 3}.$$

The inverse Fourier transform of each term can be recognized as

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t).$$

Example: Consider a system with frequency response of $H(j\xi) = \frac{j\xi + 2}{(j\xi + 1)(j\xi + 3)}$ and suppose that the input to the system is

$$x(t) = e^{-t}u(t),$$

find the output response.

The output in the frequency domain is give as

$$Y(j\xi) = H(j\xi)X(j\xi) = \left[\frac{j\xi + 2}{(j\xi + 1)(j\xi + 3)}\right]\left[\frac{1}{j\xi + 1}\right] = \frac{j\xi + 2}{(j\xi + 1)^2(j\xi + 3)},$$

Using partial-fraction expansion, we have

$$Y(j\xi) = \frac{1/4}{j\xi + 1} + \frac{1/2}{(j\xi + 1)^2} + \frac{1/4}{(j\xi + 3))},$$

By inspection, we get directly the inverse Fourier transform:

$$h(t) = \left[\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-3t}\right]u(t).$$

MODULE – III
SIGNAL TRANSMISSION THROUGH LINEAR SYSTEMS

Linear System, Impulse response, Response of a Linear System, Linear Time Invariant(LTI) System, Linear Time Variant (LTV) System, Transfer function of a LTI System, Filter characteristic of Linear System, Distortion less transmission through a system, Signal bandwidth, System Bandwidth, Ideal LPF, HPF, and BPF characteristics.
Causality and Paley-Wiener criterion for physical realization, Relationship between Bandwidth and rise time, Convolution and Correlation of Signals, Concept of convolution in Time domain and Frequency domain, Graphical representation of Convolution.

Linear systems

A system is said to be a linear if it obeys homogeneity and additivity properties. This implies that the response of a linear system to weighted sum of input signals is equal to the same weighted sum of responses of the system to each of those signals.

Homogeneity property: This property says if input signal weighted by any arbitrary constant then output signal also weighted by same arbitrary constant

$y(t)$ *is response of input signal* $x(t)$

$x(t) \rightarrow y(t) = T\,[x(t)]$

$ax(t) \rightarrow T\,[ax(t)] = a\,T\,[x(t)] = a\,y(t)$ *wher a is any arbitary constant*

Additive property: Response of system to sum of two input signals is equal to sum of individual response of the system.

$x_1(t) \rightarrow y_1(t) = T\,[x_1(t)]$

$x_2(t) \rightarrow y_2(t) = T\,[x_2(t)]$

$x_1(t) + x_2(t) \rightarrow T\,[x_1(t) + x_2(t)\,] = T\,[x_1(t)] + T\,[x_2(t)\,] = y_1(t) + y_2(t)$

Combining above two properties

$x(t) = \sum_{i=1}^{N} a_i\, x_i\,(t)$ *be the sum of N number of inpit signals wher* a_i *is*Arbitrary constant

Response

$$y(t) = T\left[\sum_{i=1}^{N} a_i\, x_i\,(t)\right] = \sum_{i=1}^{N} a_i\, T\,[x_i\,(t)\,] = \sum_{i=1}^{N} a_i\, y_i\,(t)$$

Where $y_i(t)$ is the output of the system in response to *the in puts* $x_i(t)$

Classification of linear systems

Lumped and Distributed system

Time – Invariant and Time Variant system

Lumped and Distributed system: A **Lumped System** consists of lumped elements which are interconnected in particular way. The energy in the system is considered to be stored or dissipated in distinct isolated elements. The disturbance initiated at any point is propagated instantaneously at every point in the system. The dimension of elements is very small compared to wave length of the signals to be transmitted. Lumped system obeys Ohms law and Kirchhoff laws. They can be expressed with ordinary differential equations. Examples are TVS, motors, computers, any packed sytems

Distributed systems are those in which elements are distributed over a long distances and dimensions of the circuits are small compared to the wave length of signals to be transmitted. More over such system takes finite amount of time for disturbance at one point to be propagated to the other point. They can be expressed with partial differential equations. Example are wave guides, optical fiber, transmission lines, antennas.

Linear Time Invariant (LTI) System: A system said to be LTI if it satisfies linear and invariance properties. Stated in another way, A LTI system whose parameters do not change with time. LTI system is characterized by linear equations such as algebraic, differential, or difference equations with constant coefficients.

Example: Circuits using passive elements are LTI systems

For LTI system, if input is delayed by t_0 seconds the system satisfies superposition and homogeneity principles. Also, the output delayed by the same time t_0 seconds.

$x(t) \rightarrow y(t) = T\,[x(t)]$

$x(t - t_0) \rightarrow T\,[x(t - t_0)]$

$if\ y(t, t_0) = y(t - t_0) = T\,[x(t - t_0)]\ then\ system\ said\ to\ be\ time\ invariant\ sytem$

Linear Time Variant (LTV) System: A system said to be LTV if it satisfies the linear property but not the time invariant. For LTV system, if input delayed by t_0 seconds, the system satisfies superposition and homogeneity properties but output varies with time t_0. A LTV system whose parameters change with time. The coefficients in the differential equations are time variant.

$x(t) \rightarrow y(t) = T\,[x(t)]$

$x(t - t_0) \rightarrow T\,[x(t - t_0)]$

$if\ y(t, t_0) = y(t - t_0) \neq T\,[x(t - t_0)]\ then\ system\ said\ to\ be\ time\ variant\ sytem$

Impulse response and response of LTI system

Let us consider $x(t)\ any\ arbitary\ signal$

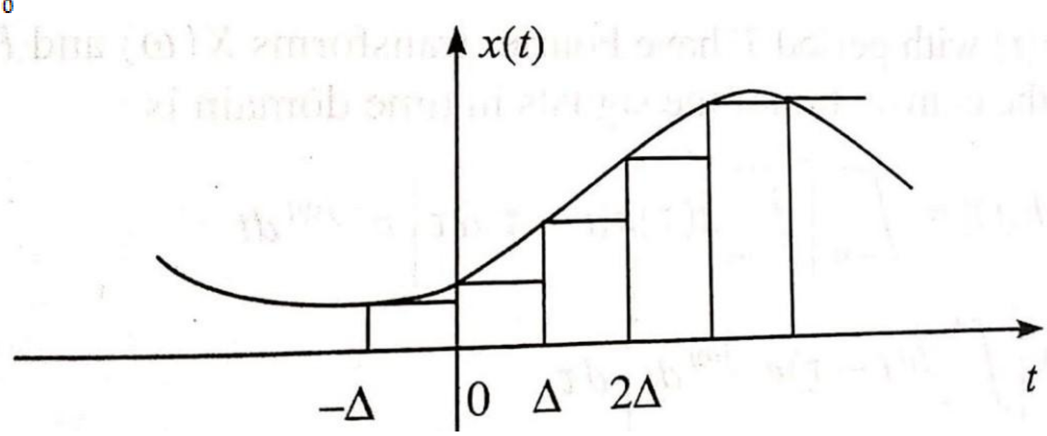
$\widehat{x(t)}\ is\ an\ approximation\ of\ x(t) and\ it\ can\ be\ expressed\ as\ liniear\ combination\ of\ shifted\ impulses$

$$\widehat{x(t)} = \ldots\ldots\ldots + x(-2\Delta)\delta_\Delta(t + 2\Delta) + x(-\Delta)\delta_\Delta(t + \Delta) + x(0)\delta_\Delta(t) + x(\Delta)\delta_\Delta(t - \Delta) + x(2\Delta)\delta_\Delta(t - 2\Delta) + \ldots\ldots\ldots + x(k\Delta)\delta_\Delta(t - k\Delta)$$

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & 0 \le t \le \Delta \\ 0 & othrt\ wise \end{cases}$$

$$\widehat{x(t)} = \sum_{k-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$

$$x(t) = \lim_{\Delta \to 0} \widehat{x(t)}$$



Signal approximation

$as\ \Delta \rightarrow 0\ , summation\ becomes\ integral, \delta_{\Delta}\ (t)\ \rightarrow\ \delta\ (t), k\Delta \rightarrow\ \tau, \Delta \rightarrow\ d\tau$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta\ (t-\tau)\, d\tau$$

$y(t)\ \ is\ response\ of\ x(t)$

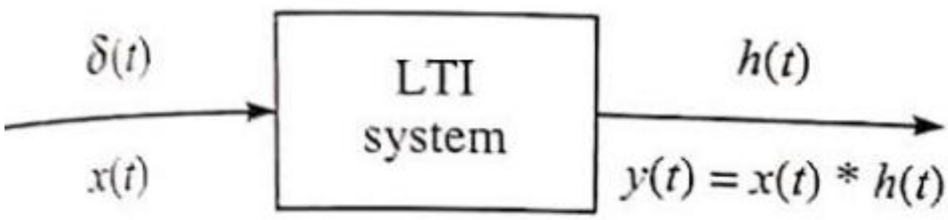
$$y(t) =\ T\ [x(t)]$$

$$y(t) =\ T\left[\int_{-\infty}^{\infty} x(\tau)\delta\ (t-\tau) d\tau\right]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\ T\ [\delta\ (t-\tau)] d\tau$$

$h(t-\tau) =\ T\ [T\ [\delta\ (t-\tau)]\ \ this\ satisfies\ time\ invariant\ property$

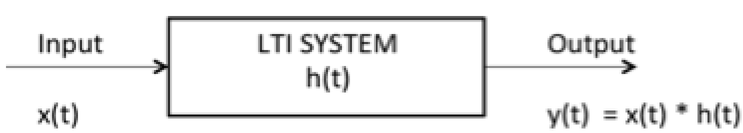
$y(t) = h(t) =\ T\ [\delta\ (t)\ ]\ this\ shows\ impulse\ response\ of\ sytem$



Impulse response of LTI system due to an impulse input applied at t=0 is h (t)

Hence $y(t)\ = \int_{-\infty}^{\infty} x(\tau)h\ (t-\tau) d\tau\ \ = x(t) * h(t)$

This is known as convolution integral and it gives relationship among input signal, output signal and impulse response of system.LTI system completely characterized by impulse response



Frequency response of LTI system:

Let us consider LTI system with impulse response h(t) and y(t) is response of input signal x(t) . Input and output relationship of system given by convolution integral.

.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\ T\ [\delta\ (t-\tau)] d\tau$$

Fourier transform of input x(t) , output y(t) and impulse response h(t) are X(ω) , Y(ω) and H(ω) respectively.

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\ dt$$

$$Y(\omega) =\ \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} x(\tau)h\ (t-\tau) d\tau] e^{-j\omega t}\ dt$$

$$Y(\omega) =\ \int_{-\infty}^{\infty} x(\tau)[\int_{-\infty}^{\infty} h\ (t-\tau)\, dt] e^{-j\omega t} d\tau$$

$$t-\tau = \lambda, t = \lambda + \tau, dt = d\lambda$$

$$Y(\omega) = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau} d\tau \int_{-\infty}^{\infty} h\ (\lambda)e^{-j\omega\lambda} d\lambda$$

$Y(\omega) = X(\omega)H(\omega)$

$H(\omega) is\ a\ complex\ valued\ function\ and\ can\ be\ expressed\ as$

$\boldsymbol{H(\omega) = |H(\omega)| \angle H(\omega)}$

$|H(\omega)|\ \ is\ magnetude\ response\ of\ sytem\ and\ \angle H(\omega)\ phase\ response\ of\ sytem$

Magnetude response is symmetric and phase response is anti symmetric.

Response to Eigen functions

If input to the system is an exponential function $x(t) = e^{j\omega t}$ then output y(t)

$y(t) = x(t) * h(t)$

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\ h(\tau)d\tau$$

$$y(t) = \int_{-\infty}^{\infty} e^{j\omega(t-\tau)}\ h(\tau)d\tau = e^{j\omega t}\int_{-\infty}^{\infty} e^{-j\omega\tau}\ h(\tau)d\tau = e^{j\omega t}H(\omega)$$

Output is a complex exponential of the same frequency as input multiplied by the complex constant $H(\omega)$. An inputs signal is called Eigen functions of the system if the corresponding output is a constant multiple of the input signal. Thus the functions $e^{j\omega t}, sin\omega t, and\ cos\omega t$ all Eigen functions as we get the same function the output as in input.

Properties of LTI system

Commutative Property

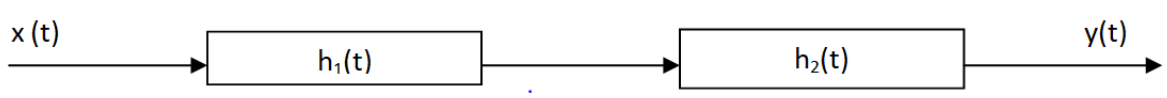
$\boldsymbol{y(t) = x(t) * h(t) = h(t) * x(t)}$

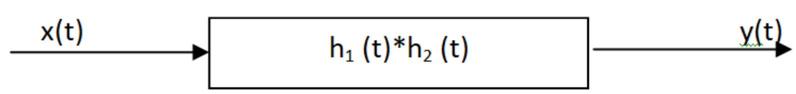
$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\ h(\tau)d\tau = \int_{-\infty}^{\infty} h(t-\tau)\ x(\tau)d\tau$$

Associate property

This implies that a cascading of two or more LTI system will results to single system with impulse response equal to the convolution of the impulse response of the cascading systems.

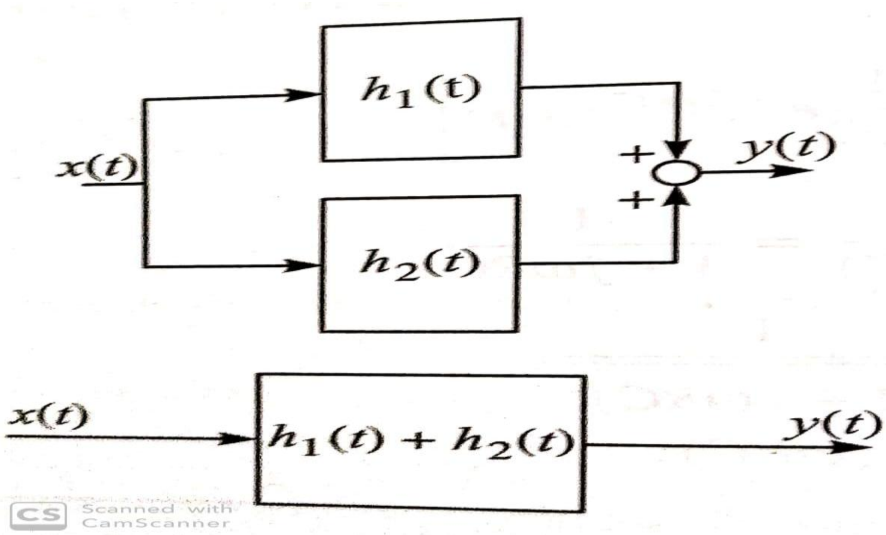
$\{x\ (t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$





Distributive Property

This property gives that addition of two or more LTI system subjected to same input will results single system with impulse response equal to the sum of impulse response of two or more individual systems.



$$x(t) * \{ h_1(t) + h_2(t)\} = x(t) * h_1(t) + x(t) * h_2(t)$$

Static and dynamic system

A system is static or memory less if its output at any time depends only on the value of its input at that instant of time. For LTI systems, this property can hold if its impulse response is itself an impulse. But convolution property, we know that the output depends on the previous samples of the input, therefore an LTI system has memory and hence it is dynamic system.

Causality

A continuous time LTI system is said to causal if and only if it impulse response is h(t) = 0 for t<0, then integral becomes

$$y(t) = \int_{0}^{\infty} h(\tau)x(t-\tau)\, d\tau$$

$$\boldsymbol{h(t)} = \begin{cases} \boldsymbol{non\ zero} & \boldsymbol{for\ t \geq 0} \\ \boldsymbol{o} & \boldsymbol{other\ wise} \end{cases}$$

Stability: a continuous time system is bounded input , bounded output stable if and only if the impulse response is absolutely Integrable.

Consider LTI system with impulse response h(t) . the output y(t) is

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\, d\tau$$

$$|y(t)| = \left|\int_{-\infty}^{\infty} h(\tau)x(t-\tau)\, d\tau\right|$$

$$|y(t)| = \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|\, d\tau$$

If x(t) is bounded and $|x(t)| \leq M_x < \infty$ then

$$|y(t)| \leq \int_{-\infty}^{\infty} M_x |h(\tau)|\, d\tau = M_x \int_{-\infty}^{\infty} |h(\tau)|\, d\tau$$

For bounded output y(t) < ꝏ , the impulse response should be absolutely integrable. Hence

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

Above equation gives necessary and sufficient condition for BIBO stability.

Inevitability:

A system T said to be invertible if and only if there exits an inverse system T^{-1} for such that T T^{-1} is an identical system . For an LTI system with impulse response $h_1(t)$, this is equivalent to the existence of another system with impulse response $h_2(t)$ such that $\mathbf{h_1(t)* h_2(t) = \delta(t)}$.

Transfer Function of LTI System:

Transfer function of LTI system defined as the ratio of Fourier transform of the output signal $Y(\omega)$ to Fourier transform of the input signal$X(\omega)$.It is expressed as

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

Inverse Fourier transforms of $H(\omega)$ gives the impulse response of the system. That is h(t) = IFT of $H(\omega)$.

In general Input and output relationship of continuous time causal LTI system described by linear constant coefficient differential equations with zero initial conditions is given by

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Where a_k and b_k are constant coefficients the order N refer to the highest derivative of y(t) in above equation.

Apply Fourier Transform on both sides of above equation

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

$$System\ function = Transfer\ function = H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

Distortion less Transmission System:

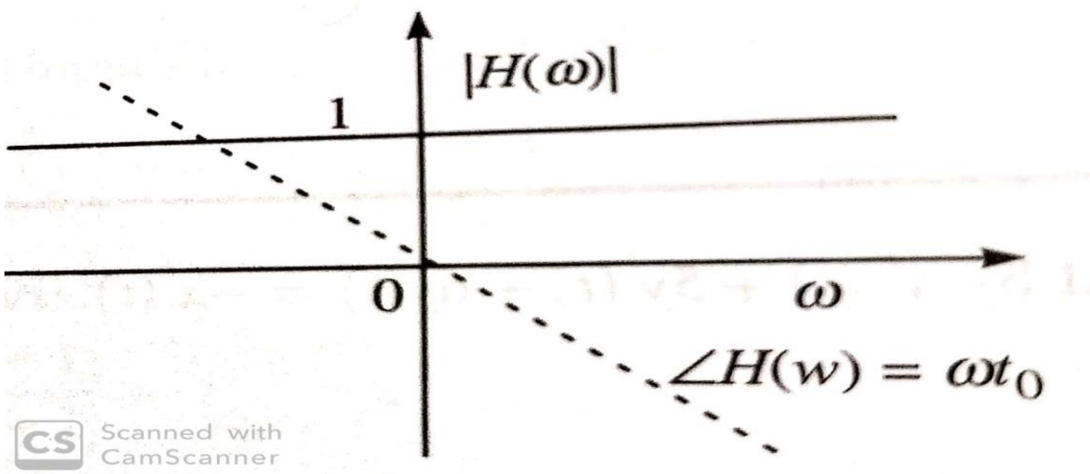
Distortion less transmission through the LTI system requires that the response be exact replica of input signal. The replica may have different magnetude and delayed in time.

Therefore, $any\ arbitary\ input\ signal\ x(t), if\ the\ output\ y(t) = k\, x(t - t_0)$

Apply the Fourier transform

$$Y(\omega) = k\, X(\omega)\, e^{-j\omega t_0}$$

$$\frac{Y(\omega)}{X(\omega)} = H(\omega) = k\, e^{-j\omega t_0}$$



$|H(\omega)| = k, \quad \angle H(\omega) = -\omega t_0 \; or \; \angle H(\omega) = n\pi - \omega t_0$ Where n is integer number

Therefore, to achieve distortion less transmission through LTI system, magnetude response of system $|H(\omega)|$ must be constant over entire frequency range and phase response of the system $\angle H(\omega)$ must be linear with frequency.

Band width of signals and System

***Band width of signals*: it** is the range of significant frequency components present in the signal. A signal may have frequency components in the entire frequency range from -∞ to ∞. For any practical signals, the energy content decreases with frequency, only some of frequency components of signals have significant amplitude within a certain frequency band; outside this band have negligible amplitude. The amplitude of significant frequency component is within the $\frac{1}{\sqrt{2}}$ times (3dB) of maximum signal amplitude.

***System Band width*:**

The band width of system is defined as the interval of frequencies over which the magnitude spectrum of $H(\omega)$ remains within $\frac{1}{\sqrt{2}}$ times (3dB) its value at the mid band. The band width of system is

$\omega_1 = lower\ cutoff\ frequency = lower frequency\ at\ which\ magnetude\ of H(\omega)\ \frac{1}{\sqrt{2}}$ times (3dB) its value at the midband

$\omega_2 = lower\ cutoff\ frequency = highest\ frequency\ at\ which\ magnetude\ of H(\omega)\ \frac{1}{\sqrt{2}}$ Times (3dB) of its value at the midband.

Band width = $\omega_2 - \omega_1$

For distortion less transmission, a system should have infinite bandwidth. But due to physical limitations it is impossible to design an ideal filters having infinite bandwidth.

For satisfactory distortion less transmission, therefore, an LTI system should have high bandwidth compared to the signal bandwidth.

The filter characteristics of linear system:

The system processes the input signal in a way that is characteristics of the system. The system modifies the spectral density function of input signal according to transfer function. It is observed that the system act as some kind of filter to various frequency components. Some frequency components are boosted in strength, some are attenuated, and some may remain unaffected. Similarly, each frequency component suffers a different amount of phase shift in the process of transmission. LTI system acts as filter depending on the transfer function of system. The transfer function acts as weighting function to different frequency components of input signal.

LTI system may be classified into five types of filters
Low pass filter
High pass filter
Band pass filter
Band reject filter
All pass filter.
The pass band of a filter the range of frequencies that allowed by the system without distortion. The stop band of filter is the range of frequencies that attenuated by the system.

Ideal filters:

An Ideal filter passes all frequency components in its pass band without distortion and completely blocks frequency components outside of pass band. There is discontinuity between pass band and stop band in frequency spectrum. But practical filters, there is gradual transition gap between pass band and stop band, The range of frequencies over which there is a gradual attenuation between pass band and stop band is called transition band. Filters with small gap are very difficult to design.

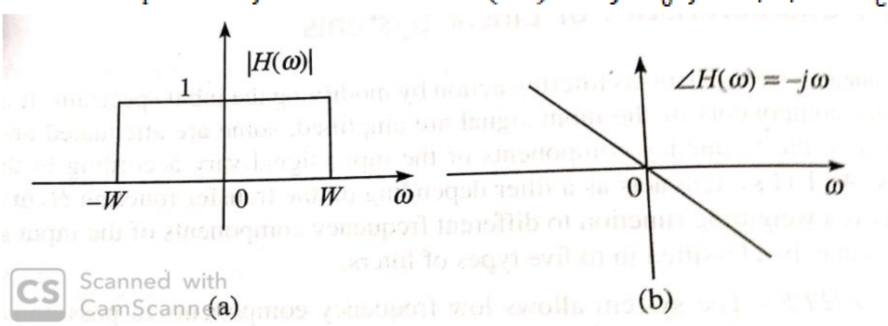
Ideal Low Pass Filter:

An ideal low pass filter transmits all frequency components below the certain frequency ω_C rad/sec called cutoff frequency, without distortion. The signal above these frequencies is filtered completely.
The transfer function of Idel Low pass filter given by

$$H(\omega) = \begin{cases} e^{-j\omega t_0} & |\omega| < \omega_C \\ 0 & |\omega| > \omega_C \end{cases}$$

$$Magnetude\ response\ of\ Ideal\ LPF\ |H(\omega)| = \begin{cases} 1 & |\omega| < W_1 \\ 0 & |\omega| > W_1 \end{cases}$$

$$Phase\ response\ of\ Ideal\ LPF\ \ \angle H(\omega) = -j\omega t_0\ for\ |\omega| < \omega_C$$



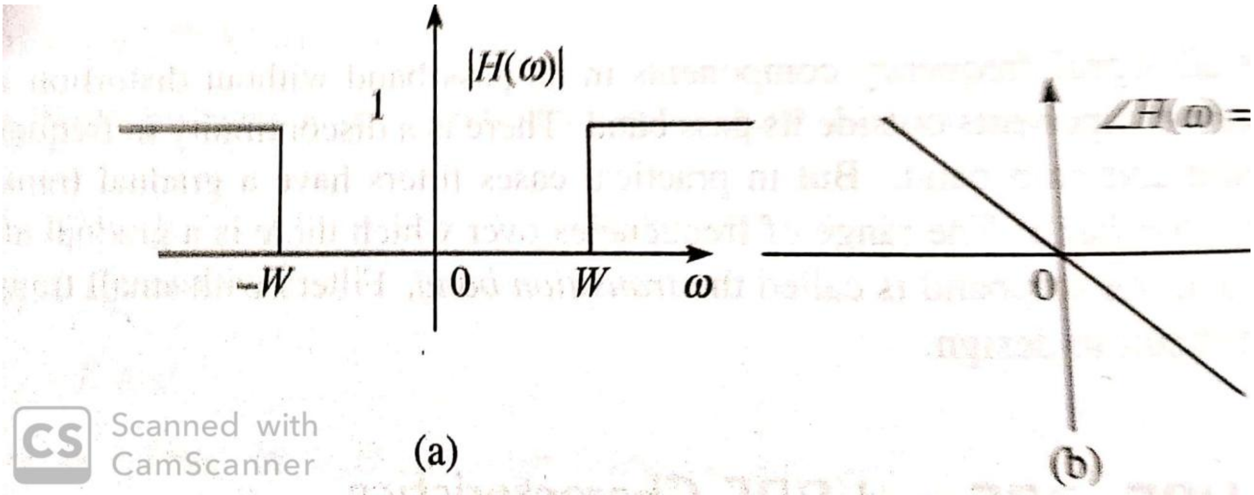
Ideal High Pass Filter:

An ideal high pass filter transmits all frequency components above the certain frequency ω_C rad/sec called cutoff frequency, without distortion. The signal below these frequencies is filtered completely.
The transfer function of Idel high pass filter given by

$$H(\omega) = \begin{cases} 0 & |\omega| < W_1 \\ e^{-j\omega t_0} & |\omega| > W_1 \end{cases}$$

$$Magnetude\ response\ of\ Ideal\ LPF\ |H(\omega)| = \begin{cases} 0 & |\omega| < \omega_C \\ 1 & |\omega| > \omega_C \end{cases}$$

$$Phase\ response\ of\ Ideal\ LPF\ \ \angle H(\omega) = -j\omega t_0\ for\ |\omega| > \omega_C$$



Ideal Band Pass Filter:

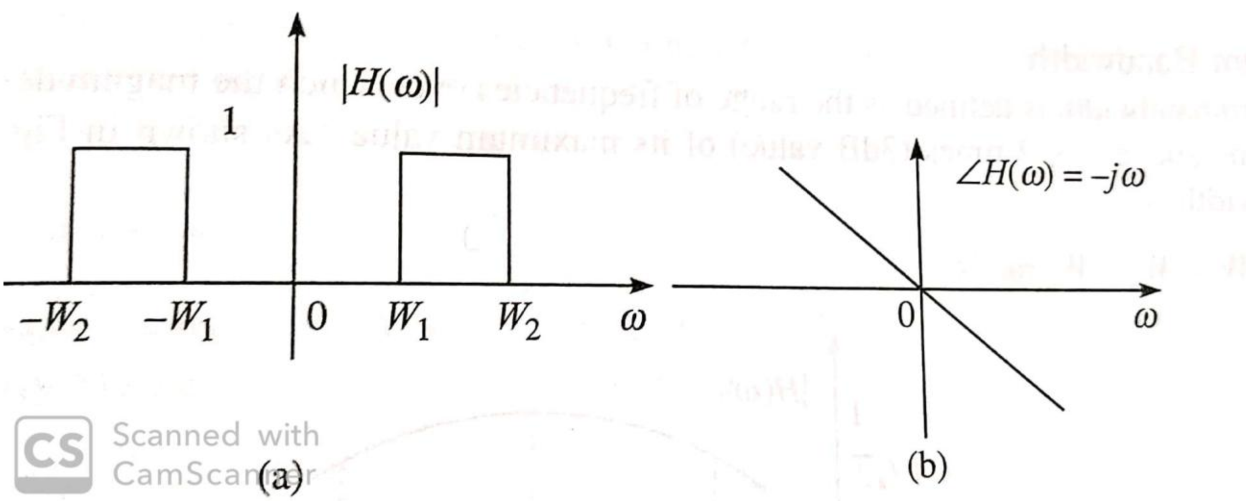
An ideal band pass filter transmits all frequency components within certain frequency band ω_{c_1} to ω_{c_2} rad/sec, without distortion. The signal with frequency outside this band is stopped completely.

The transfer function of Idel band pass filter given by

$$H(\omega) = \begin{cases} e^{-j\omega t_0} & W_1 < |\omega| < W_2 \\ 0 & otherwise \end{cases}$$

$$Magnetude\ response\ of\ Ideal\ BPF |H(\omega)| = \begin{cases} 1 & W_1 < |\omega| < W_2 \\ 0 & Otherwise \end{cases}$$

$$Phase\ response\ of\ Ideal\ BPF\ \ \angle H(\omega) = -j\omega t_0\ \ for\ \ \omega_{c_2} < |\omega| < \omega_{c_2}$$



Ideal Band Reject Filter:

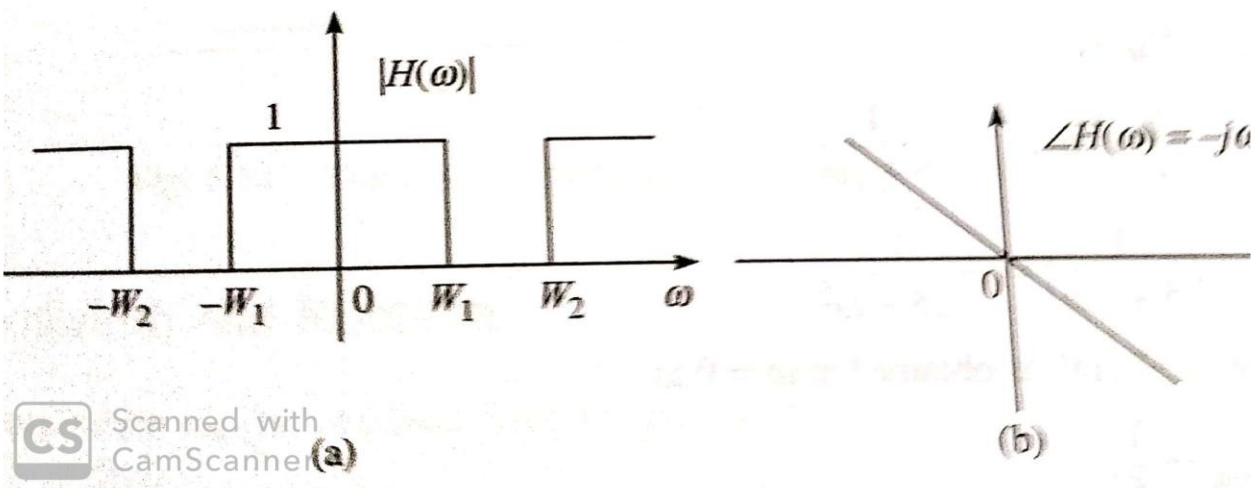
An ideal band reject filter rejects all frequency components within certain frequency band ω_{c_1} to ω_{c_2} rad/sec. The signal outside this band is transmitted without distortion.

The transfer function of Idle band reject filter given by

$$H(\omega) = \begin{cases} 0 & W_1 < |\omega| < W_2 \\ e^{-j\omega t_0} & otherwise \end{cases}$$

$$Magnetude\ response\ of\ Ideal\ BPF |H(\omega)| = \begin{cases} 0 & W_1 < |\omega| < W_2 \\ 1 & Otherwise \end{cases}$$

$$Phase\ response\ of\ Ideal\ BPF\ \ \angle H(\omega) = -j\omega t_0\ \ for\ \ |\omega| < W_1\ \ and\ \ \ |\omega| > W_2$$



Causality and Physical Realizability: **Paley – Wiener Criterion**

For physically realizable systems, that cannot have response before the input signal applied. In time domain approach the impulse response of physically realizable systems must be causal that is h(t) =0 for t< 0, this is condition known as causal condition. In frequency domain, this criterion implies that a necessary and sufficient condition for magnetude response $|H(j\omega)|$ to be physically realizable is

$$\int_{-\infty}^{\infty} \frac{|\ln\ |H(j\omega)||}{1+\omega^2}\, d\omega < \infty$$

This condition known as the Paley – Wiener criterion. To satisfy this condition the function $|H(j\omega)|$ must be square integrable that is

$$\int_{-\infty}^{\infty} |H(j\omega)|^2\, d\omega < \infty$$

All causal systems that satisfy the Paley – Wiener criterion are physically realizable.

Magnetude function $|H(j\omega)|$ may be zero at some discrete frequencies but it cannot be zero over finite band of frequencies since this will cause the integral to become infinite. Therefore Idle filters are not physically realizable. It can be concluding that magnetude function cannot fall off to zero faster than exponential order.

$|H(j\omega)| = k\, e^{-\alpha|\omega|}$ is permissible

$|H(j\omega)| = k\, e^{-\alpha\omega^2}$ this Gaussian error curve is not permissible.

But it possible to construct physically realizable filters close to the ideal filter characteristics. Low pass filter having transfer function

$$H(\omega) = \begin{cases} e^{-j\omega t_0} & |\omega| < \omega_C \\ \varepsilon & |\omega| > \omega_C \end{cases}$$

Where ε an arbitrary small value, produces nearly ideal characteristics shown in fig below

Band Width and Rise Time:

The system band width can be obtained from rise time , which can be derived from output response of the system.

Rise time : the rise time t_r of the output response is defined as the time the response takes to reach from 10% to 90% of the maximum value of the signal or in general it is the time of response to reach from zero to the final value of the signal.

$$\left.\frac{dy(t)}{dt}\right|_{t_0} = \frac{1}{t_r}$$

Relationship between Band width and rise time

Consider ideal LPF , its transfer function is given by $H(\omega) = \begin{cases} e^{-j\omega t_0} & |\omega| < \omega_C \\ 0 & |\omega| > \omega_C \end{cases}$

Where ω_C cut off frequency or 3 dB band width of filter

Apply Inverse Fourier transform

$$h(t) = \frac{1}{2\pi}\int_{-\omega_C}^{\omega_C} e^{-j\omega t_0} e^{j\omega t} d\omega$$

$$h(t) = \frac{1}{2\pi}\int_{-\omega_C}^{\omega_C} e^{j\omega(t-t_0)} d\omega = \frac{1}{\pi}\frac{\sin\omega_C(t-t_0)}{(t-t_0)}$$

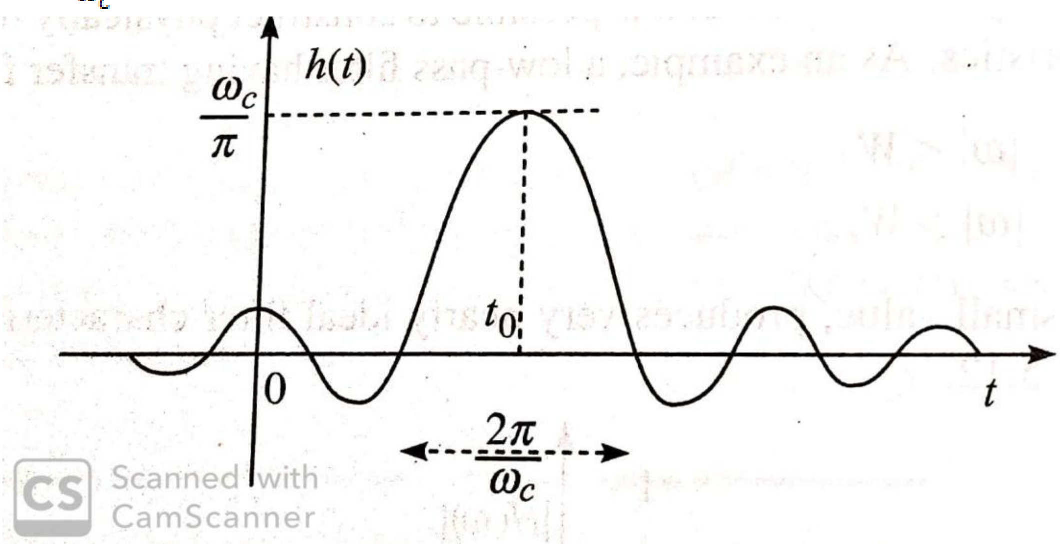
if input is impulse then output is $y(t) = h(t) * \delta(t)$ = h(t)

$$y(t) = \int_{-\infty}^{\infty} h(\tau)d\tau = \int_{-\infty}^{\infty} \frac{\omega_C}{\pi} \operatorname{sinc} \omega_C(t-t_0)\, dt$$

$$\frac{dy(t)}{dt} = \frac{\omega_C}{\pi} \sin c\ \omega_C(t-t_0)$$

$$\left.\frac{dy(t)}{dt}\right|_{t_0} = \frac{\omega_C}{\pi} = \frac{1}{t_r}$$

$$t_r = \frac{\pi}{\omega_C}$$



Product of rise time and bandwidth is constant

Rise time inversely proportional to the system band width.

Concept of convolution in time domain:

The process of expressing the output signal in terms of the superposition of weighted and time shifted impulse response is called convolution. Convolution is a particularly powerful way of characterizing the input – output relationship of LTI systems. The mathematical tool for evaluating the convolution of continuous time signals is called convolution integral; for discrete time signals, it is called convolution sum . the convolution integral plays an important role in system analysis in time and frequency domains. It is important process for signal processing and detection in communication systems.

The convolution integral

Let $x_1(t)$ and $x_2(t)$ be two continuous time signals. Then convolution of $x_1(t)$ and $x_2(t)$ can be expressed as

$$y(t) = x_1(t) * x_2(t) = y(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \quad \textit{where } \tau \textit{ is dummy variable}$$

Thus the output of any continuous LTI system is the convolution of the input x(t) with impulse response h(t) of the system.

Case I : if input signal is causal that is x(t) = 0 for t<0

$$y(t) = \int_{0}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

Case II

System is causal that is h(t) =0 for t<0 then

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)\, d\tau$$

Case III

Both input signal and system are causal then

$$y(t) = \int_{0}^{\infty} h(\tau)x(t-\tau)\, d\tau$$

Properties of convolution integral

Commutative property

Let $x_1(t)$ and $x_2(t)$ be two continuous time signals

$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$t - \tau = \lambda$$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(t-\lambda)\, x_2(\lambda)\, d\lambda = x_2(t) * x_1(t)$$

Distributive Property

$$x_1(t) * [\, x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t)$$

Associate property

$$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t) = x_1(t) * x_2(t) * x_3(t)$$

Shift property

If the signal $x_2(t)$ shifted by t_0 sec then convolution of

$x_1(t) * x_2(t - t_0) = x\,(t - t_0)$ and

If $x_1(t)$ and $x_2(t)$ shifted by t_1 and t_2 respectively

$$x_1(t - t_1) * x_2(t - t_2) = x(\, t - t_1 - t_2)$$

Convolution of function with impulse

$$x(t) * \delta(t) = x(t)$$

$$x(t) * \delta(t - t_0) = x(t - t_0\,)$$

$$x(t - t_0) * \delta(t) = x(t - t_0\,)$$

Convolution of function with unit step

Any arbitrary function x(t) with unit step function u(t)

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

Proof

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$

$$x(t) * u(t) = x(t) * \int_{-\infty}^{t} \delta(\tau)d\tau$$

$$x(t) * u(t) = x(t) * \int_{-\infty}^{t} x(\tau) * \delta(\tau)\, d\tau = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Width property

Let us consider finite duration of two signals $x_1(t)\ and\ x_2(t)$ are T_1 and T_2 respectively **then duration of y(t) =** $x_1(t) *\ x_2(t)$ is equal to the sum of duration of $x_1(t)\ and\ x_2(t)$.

$T = T_1 + T_2$

Also its area under finite signals $x_1(t)\ and\ x_2(t)$ are A_1 and A_2 respectively then the area under y (t) is product of both areas

A = area under y (t) = area under $x_1(t)$ and area under $x_2(t)$ = $A_1\ \ A_2$

Convolution property of Fourier Transform

Fourier transforms pair of two signals given by

$x(t) \leftrightarrow X(\omega) \quad h(t) \leftrightarrow H(\omega)$

$$F\ T\ of\ [x(t) * h(t)] = FT\ of \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [x(\tau) h(t-\tau) d\tau]\ e^{-j\omega t}\ dt$$

$t - \tau = \lambda$

$t = \tau + \lambda$

$$F\ T\ of\ [x(t) * h(t)] = \int_{-\infty}^{\infty} x(\tau) d\tau \int_{-\infty}^{\infty} h(\lambda)\ e^{-j\omega(\tau+\lambda)}\ d\lambda$$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau \int_{-\infty}^{\infty} h(\lambda)\ e^{-j\omega\lambda}\ d\lambda = X(\omega) H(\omega)$$

Convolution in frequency domain:

Fourier Transform of $X(\omega) * H(\omega)$ = 2Π Fourier transform of [x(t) h(t)]

Fourier transform of [x(t) h(t)] = $\int_{-\infty}^{\infty} x(t)h(t) e^{-j\omega t} dt$

$$= \int_{-\infty}^{\infty} [\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda)\ e^{j\lambda t}\ d\lambda]\ h(t) e^{-j\omega t} dt$$

$$== \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) [\int_{-\infty}^{\infty} h(t)\ \ e^{-j(\omega-\lambda)t} dt]\ d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) H(\omega-\lambda)\, d\lambda$$

$$= \frac{1}{2\pi}\ X(\omega) * H(\omega)$$

$$\boldsymbol{FT\ of}\ [\mathrm{x(t)h(t)}] == \frac{1}{2\pi}\ X(\omega) * H(\omega)$$

Thus convolution in one domain is transformed a product operation in the other domain

Graphical representation of Convolution

When two signals are provided in graphical form, the convolution can be performed by graphical method. It involves the following steps.

1. For given signals $x_1(t)\ and\ x_2(t)$, draw the signals $x_1(\tau)\ and\ x_2(\tau)$ as function of independent variable.

2. Draw the function of $x_2(-\tau)$ which is time reversal of $x_2(\tau)$ .then shift function by time t to form $x_2(t-\tau)$.
3. Draw the both signals $x_1(\tau)\ and\ x_2(t-\tau)$ on the τ axis with large time shift t along the negative axis.
4. Increase the time t along positive axis . Multiply the signals $x_1(\tau)\ and\ x_2(t-\tau)$ and integrate over the period of two signals to obtain convolution at t.
5. Increase the time shift step by step and obtain convolution using step 4.
6. Draw the convolution x (t) with the values obtained in steps 4 and 5 as function of t.

MODULE – IV
LAPLACE TRANSFORM AND Z-TRANSFORM

Laplace Transforms: Laplace Transforms (L.T), Inverse Laplace Transform, Concept of Region of Convergence (ROC) for Laplace Transforms, Properties of L.T, Relation between L.T and F.T of a signal, Laplace Transform of certain signals using waveform synthesis. Z–Transforms Concept of Z-Transform of a Discrete Sequence, Distinction between Laplace, Fourier and Z Transforms, Region of Convergence in Z-Transform, Constraints on ROC for various classes of signals, Inverse Z-transform, Properties of Z- transforms.

Complex Fourier Transform

Fourier transform is a tool which allows representing an arbitrary function $f(t)$ by continuous sum of exponential function of form of $e^{j\omega t}$.These frequencies are restricted to the $j\omega$ axis in the complex plane.

$$f(t) \leftrightarrow F(\omega)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

The variable ω always appears with j and hence the integral can also be written as function of $j\omega$

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

Let a function $\emptyset(t) = f(t)e^{-\sigma t}$

$$fourier\,transform\ of\ \emptyset(t) = \int_{-\infty}^{\infty} f(t) e^{-\sigma t}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} f(t)\, e^{-(\sigma + j\omega)t}\, dt$$

$$F\,T\ of\ \emptyset(t) = F(\sigma + j\omega)$$

$$\emptyset(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\sigma + j\omega)\, e^{j\omega t}\, d\omega$$

$$f(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\sigma + j\omega)\, e^{j\omega t}\, d\omega$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\sigma + j\omega)\, e^{(\sigma + j\omega)t}\, d\omega$$

$$\sigma + j\,\omega = s, \qquad ds = j\, d\omega, \qquad \frac{ds}{j} = d\omega$$

Limit of integration for $\omega = -\infty\ to\ \infty\ \ become\ \sigma - j\,\infty\ \ to\,\sigma + j\,\infty\ for\ variable\ s$

$$f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)\, e^{st}\, ds$$

Represent $f(t)$ as continuous sum of exponential of complex frequency $s = \sigma + j\omega$. This is special kind of Fourier Transform called as complex Fourier transform or Bilateral Laplace Transform.

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

$$f(t) \leftrightarrow F(s)$$

Unilateral Laplace transforms

Functions of interest are causal that is f(t) = 0 for t<0 , the Laplace transform of such functions are termed as unilateral or one sided Laplace Transforms.

$$F(s) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt$$

Lower limit indicates inclusions of initial conditions, impulse functions and its derivatives at t = 0^-

Convergence of Laplace Transform

The Fourier Transform of f(t) converge if f(t) is absolutely integrable , similarly the necessary condition for convergence of Laplace Transform is absolute integrability of $f(t)e^{-\sigma t}$

$$\int_{-\infty}^{\infty} |f(t)e^{-\sigma t}|\, dt < \infty$$

Existence of the Laplace Transform

Laplace Transform exists if it converge in the given interval . These fore , the condition for its existence is that the function $f(t)e^{-\sigma t}$ should be absolutely integrable.

$$\int_{-\infty}^{\infty} |f(t)e^{-\sigma t}|\, dt < \infty$$

Proof: Let $f(t)\, be\ causal\ and\ an\ exponential\ order\ function$ then it always satisfies the following inequality

$|f(t)| = M\, e^{\alpha t}\ for\ all\ t > 0$

Where M and α are real constants

$$\int_{-\infty}^{\infty} |f(t)e^{-\sigma t}|\, dt = \int_{0}^{\infty} M\, e^{\alpha t}\, e^{-\sigma t} dt = \int_{0}^{\infty} M\ e^{-(\sigma-\alpha)t} dt$$

$\int_{-\infty}^{\infty} |f(t)e^{-\sigma t}|\, dt = \frac{M}{\sigma-\alpha} e^{-(\sigma-\alpha)t}\Big|_0^{\infty} = \frac{M}{\sigma-\alpha}$ **if $\sigma > \alpha$ is finite value.thus Laplace Trnasform exists**

$\alpha < \sigma < \infty$

Region of convergence (ROC)

Region of convergence (ROC) defines the region where Laplace Transform exists. The range of values of s for which Laplace Transform converge is called as ROC .The variable s= $\sigma + j\omega$is a complex number and

display the complex plane referred to as s – plane where real part of s along the X – axis and imaginary part of s along the Y – axis. The ROC is a shaded region on the pole – zero plot, Laplace transform exists for values of s in the shaded region. Type equation here.

Poles and zeros X(s)

$$X(s) = \frac{N(s)}{D(s)}$$

N(s) :Numarator polynomial in complex variable s

D(s) : denominator Polynomial in complex variable s

$X(s)$ *will be the rational function in s then*

$$X(s) = \frac{N(s)}{D(s)} = \frac{a_0 s^m + a_1 s^{m-1} + a_2 s^{m-2} + \cdots \ldots . + a_m}{b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + \cdots \ldots . + b_n}$$

a_m *and* b_n *are real constants and m and n are positive integers .The* $X(s)$ *is called proper rational*

Function if $n > m$ *and an improper raional function if* $n \leq m$.

$$X(s) = \frac{a_0}{b_0} \frac{(s - z_1)(s - z_2)(s - z_3) \ldots \ldots \ldots \ldots \ldots \ldots (s - z_m)}{(s - p_1)(s - p_2)(s - p_3) \ldots \ldots \ldots \ldots \ldots \ldots . (s - p_n)}$$

z_k *roots of* $N(s)$ *where* $k = 1,2,3 \ldots \ldots m$

p_k *roots of* $D(s)$ *where* $k = 1,2,3 \ldots \ldots n$

Roots of of numerator polynomial are called zero of X(s) because X(s) = 0 for those values s in the same way roots of denominator polynomialt are called poles of X(s) because X(s) = ꝏ for those values of s. Therefore poles of X(s) lie outside of ROC since X(s) does not converge at poles. The zeros, on the other hand may lie inside or outside of ROC. The poles and zeros of X(s) in finite s plane characterised the algebraic expression for X(s) to within scale factor. The representation of poles and zeros in the s plane is referred to as the pole-zero plots.

Properties of ROC

A complete specification of Laplace Transform requires not only the algebraic expression for X(s) but also the associated ROC. Different signals have identical algebraic expression for X(s) , so that their Laplace transform are distinguishable only by ROC. It has been explained some specific constraint on ROC for various class of signals.

Property 1: the ROC of X(s) consists of strips parallel to $j\omega$ axis in the s plane.

The ROC of Laplace Transform of $x(t)$ consists of those values of s for which $x(t)e^{-\sigma t}$ is absolutely integrable.

$$\int_{-\infty}^{\infty} x(t)e^{-\sigma t}dt < \infty$$

This condition depends only on σ values

Property 2: For rational Laplace Transforms, the ROC does not contain any poles.

X(s) = ∞ at poles , Laplace Transform does not converge at poles and thus the ROC cannot contain values of s that are pole.

Property3: If $x(t)$ is a finite duration signal and is absolutely integrable then the ROC is the entire plane.

$$x(t) = \begin{cases} non\ zero\ T_1 \leq t \leq T_2 \\ 0 \quad other\ wise \end{cases}$$

$x(t)$is absolutely integrable $\int_{T_1}^{T_2} |x(t)| < \infty$

For s to be in the ROC, the requirement is $\int_{T_1}^{T_2} |x(t)|\, e^{-\sigma t} < \infty$

For $\sigma > 0$, the Maximum value of $e^{-\sigma t}$ over interval on which x(t) is non zero is $e^{-\sigma T_1}$

$\int_{T_1}^{T_2} |x(t)|\, e^{-\sigma t} < e^{-\sigma T_1} \int_{T_1}^{T_2} |x(t)|$ bounded

For < 0 , the Minimum value of $e^{-\sigma t}$ over interval on which x(t) is non zero is $e^{-\sigma T_2}$

$\int_{T_1}^{T_2} |x(t)|\, e^{-\sigma t} < e^{-\sigma T_2} \int_{T_1}^{T_2} |x(t)|$ bounded

$x(t)e^{-\sigma t}$ is absolutely integrable thus ROC includes entire s plane .

Property4: If $x(t)$ is right sided and if line Re { s } = σ_0 is in the ROC then all values of s for which Re { s } >σ_0 will also be in the ROC.

$$x(t) = \begin{cases} 0 \quad -\infty \leq t < T_1 \\ non\ zero\ T_1 \leq t \leq \infty \end{cases}$$

$x(t)$ *is right sided signal then* For s to be in the ROC, the requirement is

$$\int_{T_1}^{\infty} |x(t)|\, e^{-\sigma_0 t}dt < \infty$$

$if\ \sigma_1 > \sigma_0\ , e^{-\sigma_1 t} decays\ faster\ than\ e^{-\sigma_0 t}$ as t $\leftrightarrow \infty$

$$\int_{T_1}^{\infty} |x(t)|\, e^{-\sigma_1 t} dt = \int_{T_1}^{\infty} |x(t)|\, e^{-\sigma_0 t} e^{-(\sigma_0 - \sigma_1)\, t} dt$$

$$\leq \; e^{-(\sigma_0 - \sigma_1)\, t} \int_{T_1}^{\infty} |x(t)|\, e^{-\sigma_0}\, dt$$

$if\ \sigma_1 > \sigma_0\ , e^{-\sigma_1 t} diverges\ faster\ than\ \ e^{-\sigma_0 t}$ as t $\leftrightarrow -\infty$

X(t) cannot grow with out bound in –ve direction since x(t) = 0 for t < T_1

If a point s is in the ROC then all the points to the right of s that is all points larger real parts are in ROC, For this reason in this case is commonly referred to as right half s-plane.

Property 5 : if $x(t)$ is left sided and if line Re{ s } = σ_0 is in the ROC then the all values of s for which Re{ s } <σ_0 will also in the ROC.

Property 6: if $x(t)$ is two sided and if line Re{ s } = σ_0 is in the ROC then the ROC consists of a strip in the s- plane that includes line Re{ s } = σ_0.

If $x(t)$ is infinite duration signal then ROC is of the form $\sigma_1 < Re\{s\} < \sigma_2$ where $\sigma_1\ and\ \sigma_2$ are real parts of two poles of X(s) , thus ROC is a vertical strip in the s plane between the vertical line Re {s} = σ_1 and Re { s } = σ_2 . all poles lies outside the ROC.

Property 7: if the Laplace X(s) of x(t) is rational then its ROC is bounded by poles or extended to infinity . in addition , no poles of X(s) contained in the ROC.

Property 7 : if Laplace transform of x(t) is X(s) is rational.

If Laplace Transform X(s) contain more than one poles in the right side of S-plane , the ROC is the region in the plane to the right of right most pole.

If Laplace Transform X(s) contain more than one pole in the left side of s-plane , the ROC is the region in the plane to the left of left most pole.

Properties of Laplace Transform

Linearity Property

$\boldsymbol{f(t) \;\leftrightarrow F(s)\,, f_n(t) \leftrightarrow F_n(s)}$

Linear combination of signals

$$\boldsymbol{L\,\{a_1 f_1(t) +\ a_2 f_2(t) +\ a_3 f_3(t) +\ \ldots\ldots\ldots\ldots + a_n f_n(t)\}}$$
$$\boldsymbol{=\ a_1 F_1(s) + a_2 F_2(s) + a_3 F_3(s) \ldots\ldots\ldots + a_n F_n(s)}$$

Where $a_1, a_2 \ldots \ldots a_n$ are any arbitrary constants

Proof

$$L\{a_1f_1(t)+a_2f_2(t)+a_3f_3(t)+\dots\dots\dots.+a_nf_n(t)\}=\int_{-\infty}^{\infty\infty}\{a_1f_1(t)+a_2f_2(t)+a_3f_3(t)+\dots\dots\dots.+a_nf_n(t)\}e^{-st}\,dt$$

$$=\int_{-\infty}^{\infty}a_1f_1(t)e^{-st}\,dt+\int_{-\infty}^{\infty}a_2f_2(t)e^{-st}\,dt+\dots\dots.+\int_{-\infty}^{\infty}a_nf_n(t)e^{-st}\,dt$$

$$=a_1F_1(s)+a_2F_2(s)+a_3F_3(s)\dots\dots\dots.+a_nF_n(s)$$

Time Shifting Property

If signal $f(t)=\begin{cases}non\ zero & for\ t\geq 0\\ 0 & otherwise\end{cases}$

$L\{f(t)\}=F(s)$

$$L\{f(t-t_0)\}=e^{-st_0}F(s)$$

Proof

$$L\{f(t-t_0)\}=\int_0^{\infty}f(t-t_0)\,e^{-st}\,dt$$

$t-t_0=\tau,\quad t=\tau+t_0\,,dt=d\tau,\quad when\ t=0\,,$
$\tau=-t_0\,,when\ t=\infty,\tau=\infty$

$$L\{f(t-t_0)\}=\int_{-t_0}^{\infty}f(\tau)\,e^{-s(\tau+t_0)}\,d\tau$$

$$=e^{-st_0}\int_{-t_0}^{\infty}f(\tau)\,e^{-s\tau}\,d\tau$$

$$=e^{-st_0}\left\{\int_{-t_0}^{-1}f(\tau)\,e^{-s\tau}\,d\tau+\int_0^{\infty}f(\tau)\,e^{-s\tau}\,d\tau\right\}$$

$$L\{f(t-t_0)\}=e^{-st_0}\left\{\int_0^{\infty}f(\tau)\,e^{-s\tau}\,d\tau\right\}=e^{-st_0}F(s)$$

Frequency shifting Property

If signal $f(t)=\begin{cases}non\ zero & for\ t\geq 0\\ 0 & otherwise\end{cases}$

$L\{e^{-at}f(t)\}=F(s+a)$

Proof

$L\{e^{-at}f(t)\} = \int_0^{\infty} e^{-at} f(t)\, e^{-st}$ dt

$= \int_0^{\infty} f(t)\, e^{-(s+a)t}$ dt= F(s+a)

$e^{-at}f(t) \leftrightarrow F(s+a)$

Scaling Property

$f(t) \leftrightarrow F(s)$

$$f(at) \leftrightarrow \frac{1}{a}F(\frac{s}{a})$$

Proof

$$L\,\{f(at)\} = \int_0^{\infty} f(at)e^{-st}\ dt$$

$at = \tau\,, \quad t = \tau/a$

$$\mathcal{L}\,\{f(at)\} = \int_0^{\infty} f(\tau)e^{-\frac{s\tau}{a}}\frac{d\tau}{a}$$

$$\mathcal{L}\,\{f(at)\} = \frac{1}{a}\,F(\frac{s}{a})$$

Time differentiation Property

If signal $f(t) = \begin{cases} non\ zero & for\ t \geq 0 \\ 0 & otherwise \end{cases}$

$f(t) \ \leftrightarrow F(s)$

$$\frac{d}{dt}f(t)\ is\ absouletely\ \ interable$$

$\frac{d}{dt}f(t) \leftrightarrow sF(s) - f(0-)$

$$\mathcal{L}\,\left\{\frac{d}{dt}f(t)\right\} = \int_0^{\infty}\{\frac{d}{dt}f(t)\}e^{-st}\ dt$$

$$= f(t)e^{-st}|_{0-}^{\infty} + s\int_0^{\infty} f(t)\}e^{-st}\ dt$$

$sF(s) - f(0-)$

$$\frac{d^2f(t)}{dt^2} = \frac{dy(t)}{dt}$$

$$\mathcal{L}\left\{\frac{d^2f(t)}{dt^2}\right\} = \mathcal{L}\,\left\{\frac{dy(t)}{dt}\right\} = sY(s) - \ y(0-)$$

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0-) - s^{n-2} f'(0-) - \ldots\ldots\ldots\ldots s f^{n-2}(0-) - f^{n-1}(0\_)$$

For causal function , all initial conditions are zero

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s)$$

Differentiation in s – Domain

$$\mathcal{L}\{f(t)\} = F(s)$$

$$\mathcal{L}\{(-t)^n f(t)\} = \frac{d^n F(s)}{ds^n}$$

Proof

$$F(s) = \int_0^\infty f(t)\, e^{-st} dt$$

Differentiation with respect of s

$$\frac{d\,F(s)}{ds} = \int_0^\infty \{-t\, f(t)\}\, e^{-st} dt$$

$$\mathcal{L}\,\{-t\, f(t)\} = \frac{d\,F(s)}{ds}$$

2nd derivative with respect s

$$\frac{d^2\,F(s)}{ds^2} = \int_0^\infty \{t^{\,2} f(t)\}\, e^{-st} dt$$

$$\mathcal{L}\,\{t^{\,2} f(t)\} = \frac{d^2\,F(s)}{ds^2}$$

Similarly, nth derivative with respect s

$$\mathcal{L}\,\{)(-t)^{\,n} f(t)\} = \frac{d^n\,F(s)}{ds^n}$$

Time integration Property

$$\boldsymbol{\mathcal{L}\{f(t)\} = F(s)}$$

$$\boldsymbol{\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{F(s)}{s} \text{ and } \mathcal{L}\left\{\int_{-\infty}^t f(\tau)\, d\tau\right\} = \frac{F(s)}{s} + \frac{\int_{-\infty}^0 f(\tau)\, d\tau}{s}}$$

Proof

$$\boldsymbol{\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \int_0^\infty \left\{\int_0^t f(\tau)\, d\tau\right\} e^{-st} dt}$$

$$= \frac{-e^{-st}}{s} \int_0^{\tau} f(\tau)\, d\tau \Big|_0^{\infty} + \frac{1}{s}\int_0^{\infty} f(t)\ e^{-st}\, dt = \frac{F(s)}{s}$$ **for causal signal**

For non-causal signal

$$\int_{-\infty}^{t} f(\tau)\, d\tau = \int_{-\infty}^{0-} f(\tau)\, d\tau + \int_{0+}^{t} f(\tau)\, d\tau$$

$$\mathcal{L}\{ \int_{-\infty}^{t} f(\tau)\, d\tau \} = \mathcal{L}\{\int_{-\infty}^{0-} f(\tau)\, d\tau \} + \mathcal{L}\{ \int_{0+}^{t} f(\tau)\, d\tau \}$$

$$= \frac{\int_{-\infty}^{0-} f(\tau)\, d\tau}{s} + \frac{F(s)}{s}$$

Integration in S domain

$$f(t) \leftrightarrow F(s)$$

$$\int_{s}^{\infty} F(u)\, du \leftrightarrow \frac{f(t)}{t}$$

Proof

$$\int_{s}^{\infty} F(u)\, du = \int_{s}^{\infty} \{\int_{0}^{\infty} f(t)\, e^{-ut}\, dt \}du =$$

$$= \int_{0}^{\infty} f(t)\, \{\int_{s}^{\infty} e^{-ut}\, du \}dt$$

$$= \int_{0}^{\infty} f(t)\, \frac{e^{-ut}}{-t}\Big|_{s}^{\infty} dt$$

$$= \int_{0}^{\infty} \frac{f(t)}{t} e^{-st} dt$$

$$= \mathcal{L}\{\frac{f(t)}{t}$$

Time Convolution

$$f_1(t) \leftrightarrow F_1(s), f_2(t) \leftrightarrow F_2(s)$$

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t - \tau)\, d\tau$$

$$\mathcal{L}\{ f_1(t) * f_2(t)\} = F_1(s)\ F_2(s)$$

Proof

$$\mathcal{L}\{ f_1(t) * f_2(t)\} = \int_{0}^{\infty} [\int_{-\infty}^{\infty} f_1(\tau) f_2(t - \tau)\, d\tau\ ]\, e^{-st}\, dt$$

$$t - \tau = \lambda \Rightarrow t = \tau + \lambda$$

$$\mathcal{L}\{ f_1(t) * f_2(t)\} = \int_0^{\infty} f_1(\tau) \int_0^{\infty} f_2(\lambda) \quad e^{-s(\tau+\lambda)}\, d\lambda\, d\tau$$

$$= \int_0^{\infty} f_1(\tau) e^{-s\tau}\, d\tau \int_0^{\infty} f_2(\lambda) \quad e^{-s\lambda}\, d\lambda$$

$$= F_1(s)\ F_2(s)$$

Multiplication in time domain or convolution in frequency domain

$$\boldsymbol{f_1(t) \leftrightarrow F_1(s), f_2(t) \leftrightarrow F_2(s)}$$

$$\boldsymbol{\mathcal{L}\{f_1(t)\ f_2(t)\} = \frac{1}{2\pi j}\,(F_1(s) * F_2(s))}$$

Proof

$$\boldsymbol{\mathcal{L}\{f_1(t)\ f_2(t)\} = \int_0^{\infty} f_1(t)\ f_2(t) e^{-st}\, dt}$$

$$\boldsymbol{= \int_0^{\infty} f_1(t)\, [\frac{1}{2\pi j} \int_0^{\infty} F_2(\lambda) e^{\lambda t}\, d\lambda] e^{-st}\, dt}$$

$$\boldsymbol{=\frac{1}{2\pi j}\int_0^{\infty} F_2(\lambda)\, [\int_0^{\infty} \quad f_1(t) e^{(s-\lambda)t}\, dt] d\lambda}$$

$$\boldsymbol{= \frac{1}{2\pi j} \int_0^{\infty} F_2(\lambda) F_1\,(s-\lambda)\, d\lambda}$$

$$\boldsymbol{= \frac{1}{2\pi j}\,(F_1(s) * F_2(s))}$$

Initial value theorem

The initial value theorem is used to calculate f(0) from Laplace Transform of F(s) without the need of inverse Laplace Transform.It state that f(t) and it first derivative are Laplace transformable , then the initial value of f(t) is given by

$$\boldsymbol{f(0+) = \lim_{t\to 0} f(t) = \lim_{s\to\infty}[sF(s)]}$$

$$\mathcal{L}\frac{d}{dt}f(t) = \int_0^{\infty} \frac{d}{dt}f(t)e^{-st}\,dt$$

$$=\int_{0-}^{0+} \frac{d}{dt}f(t)e^{-st}\,dt \;+\; \int_{0+}^{\infty} \frac{d}{dt}f(t)e^{-st}\,dt$$

The discontinuity in f(t) at t=0 , the derivative of f(t) is an impulse function af amplitude equal in the value of discontinuity.

$$\frac{d}{dt}f(t)\Big|_{t=0} = \{ f(0+) - f(0-)\}\delta\,(t)$$

$$\int_{0-}^{0+} \frac{d}{dt}f(t)e^{-st}\,dt \;=\; \int_{0-}^{0+} [f(0+) - f(0-)]\delta\,(t)e^{-st}\,dt \;=\; f(0+) - f(0-)$$

$$\mathcal{L}\frac{d}{dt}f(t) \;=\; s\,F(s) - f(0-) \;=\; f(0+) - f(0-) \;+\; \int_{0+}^{\infty} \frac{d}{dt}f(t)e^{-st}\,dt$$

$$sF(s) = f(0+) + \int_{0+}^{\infty} \frac{d}{dt}f(t)e^{-st}\,dt$$

$$\lim_{s\to\infty}[sF(s)] \;=\; f(0+) = \lim_{t\to 0} f(t)$$

Final Value theorem

$$\lim_{s\to\infty}[sF(s)] \;=\; f(\infty)$$

Proof

$$sF(s) = f(0+) + \int_{0+}^{\infty} \frac{d}{dt}f(t)e^{-st}\,dt$$

$$\lim_{s\to 0}[sF(s)] \;=\; f(0+) + \int_{0+}^{\infty} \frac{d}{dt}f(t)\,dt$$

$$\lim_{s\to 0}[sF(s)] \;=\; f(0+) + f(\infty) \;-\; f(0+) \;=\; f(\infty)$$

Inverse Laplace Transforms

$$f(t) \;=\; \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+\infty} F(s)\,e^{st}\,ds$$

Method of finding Inverse Laplace Transform

1. Residue Method
2. Partial fraction method

1. Residue Method:

The inverse formula can be expressed as a contour integral by the residue theorem

$$f(t) = \frac{1}{j2\pi}\oint F(s)e^{st}\, ds$$

$$= \sum_{i=1}^{n} Residue\ of\ F(s)\, e^{st}\ at\ pole\ s_i\ inside\ of\ closed\ contour$$

$$whereResidue[s_i], i = 1, 2, 3 \ldots\ldots .aretheresidueofF(s)e^{st}andisaclsoedcurve$$

$$ILT\ given\ by\ f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+\infty} F(s)\, e^{st}\, ds$$

It given by a line integral along a vertical line Re {s} in the region of existence of F(s) . in this integral the real σ is to be selected such that if ROC of F(s) is Re {s} >$\sigma_1$then

$\sigma_1 < \sigma < \infty$

Finding residues

If $F(s)e^{st}$ is a rational function of s it may be expressed as

$$F(s)e^{st} = \frac{\varphi\,(s)}{(s - s_0)^n}$$

Where F(s)e^{st} has n poles at s =s_0 and $\varphi\,(s)$ has no poles at s =s_0 the residue of F(s)e^{st} at

s =s_0 is given by

Res { F(s)$e^{st}\ at\ s = s_0$ } $= = \frac{1}{(n-1)!}\left[\frac{d^{n-1}\,\varphi\,(s)}{d\,s^{n-1}}\right]_{s=s_0}$

If n=1 function has only one first order pole

Res { F(s)e^{st} $at\; s = s_0$ } =$\varphi\,(s_0)$

Partial fraction Expansion Method

(a) X(s) is proper rational function

The Partial fraction expansion of $X(s)$ $requires$ the following two conditions

(i) $X(s)$must be proper rational function that is degree of denominator polynomial in s is greater than the degree of numerator polynomial in s.

$$X(s) = \frac{N(s)}{D(s)} = \frac{a_0 s^m + a_1 s^{m-1} + a_2 s^{m-2} + \cdots \dots\dots + a_{m-1} s + a_m}{b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + \cdots \dots\dots + b_{n-1} s + b_n} \quad n > m$$

(ii) A denominator in factored form.The structure of expansion depends on the nature of the factors in Q(s). the constants in the numerator of partial fraction expansion are called residues.

$$X(s) = \frac{N(s)}{D(s)} = K\,\frac{(s-z_1)(s-z_2)(s-z_3)\dots\dots\dots(s-z_m)}{(s-p_1)(s-p_2)(s-p_3)\dots\dots\dots(s-p_n)}$$

$z_{1,}z_{2,}z_{3,}\dots\dots..z_m$ $are\ roots\ of\ N(s)$ $and\ p_{1,}p_{2,}p_{3,}\dots\dots..p_n$ $are\ roots\ of\ D(s)$

Case 1: If D(s) contain real and distinct roots.

$$X(s) = \frac{N(s)}{(s-p_1)(s-p_2)(s-p_3)\dots\dots\dots(s-p_n)}$$

$$X(s) = \frac{K_1}{s-p_1} + \frac{K_2}{s-p_2} + \frac{K_3}{s-p_3} + \cdots \dots\dots\dots + \frac{K_n}{s-p_n}$$

The coefficientK_i can be obtained as

$$K_i = X(s)(s-p_i)|_{s=p_i}$$

If D(s) contain some complex conjugate roots

$$X(s) = \frac{K_1}{s-p_1} + \frac{K_2}{s-p_2} + \frac{K_3}{s-p_3} + \cdots \dots\dots\dots + \frac{A_1}{s-C_1} + \frac{A_2}{s-C_1^*} + \frac{A_3}{s-C_2} + \frac{A_4}{s-C_2^*}$$

$A_2 = A_1^*, A_4 = A_3^*$

$C_{1,}C_2$ are complex conjugate roots

Case 2: if Denominator contain multiple roots in the form of $(s-p_1)^m$

$$X(s) = \frac{K_{11}}{s-p_1} + \frac{K_{12}}{(s-p_1)^2} + \frac{K_{13}}{(s-p_1)^3} + \cdots \dots\dots\dots + \frac{K_{1r}}{(s-p_1)^r} + \cdots \dots. + \frac{K_{1m}}{(s-p_1)^m}$$

$$K_{1r} = \frac{1}{(m-r)!}\frac{d^{m-r}}{ds^{m-r}}[(s-p_1)^r X(s)]_{s=p_1}$$

(b) If X(s) is improper rational function

Degree of N(s) greater than or equal to degree of denominator D(s) .

Degree of N(s) = m, degree of D(s) = n

$$X(s) = \frac{N(s)}{D(s)} = Q(s) + \frac{R(s)}{D(s)} \qquad m > n$$

$Q(s)$ *is quotient polynomial of order* $m-n$

R remainder has dgree less than n

Inverse Laplace transform of $\frac{R(s)}{D(s)}$ **(becomes proper rational function) and this can be evaluated by partial fraction expansion method.**

Inverse of Laplace transform of Q(s) can be computed using differentiation property.

Application of Laplace Transform on Linear Systems

The transfer function of LTI continuous system completely described the behaviour of system with any type of input. Consider LTI system with impulse response $h(t)$**. Let** $x(t), y(t)$ *and* $h(t)$ *have Laplace transform* $X(s), Y(s)$ *and* $H(s)$ *respectively*. **The transfer function of a system is defined as the ratio of the Laplace transform of the output signal to the Laplace transform of input signal with all initial conditions are zero.**

$$H(s) = \frac{Y(s)}{X(s)}$$

$$y(t) = x(t) * h(t)$$

$$\mathcal{L}\{ y(t) = \mathcal{L}\{ x(t) * h(t) \}$$

$$Y(s) = X(s)H(s)$$

$$H(s) = \frac{Y(s)}{X(s)}$$

$$\mathcal{L}^{-1}\{ H(s)\} = \mathcal{L}^{-1}\left\{\frac{Y(s)}{X(s)}\right\} = h(t)$$

Causal LTI continuous time System described by an Nth order linear constant coefficient differential equation

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1}\frac{d^{N-1} y(t)}{dt^{N-1}} a_{N-2}\frac{d^{N-2} y(t)}{dt^{N-2}} + \dots\dots\dots\dots + a_1 \frac{d\,y(t)}{dt} + a_0\, y(t) = b_M \frac{d^M x(t)}{dt^M} +$$
$$b_{M-1}\frac{d^{M-1} x(t)}{dt^{M-1}} b_{M-2}\frac{d^{M-2} x(t)}{dt^{M-2}} + \dots\dots\dots\dots + b_1 \frac{d\,x(t)}{dt} + b_0\, x(t) \qquad N > M$$

Apply Laplace transform on both sides

$$a_N s^N\, Y(s) + a_{N-1} s^{N-1}\, Y(s) + a_{N-2} s^{N-2}\, Y(s) + \cdots \ldots \ldots \ldots \ldots + a_1 s\, Y(s) + a_0\, Y(s) = b_M s^M\, X(s) + b_{M-1} s^{M-1}\, X(s) + b_{M-2} s^{M-2}\, X(s) + \cdots \ldots \ldots \ldots \ldots + b_1 s\, X(s) + b_0\, X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_M s^M + b_{M-1} s^{M-1} + b_{M-2} s^{M-2} + \cdots \ldots \ldots \ldots \ldots + b_1 s + b_0}{a_N s^N + a_{N-1} s^{N-1} + a_{N-2} s^{N-2} + \cdots \ldots \ldots \ldots \ldots + a_1 s + a_0}$$

$$H(s) = \frac{\sum_{k=0}^{M} a_k\, s^k}{\sum_{k=0}^{N} b_k\, s^k}$$

Steady state frequency response of LTI system

$$H(s) = \left.\frac{Y(s)}{X(s)}\right|_{s=j\omega} = H(\,\omega\,) = \frac{Y(j\omega)}{X(j\omega)}$$

Magnitude response = $|H(\,\omega\,)| = \left|\frac{Y(j\omega)}{X(j\omega)}\right|$

Phase Response = $\angle\, H(\,\omega\,) = \tan^{-1}\frac{Y(j\omega)}{X(j\omega)}$

Causality:
For causal system, $h(t) = 0\ for\ t < 0$ and thus right sided. Therefore, the ROC associate with the transfer function of causal system is right half plane. However, if we know that the transfer function is rational, then it suffices to check that the ROC is the right half plane to the right of right most pole in s plane to conclude that the system is causal**.**

Stability
So far, we have seen that BIBO stability 0f continuous time LTI system is equivalent to its impulse response, being absolutely integrable, in which case its Fourier transform converge. Also the stability of an LTI differential system is equivalent to having all the poles of its characteristics equation having negative real part. for the Laplace Transform, the first stability condition translates to the following.

- An LTI system is stable if and only if the ROC of transfer function contains jω axis.
- A causal system with proper rational function H(s) is stable if and only if all of its poles are in left half of s-plane.

Advantages of Laplace Transforms

- The higher order differential equations can be easily solved by using simple algebraic equations.
- It transforms higher order differential equations with initial conditions in the time domain into simple algebraic equations in the s-domain. Since the initial conditions are automatically included in the solution.
- Total solution of Differential equation can be obtained by using inverse Laplace transform.
- It is a power full tool for analysing system properties in the form of transfer function.
- It can be used to analyse many classes of signals and systems which are not absolutely integrable.
- It provides solutions for many unstable systems such as impulse functions.
- Fourier transforms can be obtained from Laplace Transforms by substituting s = jω

Limitations

- Laplace transforms does not converge for some type of signals whose amplitude grows faster than time.
- The ROC is needed to obtain to obtain inverse Laplace transforms.
- It very difficult to solve complex integrals directly in the process of inverse Laplace transform.

Z-TRANSFORMS

Analysis of continuous time LTI systems can be done using z-transforms. It is a powerful mathematical tool to convert differential equations into algebraic equations.

The bilateral (two sided) z-transform of a discrete time signal x(n) is given as.

$$Z.T[x(n)] = X(Z) = \Sigma_{n=-\infty}^{\infty} x(n) z^{-n}$$

The unilateral (one sided) z-transform of a discrete time signal x(n) is given as

$$Z.T[x(n)] = X(Z) = \Sigma_{n=0}^{\infty} x(n) z^{-n}$$

Concept of Z-Transform and Inverse Z-Transform:

Z-transform of a discrete time signal x(n) can be represented with X(Z), and it is defined as

Z-transform of a discrete time signal x(n) can be represented with X(Z), and it is defined as

If $Z = re^{j\omega}$ then equation 1 becomes

$$X(re^{j\omega}) = \Sigma_{n=-\infty}^{\infty} x(n)[re^{j\omega}]^{-n}$$

$$= \Sigma_{n=-\infty}^{\infty} x(n)[r^{-n}]e^{-j\omega n}$$

$$X(re^{j\omega}) = X(Z) = F.T[x(n)r^{-n}] \ldots \ldots (2)$$

The above equation represents the relation between Fourier transform and Z-transform.

$$X(Z)|_{z=e^{j\omega}} = F.T[x(n)].$$

Inverse Z-transform:

$$X(re^{j\omega}) = F.T[x(n)r^{-n}]$$

$$x(n)r^{-n} = F.T^{-1}[X(re^{j\omega}]$$

$$x(n) = r^n\, F.T^{-1}[X(re^{j\omega})]$$

$$= r^n \tfrac{1}{2\pi} \int X(re^{j}\omega)e^{j\omega n}\, d\omega$$

$$= \tfrac{1}{2\pi} \int X(re^{j}\omega)[re^{j\omega}]^n d\omega \ldots \;\ldots (3)$$

Substitute $re^{j\omega} = z$.

$$dz = jre^{j\omega}d\omega = jzd\omega$$

$$d\omega = \tfrac{1}{j}z^{-1}dz$$

Substitute in equation 3.

$$3 \rightarrow x(n) = \tfrac{1}{2\pi} \int X(z)z^n \tfrac{1}{j} z^{-1}dz = \tfrac{1}{2\pi j} \int X(z)z^{n-1}dz$$

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$x(n) = \frac{1}{2\pi j} \int X(z)z^{n-1}dz$$

Difference Between Laplace and Fourier Transforms:

Laplace vs Fourier Transforms:

Both Laplace transform and Fourier transform are integral transforms, which are most commonly employed as mathematical methods to solve mathematically modeled physical systems. The process is simple. A complex mathematical model is converted in to a simpler, solvable model using an integral transform. Once the simpler model is solved, the inverse integral transform is applied, which would provide the solution to the original model.

For example, since most of the physical systems result in differential equations, they can be converted into algebraic equations or to lower degree easily solvable differential equations using an integral transform. Then solving the problem will become easier.

Region of convergence in Laplace transform:

With the z-transform, the s-plane represents a set of signals (complex exponentials). For any given LTI system, some of these signals may cause the output of the system to converge, while others cause the output to diverge ("blow up"). The set of signals that cause the system's output to converge lie in the region of convergence (ROC). This module will discuss how to find this region of convergence for any discrete-time, LTI system.

The region of convergence, known as the ROC, is important to understand because it defines the region where the z-transform exists. The z-transform of a sequence is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

The ROC for a given $x[n]$, is defined as the range of z for which the z-transform converges. Since the z-transform is a power series, it converges when $x[n]z^{-n}$ is absolutely summable.

$$\sum_{n=-\infty}^{\infty} |x[n] z^{-n}| < \infty$$

must be satisfied for convergence.

Properties of the Region of Convergence:

The Region of Convergence has a number of properties that are dependent on the

characteristics of the signal, $x[n]$.

- **The ROC cannot contain any poles.** By definition a pole is a where $X(z)$ is infinite. Since $X(z)$ must be finite for all z for convergence, there cannot be a pole in the ROC.
- **If** $x[n]$ **is a finite-duration sequence, then the ROC is the entire z-plane, except possibly** $z=0$ **or** $|z|=\infty$**.** A finite-duration sequence is a sequence that is nonzero in a finite interval $n1 \leq n \leq n2$. As long as each value of $x[n]$ is finite then the sequence will be absolutely summable. When $n2>0$ there will be a z-1 term and thus the ROC will not include $z=0$. When $n1<0$ then the sum

will be infinite and thus the ROC will not include $|z|=\infty$. On the other hand, when $n2\leq 0$ then the ROC will include $z=0$, and when $n1\geq 0$ the ROC will include $|z|=\infty$. With these constraints, the only signal, then, whose ROC is the entire z-plane is $x[n]=c\delta[n]$.

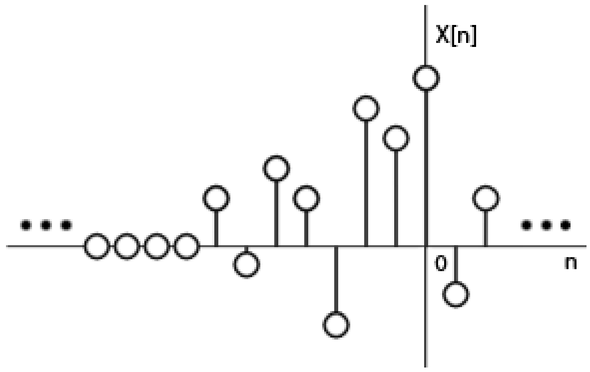


Figure 4: A left-sided sequence.

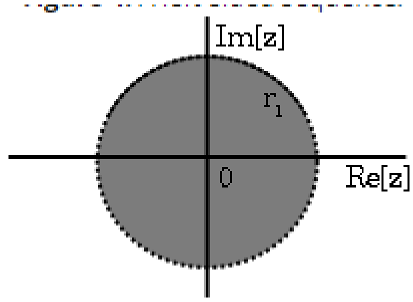


Figure 5: The ROC of a left-sided sequence.

If *x*[*n*] is a two-sided sequence, the ROC will be a ring in the z-plane that is bounded on the interior and exterior by a pole. A two-sided sequence is an sequence with infinite duration in the positive and negative directions. From the derivation of the above two properties, it follows that if $-r2<|z|<r2$ converges, then both the positive-time and negative-time portions converge and thus $X(z)$ converges as well. Therefore the ROC of a two-sided sequence is of the form $-r2<|z|<r2$.

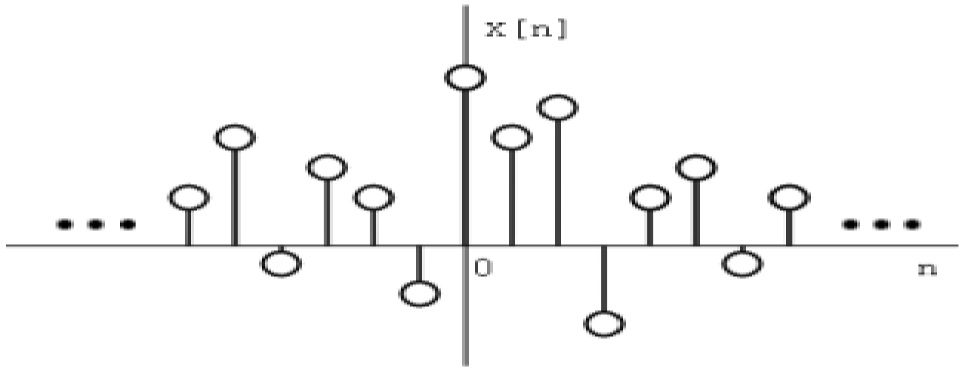


Figure 6: A two-sided sequence.

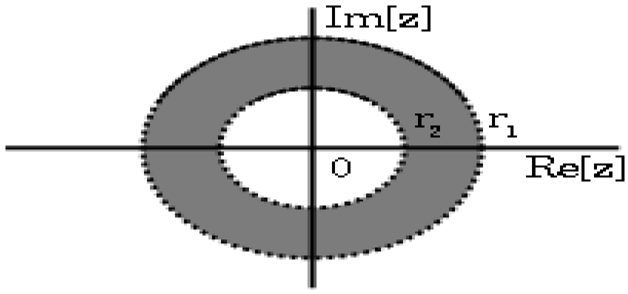


Figure 7: The ROC of a two-sided sequence.

Examples

Example 1

Lets take

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{4}\right)^n u[n]$$

The z-transform of $\left(\frac{1}{2}\right)^n u[n]$ is $\frac{z}{z-\frac{1}{2}}$ with an ROC at $|z| > \frac{1}{2}$.

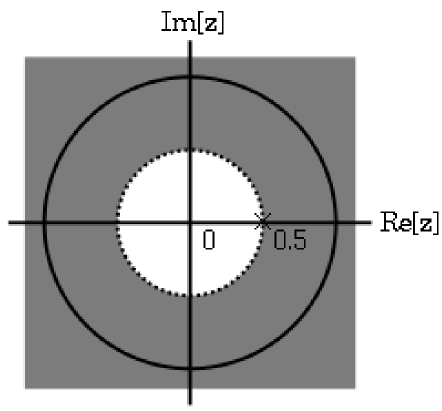


Figure 8: The ROC of $\left(\frac{1}{2}\right)^n u[n]$

Example 2

Now take

$$x_2[n] = \left(\frac{-1}{4}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[(-n)-1] \qquad [13]$$

The z-transform and ROC of $\left(\frac{-1}{4}\right)^n u[n]$ was shown in the example above. The z-transorm of $\left(-\left(\frac{1}{2}\right)^n\right) u[(-n)-1]$ is $\frac{z}{z-\frac{1}{2}}$ with an ROC at $|z| > \frac{1}{2}$.

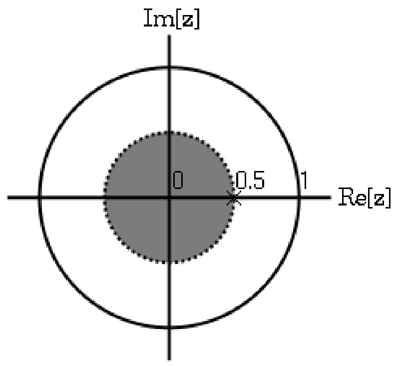


Figure 11: The ROC of $\left(-\left(\frac{1}{2}\right)^n\right) u[(-n)-1]$

Once again, by linearity,

$$\begin{aligned} X_2[z] &= \frac{z}{z+\frac{1}{4}} + \frac{z}{z-\frac{1}{2}} \\ &= \frac{z\left(2z-\frac{1}{8}\right)}{\left(z+\frac{1}{4}\right)\left(z-\frac{1}{2}\right)} \end{aligned} \qquad [14]$$

By observation it is again clear that there are two zeros, at 0 and $\frac{1}{16}$, and two poles, at $\frac{1}{2}$, and $\frac{-1}{4}$. in ths case though, the ROC is $|z| < \frac{1}{2}$.

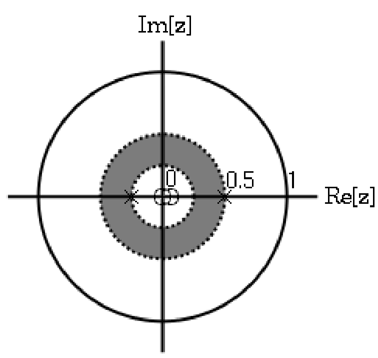


Figure 12: The ROC of $x_2[n] = \left(\frac{-1}{4}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[(-n)-1]$.

Properties of Z- transforms:

The z-transform has a set of properties in parallel with that of the Fourier transform (and Laplace transform). The difference is that we need to pay special attention to the ROCs. In the following, we always assume

$$\mathcal{Z}[x[n]] = X(z) \quad ROC = R_x$$

and

$$\mathcal{Z}[y[n]] = Y(z) \quad ROC = R_y$$

Linearity

$$\mathcal{Z}[ax[n] + by[n]] = aX(z) + bY(z), \quad ROC \supseteq (R_x \cap R_y)$$

- **Time Shifting**

$$\mathcal{Z}[x[n-n_0]] = z^{-n_0}X(z), \quad ROC = R_x$$

Proof:

$$\mathcal{Z}[x[n-n_0]] = \sum_{n=-\infty}^{\infty} x[n-n_0]z^{-n}$$

Define $m = n - n_0$, we have $n = m + n_0$ and

$$\sum_{m=-\infty}^{\infty} x[m]z^{-m}z^{-n_0} = z^{-n_0}X(z)$$

The new ROC is the same as the old one except the possible addition/deletion of the origin or infinity as the shift may change the duration of the signal.

- **Time Expansion (Scaling)**

$$\mathcal{Z}[x[n/k]] = X(z^k), \quad ROC = R_x^{1/k}$$

The discrete signal $x[n]$ cannot be continuously scaled in time as n has to be an integer (for a non-integer n $x[n]$ is zero). Therefore $x[n/k]$ is defined as

$$x[n/k] \triangleq \begin{cases} x[n/k] & \text{if } n \text{ is a multiple of } k \\ 0 & \text{else} \end{cases}$$

Example: If $x[n]$ is ramp

n	1	2	3	4	5	6
$x[n]$	1	2	3	4	5	6

then the expanded version $x[n/2]$ is

n	1	2	3	4	5	6
$n/2$	0.5	1	1.5	2	2.5	3
m		1		2		3
$x[n/2]$	0	1	0	2	0	3

where m is the integer part of n/k.

Proof: The z-transform of such an expanded signal is

$$\mathcal{Z}[x[n/k]] = \sum_{n=-\infty}^{\infty} x[n/k]z^{-n} = \sum_{m=-\infty}^{\infty} x[m]z^{-km} = X(z^k)$$

Note that the change of the summation index from n to m has no effect as the terms skipped are all zeros.

- **Convolution**

$$\mathcal{Z}[x[n] * y[n]] = X(z)Y(z), \quad ROC \supseteq (R_x \cap R_y)$$

The ROC of the convolution could be larger than the intersection of R_x and R_y, due to the possible pole-zero cancellation caused by the convolution.

- **Time Difference**

$$\mathcal{Z}[x[n] - x[n-1]] = (1 - z^{-1})X(z), \quad ROC = R_x$$

Proof:

$$\mathcal{Z}[x[n] - x[n-1]] = X(z) - z^{-1}X(z) = (1 - z^{-1})X(z) = \frac{z-1}{z}X(z)$$

Note that due to the additional zero $z = 1$ and pole $z = 0$, the resulting ROC is the same as R_x except the possible deletion of $z = 0$ caused by the added pole and/or addition of $z=1$ caused by the added zero which may cancel an existing pole.

- **Time Accumulation**

$$\mathcal{Z}[\sum_{k=-\infty}^{n} x[k]] = \frac{1}{1-z^{-1}}X(z), \quad ROC \supseteq [R_x \cap (|z|>1)]$$

Proof: The accumulation of $x[n]$ can be written as its convolution with $u[n]$:

$$u[n]*x[n] = \sum_{k=-\infty}^{\infty} u[n-k]x[k] = \sum_{k=-\infty}^{n} x[k]$$

Applying the convolution property, we get

$$\mathcal{Z}[\sum_{k=-\infty}^{n} x[k]] = \mathcal{Z}[u[n]*x[n]] = \frac{1}{1-z^{-1}}X(z)$$

as $\mathcal{Z}[u[n]] = 1/(1-z^{-1})$.

- **Time Reversal**

$$\mathcal{Z}[x[-n]] = X(1/z) \quad ROC = 1/R_x$$

Proof:

$$\mathcal{Z}[x[-n]] = \sum_{n=-\infty}^{\infty} x[-n]z^{-n} = \sum_{m=-\infty}^{\infty} x[m](\frac{1}{z})^{-m} = X(1/z)$$

where $m = -n$.

- **Scaling in Z-domain**

$$\mathcal{Z}[a^n x[n]] = X\left(\frac{z}{a}\right), \quad ROC = |a|R_x$$

Proof:

$$\mathcal{Z}[a^n x[n]] = \sum_{n=-\infty}^{\infty} x[n]\left(\frac{z}{a}\right)^{-n} = X\left(\frac{z}{a}\right)$$

In particular, if $a = e^{j\omega_0}$, the above becomes

$$\mathcal{Z}[e^{jn\omega_0} x[n]] = X(e^{-j\omega_0} z) \quad ROC = R_x$$

The multiplication by $e^{-j\omega_0}$ to z corresponds to a rotation by angle ω_0 in the z-plane, i.e., a frequency shift by ω_0. The rotation is either clockwise ($\omega_0 > 0$) or counter clockwise ($\omega_0 < 0$) corresponding to, respectively, either a left-shift or a right shift in frequency domain. The property is essentially the same as the frequency shifting property of discrete Fourier transform.

- **Conjugation**

$$\mathcal{Z}[x^*[n]] = X^*(z^*), \quad ROC = R_x$$

Proof: Complex conjugate of the z-transform of $x[n]$ is

$$X^*(z) = [\sum_{n=-\infty}^{\infty} x[n]z^{-n}]^* = \sum_{n=-\infty}^{\infty} x^*[n](z^*)^{-n}$$

Replacing z by z^*, we get the desired result.

- **Differentiation in z-Domain**

$$\mathcal{Z}[nx[n]] = -z\,\frac{d}{dz}X(z), \quad ROC = R_x$$

Proof:

$$\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} x[n]\frac{d}{dz}(z^{-n}) = \sum_{n=-\infty}^{\infty} (-n)x[n]z^{-n-1} = \frac{-1}{z}\sum_{n=-\infty}^{\infty} nx[n]z^{-n}$$

i.e.,

$$\mathcal{Z}[nx[n]] = -z\frac{d}{dz}X(z)$$

Example: Taking derivative with respect to z of the right side of

$$\mathcal{Z}[a^n u[n]] = \frac{1}{1 - az^{-1}} \quad |z| > |a|$$

we get

$$\frac{d}{dz}\left[\frac{1}{1-az^{-1}}\right] = \frac{-az^{-2}}{(1-az^{-1})^2}$$

Due to the property of differentiation in z-domain, we have

$$\mathcal{Z}[na^n u[n]] = \frac{az^{-1}}{(1-az^{-1})^2} \quad |z| > |a|$$

Note that for a different ROC $|z| < |a|$, we have

$$\mathcal{Z}[-na^n u[-n-1]] = \frac{az^{-1}}{(1-az^{-1})^2} \quad |z| < |a|$$

MODULE – V

SAMPLING THEOREM

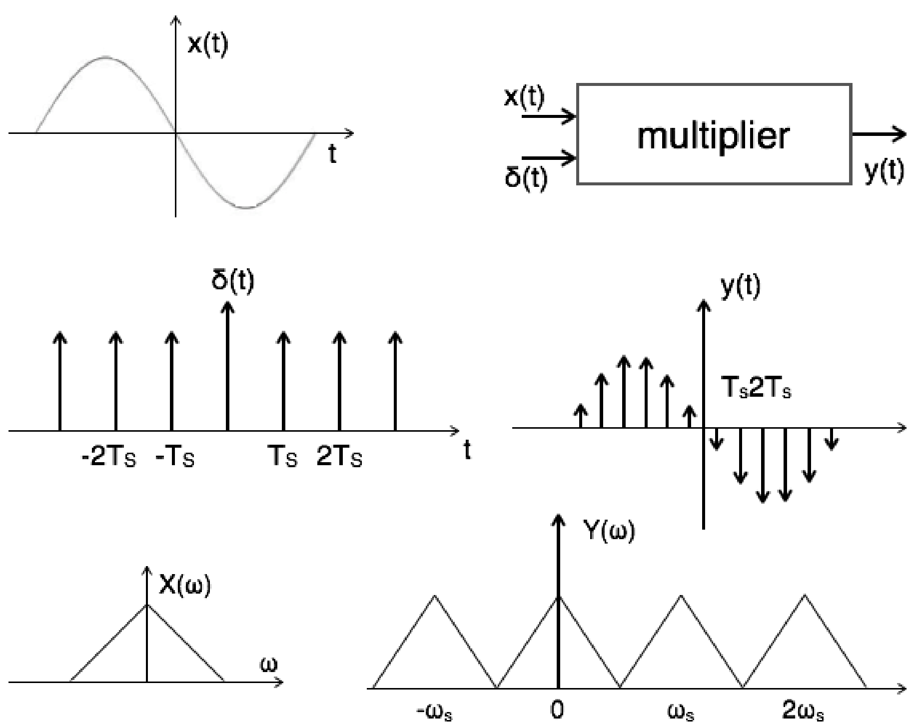
Graphical and analytical proof for Band Limited Signals, Impulse Sampling, Natural and Flat top Sampling, Reconstruction of signal from its samples, Effect of under sampling – Aliasing, Introduction to Band Pass Sampling. Correlation: Cross Correlation and Auto Correlation of Functions, Properties of Correlation Functions, Energy Density Spectrum, Parseval's Theorem, Power Density Spectrum, Relation between Autocorrelation Function and Energy/Power Spectral Density Function, Relation between Convolution and Correlation, Detection of Periodic Signals in the presence of Noise by Correlation, Extraction of Signal from Noise by filtering.

Graphical and analytical proof for Band Limited Signals:

Sampling thoerem: A continuous time signal can be represented in its samples and can be recovered back when sampling frequency f_s is greater than or equal to the twice the highest frequency component of message signal. i. e.

$$fs \geq 2fm$$

Proof: Consider a continuous time signal x(t). The spectrum of x(t) is a band limited to f_m Hz i.e. the spectrum of x(t) is zero for $|\omega|>\omega_m$.Sampling of input signal x(t) can be obtained by multiplying x(t) with an impulse train δ(t) of period T_s. The output of multiplier is a discrete signal called sampled signal which is represented with y(t) in the following diagrams:



Here, you can observe that the sampled signal takes the period of impulse. The process of sampling can be explained by the following mathematical expression:

$$\text{Sampled signal}\, y(t) = x(t).\, \delta(t) \;\ldots\ldots (1)$$

The trigonometric Fourier series representation of δ (t) is given by

$$\delta(t) = a_0 + \Sigma_{n=1}^{\infty}(a_n \cos n\omega_s t + b_n \sin n\omega_s t) \;\ldots\ldots (2)$$

Where $a_0 = \frac{1}{T_s}\int_{\frac{-T}{2}}^{\frac{T}{2}} \delta(t)dt = \frac{1}{T_s}\delta(0) = \frac{1}{T_s}$

$$a_n = \frac{2}{T_s}\int_{\frac{-T}{2}}^{\frac{T}{2}} \delta(t)\cos n\omega_s\, dt = \frac{2}{T_2}\delta(0)\cos n\omega_s 0 = \frac{2}{T}$$

$$b_n = \frac{2}{T_s}\int_{\frac{-T}{2}}^{\frac{T}{2}} \delta(t)\sin n\omega_s t\, dt = \frac{2}{T_s}\delta(0)\sin n\omega_s 0 = 0$$

Substitute above values in equation 2.

$$\therefore\; \delta(t) = \frac{1}{T_s} + \Sigma_{n=1}^{\infty}\left(\frac{2}{T_s}\cos n\omega_s t + 0\right)$$

Substitute δ(t) in equation 1.

$$\rightarrow y(t) = x(t).\, \delta(t)$$

$$= x(t)\left[\frac{1}{T_s} + \Sigma_{n=1}^{\infty}\left(\frac{2}{T_s}\cos n\omega_s t\right)\right]$$

$$= \frac{1}{T_s}\left[x(t) + 2\Sigma_{n=1}^{\infty}(\cos n\omega_s t)x(t)\right]$$

$$y(t) = \frac{1}{T_s}\left[x(t) + 2\cos\omega_s t.\, x(t) + 2\cos 2\omega_s t.\, x(t) + 2\cos 3\omega_s t.\, x(t) \ldots\ldots\right]$$

Take Fourier transform on both sides.

$$Y(\omega) = \frac{1}{T_s}[X(\omega) + X(\omega - \omega_s) + X(\omega + \omega_s) + X(\omega - 2\omega_s) + X(\omega + 2\omega_s) + \ldots]$$

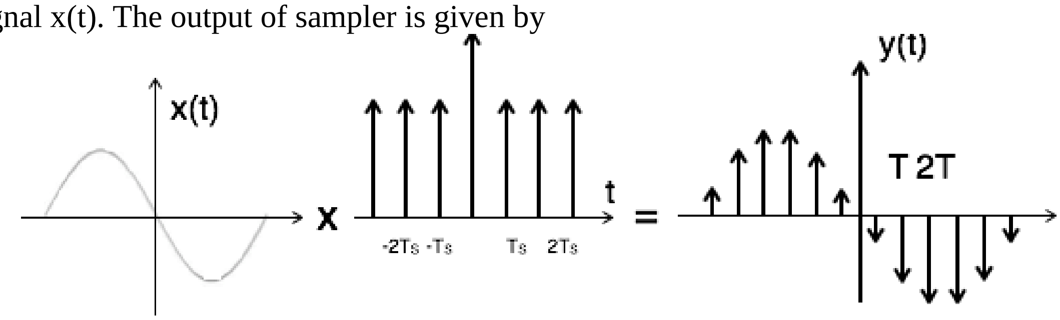
$$\therefore \; Y(\omega) = \frac{1}{T_s}\Sigma_{n=-\infty}^{\infty} X(\omega - n\omega_s) \qquad where\ n = 0, \pm 1, \pm 2, \ldots$$

To reconstruct x(t), you must recover input signal spectrum X(ω) from sampled signal spectrum Y(ω), which is possible when there is no overlapping between the cycles of Y(ω).
There are three types of sampling techniques:

- Impulse sampling.
- Natural sampling.
- Flat Top sampling.

Impulse Sampling
Impulse sampling can be performed by multiplying input signal x(t) with impulse train

$\Sigma_{n=-\infty}^{\infty}\delta(t - nT)$ of period 'T'. Here, the amplitude of impulse changes with respect to amplitude of input signal x(t). The output of sampler is given by



$$y(t) = x(t) \times \text{ impulse train}$$

$$= x(t) \times \Sigma_{n=-\infty}^{\infty}\delta(t - nT)$$

$$y(t) = y_\delta(t) = \Sigma_{n=-\infty}^{\infty} x(nt)\delta(t - nT) \ldots \ldots 1$$

To get the spectrum of sampled signal, consider Fourier transform of equation 1 on both sides

$$Y(\omega) = \frac{1}{T}\Sigma_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

This is called ideal sampling or impulse sampling. You cannot use this practically because pulse width cannot be zero and the generation of impulse train is not possible practically.

Natural Sampling

Natural sampling is similar to impulse sampling, except the impulse train is replaced by pulse train of period T. i.e. you multiply input signal x(t) to pulse train

Substitute p(t) in equation 1

$$y(t) = x(t) \times p(t)$$

$$= x(t) \times \frac{1}{T} \Sigma_{n=-\infty}^{\infty} P(n\omega_s)\, e^{jn\omega_s t}$$

$$y(t) = \frac{1}{T} \Sigma_{n=-\infty}^{\infty} P(n\omega_s)\, x(t)\, e^{jn\omega_s t}$$

To get the spectrum of sampled signal, consider the Fourier transform on both sides.

$$F.T\,[y(t)] = F.T[\frac{1}{T} \Sigma_{n=-\infty}^{\infty} P(n\omega_s)\, x(t)\, e^{jn\omega_s t}]$$

$$= \frac{1}{T} \Sigma_{n=-\infty}^{\infty} P(n\omega_s)\, F.T\,[x(t)\, e^{jn\omega_s t}]$$

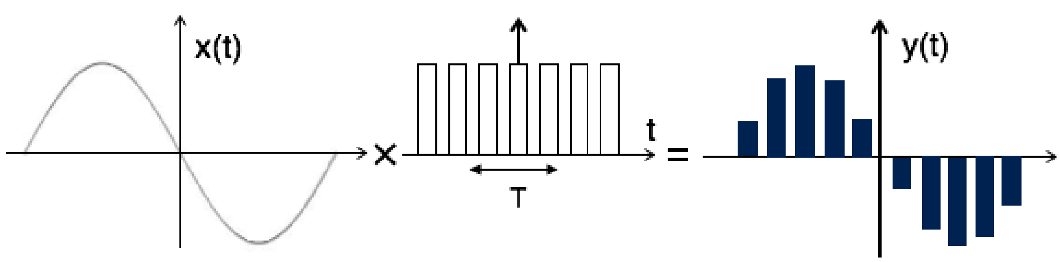
According to frequency shifting property

$$F.T\,[x(t)\, e^{jn\omega_s t}] = X[\omega - n\omega_s]$$

$$\therefore\ Y[\omega] = \frac{1}{T} \Sigma_{n=-\infty}^{\infty} P(n\omega_s)\, X[\omega - n\omega_s]$$

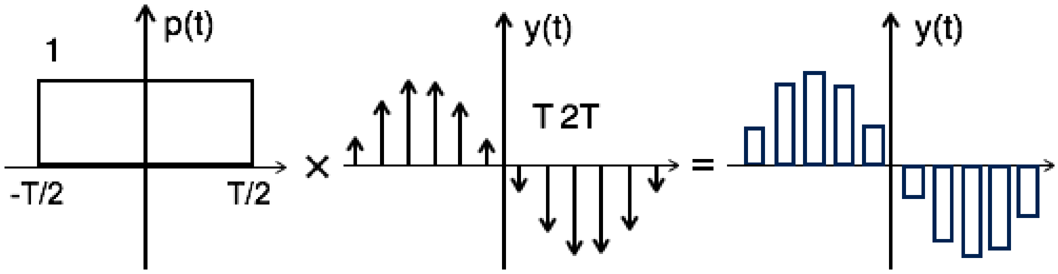
Flat Top Sampling

During transmission, noise is introduced at top of the transmission pulse which can be easily removed if the pulse is in the form of flat top. Here, the top of the samples are flat i.e. they have constant amplitude. Hence, it is called as flat top sampling or practical sampling. Flat top sampling makes use of sample and hold circuit.



Theoretically, the sampled signal can be obtained by convolution of rectangular pulse p(t) with ideally sampled signal say $y_\delta(t)$ as shown in the diagram:

i.e. $y(t) = p(t) \times y_\delta(t) \ldots\ldots (1)$



To get the sampled spectrum, consider Fourier transform on both sides for equation 1

$$Y[\omega] = F.T\,[P(t) \times y_\delta(t)]$$

By the knowledge of convolution property,

$$Y[\omega] = P(\omega)\,Y_\delta(\omega)$$

Here $P(\omega) = TSa(\frac{\omega T}{2}) = 2\sin\omega T/\omega$

Nyquist Rate

It is the minimum sampling rate at which signal can be converted into samples and can be recovered back without distortion.

Nyquist rate $f_N = 2f_m$ hz

Nyquist interval = $1/f_N$ = $1/2fm$ seconds.

Reconstruction of signal from its samples:

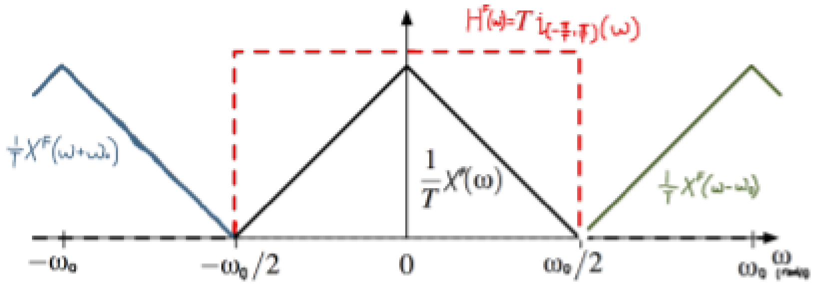
Reconstruction

Assume that the Nyquist requirement $\omega 0 > 2\omega m$ is satisfied. We consider two reconstruction schemes:

- ideal reconstruction (with ideal bandlimited interpolation),
- reconstruction with zero-order hold.

Ideal Reconstruction: Shannon interpolation formula

$$X_P(t) = \ldots + \frac{1}{T}X^F(\omega+\omega_0) + \frac{1}{T}X^F(\omega) + \frac{1}{T}X^F(\omega-\omega_0) + \ldots$$



Our ideal reconstruction filter has the frequency response:

$$H^F(\omega) = T\,\mathbb{1}_{(-\pi/T,\pi/T)}(\omega)$$

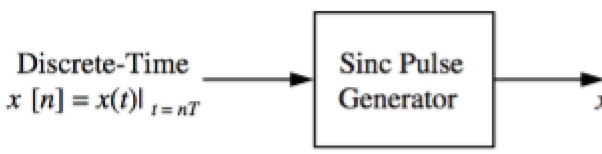
and, consequently, the impulse response

$$h(t) = \mathrm{sinc}\left(\frac{t}{T}\right).$$

Now, the reconstructed signal is

$$x(t) = \underbrace{x_P(t)}_{\text{impulse-sampled signal}} \star h(t) = \sum_{n=-\infty}^{+\infty} x(nT)\,\underbrace{\delta(t-nT)\star h(t)}_{h(t-nT),\ \text{see (3)}} = \sum_{n=-\infty}^{+\infty} x(nT)\,\mathrm{sinc}\left(\frac{t-nT}{T}\right)$$

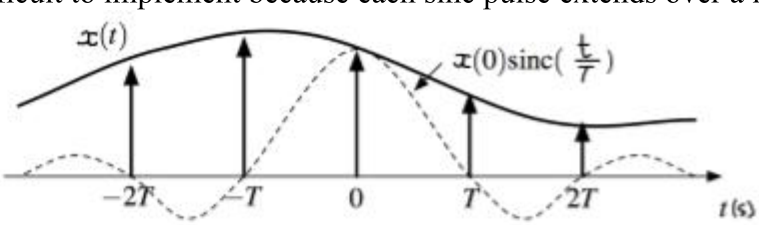
which is the Shannon interpolation (reconstruction) formula. The actual reconstruction system mixes continuous and discrete time.

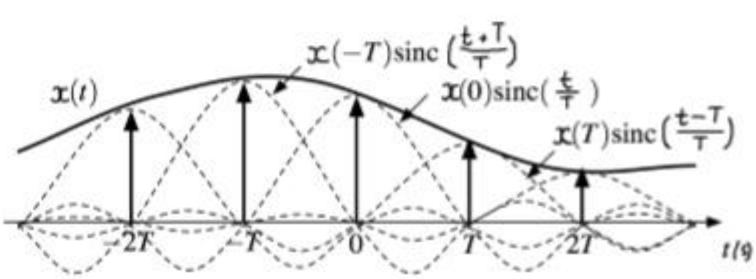


$x[n] = x(t)|_{t=nT}$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h(t-nT) = \sum_{n=-\infty}^{\infty} x[n]\,\mathrm{sinc}\left(\frac{t-nT}{T}\right)$$

The reconstructed signal xr(t) is a train of sinc pulses scaled by the samples x[n]. • This system is difficult to implement because each sinc pulse extends over a long (theoretically infinite) time interval.





A general reconstruction filter

For the development of the theory, it is handy to consider the impulse-sampled signal xP(t) and its

CTFT.

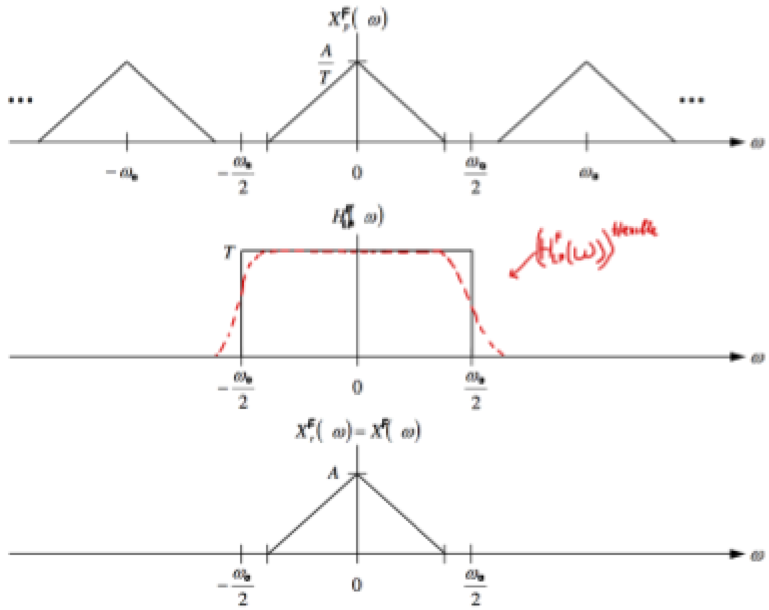


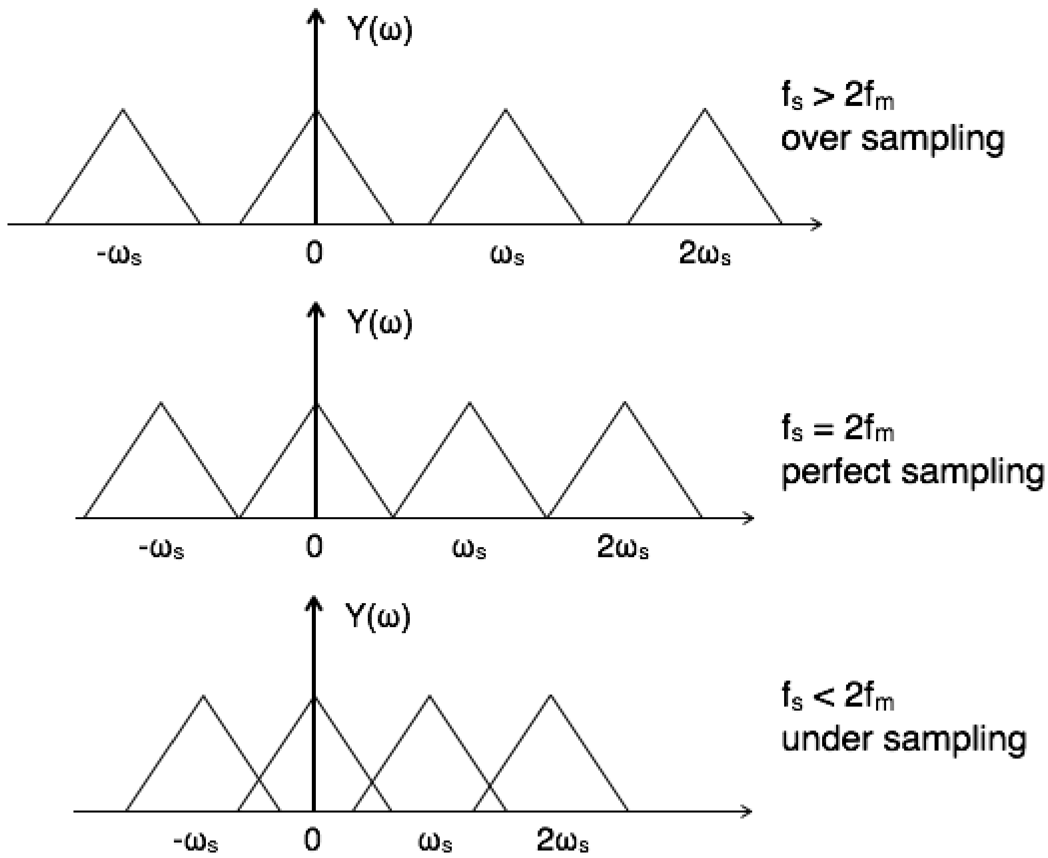
Figure : Reconstruction in the frequency domain is lowpass filtering

Here, the reconstructed signal is $x_r(t)$, with CTFT

$$X_r^F(\omega) = H_{LP}^F(\omega)\, X_p^F(\omega) \overset{\text{sampling th.}}{=} H_{LP}^F(\omega)\, \frac{1}{T} \sum_{k=-\infty}^{+\infty} X^F\Big(\omega - \underbrace{\frac{2\pi k}{T}}_{k\omega_0}\Big).$$

Effect of under sampling – Aliasing

Possibility of sampled frequency spectrum with different conditions is given by the following diagrams:



Aliasing Effect

The overlapped region in case of under sampling represents aliasing effect, which can be removed by

- considering $f_s > 2f_m$
- By using anti aliasing filters.

Samplings of Band Pass Signals

In case of band pass signals, the spectrum of band pass signal $X[\omega] = 0$ for the frequencies outside the range $f_1 \leq f \leq f_2$. The frequency f_1 is always greater than zero. Plus, there is no aliasing effect when $f_s > 2f_2$. But it has two disadvantages:

- The sampling rate is large in proportion with f_2. This has practical limitations.
- The sampled signal spectrum has spectral gaps.

To overcome this, the band pass theorem states that the input signal x(t) can be converted into its samples and can be recovered back without distortion when sampling frequency $f_s < 2f_2$.

Also,

$$f_s = \frac{1}{T} = \frac{2f_2}{m}$$

Where m is the largest integer < $\frac{f_2}{B}$

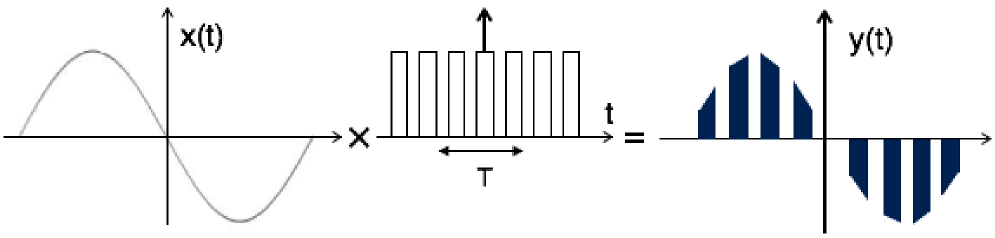
and B is the bandwidth of the signal. If f_2=KB, then

$$f_s = \frac{1}{T} = \frac{2KB}{m}$$

For band pass signals of bandwidth $2f_m$ and the minimum sampling rate f_s= 2 B = $4f_m$,

the spectrum of sampled signal is given by $Y[\omega] = \frac{1}{T}\Sigma_{n=-\infty}^{\infty} X[\omega - 2nB]$

$\Sigma_{n=-\infty}^{\infty} P(t - nT)$



The output of sampler is

$$y(t) = x(t) \times \text{pulse train}$$

$$= x(t) \times p(t)$$

$$= x(t) \times \Sigma_{n=-\infty}^{\infty} P(t - nT) \ldots \ldots (1)$$

The exponential Fourier series representation of p(t) can be given as

$$p(t) = \Sigma_{n=-\infty}^{\infty} F_n e^{jn\omega_s t} \ldots \ldots (2)$$

$$= \Sigma_{n=-\infty}^{\infty} F_n e^{j2\pi n f_s t}$$

Where $F_n = \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} p(t) e^{-jn\omega_s t} dt$

$$= \frac{1}{TP}(n\omega_s)$$

Substitute F_n value in equation 2

$$\therefore p(t) = \Sigma_{n=-\infty}^{\infty} \frac{1}{T} P(n\omega_s) e^{jn\omega_s t}$$

$$= \frac{1}{T} \Sigma_{n=-\infty}^{\infty} P(n\omega_s) e^{jn\omega_s t}$$

Correlation

Cross Correlation and Auto Correlation of Functions:

Correlation

Correlation is a measure of similarity between two signals. The general formula for correlation is

$$\int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)dt$$

There are two types of correlation:

- Auto correlation
- Cross correlation

Auto Correlation Function

It is defined as correlation of a signal with itself. Auto correlation function is a measure of similarity between a signal & its time delayed version. It is represented with R(τ).

Consider a signals x(t). The auto correlation function of x(t) with its time delayed version is given by

$$R_{11}(\tau) = R(\tau) = \int_{-\infty}^{\infty} x(t)x(t-\tau)dt \qquad [\text{+ve shift}]$$

$$= \int_{-\infty}^{\infty} x(t)x(t+\tau)dt \qquad [\text{-ve shift}]$$

Where τ = searching or scanning or delay parameter.

If the signal is complex then auto correlation function is given by

$$R_{11}(\tau) = R(\tau) = \int_{-\infty}^{\infty} x(t)x*(t-\tau)dt \qquad [\text{+ve shift}]$$

$$= \int_{-\infty}^{\infty} x(t+\tau)x*(t)dt \qquad [\text{-ve shift}]$$

Cross Correlation Function

Cross correlation is the measure of similarity between two different signals.

Consider two signals $x_1(t)$ and $x_2(t)$. The cross correlation of these two signals $R12(\tau)$R12(τ) is given by

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)\,dt \qquad [\text{+ve shift}]$$

$$= \int_{-\infty}^{\infty} x_1(t+\tau)x_2(t)\,dt \qquad [\text{-ve shift}]$$

f signals are complex then

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)x_2^*(t-\tau)\,dt \qquad [\text{+ve shift}]$$

$$= \int_{-\infty}^{\infty} x_1(t+\tau)x_2^*(t)\,dt \qquad [\text{-ve shift}]$$

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t)x_1^*(t-\tau)\,dt \qquad [\text{+ve shift}]$$

$$= \int_{-\infty}^{\infty} x_2(t+\tau)x_1^*(t)\,dt \qquad [\text{-ve shift}]$$

Properties of Correlation Functions:

- Auto correlation exhibits conjugate symmetry i.e. $R(\tau) = R^*(-\tau)$

Proof: The autocorrelation of an energy signal $x(t)$ is given by

$$R(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)\,dt$$

Taking the complex conjugate, we have

$$R^*(\tau) = \int_{-\infty}^{\infty} x^*(t)\,x(t-\tau)\,dt$$

$$\therefore \quad R^*(-\tau) = \int_{-\infty}^{\infty} x^*(t)\,x(t+\tau)\,dt = R(\tau)$$

$$\therefore \quad R(\tau) = R^*(-\tau)$$

- Auto correlation function of energy signal at origin i.e. at $\tau = 0$ is equal to total energy of that signal, which is given as:

  $R(0) = E = \int_{-\infty}^{\infty} |x(t)|^2\,dt$

Proof: We have

$$R(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)\,dt$$

Putting $\tau = 0$ gives

$$R(0) = \int_{-\infty}^{\infty} x(t)x^*(t)\,dt = \int_{-\infty}^{\infty} |x(t)|^2\,dt = E$$

- Auto correlation function is maximum at $\tau = 0$ i.e $|R(\tau)| \le R(0)\ \forall \tau$

Proof: Consider the functions $x(t)$ and $x(t+\tau)$. $[x(t) \pm x(t+\tau)]^2$ is always greater than or equal to zero since it is squared, i.e.

$$x^2(t) + x^2(t+\tau) \pm 2x(t)\,x(t+\tau) \ge 0$$

or

$$x^2(t) + x^2(t+\tau) \ge \pm 2x(t)\,x(t+\tau)$$

Integrating both the sides, we get

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt + \int_{-\infty}^{\infty} |x(t+\tau)|^2\,dt \ge 2\int_{-\infty}^{\infty} x(t)\,x(t+\tau)\,dt$$

$$\therefore \quad E + E \ge 2R(\tau) \quad \text{[If } x(t) \text{ is real valued function]}$$

$$\therefore \quad E \ge R(\tau)$$

or

$$R(0) \ge |R(\tau)| \quad (\text{Since } R(0) = E)$$

- Auto correlation function and energy spectral densities are Fourier transform pairs. i.e.

  $F.T[R(\tau)] = S_{XX}(\omega)$

  $S_{XX}(\omega) = \int R(\tau) e^{-j\omega\tau} d\tau$ where $-\infty < \tau < \infty$

- $R(\tau) = x(\tau) * x(-\tau)$

Properties of Cross Correlation Function

- Auto correlation exhibits conjugate symmetry i.e. $R_{12}(\tau)=R^*_{21}(-\tau)$.

- Cross correlation is not commutative like convolution i.e.

  $R_{12}(\tau)\neq R_{21}(-\tau)$
- If $R_{12}(0) = 0$ means, if $\int x_1(t)x^*_2(t)dt=0$ over interval$(-\infty,\infty)$, then the two signals are said to be orthogonal.

- Cross correlation function corresponds to the multiplication of spectrums of one signal to the complex conjugate of spectrum of another signal. i.e.

  $R_{12}(\tau)\longleftrightarrow X_1(\omega)X^*_2(\omega)$

  This also called as correlation theorem.

Energy Density Spectrum:

Energy spectral density describes how the energy of a signal or a time series is distributed with frequency. Here, the term energy is used in the generalized sense of signal processing;
Energy density spectrum can be calculated using the formula:

$$E=\int_{-\infty}^{\infty}|x(f)|^2 df$$

Properties of ESD: The following are the properties of ESD.

1. The total area under the energy density spectrum is equal to the total energy of the signal.

   i.e. $$E=\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(\omega)\,d\omega=\int_{-\infty}^{\infty}\psi(f)\,df$$

2. If $x(t)$ is the input to an LTI system with impulse response $h(t)$, then the input and output ESD functions are related as:

   $$\psi_y(\omega)=|H(\omega)|^2\,\psi_x(\omega)$$

   or $$\psi_y(f)=|H(f)|^2\,\psi_x(f)$$

3. The autocorrelation function $R(\tau)$ and ESD $\psi(\omega)$ form a Fourier transform pair, i.e.

   $$R(\tau)\longleftrightarrow\psi(\omega)$$

   or $$R(\tau)\longleftrightarrow\psi(f)$$

Parseval's Theorem:

Parseval's theorem for energy signals (Rayleigh's energy theorem) Parseval's theorem defines the energy of a signal in terms of its Fourier transform. Using Parseval's theorem, the energy of a signal $x(t)$ can be evaluated directly from its frequency spectrum $X(\omega)$ without the knowledge of its time domain version, i.e. $x(t)$.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

or

$$E = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

Power Density Spectrum

The above definition of energy spectral density is suitable for transients (pulse-like signals) whose energy is concentrated around one time window; then the Fourier transforms of the signals generally exist. For continuous signals over all time, such as stationary processes, one must rather define the *power spectral density* (PSD); this describes how power of a signal or time series is distributed over frequency, as in the simple example given previously. Here, power can be the actual physical power, or more often, for convenience with abstract signals, is simply identified with the squared value of the signal.

Power density spectrum can be calculated by using the formula:

$$P = \Sigma_{n=-\infty}^{\infty} |C_n|^2$$

- The spectrum of a real valued process (or even a complex process using the above definition) is real and an even function of frequency:

$$S_{xx}(-\omega) = S_{xx}(\omega).$$

- If the process is continuous and purely indeterministic, the autocovariance function can be reconstructed by using the Inverse Fourier transform
- The PSD can be used to compute the variance (net power) of a process by integrating over frequency:

$$\operatorname{Var}(X_n) = \frac{1}{\pi} \int_0^{\infty} S_{xx}(\omega) \, d\omega.$$

Proof: Consider a function $x(t)$. We know that

$$|x(t)|^2 = x(t)\, x^*(t)$$

where $x^*(t)$ is the conjugate of $x(t)$.

The average power of $x(t)$ for one cycle is:

$$P = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, x^*(t)\, dt$$

But, we have the exponential Fourier series,

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t}$$

$$\therefore \quad P = \frac{1}{T}\int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t} x^*(t)\, dt$$

Interchanging the order of summation and integration, we get

$$P = \sum_{n=-\infty}^{\infty} C_n \frac{1}{T}\int_{-T/2}^{T/2} x^*(t)\, e^{jn\omega t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} C_n C_n^* = \sum_{n=-\infty}^{\infty} |C_n|^2$$

$$\therefore \quad P = \sum_{n=-\infty}^{\infty} |C_n|^2$$

This is called Parseval's power theorem. It states that the power of a signal is equal to the sum of square of the magnitudes of various harmonics present in the discrete spectrum.

Table .1 Comparision of ESD and PSD

S.No.	ESD	PSD
1.	It gives the distribution of energy of a signal in frequency domain.	It gives the distribution of power of a signal in frequency domain.
2.	It is given by $\psi(\omega) = \lvert X(\omega)\rvert^2$	It is given by $S(\omega) = \underset{T\to\infty}{\text{Lt}} \frac{\lvert X(\omega)\rvert^2}{T}$
3.	The total energy is given by $E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\omega)\, d\omega = \int_{-\infty}^{\infty} \psi(f)\, df$	The total power is given by $P = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, d\omega = \int_{-\infty}^{\infty} S(f)\, df$
4.	The autocorrelation for an energy signal and its ESD form a Fourier transform pair. $R(\tau) \longleftrightarrow \psi(\omega)$ or $R(\tau) \longleftrightarrow \psi(f)$	The autocorrelation for a power signal and its PSD form a Fourier transform pair $R(\tau) \longleftrightarrow S(\omega)$ or $R(\tau) \longleftrightarrow S(f)$

Relation between Autocorrelation Function and Energy/Power Spectral Density Function:

1. Relation between Autocorrelation Function and Energy Spectral Density Function:

The autocorrelation function $R(\tau)$ and energy spectral density function $\psi(\omega)$ form a Fourier transform pair, i.e.

$$R(\tau) \longleftrightarrow \psi(\omega)$$

Proof: The autocorrelation of a function $x(t)$ is given as:

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

Replacing $x^*(t-\tau)$ by its inverse transform, we have

$$R(\tau) = \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega(t-\tau)}\, d\omega \right]^* dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) \left[\int_{-\infty}^{\infty} X^*(\omega)\, e^{-j\omega(t-\tau)}\, d\omega \right] dt$$

Interchanging the order of integration, we have

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \right] e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega)\, X(\omega)\, e^{j\omega\tau}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(\omega)\, e^{j\omega\tau}\, d\omega \quad [\text{since } |X(\omega)|^2 = \psi(\omega)]$$

$$= F^{-1}[\psi(\omega)]$$

$$\therefore \quad \psi(\omega) = F[R(\tau)]$$

This proves that $R(\tau)$ and $\psi(\omega)$ form a Fourier transform pair.

$$R(\tau) \longleftrightarrow \psi(\omega)$$

2. Relation between Autocorrelation Function and Power Spectral Density Function:

The autocorrelation function $R(\tau)$ and the power spectral density (PSD), $S(\omega)$ of a power signal form a Fourier transform pair, i.e.

$$R(\tau) \longleftrightarrow S(\omega)$$

Proof: The autocorrelation function of a power (periodic) signal $x(t)$ in terms of Fourier series coefficients is given as:

$$R(\tau) = \sum_{n=-\infty}^{\infty} C_n C_{-n} e^{jn\omega_0 \tau}$$

where C_n and C_{-n} are the exponential Fourier series coefficients.

$$\therefore \qquad R(\tau) = \sum_{n=-\infty}^{\infty} |C_n|^2 e^{jn\omega_0 \tau}$$

Taking Fourier transform on both sides, we have

$$F[R(\tau)] = \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} |C_n|^2 e^{jn\omega_0 \tau} \right) e^{-j\omega\tau}\, d\tau$$

Interchanging the order of integration and summation, we get

$$F[R(\tau)] = \sum_{n=-\infty}^{\infty} |C_n|^2 \int_{-\infty}^{\infty} e^{-j\tau(\omega - n\omega_0)}\, d\tau$$

$$= 2\pi \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(\omega - n\omega_0) = \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(f - nf_0)$$

The RHS is the PSD $S(\omega)$ or $S(f)$ of the periodic function $x(t)$.

$$\therefore \qquad F[R(\tau)] = S(\omega) \quad [\text{or } S(f)]$$

and $$F^{-1}[S(\omega)] \quad \{\text{or } F^{-1}[S(f)]\} = R(\tau)$$

i.e. $$R(\tau) \longleftrightarrow S(\omega) \quad [\text{or } S(f)]$$

Relation between Convolution and Correlation:

The convolution of $x_1(t)$ and $x_2(-t)$ is given by

$$x_1(t) * x_2(-t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(\tau - t)\, d\tau$$

Replacing the dummy variable τ in the above integral by another variable n, we have

$$x_1(t) * x_2(-t) = \int_{-\infty}^{\infty} x_1(n)\, x_2(n - t)\, dn$$

Changing the variable from t to τ, we get

$$x_1(\tau) * x_2(-\tau) = \int_{-\infty}^{\infty} x_1(n)\, x_2(n - \tau)\, dn = R_{12}(\tau)$$

Hence $$R_{12}(\tau) = x_1(t) * x_2(-t)\big|_{t=\tau}$$

Similarly, $$R_{21}(\tau) = x_2(t) * x_1(-t)\big|_{t=\tau}$$

Detection of Periodic Signals in the presence of Noise by Correlation:

Extraction of Signal from Noise by filtering.

Whenever we wish to use correlation for signal detection, we use a two-part system. The first part of the system performs the correlation and produces the correlation value or correlation signal, depending upon whether we are doing in-place or running correlation. The second part of the system examines the correlation or correlation signal and makes a decision or sequence of decisions. See the block diagram given in Figure