LECTURE NOTES

ON

AIRCRAFT STRUCTURAL DYNAMICS

B.Tech VII Semester

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UNIT - I
SINGLE-DEGREE-OF-FREEDOM LINEAR SYSTEMS

Introduction to Theory of Vibration:

Introduction to theory of vibration, equation of motion, free vibration, response to harmonic excitation, response to an impulsive excitation, response to a step excitation, response to periodic excitation (Fourier series), response to a periodic excitation (Fourier transform), Laplace transform (Transfer Function).

This chapter introduces the subject of vibrations in a relatively simple manner. It begins with a brief history of the subject and continues with an examination of the importance of vibration. The basic concepts of degrees of freedom and of discrete and continuous systems are introduced, along with a description of the elementary parts of vibrating systems. The various classifications of vibration namely, free and forced vibration, undamped and damped vibration, linear and nonlinear vibration, and deterministic and random vibration are indicated. The various steps involved in vibration analysis of an engineering system are outlined, and essential definitions and concepts of vibration are introduced.

The concept of harmonic motion and its representation using vectors and complex numbers is described. The basic definitions and terminology related to harmonic motion, such as cycle, amplitude, period, frequency, phase angle, and natural frequency, are given. Finally, the harmonic analysis, dealing with the representation of any periodic function in terms of harmonic functions, using Fourier series, is outlined. The concepts of frequency spectrum, time- and frequency-domain representations of periodic functions, half-range expansions, and numerical computation of Fourier coefficients are discussed in detail.

Learning Objectives:

After completing this chapter, the reader should be able to do the following:

* Describe briefly the history of vibration
* Indicate the importance of study of vibration
* Give various classifications of vibration
* State the steps involved in vibration analysis
* Compute the values of spring constants, masses, and damping constants
* Define harmonic motion and different possible representations of harmonic motion
* Add and subtract harmonic motions
* Conduct Fourier series expansion of given periodic functions
* Determine Fourier coefficients numerically using the MATLAB program

The subject of vibration is introduced here in a relatively simple manner. The chapter begins with a brief history of vibration and continues with an examination of its importance. The various steps involved in vibration analysis of an engineering system are outlined, and essential definitions and concepts of vibration are introduced. We learn here that all mechanical and structural systems can be modeled as mass-spring-damper systems. In some systems, such as an automobile, the mass, spring and damper can be identified as separate components (mass in the form of the body, spring in the form of suspension and damper in the form of shock absorbers). In some cases, the mass, spring and damper do not appear as separate components; they are inherent and integral to the system. For example, in an airplane wing, the mass of the wing is distributed throughout the wing. Also, due to its elasticity, the wing undergoes noticeable deformation during flight so that it can be modeled as a spring. In addition, the deflection of the wing introduces damping due to relative motion between components such as joints, connections and support as well as internal friction due to microstructural defects in the material. The chapter describes the modeling of spring, mass and damping elements, their characteristics and the combination of several springs, masses or damping elements appearing in a system. There follows a presentation of the concept of harmonic analysis, which can be used for the analysis of general periodic motions.

Importance of the Study of Vibration:

Most human activities involve vibration in one form or other. For example, we hear because our eardrums vibrate and see because light waves undergo vibration. Breathing is associated with the vibration of lungs and walking involves (periodic) oscillatory motion of legs and hands. Human speech requires the oscillatory motion of larynges (and tongues). In recent times, many investigations have been motivated by the engineering applications of vibration, such as the design of machines, foundations, structures, engines, turbines, and control systems.

Most prime movers have vibrational problems due to the inherent unbalance in the engines. The unbalance may be due to faulty design or poor manufacture. Imbalance in diesel engines, for example, can cause ground waves sufficiently powerful to create a nuisance in urban areas. The wheels of some locomotives can rise more than a centimeter off the track at high speeds due to imbalance. In turbines, vibrations cause spectacular mechanical failures.

Whenever the natural frequency of vibration of a machine or structure coincides with the frequency of the external excitation, there occurs a phenomenon known as resonance, which leads to excessive deflections and failure. In many engineering systems, a human being acts as an integral part of the system. The transmission of vibration to human beings results in discomfort and loss of efficiency. The vibration and noise generated by engines causes annoyance to people and, sometimes, damage to property. Vibration of instrument panels can cause their malfunction or difficulty in reading the meters. Thus one of the important purposes of vibration study is to reduce vibration through proper design of machines and their mountings.

Fig. Vibration testing of the space shuttle Enterprise.

In spite of its detrimental effects, vibration can be utilized profitably in several consumer and industrial applications. In fact, the applications of vibratory equipment have increased considerably in recent years. For example, vibration is put to work in vibratory conveyors, hoppers, sieves, compactors, washing machines, electric toothbrushes, dentist's drills, clocks, and electric massaging units. Vibration is also used in pile driving, vibratory testing of materials, vibratory finishing processes, and electronic circuits to filter out the unwanted frequencies. Vibration has been found to improve the efficiency of certain machining, casting, forging, and welding processes. It is employed to simulate earthquakes for geological research and also to conduct studies in the design of nuclear reactors.

Basic Concepts of Vibration:

Vibration: Any motion that repeats itself after an interval of time is called vibration or oscillation. The swinging of a pendulum and the motion of a plucked string are typical examples of vibration. The theory of vibration deals with the study of oscillatory motions of bodies and the forces associated with them.

Elementary Parts of Vibrating Systems: A vibratory system, in general, includes a means for storing potential energy (spring or elasticity), a means for storing kinetic energy (mass or inertia), and a means by which energy is gradually lost (damper).

The vibration of a system involves the transfer of its potential energy to kinetic energy and of kinetic energy to potential energy, alternately. If the system is damped, some energy is dissipated in each cycle

of vibration and must be replaced by an external source if a state of steady vibration is to be maintained.

Example:

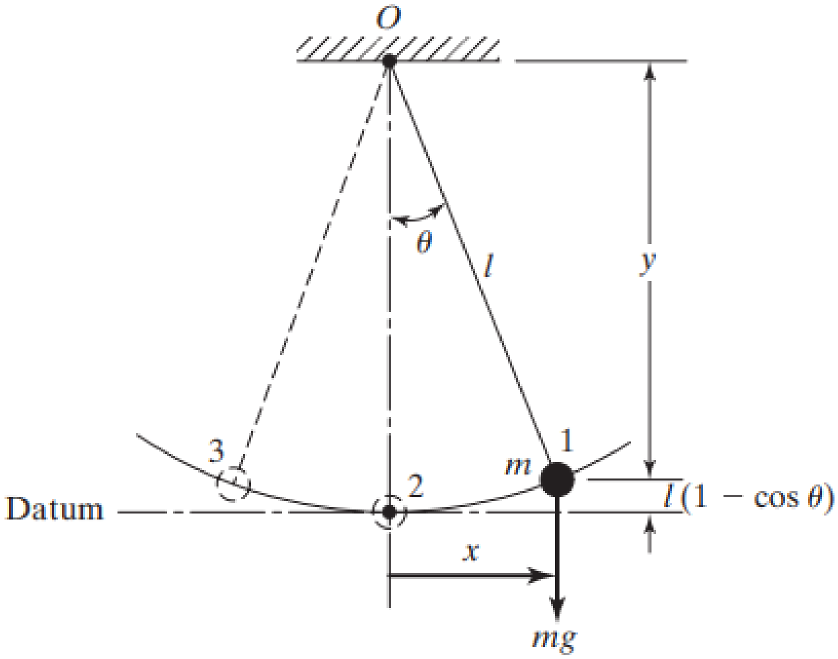


Fig. A simple pendulum.

As an example, consider the vibration of the simple pendulum shown in Fig. Let the bob of mass m be released after being given an angular displacement θ. At position 1 the velocity of the bob and hence its kinetic energy is zero. But it has a potential energy of magnitude mgl(1-cosθ) with respect to the datum position 2. Since the gravitational force mg induces a torque mglsinθ about the point O, the bob starts swinging to the left from position 1. This gives the bob certain angular acceleration in the clockwise direction, and by the time it reaches position 2, all of its potential energy will be converted into kinetic energy. Hence the bob will not stop in position 2 but will continue to swing to position 3. However, as it passes the mean position 2, a counterclockwise torque due to gravity starts acting on the bob and causes the bob to decelerate. The velocity of the bob reduces to zero at the left extreme position. By this time, all the kinetic energy of the bob will be converted to potential energy. Again due to the gravity torque, the bob continues to attain a counterclockwise velocity. Hence the bob starts swinging back with progressively increasing velocity and passes the mean position again. This process keeps repeating, and the pendulum will have oscillatory motion. However, in practice, the magnitude of oscillation gradually decreases and the pendulum ultimately stops due to the resistance (damping) offered by the surrounding medium (air). This means that some energy is dissipated in each cycle of vibration due to damping by the air.

Number of Degrees of Freedom:

The minimum number of independent coordinates required to determine completely the positions of all parts of a system at any instant of time defines the number of degrees of freedom of the system.

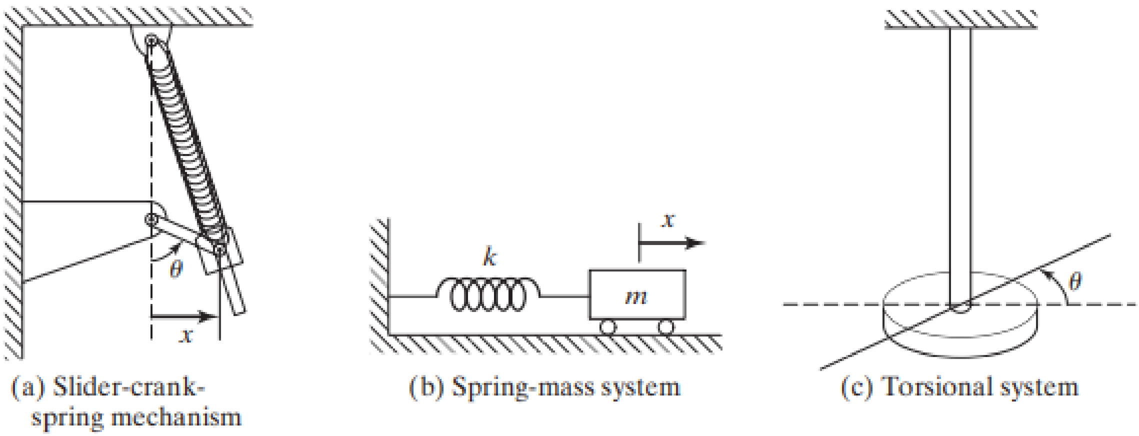


Fig. Single-degree-of-freedom systems.

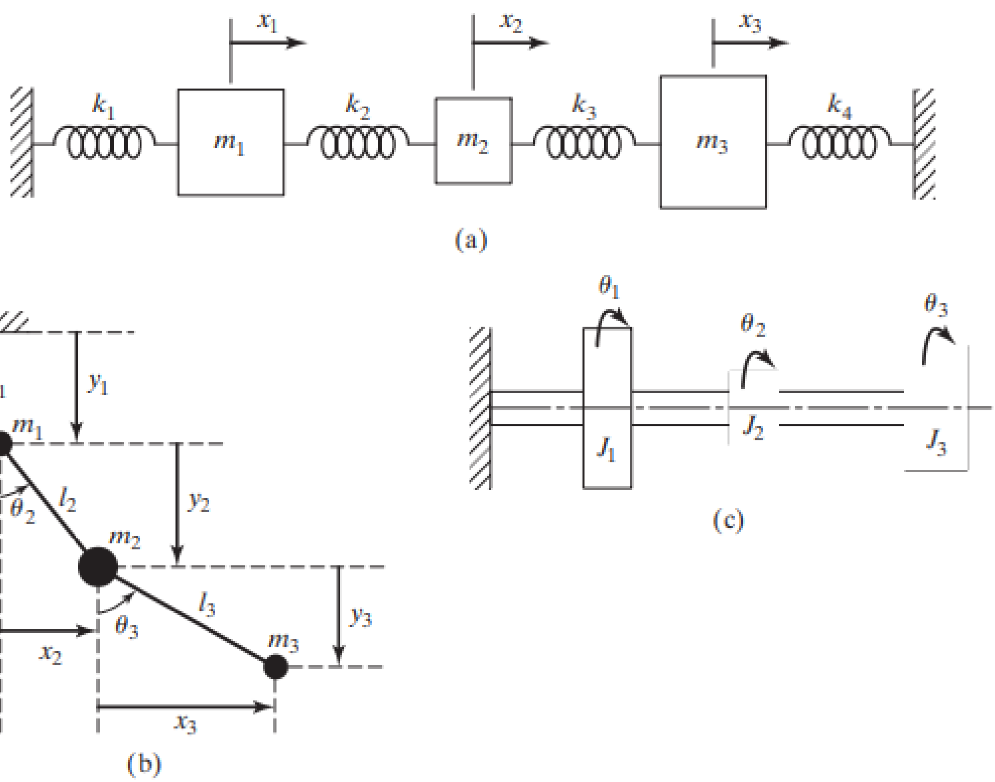


Fig. Three degree-of-freedom systems.

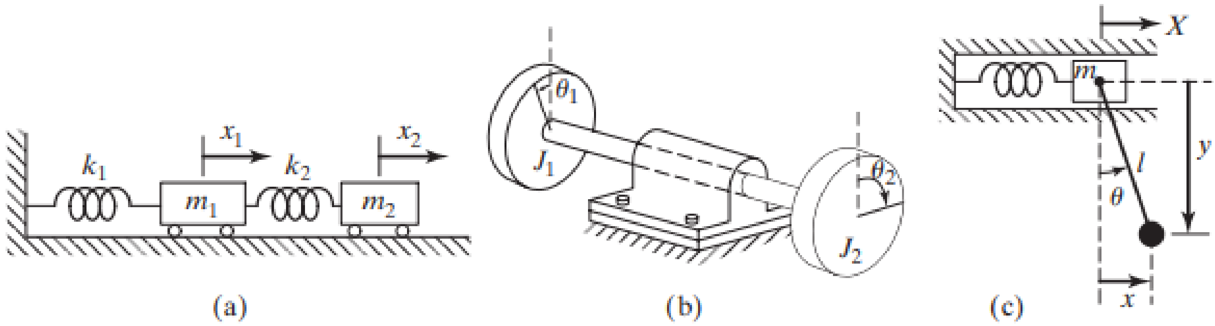


Fig. Two-degree-of-freedom systems.

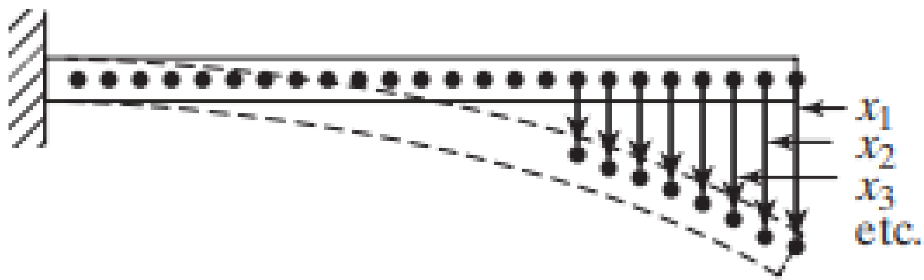


Fig. A cantilever beam (an infinite-number-of-degrees-of-freedom system).

A large number of practical systems can be described using a finite number of degrees of freedom, such as the simple systems shown in Figs. 1.3 to 1.5. Some systems, especially those involving continuous elastic members, have an infinite number of degrees of freedom. As a simple example, consider the cantilever beam shown in Fig. 1.6. Since the beam has an infinite number of mass points, we need an infinite number of coordinates to specify its deflected configuration. The infinite number of coordinates defines its elastic deflection curve. Thus the cantilever beam has an infinite number of degrees of freedom. Most structural and machine systems have deformable (elastic) members and therefore have an infinite number of degrees of freedom.

Discrete and Continuous Systems:

Systems with a finite number of degrees of freedom are called discrete or lumped parameter systems, and those with an infinite number of degrees of freedom are called continuous or distributed systems.

Classification of Vibration:

Vibration can be classified in several ways. Some of the important classifications are as follows.

1. **Free and Forced Vibration:**

Free Vibration: If a system, after an initial disturbance, is left to vibrate on its own, the ensuing vibration is known as free vibration. No external force acts on the system. The oscillation of a simple pendulum is an example of free vibration.

Forced Vibration: If a system is subjected to an external force (often, a repeating type of force), the resulting vibration is known as forced vibration. The oscillation that arises in machines such as diesel engines is an example of forced vibration.

If the frequency of the external force coincides with one of the natural frequencies of the system, a condition known as resonance occurs, and the system undergoes dangerously large oscillations. Failures of such structures as buildings, bridges, turbines, and airplane wings have been associated with the occurrence of resonance.

2. **Undamped and Damped Vibration:**

If no energy is lost or dissipated in friction or other resistance during oscillation, the vibration is known as undamped vibration. If any energy is lost in this way, however, it is called damped vibration.

In many physical systems, the amount of damping is so small that it can be disregarded for most engineering purposes. However, consideration of damping becomes extremely important in analyzing vibratory systems near resonance.

3. **Linear and Nonlinear Vibration:**

If all the basic components of a vibratory system the spring, the mass, and the damper behave linearly, the resulting vibration is known as linear vibration. If, however, any of the basic components behave nonlinearly, the vibration is called nonlinear vibration.

The differential equations that govern the behavior of linear and nonlinear vibratory systems are linear and nonlinear, respectively. If the vibration is linear, the principle of superposition holds, and the mathematical techniques of analysis are well developed. For nonlinear vibration, the superposition principle is not valid, and techniques of analysis are less well known. Since all vibratory systems tend to behave nonlinearly with increasing amplitude of oscillation, knowledge of nonlinear vibration is desirable in dealing with practical vibratory systems.

4. Deterministic and Random Vibration:

If the value or magnitude of the excitation (force or motion) acting on a vibratory system is known at any given time, the excitation is called deterministic. The resulting vibration is known as deterministic vibration.

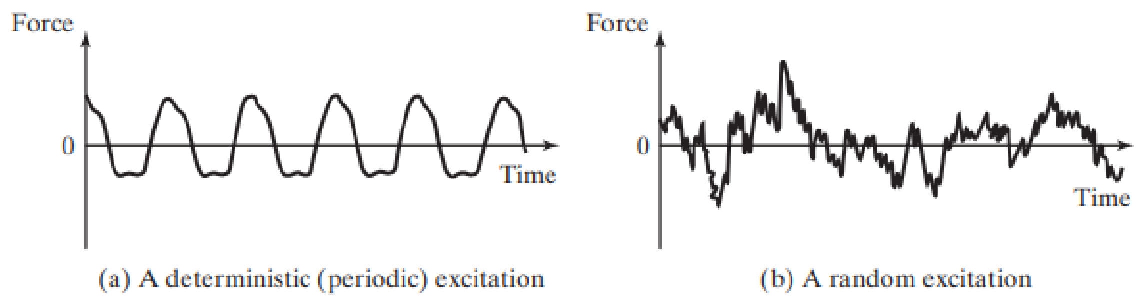


Fig. Deterministic and random excitations.

In some cases, the excitation is nondeterministic or random; the value of the excitation at a given time cannot be predicted. In these cases, a large collection of records of the excitation may exhibit some statistical regularity. It is possible to estimate averages such as the mean and mean square values of the excitation. Examples of random excitations are wind velocity, road roughness, and ground motion during earthquakes. If the excitation is random, the resulting vibration is called random vibration. In this case the vibratory response of the system is also random; it can be described only in terms of statistical quantities. Figure 1.7 shows examples of deterministic and random excitations.

Vibration Analysis Procedure:

A vibratory system is a dynamic one for which the variables such as the excitations (inputs) and responses (outputs) are time dependent. The response of a vibrating system generally depends on the initial conditions as well as the external excitations. Most practical vibrating systems are very complex, and it is impossible to consider all the details for a mathematical analysis. Only the most important features are considered in the analysis to predict the behavior of the system under specified input conditions. Often the overall behavior of the system can be determined by considering even a simple model of the complex physical system. Thus the analysis of a vibrating system usually involves mathematical modeling, derivation of the governing equations, solution of the equations, and interpretation of the results.

Step 1: Mathematical Modeling. The purpose of mathematical modeling is to represent all the important features of the system for the purpose of deriving the mathematical (or analytical) equations governing the system's behavior. The mathematical model should include enough details to allow describing the system in terms of equations without making it too complex. The mathematical model may be linear or nonlinear, depending on the behavior of the system s components. Linear models permit quick solutions and are simple to handle; however, nonlinear models sometimes reveal certain characteristics of the system that cannot be predicted using linear models. Thus a great deal of engineering judgment is needed to come up with a suitable mathematical model of a vibrating system.

To illustrate the procedure of refinement used in mathematical modeling, consider the forging hammer shown in Fig. 1.8.

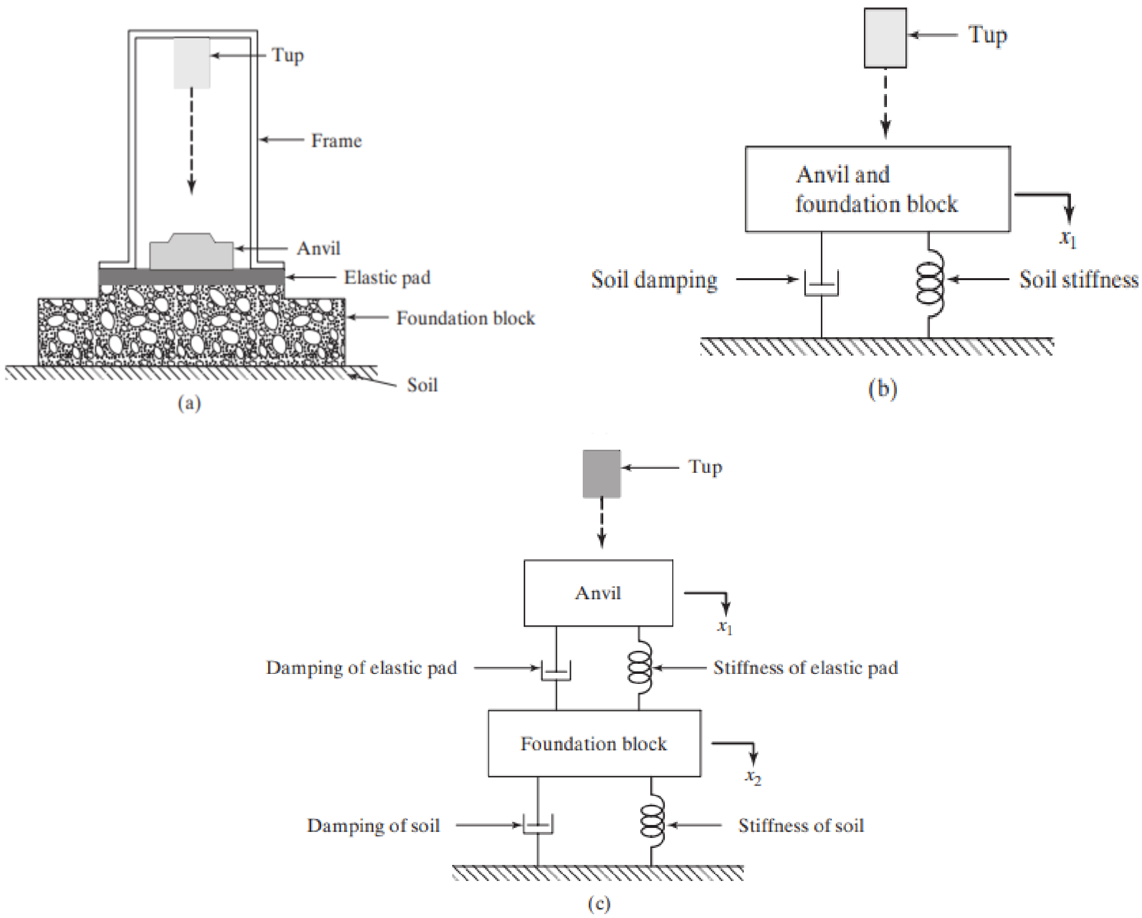


Fig. Modeling of a forging hammer.

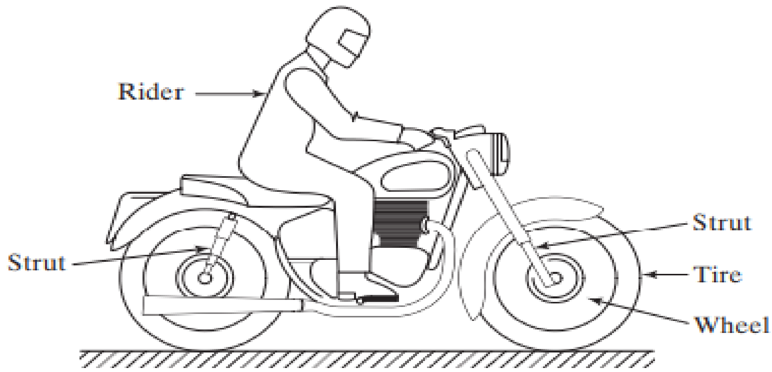
In Fig. (a). It consists of a frame, a falling weight known as the tup, an anvil, and a foundation block. The anvil is a massive steel block on which material is forged into desired shape by the repeated blows of the tup. The anvil is usually mounted on an elastic pad to reduce the transmission of vibration to the foundation block and the frame. For a first approximation, the frame, anvil, elastic pad, foundation block, and soil are modeled as a single degree of freedom system as shown in Fig. (b). For a refined approximation, the weights of the frame and anvil and the foundation block are represented separately with a two-degree-of-freedom model as shown in Fig. (c). Further refinement of the model can be made by considering eccentric impacts of the tup, which cause each of the masses shown in Fig. (c) to have both vertical and rocking (rotation) motions in the plane of the paper.

Step 2: Derivation of Governing Equations. Once the mathematical model is available, we use the principles of dynamics and derive the equations that describe the vibration of the system. The equations of motion can be derived conveniently by drawing the free-body diagrams of all the masses involved. The free-body diagram of a mass can be obtained by isolating the mass and indicating all externally applied forces, the reactive forces, and the inertia forces. The equations of motion of a vibrating system are usually in the form of a set of ordinary differential equations for a discrete system and partial differential equations for a continuous system. The equations may be linear or nonlinear, depending on the behavior of the components of the system. Several approaches are commonly used to derive the governing equations. Among them are Newton's second law of motion, D'Alembert's principle, and the principle of conservation of energy.

Step 3: Solution of the Governing Equations. The equations of motion must be solved to find the response of the vibrating system. Depending on the nature of the problem, we can use one of the following techniques for finding the solution: standard methods of solving differential equations, Laplace transform methods, matrix methods, and numerical methods. If the governing equations are nonlinear, they can seldom be solved in closed form. Furthermore, the solution of partial differential equations is far more involved than that of ordinary differential equations. Numerical methods involving computers can be used to solve the equations. However, it will be difficult to draw general conclusions about the behavior of the system using computer results.

Step 4: Interpretation of the Results. The solution of the governing equations gives the displacements, velocities, and accelerations of the various masses of the system. These results must be interpreted with a clear view of the purpose of the analysis and the possible design implications of the results.

EXAMPLE 1. Mathematical Model of a Motorcycle



(a)

Figure (a) shows a motorcycle with a rider. Develop a sequence of three mathematical models of the system for investigating vibration in the vertical direction. Consider the elasticity of the tires, elasticity and damping of the struts (in the vertical direction), masses of the wheels, and elasticity, damping, and mass of the rider.

Solution:

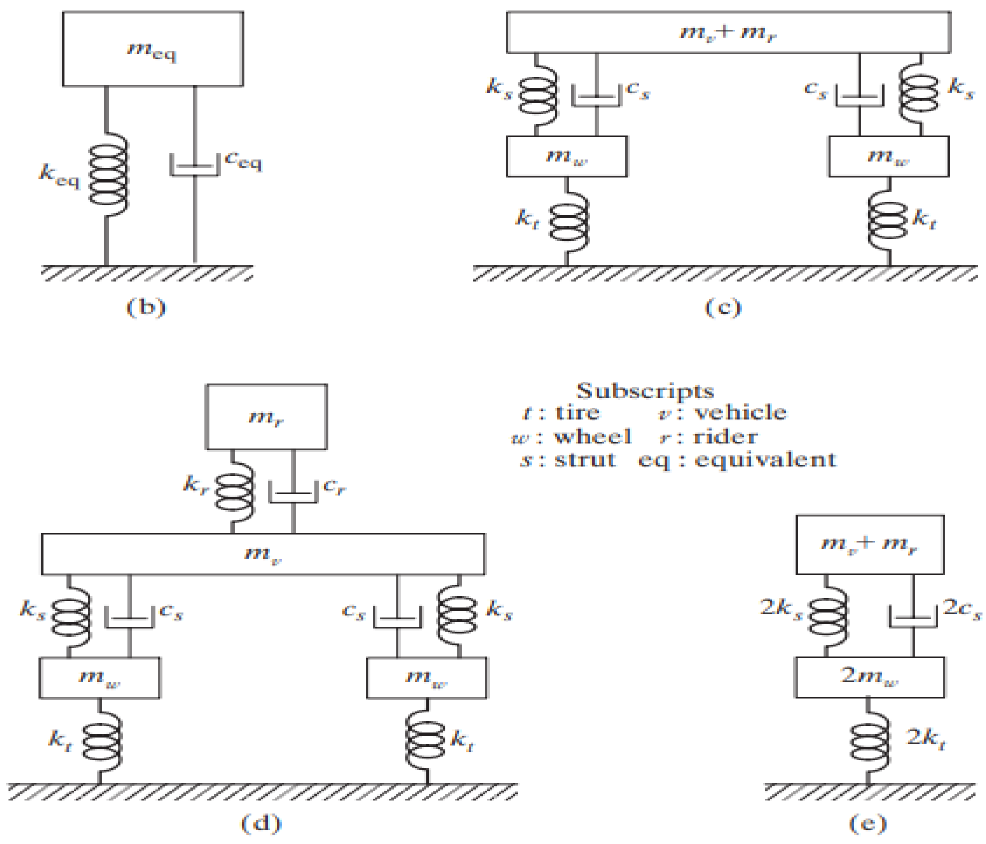


Fig. Motorcycle with a rider a physical system and mathematical model.

Source from "Structural Dynamics in Aeronautical Engineering" by Bismarck-Nasr, M.N.

We start with the simplest model and refine it gradually. When the equivalent values of the mass, stiffness, and damping of the system are used, we obtain a single-degree-of-freedom model of the motorcycle with a rider as indicated in Fig. (b).

In this model, the equivalent stiffness (keq) includes the stiffnesses of the tires, struts, and rider. The equivalent damping constant (Ceq) includes the damping of the struts and the rider. The equivalent mass includes the masses of the wheels, vehicle body, and the rider. This model can be refined by representing the masses of wheels, elasticity of the tires, and elasticity and damping of the struts separately, as shown in Fig (c). In this model, the mass of the vehicle body (mv) and the mass of the rider (mr) are shown as a single mass, mv + mr. When the elasticity (as spring constant kr) and damping (as damping constant Cr) of the rider are considered, the refined model shown in Fig.(d) can be obtained.
Note that the models shown in Figs.(b) to (d) are not unique. For example, by combining the spring constants of both tires, the masses of both wheels, and the spring and damping constants of both struts as single quantities, the model shown in Fig. (e) can be obtained instead of Fig. (c).

Equation of Motion:

Consider the single-degree-of-freedom mechanical system shown in Fig. The system consists of a concentrated mass m (kg), a spring with a spring constant k (N-m), and a dashpot having a viscous damping coefficient c (N-s/m). The external applied load is F(t)(N) and the displacement x(t)(m) is measured from the position of equilibrium.

The potential energy stored at any instance of time t, measured from the position of equilibrium, can be written as

$$U = \int_0^x kx\,\mathrm{d}x = \frac{1}{2}kx^2$$

The kinetic energy of the mass m reads

$$T = \tfrac{1}{2}mx'^2$$

Applying Lagrange's equation of motion,

$$[\mathrm{d}L/\mathrm{d}x']' - \mathrm{d}L/\mathrm{d}x + \mathrm{d}D/\mathrm{d}x' = Q$$

Where L = T — U and Q is the generalized force corresponding to the degree of freedom x, we obtain

Source from “Structural Dynamics in Aeronautical Engineering” by Bismarck-Nasr, M.N.

$$mx'' + cx' + kx = F(t)$$

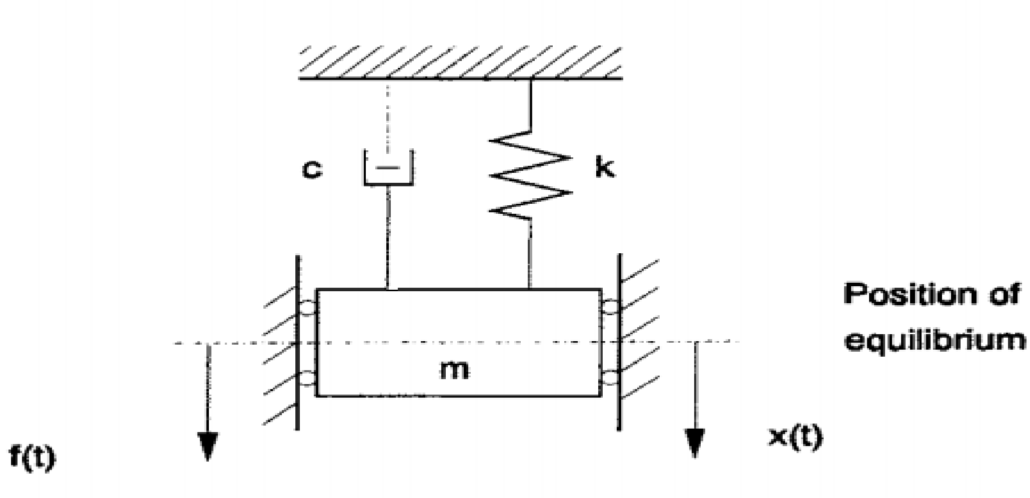


Fig. Single-degree-of-freedom mechanical systems.

Free Vibration:

We consider first the response of the system because of initial conditions x(0) and x'(0) in free vibration, i.e., F(t) = 0. The equation of motion reads

$$mx'' + cx' + kx = 0$$

is a homogeneous differential equation that admits solutions in the form

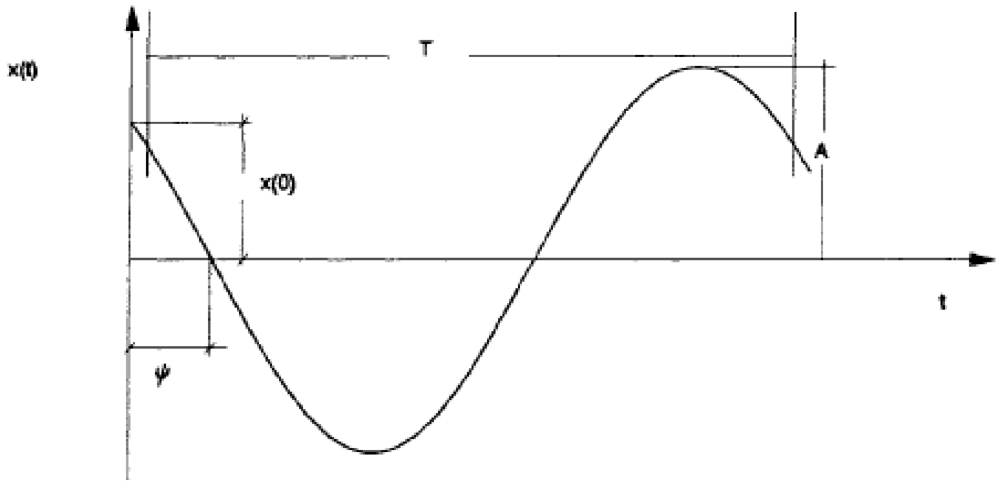
$$x = x_0 e^{pt}$$

Where x0 is an arbitrary constant to be determined from the initial conditions and p is a parameter that depends on the system properties. Substituting the solution into the equation of motion, we obtain the system characteristic equation

$$p^2 + (c/m)p + (k/m) = 0$$

and has solutions p1 and P2, given by

$$p = -(c/2m) \pm [(c/2m)^2 - (k/m)]^{\frac{1}{2}}$$



Free vibration of an undamped single-degree-of-freedom system: $T = 2\pi/\omega_n = 1/f_n$, $A = [x^2(0) + x'^2(0)/\omega_n^2]^{1/2}$, and $\psi = \phi/\omega_n = \{tg^{-1}x'(0)/[\omega_n x(0)]\}/\omega_n$.

Fig. Free vibration of an undamped single-degree-of-freedom system

Response to Harmonic Excitation:

The external force F(t) can be written as

$$F(t) = P_0 \cos \underline{\omega} t$$

the equation of motion

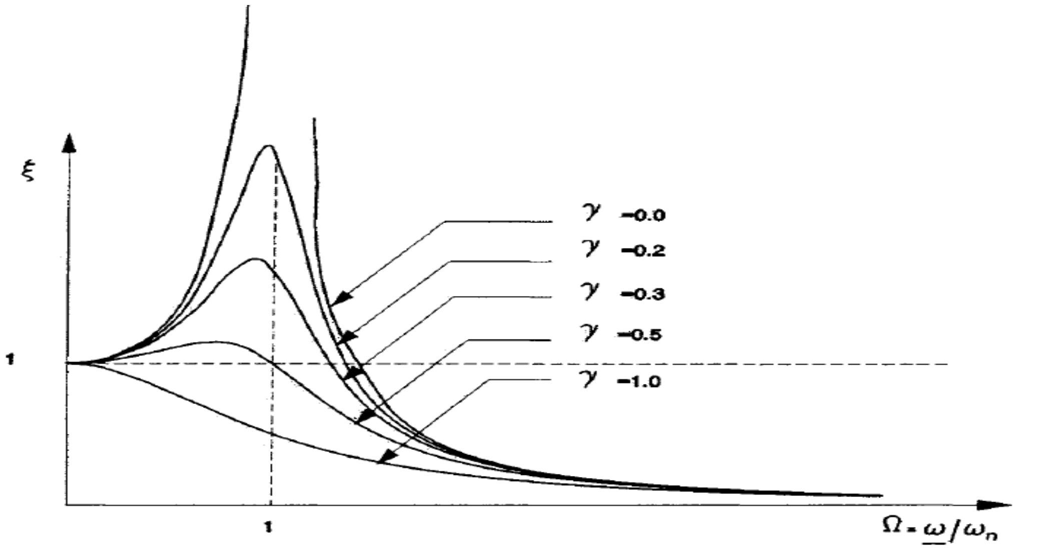
$$x'' + 2\gamma\omega_n x' + \omega_n^2 x = (P_0/m)\cos \underline{\omega} t$$

The solution can be written as

$$x = x_1 + x_2$$

$$x = e^{-\gamma\omega_n t}\left\{[x_0 - A\cos\phi]\cos\omega_d t + \frac{1}{\omega_d}[x_0 + \gamma\omega_n(x_0 - A\cos\phi) - \underline{\omega} A\sin\phi]\sin\omega_d t\right\} + A\cos(\underline{\omega} t - \phi)$$

Source from "Structural Dynamics in Aeronautical Engineering" by Bismarck-Nasr, M.N.



Curves of the dynamic amplification factor $\xi = x_{max}/x_{st}$ vs Ω for different values of γ.

Fig. Curves of the dynamic amplification factor vs Ω for different values of ϒ

Response to an Impulsive Excitation:

A Dirac-delta function or a unit impulse function *8(t — a)* is defined as

$$\delta(t-a) = 0 \qquad \text{for } t \neq a$$

$$\int_{-\infty}^{\infty} \delta(t-a)\,\mathrm{d}t = \int_{a-\varepsilon/2_{\varepsilon\to 0}}^{a+\varepsilon/2} \delta(t-a)\,\mathrm{d}t = 1$$

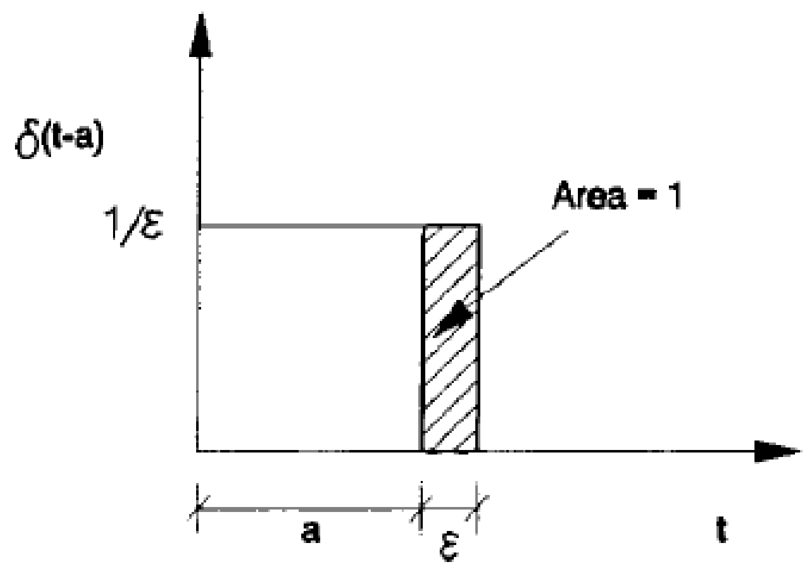


Fig. Dirac-delta function definition.

Source from "Structural Dynamics in Aeronautical Engineering" by Bismarck-Nasr, M.N.

we can write

$$mx'' + cx' + kx = \boldsymbol{F}\delta(t)$$

$$x(t) = e^{-\gamma\omega_n t}\left[\frac{x'(0^+)}{\omega_d}\sin\omega_d t\right] = \frac{\boldsymbol{F}}{m\omega_d}e^{-\gamma\omega_n t}\sin\omega_d t \qquad \text{for } t > 0$$

$$= 0 \qquad \text{for } t \preceq 0$$

Hence, if F = 1, we will have the impulsive response h(t) given by

$$h(t) = \frac{1}{m\omega_d}e^{-\gamma\omega_n t}\sin\omega_d t \qquad \text{for } t > 0$$

$$= 0 \qquad \text{for } t \preceq 0$$

And for a unit impulse applied at t = r, the response reads

$$h(t-\tau) = \frac{1}{m\omega_d}e^{-\gamma\omega_n(t-\tau)}\sin\omega_d(t-\tau) \qquad \text{for } t > \tau$$

$$= 0 \qquad \text{for } t \preceq \tau$$

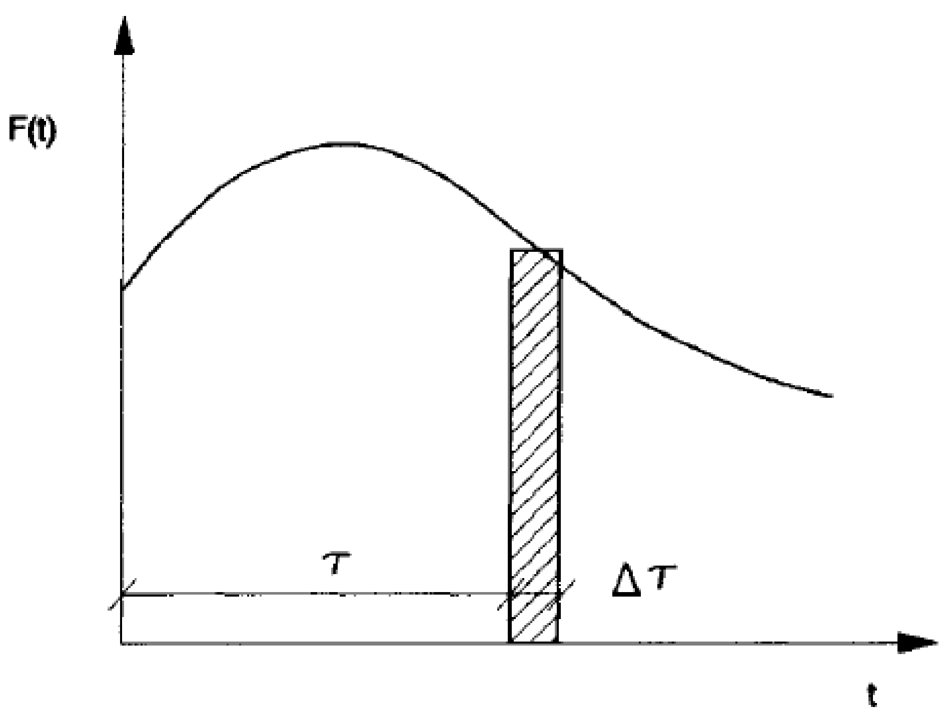


Fig. Deterministic function.

Response to a step excitation:

A unit step function is defined as

$$u(t-\tau)=0 \qquad \text{for } t \preceq \tau$$
$$u(t-\tau)=1 \qquad \text{for } t > \tau$$

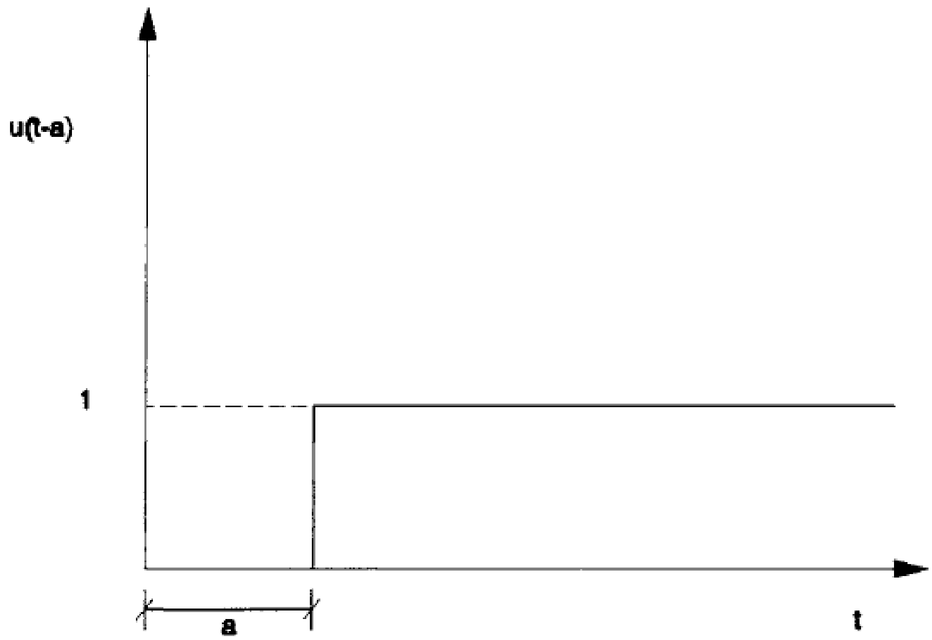


Fig. Definition of a unit step function.

Applying Duhamel's integral for the case of a step function applied at $t = 0$ with null initial conditions, we get

$$g(t)=\int_0^t u(\tau)h(t-\tau)\,\mathrm{d}\tau$$
$$=\frac{1}{m\omega_d}\int_0^t e^{-\gamma\omega_n(t-\tau)}\sin\omega_d(t-\tau)\,\mathrm{d}\tau$$

$$g(t)=\frac{1}{k}\left[1-e^{-\gamma\omega_n t}\left(\cos\omega_d t+\frac{\gamma\omega_n}{\omega_d}\sin\omega_d t\right)\right]$$

Response to periodic excitation (Fourier series):

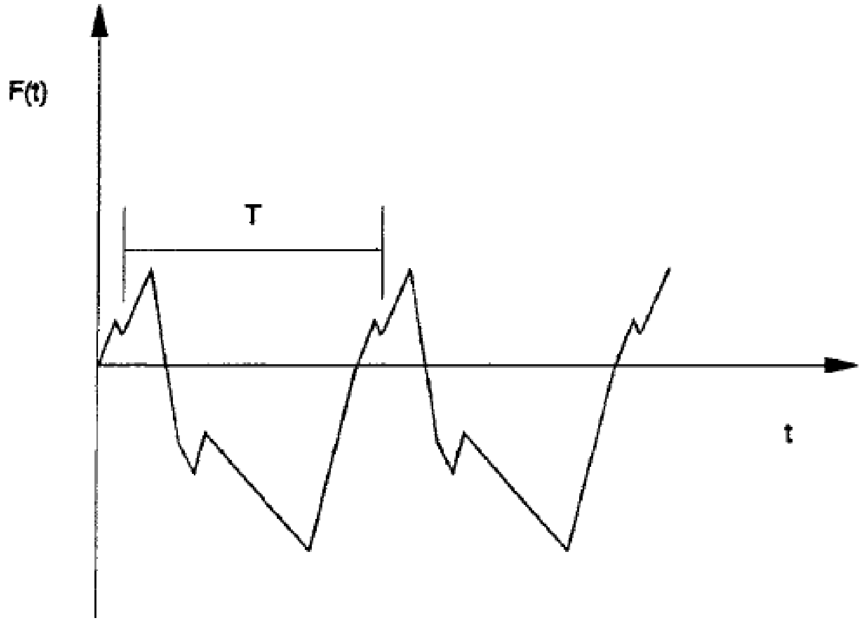


Fig. Periodic function.

Figure represents a periodic external applied load F(t) with a period T. We call 2n/ T the fundamental frequency of excitation and denote it by

$$\omega_0 = 2\pi / T$$

Now, if the function F(t) is periodic and possesses a finite number of discontinuities and if the following relation is satisfied:

$$\int_0^T |F(t)|\,\mathrm{d}t < \infty$$

Then from the theory of Fourier analysis, we can write F(t) as

$$F(t) = \frac{a_0}{T} + \frac{2}{T}\left(\sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right)$$

$$a_0 = \int_0^T F(t)\,\mathrm{d}t \qquad a_n = \int_0^T F(t)\cos n\omega_0 t\,\mathrm{d}t \qquad b_n = \int_0^T F(t)\sin n\omega_0 t\,\mathrm{d}t$$

we can write the permanent solution response as

$$x(t) = \frac{1}{kT}\left(a_0 + 2\sum_{n=1}^{\infty}\left[\left(1-\Omega_n^2\right)^2 + (2\gamma\Omega_n)^2\right]^{-1}\left\{\left[2\gamma\Omega_n a_n + \left(1-\Omega_n^2\right)b_n\right]\right.\right.$$

$$\left.\left.\times \sin n\omega_0 t + \left[\left(1-\Omega_n^2\right)a_n - 2\gamma\Omega_n b_n\right]\cos n\omega_0 t\right\}\right)$$

Response to a periodic excitation (Fourier transform):

Consider again the exponential expansion:

$$F(t) = \sum_{-\infty}^{\infty} c_n e^{in\omega_0 t} \qquad n = \pm 1, \pm 2, \pm 3, \ldots \qquad \omega_0 = 2\pi/T$$

And the series coefficient given by

$$c_n = \frac{1}{T}\int_0^T F(t)e^{-in\omega_0 t}\,\mathrm{d}t \qquad n = \pm 1, \pm 2, \pm 3, \ldots$$

We obtain

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)F(\omega)e^{i\omega t}\,\mathrm{d}\omega$$

Laplace transforms (Transfer Function):

The Laplace transform of a function *x(t)* is defined as

$$\boldsymbol{x}(s) = \boldsymbol{L}[x(t)] = \int_0^{\infty} e^{-st}x(t)\,\mathrm{d}t$$

we can obtain the Laplace transform of the velocity and the acceleration as

$$\begin{aligned} \boldsymbol{x}'(s) = \boldsymbol{L}[x'(t)] &= \int_0^{\infty} e^{-st}\frac{\mathrm{d}x(t)}{\mathrm{d}t}\,\mathrm{d}t \\ &= [e^{-st}x(t)]_0^{\infty} + s\int_0^{\infty} e^{-st}x(t)\,\mathrm{d}t \\ &= s\boldsymbol{x}(s) - x(0) \end{aligned}$$

And

$$\begin{aligned} \boldsymbol{x}''(s) = \boldsymbol{L}[x''(t)] &= \int_0^{\infty} e^{-st}\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}\mathrm{d}t \\ &= \left[e^{-st}\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right]_0^{\infty} + s\int_0^{\infty} e^{-st}\frac{\mathrm{d}x(t)}{\mathrm{d}t}\mathrm{d}t \\ &= s^2\boldsymbol{x}(s) - sx(0) - x'(0) \end{aligned}$$

UNIT - II

MULTI-DEGREE-OF-FREEDOM LINEAR SYSTEMS

Equations of Motion:

Applying Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i'}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial q_i'} = Q_i$$

We obtain the equations of motion of a discrete elastic mechanical system of n degrees of freedom written in matrix form as

$$[M]\{q''\} + [B]\{q'\} + [K]\{q\} = \{Q\}$$

Where {Q} is the column of the generalized external forces.

Free Vibration: The Eigen Value Problem

- Undamped Systems:

Equations of motion for undamped free vibration read

$$[M]\{q''\} + [K]\{q\} = \{0\}$$

The system of equations is a system of second-order differential equations with constant coefficients, whose solution can be written as

$$\{q\} = \{q_0\}e_i^{Pt}$$

$$P_i^2[M]\{q_0\} + [K]\{q_0\} = \{0\}$$

Defining

$$\lambda_i = -P_i^2$$

We get

$$[[K] - \lambda_i[M]]\{q_0\} = \{0\}$$

This represents an eigenvalue problem.

- Damped Systems:

For free vibration of a damped system, the equations of motion read

$$[M]\{q''\} + [C]\{q'\} + [K]\{q\} = \{0\}$$

Source from "Structural Dynamics in Aeronautical Engineering" by Bismarck-Nasr, M.N.

The system of equations admits solutions in the form

$$\{q\} = e^{st}\{q_0\}$$

Where s and {q0} are in general complex. After substitution we obtain

$$s^2[M]\{q_0\} + s[C]\{q_0\} + [K]\{q_0\} = \{0\}$$

This represents an eigenvalue problem of the second order.

Response to an External Applied Load:

For an externally applied load, the equations of motion read

$$[M]\{q''\} + [C]\{q'\} + [K]\{q\} = \{F(t)\}$$

The solution falls into two categories, the modal superposition technique and numerical methods.

- The Modal Superposition Technique:

The modal superposition technique consists of transforming the equations of motion into the modal base of the associated conservative system. The associated conservative system is obtained by the elimination of the damping from the equations of motion. For free vibration, the equations of motion of the associated conservative system read

$$[M]\{q''\} + [K]\{q\} = \{0\}$$

The solution will give the eigenvalue matrix [λ] and the eigenvector matrix [Q]. Making the transformation

$$\{q\} = [Q]\{\eta\}$$

Where {η} is the vector of the modal amplitude, the equations of motion read

$$[M][Q]\{\eta''\} + [C][Q]\{\eta'\} + [K][Q]\{\eta\} = \{F\}$$

$$\lceil \mu \rfloor\{\eta''\} + \lceil \beta \rfloor\{\eta'\} + \lceil \gamma \rfloor\{\eta\} = \{\phi\}$$

The result reads

$$\eta_i(t) = e^{-\omega_i \xi_i t}\left[\frac{1}{\omega_{d_i}}\{\eta_i'(0) + \omega_i \xi_i \eta_i(0)\}\sin\omega_{d_i}t + \eta_i(0)\cos\omega_{d_i}t\right]$$

$$+ \frac{1}{\omega_{d_i}\mu_i}\int_0^t e^{-\omega_i \xi_i (t-\sigma)}\phi_i(\sigma)\sin\omega_{d_i}(t-\sigma)\,\mathrm{d}\sigma \qquad i = 1, 2, \ldots, n$$

- Numerical Methods:

The modal superposition technique described needs the determination of the modal values of the associated conservative system as a first step in the solution procedure, which is a time-consuming process, especially if such information will not be used in further analyses. Numerical methods, on the other hand, work directly on the coupled equations of motion and can be basically described as a step-by-step successive extrapolation procedure.

Damping Effect:

To include a damping effect in the dynamic formulation, we need to consider the work done by the damping forces and include it in Hamilton's principle. Damping forces are difficult, if not impossible, to calculate. However, two types of damping forces have been extensively used and will be treated here, namely viscous damping and structural damping.

- Viscous Damping:

A viscous damping arises when a body is moving in a fluid (e.g., a dashpot); in such a case, we can assume that the damping force is proportional to the velocity, and we write

$$F_D = \gamma q'$$

$$W_D = \int_V \{q\}^T \{F_D\} \, dv$$

The equations of motion of the whole structure read

$$[M]\{q''\} + [C]\{q'\} + [K]\{q\} = \{F\}$$

- Structural Damping:

Structural damping, also known as hysteretic or solid damping, is due to internal friction or friction among components of the system and is proportional to elastic internal forces and acts in the velocity direction. In such cases, if a harmonic motion was assumed for the solution of the problem, we can write the damping force as

$$F_D = igF_E$$

Modeling of continuous systems as multi-degree-of-freedom systems:

Different methods can be used to approximate a continuous system as a multi degree-of-freedom system. A simple method involves replacing the distributed mass or inertia of the system by a finite number of lumped masses or rigid bodies. The lumped masses are assumed to be connected by mass less elastic and damping members. Linear (or angular) coordinates are used to describe the motion of the lumped masses (or rigid bodies). Such models are called lumped-parameter or lumped-mass or discrete-mass systems.

The minimum number of coordinate's necessary to describe the motion of the lumped masses and rigid bodies define the number of degrees of freedom of the system. Naturally, the larger the number of lumped masses used in the model, the higher the accuracy of the resulting analysis.

Using Newton's second law to derive equations of motion:

The following procedure can be adopted to derive the equations of motion of a multi degree of- freedom system using Newton s second law of motion:

1. Set up suitable coordinates to describe the positions of the various point masses and rigid bodies in the system. Assume suitable positive directions for the displacements, velocities, and accelerations of the masses and rigid bodies.

2. Determine the static equilibrium configuration of the system and measure the displacements of the masses and rigid bodies from their respective static equilibrium positions.

3. Draw the free-body diagram of each mass or rigid body in the system. Indicate the spring, damping, and external forces acting on each mass or rigid body when positive displacement and velocity are given to that mass or rigid body.

4. Apply Newton s second law of motion to each mass or rigid body shown by the free body diagram as

$$m_i \ddot{x}_i = \sum_j F_{ij} \text{ (for mass } m_i)$$

or

$$J_i \ddot{\theta}_i = \sum_j M_{ij} \text{ (for rigid body of inertia } J_i)$$

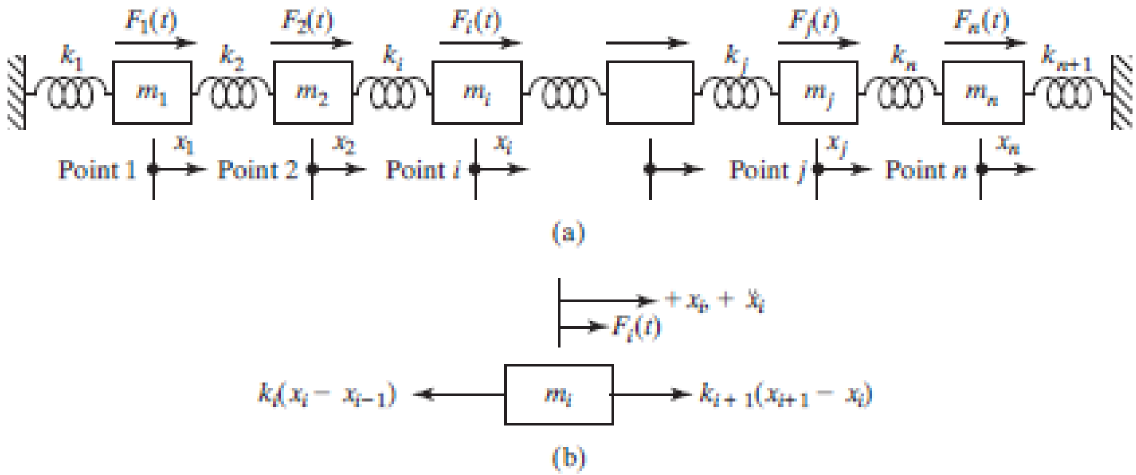
Influence coefficients:

The equations of motion of a multi degree-of-freedom system can also be written in terms of influence coefficients, which are extensively used in structural engineering. Basically, one set of influence coefficients can be associated with each of the matrices involved in the equations of motion. The influence coefficients associated with the stiffness and mass matrices are, respectively, known as the stiffness and inertia influence coefficients. In some cases, it is more convenient to rewrite the equations of motion using the inverse of the stiffness matrix (known as the flexibility matrix) or the inverse of the mass matrix. The influence coefficients corresponding to the inverse stiffness matrix are called the flexibility influence coefficients, and those corresponding to the inverse mass matrix are known as the inverse inertia coefficients.

Stiffness influence coefficients:

For a simple linear spring, the force necessary to cause a unit elongation is called the stiffness of the spring. In more complex systems, we can express the relation between the displacements at a point and the forces acting at various other points of the system by means of stiffness influence coefficients.

$$F_i = \sum_{j=1}^{n} k_{ij} x_j, \qquad i = 1, 2, \ldots, n$$



Multi degree-of-freedom spring-mass system.

The following aspects of stiffness influence coefficients are to be noted:

1. Since the force required at point i to cause a unit deflection at point j and zero deflection at all other points is the same as the force required at point j to cause a unit deflection at point i and zero deflection at all other points.

2. The stiffness influence coefficients can be calculated by applying the principles of statics and solid mechanics.

3. The stiffness influence coefficients for torsional systems can be defined in terms of unit angular displacement and the torque that causes the angular displacement.

Flexibility influence coefficients:

The generation of the flexibility influence coefficients, proves to be simpler and more convenient than stiffness influence coefficients.

$$x_i = \sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} a_{ij} F_j, \quad i = 1, 2, \ldots, n$$

$$\vec{x} = [a]\,\vec{F}$$

$$[a] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The flexibility influence coefficients of a multi degree-of-freedom system can be determined as follows:

1. Assume a unit load at point j. By definition, the displacements of the various points resulting from this load give the flexibility influence coefficients, thus can be found by applying the simple principles of statics and solid mechanics.
2. After completing Step 1 for the procedure is repeated for
3. Instead of applying Steps 1 and 2, the flexibility matrix, [a], can be determined by finding the inverse of the stiffness matrix, [k], if the stiffness matrix is available.

Inertia influence coefficients:

The elements of the mass matrix, mij, are known as the inertia influence coefficients.

$$F_i = \sum_{j=1}^{n} m_{ij} \dot{x}_j$$

In matrix form

$$\vec{F} = [m]\dot{\vec{x}}$$

The velocity and impulse vectors given by

$$\dot{\vec{x}} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{Bmatrix}, \qquad \vec{F} = \begin{Bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ F_n \end{Bmatrix}$$

The mass matrix given by

$$[m] = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix}$$

Potential and kinetic energy expressions in matrix form:

The elastic potential energy (also known as strain energy or energy of deformation) of the ith spring is given by

$$V_i = \frac{1}{2} F_i x_i$$

The total potential energy can be expressed as

$$V = \sum_{i=1}^{n} V_i = \frac{1}{2} \sum_{i=1}^{n} F_i x_i$$

In matrix form as

$$V = \frac{1}{2} \vec{x}^T [k] \vec{x}$$

The stiffness matrix is given by

$$[k] = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & & & \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix}$$

The kinetic energy associated with mass mi is, by definition, equal to

$$T_i = \frac{1}{2} m_i \dot{x}_i^2$$

In matrix form as

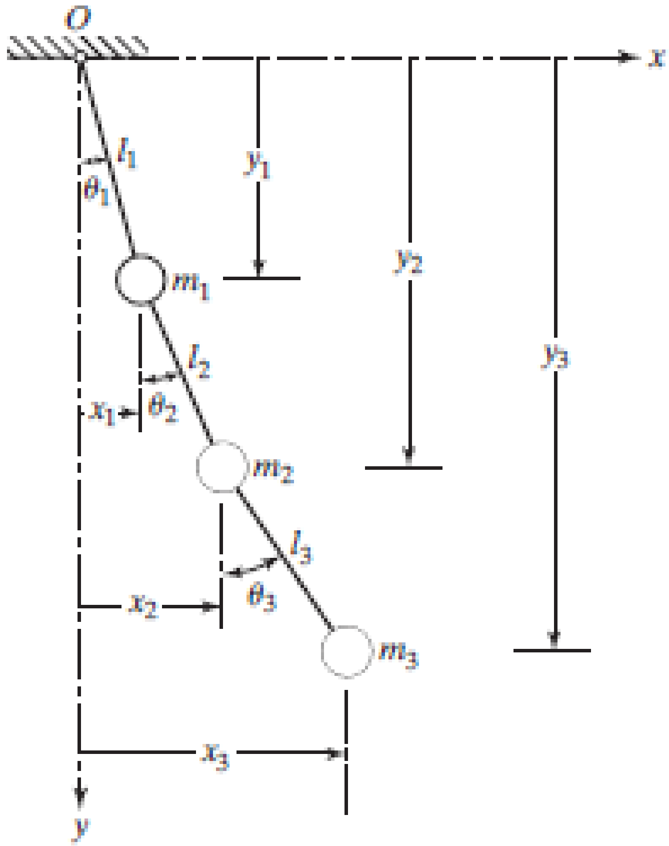
$$T = \frac{1}{2} \dot{\vec{x}}^T [m] \dot{\vec{x}}$$

It can be seen that the potential energy is a quadratic function of the displacements, and the kinetic energy is a quadratic function of the velocities. Hence they are said to be in quadratic form. Since kinetic energy, by definition, cannot be negative and vanishes only when all the velocities vanish, and are called positive definite quadratic forms and the mass matrix [m] is called a positive definite matrix.

Generalized Coordinates and Generalized Forces:

The equations of motion of a vibrating system can be formulated in a number of different coordinate systems. As stated earlier, n independent coordinates are necessary to describe the motion of a system having n degrees of freedom. Any set of n independent coordinates is called generalized coordinates, usually designated by q1, q2, q3…..qn. The generalized coordinates may be lengths, angles, or any other set of numbers that define the configuration of the system at any time uniquely. They are also independent of the conditions of constraint.

The configuration of the system can be specified by the six coordinates



Triple pendulum

(xj, yj), j = 1, 2, 3. However, these coordinates are not independent but are constrained by the relations

$$x_1^2 + y_1^2 = l_1^2$$
$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2$$
$$(x_3 - x_2)^2 + (y_3 - y_2)^2 = l_3^2$$

Lagrange s Equations to Derive Equations of Motion:

The equations of motion of a vibrating system can often be derived in a simple manner in terms of generalized coordinates by the use of Lagrange s equations. Lagrange equations can be stated, for an n-degree-of-freedom system, as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = Q_j^{(n)}, \qquad j = 1, 2, \ldots, n$$

The generalized force can be computed as follows:

$$Q_j^{(n)} = \sum_k \left(F_{xk}\frac{\partial x_k}{\partial q_j} + F_{yk}\frac{\partial y_k}{\partial q_j} + F_{zk}\frac{\partial z_k}{\partial q_j} \right)$$

Thus the equations of motion of the vibrating system can be derived, provided the energy expressions are available.

Equations of Motion of Undamped Systems in Matrix Form:

The equations of motion of a multi degree-of-freedom system in matrix form from Lagrange s equations.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) - \frac{\partial T}{\partial x_i} + \frac{\partial V}{\partial x_i} = F_i, \quad i = 1, 2, \ldots n$$

The kinetic and potential energies of a multi degree-of-freedom system can be expressed in matrix form as

$$T = \frac{1}{2}\dot{\vec{x}}^T[m]\dot{\vec{x}}$$

$$V = \frac{1}{2}\vec{x}^T[k]\vec{x}$$

Where the column vector of the generalized coordinates

$$\vec{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{Bmatrix}$$

From the theory of matrices,

$$\frac{\partial T}{\partial \dot{x}_i} = \frac{1}{2}\vec{\delta}^T[m]\dot{\vec{x}} + \frac{1}{2}\dot{\vec{x}}^T[m]\vec{\delta} = \vec{\delta}^T[m]\dot{\vec{x}}$$

$$= \vec{m}_i^T\dot{\vec{x}}, \quad i = 1, 2, \ldots, n$$

All the relations represented can be expressed as

$$\frac{\partial T}{\partial \dot{x}_i} = \vec{m}_i^T\dot{\vec{x}}$$

Differentiation of Equation with respect to time gives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) = \vec{m}_i^T \ddot{\vec{x}}, \qquad i = 1, 2, \ldots, n$$

So the equations of motion become

$$[m]\ddot{\vec{x}} + [k]\vec{x} = \vec{0}$$

Eigen value Problem:

Assuming a solution of the form

$$x_i(t) = X_i T(t), \qquad i = 1, 2, \ldots, n$$

The configuration of the system, given by the vector

$$\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_n \end{Bmatrix}$$ is known as the mode shape of the system.

$$[m]\vec{X}\ddot{T}(t) + [k]\vec{X}T(t) = \vec{0}$$

From which we can obtain the relations

$$-\frac{\ddot{T}(t)}{T(t)} = \frac{\left(\sum_{j=1}^{n} k_{ij}X_j\right)}{\left(\sum_{j=1}^{n} m_{ij}X_j\right)}, \qquad i = 1, 2, \ldots, n$$

The solution of Equation can be expressed as

$$T(t) = C_1 \cos(\omega t + \phi)$$

Where constants known as the amplitude and the phase angle, respectively.

Solution of the Eigen value Problem:

Equation

$$\left[[k] - \omega^2[m]\right]\vec{X} = \vec{0}$$

Can also be expressed as

$$\left[\lambda[k] - [m]\right]\vec{X} = \vec{0}$$

By premultiplying we obtain

$$\left[\lambda[I] - [D]\right]\vec{X} = \vec{0}$$

Where [I] is the identity matrix and

$$[D] = [k]^{-1}[m]$$

is called the dynamical matrix. The eigenvalue problem is known as the standard eigenvalue problem.

Expansion Theorem:

The eigenvectors, due to their property of orthogonality, are linearly independent. If x is an arbitrary vector in n-dimensional space, it can be expressed as

$$\vec{x} = \sum_{i=1}^{n} c_i \vec{X}^{(i)}$$

The value of the constant Ci can be determined as

$$c_i = \frac{\vec{X}^{(i)T}[m]\,\vec{x}}{\vec{X}^{(i)T}[m]\,\vec{X}^{(i)}} = \frac{\vec{X}^{(i)T}[m]\,\vec{x}}{M_{ii}}, \qquad i = 1, 2, \ldots, n$$

$c_i = \vec{X}^{(i)T}[m]\,\vec{x}, \qquad i = 1, 2, \ldots, n$ is known as the expansion theorem .

It is very useful in finding the response of multi degree-of-freedom systems subjected to arbitrary forcing conditions according to a procedure called modal analysis.

Unrestrained Systems:

Consider the equation of motion for free vibration in normal coordinates:

$$\ddot{q}(t) + \omega^2 q(t) = 0$$

The eigenvalue problem can be expressed as

$$\omega^2[m]\vec{X}^{(0)} = [k]\vec{X}^{(0)}$$

That is,

Source from “Structural Dynamics in Aeronautical Engineering” by Bismarck-Nasr, M.N.

$$k_{11}X_1^{(0)} + k_{12}X_2^{(0)} + \cdots + k_{1n}X_n^{(0)} = 0$$

$$k_{21}X_1^{(0)} + k_{22}X_2^{(0)} + \cdots + k_{2n}X_n^{(0)} = 0$$

.

.

.

$$k_{n1}X_1^{(0)} + k_{n2}X_2^{(0)} + \cdots + k_{nn}X_n^{(0)} = 0$$

If the system under goes rigid-body translation. The potential energy is given by

$$V = \frac{1}{2}\vec{X}^{(0)T}[k]\vec{X}^{(0)}$$

An unrestrained system is also called a semidefinite system.

Free Vibration of Undamped Systems:

The equation of motion for the free vibration of an undamped system can be expressed in matrix form as

$$[m]\ddot{\vec{x}} + [k]\vec{x} = \vec{0}$$

The most general solution can be expressed as a linear combination of all possible solutions given by

$$\vec{x}(t) = \sum_{i=1}^{n} \vec{X}^{(i)} A_i \cos(\omega_i t + \phi_i)$$

If

$$\vec{x}(0) = \begin{Bmatrix} x_1(0) \\ x_2(0) \\ \cdot \\ \cdot \\ \cdot \\ x_n(0) \end{Bmatrix} \quad \text{and} \quad \dot{\vec{x}}(0) = \begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n(0) \end{Bmatrix}$$

Denote the initial displacements and velocities given to the system,

$$\vec{x}(0) = \sum_{i=1}^{n} \vec{X}^{(i)} A_i \cos\phi_i$$

$$\dot{\vec{x}}(0) = -\sum_{i=1}^{n} \vec{X}^{(i)} A_i \omega_i \sin\phi_i$$

It can be solved to find the n values of Ai.

Forced Vibration of Undamped Systems Using Modal Analysis:

When external forces act on a multi degree-of-freedom system, the system undergoes forced vibration. For a system with n coordinates or degrees of freedom, the governing equations of motion are a set of n coupled ordinary differential equations of second order. The solution of these equations becomes more complex when the degree of freedom of the system (n) is large and/or when the forcing functions are nonperiodic.

In such cases, a more convenient method known as modal analysis can be used to solve the problem. In this method, the expansion theorem is used, and the displacements of the masses are expressed as a linear combination of the normal modes of the system. This linear transformation uncouples the equations of motion so that we obtain a set of n uncoupled differential equations of second order. The solution of these equations, which is equivalent to the solution of the equations of n single-degree-of-freedom systems, can be readily obtained.

Modal Analysis:-

The equations of motion of a multi degree-of-freedom system under external forces are given by

$$[m]\ddot{\vec{x}} + [k]\vec{x} = \vec{F}$$

To solve Equation by modal analysis, it is necessary first to solve the eigenvalue problem.

$$\omega^2[m]\vec{X} = [k]\vec{X}$$

the solution vector of Equation can be expressed by a linear combination of the normal modes

$$\vec{x}(t) = q_1(t)\vec{X}^{(1)} + q_2(t)\vec{X}^{(2)} + \cdots + q_n(t)\vec{X}^{(n)}$$

can be rewritten as

$$\vec{x}(t) = [X]\vec{q}(t)$$

Where

$$\vec{q}(t) = \begin{Bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{Bmatrix}$$

The initial generalized displacements and the initial generalized velocities can be obtained from the initial values of the physical displacements and physical velocities as:

$$\vec{q}(0) = [X]^T[m]\vec{x}(0)$$

$$\dot{\vec{q}}(0) = [X]^T[m]\dot{\vec{x}}(0)$$

Where

$$\vec{q}(0) = \begin{Bmatrix} q_1(0) \\ q_2(0) \\ \vdots \\ q_n(0) \end{Bmatrix}, \qquad \vec{x}(0) = \begin{Bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{Bmatrix},$$

$$\dot{\vec{q}}(0) = \begin{Bmatrix} \dot{q}_1(0) \\ \dot{q}_2(0) \\ \vdots \\ \dot{q}_n(0) \end{Bmatrix}, \qquad \dot{\vec{x}}(0) = \begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \vdots \\ \dot{x}_n(0) \end{Bmatrix}$$

Forced Vibration of Viscously Damped Systems:

Modal analysis, applies only to undamped systems. In many cases, the influence of damping upon the response of a vibratory system is minor and can be disregarded. However, it must be considered if the response of the system is required for a relatively long period of time compared to the natural periods of the system. Further, if the frequency of excitation (in the case of a periodic force) is at or near one of the natural frequencies of the system, damping is of primary importance and must be taken into account. In general, since the effects are not known in advance, damping must be considered in the vibration analysis of any system. In this section, we shall consider the equations of motion of a damped multidegree-of-freedom system and their solution using Lagrange s equations. If the system has viscous damping, its motion will be resisted by a force whose magnitude is proportional to that of the velocity but in the opposite direction.

It is convenient to introduce a function R, known as Rayleigh s dissipation function, in deriving the quations of motion by means of Lagrange s equations. This function is defined as

$$R = \frac{1}{2}\dot{\vec{x}}^T[c]\dot{\vec{x}}$$

Where the matrix [c] is called the damping matrix and is positive definite, like the mass and stiffness matrices. Lagrange s equations can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) - \frac{\partial T}{\partial x_i} + \frac{\partial R}{\partial \dot{x}_i} + \frac{\partial V}{\partial x_i} = F_i, \qquad i = 1, 2, \ldots, n$$

The equations of motion of a damped multi degree-of-freedom system in matrix form:

$$[m]\ddot{\vec{x}} + [c]\dot{\vec{x}} + [k]\vec{x} = \vec{F}$$

After substitution, we obtain

$$[m]\ddot{\vec{x}} + [\alpha[m] + \beta[k]]\dot{\vec{x}} + [k]\vec{x} = \vec{F}$$

can be rewritten as

$$[m][X]\ddot{\vec{q}}(t) + [\alpha[m] + \beta[k]][X]\dot{\vec{q}}(t) + [k][X]\vec{q}(t) = \vec{F}(t)$$

The solution can be expressed as

$$q_i(t) = e^{-\zeta_i\omega_i t}\left\{\cos\omega_{di}t + \frac{\zeta_i}{\sqrt{1-\zeta_i^2}}\sin\omega_{di}t\right\}q_i(0) + \left\{\frac{1}{\omega_{di}}e^{-\zeta_i\omega_i t}\sin\omega_{di}t\right\}\dot{q}_0(0) + \frac{1}{\omega_{di}}\int_0^t Q_i(\tau)e^{-\zeta_i\omega_i(t-\tau)}\sin\omega_{di}(t-\tau)\,d\tau,$$

$$i = 1, 2, \ldots, n$$

Where

$$\omega_{di} = \omega_i\sqrt{1-\zeta_i^2}$$

UNIT - III

NONLINEAR AND RANDOM VIBRATION

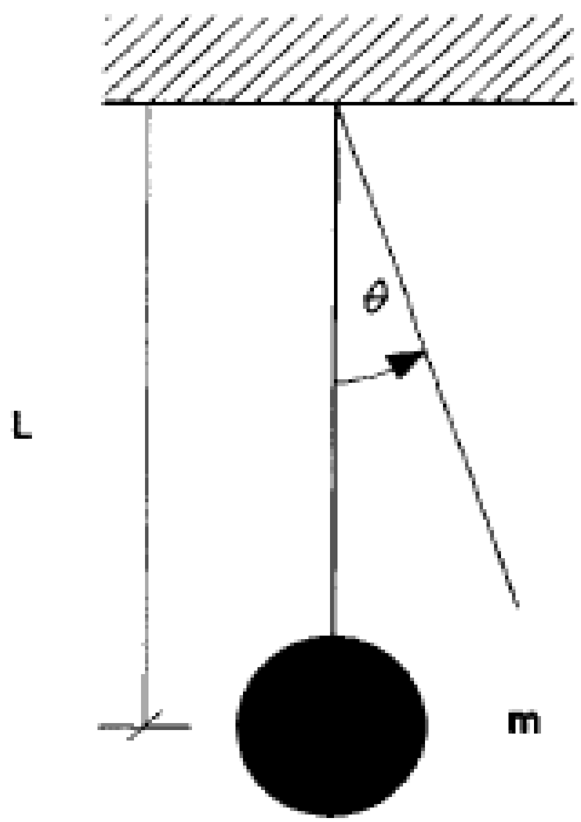
Introduction to Nonlinear Vibrations:

The progress achieved in the past decades in the applied mechanics field is attributed to the representation of complex physical problems by simple mathematical equations. In many applications, these equations are nonlinear. In spite of this fact, simplifications consistent with the physical situation permit, in most cases, a linearization process that simplifies the mathematical solution of the problem while conserving the precision of the physical results. However, in few cases, the linear solutions are not sufficient to describe adequately the problem at hand because new physical phenomena are introduced and can be explained only if nonlinearity is considered.

Simple Examples of Nonlinear Systems:

- Simple Pendulum in Free Vibrations:-

Consider the simple pendulum in free vibrations shown in Figure.



Simple pendulum in free vibrations

The equation of motion of the pendulum can be written as

$$mL^2\theta'' + mgL\sin\theta = 0$$

Can be written as

$$mL^2\theta'' + mgL\,\theta - mgL\frac{\theta^3}{3!} + mgL\frac{\theta^5}{5!} - \cdots = 0$$

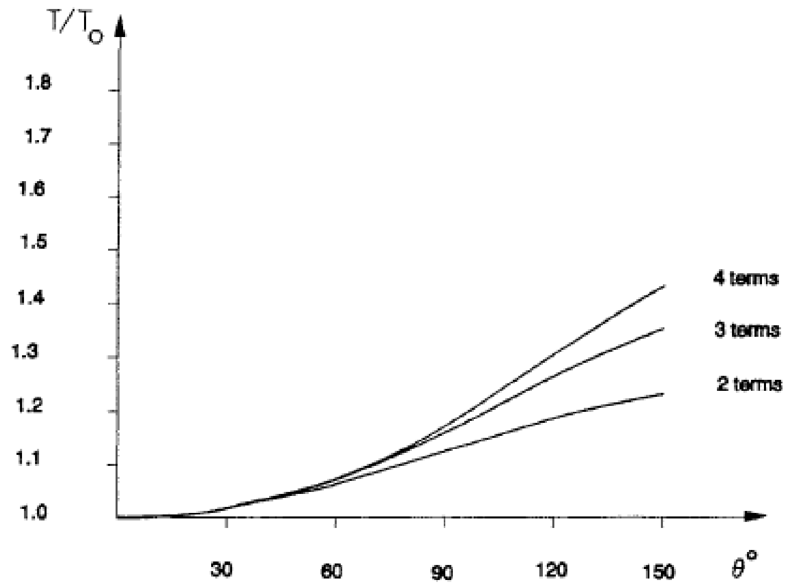
Physical Properties of Nonlinear Systems:

- Undamped Free Vibrations:

Physical considerations reveal that, for a mechanical system with nonlinear stiffness in free vibrations, the period (and thus the frequency) of the response will be a function of the amplitude of vibration. This is expected mathematically since k = k(x) and therefore T = T(x). It is to be emphasized that the natural frequency is a constant and is a property of the mechanical system, despite whether the system is linear. The frequency of response in free vibration of a linear system is constant and is equal to the natural frequency of the system, while a nonlinear system in free vibration responds with a frequency that is a function of the amplitude of vibration. As an example (the proof will be given in the next sections), for the dependence of the period of free vibration on the amplitude of the response, it can be shown that the period of the simple pendulum of Fig. is given by

$$T = T_0\left[1 + \frac{1}{4}\left(\sin\frac{\theta}{2}\right)^2 + \frac{9}{64}\left(\sin\frac{\theta}{2}\right)^4 + \frac{25}{256}\left(\sin\frac{\theta}{2}\right)^6 + \cdots\right]$$

Where TO is the period of the linear system. A plot of T / To vs 0 is shown in Fig.



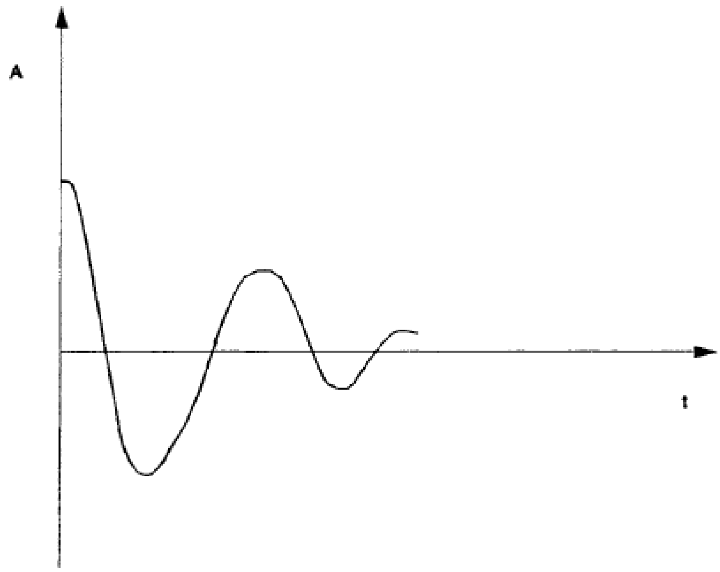
Period of free vibrations of a simple pendulum

- Damped Free Vibrations

Consider a nonlinear damped system having a hard spring nonlinearity characteristic in free vibrations. The system equation of motion can be written as

$$mx'' + cx' + k_0 x + k_1 x^3 = 0$$

With initial conditions different from zero and an initial displacement value in the nonlinear regime, physical considerations and Eq. (5.10) reveal that the response will appear as the curve sketched in Fig. We notice that, for nonlinear amplitude values, we will have smaller periods of response (thus higher frequencies) compared to the linear part. Thus, we expect that the amplitude of the response will begin with a certain value in the nonlinear regime, and the system will oscillate with frequencies higher than the damped natural frequency; with the increase of time, the amplitude of the response will decrease due to the system damping. As a result, we will have an amplitude response oscillating with a decrease in amplitude and frequency values until it reaches the linear amplitude where the system responds with damped amplitude and a constant frequency equal to the system damped natural frequency.



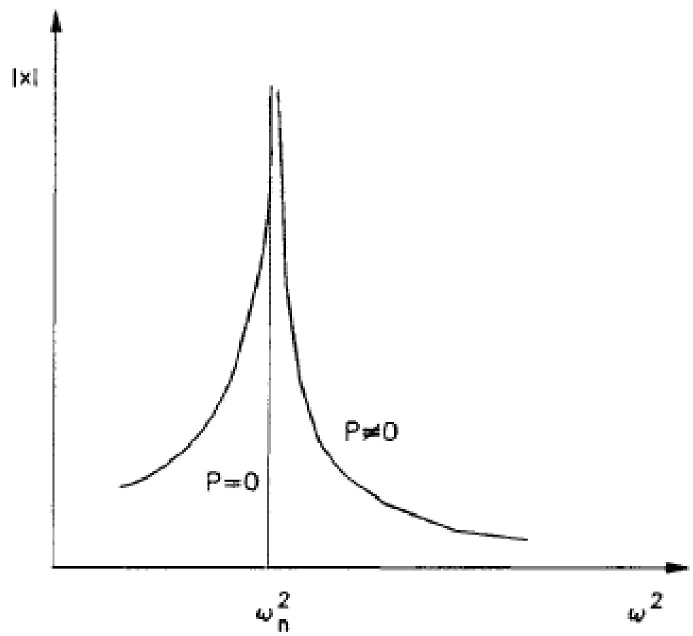
Damped free vibration response of a nonlinear system

- Forced Vibrations

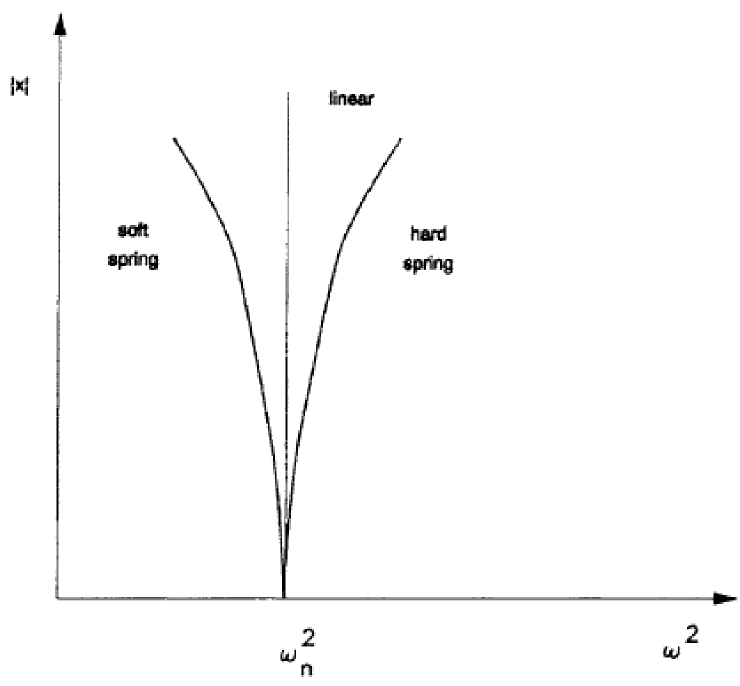
Consider an undamped linear single degree of freedom with a harmonic external excitation. The equation of motion of the system reads

$$x'' + \omega_n^2 x = \frac{P}{m} \cos \omega t$$

The amplitude of the permanent response is sketched in Fig. We notice that for P = 0, i.e., for free vibration, we will have a harmonic response with a frequency of response equal to the undamped natural frequency of the system.



Permanent response amplitude of a linear undamped system due to harmonic external excitation. We expect that the amplitude of the response when plotted against the frequency of excitation will have the form sketched in Fig. for soft and hard springs, respectively.



Free vibration response of linear and nonlinear systems

Solutions of the Equation of Motion of a Single-Degree-of-Freedom Nonlinear System:

- Exact Solutions:

Very few nonlinear differential equations have exact solutions. Exact mathematical solutions of nonlinear systems are studied not only because of their importance for the cases where they exist but also because these exact solutions can be used in the studies of the performance and convergence of nonlinear numerical algorithm solvers that are to be used for the solution of the problems that do not have exact solutions.

- Free vibration:

Consider an undamped single-degree-of- freedom system with stiffness nonlinearity in free vibration. The related equation of motion can be written as

$$x'' + \phi^2 f(x) = 0$$

Can be written as

$$\frac{\mathrm{d}(x')^2}{\mathrm{d}x} + 2\phi^2 f(x) = 0$$

Integrating, we obtain

$$(x')^2 = 2\phi^2 \int_x^X f(\xi)\,\mathrm{d}\xi$$

We now consider the case when f (x) is given by

$$f(x) = x^n + \mu x^m \qquad m > n > 0$$

$$m = 3, 5, 7, \ldots \qquad n = 1, 3, 5, \ldots$$

We obtain

$$T = \frac{4}{\phi\sqrt{X^{n-1}}}\left[\sqrt{\frac{n+1}{2}}\int_0^1 \frac{\mathrm{d}u}{\sqrt{(1+\nu) - (u^{n+1} + \nu u^{m+1})}}\right]$$

Where

$$\nu = \mu X^{m-n}\left\{\frac{n+1}{m+1}\right\}$$

The extension to the case of a higher-order polynomial is straightforward.

- Forced vibration:

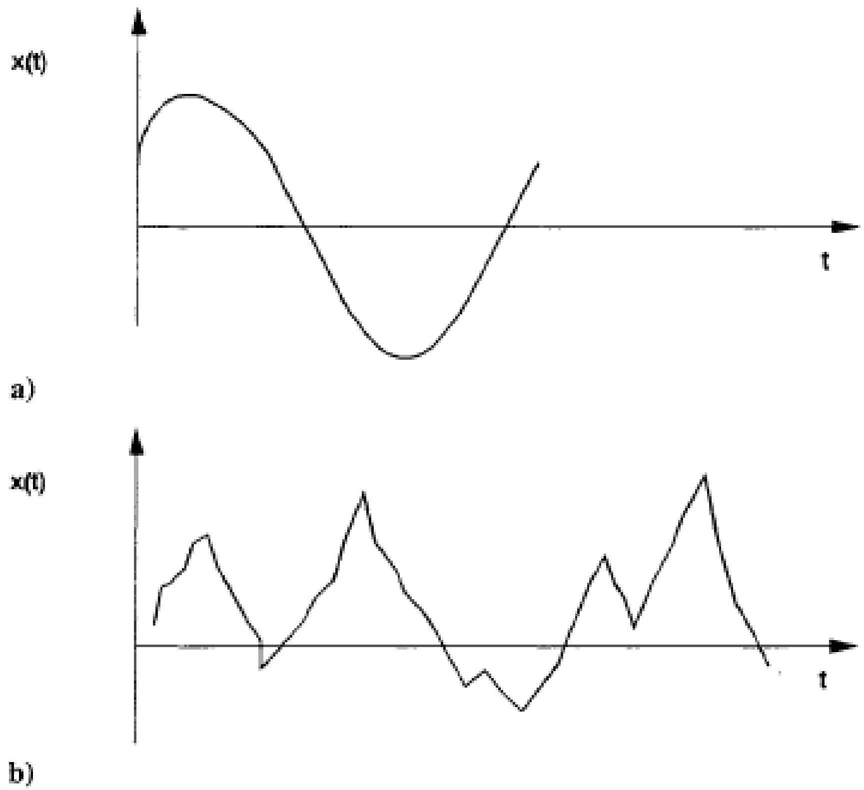
There is no exact solution for the general case of forced vibration of a nonlinear dynamic single-degree-of-freedom system. The solutions are therefore obtained using numerical methods that will be discussed in the next section.

Multi degree-of-Freedom Nonlinear Systems:

The step-by-step numerical integration methods given in previous chapter are directly extended for the analysis of arbitrary nonlinear systems with multiple degrees of freedom. As in the linear case, the time-history response is divided into short, normally equal time increments, and the response is calculated at the end of the time interval for a linearized system having properties determined at the beginning of the interval. The system nonlinear properties are then modified at the end of the interval to conform to the state of deformations and stresses at that time. The mass matrix is usually constant in most practical applications so that its inverse is evaluated once at the beginning of the solution procedure. The stiffness and the damping matrices are modified at the beginning of each step. Therefore, during each step of the nonlinear solution, a triangular decomposition of the equivalent stiffness matrix must be done to obtain the end displacements and velocities. As in the linear case, the acceleration vectors are obtained solving the equations of motion at the beginning of the interval to avoid accumulation of errors during the solution procedure. The modal transformation technique can be used in the solution of the nonlinear system with multiple degrees of freedom; however, in this case, the related matrices are coupled, but the system will have a smaller number of equations compared to the original system written in the physical coordinates. The step-by-step integration procedures are applied to the transformed smaller system of equations.

Introduction to random vibrations:

Consider the record of a measured variable x (t), illustrated in Fig.1, which can represent for instance the displacement of a point in a structure as a function of time. In Fig.la, we can conclude that the variable x(t) is predominantly harmonic, while x(t) of Fig.1b is predominantly irregular. If we repeat the process of measuring and recording the response of the displacement several times and if in all cases we obtain the same responses in both processes, we define such processes as being deterministic processes. Now if, in the process of Fig.la, during the repeated measurements of the records at each time, we obtain a different angle of phase and if, in the process of Fig.1b, the responses are different from each other during the repeated measurements, we call such processes random processes. Random processes are characterized by the fact that their behavior cannot be predicted in advance and therefore can be treated only in a statistical manner. We will begin this chapter by studying random processes and their statistical properties. In the sequence, we will study the response of linear systems due to random excitations.

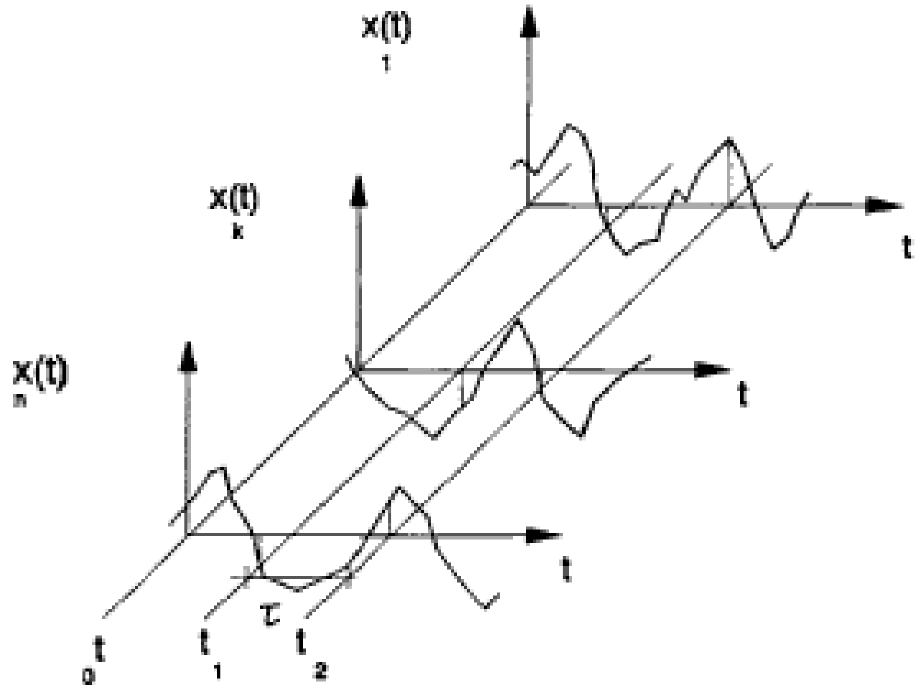


1 Record of a variable as a function of time

Classification of Random Processes:

- Stationary Random Processes:

Consider n records of a random variable as given in Fig.2. We define the complete set of xk(t),k = 1 , 2 , . . . , n as a random process, and each record of the set will be called a sample of the random process. Consider now the values of xk(t) for the instant of time t = t\; we can write the mean value of the random process at that instant of time as



2 Time history of a random process

$$\mu_x(t_1) = \frac{1}{n}\sum_{k=1}^{n} x_k(t_1)$$

For an instant of time t = t^ separated from t\ by an interval of time r, we can write a statistical measurement of the behavior of the mean value in relation to a shift r as a function Rx(t\, t\ + r), given by

$$R_x(t_1, t_1+\tau) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} x_k(t_1)x_k(t_1+\tau)$$

For example, for two shifts, we can write an expression in the form

$$R_x(t_1, t_1+\tau, t_1+\sigma) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} x_k(t_1)x_k(t_1+\tau)x_k(t_1+\sigma)$$

In general, *iix(t\\ Rx(t* , *t* + T), *Rx(t\ ,ti+T,t*+ cr), etc., will be functions of *t* where the mean values have been calculated. Now if in a random process these mean values do not depend on *t*, i.e., *^x(t\) = JJLX* = const and *Rx(t*, *t\ + r) = Rx(r)* and *Rx(t*, *t* + r, *t\ + a) = Rx(r*, a), etc., we call the random process a process that is heavily stationary.

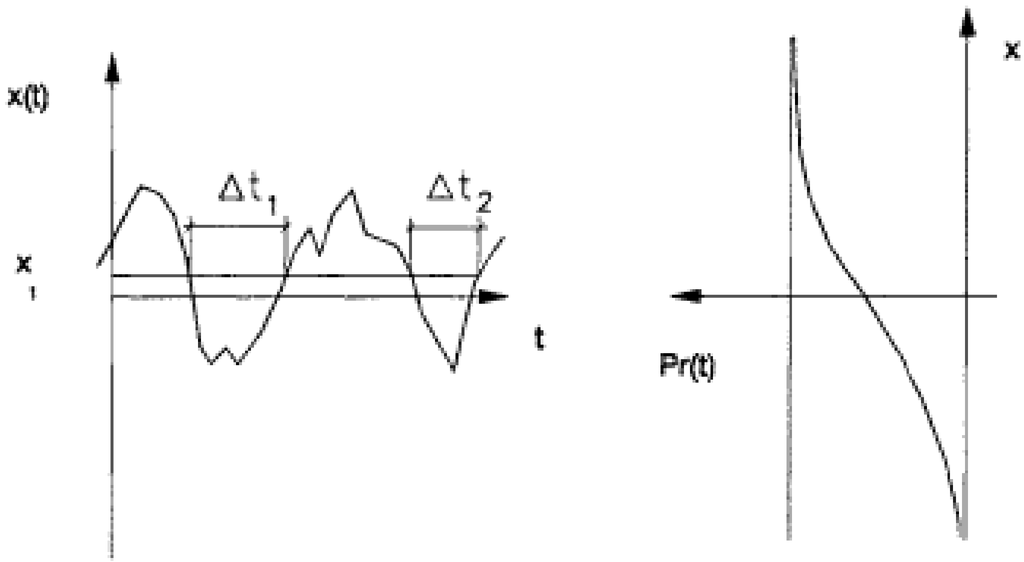
Probability Distribution and Density Functions:

Consider a sample of an ergodic process as shown in Fig. 3. We define the probability distribution function as

$$P(x) = \text{Prob}\,[x(t) < x] = \lim_{T\to\infty} \frac{1}{T} \sum_i \Delta t_i$$

We will define the probability density function as

$$p(x) = \lim_{\Delta x\to 0} \frac{P(x+\Delta x) - P(x)}{\Delta x} = \frac{\mathrm{d}P(x)}{\mathrm{d}x}$$



3 Probability distribution function

We verify the following relations:

$$P(x_1 < x < x_2) = \int_{x_1}^{x_2} p(x)\,\mathrm{d}x$$

$$p(-\infty) = p(\infty) = 0$$

$$P(x) \geq 0$$

$$P(x) = \int_{-\infty}^{x} p(\xi)\,\mathrm{d}\xi$$

$$P(\infty) = \int_{-\infty}^{\infty} p(\xi)\,\mathrm{d}\xi = 1$$

In many statistical applications where the number of samples is very great and none of the samples represents a significant weight in the process, the probability density function can be represented by the so-called Gaussian distribution. The probability density function for the Gaussian or normal distribution reads

$$p(x) = \frac{1}{\sigma_x\sqrt{2\pi}} e^{-\frac{(x-\underline{x})^2}{2\sigma_x^2}}$$

And thus the probability distribution function is given by

$$P(x) = \frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(x-\underline{x})^2}{2\sigma_x^2}} \, \mathrm{d}x$$

Description of the Mean Values in Terms of the Probability Density Function:

Considering a stationary random process *(x(t)}* for a continuous function *g(x)*, we can write the mean value *g(x)* as

$$\underline{g(x)} = \frac{1}{n}\sum_{1}^{n}{}_{n\to\infty} g(x) = \sum_{1}^{n}{}_{n\to\infty} g(x)\frac{1}{n}$$

We note that *{ l / n }* represents the probability of the process to have the value of g(x). Thus, we can write

$$\underline{g(x)} = \sum_{-\infty}^{\infty}{}_{\Delta x\to 0} g(x)p(x)\Delta x = \int_{-\infty}^{\infty} g(x)p(x)\,\mathrm{d}x$$

We call *g(x)* the mean value or the mathematical expectation, and we write

$$\underline{g(x)} = E[g(x)]$$

Thus, we can write for the mean values the following expressions in terms of the probability density function:

1) For the mean value *g(x) = x,*

$$E[x] = \underline{x} = \int_{-\infty}^{\infty} x p(x)\,\mathrm{d}x$$

2) For the mean square value *g(x) = x2,*

$$E[x^2] = \underline{x}^2 = \int_{-\infty}^{\infty} x^2 p(x)\,\mathrm{d}x$$

3) For the variance *g(x) = (x — x)2,*

$$\sigma_x^2 = E[(x-\underline{x})^2] = \int_{-\infty}^{\infty} (x-\underline{x})^2 p(x)\,\mathrm{d}x$$

$$\underline{x}^2 - (\underline{x})^2 = E[x^2] - (E[x])^2$$

Properties of the Autocorrelation Function:

The autocorrelation function for an ergodic process reads

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)\,dt$$

Making the transformation *t* — *r* = X, we get

$$R_x(-\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2-\tau}^{T/2-\tau} x(\lambda+\tau)x(\lambda)\,d\lambda$$

and because the integration is made for *T* -> ∞, we can write

$$R_x(-\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(\lambda+\tau)x(\lambda)\,d\lambda$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t+\tau)x(t)\,dt$$

Hence we conclude that the autocorrelation function is an even function.

Power Spectral Density Function:

Consider the sample *f (t)* of an ergodic process and its autocorrelation function, which can be written as

$$R_f(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t)f(t+\tau)\,dt$$

This implies that the autocorrelation function is the inverse Fourier transform of *Sf(ω),* or

$$R_f(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_f(\omega)e^{i\omega\tau}\,d\omega$$

and we observe the following:

Sf(a>) does not furnish any new information since *Rf(ω)* is its Fourier transform, and thus the information contained in one is the same as the information contained in its transform. However, *Sf(co)* gives us the information in the frequency domain while *Rf(r)* gives us the information in the time domain, and depending on the application, one may be more convenient than the other.

Properties of the Power Spectral Density Function:

1 The Power Spectral Density Function Is a Positive Function

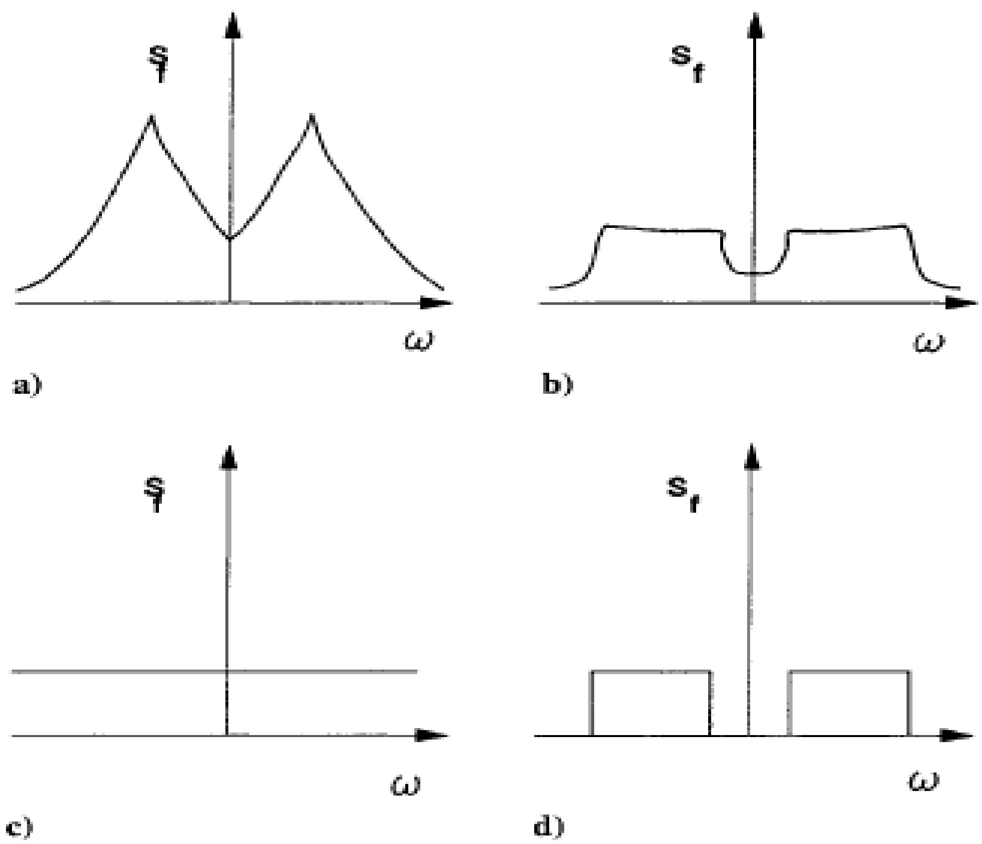
2 The Power Spectral Density Function Is an Even Function

3 Representation of the Power Spectral Density Function in the Positive Domain

White Noise and Narrow and Large Bandwidth:

The power spectral density function provides the necessary information on the frequency decomposition of a random process. Now if the frequency decomposition is concentrated in turns of a peak frequency a)Q as shown in Fig.a, we call such distribution a narrow bandwidth distribution. This is in contrast to the distribution given in Fig.4b, where we have an equal frequency distribution in a large band, and we call such distribution a large bandwidth distribution. Now, if Sf(a>) is a constant for all the frequency decompositions, i.e., from -∞ to ∞ as shown in Fig.4c,

We define such distribution as white noise; this is in comparison with the white light distribution, which has a plain spectral distribution in the large visible band frequency. In many practical cases, processes having distributions as shown in Fig.4d with an equal distribution in a large band of frequency can be considered as white noise distribution for practical purposes.



Narrow, large bandwidth and white noise distributions

Single-Degree-of-Freedom Response:

The response *x(t)* of a linear single-degree-of-freedom system due to an external applied load *f(t)*, whether a deterministic or random excitation, can be written in terms of Duhamel's convolution integral as

$$x(t) = \int_0^t f(t)h(t-\tau)\,\mathrm{d}\tau = \int_0^t f(t-\lambda)h(\lambda)\,\mathrm{d}\lambda$$

Now, for random excitation, we can extend the integration to —∞, and we write

$$x(t) = \int_{-\infty}^t f(t-\lambda)h(\lambda)\,\mathrm{d}\lambda$$

The Fourier transform of the response reads

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\,\mathrm{d}t$$

Considering now a random ergodic excitation *f (t)* to a single-degree-of- freedom mechanical system, we can write the mean value of the response jc as

$$\underline{x} = E[x(t)] = E\int_{-\infty}^{\infty} h(\lambda)f(t-\lambda)\,\mathrm{d}\lambda$$

And, because the system is linear, we can invert the order of the mean and the integration operations to write

$$\underline{x} = E[x(t)] = \int_{-\infty}^{\infty} E[h(\lambda)f(t-\lambda)\,\mathrm{d}\lambda]$$

In the sequel, we will calculate the autocorrelation function of the response to a single degree of freedom due to an ergodic external excitation. Using Eq. we can write

$$x(t) = \int_{-\infty}^{\infty} f(\lambda_1)h(t-\lambda_1)\,\mathrm{d}\lambda_1$$

$$x(t+\tau) = \int_{-\infty}^{\infty} f(\lambda_2)h(t+\tau-\lambda_2)\,\mathrm{d}\lambda_2$$

Using the definition of the power spectral density function and Eq. we can write

$$S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau)e^{-i\omega\tau}\,\mathrm{d}\tau$$

$$= \int\int_{-\infty}^{\infty} e^{-i\omega\tau}[h(\lambda_1)h(\lambda_2)R_f(\tau+\lambda_1-\lambda_2)\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2]\,\mathrm{d}\tau$$

We conclude that

$$S_x(\omega) = |H(\omega)|^2 S_f(\omega)$$

It represents an algebraic relation between three functions, is a very important relation in structural dynamics.

Response to a White Noise:

Consider a single-degree-of-freedom mechanical system subjected to an external random ergodic excitation having a power spectral density function given by a white noise with intensity so- Thus, we can write

$$S_f(\omega) = S_0$$

Now, for a single-degree-of-freedom system, the complex frequency response function H(ω) reads

$$H(\omega) = \frac{1/k}{(1-\Omega^2) + 2i\gamma\Omega}$$

The autocorrelation function of the response can be obtained from the inverse Fourier transform of *Sx(a>)* and reads

$$R_x(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(1/k)^2 S_0 e^{i\omega\tau}\,\mathrm{d}\omega}{[(1-\Omega^2)^2 + 4\gamma^2\Omega^2]}$$
$$= \frac{1}{\pi}\int_{0}^{\infty} \frac{(1/k)^2 S_0 e^{i\omega\tau}\,\mathrm{d}\omega}{[(1-\Omega^2)^2 + 4\gamma^2\Omega^2]} \qquad \text{for } \tau \geq 0$$

Integrating, we obtain

$$R_x(\tau) = \frac{S_0\omega_n e^{-\gamma\omega_n\tau}}{4\gamma k^2}\left[\cos\omega_d\tau + \frac{\gamma}{(1-\gamma^2)^{\frac{1}{2}}}\sin\omega_d\tau\right] \qquad \text{for } \tau \geq 0$$

And the mean square value of the response reads

$$\psi_x^2 = R_x(0) = \frac{S_0\omega_n}{4\gamma k^2}$$

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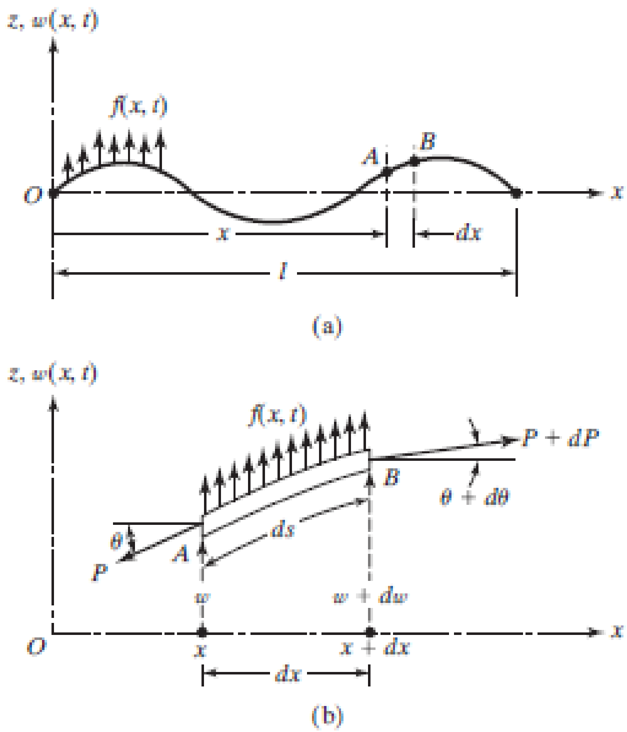
UNIT – IV

DYNAMICS OF CONTINUOUS ELASTIC BODIES

Introduction:

We have so far dealt with discrete systems where mass, damping, and elasticity were assumed to be present only at certain discrete points in the system. In many cases, known as *distributed* or *continuous systems*, it is not possible to identify discrete masses, dampers, or springs. We must then consider the continuous distribution of the mass, damping, and elasticity and assume that each of the infinite number of points of the system can vibrate. This is why a continuous system is also called a *system of infinite degrees of freedom.*

If a system is modeled as a discrete one, the governing equations are ordinary differential equations, which are relatively easy to solve. On the other hand, if the system is modeled as a continuous one, the governing equations are partial differential equations, which are more difficult. However, the information obtained from a discrete model of a system may not be as accurate as that obtained from a continuous model. The choice between the two models must be made carefully, with due consideration of factors such as the purpose of the analysis, the influence of the analysis on design, and the computational time available.



A vibrating string

Transverse Vibration of a String or Cable:

Consider a tightly stretched elastic string or cable of length l subjected to a transverse force f(x, t) per unit length, as shown in Fig.(a). The transverse displacement of the string, w(x, t), is assumed to be small. Equilibrium of the forces in the z direction gives

The net force acting on an element is equal to the inertia force acting on the element, or

$$(P + dP)\sin(\theta + d\theta) + f\,dx - P\sin\theta = \rho\,dx\frac{\partial^2 w}{\partial t^2}$$

For an elemental length dx,

$$dP = \frac{\partial P}{\partial x}dx$$

$$\sin\theta \simeq \tan\theta = \frac{\partial w}{\partial x}$$

And

$$\sin(\theta + d\theta) \simeq \tan(\theta + d\theta) = \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}dx$$

Hence the forced-vibration equation of the non uniform string, Equation, can be simplified to

$$\frac{\partial}{\partial x}\left[P\frac{\partial w(x,t)}{\partial x}\right] + f(x,t) = \rho(x)\frac{\partial^2 w(x,t)}{\partial t^2}$$

If the string is uniform and the tension is constant, Equation reduces to

$$P\frac{\partial^2 w(x,t)}{\partial x^2} + f(x,t) = \rho\frac{\partial^2 w(x,t)}{\partial t^2}$$

We obtain the free-vibration equation

$$P\frac{\partial^2 w(x,t)}{\partial x^2} = \rho\frac{\partial^2 w(x,t)}{\partial t^2}$$

Or

$$c^2\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial t^2}$$

Is also known as the wave equation.

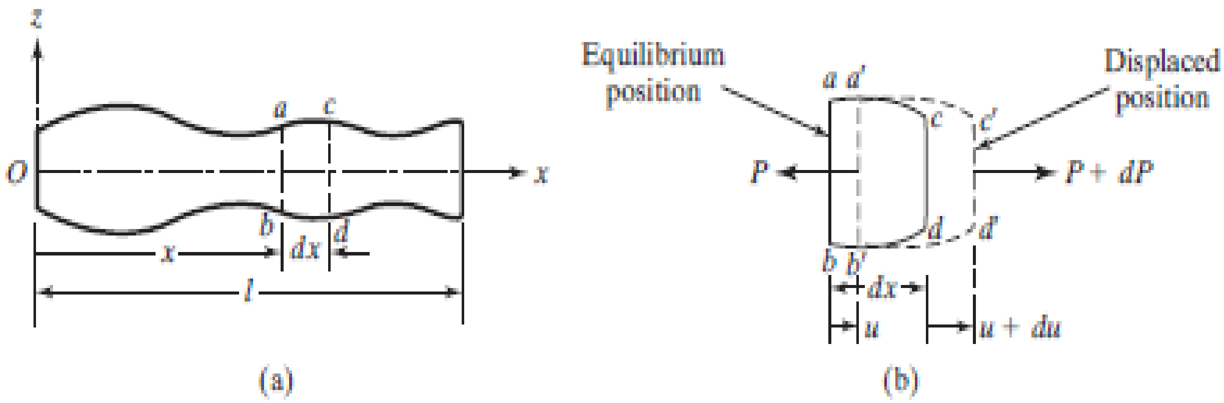
Longitudinal Vibration of a Bar or Rod:

Consider an elastic bar of length l with varying cross-sectional area A(x), The forces acting on the cross sections of a small element of the bar are given by P and P + dP with

$$P = \sigma A = EA\frac{\partial u}{\partial x}$$

Where σ is the axial stress, E is Young s modulus, u is the axial displacement, and du/dx is the axial strain. If f(x, t) denotes the external force per unit length, the summation of the forces in the x direction gives the equation of motion

$$(P + dP) + f\,dx - P = \rho A\,dx\,\frac{\partial^2 u}{\partial t^2}$$



Longitudinal vibration of a bar

The equation of motion for the forced longitudinal vibration of a non uniform bar, Equation, can be expressed as

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u(x,t)}{\partial x}\right] + f(x,t) = \rho(x)A(x)\frac{\partial^2 u}{\partial t^2}(x,t)$$

For a uniform bar, Equation reduces to

$$EA\frac{\partial^2 u}{\partial x^2}(x,t) + f(x,t) = \rho A\frac{\partial^2 u}{\partial t^2}(x,t)$$

The free-vibration equation can be obtained from Equation, by setting f = 0, as

$$c^2\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t)$$

Where

$$c = \sqrt{\frac{E}{\rho}}$$

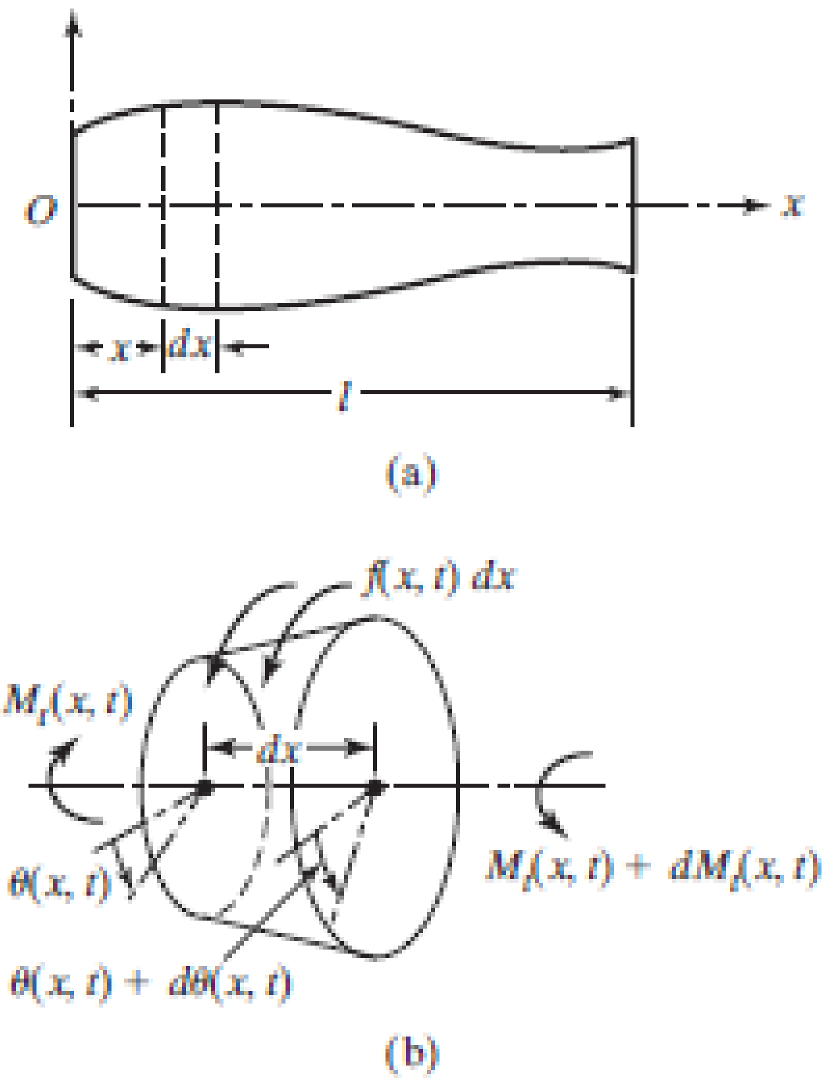
Torsional Vibration of a Shaft or Rod:

Figure, represents a non uniform shaft subjected to an external torque f(x, t) per unit length. If u(x, t) denotes the angle of twist of the cross section, the relation between the torsional deflection and the twisting moment Mt(x, t) is given by

$$M_t(x,t) = GJ(x)\frac{\partial \theta}{\partial x}(x,t)$$

Where G is the shear modulus and GJ(x) is the torsional stiffness, with J(x) denoting the polar moment of inertia of the cross section in the case of a circular section. If the mass polar moment of inertia of the shaft per unit length is the inertia torque acting on an element of length dx becomes

$$I_0\, dx \frac{\partial^2 \theta}{\partial t^2}$$



Torsional vibration of a shaft

If an external torque f (x, t) acts on the shaft per unit length, the application of Newton second law yields the equation of motion:

$$(M_t + dM_t) + f\, dx - M_t = I_0\, dx \frac{\partial^2 \theta}{\partial t^2}$$

By expressing dMt as

$$\frac{\partial M_t}{\partial x} dx$$

The forced torsional vibration equation for a nonuniform shaft can be obtained:

$$\frac{\partial}{\partial x}\left[GJ(x) \frac{\partial \theta}{\partial x}(x, t) \right] + f(x, t) = I_0(x) \frac{\partial^2 \theta}{\partial t^2}(x, t)$$

For a uniform shaft, takes the form

$$GJ \frac{\partial^2 \theta}{\partial x^2}(x, t) + f(x, t) = I_0 \frac{\partial^2 \theta}{\partial t^2}(x, t)$$

Which, in the case of free vibration, reduces to

$$c^2 \frac{\partial^2 \theta}{\partial x^2}(x, t) = \frac{\partial^2 \theta}{\partial t^2}(x, t)$$

Where

$$c = \sqrt{\frac{GJ}{I_0}} .$$

Lateral Vibration of Beams:

Consider the free-body diagram of an element of a beam shown in Figure, where M(x, t) is the bending moment, V(x, t) is the shear force, and f(x, t) is the external force per unit length of the beam. Since the inertia force acting on the element of the beam is

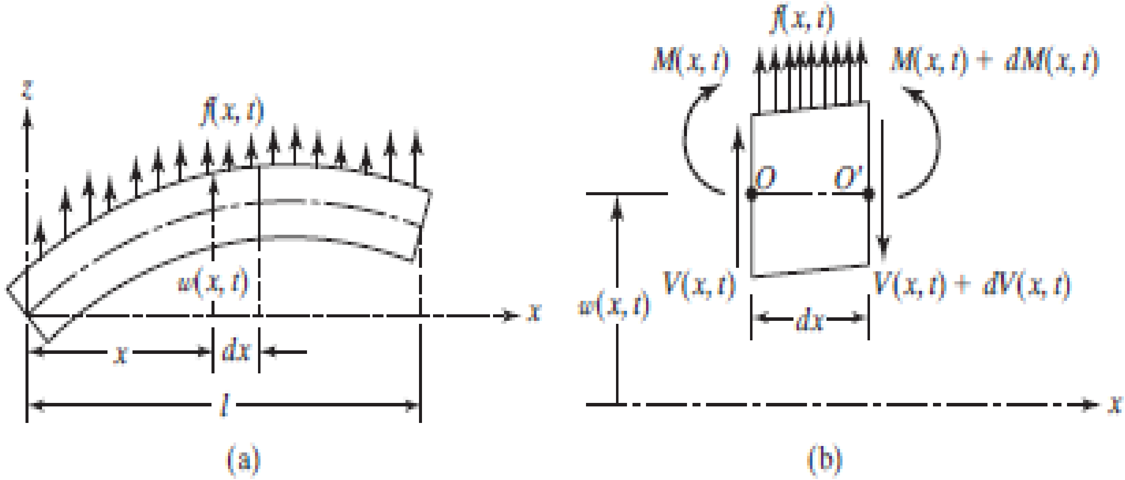
$$\rho A(x)\, dx \frac{\partial^2 w}{\partial t^2}(x, t)$$

The force equation of motion in the z direction gives

$$-(V + dV) + f(x, t)\, dx + V = \rho A(x)\, dx \frac{\partial^2 w}{\partial t^2}(x, t)$$

Where ρ is the mass density and A(x) is the cross-sectional area of the beam. The moment equation of motion about the y-axis passing through point O in Figure leads to

$$(M + dM) - (V + dV)\, dx + f(x, t)\, dx \frac{dx}{2} - M = 0$$



A beam in bending

By writing

$$dV = \frac{\partial V}{\partial x} dx \qquad \text{and} \qquad dM = \frac{\partial M}{\partial x} dx$$

By using the relation V = dM/dx

$$-\frac{\partial^2 M}{\partial x^2}(x, t) + f(x, t) = \rho A(x) \frac{\partial^2 w}{\partial t^2}(x, t)$$

From the elementary theory of bending of beams (also known as the Euler-Bernoulli or thin beam theory), the relationship between bending moment and deflection can be expressed as

$$M(x, t) = EI(x) \frac{\partial^2 w}{\partial x^2}(x, t)$$

Where E is Young s modulus and I(x) is the moment of inertia of the beam cross section about the y-axis. We obtain the equation of motion for the forced lateral vibration of a nonuniform beam:

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w}{\partial x^2}(x, t)\right] + \rho A(x) \frac{\partial^2 w}{\partial t^2}(x, t) = f(x, t)$$

Reduces to

$$EI\frac{\partial^4 w}{\partial x^4}(x, t) + \rho A \frac{\partial^2 w}{\partial t^2}(x, t) = f(x, t)$$

For free vibration, f(x, t) = 0, and so the equation of motion becomes

$$c^2\frac{\partial^4 w}{\partial x^4}(x, t) + \frac{\partial^2 w}{\partial t^2}(x, t) = 0$$

Where

$$c = \sqrt{\frac{EI}{\rho A}}$$
.

Rayleigh s Method:

Rayleigh s method can be applied to find the fundamental natural frequency of continuous systems. This method is much simpler than exact analysis for systems with varying distributions of mass and stiffness. Although the method is applicable to all continuous systems, we shall apply it only to beams in this section.4 Consider the beam shown in Figure. In order to apply Rayleigh s method, we need to derive expressions for the maximum kinetic and potential energies and Rayleigh s quotient. The kinetic energy of the beam can be expressed as

$$T = \frac{1}{2}\int_0^l \dot{w}^2\, dm = \frac{1}{2}\int_0^l \dot{w}^2 \rho A(x)\, dx$$

The maximum kinetic energy can be found by assuming a harmonic variation w(x, t) = W(x) cos vt:

$$T_{\max} = \frac{\omega^2}{2}\int_0^l \rho A(x) W^2(x)\, dx$$

The potential energy of the beam V is the same as the work done in deforming the beam. By disregarding the work done by the shear forces, we have

$$V = \frac{1}{2}\int_0^l M\, d\theta$$

Can be rewritten as

$$V = \frac{1}{2}\int_0^l \left(EI\frac{\partial^2 w}{\partial x^2}\right)\frac{\partial^2 w}{\partial x^2}\, dx = \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx$$

Since the maximum value of w(x, t) is W(x), the maximum value of V is given by

$$V_{\max} = \frac{1}{2}\int_0^l EI(x)\left(\frac{d^2W(x)}{dx^2}\right)^2 dx$$

By equating Tmax to Vmax, we obtain Rayleigh s quotient:

$$R(\omega) = \omega^2 = \frac{\int_0^l EI\left(\frac{d^2W(x)}{dx^2}\right)^2 dx}{\int_0^l \rho A(W(x))^2\, dx}$$

For a stepped beam, can be more conveniently written as

$$R(\omega) = \omega^2$$

$$= \frac{E_1 I_1 \int_0^{l_1} \left(\frac{d^2W}{dx^2}\right)^2 dx + E_2 I_2 \int_{l_1}^{l_2} \left(\frac{d^2W}{dx^2}\right)^2 dx + \cdots}{\rho A_1 \int_0^{l_1} W^2\, dx + \rho A_2 \int_{l_1}^{l_2} W^2\, dx + \cdots}$$

Where and correspond t Ei, Ii, Ai, li o the ith step (i = 1, 2, Á).

The Rayleigh-Ritz Method:

The Rayleigh-Ritz method can be considered an extension of Rayleigh s method. It is based on the premise that a closer approximation to the exact natural mode can be obtained by superposing a number of assumed functions than by using a single assumed function, as in Rayleigh s method. If the assumed functions are suitably chosen, this method provides not only the approximate value of the fundamental frequency but also the approximate values of the higher natural frequencies and the mode shapes. An arbitrary number of functions can be used, and the number of frequencies that can be obtained is equal to the number of functions used. A large number of functions, although it involves more computational work, leads to more accurate results.

In the case of transverse vibration of beams, if n functions are chosen for approximating the deflection W(x), we can write

$$W(x) = c_1 w_1(x) + c_2 w_2(x) + \cdots + c_n w_n(x)$$

Where w1(x), w2(x), Á , wn(x) are known linearly independent functions of the spatial coordinate x, which satisfy all the boundary conditions of the problem, and c1, c2, Á , cn are coefficients to be found. To make the natural frequency stationary, we set each of the partial derivatives equal to zero and obtain

$$\frac{\partial(\omega^2)}{\partial c_i} = 0, \qquad i = 1, 2, \ldots, n$$

.

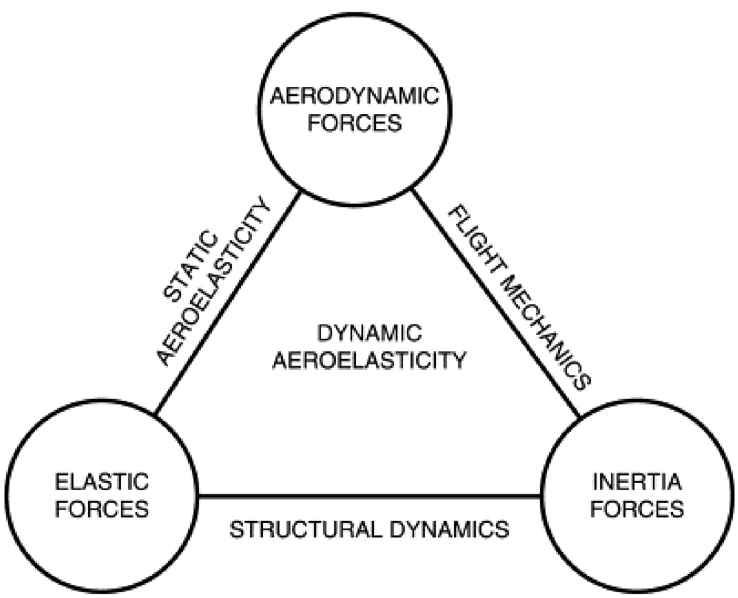
UNIT – V

INTRODUCTION TO AEROELASTICITY

Introduction:

Aeroelasticity is a notably new branch of applied mechanics that studies the interaction between fluid matters and flexible solid bodies. The typical application of aeroelasticity is in the branch of aircraft engineering. However, aeroelastic issues are applicable also for civil engineering (e.g., slender buildings, towers, smokestacks, suspension bridges, electric lines, and pipelines) or transportation engineering (cars, ships, submarines). Also important are its applications in machine engineering (compressors, turbines).
In the following text we will focus on aerospace aeroelasticity. Aeroelasticity in regard to aircraft structures is defined as the branch that investigates the phenomena that emerge due to the interaction of aerodynamic (in particular unsteady), inertial and elastic forces emerging during the relative movement of a fluid (air) and a flexible body (aircraft).

Two factors drive aviation development: 1) the quest for speed; and, 2) the competition for new air vehicle military and commercial applications. These factors trigger the appearance of new aircraft shapes, devices and materials, as well as applications of new technologies such as avionics. These factors have created and continue to create new challenges for the engineering discipline known as aeroelasticity.



Three-ring aeroelastic interaction Venn diagram.

Collar's aeroelastic triangle:

Aeroelastic phenomena may be divided according to the diagrammed definition of aeroelasticity (Collar's triangle of forces – Figure 1.1). The sides of the triangle represent the relationships among the particular pairs of forces representing specific areas of mechanics, including aeroelasticity, whereas the triangle's interior represents the interference of all three groups of forces typical for dynamic aeroelastic phenomena. Static aeroelastic phenomena that exclude inertial forces are characterised by the unidirectional deformation of the structure, whereas dynamic aeroelastic phenomena that include inertial forces are typical in their oscillatory property of structure deformation.

Collar's triangle of forces:

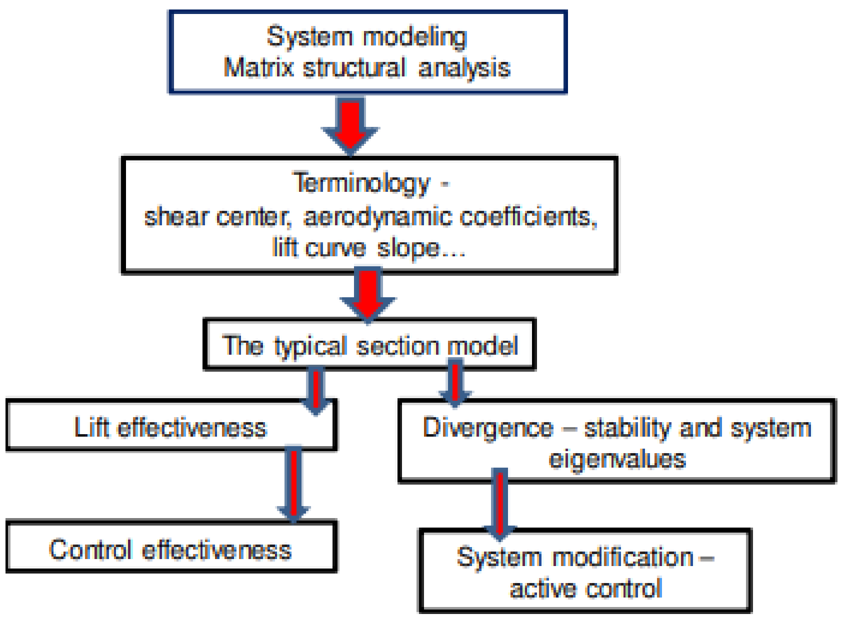
Problems with aeroelasticity have been occurring since the birth of aviation. The first famous event caused by an aeroelastic phenomenon was the crash of Langley's monoplane, which occurred only eight days before the Wright brothers' first successful flight. Thus, the Wrights became famous as the first fliers and Langley is only remarked in aeroelastic textbooks. The cause of the crash was the torsional divergence of the wing with low torsional stiffness. Therefore, the early stage of aviation is characterised by biplanes that allow for the design of a torsionally stiffer structure. At this time, torsional divergence was the dominant aeroelastic phenomenon. Torsional divergence was also the cause of several crashes of Fokker's high-wing monoplane D-8. The low stiffness of the fuselage and tail planes, as well as the unsuitable design of the control system, caused the crashes of the British Handley-Page O/400 twin-engine biplane bomber and the DH-9 biplane fighter during the First World War.

After the First World War, during tests in the USA, a further aeroelastic phenomenon emerged with the D-8 aircraft – wing aileron flutter. This phenomenon was eliminated using static balancing of the ailerons. With increasing flight velocities, as well as the design of monoplanes, the flutter phenomenon became increasingly important. The basic theories of flutter were formulated by Küssner in Germany and by Frazer and Duncan in the UK at the end of the 1920s. In the 1930s Theodorsen formulated the theory of unsteady forces distribution on a harmonically vibrating airfoil with a control surface. Similar problems were also solved in Russia.

With the attainment of sonic speed after the Second World War, qualitatively new aeroelastic issues, such as panel flutter or control buzz, gained prominence. These phenomena are caused by shock waves and pressure oscillations over the airfoil in the transonic region. In the supersonic velocity range, the aerodynamic heat effect also became an important contributing factor (aero-thermo-elasticity). Apart from this, further phenomena are connected to the effects of servo-systems in the control circuits (aero-servoelasticity). Currently, new aeroelastic problems are emerging in connection with the active control systems that are used for gust alleviation and the suppression of structural loads. Additionally, aeroelastic optimisation methods enable the potential of structural adjustments to, for instance, decrease mass and increase flight performance.

In general, a new aircraft design concept or a part or system may cause new aeroelastic problems. A typical example is the whirl flutter phenomenon that is the subject of this book. In the late 1950s, a new aircraft category (turboprop airliners) emerged. These aircraft were characterised by wing-mounted gas turbine engines and heavy propellers placed somewhat far in front of the wing. The gyroscopic effect of the rotating masses of the propeller, compressor and turbine combined with the wing dynamics, consequently created a new aeroelastic issue – whirl flutter.

Aeroelastic interactions determine airplane loads and influence flight performance in four primary areas: 1) wing and tail surface lift redistribution that change external loads from preliminary loads computed on rigid surfaces; 2) stability derivatives, including lift effectiveness, that affects flight static and dynamic control features such as aircraft trim and dynamic response; 3) control effectiveness, including aileron reversal, that limits maneuverability; 4) aircraft structural dynamic response to atmospheric turbulence and buffeting, as well as structural stability, in particular flutter.



As indicated in Figure this chapter begins with a discussion of aeroelastic models and the introduction to special terminology required to define the features of these models. This includes a brief discussion of structural analysis matrix methods and concepts such as the shear center, aerodynamic coefficients and aerodynamic center of pressure.

DEFINITIONS:

Aeroelasticity:

1. Aeroelastic problems would not exist if airplane structures where perfectly rigid.
2. Many important aeroelastic phenomena involve inertia forces as well as aerodynamic and elastic forces.

Static Aeroelasticity:

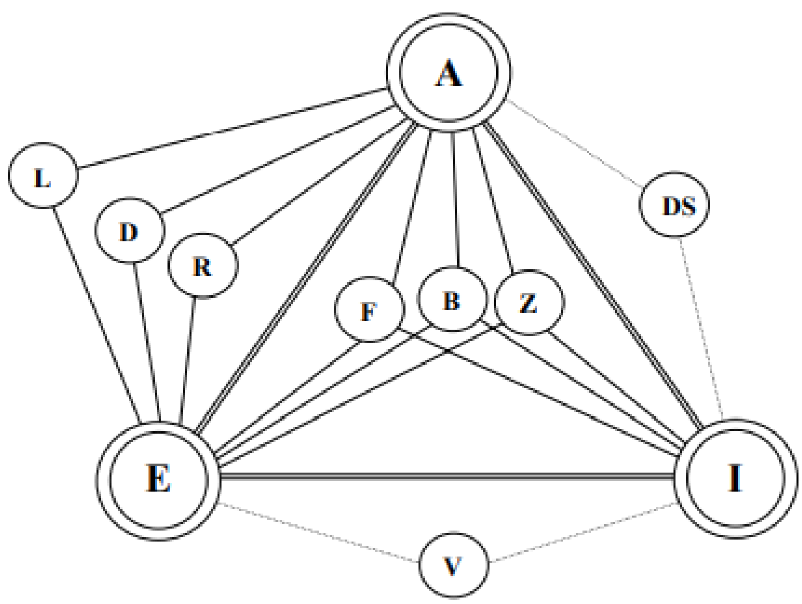
Science which studies the mutual interaction between aerodynamic forces and elastic forces, and the influence of this interaction on airplane design.

Dynamic Aeroelasticity:

Phenomena involving interactions of inertial, aerodynamic, and elastic forces.

Collar diagram:

Describes the aeroelastic phenomena by means of a triangle of forces



A – Aeroelastic force
E – Elastic force
I – Inertial force

DYNAMIC AEROELASTICITY: - Phenomena involving all three types of forces:

1. F – Flutter: dynamic instability occurring for aircraft in flight at a speed called flutter speed
2. B – Buffeting: transient vibrations of aircraft structural components due to aerodynamic impulses produced by wake behind wings, nacelles, fuselage pods, or other components of the airplane
3. Z – Dynamic response: transient response of aircraft structural components produced by rapidly applied loads due to gusts, landing, gun reactions, abrupt control motions, and moving shock waves

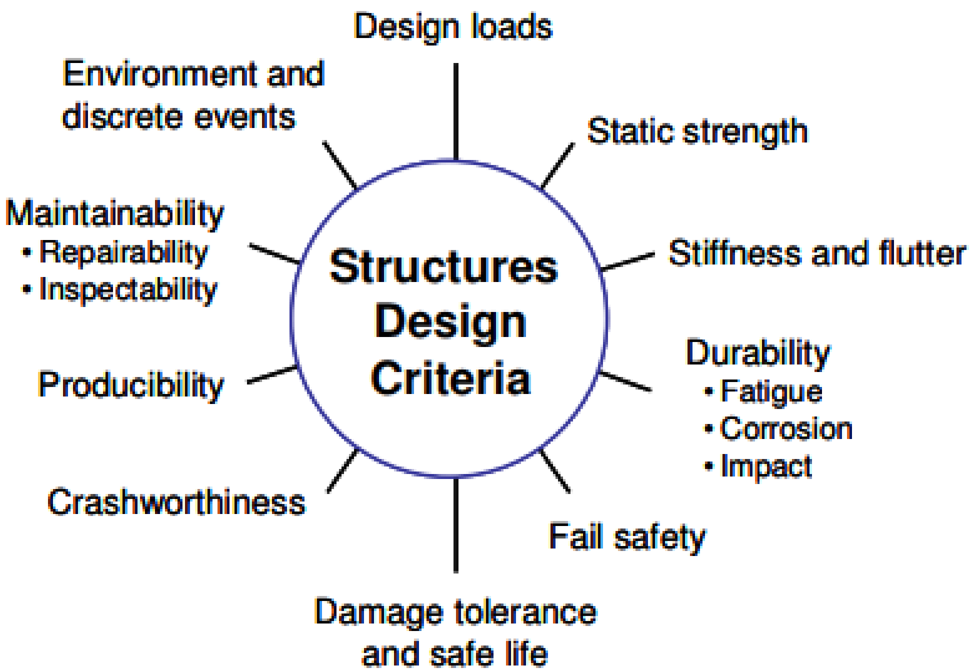
STATIC AEROELASTICITY: - Phenomena involving only elastic and aerodynamic forces:

1. L – Load distribution: influence of elastic deformations of the structure on the distribution of aerodynamic pressures over the structure
2. D – Divergence: a static instability of a lifting surface of an aircraft in flight, at a speed called the divergence speed, where elasticity of the lifting surface plays an essential role in the instability.
3. R – Control system reversal: A condition occurring in flight, at a speed called the control reversal speed, at which the intended effect of displacing a given component of the control system are completely nullified by elastic deformations of the structure.

The structures enterprise and its relation to aeroelasticity:

Every aircraft company has a large engineering division with a name such as “Structures Technology.” The purpose of the structures organization is to create an airplane flight structure with structural integrity. This organization also has the responsibility for determining and fulfilling structural design objectives and structural certification of production aircraft. In addition, the organization conducts research and develops or identifies new materials, techniques and information that will lead to new aircraft or improvements in existing aircraft. The structures group has primary responsibility for loads prediction, component strength analysis and structural component stability prediction. There is strong representation within this group of people with expertise in structural mechanics, metallurgy, aerodynamics and academic disciplines such as civil engineering, mechanical engineering, chemical engineering, and engineering mechanics, as well as the essential aeronautical engineering representation.

The structural design process begins with very general, sometimes “fuzzy,” customer requirements that lead to clearly stated engineering design criteria with numbers attached. A summary of these general design criteria is shown in Figure.



Structural design requirements

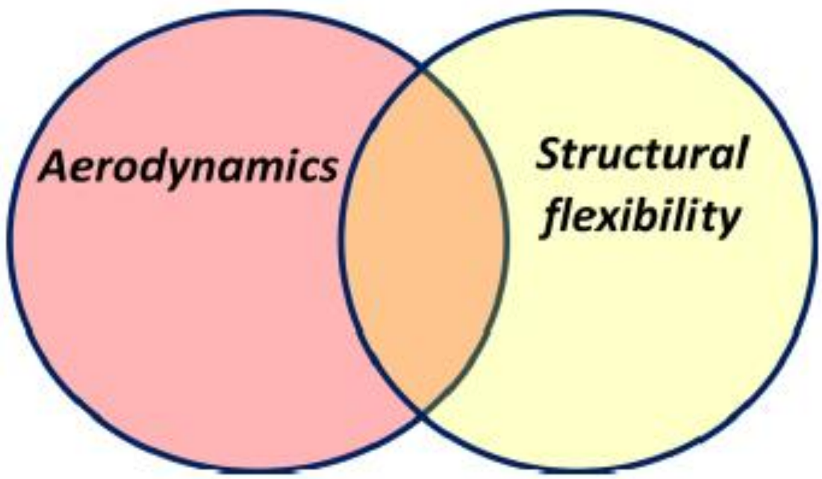
Beginning at the top of the “wheel” we have design loads. These loads include airframe loads encountered during landing and take-off, launch and deployment as well as in-flight loads and other operational loadings. There are thousands of such “load sets.” Once these load sets are identified, there are at least nine design criteria that must be taken into account. On the wheel in Figure stiffness and flutter are one important set of criteria that must be addressed.

The traditional airframe design and development process can be viewed as six interconnected blocks, shown in Figure. During Block 1 the external shape is chosen with system performance objectives in mind (e.g. range, lift and drag). Initial estimates of aircraftft component weights use empirical data gathered from past experience. On the other hand, if the designs considered at this early stage have radical new forms, these estimates may be in error; but these errors will only be discovered later.

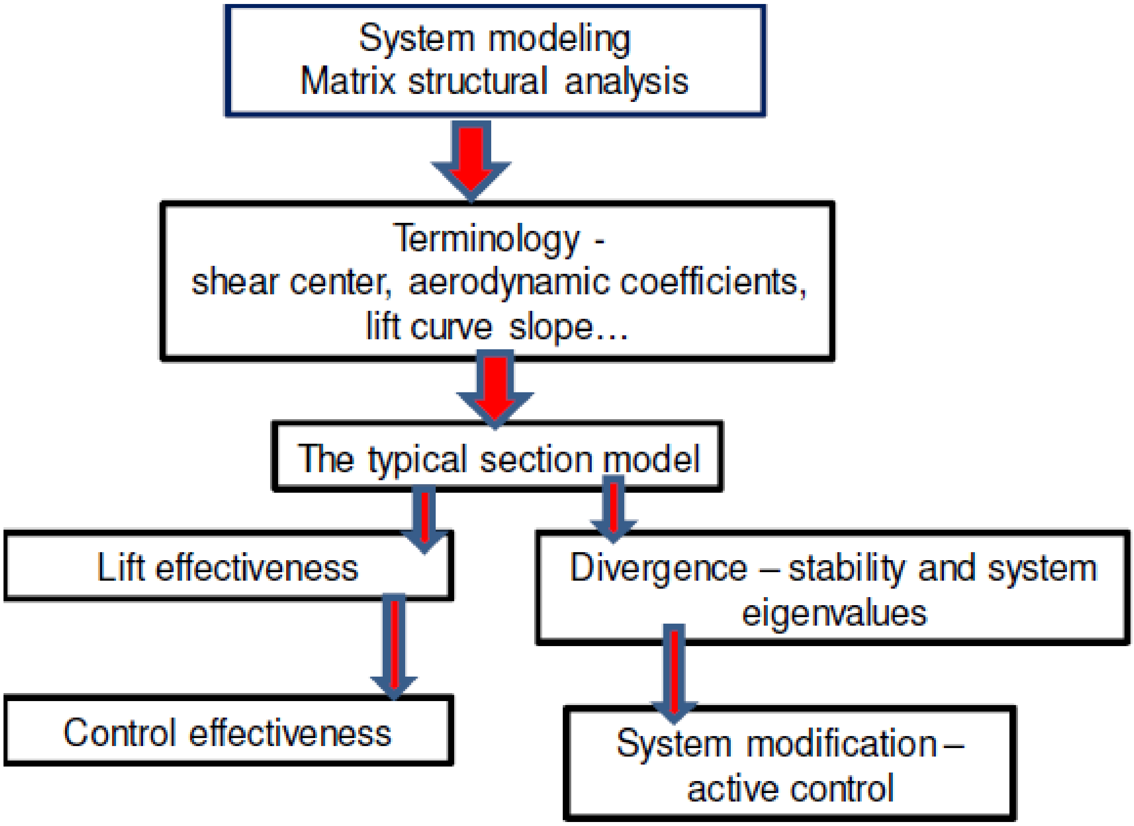
Static aeroelasticity phenomena:

All structures deform when external loads are applied although the deflections may be barely discernible. In most cases, the external and internal loads do not depend on the structural deformation. From an analysis perspective this means we can compute the internal loads and the external deflections independently.

These structural analysis problems are called statically determinate and include structural stability problems such as column buckling. However, if the loads and structural deflection interact the structural analysis problem becomes very different, both physically and computationally, because the problem is statically indeterminate. Both loads and deflections must be determined simultaneously. This load/deflection interaction is represented graphically by the Venn diagram in Figure in which the overlapping orange area represents the statically indeterminate problem area.



Static aeroelasticity encompasses problems involving the intersection between steady-state aerodynamic and structural deformation interactions.



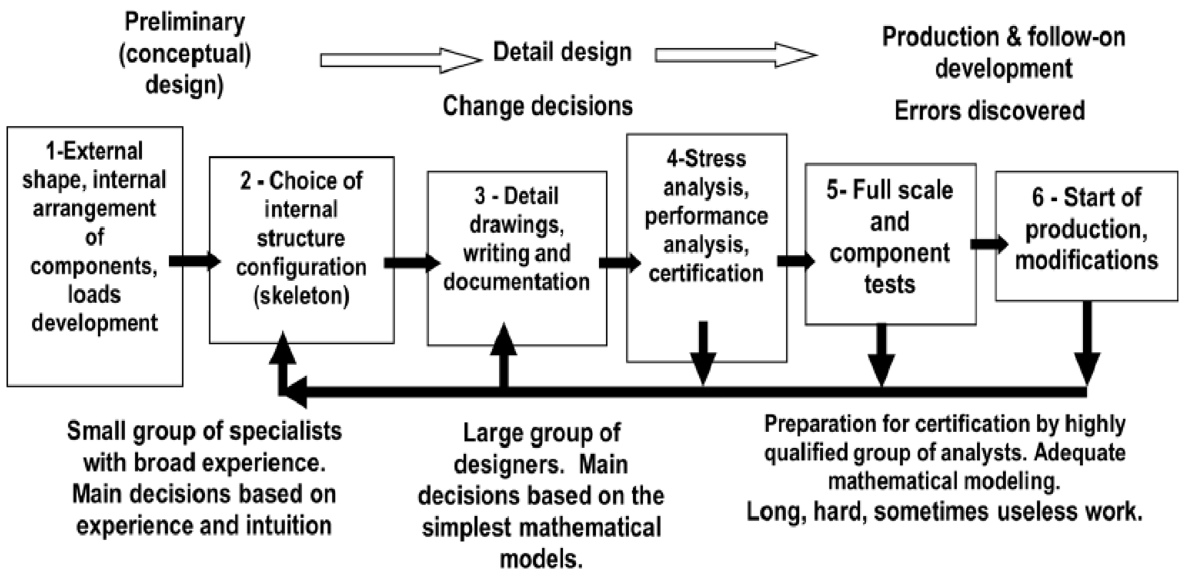
Scope and purpose:

As indicated in Figure, this chapter begins with a discussion of aero elastic models and the introduction to special terminology required defining the features of these models. This includes a brief discussion of structural analysis matrix methods and concepts such as the shear center, aerodynamic coefficients and aerodynamic center of pressure.

STRUCTURAL DESIGN REQUIREMENTS:-

Every aircraft company has a large engineering division with a name such as "Structures Technology." The purpose of the structures organization is to create an airplane flight structure with

Structural integrity: This organization also has the responsibility for determining and fulfilling structural design objectives and *structural certification* of production aircraft. In addition, the organization conducts research and develops or identifies new materials, techniques and information that will lead to new aircraft or improvements in existing aircraft.

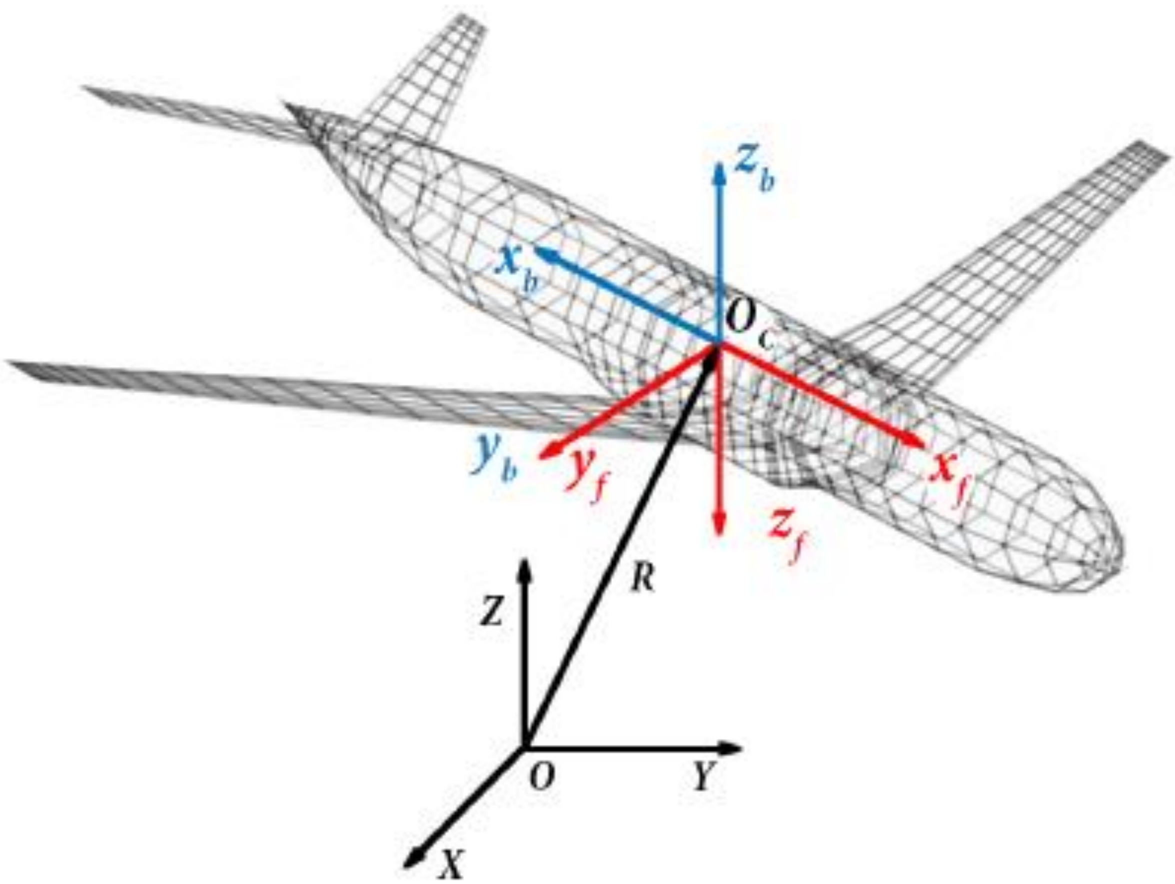


The structural design and development process requires testing, analysis and feedback

Modern aircraftft are increasingly designed to be highly maneuverable in order to achieve high-performance mission objectives. Toward this goal, aircraft designers have been adopting light-weight, flexible, high aspect ratio wings in modern aircraft. Aircraft design concepts that take advantage of wing flexibility to increase maneuverability have been investigated. By twisting a wing structure, an aerodynamic moment can be generated to enable an aircraft to execute a maneuver in place of the use of traditional control surfaces. For example, a rolling moment can be induced by twisting the left and right wings in the opposite direction. Similarly, a pitching moment can be generated by twisting both wings in the same direction. Wing twisting or warping for flight control is not a new concept and was used in the Wright Flyer in the 1903. The U.S. Air Force conducted the Active Flexible Wing program in the 1980's and 1990's to explore potential use of leading edge slats and trailing edge flaps to increase control effectiveness of F-16 aircraft for high speed maneuvers.1 In the recent years, the Active Aeroelastic Wing research program also investigated a similar technology to induce wing twist in order to improve roll maneuverability of F/A-18 aircraftft.2

Structural deflections of lifting surfaces interact with aerodynamic forces to create aeroelastic coupling that can affect aircraft performance. Understanding these effects can improve the prediction of aircraft flight dynamics and can provide insight into how to design a flight control system that can reduce aeroelastic interactions with a rigid-body flight controller. Generally, high aspect ratio lifting surfaces undergo a greater degree of structural deflections than low aspect ratio lifting surfaces. In general, a wing section possesses a lower stiffness than a horizontal stabilizer or a vertical stabilizer. As a result, its natural frequency is normally present inside a flight control frequency bandwidth that potentially can result in flight control interactions. For example, when a pilot commands a roll maneuver, the aileron deflections can cause one or more elastic modes of the wings to excite. The wing elastic modes can result in changes to the intended aerodynamics of the wings, thereby potentially causing undesired aircraft responses.

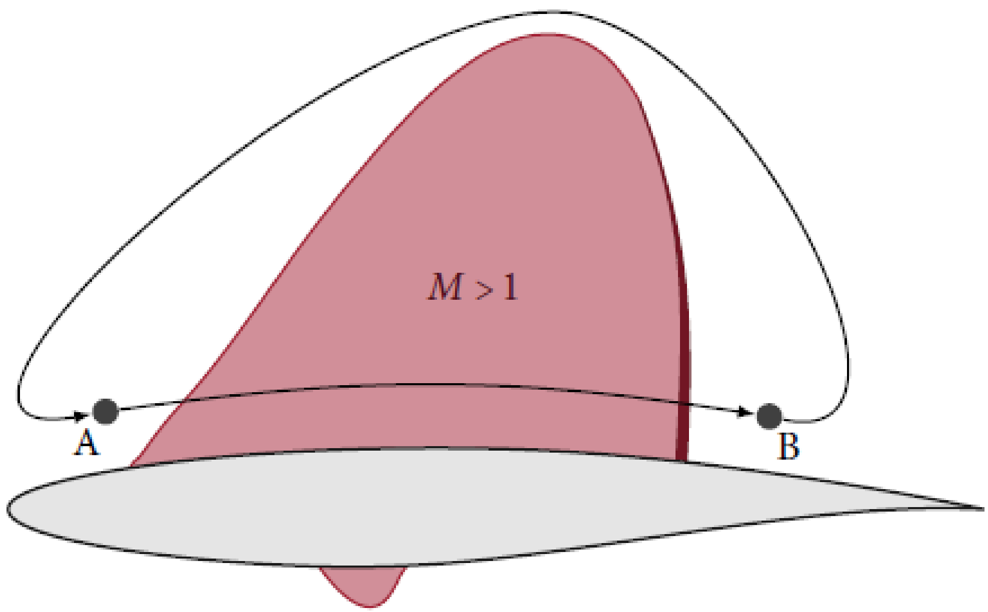
Aero-servoelastic filtering is a traditional method for suppressing elastic modes, but this usually comes at an expense in terms of reducing the phase margin in a flight control system.3 if the phase margin is significantly reduced, aircraft responses may become more sluggish to pilot commands. Consequently, with a phase lag in the control inputs, potential pilot induced oscillations (PIOs) can occur. Numerous studies have been made to increase the understanding of the role of aero-servoelasticity in the design of flight control systems.



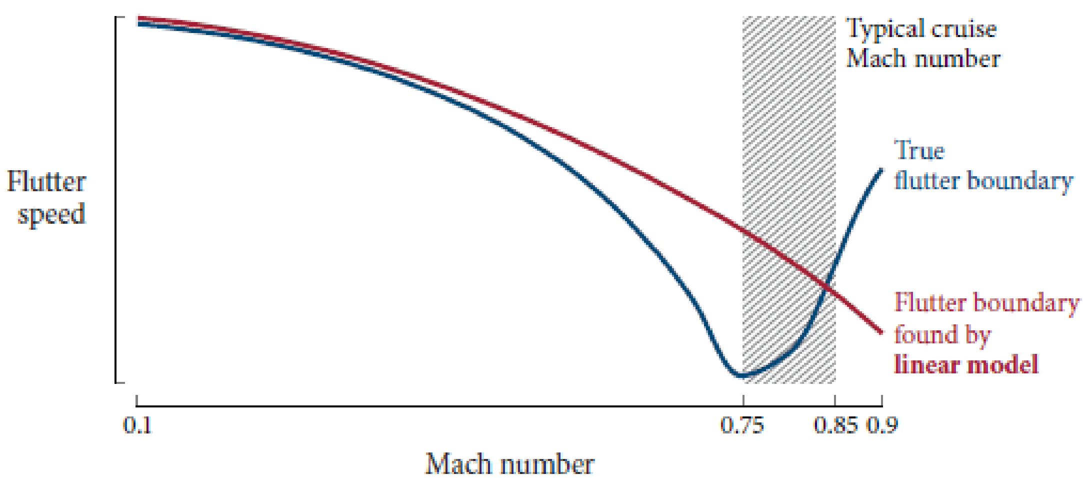
TRANSONIC FLUTTER:

Aeroelasticity is the study of aerodynamic, elastic, and inertial forces on a body in a fluid flow. Flutter is an aeroelastic phenomenon where these forces start exciting each other, leading to instability in the structure. In an aircraft wing, this results in large cyclic bending and twisting motions of the wing, likely leading to structural failure of the wing. The onset of flutter, therefore, has to be avoided at all times and rigorous flight testing procedures are in place to ensure that flutter does not occur within an aircraft's operational envelope. Ideally, engineers design around aeroelastic issues as early as possible in the design process by analyzing the aeroelastic behavior of aircraft wings well before such flight tests. Incorporating flutter prediction into early-stage design, however, is a considerable challenge for the transonic flow regime relevant to most civil transport aircraft designs, since existing accurate models for transonic flutter prediction typically require extensive Computational Fluid Dynamics (cfd) analyses. Such simulations are too expensive to conduct in early design stages where potentially thousands of wing designs might be considered. A wealth of literature is available on static aeroelastic optimization of aircraft wings for both conventional tube-wing configurations, and next-generation con_gurations.10 Dynamic aeroelasticity, however, is o en not included in these studies. Flutter constraints can be included in state-of-the-art multidisciplinary design optimization settings, but even if the static aeroelastic methods are high fidelity, the methods for unsteady (compressible) flow usually either linearize the unsteady response or use a Prandtl-Glauert correction, which is limited to subsonic flow. Moreover, all afore mentioned methods use high-fidelity aerodynamic and structural analysis methods, and are therefore too expensive to use in conceptual aircraft design. Conceptual aircraft design tools, such as suave, typically do not include flutter constraints.

While flutter for low subsonic flows can, in general, is accurately predicted with linearized small-disturbance theories, such methods fail for transonic flow where the traditional small-disturbance formulation is inherently nonlinear. For example, linear theory predicts that thinner wings would be less susceptible to flutter, but wind tunnel tests have shown the opposite to be true. Another interesting phenomenon that occurs in transonic flows is the transonic dip in the flutter boundary. In subsonic flow, information travels in all directions, but in supersonic flow, information can only travel in the direction of the flow. Thus, for an airfoil in transonic flow, pressure perturbations from e.g. the trailing edge take a much longer time to reach the leading edge of the airfoil compared to subsonic flow. This is illustrated in Figure 1.1, where in transonic flow, the information travel from B to A has to take a much longer route than the information travel from A to B. this results in phase lag in the aerodynamic response, which in turn—together with other processes at work—leads to a very different flutter response: a dip in the transonic flutter boundary.



Information travel for airfoil in transonic flow.
The path from B to A is much longer than from A to B.



Transonic flutter boundary with typical transonic dip, which linear theory cannot predict.

Source from "Structural Dynamics in Aeronautical Engineering" by Bismarck-Nasr, M.N.

Aeroelastic tailoring:

Novel manufacturing techniques open up additional design space for aerospace vehicles. High aspect ratio wings in novel aircraft concepts, for example, over the benefits of higher aerodynamic efficiency, but present the challenge of being more susceptible to aero elastic problems such as flutter. Novel manufacturing techniques present an opportunity to address this challenge via improved material properties (e.g., increased stiffness) and also by enabling unconventional internal wing layouts. This thesis examines the possibility of designing the internal structure of aircraft wings as a lattice structure, while mitigating flutter. The fabrication of these lattice structures is enabled by advances in additive manufacturing technology.

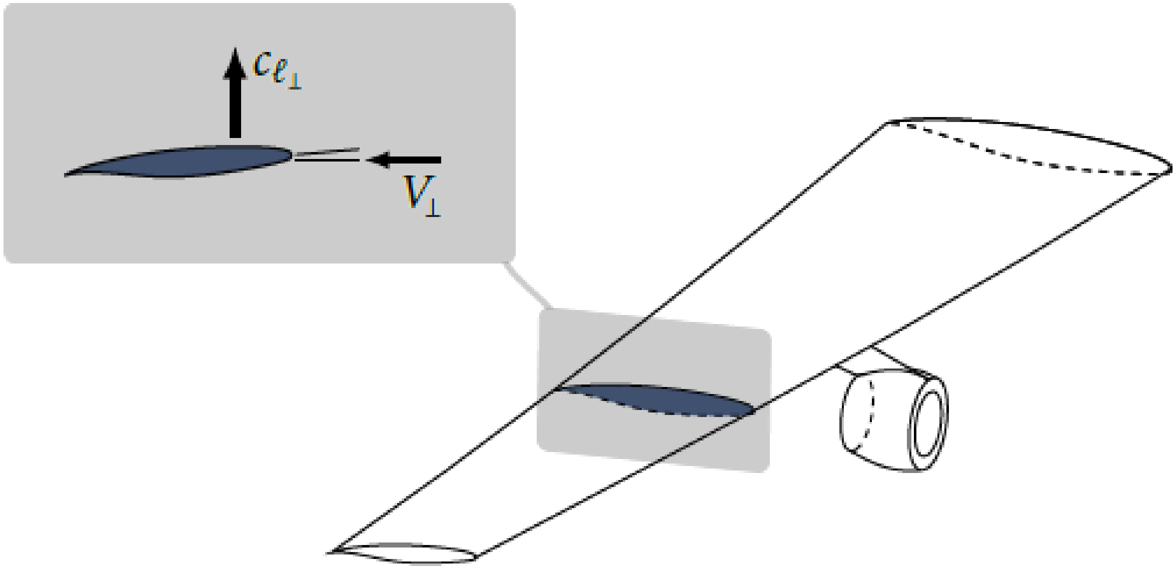
To date, the aeroelastic optimization of aircraft wings has mostly focused on conventional internal wing structures—i.e., an orthogonal array of ribs and spars. The structural efficiency of the wing can be further improved by using novel manufacturing techniques, which allow for moving away from the conventional orthogonal rib-spar lay- out. Several ways to parametrize the internal structure of the wing have been demonstrated.

Active flutter suppression:

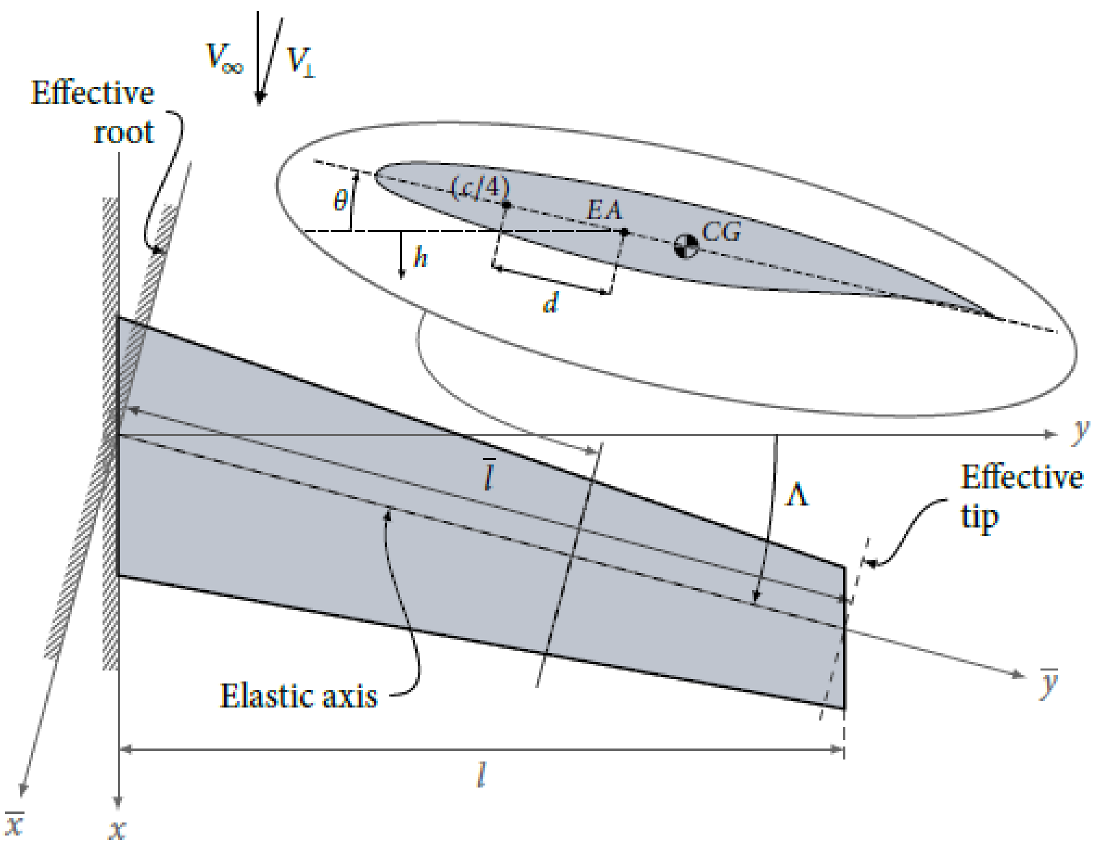
1. **Wing flutter model aerodynamic model:**

   Three of the most common methods to predict unsteady loads on an aircraft are strip theory (2D unsteady airfoil theory with 3D corrections), the doublet-lattice method, and the unsteady vortex-lattice method (uvlm). Considering that our goal is to investigate aircraft concepts with high aspect ratios, the use of strip theory here is appropriate.

In strip theory, one assumes that the flow along a cut in the wing perpendicular to a span-wise axis of the wing is two-dimensional (Figure 3.1). For this work, that means we can use our calibrated two- dimensional flutter model (Chapter 2) also for three-dimensional wings.



In strip theory, one assumes that the flow is two dimensional in cuts perpendicular to the span.



Swept wing considered in model

Source from "Structural Dynamics in Aeronautical Engineering" by Bismarck-Nasr, M.N.

2. Structural model:

For the structural part of the model, we use Bernoulli-Euler beam theory.

The beam equations are,

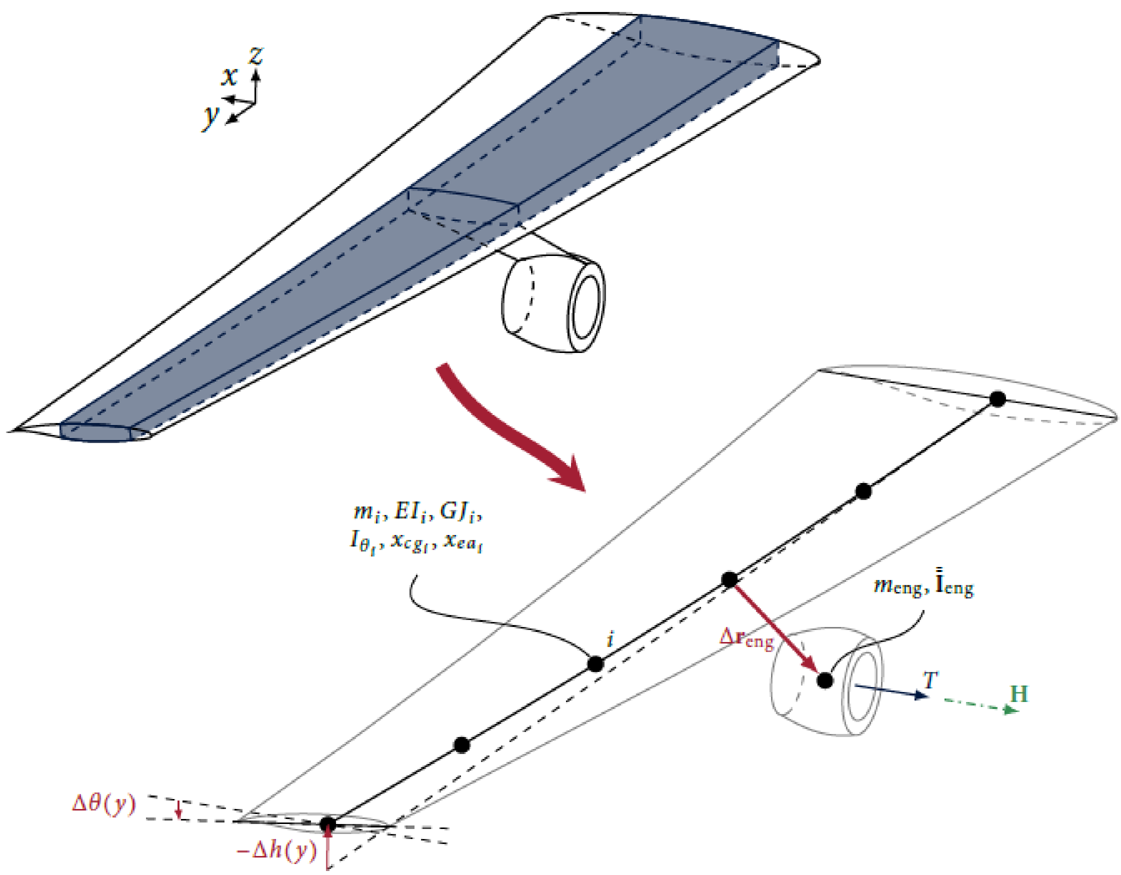
$$m(\bar{y})\Delta\ddot{h}(\bar{y},t) + S_{\bar{y}}(\bar{y})\Delta\ddot{\theta}(\bar{y},t) + \frac{\partial^2}{\partial\bar{y}^2}\left[EI(\bar{y})\frac{\partial^2\Delta h(\bar{y},t)}{\partial\bar{y}^2}\right] = -\Delta L(\bar{y},t)$$

$$I_{\bar{y}}(\bar{y})\Delta\ddot{\theta}(\bar{y},t) + S_{\bar{y}}(y)\Delta\ddot{h}(\bar{y},t) - \frac{\partial}{\partial\bar{y}}\left[GJ(\bar{y})\frac{\partial\Delta\theta(\bar{y},t)}{\partial\bar{y}}\right] = \Delta M_{ea}(\bar{y},t)$$

The appropriate boundary conditions here are,

$$\Delta h(0,t) = 0, \quad \frac{\partial\Delta h}{\partial\bar{y}}(0,t) = 0, \quad \frac{\partial^2\Delta h}{\partial\bar{y}^2}(\bar{l},t) = 0, \quad \frac{\partial^3\Delta h}{\partial\bar{y}^3}(\bar{l},t) = 0,$$

$$\Delta\theta(0,t) = 0, \quad \frac{\partial\Delta\theta}{\partial\bar{y}}(\bar{l},t) = 0.$$



Discretized beam model used in structural part of the flutter model.
The beam equations can be rewritten as

$$m(\bar{y})\Delta\ddot{h}(\bar{y},t) - S_{\bar{y}}(\bar{y})\Delta\ddot{\theta}(\bar{y},t) + \frac{\partial \Delta \mathcal{S}(\bar{y},t)}{\partial \bar{y}} = -\Delta L(\bar{y},t)$$

$$I_{\bar{y}}(\bar{y})\Delta\ddot{\theta}(\bar{y},t) + S_{\bar{y}}(y)\Delta\ddot{h}(\bar{y},t) - \frac{\partial \Delta \mathcal{T}(\bar{y},t)}{\partial \bar{y}} = \Delta M_{ea}(\bar{y},t)$$

Where

$$\frac{\partial \Delta \mathcal{M}}{\partial \bar{y}} = \Delta \mathcal{S}, \quad \frac{\partial \Delta y}{\partial \bar{y}} = \frac{\Delta \mathcal{M}}{EI}, \quad \frac{\partial \Delta h}{\partial \bar{y}} = \Delta y, \quad \frac{\partial \Delta \theta}{\partial \bar{y}} = \frac{\Delta \mathcal{T}}{GJ},$$

With boundary conditions

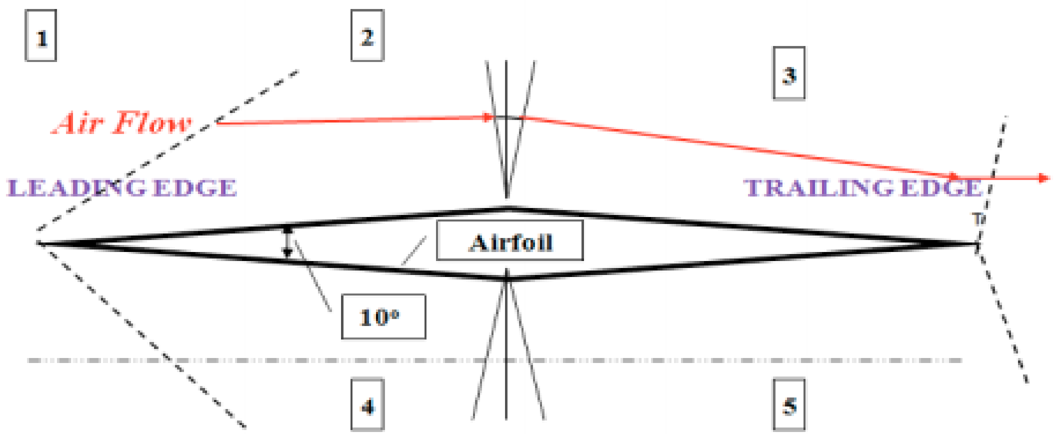
$$\Delta h(0,t) = 0, \quad \Delta y(0,t) = 0, \quad \Delta \mathcal{M}(\bar{l},t) = 0, \quad \Delta \mathcal{S}(\bar{l},t) = 0,$$
$$\Delta \theta(0,t) = 0, \quad \Delta \mathcal{T}(\bar{l},t) = 0,$$

Effect of aeroelasticity in flight vehicle design:-

STRUCTURAL ANALYSIS:

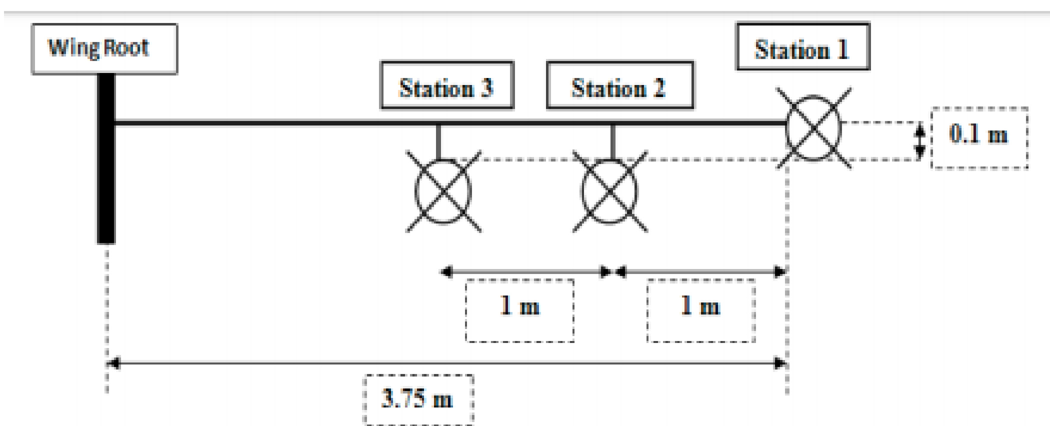
i. **Supersonic Wing Characteristics:**

The present work utilizes a wing platform with an aspect ratio of 5 and taper ratio of 0.5. The wing leading edge swept back angle is 30°. The airfoil of this supersonic wing is a double wedge shape as shown in Fig. 1. The wedge angle of the airfoil is 10°. Along the wing span, the airfoil is divided into three parts which are the main wing box and two control surfaces at the leading and trailing edge. The portions of the leading and trailing edge have been specified as 15% and 20% of the chord length, respectively. The performance of the selected airfoil uses the characteristics provided by for higher supersonic region analysis. The present wing design is used as a baseline for further work where the wing geometry as well as wing composite structure is set as the sensitivity parameter to obtain an optimum supersonic wing design.



Double wedge airfoil

For the external store, Fig. shows the configuration of the loaded missile on the wing. The external stores for each station of wing are specified in Table 1. There are two types of missiles used which are AMRAAM and Sidewinder.



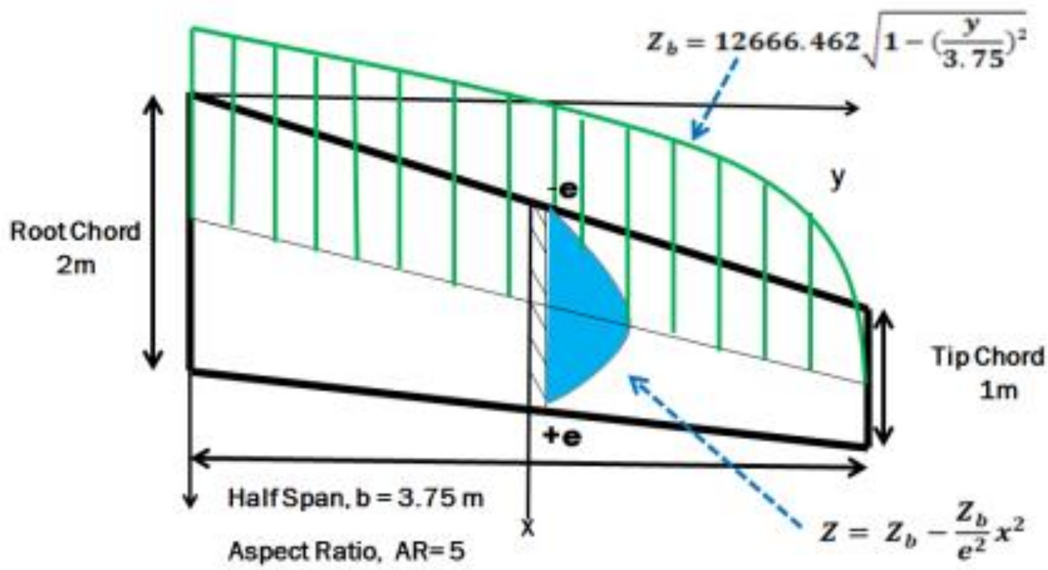
External stores configuration of the wing

Station	Missile Type	Length [m]	Diameter [m]	Mass [kg]
1	AIM-9M	2.85	0.128	86.0
2	AIM-120 A	3.66	0.178	157.89
3	AIM-120 A	3.66	0.178	157.89

External stores technical data

ii. Wing Loading:

Based on, the load factor for the fighter aircraft is set atnz = 5.5. The load for this wing, as shown in Fig. 3, is assumed to be elliptic load acting along the wing span wise (y-axis) direction and symmetric quadratic load along chord wise (x-axis) direction. With this load assumption, the sizing of the wing box can be conducted.



Wing loading estimation equation

The formula to calculate the load factor is given by Eq. (1) in which L is the lift and W is the weight of one side of the aircraft wing based on;

$$n_z = \frac{L}{W}$$

Here the lift can be calculated

$$L = n_z W$$

The span wise elliptic load can be formulated as:

$$\left(\frac{y}{a}\right)^2 + \left(\frac{z_b}{b}\right)^2 = 1$$

Where the value of parameter a is half the span length since it is the length of the major axis, and parameter b is the minor axis. The value of b can be calculated using (6). The chord wise quadratic load distribution is given by:

$$z = C_0 + C_1 x + C_2 x^2$$

The area of the quadratic load in chord wise direction can be calculated by integrating Eq. acting along the x axis.

$$A_{quadratic} = \int_{-e}^{e} z_b \left[1 - \left(\frac{x}{e}\right)^2\right] dx$$

Then, the volume of the elliptic load can be found by integrating Eq. (5) along the y axis in Eq

$$V = \frac{4}{3}\int_0^a z_b(y)e(y)dy$$

To find the minor axis of the elliptic equation, Eq. (3), equation (2) is divided by 2 since this is only applicable for the half wing, equal to the volume found in Eq. (3). This expression can be written as:

$$V = \frac{L}{2}$$

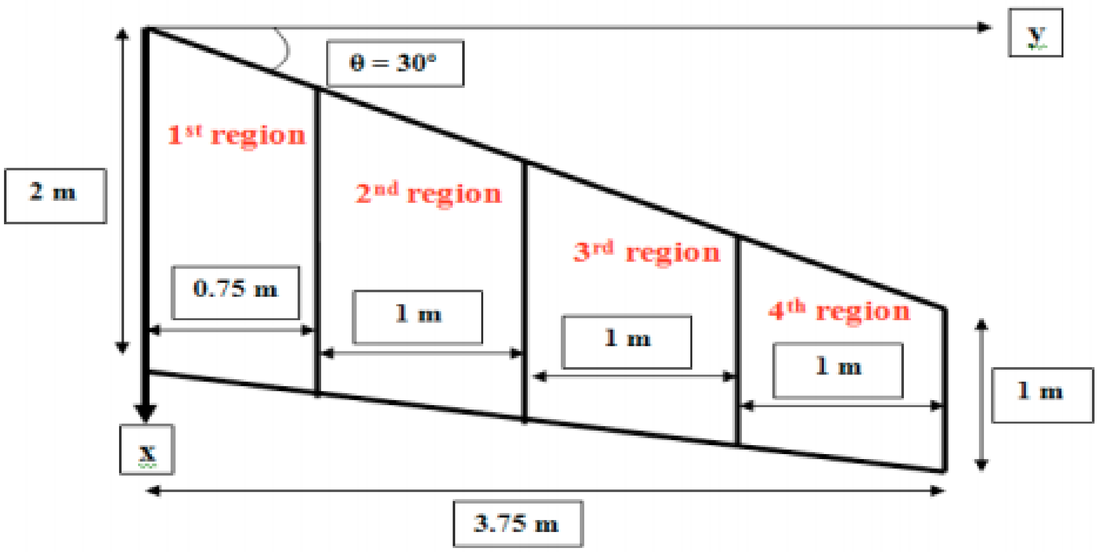
The wing can be assumed as a beam along y axis to find the shear force Q and moment M of each section as denoted in Eq. (8) and Eq. (9), respectively.

$$Q(y) = \int dQ$$

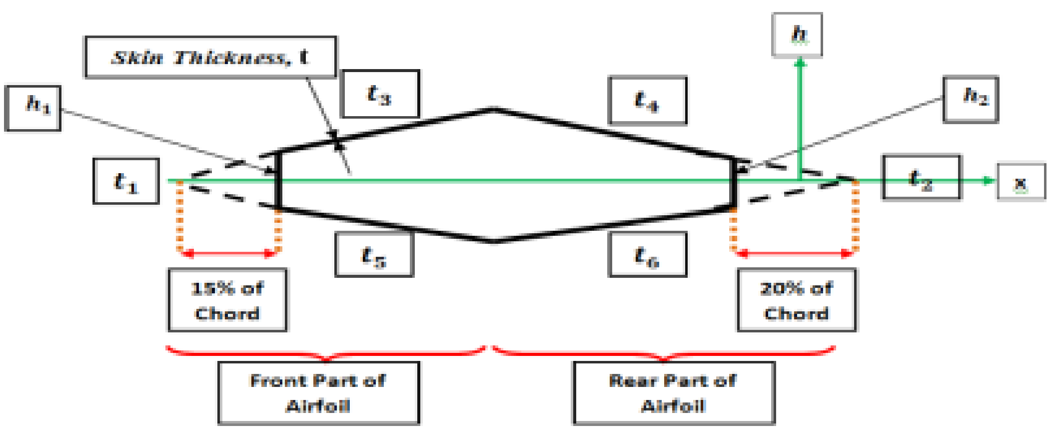
$$M(y) = \int dM$$

iii. Wing Sizing:

To calculate the skin thickness of the overall wing, the wing is divided into 4 regions along the span and the thickness in each region is calculated based on the maximum load in the respective region as shown in Fig. 4 and Fig. 5.



Top view of skin thickness division region



Skin thickness segment in a region

The moment of inertia formula is given as:

$$I = \int h^2 \, dA$$

The moment of inertia for the front and rear spar are calculated as a vertical segment in

$$I_{xx}= \frac{1}{12}\, th^3$$

The moment of inertia for the inclined segments which has an inclination angle of θ, can be derived using Eq. (10). This can be done by setting the limit for integration along z axis starting from 0 to the end of each inclination segment denoted as./. The final formula is given by Eq.

$$I = \frac{t}{\cos\theta}\int_0^{b'} (\frac{h}{2} + x\tan\theta)^2 dz$$

By assuming the thickness to be constant at every skin and spar, equation (12) reduces to form an equation to find the thickness in any region based on the associated moment Macting in that region as shown in Eq.

$$\sigma_y = \frac{M\, h_{middle}}{I(t)}$$

Where 7 is calculated based on Eq. (9), 89: is the height of the inclination for the front spar of the wing only and (is the summation moment of inertia of the wing box in terms of t as given in Eq. (11) and Eq. (12).

iv. Safety Factor:

The safety factor FS for the structural strength analysis

$$\tau_y = \frac{QS}{b\, I}$$

$$\frac{\tau_{allow}}{\tau_y} \geq F.S$$

$$\frac{\sigma_{allow}}{\sigma_y} \geq F.S$$