ADVANCED STRUCTURAL ANALYSIS
(COURSE CODE : BSTB01)
REGULATION : IARE-R18
M.TECH- I SEM

PREPARED BY
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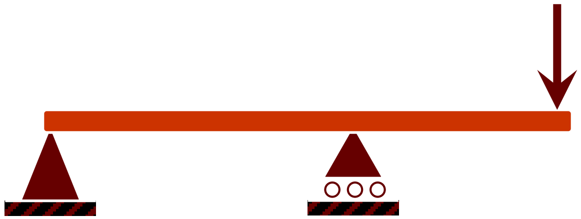
UNIT-I
INFLUENCE COEFFICIENTS

Introduction

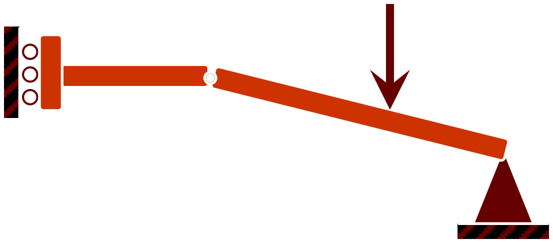
- What is statically ***DETERMINATE*** structure?
  - When all the forces (reactions) in a structure can be determined from the equilibrium equations its called statically determinate structure
  - Structure having unknown forces equal to the available equilibrium equations



No. of unknown = 3

No. of equilibrium equations = 3

3 = 3 thus statically determinate



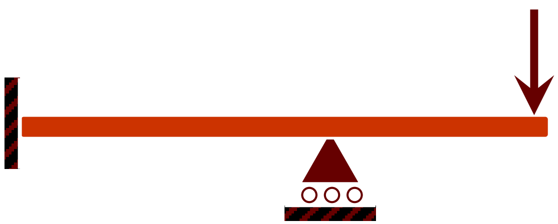
No. of unknown = 6

No. of equilibrium equations = 6

6 = 6 thus statically determinate

Introduction

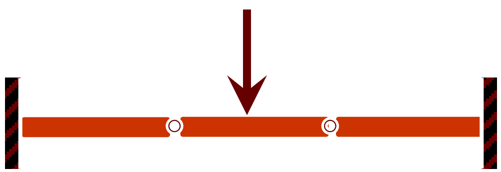
- What is statically ***INETERMINATED*** structure
  - Structure having more unknown forces than available equilibrium equations
  - Additional equations needed to solve the unknown reactions



No. of unknown = 4

No. of equilibrium equations = 3

4 > 3 thus statically Indeterminate



No. of unknown = 10

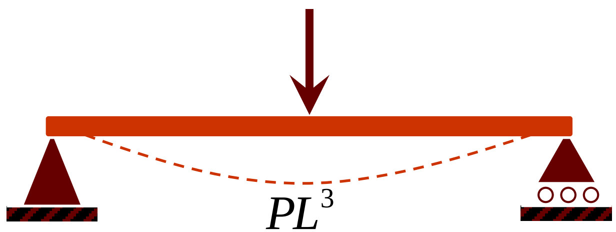
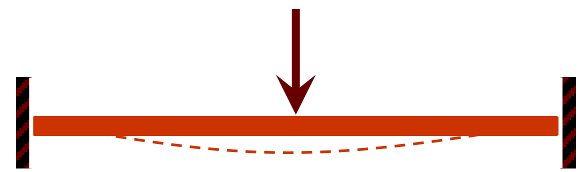
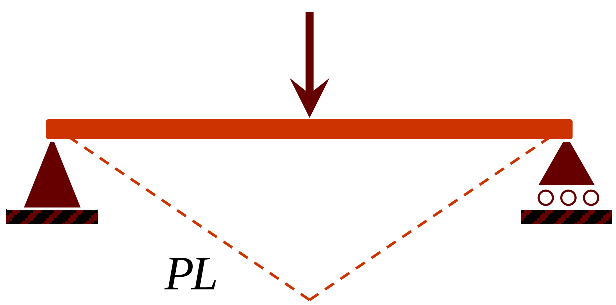
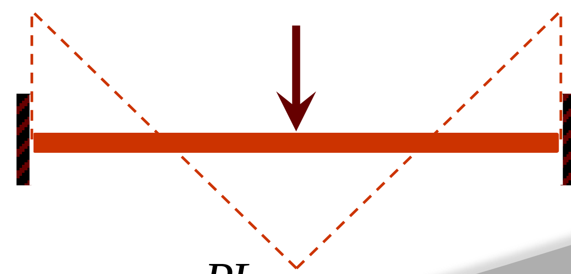
No. of equilibrium equations = 9

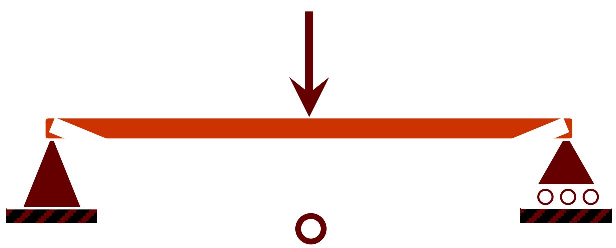
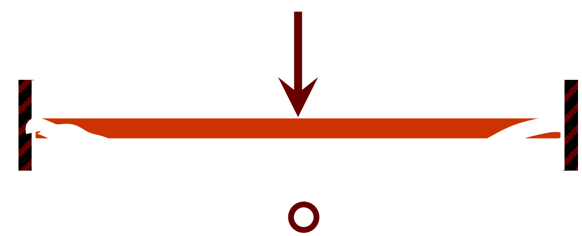
10 > 9 thus statically

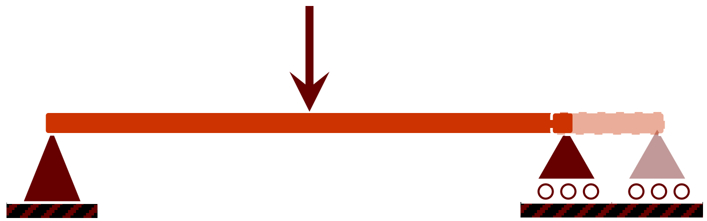
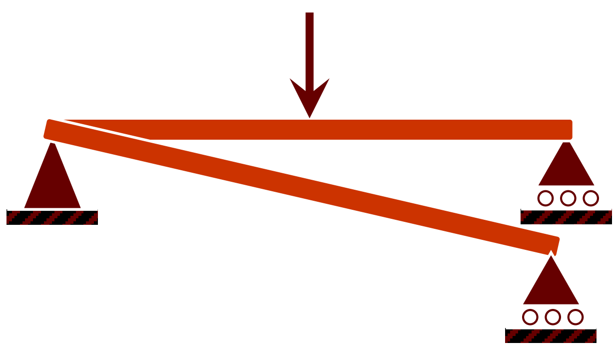
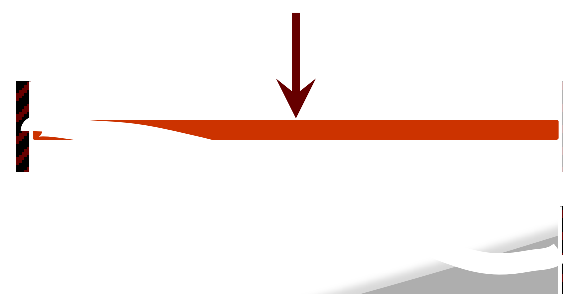
Indeterminate Structure

Why we study indeterminate structure

- Most of the structures designed today are statically indeterminate
- Reinforced concrete buildings are considered in most cases as a statically indeterminate structures since the columns & beams are poured as continuous member through the joints & over the supports
- More stable compare to determinate structure or in another word safer.
- In many cases more economical than determinate.
- The comparison in the next page will enlighten more

	Determinate Structure	Indeterminate Structure
Deflection	 $\frac{PL^3}{48EI}$	 $\frac{PL^3}{192EI}$
Stress	 $\frac{PL}{4}$	 $\frac{PL}{8}$

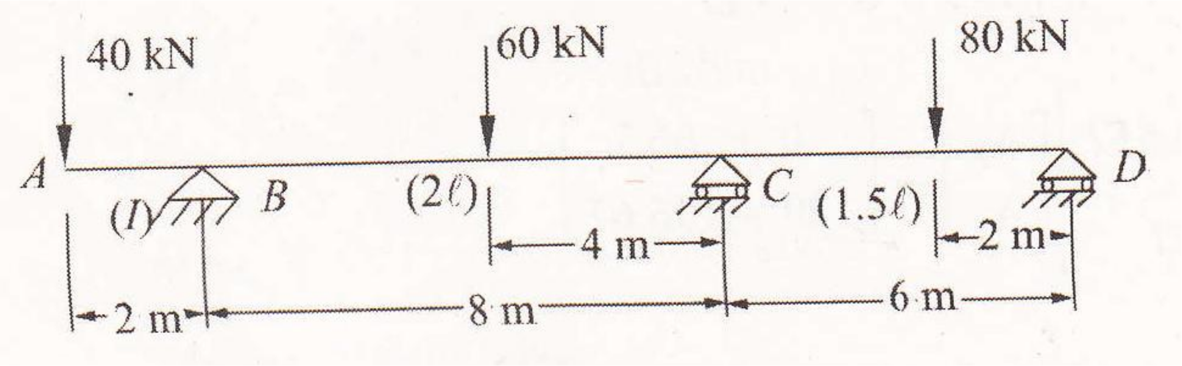
	Determinate Structure	Indeterminate Structure
Stability in case of over load	✶Support will not develop the horizontal force & moments that necessary to prevent total collapse ✶No load redistribution ✶When the plastic hinge formed certain collapse for the system 	✶Will develop horizontal force & moment reactions that will hold the beam ✶Has the tendency to redistribute its load to ✶When the plastic hinge formed the system would be a determinate structure 

	Determinate Structure	Indeterminate Structure
Temperature	No effect & no stress would be developed in the beam 	Serious effect and stress would be developed in the 
Differential Displacement	No effect & no stress would be developed 	Serious effect and stress would be developed 

STIFFNESS METHOD

Example:

Analyze the continuous beam as shown in figure using stiffness method.



Solution:

Solution Figure 11.28(*b*) shows an equivalent beam. The coordinates selected are shown in Figure 11.28(*c*) and the fully restrained structure is shown in Figure 11.28(*d*).

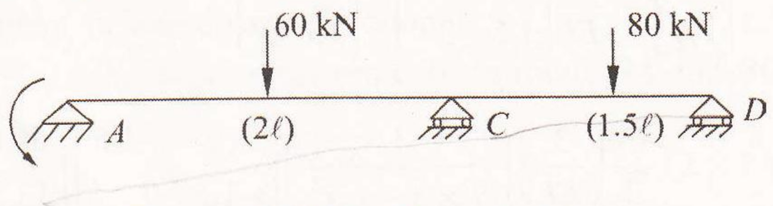


Figure 11.28(b): Equivalent beam.

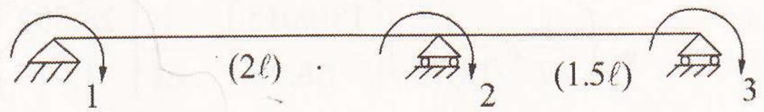


Figure 11.28(c): Coordinates selected.

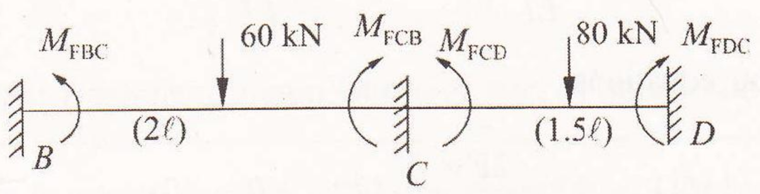


Figure 11.28(d): Fully restrained structure.

Final force vector

$$[P] = \begin{bmatrix} -80 \\ 0 \\ 0 \end{bmatrix}$$

$$M_{\text{FBC}} = -60 \times \frac{8}{8} = -60 \text{ kNm}$$

$$M_{\text{FCB}} = 60 \text{ kNm}$$

$$M_{\text{FCD}} = -\frac{80 \times 4 \times 2^2}{6^2} = -35.55 \text{ kNm}$$

$$M_{\text{FDC}} = \frac{80 \times 4^2 \times 2}{6^2} = 17.11 \text{ kNm}$$

$$[P_{\text{L}}] = \begin{bmatrix} -60.0 \\ 60 - 35.55 \\ 71.11 \end{bmatrix} = \begin{bmatrix} -60 \\ 24.45 \\ 71.11 \end{bmatrix}$$

Stiffness Matrix

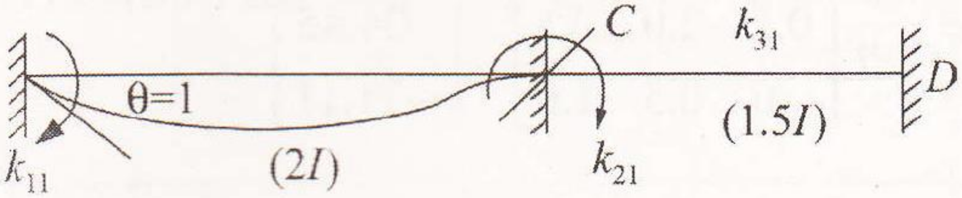
(*a*) ***Unit displacement at B:***

Referring to Figure 11.29 given below

$$k_{11} = \frac{4E \times 2I}{8} = EI$$

$$k_{21} = \frac{2E(2I)}{8} = 0.5\,EI$$

$$k_{31} = 0$$



(*b*) ***Unit displacement at C:***

Referring to Figure 11.30 given below

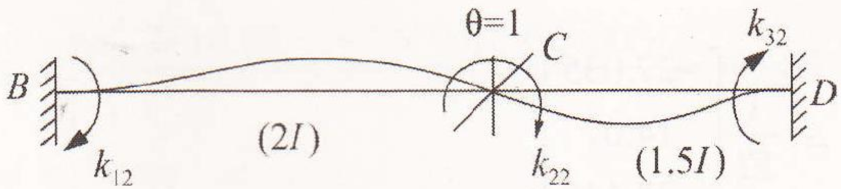


Figure 11.30: Restrained structure with unit displacement in coordinate direction 2.

$$k_{12} = \frac{2E\,(2I)}{8} = 0.5EI$$

$$k_{22} = \frac{4E\,(2I)}{8} + \frac{2\,E\,(1.5\,I)}{6} = 2\,EI$$

$$k_{32} = \frac{2E\,(1.5I)}{6} = 0.5\,EI$$

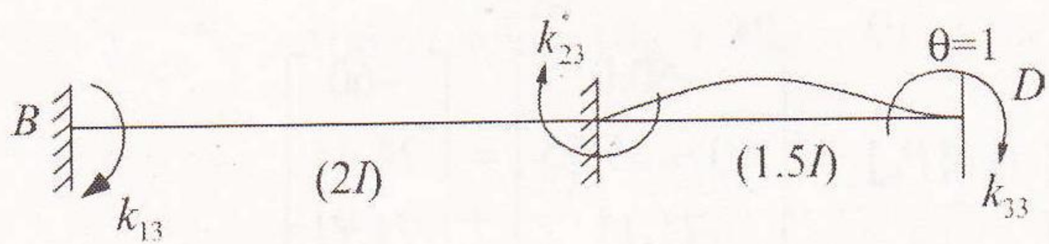
(c) ***Unit displacement at D:***

Referring to Figure 11.31 given below

$$k_{13} = 0$$

$$k_{23} = \frac{2E\,(1.5I)}{6} = 0.5\,EI$$

$$k_{33} = \frac{4E\,(1.5I)}{6} = EI$$



Stiffness Matrix equation is

$$[k]\ [\Delta] = [P - P_L]$$

$$EI \begin{bmatrix} 1.0 & 0.5 & 0 \\ 0.5 & 2.0 & 0.5 \\ 0 & 0.5 & 1.0 \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \begin{bmatrix} -80 + 60 \\ 0 - 24.45 \\ 0 - 71.11 \end{bmatrix}$$

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} 1.0 & 0.5 & 0 \\ 0.5 & 2.0 & 0.5 \\ 0 & 0.5 & 1.0 \end{bmatrix}^{-1} \begin{bmatrix} -20 \\ -24.45 \\ -71.11 \end{bmatrix}$$

$$= \frac{1}{EI}\left(\frac{1}{2\times 1-0.5^2-0.5\,(0.5-0)}\right)\times\begin{bmatrix} 1.75 & -0.5 & 0.25 \\ -0.5 & 1.0 & -0.5 \\ 0.25 & -0.5 & 7.75 \end{bmatrix}\begin{bmatrix} -20 \\ -24.45 \\ -71.11 \end{bmatrix}$$

$$= \frac{1}{EI}\begin{bmatrix} -27.035 \\ 14.07 \\ -78.145 \end{bmatrix}$$

$$\begin{bmatrix} \theta_B \\ \theta_C \\ \theta_D \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} -27.035 \\ 14.07 \\ -78.145 \end{bmatrix}$$

$$M_{AB} = -80 \text{ kNm}$$

$$M_{BC} = -60 + \frac{2E\,(2I)}{8}\left[2\frac{(-27.035)}{EI} + \frac{14.07}{EI}\right] = -80 \text{ kNm}$$

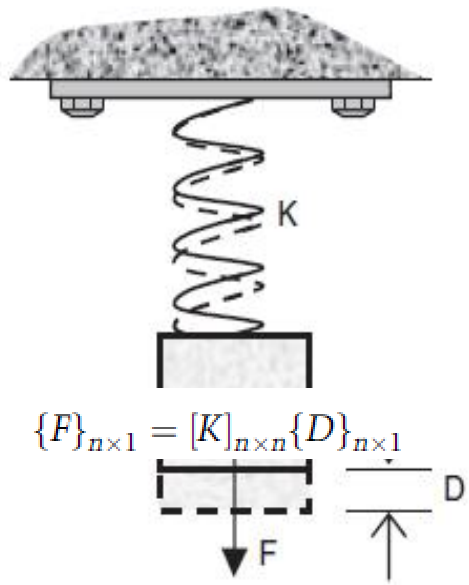
$$M_{CB} = 60 + \frac{2E(2I)}{8}\left[\frac{-27.035 + 2 \times 14.07}{EI}\right] = 60.55 \text{ kNm}$$

$$M_{CD} = -35.55 + \frac{2E(1.5I)}{6}\left[\frac{2 \times 14.07 - 78.145}{EI}\right] = -60.55 \text{ kNm}$$

$$M_{DC} = 71.11 + \frac{2E(1.5I)}{6}\left[\frac{14.07 - 2 \times 78.145}{EI}\right] = 0$$

UNIT-II
STIFFNESS METHOD APPLIED TO LARGE FRAMES

MATRIX STIFFNESS METHOD OF ANALYSIS

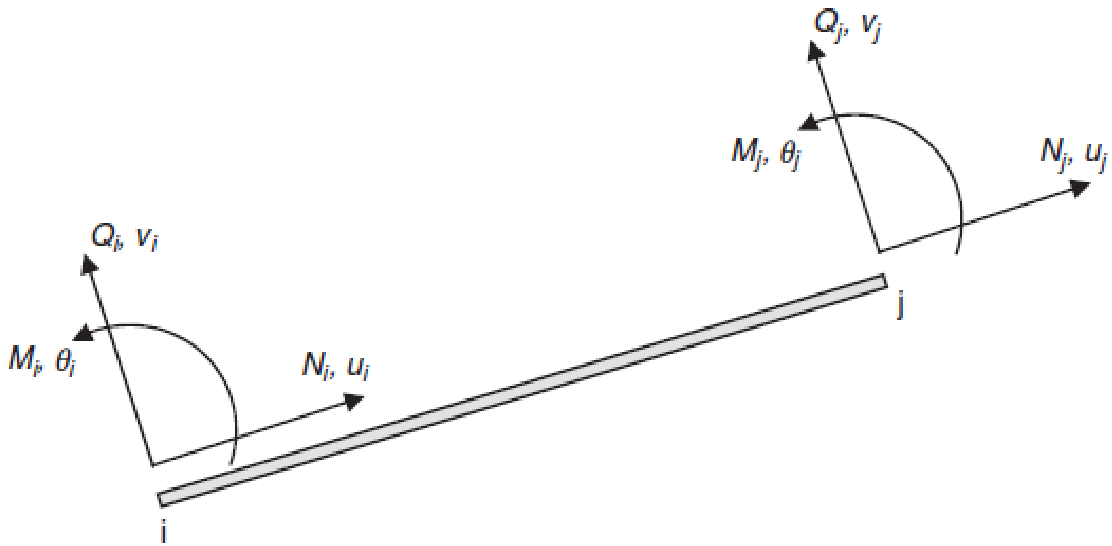


$$\{F\}_{n\times1} = [K]_{n\times n}\{D\}_{n\times1}$$

$$F = KD \qquad \{F\}_{n\times1} = [K]_{n\times n}\{D\}_{n\times1}$$

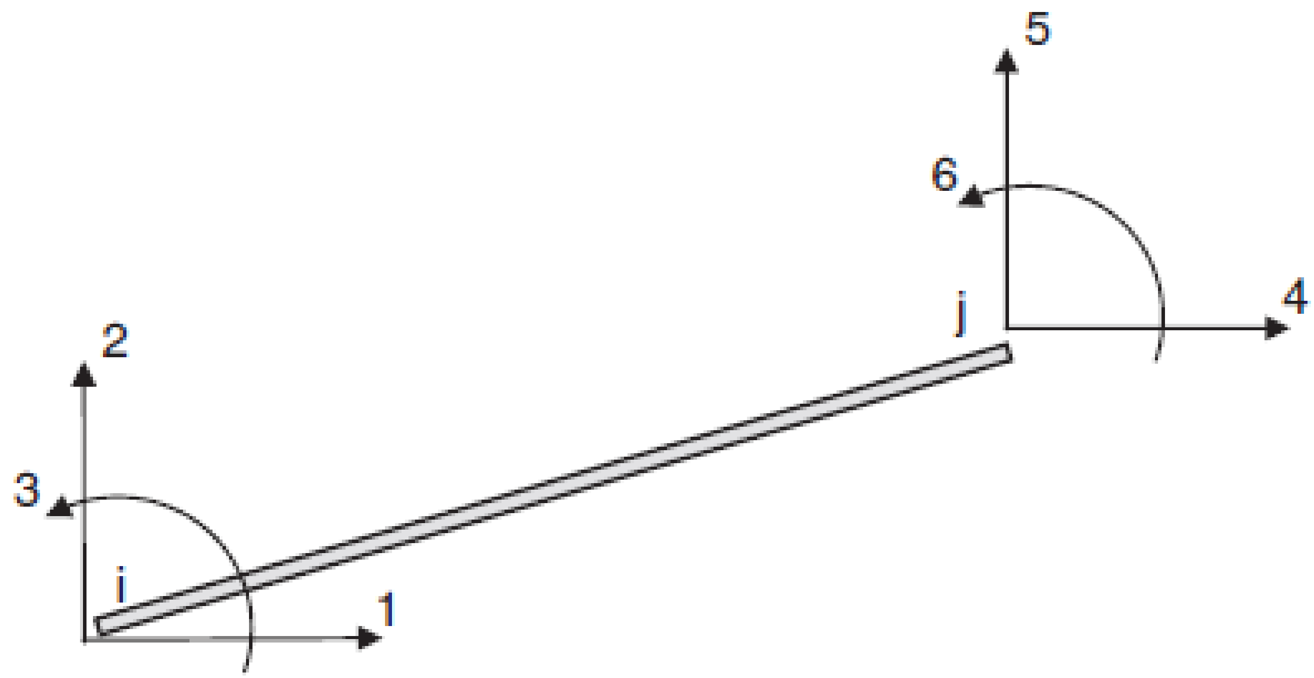
$$D = F/K \qquad \{D\} = [K]^{-1}\{F\}$$

LOCAL COORDINATE SYSTEM



$M_{i,j}$, $\theta_{i,j}$ = bending moments and corresponding rotations at ends i, j, respectively; $N_{i,j}$, $u_{i,j}$ are axial forces and corresponding axial deformations at ends i, j, respectively; and $Q_{i,j}$, $v_{i,j}$ are shear forces and corresponding transverse displacements at ends i, j, respectively. The directions of the actions and movements are positive when using the stiffness method.

DEGREES OF FREEDOM



MEMBER STIFFNESS MATRIX

The structure stiffness matrix $[K]$ is assembled on the basis of the equilibrium and compatibility conditions between the members. For a general frame, the equilibrium matrix equation of a member is

$$\{P\} = [K_e]\{d\} \tag{1.9}$$

where $\{P\}$ is the member force vector, $[K_e]$ is the member stiffness matrix, and $\{d\}$ is the member displacement vector, all in the member's local coordinate system. The elements of the matrices in Equation (1.9) are given as

$$\{P\} = \begin{Bmatrix} N_i \\ Q_i \\ M_i \\ N_j \\ Q_j \\ M_j \end{Bmatrix}; [K_e] = \begin{bmatrix} K_{11} & 0 & 0 & K_{14} & 0 & 0 \\ 0 & K_{22} & K_{23} & 0 & K_{25} & K_{26} \\ 0 & K_{32} & K_{33} & 0 & K_{35} & K_{36} \\ K_{41} & 0 & 0 & K_{44} & 0 & 0 \\ 0 & K_{52} & K_{53} & 0 & K_{55} & K_{56} \\ 0 & K_{62} & K_{63} & 0 & K_{65} & K_{66} \end{bmatrix}; \{d\} = \begin{Bmatrix} u_i \\ v_i \\ \theta_i \\ u_j \\ v_j \\ \theta_j \end{Bmatrix}$$

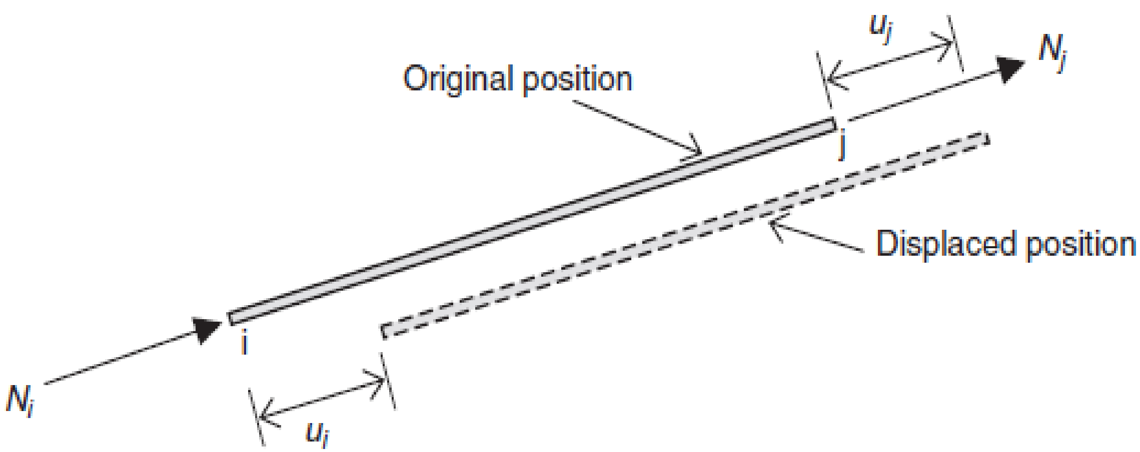
AXIAL LOADING

A member under axial forces N_i and N_j acting at its ends produces axial displacements u_i and u_j as shown in Figure 1.10. From the stress-strain relation, it can be shown that

$$N_i = \frac{EA}{L}(u_i - u_j) \tag{1.10a}$$

$$N_j = \frac{EA}{L}(u_j - u_i) \tag{1.10b}$$

where E is Young's modulus, A is cross-sectional area, and L is length of the member. Hence, $K_{11} = -K_{14} = -K_{41} = K_{44} = \frac{EA}{L}$.



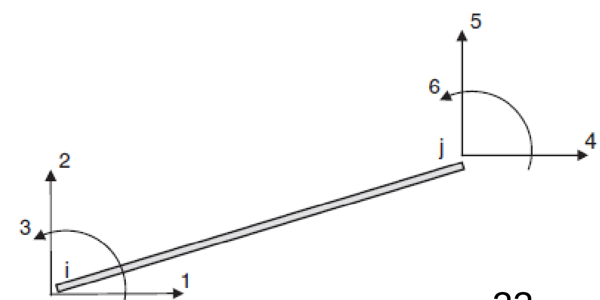


FIGURE 1.10. Member under axial forces.

For a member with shear forces Q_i, Q_j and bending moments M_i, M_j acting at its ends as shown in Figure 1.11, the end displacements and rotations are related to the bending moments by the slope-deflection equations as

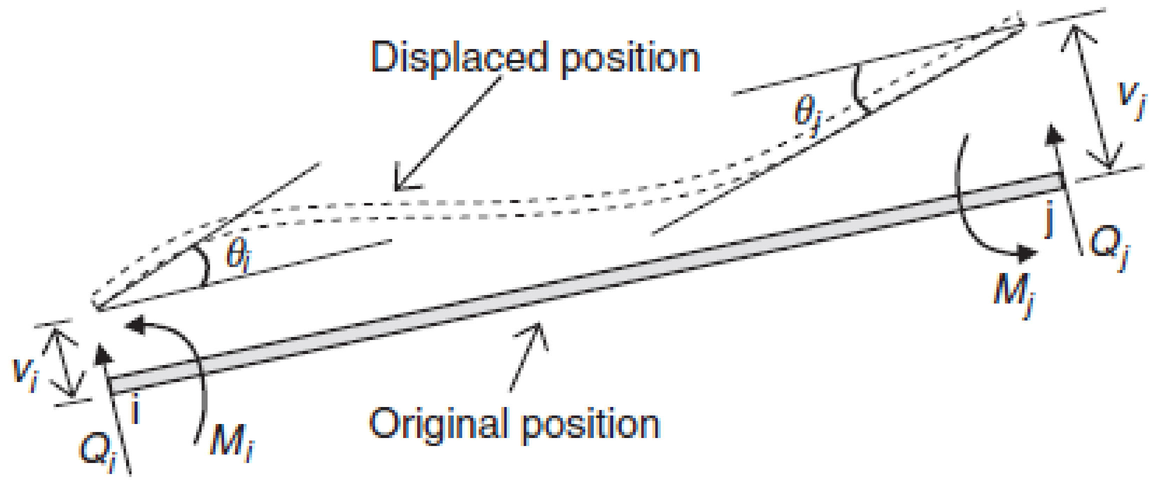


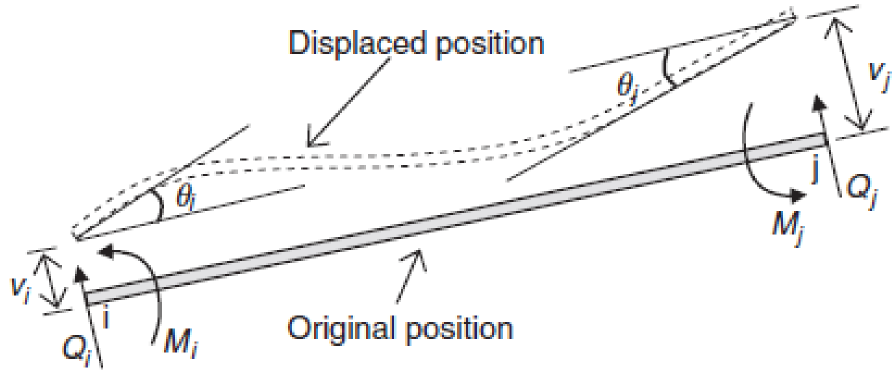
FIGURE 1.11. Member under shear forces and bending moments.

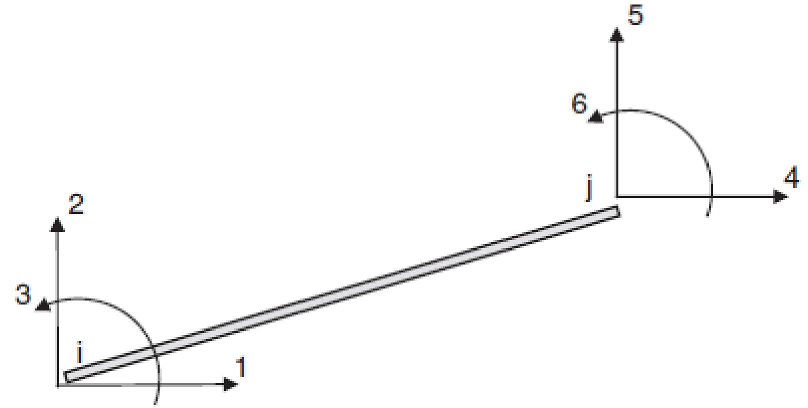
BENDING MOMENTS AND SHEAR

$$M_i = \frac{2EI}{L}\left[2\theta_i + \theta_j - \frac{3(v_j - v_i)}{L}\right]$$

$$M_j = \frac{2EI}{L}\left[2\theta_j + \theta_i - \frac{3(v_j - v_i)}{L}\right]$$

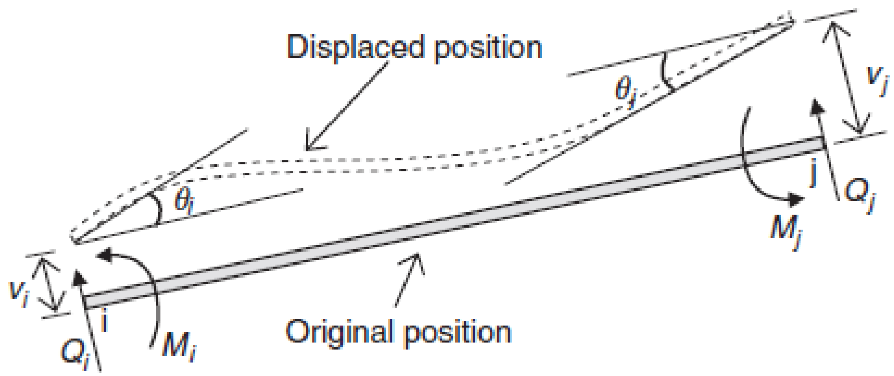
Hence, $K_{62} = -K_{65} = \frac{6EI}{L^2}$, $K_{63} = \frac{2EI}{L}$, and $K_{66} = \frac{4EI}{L}$.





By taking the moment about end j of the member in Figure 1.11, we obtain

$$Q_i = \frac{M_i + M_j}{L} = \frac{2EI}{L^2}\left[3\theta_i + 3\theta_j - \frac{6(v_j - v_i)}{L}\right] \qquad (1.12a)$$



Also, by taking the moment about end i of the member, we obtain

$$Q_j = -\left(\frac{M_i + M_j}{L}\right) = -Q_i \qquad (1.12b)$$

$$K_{22} = K_{55} = -K_{25} = -K_{52} = \frac{12EI}{L^3} \text{ and } K_{23} = K_{26} = -K_{53} = -K_{66} = \frac{6EI}{L^2}.$$

In summary, the resulting member stiffness matrix is symmetric about the diagonal:

$$[K_e] = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \tag{1.13}$$

COORDINATES TRANSFORMATION

In order to establish the equilibrium conditions between the member forces in the local coordinate system and the externally applied loads in the global coordinate system, the member forces are transformed into the global coordinate system by force resolution. Figure 1.12 shows a member inclined at an angle α to the horizontal.

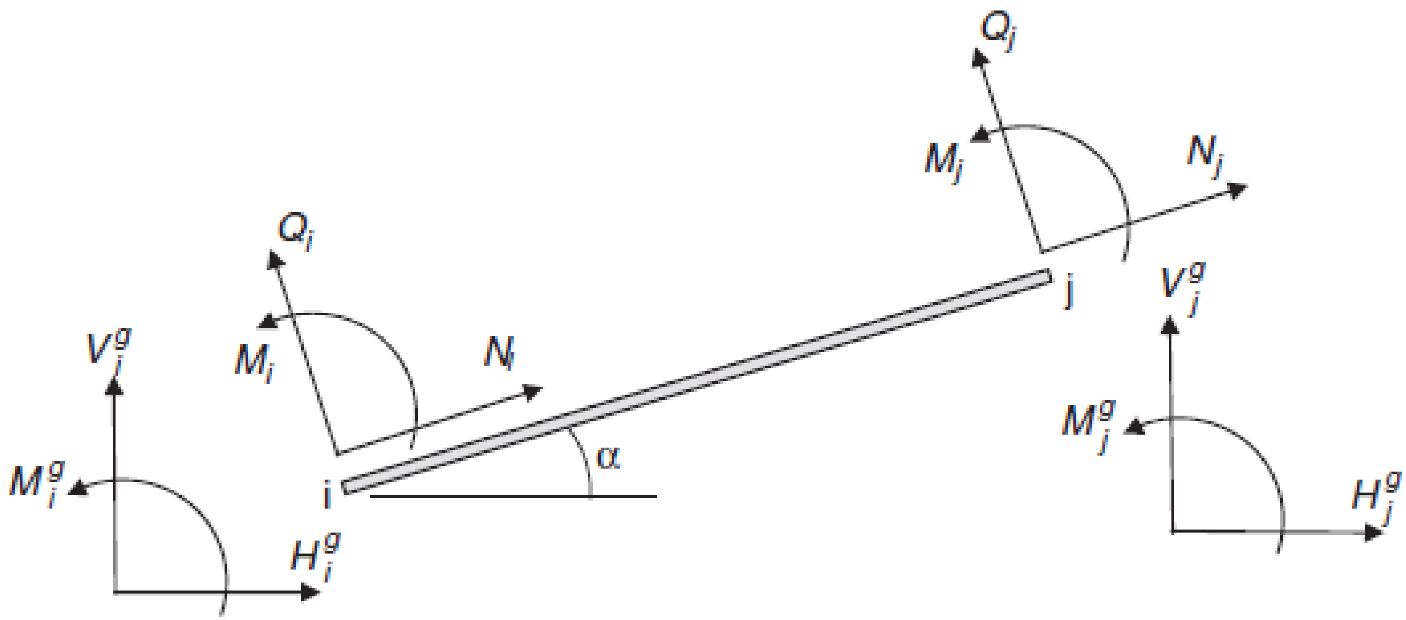
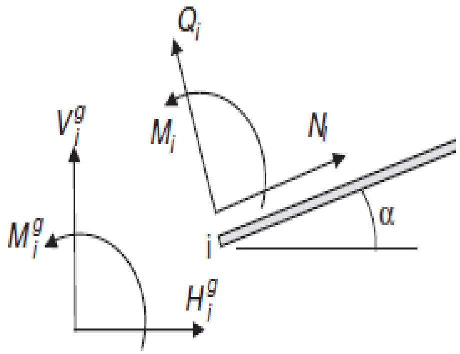


FIGURE 1.12. Forces in the local and global coordinate systems.

LOAD TRANSFORMATION

The forces in the global coordinate system shown with superscript "g" in Figure 1.12 are related to those in the local coordinate system by

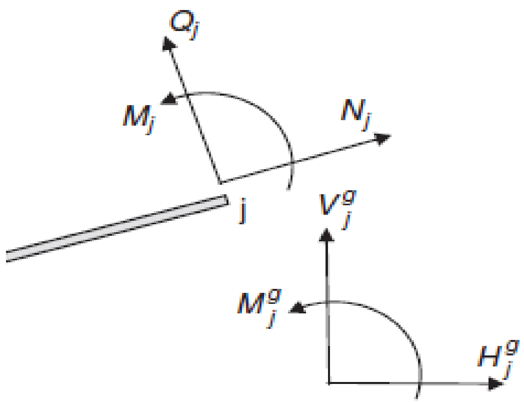


$$H_i^g = N_i \cos\alpha - Q_i \sin\alpha \tag{1.14a}$$

$$V_i^g = N_i \sin\alpha + Q_i \cos\alpha \tag{1.14b}$$

$$M_i^g = M_i \tag{1.14c}$$

Similarly,



$$H_j^g = N_j \cos\alpha - Q_j \sin\alpha \tag{1.14d}$$

$$V_j^g = N_j \sin\alpha + Q_j \cos\alpha \tag{1.14e}$$

$$M_j^g = M_j \tag{1.14f}$$

In matrix form, Equations (1.14a) to (1.14f) can be expressed as

$$\{F_e^g\} = [T]\{P\} \tag{1.15}$$

LOAD TRANSFORMATION

In matrix form, Equations (1.14a) to (1.14f) can be expressed as

$$\{F_e^g\} = [T]\{P\} \tag{1.15}$$

where $\{F_e^g\}$ is the member force vector in the global coordinate system and $[T]$ is the transformation matrix, both given as

$$\{F_e^g\} = \begin{Bmatrix} H_i^g \\ V_i^g \\ M_i^g \\ H_j^g \\ V_j^g \\ M_j^g \end{Bmatrix} \text{ and } [T] = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & 0 & 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

DISPLACEMENT TRANSFORMATION

The displacements in the global coordinate system can be related to those in the local coordinate system by following the procedure similar to the force transformation. The displacements in both coordinate systems are shown in Figure 1.13.

From Figure 1.13,

$$u_i = u_i^g \cos\alpha + v_i^g \sin\alpha \tag{1.16a}$$

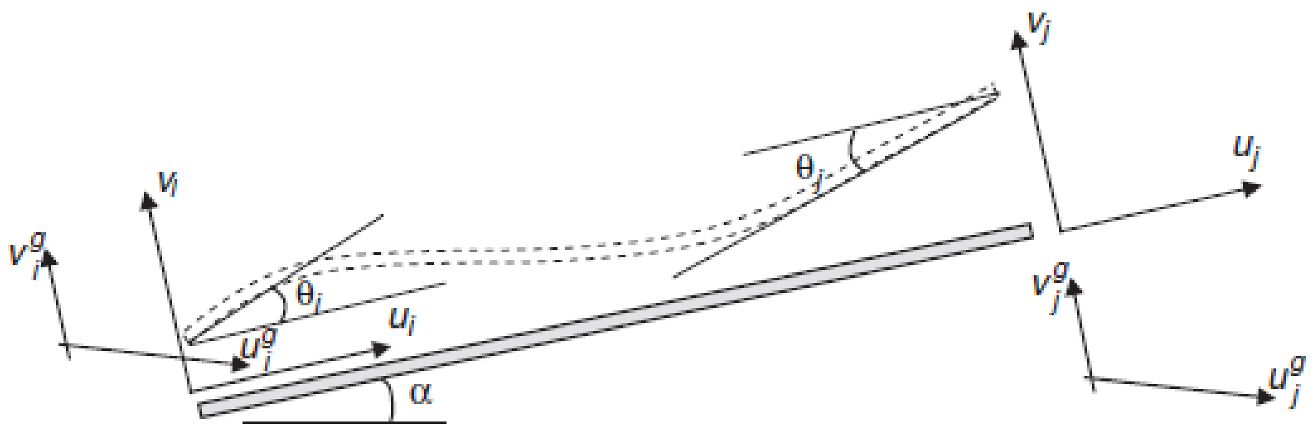


FIGURE 1.13. Displacements in the local and global coordinate systems.

DISPLACEMENT TRANSFORMATION

$$v_i = -u_i^g \sin\alpha + v_i^g \cos\alpha \tag{1.16b}$$

$$\theta_i = \theta_i^g \tag{1.16c}$$

$$u_j = u_j^g \cos\alpha + v_j^g \sin\alpha \tag{1.16d}$$

$$v_j = -u_j^g \sin\alpha + v_j^g \cos\alpha \tag{1.16e}$$

$$\theta_j = \theta_j^g \tag{1.16f}$$

DISPLACEMENT TRANSFORMATION

In matrix form, Equations (1.16a) to (1.16f) can be expressed as

$$\{d\} = [T]^t \{D_e^g\} \tag{1.17}$$

where $\{D_e^g\}$ is the member displacement vector in the global coordinate system corresponding to the directions in which the freedom codes are specified and is given as

$$\{D_e^g\} = \begin{Bmatrix} u_i^g \\ v_i^g \\ \theta_i^g \\ u_j^g \\ v_j^g \\ \theta_j^g \end{Bmatrix}$$

and $[T]^t$ is the transpose of $[T]$.

Member Stiffness in Global Coordinate System

From Equation (1.15),

$$
\begin{aligned}
\{F_e^g\} &= [T]\{P\} \\
&= [T][K_e]\{d\} \qquad \text{from Equation (1.9)} \\
&= [T][K_e][T]^t\{D_e^g\} \quad \text{from Equation (1.17)} \\
&= [K_e^g]\{D_e^g\}
\end{aligned}
$$

where $[K_e^g] = [T][K_e][T]^t$ = member stiffness matrix in the global coordinate system.

Member Stiffness in Global Coordinate System

An explicit expression for $[K_e^g]$ is

$$[K_e^g] = \begin{bmatrix} C^2\frac{EA}{L} + S^2\frac{12EI}{L^3} & SC\left(\frac{EA}{L} - \frac{12EI}{L^3}\right) & -S\frac{6EI}{L^2} & -\left(C^2\frac{EA}{L} + S^2\frac{12EI}{L^3}\right) & -SC\left(\frac{EA}{L} - \frac{12EI}{L^3}\right) & -S\frac{6EI}{L^2} \\ & S^2\frac{EA}{L} + C^2\frac{12EI}{L^3} & C\frac{6EI}{L^2} & -SC\left(\frac{EA}{L} - \frac{12EI}{L^3}\right) & -\left(S^2\frac{EA}{L} + C^2\frac{12EI}{L^3}\right) & C\frac{6EI}{L^2} \\ & & \frac{4EI}{L} & S\frac{6EI}{L^2} & -C\frac{6EI}{L^2} & \frac{2EI}{L} \\ & & & C^2\frac{EA}{L} + S^2\frac{12EI}{L^3} & SC\left(\frac{EA}{L} - \frac{12EI}{L^3}\right) & S\frac{6EI}{L^2} \\ & \textit{Symmetric} & & & S^2\frac{EA}{L} + C^2\frac{12EI}{L^3} & -C\frac{6EI}{L^2} \\ & & & & & \frac{4EI}{L} \end{bmatrix} \quad (1.19)$$

where C = cos α; S = sin α.

Assembly of Structure Stiffness Matrix

Consider part of a structure with four externally applied forces, F1, F2, F4, and F5, and two applied moments, M3 and M6, acting at the two joints *p* and *q* connecting three members A, B, and C as shown in Figure 1.14. The freedom codes at joint *p* are {1, 2, 3} and at joint *q* are {4, 5, 6}. The structure stiffness matrix [K] is assembled on the basis of two conditions: compatibility and equilibrium conditions at the joints.

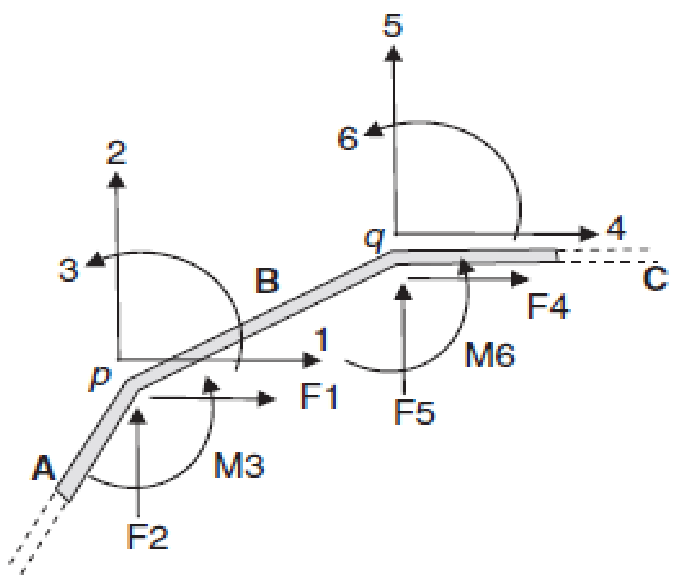


FIGURE 1.14. Assembly of structure stiffness matrix [*K*].

Compatibility Condition

At joint p, the global displacements are $D1$ (horizontal), $D2$ (vertical), and $D3$ (rotational). Similarly, at joint q, the global displacements are $D4$ (horizontal), $D5$ (vertical), and $D6$ (rotational). The compatibility condition is that the displacements ($D1$, $D2$, and $D3$) at end p of member A are the same as those at end p of member B. Thus, $(u_j^g)_A = (u_i^g)_B = D1$, $(v_j^g)_A = (v_i^g)_B = D2$, and $(\theta_j^g)_A = (\theta_i^g)_B = D3$. The same condition applies to displacements ($D4$, $D5$, and $D6$) at end q of both members B and C.

The member stiffness matrix in the global coordinate system given in Equation (1.19) can be written as

$$[K_e^g] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix} \tag{1.20}$$

where $k_{11} = C^2 \dfrac{EA}{L} + S^2 \dfrac{12EI}{L^3}$, etc.

Compatibility Condition

For member A, from Equation (1.18),

$$\left(H_j^g\right)_A = \ldots.. + \ldots.. + \ldots.. + (k_{44})_A D_1 + (k_{45})_A D_2 + (k_{46})_A D_3 \qquad (1.21a)$$

$$\left(V_j^g\right)_A = \ldots.. + \ldots.. + \ldots.. + (k_{54})_A D_1 + (k_{55})_A D_2 + (k_{56})_A D_3 \qquad (1.21b)$$

$$\left(M_j^g\right)_A = \ldots.. + \ldots.. + \ldots.. + (k_{64})_A D_1 + (k_{65})_A D_2 + (k_{66})_A D_3 \qquad (1.21c)$$

Similarly, for member B,

$$(H_i^g)_B = (k_{11})_B D_1 + (k_{12})_B D_2 + (k_{13})_B D_3 + (k_{14})_B D_4 + (k_{15})_B D_5 + (k_{16})_B D_6 \qquad (1.21d)$$

$$(V_i^g)_B = (k_{21})_B D_1 + (k_{22})_B D_2 + (k_{23})_B D_3 + (k_{24})_B D_4 + (k_{25})_B D_5 + (k_{26})_B D_6 \qquad (1.21e)$$

$$(M_i^g)_B = (k_{31})_B D_1 + (k_{32})_B D_2 + (k_{33})_B D_3 + (k_{34})_B D_4 + (k_{35})_B D_5 + (k_{36})_B D_6 \qquad (1.21f)$$

Compatibility Condition

$$\left(H_j^g\right)_B = (k_{41})_B D_1 + (k_{42})_B D_2 + (k_{43})_B D_3 + (k_{44})_B D_4 + (k_{45})_B D_5 + (k_{46})_B D_6 \qquad (1.21g)$$

$$\left(V_j^g\right)_B = (k_{51})_B D_1 + (k_{52})_B D_2 + (k_{53})_B D_3 + (k_{54})_B D_4 + (k_{55})_B D_5 + (k_{56})_B D_6 \qquad (1.21h)$$

$$\left(M_j^g\right)_B = (k_{61})_B D_1 + (k_{62})_B D_2 + (k_{63})_B D_3 + (k_{64})_B D_4 + (k_{65})_B D_5 + (k_{66})_B D_6 \qquad (1.21i)$$

Similarly, for member C,

$$(H_i^g)_C = (k_{11})_C D_1 + (k_{12})_C D_2 + (k_{13})_C D_3 + ... + ... + ... \qquad (1.21j)$$

$$(V_i^g)_C = (k_{21})_C D_1 + (k_{22})_C D_2 + (k_{23})_C D_3 + ... + ... + ... \qquad (1.21k)$$

$$(M_i^g)_C = (k_{31})_C D_1 + (k_{32})_C D_2 + (k_{33})_C D_3 + ... + ... + ... \qquad (1.21l)$$

Equilibrium Condition

Any of the externally applied forces or moments applied in a certain direction at a joint of a structure is equal to the sum of the member forces acting in the same direction for members connected at that joint in the global coordinate system. Therefore, at joint p,

$$F1 = \left(H_j^g\right)_A + \left(H_i^g\right)_B \quad (1.22a)$$

$$F2 = \left(V_j^g\right)_A + \left(V_i^g\right)_B \quad (1.22b)$$

$$M3 = \left(M_j^g\right)_A + \left(M_i^g\right)_B \quad (1.22c)$$

Also, at joint q,

$$F4 = \left(H_j^g\right)_B + \left(H_i^g\right)_C \quad (1.22d)$$

$$F5 = \left(V_j^g\right)_B + \left(V_i^g\right)_C \quad (1.22e)$$

$$M6 = \left(M_j^g\right)_B + \left(M_i^g\right)_C \quad (1.22f)$$

Equilibrium Condition

$$
\begin{Bmatrix} \bullet \\ F1 \\ F2 \\ M3 \\ F4 \\ F5 \\ M6 \\ \bullet \end{Bmatrix} = \begin{bmatrix}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & (k_{44})_A + (k_{11})_B & (k_{45})_A + (k_{12})_B & (k_{46})_A + (k_{13})_B & (k_{14})_B & (k_{15})_B & (k_{16})_B & \bullet \\
\bullet & (k_{54})_A + (k_{21})_B & (k_{55})_A + (k_{22})_B & (k_{56})_A + (k_{23})_B & (k_{24})_B & (k_{25})_B & (k_{26})_B & \bullet \\
\bullet & (k_{64})_A + (k_{31})_B & (k_{65})_A + (k_{32})_B & (k_{66})_A + (k_{33})_B & (k_{34})_B & (k_{35})_B & (k_{36})_B & \bullet \\
\bullet & (k_{41})_B & (k_{42})_B & (k_{43})_B & (k_{44})_B + (k_{11})_C & (k_{45})_B + (k_{12})_C & (k_{46})_B + (k_{13})_C & \bullet \\
\bullet & (k_{51})_B & (k_{52})_B & (k_{53})_B & (k_{54})_B + (k_{21})_C & (k_{55})_B + (k_{22})_C & (k_{56})_B + (k_{23})_C & \bullet \\
\bullet & (k_{61})_B & (k_{62})_B & (k_{63})_B & (k_{64})_B + (k_{31})_C & (k_{65})_B + (k_{32})_C & (k_{66})_B + (k_{33})_C & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{bmatrix} \begin{Bmatrix} \bullet \\ \bullet \\ D1 \\ D2 \\ D3 \\ D4 \\ D5 \\ D4 \\ \bullet \\ \bullet \end{Bmatrix} \tag{1.23}
$$

where the "•" stands for matrix coefficients contributed from the other parts of the structure. In simple form, Equation (1.23) can be written as

$$\{F\} = [K]\{D\}$$

which is identical to Equation (1.7). Equation (1.23) shows how the structure equilibrium equation is set up in terms of the load vector $\{F\}$, structure stiffness matrix $[K]$, and the displacement vector $\{D\}$.

Assembly of Structure Stiffness Matrix

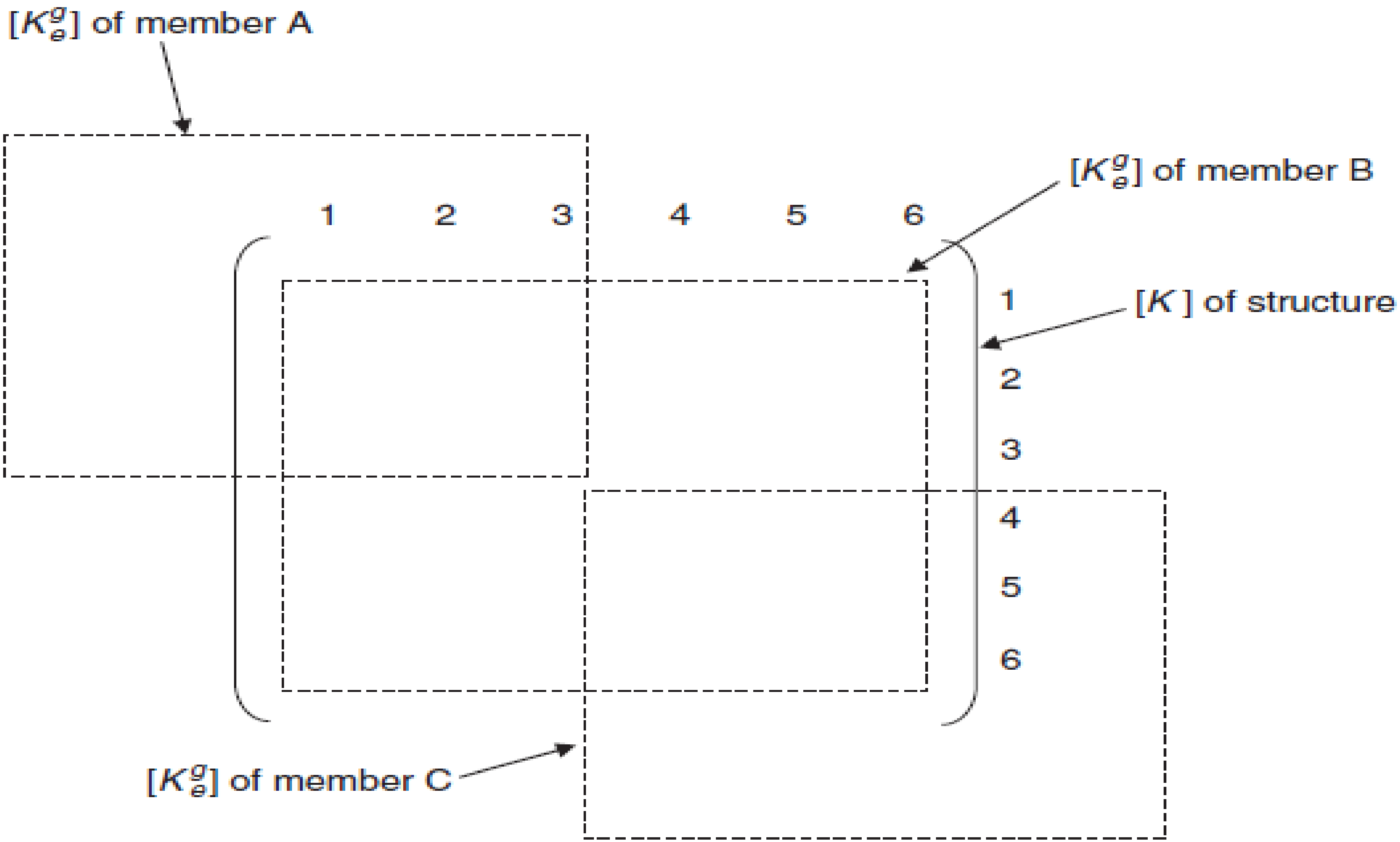


FIGURE 1.16. Assembly of structure stiffness matrix.

ASSEMBLY OF LOAD VECTOR

Freedom codes

$$\{F\} = \begin{array}{c} \left\{ \begin{array}{c} \bullet \\ F1 \\ F2 \\ F3 \\ F4 \\ F5 \\ F6 \\ \bullet \end{array} \right\} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ \\ \end{array}$$

FIGURE 1.17. Assembly of load vector.

METHODS OF SOLUTION

The displacements of the structure can be found by solving Equation (1.23). Because of the huge size of the matrix equation usually encountered in practice, Equation (1.23) is solved routinely by numerical methods such as the Gaussian elimination method and the iterative Gauss–Seidel method. It should be noted that in using these

numerical methods, the procedure is analogous to inverting the structure stiffness matrix, which is subsequently multiplied by the load vector as in Equation (1.8):

$$\{D\} = [K]^{-1}\{F\} \tag{1.8}$$

METHODS OF SOLUTION

The numerical procedure fails only if an inverted $[K]$ cannot be found. This situation occurs when the determinant of $[K]$ is zero, implying an unstable structure. Unstable structures with a degree of statically indeterminacy, f_r, greater than zero (see Section 1.2) will have a zero determinant of $[K]$. In numerical manipulation by computers, an exact zero is sometimes difficult to obtain. In such cases, a good indication of an unstable structure is to examine the displacement vector $\{D\}$, which would include some exceptionally large values.

CALCULATION OF MEMBER FORCES

Member forces are calculated according to Equation (1.9). Hence,

$$\begin{aligned}\{P\} &= [K_e]\{d\} \\ &= [K_e]\,[T]^t\{D_e^g\}\end{aligned} \tag{1.24}$$

where $\{D_e^g\}$ is extracted from $\{D\}$ for each member according to its freedom codes and

$$[K_e][T]^t = \begin{bmatrix} C\dfrac{EA}{L} & S\dfrac{EA}{L} & 0 & -C\dfrac{EA}{L} & -S\dfrac{EA}{L} & 0 \\ -S\dfrac{12EI}{L^3} & C\dfrac{12EI}{L^3} & \dfrac{6EI}{L^2} & S\dfrac{12EI}{L^3} & -C\dfrac{12EI}{L^3} & \dfrac{6EI}{L^2} \\ -S\dfrac{6EI}{L^2} & C\dfrac{6EI}{L^2} & \dfrac{4EI}{L} & S\dfrac{6EI}{L^2} & -C\dfrac{6EI}{L^2} & \dfrac{2EI}{L} \\ -C\dfrac{EA}{L} & -S\dfrac{EA}{L} & 0 & C\dfrac{EA}{L} & S\dfrac{EA}{L} & 0 \\ S\dfrac{12EI}{L^3} & -C\dfrac{12EI}{L^3} & -\dfrac{6EI}{L^2} & -S\dfrac{12EI}{L^3} & C\dfrac{12EI}{L^3} & -\dfrac{6EI}{L^2} \\ -S\dfrac{6EI}{L^2} & C\dfrac{6EI}{L^2} & \dfrac{2EI}{L} & S\dfrac{6EI}{L^2} & -C\dfrac{6EI}{L^2} & \dfrac{4EI}{L} \end{bmatrix}$$

MEMBER FORCES

$$\{P\} = \begin{Bmatrix} N_i \\ Q_i \\ M_i \\ N_j \\ Q_j \\ M_j \end{Bmatrix} = \begin{bmatrix} C\frac{EA}{L} & S\frac{EA}{L} & 0 & -C\frac{EA}{L} & -S\frac{EA}{L} & 0 \\ -S\frac{12EI}{L^3} & C\frac{12EI}{L^3} & \frac{6EI}{L^2} & S\frac{12EI}{L^3} & -C\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ -S\frac{6EI}{L^2} & C\frac{6EI}{L^2} & \frac{4EI}{L} & S\frac{6EI}{L^2} & -C\frac{6EI}{L^2} & \frac{2EI}{L} \\ -C\frac{EA}{L} & -S\frac{EA}{L} & 0 & C\frac{EA}{L} & S\frac{EA}{L} & 0 \\ S\frac{12EI}{L^3} & -C\frac{12EI}{L^3} & -\frac{6EI}{L^2} & -S\frac{12EI}{L^3} & C\frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ -S\frac{6EI}{L^2} & C\frac{6EI}{L^2} & \frac{2EI}{L} & S\frac{6EI}{L^2} & -C\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix}$$

SUMMARY

1. Assign freedom codes to each joint indicating the displacement freedom at the ends of the members connected at that joint. Assign a freedom code of "zero" to any restrained displacement.
2. Assign an arrow to each member so that ends *i* and *j* are defined. Also, the angle of orientation α for the member is defined in Figure 1.18 as:

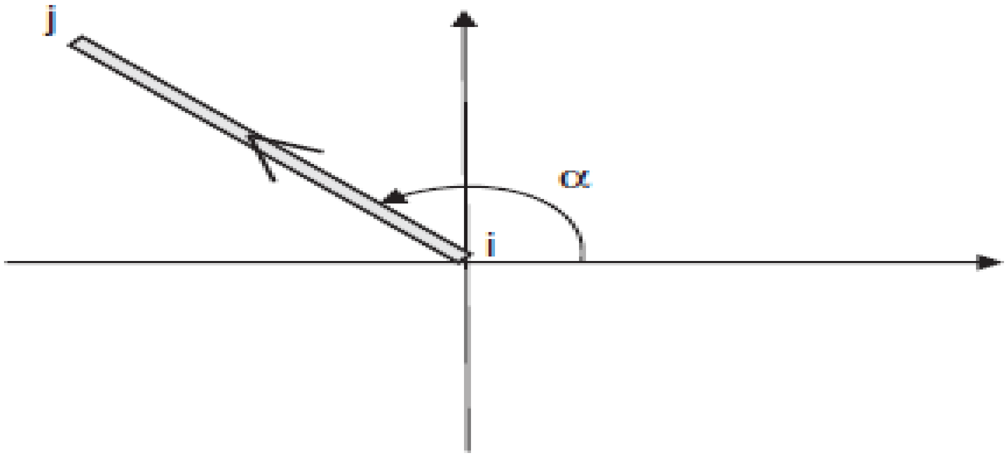


FIGURE 1.18. Definition of angle of orientation for member.

SUMMARY

3. Assemble the structure stiffness matrix $[K]$ from each of the member stiffness matrices.

4. Form the load vector $\{F\}$ of the structure.

5. Calculate the displacement vector $\{D\}$ by solving for $\{D\} = [K]^{-1}\{F\}$.

6. Extract the local displacement vector $\{D_e^g\}$ from $\{D\}$ and calculate the member force vector $\{P\}$ using $\{P\} = [K_e][T]^t\{D_e^g\}$.

Sign Convention for Member Force

Positive member forces and displacements obtained from the stiffness method of analysis are shown in Figure 1.19. To plot the forces in conventional axial force, shear force, and bending moment diagrams, it is necessary to translate them into a system commonly adopted for plotting.

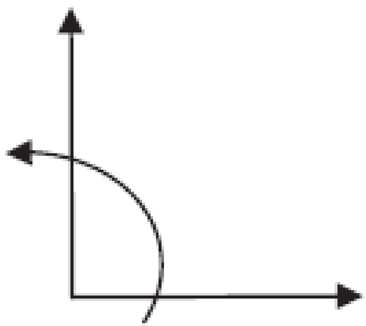


FIGURE 1.19. Direction of positive forces and displacements using stiffness method.

Axial Force

For a member under compression, the axial force at end *i* is positive (from analysis) and at end *j* is negative (from analysis), as shown in Figure 1.20.

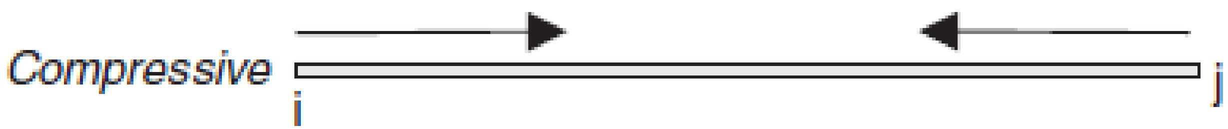


FIGURE 1.20. Member under compression.

Shear Force

A shear force plotted positive in diagram is acting upward (positive from analysis) at end *i* and downward (negative from analysis) at end *j* as shown in Figure 1.21. Positive shear force is usually plotted in the space above the member.

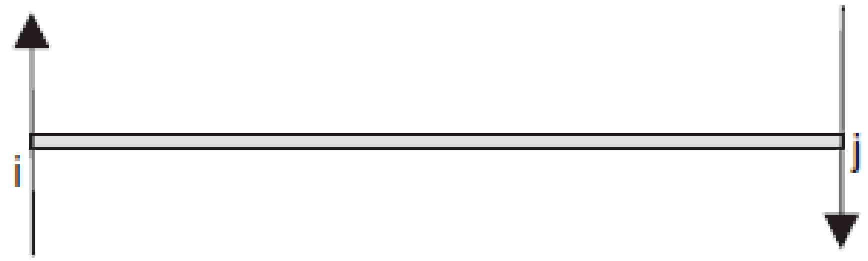


FIGURE 1.21. Positive shear forces.

A member under sagging moment is positive in diagram (clockwise and negative from analysis) at end *i* and positive (anticlockwise and positive from analysis) at end *j* as shown in Figure 1.22. Positive bending moment is usually plotted in the space beneath the member. In doing so, a bending moment is plotted on the tension face of the member.

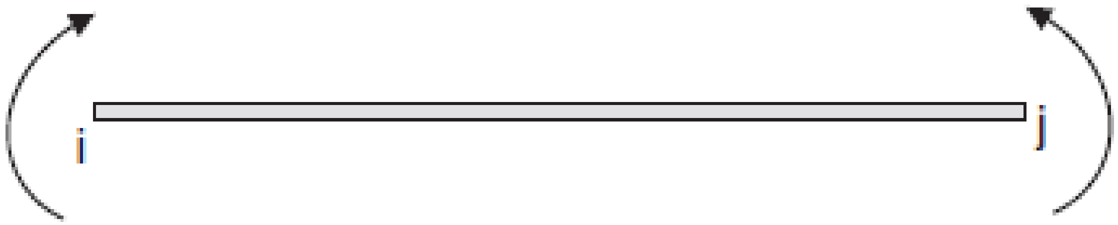


FIGURE 1.22. Sagging moment of a member.

UNIT-III
STIFFNESS MATRIX ASSEMBLY OF STRUCTURES AND APPLICATIONS TO SIMPLE PROBLEMS

Introduction

For a two-dimensional frame element each node has the capability of translating in two directions and rotating about one axis. Thus each node of plane frame has three degrees of freedom. Similarly three structure forces (vertical force, shear force and bending moment) act at a node.

For a two-dimensional beam element each node has two degrees of freedom (one rotation and one translation). Similarly two structure forces (vertical force and a bending moment) act at a node. However in some structures a node has one degree of freedom either rotation or translation. Therefore they are subjected to a moment or a force as the case may be.

The structure stiffness matrices for all these cases have been developed in the previous chapter which can be summarized as under:-

i) Beams and frames subjected to bending moment

ii) Beams and frames subjected to shear force and bending moment

iii) Beams and frames subjected to shear force, bending moment and axial forces

Following steps provide a procedure for the determination of unknown deformation, support reactions and element forces (axial forces, shear forces and bending moment) using the force displacement relationship **(W=KΔ)**. The same procedure applies both to determinate and indeterminate structures.

PROCEDURE TO ANALYSE BEAMS AND FRAMES USING DIRECT STIFFNESS METHOD

Identifying the components of the structural system or labeling the Structures & Elements.

As a first step, divide the structure into some finite number of elements by defining nodes or joints. Nodes may be points of supports, points of concentrated loads, corners or bends or the points where the internal forces or displacements are to be determined. Each element extends between the nodes and is identified by arbitrary numbers (1,2,3).

a) Structure Forces and Deformations

At a node structure forces are assumed to act in their positive direction. The positive direction of the forces is to the right and upward and positive moments and rotations are clockwise. Start numbering the known forces first and then the unknown forces.

Structures Forces not acting at the joints

Stiffness method is applicable to structures with structure forces acting at nodes only. However if the structure is subjected to concentrated loads which are not acting at the joints or nodal points or if it is subjected to distributed loads then equivalent joint loads are calculated using the following procedure.

i) All the joints are considered to be fixed. [Figure-5(b)]

ii) Fixed End Moments (FEM's) and Reactions are calculated using the formulae given in the table as annex-I

iii) If more than one FEM and reactions are present then the net FEM and Reaction is calculated. This is done by algebric summation. [Figure-5(c)]

Equivalent structure forces or loads at the joints/nodes are obtained by reversing the signs of net FEM's & Reactions. [Figure-5(d)]

or

Reversing the signs of Net FEM's or reaction gives the equivalent structure loads on the joints/nodes.

v) Equivalent element forces are calculated from these equivalent structure loads using equation 5.2, 5.3 and 5.4 as explained in article number 5.

vi) Final element forces are obtained by the following equation

$$w = w_E + w_F$$

where

w_E = Equivalent element forces

w_F = Element forces while considering the elements to be fixed.

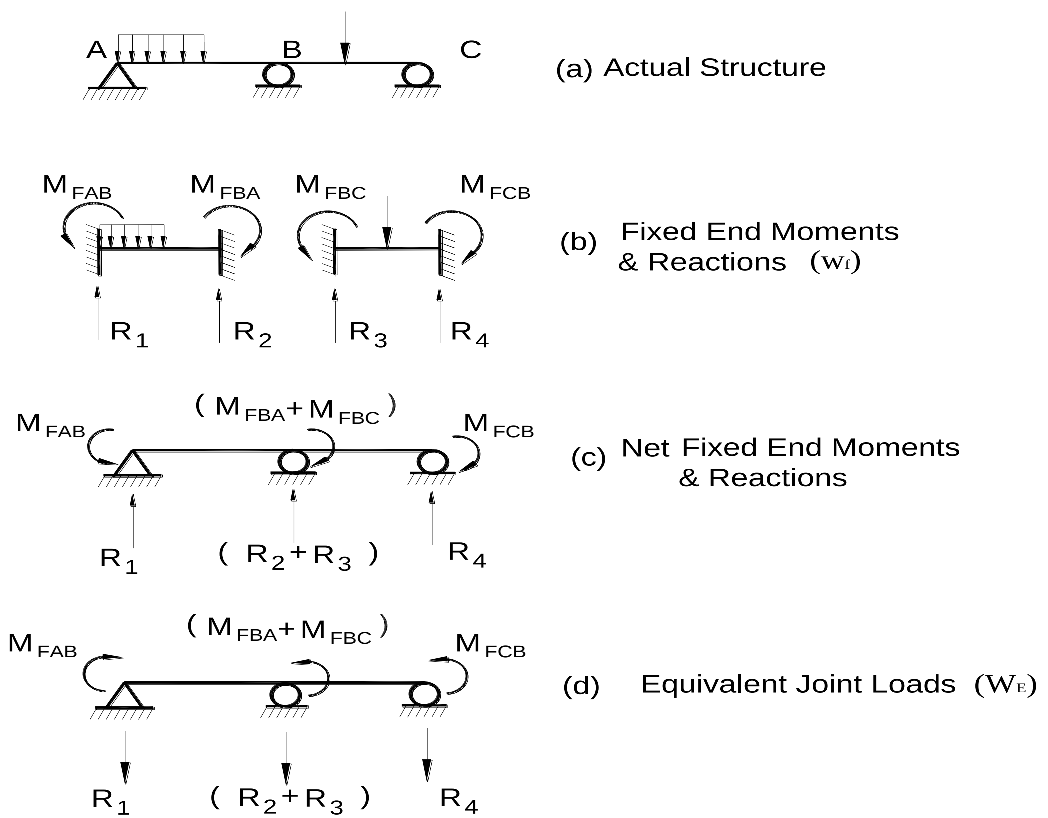


Fig.5.1

b) Element Forces

Specify the near and far end of each element. Draw free body diagram of each member showing its local co-ordinate and element forces. Arbitrary numbers can identify all the element forces of the structure.

Calculation of Structure Stiffness Matrices of the members

Properties of each element like its length, cross-sectional area, moment of inertia, direction cosines, and numbers identifying the structure forces acting at its near and far ends can be systematically tabulated. Using values of these parameters in equation 4.53 structure stiffness matrix of each member can be formed by applying equation 4.54,4.55 and 4.56 depending upon the situation.

Formation of Structure Stiffness Matrix of the Entire Structure

According to the procedure discussed in chapter 3 article 3.1.3 stiffness matrix [K] of the entire structure is formed.

Calculation of Unknown Structure Forces and Displacements

Following relation expresses the force-displacement relationship of the structure in the global coordinate system:

$$[W] = [K]\,[\Delta]$$

Where

[W] is the structure load vector

[K] is the structure stiffness matrix

[Δ] is the displacement vector

Partitioning the above equation into known and unknown portions as shown below:

Where
W_k = known loads
W_u = unknown loads
Δ_u = unknown

$$\begin{bmatrix} [W_k] \\ \hdashline [W_u] \end{bmatrix} = \left[\begin{array}{c:c} [K_{11}] & [K_{12}] \\ \hdashline [K_{21}] & [K_{22}] \end{array}\right] \begin{bmatrix} [\Delta_u] \\ \hdashline [\Delta_k] \end{bmatrix}$$

$[Wk] = [K_{11}]\ [\Delta u] + [K_{12}]\ [\Delta_k]$ -------------------- (A)
$[Wu] = [K_{21}]\ [\Delta u] + K_{22}\ \Delta_k$ -------------------- (B)
As $\Delta_k = 0$
So, unknown structure displacement $[\Delta u]$ can be calculated by solving the relation (A), which takes the following form.
$[\Delta u] = [K_{11}]^{-1}\ [W_k]$ --------------------- (C)

$Wu = [K_{21}]\ [\Delta u]$ -------------------- (D)

Calculation of element forces:

Finally element forces at the end of the member are computed using the following equation (E).

$w = k\delta$

$\delta = T\Delta$

$w = kT\Delta$ --------------- (E)

where [w] is the element force vector

[kT] is the product of [k] and [T] matrices of the element

where [w] is the element force vector

[kT] is the product of [k] and [T] matrices of the element

[Δ] is the structure displacement vector for the element.

Following are the [kT] matrices for different elements used in the subsequent examples.

Case-I Beam/frame subjected to bending moment only

$$[kT] = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix}$$

Case-II Beams subjected to Shear Forces & Bending Moment

$$[kT] = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} & \frac{-6EI}{L^2} & \frac{6EI}{L^2} \\ \frac{2EI}{L} & \frac{4EI}{L} & \frac{-6EI}{L^2} & \frac{6EI}{L^2} \\ \frac{-6EI}{L^2} & \frac{-6EI}{L^2} & \frac{12EI}{L^3} & \frac{-12EI}{L^3} \\ \frac{6EI}{L^2} & \frac{6EI}{L^2} & \frac{-12EI}{L^3} & \frac{12EI}{L^3} \end{bmatrix}$$

Case-IIIFor frame element subjected to axial force, shear force and bending moment.

$$[kT]_m = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} & \frac{-6EI.l}{L^2} & \frac{6EI.l}{L^2} & \frac{6EI.m}{L^2} & \frac{-6EI.m}{L^2} \\ \frac{2EI}{L} & \frac{4EI}{L} & \frac{-6EI.l}{L^2} & \frac{6EI.l}{L^2} & \frac{6EI.m}{L^2} & \frac{-6EI.m}{L^2} \\ \frac{-6EI}{L^2} & \frac{-6EI}{L^2} & \left(\frac{12EI.l}{L^3}\right) & -\left(\frac{12EI.l}{L^3}\right) & \left(\frac{-12EI.m}{L^3}\right) & -\left(\frac{-12EI.m}{L^3}\right) \\ \frac{6EI}{L^2} & \frac{6EI}{L^2} & -\left(\frac{12EI.l}{L^3}\right) & \left(\frac{12EI.l}{L^3}\right) & -\left(\frac{-12EI.m}{L^3}\right) & \left(\frac{-12EI.m}{L^3}\right) \\ 0 & 0 & \left(\frac{AE}{L}\right)m & -\left(\frac{AE}{L}\right)m & \left(\frac{AE.l}{L}\right) & -\left(\frac{AE.l}{L}\right) \\ 0 & 0 & -\left(\frac{AE}{L}\right)m & \left(\frac{AE}{L}\right)m & -\left(\frac{AE.l}{L}\right) & \left(\frac{AE.l}{L}\right) \end{bmatrix}$$

Plotting bending moment and shearing force diagrams:

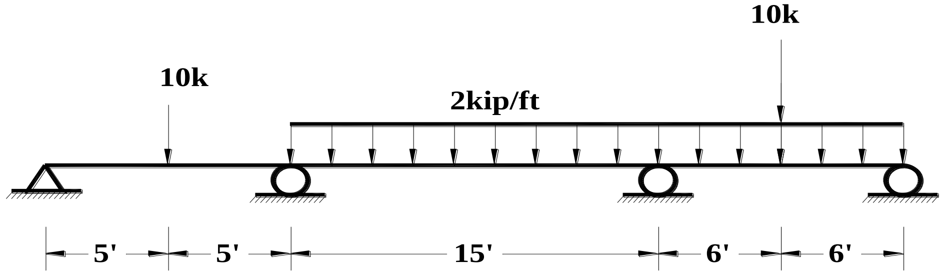
Bending moment **and** ***shearing force*** **diagrams of the structure are plotted using the element forces calculated in step-5. Examples on the next pages have been solved using the above-mentioned procedure.**

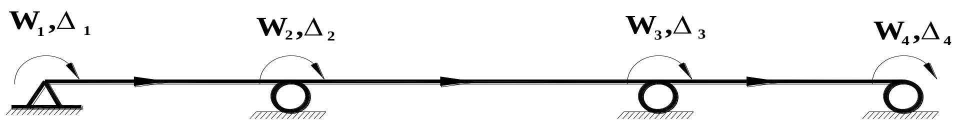
ILLUSTRATIVE EXAMPLES

EXAMPLE Solve the beam shown in the figure using stiffness method.

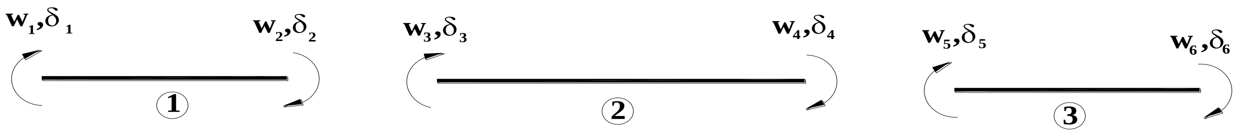
Solution

Numbering element and structure forces





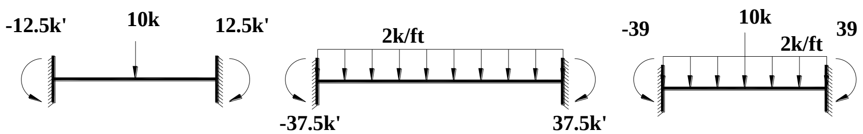
Structure loads and deformations



Element forces and deformation

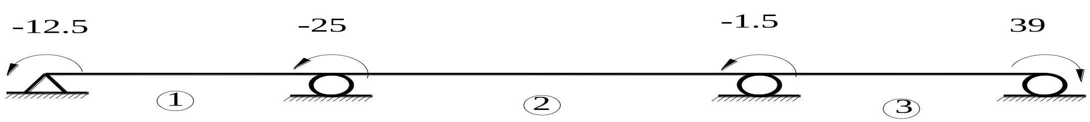
Calculating fixed end moments and equivalent joint loads

$$[FEF's] = \begin{bmatrix} -12.5 \\ 12.5 \\ -37.5 \\ 37.5 \\ -39 \\ 39 \end{bmatrix}$$



Fixed end moments

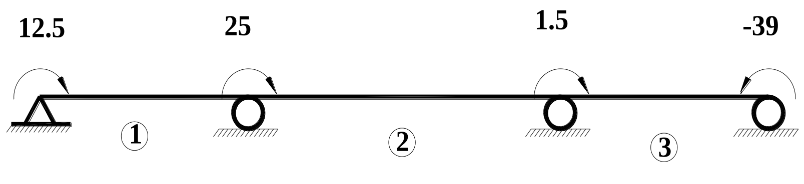
$$[\mathrm{W}]_{\mathbf{F}} = \text{Net Fixed End Moments} = \begin{bmatrix} W_{1F} \\ W_{2F} \\ W_{3F} \\ W_{4F} \end{bmatrix} = \begin{bmatrix} -12.5 \\ -25 \\ -1.5 \\ 39 \end{bmatrix}$$



Net fixed end moment

Equivalent joint loads are :

$$[W]_K = \begin{bmatrix} W_{1E} \\ W_{2E} \\ W_{3E} \\ W_{4E} \end{bmatrix} = \begin{bmatrix} 12.5 \\ 25 \\ 1.5 \\ -39 \end{bmatrix}$$



Equivalent joint Moments

Calculating structure stiffness matrices of element

Following table lists the properties needed to form structure stiffness matrices of elements.

Member	Length (ft)	I	J
1	10	1	2
2	15	2	3
3	12	3	4

Structure stiffness matrices are:

$$[K]_1 = \frac{EI}{10}\begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array} = \mathrm{EI}\begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.4 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

$$[K]_2 = \frac{EI}{15}\begin{array}{cc} \begin{array}{cc} 2 & 3 \end{array} & \\ \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array} = \mathrm{EI}\begin{array}{cc} \begin{array}{cc} 2 & 3 \end{array} & \\ \begin{bmatrix} 0.267 & 0.13 \\ 0.13 & 0.267 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array}$$

$$[\mathrm{K}]_3 = \frac{\mathrm{EI}}{12}\begin{array}{cc} \begin{array}{cc} 3 & 4 \end{array} & \\ \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} & \begin{array}{c} 3 \\ 4 \end{array} \end{array} = \mathrm{EI}\begin{array}{cc} \begin{array}{cc} 3 & 4 \end{array} & \\ \begin{bmatrix} 0.34 & 0.167 \\ 0.167 & 0.34 \end{bmatrix} & \begin{array}{c} 3 \\ 4 \end{array} \end{array}$$

Forming structure stiffness matrix of the entire structure

Using relation $[K] = [K]_1 + [K]_2 + [K]_3$ structure stiffness matrix of the entire structure is:

$$
\begin{array}{cc}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
[K] = EI & \begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4+0.267 & 0.13 & 0 \\ 0 & 0.13 & 0.267+0.34 & 0.167 \\ 0 & 0 & 0.167 & 0.33 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}
\end{array}
$$

$$
\begin{array}{cc}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
[\mathrm{K}] = \mathrm{EI} & \begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.667 & 0.13 & 0 \\ 0 & 0.13 & 0.597 & 0.167 \\ 0 & 0 & 0.167 & 0.33 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}
\end{array}
$$

Finding unknown deformations

Unknown deformations are obtained by using the following equation

$[\Delta] = [K]^{-1} [W]$

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.667 & 0.13 & 0 \\ 0 & 0.13 & 0.597 & 0.167 \\ 0 & 0 & 0.167 & 0.33 \end{bmatrix}^{-1} \begin{bmatrix} 12.5 \\ 25 \\ 1.5 \\ -39 \end{bmatrix}$$

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} 2.97 & -0.94 & 0.24 & -0.123 \\ -0.94 & 1.88 & -0.49 & -0.25 \\ 0.24 & -0.49 & 2.08 & -1.05 \\ -0.123 & -0.25 & -1.05 & 3.56 \end{bmatrix} \begin{bmatrix} 12.5 \\ 25 \\ 1.5 \\ -39 \end{bmatrix}$$

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} 18.8 \\ 24.85 \\ 35 \\ -132.736 \end{bmatrix}$$

Calculating element forces

Using relation [*w*]=[*kT*][*Δ*] we get

$$\begin{bmatrix} w_{1E} \\ w_{2E} \\ w_{3E} \\ w_{4E} \\ w_{5E} \\ w_{6E} \end{bmatrix} = EI \begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 \\ 0 & 0.267 & 0.13 & 0 \\ 0 & 0.13 & 0.267 & 0 \\ 0 & 0 & 0.34 & 0.167 \\ 0 & 0 & 0.167 & 0.34 \end{bmatrix} \begin{bmatrix} 18.8 \\ 24.85 \\ 35 \\ -132.74 \end{bmatrix} \frac{1}{EI} = \begin{bmatrix} 12.5 \\ 13.7 \\ 11.325 \\ 12.67 \\ -10.27 \\ -39.88 \end{bmatrix}$$

Actual forces on the structure are obtained by superimposing the fixed end reactions on above calculated forces.

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix} = \begin{bmatrix} w_{1F} \\ w_{2F} \\ w_{3F} \\ w_{4F} \\ w_{5F} \\ w_{6F} \end{bmatrix} + \begin{bmatrix} w_{1E} \\ w_{2E} \\ w_{3E} \\ w_{4E} \\ w_{5E} \\ w_{6E} \end{bmatrix} = \begin{bmatrix} -12.5 \\ 12.5 \\ -37.5 \\ 37.5 \\ -39 \\ 39 \end{bmatrix} + \begin{bmatrix} 12.5 \\ 13.7 \\ 11.325 \\ 12.67 \\ -10.27 \\ -39.08 \end{bmatrix} = \begin{bmatrix} 0 \\ 26.2 \\ -26.2 \\ 50.1 \\ -50.1 \\ 0 \end{bmatrix}$$

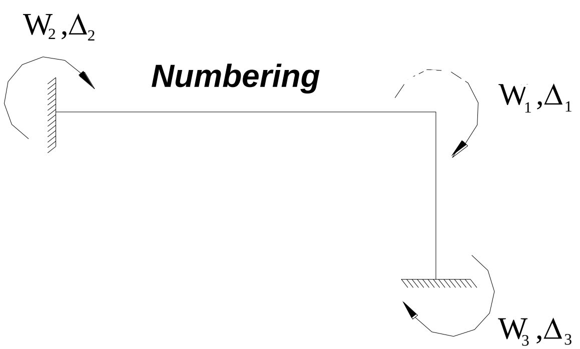
$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix} = \begin{bmatrix} w_{1F} \\ w_{2F} \\ w_{3F} \\ w_{4F} \\ w_{5F} \\ w_{6F} \end{bmatrix} + \begin{bmatrix} w_{1E} \\ w_{2E} \\ w_{3E} \\ w_{4E} \\ w_{5E} \\ w_{6E} \end{bmatrix} = \begin{bmatrix} -12.5 \\ 12.5 \\ -37.5 \\ 37.5 \\ -39 \\ 39 \end{bmatrix} + \begin{bmatrix} 12.5 \\ 13.7 \\ 11.325 \\ 12.67 \\ -10.27 \\ -39.08 \end{bmatrix} = \begin{bmatrix} 0 \\ 26.2 \\ -26.2 \\ 50.1 \\ -50.1 \\ 0 \end{bmatrix}$$

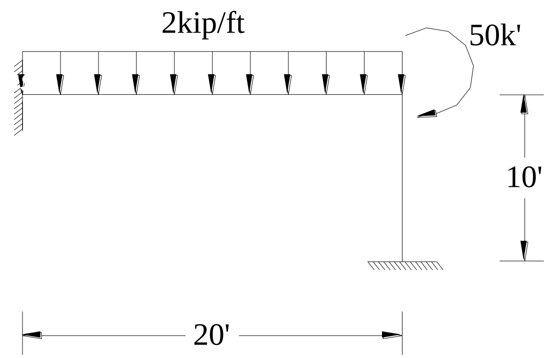
UNIT-IV
BOUNDARY VALUE PROBLEMS(BVP)

EXAMPLE

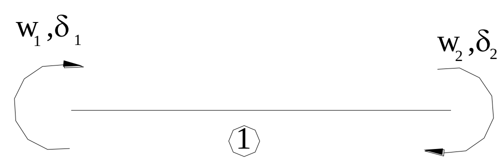
Analyse the frame shown in the figure using stiffness method

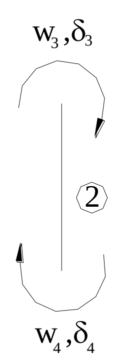
Solution





Structure forces and deformations



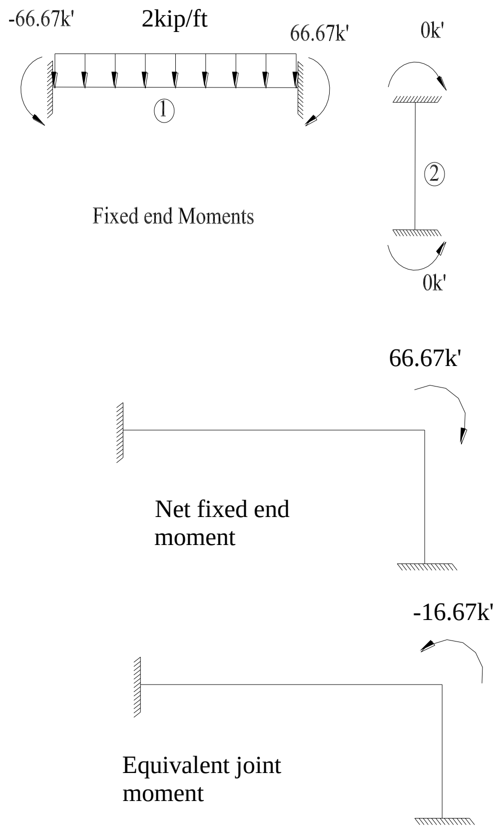


Element forces and deformations

Finding Fixed end moments and equivalent structure load

$$[FEM's] = \begin{bmatrix} w_{1F} \\ w_{2F} \\ w_{3F} \\ w_{4F} \end{bmatrix} = \begin{bmatrix} -66.67 \\ 66.67 \\ 0 \\ 0 \end{bmatrix}$$

$$[W]_F = \text{Net fixed end moments} = \begin{bmatrix} W_{1F} \\ W_{2F} \\ W_{3F} \end{bmatrix} = \begin{bmatrix} 66.67 \\ -66.67 \\ 0 \end{bmatrix}$$



Equivalent Joint Loads

$$[W]_{1E} = [-16.67]$$

Calculating structure stiffness matrices of elements.

Following table shows the properties of the elements required to form structure stiffness matrices of elements.

Members	*Length (ft)*	*i*	*j*
1	20	2	1
2	10	1	3

Structure stiffness matrices of both elements

$$[K]_1 = \frac{EI}{20}\begin{array}{cc} & \begin{array}{cc} 2 & 1 \end{array} \\ & \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{array}{c} 2 \\ 1 \end{array} \end{array} = EI \begin{array}{cc} & \begin{array}{cc} 2 & 1 \end{array} \\ & \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix} \begin{array}{c} 2 \\ 1 \end{array} \end{array}$$

$$[K]_2 = \frac{EI}{10}\begin{array}{cc} & \begin{array}{cc} 1 & 3 \end{array} \\ & \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{array}{c} 1 \\ 3 \end{array} \end{array} = EI \begin{array}{cc} & \begin{array}{cc} 1 & 3 \end{array} \\ & \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.4 \end{bmatrix} \begin{array}{c} 1 \\ 3 \end{array} \end{array}$$

Forming structure stiffness matrix for the entire structure.

Structure stiffness matrix for the entire frame is obtained using relation

$[K] = [K]_1 + [K]_2$

$$[K] = EI \begin{array}{c c} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \left[\begin{array}{c:c:c} 0.6 & 0.1 & 0.2 \\ \hdashline 0.1 & 0.2 & 0 \\ 0.2 & 0 & 0.4 \end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

finding unknown deformation

$[\Delta]_u = [K_{11}]^{-1} [W]_k$

This can be done by partitioning the structure stiffness matrix into known and unknown deformations and forces

$$\begin{bmatrix} W_k \\ \text{-- -- -} \\ W_u \end{bmatrix} = \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] \begin{bmatrix} \Delta_u \\ \text{-- -- -} \\ \Delta_k \end{bmatrix}$$

$[\Delta_u] = [K_{11}]^{-1}[W_u]$

$$\begin{bmatrix} -16.67 \\ W_{2E} \\ W_{3E} \end{bmatrix} = EI \begin{bmatrix} 0.6 & 0.1 & 0.2 \\ 0.1 & 0.2 & 0 \\ 0.2 & 0 & 0.4 \end{bmatrix} \begin{bmatrix} \Delta_1 \\ 0 \\ 0 \end{bmatrix} \frac{1}{EI}$$

$$[\Delta_1] = \frac{1}{EI}\left(\frac{1}{0.6}\right)(-16.667) = \frac{-27.78}{EI}$$

Unknown reactions can be calculated using the following equation.

$$[W]_u = [K]_{21}[\Delta]_u$$

$$\begin{bmatrix} W_{2E} \\ W_{3E} \end{bmatrix} = EI\begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} 1/EI\begin{bmatrix} -27.78 \end{bmatrix}$$

$$\begin{bmatrix} W_{2E} \\ W_{3E} \end{bmatrix} = \begin{bmatrix} -2.78 \\ -5.556 \end{bmatrix}$$

$W = W_F + W_E$

$$\begin{bmatrix} W_2 \\ W_3 \end{bmatrix} = \begin{bmatrix} -66.67 \\ 0 \end{bmatrix} + \begin{bmatrix} -2.78 \\ -5.556 \end{bmatrix} = \begin{bmatrix} -69.45 \\ -5.56 \end{bmatrix}$$

Calculating element forces(Moments

$$[w] = [kT][\Delta]$$

$$\begin{bmatrix} w_{1E} \\ w_{2E} \\ w_{3E} \\ w_{4E} \end{bmatrix} = EI\begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.1 & 0.2 & 0 \\ 0 & 0.4 & 0.2 \\ 0 & 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 0 \\ -27.78 \\ 0 \end{bmatrix}\frac{1}{EI} = \begin{bmatrix} -2.78 \\ -5.56 \\ -11.11 \\ -5.56 \end{bmatrix}$$

OR

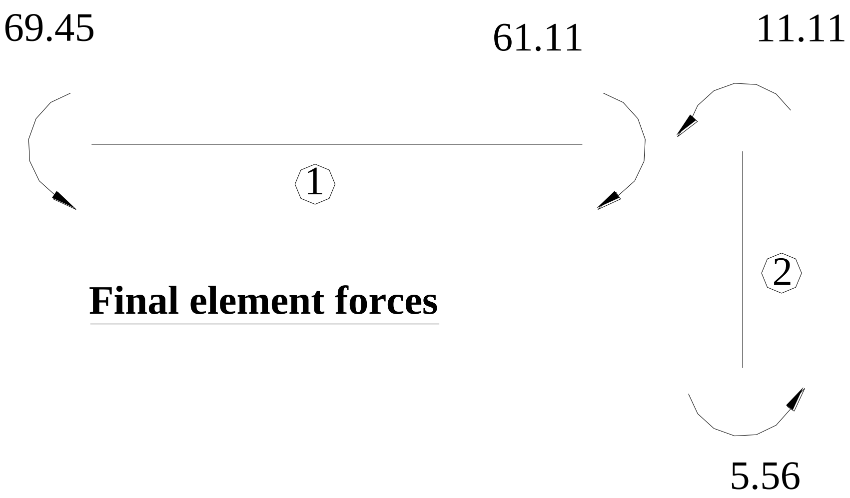
$$\begin{bmatrix} w_{1E} \\ w_{2E} \end{bmatrix} = EI\begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}\begin{bmatrix} 0 \\ -27.78 \end{bmatrix} 1/EI = \begin{bmatrix} -2.78 \\ -5.56 \end{bmatrix}$$

$$\begin{bmatrix} w_{3E} \\ w_{4E} \end{bmatrix} = EI\begin{bmatrix} O.4 & 0.2 \\ 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} -27.78 \\ 0 \end{bmatrix} 1/EI = \begin{bmatrix} -11.11 \\ -5.56 \end{bmatrix}$$

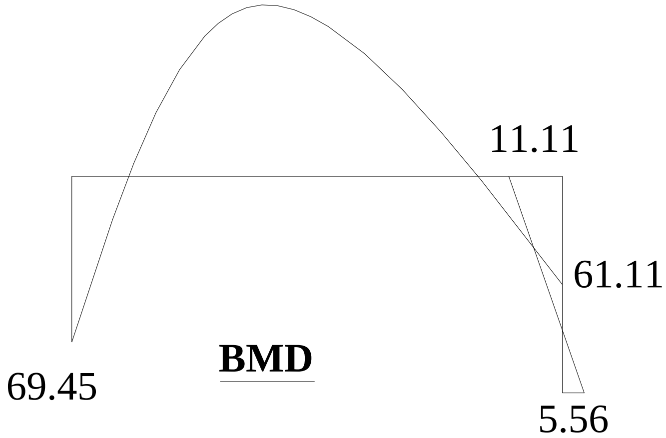
Actual forces acting on the structure can be found by superimposing the fixed end reactions on the forces calculated above.

$$w = w_F + w_E$$

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} -2.78 \\ -5.56 \\ -11.11 \\ -5.56 \end{bmatrix} + \begin{bmatrix} -66.667 \\ 66.667 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -69.45 \\ 61.11 \\ -11.11 \\ -5.56 \end{bmatrix}$$



Final element forces

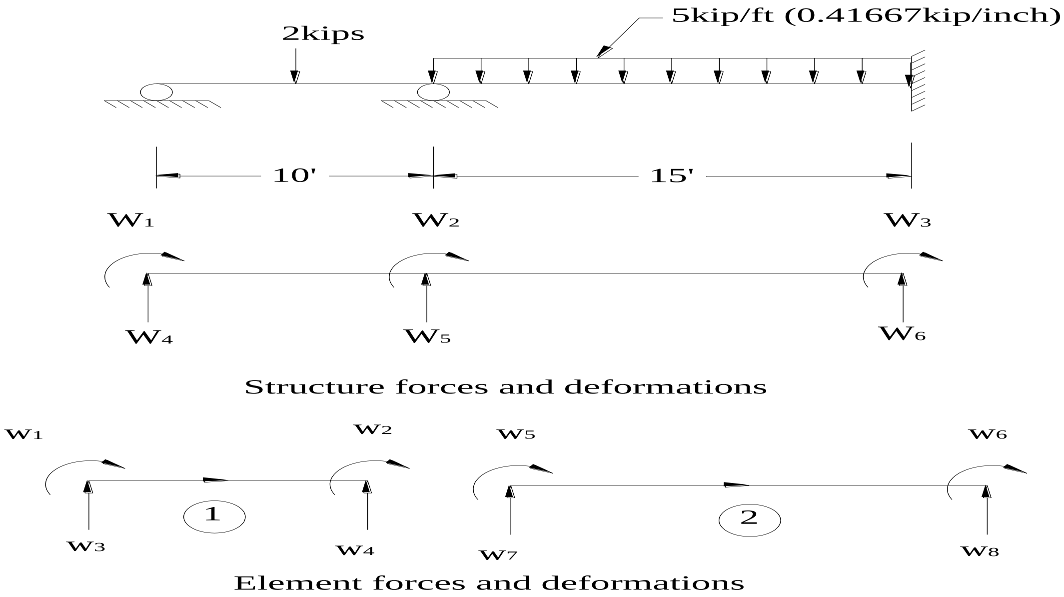


BMD

<u>EXAMPLE</u>

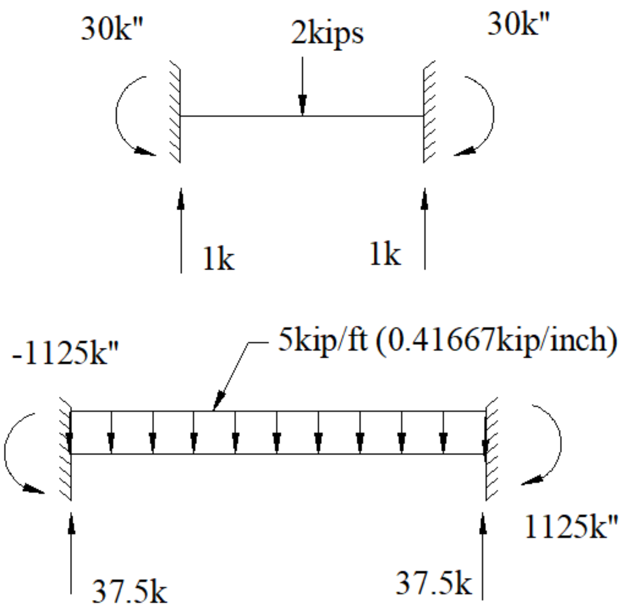
Analyse the shown beam using direct stiffness method.
Beam subjected to shear and moment

STEP-1 Numbering the forces and deformations

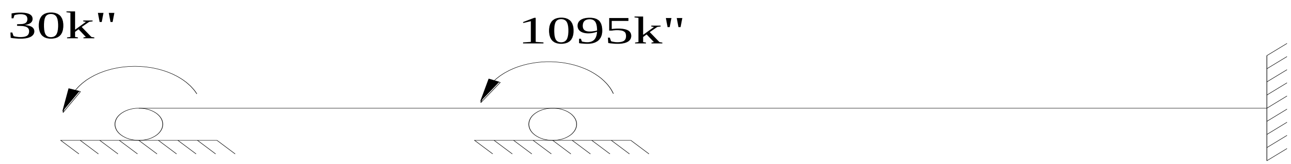


Finding fixed end forces and equivalent joint loads

$$[FEF''s] = \begin{bmatrix} w_{1F} \\ w_{2F} \\ w_{3F} \\ w_{4F} \\ w_{5F} \\ w_{6F} \\ w_{7F} \\ w_{8F} \end{bmatrix} = \begin{bmatrix} -30 \\ 30 \\ 1 \\ 1 \\ -1125 \\ 1125 \\ 37.5 \\ 37.5 \end{bmatrix}$$



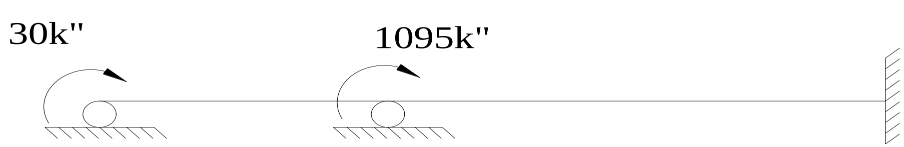
Fixed end forces



Net fixed end forces

[W]$_F$=Net fixed end moments and forces =

$$\begin{bmatrix} W_{1F} \\ W_{2F} \\ W_{3F} \\ W_{4F} \\ W_{5F} \\ W_{6F} \end{bmatrix} = \begin{bmatrix} -30 \\ -1095 \\ 1125 \\ 1 \\ 38.5 \\ 37.5 \end{bmatrix}$$



Equivalent joint loads

Equivalent joint loads = $\begin{bmatrix} W_{1E} \\ W_{2E} \end{bmatrix} = \begin{bmatrix} 30 \\ 1095 \end{bmatrix}$

Calculating Structure Stiffness Matrices of Elements

Following table shows the properties of the elements required to form structure stiffness matrices of elements

M	L	I	J	K	L
1	120	1	2	4	5
2	180	2	3	5	6

From structural elements

Member-1

$$
[K]_1 = EI\;\begin{array}{cc}
\begin{array}{cccc} 1 & 2 & 4 & 5 \end{array} & \\
\begin{bmatrix}
0.0333 & 0.016667 & -0.00041667 & 0.00041667 \\
0.016667 & 0.033333 & -0.00041667 & 0.00041667 \\
-0.00041667 & -0.00041667 & 0.00000694 & -0.00000694 \\
0.00041667 & 0.00041667 & -0.00000694 & 0.00000694
\end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 4 \\ 5 \end{array}
\end{array}
$$

Member-2

$$
[K]_2 = EI\;\begin{array}{cc}
\begin{array}{cccc} 2 & 3 & 5 & 6 \end{array} & \\
\begin{bmatrix}
0.0222 & 0.0111 & -0.000185 & 0.000185 \\
0.0111 & 0.0222 & -0.000185 & 0.000185 \\
-0.000185 & -0.000185 & 0.00000206 & 0.00000206 \\
0.000185 & 0.000185 & 0.00000206 & 0.00000206
\end{bmatrix} & \begin{array}{c} 2 \\ 3 \\ 5 \\ 6 \end{array}
\end{array}
$$

Forming structure stiffness matrix for the entire structure

Structure stiffness matrix for the entire beam is obtained using the relation

$[K] = [K]_1 + [K]_2$

$$[K] = EI \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cc:cccc} 0.0333 & 0.016667 & 0 & -0.00041667 & 0.00041667 & 0 \\ 0.016667 & 0.0555 & 0.0111 & -0.00041667 & 0.00023148 & 0.00018519 \\ \hdashline 0 & 0.0111 & 0.0222 & 0 & -0.000185 & 0.000185 \\ -0.00041667 & -0.00041667 & 0 & -0.00000694 & -0.000000694 & 0 \\ 0.00041667 & 0.00023148 & -0.000185 & -0.00000694 & 0.000009 & 0.00000206 \\ 0 & 0.00018519 & 0.000185 & 0 & 0.0000206 & 0.00000206 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Finding unknown deformations

Unknown deformation can be calculated using equation

$[\Delta_u] = [K_{11}]^{-1} [W_k]$

This can be done by partitioning the structure stiffness matrix into known and unknown deformations and

$$\begin{bmatrix} W_k \\ \hdashline W_u \end{bmatrix} = \left[\begin{array}{c:c} K_{11} & K_{12} \\ \hdashline K_{21} & K_{22} \end{array}\right] \begin{bmatrix} \Delta_u \\ \hdashline \Delta_k \end{bmatrix}$$

$$\begin{bmatrix} W_{1E} \\ W_{2E} \\ W_{3E} \\ W_{4E} \\ W_{5E} \\ W_{6E} \end{bmatrix} = EI \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0.0333 & 0.016667 & 0 & -0.00041667 & 0.00041667 & 0 \\ 0.016667 & 0.0555 & 0.0111 & -0.00041667 & 0.00023148 & 0.00018519 \\ 0 & 0.0111 & 0.0222 & 0 & -0.000185 & 0.000185 \\ -0.00041667 & -0.00041667 & 0 & -0.00000694 & -0.000000694 & 0 \\ 0.00041667 & 0.00023148 & -0.000185 & -0.00000694 & 0.000009 & 0.00000206 \\ 0 & 0.00018519 & 0.000185 & 0 & 0.0000206 & 0.00000206 \end{bmatrix} \end{array} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \end{bmatrix}$$

Using Equation $[\Delta_u] = [K_{11}]^{-1} [W_k]$

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} 0.0333 & 0.016667 \\ 0.016667 & 0.0555 \end{bmatrix}^{-1} \begin{bmatrix} 30 \\ 1095 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} -10535.2946 \\ 22870.5886 \end{bmatrix}$$

Finding Unknown Reactions

$[W]_u = [K]_{21} [\Delta]_u$

$$\begin{bmatrix} W_{3E} \\ W_{4E} \\ W_{5E} \\ W_{6E} \end{bmatrix} = EI \begin{bmatrix} 0 & 0.0111 \\ -0.00041667 & -0.00041667 \\ 0.00041667 & 0.00023148 \\ 0 & 0.00018519 \end{bmatrix} \begin{bmatrix} -10535.2946 \\ 22870.5886 \end{bmatrix} \frac{1}{EI} = \begin{bmatrix} 253.86 \\ -5.14 \\ 0.91 \\ 4.235 \end{bmatrix}$$

$W = W_E + W_F$

$$\begin{bmatrix} W_3 \\ W_4 \\ W_5 \\ W_6 \end{bmatrix} = \begin{bmatrix} 253.86 \\ -5.14 \\ 0.91 \\ 4.235 \end{bmatrix} + \begin{bmatrix} 1125 \\ 1 \\ 38.5 \\ 37.5 \end{bmatrix} = \begin{bmatrix} 1378.86 \\ -4.14 \\ 39.41 \\ 41.735 \end{bmatrix}$$

Since all the deformations are known to this point we can find the

$[w]_m = [kT]_m [\Delta]_m$

Member-1:

$$\begin{bmatrix} w_{1E} \\ w_{2E} \\ w_{3E} \\ w_{4E} \end{bmatrix} = EI \begin{bmatrix} 0.0333 & 0.016667 & -0.00041667 & 0.0001667 \\ 0.016667 & 0.033333 & -0.00041667 & 0.00041667 \\ -0.000416667 & -0.00041667 & 0.00000694 & -0.00000694 \\ 0.00041667 & 0.00041667 & -0.00000694 & 0.00000694 \end{bmatrix} \frac{1}{EI} \begin{bmatrix} -10535.2946 \\ 22870.58863 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 30 \\ 586.7647 \\ -5.1397 \\ 5.1397 \end{bmatrix}$$

Superimposing the fixed end forces for member-1 on the above w's we get

$w = w_E + w_F$

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 30 \\ 586.7647 \\ -5.1397 \\ 5.1397 \end{bmatrix} + \begin{bmatrix} -30 \\ 30 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 616.764 \\ -4.1397 \\ 6.1397 \end{bmatrix}$$

Member 2 :

$$\begin{bmatrix} w_{5E} \\ w_{6E} \\ w_{7E} \\ w_{8E} \end{bmatrix} = EI \begin{bmatrix} 0.0222 & 0.0111 & -0.000185 & 0.000185 \\ 0.0111 & 0.0222 & -0.000185 & 0.000185 \\ -0.000185 & -0.000185 & 0.00000206 & 0.00000206 \\ 0.000185 & 0.000185 & 0.00000206 & 0.00000206 \end{bmatrix} \frac{1}{EI} \begin{bmatrix} 22870.5886 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 508.2353 \\ 254.1176 \\ -4.23529 \\ 4.23529 \end{bmatrix}$$

Superimposing the fixed end forces

$$w = w_E + w_F$$

$$\begin{bmatrix} w_5 \\ w_6 \\ w_7 \\ w_8 \end{bmatrix} = \begin{bmatrix} 508.2353 \\ 254.1176 \\ -4.23529 \\ 4.23529 \end{bmatrix} + \begin{bmatrix} -1125 \\ 1125 \\ 37.5 \\ 37.5 \end{bmatrix} = \begin{bmatrix} -616.764 \\ 1379.117 \\ 33.2647 \\ 41.73529 \end{bmatrix}$$

OR

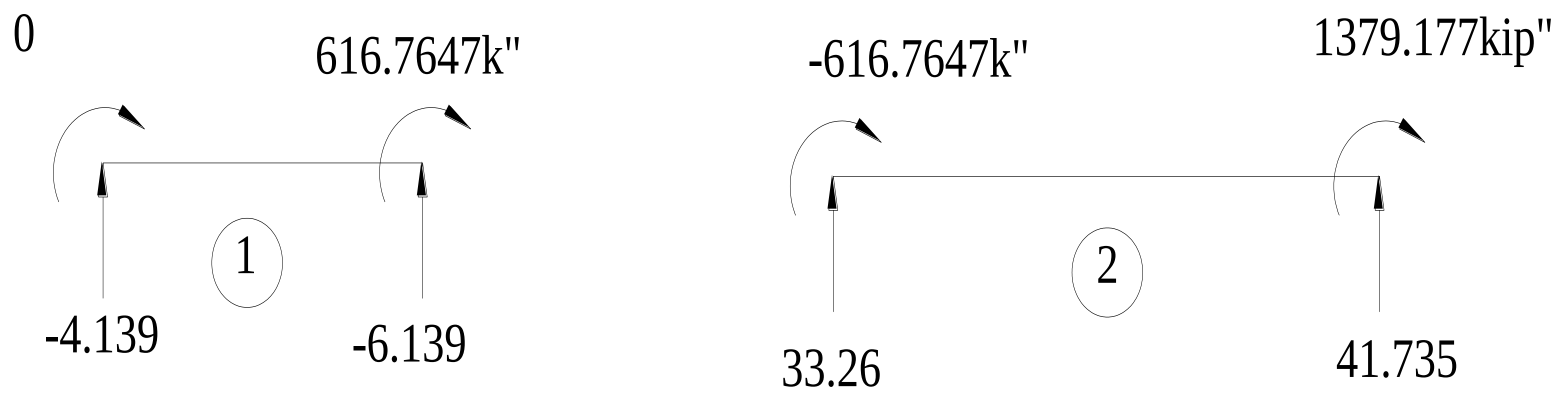
$$[w]_E = \begin{bmatrix} w_{1E} \\ w_{2E} \\ w_{3E} \\ w_{4E} \\ w_{5E} \\ w_{6E} \\ w_{7E} \\ w_{8E} \end{bmatrix} = EI \begin{bmatrix} 0.0333 & 0.01667 & 0 & -0.00041667 & 0.00041667 & 0 \\ 0.01667 & 0.0333 & 0 & -0.00041667 & 0.00041667 & 0 \\ -0.00041667 & -0.00041667 & 0 & 0.00000694 & -0.00000694 & 0 \\ 0.00041667 & 0.00041667 & 0 & -0.00000694 & 0.000000694 & 0 \\ 0 & 0.0222 & 0.0111 & 0 & -0.000185 & 0.000185 \\ 0 & 0.0111 & 0.0222 & 0 & -0.000185 & 0.000185 \\ 0 & -0.000185 & -0.000185 & 0 & 0.00000206 & 0.00000206 \\ 0 & 0.000185 & 0.000185 & 0 & 0.00000206 & 0.00000206 \end{bmatrix} \begin{bmatrix} -10535.2946 \\ 22870.5886 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \frac{1}{EI}$$

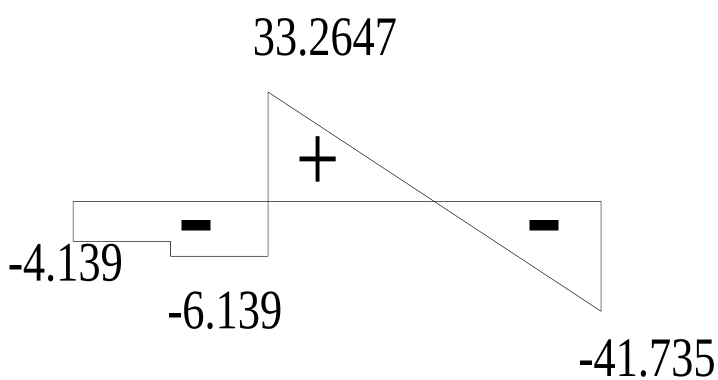
$$[w]_E = \begin{bmatrix} w_{1E} \\ w_{2E} \\ w_{3E} \\ w_{4E} \\ w_{5E} \\ w_{6E} \\ w_{7E} \\ w_{8E} \end{bmatrix} = \begin{bmatrix} 30 \\ 586.53 \\ -5.14 \\ 5.14 \\ 508.42 \\ 254.21 \\ -4.24 \\ 4.24 \end{bmatrix}$$

$w = w_E + w_F$

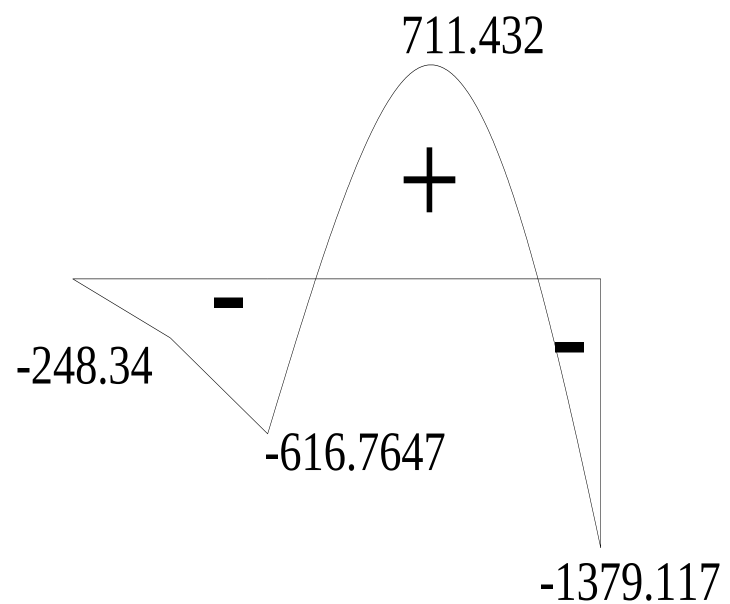
$$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \end{bmatrix} = \begin{bmatrix} 30 \\ 586.53 \\ -5.14 \\ 5.14 \\ 508.42 \\ 254.21 \\ -4.24 \\ 4.24 \end{bmatrix} + \begin{bmatrix} -30 \\ 30 \\ 1 \\ 1 \\ -1125 \\ 1125 \\ 37.5 \\ 37.5 \end{bmatrix}$$

$$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 616.53 \\ -4.14 \\ 6.14 \\ -616.58 \\ 1379.2 \\ 33.26 \\ 41.74 \end{bmatrix}$$





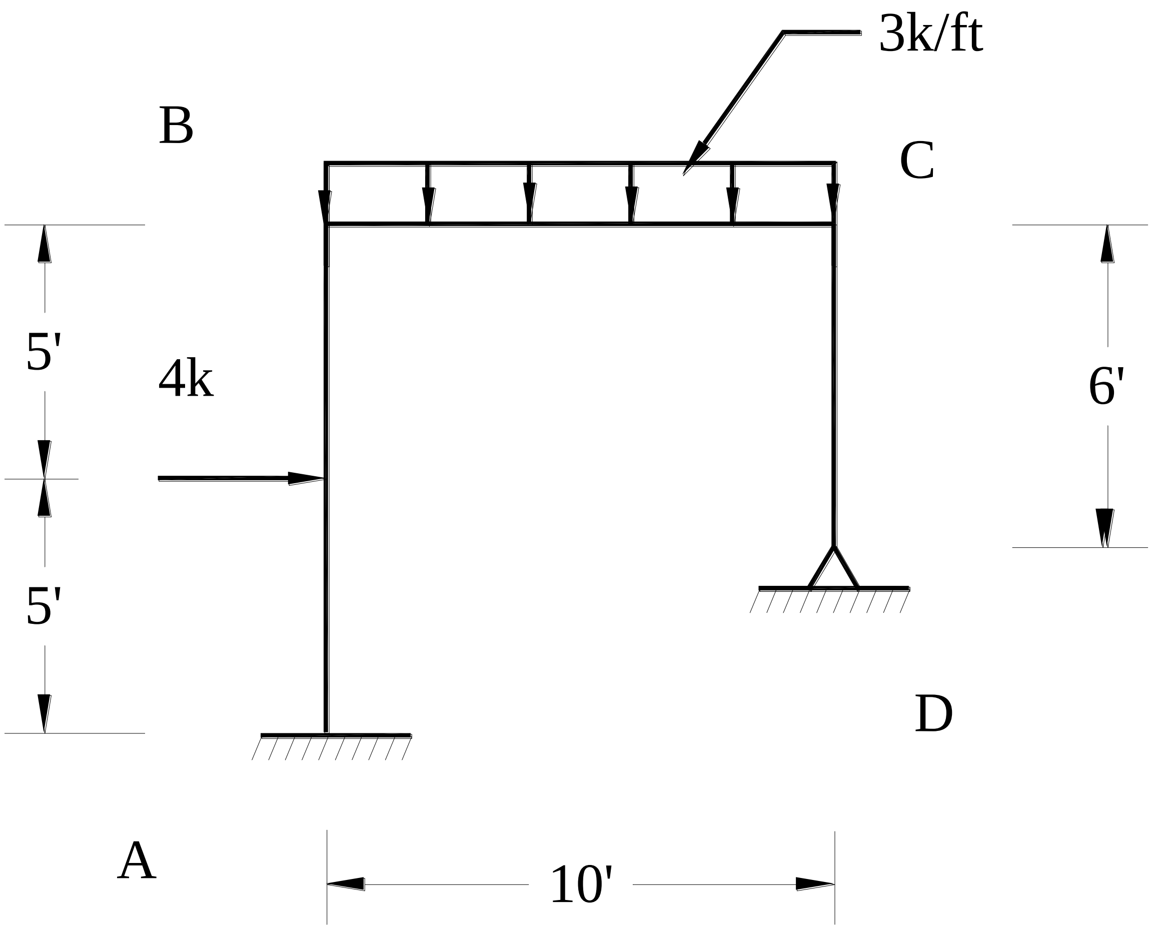
Shear Force Diagram

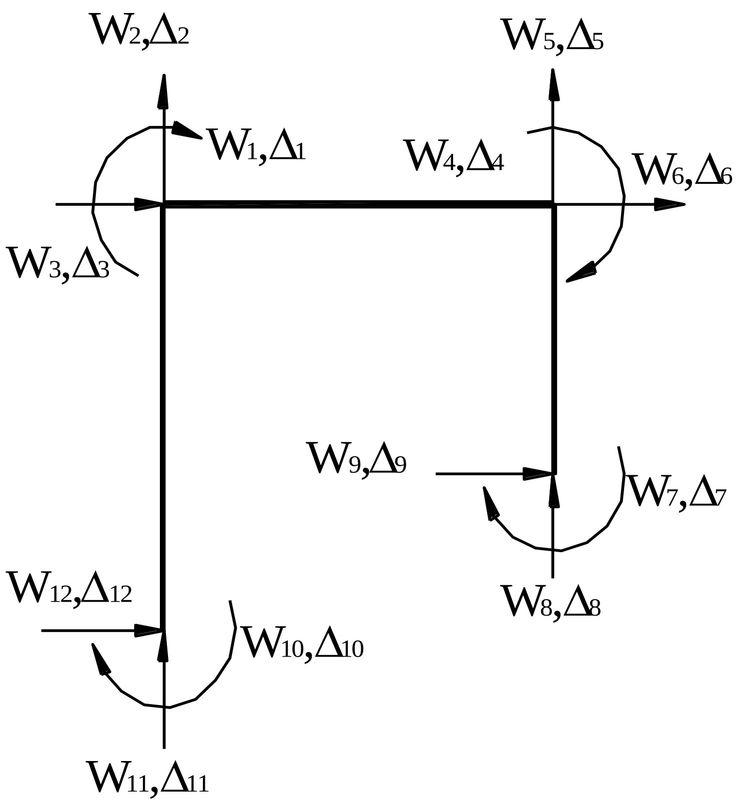


Bending Moment Diagram

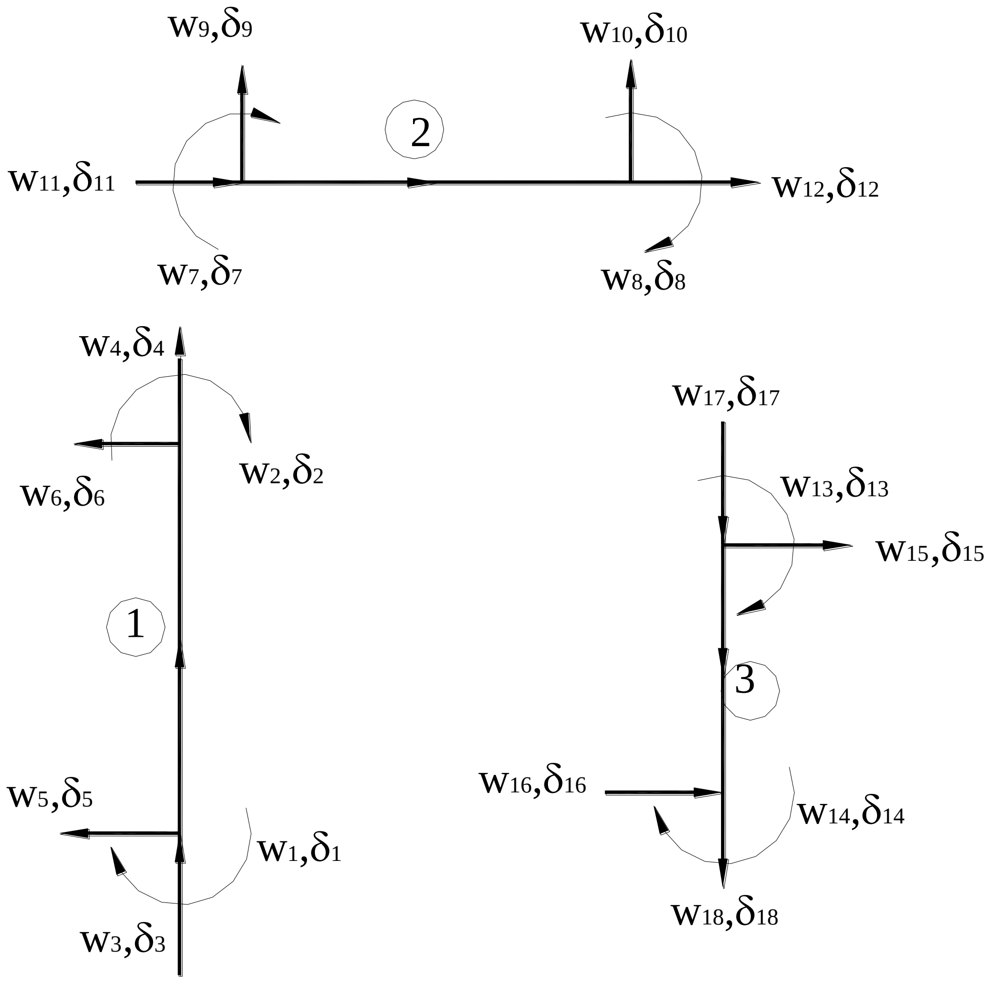
EXAMPLE

Analyse the frame by STIFFNESS METHOD



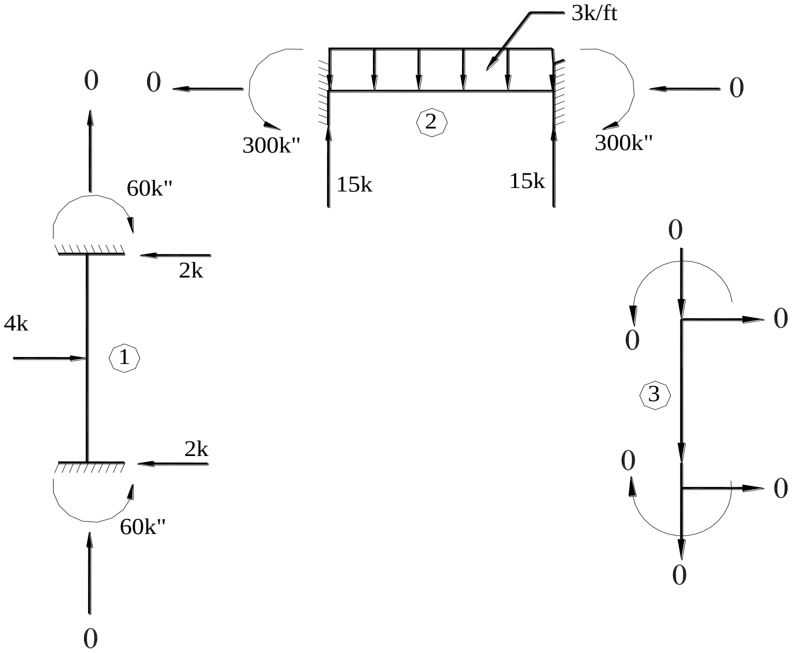


Structure Forces and Deformation



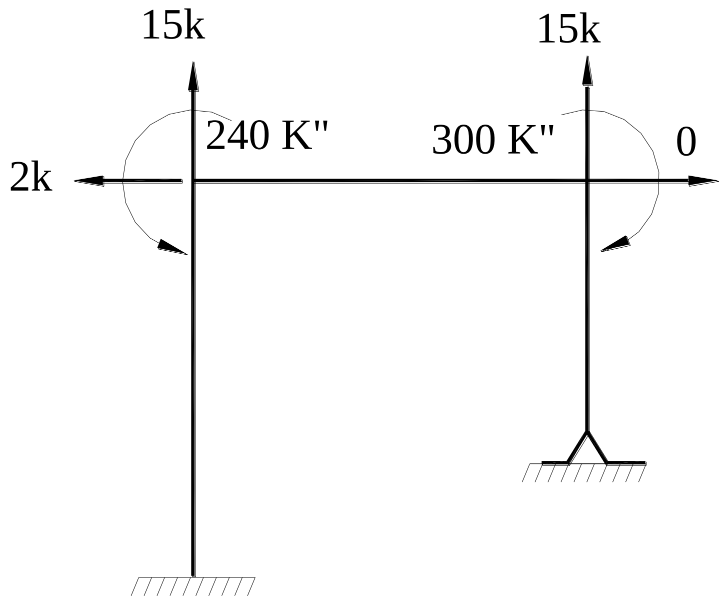
Element Forces and Deformations

$$[\mathbf{FEM's}] = \begin{bmatrix} w_{1F} \\ w_{2F} \\ w_{3F} \\ w_{4F} \\ w_{5F} \\ w_{6F} \\ w_{7F} \\ w_{8F} \\ w_{9F} \\ w_{10F} \\ w_{11F} \\ w_{12F} \\ w_{13F} \\ w_{14F} \\ w_{15F} \\ w_{16F} \\ w_{17F} \\ w_{18F} \end{bmatrix} = \begin{bmatrix} -60 \\ 60 \\ 0 \\ 0 \\ 2 \\ 2 \\ -300 \\ 300 \\ 15 \\ 15 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



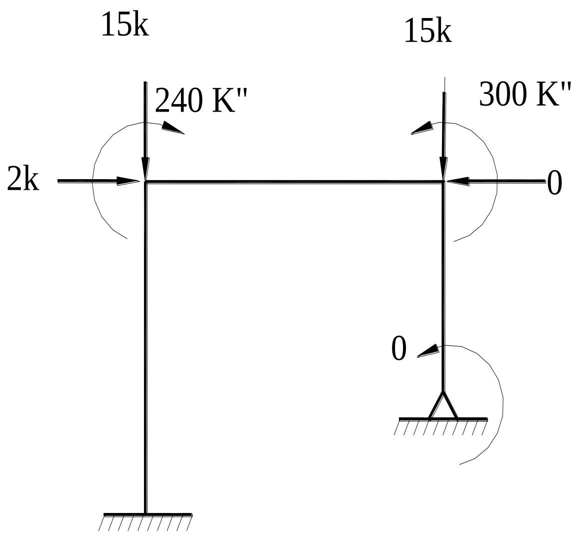
Fixed End Moments and reactions

$$[W]_F = \begin{bmatrix} W_{1F} \\ W_{2F} \\ W_{3F} \\ W_{4F} \\ W_{5F} \\ W_{6F} \\ W_{7F} \\ W_{8F} \\ W_{9F} \\ W_{10F} \\ W_{11F} \\ W_{12F} \end{bmatrix} = \begin{bmatrix} -240 \\ 15 \\ -2 \\ 300 \\ 15 \\ 0 \\ 0 \\ 0 \\ 0 \\ -60 \\ 0 \\ -2 \end{bmatrix}$$



Net fixed end moments

$$[W]_E = \begin{bmatrix} W_{1E} \\ W_{2E} \\ W_{3E} \\ W_{4E} \\ W_{5E} \\ W_{6E} \\ W_{7E} \end{bmatrix} = \begin{bmatrix} 240 \\ -15 \\ 2 \\ -300 \\ -15 \\ 0 \\ 0 \end{bmatrix}$$



Equivalent joint loads

Finding the structure stiffness matrices of elements.

Next we will calculate the structure stiffness matrices for each element using the properties of members tabulated

E = 29000 ksi   I = 100 $inch^4$ A = 5 $inch^2$

Member	Length (in.)	*l*	m	I	J	K	*L*	M	N
1	120	0	1	10	1	11	2	12	3
2	120	1	0	1	4	2	5	3	6
3	72	0	-1	4	7	5	8	6	9

For member-1 we have

$$
[K]_1 = \begin{array}{cc}
\begin{array}{cccccc} 10 & 1 & 11 & & 2 & 12 \end{array} & \\
\begin{bmatrix}
96666.667 & 48333.333 & 0 & 0 & 1208.333 & -1208.333 \\
48333.333 & 96666.667 & 0 & 0 & 1208.333 & -1208.333 \\
0 & 0 & 1208.333 & -1208.333 & 0 & 0 \\
0 & 0 & -1208.333 & 1208.333 & 0 & 0 \\
1208.333 & 1208.333 & 0 & 0 & 20.13889 & -20.13889 \\
-1208.333 & -1208.333 & 0 & 0 & -20.13889 & 20.13889
\end{bmatrix} & \begin{array}{c} 10 \\ 1 \\ 11 \\ 2 \\ 12 \\ 3 \end{array}
\end{array}
$$

For member-2

$$
\begin{array}{ccccccc}
 & 1 & 4 & 2 & 5 & 3 & 6 \\
[K]_2 = & \multicolumn{6}{c}{\left[\begin{array}{rrrrrr}
96666.667 & 48333.333 & -1208.333 & 1208.333 & 0 & 0 \\
48333.333 & 96666.667 & -1208.333 & 1208.333 & 0 & 0 \\
-1208.333 & -1208.333 & 20.13889 & -20.13889 & 0 & 0 \\
1208.333 & 1208.333 & -20.13889 & 20.13889 & 0 & 0 \\
0 & 0 & 0 & 0 & 1208.333 & -1208.333 \\
0 & 0 & 0 & 0 & -1208.333 & 1208.333
\end{array}\right]}
\end{array}
\begin{array}{c} 1 \\ 4 \\ 2 \\ 5 \\ 3 \\ 6 \end{array}
$$

For member-3

$$
\begin{array}{ccccccc}
 & 4 & 7 & 5 & 8 & 6 & 9 \\
[K]_3 = & \left[\begin{array}{cccccc}
161111.111 & 80555.556 & 0 & 0 & -3356.481 & 3356.481 \\
80555.556 & 161111.111 & 0 & 0 & -3356.481 & 3356.481 \\
0 & 0 & 2013.889 & -2013.889 & 0 & 0 \\
0 & 0 & -2013.889 & 2013.889 & 0 & 0 \\
-3356.481 & -3356.481 & 0 & 0 & 93.2356 & -93.2356 \\
3356.481 & 3356.481 & 0 & 0 & -93.2356 & 93.2356
\end{array}\right] & \begin{array}{c} 4 \\ 7 \\ 5 \\ 8 \\ 6 \\ 9 \end{array}
\end{array}
$$

Finding structure stiffness matrix for the entire frame.

Using relation $[K] = [K]_1+[K]_2+[K]_3$ we get the following structure stiffness matrix.

$$
[K] = \begin{array}{cc}
 & \begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} &
\left[\begin{array}{ccccccc|ccccc}
193333.3 & -1208.3 & -1208.33 & 48333.3 & 1208.3 & 0 & 0 & 0 & 0 & 48333.3 & 0 & 1208.3 \\
-1208.3 & 1228.47 & 0 & -1208.3 & -20.138 & 0 & 0 & 0 & 0 & 0 & -1208.3 & 0 \\
-1208.3 & 0 & 1228.47 & 0 & 0 & -1208.3 & 0 & 0 & 0 & -1208.3 & 0 & -20.139 \\
48333.3 & -1208.3 & 0 & 257777.78 & 1208.3 & -3356.48 & 80555.56 & 0 & 3356.48 & 0 & 0 & 0 \\
1208.3 & -20.139 & 0 & 1208.3 & 2034.03 & 0 & 0 & -2013.89 & 0 & 0 & 0 & 0 \\
0 & 0 & -1208.3 & -3356.48 & & 1301.568 & -3356.48 & 0 & -93.2356 & 0 & 0 & 0 \\
0 & 0 & 0 & -80555.56 & 0 & -3356.48 & 161111.1 & 0 & 3356.48 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & -2013.89 & 0 & 0 & 2013.89 & 01 & 0 & 0 & 0 \\
0 & 0 & 0 & 3356.48 & 0 & -93.2356 & 3356.48 & 0 & 93.2356 & 0 & 0 & 0 \\
48333.3 & 0 & -1208.33 & 0 & 0 & 0 & 0 & 0 & 0 & 96666.67 & 0 & 1208.3 \\
0 & -1208.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1208.3 & 0 \\
1208.3 & 0 & -20.139 & 0 & 0 & 0 & 0 & 0 & 0 & 1208.3 & 0 & 20.139
\end{array}\right]
\end{array}
$$

Finding unknown deformations

Unknown deformation can be calculated using equation

$[\Delta u] = [K_{11}]^{-1}[Wk]$

This can be done by partitioning the structure stiffness matrix into known and unknown deformations and forces.

$$\begin{bmatrix} \mathbf{W}_k \\ \hline \mathbf{W}_u \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \hline \mathbf{K}_{21} & \mathbf{K}_{22} \end{array}\right] \begin{bmatrix} \Delta_u \\ \hline \Delta_k \end{bmatrix}$$

Unknown deformations can be calculated using the equation ;

$[\Delta]u=[K_{11}]^{-1}[W]k$

Solving the above equation we get ,

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \\ \Delta_7 \end{bmatrix} = \begin{bmatrix} 193333.333 & -1208.333 & -1208.333 & 48333.333 & 1208.333 & 0 & 0 \\ -1208.333 & 1228.4718 & 0 & -1208.333 & -20.13889 & 0 & 0 \\ -1208.333 & 0 & 1228.722 & 0 & 0 & -1208.333 & 0 \\ 48333.333 & -1208.333 & 0 & 257777.778 & 1208.333 & -3356.481 & 80555.556 \\ 1208.333 & -20.13889 & 0 & 1208.333 & 2034.0278 & 0 & 0 \\ 0 & 0 & -1208.333 & -3356.4814 & 0 & 1301.568 & -3356.481 \\ 0 & 0 & 0 & 80555.556 & 0 & -3356.481 & 161111.111 \end{bmatrix}^{-1} \begin{bmatrix} 240 \\ -15 \\ 2 \\ -300 \\ -15 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \\ \Delta_7 \end{bmatrix} = \begin{bmatrix} .00184 \\ -.01202 \\ .039859 \\ -.001527 \\ -.007682 \\ .037023 \\ .001535 \end{bmatrix}$$

Finding Unknown reactions

Unknown reactions can be calculated using the following equation:

$$[W_u] = [K_{21}][\Delta_u]$$

$$\begin{bmatrix} W_{8E} \\ W_{9E} \\ W_{10E} \\ W_{11E} \\ W_{12E} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & -2013.889 & 0 & 0 \\ 0 & 0 & 0 & 3356.481 & 0 & -93.2356 & 3356.481 \\ 48333.333 & 0 & -1208.333 & 0 & 0 & 0 & 0 \\ 0 & -1208.333 & 0 & 0 & 0 & 0 & 0 \\ 1208.333 & 0 & -20.13889 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.00184 \\ -0.01202 \\ 0.039859 \\ -0.001527 \\ -0.007682 \\ 0.037023 \\ 0.001535 \end{bmatrix}$$

$$= \begin{bmatrix} 15.4707 \\ -3.42501 \\ 40.7704 \\ 14.5242 \\ 1.42062 \end{bmatrix}$$

Now we will superimpose the fixed end reactions on the above calculated structure forces.

$[W]=[W]_E+[W]_F$

$$\begin{bmatrix} W_8 \\ W_9 \\ W_{10} \\ W_{11} \\ W_{12} \end{bmatrix} = \begin{bmatrix} 15.47 \\ -3.42 \\ 40.77 \\ 14.52 \\ 1.42 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -60 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 15.47 \\ -3.42 \\ -19.23 \\ 14.52 \\ -0.58 \end{bmatrix}$$

Finding the unknown element forces.

Up to this point all the deformations are known to us, we can find the element forces in each element using relation

$[w]_m = [kT]_m[\Delta]_m$

For member-1

$$\begin{bmatrix} w_1 \\ w_2 \\ w_5 \\ w_6 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 96666.667 & 48333.333 & 0 & 0 & 1208.333 & -1208.333 \\ 48333.333 & 96666.667 & 0 & 0 & 1208.333 & -1208.333 \\ -1208.333 & -1208.333 & 0 & 0 & -20.13889 & 20.13889 \\ 1208.333 & 1208.333 & 0 & 0 & 20.13889 & -20.13889 \\ 0 & 0 & 1208.3330 & -1208.333 & 0 & 0 \\ 0 & 0 & -1208.333 & 1208.333 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.00184 \\ 0 \\ -0.01202 \\ 0 \\ 0.039859 \end{bmatrix}$$

Equation gives

$w1$ = 40.77 kips-inch	c.w.
$w2$ = 129.704 kips-inch	c.w.
$w5$ = –1.421 kips	Downward
$w6$ = 1.421 kips	Upward
$w3$ =14.524 kips	Rightward
$w4$ = –14.524 kips	Leftward

Superimposing the fixed end reactions in their actual direction we get

$w1$ = 41.0269 – 60 = -18.973 kips-inch	c.c.w.
$w2$ = 130.2173 + 60 = 190.2173 kips-inch	c.w.
$w5$ = 1.427036 – 2 = -3.1427036 kips	Rightward
$w6$ = -1.427036 – 2 =0.57296 kips	Rightward
$w3$ = 14.5289 kips	Upward
$w4$ = -14.5289 kips	Downward

For member-2

$$
\begin{bmatrix} w_7 \\ w_8 \\ w_9 \\ w_{10} \\ w_{11} \\ w_{12} \end{bmatrix} = \begin{bmatrix} 96666.667 & 48333.333 & -1208.333 & 1208.333 & 0 & 0 \\ 48333.333 & 96666.667 & -1208.333 & 1208.333 & 0 & 0 \\ -1208.333 & -1208.333 & 20.13889 & -20.13889 & 0 & 0 \\ 1208.333 & 1208.333 & -20.13889 & 20.13889 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1208.333 & -1208.333 \\ 0 & 0 & 0 & 0 & -1208.333 & 1208.333 \end{bmatrix} \begin{bmatrix} 0.001845 \\ -0.001528 \\ -0.012024 \\ -0.0076822 \\ 0.0398595 \\ 0.0370233 \end{bmatrix}
$$

Above equation gives

***w*7 = 109.782 kips-inch**	**c.w.**
***w*8 = -53.25353 kips-inch**	**c.c.w.**
***w*9 = - 0.47048 kips**	**Downward**
***w*10 = 0.47048 kips**	**Upward**
***w*11 = 3.427 kips**	**Rightward**
***w*12 = -3.427 kips**	**Leftward**

Similarly for member-2 we have to superimpose the fixed end reactions

*w*7 = 109.782 – 300 = -190.218 kips-inch c.c.w.

*w*8 = –53.25353 + 300 = 246.746 kips-inch c.w.

*w*9 = –0.471071+15 = 14.5289 kips Upward

*w*10 = 0.471071 +15 = 15.471 kips Upward

*w*11 = 3.427 kips Rightward

*w*12 = –3.427 kips Leftward

And finally for member-3 we get

$$\begin{bmatrix} w_{13} \\ w_{14} \\ w_{15} \\ w_{16} \\ w_{17} \\ w_{18} \end{bmatrix} = \begin{bmatrix} 161111.111 & 80555.556 & 0 & 0 & -3356.481 & 3356.481 \\ 80555.556 & 161111.111 & 0 & 0 & -3356.481 & 3356.481 \\ -3356.481 & -3356.481 & 0 & 0 & 93.2356 & -93.2356 \\ 3356.481 & 3356.481 & 0 & 0 & -93.2356 & 93.2356 \\ 0 & 0 & -2013.889 & 2013.889 & 0 & 0 \\ 0 & 0 & 2013.889 & -2013.889 & 0 & 0 \end{bmatrix} \begin{bmatrix} -0.0015278 \\ 0.00153523 \\ -0.00768219 \\ 0 \\ 0.0370233 \\ 0 \end{bmatrix}$$

Solving the above equation we get

***w*13 = -246.74227 kips-inch** **c.c.w.**

***w*14 = 0 kips-inch**

***w*15 = 3.427kips** **Upward**

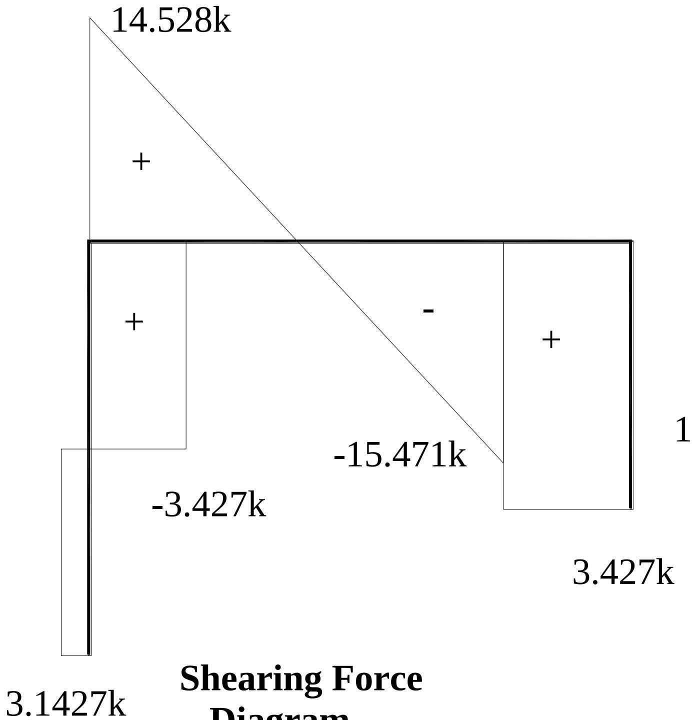
***w*16 = -3.427 kips** **Downward**

***w*17 = 15.4711 kips** **Rightward**

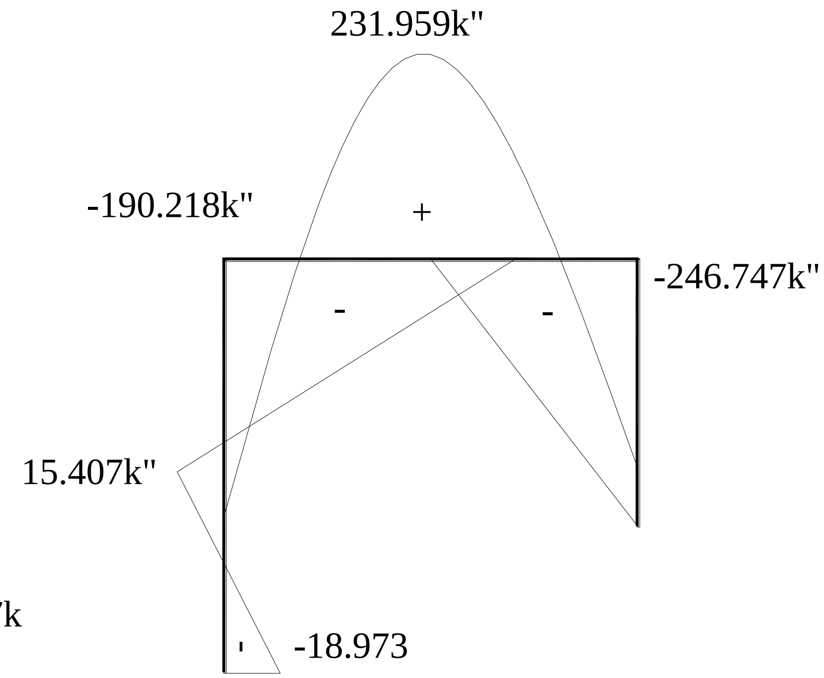
***w*18 = -15.4711kips** **Leftward**

PLOTTING THE BENDING MOMENT AND SHEARING FORCE DIAGRAM.

According to the forces calculated above bending moment and shearing force diagrams are plotted below:



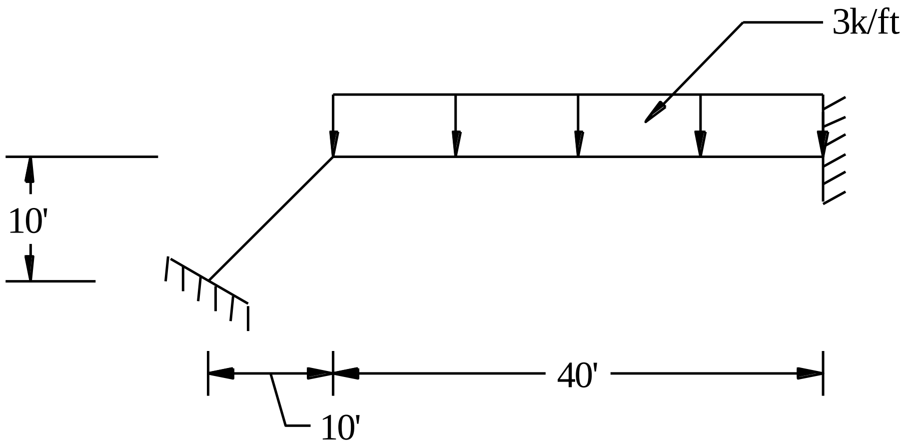
Shearing Force Diagram

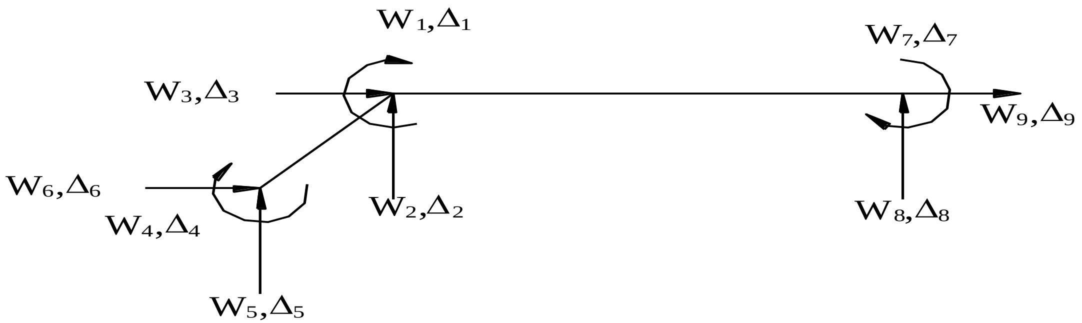


Bending moment diagram

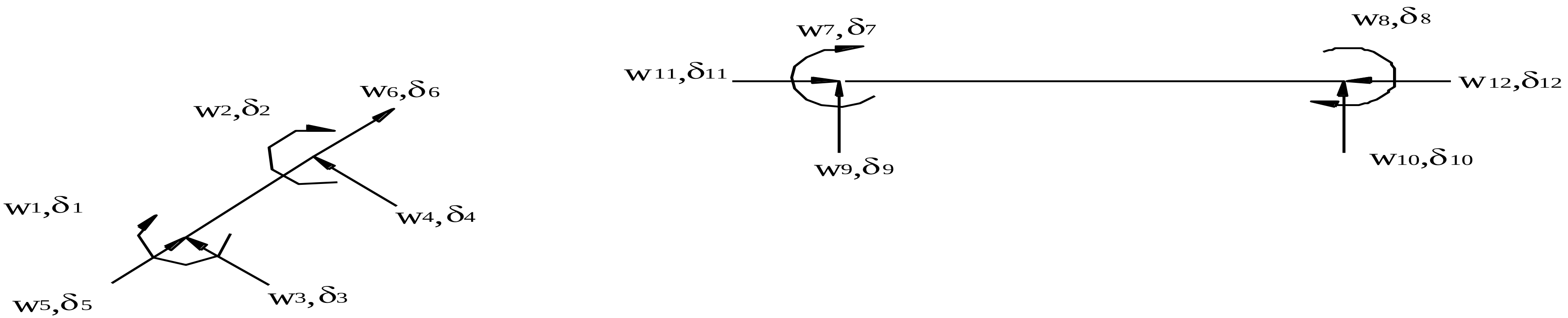
UNIT-V
LINEAR ELEMENT

EXAMPLE **To analyse the frame shown in the figure using direct stiffness method.**





Structure Forces and Deformation



Element Forces and Deformation

Fixed end moments

$$\begin{bmatrix} W_{1F} \\ W_{2F} \\ W_{3F} \\ W_{4F} \\ W_{5F} \\ W_{6F} \\ W_{7F} \\ W_{8F} \\ W_{9F} \end{bmatrix} = \begin{bmatrix} -4800 \\ 60 \\ 0 \\ 0 \\ 0 \\ 0 \\ 4800 \\ 60 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} w_{1F} \\ w_{2F} \\ w_{3F} \\ w_{4F} \\ w_{5F} \\ w_{6F} \\ w_{7F} \\ w_{8F} \\ w_{9F} \\ w_{10F} \\ w_{11F} \\ w_{12F} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -4800 \\ 4800 \\ 60 \\ 60 \\ 0 \\ 0 \end{bmatrix}$$

Equivalent joint loads= $$\begin{bmatrix} W_{1E} \\ W_{2E} \\ W_{3E} \end{bmatrix} = \begin{bmatrix} 4800 \\ -60 \\ 0 \end{bmatrix}$$

The properties of each member are shown in the table below.

E = 29 x 103 ksi , I = 1000 inch4 and A = 10 inch2 are same for all members.

Member	Length	l	m	I	J	K	L	M	N
1	169.7056 inch	0.707	0.707	4	1	5	2	6	3
2	480 inch	1	0	1	7	2	8	3	9

Using the structure stiffness matrix for frame element in general form we ge
structure stiffness matrix for member-1.

$$[K]_1 = \begin{bmatrix} 683536.666 & 341768.33 & -4272.064 & 4272.064 & 4272.064 & -4272.064 \\ 341768.33 & 683536.666 & -4272.064 & 4272.064 & 4272.064 & -4272.064 \\ -4272.064 & -4272.064 & 890.0045 & -890.0045 & 818..804 & -818.804 \\ 4272.064 & 4272.064 & -890.0045 & 890.045 & -818.804 & 818.804 \\ 4272.064 & 4272.064 & 818.804 & -818.804 & 890.0045 & -890.0045 \\ -4272.064 & -4272.064 & -818.804 & 818.804 & -890.0045 & 890.0045 \end{bmatrix} \begin{array}{c} 4 \\ 1 \\ 5 \\ 2 \\ 6 \\ 3 \end{array}$$

Similarly for member-2 we get,

$$[K]_2 = \begin{bmatrix} 241666.667 & 120833.33 & -755.21 & 755.21 & 0 & 0 \\ 120833.33 & 241666.667 & -755.21 & 755.21 & 0 & 0 \\ -755.21 & -755.21 & 3.15 & -3.15 & 0 & 0 \\ 755.21 & 755.21 & -3.15 & 3.15 & 0 & 0 \\ 0 & 0 & 0 & 0 & 604.167 & -604.167 \\ 0 & 0 & 0 & 0 & -604.167 & 604.167 \end{bmatrix} \begin{array}{c} 1 \\ 7 \\ 2 \\ 8 \\ 3 \\ 9 \end{array}$$

Calculating the structure stiffness matrix for the entire frame.

After getting the structure stiffness matrices for each element we can find the structure stiffness matrix for the whole structure using following relation:

$$[K]_1+[K]_2 = [K]$$

$$[K]= \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \left[\begin{array}{ccc|cccccc} 925203.32 & 3516.854 & -4272.064 & 341678.33 & -4272.06 & 4272.064 & 120833.33 & 755.21 & 0 \\ 3516.854 & 893.195 & 818.804 & 4272.064 & -890.0045 & -818.804 & -755.21 & -3.15 & 0 \\ -4272.064 & 818.804 & 1494.1715 & -4272.064 & -818.804 & -890.0045 & 0 & 0 & -604.167 \\ \hline 341768.33 & 4272.064 & -4272.064 & 683536.66 & -4272.064 & 4272.064 & 0 & 0 & 0 \\ -4272.064 & -890.0045 & -818.804 & -4272.064 & 890.0045 & 818.804 & 0 & 0 & 0 \\ 4272.064 & -818.804 & -890.0045 & 4272.064 & 818.804 & 890.0045 & 0 & 0 & 0 \\ 120833.33 & -755.21 & 0 & 0 & 0 & 0 & 241666.66 & 755.21 & 0 \\ 755.21 & -3.15 & 0 & 0 & 0 & 0 & 755.21 & 3.15 & 0 \\ 0 & 0 & -604.167 & 0 & 0 & 0 & 0 & 0 & 604.167 \end{array}\right] \end{array}$$

Finding unknown deformations

Unknown deformation can be calculated using equation

$[\Delta u] = [K11]^{-1}[Wk]$

This can be done by partitioning the structure stiffness matrix into known and unknown deformations and forces.

$$\begin{bmatrix} \mathbf{W}_k \\ \text{-- -- --} \\ \mathbf{W}_u \end{bmatrix} = \left[\begin{array}{c:c} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \hdashline \mathbf{K}_{21} & \mathbf{K}_{22} \end{array}\right] \begin{bmatrix} \Delta_u \\ \text{-- --} \\ \Delta_k \end{bmatrix}$$

Unknown deformations can be calculated using the equation ;

[Δu] = [K11]-1 [Wk]

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \begin{bmatrix} 925203.333 & 3516.854 & -4272.064 \\ 3516.854 & 893.195 & 818.804 \\ -4272.064 & 818.804 & 1494.1715 \end{bmatrix}^{-1} \begin{bmatrix} 4800 \\ -60 \\ 0 \end{bmatrix}$$

Solving above equation we get

Δ1 = 0.00669 c.w.
Δ2 = - 0.223186 downward
Δ3 = 0.141442 Rightward

$$[W_u] = [K_{21}][\Delta_u]$$

$$\begin{bmatrix} W_{4E} \\ W_{5E} \\ W_{6E} \\ W_{7E} \\ W_{8E} \\ W_{9E} \end{bmatrix} = \begin{bmatrix} 341768.33 & 4272.064 & -4272.064 \\ -4272.064 & -890.0045 & -818.804 \\ 4272.064 & -818.804 & -890.0045 \\ 120833.33 & -755.21 & 0 \\ 755.21 & -3.15 & 0 \\ 0 & 0 & -604.167 \end{bmatrix} \begin{bmatrix} 0.00669 \\ -0.223186 \\ 0.141442 \\ \\ \\ \\ \end{bmatrix}$$

$$= \begin{bmatrix} 728.716 \\ 54.2432 \\ 85.4417 \\ 976.927 \\ 5.75539 \\ -85.4546 \end{bmatrix}$$

Now we will superimpose the fixed end forces on the above calculated equivalent forces.

$W = W_E + W_F$

$$\begin{bmatrix} W_4 \\ W_5 \\ W_6 \\ W_7 \\ W_8 \\ W_9 \end{bmatrix} = \begin{bmatrix} 728.716 \\ 54.2432 \\ 85.4417 \\ 976.927 \\ 5.75539 \\ -85.4546 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4800 \\ 60 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 728.716 \\ 54.2432 \\ 85.4417 \\ 5776.927 \\ 65.75539 \\ -85.4546 \end{bmatrix}$$

Calculating the unknown element forces

All the deformations are known up to this point, therefore we can calculate the forces in all elements using relation

[*w*] =[*kT*][*Δ*]

For member-1 we get

$$\begin{bmatrix} w_{1E} \\ w_{2E} \\ w_{3E} \\ w_{4E} \\ w_{5E} \\ w_{6E} \end{bmatrix} = \begin{bmatrix} 683536.665 & 341768.332 & -4272.063 & 4272.063 & 4272.063 & -4272.063 \\ 341768.332 & 683536.665 & -4272.063 & 4272.063 & 4272.063 & -4272.063 \\ -6041.668 & -6041.668 & 50.34676 & -50.34676 & -50.34676 & 50.34676 \\ 6041.668 & 6041.668 & -50.34676 & 50.34676 & 50.34676 & -50.34676 \\ 0 & 0 & 1208.321 & -1208.321 & 1208.321 & -1208.321 \\ 0 & 0 & -1208.321 & 1208.321 & -1208.321 & 1208.321 \end{bmatrix} \begin{bmatrix} 0 \\ 0.00669 \\ 0 \\ -0.22319 \\ 0 \\ 0.14144 \end{bmatrix}$$

Solving the above equation we get following values of w's for the equivalent loading condition provided by member-1

w_{1E}=	729.0578 kip-inch	c.w.
w_{2E}=	3015.829 kip-inch	c..w.
w_{3E}=	–22.066 kip	Downward
w_{4E}=	22.0669 kip	Upward
w_{5E}=	98.773 kip	Rightward
w_{6E}=	–98.773 kip	Leftward

$w = w_E + w_F$

As w_F are zero therefore w's will be having the same values as those of w_E 's

For member-2 we get

$[w]2=[kT]2[\Delta]2$

$$\begin{bmatrix} w_{7E} \\ w_{8E} \\ w_{9E} \\ w_{10E} \\ w_{11E} \\ w_{12E} \end{bmatrix} = \begin{bmatrix} 241666.667 & 120833.333 & -755.208 & 755.208 & 0 & 0 \\ 120833.333 & 241666.667 & -755.208 & 755.208 & 0 & 0 \\ -755.208 & -755.208 & 3.1467 & -3.1467 & 0 & 0 \\ 755.208 & 755.208 & -3.1467 & 3.1467 & 0 & 0 \\ 0 & 0 & 0 & 0 & 604.16667 & -604.16667 \\ 0 & 0 & 0 & 0 & -604.16667 & 604.16667 \end{bmatrix} \begin{bmatrix} 0.00669 \\ 0 \\ -0.223186 \\ 0 \\ 0.141442 \\ 0 \end{bmatrix}$$

solving the above equation we get the structure forces for the equivalent structure loads provided by member-2

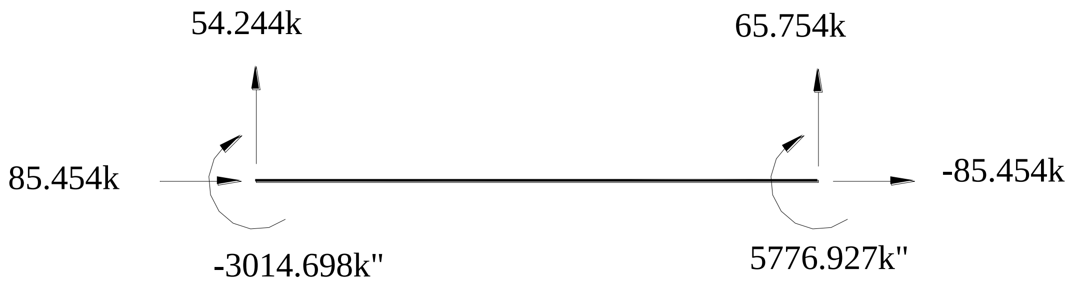
*w*7E=	1785.543 kip-inch	c.w.
*w*8E=	977.047 kip-inch	c..w.
*w*9E=	–5.7553 kip	Downward
*w*10E=	5.7553 kip	Upward
*w*11E=	85.454 kip	Rightward
*w*12E=	–85.454 kip	

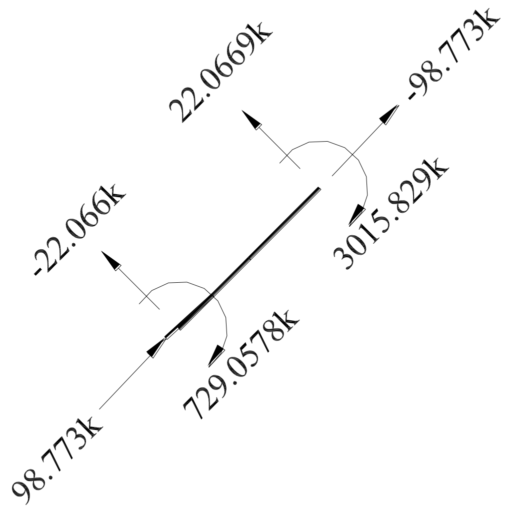
Superimposing the fixed end forces in their actual direction we get the element forces for the actual loading condition.

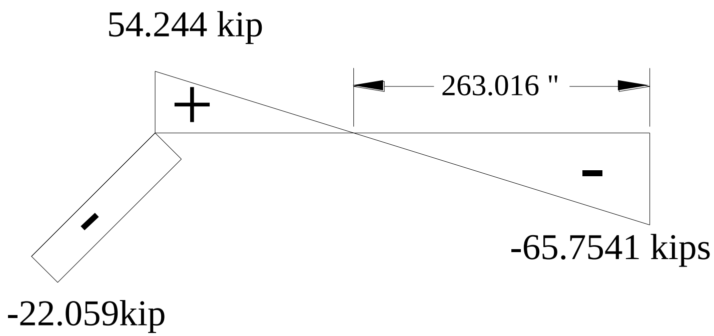
***w*7= 1785.543 – 4800 = -3014.698**	**kip-inch**	**c.c.w.**
***w*8= 977.047 + 4800 = 5776.927**	**kip-inch**	**c..w.**
***w*9= –5.7553 +60 = 54.244 kip**		**Upward**
***w*10= 5.7553 + 60 = 65.754 kip**		**Upward**
***w*11 = 85.454 kip**		**Rightward**
***w*12= – 85.454 kip**		**Leftward**

Plotting the bending moment and shearing force diagrams.

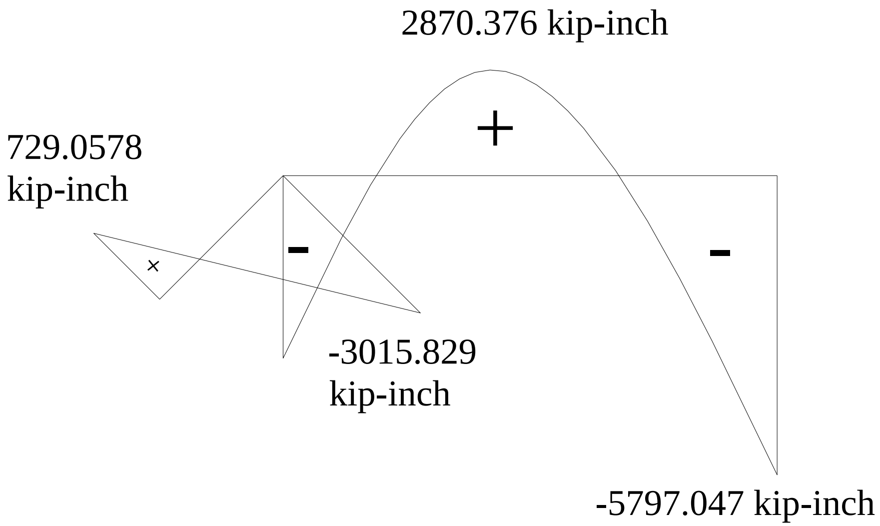
Bending moment and the shearing force diagram can be drawn according to the element forces calculated


54.244k
65.754k
85.454k
-85.454k
-3014.698k"
5776.927k"

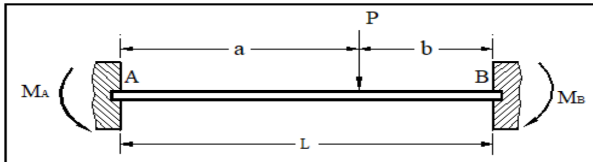

22.0669k
-98.773k
3015.829k
-22.066k
729.0578k
98.773k



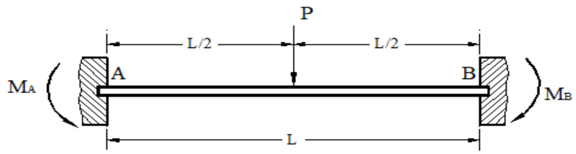
Shearing force Diagram



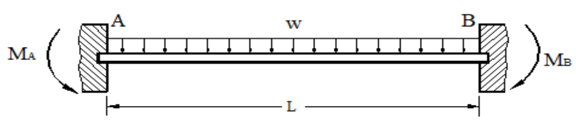
Bending Moment Diagram



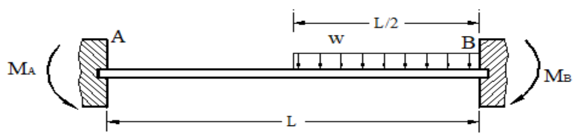
$$M_A = -\frac{Pab^2}{L^2} \qquad M_B = \frac{Pa^2b}{L^2}$$



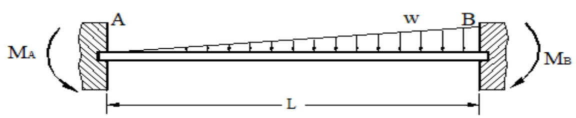
$$M_A = -\frac{Pab^2}{L^2} \qquad M_B = \frac{Pa^2b}{L^2}$$



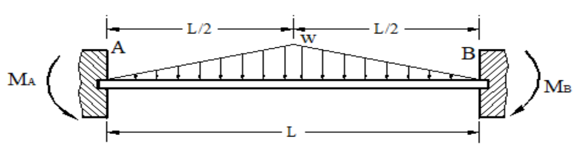
$$M_A = -\frac{wL^2}{12} \qquad M_B = \frac{wL^2}{12}$$



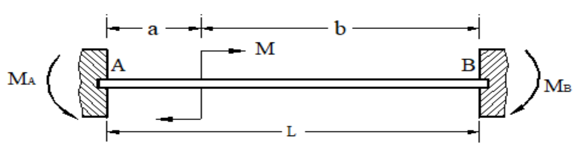
$$M_A = -\frac{5}{192}wL^2 \qquad M_B = \frac{11}{192}wL^2$$



$$M_A = -\frac{wL^2}{30} \qquad M_B = \frac{wL^2}{20}$$



$$M_A = -\frac{5wL^2}{96} \qquad M_B = \frac{5wL^2}{96}$$



$$M_A = -\frac{Mb}{L}\left(\frac{3a}{L} - 1\right) \qquad M_B = \frac{Ma}{L}\left(\frac{3b}{L} - 1\right)$$