**INSTITUTE OF AERONAUTICAL ENGINEERING**
**(Autonomous)**
Dundigal, Hyderabad -500 043

# ELECTRONICS AND COMMUNICATION ENGINEERING

## COURSE LECTURE NOTES

| | |
|---|---|
| **Course Name** | **CONTROL SYSTEMS** |
| **Course Code** | AEEB16 |
| **Programme** | B.Tech |
| **Semester** | IV |
| **Course Coordinator** | Dr. M Pala Prasad Reddy, Associate Professor |
| **Course Faculty** | Dr. P Sridhar, Professor<br>Dr. M Pala Prasad Reddy, Associate Professor |
| **Lecture Numbers** | 1-60 |
| **Topic Covered** | All |

## COURSE OBJECTIVES (COs):

| **The course should enable the students to:** | |
|---|---|
| I | Organize modeling and analysis of electrical and mechanical systems. |
| II | Analyze control systems by block diagrams and signal flow graph technique. |
| III | Demonstrate the analytical and graphical techniques to study the stability. |
| IV | Illustrate the frequency domain and state space analysis. |

## COURSE LEARNING OUTCOMES (CLOs):

**Students, who complete the course, will have demonstrated the ability to do the following:**

| | |
|---|---|
| AEEB16.01 | Differentiate between open loop, closed loop system and their importance in real time applications. |
| AEEB16.02 | Predict the transfer function of translational and rotational mechanical, electrical system using differential equation method. |
| AEEB16.03 | Analyze the analogy between translation and rotational mechanical systems. |
| AEEB16.04 | Apply the block diagram and signal flow graph technique to determine transfer function of an control systems. |
| AEEB16.05 | Demonstrate the response of first order and second order systems with various standard test signals. |
| AEEB16.06 | Estimate the steady state error and its effect on the performance of control systems and gives the importance of PID controllers. |

| | |
|---|---|
| AEEB16.07 | Summarize the procedure of routh – hurwirtz criteria to steady the stability of electrical systems. |
| AEEB16.08 | List the steps required to draw root – locus of any control system and their by predict the stability. |
| AEEB16.09 | Explain the effect of adding zeros and poles to the transfer function of control system for improving stability. |
| AEEB16.10 | Discuss the method of bode plot and polar plot to calculate gain margin and phase margin of control system. |
| AEEB16.11 | Describe the characteristics of control system and its stability by drawing nyquist plot. |
| AEEB16.12 | Compare the behaviour of control system in terms of time domain and frequency domain response. |
| AEEB16.13 | Define the state model of control system using its block diagram and give the role of diagonalization in state space analysis. |
| AEEB16.14 | Formulate state transmission matrix and explain the concept of controllability and observability. |
| AEEB16.15 | Design lag, lead, lead – lag compensation to improve stability of control system/ |
| AEEB16.16 | Apply the concept of electromagnetic and electrostatic fields to solve real time world applications. |
| AEEB16.17 | Explore the knowledge and skills of employability to succeed in national and international level competitive examinations. |

## SYLLABUS

| MODULE-I | INTRODUCTION AND MODELING OF PHYSICAL SYSTEMS |
|---|---|
| Control systems: Introduction, open loop and closed loop systems, examples, comparison, mathematical modeling and differential equations of physical systems, concept of transfer function, translational and rotational mechanical systems, electrical systems, force, voltage and force, current analogy. | |
| **MODULE-II** | **BLOCK DIAGRAM REDUCTION AND TIME RESPONSE ANALYSIS** |
| Block Diagrams: Block diagram representation of various systems, block diagram algebra, characteristics of feedback systems, AC servomotor, signal flow graph, Mason‟s gain formula; Time response analysis: Standard test signals, shifted MODULE step, shifting theorem, convolution integral, impulse response, MODULE step response of first and second order systems, time response specifications, steady state errors and error constants, dynamic error coefficients method, effects of proportional, derivative and proportional derivative, proportional integral and PID controllers. | |
| **MODULE-III** | **CONCEPT OF STABILITY AND ROOT LOCUS TECHNIQUE** |

Concept of stability: Necessary and sufficient conditions for stability, Routh‟s and Routh Hurwitz stability criterions and limitations.

Root locus technique: Introduction, root locus concept, construction of root loci, graphical determination of „k‟ for specified damping ratio, relative stability, effect of adding zeros and poles on stability.

| MODULE-IV | FREQUENCY DOMAIN ANALYSIS |
|---|---|

Frequency domain analysis: Introduction, frequency domain specifications, stability analysis from Bode plot, Nyquist plot, calculation of gain margin and phase margin, determination of transfer function, correlation between time and frequency responses.

| MODULE-V | STATE SPACE ANALYSIS AND COMPENSATORS |
|---|---|

State Space Analysis: Concept of state, state variables and state model, derivation of state models from block diagrams, diagonalization, solving the time invariant state equations, state transition matrix and properties, concept of controllability and observability; Compensators: Lag, lead, lead - lag networks.

**Text Books:**


1. I J Nagrath, M Gopal, “Control Systems Engineering”, New Age International Publications, 3rd Edition, 2007.
2. K Ogata, “Modern Control Engineering”, Prentice Hall, 4th Edition, 2003.
3. N C Jagan, “Control Systems”, BS Publications, 1st Edition, 2007.


**Reference Books:**


1. Anand Kumar, “Control Systems”, PHI Learning, 1st Edition, 2007.
2. S Palani, “Control Systems Engineering”, Tata McGraw-Hill Publications, 1st Edition, 2001.
3. N K Sinha, “Control Systems”, New Age International Publishers, 1st Edition, 2002.

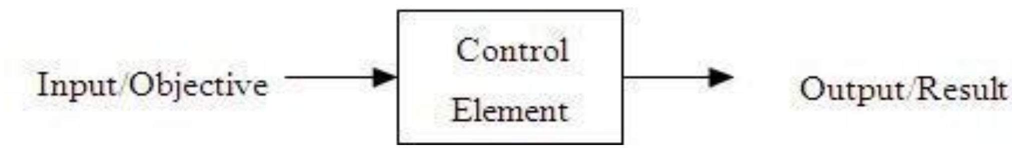
## MODULE - I
## INTRODUCTION AND MODELING OF PHYSICAL SYSTEMS

### 1.1 Basic elements of control system

In recent years, control systems have gained an increasingly importance in the development and advancement of the modern civilization and technology. Figure shows the basic components of a control system. Disregard the complexity of the system; it consists of an input (objective), the control system and its output (result). Practically our day-to-day activities are affected by some type of control systems. There are two main branches of control systems:

1) Open-loop systems and

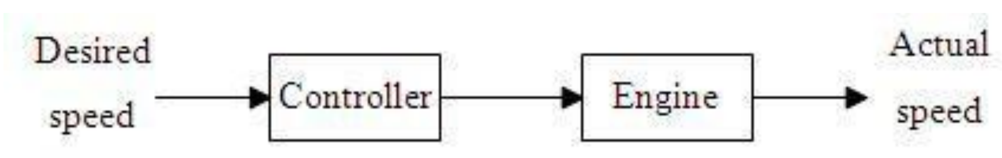
2) Closed-loop systems.




**Basic Components of Control System**

### 1.2 Open-loop systems:

The open-loop system is also called the non-feedback system. This is the simpler of the two systems. A simple example is illustrated by the speed control of an automobile as shown in Figure 1-2. In this open-loop system, there is no way to ensure the actual speed is close to the desired speed automatically. The actual speed might be way off the desired speed because of the wind speed and/or road conditions, such as uphill or downhill etc.




Basic Open Loop System

### Closed-loop systems:

The closed-loop system is also called the feedback system. A simple closed-system is shown in Figure 1-3. It has a mechanism to ensure the actual speed is close to the desired speed automatically.

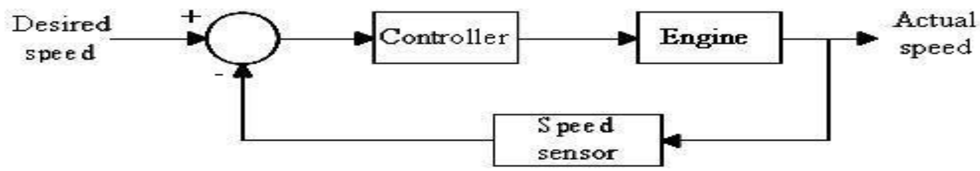



**Fig. 1-3.** Basic closed-loop **system**.

**Examples** − Traffic lights control system, washing machine

**Traffic lights control system** is an example of control system. Here, a sequence of input signal is applied to this control system and the output is one of the three lights that will be on for some duration of time. During this time, the other two lights will be off. Based on the traffic study at a particular junction, the on and off times of the lights can be determined. Accordingly, the input signal controls the output. So, the traffic lights control system operates on time basis.

**Classification of Control Systems**

Based on some parameters, we can classify the control systems into the following ways.

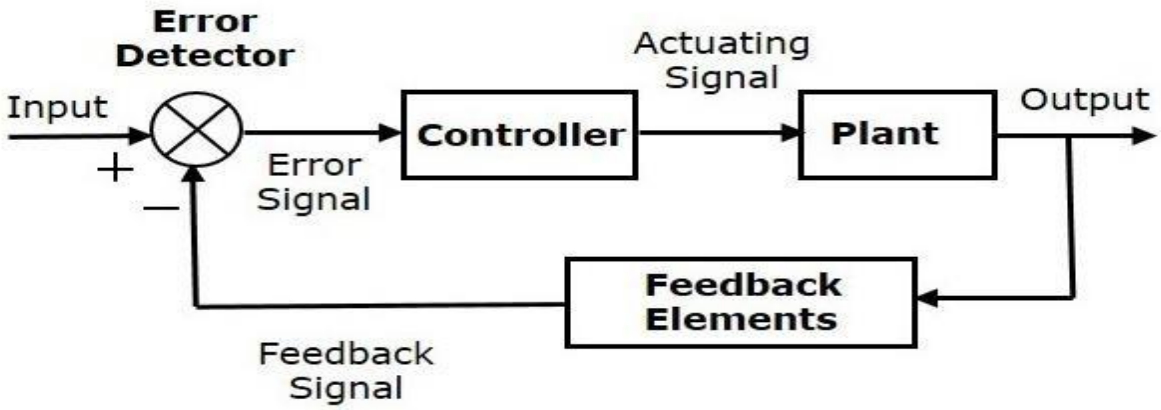
**Continuous time and Discrete-time Control Systems**

- Control Systems can be classified as continuous time control systems and discrete time control systems based on the **type of the signal** used.
- In **continuous time** control systems, all the signals are continuous in time. But, in **discrete time** control systems, there exists one or more discrete time signals.

**SISO and MIMO Control Systems**

- Control Systems can be classified as SISO control systems and MIMO control systems based on the **number of inputs and outputs** present.
- **SISO** (Single Input and Single Output) control systems have one input and one output. Whereas, **MIMO** (Multiple Inputs and Multiple Outputs) control systems have more than one input and more than one output.

The following figure shows the block diagram of negative feedback closed loop control system.




The error detector produces an error signal, which is the difference between the input and the feedback signal. This feedback signal is obtained from the block (feedback elements) by considering the output of the overall system as an input to this block. Instead of the direct input, the error signal is applied as an input to a controller.

So, the controller produces an actuating signal which controls the plant. In this combination, the output of the control system is adjusted automatically till we get the desired response. Hence, the closed loop control systems are also called the automatic control systems. Traffic lights control system having sensor at the input is an example of a closed loop control system.

The differences between the open loop and the closed loop control systems are mentioned in the following table.

| Open Loop Control Systems | Closed Loop Control Systems |
|---|---|
| Control action is independent of the desired output. | Control action is dependent of the desired output. |
| Feedback path is not present. | Feedback path is present. |
| These are also called as **non-feedback control systems**. | These are also called as **feedback control systems**. |
| Easy to design. | Difficult to design. |
| These are economical. | These are costlier. |
| Inaccurate. | Accurate. |

If either the output or some part of the output is returned to the input side and utilized as part of the system input, then it is known as **feedback**. Feedback plays an important role in order to improve the performance of the control systems. In this chapter, let us discuss the types of feedback & effects of feedback.

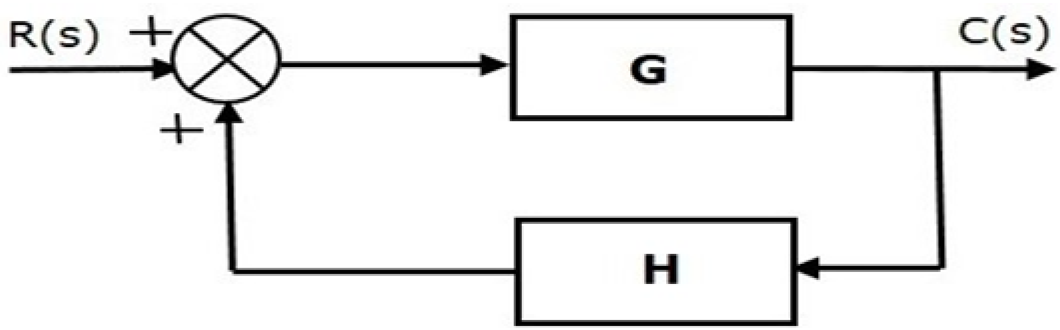
### Types of Feedback

There are two types of feedback −

- Positive feedback
- Negative feedback

### Positive Feedback

The positive feedback adds the reference input, R(s)R(s) and feedback output. The following figure shows the block diagram of **positive feedback control system**




The concept of transfer function will be discussed in later chapters. For the time being, consider the transfer function of positive feedback control system is,

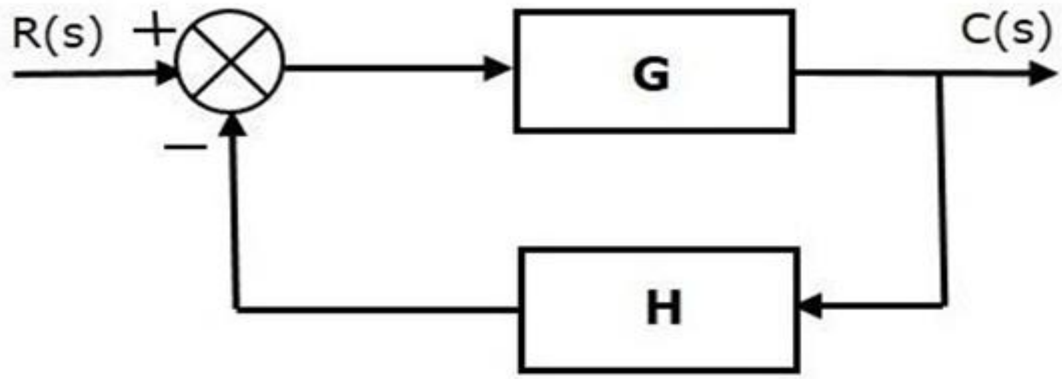
$$T = \frac{G}{1-GH} \qquad \text{(Equation 1)}$$

Where,

- **T** is the transfer function or overall gain of positive feedback control system.
- **G** is the open loop gain, which is function of frequency.A
- **H** is the gain of feedback path, which is function of frequency.

### Negative Feedback

Negative feedback reduces the error between the reference input, R(s)R(s) and system output. The following figure shows the block diagram of the **negative feedback control system**




Transfer function of negative feedback control system is,

$$T = \frac{G}{1+GH} \qquad \text{(Equation 2)}$$

Where,

- **T** is the transfer function or overall gain of negative feedback control system.
- **G** is the open loop gain, which is function of frequency.
- **H** is the gain of feedback path, which is function of frequency.

## Effects of Feedback

Let us now understand the effects of feedback.

### Effect of Feedback on Overall Gain

- From Equation 2, we can say that the overall gain of negative feedback closed loop control system is the ratio of 'G' and (1+GH). So, the overall gain may increase or decrease depending on the value of (1+GH).
- If the value of (1+GH) is less than 1, then the overall gain increases. In this case, 'GH' value is negative because the gain of the feedback path is negative.
- If the value of (1+GH) is greater than 1, then the overall gain decreases. In this case, 'GH' value is positive because the gain of the feedback path is positive.

In general, 'G' and 'H' are functions of frequency. So, the feedback will increase the overall gain of the system in one frequency range and decrease in the other frequency range.

### Effect of Feedback on Sensitivity

**Sensitivity** of the overall gain of negative feedback closed loop control system (**T**) to the variation in open loop gain (**G**) is defined as

$$S_G^T = \frac{\frac{\partial T}{T}}{\frac{\partial G}{G}} = \frac{Percentage\ change\ in\ T}{Percentage\ change\ in\ G} \quad \text{(Equation 3)}$$

Where, ∂T is the incremental change in T due to incremental change in G.

We can rewrite Equation 3 as

$$S_G^T = \frac{\partial T}{\partial G}\frac{G}{T} \quad \text{(Equation 4)}$$

Do partial differentiation with respect to G on both sides of Equation 2.

$$\frac{\partial T}{\partial G} = \frac{\partial}{\partial G}\left(\frac{G}{1+GH}\right) = \frac{(1+GH).1-G(H)}{(1+GH)^2} = \frac{1}{(1+GH)^2} \quad \text{(Equation 5)}$$

From Equation 2, you will get

$$\frac{G}{T} = 1 + GH \quad \text{(Equation 6)}$$

Substitute Equation 5 and Equation 6 in Equation 4.

$$S_G^T = \frac{1}{(1+GH)^2}(1+GH) = \frac{1}{1+GH}$$

So, we got the **sensitivity** of the overall gain of closed loop control system as the reciprocal of (1+GH). So, Sensitivity may increase or decrease depending on the value of (1+GH).

- If the value of (1+GH) is less than 1, then sensitivity increases. In this case, 'GH' value is negative because the gain of feedback path is negative.
- If the value of (1+GH) is greater than 1, then sensitivity decreases. In this case, 'GH' value is positive because the gain of feedback path is positive.

In general, 'G' and 'H' are functions of frequency. So, feedback will increase the sensitivity of the system gain in one frequency range and decrease in the other frequency range. Therefore, we have to choose the values of 'GH' in such a way that the system is insensitive or less sensitive to parameter variations.

## Effect of Feedback on Stability

- A system is said to be stable, if its output is under control. Otherwise, it is said to be unstable.
- In Equation 2, if the denominator value is zero (i.e., GH = -1), then the output of the control system will be infinite. So, the control system becomes unstable.

Therefore, we have to properly choose the feedback in order to make the control system stable.

## Effect of Feedback on Noise

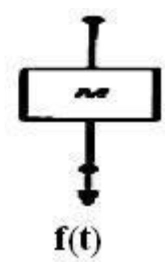
To know the effect of feedback on noise, let us compare the transfer function relations with and without feedback due to noise signal alone.

## 1.3 Mechanical Translational systems

The model of mechanical translational systems can obtain by using three basic elements mass, spring and dashpot. When a force is applied to a translational mechanical system, it is opposed by opposing forces due to mass, friction and elasticity of the system. The force acting on a mechanical body is governed by Newtons second law of motion. For translational systems it states that the sum of forces acting on a body is zero.

**Force balance equations of idealized elements:**

Consider an ideal mass element shown in fig. which has negligible friction and elasticity. Let a force be applied on it. The mass will offer an opposing force which is proportional to acceleration of a body.



**Mass**

Let f = applied force

fm =opposing force due to mass

Here $f_m \alpha M d^2x / dt^2$

By Newton's second law, $f = f_m = M d^2x / dt^2$

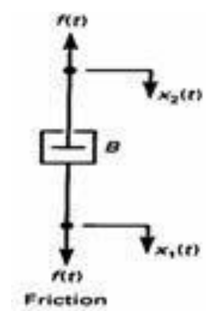
Consider an ideal frictional element dash-pot shown in fig. which has negligible mass and elasticity. Let a force be applied on it. The dashpot will be offer an opposing force which is proportional to velocity of the body.

Let f = applied force

f b = opposing force due to

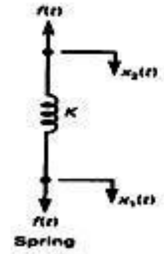
friction Here, f b α B dx / dt

By Newton's second law, $f = f_b = M\, d\, x / dt$



Friction

Consider an ideal elastic element spring is shown in fig. This has negligible mass and friction.




Let f = applied force
f k = opposing force due to elasticity

Here, f k α x
By Newton_s second law, f = f k = *x*

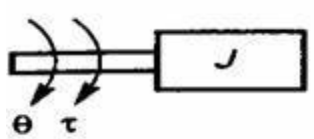
**Mechanical Rotational Systems:**

The model of rotational mechanical systems can be obtained by using three elements, moment of inertia [J] of mass, dash pot with rotational frictional coefficient [B] and torsional spring with stiffness[k].

When a torque is applied to a rotational mechanical system, it is opposed by opposing torques due to moment of inertia, friction and elasticity of the system. The torque acting on rotational mechanical bodies is governed by Newton_s second law of motion for rotational systems.

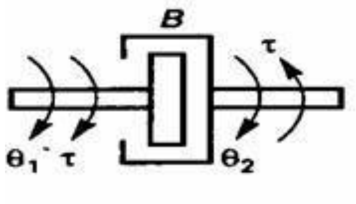
**Torque balance equations of idealized elements**

Consider an ideal mass element shown in fig. which has negligible friction and elasticity. The opposing torque due to moment of inertia is proportional to the angular acceleration.




Let T = applied torque
Tj =opposing torque due to moment of inertia of the body
Here $T_j = \alpha\, J\, d^2\theta / dt^2$
By Newton_s law
$T = T_j = J\, d^2\theta / dt^2$

Consider an ideal frictional element dash pot shown in fig. which has negligible moment of inertia and elasticity. Let a torque be applied on it. The dash pot will offer an opposing torque is proportional to angular velocity of the body.




Let T = applied torque

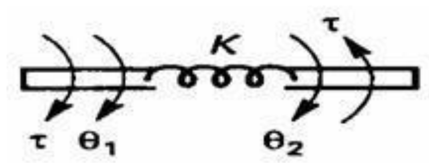
Tb =opposing torque due to friction

Here Tb = α B d / dt ($\theta_1$- $\theta_2$)

By Newton_s law

T= Tb = B d / dt ($\theta_1$- $\theta_2$)

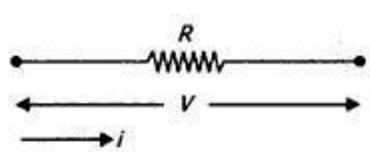
. Consider an ideal elastic element, torsional spring as shown in fig. which has negligible moment of inertia and friction. Let a torque be applied on it. The torsional spring will offer an opposing torque which is proportional to angular displacement of the body




Let T = applied torque
Tk =opposing torque due to friction
Here Tk α K ($\theta_1$- $\theta_2$)
By Newton_s law
T = Tk = K ($\theta_1$- $\theta_2$)

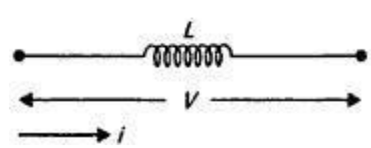
**Modeling of electrical system**

- Electrical circuits involving resistors, capacitors and inductors are considered. The behaviour of such systems is governed by Ohm's law and Kirchhoff's laws
- Resistor: Consider a resistance of R Ω carrying current i Amps as shown in Fig (a), then the

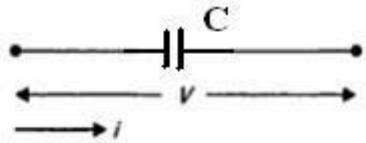



voltage drop across it is v = R I

- **Inductor:** Consider an inductor — L H carrying current i Amps as shown in Fig (a), then the voltage drop across it can be written as v = L di/dt



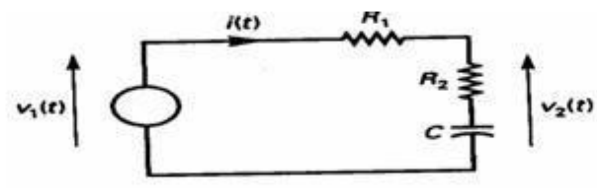
- **Capacitor:** Consider a capacitor _C_ F carrying current _i_ Amps as shown in Fig (a), then the voltage drop across it can be written as v = (1/C)∫ i dt




**Steps for modeling of electrical system**

- Apply Kirchhoff's voltage law or Kirchhoff's current law to form the differential equations describing electrical circuits comprising of resistors, capacitors, and inductors.
- Form Transfer Functions from the describing differential equations.
- Then simulate the model.

**Example**

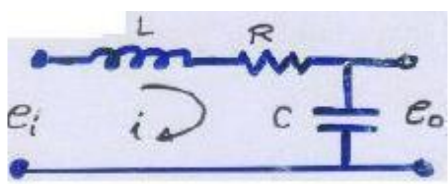



$$R_1\, i(t) + R_2\, i(t) + 1/C \int i(t)\, dt = V_1(t)$$
$$R_2\, i(t) + 1/C \int i(t)\, dt = V_2(t)$$

Electrical systems

RLC circuit. Applying Kirchhoff's voltage law to the system shown. We obtain the following equation;

**Resistance circuit**




$L(di/dt) + Ri + 1/C \int i(t)\, dt = e_i$ .................................. (1)

$1/C \int i(t)\, dt = e_0$ ........................................................... (2)

Equation (1) & (2) give a mathematical model of the circuit. Taking the L.T. of equations (1)&(2), assuming zero initial conditions, we obtain

$$LsI(s) + RI(s) + \frac{1}{C}\frac{1}{s}I(s) = E_i(s)$$

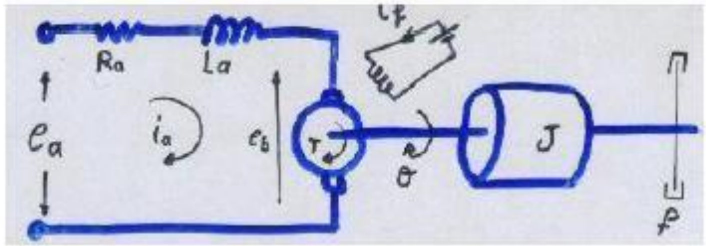
$$\frac{1}{C}\frac{1}{s}I(s) = E_o(s)$$

$$\textit{the transfer function} \quad \frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

**Armature-Controlled dc motors**

The dc motors have separately excited fields. They are either armature-controlled with fixed field or field-controlled with fixed armature current. For example, dc motors used in instruments employ a fixed permanent-magnet field, and the controlled signal is applied to the armature terminals.

Consider the armature-controlled dc motor shown in the following figure.




Ra = armature-winding resistance, ohms

La = armature-winding inductance, henrys

ia = armature-winding current, amperes

if = field current, a-pares

ea = applied armature voltage, volt

eb = back emf, volts

θ = angular displacement of the motor shaft, radians

T = torque delivered by the motor, Newton*meter

J = equivalent moment of inertia of the motor and load referred to the motor shaft kg.m2

f = equivalent viscous-friction coefficient of the motor and load referred to the motor shaft. Newton*m/rad/s
T = k1 ia ψ where ψ is the air gap flux, ψ = kf if , k1 is constant
For the constant flux

$$e_b = k_b \frac{d\vartheta}{dt}$$

Where Kb is a back emf constant -------------- (1)
The differential equation for the armature circuit

$$L_a \frac{di_a}{dt} + R_a i_a + e_b = e_a \text{........(2)}$$

The armature current produces the torque which is applied to the inertia and friction; hence

$$\frac{Jd^2\vartheta}{dt^2} + f\frac{d\vartheta}{dt} = T = Ki_a \text{..........(3)}$$

Assuming that all initial conditions are condition are zero/and taking the L.T. of equations (1), (2) & (3), we obtain

$K_p s\ \theta\ (s) = E_b\ (s)$

$(L_a s+Ra\ )\ I_a(s) + E_b\ (s) = E_a\ (s)\ (Js^2$

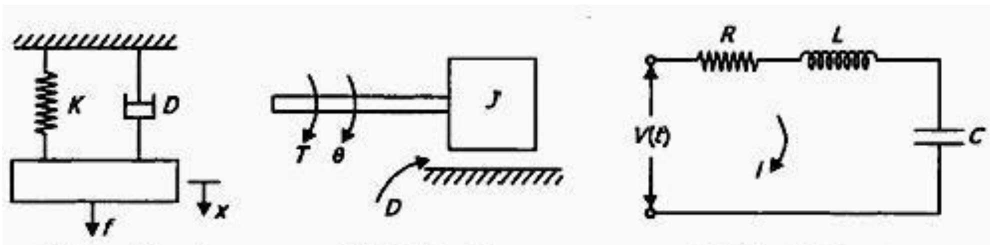
$\theta(s) = T(s) = K\,I_a(s)$

The T.F can be obtained is +fs)

$$\frac{\theta(s)}{E_a(s)} = \frac{K}{s(L_aJs^2 + (L_af + R_aJ)s + R_af + KK_b)}$$

**Analogous Systems**

Let us consider a mechanical (both translational and rotational) and electrical system as shown in the fig.




(a) Translational (b) Rotational (c) Electrical system

From the fig (a)

We get $M\,d^2x/dt^2 + D\,dx/dt + Kx = f$

From the fig (b)

We get $M\,d^2\theta/dt^2 + D\,d\theta/dt + K\theta = T$

From the fig (c)

We get $L\,d^2q/dt^2 + R\,dq/dt + (1/C)\,q = V(t)$

Where $q = \int i\,dt$

They are two methods to get analogous system. These are (i) force- voltage (f-v) analogy and (ii) force-current (f-c) analogy

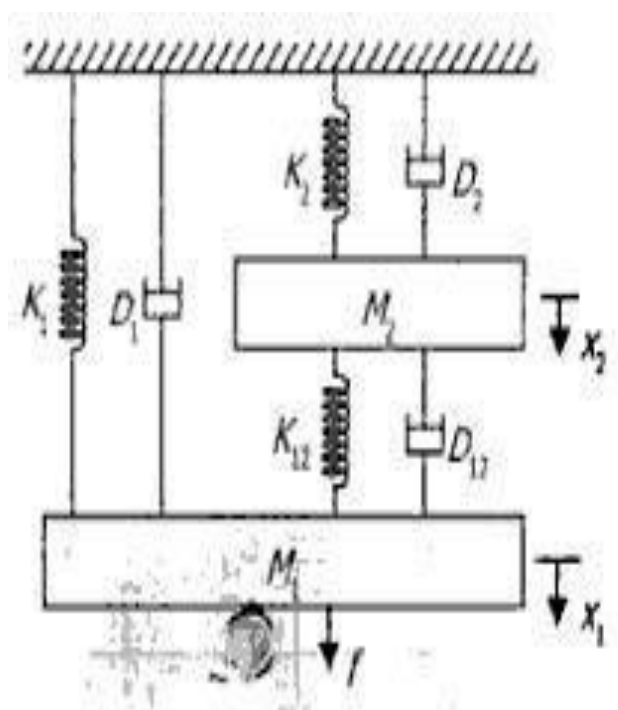
| Translational | Electrical | Rotational |
|---|---|---|
| Force ($f$) | Voltage ($v$) | Torque ($T$) |
| Mass ($M$) | Inductance ($L$) | Inertia ($J$) |
| Damper ($D$) | Resistance ( $R$) | Damper ($D$) |
| Spring ($K$) | Elastance $\left(\frac{1}{C}\right)$ | Spring ($K$) |
| Displacement ($x$) | Charge ($q$) | Displacement ($\theta$) |
| Velocity ($u$) | Current ($i$) | Velocity ($\omega$) |

**Force –Voltage Analogy**
**Force – Current Analog**

| Translational | Electrical | Rotational |
|---|---|---|
| Force ($f$) | Current ($i$) | Torque ($T$) |
| Mass ($M$) | Capacitance ($C$) | Inertia ($J$) |
| Spring ($K$) | Reciprocal of Inductance $\left(\frac{1}{L}\right)$ | Damper ($D$) |
| Damper ($D$) | Conductance $\left(\frac{1}{K}\right)$ | Spring ($K$) |
| Displacement ($x$) | Flux Linkage ($\psi$) | Displacement ($\theta$) |
| Velocity $\left(u = \frac{dx}{dt}\right)$ | Voltage ($v$) = $\frac{d\psi}{dt}$ | Velocity $\left(\omega = \frac{d\theta}{dt}\right)$ |

**Problem**

1. Find the system equation for system shown in the fig. And also determine f-v and f-i analogies



For free body diagram M1

$$f = M_1 \frac{d^2x_1}{dt^2} + D_1 \frac{dx_1}{dt} + K_1 x_1 + D_{12} \frac{d}{dt}(x_1 - x_2) + K_{12}(x_1 - x_2) \qquad (1)$$

For free body diagram M2

$$K_{12}(x_1 - x_2) + D_{12} \frac{d}{dt}(x_1 - x_2) = M_2 \frac{d^2x_2}{dt^2} + D_2 \frac{dx_2}{dt} + K_2 x_2 \qquad (2)$$

Force –voltage analogy

$$f \to v,\ M \to L,\ D \to R,\ K \to \frac{1}{C},\ x \to q$$

From eq (1) we get

$$v = L_1 \frac{d^2q_1}{dt^2} + R_1 \frac{dq_1}{dt} + \frac{1}{C_1} q_1 + R_{12} \frac{d}{dt}(q_1 - q_2) + \frac{1}{C_{12}}(q_1 - q_2)$$

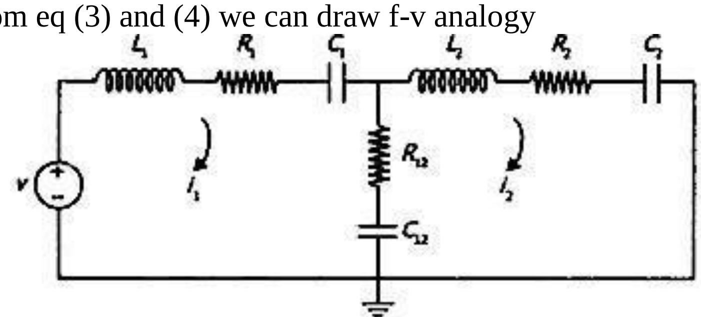
$$v = L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int i_1 dt + R_{12}(i_1 - i_2) + \frac{1}{C_{12}}\int (i_1 - i_2) dt \qquad (3)$$

From eq (2) we get

$$\frac{1}{C_{12}}(q_1 - q_2) + R_{12} \frac{d}{dt}(q_1 - q_2) = L_2 \frac{d^2q_2}{dt^2} + R_2 \frac{dq_2}{dt} + \frac{1}{C_2} q_2$$

$$\frac{1}{C_{12}}\int (i_1 - i_2) dt + R_{12}(i_1 - i_2) = L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_2}\int i_2 dt \qquad \text{…..(4)}$$

From eq (3) and (4) we can draw f-v analogy




Force–current analogy

$$f \to i,\ M \to C,\ D \to \frac{1}{R},\ K \to \frac{1}{L},\ x \to \psi$$

From eq (1) we get

$$i = C_1 \frac{d^2\psi_1}{dt^2} + \frac{1}{R_1}\frac{d\psi_1}{dt} + \frac{1}{L_1}\psi_1 + \frac{1}{R_{12}}\frac{d}{dt}(\psi_1 - \psi_2) + \frac{1}{L_{12}}(\psi_1 - \psi_2)$$

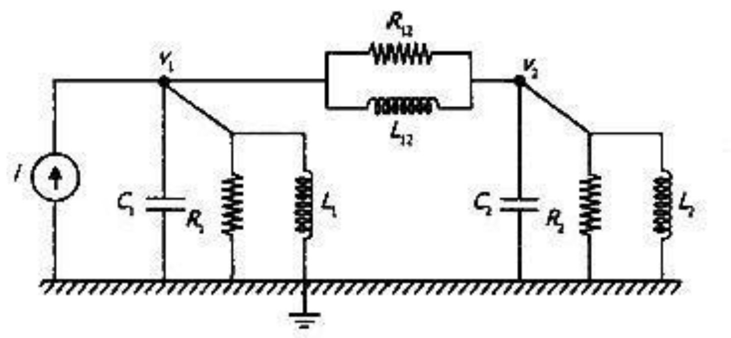
$$i = C_1 \frac{dv_1}{dt} + \frac{1}{R_1} v_1 + \frac{1}{L_1}\int i_1 dt + \frac{v_1 - v_2}{R_{12}} + \frac{1}{L_{12}}\int (v_1 - v_2) dt \qquad \text{……..(5)}$$

From eq (2) we get

$$\frac{1}{L_{12}}(\psi_1 - \psi_2) + \frac{1}{R_{12}}\frac{d}{dt}(\psi_1 - \psi_2) = C_2 \frac{d^2\psi_2}{dt^2} + \frac{1}{R_2}\frac{d\psi_2}{dt} + \frac{1}{L_{12}}\psi_2$$

$$\frac{1}{L_{12}}\int (v_1 - v_2) dt + \frac{1}{R_{12}}(v_1 - v_2) = C_2 \frac{dv_2}{dt^2} + \frac{v_2}{R_2} + \frac{1}{L_{12}}\int v_2 dt \qquad \text{…………(6)}$$

From eq (5) and (6) we can draw force-current analogy




The system can be represented in two forms:

- Block diagram representation
- Signal flow graph

**1.4 Transfer Function**

- A simpler system or element maybe governed by first order or second order differential equation. When several elements are connected in sequence, say ―n‖ elements, each one with first order, the total order of the system will be nth order
- In general, a collection of components or system shall be represented by nth order differential equation.

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + \cdots + b_0 u(t)$$

- In control systems, transfer function characterizes the input output relationship of components or systems that can be described by Liner Time Invariant Differential Equation
- In the earlier period, the input output relationship of a device was represented graphically.
- In a system having two or more components in sequence, it is very difficult to find graphical relation between the input of the first element and the output of the last element. This problem is solved by transfer function

**Definition of Transfer Function:**

Transfer function of a LTIV system is defined as the ratio of the Laplace Transform of the output variable to the Laplace Transform of the input variable assuming all the initial condition as zero.

**Properties of Transfer Function:**

- The transfer function of a system is the mathematical model expressing the differential equation that relates the output to input of the system.

- The transfer function is the property of a system independent of magnitude and the nature of the input.
- The transfer function includes the transfer functions of the individual elements. But at the same time, it does not provide any information regarding physical structure of the system.
- The transfer functions of many physically different systems shall be identical.
- If the transfer function of the system is known, the output response can be studied for various types of inputs to understand the nature of the system.
- If the transfer function is unknown, it may be found out experimentally by applying known inputs to the device and studying the output of the system.

**How you can obtain the transfer function (T. F.):**

- Write the differential equation of the system.
- Take the L. T. of the differential equation, assuming all initial condition to be zero.
- Take the ratio of the output to the input. This ratio is the T. F.

**Mathematical Model of control systems**

A control system is a collection of physical object connected together to serve an objective. The mathematical model of a control system constitutes a set of differential equation.

**1.5 Synchros**

A commonly used error detector of mechanical positions of rotating shafts in AC control systems is the Synchro.

It consists of two electro mechanical devices.

- Synchro transmitter
- Synchro receiver or control transformer.

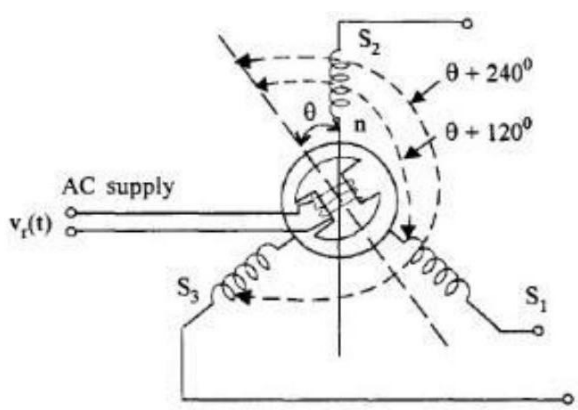
The principle of operation of these two devices is sarne but they differ slightly in their construction.

- The construction of a Synchro transmitter is similar to a phase alternator.
- The stator consists of a balanced three phase winding and is star connected.

The rotor is of dumbbell type construction and is wound with a coil to produce a magnetic field.

When a no voltage is applied to the winding of the rotor, a magnetic field is produced.

- 
- The coils in the stator link with this sinusoidal distributed magnetic flux and voltages are induced in the three coils due to transformer action.
- Than the three voltages are in time phase with each other and the rotor voltage.
- The magnitudes of the voltages are proportional to the cosine of the angle between the rotor position and the respective coil axis.
- The position of the rotor and the coils are shown in Fig.




$$v_R(t) = v_r \text{ Sin } \omega_r t$$
$$v_{s_{1n}} = KV_r \text{ Sin } \omega_r t \cos(\theta + 120)$$
$$v_{s_{2n}} = KV_r \text{ Sin } \omega_r t \cos\theta$$
$$v_{s_{3n}} = KV_r \text{ Sin } \omega_r t \cos(\theta + 240)$$
$$v_{s_1s_2} = v_{s_1n} - v_{s_2n} = \sqrt{3}\ KV_r \text{ Sin } (\theta + 240) \text{ Sin } \omega_r t$$
$$v_{s_2s_3} = v_{s_2n} - v_{s_3n} = \sqrt{3}\ KV_r \text{ Sin } (\theta + 120) \text{ Sin } \omega_r t$$
$$v_{s_3s_1} = v_{s_3n} - v_{s_1n} = \sqrt{3}\ KV_r \text{ Sin } \theta \text{ Sin } \omega_r t$$

- when x$\theta$is of the magnetic field coincides with the axis of coil S2 and maximum voltage is induced in it as seen.
- For this position of the rotor, the voltage c, is zero, this position of the rotor is known as the 'Electrical Zero' of die transmitter and is taken as reference for specifying the rotor position.
- In summary, it can be seen that the input to the transmitter is the angular position of the rotor and the set of three single phase voltages is the output.

W The magnitudes of these voltages depend on the angular position of the rotor as given

Now consider these three voltages to he applied to the stator of a similar device called control transformer or synchro receiver.

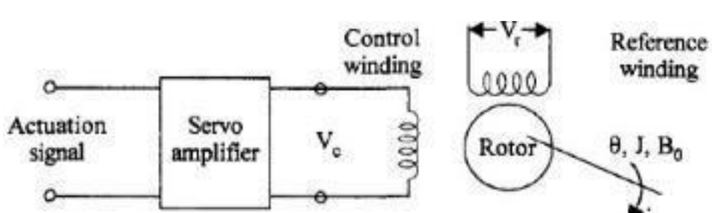
$$e_r(t) = K_1 V_r \text{Cos } \phi \sin \omega_r t$$

- The construction of a control transformer is similar to that of the transmitter except that the rotor is made cylindrical in shape whereas the rotor of transmitter is dumbbell in shape.
- Since the rotor is cylindrical, the air gap is uniform and the reluctance of the magnetic path is constant.

  This makes the output impedance of rotor to be a constant.
- 
- Usually the rotor winding of control transformer is connected teas amplifier which requires signal with constant impedance for better performance.
- A synchro transmitter is usually required to supply several control transformers and hence the stator winding of control transformer is wound with higher impedance per phase.
- Since the some currents flow through the stators of the synchro transmitter and receiver, the same pattern of flux distribution will be produced in the air gap of the control transformer.
- The control transformer flux axis is in the same position as that of the synchro transmitter.
- Thus the voltage induced in the rotor coil of control transformer is proportional to the cosine of the angle between the two rotors.

**1.6 AC Servo Motors**

- An AC servo motor is essentially a two phase induction motor with modified constructional features to suit servo applications.
- The schematic of a two phase or servo motor is shown

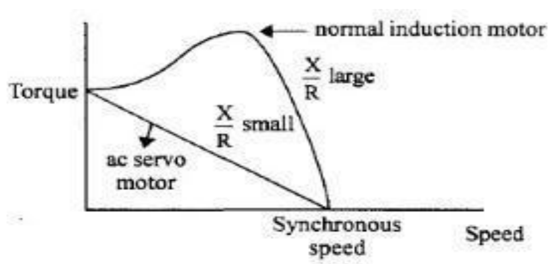



- It has two windings displaced by $90^o$ on the stator One winding, called as reference winding, is supplied with a constant sinusoidal voltage.

- The second winding, called control winding, is supplied with a variable control voltage which is displaced by -- 90 out of phase from the reference voltage.

The major differences between the normal induction motor and an AC servo motor are

- The rotor winding of an ac servo motor has high resistance (R) compared to its inductive reactance (X) so that its X / *R* ratio is very low.
- For a normal induction motor, X / *R* ratio is high so that the maximum torque is obtained in normal operating region which is around 5% of slip.
- The torque speed characteristics of a normal induction motor and an ac servo motor are shown in fig

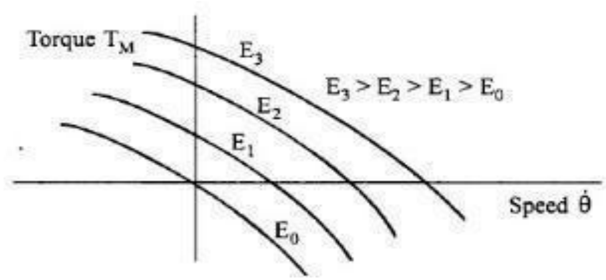



- The Torque speed characteristic of a normal induction motor is highly nonlinear
- and has a positive slope for some portion of the curve.
- This is not desirable for control applications. as the positive slope makes the systems unstable. The torque speed characteristic of an ac servo motor is fairly linear and has
- negative slope throughout.
- The rotor construction is usually squirrel cage or drag cup type for an ac servo motor. The diameter is small compared to the length of the rotor which reduces inertia of the moving parts.

  Thus it has good accelerating characteristic and good dynamic response.

- 
- The supplies to the two windings of ac servo motor are not balanced as in the case of a normal induction motor.
- The control voltage varies both in magnitude and phase with respect to the constant reference vulture applied to the reference winding.
- The direction of rotation of the motor depends on the phase (± 90°) of the control voltage with respect to the reference voltage.

- The torque varies approximately linearly with respect to speed and also controls voltage. The torque speed characteristics can be linearised at the operating point and the transfer function of the motor can be obtained.

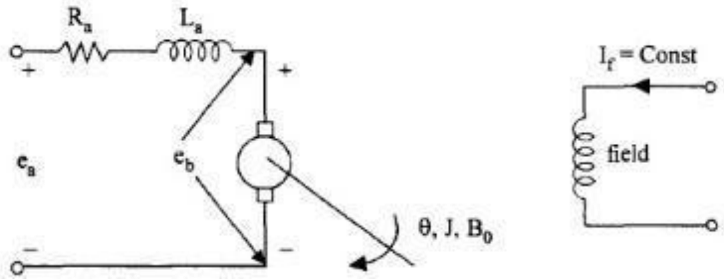



**DC Servo Motor**

- A DC servo motor is used as an actuator to drive a load. It is usually a DC motor of low power rating.
- DC servo motors have a high ratio of starting torque to inertia and therefore they have a faster dynamic response.
- DC motors are constructed using rare earth permanent magnets which have high residual flux density and high coercively.
- As no field winding is used, the field copper losses am zero and hence, the overall efficiency of the motor is high.
- The speed torque characteristic of this motor is flat over a wide range, as the armature reaction is negligible.
- Moreover speed in directly proportional to the armature voltage for a given torque.
- Armature of a DC servo motor is specially designed to have low inertia.
- In some application DC servo motors are used with magnetic flux produced by field windings.
- The speed of PMDC motors can be controlled by applying variable armature voltage.
- These are called armature voltage controlled DC servo motors.
- Wound field DC motors can be controlled by either controlling the armature voltage or controlling rho field current. Let us now consider modelling of these two types or DC servo motors.

**(a) Armature controlled DC servo motor**

The physical model of an armature controlled DC servo motor is given in




The armature winding has a resistance R a and inductance La.

- The field is produced either by a permanent magnet or the field winding is separately excited and supplied with constant voltage so that the field current If is a constant.
- When the armature is supplied with a DC voltage of e a volts, the armature rotates and produces a back e.m.f eb.
- The armature current ia depends on the difference of eb and en. The armature has a permanent of inertia J, frictional coefficient B0

  The angular displacement of the motor is 8.
- 
- The torque produced by the motor is given by

$$T = K_T\, i_a$$

Where K T is the motor torque constant.

The back emf is proportional to the speed of the motor and hence

$$e_b = K_b\, \dot{\theta}$$

The differential equation representing the electrical system is given by

$$R_a\, i_a + L_a \frac{di_a}{dt} + e_b = e_a$$

Taking Laplace transform of equation from above equation

$$T(s) = K_T\, I_a(s)$$

$$E_b(s) = K_b\, s\, \theta(s)$$

$$(R_a + s\, L_a)\, I_a(s) + E_b(s) = E_a(s)$$

$$I_a(s) = \frac{E_a(s) - K_b s\, \theta(s)}{R_a + sL_a}$$

The mathematical model of the mechanical system is given by

$$J\, \frac{d^2\theta}{dt^2} + B_0 \frac{d\theta}{dt} = T$$

Taking Laplace transform

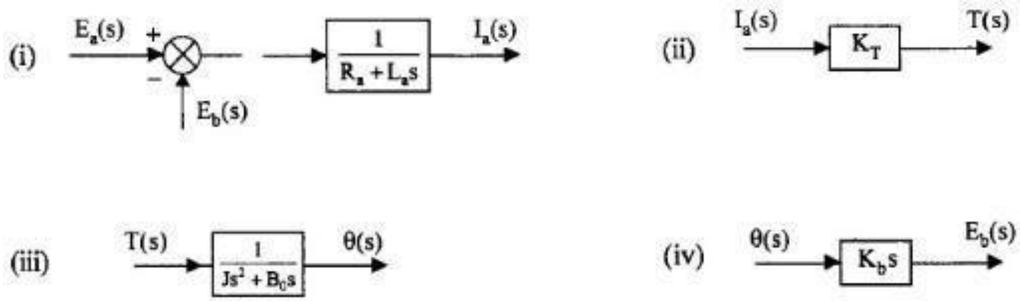
$$(Js^2 + B_0 s)\, \theta(s) = T(s)$$

$$\theta(s) = K_T\, \frac{E_a(s) - K_b s\, \theta(s)}{(R_a + sL_a)(Js^2 + B_0 s)}$$

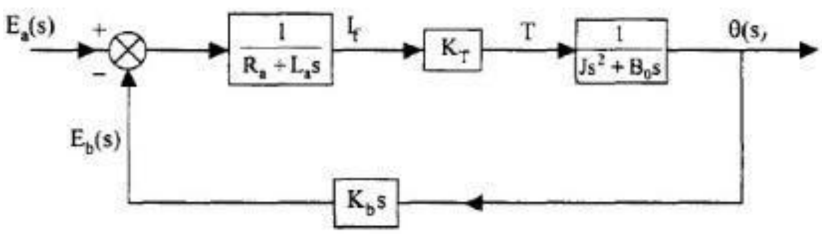
Solving for θ (*s*),we get

$$\theta(s) = \frac{K_T\, E_a(s)}{s[(R_a + sL_a)(Js + B_0) + K_T\, K_b]}$$

The block diagram representation of the armature controlled DC servo motor is developed in Steps




Combining these blocks we have




Usually the inductance of the armature winding is small and hence neglected

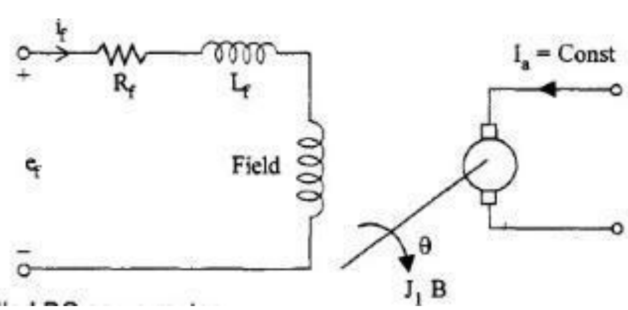
$$T(s) = \frac{\theta(s)}{E_a(s)} = \frac{K_T / R_a}{s\left[Js + B_0 + \frac{K_b K_T}{R_a}\right]}$$

$$= \frac{K_T / R_a}{s(Js + B)}$$

Where

$$B = B_0 + \frac{K_b K_T}{R_a}$$

**Field Controlled Dc Servo Motor**
**The field servo motor**




The electrical circuit is modeled as

$$I_f(s) = \frac{E_f(s)}{R_f + L_f s}$$

$$T(s) = K_T\, I_f(s)$$

$$(Js^2 + B_0)\,\theta(s) = T(s)$$

$$\frac{\theta(s)}{E_f(s)} = \frac{K_T}{s(Js + B_0)(R_f + L_f s)}$$

$$= \frac{K_T / R_f B_0}{s\left(\frac{J}{B_0} s + 1\right)\left(\frac{L_f}{R_f} s + 1\right)}$$

$$= \frac{K_m}{s(\tau_m s + 1)(\tau_f s + 1)}$$

Motor gain constant

$$K_m = K_T / R_f B_0$$

Motor time constant

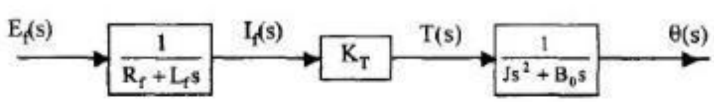
$$\tau_m = J/B_0$$

Field time constant

Where

$$\tau_f = L_f/R_f$$

The block diagram is as shown as



## MODULE – II

## BLOCK DIAGRAM REDUCTION AND TIME RESPONSE ANALYSIS

**Block diagram**

A pictorial representation of the functions performed by each component and of the flow of signals.

**Basic elements of a block diagram**

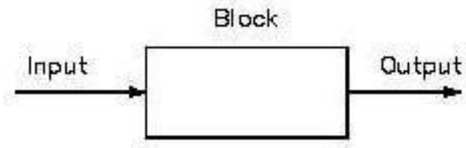
- Blocks
- Transfer functions of elements inside the blocks
- Summing points
- Take off points
- Arrow

**Block diagram**

A control system may consist of a number of components. A block diagram of a system is a pictorial representation of the functions performed by each component and of the flow of signals. The elements of a block diagram are block, branch point and summing point.

**Block**

In a block diagram all system variables are linked to each other through functional blocks. The functional block or simply block is a symbol for the mathematical operation on the input signal to the block that produces the output.




**Summing point**

Although blocks are used to identify many types of mathematical operations, operations of addition and subtraction are represented by a circle, called a summing point. As shown in Figure a summing point may have one or several inputs. Each input has its own appropriate plus or minus sign.

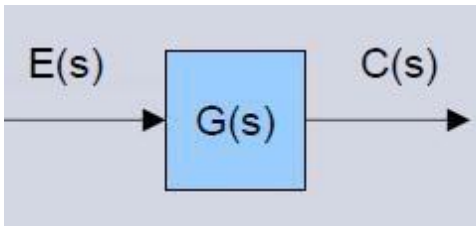
A summing point has only one output and is equal to the algebraic sum of the inputs.

A takeoff point is used to allow a signal to be used by more than one block or summing point. The transfer function is given inside the block

• The input in this case is E(s)

• The output in this case is C(s)

- C(s) = G(s) E(s)




**Functional block** – each element of the practical system represented by block with its T.F.

**Branches** – lines showing the connection between the blocks

**Arrow** – associated with each branch to indicate the direction of flow of signal

**Closed loop system**

**Summing point** – comparing the different signals

**Take off point** – point from which signal is taken for feed back

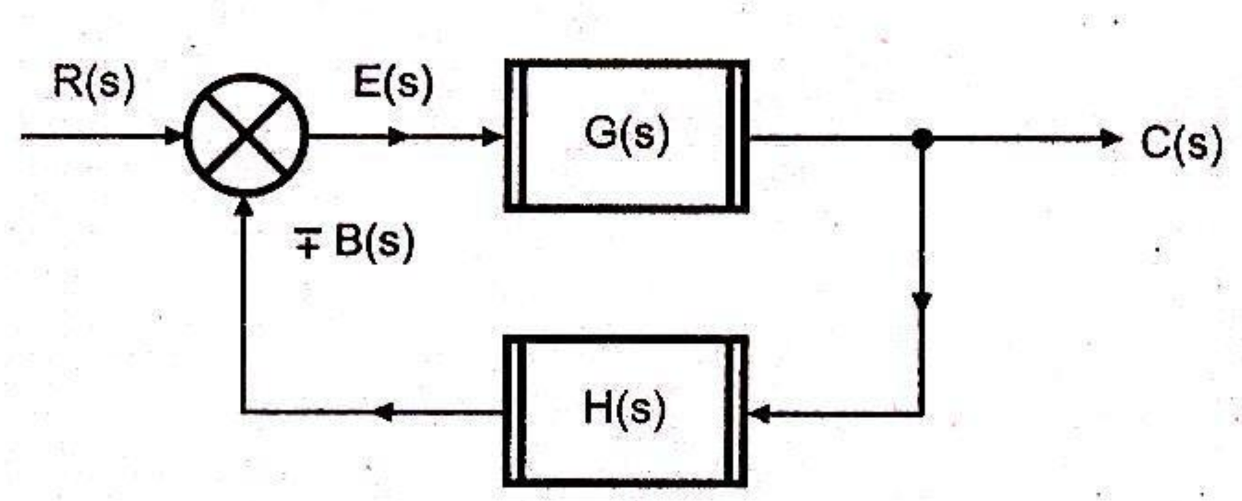
**Advantages of Block Diagram Representation**

- Very simple to construct block diagram for a complicated system
- Function of individual element can be visualized
- Individual & Overall performance can be studied
- Over all transfer function can be calculated easily.

**Disadvantages of Block Diagram Representation**

- No information about the physical construction
- Source of energy is not shown

**Simple or Canonical form of closed loop system**




R(s) – Laplace of reference input r(t)

C(s) – Laplace of controlled output c(t)

E(s) – Laplace of error signal e(t)

B(s) – Laplace of feed back signal b(t)

G(s) – Forward path transfer function

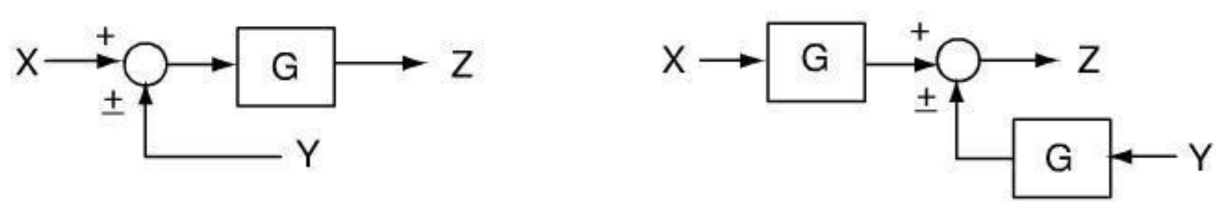
H(s) – Feed back path transfer function

**Block diagram reduction technique**

Because of their simplicity and versatility, block diagrams are often used by control engineers to describe all types of systems. A block diagram can be used simply to represent the composition and interconnection of a system. Also, it can be used, together with transfer functions, to represent the cause-and-effect relationships throughout the system. Transfer Function is defined as the relationship between an input signal and an output signal to a device.

**Block diagram rules**

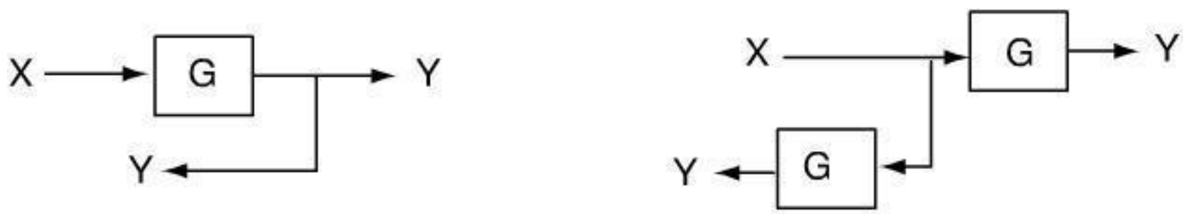
Cascaded blocks




Moving a summer beyond the block

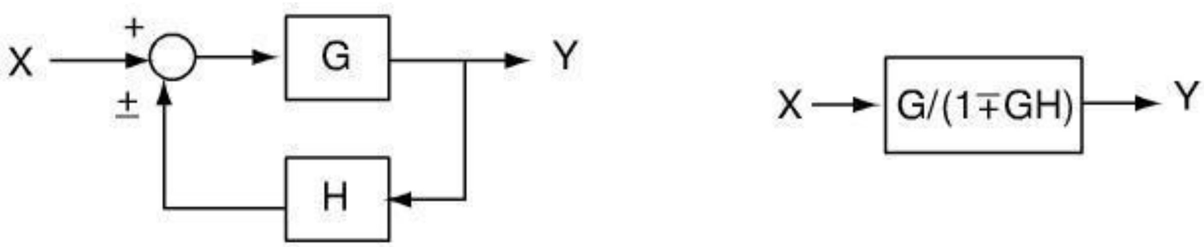
Moving a summer ahead of block


Moving a pick-off ahead of block

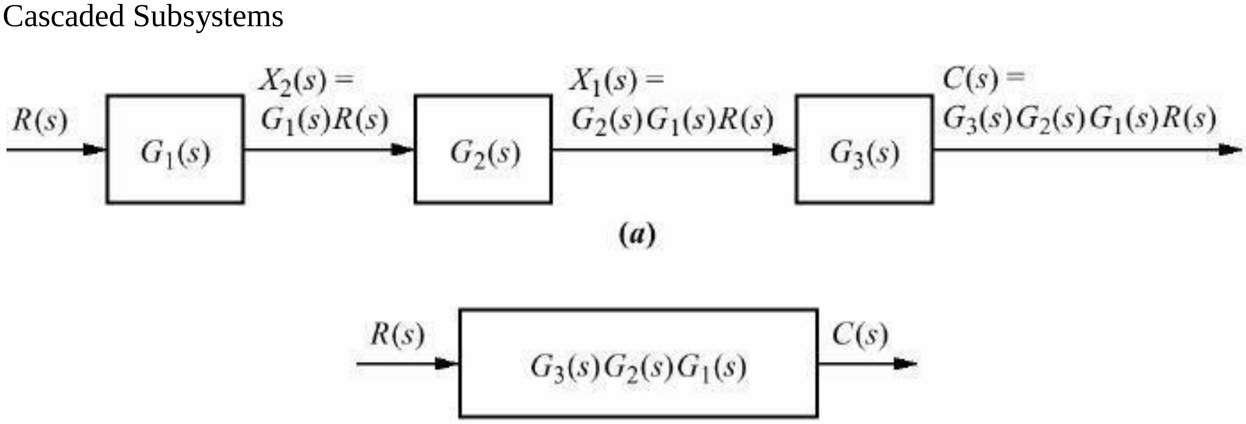



Moving a pick-off behind a block

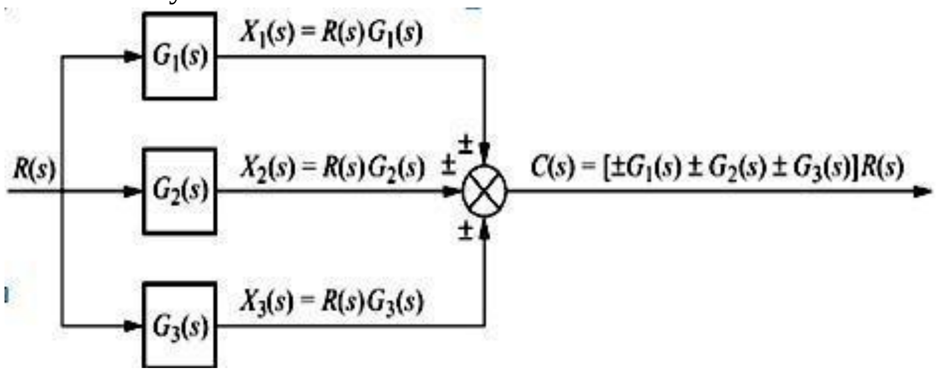

Eliminating a feedback loop



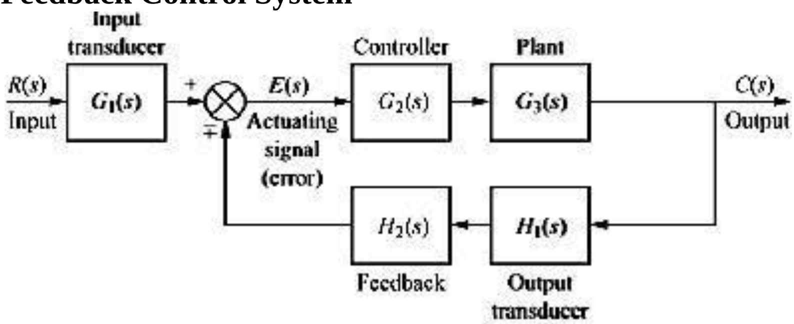

Cascaded Subsystems



Parallel Subsystems




**Feedback Control System**




**Procedure to solve Block Diagram Reduction**

**Problems** Step 1: Reduce the blocks connected in series

Step 2: Reduce the blocks connected in parallel

Step 3: Reduce the minor feedback loops

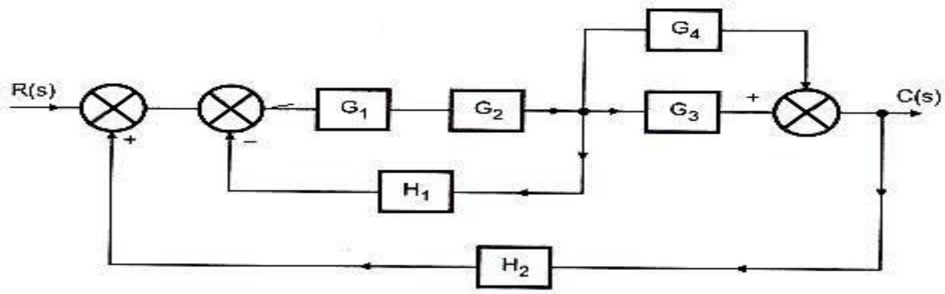
Step 4: Try to shift take off points towards right and Summing point towards left

Step 5: Repeat steps 1 to 4 till simple form is obtained

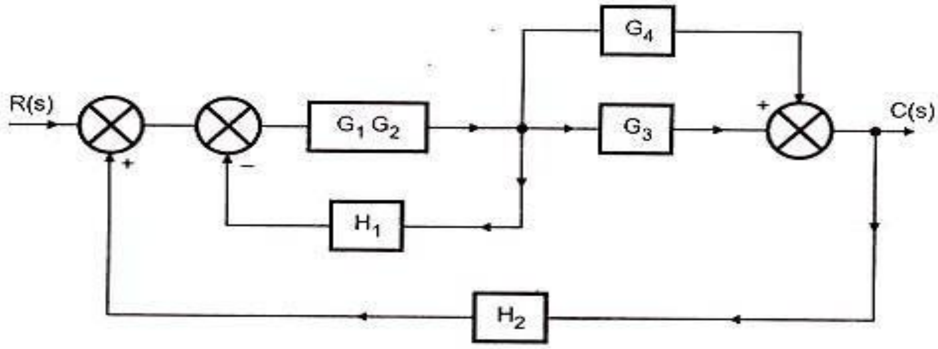
Step 6: Obtain the Transfer Function of Overall System

**Problem 1**

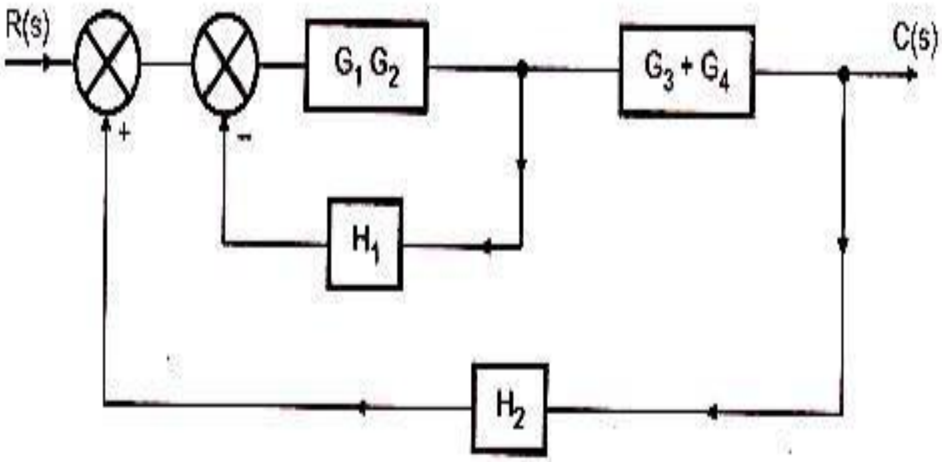
**Obtain the Transfer function of the given block diagram**



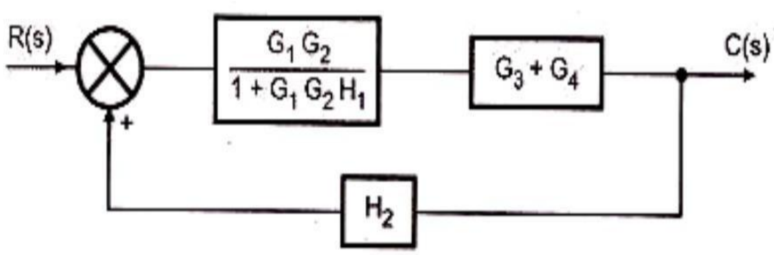

Combine G1, G2 which are in series

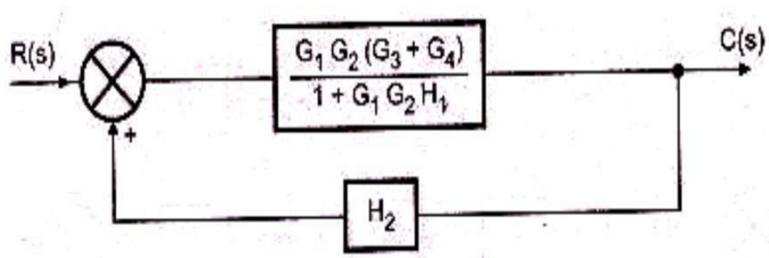


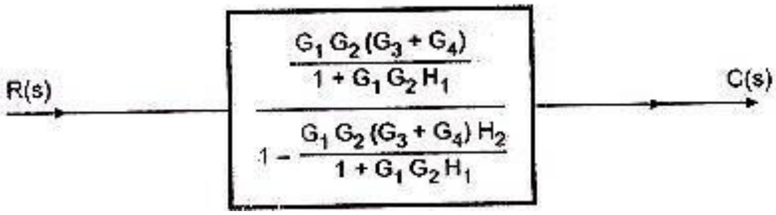

Combine G3, G4 which are in Parallel




Reduce minor feedback loop of G1, G2 and H1



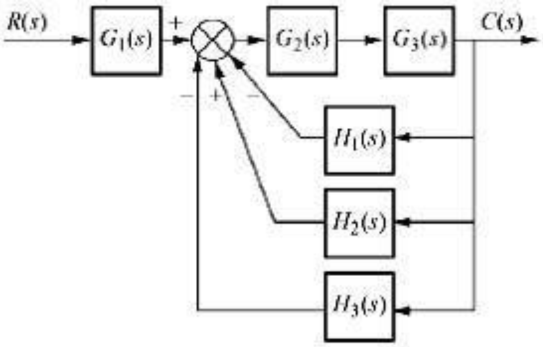





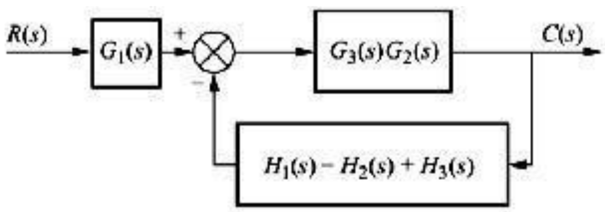

**Transfer function**

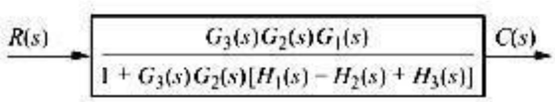
$$\frac{C(s)}{R(s)} = \frac{G_1 \ G_2 (G_3 + G_4)}{1 + G_1 G_2 H_1 - G_1 G_2 (G_3 + G_4) H_2}$$

**2. Obtain the transfer function for the system shown in the fig**

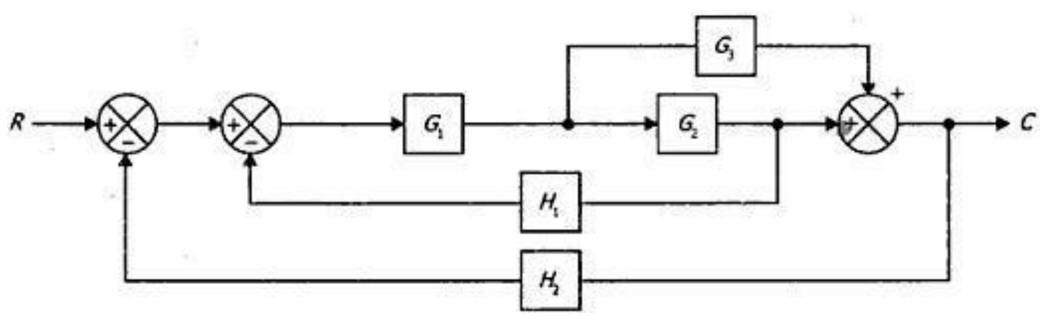



**Solution**



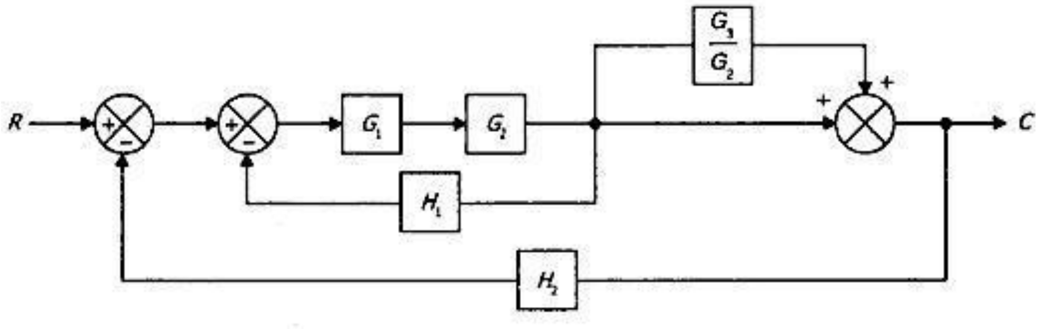




**3. Obtain the transfer function C/R for the block diagram shown in the fig**

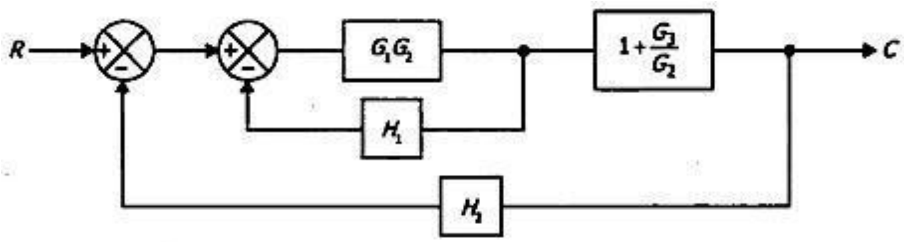


**Solution**

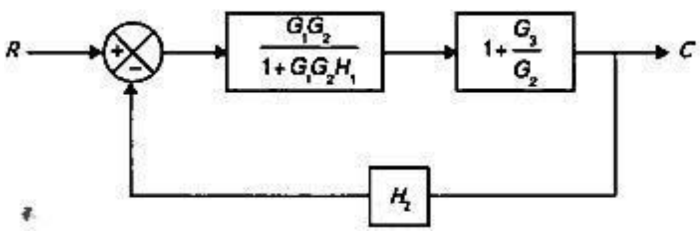
The take-off point is shifted after the block G2



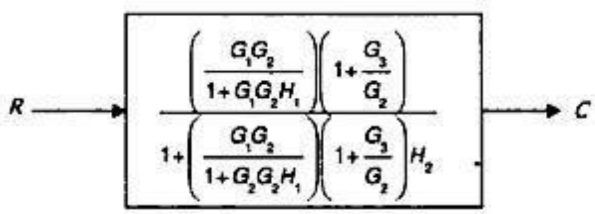

Reducing the cascade block and parallel block




Replacing the internal feedback loop




Equivalent block diagram




Transfer function

$$\frac{C}{R} = \frac{\dfrac{G_1\,(G_2+G_3)}{1+G_1G_2H_1}}{1+\dfrac{G_1(G_2+G_3)\,H_2}{1+G_1G_2H_1}}$$

$$= \frac{G_1(G_2+G_3)}{1+G_1G_2(H_1+H_2)+G_1G_3H_2}$$

## Signal Flow Graph Representation

Signal Flow Graph Representation of a system obtained from the equations, which shows the flow of the signal

### Signal flow graph

A signal flow graph is a diagram that represents a set of simultaneous linear algebraic equations. By taking Laplace transfer, the time domain differential equations governing a control system can be transferred to a set of algebraic equation in s-domain. A signal-flow graph consists of a network in which nodes are connected by directed branches. It depicts the flow of signals from one point of a system to another and gives the relationships among the signals.

### Basic Elements of a Signal flow graph

**Node -** a point representing a signal or variable.

**Branch** – unidirectional line segment joining two nodes.

**Path** – a branch or a continuous sequence of branches that can be traversed from one node to another node.

**Loop** – a closed path that originates and terminates on the same node and along the path no node is met twice.

**Nontouching loops** – two loops are said to be nontouching if they do not have a common node.

Mason’s gain formula

The relationship between an input variable and an output variable of signal flow graph is given by the net gain between the input and the output nodes is known as overall gain of the

$$M = \frac{1}{\Delta}\sum_{k=1}^{N} P_k \Delta_k = \frac{X_{out}}{X_{in}}$$

system. Mason_s gain rule for the determination of the overall system gain is given below.

Where M= gain between Xin and Xout

Xout =output node variable

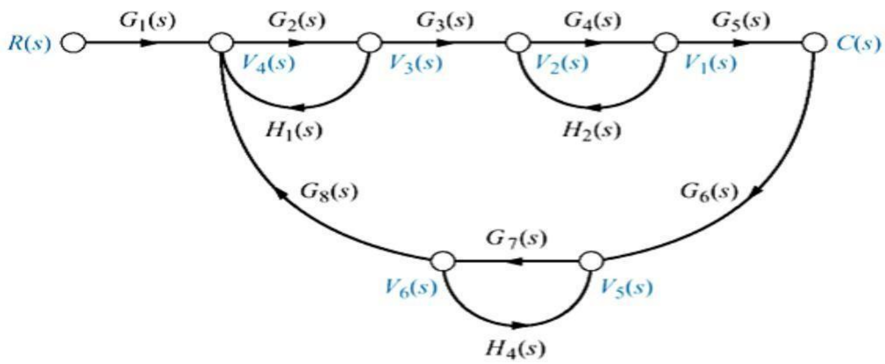
Xin= input node variable

N = total number of forward paths

Pk= path gain of the kth forward path

Δ=1-(sum of loop gains of all individual loop) + (sum of gain product of all possible combinations of two nontouching loops) – (sum of gain products of all possible combination of three nontouching loops)

**Problem**




- Forward path gain: $T_1 = G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)$
- Closed loop gain

(1) $G_2(s)H_1(s)$ (2) $G_4(s)H_2(s)$

(3) $G_7(s)H_4(s)$ (4) $G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)$

- Nontouching loops taken two at a time

(5) loop (1) and loop (2): $G_2(s)H_1(s)G_4(s)H_2(s)$

(6) loop (1) and loop (3): $G_2(s)H_1(s)G_7(s)H_4(s)$

(7) loop (2) and loop (3): $G_4(s)H_2(s)G_7(s)H_4(s)$

- Nontouching loops taken three at a time

(8) loops (1), (2), (3): $G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)$

- Now, $\Delta = 1 - \{(1) + (2) + (3) + (4)\} + \{(5) + (6) + (7)\} - (8)$
- Portion of $\Delta$ not touching the forward path

$$\Delta_1 = 1 - G_7(s)H_4(s)$$

- Hence,

$$G(s) = \frac{C(s)}{R(s)} = \frac{T_1\Delta_1}{\Delta}$$

$$= \frac{G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)[1 - G_7(s)H_4(s)]}{\Delta}$$

# TIME RESPONSE

## Introduction

- After deriving a mathematical model of a system, the system performance analysis can be done in various methods.
- In analyzing and designing control systems, a basis of comparison of performance of various control systems should be made. This basis may be set up by specifying particular test input signals and by comparing the responses of various systems to these signals.
- The system stability, system accuracy and complete evaluation are always based on the time response analysis and the corresponding results.
- Next important step after a mathematical model of a system is obtained.
- To analyze the system_s performance.
- Normally use the standard input signals to identify the characteristics of system_s
  - response Step function
  - Ramp function
  - Impulse function
  - Parabolic function
  - Sinusoidal function

## 2.10 Time response analysis

It is an equation or a plot that describes the behavior of a system and contains much information about it with respect to time response specification as overshooting, settling time, peak time, rise time and steady state error. Time response is formed by the transient response and the steady state response.

**Time response = Transient response + Steady state response**

**Transient time response** (Natural response) describes the behavior of the system in its first short time until arrives the steady state value and this response will be our study focus. If the input is step function then the output or the response is called step time response and if the input is ramp, the response is called ramp time response ... etc.

**Classification of Time Response**

- Transient response
- Steady state response

$y(t) = yt(t) + yss(t)$

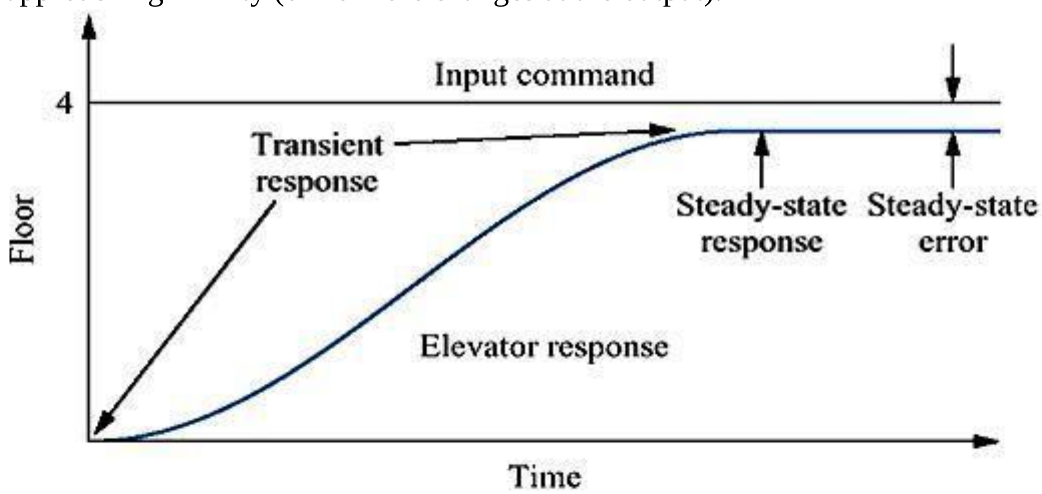
**Transient Response**

The transient response is defined as the part of the time response that goes to zero as time becomes very large. Thus yt(t) has the property

**Lim yt(t) = 0**

**t -->∞**

The time required to achieve the final value is called transient period. The transient response may be exponential or oscillatory in nature. Output response consists of the sum of forced response (form the input) and natural response (from the nature of the system).The transient response is the change in output response from the beginning of the response to the final state of the response and the steady state response is the output response as time is approaching infinity (or no more changes at the output).

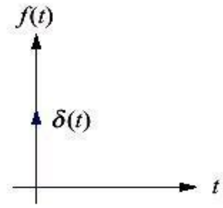
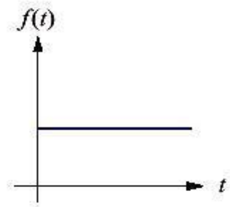
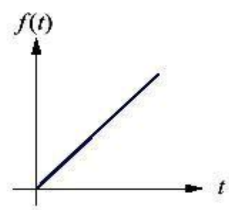
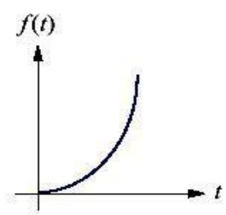
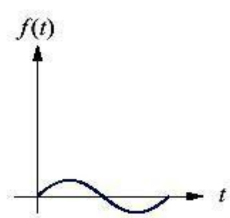



**Steady State Response**

The steady state response is the part of the total response that remains after the transient has died out. For a position control system, the steady state response when compared to with the desired reference position gives an indication of the final accuracy of the system. If the steady state response of the output does not agree with the desired reference exactly, the system is said to have steady state error.

**2.3 Typical Input Signals**

- Impulse Signal
- Step Signal
- Ramp Signal
- Parabolic Signal

| Input | Function | Description | Sketch | Use |
|---|---|---|---|---|
| Impulse | $\delta(t)$ | $\delta(t) = \infty$ for $0- < t < 0+$ <br> $= 0$ elsewhere <br> $\int_{0-}^{0+} \delta(t)\,dt = 1$ |  | Transient response <br> Modeling |
| Step | $u(t)$ | $u(t) = 1$ for $t > 0$ <br> $= 0$ for $t < 0$ |  | Transient response <br> Steady-state error |
| Ramp | $tu(t)$ | $tu(t) = t$ for $t \geq 0$ <br> $= 0$ elsewhere |  | Steady-state error |
| Parabola | $\frac{1}{2}t^2u(t)$ | $\frac{1}{2}t^2u(t) = \frac{1}{2}t^2$ for $t \geq 0$ <br> $= 0$ elsewhere |  | Steady-state error |
| Sinusoid | $\sin \omega t$ | |  | Transient response <br> Modeling <br> Steady-state error |

**Time Response Analysis & Design**

Two types of inputs can be applied to a control system.

Command Input or Reference Input yr(t).

Disturbance Input w(t) *(*External disturbances w(t) are typically uncontrolled variations in the load on a control system).

In systems controlling mechanical motions, load disturbances may represent forces.

In voltage regulating systems, variations in electrical load area major source of disturbances.

**Test Signals**

| Input | r(t) | R(s) |
|---|---|---|
| Step Input | A | A/s |
| Ramp Input | At | A/s |

| | | |
|---|---|---|
| Parabolic Input | $At^2 / 2$ | $A/s^3$ |
| Impulse Input | δ(t) | 1 |

**Transfer Function**

- One of the types of Modeling a system
- Using first principle, differential equation is obtained
- Laplace Transform is applied to the equation assuming zero initial conditions
- Ratio of LT (output) to LT (input) is expressed as a ratio of polynomial in *s* in the transfer function.

**Order of a system**

- The Order of a system is given by the order of the differential equation governing the system
- Alternatively, order can be obtained from the transfer function
- In the transfer function, the maximum power of s in the denominator polynomial gives the order of the system.

**Dynamic Order of Systems**

- Order of the system is the order of the differential equation that governs the dynamic behaviour
- Working interpretation: Number of the dynamic elements / capacitances or holdup elements between a
- manipulated variable and a controlled variable
- Higher order system responses are usually very difficult to resolve from one another The response generally becomes sluggish as the order increases.

**System Response**

First-order system time response

**First Order System**

$Y s / R(s) = K / (1+ K+sT) = K / (1+sT)$

**Step Response of First Order System**

Evolution of the transient response is determined by the pole of the transfer function at s=-1/t where t is the time constant

Also, the step response can be found:

| | | |
|---|---|---|
| **Impulse response** | K / (1+sT) | **Exponential** |
| **Step response** | (K/S) – (K / (S+(1/T))) | **Step, exponential** |
| **Ramp response** | $(K/S^2)$-(KT / S)- (KT / (S+1/T)) | **Ramp, step, exponential** |

**Second-order systems**

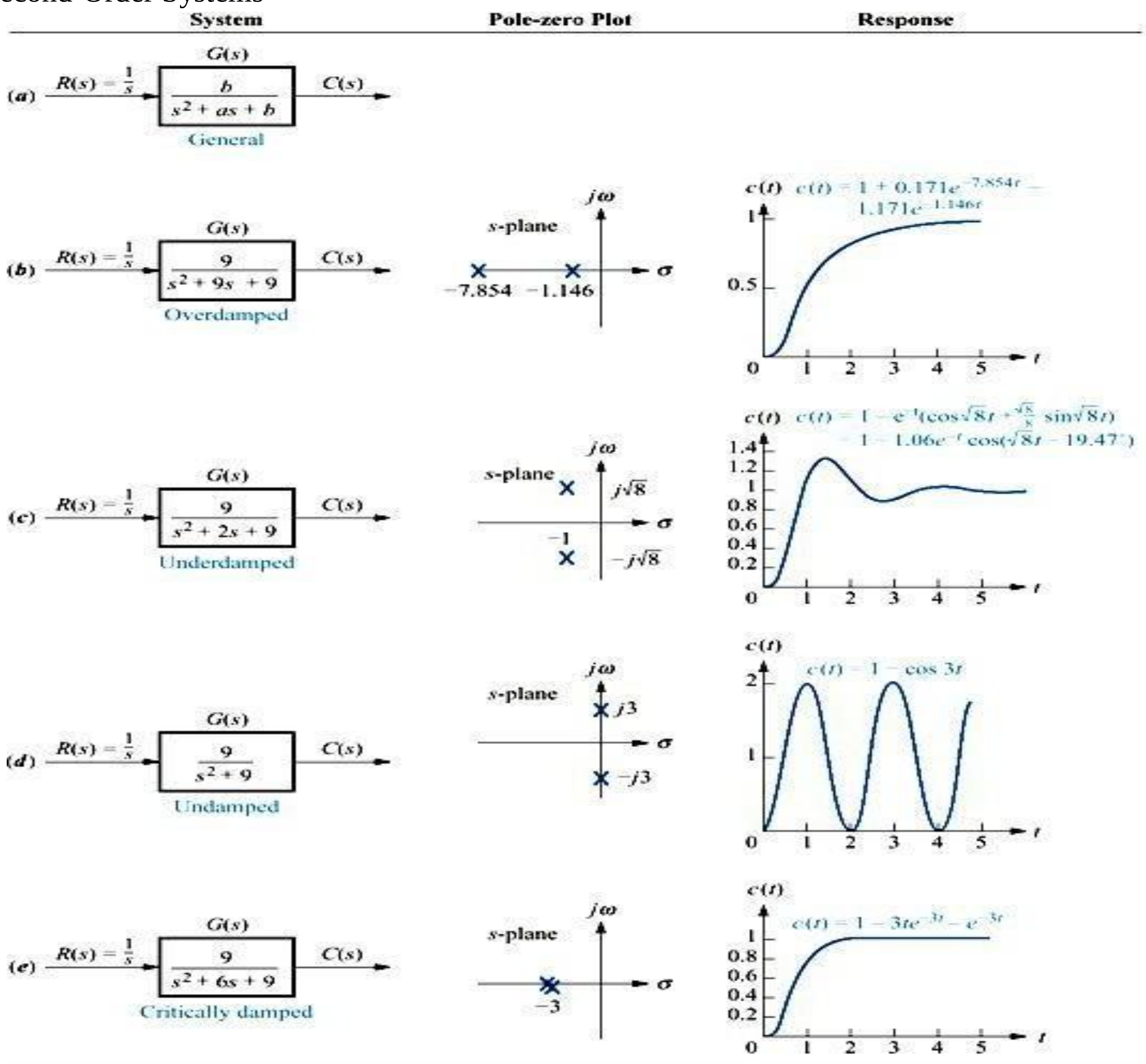
LTI second-order system

$$G(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$(s^2 + 2\zeta\omega_n s + \omega_n^2)\,C(s) = \omega_n^2 R(s)$$

$$c(t) + 2\zeta\omega_n c(t) + \omega_n^2\, c(t) = \omega_n^2\, r(t)$$

Second-Order Systems



**Second order system responses**

Overdamped response:

Poles: Two real at

$-\zeta_1$ - $-\zeta_2$

Natural response: Two exponentials with time constants equal to the reciprocal of the pole location

$C(t) = k_1 e^{-\zeta 1} + k_2 e^{-\zeta 2}$

Poles: Two complex at

**Underdamped response:**

$$-\zeta_1 \pm jW_d$$

**Natural response:** Damped sinusoid with an exponential envelope whose time constant is equal to the reciprocal of the pole_s radian frequency of the sinusoid, the damped frequency of oscillation, is equal to the imaginary part of the poles

**Undamped Response:**

Poles: Two imaginary at

$$\pm jW1$$

**Natural response:** Undamped sinusoid with radian frequency equal to the imaginary part of the poles

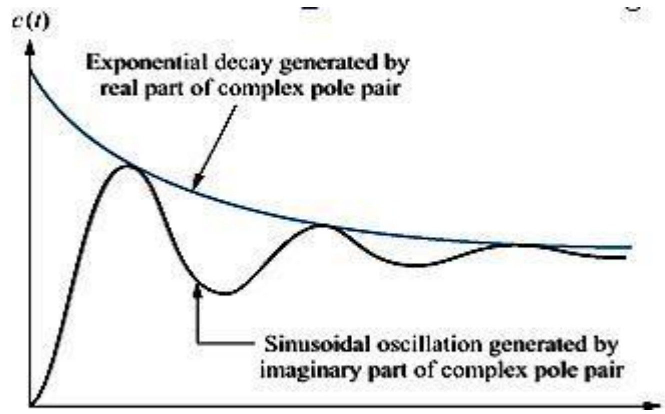
$$C(t) = A\cos(w_1 t - \varphi)$$

**Critically damped responses:**

**Poles: Two real at**

**Natural response:** One term is an exponential whose time constant is equal to the reciprocal of the pole location. Another term product of time and an exponential with time constant equal to the reciprocal of the pole location.

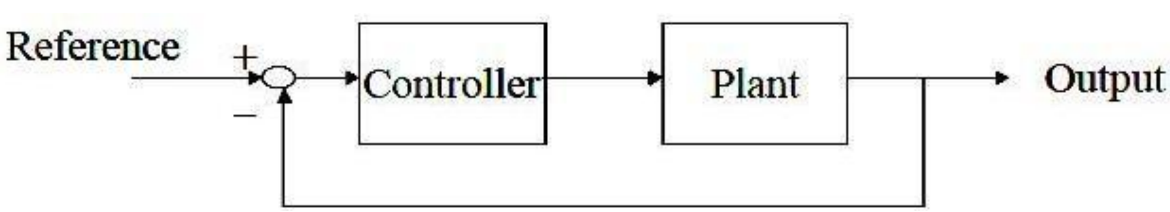
**Second- order step response**
Complex poles




**Steady State Error**
Consider a MODULEy feedback
system Transfer function between e(t)
and r(t)

| Type of system | Error constants | | | Steady state error $e_{ss}$ | | |
|---|---|---|---|---|---|---|
| | $K_p$ | $K_v$ | $K_a$ | Unit step input | Unit ramp input | Unit parabolic input |
| 0 | K | 0 | 0 | 1/(1+K) | ∞ | ∞ |
| 1 | ∞ | K | 0 | 0 | 1/K | ∞ |
| 2 | ∞ | ∞ | K | 0 | 0 | 1/K |
| 3 | ∞ | ∞ | ∞ | 0 | 0 | 0 |

Output Feedback Control Systems




Feedback only the output signal
– Easy access
– Obtainable in practice

**PID Controllers**

Proportional controllers
– pure gain or attenuation

Integral controllers
– integrate error

Derivative controllers
– differentiate error

**Proportional Controller**

**U = $K_p$ e**

- Controller input is error (reference output)
- Controller output is control signal
- P controller involves only a proportional gain (or attenuation)

**Integral Controller**
- Integral of error with a constant gain
- Increase system type by 1
- Infinity steady-state gain
- Eliminate steady-state error for a MODULE step input

**Integral Controller**

$$\frac{Y(s)}{R(s)} = \frac{G_p(s)}{1 + G_p(s)}$$

$$Y(s) = E(s)G_p(s)$$

$$E(s) = \frac{R(s)}{1 + G_p(s)}$$

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G_p(s)} = \lim_{s \to 0} \frac{1}{1 + G_p(s)} = \frac{1}{1 + \infty} = 0$$

**Derivative Control**

$$u = K_d \frac{de}{dt}$$

- Differentiation of error with a constant gain
- Reduce overshoot and oscillation
- Do not affect steady-state response
- Sensitive to noise

**Controller Structure**

Single controller

P controller, I controller, D controller

ombination of controllers

PI controller, PD controller

PID controller

**Controller Performance**

- P controller PI
- controller PD
- Controller PID
- Controller

**Design of PID Controllers**

- Based on the knowledge of P, I and D
- – trial and error
- – manual tuning
- – simulation

**Design of PID Controllers**

Time response measurements are particularly simple.

- A step input to a system is simply a suddenly applied input - often just a constant voltage applied through a switch.
- The system output is usually a voltage, or a voltage output from a transducer measuring the output.
- A voltage output can usually be captured in a file using a C program or a Visual Basic program.
- You can use responses in the time domain to help you determine the transfer function of a system.
- First we will examine a simple situation. Here is the step response of a system. This is an example of really "clean" data, better than you might have from measurements. The input to the system is a step of height 0.4. The goal is to determine the transfer function of the system.

**Impulse Response of A First Order System**

- The impulse response of a system is an important response. The impulse response is the response to a MODULE impulse.
- The MODULE impulse has a Laplace transform of MODULEy (1).That gives the MODULE impulse a unique stature. If a system has a MODULE impulse input, the output transform is G(s), where G(s) is the transfer function of the system. The MODULE impulse response is therefore the inverse transform of G(s), i.e. g(t), the time function you get by inverse transforming G(s). If you haven't begun to study Laplace transforms yet, you can just file these last statements away until you begin to learn about Laplace transforms. Still there is an important fact buried in all of this.
- Knowing that the impulse response is the inverse transform of the transfer function of a system can be useful in identifying systems (getting system parameters from measured responses).

In this section we will examine the shapes/forms of several impulse responses. We will start with simple first order systems, and give you links to modules that discuss other, higher order responses.

A general first order system satisfies a differential equation with this general form

If the input, u(t), is a MODULE impulse, then for a short instant around t = 0 the input is infinite. Let us assume that the state, x(t), is initially zero, i.e. x(0) = 0. We will integrate both sides of the differential equation from a small time, , before t = 0, to a small time, after t = 0. We are just taking advantage of one of the properties of the MODULE impulse.

The right hand side of the equation is just G**dc** since the impulse is assumed to be a MODULE impulse - one with MODULE area. Thus, we have:

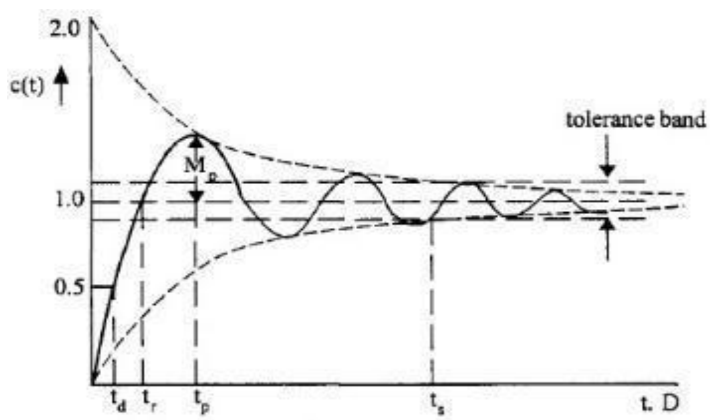
We can also note that x(0) = 0, so the second integral on the right hand side is zero. In other words, what the impulse does is it produces a calculable change in the state, x(t), and this change occurs in a negligibly short time (the duration of the impulse) after t = 0 That leads us to a simple strategy for getting the impulse response. Calculate the new initial condition after the impulse passes. Solve the differential equation - with zero input - starting from the newly calculated initial condition.

**Time Domain Specifications of a Second Order System**

The performance of a system is usually evaluated in terms of the following qualities. .

- How fast it is able to respond to the input.
- How fast it is reaching the desired output
- What is the error between the desired output and the actual output, once the transients die down and steady slate is achieved.
- Does it oscillate around the desired value, and
- Is the output continuously increasing with time or is it bounded.
- The last aspect is concerned with the stability of the system and we would require the system to be stable. This aspect will be considered later. The first four questions will be answered in terms of time domain specifications of the system based on its response to a MODULE step input.

  These are the specifications to be given for the design of a controller for a given system.
- 
- We have obtained the response of a type 1 second order system to a MODULE step input. The step
- Response of a typical underdamped second order system is plotted in Fig.




It is observed that, for an underdamped system, there are two complex conjugate poles.
Usually, even if a system is of higher order, the two complex conjugate poles nearest to the $j\omega$ - axis (called dominant poles) are considered and the system is approximated by a second order system. Thus, in designing any system, certain design specifications are given based on the typical underdamped step response shown as Fig.

The design specifications are

**Delay time $T_d$: It** is the time required for the response to reach 50% of the steady state value for the first time.

**Rise time $T_r$ :** It is the time required for the response to mach 100% of the steady state value for under damped systems. However, for over damped systems, it is taken as the time required for the response to rise from 10% to 90% of the steady state value.

**Peak time $T_p$ :** It is the time required for the response to reach the maximum or Peak value of the response.

**Peak overshoot $M_p$ :** It is defined as the difference between the peak value of the response and the steady state value. Iris usually expressed in percent of the steady state value. If the time for the peak is $t_p$ , percent peak overshoot is given by,

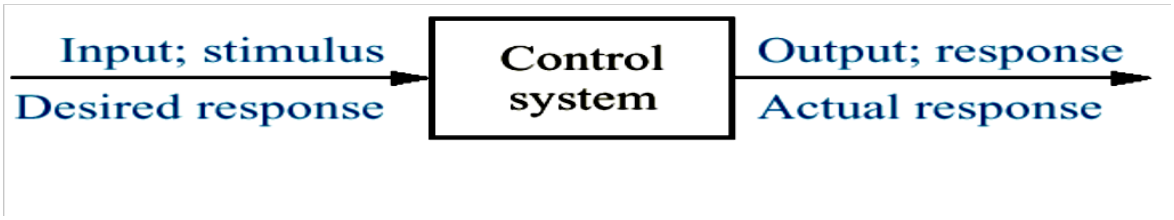
$$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100.$$

## MODULE-III

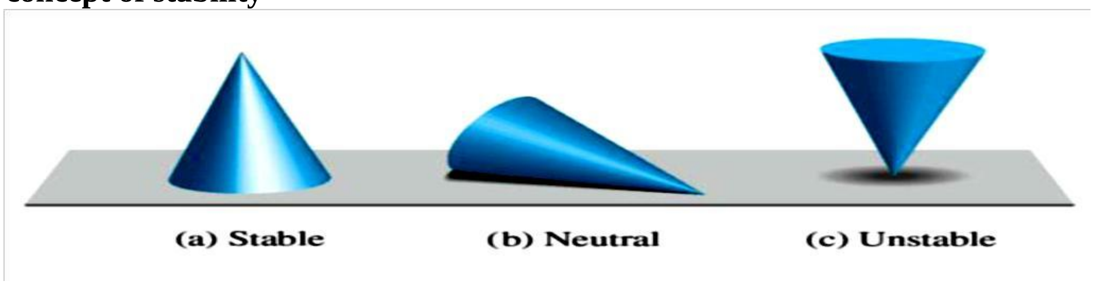
## Concept of Stability and Root Locus Technique

**Stability**

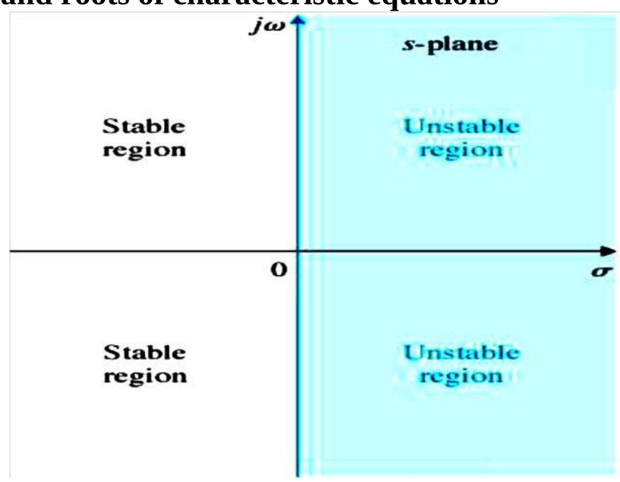
A system is stable if any bounded input produces a bounded output for all bounded initial conditions.




**Basic concept of stability**




**Stability of the system and roots of characteristic equations**




**Characteristic Equation**

Consider an nth-order system whose the characteristic equation (which is also the denominator of the transfer function) is

$$a(S) = S^n + a_1 S^{n-1} + a_2 S^{n-2} + \ldots\ldots + a_{n-1} S^1 + a_0 S^0$$

**Routh Hurwitz Criterion**

Goal: Determining whether the system is stable or unstable from a characteristic equation in polynomial form without actually solving for the roots Routh's stability criterion is useful for determining the ranges of coefficients of polynomials for stability, especially when the coefficients are in symbolic (non numerical) form.

To find K mar & ω

**A necessary condition for Routh's Stability**

- A necessary condition for stability of the system is that all of the roots of its characteristic equation have negative real parts, which in turn requires that all the coefficients be positive.
- A necessary (but not sufficient) condition for stability is that all the coefficients of the polynomial characteristic equation are positive & none of the co-efficient vanishes.
- Routh's formulation requires the computation of a triangular array that is a function of the coefficients of the polynomial characteristic equation.
- A system is stable if and only if all the elements ofthe first column of the Routh array are positive

**Method for determining the Routh array**

Consider the characteristic equation

$$a(S) = 1 X S^n + a_1 S^{n-1} + a_2 S^{n-2} + \ldots\ldots + a_{n-1} S^1 + a_0 S^0$$

**Routh array method**

Then add subsequent rows to complete the Routh array

**Compute elements for the 3rd row:**

$$b_1 = -\frac{1 \times a_3 - a_2 a_1}{a_1},$$

$$b_2 = -\frac{1 \times a_5 - a_4 a_1}{a_1},$$

$$b_3 = -\frac{1 \times a_7 - a_6 a_1}{a_1}$$

$$\ldots$$

**Given the characteristic equation,**

$$a(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$$

**Is the system described by this characteristic equation stable?**

Answer:

- All the coefficients are positive and nonzero
- Therefore, the system satisfies the necessary condition for stability

**We should determine whether any of the coefficients of the first column of the Routh array are Negative**

| | | | | |
|---|---|---|---|---|
| $s^6$ : | 1 | 3 | 1 | 4 |
| $s^5$ : | 4 | 2 | 4 | 0 |
| $s^4$ : | 5/2 | 0 | 4 | |
| $s^3$ : | 2 | −12/5 | 0 | |
| $s^2$ : | ? | ? | | |
| $s^1$ : | ? | ? | | |
| $s^0$ : | ? | | | |

The elements of the 1st column are not all positive. Then the system is unstable

**Special cases of Routh's criteria:**

Case 1: All the elements of a row in a RA are zero

- Form Auxiliary equation by using the co-efficient of the row which is just above the row of zeros.
- Find derivative of the A.E.
- Replace the row of zeros by the co-efficient of dA(s)/ds
- Complete the array in terms of these coefficients.
- analyze for any sign change, if so, unstable
- no sign change, find the nature of roots of AE
- non-repeated imaginary roots - marginally stable
- repeated imaginary roots – unstable

Case 2:

- First element of any of the rows of RA is
- Zero and the same remaining row contains atleast one non-zero element
- Substitute a small positive no. ‗ε' in place of zero and complete the array.
- Examine the sign change by taking Lt $\varepsilon = 0$

**Root Locus Technique**

- Introduced by W. R. Evans in 1948
- Graphical method, in which movement of poles in the s-plane is sketched when some parameter is varied The path taken by the roots of the characteristic equation when open loop gain K is varied from 0 to ∞ are called root loci
- Direct Root Locus = $0 < k < \infty$
- Inverse Root Locus = $-\infty < k < 0$

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Direct Root Locus = $0 < k < \infty$

- 
- Inverse Root Locus = $-\infty < k < 0$

**Root Locus Analysis:**

- The roots of the closed-loop characteristic equation define the system characteristic responses

- Their location in the complex s-plane lead to prediction of the characteristics of the time domain responses in terms of:
  damping ratio ζ,
- 
- natural frequency, wn- damping constant ζ, first-order modes
- 
- Consider how these roots change as the loop gain is varied from 0 to∞

**Basics of Root Locus:**

- Symmetrical about real axis
- RL branch starts from OL poles and terminates at OL zeroes No. of RL branches = No. of poles of OLTF
- Centroid is common intersection point of all the asymptotes on the real axis Asymptotes are straight lines which are parallel to RL going to ∞ and meet the RL at ∞ No. of asymptotes = No. of branches going to ∞
- At Break Away point , the RL breaks from real axis to enter into the complex plane At BI point, the RL enters the real axis from the complex plane

**Constructing Root Locus:**

- Locate the OL poles & zeros in the plot
- Find the branches on the real axis
- Find angle of asymptotes & centroid
- $\Phi a = \pm 180^\circ(2q+1) / (n-m)$
- $\zeta a = (\Sigma poles - \Sigma zeroes) / (n-m)$
- Find BA and BI points
- Find Angle Of departure (AOD) and Angle Of Arrival (AOA)
- AOD = 180º- (sum of angles of vectors to the complex pole from all other poles) + (Sum of angles of vectors to the complex pole from all zero)
- AOA = 180º- (sum of angles of vectors to the complex zero from all other zeros) + (sum of angles of vectors to the complex zero from poles)
- Find the point of intersection of RL with the imaginary axis.

**Application of the Root Locus Procedure**

Step 1: Write the characteristic equation as

$$1 + F(s) = 0$$

Step 2: Rewrite preceding equation into the form of poles and zeros as follows

$$1 + K\frac{\prod_{j=1}^{m}(s - z_j)}{\prod_{i=1}^{n}(s - p_i)} = 0$$

Step 3:

- Locate the poles and zeros with specific symbols, the root locus begins at the open-loop poles and ends at the open loop zeros as K increases from 0 to infinity
- If open-loop system has n-m zeros at infinity, there will be n-m branches of the root locus approaching the n-m zeros at infinity

Step 4:

- The root locus on the real axis lies in a section of the real axis to the left of an odd number of real poles and zeros

Step 5:

- The number of separate loci is equal to the number of open-loop poles

Step 6:

- The root loci must be continuous and symmetrical with respect to the horizontal real axis

Step 7:

The loci proceed to zeros at infinity along asymptotes centered at centroid and with angles

$$\sigma_a = \frac{\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} z_j}{}$$

$$\phi_a = \frac{(2k+1)\pi}{n-m} \qquad (k = 0, 1, 2, \cdots n-m-1)$$

Step 8: The actual point at which the root locus crosses the imaginary axis is readily evaluated by using Routh‘s criterion

Step 9: Determine the breakaway point d (usually on the real axis)

Step 10: • Plot the root locus that satisfy the phase criterion

$$\angle P(s) = (2k+1)\pi \qquad k = 1, 2, \cdots$$

Step 11:

Determine the parameter value *K1* at a specific root using the magnitude criterion

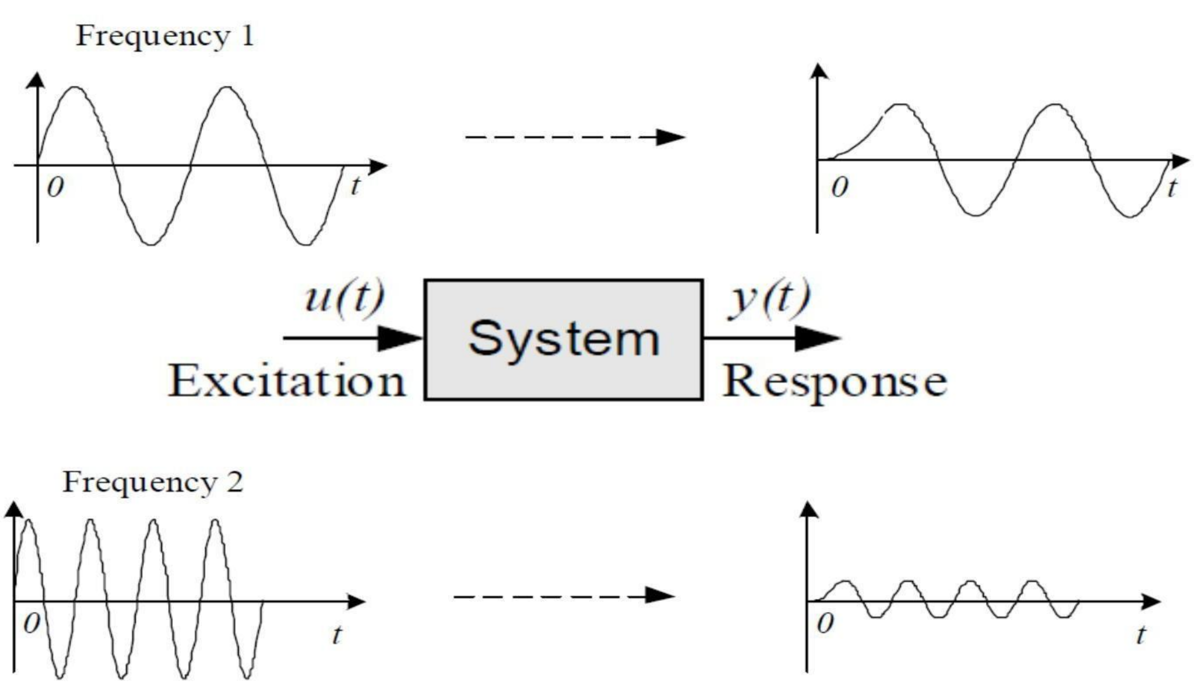
$$K_1 = \left.\frac{\prod_{i=1}^{n}\left|(s - p_i)\right|}{\prod_{j=1}^{m}\left|(s - z_j)\right|}\right|_{s=s_1}$$

**MODULE -IV**
**FREQUENCY DOMAIN ANALYSIS**

## Frequency Response

The frequency response of a system is a frequency dependent function which expresses how a sinusoidal signal of a given frequency on the system input is transferred through the system. Time-varying signals at least periodical signals —which excite systems, as the reference (set point) signal or a disturbance in a control system or measurement signals which are inputs signals to signal filters, can be regarded as consisting of a sum of frequency components. Each frequency component is a sinusoidal signal having certain amplitude and a certain frequency. (The Fourier series expansion or the Fourier transform can be used to express these frequency components quantitatively.) The frequency response expresses how each of these frequency components is transferred through the system. Some components may be amplified, others may be attenuated, and there will be some phase lag through the system.

The frequency response is an important tool for analysis and design of signal filters (as low pass filters and high pass filters), and for analysis, and to some extent, design, of control systems. Both signal filtering and control systems applications are described (briefly) later in this MODULE. The definition of the frequency response — which will be given in the next section — applies only to linear models, but this linear model may very well be the local linear model about some operating point of a non-linear model. The frequency response can found experimentally or from a transfer function model. It can be presented graphically or as a mathematical function.



**Bode plot**

• Plots of the magnitude and phase characteristics are used to fully describe the frequency response

• A **Bode plot** is a (semilog) plot of the transfer function magnitude and phase angle as a function of frequency.

The gain magnitude is many times expressed in terms of decibels (dB)

$$db = 20 \log 10\ A$$

**BODE PLOT PROCEDURE:**

There are 4 basic forms in an open-loop transfer function G(*jω*)H(*jω*)

- Gain Factor K
- $(j\omega)^{\pm p}$ factor: pole and zero at origin
- $(1+j\omega T)^{\pm q}$ factor
- Quadratic factor

  $1+j2\zeta(W / W_n)-(W / W_n)^2$

**Gain margin and Phase margin**

**Gain margin:**

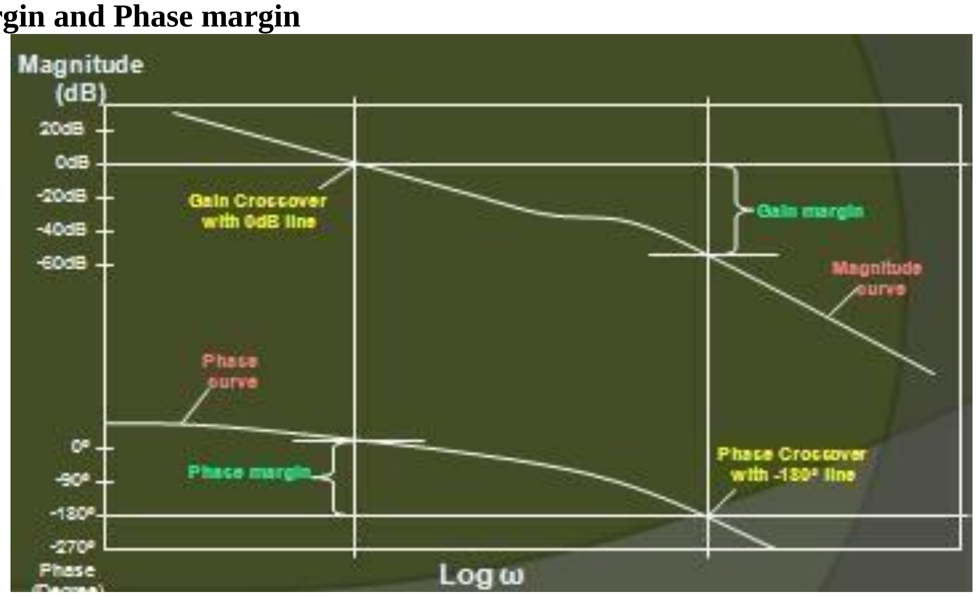
The gain margin is the number of dB that is below 0 dB at the phase crossover frequency (ø=-180º). It can also be increased before the closed loop system becomes unstable

| Term | Corner Frequency | Slope db /dec | Change in slope |
|---|---|---|---|
| 20/jW | ----- | -20 | |
| 1/ (1+4jW) | $WC_1$=1/4 = 0.25 | -20 | -20-20=-40 |
| 1/(1+j3w) | wc2=1/3=0.33 | -20 | -40-20=-60 |

**Phase margin:**

The phase margin is the number of degrees the phase of that is above -180º at the gain crossover frequency

**Gain margin and Phase margin**



**Bode Plot – Example**

**For the following T.F draw the Bode plot and obtain Gain cross over frequency (wgc) , Phase cross over frequency , Gain Margin and Phase Margin. G(s) = 20 / [s (1+3s) (1+4s)]**

**Solution:**

The sinusoidal T.F of G(s) is obtained by replacing s by jw in the given T.F

G(jw) = 20 / [jw (1+j3w) (1+j4w)]

**Corner frequencies:**

wc1= 1/4 = 0.25 rad /sec ;

wc2 = 1/3 = 0.33 rad /sec

Choose a lower corner frequency and a higher Corner frequency

wl= 0.025 rad/sec ;

wh = 3.3 rad / sec

Calculation of Gain (A) (MAGNITUDE PLOT)

A @ wl ; A= 20 log [ 20 / 0.025 ] = 58 .06 dB

A @ wc1 ; A = [Slope from wl to wc1 x log (wc1 / wl ] + Gain (A)@wl

= - 20 log [ 0.25 / 0.025 ] + 58.06

= 38.06 dB

A @ wc2 ; A = [Slope from wc1 to wc2 x log (wc2 / wc1 ] + Gain (A)@ wc1

= - 40 log [ 0.33 / 0.25 ] + 38

= 33 dB

A @ wh ; A = [Slope from wc2 to wh x log (wh / wc2 ] + Gain (A) @ wc2

= - 60 log [ 3.3 / 0.33 ] +

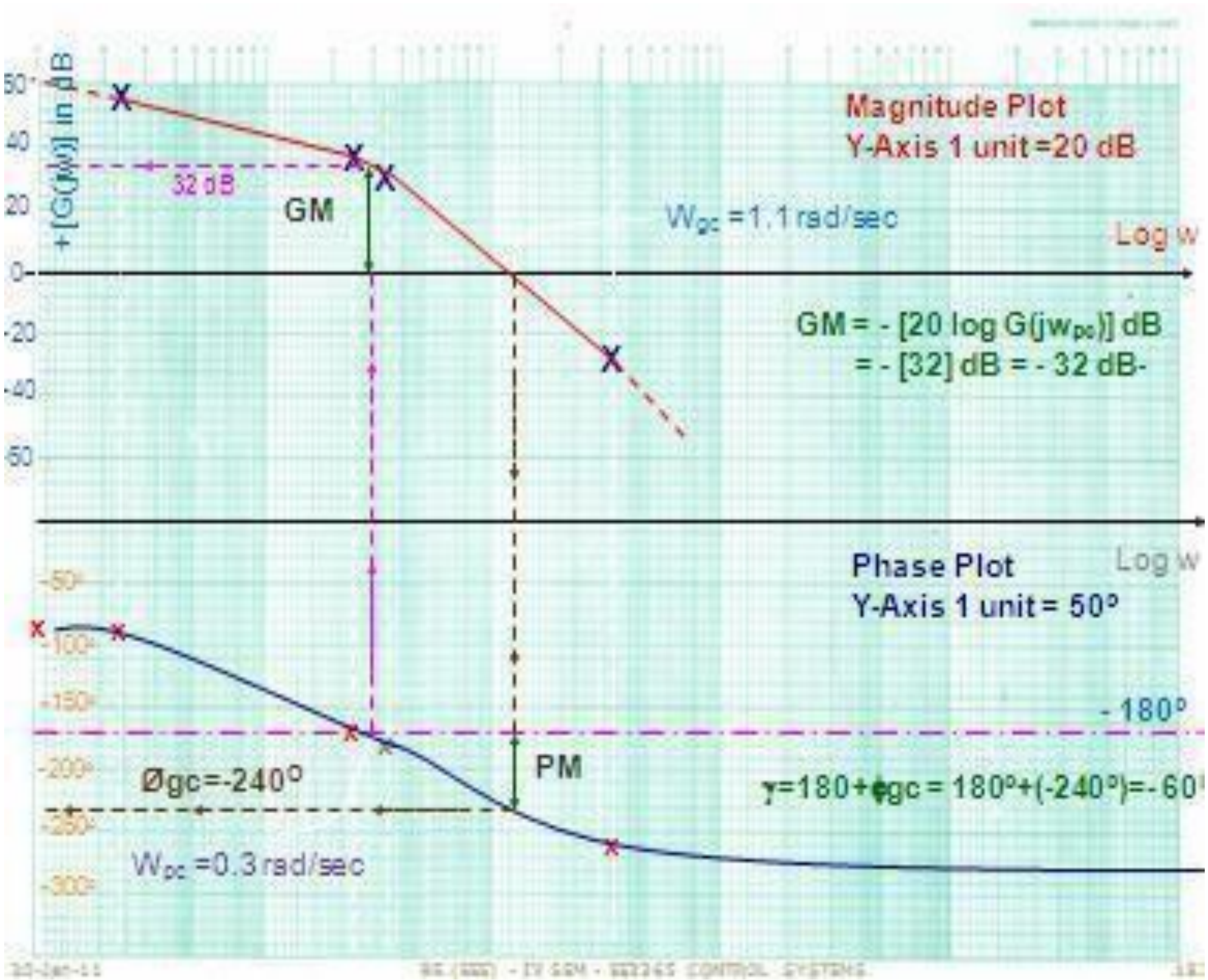
33 =-27 dB

Calculation of Phase angle for different values of frequencies [PHASE PLOT]

$Ø = -90^{O} - \tan^{-1} 3w - \tan^{-1}$

4w When

| Frequency in rad / sec | Phase angles in Degree |
|---|---|
| w=0 | $Ø= -90^{o}$ |
| w = 0.025 | $Ø= -99^{o}$ |
| w = 0.25 | $Ø= -172^{o}$ |
| w = 0.33 | Ø= - 188 |
| w =3.3 | $Ø= -259^{o}$ |
| w =∞ | $Ø= -270^{o}$ |




- **Calculations of Gain cross over frequency**

The frequency at which the dB magnitude is Zero

wgc = 1.1 rad / sec

- **Calculations of Phase cross over frequency** The

frequency at which the Phase of the system is - 180o

wpc = 0.3 rad / sec

- Gain Margin

  The gain margin in dB is given by the negative of dB magnitude of G(jw) at phase cross over frequency

  GM = - { 20 log [G( jwpc )] = - { 32 } = -32 dB

- **Phase Margin**

  Ɣ = 180$^0$ + Øgc= 180$^0$ + (- 2400$^o$ ) = -60$^0$

- **Conclusion**

  For this system GM and PM are negative in values. Therefore the system is unstable in nature.

**Polar plot**

To sketch the polar plot of $G(j\omega)$ for the entire range of frequency ω, i.e., from 0 to infinity, there are four key points that usually need to be known:
(1) the start of plot where ω = 0,
(2) the end of plot where ω = ∞,
(3) where the plot crosses the real axis, i.e., $\mathrm{Im}(G(j\omega)) = 0$, and
(4) where the plot crosses the imaginary axis, i.e., $\mathrm{Re}(G(j\omega)) = 0$.

**BASICS OF POLAR PLOT:**

- The polar plot of a sinusoidal transfer function G(jω) is a plot of the magnitude of G(jω) Vs the phase of G(jω) on polar co-ordinates as ω is varied from 0 to ∞. (ie) |G(jω)| Vs angle G(jω) as ω → 0 to ∞.
- Polar graph sheet has concentric circles and radial lines.
- Concentric circles represents the magnitude.
- Radial lines represents the phase angles.
- In polar sheet
  +ve phase angle is measured in ACW from $0^0$ -
  ve phase angle is measured in CW from $0^0$

**PROCEDURE**

- Express the given expression of OLTF in (1+sT) form.
- Substitute s = jω in the expression for G(s)H(s) and get G(jω)H(jω).
- Get the expressions for | G(jω)H(jω)| & angle G(jω)H(jω).
- Tabulate various values of magnitude and phase angles for different values of ω ranging from 0 to ∞.
- Usually the choice of frequencies will be the corner frequency and around corner frequencies.
- Choose proper scale for the magnitude circles.
- Fix all the points in the polar graph sheet and join the points by a smooth curve.
- Write the frequency corresponding to each of the point of the plot.

**MINIMUM PHASE SYSTEMS:**

- Systems with all poles & zeros in the Left half of the s-plane – Minimum Phase Systems.
- For Minimum Phase Systems with only poles
- Type No. determines at what quadrant the polar plot starts.
- Order determines at what quadrant the polar plot ends.
- Type No. → No. of poles lying at the origin
- Order → Max power of‗s' in the denominator polynomial of the transfer function.

**GAIN MARGIN**

- Gain Margin is defined as ―the factor by which the system gain can be increased to drive the system to the verge of instability‖.
- For stable systems,

$$\omega gc < \omega pc$$

Magnitude of G(j )H(j ) at ω=ωpc < 1

GM = in positive dB
More positive the GM, more stable is the system.

- For marginally stable systems,

$\omega gc = \omega pc$

magnitude of G(j )H(j ) at $\omega=\omega pc = 1$
GM = 0 dB
For Unstable systems,

$\omega gc > \omega pc$

magnitude of G(j )H(j ) at $\omega=\omega pc > 1$
GM = in negative dB
Gain is to be reduced to make the system stable

Note:

- If the gain is high, the GM is low and the system's step response shows high overshoots and long settling time.
- On the contrary, very low gains give high GM and PM, but also causes higher ess, higher values of rise time and settling time and in general give sluggish response.
- Thus we should keep the gain as high as possible to reduce ess and obtain acceptable response speed and yet maintain adequate GM & PM.
- An adequate GM of 2 i.e. (6 dB) and a PM of 30 is generally considered good enough as a thumb rule.

At w=w pc , angle of G(jw )H(jw ) = $-180^0$

- Let magnitude of G(jw)H(jw ) at w = $w_{pc}$ be taken a B
- If the gain of the system is increased by factor 1/B, then the magnitude of G(jw)H(j w) at w = wpc becomes B(1/B) = 1 and hence the G(jw)H(jw) locus pass through -1+j0 point driving the system to the verge of instability.
- GM is defined as the reciprocal of the magnitude of the OLTF evaluated at the phase cross over frequency.

GM in dB = 20 log (1/B) = - 20 log B

PHASE MARGIN

Phase Margin is defined as ― the additional phase lag that can be introduced before the system becomes unstable‖.

‗A' be the point of intersection of G(j )H(j ) plot and a MODULE circle centered at the origin.

Draw a line connecting the points ‗O' & ‗A' and measure the phase angle between the line OA and

+ve real axis.

This angle is the phase angle of the system at the gain cross over frequency. Angle of G(jwgc)H(jw gc) =φ gc

If an additional phase lag of φ PM is introduced at this frequency, then the phase angle G(jwgc)H(jw gc) will become 180 and the point ‗A‗ coincides with (-1+j0) driving the system to the verge of instability.

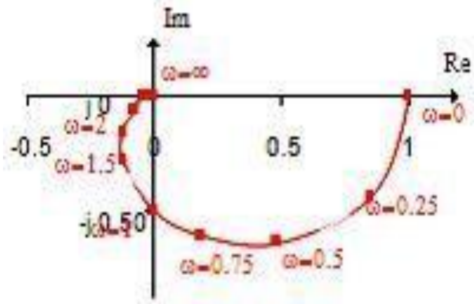
This additional phase lag is known as the Phase Margin.

$\gamma = 180^0 + $ angle of G(jwgc)H(jw gc)

$\gamma = 180^0 + \varphi gc$

[Since φ gc is measured in CW direction, it is taken as negative] For a stable system, the phase margin is positive.

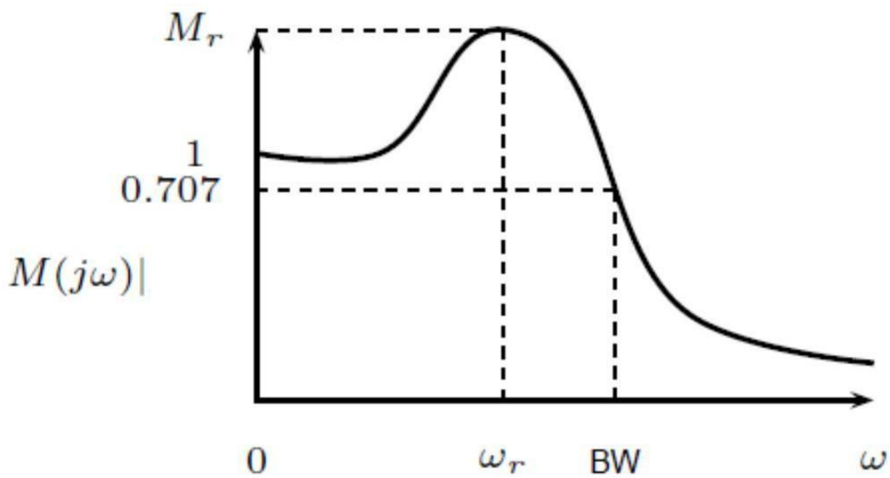
- A Phase margin close to zero corresponds to highly oscillatory system.




- A polar plot may be constructed from experimental data or from a system transfer function
- If the values of w are marked along the contour, a polar plot has the same information as a bode plot.
- Usually, the shape of a polar plot is of most interest.

**Frequency domain specifications**

- The resonant peak Mr is the maximum value of jM(jw)j.
- The resonant frequency !r is the frequency at which the peak resonance Mr occurs.
- The bandwidth BW is the frequency at which(jw) drops to 70:7% (3 dB) of its zero-frequency value.




- Mr indicates the relative stability of a stable closed loop system.
- A large Mr corresponds to larger maximum overshoot of the step
- response. Desirable value: 1.1 to 1.5
- BW gives an indication of the transient response properties of a control system.
- A large bandwidth corresponds to a faster rise time. BW and rise time tr are inversely proportional.
- BW also indicates the noise-filtering characteristics and robustness of the system.
- Increasing wn increases BW.
- BW and Mr are proportional to each other.

.

**Nyquist Stability Criteria:**

The Routh-Hurwitz criterion is a method for determining whether a linear system is stable or not by examining the locations of the roots of the characteristic equation of the system. In fact, the method determines only if there are roots that lie outside of the left half plane; it does not actually compute the roots. Consider the characteristic equation.

To determine whether this system is stable or not, check the following conditions

$$1+GH(s)=D(s)=a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = 0$$

1. Two necessary but not sufficient conditions that all the roots have negative real parts are a) All the polynomial coefficients must have the same sign.
b) All the polynomial coefficients must be nonzero.
2. If condition (1) is satisfied, then compute the Routh-Hurwitz array as follows

$$\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & & \cdots \\
s^{n-4} & & & \vdots & & \\
\vdots & & & \vdots & & \\
s^1 & & & \vdots & & \\
s^0 & & & \vdots & &
\end{array}$$

where the $a_i$'S are the polynomial coefficients, and the coefficients in the rest of the table are computed using the following pattern

$$b_1 = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} = \frac{-1}{a_{n-1}} (a_n a_{n-3} - a_{n-2} a_{n-1})$$

$$b_2 = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

$$b_3 = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-6} \\ a_{n-1} & a_{n-7} \end{vmatrix} \quad \cdots$$

$$c_1 = \frac{-1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix}$$

$$c_2 = \frac{-1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix} \quad \cdots$$

3. The necessary condition that all roots have negative real parts is that all the elements of the first column of the array have the same sign. The number of changes of sign equals the number of roots with positive real parts.
4. Special Case 1: The first element of a row is zero, but some other elements in that row are nonzero. In this case, simply replace the zero elements by " ", complete the table development, and then interpret the results assuming that " " is a small number of the same sign as the element above it. The results must be interpreted in the limit as ε to 0.
5. Special Case 2: All the elements of a particular row are zero. In this case, some of the roots of the polynomial are located symmetrically about the origin of the *s*-plane, e.g., a pair of purely imaginary roots. The zero rows will always occur in a row associated with an odd power of *s*. The row just above the zero rows holds the coefficients of the auxiliary polynomial. The roots of the auxiliary polynomial are the symmetrically placed roots. Be careful to remember that the coefficients in the array skip powers of *s* from one coefficient to the next.

Let P = no. of poles of q(s)-plane lying on Right Half of s-plane and encircled by s-plane contour.

Let Z = no. of zeros of q(s)-plane lying on Right Half of s-plane and encircled by s-plane contour.

For the CL system to be stable, the no. of zeros of q(s) which are the CL poles that lie in the right half of s-plane should be zero. That is Z = 0, which gives N = -P.

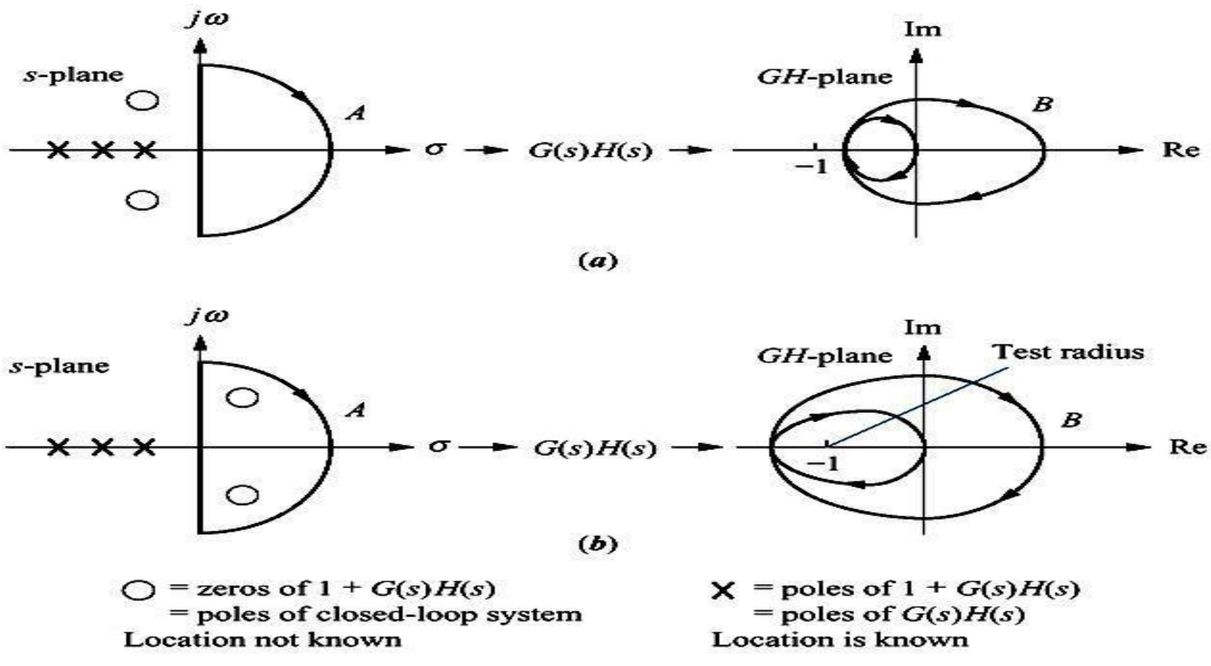
Therefore, for a stable system the no. of ACW encirclements of the origin in the q(s)-plane by the contour Cq must be equal to P.

**Nyquist modified stability criteria**

We know that q(s) = 1+G(s)H(s)

Therefore G(s)H(s) = [1+G(s)H(s)] – 1

- The contour Cq, which has obtained due to mapping of Nyquist contour from s-plane to q(s)-plane (ie)[1+G(s)H(s)] -plane, will encircle about the origin.
- The contour CGH, which has obtained due to mapping of Nyquist contour from s-plane to G(s)H(s) -plane, will encircle about the point (-1+j0).
- Therefore encircling the origin in the q(s)-plane is equivalent to encircling the point -1+j0 in the G(s)H(s)-plane.



**Problem**

Sketch the Nyquist stability plot for a feedback system with the following open-loop transfer function

$$G(s)H(s) = \frac{1}{s(s^2+s+1)}$$

**Solution**

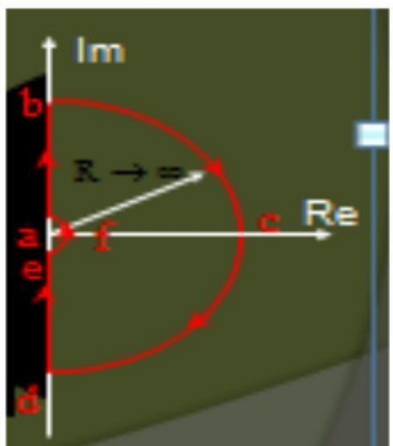
For section $\underline{ab}$, $s = j\omega$, $\omega : 0 \to \infty$

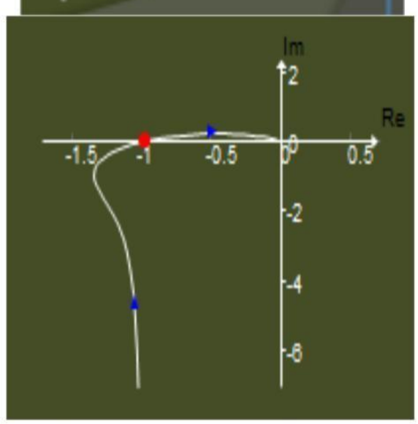
$$G(j\omega)H(j\omega) = \frac{1}{j\omega(1-\omega^2+j\omega)}$$

(i) $\omega \to 0 : G(j\omega)H(j\omega) \to -1 - j\infty$

(ii) $\omega = 1 : G(j\omega)H(j\omega) \to -1 + j0$

(iii) $\omega \to \infty : G(j\omega)H(j\omega) \to 0\angle -270^\circ$

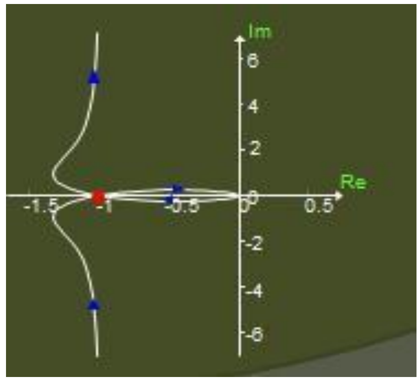



On section $\underline{bcd}$, $s = Re^{j\theta}|_{R\to\infty}$ ; therefore i.e. section $\underline{bcd}$ maps onto the origin of the G(s)H(s)-plane

$$|G(s)H(s)| \to \frac{1}{R^3} \to 0$$

Section de maps as the complex image of the polar plot as before



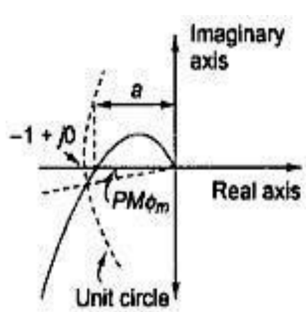

**Relative stability**

The main disadvantage of a Bode plot is that we have to draw and consider two different curves at a time, namely, magnitude plot and phase plot. Information contained in these two plots can be combined into one named polar plot. The polar plot is for a frequency range of 0<w<α . while the Nyquist plot is in the frequency range of - α<w<α. The information on the negative frequency is redundant because the magnitude and real part of *G*( *jw*) *an* are even functions. In this section. We consider how to evaluate the system performance in terms of relative stability using a Nyquist plot. The open-loop system represented by this plot will become unstable beyond a certain value. As shown in the Nyquist plot of Fig. the intercept of magnitude 'a on the negative real axis corresponds lost phase shift of - 180° and - 1 represents the amount of increase in gain that can be tolerated before closed-loop system tends toward instability. As *'a'* approaches (-1+ *j*0) point the relative stability is reduced; the gain and phase margins are represented as follows in the Nyquist plot.

**Gain margin**

As system gain is increased by a factor 1/a, the open loop magnitude of *G* ( *jw*)*H*( *jw*) will increase by a factor a( 1/a) = 1 and the system would be driven to instability. Thus, the gain margin is the reciprocal of the gain at the frequency at which the phase angle of the Nyquist plot is - 180 . The gain rnargin, usually measured in dB, is a positive quantity given by

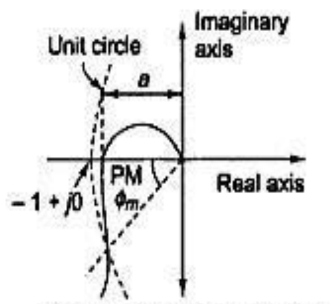
GM = -20log *a dB*




*Gain margin representation in Nyquist plot with small PM*

**Phase Margin $\phi_m$**

Importance of the phase margin has already in the content of Bode. Phase margin is defined as the change in open-loop phase shift required al MODULEy gain to make a closed loop system unstable. A closed-loop system will be unstable if the Nyquist plot encircles -1 +*j*0 point. Therefore, the angle required to make this system marginally stable in a closed loop is the phase margin .In order to measure this angle, we draw a circle with a radius of 1, and find the point of intersection of the Nyquist plot with this circle, and measure the phase shift needed for this point to be at an angle of 1800. If may be appreciated that the system having plot of Fig with larger PM is more stable than the one with plot of Fig.




*Phase margin representation*

## MODULE-V
## STATE VARIABLE ANALYSIS AND COMPENSATORS

State space analysis is an excellent method for the design and analysis of control systems. The conventional and old method for the design and analysis of control systems is the transfer function method. The transfer function method for design and analysis had many drawbacks.

**Advantages of state variable analysis.**

- It can be applied to non linear system.
- It can be applied to tile invariant systems.
- It can be applied to multiple input multiple output systems.
- Its gives idea about the internal state of the system.

**State Variable Analysis and Design**

**State:** The state of a dynamic system is the smallest set of variables called state variables such that the knowledge of these variables at time $t=t_o$ (Initial condition), together with the knowledge of input for $\geq t_0$, completely determines the behaviour of the system for any time $t \geq t_0$.

**State vector:** If n state variables are needed to completely describe the behaviour of a given system, then these n state variables can be considered the n components of a vector X. Such a vector is called a state vector.

**State space:** The n-dimensional space whose co-ordinate axes consists of the $x_1$ axis, $x_2$ axis,.... $x_n$ axis, where x1,x2……Xn are state variables.

State Model

Lets consider a multi input & multi output system is having

r inputs $u_1(t), u_2(t), \ldots \ldots . u_r(t)$

m no of outputs $y_1(t), y_2(t), \ldots \ldots . y_m(t)$

n no of state variables $x_1(t), x_2(t), \ldots \ldots . x_n(t)$

Then the state model is given by state & output equation

$$\dot{X}(t) = AX(t) + BU(t) \ldots \ldots \ldots \ldots \text{state equation}$$

$$Y(t) = CX(t) + DU(t) \ldots \ldots \ldots \text{output equation}$$

A is state matrix of size (n×n)

B is the input matrix of size (n×r)

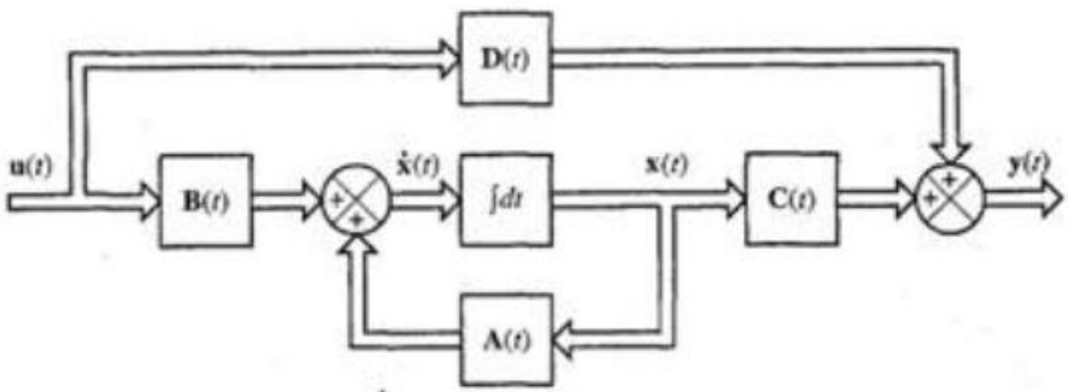
C is the output matrix of size (m×n)

D is the direct transmission matrix of size (m×r)

X(t) is the state vector of size (n×1)

Y(t) is the output vector of size (m×1)

U(t) is the input vector of size (r×1)




(Block diagram of the linear, continuous time control system represented in state space)

$$\dot{\mathbf{X}}(\mathbf{t}) = \mathbf{AX}(\mathbf{t}) + \mathbf{Bu}(\mathbf{t})$$

$$\mathbf{Y}(\mathbf{t}) = \mathbf{CX}(\mathbf{t}) + \mathbf{Du}(\mathbf{t})$$

STATE SPACE REPRESENTATION OF NTH ORDER SYSTEMS OF LINEAR DIFFERENTIAL EQUATION IN WHICH FORCING FUNCTION DOES NOT INVOLVE DERIVATIVE TERM

Consider following nth order LTI system relating the output y(t) to the input u(t).

$$y^n + a_1 y^{n-1} + a_2 y^{n-2} + \cdots + a_{n-1} y^1 + a_n y = u$$

Phase variables: The phase variables are defined as those particular state variables which are obtained from one of the system variables & its (n-1) derivatives. Often the variables used is the system output & the remaining state variables are then derivatives of the output. Let us define the state variables as

$$x_1 = y$$

$$x_2 = \frac{dy}{dt} = \frac{dx}{dt}$$

$$x_3 = \frac{d\dot{y}}{dt} = \frac{dx_2}{dt}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$x_n = y^{n-1} = \frac{dx_{n-1}}{dt}$$

From the above equations we can write

$$\dot{x_1} = x_2$$

$$\dot{x_2} = x_3$$

$$\vdots \qquad \vdots$$

$$\dot{x_{n-1}} = x_n$$

$$\dot{x_n} = -a_n x_1 - a_{n-1} x_2 - \cdots \ldots \ldots \ldots - a_1 x_n + u$$

Writing the above state equation in vector matrix form

$$\dot{X}(t) = AX(t) + Bu(t)$$

$$\text{Where } X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}, \; A = \begin{bmatrix} 0 & 1 & 0 \ldots \ldots & 0 \\ 0 & 0 & 1 \ldots \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \ldots \ldots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} \ldots \ldots. & -a_1 \end{bmatrix}_{n\times n}$$

Output equation can be written as

$$Y(t) = CX(t)$$

$$C = [1 \quad 0 \ldots \ldots. \quad 0]_{1\times n}$$

## State Space to Transfer Function

Consider the state space system:

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{q}(t) + Du(t)$$

Now, take the Laplace Transform (with zero initial conditions since we are finding a transfer function):

$$s\mathbf{Q}(s) = \mathbf{A}\mathbf{Q}(s) + \mathbf{B}U(s)$$
$$Y(s) = \mathbf{C}\mathbf{Q}(s) + DU(s)$$

We want to solve for the ratio of Y(s) to U(s), so we need so remove **Q**(s) from the output equation. We start by solving the state equation for **Q**(s)

$$s\mathbf{Q}(s) - A\mathbf{Q}(s) = \mathbf{B}U(s)$$
$$\left(s\mathbf{I} - \mathbf{A}\right)\mathbf{Q}(s) = \mathbf{B}U(s)$$
$$\mathbf{Q}(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}U(s) = \mathbf{\Phi}(s)\mathbf{B}U(s); \quad \text{where } \mathbf{\Phi}(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1}$$

The matrix **Φ**(s) is called the state transition matrix. Now we put this into the output equation

$$\begin{aligned} Y(s) &= \mathbf{C}\mathbf{\Phi}(s)\mathbf{B}U(s) + DU(s) \\ &= \left(\mathbf{C}\mathbf{\Phi}(s)\mathbf{B} + D\right)U(s) \end{aligned}$$

Now we can solve for the transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \mathbf{C}\mathbf{\Phi}(s)\mathbf{B} + D = \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + D$$

Note that although there are many state space representations of a given system, all of those representations will result in the same transfer function (i.e., the transfer function of a system is unique; the state space representation is not).

## Transfer Function to State Space

Recall that state space models of systems are not unique; a system has many state space representations. Therefore we will develop a few methods for creating state space models of systems. Before we look at procedures for converting from a transfer function to a state space model of a system, let's first examine going from a differential equation to state space. We'll do this first with a simple system, then move to a more complex system that will demonstrate the usefulness of a standard technique. First we start with an example demonstrating a simple way of converting from a single differential equation to state space, followed by a conversion from transfer function to state space.

Example: Differential Equation to State Space (simple)

Consider the differential equation with no derivatives on the right hand side. We'll use a third order equation, thought it generalizes to $n_{th}$ order in the obvious way.

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0 u$$

For such systems (no derivatives of the input) we can choose as our n state variables the variable y and its first n-1 derivatives (in this case the first two derivatives)

$$\begin{aligned} q_1 &= y \\ q_2 &= \dot{y} \\ q_3 &= \ddot{y} \end{aligned}$$

Taking the derivatives we can develop our state space model

$$\begin{aligned}
\dot{q}_1 &= q_2 = \dot{y} \\
\dot{q}_2 &= q_3 = \ddot{y} \\
\dot{q}_3 &= \dddot{y} = -a_3 y - a_2\dot{y} - a_1\ddot{y} + b_0 u \\
&= -a_3 q_1 - a_2 q_2 - a_1 q_3 + b_0 u \\
\dot{\mathbf{q}} &= \mathbf{A}\mathbf{q} + \mathbf{B}u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}\mathbf{q} + \begin{bmatrix} 0 \\ 0 \\ b_0 \end{bmatrix}u \\
y &= \mathbf{C}\mathbf{q} + Du = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{q} + 0 \cdot u
\end{aligned}$$

Note: For an nth order system the matrices generalize in the obvious way (**A** has ones above the main diagonal and the differential equation constants for the last row, **B** is all zeros with $b_0$ in the bottom row, **C** is zero except for the leftmost element which is one, and D is zero)

Repeat Starting from Transfer Function

Consider the transfer function with a constant numerator (note: this is the same system as in the preceding example). We'll use a third order equation, thought it generalizes to $n^{th}$ order in the obvious way.

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^3 + a_1 s^2 + a_2 s + a_3}$$

$$\left(s^3 + a_1 s^2 + a_2 s + a_3\right)Y(s) = b_0 U(s)$$

For such systems (no derivatives of the input) we can choose as our n state variables the variable y and its first n-1 derivatives (in this case the first two derivatives)

$$\begin{aligned} q_1(t) &= y(t) & \qquad Q_1(s) &= Y(s) \\ q_2(t) &= \dot{y}(t) & \qquad Q_2(s) &= sY(s) \\ q_3(t) &= \ddot{y}(t) & \qquad Q_3(s) &= s^2Y(s) \end{aligned}$$

Taking the derivatives we can develop our state space model (which is exactly the same as when we started from the differential equation).

$$
\begin{aligned}
sQ_1(s) &= Q_2(s) = sY(s) \\
sQ_2(s) &= Q_3(s) = s^2Y(s) \\
sQ_3(s) &= s^3Y(s) = -a_1s^2Y(s) - a_2sY(s) - a_3Y(s) + b_0u \\
&= -a_1s^2Q_3(s) - a_2sQ_2(s) - a_3Q_1(s) + b_0u
\end{aligned}
$$

$$
s\mathbf{Q}(s) = \mathbf{A}\mathbf{Q}(s) + \mathbf{B}U(s) == \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}\mathbf{Q}(s) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}U(s)
$$

$$
Y(s) = \mathbf{C}\mathbf{Q}(s) + DU(s) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{Q}(s) + 0 \cdot U(s)
$$

Note: For an nth order system the matrices generalize in the obvious way (**A** has ones above the main diagonal and the coefficients of the denominator polynomial for the last row, **B** is all zeros with b0 (the numerator coefficient) in the bottom row, **C** is zero except for the leftmost element which is one, and D is zero) If we try this method on a slightly more complicated system, we find that it initially fails (though we can succeed with a little cleverness).

Example: Differential Equation to State Space (harder)

Consider the differential equation with a single derivative on the right hand side.

$$
\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3y = b_0\dot{u} + b_1u
$$

We can try the same method as before:

$$
\begin{aligned}
q_1 &= y \\
q_2 &= \dot{y} \\
q_3 &= \ddot{y}
\end{aligned}
$$

$$
\begin{aligned}
\dot{q}_1 &= q_2 = \dot{y} \\
\dot{q}_2 &= q_3 = \ddot{y} \\
\dot{q}_3 &= \dddot{y} = -a_3y - a_2\dot{y} - a_1\ddot{y} + b_0\dot{u} + b_1u \\
&= -a_3q_1 - a_2q_2 - a_1q_3 + b_0\dot{u} + b_1u
\end{aligned}
$$

The method has failed because there is a derivative of the input on the right hand, and that is not allowed in a state space model. Fortunately we can solve our problem by revising our choice of state variables.

$$
\begin{aligned}
q_1 &= y \\
q_2 &= \dot{y} \\
q_3 &= \ddot{y} - b_0u
\end{aligned}
$$

Now when we take the derivatives we get:

$$\begin{aligned}\dot{q}_1 &= q_2 = \dot{y} \\ \dot{q}_2 &= \ddot{y} \\ \dot{q}_3 &= \dddot{y} - b_0\dot{u} = -a_3 y - a_2\dot{y} - a_1\ddot{y} + b_1 u\end{aligned}$$

The second and third equations are not correct, because $\ddot{y}$ is not one of the state variables. However we can make use of the fact:

$$\begin{aligned} q_3 &= \ddot{y} - b_0 u, \quad \text{so} \\ \ddot{y} &= q_3 + b_0 u\end{aligned}$$

The second state variable equation then becomes

$$\dot{q}_2 = q_3 + b_0 u$$

In the third state variable equation we have successfully removed the derivative of the input from the right side of the third equation, and we can get rid of the $\ddot{y}$ term using the same substitution we used for the second state variable.

$$\begin{aligned}\dot{q}_3 &= -a_3 y - a_2\dot{y} - a_1\ddot{y} + b_1 u \\ &= -a_3 q_1 - a_2 q_2 - a_1\left(q_3 + b_0 u\right) + b_1 u \\ &= -a_3 q_1 - a_2 q_2 - a_1 q_3 + \left(b_1 - a_1 b_0\right) u\end{aligned}$$

$$\begin{aligned}\dot{\mathbf{q}} &= \mathbf{A}\mathbf{q} + \mathbf{B}u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}\mathbf{q} + \begin{bmatrix} 0 \\ b_0 \\ b_1 - a_1 b_0 \end{bmatrix} u \\ y &= \mathbf{C}\mathbf{q} + Du = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{q} + 0 \cdot u\end{aligned}$$

The process described in the previous example can be generalized to systems with higher order input derivatives but unfortunately gets increasingly difficult as the order of the derivative increases. When the order of derivatives is equal on both sides, the process becomes much more difficult (and the variable "D" is no longer equal to zero). Clearly more straightforward techniques are necessary. Two are outlined below, one generates a state space method known as the "controllable canonical form" and the other generates the "observable canonical form (the meaning of these terms derives from Control Theory but is not important to us).

## Controllable Canonical Form (CCF)

Probably the most straightforward method for converting from the transfer function of a system to a state space model is to generate a model in "controllable canonical form." This term comes from Control Theory but its exact meaning is not important to us. To see how this method of generating a state space model works, consider the third order differential transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^2 + b_1 s + b_2}{s^3 + a_1 s^2 + a_2 s + a_3}$$

We start by multiplying by Z(s)/Z(s) and then solving for Y(s) and U(s) in terms of Z(s). We also convert back to a differential equation.

$$\begin{aligned} Y(s) &= \left(b_0 s^2 + b_1 s + b_2\right) Z(s) \qquad & y &= b_0 \ddot{z} + b_1 \dot{z} + b_2 z \\ U(s) &= \left(s^3 + a_1 s^2 + a_2 s + a_3\right) Z(s) \qquad & u &= \dddot{z} + a_1 \ddot{z} + a_2 \dot{z} + a_3 z \end{aligned}$$

We can now choose z and its first two derivatives as our state variables

$$\begin{aligned} q_1 &= z \qquad & \dot{q}_1 &= \dot{z} = q_2 \\ q_2 &= \dot{z} \qquad & \dot{q}_2 &= \ddot{z} = q_3 \\ q_3 &= \ddot{z} \qquad & \dot{q}_3 &= \dddot{z} = u - a_1 \ddot{z} - a_2 \dot{z} - a_3 z \\ & & &= u - a_1 q_3 - a_2 q_2 - a_3 q_1 \end{aligned}$$

Now we just need to form the output

$$\begin{aligned} y &= b_0 \ddot{z} + b_1 \dot{z} + b_2 z \\ &= b_0 q_3 + b_1 q_2 + b_2 q_1 \end{aligned}$$

From these results we can easily form the state space model:

$$\begin{aligned} \dot{\mathbf{q}} &= \mathbf{A}\mathbf{q} + \mathbf{B}u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \mathbf{q} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= \mathbf{C}\mathbf{q} + Du = \begin{bmatrix} b_2 & b_1 & b_0 \end{bmatrix} \mathbf{q} + 0 \cdot u \end{aligned}$$

In this case, the order of the numerator of the transfer function was less than that of the denominator. If they are equal, the process is somewhat more complex. A result that works in all cases is given below; the details are here. For a general $n_{th}$ order transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

the controllable canonical state space model form is

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q} + \mathbf{B}u; \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$y = \mathbf{C}\mathbf{q} + Du \quad \mathbf{C} = \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1} b_0 & \cdots & b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix} \quad D = b_0$$

## Observable Canonical Form (OCF)

Another commonly used state variable form is the "observable canonical form." This term comes from Control Theory but its exact meaning is not important to us. To understand how this method works consider a third order system with transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^2 + b_1 s + b_2}{s^3 + a_1 s^2 + a_2 s + a_3}$$

We can convert this to a differential equation and solve for the highest order derivative of y:

$$\begin{aligned} \left(s^3 + a_1 s^2 + a_2 s + a_3\right) Y(s) &= \left(b_0 s^2 + b_1 s + b_2\right) U(s) \\ s^3 Y(s) &= \left(b_0 s^2 + b_1 s + b_2\right) U(s) - \left(a_1 s^2 + a_2 s + a_3\right) Y(s) \\ \dddot{y} &= b_0 \ddot{u} + b_1 \dot{u} + b_2 u - a_1 \ddot{y} - a_2 \dot{y} - a_3 y \end{aligned}$$

Now we integrate twice (the reason for this will be apparent soon), and collect terms according to order of the integral:

$$\dot{y} = b_0u + b_1\int u \cdot dt + b_2\int\int u \cdot dt \cdot dt - a_1y - a_2\int y \cdot dt - a_3\int\int y \cdot dt \cdot dt$$
$$= b_0u - a_1y + \int(b_1u - a_2y)\,dt + \int\int(b_2u - a_3y) \cdot dt \cdot dt$$

Choose the output as our first state variable

$$q_1 = y \qquad \dot{q}_1 = \dot{y} = b_0u - a_1y + \int(b_1u - a_2y)\,dt + \int\int(b_2u - a_3y) \cdot dt \cdot dt$$

Looking at the right hand side of the differential equation we note that $y=q_1$ and we call the two integral terms $q_2$:

$$q_2 = \int(b_1u - a_2y)\,dt + \int\int(b_2u - a_3y)\,dt \cdot dt$$

Now let's examine $q_2$ and its derivative

$$q_2 = \int(b_1u - a_2y)\,dt + \int\int(b_2u - a_3y)\,dt \cdot dt$$
$$\dot{q}_2 = b_1u - a_2y + \int(b_2u - a_3y)\,dt$$

Again we note that $y=q_1$ and we call the integral terms $q_3$:

$$q_3 = \int(b_2u - a_3y)\,dt$$

Our state space model now becomes

$$\dot{\mathbf{q}} = \mathbf{Aq} + \mathbf{B}u = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix}\mathbf{q} + \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}u$$
$$y = \mathbf{Cq} + Du = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{q} + 0 \cdot u$$

In this case, the order of the numerator of the transfer function was less than that of the denominator. If they are equal, the process is somewhat more complex.

## Concept of Eigen Values and Eigen Vectors

The roots of characteristic equation that we have described above are known as eigen values of matrix A. Now there are some properties related to eigen values and these properties are written below-

1. Any square matrix A and its transpose $A^T$ have the same eigen values.

2. Sum of eigen values of any matrix A is equal to the trace of the matrix A.

3. Product of the eigen values of any matrix A is equal to the determinant of the matrix A.

4. If we multiply a scalar quantity to matrix A then the eigen values are also get multiplied by the same value of scalar.

5. If we inverse the given matrix A then its eigen values are also get inverses.

6. If all the elements of the matrix are real then the eigen values corresponding to that matrix are either real or exists in complex conjugate pair.

**Eigen Vectors**

Any non zero vector $m_i$ that satisfies the matrix equation $\lambda_i I - A\ m_i = 0$ is called the eigen vector of A associated with the eigen value $\lambda_i$,. Where $\lambda_i$, i = 1, 2, 3, ……..n denotes the $i^{th}$ eigen values of A.

This eigen vector may be obtained by taking cofactors of matrix $\lambda_i I - A$ along any row & transposing that row of cofactors.

## Diagonalization

Let $m_1, m_2, \dots\dots. m_n$ be the eigenvectors corresponding to the eigen value $\lambda_1, \lambda_2, \dots\dots. \lambda_n$ respectively.

Then $M = [m_1 \ \vdots \ m_2 \ \vdots \cdots \ \vdots \ m_n]$ is called diagonalizing or **modal matrix** of A.

Consider the n[th] order MIMO state model

$$\dot{X}(t) = AX(t) + BU(t)$$

$$Y(t) = CX(t) + DU(t)$$

System matrix A is non diagonal, so let us define a new state vector V(t) such that X(t)=MV(t).

Under this assumption original state model modifies to

$$\dot{V}(t) = \tilde{A}V(t) + \tilde{B}U(t)$$

$$Y(t) = \check{C}V(t) + DU(t)$$

Where $\tilde{\boldsymbol{A}} = \boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M} = diagonal\ matrix, \quad \tilde{\boldsymbol{B}} = \boldsymbol{M}^{-1}\boldsymbol{B}, \quad \tilde{\boldsymbol{C}} = \boldsymbol{C}\boldsymbol{M}$

The above transformed state model is in canonical state model. The transformation described above is called similarity transformation. If the system matrix A is in companion form & if all its n eigen values are distinct, then modal matrix will be special matrix called the **Vander Monde matrix.**

$$Vander\ Monde\ matrix\ V = \begin{bmatrix} 1 & 1 & 1 \dots\dots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \dots\dots & \lambda_n \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{n-2} & \lambda_2^{n-2} & \lambda_3^{n-2} \dots\dots & \lambda_n^{n-2} \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} \dots\dots. & \lambda_n^{n-1} \end{bmatrix}_{n \times n}$$

## State Transition Matrix and Zero State Response

We are here interested in deriving the expressions for the state transition matrix and zero state response. Again taking the state equations that we have derived above and taking their Laplace transformation we have,

$$sX(s) - X(0) = AX(s) + BU(s)$$

Now on rewriting the above equation we have

$$X(s) = [sI - A]^{-1} \times x(0) + [sI - A]^{-1} \times BU(s)$$

Let $[sI-A]^{-1} = \theta(s)$ and taking the inverse Laplace of the above equation we have

$$X(t) = \theta(t).x(0) + L^{-1} \times \theta(s)BU(s)$$

The expression θ(t) is known as **state transition matrix(STM).**

$L^{-1}.\theta(t)BU(s)$ = zero state response.

Now let us discuss some of the properties of the state transition matrix.

1. If we substitute t = 0 in the above equation then we will get 1.Mathematically we can write θ(0) =1.
2. If we substitute t = -t in the θ(t) then we will get inverse of θ(t). Mathematically we can write $\theta(-t) = [\theta(t)]^{-1}$.
3. We also another important property $[\theta(t)]^n = \theta(nt)$.

**Computation of STN using Cayley-Hamilton Theorem:**

The Cayley–hamilton theorem states that every square matrix A satisfies its own characteristic equation. This theorem provides a simple procedure for evaluating the functions of a matrix. To determine the matrix polynomial

$$f(A) = k_0 I + k_1 A + k_2 A^2 + \cdots + k_n A^n + k_{n+1} A^{n+1} + \cdots$$

Consider the scalar polynomial

$$f(\lambda) = k_0 + k_1\lambda + k_2\lambda^2 + \cdots + k_n\lambda^n + k_{n+1}\lambda^{n+1} + \cdots$$

Here A is a square matrix of size (n×n). Its characteristic equation is given by

$$q(\lambda) = |\lambda I - A| = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0$$

If $f(A)$ is divided by the characteristic polynomial $q(\lambda)$ , then

$$\frac{f(\lambda)}{q(\lambda)} = Q(\lambda) + \frac{R(\lambda)}{q(\lambda)}$$

$$f(\lambda) = Q(\lambda)q(\lambda) + R(\lambda) \quad \ldots\ldots\ldots\ldots\ldots..(1)$$

Where $R(\lambda)$ is the remainder polynomial of the form

$$R(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_{n-1}\lambda^{n-1} \quad \ldots\ldots\ldots\ldots\ldots(2)$$

If we evaluate $f(A)$ at the eigen values $\lambda_1, \lambda_2, \ldots\ldots\ldots \lambda_n$, then $q(\lambda) = 0$ and we have from equation (1) $f(\lambda_i) = R(\lambda_i); \quad i = 1,2, \ldots . n \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$

The coefficients $a_0, a_1, \ldots\ldots\ldots a_{n-1}$, can be obtained by successfully substituting $\lambda_1, \lambda_2, \ldots\ldots\ldots \lambda_n$ into equation (3).

Substituting A for the variable $\lambda$ in equation (1), we get

$$f(A) = Q(A)q(A) + R(A)$$

As $q(A)$ *is zero, so* $f(A) = R(A)$

$$\Rightarrow \quad f(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_{n-1} A^{n-1}$$

*which is the desired result.*

## CONCEPTS OF CONTROLLABILITY & OBSERVABILITY

**State Controllability** A system is said to be completely state controllable if it is possible to transfer the system state from any initial state X(t₀) to any desired state X(t) in specified finite time by a control vector u(t).

Kalman‟s test

Consider nth order multi input LTI system with m dimensional control vector

$$\dot{X}(t) = AX(t) + BU(t)$$

is completely controllable if & only if the rank of the composite matrix $Q_c$ is n.

$$Q_c = [B \; \vdots \; AB \; \vdots \cdots \vdots \; A^{n-1}B]$$

**Observability**

A system is said to be completely observable, if every state $X(t_o)$ can be completely identified by measurements of the outputs y(t) over a finite time interval $(t_o \le t \le t_1)$.

Kalman's test

Consider $n^{th}$ order multi input LTI system with m dimensional output vector

$$\dot{X}(t) = AX(t) + BU(t)$$

$$Y(t) = CX(t) + DU(t)$$

is completely observable if & only if the rank of the observability matrix $Q_o$ is n.

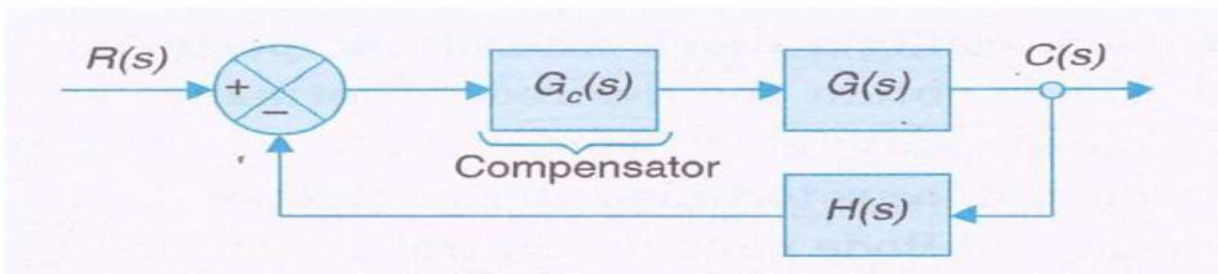
$$Q_o = [C^T \; \vdots \; A^T C^T \; \vdots \cdots \vdots \; (A^T)^{n-1} C^T]$$

## COMPENSATOR DESIGN

Every control system which has been designed for a specific application should meet certain performance specification. There are always some constraints which are imposed on the control system design in addition to the performance specification. The choice of a plant is not only dependent on the performance specification but also on the size , weight & cost. Although the designer of the control system is free to choose a new plant, it is generally not advised due to the cost & other constraints. Under this circumstances it is possible to introduce some kind of corrective sub-systems in order to force the chosen plant to meet the given specification. We refer to these sub-systems as **compensator** whose job is to compensate for the deficiency in the performance of the plant.

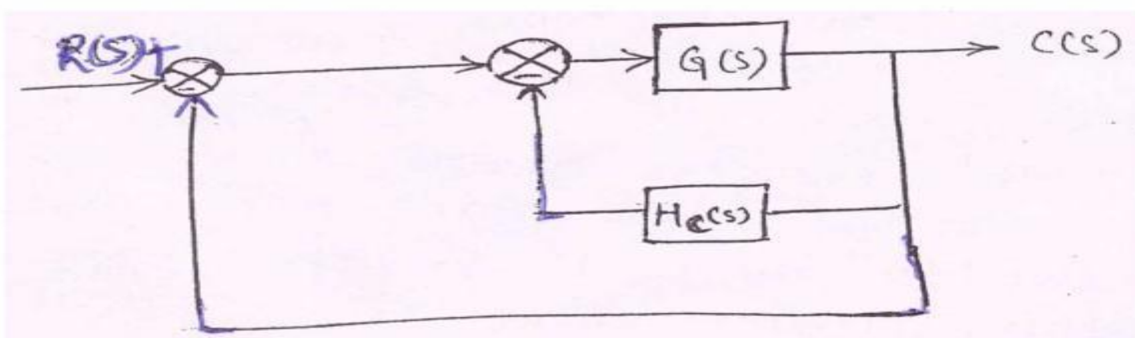
**REALIZATION OF BASIC COMPENSATORS**

Compensation can be accomplished in several ways. Series or Cascade compensation Compensator can be inserted in the forward path as shown in fig below. The transfer function of compensator is denoted as Gc(s), whereas that of the original process of the plant is denoted by G(s).

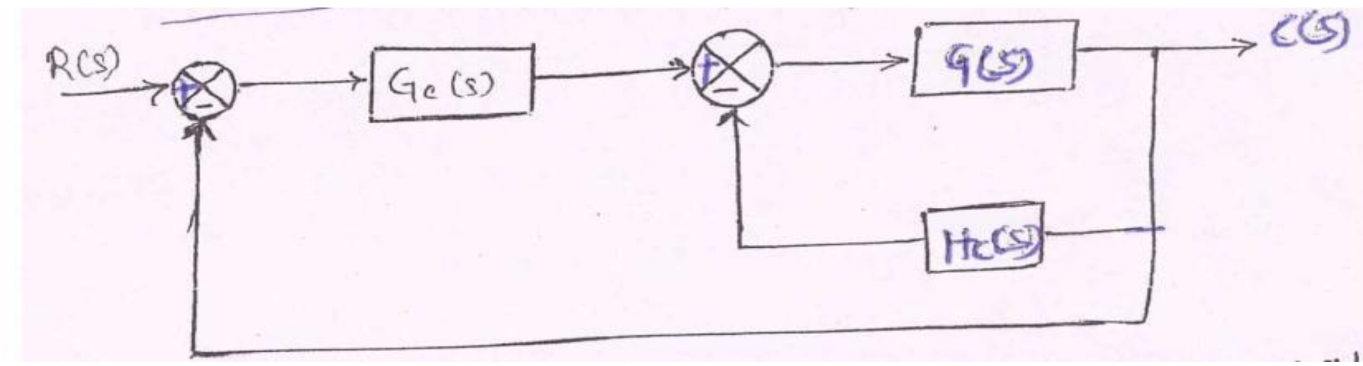


(a) Cascade compensation

Feedback compensation



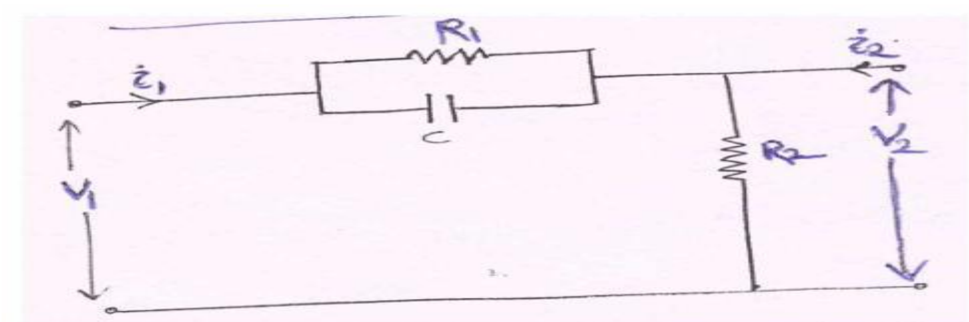

Combined Cascade & feedback compensation




Compensator can be electrical, mechanical, pneumatic or hydrolic type of device. Mostly electrical networks are used as compensator in most of the control system. The very simplest of these are Lead, lag & lead-lag networks.

**Lead Compensator**

Lead compensator are used to improve the transient response of a system.




Taking $i_2$=0 & applying Laplace Transform, we get

$$\frac{V_2(s)}{V_1(s)} = \frac{R_2(R_1Cs+1)}{R_2+R_2R_1Cs+R_1}$$

Let $\tau = R_1C$ , $\alpha = \frac{R_2}{R_1+R_2} < 1$

$$\frac{V_2(s)}{V_1(s)} = \frac{\alpha(\tau s+1)}{(1+\tau\alpha s)}$$ Transfer function of Lead Compensator

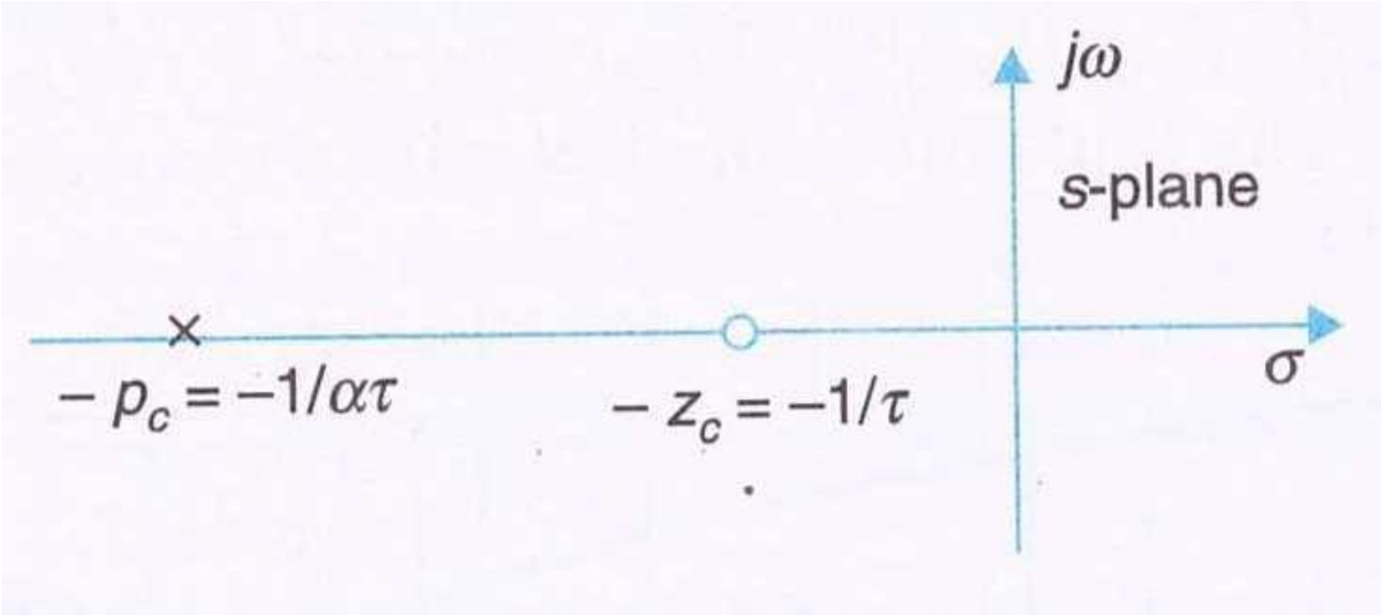



Fig: S-Plane representation of Lead Compensator

Bode plot for Lead Compensator

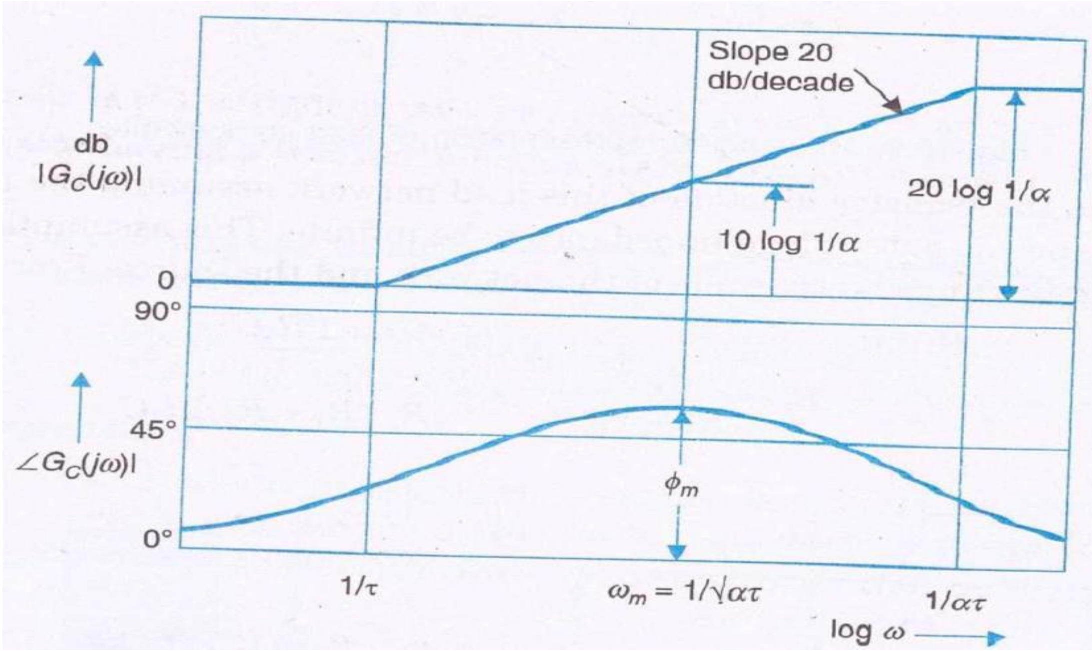
Maximum phase lead occurs at $w_m = \frac{1}{\tau\sqrt{\alpha}}$

Let $\phi_m$ = maximum phase lead

$$\sin\phi_m = \frac{1-\alpha}{1+\alpha}$$

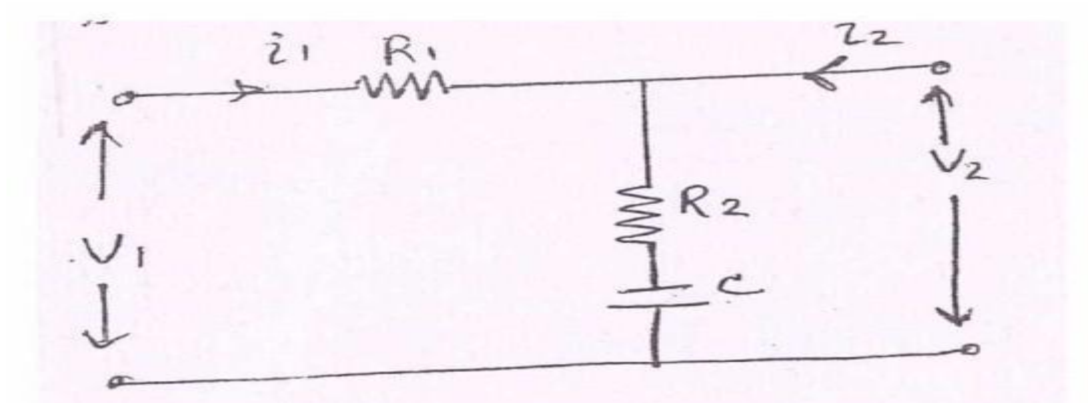
$$\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m}$$

Magnitude at maximum phase lead $|G_c(jw)| = \frac{1}{\sqrt{\alpha}}$




**Lag Compensator**

Lag compensator are used to improve the steady state response of a system.



Taking $i_2$=0 & applying Laplace Transform, we get

$$\frac{V_2(s)}{V_1(s)} = \frac{R_2Cs + 1}{(R_2 + R_1)Cs + 1}$$

Let $\tau = R_2C$ , $\beta = \frac{R_1+R_2}{R_2} > 1$

$\frac{V_2(s)}{V_1(s)} = \frac{\tau s+1}{1+\tau\beta s}$ Transfer function of Lag Compensator

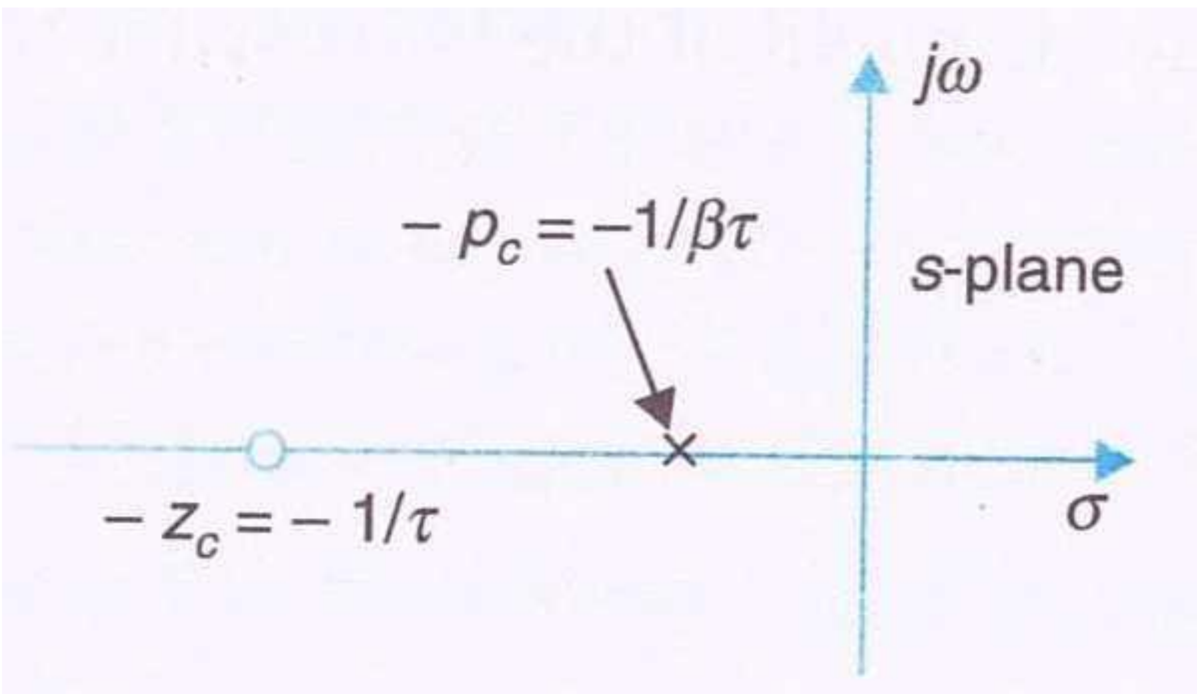



Fig: S-Plane representation of Lag Compensator

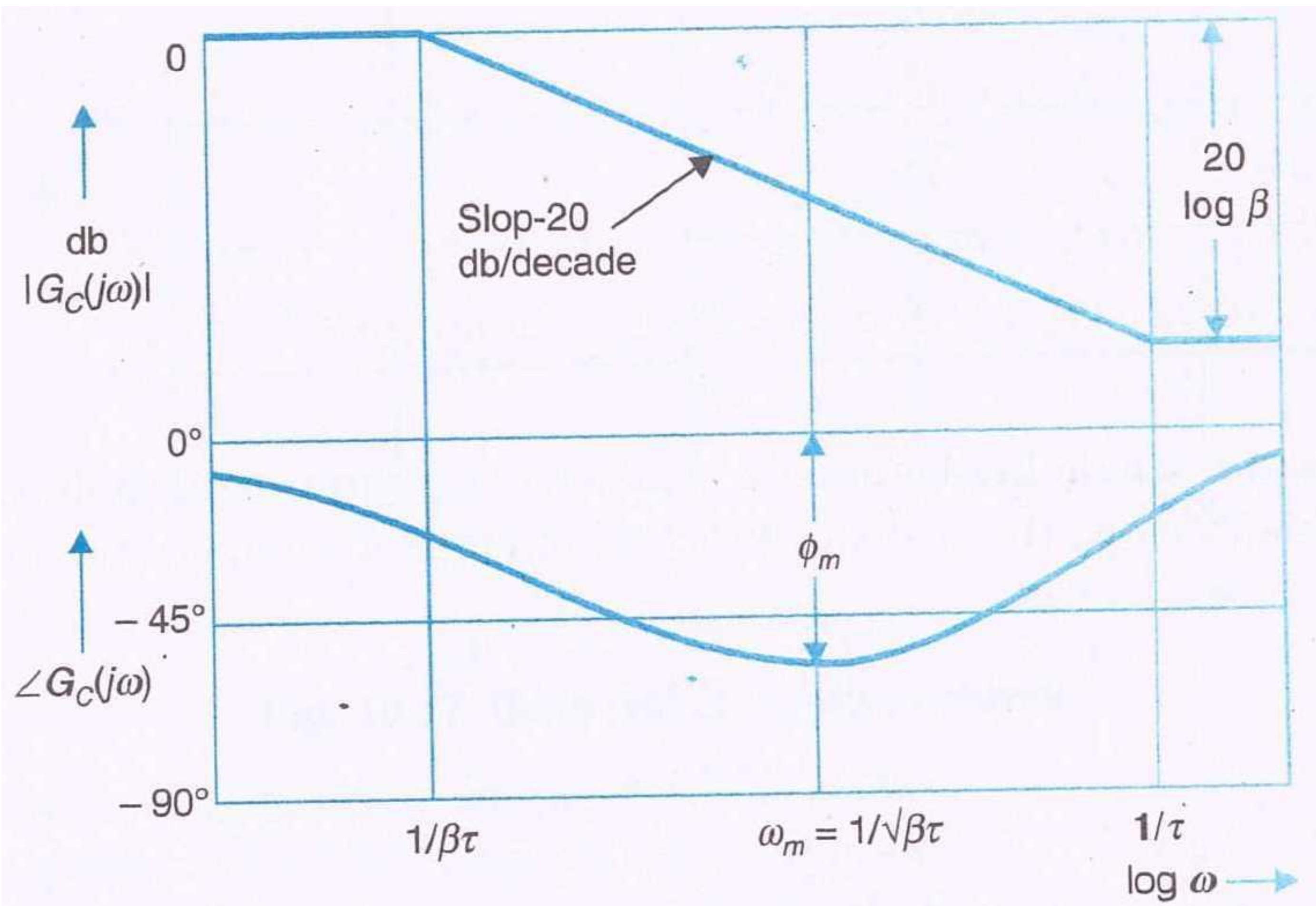
Bode plot for Lag Compensator

Maximum phase lag occurs at $w_m = \frac{1}{\tau\sqrt{\beta}}$

Let $\phi_m$ = maximum phase lag

$$\sin\phi_m = \frac{1-\beta}{1+\beta}$$

$$\beta = \frac{1-\sin\phi_m}{1+\sin\phi_m}$$




## Cascade compensation in Time domain

Cascade compensation in time domain is conveniently carried out by the root locus technique. In this method of compensation, the original design specification on dynamic response are converted into $\zeta$ & $wn$ of a pair of desired complex conjugate closed loop pole based on the assumption the system would be dominated by these two complex pole therefore its dynamic behavior can be approximated by that of a second order system. A compensator is now designed so that the least damped complex pole of the resulting transfer function correspond to the desired dominant pole & all other closed loop poles are located very close to the open loop zeros or relatively far away from the jw axis. This ensures that the poles other than the dominant poles make negligible contribution to the system dynamics.

**Lead Compensation**

- Consider a unity feedback system with a forward path unalterable Transfer function $G_f(s)$, then let the dynamic response specifications are translated into desired location $S_d$ for the dominant complex closed loop poles.
- If the angle criteria as $S_d$ is not meet i.e $\angle G_f(s) \neq \pm 180°$ the uncompensated Root Locus with variable open loop gain will not pass through the desired root location, indicating the need for the compensation.
- The lead compensator $G_c(s)$ has to be designed that the compensated root locus passes through $S_d$ . In terms of angle criteria this requires that

$$\angle G_c(s_d)G_f(s_d) = \angle G_c(s_d) + \angle G_f(s_d) \pm 180°$$

$$\angle G_c(s_d) = \phi = \pm 180° - \angle G_f(s_d)$$

- Thus for the root locus for the compensated system to pass through the desired root location the lead compensator pole-zero pair must contribute an angle $\phi$ .
- For a given angle $\phi$ required for lead compensation there is no unique location for pole-zero pair. The best compensator pole-zero location is the one which gives the largest value of .

  Where $\alpha = \frac{Z_c}{P_c}$

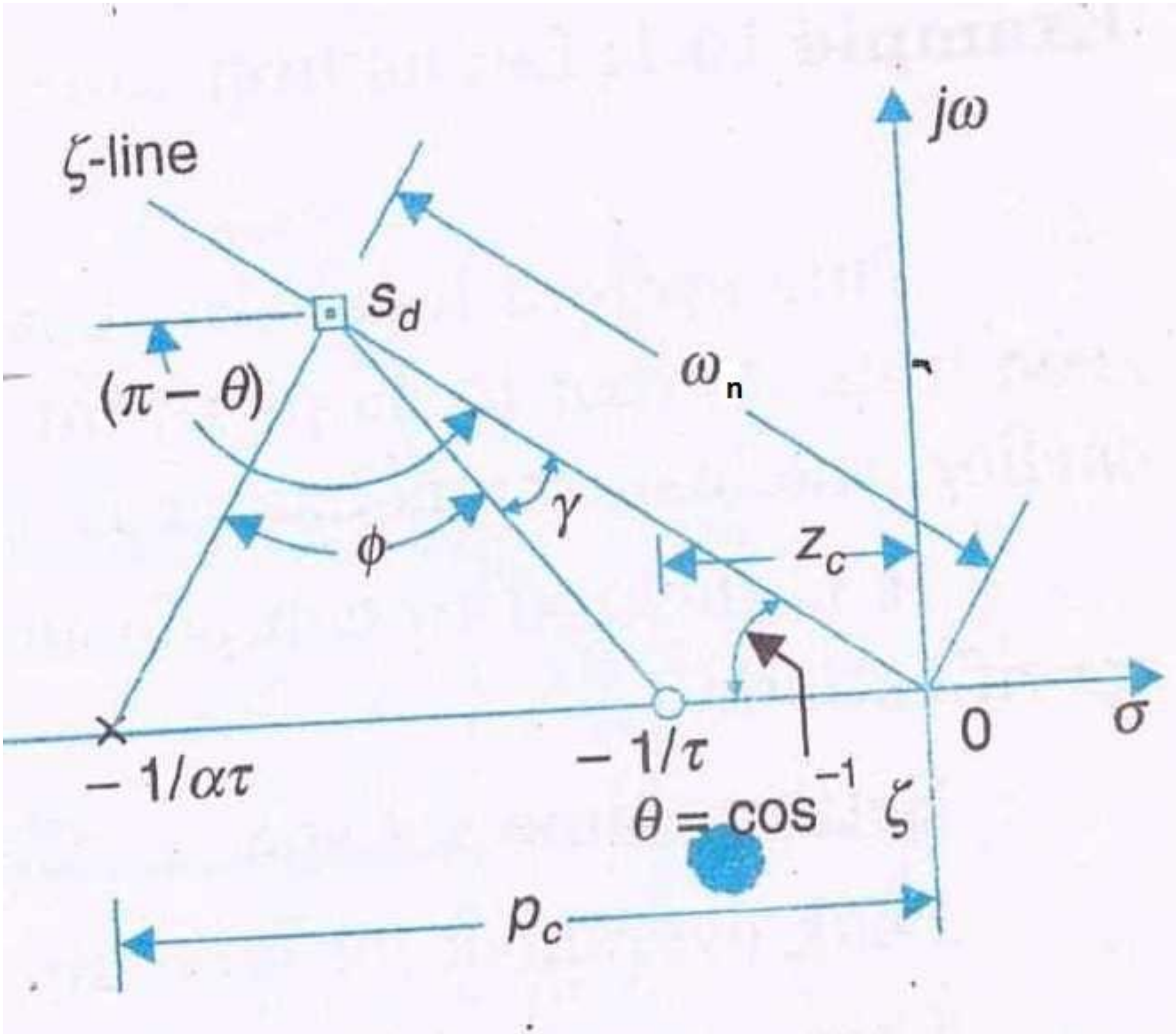



Fig: Angle contribution of Lead compensator

- The compensator zero is located by drawing a line from $S_d$ making an angle $\gamma$ with $\zeta$ line.
- The compensator pole then located by drawing a further requisite angle $\phi$ to be contribute at $S_d$ by the pole zero pair. From the geometry of the figure

$$\frac{z_c}{\sin\gamma} = \frac{w_n}{\sin(\pi - \theta - \gamma)}$$

$$\Rightarrow z_c = \frac{w_n \sin\gamma}{\sin(\pi - \theta - \gamma)}$$

Assuming big triangle

$$\frac{p_c}{\sin(\phi+\gamma)} = \frac{w_n}{\sin(\pi-\theta-\gamma-\phi)}$$

$$\Rightarrow p_c = \frac{w_n \sin(\phi+\gamma)}{\sin(\pi-\theta-\gamma-\phi)}$$

$$\alpha = \frac{Z_c}{P_c} == \frac{\sin(\pi-\theta-\gamma-\phi)\sin\gamma}{\sin(\pi-\theta-\gamma)\sin(\phi+\gamma)}$$

To find $\alpha_{max}$ , $\frac{d\alpha}{d\gamma} = 0$

$$\Rightarrow \gamma = \frac{1}{2}(\pi-\theta-\phi)$$

Though the above method of locating the lead compensator pole-zero yields the largest value of $\alpha$, it does not guarantee the dominance of the desired closed loop poles in the compensated root-locus. The dominance condition must be checked before completing the design. With compensator pole-zero so located the system gain at Sd is computed to determine the error constant. If the value of the error constant so obtained is unsatisfactory the above procedure is repeated after readjusting the compensator pole-zero location while keeping the angle contribution fixed as $\phi$.

**Lag Compensation**

Consider a MODULE feedback system with forward path transfer function

$$G_f(s) = \frac{k\prod_{i=1}^{m}(s+z_i)}{s^r\prod_{j=r+1}^{n}(s+p_j)}$$

At certain value of K, this system has satisfactory transient response i.e its root locus plot passes through(closed to) the desired closed loop poles location Sd .

It is required to improve the system error constant to a specified value $Kec$ without damaging its transient response. This requires that after compensation the root locus should continue to pass through Sd while the error constant at Sd is raised to $Kec$. To accomplish this consider adding a lag compensator pole-zero pair with zero the left of the pole. If this pole-zero pair is located closed to each other it will contribute a negligible angle at $S_d$ such that $S_d$ continues to lie on the root locus of the compensated system.

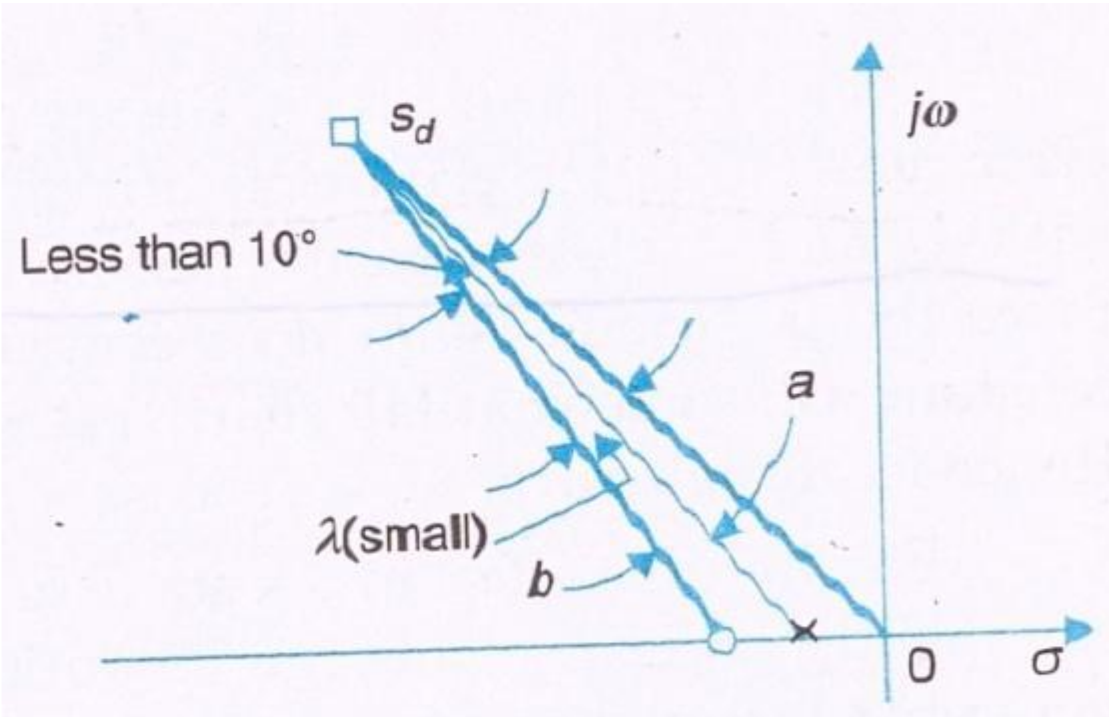



Fig:Locating the Lag Compensator Pole-zero

From the above fig. that apart from being close to each other the pole-zero pair close to origin, the reason which will become obvious from discussion below. The gain of the uncompensated system at $S_d$ is given by

$$K^{uc}(s_d) = \frac{|s_d|^r \prod_{j=r+1}^{n}(s_d + p_j)}{\prod_{i=1}^{m}(s_d + z_i)}$$

For compensated system the system gain at $S_d$ is given by

$$K^{c}(s_d) = \frac{|s_d|^r \prod_{j=r+1}^{n}(s_d + p_j)}{\prod_{i=1}^{m}(s_d + z_i)} \quad \frac{a}{b}$$

Since the pole & zero are located close to each other they are nearly equidistance from $S_d$
i.e $a \approx b$
i.e $K^c(s_d) \cong K^{uc}(s_d)$

The error constant for the compensated system is given by

$$K_e^c = K^c(s_d) \frac{\prod_{i=1}^{m} z_i}{\prod_{j=r+1}^{n} p_j} \frac{z_c}{p_c}$$

$$K_e^c \cong K^{uc}(s_d) \frac{\prod_{i=1}^{m} z_i}{\prod_{j=r+1}^{n} p_j} \frac{z_c}{p_c}$$

$$K_e^c = K_e^{uc} \frac{z_c}{p_c}$$

$$\beta = \frac{z_c}{p_c} = \frac{K_e^c}{K_e^{uc}} \ldots\ldots\ldots\ldots\ldots..(1)$$

The $\beta$ parameter of lag compensator is nearly equal to the ratio of specified error constant to the error constant of the uncompensated system.
Any value of $\beta = \frac{z_c}{p_c} > 1$ with $-p_c$ & $-z_c$ close to each other can be realized by keeping the pole-zero pair close to origin.

Since the Lag compensator does contribute a small negative angle $\lambda$ at $S_d$ , the actual error constant will some what fall short of the specified value if $\beta$ obtained from equation(1) is used. Hence for design purpose we choose $\beta$ somewhat larger than that the given by this equation(1).

For the effect of the small lag angle $\lambda$ is to give the closed loop pole Sd with specified $\zeta$ but slightly lower . This can be anticipated & counteracted by taking the $wn$ of Sd to be somewhat larger than the specified value.

Procedure of Lead Compensation

Step1: Determine the value of loop gain K to satisfy the specified error constant. Usually the error constant ($K_p$,$K_v$,$K_a$) & Phase margin are the specification given.

Step2: For this value of K draw the bode plot & determine the phase margin $\phi$ for the system.

Step3: If $\phi_s$ =specified phase margin &

$\phi$ = phase margin of uncompensated system (found out from the bode plot drawn)

$\epsilon$ =margin of safety (since crossover frequency may increase due to compensation)

- $\epsilon$ is the unknown reduction in phase angle $\angle G_f(s)$ on account of the increase in cross-over frequency. A guess is made on the value of $\epsilon$ depending on the slope in this region of the dB-log w plot of the uncompensated system.
- For a slope of -40dB/decade $\epsilon = 5° - 10°$ is a good guess. The guess value may have to be as high as $15°\ to\ 20°$ for a slope of -60dB/decade.
- Phase lead required $\phi_l = \phi_s - \phi + \epsilon$

Step4: Let $\phi_l = \phi_m$

Determine

$$\alpha = \frac{1 - \sin\phi_m}{1 + \sin\phi_m}$$

If $\phi_m > 60°$, two identical networks each contributing a maximum lead of $\phi_l/2$ are used.

Step5: Find the frequency $\omega_m$ at which the uncompensated system will have a gain equals to $-10\log\frac{1}{\alpha}$ from the bode plot drawn.

Take $\omega_{c2} = \omega_m$ = cross-over frequency of compensated system.

Step6: Corner frequency of the network are calculated as

$$\omega_1 = \frac{1}{\tau} = \omega_m\sqrt{\alpha} \quad ,\omega_2 = \frac{1}{\tau\alpha} = \frac{\omega_m}{\sqrt{\alpha}}$$

Transfer function for compensated system in Lead network $G_c(s) = \frac{s+\frac{1}{\tau}}{s+\frac{1}{\tau\alpha}}$

Step7: Draw the magnitude & Phase plot for the compensated system & check the resulting phase margin. If the phase margin is still low raise the value of $\epsilon$ & repeat the procedure.

Procedure of Lead Compensation

Step1: Determine the value of loop gain K to satisfy the specified error constant.

Step2: For this value of K draw the bode plot & determine the phase margin $\phi$ for the system.

Step3: If $\phi_s$ =specified phase margin &

$\phi$ = phase margin of uncompensated system (found out from the bode plot drawn)

$\epsilon$ =margin of safety ($5° - 10°$)

- For a suitable $\epsilon$ find $\phi_2 = \phi_s + \epsilon$ ,where $\phi_2$ is measured above $-180°$ line.

Step4: Find the frequency $\omega_{c2}$ where the uncompensated system makes a phase margin contribution of $\phi_2$.

Step5: Measure the gain of uncompensated system at $\omega_{c2}$. Find $\beta$ from the equation

$$gain\ at\ \omega_{c2} = 20 \log \beta$$

Step6: Choose the upper corner frequency $\omega_2 = \frac{1}{\tau}$ of the network one octave to one decade below $\omega_{c2}$ ($i.e\ between\ \frac{\omega_{c2}}{2}\ to\ \frac{\omega_{c2}}{10}$)

Step7: Thus $\beta$ & $\tau$ are determined which can be used to find the transfer function of Lag compensator.

$$G_c(s) = \frac{1}{\beta}\left[\frac{s + \frac{1}{\tau}}{s + \frac{1}{\tau\alpha}}\right]$$

Compensated Transfer function $G(s) = G_f\ G_c$

Draw the bode plot of the compensated system & check if the given specification are met.