Dynamics of Machinery

V semester

Department of Mechanical Engineering

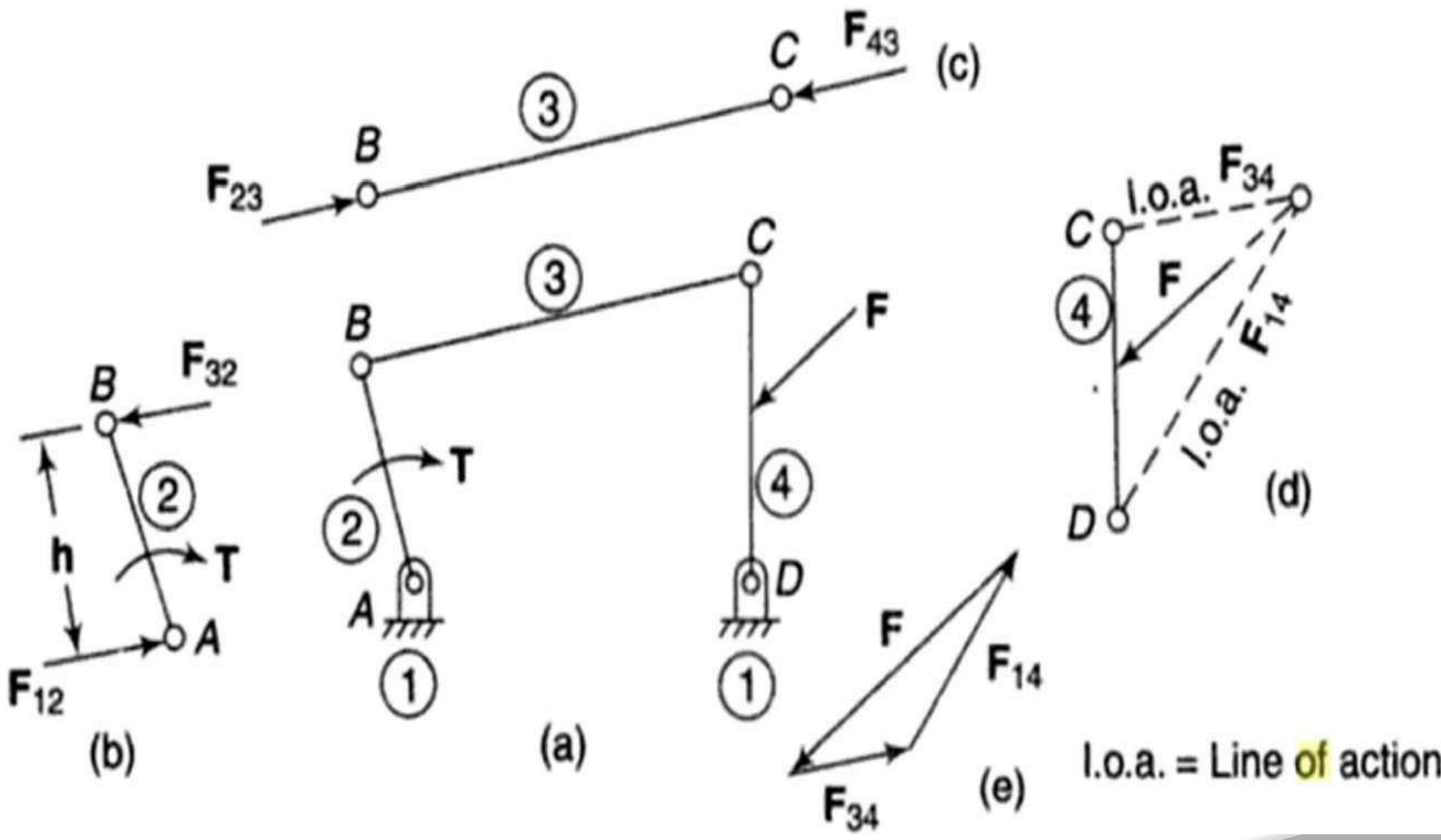
Prepared by

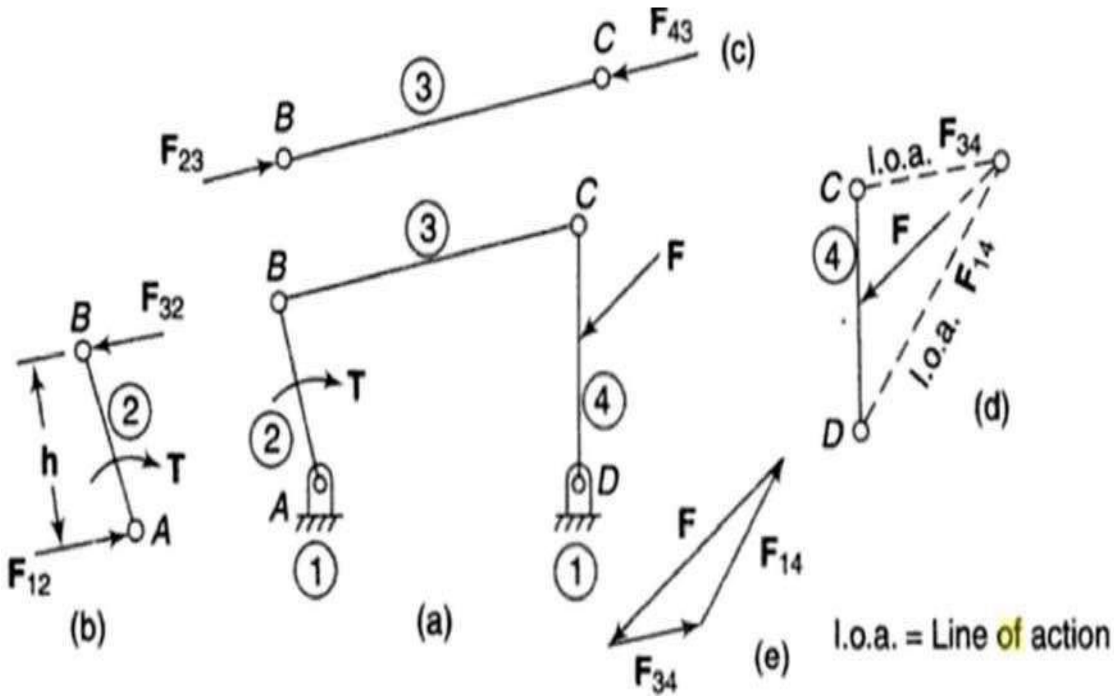
Dr K Viswanath Allamraju

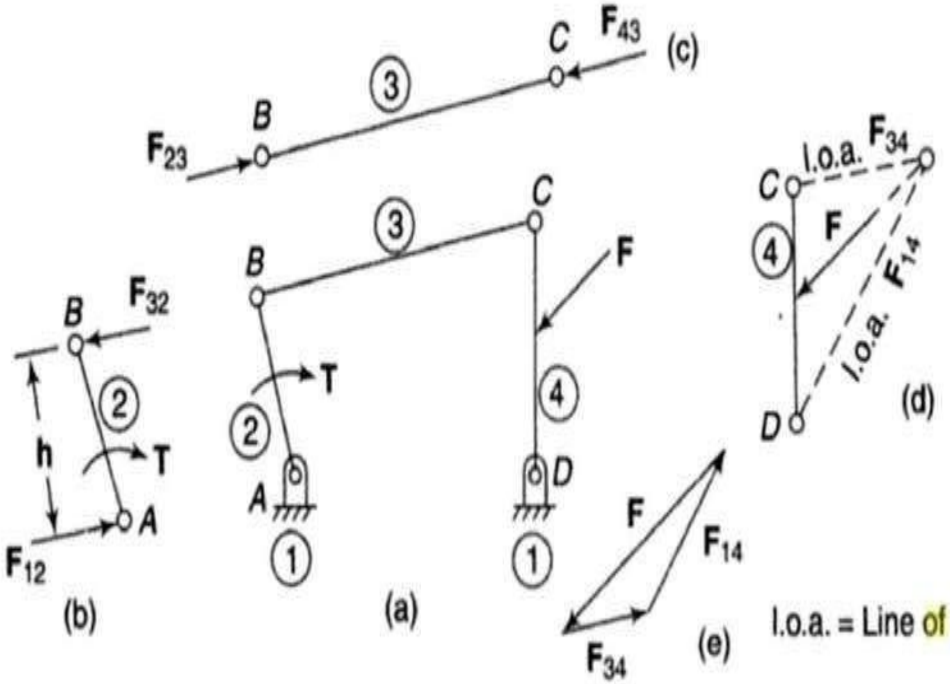
Prof V.V.S.H. Prasad

Unit-I: Precisional Motion

- Static force analysis
  - Static equilibrium
  - Equilibrium of two and three force members
  - Members with two forces and torque
  - Free body diagrams
  - principle of virtual work
- Static force analysis of
  - four bar mechanism
  - slider-crank mechanism with and without friction


F43
C
(c)
3
B
F23
F34
l.o.a.
C
F
4
F14
l.o.a.
D
(d)
C
3
B
F
F32
B
2
T
h
2
T
A
4
D
A
F12
1
1
(b)
(a)
F
F14
F34
(e)
l.o.a. = Line of action


F43
C
(c)
3
B
F23
C
3
B
F
F32
B
2
T
h
A
F12
(b)
2
T
2
A
1
(a)
4
D
1
F34
l.o.a.
C
F
4
F14
l.o.a.
(d)
D
F
F14
F34
(e)
l.o.a. = Line of action



l.o.a. = Line of action

Gyroscope

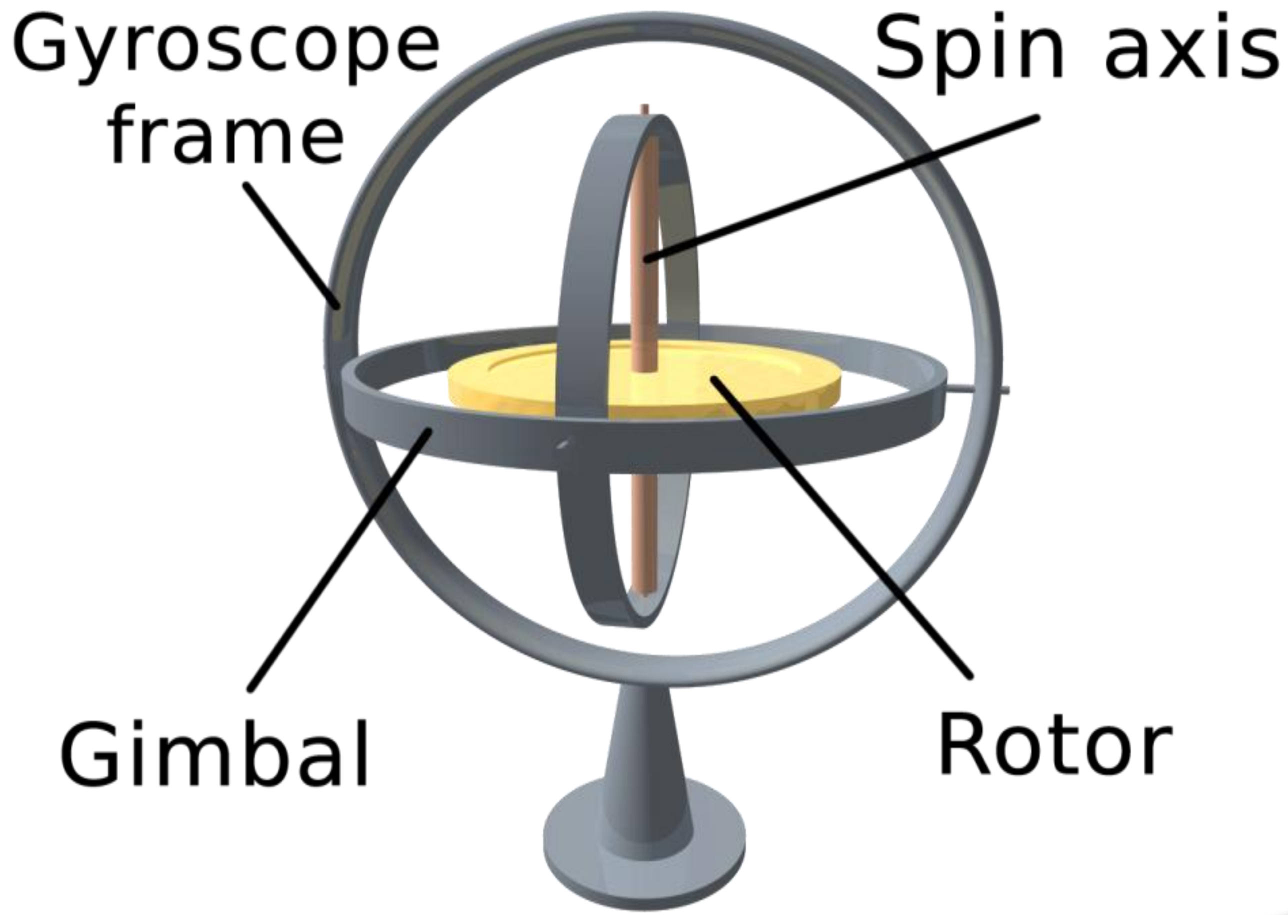
- A gyroscope is a device for measuring or maintaining orientation, based on the principles of angular momentum.
- Mechanically, agyroscope is a spinning wheel or disk in which the axle is free to assume any orientation. Although this orientation does not remain fixed, it changes in response toan external torque much less and in a different direction than it would without the large angular momentum associated with the disk's high rate of spin and moment of inertia.

- Since external torque is minimized by mounting the device in gimbals, its orientation remains nearly fixed, regardless of any motion of the platform on which itis mounted.

Gyroscope

Gyroscope

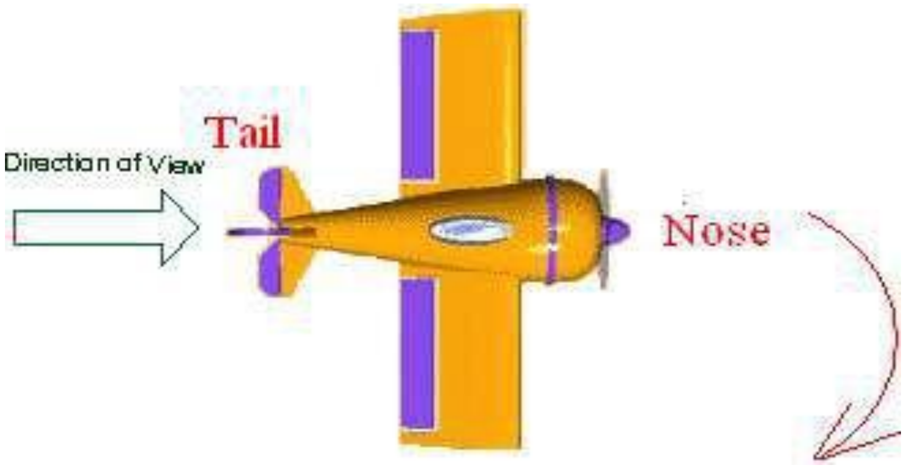
A mechanical gyroscope is essentially a spinning wheel or disk whose axle is free to take any orientation. This orientation changes much less in response to a given external torque than it would without the large angular momentum associated with the gyroscope's high rate of spin.

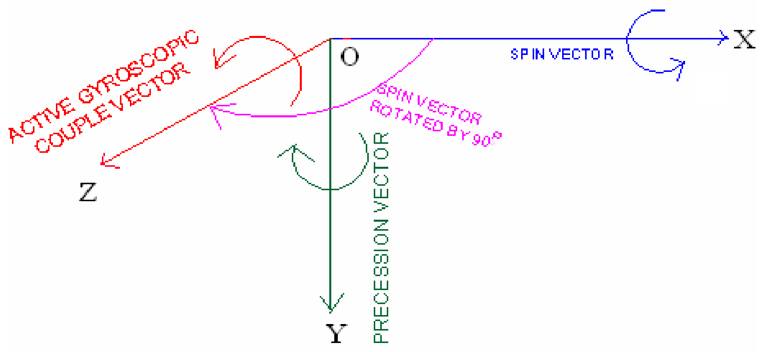

Gyroscope frame
Spin axis
Gimbal
Rotor

PROPERTIES OF GYROSCOPES

Gyroscopes have two basic properties:
Rigidity and Precession
These properties are defined as follows:
RIGIDITY: The axis of rotation (spin axis) of the gyro wheel tends to remain in a fixed direction in space if no force is applied to it.
PRECESSION: The axis of rotation has a tendency to turn at a right angle to the direction of an applied force.

PROPELLER ROTATES ANTICLOCKWISE



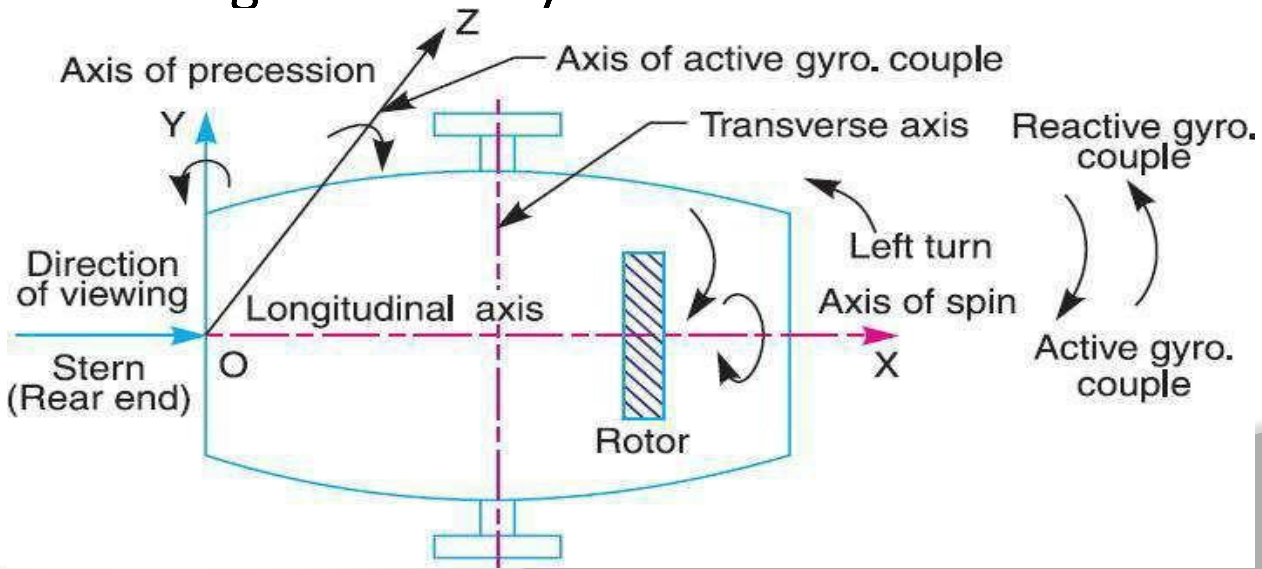


Effect of Gyroscopic Couple on a Naval Ship

•Steering is the turning of a complete ship in a curve towards left or right, while it moves forward.

•Consider the ship taking a left turn, and rotor rotates in the clockwise direction when viewed from the stern, as shown inFig.

•The effect of gyroscopic couple on a naval ship during steering taking left or right turn may be obtained



Stability of a Four Wheel Drive Moving in a Curved Path

Let m = Mass of the vehicle in kg,

W = Weight of the vehicle in newtons = $m.g$,

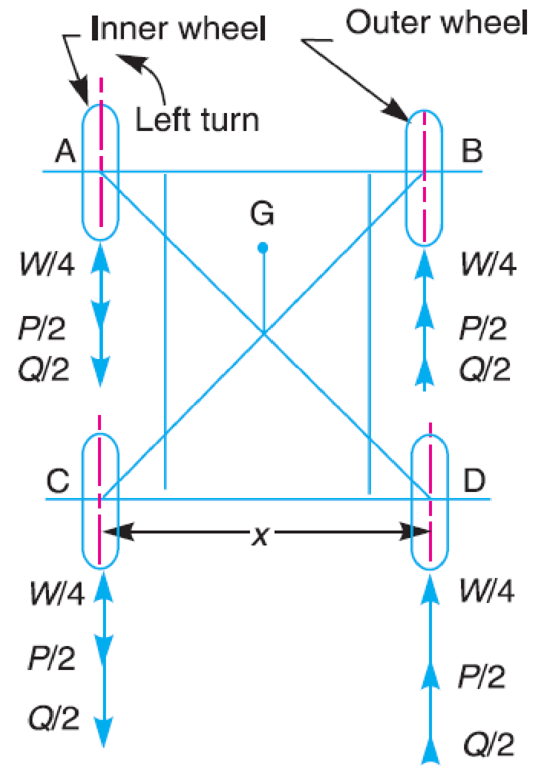
r_W = Radius of the wheels in metres,

R = Radius of curvature in metres ($R > r_W$),

h = Distance of centre of gravity, vertically above the road surface in metres,

x = Width of track in metres,

I_W = Mass moment of inertia of one of the wheels in kg-m^2,



Stability of a Four Wheel Drive Moving in a Curved Path

1. *Effect of the gyroscopic couple*

Since the vehicle takes a turn towards left due to the precession and other rotating parts, therefore a gyroscopic couple will act.

We know that velocity of precession,

$$\omega_P = v/R$$

$\therefore$ Gyroscopic couple due to 4 wheels,

$$C_W = 4\, I_W . \omega_W . \omega_P$$

and gyroscopic couple due to the rotating parts of the engine,

$$C_E = I_E . \omega_E . \omega_P = I_E . G . \omega_W . \omega_P \qquad \ldots (\because G = \omega_E / \omega_W)$$

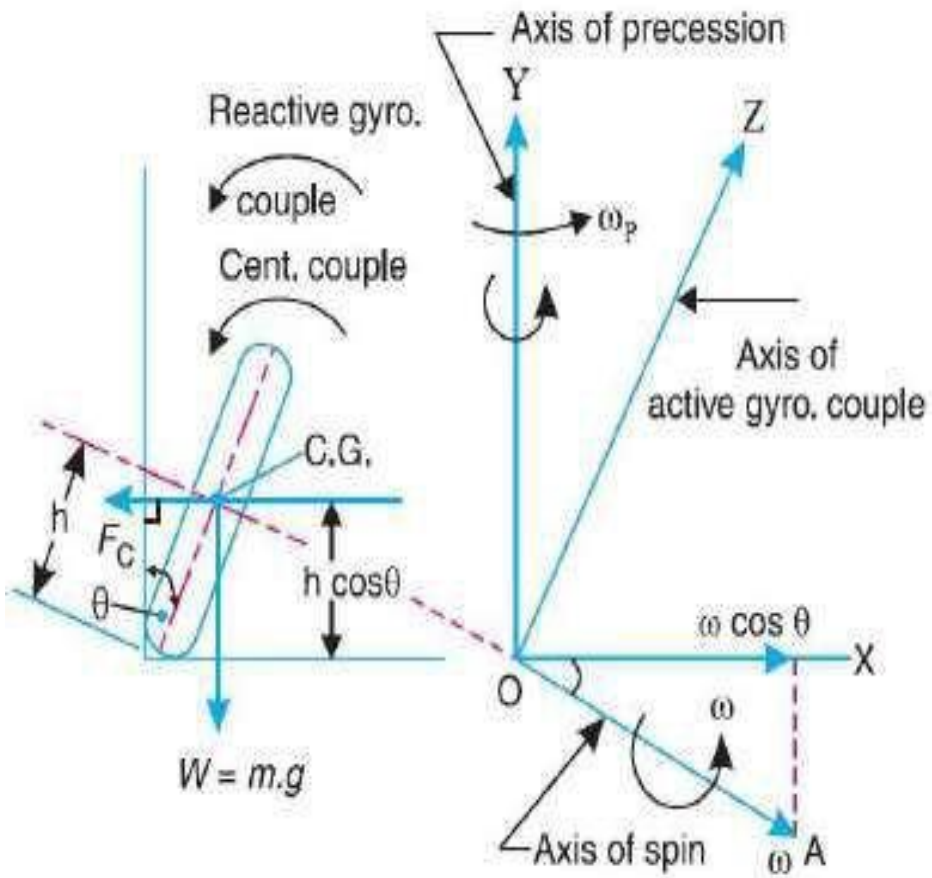
$\therefore$ Net gyroscopic couple,

$$C = C_W \pm C_E = 4\, I_W . \omega_W . \omega_P \pm I_E . G . \omega_W . \omega_P$$
$$= \omega_W . \omega_P \, (4\, I_W \pm G . I_E)$$

$\therefore$ The couple tending to overturn the vehicle or overturning couple,

$$C_O = F_C \times h = \frac{m . v^2}{R} \times h$$

Stability of a Two Wheel Drive Moving in a Curved Path



I_W = Mass moment of inertia of each wheel,

I_E = Mass moment of inertia of the rotating parts of the engine,

ω_W = Angular velocity of the wheels,

ω_E = Angular velocity of the engine,

G = Gear ratio = ω_E / ω_W,

v = Linear velocity of the vehicle = $\omega_W \times r_W$,

θ = Angle of heel. It is inclination of the vehicle

1. *Effect of gyroscopic couple*

We know that $v = \omega_W \times r_W$ or $\omega_W = v / r_W$

and $\omega_E = G.\omega_W = G \times \dfrac{v}{r_W}$

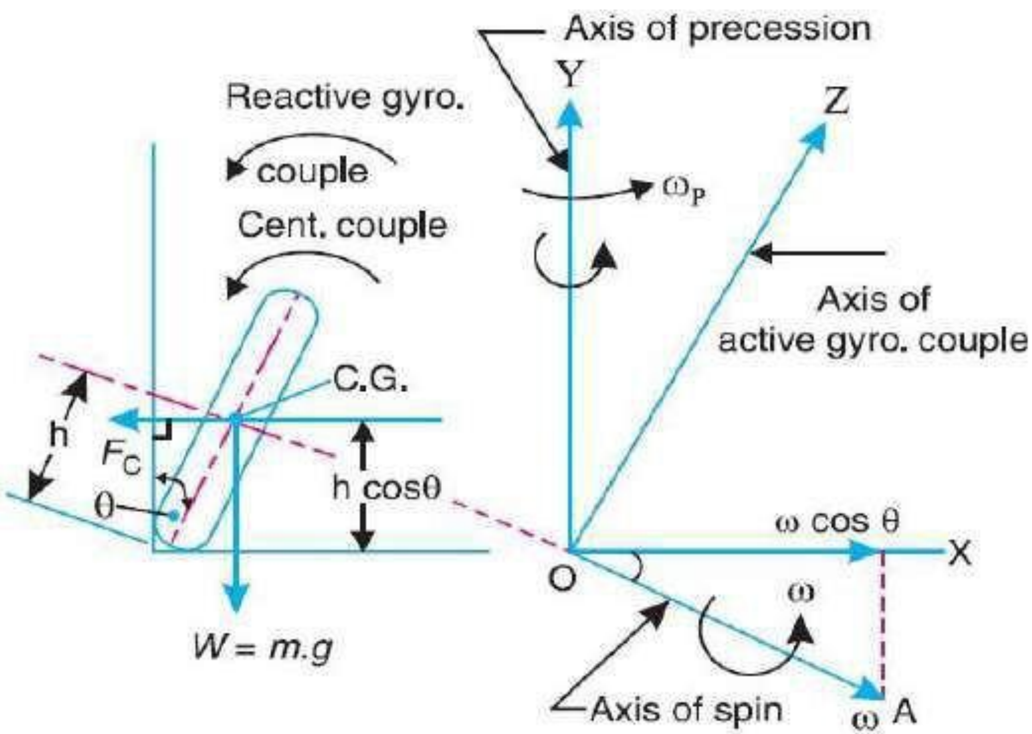
$\therefore$ Total $(I \times \omega) = 2\,I_W \times \omega_W \pm I_E \times \omega_E$

$$= 2\,I_W \times \frac{v}{r_W} \pm I_E \times G \times \frac{v}{r_W} = \frac{v}{r_W}(2\,I_W \pm G.I_E)$$

and velocity of precession, $\omega_P = v / R$

$$C_1 = I.\omega \cos\theta \times \omega_P = \frac{v}{r_W}(2\, I_W \pm G.I_E)\cos\theta \times \frac{v}{R}$$

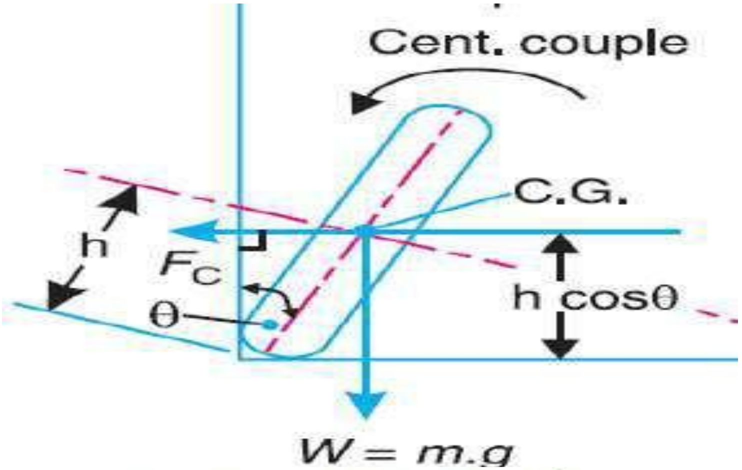
$$= \frac{v^2}{R.r_W}(2\, I_W \pm G.I_E)\cos\theta$$



This force acts horizontally through the centre of gravity (*C.G.*) along the outward direction.

∴ Centrifugal couple,

$$C_2 = F_C \times h \cos\theta = \left(\frac{m.v^2}{R}\right) h \cos\theta$$



Since the centrifugal couple has a tendency to overturn the vehicle, therefore

Total overturning couple,

$$C_O = \text{Gyroscopic couple} + \text{Centrifugal couple}$$

$$= \frac{v^2}{R.r_W}(2\,I_W + G.I_E)\cos\theta + \frac{m.v^2}{R} \times h\cos\theta$$

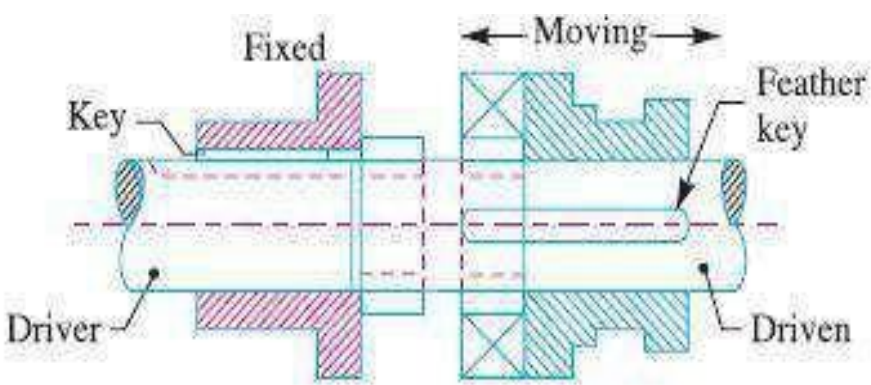
$$= \frac{v^2}{R}\left[\frac{2\,I_W + G.I_E}{r_W} + m.h\right]\cos\theta$$

- A clutch is designed with the following requirements
  - Allow the vehicle to come to a stop while the transmission remains in gear
  - Allow the driver to smoothly take off from a dead stop
  - Allow the driver to smoothly change gears
  - Must not slip under heavy loads and full engine power

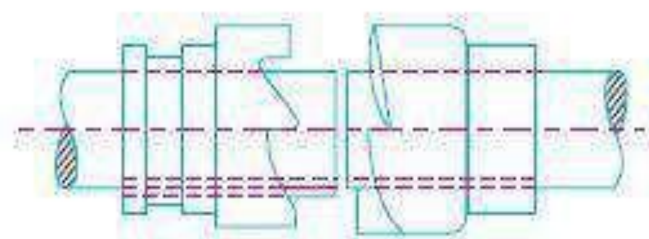
Types of clutches

Positive clutches

- When positive drive is required then positive clutches are used. The simplest type of positive clutch is the jaw clutch which transmits the torque from one shaft to another shaft through interlocking jaws.



(a) Square jaw clutch.



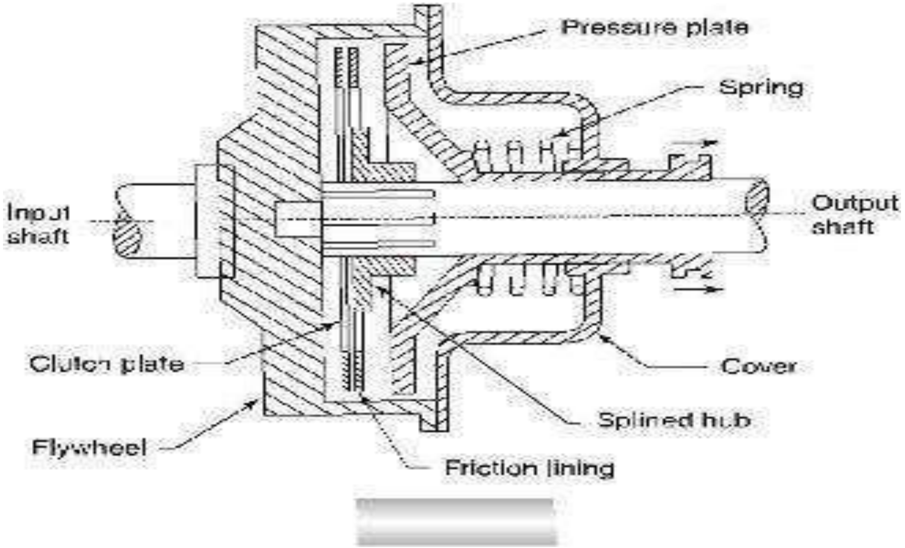
(b) Spiral jaw clutch.

Single plate clutch

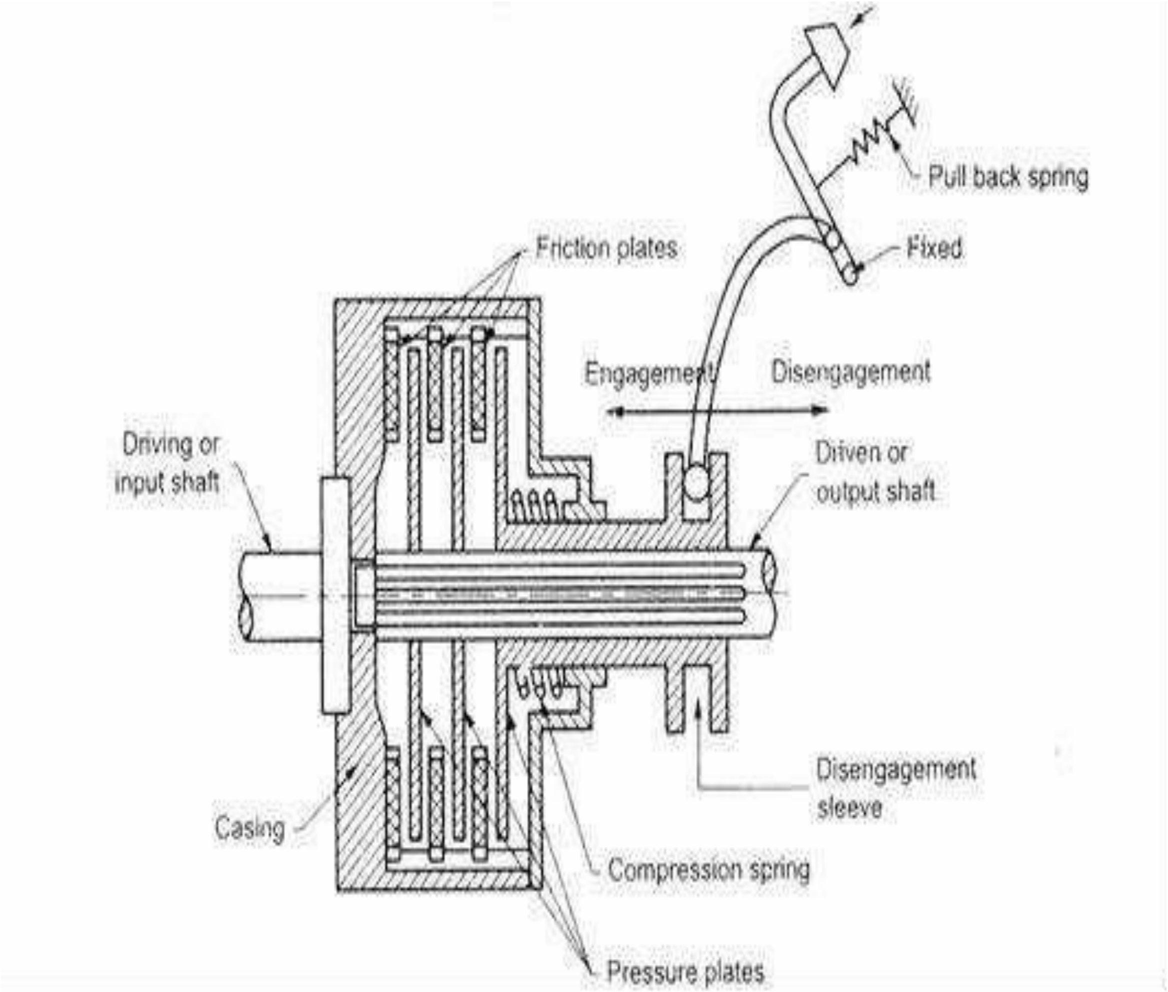
Single plate clutch

It consists of various elements

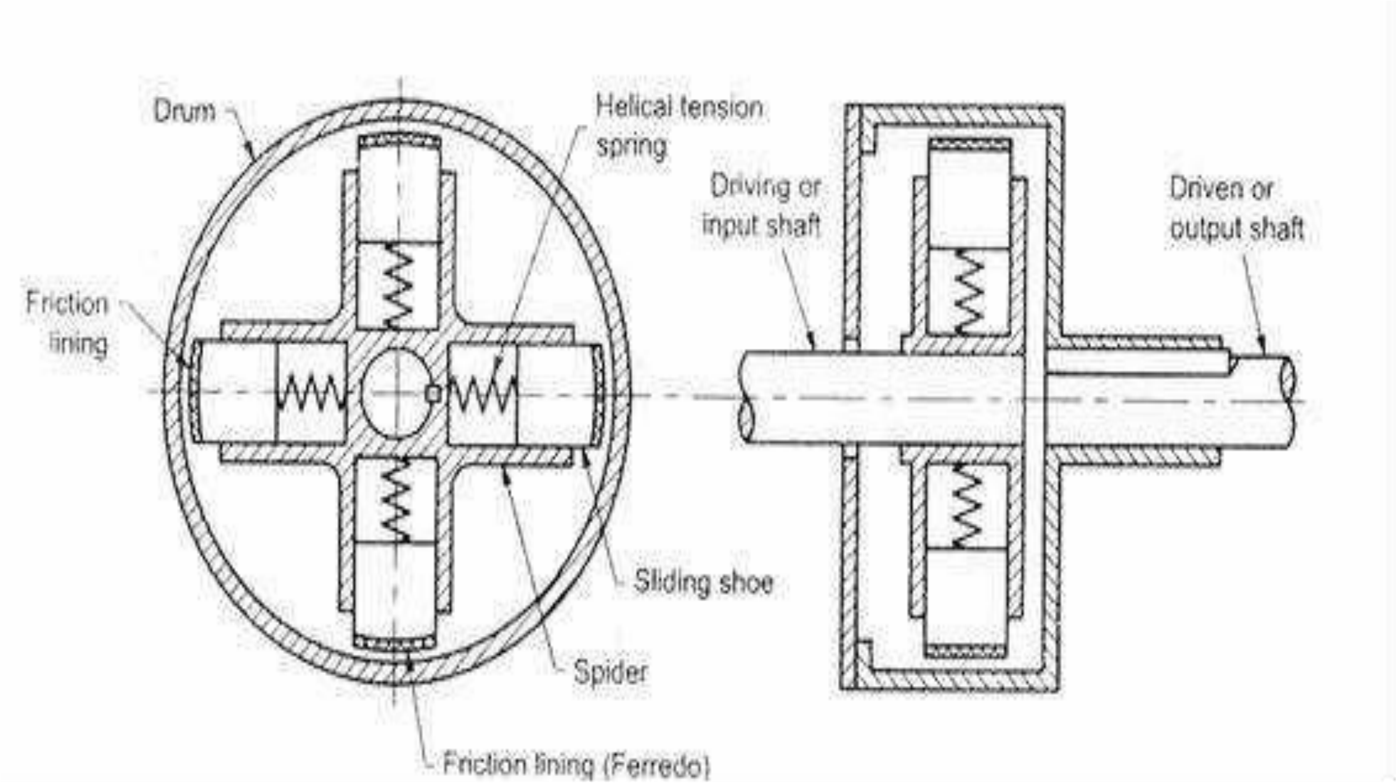
- pressure plate,
- friction plate (clutch plate),
- driving shaft,
- splined driven shaft,
- splined hub,
- brass bush etc.



Multi plate clutch



Centrifugal clutch



Brakes

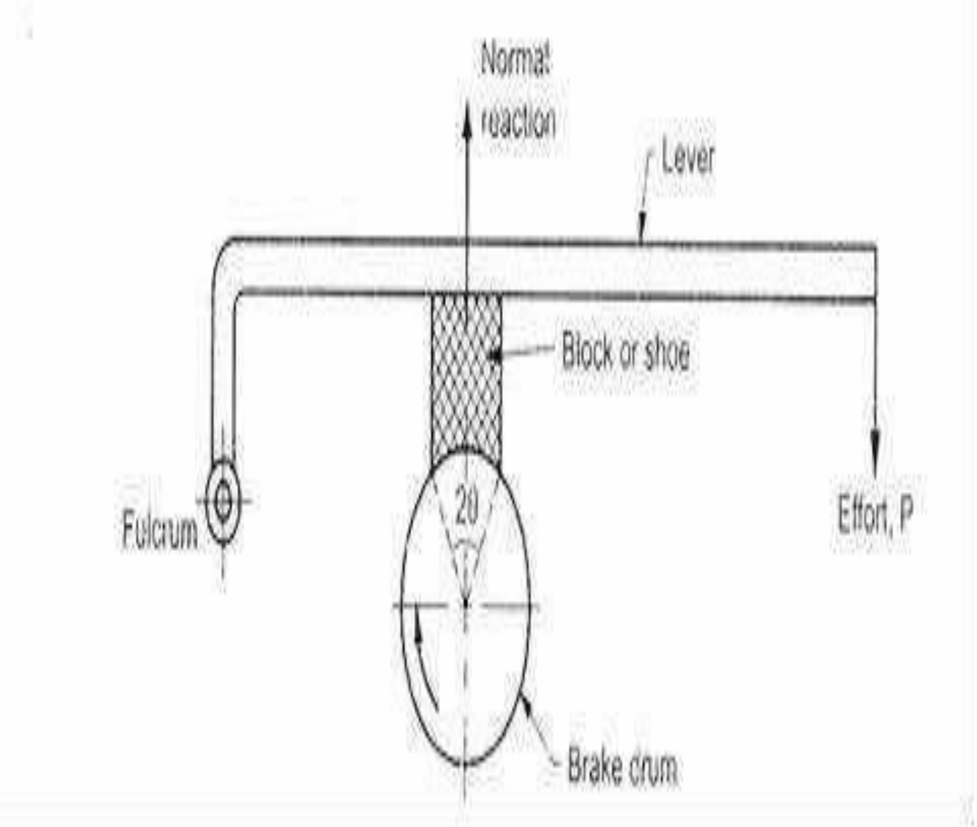
- Brake is a device by means of which an artificial frictional resistance is applied to a moving body in order to retard or stop the motion of a body. During braking process, the brake absorbs either kinetic energy or potential energy or both by an object.

(a) Single block or shoe brake Construction

- It consists of blocks which are pressed against the surface of a rotating drum by means of lever. The friction between friction lining on the block and drum retards the rotation of the drum. The block or shoe is made up of softer material than the rim of the drum.

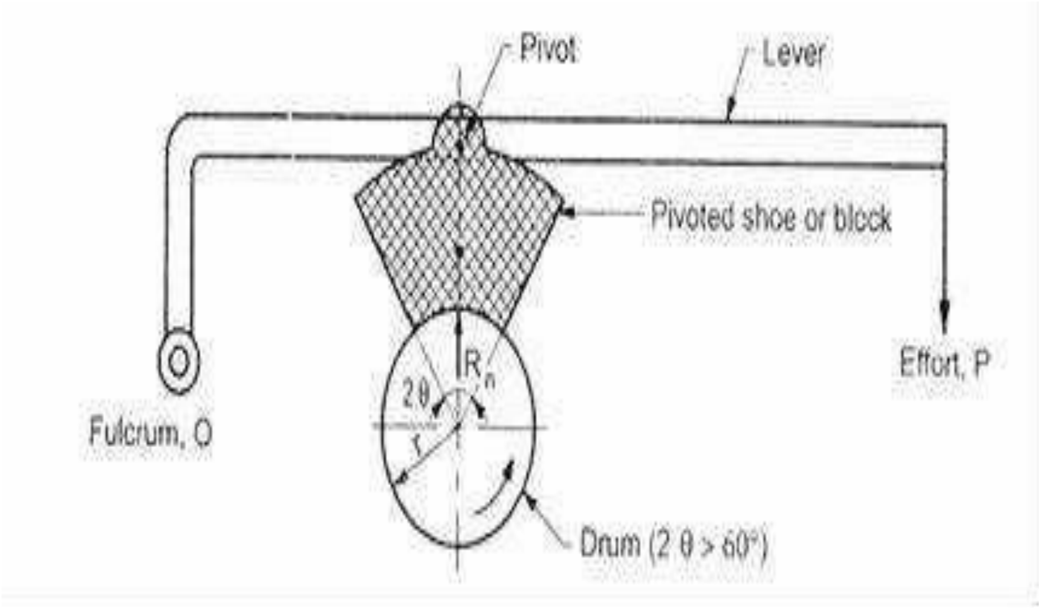
- The material of the block for light and slow vehicles is wood and rubber and for heavy and fast vehicles it is cast steel.

Shoe brake


Normal
reaction
Lever
Block or shoe
Fulcrum
2θ
Effort, P
Brake drum

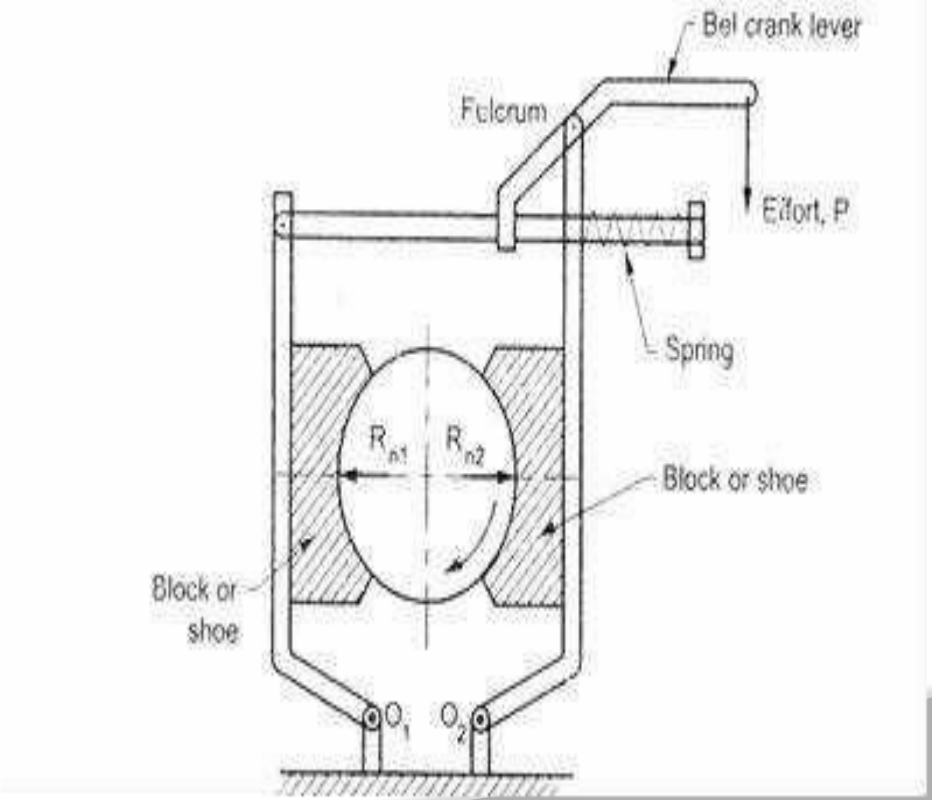
b) Pivoted block brake

- **Construction**
- A pivoted block brake is shown in figure. Unlike single block brake, in this the shoe is pivoted to lever to get uniform wear.



Double block or shoe brake

- This load produces the bending of the shaft. It can be prevented by using a double block or shoe brake having two blocks on the two sides of the drum.



Band brake Construction

- It consists of a rope, belt or flexible steel band lined with frictional material which is wrapped partly round the drum.

Working

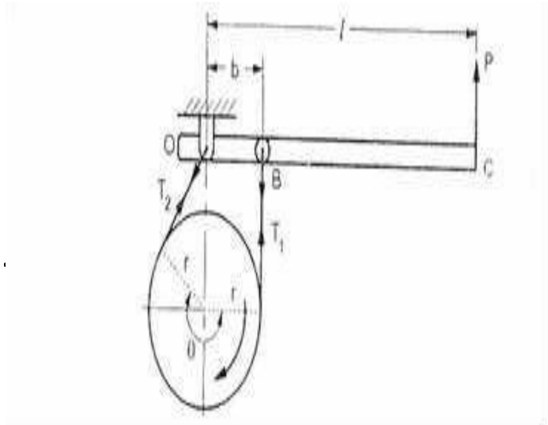
- When band is pressed against the external surface of drum, the frictional force between drum and band will induce braking torque on the drum.
- There are two types of band brake,
- (a) Simple band brake
- (b) Differential band brake

(a) Simple band brake

Construction

In this brake one end of the band is attached at the fulcrum of the lever while the other end is a

a distance 'b' from fulcrum

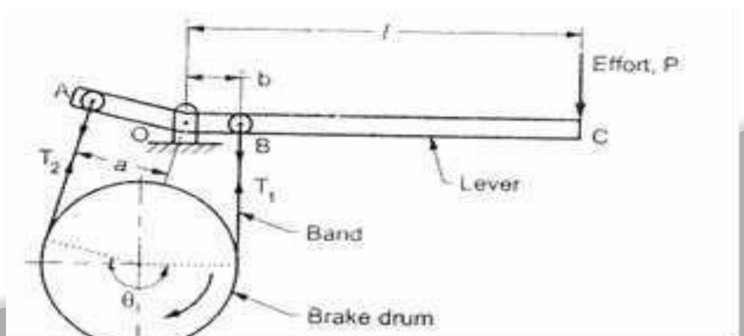


(b) Differential band brake

In a differential band brake, neither end of the band is attached to the fulcrum of the lever.

The two ends of band are attached to the two points on opposite side of the fulcrum as shown in figure

The lever AOC is pivoted at fulcrum '0' and two ends of band are attached at points A and B.



Dynamometers

Definition:

- Dynamometer is a device which is used to measure the frictional resistance. By knowing frictional resistance we can determine the torque transmitted and hence the power of the engine.

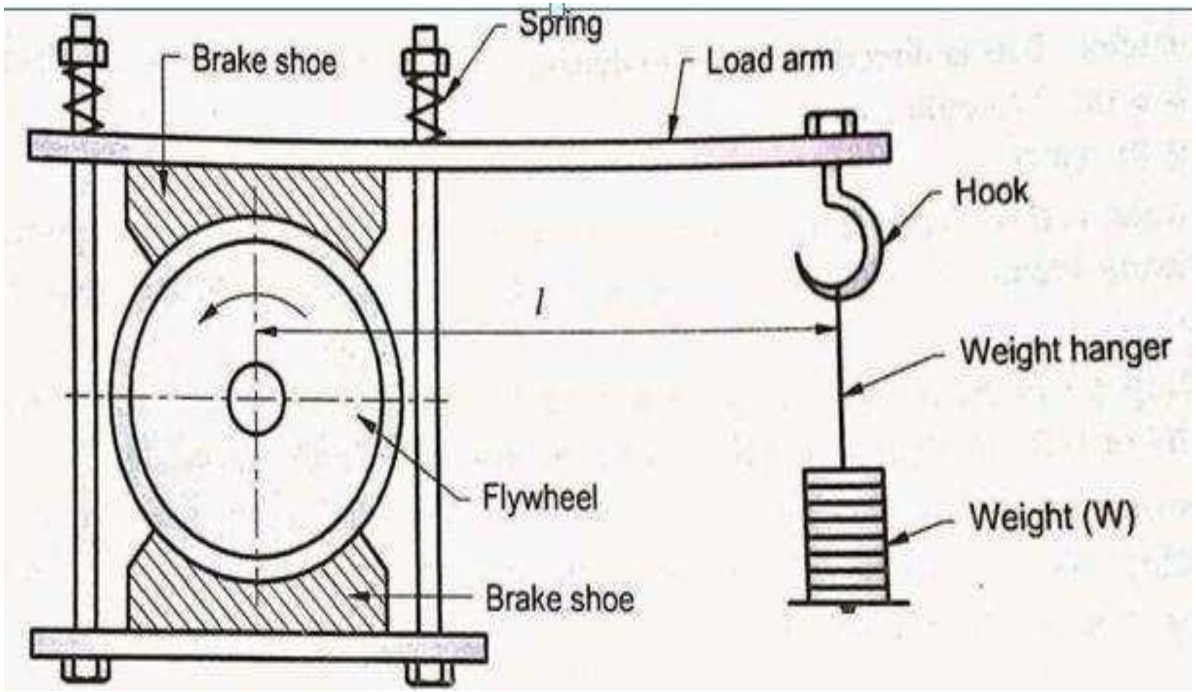
Types of dynamometers:

1)Absorption dynamometer: Prony brake dynamometer Rope brake dynamometer Hydraulic dynamometer

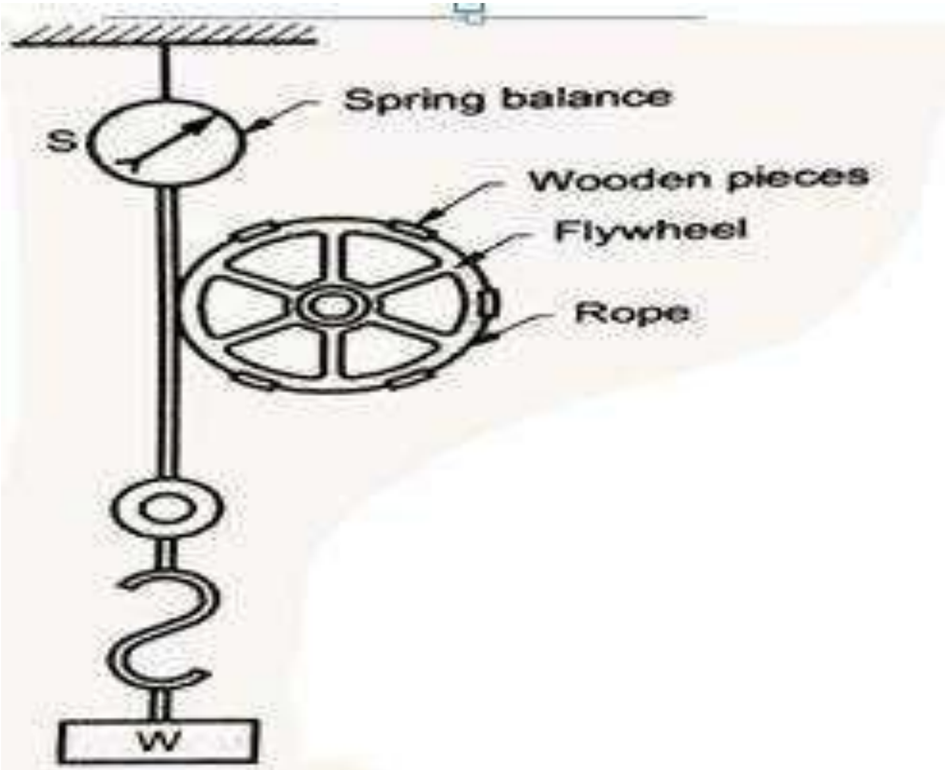
2) Transmission dynamometer: Belt transmission dynamometer Epicyclic dynamometer

Torsion dynamometer

Prony brake dynamometer

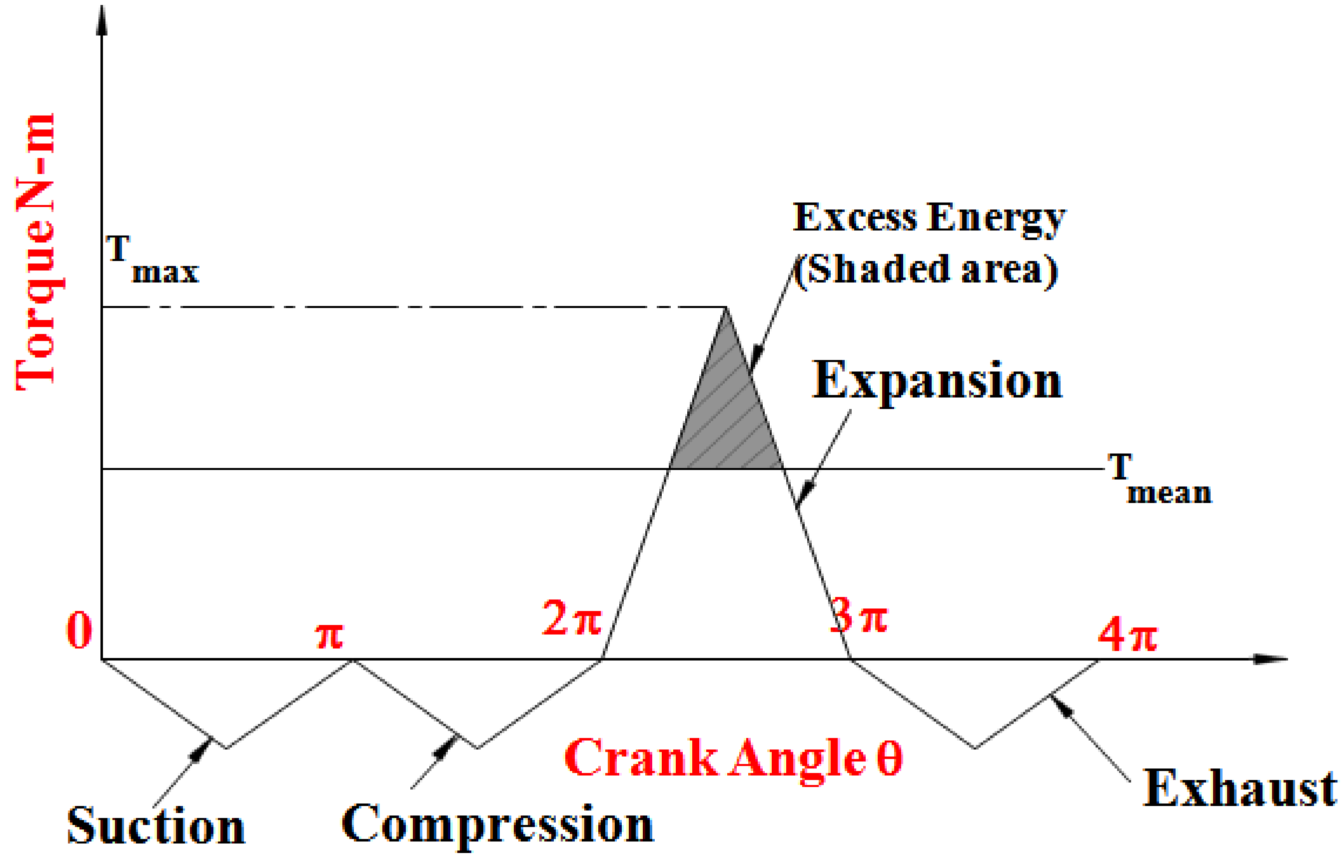


Rope brake dynamometer



Rope brake friction dynamometer

Unit-III Turning moment diagram



Turning Moment(Or Crank Effort) Diagram (TMD)

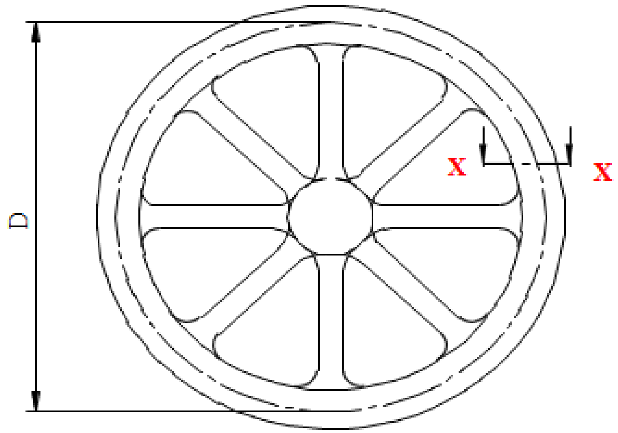
- Turning moment diagram is a graphical representation of turning moment or torque (along Y-axis)versus crank angle(X-axis)for various positions of crank.
- **Uses of TMD**

1. The area under the TMD gives the **work done per cycle.**
2. The work done per cycle when divided by the crank angle per cycle

gives the **mean torque T_m.**

FLYWHEEL

- Flywheels are used in IC engines, Pumps, Compressors & in machines performing intermittent operations such as punching, shearing, riveting, etc.
- A Flywheel may be of Disk type or Rim Type Flywheels help in smoothening out the fluctuations of the torque on the crankshaft & maintain the speed within the prescribed limits.



RIM TYPE FLYWHEEL

Governors

The function of governor is to regulate the speed of an engine when there are variation in the load

Eg. When the load on an engine increases, its speed decreases, therefore it is necessary to increase the supply of working fluid & vice-versa. Thus, Governor automatically controls the speed under varying load.

Types of Governors:

The governors may broadly be classified as

1)Centrifugal governors

2)Inertia governors

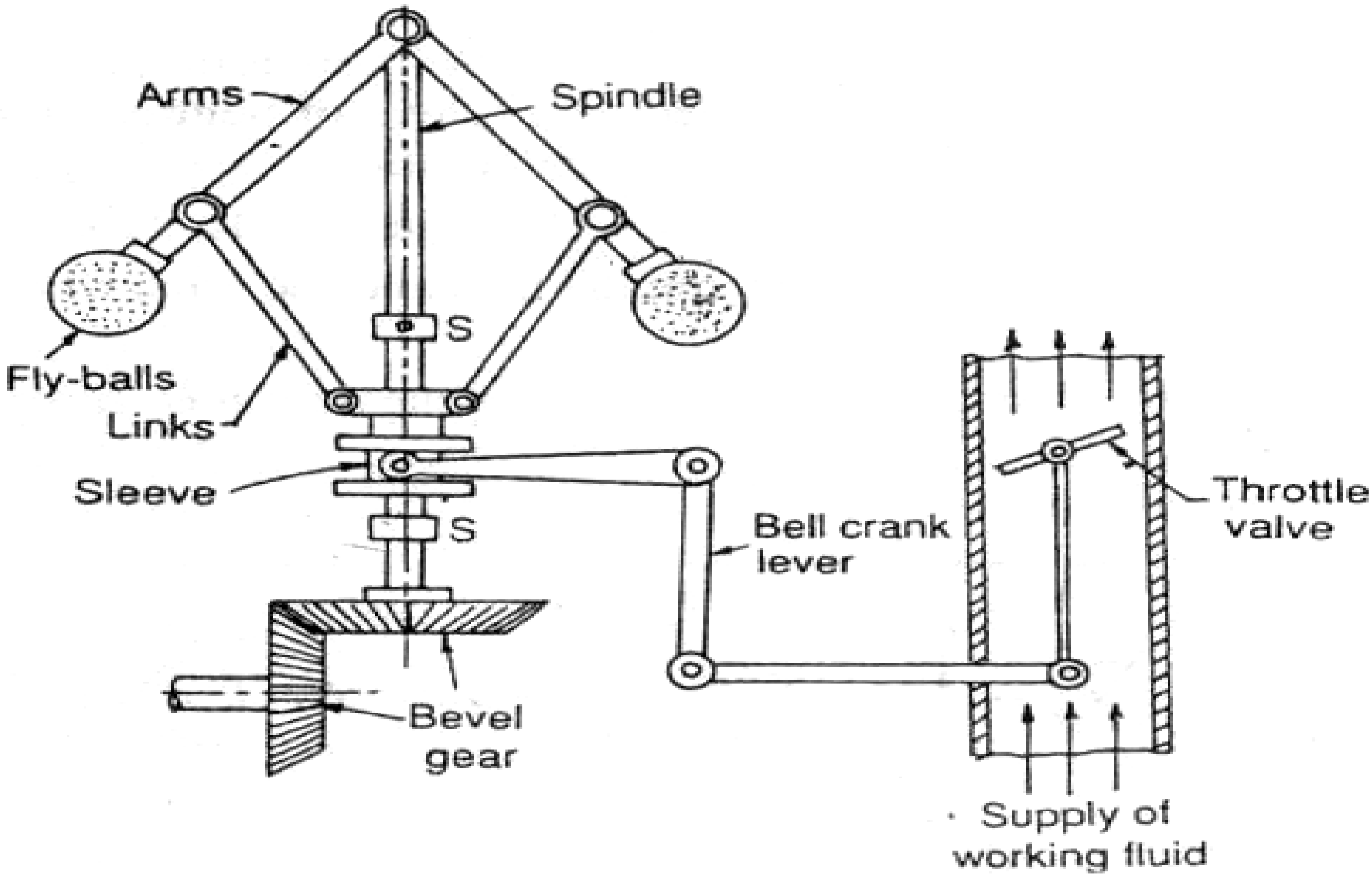


Fig. 18.1. Centrifugal governor.

When the load on the engine decreases, the engine and governor speed increased, which results in the increase of centrifugal force on the balls. Thus the ball move outwards and sleeve rises upwards. This upward movement of sleeve reduces the supply of the working fluid and hence the speed is decreased. In this case power output is reduced.

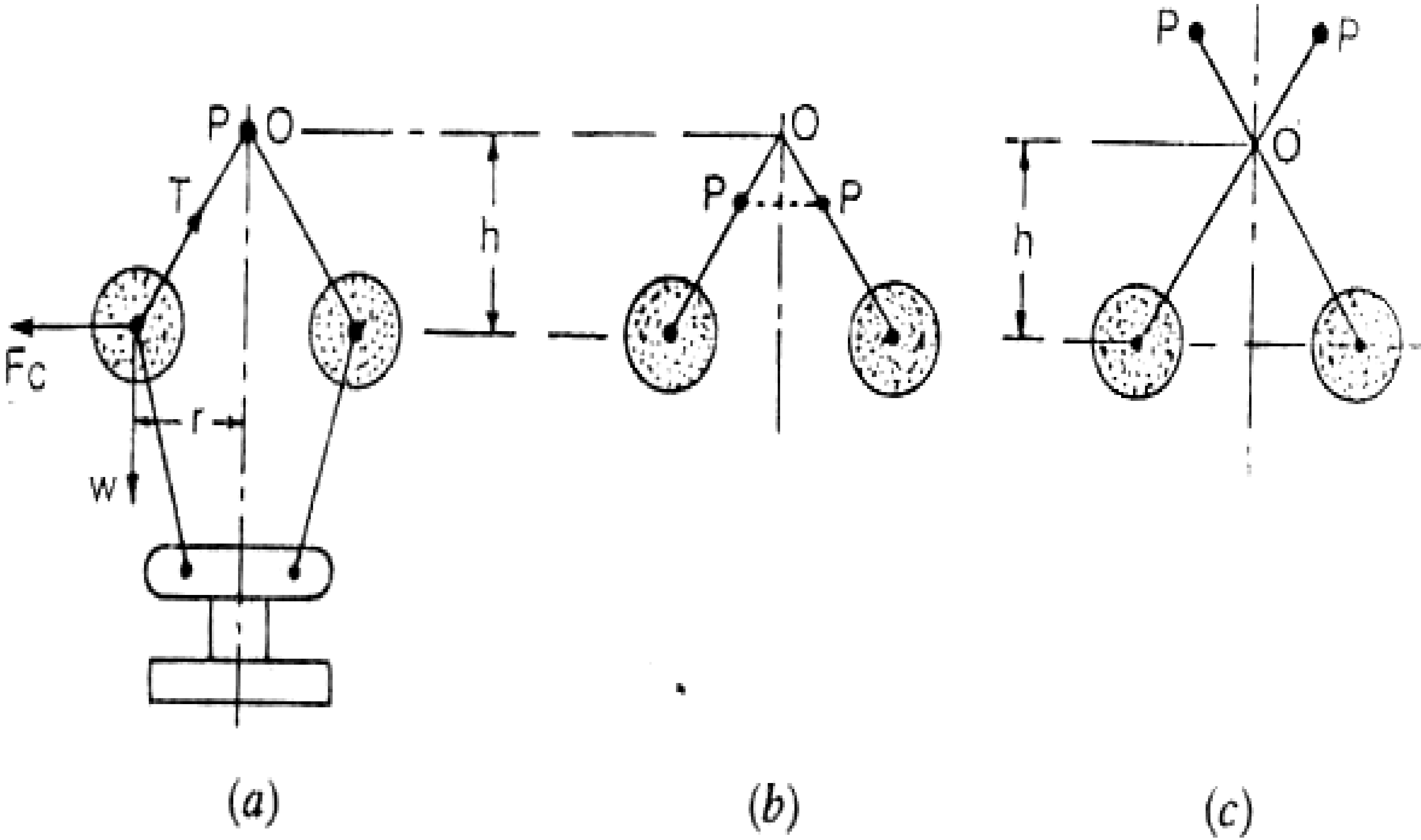


Fig. 18.2. Watt governor.

$Fc \times h = W \times r$

$mr\,\omega^2 \times h = m.g.r$

$h = g / \omega^2$

When g is in m/s2 and ω is in rad/sec, then h is in mtrs.
If N is the speed in r.p.m. then

$\omega = 2\Pi N / 60$

$H = 9.81 / (2\Pi N / 60)$

$= 895 / N^2$ mtrs.

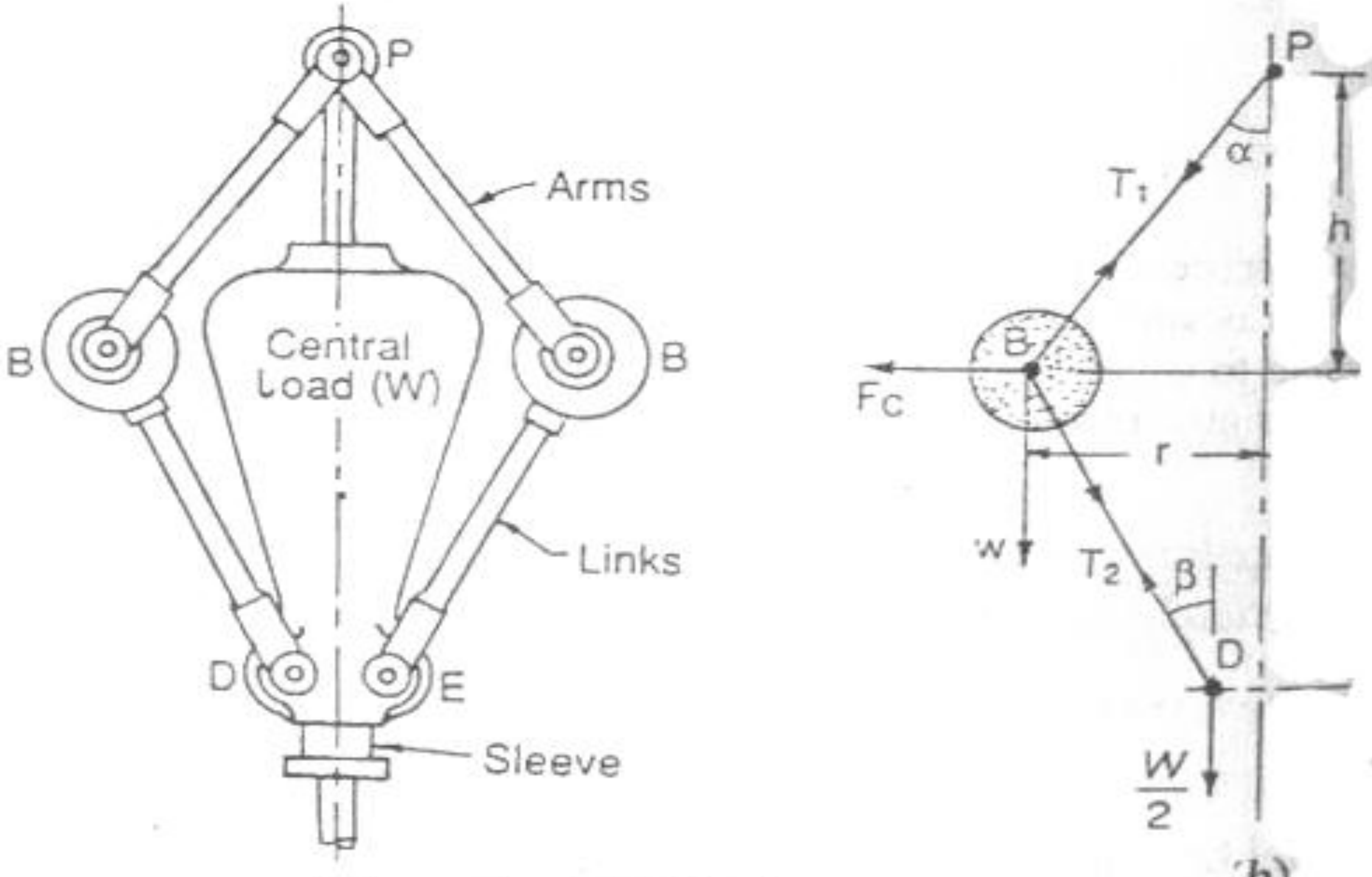


Fig. 18.3. Porter governor.

The porter governor is a modification of a Watt's governor, with central load attached to the sleeve. The load moves up and down the central spindle. This additional downward force increases the speed of revolution required to enable the balls to rise to any predetermined level.

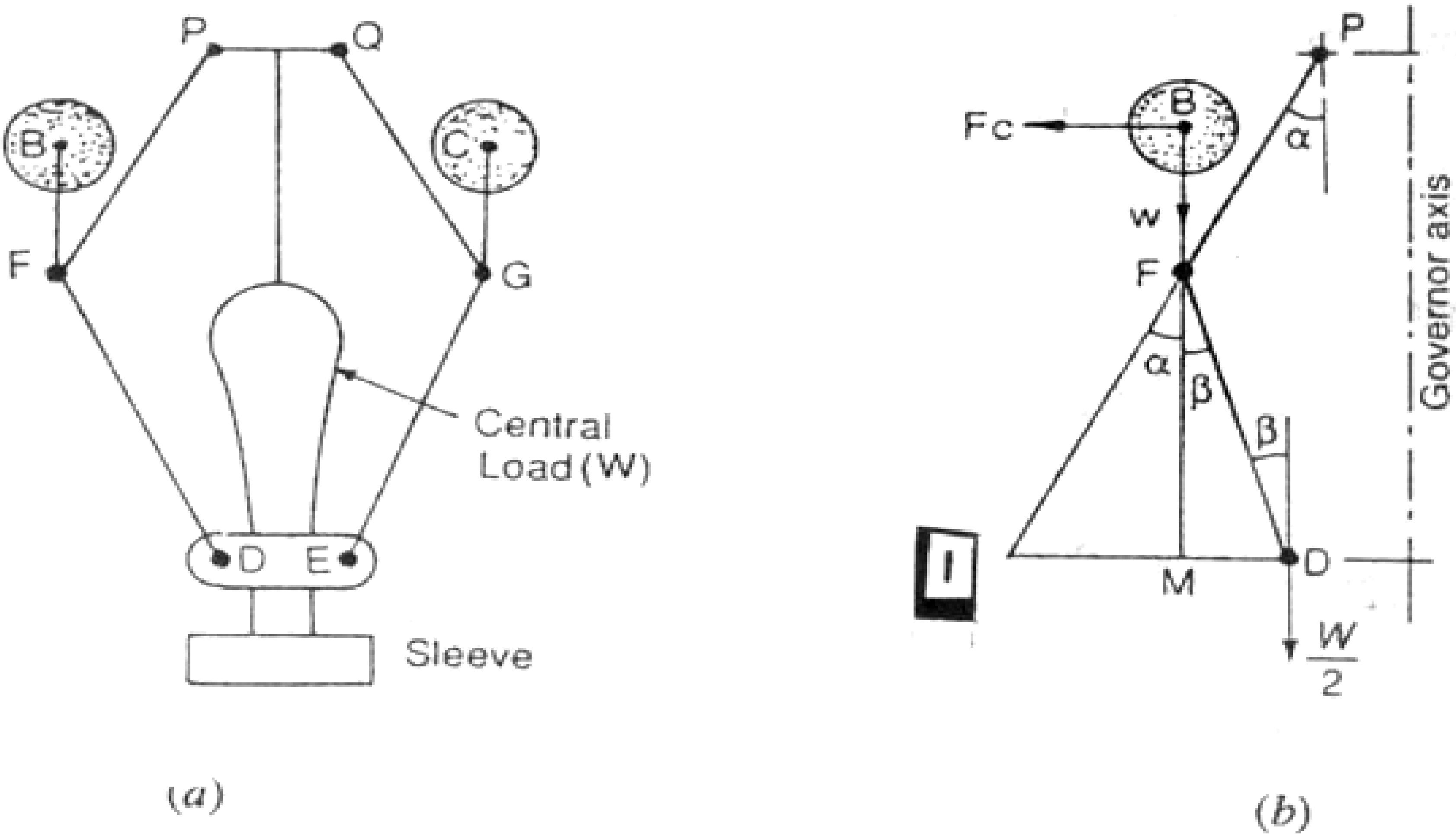


Fig. 18.12. Proell governor.

The Proell governor has the balls fixed at B & C to the extension of the links DF & EG, as shown. The arms FP & GQ are pivoted at p & Q respectively.

Consider the equilibrium of the forces on one half of the governor. The instantaneous centre (I) lies on the intersection of the line PF produced and the line from the D drawn perpendicular to the spindle axis. The perpendicular BM is drawn on ID

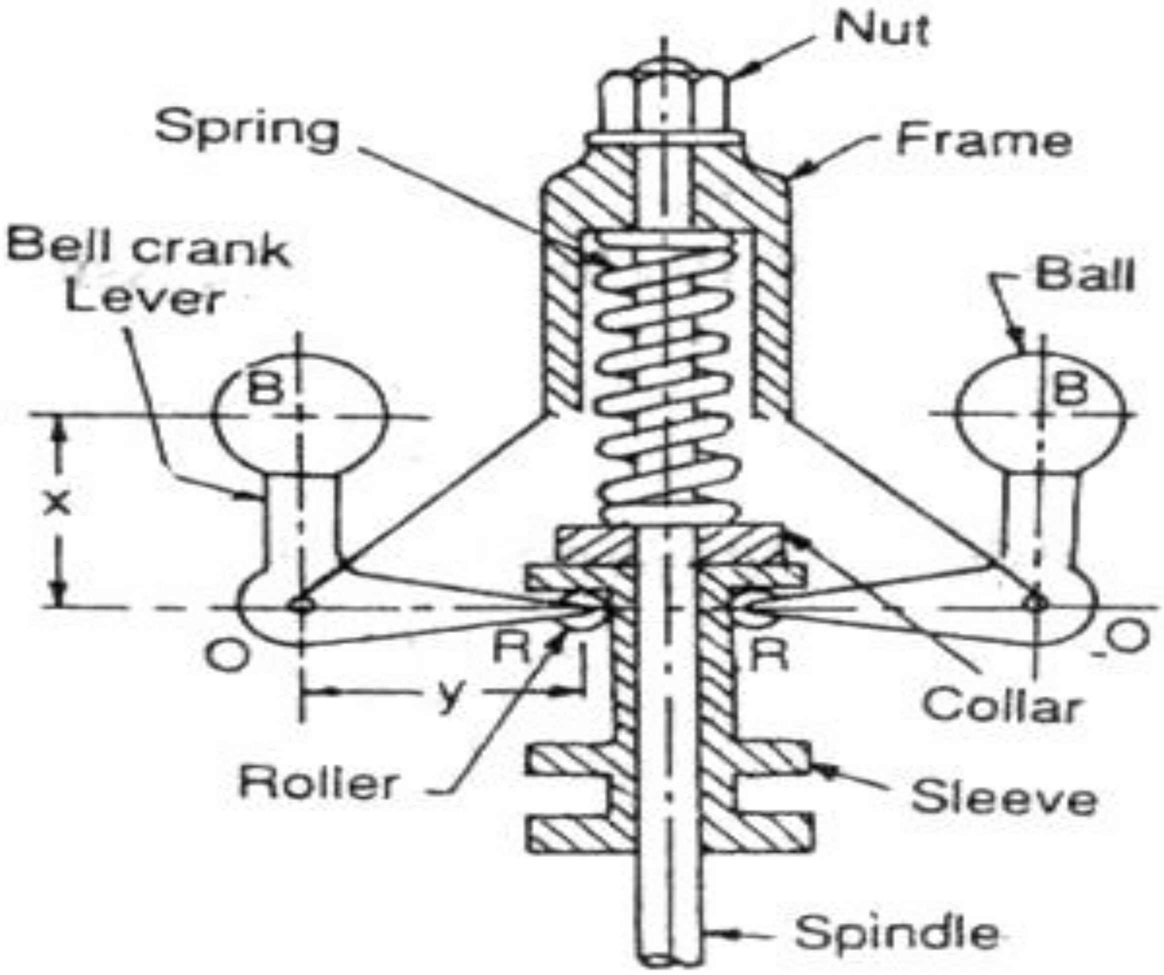


Fig. 18.18. Hartnell governor.

It is a spring loaded governor, consists of two bell crank levers pivoted at the pts. O, O to the frame. Frame is attached to the governor spindle and therefore rotates with it. Each lever carries a ball at the end of the vertical arm OB & a roller at the end of horizontal arm OR. A helical spring in compression provides equal downward forces on two rollers through collar on the sleeve. The spring force may be adjusted by screwing a nut up or down on the sleeve.

Unit-IV Balancing

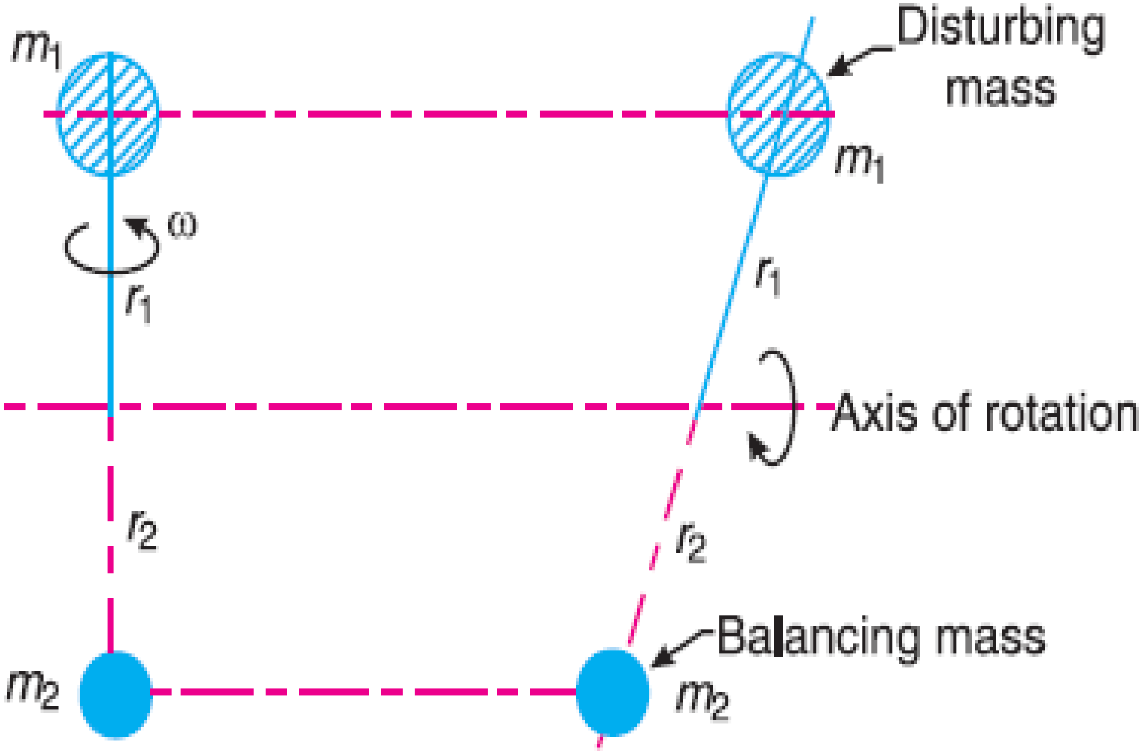
Why Balancing is necessary?

- The high speed of engines and other machines is a common phenomenon now-a-days.
- It is, therefore, very essential that all the rotating and reciprocating parts should be completely balanced as far as possible.
- If these parts are not properly balanced, the dynamic forces are set up.
- These forces not only increase the loads on bearings and stresses in the various members, but also produce unpleasant and even dangerous vibrations.

Balancing of Rotating Masses

- Whenever a certain mass is attached to a rotating shaft, it exerts some centrifugal force, whose effect is to bend the shaft and to produce vibrations in it.
- In order to prevent the effect of centrifugal force, another mass is attached to the opposite side of the shaft, at such a position so as to balance the effect of the centrifugal force of the first mass.
- This is done in such a way that the centrifugal force of both the masses are made to be equal and opposite.
- The process of providing the second mass in order to counteract the effect of the centrifugal force of the first mass, is called ***balancing of rotating masses.***

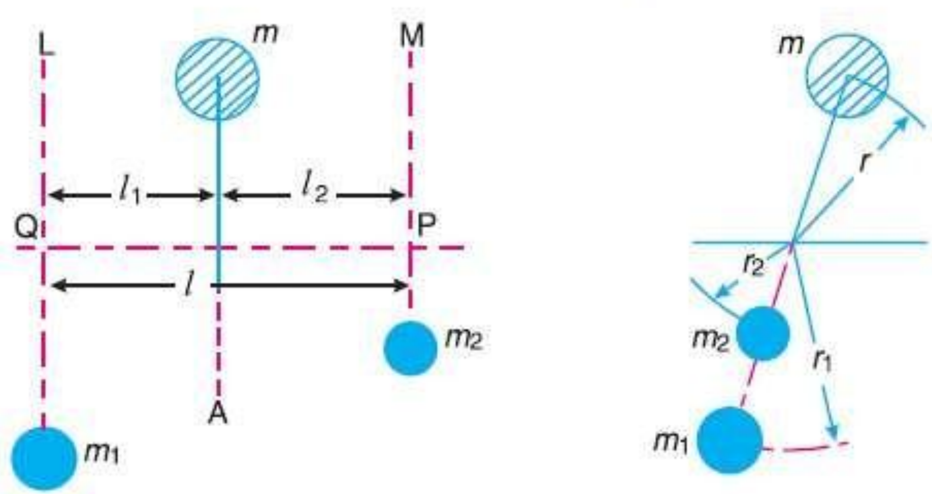
- Consider a disturbing mass m_1 attached to a shaft rotating at ω rad/s as shown.
- Let r_1 be the radius of rotation of the mass m_1 (*i.e. distance between the axis of rotation of the shaft* and the centre of gravity of the mass m_1).
- We know that the centrifugal force exerted by the mass m_1 on the shaft,This centrifugal force acts radially outwards and thus produces bending moment on the shaft.
- In order to counteract the effect of this force, a balancing mass (m_2) may be attached in the same plane of rotation as that of disturbing mass (m_1) such that the centrifugal forces due to the two masses are equal and opposite.


m1
Disturbing mass
m1
ω
r1
r1
Axis of rotation
r2
r2
Balancing mass
m2
m2

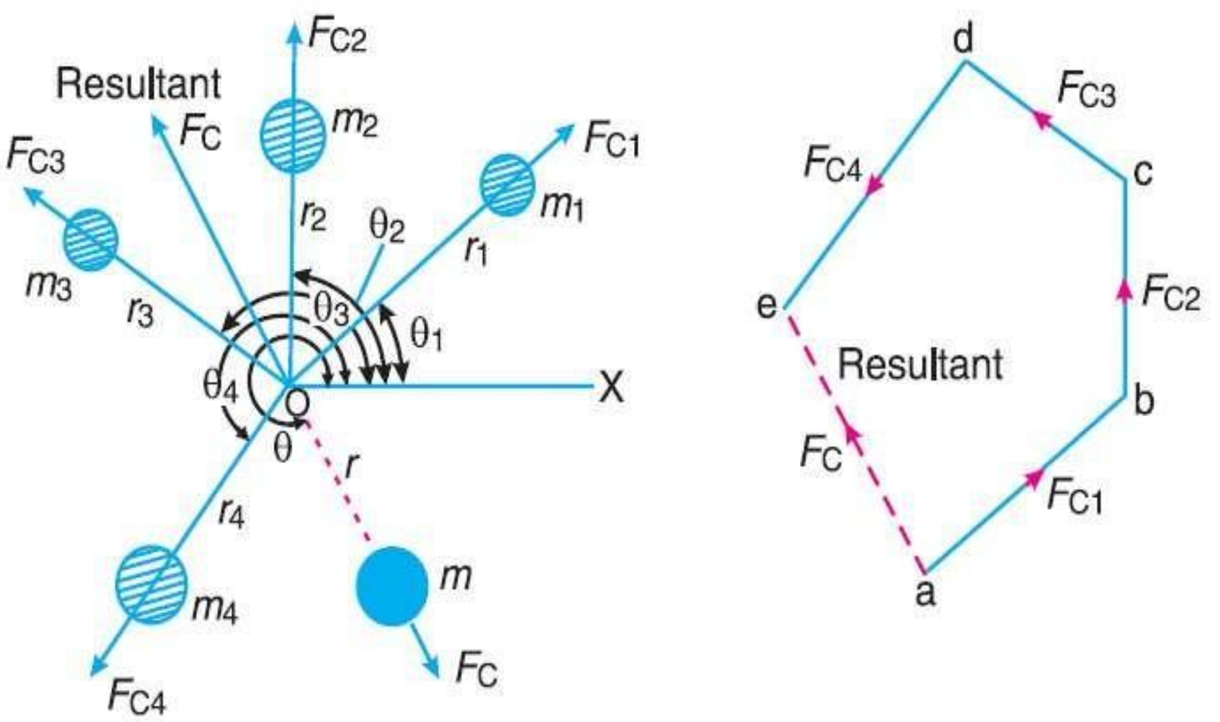
1. When the plane of the disturbing mass lies in between the planes *of the two balancing masses*

- Consider a disturbing mass *m lying in a plane A to be balanced by two rotating masses m*1 and *m*2 lying in two different planes *L and M as shown in Fig.*
- *Let r, r*1 and *r*2 be the radii of rotation of the masses in planes *A, L and M*

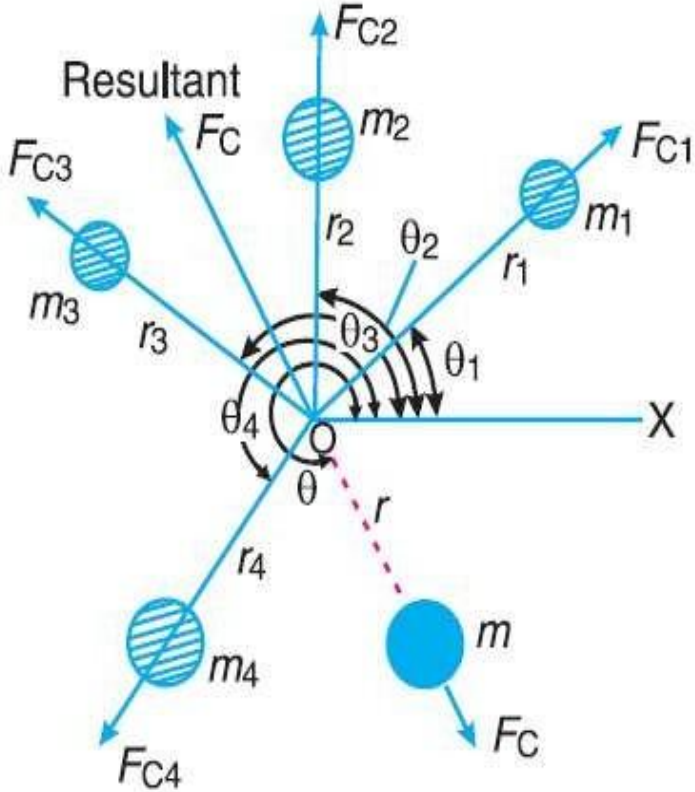
Let l_1 = Distance between the planes *A* and *L*,
l_2 = Distance between the planes *A* and *M*, and
l = Distance between the planes *L* and *M*.



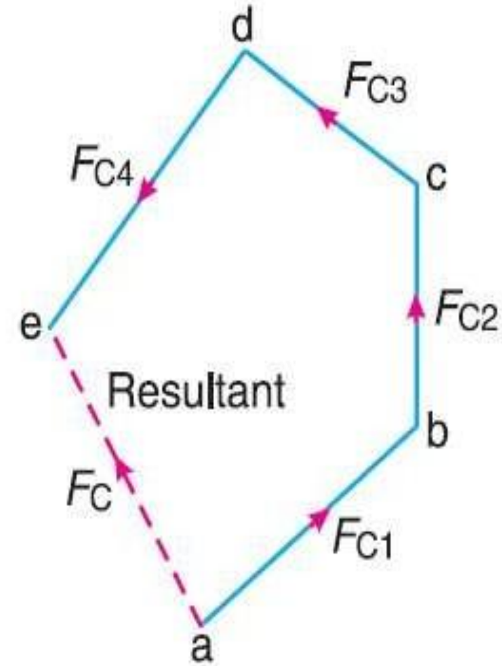
2. Graphical method



(a) Space diagram. (b) Vector diagram.



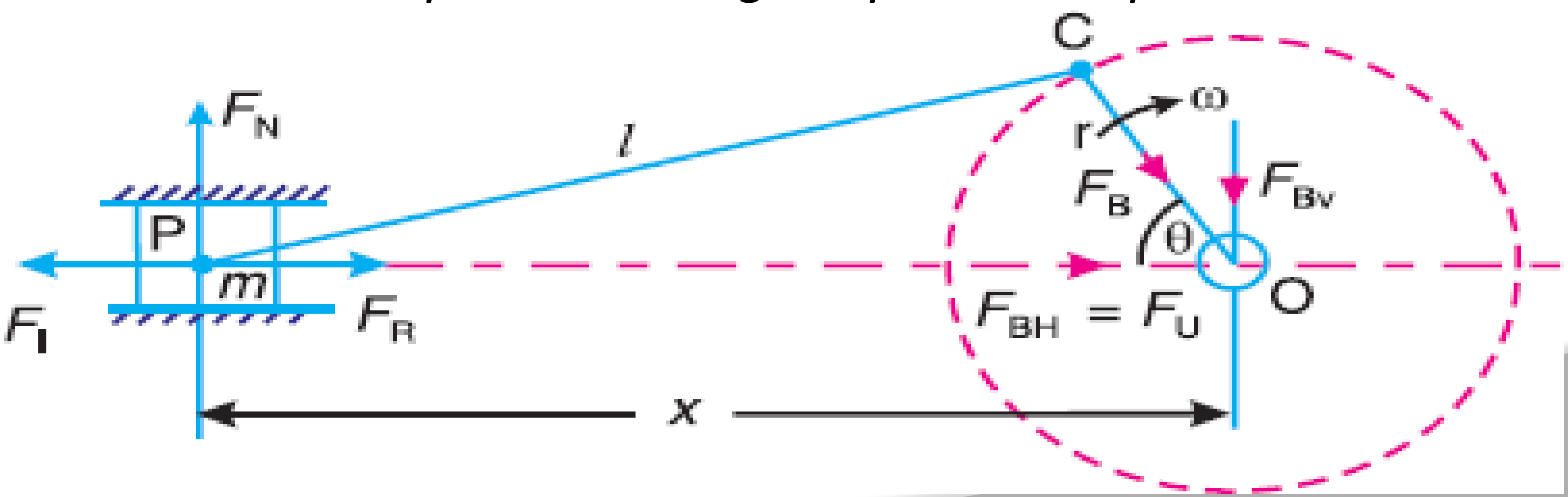
(a) Space diagram.



(b) Vector diagram.

Balancing of Reciprocating masses

- Various forces acting on the reciprocating parts of an engine.
- The resultant of all the forces acting on the body of the engine due to inertia forces only is known as *unbalanced force or shaking force.*
- *Thus if the resultant of all the forces due to inertia effects is zero, then there will be no unbalanced force, but even then an unbalanced couple or shaking couple will be present.*



F_R = Force required to accelerate the reciprocating parts.

F_I = Inertia force due to reciprocating parts,

F_N = Force on the sides of the cylinder walls or normal force acting on the cross-head guides, and F_B = Force acting on the crankshaft bearing or main bearing.

- Since F_R and F_I, are equal in magnitude but opposite in direction, therefore they balance each other.
- The horizontal component of F_B *(i.e. F_{BH}) acting along the line of reciprocation is also equal and opposite to F_I .*
- *This force F_{BH} = F_U is an unbalanced force or shaking force and required to be properly balanced.*

Velocity of the piston

From the geometry of Fig. 15.7,

$$x = P'P = OP' - OP = (P'C' + C'O) - (PQ + QO)$$

$$= (l + r) - (l \cos \phi + r \cos \theta) \qquad \cdots \begin{pmatrix} \because \quad PQ = l \cos \phi, \\ \text{and } QO = r \cos \theta \end{pmatrix}$$

$$= r\,(1-\cos\theta) + l\,(1-\cos\phi) = r\left[(1-\cos\theta) + \frac{l}{r}(1-\cos\phi)\right]$$

$$= r\,[(1-\cos\theta) + n\,(1-\cos\phi)] \qquad \text{...}(i)$$

From triangles *CPQ* and *CQO*,

$$CQ = l\sin\phi = r\sin\theta \quad \text{or} \quad l/r = \sin\theta/\sin\phi$$

$$\therefore \quad n = \sin\theta/\sin\phi \quad \text{or} \quad \sin\phi = \sin\theta/n \qquad \text{...}(ii)$$

We know that, $\cos\phi = \left(1-\sin^2\phi\right)^{\frac{1}{2}} = \left(1-\frac{\sin^2\theta}{n^2}\right)^{\frac{1}{2}}$

Expanding the above expression by binomial theorem, we get

$$\cos\phi = 1 - \frac{1}{2}\times\frac{\sin^2\theta}{n^2} + \ldots\ldots \qquad \text{...(Neglecting higher terms)}$$

$$1-\cos\phi = \frac{\sin^2\theta}{2n^2} \qquad \text{...}(iii)$$

Substituting the value of $(1 - \cos\phi)$ in equation (*i*), we have

$$x = r\left[(1 - \cos\theta) + n \times \frac{\sin^2\theta}{2n^2}\right] = r\left[(1 - \cos\theta) + \frac{\sin^2\theta}{2n}\right] \quad ..(iv)$$

Differentiating equation (*iv*) with respect to θ,

$$\frac{dx}{d\theta} = r\left[\sin\theta + \frac{1}{2n} \times 2\sin\theta.\cos\theta\right] = r\left(\sin\theta + \frac{\sin 2\theta}{2n}\right) \quad ...(v)$$

$(\because 2\sin\theta.\cos\theta = \sin 2\theta)$

∴ Velocity of *P* with respect to *O* or velocity of the piston *P*,

$$v_{PO} = v_P = \frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt} = \frac{dx}{d\theta} \times \omega$$

...($\because$ Ratio of change of angular velocity $= d\theta / dt = \omega$)

Substituting the value of $dx/d\theta$ from equation (*v*), we have

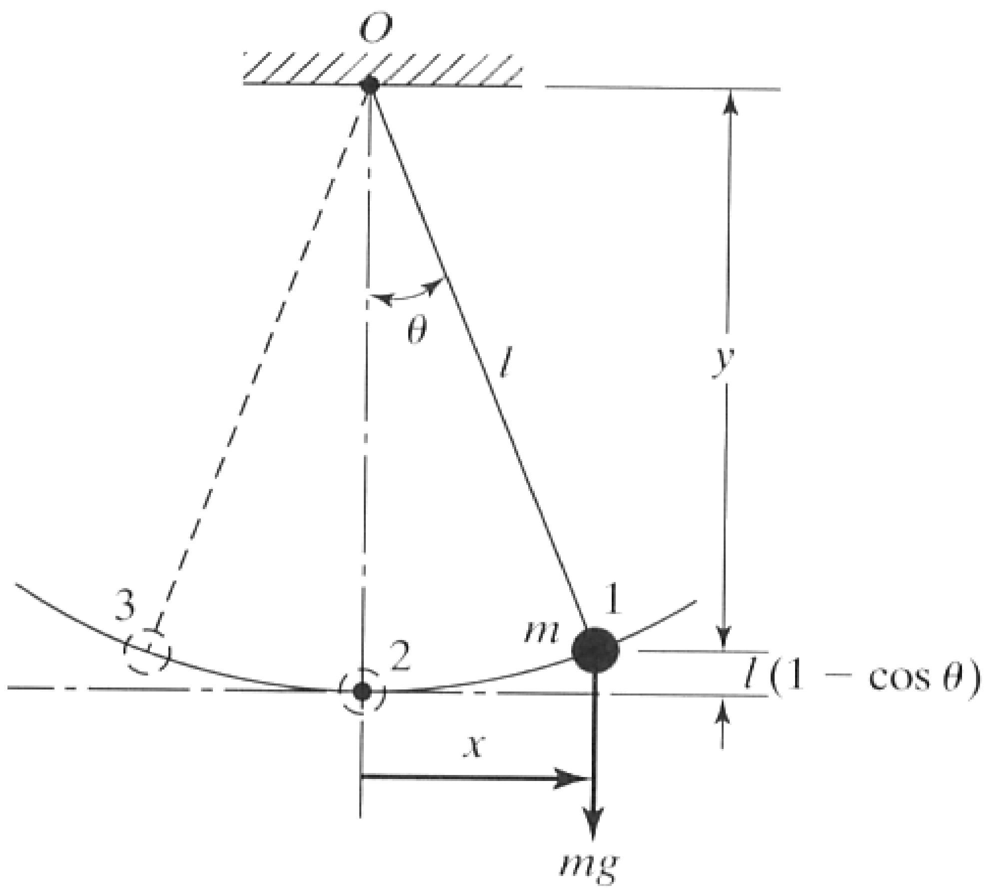
$$v_{PO} = v_P = \omega r\left(\sin\theta + \frac{\sin 2\theta}{2n}\right) \quad ...(vi)$$

- Defined as oscillatory motion of bodies in response to disturbance. Oscillations occur due to the presence of a restoring force
- Vibrations are everywhere:
    - Human body: eardrums, vocal cords, walking and running
    - Vehicles: residual imbalance of engines, locomotive wheels
    - Rotating machinery: Turbines, pumps, fans, reciprocating machines, Musical instruments
- Excessive vibrations can have detrimental effects:
    - Noise,Loosening of fasteners,Tool chatter,Fatigue failure
    - Discomfort

Fundamentals

- In simple terms, a vibratory system involves the transfer of potential energy to kinetic energy and vice-versa in alternating fashion.
- When there is a mechanism for dissipating energy (damping) the oscillation gradually diminishes.
- In general, a vibratory system consists of three basic components:
  - A means of storing potential energy (spring, gravity)
  - A means of storing kinetic energy (mass, inertial component)
  - A means to dissipate vibrational energy (damper)

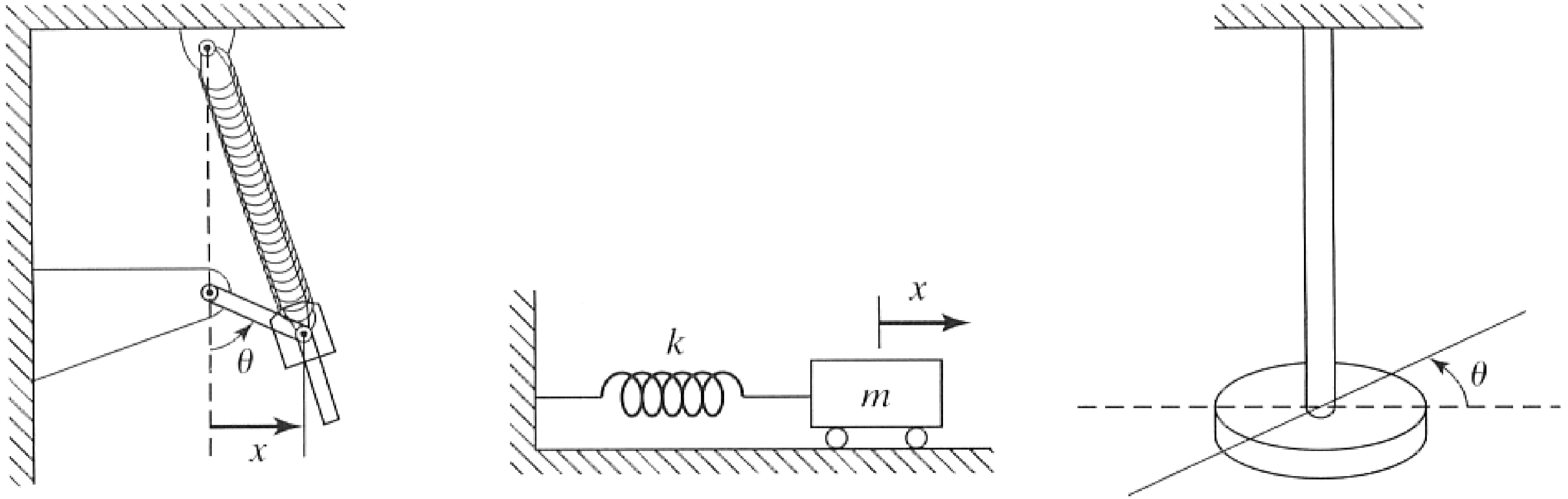
- This can be observed with a pendulum:
- At position 1: the kinetic energy is zero and the potential energy is $mgl(1 - \cos\theta)$
- At position 2: the kinetic energy is at its maximum
- At position 3: the kinetic energy is again zero and the potential energy at its maximum.
- In this case the oscillation will eventually stop due to aerodynamic drag and pivot friction $\rightarrow$ HEAT


O
θ
l
y
1
m
3
2
l (1 − cos θ)
x
mg

Degrees of Freedom

The number of degrees of freedom : number of independent coordinates required to completely determine the motion of all parts of the system at any time.

Examples of single degree of freedom systems:



(a) Slider-crank-spring mechanism (b) Spring-mass system (c) Torsional system

Classification of Vibration

- **Free and Forced vibrations**
  - ***Free vibration***: Initial disturbance, system left to vibrate without influence of external forces.

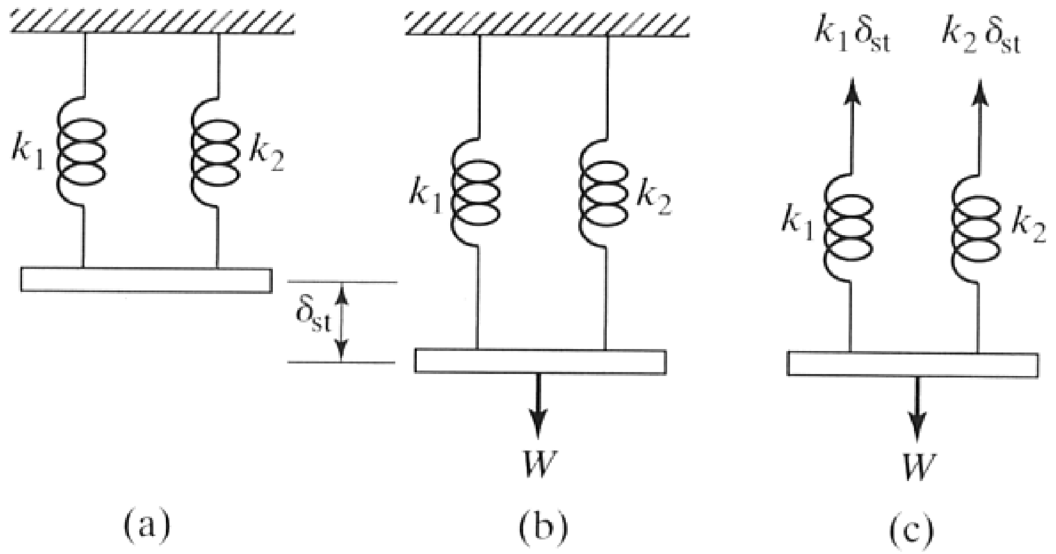
    Forced vibration: Vibrating system is stimulated by external forces. If *excitation* frequency coincides with *natural* frequency, resonance occurs.
- **Undamped and damped vibration:*Undamped vibration***: No dissipation of energy. In many cases, damping is (negligibly) small (steel 1 – 1.5%). However small, damping has critical importance when analysing systems at or near resonance. ***Damped vibration***: Dissipation of energy occurs - vibration amplitude decays.

Classification of Vibration

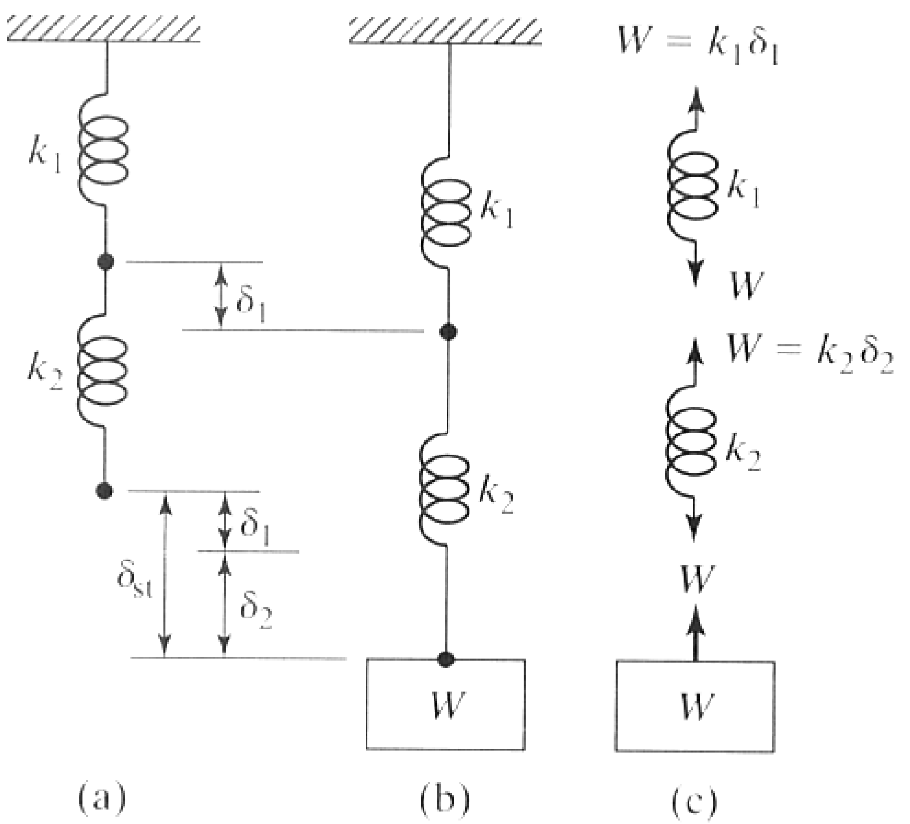
- **Linear and nonlinear vibration**
  - ***Linear vibration***: Elements (mass, spring, damper) behave linearly. Superposition holds - double excitation level = double response level, mathematical solutions well defined.
  - ***Nonlinear vibration***: One or more element behave in nonlinear fashion (examples). Superposition does not hold, and analysis technique not clearly defined.

Spring Elements

- Pure spring element considered to have negligible mass and damping
- Force proportional to spring deflection (relative motion between ends): $F = k\Delta x$
- For linear springs, the potential energy stored is: $U = \frac{1}{2}k(\Delta x)^2$
- Actual springs sometimes behave in nonlinear fashion
- Important to recognize the presence and significance (magnitude) of nonlinearity
- Desirable to generate linear estimate



$$k_{eq} = k_1 + k_2$$

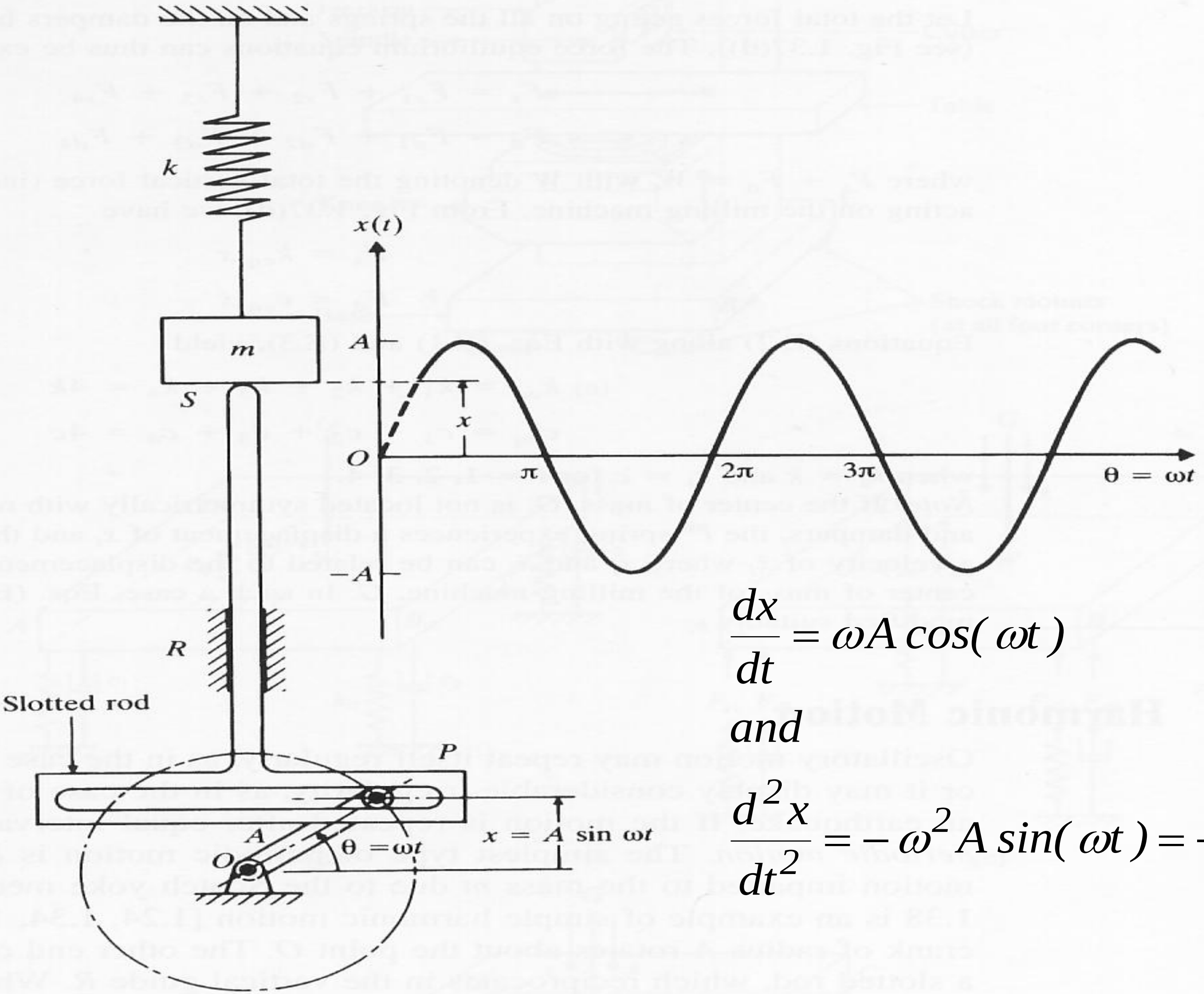


$$k_1\delta_1 = k_2\delta_2 = k_{eq}\delta_t$$

Damping Elements

- Absorbs energy from vibratory system $\rightarrow$ vibration amplitude decays.Damping element considered to have no mass or elasticity. Real damping systems very complex, damping modelled as:
  - **Viscous damping:**Based on viscous fluid flowing through gap or orifice.Eg: film between sliding surfaces, flow b/w piston & cylinder, flow thru orifice, film around journal bearing. Damping force $\propto$ relative velocity between ends
  - **Coulomb (dry Friction) damping:**Based on friction between unlubricated surfaces.Damping force is constant and opposite the direction of motion

Harmonic Motion



$$\frac{dx}{dt} = \omega A \cos(\omega t)$$

and

$$\frac{d^2x}{dt^2} = -\omega^2 A \sin(\omega t) = -\omega^2 x$$

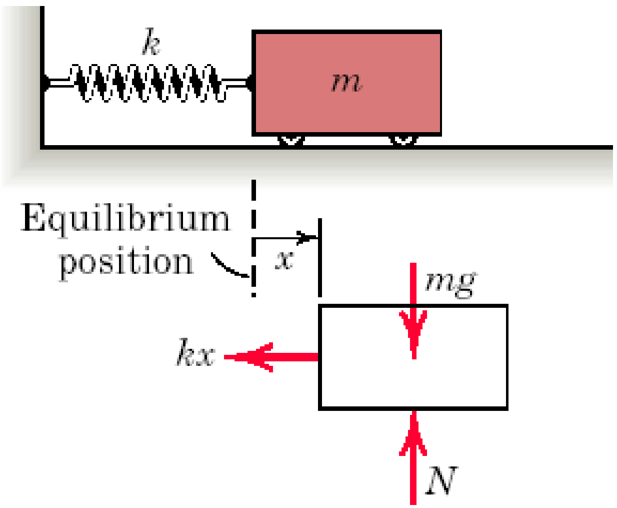
- **Cycle**: motion of body from equilibrium position $\rightarrow$ extreme position $\rightarrow$ equilibrium position $\rightarrow$ extreme position in other direction $\rightarrow$ equilibrium position .
- **Amplitude**: Maximum value of motion from equilibrium. (Peak – Peak = 2 x amplitude)
- **Period**: Time taken to complete one cycle

$$f = \frac{1}{\tau} = \frac{\omega}{2\pi}$$

$$\tau = \frac{2\pi}{\omega}$$

Free undamped vibration single DoF

- Single DoF:
  - mass treated as rigid, limped (particle)
  - Elasticity idealised by single sprin
  - only one natural frequency.
- The equation of motion can be derived using Newton's second law of motion
  - D'Alembert's Principle,
  - The principle of virtual displacements and,The principle of conservation



$$F(t) = -kx = m\ddot{x}$$

or

$$m\ddot{x} + kx = 0$$

Free undamped vibration single DoF

$$x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$$

or

$$x(t) = Ae^{i\omega_n t} + Be^{-i\omega_n t}$$

$alternatively, if\ we\ let\ s = \pm i\omega_n$

$$x(t) = C\,e^{\pm st}$$

$x_{(t=0)} = A = x_0$ *initial displacement*

$\dot{x}_{(t=0)} = B\omega_n = \dot{x}_0$ *initial velocity*

Free undamped vibration single DoF

$$x(t) = x_0 \cos(\omega_n t) + \frac{\dot{x}_o}{\omega_n} \sin(\omega_n t)$$

if we let $A_0 = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right]^{1/2}$ *and* $\phi = a\tan\left(\frac{x_0 \omega_n}{\dot{x}_o} \right)$ *then*

$$x(t) = A_0 \sin(\omega_n t + \phi)$$

Free undamped vibration single DoF

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\text{since} \quad k = \frac{mg}{\delta_{st}}$$

$$\omega_n = \sqrt{\frac{g}{\delta_{st}}} \quad \text{or} \quad f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}}$$

Natural frequency :

$$\omega_n = \sqrt{\frac{mgd}{J_o}}$$

since for a simple pendulum

$$\omega_n = \sqrt{\frac{g}{l}}$$

Then, $l = \frac{J_o}{md}$ *and since* $J_o = mk_o^2$ *then*

$$\omega_n = \sqrt{\frac{gd}{k_o^2}} \quad \text{and} \quad l = \frac{k_o^2}{d}$$

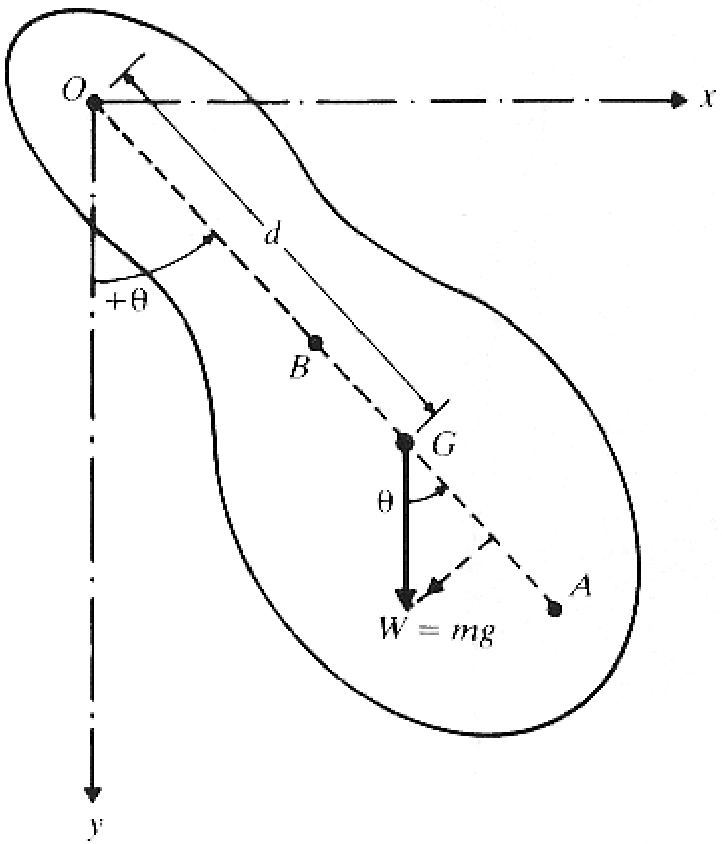
Applying the parallel axis theorem $k_o^2 = k_G^2 + d^2$

$$l = \frac{k_G^2}{d} + d$$

Let $l = GA + d = OA$

$$\omega_n = \sqrt{\frac{g}{k_o^2/d}} = \sqrt{\frac{g}{l}} = \sqrt{\frac{g}{OA}}$$

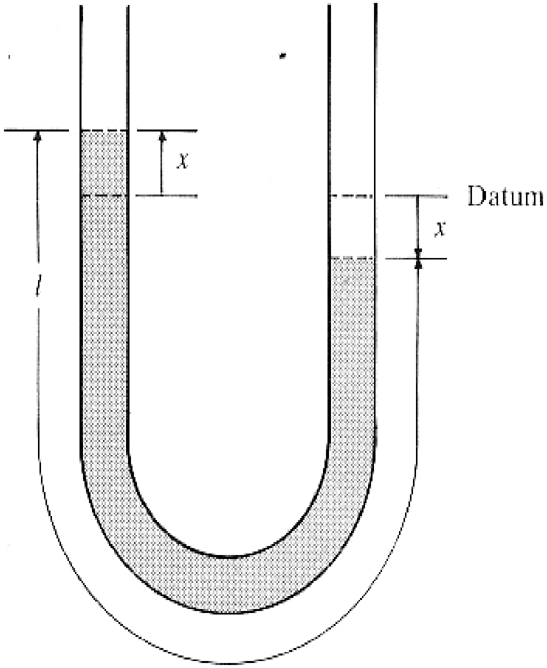
The location A $\left(GA = \frac{k_G^2}{d} \right)$ *is the "centre of percussion"*



$$U_{max} = A\gamma X^2 \quad and \quad T_{max} = \frac{1}{2}\left(\frac{Al\gamma}{g}\right)(2\pi f_n)^2 X^2$$

$$U_{max} = T_{max} \quad \therefore \quad A\gamma X^2 = \frac{1}{2}\left(\frac{Al\gamma}{g}\right)(2\pi f_n)^2 X^2$$

$$f_n = \frac{1}{2\pi}\sqrt{\left(\frac{2g}{l}\right)}$$



Free single DoF vibration + viscous damping

$F = -c\dot{x}$ $\quad$ $c = \text{damping constant or coefficient}\ [Ns/m]$

Applying Newton's second law of motion to obtain the eqn. of motion :

$$m\ddot{x} = -c\dot{x} - kx \quad \text{or} \quad m\ddot{x} + c\dot{x} + kx = 0$$

If the solution is assumed to take the form :

$$x(t) = Ce^{st} \quad \text{where } s = \pm i\omega_n$$

then : $\dot{x}(t) = sCe^{st}$ and $\ddot{x}(t) = s^2Ce^{st}$

Substituting for $x, \dot{x}$ and $\ddot{x}$ in the eqn. of motion

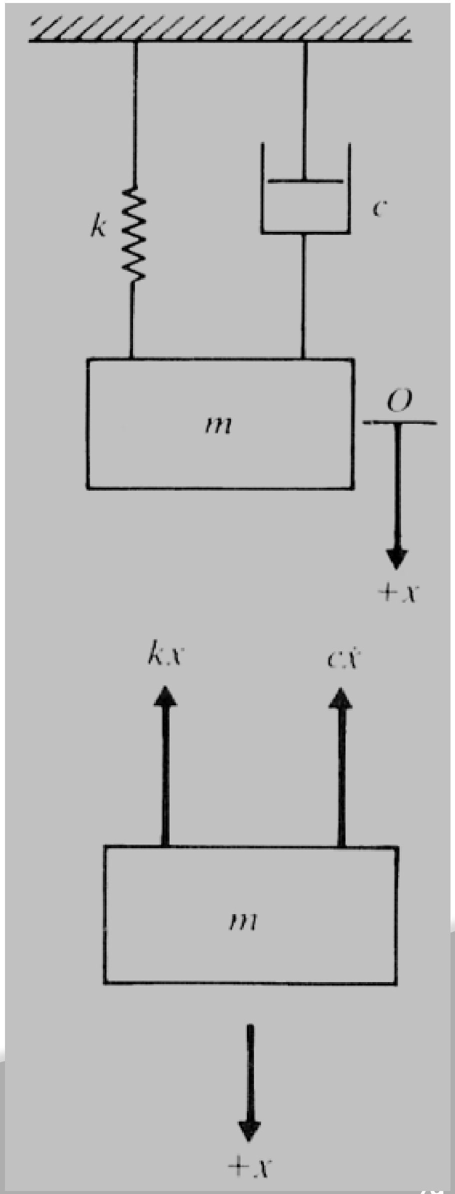
$$ms^2 + cs + k = 0$$

The root of the characteristic eqn. are :

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}$$

The two solutions are :

$$x_1(t) = C_1e^{s_1t} \quad \text{and} \quad x_2(t) = C_2e^{s_2t}$$



Free single DoF vibration + viscous damping

$$\left(\frac{c_c}{2m}\right)^2 - \left(\frac{k}{m}\right) = 0 \qquad or \qquad c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n = 2\sqrt{km}$$

$$\zeta = \frac{c}{c_c} \qquad or \qquad \frac{c}{2m} = \frac{c}{c_c}\frac{c_c}{2m} = \zeta\omega_n$$

The roots can be re – written :

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

And the solution becomes :

$$x(t) = C_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}$$

$$x(t=0)=x_0 \quad and \quad \dot{x}(t=0)=\dot{x}_0$$

Then

$$C_1^{'}=x_0 \quad and \quad C_2^{'}=\frac{\dot{x}_0+\zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}$$

Therefore the solution becomes

$$x(t)=e^{-\zeta\omega_n t}\left\{x_0\cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right)+\frac{\dot{x}_0+\zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)\right\}$$

Forced (harmonically excited) single DoF vibration – undamped.

$$m\ddot{x} + kx = F_0 \cos(\omega t)$$

$$x(t) = x_h(t) + x_p(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t) + \frac{F_0}{k - m\omega^2} \cos(\omega t)$$

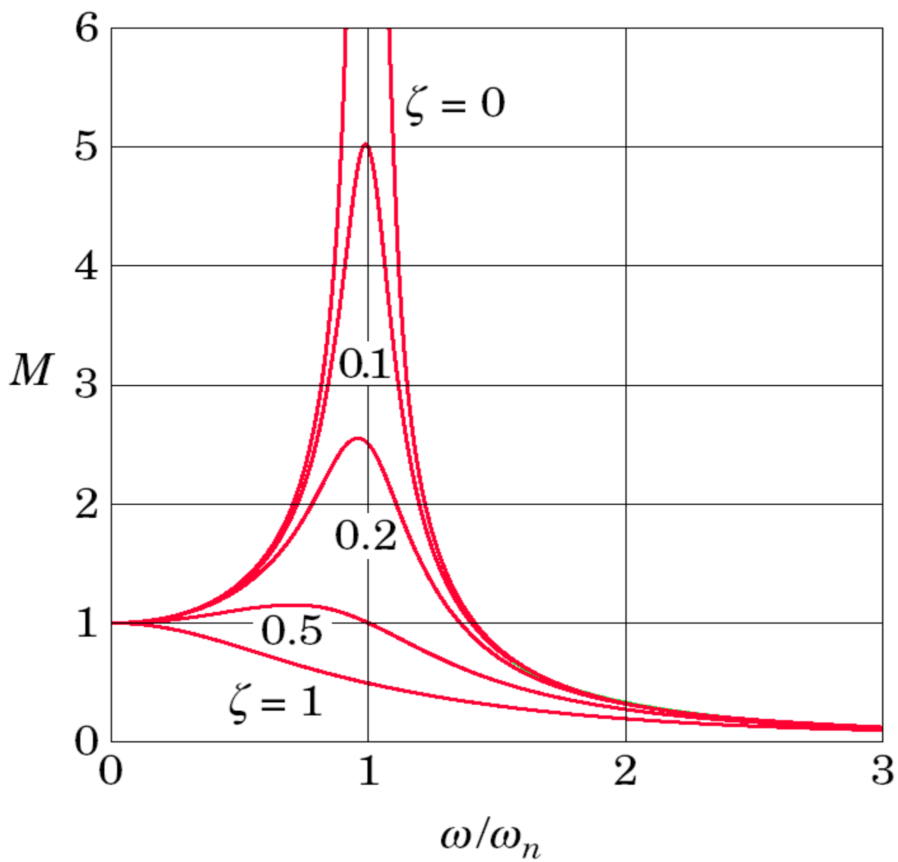
$$x(t) = A\cos(\omega_n t + \phi) + \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \cos(\omega t) \qquad for\ \omega / \omega_n < 1$$

$$x(t) = A\cos(\omega_n t + \phi) - \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \cos(\omega t) \qquad for\ \omega / \omega_n > 1$$

where A *and* ϕ *are functions of* x_0 *and* $\dot{x}_0$ *as before.*

Forced (harmonically excited) single DoF vibration – Damped.

$$\frac{X}{\delta_{st}} = \frac{1}{\left\{\left[1-r^2\right]^2 + \left[2\zeta r\right]^2\right\}^{1/2}} \qquad \phi = a\tan\left(\frac{2\zeta r}{1-r^2}\right)$$



Forced (harmonically excited) single DoF vibration – Damped.

$$X = \frac{F_0}{\left[\left(k - m\omega^2\right) + ic\omega\right]} \quad \leftarrow \; X / F_0 \textit{ is called the RECEPTANCE (Dynamic compliance)}$$

multiplying the numerator & denominator on the RHS by $\left(k - m\omega^2\right) - ic\omega$
and separating real and imaginary components :

$$X = F_0\left[\frac{k - m\omega^2}{\left(k - m\omega^2\right)^2 + c^2\omega^2} - i\frac{c\omega}{\left(k - m\omega^2\right)^2 + c^2\omega^2}\right]$$

applying the complex relationships : $x + iy = Ae^{i\phi}$ *where* $A = \sqrt{x^2 + y^2}$ *and* $\phi = a\tan\left(\frac{y}{x}\right)$

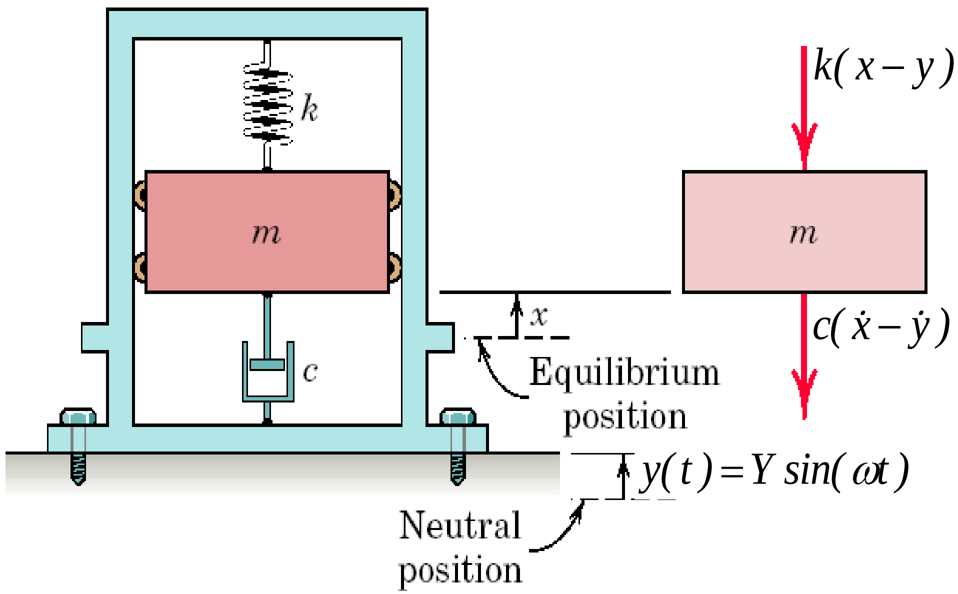
The magnitude of the response can be written as :

$$X = \frac{F_0}{\left[\left(k - m\omega^2\right)^2 + c^2\omega^2\right]^{1/2}} e^{-i\phi} \quad \textit{where} \quad \phi = a\tan\left(\frac{c\omega}{k - m\omega^2}\right)$$

And the steady – state solution becomes :

$$x_p(t) = \frac{F_0}{\left[\left(k - m\omega^2\right)^2 + c^2\omega^2\right]^{1/2}} e^{i(\omega t - \phi)}$$

Response due to base motion (harmonic)



Response due to base motion (harmonic)

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$

If $y(t) = Y\sin(\omega t)$ *the eqn. of motion becomes :*

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$$

$$= c\omega Y\cos(\omega t) + kY\sin(\omega t)$$

$$= A\sin(\omega t - \alpha)$$

where $A = Y\sqrt{k^2 + (c\omega)^2}$ *and* $\alpha = a\tan\left(-\dfrac{c\omega}{k}\right)$

Response due to base motion (harmonic)

$$x_p(t) = \frac{Y\sqrt{k^2+(c\omega)^2}}{\left[\left(k-m\omega^2\right)^2+c^2\omega^2\right]^{1/2}} \sin(\omega t-\phi_1-\alpha)$$

where $\alpha = a\tan\left(-\frac{c\omega}{k}\right)$ *and* $\phi_1 = a\tan\left(\frac{c\omega}{k-m\omega^2}\right)$

The solution can be simplified to :

$$x_p(t) = X\sin(\omega t-\phi)$$

where

$$\frac{X}{Y} = \left[\frac{k^2+(c\omega)^2}{\left(k-m\omega^2\right)^2+c^2\omega^2}\right]^{1/2} = \left[\frac{1+(2\zeta r)^2}{\left(1-r^2\right)^2+(2\zeta r)^2}\right]^{1/2} \quad \leftarrow \text{Displacement Transmissibility}$$

and

$$\phi = a\tan\left(\frac{mc\omega^3}{k\left(k-m\omega^2\right)+(c\omega)^2}\right) = a\tan\left(\frac{2\zeta r^3}{1+(4\zeta^2-1)r^2}\right)$$

Rotating Imbalance Excitation

$$F(t) = me\omega^2 \sin(\omega t)$$

The eqn. of motion is :

$$M\ddot{x} + c\dot{x} + kx = me\omega^2 \sin(\omega t)$$

and the steady – state solution becomes :

$$x_p(t) = X \sin(\omega t - \phi) = Im\left[\frac{me}{M}\left(\frac{\omega}{\omega_n}\right)^2 \left|H(i\omega)\right| e^{i(\omega t - \phi)}\right]$$

The response amplitude and phase are given by :

$$X = \frac{me\omega^2}{\left[\left(k - M\omega^2\right)^2 + (c\omega)^2\right]^{1/2}} = \frac{me}{M}\left(\frac{\omega}{\omega_n}\right)^2 \left|H(i\omega)\right| \quad or \quad \frac{MX}{me} = \frac{r^2}{\left[\left(1 - r^2\right)^2 + (2\zeta r)^2\right]^{1/2}} = r^2 \left|H(i\omega)\right|$$

$$\phi = a\tan\left(\frac{c\omega}{k - M\omega^2}\right) = a\tan\left(\frac{2\zeta r}{1 - r^2}\right)$$

SDoF systems – General forcing functions – Nonperiod

- When the forcing function is arbitrary and nonperiodic (aperiodic) it cannot be represented with a Fourier series
- Alternative methods for determining the response must be used:
  - Representation of the excitation function with a ***Convolution integral***
  - Using ***Laplace Transformations***
  - Approximating F(t) with a suitable ***interpolation method*** then using a numerical procedure
  - ***Numerical integration*** of the equations of motion.

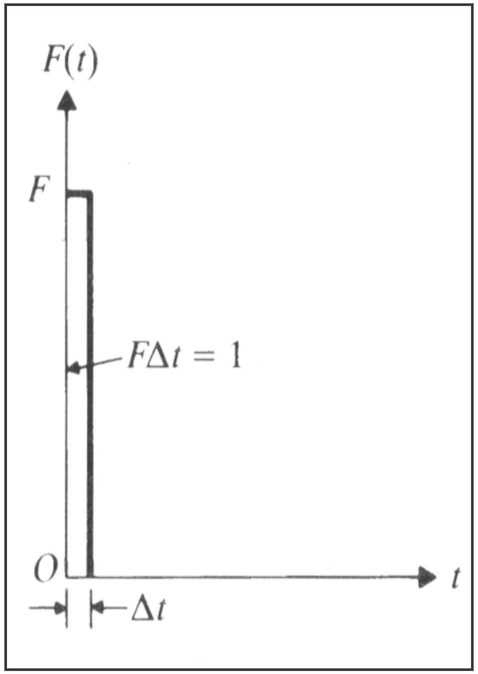
- **Convolution integral**
- Consider one of the simplest nonperiodic exciting force: Impulsive force: which has a large magnitude F which acts for a very short time Δt.
- An impulse can be measured by the resulting change in momentum

$$Impulse = F\Delta t = m\dot{x}_2 - m\dot{x}_1$$

where $\dot{x}_1$ and $\dot{x}_2$ represent the velocity of the lumped mass before and after the impulse.

$$\underset{\sim}{F} = \int_{t}^{t+\Delta t} F\,dt$$

and a unit impulse is defined as

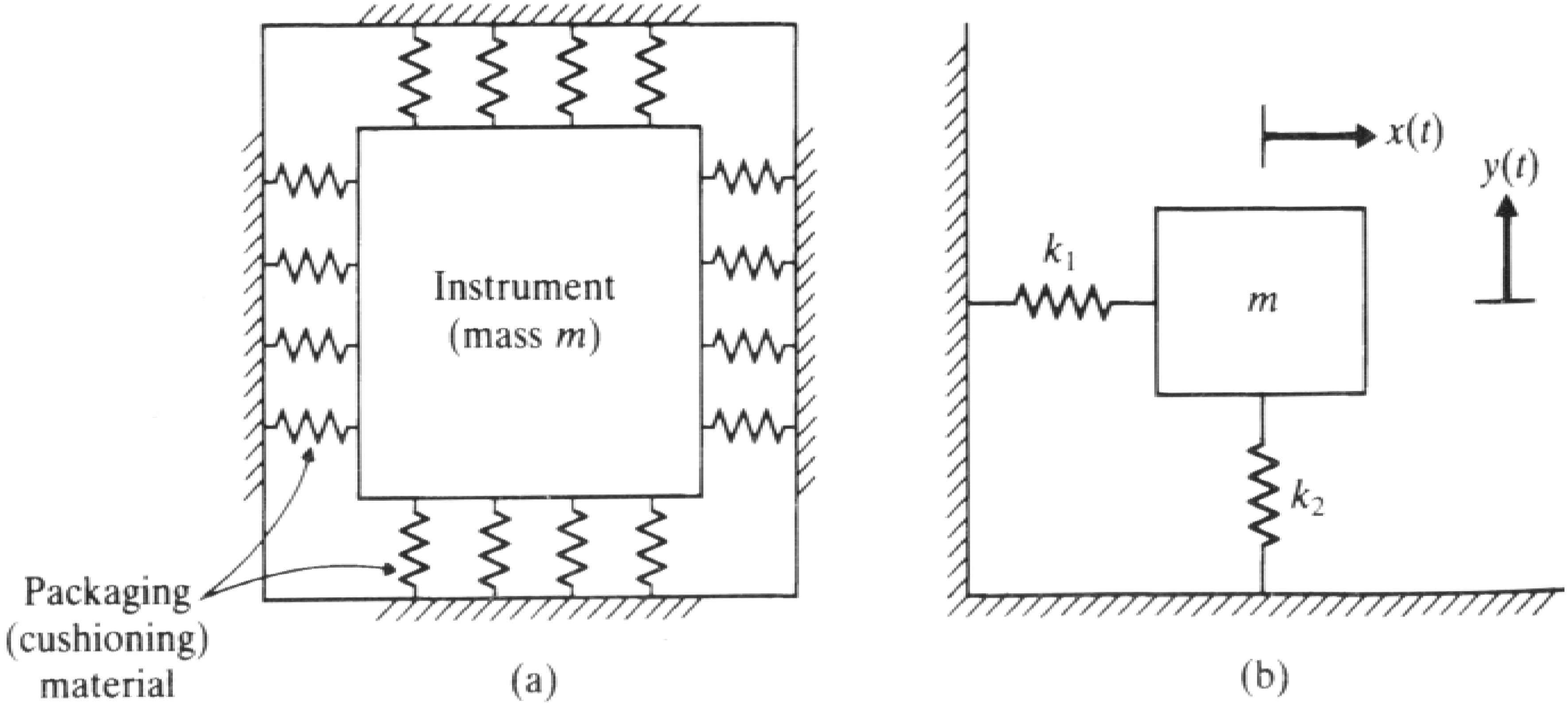


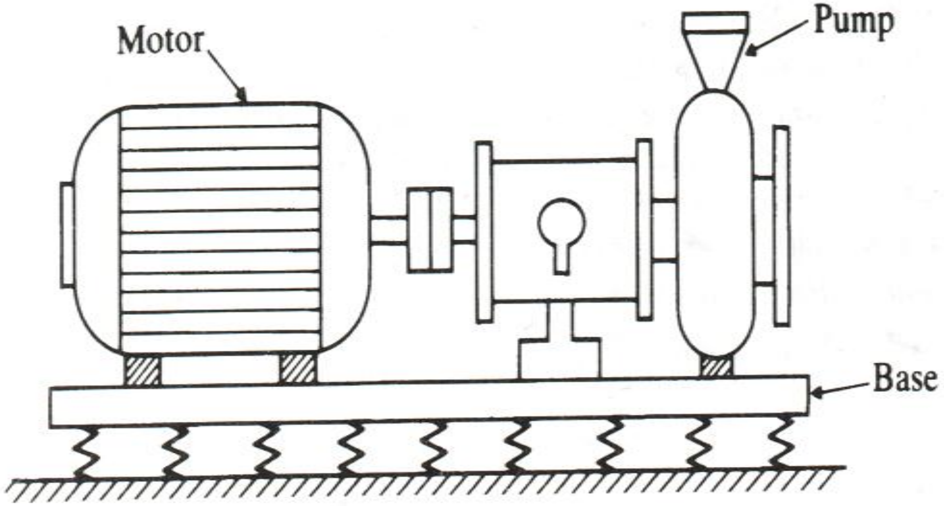
$$\underset{\sim}{f} = \lim_{\Delta t \to 0} \int_{t}^{t+\Delta t} F\,dt = F dt = 1$$

$$x(t<0)=0 \text{ and } \dot{x}(t<0)=0 \quad \text{or} \quad x(t=0^-)=0 \text{ and } \dot{x}(t=0^-)=0$$

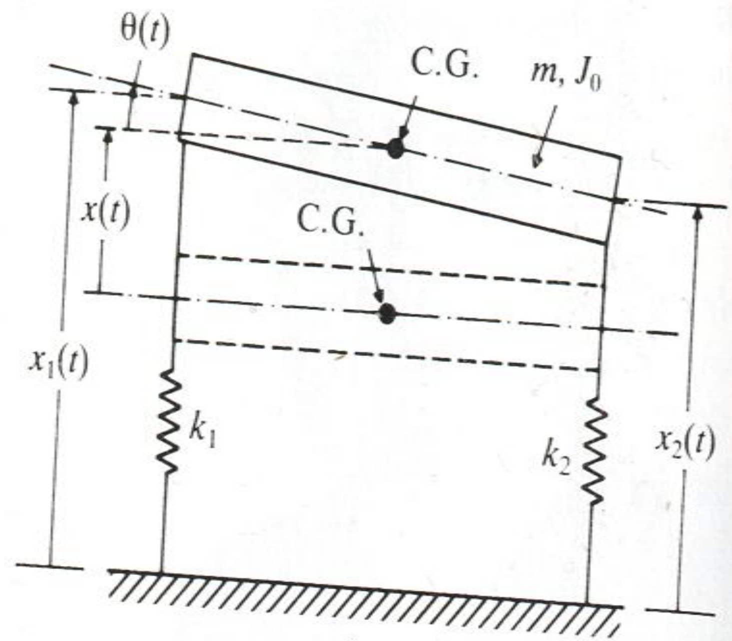
$$x(t)=g(t)=\frac{e^{-\zeta\omega_n t}}{m\omega_d}\sin\left(\omega_d t\right)$$

Two DOF systems





(a)

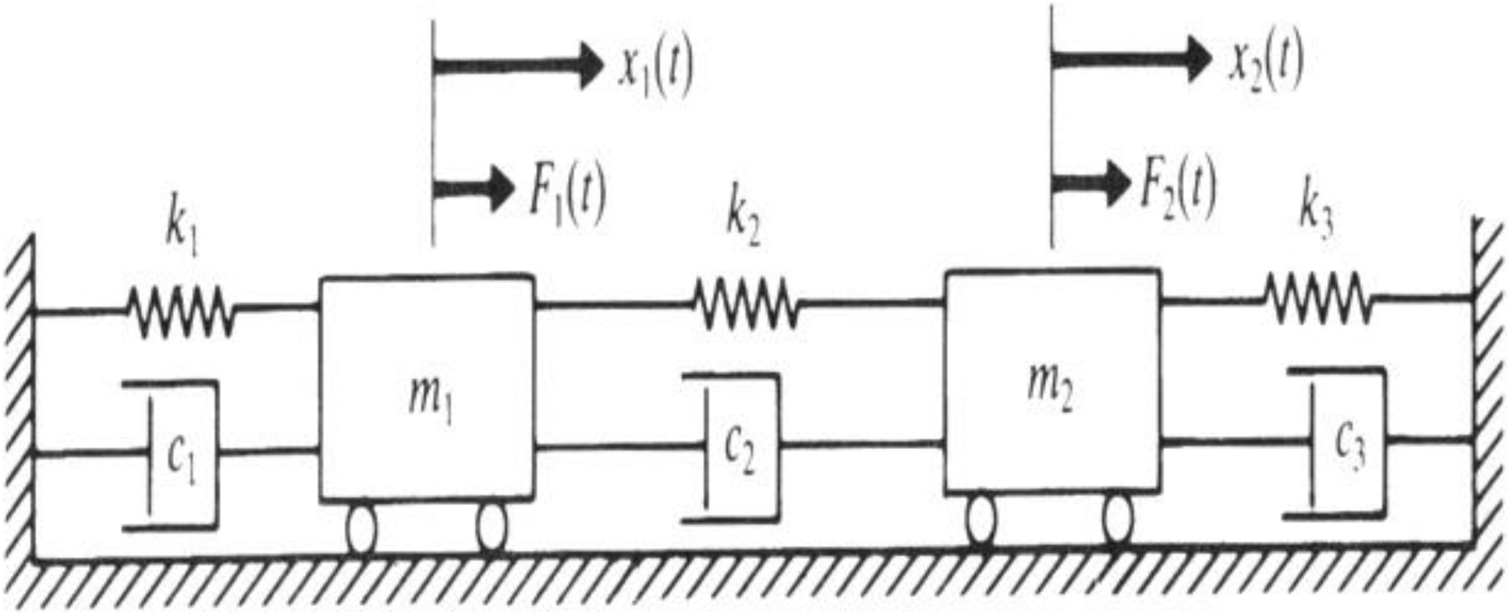


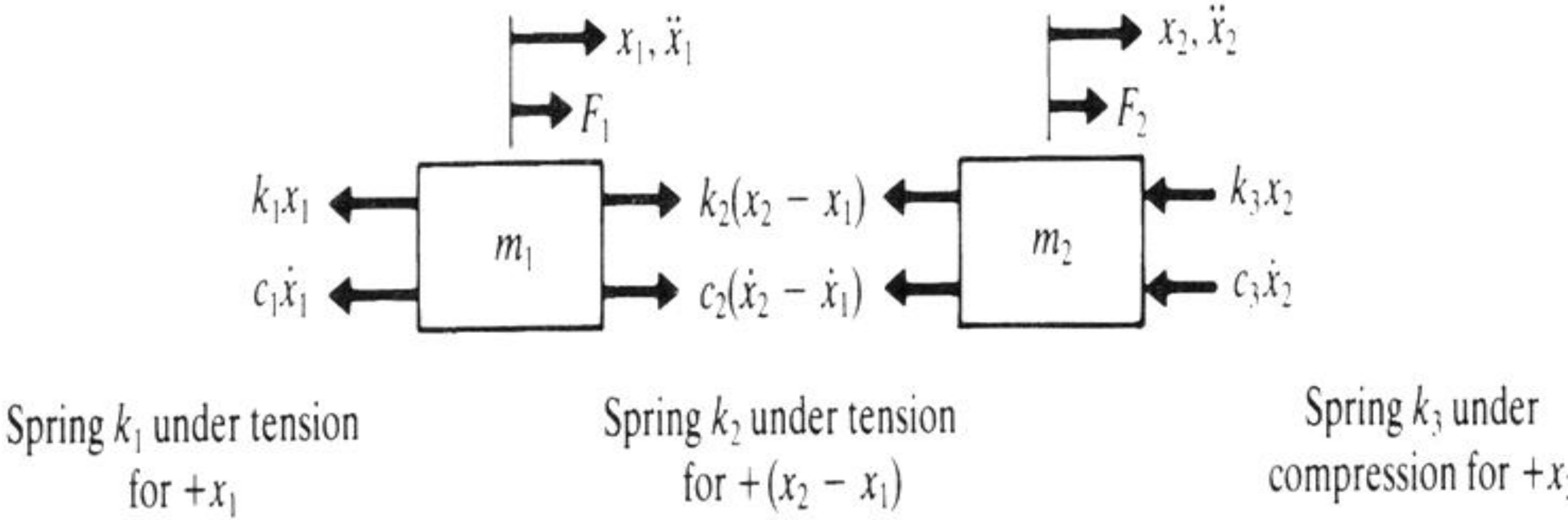
(b)

- No. of DoF of system = No. of mass elements x number of motion types for each mass
- For each degree of freedom there exists an equation of motion – usually **coupled** differential equations.
- Coupled means that the motion in one coordinate system depends on the other
- If harmonic solution is assumed, the equations produce two natural frequencies and the amplitudes of the two degrees of freedom are related by the *natural, principal or normal* mode of vibration.

- Under an arbitrary initial disturbance, the system will vibrate freely such that the two normal modes are superimposed.
- Under sustained harmonic excitation, the system will vibrate at the excitation frequency. Resonance occurs if the excitation frequency corresponds to one of the natural frequencies of the system

Two DOF systems




x1, ẍ1
F1
x2, ẍ2
F2
k1x1
c1ẋ1
m1
k2(x2 − x1)
c2(ẋ2 − ẋ1)
m2
k3x2
c3ẋ2
Spring k1 under tension for +x1
Spring k2 under tension for +(x2 − x1)
Spring k3 under compression for +x2

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 - c_2(\dot{x}_2 - \dot{x}_1) - k_2(x_2 - x_1) = F_1$$

$$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) + c_3\dot{x}_2 + k_3x_2 = F_2$$

or

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = F_1$$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = F_2$$

Mode shapes of Undamped System

- **Equations of motion**

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0$$

$$m_2\ddot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 = 0$$

$$\left[\left\{-m_1\omega^2 + (k_1 + k_2)\right\}X_1 - k_2 X_2\right]\cos(\omega t + \phi) = 0$$

$$\left[-k_2 X_1 + \left\{-m_2\omega^2 + (k_2 + k_3)\right\}X_2\right]\cos(\omega t + \phi) = 0$$

As these equations must be zero for all values of t, the cosine terms cannot be zero. Therefore:

$$\left\{-m_1\omega^2 + (k_1 + k_2)\right\}X_1 - k_2 X_2 = 0$$

$$-k_2 X_1 + \left\{-m_2\omega^2 + (k_2 + k_3)\right\}X_2 = 0$$


Thank you