INSTITUTE OF AERONAUTICAL ENGINEERING

(Autonomous)
Dundigal, Hyderabad -500 043

ELECTRONICS AND COMMUNICATION ENGINEERING

COURSE LECTURE NOTES

Course Name	**ELECTROMAGNETIC WAVES AND TRANSMISSION LINES**
Course Code	AECB13
Programme	B.Tech
Semester	IV
Course Coordinator	Ms. K C Koteswaramma, Assistant Professor, ECE
Course Faculty	Dr. P Ashok Babu, Professor & HoD, ECE Ms. A Usha Rani, Assistant Professor, ECE Ms. M Sreevani, Assistant Professor, ECE
Lecture Numbers	1-60
Topic Covered	All

COURSE OBJECTIVES:

The course should enable the students to:

I	Understand the 3D vector co-ordinate systems and electromagnetic field concepts.
II	Analyze the importance of Maxwell's equations in electromagnetic theory and wave propagation.
III	Study the propagation characteristics of electromagnetic waves at boundary.
IV	Demonstrate the ability to compute various parameters for transmission lines using smith chart and classical theory.

COURSE LEARNING OUTCOMES:

Students, who complete the course, will have demonstrated the ability to do the following:

SNO	DESCRIPTION
AECB13.01	Understand the different types of 3D co- ordinate systems, scalars and vectors, physical significance of divergence, curl and gradient
AECB13.02	Illustrate the concepts of coulomb's law and gauss's law to different charge distributions like point charge, line charge, surface charge and volume charge. Analyze its applications.

AECB13.03	Understand the applications of Laplace's and Poisson's equations to solve problems on capacitance of different charge distributions.
AECB13.04	Illustrate the physical significance of Biot- Savart's law and Ampere's Circuit law for different current distributions and analyze its applications
AECB13.05	Evaluate the physical interpretation of Maxwell's equations and applications for various fields like antennas and wave guides.
AECB13.06	Derive the boundary conditions between different media like dielectric to conductor, conductor to free space.
AECB13.07	Analyze and apply the Maxwell's equations to derive electromagnetic wave equations for different media.
AECB13.08	Understand the behavior of electromagnetic waves incident on the interface between two different media.
AECB13.09	Formulate and analyze problems in different media such as lossy, lossless with boundaries using uniform plane waves.
AECB13.10	Understand the significance of transmission lines and its types, derive their primary constants and secondary constants.
AECB13.11	Understand the concept of attenuation, loading, and analyze the loading technique to the transmission lines.
AECB13.12	Understand the design of various transmission lines with respect to distortion, loss, impedance matching, and VSWR and reflection coefficient.
AECB13.13	Summarize the impedance transformation for different lengths such as $\lambda/4,\lambda/2,\lambda/8$ transmission lines.
AECB13.14	Understand the design of ultra high frequency transmission lines for different applications by using single and double stub matching techniques.
AECB13.15	Formulate and analyze the smith chart to estimate impedance, VSWR, reflection coefficient, OC and SC lines.
AECB13.16	Apply the concept of electromagnetic fields to understand and analyze land mobile communications.
AECB13.17	Acquire the knowledge and develop capability to succeed national and international level competitive examinations.

MODULE -I	ELECTROSTATICS	Classes: 10

Electrostatics: Coulomb"s law, electric field intensity, fields due to different charge distributions; Electric flux density, Gauss law and its applications; Scalar electric potential; Energy density, illustrative problems; Conductors and dielectrics-characterization; Convection and conduction currents; Dielectric constant, isotropic and homogeneous dielectrics; Continuity equation and relaxation time, conductivity, power absorbed in conductor, Poisson"s and Laplace"s equations; Capacitance: Parallel plate, co axial, spherical capacitors; Method of images; Illustrative problems.

MODULE – II	MAGNETOSTATICS	Classes: 10

Magnetostatics: Biot-savart law; Ampere"s circuital law and applications; Magnetic flux density; Magnetic scalar and vector potentials; Forces due to magnetic fields; Ampere"s force law; Boundary conditions: Dielectric- dielectric, dielectric conductor interfaces; Inductances and magnetic energy; Illustrative problems;

Maxwell's equations (Time varying fields): Faraday"s law; Inconsistency of ampere"s law for time varying fields and definition for displacement current density; Maxwell"s equations in differential form, integral form and word

Statements.		

MODULE - III	**UNIFORM PLANE WAVES**	**Classes: 08**

Uniform plane waves: Wave equations for conducting and perfect dielectric media; Relation between E and H; Wave propagation in lossless and conducting media, Loss tangent, Intrinsic impedance; Skin depth; Polarization, Illustrative problems.

Reflection/refraction of plane waves: Reflection and refraction at normal incidence, reflection and refraction at oblique incidence; Standing waves; Brewster angle, critical angle, total internal reflection, surface impedance; Poynting vector and poynting theorem-applications; Power loss in plane conductor; Illustrative problems.

MODULE - IV	**TRANSMISSION LINE CHARACTERISTICS**	**Classes: 09**

Transmission line characteristics: Types; Transmission line parameters; Transmission line equations; Characteristic impedance, propagation constant; Phase and group velocities; Infinite line concepts, Loss less/low loss transmission line characterization; condition for distortion less and minimum attenuation in transmission lines; Loading: Types of loading; Illustrative problems.

MODULE - V	**UHF TRANSMISSION LINES AND APPLICATIONS**	**Classes: 08**

UHF transmission lines and applications: Input impedance relations; SC and OC lines; Reflection coefficient, VSWR; UHF lines as circuit elements, $\lambda/4$, $\lambda/2$ and $\lambda/8$ lines, impedance transformations, significance of Zmin and Zmax; Smith chart: Configuration and applications; Single and double stub matching; Illustrative problems.

Coordinate Systems:

In order to describe the spatial variations of the quantities, appropriate coordinate system is required.

A point or vector can be represented in a curvilinear coordinate system that may be orthogonal or non-orthogonal.

- A way of addressing the points in the space ,,,,

- An orthogonal system is one in which the coordinates are mutually perpendicular to each other.

- The different co-ordinate system available are:

  - Cartesian or Rectangular.

  - Circular cylindrical.

  - Spherical.

  - Elliptical Cylindrical.

  - Hyperbolic Cylindrical.

  - Parabolic Cylindrical.

  The choice depends on the geometry of the application

The frequently used and hence discussed herein are

Rectangular Co-ordinate system.(Example: Cube, Cuboids)

Cylindrical Co-ordinate system.(Example : Cylinder)

Spherical Co-ordinate system.(Example : Sphere)

A set of 3 scalar values that define position and a set of unit vectors that define direction form a co-ordinate system.
The 3 scalar values used to define position are called co-ordinates. All coordinates are defined with respect to an arbitrary point called the origin.
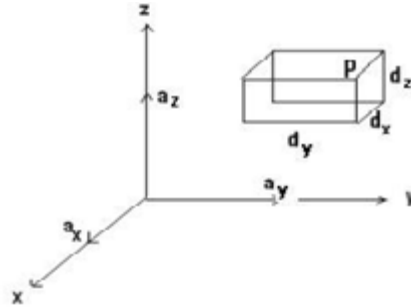
Cartesian Co-ordinate System / Rectangular Co-ordinate System (X, Y, Z)

A Vector in Cartesian system is represented as

(A_x, A_y, A_z)

Or

$A = A_x a_x + A_y a_y + A_z a_z$

Where a_x, a_y and a_z are the unit vectors in x, y, z direction respectively.

Range of the variables:

It defines the minimum and the maximum value that x, y and z can have in Cartesian system.

$-\infty < x, y, z < \infty$

Differential Displacement / Differential Length (dl):

It is given as $dl = dx a_x + dy a_y + dz a_z$
Differential length for a surface is given as:

$dl = dx a_x + dy a_y$ --- (For XY Plane or Z Constant Plane). $dl = dy a_y + dz a_z$, ---(For YZ Plane or X Constant Plane). $dl = dx a_x + dz a_z$ ---(For XZ Plane or Y Constant Plane). Differential length for a line parallel to x, y and z axis are respectively given as:

$dl = dx a_x$ ---(For a line parallel to x-axis). $dl = dy a_y$ ---(For a line Parallel to y-axis). $dl = dz a_z$, ---(For a line parallel to z-axis). Differential Normal Surface ds):

The differential surface (area element) is defined as

$ds = ds \, a_n$
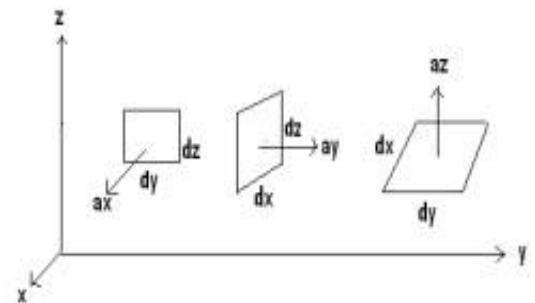
where a_n is the unit vector perpendicular to the
surface. For the 1st figure, ds = dydz a_x

2nd   figure,

ds = dxdz a_y

3rd   figure,

ds = dxdy a_z

Differential surface is basically a cross product between two parameters of the surface. For example, consider the first figure. The surface has two differential lengths, one is dy and dz. The differential surface (ds) is hence given as:

dS = dy * dz

$$= |dy|\,|dz|\,\sin\square_{AB}\,a_n$$

$$=|dy|\,|dz|\ \ a_n$$

Where a_n is the unit vector normal to both dy and dz

i.e. $a_n = a_y * a_z = a_x$

In other words the differential surface element (ds) has an area equal to product dydz, and a normal vector that points in a_x direction.

Differential Volume element (dv)

The differential volume element (dv) can be expressed in terms of the triple product.

dv = dx . (dy * dz)

Consider a cubical surface having dimension x * y * z. The differential volume (dv) of the cubical surface is given as the triple product of the dimensions.

dv = dx . (dy * dz)

= dx a_x . (dy dz a $_x$sin θ $_{AB}$)

= dx a_x . (dy dz a $_x$)

= dx dy dz

Where dy and dz are mutually perpendicular to each other. Therefore the angle between them is 90°.

$a_x.a_x= 1$	$a_x.a_y= 0$	$a_x.a_z= 0$
$a_y.a_x= 0$	$a_y.a_y= 1$	$a_y.a_z= 0$
$a_z.a_x= 0$	$a_z.a_y= 0$	$a_z.a_z= 1$

One thing to remember is that, the three parameters of Cartesian

coordinate system i.e. X, Y, Z are all mutually perpendicular to each other.

Therefore a_x, a_y and a_z are all mutually perpendicular to each other.

Circular Cylindrical Co-ordinate System...

A Vector in Cylindrical system is represented as

$$(A_\rho, A_\varphi, A_z)$$

or

$A = A_\rho a_\rho + A_\varphi a_\varphi + A_z a_z$

Where a_ρ, a_φ and a_z are the unit vectors in ρ, φ and z direction respectively.

The physical significance of each parameter of cylindrical coordinates:

- The value ρ indicates the distance of the point from the z-axis. It is the radius of the cylinder.

- The value φ, also called the azimuthal angle, indicates the rotation angle around the z-axis. It is basically measured from the x axis in the x-y plane. It is measured anti-clockwise.
- The value z indicates the distance of the point from z-axis. It is the same as in the Cartesian system. In short, it is the height of the cylinder.

Range of the variables:

It defines the minimum and the maximum value that ρ, φ and z can have in Cartesian system.
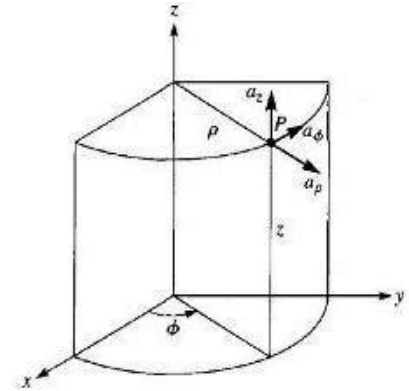
$0 \leq \rho < \infty$

$0 \leq \varphi < 2\pi$

$-\infty < z < \infty$

Cylindrical System - Unit vectors:

Since the co-ordinate system is orthogonal, the unit vectors a_ρ, a_φ and a_z are mutually perpendicular to each other.
- a_ρ points in the direction of increasing ρ, i.e a_ρ points away from the z-axis.

- a_φ points in the direction of increasing φ (anticlockwise).

- a_z points in the direction of increasing z.

Introduction to Co-Ordinate System:

Line, surface and volume integrals:

In electromagnetic theory, we come across integrals, which contain vector functions. Some representative integrals are listed below:

$$\int_V \vec{F}\,dv \qquad \int_C \phi\,\vec{dl} \qquad \int_C \vec{F}.\vec{dl} \qquad \int_S \vec{F}.\vec{ds}$$

n the above integrals, $\vec{F}$ and ϕ respectively represent vector and scalar function of space coordinates. C,S and V represent path, surface and volume of integration. All these integrals are evaluated using extension of the usual one-dimensional integral as the limit of a sum, i.e., if a function f(*x*) is defined over arrange *a* to *b* of values of *x*, then the integral is given by

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f_i \delta x_i$$

...............................(1.42)

where the interval (*a,b*) is subdivided into n continuous interval of lengths $\delta x_1, \ldots\ldots, \delta x_n$.

Line Integral: Line integral $\int_C \vec{E} \cdot \vec{dl}$ is the dot product of a vector with a specified *C*; in other words it is the integral of the tangential component $\vec{E}$ along the curve *C*.
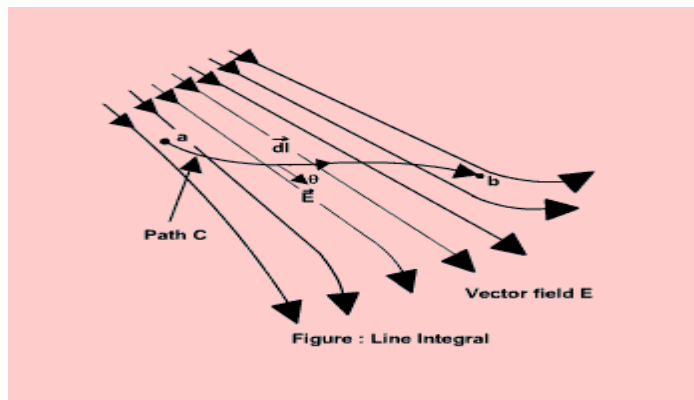

Fig 1.14: Line Integral

As shown in the figure 1.14, given a vector $\vec{E}$ around *C*, we define the integral $\int_C \vec{E}.\vec{dl} = \int_a^b E \cos \theta \, dl$ as the line integral of E along the curve C.

f the path of integration is a closed path as shown in the figure the line integral becomes a closed line integral and

is called the circulation of $\vec{E}$ around *C* and denoted as $\oint_C \vec{E}.\vec{dl}$ as shown in the figure 1.15.
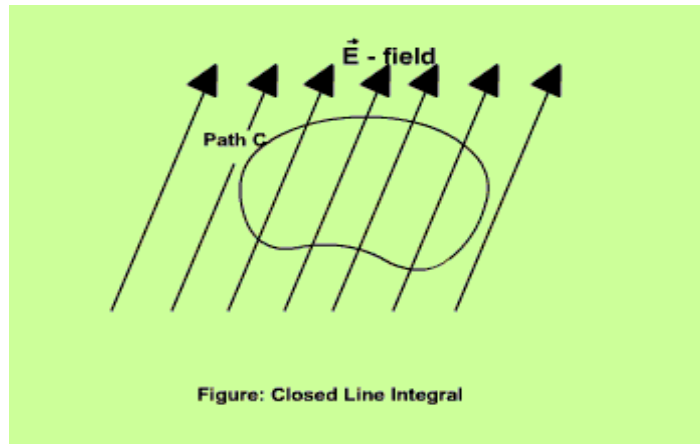
Fig 1.15: Closed Line Integral

Surface Integral :

Given a vector field $\vec{A}$, continuous in a region containing the smooth surface S, we define the surface integral or

the flux of $\vec{A}$ through S as $\psi = \int_S A \cos \theta \, dS = \int_S \vec{A} . \hat{a}_n \, dS = \int_S \vec{A} \, d\vec{S}$ as surface integral over surface S.
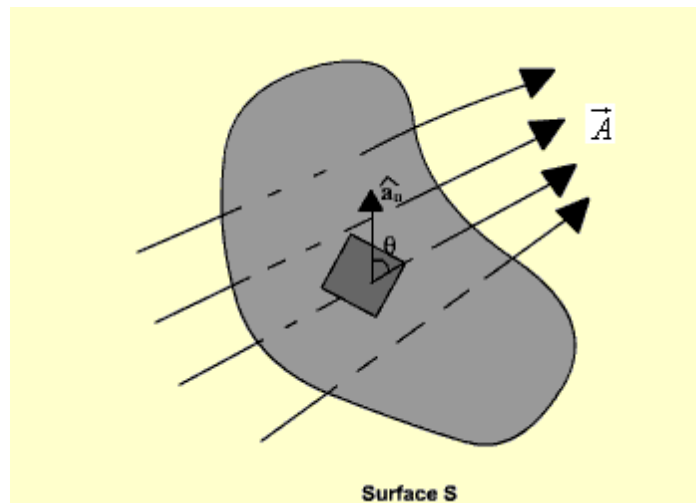

Fig 1.16 : Surface Integral

Volume Integrals:

We define $\int_V f dV$ or $\iiint_V f dV$ as the volume integral of the scalar function f(function of spatial coordinates) over the volume V. Evaluation of integral of the form $\int_V \vec{F} dV$ can be carried out as a sum of three scalar volume integrals, where each scalar volume integral is a component of the vector $\vec{F}$

The Del Operator :

The vector differential operator ∇ was introduced by Sir W. R. Hamilton and later on developed by P. G. Tait.

Mathematically the vector differential operator can be written in the general form as:

$$\nabla = \frac{1}{h_1}\frac{\partial}{\partial u}\hat{a}_u + \frac{1}{h_2}\frac{\partial}{\partial v}\hat{a}_v + \frac{1}{h_3}\frac{\partial}{\partial w}\hat{a}_w$$ ....(1.43)

In Cartesian coordinates:

$$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$$ ....(1.44)

In cylindrical coordinates:

$$\nabla = \frac{\partial}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z$$ ....(1.45)

and in spherical polar coordinates:

$$\nabla = \frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{a}_\phi$$ ....(1.46)

Gradient of a Scalar function:

Let us consider a scalar field V(u,v,w) , a function of space coordinates.

Gradient of the scalar field V is a vector that represents both the magnitude and direction of the maximum space rate of increase of this scalar field V.
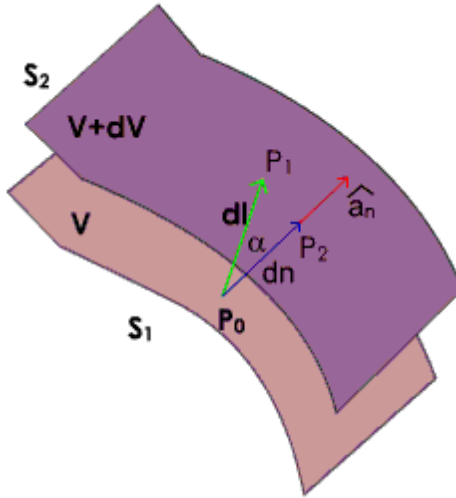
Fig 1.17 : Gradient of a scalar function

By our definition of gradient we can write:

$$\text{grad}V = \frac{dV}{dn}\hat{a}_n = \nabla V$$

....(1.47)

Also we can write,

$$dV = \frac{\partial V}{\partial l_u}dl_u + \frac{\partial V}{\partial l_v}dl_v + \frac{\partial V}{\partial l_w}dl_w$$

$$= \left(\frac{\partial V}{\partial l_u}\hat{a}_u + \frac{\partial V}{\partial l_v}\hat{a}_v + \frac{\partial V}{\partial l_w}\hat{a}_w\right) \cdot (dl_u\hat{a}_u + dl_v\hat{a}_v + dl_w\hat{a}_w)$$

$$= \left(\frac{\partial V}{h_1\partial u}\hat{a}_u + \frac{\partial V}{h_2\partial v}\hat{a}_v + \frac{\partial V}{h_3\partial w}\hat{a}_w\right) \cdot (h_1 du\hat{a}_u + h_2 dv\hat{a}_v + h_3 dw\hat{a}_w)$$

...........................(1.51)

By comparison we can write,

$$\nabla V = \frac{1}{h_1}\frac{\partial V}{\partial u}\hat{a}_u + \frac{1}{h_2}\frac{\partial V}{\partial v}\hat{a}_v + \frac{1}{h_3}\frac{\partial V}{\partial w}\hat{a}_w$$

....(1.52)

Hence for the Cartesian, cylindrical and spherical polar coordinate system, the expressions for gradient can be written   as:
In Cartesian coordinates:

$$\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z$$

....(1.53)

In cylindrical coordinates:

$$\nabla V = \frac{\partial V}{\partial \rho} \hat{a}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \hat{a}_\phi + \frac{\partial V}{\partial z} \hat{a}_z$$ ....(1.54)

and in spherical polar coordinates:

$$\nabla V = \frac{\partial V}{\partial r} \hat{a}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \hat{a}_\phi$$ ....(1.55)

Divergence theorem :

Divergence theorem states that the volume integral of the divergence of vector field is equal to the net outward flux of the vector through the closed surface that bounds the volume. Mathematically, $\int_V \nabla \cdot \vec{A} dv = \oint_S \vec{A} \cdot d\vec{s}$

Proof:

Let us consider a volume V enclosed by a surface S . Let us subdivide the volume in large number of cells. Let the kth cell has a volume ΔV_k and the corresponding surface is denoted by S_k. Interior to the volume, cells have common surfaces. Outward flux through these common surfaces from one cell becomes the inward flux for the neighboring cells. Therefore when the total flux from these cells are considered, we actually get the net outward flux through the surface surrounding the volume. Hence we can write:

$$\oint_S \vec{A} \cdot d\vec{s} = \sum_k \oint_{S_k} \vec{A} \cdot d\vec{s} = \sum_k \frac{\oint_{S_k} \vec{A} \cdot d\vec{s}}{\Delta V_k} \Delta V_k$$

In the limit, that is when $K \to \infty$ and $\Delta V_k \to 0$ the right hand of the expression can be written as $\int_V \nabla \cdot A dV$.

Hence we get $\oint_S \vec{A} \cdot d\vec{S} = \int_V \nabla \cdot A dV$, which is the divergence theorem.

Curl of a vector field:

We have defined the circulation of a vector field *A* around a closed path as $\oint \vec{A} \cdot d\vec{l}$

Curl of a vector field is a measure of the vector field's tendency to rotate about a point. Curl $\vec{A}$, also written as $\nabla \times \vec{A}$ is defined as a vector whose magnitude is maximum of the net circulation per unit area when the area tends to zero and its direction is the normal direction to the area when the area is oriented in such a way so as to make the circulation maximum.

Therefore, we can write:

$$Curl\ \vec{A} = \nabla \times \vec{A} = \lim_{\Delta S \to 0} \frac{\hat{a}_n}{\Delta S} \left[\oint_L \vec{A} \cdot dl \right]_{max}$$ ....(1.68)

This can be written as in three coordinate systems

In Cartesian coordinates:

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

In Cylindrical coordinates,

$$\nabla \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$.

In Spherical polar coordinates,

$$\nabla \times \vec{A} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r \sin \theta \hat{a}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & r A_\theta & r \sin \theta A_\phi \end{vmatrix}$$

Stoke's theorem :

It states that the circulation of a vector field $\vec{A}$ around a closed path is equal to the integral of $\nabla \times \vec{A}$ over the surface bounded by this path. It may be noted that this equality holds provided $\vec{A}$ and $\nabla \times \vec{A}$ are continuous on the surface.

i.e,

$$\oint_L \vec{A} \cdot d\vec{l} = \int_S \nabla \times \vec{A} \cdot d\vec{s}$$

Proof: Let us consider an area S that is subdivided into large number of cells as shown in the figure 1.21.
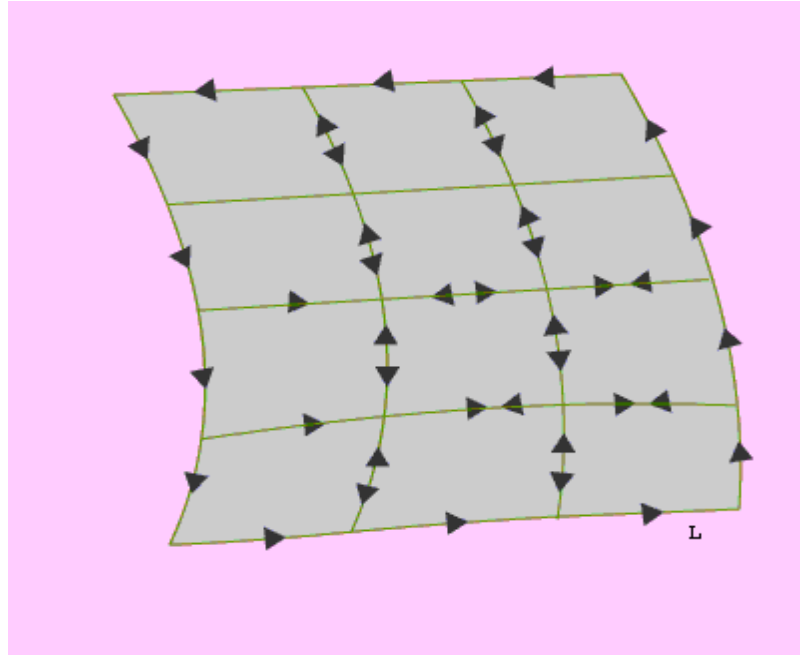
Fig 1.21: Stokes theorem

Let k^{th} cell has surface area ΔS_k and is bounded path L_k while the total area is bounded by path L. As seen from the figure that if we evaluate the sum of the line integrals around the elementary areas, there is cancellation along every interior path and we are left the line integral along path *L*. Therefore we can write,

$$\oint_L \vec{A} \cdot d\vec{l} = \sum_k \oint_{L_k} \vec{A} \cdot d\vec{l} = \sum_k \frac{\oint_{L_k} \vec{A} \cdot d\vec{l}}{\Delta S_k} \Delta S_k$$

..............(1.83)

As $\Delta S_k \to 0$

$$\oint_L \vec{A} \cdot d\vec{l} = \int_S \nabla \times \vec{A} \cdot d\vec{s}$$.

..............(1.84)

which is the stoke's theorem.

Coulomb's law:

Coulomb's Law states that the force between two point charges Q_1 and Q_2 is directly proportional to the product of the charges and inversely proportional to the square of the distance between them.

Point charge is a hypothetical charge located at a single point in space. It is an idealised model of a particle having an electric charge.

Mathematically, $$F = \frac{kQ_1Q_2}{R^2}$$,where k is the proportionality constant.

In SI units, Q_1 and Q_2 are expressed in Coulombs(C) and R is in meters.

Force F is in Newtons (N) and $$k = \frac{1}{4\pi\varepsilon_0}$$, ε_0 is called the permittivity of free space.

(We are assuming the charges are in free space. If the charges are any other dielectric medium, we will use $\varepsilon = \varepsilon_0\varepsilon_r$ instead where ε_r is called the relative permittivity or the dielectric constant of the medium).

Therefore $$F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{R^2}$$

As shown in the Figure 1 let the position vectors of the point charges Q_1 and Q_2 are given by $\vec{r_1}$ and $\vec{r_2}$. Let $\overrightarrow{F_{12}}$ represent the force on Q_1 due to charge Q_2.
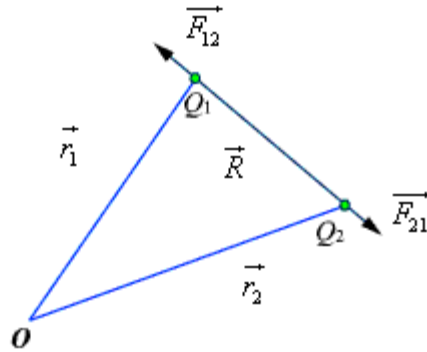
Fig 1: Coulomb's Law

The charges are separated by a distance of $$R = \left|\vec{r_1} - \vec{r_2}\right| = \left|\vec{r_2} - \vec{r_1}\right|$$. We define the unit vectors as

$$\widehat{a_{12}} = \frac{\left(\vec{r_2} - \vec{r_1}\right)}{R} \quad \text{and} \quad \widehat{a_{21}} = \frac{\left(\vec{r_1} - \vec{r_2}\right)}{R}$$

$\overrightarrow{F_{12}}$ can be defined as $$\overrightarrow{F_{12}} = \frac{Q_1Q_2}{4\pi\varepsilon_0 R^2}\widehat{a_{12}} = \frac{Q_1Q_2}{4\pi\varepsilon_0 R^2}\frac{\left(\vec{r_2} - \vec{r_1}\right)}{\left|\vec{r_2} - \vec{r_1}\right|^3}$$. Similarly the force on Q_1 due to charge Q_2 can be calculated and if $\overrightarrow{F_{21}}$ represents this force then we can write $\overrightarrow{F_{21}} = -\overrightarrow{F_{12}}$

When we have a number of point charges, to determine the force on a particular charge due to all other charges, we apply principle of superposition. If we have N number of charges $Q_1,Q_2,.........Q_N$ located

respectively at the points represented by the position vectors $\vec{r_1}, \vec{r_2}, ... \vec{r_N}$, the force experienced by a charge Q located at $\vec{r}$ is given by,

$$\vec{F} = \frac{Q}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{Q_i(\vec{r} - \vec{r_i})}{\left|\vec{r} - \vec{r_i}\right|^3}$$

Electric Field Intensity:

The electric field intensity or the electric field strength at a point is defined as the force per unit charge. That is

$$\vec{E} = \lim_{Q \to 0} \frac{\vec{F}}{Q} \quad \text{or,} \quad \vec{E} = \frac{\vec{F}}{Q}$$

The electric field intensity E at a point r (observation point) due a point charge Q located at $\vec{r'}$ (source point) is given by:

$$\vec{E} = \frac{Q(\vec{r} - \vec{r'})}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r'}\right|^3}$$

For a collection of N point charges $Q_1, Q_2, ...Q_N$ located at $\vec{r_1}, \vec{r_2}, ... \vec{r_N}$, the electric field intensity at point $\vec{r'}$ is obtained as

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{Q_k(\vec{r} - \vec{r_i})}{\left|\vec{r} - \vec{r_i}\right|^3}$$
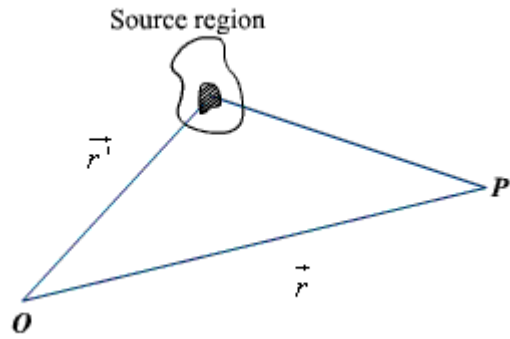
Field due to Different Types of Charges:

Eelectric filed due to a continuous distribution of charges.:

In figure 2.2 we consider a continuous volume distribution of charge $r(t)$ in the region denoted as the source region.

For an elementary charge $dQ = \rho(\vec{r'})dv'$, i.e. considering this charge as point charge, we can write the field expression as:

$$d\vec{E} = \frac{dQ(\vec{r} - \vec{r'})}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r'}\right|^3} = \frac{\rho(\vec{r'})dv'(\vec{r} - \vec{r'})}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r'}\right|^3}$$

Source region

When this expression is integrated over the source region, we get the electric field at the point *P* due to this distribution of charges. Thus the expression for the electric field at *P* can be written as:

$$\overline{E(r)} = \int_V \frac{\rho(\vec{r'})(\vec{r}-\vec{r'})}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r'}\right|^3}dv'$$

Similar technique can be adopted when the charge distribution is in the form of a line charge density or a surface charge density.

$$\overline{E(r)} = \int_L \frac{\rho_L(\vec{r'})(\vec{r}-\vec{r'})}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r'}\right|^3}dl'$$

$$\overline{E(r)} = \int_S \frac{\rho_s(\vec{r'})(\vec{r}-\vec{r'})}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r'}\right|^3}ds'$$

Electric Flux Density:

Electric field intensity or simply 'Electric field' gives the strength of the field at a particular point. The electric field depends on the material media in which the field is being considered. The flux density vector is defined to be independent of the material media (as we'll see that it relates to the charge that is producing it).For a linear isotropic medium under consideration; the flux density vector is defined as:

$$\vec{D} = \varepsilon\vec{E}$$

We define the electric flux Y as

$$\psi = \int_S \vec{D}.d\vec{s}$$

Consider a point charge at the origin:
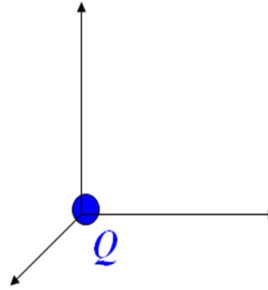
Electric Flux Density of a Point Charge

Assume from symmetry the form of the field

$$\underline{D} = \hat{a}_r \, D_r(r)$$

Construct a family of Gaussian surfaces

$$0 \leq r \leq \infty$$

Evaluate the total charge within the volume   enclosed by each Gaussian surface

$$Q_{encl} = \int_V q_{ev} \, dv$$

For each Gaussian surface, evaluate the integral

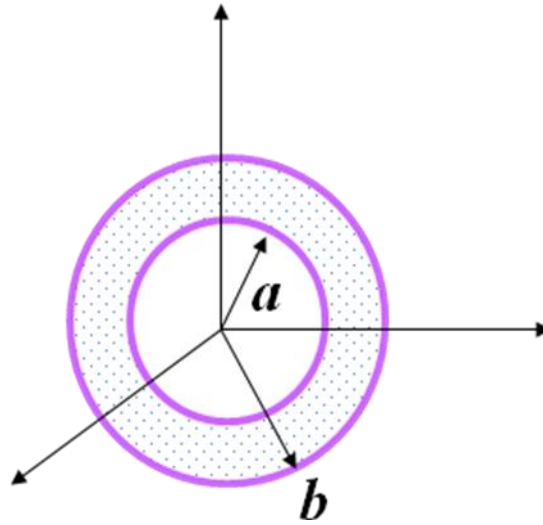$$\oint_S \underline{D} \cdot d\underline{s} = DS$$

$$\oint_S \underline{D} \cdot d\underline{s} = D_r(r) \, 4\pi \, r^2$$

Solve for D on each Gaussian surface

$$D = \frac{Q_{encl}}{S} \quad \boxed{\underline{D} = \hat{a}_r \, \frac{Q}{4\pi \, r^2}} \quad \Rightarrow \quad \underline{E} = \frac{\underline{D}}{\varepsilon_0} = \hat{a}_r \, \frac{Q}{4\pi\varepsilon_0 \, r^2}$$

Consider a spherical shell of uniform charge density:

$$q_{ev} = \begin{cases} q_0, & a \leq r \leq b \\ 0, & \text{otherwise} \end{cases}$$

Gauss's Law:

Gauss's law is one of the fundamental laws of electromagnetism and it states that the total electric flux through a closed surface is equal to the total charge enclosed by the surface.
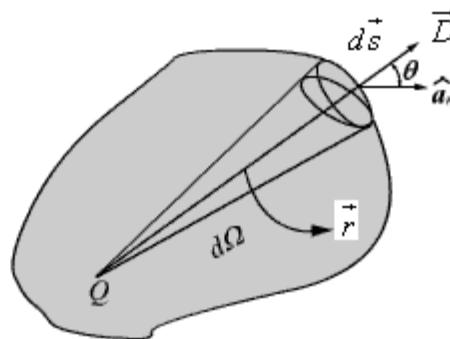

Fig 1: Gauss's Law

Let us consider a point charge Q located in an isotropic homogeneous medium of dielectric constant e. The flux density at a distance r on a surface enclosing the charge is given by

$$\vec{D} = \varepsilon \vec{E} = \frac{Q}{4\pi r^2} \hat{a}_r$$

If we consider an elementary area ds, the amount of flux passing through the elementary area is given by

$$d\psi = \vec{D}.ds = \frac{Q}{4\pi r^2} ds \cos\theta$$

But $\dfrac{ds \cos\theta}{r^2} = d\Omega$, is the elementary solid angle subtended by the area $\vec{ds}$ at the location of Q. Therefore we can write $d\psi = \dfrac{Q}{4\pi} d\Omega$

For a closed surface enclosing the charge, we can write $\psi = \oint_S d\psi = \dfrac{Q}{4\pi} \oint_S d\Omega = Q$

which can seen to be same as what we have stated in the definition of Gauss's Law.

Application of Gauss's Law:

Gauss's law is particularly useful in computing $\vec{E}$ or $\vec{D}$ where the charge distribution has some symmetry. We shall illustrate the application of Gauss's Law with some examples.

1.An infinite line charge

As the first example of illustration of use of Gauss's law, let consider the problem of determination of the electric field produced by an infinite line charge of density r_L C/m. Let us consider a line charge positioned along the *z*-axis as shown in Fig. 2(a). Since the line charge is assumed to be infinitely long, the electric field will be of the form as shown in Fig. 2(b) .

If we consider a close cylindrical surface as shown in Fig. 2.4(a), using Gauss's theorm we can write,

$$\rho_L l = Q = \oint_S \varepsilon_0 \vec{E}.\vec{ds} = \int_{S_1} \varepsilon_0 \vec{E}.\vec{ds} + \int_{S_2} \varepsilon_0 \vec{E}.\vec{ds} + \int_{S_3} \varepsilon_0 \vec{E}.\vec{ds}$$

Considering the fact that the unit normal vector to areas S_1 and S_3 are perpendicular to the electric field, the surface integrals for the top and bottom surfaces evaluates to zero. Hence we can write, $\rho_L l = \varepsilon_0 E.2\pi\rho l$
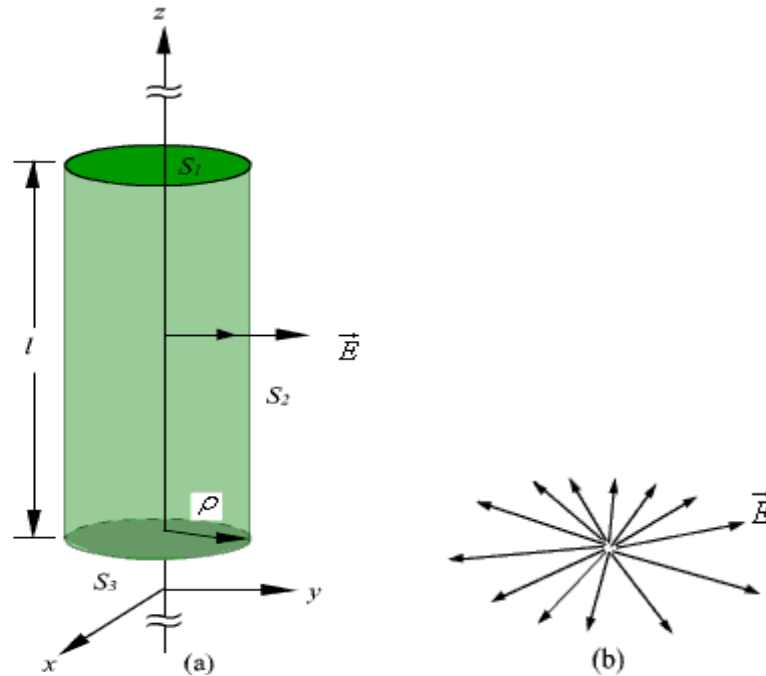
Fig 2: Infinite Line Charge

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\hat{a}_\rho$$

2. Infinite Sheet of Charge

As a second example of application of Gauss's theorem, we consider an infinite charged sheet covering the *x-z* plane as shown in figure 3.

Assuming a surface charge density of ρ_s for the infinite surface charge, if we consider a cylindrical volume having sides Δs placed symmetrically as shown in figure 5, we can write:

$$\oint_s \vec{D}\cdot d\vec{s} = 2D\Delta s = \rho_s \Delta s$$

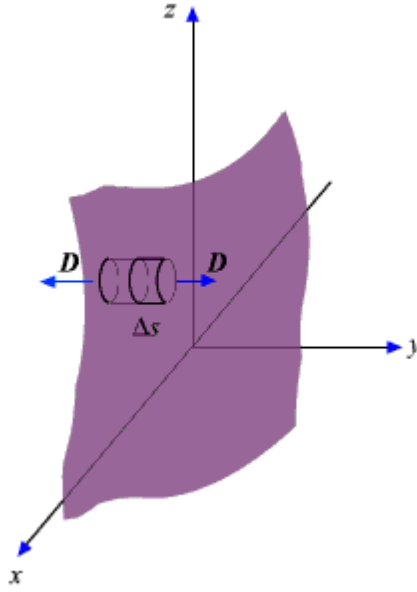$$\therefore \quad \vec{E} = \frac{\rho_s}{2\varepsilon_0}\hat{a}_y$$

Fig 3: Infinite Sheet of Charge

It may be noted that the electric field strength is independent of distance. This is true for the infinite plane of charge; electric lines of force on either side of the charge will be perpendicular to the sheet and extend to infinity as parallel lines. As number of lines of force per unit area gives the strength of the field, the field becomes independent of distance. For a finite charge sheet, the field will be a function of distance.

3. Uniformly Charged Sphere

Let us consider a sphere of radius r_0 having a uniform volume charge density of r_v C/m^3. To determine $\vec{D}$ everywhere, inside and outside the sphere, we construct Gaussian surfaces of radius $r < r_0$ and $r > r_0$ as shown in Fig. 4(a) and Fig. 4(b).

For the region $r \leq r_0$; the total enclosed charge will be
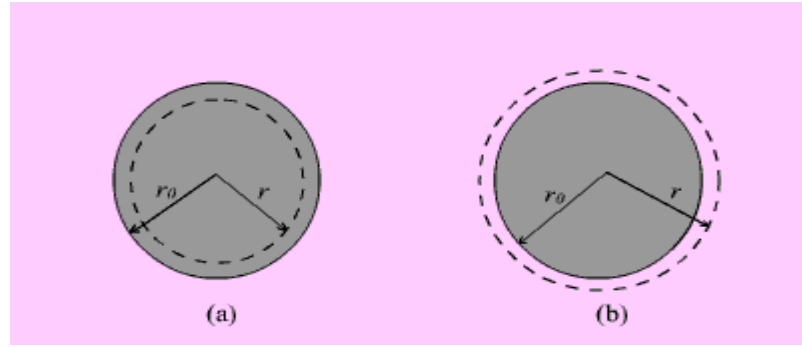
$$Q_{en} = \rho_v \frac{4}{3}\pi r^3$$

Fig 4: Uniformly Charged Sphere

By applying Gauss's theorem,

$$\oint_S \vec{D} \cdot d\vec{s} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} D_r r^2 \sin\theta \, d\theta \, d\phi = 4\pi r^2 D_r = Q_{en}$$

Therefore

$$\vec{D} = \frac{r}{3} \rho_v \hat{a}_r, \qquad 0 \le r \le r_0$$

For the region $r \ge r_0$; the total enclosed charge will be

$$Q_{en} = \rho_v \frac{4}{3} \pi r_0^3$$

By applying Gauss's theorem,

$$\vec{D} = \frac{r_0^3}{3r^2} \rho_v \hat{a}_r, \qquad r \ge r_0$$

Electrostatic Potential:

In the previous sections we have seen how the electric field intensity due to a charge or a charge distribution can be found using Coulomb's law or Gauss's law. Since a charge placed in the vicinity of another charge (or in other words in the field of other charge) experiences a force, the movement of the charge represents energy exchange. Electrostatic potential is related to the work done in carrying a charge from one point to the other in the presence of an electric field.

Let us suppose that we wish to move a positive test charge Δq from a point P to another point Q as shown in the Fig. 1.

The force at any point along its path would cause the particle to accelerate and move it out of the region if unconstrained. Since we are dealing with an electrostatic case, a force equal to the negative of that acting on the charge is to be applied while Δq moves from P to Q. The work done by this external agent in moving the charge by a distance $\vec{dl}$ is given by:
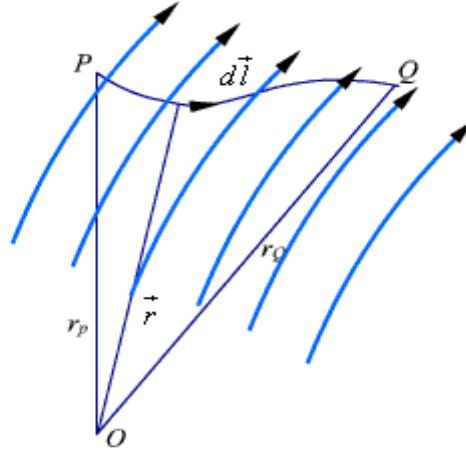

Fig 1: Movement of Test Charge **in Electric Field**

$$dW = -\Delta q\,\vec{E}.\vec{dl}$$

The negative sign accounts for the fact that work is done on the system by the external agent.

$$W = -\Delta q \int_{P}^{Q} \vec{E} \cdot \vec{dl}$$

The potential difference between two points P and Q, V_{PQ}, is defined as the work done per unit charge, i.e.

$$V_{PQ} = \frac{W}{\Delta Q} = -\int_{P}^{Q} \vec{E} \cdot \vec{dl}$$

It may be noted that in moving a charge from the initial point to the final point if the potential difference is positive, there is a gain in potential energy in the movement, external agent performs the work against the field. If the sign of the potential difference is negative, work is done by the field.

We will see that the electrostatic system is conservative in that no net energy is exchanged if the test charge is moved about a closed path, i.e. returning to its initial position. Further, the potential difference between two points in an electrostatic field is a point function; it is independent of the path taken. The potential difference is measured in Joules/Coulomb which is referred to as Volts.

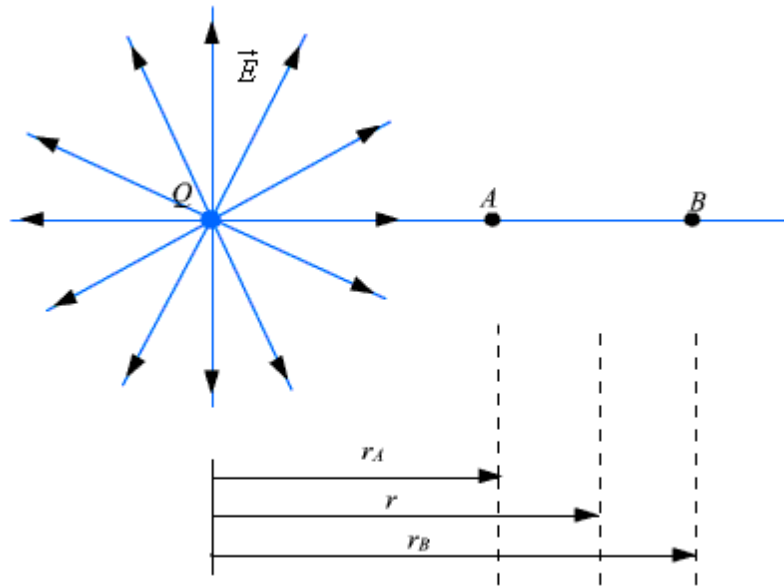Let us consider a point charge Q as shown in the Fig. 2

Fig 2: Electrostatic Potential calculation for a point charge

Further consider the two points A and B as shown in the Fig. 2.9. Considering the movement of a unit positive test charge from B to A, we can write an expression for the potential difference as:

$$V_{BA} = -\int_{B}^{A} \vec{E} \cdot \vec{dl} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 r^2} \hat{a}_r \cdot dr\hat{a}_r = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] = V_A - V_B$$

It is customary to choose the potential to be zero at infinity. Thus potential at any point ($r_A = r$) due to a point charge Q can be written as the amount of work done in bringing a unit positive charge from infinity to that point (i.e. $r_B = 0$).

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

Or, in other words,

$$V = -\int_{\infty}^{r} E.dl$$

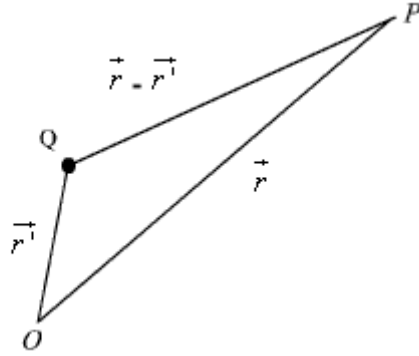Let us now consider a situation where the point charge Q is not located at the origin as shown in Fig. 2.10.

Fig 2.10: Electrostatic Potential due a Displaced Charge

The potential at a point P becomes

$$V(r) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{\left|\vec{r}-\vec{r'}\right|}$$

So far we have considered the potential due to point charges only. As any other type of charge distribution can be considered to be consisting of point charges, the same basic ideas now can be extended to other types of charge distributionalso.

Let us first consider N point charges Q_1, Q_2,.....Q_N located at points with position vectors $\vec{r_1}$, $\vec{r_2}$,...... $\vec{r_N}$. The potential at a point having position vector $\vec{r}$ can be written as:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1}{\left|\vec{r}-\vec{r_1}\right|} + \frac{Q_2}{\left|\vec{r}-\vec{r_2}\right|} + \cdots \frac{Q_N}{\left|\vec{r}-\vec{r_N}\right|}\right)$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=n}^{N}\frac{Q_n}{\left|\vec{r}-\vec{r_n}\right|}$$

Potential Field Due To Different Types of Charges:

For continuous charge distribution, we replace point charges Q_n by corresponding charge elements $\rho_L dl$ or $\rho_S ds$ or $\rho_V dv$ depending on whether the charge distribution is linear, surface or a volume charge distribution and the summation is replaced by an integral. With these modifications we can write:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_L \frac{\rho_L(\vec{r'})dl'}{\left|\vec{r} - \vec{r_n}\right|}$$

For line charge,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\rho_S(\vec{r'})ds'}{\left|\vec{r} - \vec{r_n}\right|}$$

For surface charge,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho_V(\vec{r'})dv'}{\left|\vec{r} - \vec{r_n}\right|}$$

For volume charge,

It may be noted here that the primed coordinates represent the source coordinates and the unprimed coordinates represent field point.

Further, in our discussion so far we have used the reference or zero potential at infinity. If any other point is chosen as reference, we can write:

$$V = \frac{Q}{4\pi\varepsilon_0 r} + C$$

where C is a constant. In the same manner when potential is computed from a known electric field we can write:

$$V = -\int \vec{E} \cdot d\vec{l} + C$$

Potential Gradient: The potential difference is however independent of the choice of reference then

$$V_{AB} = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{l} = \frac{W}{Q}$$

We have mentioned that electrostatic field is a conservative field; the work done in moving a charge from one point to the other is independent of the path. Let us consider moving a charge from point P_1 to P_2 in one path and then from point P_2 back to P_1 over a different path. If the work done on the two paths were different, a net positive or negative amount of work would have been done when the body returns to its original position P_1. In a conservative field there is no mechanism for dissipating energy corresponding to any positive work neither any source is present from which energy could be absorbed in the case of negative work. Hence the question of different works in two paths is untenable; the work must have to be independent of path and depends on the initial and final positions.

Since the potential difference is independent of the paths taken, $V_{AB} = - V_{BA}$, and over a closed path,

$$V_{BA} + V_{AB} = \oint \vec{E} \cdot d\vec{l} = 0$$

Applying Stokes's theorem, we can write:

$$\oint \vec{E} \cdot d\vec{l} = \int_s \left(\nabla \times \vec{E} \right) \cdot d\vec{s} = 0$$

from which it follows that for electrostatic field,

$$\nabla \times \vec{E} = 0$$

Any vector field $\vec{A}$ that satisfies $\nabla \times \vec{A} = 0$ is called an irrotational field.

From our definition of potential, we can write

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial x} dz = -\vec{E} \cdot d\vec{l}$$

$$\left(\frac{\partial V}{\partial x} \hat{a}_x + \frac{\partial V}{\partial y} \hat{a}_y + \frac{\partial V}{\partial z} \hat{a}_z \right) \cdot \left(dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z \right) = -\vec{E} \cdot d\vec{l}$$

$$\nabla V \cdot d\vec{l} = -\vec{E} \cdot d\vec{l}$$

from which we obtain,

$$\vec{E} = -\nabla V$$

From the foregoing discussions we observe that the electric field strength at any point is the negative of the potential gradient at any point, negative sign shows that $\vec{E}$ is directed from higher to lower values of $\vec{V}$. This gives us another method of computing the electric field, i. e. if we know the potential function, the electric field may be computed. We may note here that that one scalar function $\vec{V}$ contain all the information that three

components of $\vec{E}$ carry, the same is possible because of the fact that three components of $\vec{E}$ are interrelated by the relation $\nabla \times \vec{E}$.

Dipole field due to Dipole :

An electric dipole consists of two point charges of equal magnitude but of opposite sign and separated by a small distance.

As shown in figure 1, the dipole is formed by the two point charges Q and $-Q$ separated by a distance d, the charges being placed symmetrically about the origin. Let us consider a point P at a distance r, where we are interested to find the field.
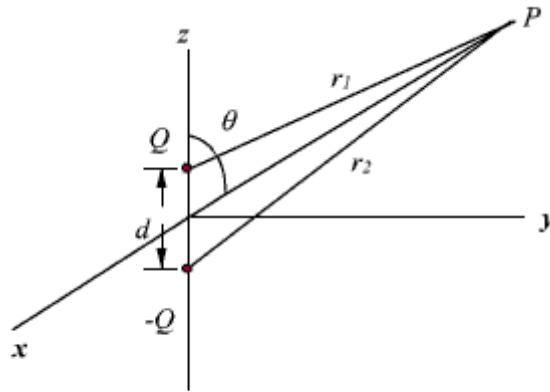

Fig 1: Electric Dipole

The potential at P due to the dipole can be written as:

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r_1} - \frac{Q}{r_2}\right] = \frac{Q}{4\pi\varepsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right]$$

When r_1 and $r_2 >> d$, we can write $r_2 - r_1 = 2\times\frac{d}{2}\cos\theta = d\cos\theta$ and $r_1 \cong r_2 \cong r$.

Therefore,

$$V = \frac{Q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2}$$

We can write,

$$Qd\cos\theta = Qd\hat{a}_z \cdot \hat{a}_r,$$

The quantity $\vec{P} = Q\vec{d}$ is called the dipole moment of the electric dipole.

Hence the expression for the electric potential can now be written as:

$$V = \frac{\vec{P} \cdot \hat{a}_r}{4\pi\varepsilon_0 r^2}$$

It may be noted that while potential of an isolated charge varies with distance as $1/r$ that of an electric dipole varies as $1/r^2$ with distance. If the dipole is not centered at the origin, but the dipole center lies at $\vec{r'}$, the expression for the potential can be written as:

$$V = \frac{\vec{P} \cdot (\vec{r} - \vec{r'})}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r'}\right|^3}$$

The electric field for the dipole centered at the origin can be computed as

$$\vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta\right]$$

$$= \frac{Qd\cos\theta}{2\pi\varepsilon_0 r^3}\hat{a}_r + \frac{Qd\sin\theta}{4\pi\varepsilon_0 r^3}\hat{a}_\theta$$

$$= \frac{Qd}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{a}_r + \sin\theta\hat{a}_\theta\right)$$

$$\vec{E} = \frac{\vec{P}}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{a}_r + \sin\theta\hat{a}_\theta\right)$$

$\vec{P} = Q\vec{d}$ is the magnitude of the dipole moment. Once again we note that the electric field of electric dipole varies as $1/r^3$ where as that of a point charge varies as $1/r^2$.

Energy Density in Electrostatic Field:

Electrostatic Energy and Energy Density

We have stated that the electric potential at a point in an electric field is the amount of work required to bring a unit positive charge from infinity (reference of zero potential) to that point.

To determine the energy that is present in an assembly of charges, let us first determine the amount of work required to assemble them. Let us consider a number of discrete charges Q_1, Q_2,......., Q_N are brought from infinity to their present position one by one.

Since initially there is no field present, the amount of work done in bring Q_1 is zero. Q_2 is brought in the presence of the field of Q_1, the work done $W_1 = Q_2 V_{21}$ where V_{21} is the potential at the location of Q_2 due to Q_1. Proceeding in this manner, we can write, the total work done

$$W = V_{21}Q_2 + (V_{31}Q_3 + V_{32}Q_3) + \dots\dots\dots + (V_{N1}Q_N + \dots\dots + V_{N(N-1)}Q_N)$$

Had the charges been brought in the reverse order,

$$W = (V_{1N}Q_1 + \dots\dots + V_{12}Q_1) + \dots\dots\dots\dots\dots + (V_{(N-2)(N-1)}Q_{N-2} + V_{(N-2)N}Q_{N-2}) + V_{(N-1)N}Q_{N-1}$$

Therefore,

$$2W = (V_{1N} + V_{1(N-1)} + \dots\dots + V_{12})Q_1 + (V_{2N} + V_{2(N-1)} + \dots\dots + V_{23} + V_{21})Q_1 \dots\dots$$
$$\dots\dots\dots\dots\dots + (V_{N1} + \dots\dots V_{N2} + V_{N(N-1)})Q_N$$

Here V_{IJ} represent voltage at the Ith charge location due to J^{th} charge. Therefore,

$$2W = V_1Q_1 + \dots\dots\dots + V_NQ_N = \sum_{I=1}^{N} V_I Q_I$$

Or,
$$W = \frac{1}{2}\sum_{I=1}^{N} V_I Q_I$$

If instead of discrete charges, we now have a distribution of charges over a volume v then we can write,

$$W = \frac{1}{2}\int_v V \rho_v dv$$

where ρ_v is the volume charge density and V represents the potential function.

Since, $\rho_v = \nabla \cdot \vec{D}$, we can write

$$W = \frac{1}{2}\int_V (\nabla \cdot \vec{D})V \, dv$$

Using the vector identity,

$$\nabla.(V\vec{D}) = \vec{D}\cdot\nabla V + V\nabla\cdot\vec{D}$$, we can write

$$W = \frac{1}{2}\int_V \left(\nabla.(V\vec{D}) - \vec{D}\cdot\nabla V\right)dv$$

$$= \frac{1}{2}\oint_S (V\vec{D})d\vec{s} - \frac{1}{2}\int_V (\vec{D}\cdot\nabla V)dv$$

In the expression $\frac{1}{2}\oint_S (V\vec{D})d\vec{s}$, for point charges, since V varies as $\frac{1}{r}$ and D varies as $\frac{1}{r^2}$, the term $V\vec{D}$ varies as $\frac{1}{r^3}$ while the area varies as r^2. Hence the integral term varies at least as $\frac{1}{r}$ and the as surface becomes large (i.e. $r \rightarrow \infty$) the integral term tends to zero.

Thus the equation for W reduces to

$$W = -\frac{1}{2}\int_V (\vec{D}\nabla V)dv = \frac{1}{2}\int_V (\vec{D}.\vec{E})dv = \frac{1}{2}\int_V (\varepsilon E^2)dv = \int_V w_e dv$$

$$w_e = \frac{1}{2}\varepsilon E^2$$, is called the energy density in the electrostatic field

Illustrative Problems:

1. Consider a vector function given by $D = 2z\rho\hat{a}_\rho$. **Find** $\oint_s D.dS$ **and** $\int_v \nabla.D dV$ **over the region defined by** $0 \leq \rho \leq 1$, $0 \leq z \leq 1$ **and** $0 \leq \phi \leq 2\pi$. **Is the divergence theorem satisfied?**

Solution:

Given $D = 2z\rho\hat{a}_\rho$.

As shown in Figure 1 (next slide) , the closed cylindrical surface can be split into three surfaces S_1 S_2 and S_3

$$\therefore \oint D.dS = \oint_{S_1} D.dS_1 + \oint_{S_2} D.dS_2 + \oint_{S_3} D.dS_3$$

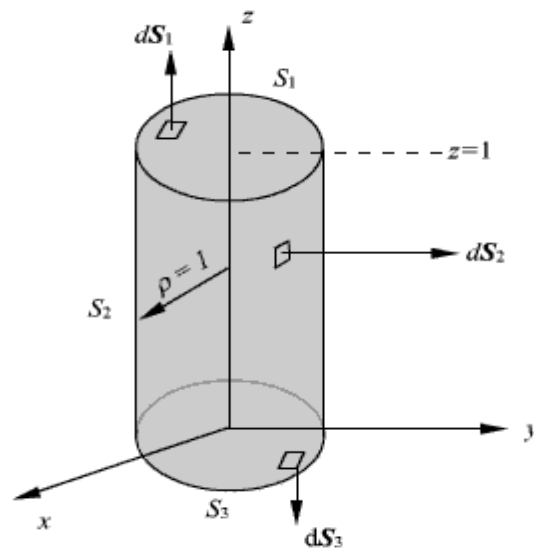**Figure 1**

Since the unit vector perpendicular to S_1 & S_3 are directed along $\hat{a}_3$ & - $\hat{a}_3$, the contribution from the 1 st and 3 rd integrals are zero.

$$\oint D.dS = \int_0^{2\pi} \int_0^1 \left(2z\,\hat{a}_\rho\right).\left(1dzd\,\phi\hat{a}_\rho\right) \quad [\text{Since } \rho = 1]$$

$$= \int_0^{2\pi} d\phi \int_0^1 2zdz$$

$$= 2\pi \left[\frac{2z^2}{2}\right]_0^1 = 2\pi$$

$D = 2z\rho\hat{a}_\rho$ has only $\hat{a}_\rho$ component.

$$\therefore \nabla . \mathbf{D} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho 2z\rho) = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left[2z\rho^2 \right] = 4z$$

$$\therefore \int_V \nabla . \mathbf{D} \, dV = \int_0^1 \int_0^{2\pi} \int_0^1 4z \rho \, d\rho \, d\phi \, dz$$

$$= \int_0^1 4z \, dz \int_0^{2\pi} d\phi \int_0^1 \rho \, d\rho$$

$$= \left[\frac{4z^2}{2} \right]_0^1 \times 2\pi \times \left[\frac{\rho^2}{2} \right]_0^1$$

$$= 2 \times 2\pi \times \frac{1}{2} = 2\pi$$

Since $\oint_S \mathbf{D}.d\mathbf{S} = \int_V \nabla . \mathbf{D} \, dV$, the divergence theorem is satisfied.

2. Given $\mathbf{A} = \rho \cos \phi \hat{a}_\rho + \rho^2 \hat{a}_z$. **Compute** $\nabla \times \mathbf{A}$ **and** $\int_S \nabla \times \mathbf{A}.d\mathbf{S}$ **over the area S as shown in the figure 2 below.**
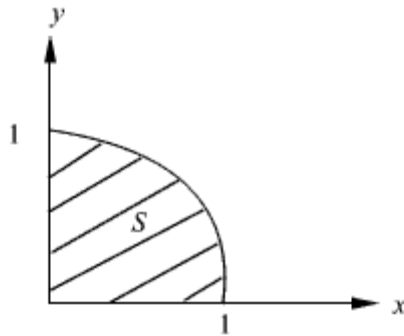

Figure 2

Solution:

$$\mathbf{A} = \rho \cos \phi \hat{a}_\rho + \rho^2 \hat{a}_z$$

In cylindrical coordinate; the curl of a vector *A* is given by

$$\nabla \times A = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_3 \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ \rho\cos\phi & 0 & \rho^2 \end{vmatrix} = \frac{1}{\rho}\left[-\rho\hat{a}_\phi(2\rho) + \hat{a}_z\,\rho\sin\phi\right]$$

$$dS = \rho\, d\rho\, d\phi\, \hat{a}_z$$

$$\therefore \nabla \times A.dS = \rho\sin\phi\, d\rho\, d\phi$$

$$\therefore \int_S \nabla \times A.dS = \int_0^1 \rho\, d\rho \int_0^{\pi/2} \sin\phi\, d\phi = \frac{1}{2}$$

Convection and Conduction Currents: -

- Current (in amperes) through a given area is the electric charge passing through the area per unit time

$$\text{Current } I = \frac{dQ}{dt}$$

- Current density is the amount of current flowing through a surface, A/m^2, or the current through a unit normal area at that point
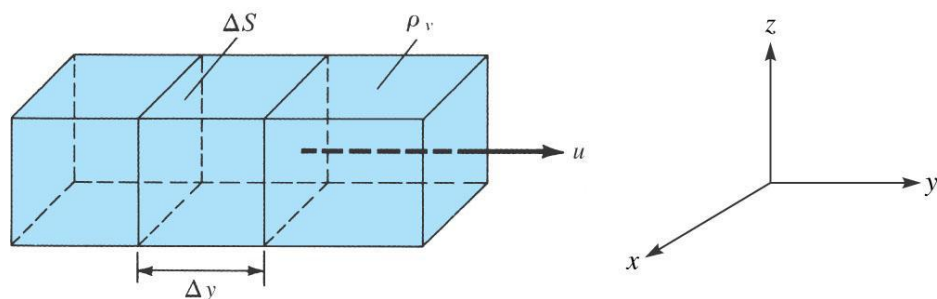
- Current density

$$J = \frac{\Delta I}{\Delta S}$$

- Where

$$I = \int_S J.dS$$

- Depending on how the current is produced, there are different types of current density

  ✓ **Convection current density**

  ✓ **Conduction current density**

  ✓ **Displacement current density**

  - Current generated by a magnetic field

Convection current density

- Does not involve conductors and does not obey Ohm's law

- Occurs when current flows through an insulating medium such as liquid, gas, or vacuum

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \Delta S \frac{\Delta y}{\Delta t} = \rho_v \Delta S v_y$$

- Where v is the velocity vector of the fluid

$$J_y = \frac{\Delta I}{\Delta S} = \rho_v v_y$$

Conduction current density

- Current in a conductor

- Obeys Ohm's law

- Consider a large number of free electrons travelling in a metal with mass (m), velocity (v), and scattering time (time between electron collisions), τ

$$F = -qE = \frac{mv}{\tau}$$

- The carrier density is determined by the number of electrons, n, with charge, e

$$\rho_v = -ne$$

- Conduction current density can then be calculated as

$$J = \rho_v v = \frac{ne^2\tau}{m} E = \sigma E$$

- Where σ is the conductivity of the conductor. This relationship between current concentration and electric field is known as Ohm's Law.

Continuity Equation and Relaxation Time: -

- Due to the principle of charge conservation, the time rate of decrease of charge within a given volume must be equal to the net outward current flow through the closed surface of the volume.

- Thus, the current coming out of the closed surface is

$$I_{out} = \int_S J.dS = -\frac{dQ_{in}}{dt}$$

- Where Qin is the total charge enclosed by the closed surface. Invoking divergence theorem

$$\oint_S J.dS = \int_v \nabla.J dv =$$

- But,

$$-\frac{dQ_{in}}{dt} = -\frac{d}{dt}\int_v \rho_v dv = -\int_v \frac{d\rho_v}{dt} dv$$

- From the above three equations, we can write as

$$\int_v \nabla.J dv = -\int_v \frac{\partial \rho_v}{\partial t} dv$$

$$\nabla.J = \frac{\partial \rho_v}{\partial t}$$

- which is called the **continuity of current equation**.

- The continuity equation is derived from the principle of conservation of charge and essentially states that there can be no accumulation of charge at any point. For steady currents, $\frac{\partial \rho_v}{\partial t} = 0$ and hence $\nabla.J = 0$ showing that the total charge leaving a volume is the same as the total charge entering it.

Relaxation time

- Utilizing the continuity equation and material properties such as permittivity and conductivity, one can derive a time constant (in seconds) by which to measure the relaxation time associated with the decay of charge from the point at which it was introduced within a material to the surface of that material.

- We start with Ohm's and Gauss' Laws

$$J = \sigma E$$

$$\nabla.E = \frac{\rho_v}{\varepsilon}$$

$$\nabla.J = \nabla.\sigma E = \frac{\sigma \rho_v}{\varepsilon} = -\frac{\partial \rho_v}{\partial t}$$

$$\frac{\sigma \rho_v}{\varepsilon} + \frac{\partial \rho_v}{\partial t} = 0$$

$$\frac{\partial \rho_v}{\rho_v} = -\frac{\sigma \partial t}{\varepsilon}$$

$$\ln \rho_v = -\frac{\sigma t}{\varepsilon} + \ln \rho_{v0}$$

$$\rho_v = \rho_{v0} e^{-\frac{t}{T_r}}$$

$$T_r = \frac{\varepsilon}{\sigma}$$

- ρ_{v0} is the initial charge density. The relaxation time(T_r) is the time it takes a charge placed in the interior of a material to drop by e^{-1} (=36.8%) of its initial value.

- For good conductors T_r is approx. $2*10^{-19}$ s.

- For good insulators T_r can be days

Poisson's and Laplace's Equations: -

- We have determined the electric field E in a region using Coulomb's law or Gauss law when the charge distribution is specified in the region or using the relation $E = -\nabla V$ when the potential V is specified throughout the region.

- However, in practical cases, neither the charge distribution nor the potential distribution is specified only at some boundaries. These types of problems are known as electrostatic boundary value problems.

- For these type of problems, the field and the potential V are determined by using Poisson's equation or Laplace's equation.

- Laplace's equation is the special case of Poisson's equation.

- For the Linear materials Poisson's and Laplace's equation can be easily derived from Gauss's equation

$$\nabla \cdot D = \rho_v$$

- But,

$$D = \varepsilon E$$

- Putting the value of D in Gauss Law,

$$\nabla \cdot \varepsilon E = \rho_v$$

- From homogeneous medium for which ε is a constant, we write

$$\nabla \cdot E = \frac{\rho_v}{\varepsilon}$$

- Also, $E = -\nabla V$
- Then the previous equation becomes,

$$\nabla \cdot -\nabla V = \frac{\rho_v}{\varepsilon}$$

or

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

- This equation is known as Poisson's equation which state that the potential distribution in a region depend on the local charge distribution.
- In many boundary value problems, the charge distribution is involved on the surface of the conductor for which the free volume charge density is zero, i.e., $\rho_v \varrho = 0$. In that case, Poisson's equation reduces to,

$$\nabla^2 V = 0$$

- This equation is known as Laplace's equation.

Laplace Equation in Three Coordinate System
In Cartesian coordinates:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

In cylindrical coordinates

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

In spherical coordinates:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Application of Laplace's and Poisson's Equation

- Using Laplace or Poisson's equation we can obtain:

- Potential at any point in between two surfaces when potential at two surfaces are given.

- We can also obtain capacitance between these two surfaces.

General Procedure for solving Laplace or Poisson Equation:

- Solve Laplace or Poisson equations for V by (a) direct substitution for single variable or (b) by method of separation of variables for more than one variable. The solution at this point is not unique because of the integration constants

- Apply the boundary conditions to determine the integration constants giving a unique solution for V. Having found V, find $E = -\nabla V$ and $D = \varepsilon E$.

- If desired find the charge Q induced on a conductor surface using $Q = \int_s \rho_s dS = \varepsilon E$ and $\rho_s = D_n$ where

  D_n is the component of D normal to the conductor. If necessary the capacitance between two conductors can be found using $C = \frac{Q}{V}$.

Capacitance: -

- Capacitance is an intuitive characterization of a capacitor. It tells you, how much charge a capacitor can hold for a given voltage.
- The property of a capacitor to 'store electricity' may be called its capacitance.
- A capacitor essentially consists of two conducting surfaces separated by a layer of an insulating medium called dielectric. The conducting surfaces may be in the form of either circular (or rectangular) plates or be of spherical or cylindrical shape.
- Generally speaking, to have a capacitor we must have two (or more) conductors carrying equal but opposite charges.
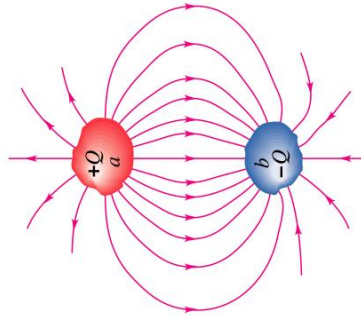

Figure 1. Charge carriers of conductor with opposite polarity.

- This implies that all the flux lines leaving one conductor must necessarily terminate at the surface of the other conductor. The conductors are sometimes referred to as the plates of the capacitor. The plates may be separated by free space or a dielectric.
- Suppose we give Q coulomb of charge to one of the two plates of capacitor, the potential difference V is established between the two plates, then its capacitance is

$$C = \frac{Q}{V_{ab}}$$

- The capacitance C is a physical property of the capacitor and in measured in farads (F).

- The charge Q on the surface of the plate and the potential difference V_{ab} between the plates can be represented in terms of electric field

$$\oint \varepsilon E.\ dS = Q$$

$$V_{ab} = V_a - V_b = \int_a^b E.\ dl$$

- Therefore, the capacitance C can be written as

$$C = \frac{Q}{V_{ab}} = \frac{\oint \varepsilon E.\ dS}{\int_a^b E.\ dl}$$

- From the above expression, Capacitance can be obtained for any given two-conductor capacitance by following either of these methods:
- Assuming Q and determining V in terms of Q (involving Gauss's law)
- Assuming V and determining Q in terms of V (involving solving Laplace's equation)

- In this lecture we will discuss the former method and capacitance can be determined using later method in the following lectures.
- The following steps used for finding the capacitance by using former method.

    ✓ Choose a suitable coordinate system.

    ✓ Let the two conducting plates carry charges + Q and — Q.

    ✓ Determine E using Coulomb's or Gauss's law and find V from $\int E.\,dl$.

    ✓ Finally, obtain C from $C = \dfrac{Q}{V}$

- Now, we will apply this procedure to determine the capacitance of some important two-conductor configurations.

Parallel-Plate Capacitor: -

- A parallel-plate capacitor consisting of two plates each of area A m^2 separated by a thickness d meters of a medium of relative permittivity ε_r is shown in Figure 2. Assume the charge of + Q coulomb and - Q coulomb is distributed on top and bottom plates, respectively. So that the charge density is given by
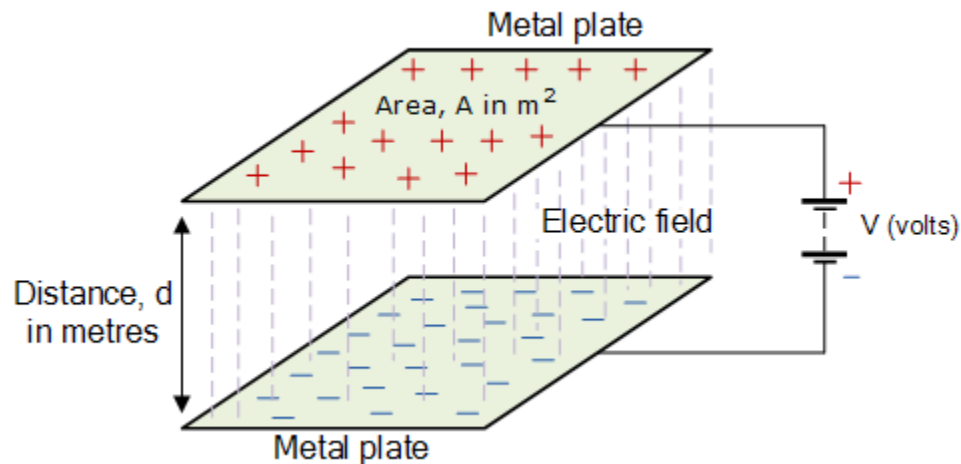
$$\rho_S = \frac{Q}{A}$$

Figure 2. Parallel plate conductors.

- If the space between the plates is filled with a homogeneous dielectric with permittivity ε_r and we ignore flux fringing at the edges of the plates, then the flux passing through the medium is $\Psi = Q$ coulomb and flux density is given by the medium is

$$\psi = \int_S D_n.dA = Q = \int_S \rho_S.dA$$

$$D_n = \rho_S$$

- But, we know

$$D = \varepsilon E$$

- From the above relation, we can write the charge density in terms of electric field as

$$E = \frac{\rho_S}{\varepsilon}$$

- Where, $\rho_S = \frac{Q}{A}$, then the above equation modifies to

$$E = \frac{Q}{A\varepsilon}$$

- Also, from the relation between electric field and electric potential, we write

$$V = \int_0^d E.\,dl = Ed$$

$$V = \frac{Q}{A\varepsilon}d$$

- Thus, the parallel plate capacitor $C = \frac{Q}{V}$ can be written as

$$C = \frac{Q}{\dfrac{Q}{A\varepsilon}d} = \frac{A\varepsilon}{d}$$

- This formula offers a means of measuring the dielectric constant ε_r of a given dielectric. By measuring the capacitance C of a parallel-plate capacitor with the space between the plates filled with the dielectric and the capacitance Co with air between the plates, we find ε_r from

$$\varepsilon_r = \frac{C}{C_0}$$

Coaxial Capacitor: -

- This is essentially a coaxial cable or coaxial cylindrical capacitor. Consider length l of two coaxial conductors of inner radius a and outer radius b $(b > a)$ as shown in Figure 3.
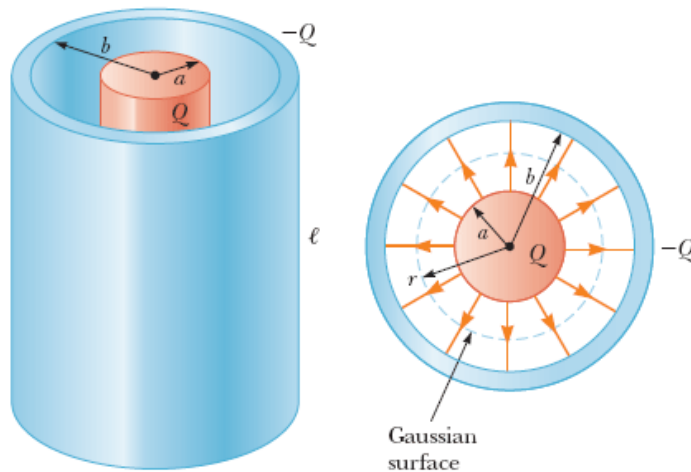

Figure 3: Cylindrical conductors.

- Let the space between the conductors be filled with a homogeneous dielectric with permittivity ε. We assume that inner and outer conductors, respectively, carry +Q and -Q uniformly distributed on them. By applying Gauss's law to an arbitrary Gaussian cylindrical surface of radius ρ $(a < \rho < b)$, we obtain

$$Q = \Psi = \oint_S D.dA = \oint_S \varepsilon E.dA$$

$$Q = \oint_S \varepsilon E.dA = \varepsilon E \int_0^{2\pi} \rho d\phi \int_0^l dl$$

$$Q = \varepsilon E (2\pi\rho l)$$

- Neglecting flux fringing at the cylinder ends, the potential difference between the inner and outer conductors can be written as

$$V = -\int_-^+ E.dr = -\frac{Q}{2\pi\varepsilon l} \int_b^a \frac{d\rho}{\rho}$$

$$= -\frac{Q}{2\pi\varepsilon l} \left[\ln(\rho)\right]_b^a$$

$$V = -\frac{Q}{2\pi\varepsilon l} \left[\ln(\rho)\right]_b^a$$

$$V = -\frac{Q}{2\pi\varepsilon l} \left[\ln(a) - \ln(b)\right] = \frac{Q}{2\pi\varepsilon l} \left[\ln(b) - \ln(a)\right]$$

$$V = \frac{Q}{2\pi\varepsilon l} \ln\left(\frac{b}{a}\right)$$

- Thus, the capacitance of a coaxial cylinder is given by

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{2\pi\varepsilon l} \ln\left(\frac{b}{a}\right)} = \frac{2\pi\varepsilon l}{\ln\left(\frac{b}{a}\right)}$$
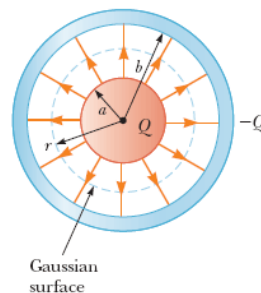

Gaussian
surface

Figure 4. Spherical conductor

Spherical Capacitor:-

- This is the case of two concentric spherical conductors. Consider the inner sphere of radius *a* and outer sphere of radius *b(b> a)* separated by a dielectric medium with permittivity ε as shown in Figure 4.

- We assume charges +*Q* and -*Q* on the inner and outer spheres, respectively. By applying Gauss's law to an arbitrary Gaussian spherical surface of radius *r(a<r<b)*, we obtain

$$Q = \Psi = \oint_S D.dA = \oint_S \varepsilon E.dA$$

$$Q = \oint_S \varepsilon E.dA = \varepsilon E r^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta$$

$$Q = \varepsilon E(4\pi r^2)$$

- Therefore,

$$E = \frac{Q}{4\pi\varepsilon r^2}$$

- The potential difference between the inner and outer sphere can be written as

$$V = -\int_{-}^{+} E.dl = -\frac{Q}{4\pi\varepsilon} \int_b^a \frac{dr}{r^2}$$

$$= -\frac{Q}{4\pi\varepsilon} \left[-\frac{1}{r} \right]_b^a$$

$$V = \frac{Q}{4\pi\varepsilon} \left(\frac{1}{a} - \frac{1}{b} \right)$$

- Thus, the capacitance of a spherical capacitor is given by

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi\varepsilon} \left(\frac{1}{a} - \frac{1}{b} \right)} = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}$$

- By letting b→ ∞, $C \rightarrow 4\pi\varepsilon a$ which is the capacitance of a spherical capacitor whose outer plate is infinitely large. Such is the case of a spherical conductor at a large distance from other conducting bodies—the isolated sphere.

Poisson's and Laplace's Equations: -

- The method of images, introduced by Lord Kelvin in 1848, is commonly used to determine *V,* E, D, and ρ_s due to charges in the presence of conductors.
- Method of images replaces the original boundary by an appropriate image charges. These equivalent charges are at the image positions of the original charges, and are called image charges, and this method is called the method of images
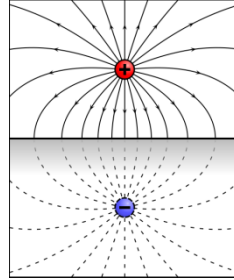

Figure 1. Point charge and its Image charge.

- By this method, we avoid solving Poisson's or Laplace's equation but rather utilize the fact that a conducting surface is an equipotential.
- Although the method does not apply to all electrostatic problems, it can reduce a formidable problem to a simple one.
- In applying the image method, two conditions must always be satisfied
- The image charge(s) must be located in the conducting region.
- The image charge(s) must be located such that on the conducting surface(s) the potential is zero or constant.
- The first condition is necessary to satisfy Poisson's equation, and the second condition ensures that the boundary conditions are satisfied. Let us now apply the image theory to two specific problems.

A Point Charge Above a Grounded Conducting Plane: -

- Consider a point charge *Q* placed at a distance *d* from a perfect conducting plane of infinite extent as in Figure.
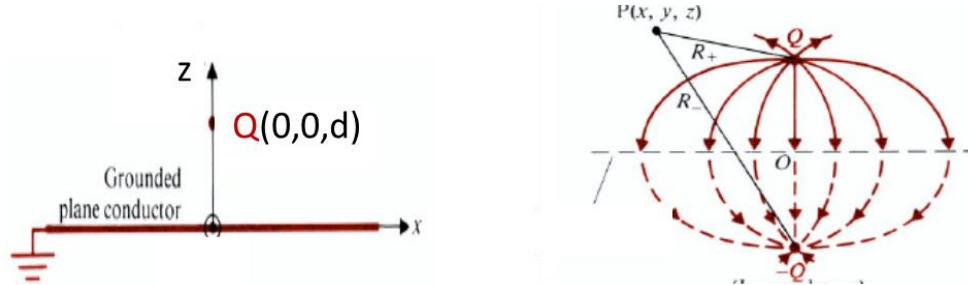

Figure 2. Point charge at a distance d on z-axis

- The electric field at point *P (x, y, z)* is given by

$$E = E_+ + E_-$$

$$E = \frac{QR_+}{4\pi\varepsilon_0 R_+^{\,3}} + \frac{-QR_-}{4\pi\varepsilon_0 R_-^{\,3}}$$

- The distance vectors R_+ and R_- are given by

$$R_+ = (x, y, z) - (0,0,d) = (x, y, z-d)$$
$$R_- = (x, y, z) - (0,0,-d) = (x, y, z+d)$$

- So, the electric field becomes

$$E = \frac{Q}{4\pi\varepsilon_0} \left[\frac{xa_x + ya_y + (z-d)a_z}{\left[x^2 + y^2 + (z-d)^2 \right]^{\frac{3}{2}}} - \frac{xa_x + ya_y + (z+d)a_z}{\left[x^2 + y^2 + (z+d)^2 \right]^{\frac{3}{2}}} \right]$$

- It should be noted that when $z = 0$, E has only the z-component, confirming that E is normal to the conducting surface.

- The potential at P is easily obtained from using $V = -\int_-^+ E.dl$

- Thus

$$V = V_+ + V_-$$

$$V = \frac{Q}{4\pi\varepsilon_0 R_+} + \frac{-Q}{4\pi\varepsilon_0 R_-}$$

$$V = \frac{Q}{4\pi\varepsilon_0} \left[\frac{1}{\left[x^2 + y^2 + (z-d)^2 \right]^{\frac{1}{2}}} - \frac{1}{\left[x^2 + y^2 + (z+d)^2 \right]^{\frac{1}{2}}} \right]$$

- The surface charge density of the induced charge can also be obtained from E as

$$\rho_S = D_n = \varepsilon_0 E\big|_{z=0}$$

$$= \frac{-Qd}{2\pi} \left[\frac{1}{\left[x^2 + y^2 + d^2 \right]^{\frac{3}{2}}} \right]$$

- The total induced charge on the conducting plane is

$$Q_i = \int \rho_S dS = \frac{-Qd}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\frac{dxdy}{\left[x^2 + y^2 + d^2 \right]^{\frac{3}{2}}} \right]$$

- By changing variables, $x^2 + y^2 = \rho^2 \quad dxdy = \rho d\rho d\phi$

$$Q_i = \frac{-Qd}{2\pi\varepsilon_0} \int_0^{2\pi} \int_0^{\infty} \frac{d\phi d\rho}{\left[\rho^2 + d^2 \right]^{\frac{3}{2}}}$$

- Above integration will give

$$Q_i = -Q$$

- as expected, because all flux lines terminating on the conductor would have terminated on the image charge if the conductor were absent.

A Line Charge above a Grounded Conducting Plane: -

- Consider an infinite charge with density ρ_l C/m located at a distance d from the grounded conducting plane $z = 0$. The infinite line charge ρ_l may be assumed to be at $x = 0$, $z = d$ and the image charge density $- \rho_l$ at $x = 0$, $z = -d$ so that the two are parallel to the y-axis.

- The electric field at point P (x, y, z) is given by

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-$$

$$E = \frac{\rho_l}{4\pi\varepsilon_0 \rho_1} + \frac{\rho_l}{4\pi\varepsilon_0 \rho_2}$$

- The distance vectors ρ_1 and ρ_2 are given by

$$\rho_1 = (x, y, z) - (0, y, d) = (x, 0, z - d)$$
$$\rho_2 = (x, y, z) - (0, y, -d) = (x, 0, z + d)$$

- So, the electric field becomes

$$E = \frac{\rho_l}{2\pi\varepsilon_0} \left[\frac{xa_x + (z-d)a_z}{x^2 + (z-d)^2} - \frac{xa_x + (z+d)a_z}{x^2 + (z+d)^2} \right]$$

- It should be noted that when $z = 0$, E has only the y-component, confirming that E is normal to the conducting surface.

- The potential at P is easily obtained from using $V = -\int_-^+ E.dl$

- Thus

$$V = V_+ + V_-$$

$$V = -\frac{\rho_l}{2\pi\varepsilon_0} \ln \rho_1 - \frac{-\rho_l}{2\pi\varepsilon_0} \ln \rho_1$$

$$V = -\frac{\rho_l}{2\pi\varepsilon_0} \ln \frac{\rho_1}{\rho_2}$$

- Substituting $\rho_1 = |\rho_1|$ and $\rho_2 = |\rho_2|$, the above equation modifies to

$$V = -\frac{\rho_l}{2\pi\varepsilon_0} \ln \left[\frac{x^2 + (z-d)^2}{x^2 + (z+d)^2} \right]^{\frac{1}{2}}$$

- The surface charge density of the induced charge can also be obtained from E as

- $\rho_S = D_n = \varepsilon_0 E\big|_{z=0}$

$$= -\frac{\rho_l d}{\pi(x^2 + d^2)}$$

- The total induced charge on the conducting plane (z=0) is

$$\rho_i = \int \rho_S dx = -\frac{\rho_l d}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{x^2 + d^2}$$

- By letting, $x = d \tan\alpha$, the above equation becomes

$$\rho_i = -\frac{\rho_l d}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\alpha}{d}$$

- Above integration will give

$$\rho_i = -\rho_l$$

as expected.

Magnetic Scalar and Vector Potentials:

In studying electric field problems, we introduced the concept of electric potential that simplified the computation of electric fields for certain types of problems. In the same manner let us relate the magnetic field intensity to a scalar magnetic potential and write:

$$\vec{H} = -\nabla V_m$$ ....(1)

From Ampere's law , we know that

$$\nabla \times \vec{H} = \vec{J}$$ ....(2)

Therefore, $$\nabla \times \left(-\nabla V_m\right) = \vec{J}$$ ....(3)

But using vector identity, $\nabla \times \left(\nabla V\right) = 0$ we find that $\vec{H} = -\nabla V_m$ is valid only where $\vec{J} = 0$. Thus the scalar magnetic potential is defined only in the region where $\vec{J} = 0$. Moreover, V_m in general is not a single valued function of position.

This point can be illustrated as follows. Let us consider the cross section of a coaxial line as shown in fig 1.

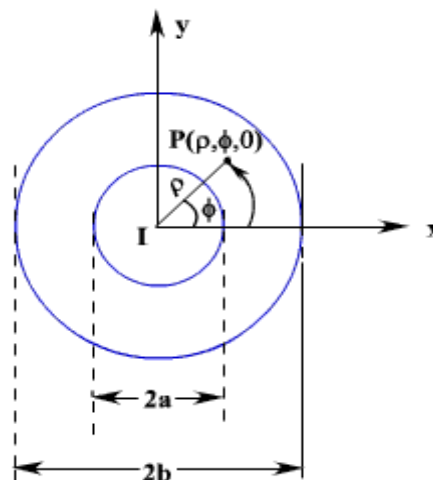In the region $a < \rho < b$, $\vec{J} = 0$ and $\vec{H} = \dfrac{I}{2\pi\rho}\hat{a}_\phi$

If V_m is the magnetic potential then,

$$-\nabla V_m = -\frac{1}{\rho}\frac{\partial V_m}{\partial \phi}$$

$$= \frac{I}{2\pi\rho}$$

$$\therefore V_m = -\frac{I}{2\pi}\phi + c$$

If we set $V_m = 0$ at $\phi = 0$ then c=0 and $V_m = -\frac{I}{2\pi}\phi$

$$\therefore \text{At } \phi = \phi_0 \qquad V_m = -\frac{I}{2\pi}\phi_0$$

We observe that as we make a complete lap around the current carrying conductor , we reach ϕ_0 again but V_m this time becomes

$$V_m = -\frac{I}{2\pi}(\phi_0 + 2\pi)$$

We observe that value of V_m keeps changing as we complete additional laps to pass through the same point. We introduced V_m analogous to electostatic potential V. But for static electric fields, $\nabla \times \vec{E} = 0$ and $\oint \vec{E} \cdot \vec{dl} = 0$, whereas for steady magnetic field $\nabla \times \vec{H} = 0$ wherever $\vec{J} = 0$ but $\oint \vec{H} \cdot \vec{dl} = I$ even if $\vec{J} = 0$ along the path of integration.

We now introduce the vector magnetic potential which can be used in regions where current density may be zero or nonzero and the same can be easily extended to time varying cases. The use of vector magnetic potential provides elegant ways of solving EM field problems.

Since $\nabla \cdot \vec{B} = 0$ and we have the vector identity that for any vector $\vec{A}$, $\nabla \cdot (\nabla \times \vec{A}) = 0$, we can write $\vec{B} = \nabla \times \vec{A}$.

Here, the vector field $\vec{A}$ is called the vector magnetic potential. Its SI unit is Wb/m. Thus if can find $\vec{A}$ of a given current distribution, $\vec{B}$ can be found from $\vec{A}$ through a curl operation.

We have introduced the vector function $\vec{A}$ and related its curl to $\vec{B}$. A vector function is defined fully in terms of its curl as well as divergence. The choice of $\nabla \cdot \vec{A}$ is made as follows.

$$\nabla \times \nabla \times \vec{A} = \mu \nabla \times \vec{H} = \mu \vec{J} \qquad ....(4)$$

By using vector identity, $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$ ....(5)

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu \vec{J} \qquad ....(6)$$

Great deal of simplification can be achieved if we choose $\nabla \cdot \vec{A} = 0$. Putting $\nabla \cdot \vec{A} = 0$, we get $\nabla^2 \vec{A} = -\mu \vec{J}$ which is vector poisson equation. In Cartesian coordinates, the above equation can be written in terms of the components as

$$\nabla^2 A_x = -\mu J_x \qquad ....(7a)$$

$$\nabla^2 A_y = -\mu J_y \qquad ....(7b)$$

$$\nabla^2 A_z = -\mu J_z \qquad ....(7c)$$

The form of all the above equation is same as that of

$$\nabla^2 V = -\frac{\rho}{\varepsilon} \qquad ....(8)$$

for which the solution is

$$V = \frac{1}{4\pi\varepsilon} \int_v \frac{\rho}{R} dv', \qquad R = |\vec{r} - \vec{r}'| \qquad ....(9)$$

In case of time varying fields we shall see that $\nabla \cdot \vec{A} = \mu\varepsilon \frac{\partial V}{\partial t}$, which is known as Lorentz condition, V being the electric potential. Here we are dealing with static magnetic field, so $\nabla \cdot \vec{A} = 0$.

By comparison, we can write the solution for Ax as

$$A_x = \frac{\mu}{4\pi} \int_v \frac{J_x}{R} dv' \qquad ....(10)$$

Computing similar solutions for other two components of the vector potential, the vector potential can be written as

$$\vec{A} = \frac{\mu}{4\pi} \int_{v'} \frac{\vec{J}}{R} dv'$$

....(11)

This equation enables us to find the vector potential at a given point because of a volume current density $\vec{J}$. Similarly for line or surface current density we can write

$$\vec{A} = \frac{\mu}{4\pi} \int_{L'} \frac{I}{R} d\vec{l}'$$

....(12)

$$\vec{A} = \frac{\mu}{4\pi} \int_{S'} \frac{\vec{K}}{R} ds'$$

respectively. ....(13)

The magnetic flux ψ through a given area S is given by

$$\psi = \int_{S} \vec{B}.d\vec{s}$$

....(14)

Substituting $\vec{B} = \nabla \times \vec{A}$

$$\psi = \int_{S} \nabla \times \vec{A}.d\vec{s} = \oint_{C} \vec{A}.d\vec{l}$$

....(15)

Vector potential thus have the physical significance that its integral around any closed path is equal to the magnetic flux passing through that path.

Forces due to Magnetic Fields, Ampere's Force Law:

**Magnetic field due to a solenoid and a toroid:**

A solenoid is essentially a long current loop with closely packed circular turns. The length of the solenoid is very large compared to the diameter of the turns. In diagram shown below, current enters the page from the top and goes behing the page at the bottom. It leads to addition of the magnetic field due to the turns inside the solenoid and cancellation outside.
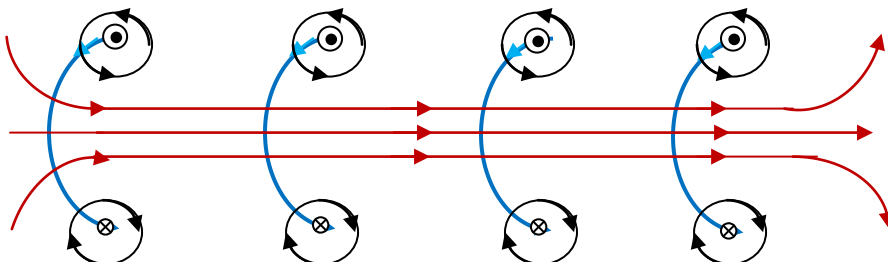
The magnetic field can be calculated by Ampere's law and it gives,

Field inside = $\mu_0 n I$, where n is the number of turns per unit length of the solenoid. The field outside thec solenoid is zero. The same result can be obtained by use of Biot Savart's law, which is left as an exercise.

A toroid looks like a doughnut with a central hole and current carrying wires wound over its core. The core is usually made of iron or some such magnetic metal.
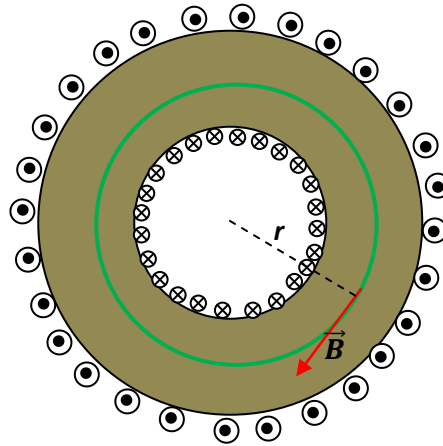
Take a circular loop of radius r concentric with the toroid. The amount of current enclosed by the loop is NI where N is the total number of turns and I is the current in each turn. By Ampere's law, the field is given by

$$2\pi R B = \mu_0 N I$$
$$\vec{B} = \frac{\mu_0 N I}{2r} \hat{\theta}$$

where in the last step we have explicitly shown that the field is cuircumferential. The field outside the toroid is zero because the Amperian loop would enclose zero current as current both goes in and comes out through the loop. Likewise, inside the hole the field would also be zero as it would not thread any current. Note that unlike in the case of a solenoid, the field is not uniform.

Force Between Current Loops

We have seen that a magnetic field exerts a force on a moving charge. Since a current loop contains moving charge, it follows that such a loop would experience a force in a magnetic field. Further, since the magnetic field in which a loop experiences a force must have its origin in another current source, two current carrying circuits would exert force on each other.

Consider two such circuits, numbered 1 and 2.
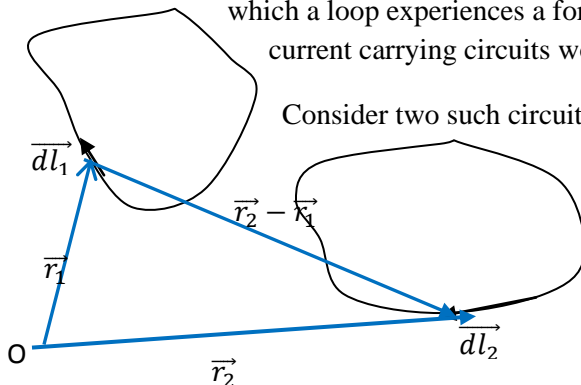
Consider the field due to the current element $\overrightarrow{Idl_1}$ in circuit 1. At a position $\overrightarrow{r_2} - \overrightarrow{r_1}$, the field is given by Biot Savart's law,

$$d\overrightarrow{B_1} = \frac{\mu_0}{4\pi} I_1 \frac{\overrightarrow{dl_1} \times (\overrightarrow{r_2} - \overrightarrow{r_1})}{|\overrightarrow{r_2} - \overrightarrow{r_1}|^3}$$

The net magnetic field at this point is given by summing over the contribution due to all elemental current comprising the circuit. Since the circuit 2 contains moving charges, the element $\overrightarrow{Idl_2}$ experiences a force due to this field which is given by $\oint I_2 \overrightarrow{dl_2} \times d\overrightarrow{B_1}$. By summing over all the current elements in circuit 2, we get the force exerted by circuit 1 on circuit 2 is given by

$$\vec{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint \oint \frac{\overrightarrow{dl_2} \times (\overrightarrow{dl_1} \times (\overrightarrow{r_2} - \overrightarrow{r_1}))}{|\overrightarrow{r_2} - \overrightarrow{r_1}|^3}$$

This is a very clumsy expression and can be evaluated only in cases of simple geometry.

By symmetry it follows that the force on circuit 1 due to current in circuit 2 is given by

$$\vec{F}_{21} = \frac{\mu_0}{4\pi} I_1 I_2 \oint \oint \frac{\overrightarrow{dl_1} \times (\overrightarrow{dl_2} \times (\overrightarrow{r_1} - \overrightarrow{r_2}))}{|\overrightarrow{r_2} - \overrightarrow{r_1}|^3}$$

Inductances and Magnetic Energy:

Inductance and Inductor:

Resistance, capacitance and inductance are the three familiar parameters from circuit theory. We have already discussed about the parameters resistance and capacitance in the earlier chapters. In this section, we discuss about the parameter inductance. Before we start our discussion, let us first introduce the concept of flux linkage. If in a coil with N closely wound turns around where a current I produces a flux ϕ and this flux links or encircles each of the N turns, the flux linkage Λ is defined as $\Lambda = N\phi$. In a linear medium, where the flux is proportional to the current, we define the self inductance L as the ratio of the total flux linkage to the current which they link.

i.e., $L = \frac{\Lambda}{I} = \frac{N\phi}{I}$ ....(4.47)

To further illustrate the concept of inductance, let us consider two closed loops C_1 and C_2 as shown in the figure 1, S_1 and S_2 are respectively the areas of C_1 and C_2 .
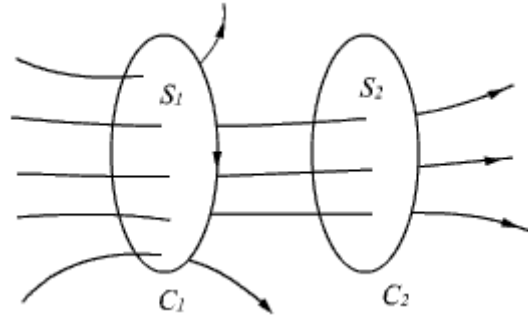
Fig 1

If a current I_1 flows in C_1 , the magnetic flux B_1 will be created part of which will be linked to C_2 as shown in Figure 4.10.

$$\phi_{12} = \int_{S_2} \vec{B}_1 . d\vec{S}_2$$

In a linear medium, ϕ_{12} is proportional to I_1. Therefore, we can write

$$\phi_{12} = L_{12} I_1$$

where L_{12} is the mutual inductance. For a more general case, if C_2 has N_2 turns then

$$\Lambda_{12} = N_2 \phi_{12}$$

and $\quad \Lambda_{12} = L_{12} I_1$

or $\quad L_{12} = \dfrac{\Lambda_{12}}{I_1}$

i.e., the mutual inductance can be defined as the ratio of the total flux linkage of the second circuit to the current flowing in the first circuit.

As we have already stated, the magnetic flux produced in C_1 gets linked to itself and if C_1 has N_1 turns then $\Lambda_{11} = N_1 \phi_{11}$, where ϕ_{11} is the flux linkage per turn.

Therefore, self inductance

$$L_{11} (\text{or } L \text{ as defined earlier}) = \dfrac{\Lambda_{11}}{I_1}$$

As some of the flux produced by I_1 links only to C_1 & not C_2.

$$\Lambda_{11} = N_1 \phi_{11} > N_2 \phi_{12} = \Lambda_{12}$$

Further in general, in a linear medium, $L_{12} = \dfrac{d\Lambda_{12}}{dI_1}$ and $L_{11} = \dfrac{d\Lambda_{11}}{dI_1}$

Inductance per unit length of a very long solenoid:

Let us consider a solenoid having n turns/unit length and carrying a current *I*. The solenoid is air cored.
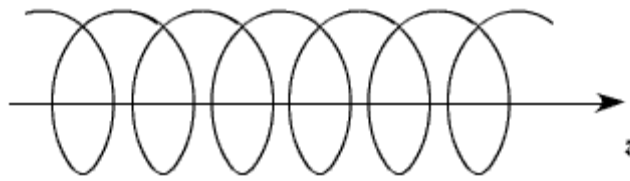

Fig 2: A long current carrying solenoid

The magnetic flux density inside such a long solenoid can be calculated as

$$\vec{B} = \mu_0 n I \, \hat{a}_3$$

where the magnetic field is along the axis of the solenoid.

If S is the area of cross section of the solenoid then

$$\phi = BS = \mu_0 n I S$$

The flux linkage per unit length of the solenoid

$$\Lambda = n\phi = \mu_0 n^2 I S$$

∴ The inductance per unit length of the solenoid

$$L = \frac{\Lambda}{I} = \mu_0 n^2 S$$

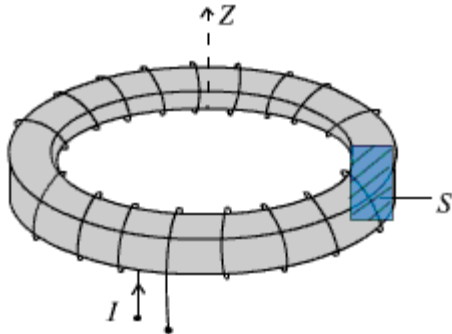Inductance of an *N* turn toroid carrying a filamentary current *I*.

Fig 3: *N* turn toroid carrying filamentary current *I*.

Solution: Magnetic flux density inside the toroid is given by

$$\vec{B} = \frac{\mu I}{2\pi\rho}\,\hat{a}_\phi$$

Let the inner radius is 'a' and outer radius is 'b'. Let the cross section area 'S' is small compared to the mean radius

of the toroid $\rho_0 \left[= \dfrac{a+b}{2} \right]$

Then total flux

$$\phi = N\frac{\mu I}{2\pi\rho_0}S$$

and flux linkage

$$\Lambda = \frac{\mu N^2 I S}{2\pi\rho_0}$$

The inductance

$$L = \frac{\Lambda}{I} = \frac{\mu N^2 S}{2\pi\rho_0}$$

Energy stored in Magnetic Field:

So far we have discussed the inductance in static forms. In earlier chapter we discussed the fact that work is required to be expended to assemble a group of charges and this work is stated as electric energy. In the same manner energy needs to be expended in sending currents through coils and it is stored as magnetic energy. Let us consider a scenario where we consider a coil in which the current is increased from 0 to a value I. As mentioned earlier, the self inductance of a coil in general can be written as

$$L = \frac{d\Lambda}{di} = N\frac{d\phi}{di}$$

or $L\,di = N\,d\phi$

If we consider a time varying scenario,

$$L\frac{di}{dt} = N\frac{d\phi}{dt}$$

We will later see that $N\dfrac{d\phi}{dt}$ is an induced voltage.

$$\therefore v = L\frac{di}{dt}$$ is the voltage drop that appears across the coil and thus voltage opposes the change of current.

Therefore in order to maintain the increase of current, the electric source must do an work against this induced voltage.

$$dW = vi\,dt$$
$$= Li\,di .$$

& $$W = \int_0^I Li\,di = \frac{1}{2}LI^2 \quad \text{(Joule)}$$

which is the energy stored in the magnetic circuit.

Maxwell's Equations: For Steady Fields in Point Form and Integral Form:

In our study of static fields so far, we have observed that static electric fields are produced by electric charges, static magnetic fields are produced by charges in motion or by steady current. Further, static electric field is a conservative field and has no curl, the static magnetic field is continuous and its divergence is zero. The fundamental relationships for static electric fields among the field quantities can be summarized as:

$$\nabla \times \vec{E} = 0 \qquad (5.1a)$$

$$\nabla \cdot \vec{D} = \rho_v \qquad (5.1b)$$

For a linear and isotropic medium,

$$\vec{D} = \varepsilon \vec{E} \qquad (5.1c)$$

Similarly for the magnetostatic case

$$\nabla \cdot \vec{B} = 0 \qquad (5.2a)$$

$$\nabla \times \vec{H} = \vec{J} \qquad (5.2b)$$

$$\vec{B} = \mu \vec{H} \qquad (5.2c)$$

It can be seen that for static case, the electric field vectors $\vec{E}$ and $\vec{D}$ and magnetic field vectors $\vec{B}$ and $\vec{H}$ form separate pairs.

In this chapter we will consider the time varying scenario. In the time varying case we will observe that a changing magnetic field will produce a changing electric field and vice versa.

We begin our discussion with Faraday's Law of electromagnetic induction and then present the Maxwell's equations which form the foundation for the electromagnetic theory.

Faraday's Law of electromagnetic Induction:

Michael Faraday, in 1831 discovered experimentally that a current was induced in a conducting loop when the magnetic flux linking the loop changed. In terms of fields, we can say that a time varying magnetic field produces an electromotive force (emf) which causes a current in a closed circuit. The quantitative relation between the induced emf (the voltage that arises from conductors moving in a magnetic field or from changing magnetic fields) and the rate of change of flux linkage developed based on experimental observation is known as Faraday's law. Mathematically, the induced emf can be written as

$$\text{Emf} = -\frac{d\phi}{dt} \quad \text{Volts} \qquad (1)$$

where ϕ is the flux linkage over the closed path.

A non zero $\frac{d\phi}{dt}$ may result due to any of the following:

(a) time changing flux linkage a stationary closed path.

(b) relative motion between a steady flux a closed path.

(c) a combination of the above two cases.

The negative sign in equation (1) was introduced by Lenz in order to comply with the polarity of the induced emf. The negative sign implies that the induced emf will cause a current flow in the closed loop in such a direction so as to oppose the change in the linking magnetic flux which produces it. (It may be noted that as far as the induced emf is concerned, the closed path forming a loop does not necessarily have to be conductive).

If the closed path is in the form of N tightly wound turns of a coil, the change in the magnetic flux linking the coil induces an emf in each turn of the coil and total emf is the sum of the induced emfs of the individual turns, i.e.,

$$Emf = -N\frac{d\phi}{dt} \quad \textit{Volts} \qquad (2)$$

By defining the total flux linkage as

$$\lambda = N\phi \qquad (3)$$

The emf can be written as

$$\text{Emf} = -\frac{d\lambda}{dt} \qquad (4)$$

Continuing with equation (1), over a closed contour '*C*' we can write

$$\text{Emf} = \oint_c \vec{E}.\vec{dl} \qquad (5)$$

where $\vec{E}$ is the induced electric field on the conductor to sustain the current.

Further, total flux enclosed by the contour 'C ' is given by

$$\phi = \int_S \vec{B}.\vec{ds} \qquad (6)$$

Where S is the surface for which 'C' is the contour.

From (5) and using (6) in (1) we can write

$$\oint_c \vec{E}.\vec{dl} = -\frac{\partial}{\partial t} \oint_s \vec{B}.\vec{ds} \qquad (7)$$

By applying stokes theorem

we can write

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (8)$$

which is the Faraday's law in the point form

Transformer EMF: We have said that non zero $\frac{d\phi}{dt}$ can be produced in a several ways. One particular case is when a time varying flux linking a stationary closed path induces an emf. The emf induced in a stationary closed path by a time varying magnetic field is called a transformer emf **.**

As shown in figure 1, a transformer consists of two or more numbers of coils coupled magnetically through a common core. Let us consider an ideal transformer whose winding has zero resistance, the core having infinite permittivity and magnetic losses are zero.
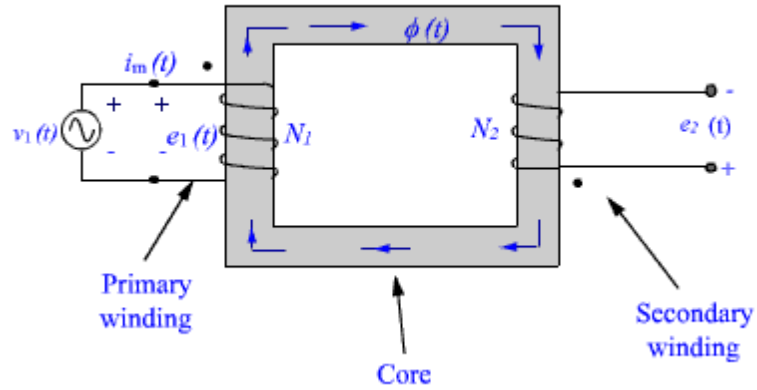
Fig 1: Transformer with secondary open

These assumptions ensure that the magnetization current under no load condition is vanishingly small and can be ignored. Further, all time varying flux produced by the primary winding will follow the magnetic path inside the core and link to the secondary coil without any leakage. If N_1 and N_2 are the number of turns in the primary and the secondary windings respectively, the induced emfs are

$$e_1 = N_1 \frac{d\phi}{dt}$$

(9a)

$$e_2 = N_2 \frac{d\phi}{dt}$$

(9b)

(The polarities are marked, hence negative sign is omitted. The induced emf is +ve at the dotted end of the winding.)

$$\therefore \frac{e_1}{e_2} = \frac{N_1}{N_2}$$

(10)

i.e., the ratio of the induced emfs in primary and secondary is equal to the ratio of their turns. Under ideal condition, the induced emf in either winding is equal to their voltage rating.

$$\frac{v_1}{v_2} = \frac{N_1}{N_2} = a$$

(11)

where 'a' is the transformation ratio. When the secondary winding is connected to a load, the current flows in the secondary, which produces a flux opposing the original flux. The net flux in the core decreases and induced emf will tend to decrease from the no load value. This causes the primary current to increase to nullify the decrease in the flux and induced emf. The current continues to increase till the flux in the core and the induced emfs are restored to the no load values. Thus the source supplies power to the primary winding and the secondary winding delivers the power to the load. Equating the powers

$$i_1 v_1 = i_2 v_2 \qquad (12)$$

$$\frac{i_2}{i_1} = \frac{v_1}{v_2} = \frac{e_1}{e_2} = \frac{N_1}{N_2} \qquad (13)$$

Further,

$$i_2 N_2 - i_1 N_1 = 0 \qquad (14)$$

i.e., the net magnetomotive force (mmf) needed to excite the transformer is zero under ideal condition.

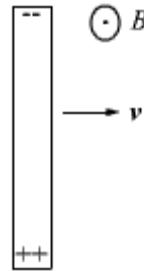**Motional EMF:** Let us consider a conductor moving in a steady magnetic field as shown in the fig 2.

Fig 2

If a charge Q moves in a magnetic field $\vec{B}$, it experiences a force

$$\vec{F} = Q\vec{v} \times \vec{B} \qquad (15)$$

This force will cause the electrons in the conductor to drift towards one end and leave the other end positively charged, thus creating a field and charge separation continuous until electric and magnetic forces balance and an equilibrium is reached very quickly, the net force on the moving conductor is zero.

$$\frac{\vec{F}}{Q} = \vec{v} \times \vec{B}$$
can be interpreted as an induced electric field which is called the motional electric field

$$\vec{E}_m = \vec{v} \times \vec{B} \qquad (16)$$

If the moving conductor is a part of the closed circuit C, the generated emf around the circuit is $\oint_c \vec{v} \times \vec{B}.d\vec{l}$. This emf is called the motional emf.

UNIT – III
UNIFORM PLANE WAVES

### Reflection/ Refraction of Plane waves

A **plane wave** is a **wave** of constant frequency and amplitude with wavefronts that are an infinitely long straight line. **Plane waves** travel in the direction perpendicular to the wavefronts. Although they are a mathematical abstraction, many physical **waves** approximate **plane waves** far from their source. Consider the figure given below which shows incident and reflected wave fronts when a plane wave fronts travels towards a plane reflecting surface.

From the figure we define Reflection and Refraction.

Reflection:
It is the change in direction of a wavefront at an interface between two different media so that the wave front returns into the medium from which it originated.

The laws of reflections are verified using Huygens's Principle. The incident ray, the reflected ray and normal to the reflecting surface lie in one plane which is perpendicular to the reflecting surface. The angle of incidence equals angle of reflection.
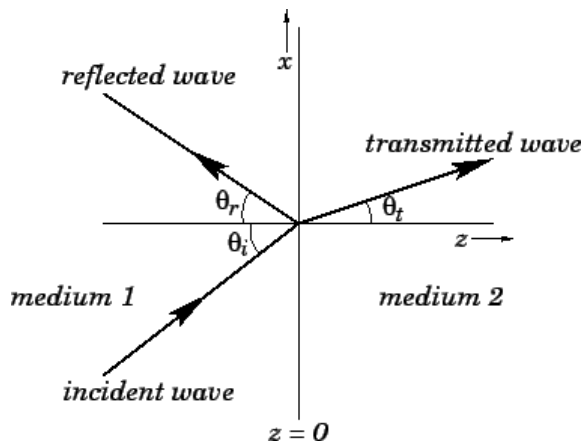
Refraction:

When a wave travels from one medium to another at a specific angle other than 90^0 or 0^0, the line of the travel of the wave changes at the interface because of a change in wave velocity. This phenomenon is called refraction.

Refraction is the bending of waves when passes through a different medium. Reflection is the process in which light waves falls on a surface and bounces back.

In refraction, the sine of angle between the incident ray and normal maintains a constant ratio with the sine of angle of refracted ray and normal. **Snell's Law** of refraction is proved using Huygens's principle

The incident ray, the refracted ray and the normal to the refracting surface lie in the same plane.

Snell's Laws

1. Law of reflection
2. Law of refraction

Snell's law:
$n_1 \sin i = n_2 \sin r$
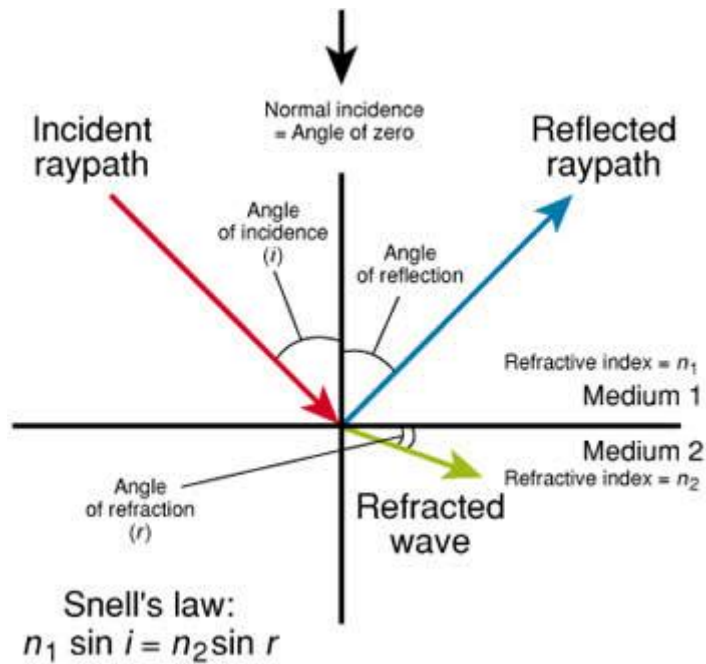
Law of reflection

According to this law, sine of angle of incidence is equal to the sine of angle of reflection.

$$sin\theta_i = sin\theta_r$$

or

$$\theta_i = \theta_r$$

Law of refraction

According to this law,

$$n_1 sin\theta_i = n_2 sin\theta_t$$

$$\frac{sin\theta_t}{sin\theta_i} = \frac{n_1}{n_2}$$

where 'n' is the refractive index and it is defined as the ratio of speed of light in free space to the speed of light in medium.i.e $n = \frac{c}{v}$

and

$$n_1 = \frac{c}{v_1}, \quad n_2 = \frac{c}{v_2}$$

$$\frac{n_1}{n_2} = \frac{c/v_1}{c/v_2} = \frac{v_2}{v_1}$$

Therefore $\qquad \frac{sin\theta_t}{sin\theta_i} = \frac{n_1}{n_2} = \frac{v_2}{v_1}$

Finally we write,

$$\frac{sin\theta_t}{sin\theta_i} = \frac{n_1}{n_2} = \frac{v_2}{v_1} = \frac{\eta_2}{\eta_1} = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}$$

When a plane wave propagating in a homogeneous medium encounters an interface with a different medium, a portion of the wave is reflected from the interface while the remainder of the wave is transmitted. The reflected and transmitted waves can be determined by enforcing the fundamental electromagnetic field boundary conditions at the media interface.

The proportion of reflection and transmission depends on the constitutive parameters of the media such as η, ε, μ, and σ.There are two cases of the incidence by which the uniform plane wave is incident at the boundary.

   I.   Normal Incidence
  II.   Oblique Incidence

Normal Incidence

   When a uniform plane wave incidences normally to the boundary between the two media, it is known as normal incidence.

Oblique Incidence

   When a uniform plane wave incidences obliquely to the boundary between the two media, it is known as oblique incidence.

Reflection of a Plane Wave at Normal Incidence - Dielectric Boundary :

In case of plane wave in air incident normally upon the surface of conductor the wave is entirely reflected. Since there can be no loss within a perfect conductor ,none of the energy is observed. As a result amplitude of E and H in the reflected and incident are same and only difference is in the direction of power flow.
The expression of incident wave

$$E_t e^{-j\beta x}$$

The expression for reflected wave

$$E_r e^{j\beta x}$$

E_r must be determined from boundary conditions tangential component of E must be continuous across the boundary and E is zero within the conductor. The tangential component of E just outside the conductor must also be zero. The amplitude of reflected electric field strength is equal to that of the of the incident electric field strength, but its phase has been reversed on reflection.

$$E_r = -E_t$$
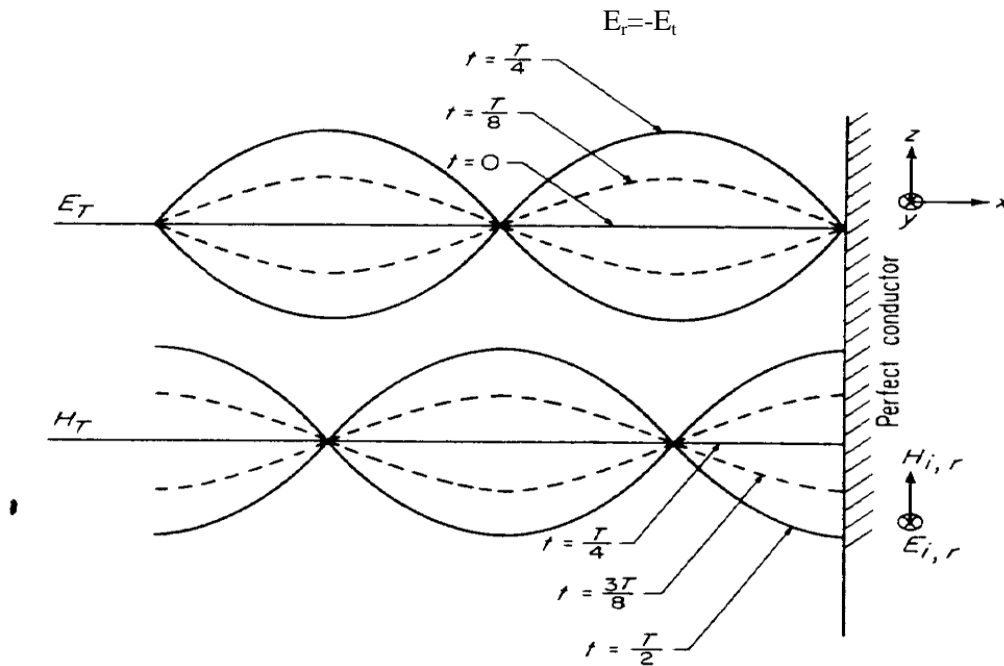
The resultant electric field strength at any point a distance –x from the x=0 plane will be the sum of field strengths of the incident and reflected waves at that point and will be given by

$$E_T(x) = E_i e^{-j\beta x} + E_r e^{j\beta x}$$
$$= E_i(e^{-j\beta x} - e^{j\beta x})$$
$$= -2j\ E_i \sin\beta x$$
$$\breve{E}_T(x,t) = Re\{-2jE_t \sin\beta x\ e^{j\omega t}\}$$

If E_t is chosen to be real

$$\breve{E}_T(x,t) = 2E_t \sin\beta x\ \sin\omega t$$

The incident and reflected waves combine to produce a standing wave. The magnitude of electric field varies sinusoidally with distance from the reflecting plane.it has a maximum value of twice the electric field strength of the incident wave at distance from the surface that are odd multiples of a quarter wavelength

The magnetic field strength must be reflected without reversal of phase. If both magnetic and electric field strengths were reversed, there would be no reversal of direction of energy propagation. The amplitude of reflected electric field strength is equal to that of the of the incident electric field strength, but its phase has been same on reflection

$$H_T(x) = H_i e^{-j\beta x} + H_r e^{j\beta x}$$
$$= H_i(e^{-j\beta x} + e^{j\beta x})$$
$$= 2H_i \cos\beta x$$

If H_t is chosen to be real

$$\tilde{H}_T(x,t) = Re\{H_T(x)e^{j\omega t}\}$$
$$= 2H_i \cos\beta x \cos\omega t$$

The incident and reflected waves combine to produce a standing wave. The magnitude of magnetic field varies sinusoidally with distance from the reflecting plane.it has a maximum value at the surface of the conductor and at multiples of a half wavelength from the surface where as the zero points occur at odd multiples of a quarter wavelength from the surface. From the boundary conditions for H it follows that there must be a surface current of J_s amperes per meter such that Js=Ht(at x=0)
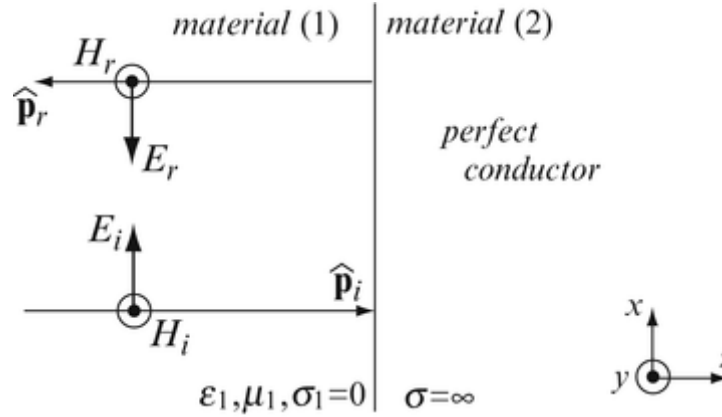
That ET and HT are 90 degrees apart in time phase ,replacing –j by its equivalent $e^{-j(\pi/2)}$ and combining this with $e^{j\omega t}$ term to give $e^{j[\omega t-(\pi/2)]}$

$$\breve{E}_T(x,t) = Re\{2E_t \sin\beta x\ e^{-j(\pi/2)}\ e^{j\omega t}\}$$
$$= 2E_t \sin\beta x\ \cos(\omega t - \pi/2)$$

Like $H_T = 2H_i \cos\beta x \cos\omega t$ shows that ET and HT differ in time phase by $\pi/2$ radians

Consider a uniform plane wave striking interface between perfect dielectric and perfect conductor.

For medium 2 the intrinsic impedance is 0. i.e $\eta_2 = 0$.

Therefore, the transmission coefficient is given by

$$\tau = \frac{2\eta_2}{\eta_2 + \eta_1} = 0$$

And the reflection coefficient is given by

$$\gamma = \frac{-\eta_1 + \eta_2}{\eta_1 + \eta_2} = -1$$

From the values of transmission coefficient and reflection coefficient, it is clear that the wave is totally reflected and there is no transmitted wave in medium 2.

Since magnitude of reflection coefficient is equal to one, the entire power is reflected from the conducting boundary.

This case is exactly identical to the Transmission Line with the short circuit load.

We therefore have two waves with equal amplitude travelling in opposite directions. Therefore in dielectric medium, we get Standing waves. The electric field becomes zero in the planes which are parallel to the conducting boundary and are located at distances which are multiple of $\lambda/2$.

Standing Waves

Standing wave is also known as stationary wave.

It occurs when incident wave combines with reflected wave.

Two sine waves with the same Amplitude, Wavelength and Frequency travelling in opposite directions will interfere and produce a combined wave . The waveform of the combined waves, on average has No Net Propagation of Energy and is known as standing wave.
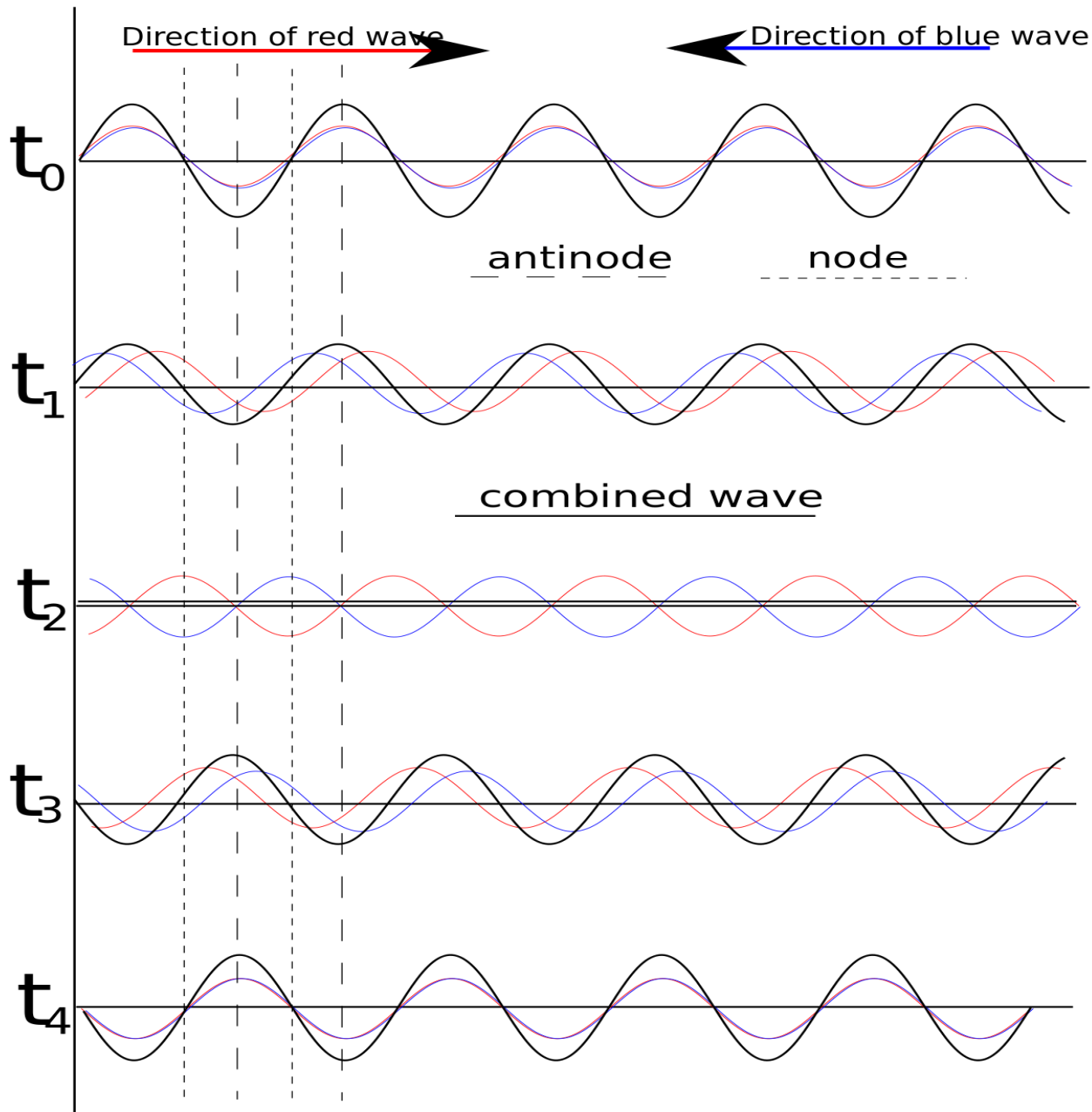
In the above figure red wave represents incident wave, and blue wave represents reflected wave.

A standing wave pattern is an interference phenomenon. It is formed as the result of the perfectly timed interference of two waves passing through the same medium. A standing wave pattern is not actually a wave; rather it is the pattern resulting from the presence of two waves (sometimes more) of the same frequency with different directions of travel within the same medium.

Standing wave is denoted by E_{1s}.

It is given by

$$E_{1s} = E_i + E_r$$

Problem:

A uniform plane wave in air is normally incident on an infinitely thick slab. If the refractive index of glass slab is 1.5, then find the percentage of the incident power that is reflected from the air – glass interface.

Sol.

Medium 1	**Medium 2**
n_1	$n_2 = 1.5$
$\mu_1 = \mu_0$	$\mu_2 = \mu_0$
$\epsilon_1 = \epsilon_0$	$\epsilon_2 = \epsilon_0 \epsilon_r$

If n_1, n_2 are refractive indices and v_1, v_2 are the velocities

$$\frac{n_1}{n_2} = \frac{v_2}{v_1} = \frac{\sqrt{\mu_1 \epsilon_1}}{\sqrt{\mu_2 \epsilon_2}} = \frac{1}{1.5}$$

Reflection of a Plane Wave Oblique Incidence - Conducting Boundary (Perpendicular Polarization)

We have discussed the case of normal incidence where electromagnetic wave traveling in a lossless medium impinges normally at the interface of a second medium. In this section we shall consider the case of oblique incidence. As before, we consider two cases

- When the second medium is a perfect conductor.
- When the second medium is a perfect dielectric.

A plane incidence is defined as the plane containing the vector indicating the direction of propagation of the incident wave and normal to the interface. We study two specific cases when the incident electric field Ei is perpendicular to the plane of incidence (perpendicular polarization) and Ei is parallel to the plane of incidence (parallel polarization). For a general case, the incident wave may have arbitrary polarization but the same can be expressed as a linear combination of these two individual cases.

Oblique Incidence at a plane conducting boundary

i.   **PerpendicularPolarization**
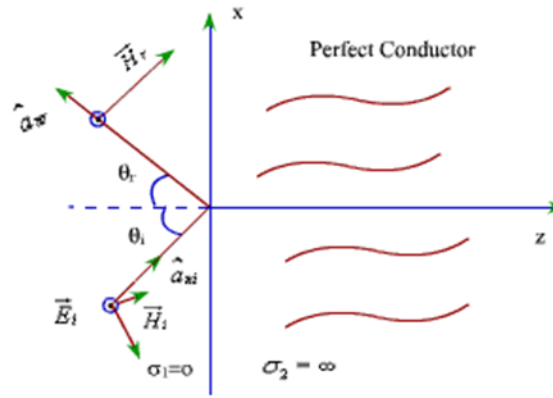
As the EM field inside the perfect conductor is zero, the interface reflects the incident plane wave. axi

And a_x respectively represent the unit vector in the direction of propagation of the incident and reflected waves Θi is the angle of incidence and Θr is the angle of reflection.

We find that

$$\hat{a}_{ni} = \hat{a}_x \sin\theta_i + \hat{a}_z \cos\theta_i$$

Since the incident wave is considered to be perpendicular to the plane of incidence, which for the present case happens to be xz plane, the electric field has only y-component.

$$\overset{\text{1}}{E}_i(x,z) = \hat{a}_y E_{i0} e^{-j\beta_1 \hat{a}_{ni} \cdot \overset{\text{1}}{R}}$$

$$= \hat{a}_y E_{i0} e^{-j\beta_1 (x\sin\theta_i + z\cos\theta_i)}$$

Therefore,
The corresponding magnetic field is given by

$$\overset{\text{r}}{H}_i(x,z) = \frac{1}{\eta_1}\left[\hat{a}_{ni} \times \overset{\text{r}}{E}_i(x,z)\right] \qquad = \frac{E_{i0}}{\eta_1}\left(-\hat{a}_x \cos\theta_i + \hat{a}_z \sin\theta_i\right) e^{-j\beta_1 (x\sin\theta_i + z\cos\theta_i)}$$

Direction of propagation of reflected wave

$$\hat{a}_{nr} = \hat{a}_x \sin\theta_r - \hat{a}_z \cos\theta_r$$

Reflected wave inside medium 1

$$\overset{1}{E}_r(x,z) = \hat{a}_y E_{r0} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

Boundary condition at z=0

$$\overset{1}{E}_1(x,0) = \overset{1}{E}_i(x,0) + \overset{1}{E}_r(x,0)$$
$$= \hat{a}_y (E_{i0} e^{-j\beta_1 x\sin\theta_i} + E_{r0} e^{-j\beta_1 x\sin\theta_r}) = 0$$

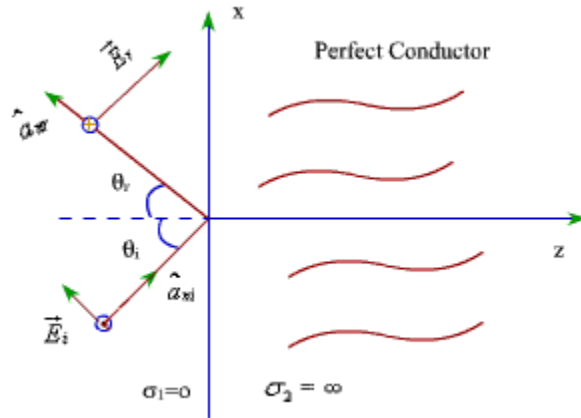$$\therefore E_{r0} = -E_{i0} \quad \& \quad \theta_i = \theta_r$$

Total field in medium 1,

$$\overset{1}{E}_1(x,z) = \overset{1}{E}_i(x,z) + \overset{1}{E}_r(x,z)$$

$$= \hat{a}_y E_{i0}(e^{-j\beta_1 z\cos\theta_i} - e^{j\beta_1 z\cos\theta_i}) e^{-j\beta_1 x\sin\theta_i}$$

$$= -\hat{a}_y j2E_{i0} \sin(\beta_1 z\cos\theta_i) e^{-j\beta_1 x\sin\theta_i}$$

$$\overset{r}{H}_1(x,y) = -2\frac{E_{i0}}{\eta_1} \Big[\hat{a}_x \cos\theta_i \cos(\beta_1 z\cos\theta_i) e^{-j\beta_1 x\sin\theta_i}$$
$$+ \hat{a}_z j\sin\theta_i \sin(\beta_1 z\cos\theta_i) e^{-j\beta_1 x\sin\theta_i} \Big]$$

Reflection of a Plane Wave Oblique Incidence -Conducting Boundary (Parallel Polarization)

In this case also axi and axr are given by equations above. Here Hi and Hr have only y component

With reference to above fig the field components can be written as:

Direction of propagation of incident wave

$$\hat{a}_{ni} = \hat{a}_x \sin\theta_i + \hat{a}_z \cos\theta_i$$

Incident wave inside medium 1,

$$\vec{E}_i(x,z) = E_{i0}(\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i)e^{-j\beta_1 \hat{a}_{ni} \cdot \vec{R}}$$

$$= E_{i0}(\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_i(x,z) = \hat{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

Direction of propagation of reflected wave

$$\hat{a}_{nr} = \hat{a}_x \sin\theta_i - \hat{a}_z \cos\theta_i$$

Reflected wave inside medium 1

$$\vec{E}_r(x,z) = E_{r0}(\hat{a}_x \cos\theta_i + \hat{a}_z \sin\theta_r)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r(x,z) = -\hat{a}_y \frac{E_{r0}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

Boundary condition at z=0

$$E_{1t} = 0 \quad (z=0)$$

$$\rightarrow (E_{io} \cos\theta_i)e^{-j\beta_1 x\sin\theta_i} + (E_{ro} \cos\theta_r)e^{-j\beta_1 x\sin\theta_r} = 0 \text{ for all x,}$$

$$\therefore E_{ro} = -E_{io} \qquad \theta_r = \theta_i$$

Hence the total field in medium 1,

$$\vec{E}_1(x,z) = \vec{E}_i(x,z) + \vec{E}_r(x,z)$$

$$= -2E_{i0}[\hat{a}_x j\cos\theta_i \sin(\beta_1 z\cos\theta_i) + \hat{a}_z \sin\theta_i \cos(\beta_1 z\cos\theta_i)]e^{-j\beta_1 x\sin\theta_i}$$

$$\vec{H}_1(x,z) = \vec{H}_i(x,z) + \vec{H}_r(x,z)$$

$$= \hat{a}_y 2\frac{E_{i0}}{\eta_1} \cos(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}$$

Once again, we find a standing wave pattern along z for the x and y components of E and H,while a_n on uniform plane wave propagates along x with a phase velocity Since, for this propagating wave, magnetic field is in transverse direction, such waves are called transverse magnetic or TM waves

Oblique incidence: Interface between dielectric media:

- We will consider the problem of a plane wave obliquely incident on a plane interface between two dielectric media.
- If a plane wave is incident at an angle from medium 1. The interface plane defines the boundary between the media. The plane of incidence contains the propagation vector and is both perpendiculars to the interface plane and to the phase planes of the wave.
- We will first consider two particular cases (polarizations) of this problem as follows:
  - The electric field is in the x-z plane (parallel polarization)
  - The electric field is in normal to the x-z plane (perpendicular polarization)
- Any arbitrary incident plane wave can be expressed as a linear combination of these two principal polarizations. The plane of incidence is that plane containing
  - The normal vector to the interface and
  - The direction of propagation vector of the incident wave
- **Perpendicular Polarization** The electric field is perpendicular to the plane of incidence and the magnetic field is parallel to the plane of incidence. The fields are configured as in the Transverse Electric (TE) modes.
- **Parallel Polarization** The magnetic field is perpendicular to the plane of incidence and the electric field is parallel to the plane of incidence. The fields are configured as in the Transverse Magnetic (TM) modes.
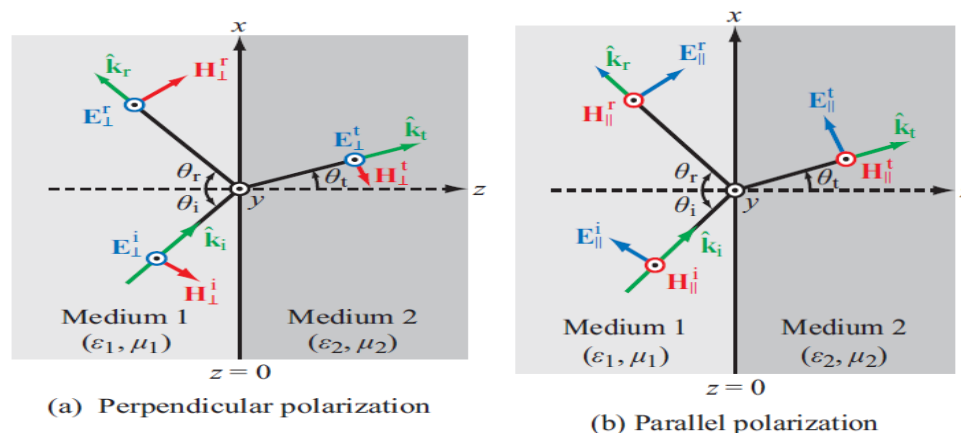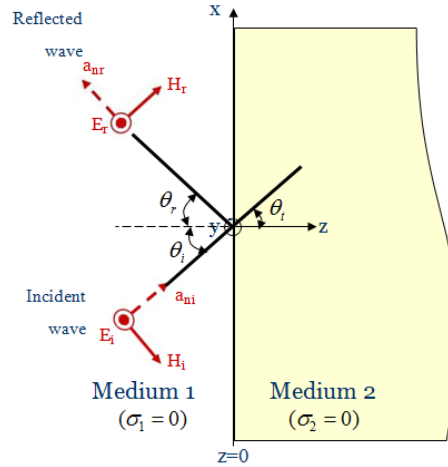

Fig.1 Oblique incidence of plane EM wave at a media interface.

Perpendicular polarization (TE):

- In this case, electric field vector is perpendicular to the x-z plane, hence, it will have component along the y-axis. Since the electric field is transversal to the plane of incidence.

- Let us assume that the incident wave propagates with a propagation vector $\gamma_1^i = \beta_1^i$ (loss less medium) in the x-z plane and makes an angle θ_i (incident propagation vector) with the normal.
  - Using direction cosines the direction of propagation of incidence wave can be written as

$$\vec{a}_{ni} = \vec{a}_x \sin\theta_i + \vec{a}_z \cos\theta_i$$

$$\vec{\gamma}_1^i.\vec{r} = \vec{\beta}_1^i.\vec{r} = (\vec{a}_x \sin\theta_i + \vec{a}_z \cos\theta_i).(x\,\vec{a}_x + z\,\vec{a}_z)$$

  - Similarly the direction of propagation of reflected wave (-ve z direction) and transmitted wave can be written as can be written as

$$\vec{a}_{ni} = \vec{a}_x \sin\theta_r - \vec{a}_z \cos\theta_r$$
$$\vec{\gamma}_1^r.\vec{r} = \vec{\beta}_1^r.\vec{r} = (\vec{a}_x \sin\theta_r - \vec{a}_z \cos\theta_r).(x\,\vec{a}_x + z\,\vec{a}_z)$$

$$\vec{a}_{ni} = \vec{a}_x \sin\theta_t + \vec{a}_z \cos\theta_t$$
$$\vec{\gamma}_1^t.\vec{r} = \vec{\beta}_1^t.\vec{r} = (\vec{a}_x \sin\theta_t + \vec{a}_z \cos\theta_t).(x\,\vec{a}_x + z\,\vec{a}_z)$$

- The electric field phasors for the perpendicular polarization can be written as

$$\vec{E}_i(x,z) = \vec{a}_y E_{io} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$
$$\vec{E}_r(x,z) = \vec{a}_y \Gamma E_{io} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$
$$\vec{E}_t(x,z) = \vec{a}_y \tau E_{to} e^{-j\beta_1(x\sin\theta_t + z\cos\theta_t)}$$

- Similarly the magnetic field phasors can be obtained by using the relation

$$\overset{\text{\tiny UⅢ}}{H}_i(x,z) = \frac{1}{\eta_1}\left[\nabla \times \overset{\text{\tiny r}}{E}_i(x,z)\right] = \frac{E_{io}}{\eta_1}(-\overset{\text{\tiny r}}{a}_x cos\theta_i + \overset{\text{\tiny r}}{a}_z sin\theta_i\,)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\overset{\text{\tiny UⅢ}}{H}_r(x,z) = \frac{1}{\eta_1}\left[\nabla \times \overset{\text{\tiny r}}{E}_r(x,z)\right] = \frac{\Gamma E_{io}}{\eta_1}(\overset{\text{\tiny r}}{a}_x cos\theta_r + \overset{\text{\tiny r}}{a}_z sin\theta_r\,)e^{-j\beta_1(x\sin\theta_r + z\cos\theta_r)}$$

$$\overset{\text{\tiny UⅢ}}{H}_t(x,z) = \frac{1}{\eta_1}\left[\nabla \times \overset{\text{\tiny r}}{E}_t(x,z)\right] = \frac{\tau E_{io}}{\eta_2}(-\overset{\text{\tiny r}}{a}_x cos\theta_t + \overset{\text{\tiny r}}{a}_z sin\theta_t\,)e^{-j\beta_1(x\sin\theta_t + z\cos\theta_t)}$$

At the boundary interface (z=0) equating the tangential components of electric field and magnetic field. The electric field has tangential E_y component, but the magnetic field has x- and z- components and tangential component of magnetic field is only x- component at the interface.

$$\overset{\text{\tiny I}}{E}_{iy}(x,0) + \overset{\text{\tiny I}}{E}_{ry}(x,0) = \overset{\text{\tiny I}}{E}_{ty}(x,0)$$
$$\overset{\text{\tiny UⅢ}}{H}_{ix}(x,0) + \overset{\text{\tiny UⅢ}}{H}_{rx}(x,0) = \overset{\text{\tiny UⅢ}}{H}_{tx}(x,0)$$

- Boundary at z=0 gives

$$e^{-j\beta_1 x\sin\theta_i} + \Gamma e^{-j\beta_1 x\sin\theta_r} = \tau e^{-j\beta_2 x\sin\theta_t}$$

$$-\frac{1}{\eta_1}\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + \frac{\Gamma}{\eta_1}\cos\theta_r e^{-j\beta_1 x\sin\theta_r} = \frac{\tau}{\eta_2}\cos\theta_t e^{-j\beta_2 x\sin\theta_t}$$

- If E_x and H_y are to be continuous at the interface $z = 0$ for all x, then, this x variation must be the same on both sides of the equations (also known as phase matching condition)

$$\beta_1 \sin\theta_i = \beta_1 \sin\theta_r = \beta_2 \sin\theta_t$$

- Follows Snell's law

$$\theta_i = \theta_r$$

- Now we can simplify above two equations by applying Snell's

$$1 + \Gamma = \tau$$

$$-\frac{1}{\eta_1}\cos\theta_i + \frac{\Gamma}{\eta_1}\cos\theta_r = -\frac{\tau}{\eta_2}\cos\theta_t$$

- The above two equations has two unknowns τ and Γ and it can be solved easily as follows

$$\left(\frac{1}{\eta_1}\cos\theta_i - \frac{\Gamma}{\eta_1}\cos\theta_r\right)\frac{\eta_2}{\cos\theta_t} = \tau$$

$$\Gamma_\perp = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

$$\tau_\perp = \frac{E_{to}}{E_{io}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

- For normal incidence, it is a particular case and put

$$\theta_i = \theta_r = \theta_t$$

Parallel polarization (TM):

▪ Assume the wave is propagating obliquely along an arbitrary direction a_r and the electric field vector is parallel to the plane of incidence (x-z plane) and the magnetic field vector is normal to the plane of incidence
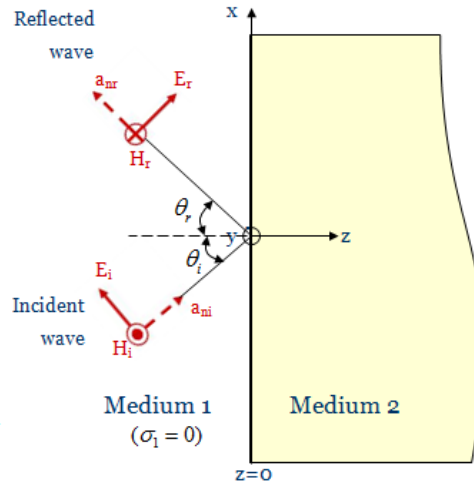
Reflected wave — a_{nr}, E_r, H_r
θ_r, θ_i
E_i, a_{ni}
Incident wave — H_i
Medium 1 ($\sigma_1 = 0$)   Medium 2
z=0

• Let us assume that the incident wave propagates with a propagation vector $\gamma_1^i = \beta_1^i$ (loss less medium) in the x-z plane and makes an angle θ_i (incident propagation vector) with the normal.

▪ Using direction cosines the direction of propagation of incidence wave can be written as

$$a_{ni} = a_x \sin \theta_i + a_z \cos \theta_i$$

$$\gamma_1^i . r = \beta_1^i . r = (a_x \sin \theta_i + a_z \cos \theta_i).(x\, a_x + z\, a_z)$$

▪ Similarly the direction of propagation of reflected wave (-z direction) and transmitted wave can be written as can be written as

$$a_{ni} = a_x \sin \theta_r - a_z \cos \theta_r$$
$$\gamma_1^r . r = \beta_1^r . r = (a_x \sin \theta_r - a_z \cos \theta_r).(x\, a_x + z\, a_z)$$

$$a_{ni} = a_x \sin \theta_t + a_z \cos \theta_t$$
$$\gamma_1^t . r = \beta_1^t . r = (a_x \sin \theta_t + a_z \cos \theta_t).(x\, a_x + z\, a_z)$$

• The Magnetic field phasors for the perpendicular polarization can be written as

$$\vec{H}_i(x,z) = \vec{a}_y \frac{E_{io}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_r(x,z) = -\vec{a}_y \frac{\Gamma E_{io}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_t(x,z) = \vec{a}_y \frac{\tau E_{io}}{\eta_2} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$\vec{E}_t(x,z) = E_{to}(\vec{a}_x\cos\theta_t - \vec{a}_z\sin\theta_t) e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

- Similarly the electric field phasors can be obtained by using the relation

$$\vec{E}_i(x,z) = \frac{1}{j\omega\varepsilon_1}\left[\nabla \times \vec{H}_i(x,z)\right] = E_{io}(\vec{a}_x\cos\theta_i - \vec{a}_z\sin\theta_i) e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{E}_r(x,z) = \frac{1}{j\omega\varepsilon_1}\left[\nabla \times \vec{H}_r(x,z)\right] = \Gamma E_{io}(\vec{a}_x\cos\theta_r + \vec{a}_z\sin\theta_i) e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{E}_t(x,z) = \frac{1}{j\omega\varepsilon_1}\left[\nabla \times \vec{H}_t(x,z)\right] = \tau E_{io}(\vec{a}_x\cos\theta_t - \vec{a}_z\sin\theta_t) e^{-j\beta_1(x\sin\theta_t + z\cos\theta_t)}$$

- At the boundary interface (z=0) equating the tangential components of magnetic field and electric field. The magnetic field has tangential H_y component, but the electric field has x- and z- components and tangential component of electric field is only x- component at the interface.

$$\vec{H}_{iy}(x,0) + \vec{H}_{ry}(x,0) = \vec{H}_{ty}(x,0)$$
$$\vec{E}_{ix}(x,0) + \vec{E}_{rx}(x,0) = \vec{E}_{tx}(x,0)$$

- Boundary at z=0 gives

$$\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + \Gamma\cos\theta_r e^{-j\beta_1 x\sin\theta_r} = \tau\cos\theta_t e^{-j\beta_2 x\sin\theta_t}$$

$$\frac{1}{\eta_1} e^{-j\beta_1 x\sin\theta_i} - \frac{\Gamma}{\eta_1} e^{-j\beta_1 x\sin\theta_r} = \frac{\tau}{\eta_2} e^{-j\beta_2 x\sin\theta_t}$$

- If E_x and H_y are to be continuous at the interface z = 0 for all x, then, this x variation must be the same on both sides of the equations (also known as phase matching condition)

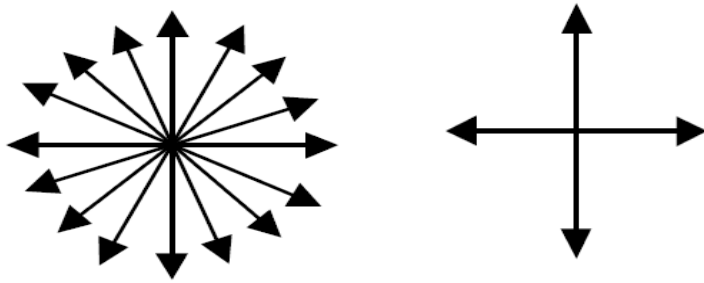$$\beta_1\sin\theta_i = \beta_1\sin\theta_r = \beta_2\sin\theta_t$$

- Follows Snell's law

$$\theta_i = \theta_r$$

BREWSTER ANGLE

Before going to Brewster angle we have to know the difference between polarized wave and unpolarized waves.

For unpolarized waves the electric fields are in many directions as shown in figure, e.g. unpolarized light
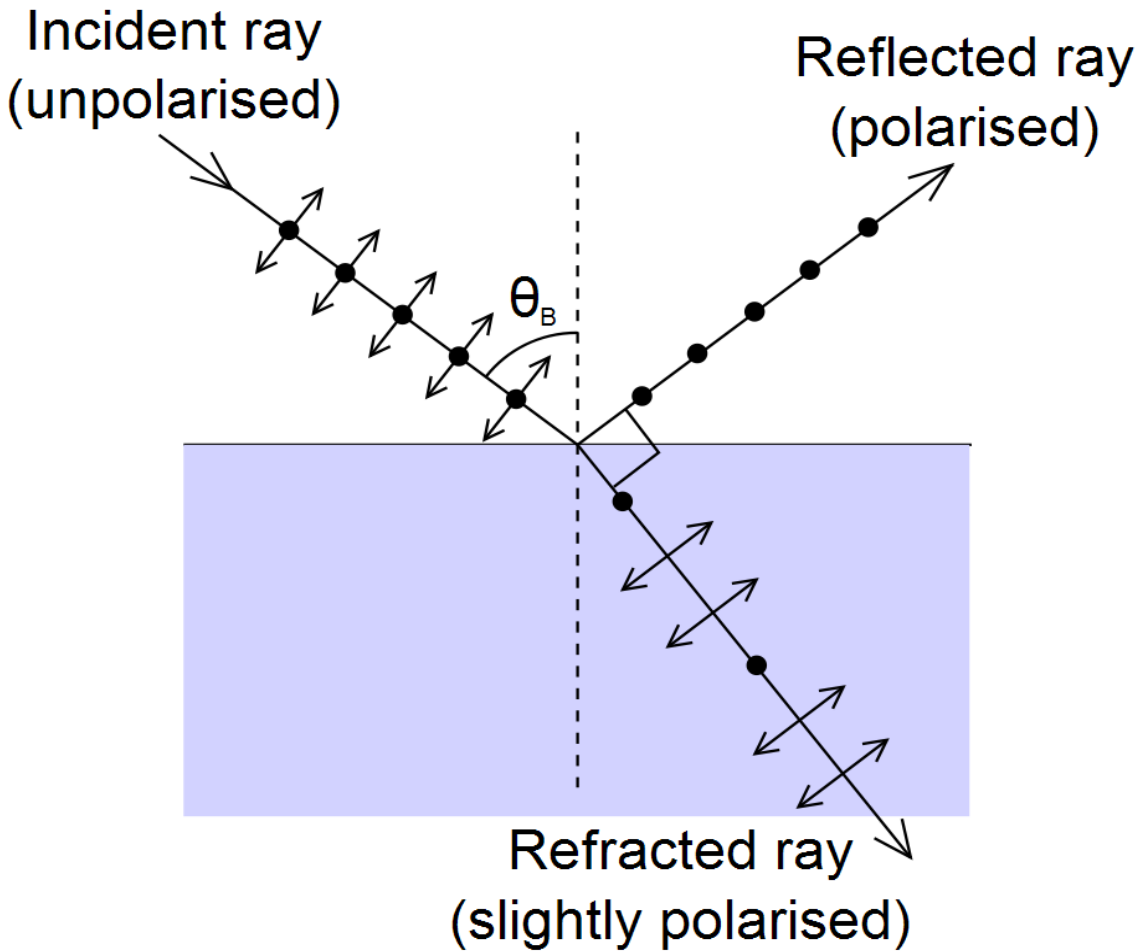
Where as in polarized waves the electric field vector is either vertical or horizontal as shown in figure below.

We can convert unpolarized light into polarized light by passing the light through a polarizing filter.

Incident ray
(unpolarised)

Reflected ray
(polarised)

θ_B

Refracted ray
(slightly polarised)
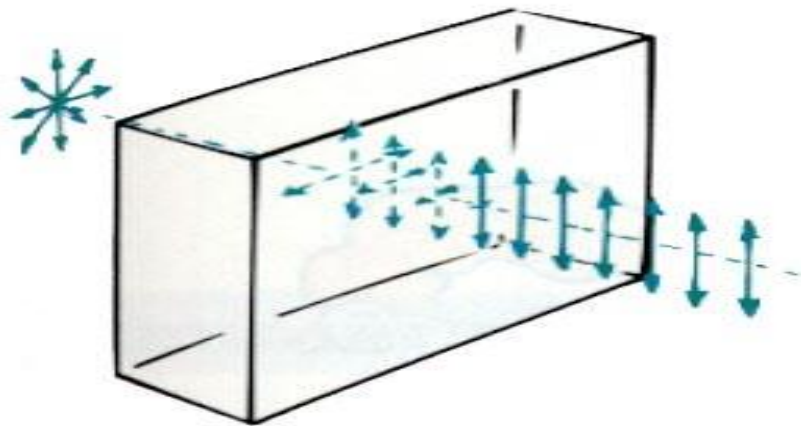
When unpolarized wave is incident obliquely at Brewster angle θ_B, only the component with perpendicular polarization (Horizontal polarization) will be reflected, while component with parallel polarization will not be reflected.

It is also called as Polarizing angle.

At Brewster angle, the angle between reflected ray and refracted ray is 90^0.
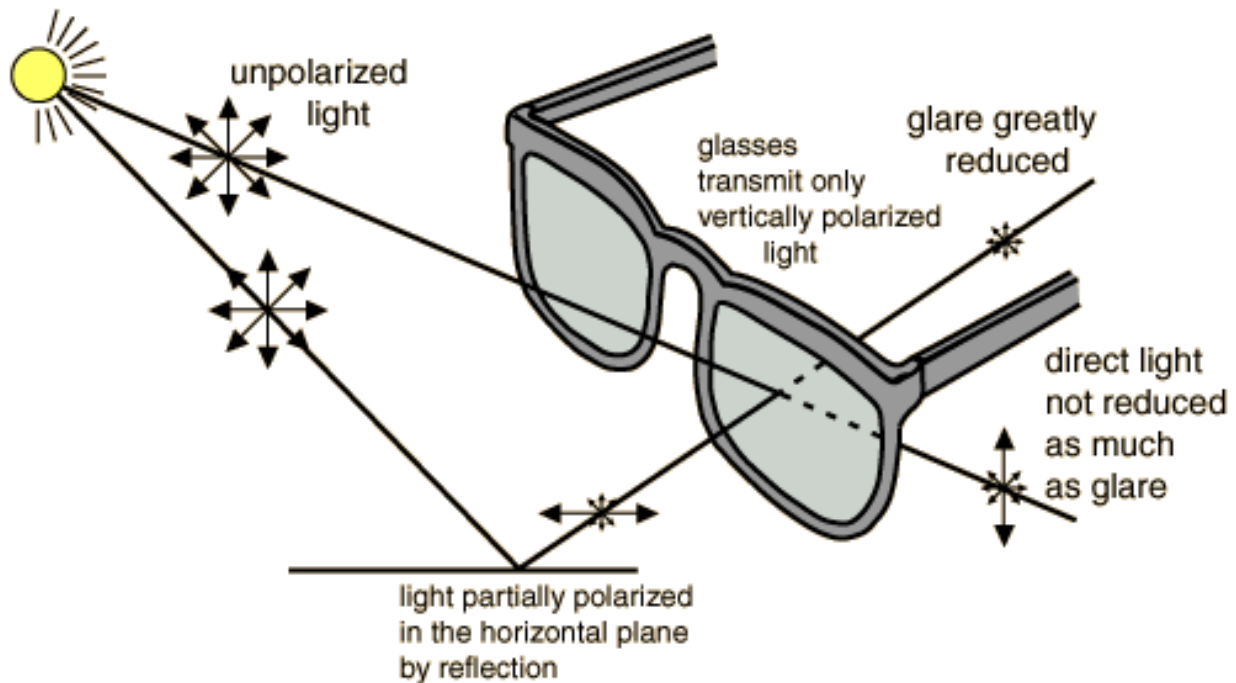
It is denoted by θ_B.

It is given by

$$\tan \theta_B = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1}$$

$$\theta_B = \tan^{-1} \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \tan^{-1} \frac{n_2}{n_1}$$

Application:

Polaroid sunglasses: The polarization axes of the lenses are vertical as the glare usually comes from reflection off horizontal surfaces**.**
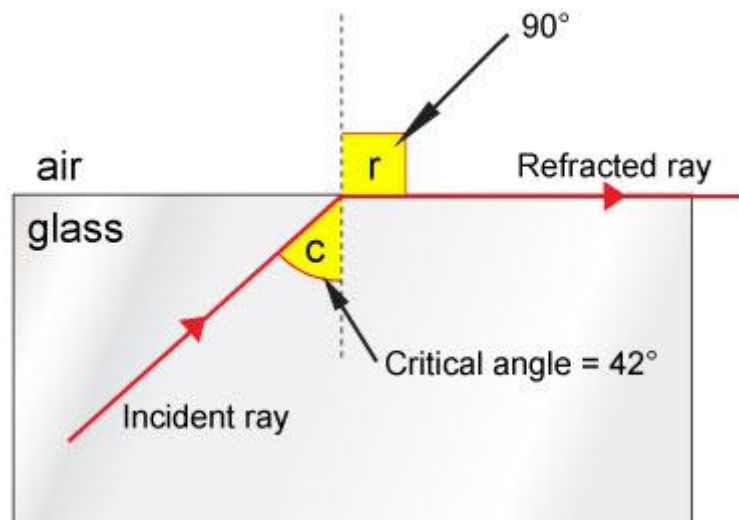

CRITICAL ANGLE

When light travels from one medium to another it changes speed and is refracted. If the light rays are travelling for a less dense material to a dense medium they are refracted towards the normal and if they are travelling from a dense to less dense medium they are refracted away from the normal.

For total internal reflection to occur the light must travel from a dense medium to a less dense medium (e.g. glass to air or water to air).

As the angle of incidence increases so does the angle of refraction. When the angle of incidence reaches a value known as the **critical angle** the refracted rays travel along the surface of the medium or in other words are refracted to an angle of 90°. The critical angle for the angle of incidence in glass is 42°.

When the angle of incidence of the light ray reaches the critical angle (42^0) the angle of refraction is 90^0. The refracted ray travels along the surface of the denser medium in this case the glass.

According to the law of refraction,

$$n_1 sin\theta_i = n_2 sin\theta_t$$

$$\frac{sin\theta_t}{sin\theta_i} = \frac{n_1}{n_2} = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}$$

$$if\ \theta_i = \theta_c\ then\ \theta_t = \frac{\pi}{2}$$

$$\theta_c = sin^{-1}\frac{n_2}{n_1}\ \ or\ sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

TOTAL INTERNAL REFLECTION

When the angle of incidence of the light ray is greater than the critical angle then no refraction takes place. Instead, all the light is reflected back into the denser material in this case the glass. This is called **total internal reflection.**
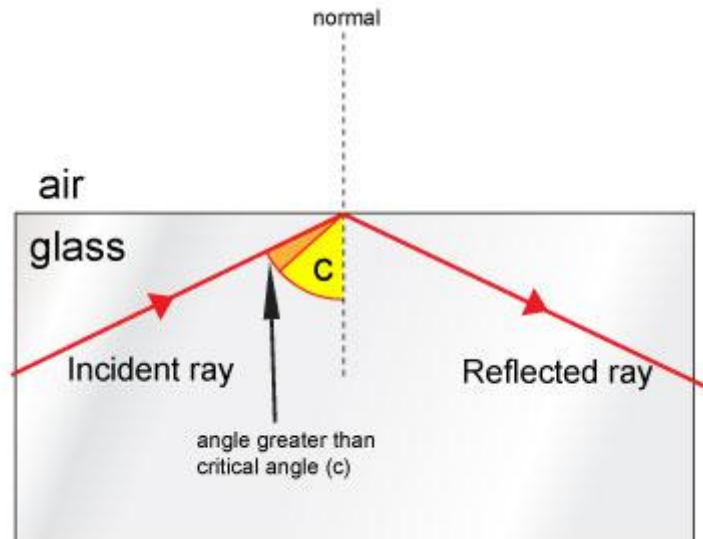
SURFACE IMPEDANCE

It is defined as the ratio of the tangential component of the electric field to the surface current density at the conductor surface.

It is given by

$$Z_s = \frac{E_{tan}}{J_s} \ \Omega$$

Where E_{tan} is the tangential component parallel to the surface of the conductor.

And J_s is the surface current density.

Poynting vector and theorem:

- Electromagnetic waves transport throughout space the energy and momentum arising from a set of charges and currents (the sources).

- If the electromagnetic waves interact with another set of charges and currents in a receiver, information (energy) can be delivered from the sources to another location in space.

- The energy and momentum exchange between waves and charges and currents is described by the Lorentz force equation.

Derivation of Poynting's Theorem

- Poynting's theorem concerns the conservation of energy for a given volume in space. Poynting's theorem is a consequence of Maxwell's equations

- Time-Domain Maxwell's curl equations in differential form

$$\nabla \times \underline{E} = -\underline{K}_i - \underline{K}_c - \frac{\partial \underline{B}}{\partial t}$$

$$\nabla \times \underline{H} = \underline{J}_i + \underline{J}_c + \frac{\partial \underline{D}}{\partial t}$$

- Recall a vector identity

$$\nabla \cdot (\underline{E} \times \underline{H}) = \underline{H} \cdot \nabla \times \underline{E} - \underline{E} \cdot \nabla \times \underline{H}$$

$$-\underline{E} \cdot \nabla \times \underline{H} = -\underline{E} \cdot \underline{J}_i - \underline{E} \cdot \underline{J}_c - \underline{E} \cdot \frac{\partial \underline{D}}{\partial t}$$

$$\underline{H} \cdot \nabla \times \underline{E} = -\underline{H} \cdot \underline{K}_i - \underline{H} \cdot \underline{K}_c - \underline{H} \cdot \frac{\partial \underline{B}}{\partial t}$$

$$\nabla \cdot (\underline{E} \times \underline{H}) = \underline{H} \cdot \nabla \times \underline{E} - \underline{E} \cdot \nabla \times \underline{H}$$

$$= -\underline{H} \cdot \underline{K}_i - \underline{H} \cdot \underline{K}_c - \underline{H} \cdot \frac{\partial \underline{B}}{\partial t}$$

$$-\underline{E} \cdot \underline{J}_i - \underline{E} \cdot \underline{J}_c - \underline{E} \cdot \frac{\partial \underline{D}}{\partial t}$$

- For simple, lossless media, we have

$$\int_V (\underline{E} \cdot \underline{J}_i + \underline{H} \cdot \underline{K}_i) dv = -\int_V \left(\varepsilon \underline{E} \cdot \frac{\partial \underline{E}}{\partial t} + \mu \underline{H} \cdot \frac{\partial \underline{H}}{\partial t} \right) dv$$

$$-\oint_S (\underline{E} \times \underline{H}) \cdot d\underline{s}$$

- Hence, we have the form of Poynting's theorem valid in simple, lossless media:

$$\underline{A} \cdot \frac{\partial \underline{A}}{\partial t} = A \frac{\partial A}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} \left(A^2 \right)$$

$$\int_V \left(\underline{E} \cdot \underline{J}_i + \underline{H} \cdot \underline{K}_i \right) dv = -\frac{\partial}{\partial t} \int_V \left(\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right) dv$$
$$- \oint_S \left(\underline{E} \times \underline{H} \right) \cdot d\underline{s}$$

Physical Interpretation of the Terms in Poynting's Theorem

$$\int_V \sigma E^2 dv + \int_V \sigma_m H^2 dv$$

represent the instantaneous power dissipated in the electric and magnetic conductivity losses, respectively, in volume *V*. represent the total electromagnetic energy stored in the volume *V*.

$$\int_V \left(\frac{1}{2} \varepsilon' E^2 + \frac{1}{2} \mu' H^2 \right) dv$$

$$\oint_S \left(\overline{E} \times \overline{H} \right) \cdot d\overline{s}$$

represents the flow of instantaneous power out of the volume *V* through the surface *S*

- In words the Poynting vector can be stated as "The sum of the power generated by the sources, the imaginary power (representing the time-rate of increase) of the stored electric and magnetic energies, the power leaving, and the power dissipated in the enclosed volume is equal to zero."

$$\overline{S} = \overline{E} \times \overline{H}$$

We define a new vector called the (instantaneous) ***Poynting vector*** as

- Poynting vector has units of W/m^2.

- The Poynting vector has the same direction as the direction of propagation.

- The Poynting vector at a point is equivalent to the power density of the wave at that point.

Time-Average Poynting Vector

- The time-average Poynting vector can be computed from the instantaneous Poynting vector as

$$\underline{S}_{av}(\underline{r}) = \frac{1}{T_p} \int\limits_0^{T_p} S(\underline{r},t)\,dt$$

- The time-average Poynting vector can also be computed as

$$\underline{S}_{av}(\underline{r}) = \frac{1}{2}\,\mathrm{Re}\left[\underline{E} \times \underline{H}^*\right]$$

Power Loss in a Conductor:

- The resistance of the slab along the direction of the current I is

$$dR = \frac{\rho l}{A} = \frac{1}{\sigma \, dz}$$

-
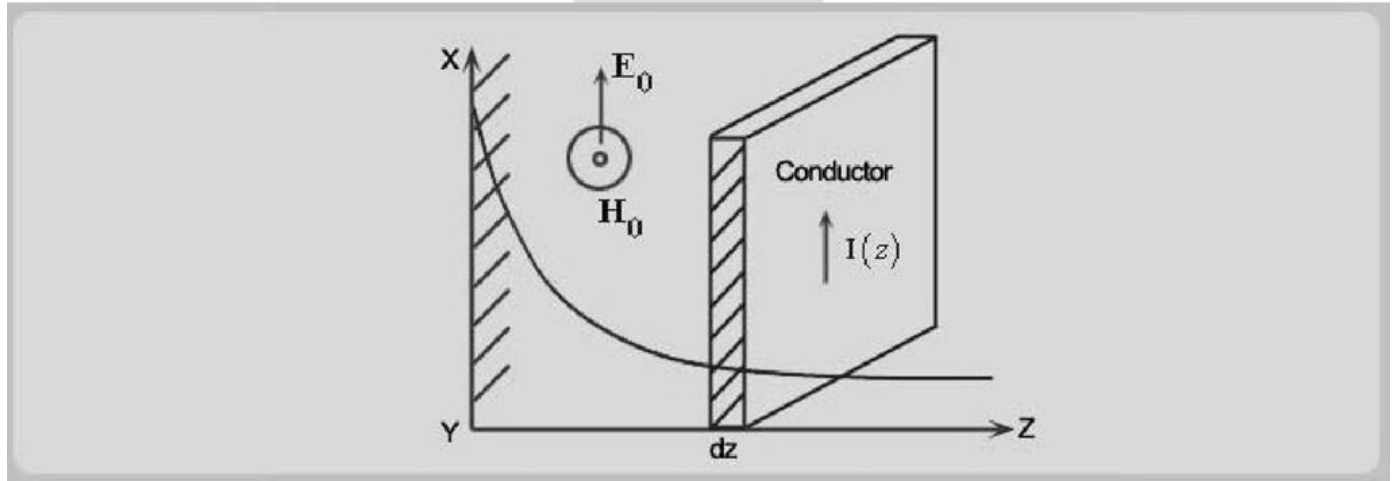
$dR = \rho l / A = I/\sigma\ dz$

The ohmic loss in the slab is

$$dW = |I(Z)|^2 dR$$

Substituting for I(z) we get

$$dW = |\sigma E_0 e^{-\gamma z} dz|^2 \ (1/\ \sigma dz)$$

$$= \sigma |E_0|^2 \ e^{-2az} dz$$

The total loss per unit area of the conductor surface therefore is

$W = \int \sigma |E_0|^2 e^{-2az} dz$

$\quad = \sigma |E_0|^2 [e^{-2az}/-2\alpha]$

$W = \sigma |E_0|^2/2\alpha \ = \sigma/2\alpha(|\gamma^2|/\ \sigma^2)\ |J_s|^2$

- Substituting for γ and α , the loss per unit area of the conducting surface is
  $W = R_s |Js|^2$

The power loss is proportional to the surface resistance which increases with frequency and decreases with conductivity. Higher the conductivity lesser the loss and for ideal conductor when the conductivity is infinite the ohmic loss is zero.

PROBLEMS:

1. A uniform plane wave is travelling at a velocity $2.5*10^5$ m/s having wavelength $\lambda=0.25$nm in a non magnetic good conductor .calculate the frequency of the wave and the conductivity of the medium $1.6*10^5$ simen/m.

Solution formulae:

   1.$f=(\upsilon/\lambda)$

   2.$\upsilon=(\omega/\beta)$

   3.$\beta=2\pi f/\upsilon$

2. A 300MHZ uniform plane wave propagates through fresh water for which $\mu_r=1, \varepsilon_r=78, \sigma=0$ Find attenuation constant, phase constant, wavelength , intrinsic impedance.

Solution Formulae:

   1.Phase constant $\beta=\omega\sqrt{\mu\varepsilon}$

   2.$\eta=\sqrt{\mu/\varepsilon}$

3. Find the skin depth and surface resistance of a copper conductor at 100MHz having conductivity $\sigma=5.8*10^7$, $\mu_r=100$.

Solution Formulae:

   Skin depth $s=\sqrt{(2/\omega\mu\sigma)}$

4. For a wave travelling in air the electric field is given by $E=6\cos(\omega t-\beta z)a_x$ at frequency 10MHZ calculate β,H and average pointing vector .

Solution Formulae:

   $\beta=2\pi/\lambda$

   $\lambda=\upsilon/f$

   $E/H=\sqrt{\mu/\varepsilon}$

   $H=(E/\eta_0)$

Transmission Lines and Types:

Definition: Any physical structure that will guide an electromagnetic wave place to place is called a Transmission Line.

Types of Transmission Lines:-

1)Open wire line
2)Coaxial cable
3)Waveguide s   a)Rectangular
                        b)Circular
4)Optical Fibre Cable

Open wire line:

A transmission line consisting of two spaced parallel wires supported by insulators, at the proper distance to give adesired value of surge impedance. Also known as open-wire feeder. Open wire cable is a two-conductor flat cable used as a balanced transmission line to carry radio frequency (RF) signals. It is constructed of two stranded copper or copper-clad steel wires, held a precise distance apart by a plastic (usually polyethylene) ribbon. The uniform spacing of the wires is the key to the cable's function as a transmission line; any abrupt changes in spacing would reflect some of the signal back toward the source. The plastic also covers and insulates the wires.
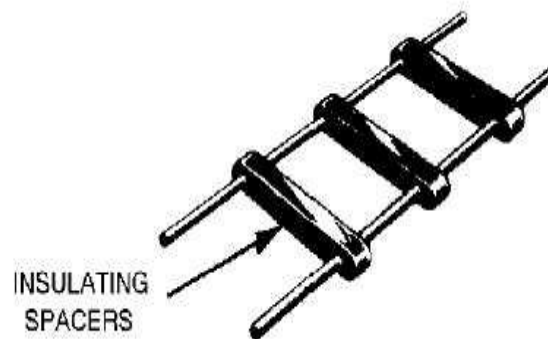
Fig: Open wire line

Coaxial cable:

Coaxial cable, or coax is a type of electrical cable that has an inner conductor surrounded by a tubular insulating layer, surrounded by a tubular conducting shield. Many coaxial cables also have an insulating outer sheath or jacket. The term coaxial comes from the inner conductor and the outer shield sharing a geometric axis. Coaxial cable was invented by English engineer and mathematician Oliver Heaviside, who patented the design in 1880. Coaxial cable differs from other shielded cables because the dimensions of the cable are controlled to give a precise, constant conductor spacing, which is needed for it to function efficiently as a transmission line.

Fig: Coaxial cable

Waveguide:

A waveguide is a structure that guides waves, such as electromagnetic waves or sound, with minimal loss of energy by restricting expansion to one dimension or two. There is a similar effect in water waves constrained within a canal, or guns that have barrel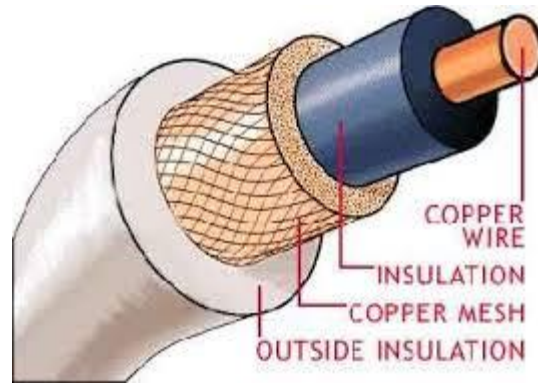s which restrict hot gas expansion to maximize energy transfer to their bullets. Without the physical constraint of a waveguide, wave amplitudes decrease according to the inverse square law as they expand into three dimensional space.

There are different types of waveguides for each type of wave. The original and most common meaning is a hollow conductive metal pipe used to carry high frequency radio waves, particularly microwaves.

The geometry of a waveguide reflects its function. Slab waveguides confine energy in one dimension, fiber or channel waveguides in two dimensions. The frequency of the transmitted wave also dictates the shape of a waveguide: an optical fiber guiding high-frequency light will not guide microwaves of a much lower frequency. As a rule of thumb, the width of a waveguide needs to be of the same order of magnitude as the wavelength of the guided wave.
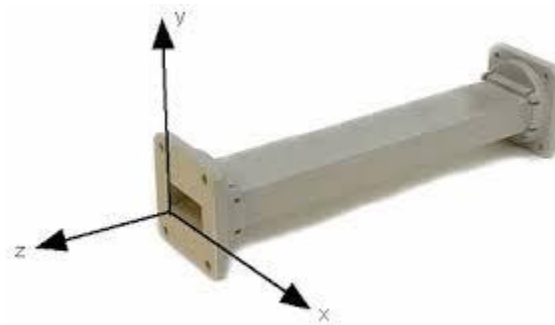
Fig: wave guide

Optical Fibre:

An optical fiber or optical fibre is a flexible, transparent fiber made by drawing glass (silica) or plastic to a diameter slightly thicker than that of a human hair. Optical fibers are used most often as a means to transmit light between the two ends of the fiber and find wide usage in fiber-optic communications, where they permit transmission over longer distances and at higher bandwidths (data rates) than wire cables. Fibers are used instead

of metal wires because signals travel along them with less loss; in addition, fibers are immune to electromagnetic interference, a problem from which metal wires suffer excessively. Fibers are also used for illumination and imaging, and are often wrapped in bundles so that they may be used to carry light into, or images out of confined spaces, as in the case of a fiberscope. Specially designed fibers are also used for a variety of other applications, some of them being fiber optic sensors and fiber lasers.

Fig: optical fibre

At low frequencies, the circuit elements are Lumped voltage and current waves affect the entire circuit at the same time.

At microwave frequencies Voltage and   current waves do not affect the entire circuit at the same time

The circuit must be broken down into unit sections within which the circuit elements are considered to be lumped

Basic Transmission Line:

It is convenient to describe a transmission line in terms of its line parameters, which are

1)Resistance per unit length **R**,

2)Inductance per unit length **L**,

3)Conductance per unit length **G**,

4)Capacitance per unit length **C**.

For Each Line

$$LC = \mu\varepsilon \quad \text{and} \quad G/C = \sigma/\varepsilon$$

The line parameters *R, L, G, and C are not discrete or lumped but distributed as* shown in Figure.
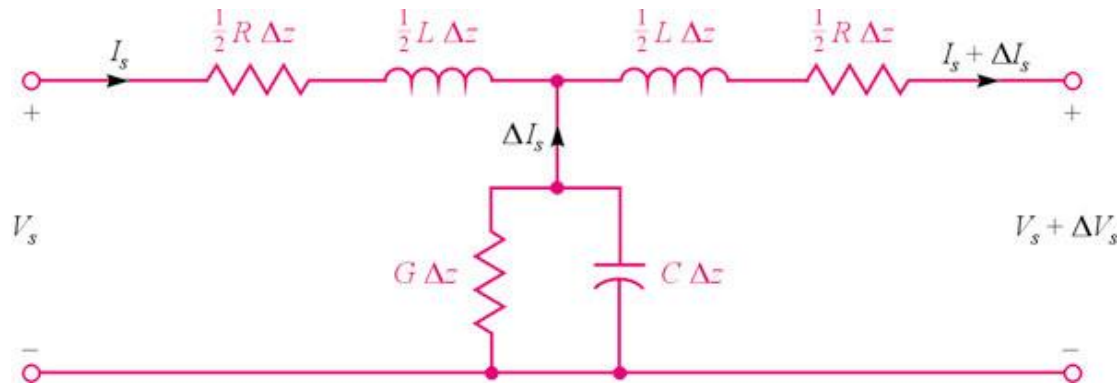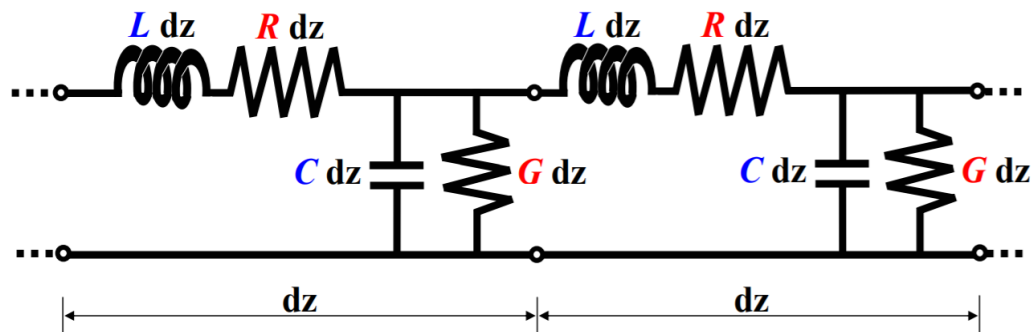

Fig: Basic Transmission line

Transmission line Equations:

- If we are familiar with low frequency circuits and the circuit consists of lumped impedance elements (R, L, C), we treat all lines(wires) connecting the various circuit elements as perfect wires, with no voltage drop and no impedance associated to them.

- This is a reasonable procedure as long as the length of the wires is much smaller than the wavelength of the signal and at any given time, the measured voltage and current are the same for each location on the same wire.

- Let us try to explore what happens, when the signal propagates as a wave of voltage and current along the line at sufficiently high frequencies

- For sufficiently high frequencies, wavelength is comparable to the length of a conductor, so the positional dependence impedance properties (position dependent voltage and current) of wire cannot be neglected, because it cannot change instantaneously at all locations

- Consider a uniform transmission line is a "distributed circuit" that we can describe as a cascade of identical cells with infinitesimal length. So, our first goal is to represent the uniform transmission line as a distributed circuit and determine the differential voltage and current behavior of an elementary cell of the distributed circuit.

- Once that is known, we can find a global differential equation that describes the entire transmission line by considering a cascaded network (subsections) of these equivalent models.

- In order to analyze the transmission wave behavior and positional dependence impedance properties, the line possess a certain series inductance and resistance. In addition, there is a shunt capacitance between the

conductors, and even a shunt conductance if the medium insulating the wires is not perfect. We use the concept of shunt conductance, rather than resistance, because it is more convenient for adding the parallel elements of the shunt.

- We can represent the uniform transmission line with the distributed circuit below (general lossy line) and the impedance parameters L, R, C, and G represent:

  - L = series inductance per unit length
  - R = series resistance per unit length
  - C = shunt capacitance per unit length
  - G = shunt conductance per unit length.

  ▪ Each cell of the distributed circuit will have impedance elements with values: Ldz, Rdz, Cdz, and Gdz, where dz is the infinitesimal length of the cells
  ▪ Let us assume a differential length dz of transmission line ,V(z),I(z) are voltage and current at point P and V(z)+dV, I(z)+dI are voltage and current at point Q
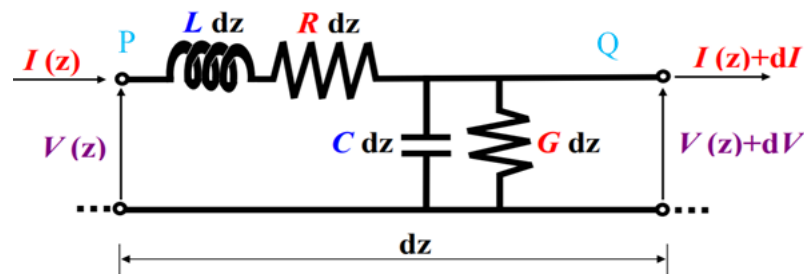
  ▪ The series inductance determines the variation of the voltage from input to output of the cell, according to the sub-circuit below
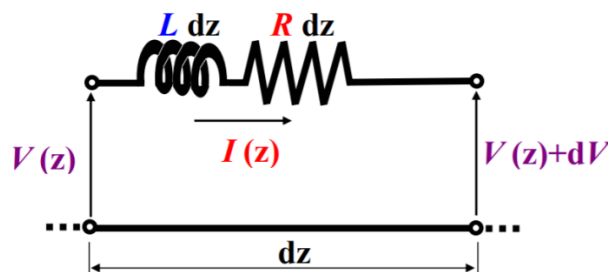
  ▪ Similarly, the shunt admittance determines the variation of the voltage and current from input to output of the cell shown below
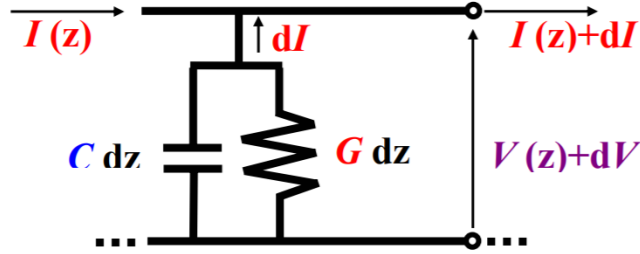
- For a differential length dz of transmission line ,the series impedance and shunt admittance of the elemental length can be written as

$$(R + j\omega L)dz$$

$$(G + j\omega C)dz$$

- The corresponding circuit voltage and current equation is

$$V + dV - V = -I(R + j\omega L)dz \qquad (1)$$
$$I + dI - I = -V(G + j\omega C)dz \qquad (2)$$

- From which we obtain a first order differential equation

$$\frac{dV}{dz} = -I(R + j\omega L) \qquad (3)$$

$$\frac{dI}{dz} = -V(G + j\omega C) \qquad (4)$$

- We have a system of coupled first order differential equations that describe the behavior of voltage and current on the transmission line
- It can be easily obtained by a set of uncoupled equations by differentiating equation (3) and (4) with respect to the coordinate z

$$\frac{d^2V}{dz^2} = -\frac{dI}{dz}(R + j\omega L) \qquad (5)$$

$$\frac{d^2I}{dz^2} = -\frac{dV}{dz}(G + j\omega C) \qquad (6)$$

- Substitute equation (4) in equation (5) and equation (3) in equation (6), we obtain the second order differential equations

$$\frac{d^2V}{dz^2} = (G + j\omega C)(R + j\omega L)V \qquad (7)$$

$$\frac{d^2I}{dz^2} = (G + j\omega C)(R + j\omega L)I \qquad (8)$$

- These equations can be written in terms of propagation constant γ

$$\frac{d^2V}{dz^2} = \gamma^2 V \qquad (9)$$

$$\frac{d^2I}{dz^2} = \gamma^2 I \qquad (10)$$

▪ Let the constant term γ can be represented as propagation constant, which is written as

$$\gamma = \alpha + j\beta = \sqrt{(G + j\omega C)(R + j\omega L)}$$

▪ γ is the complex propagation constant, which is function of frequency
▪ *α* is the attenuation constant in nepers per unit length, *β* is the phase constant in radians per unit length

▪ The solution of the second order transmission line equation is

$$V = ae^{+\gamma z} + be^{-\gamma z} \qquad (11)$$

$$I = ce^{+\gamma z} + de^{-\gamma z} \qquad (12)$$

▪ Where, a, b, c, and d are the constants
▪ Equation (11) and (12) represent the standard solutions of the wave equations, which are similar to the solution of uniform plane wave equations.

▪ The terms $e^{+\gamma z}$ and $e^{-\gamma z}$ can be represented as backward and forward wave along z-direction.

Determination of the constant terms:

- Let the solutions of the transmission line wave equations can written as

$$V = ae^{+\gamma z} + be^{-\gamma z} \qquad (1)$$

$$I = ce^{+\gamma z} + de^{-\gamma z} \qquad (2)$$

- To determine the constants a,b,c, and d , the above equations can be written in terms of hyperbolic functions, where substitute

$$e^{\gamma z} = \cosh \gamma z + \cosh \gamma z \qquad (3)$$

$$e^{-\gamma z} = \cosh \gamma z - \cosh \gamma z \qquad (4)$$

- Substitute equation (3) and (4) in equations (1),(2)

$$V = a(\cosh \gamma z + \cosh \gamma z) + b(\cosh \gamma z - \cosh \gamma z) \qquad (5)$$

$$I = c(\cosh \gamma z + \cosh \gamma z) + d(\cosh \gamma z - \cosh \gamma z) \qquad (6)$$

$$V = (a+b)\cosh \gamma z + (a-b)\sinh \gamma z \qquad (7)$$

$$I = (c+d)\cosh \gamma z + (c-d)\sinh \gamma z \qquad (8)$$

- The constants a+b, a-b, c+d, and c-d can be replaced by another constant terms A, B,C, and D respectively.
- So, the equations (7) and (8) can be written as

$$V = A\cosh \gamma z + B\sinh \gamma z \qquad (9)$$

$$I = C\cosh \gamma z + D\sinh \gamma z \qquad (10)$$

- In order to reduce the four constant terms to two constant terms, we write the relation between V and I by considering the following basic differential equations

$$-\frac{dV}{dz} = I(R + j\omega L) \qquad (11)$$

$$-\frac{dI}{dz} = V(G + j\omega C) \qquad (12)$$

- Substitute equation (9) in equation (11)

$$-\frac{d(A\cosh \gamma z + B\sinh \gamma z)}{dz} = I(R + j\omega L) \qquad (13)$$

- Differentiating equation (13) in terms of z, we obtain

$$-\gamma(A\sinh \gamma z + B\cosh \gamma z = I(R + j\omega L) \qquad (14)$$

  - From the above equation(14) the current I can be written in terms of constants A and B as

$$-\frac{\gamma}{(R + j\omega L)}(A\sinh \gamma z + B\cosh \gamma z = I \qquad (15)$$

- Where γ is propagation constant, which is $\gamma = \sqrt{(G + j\omega C)(R + j\omega L)}$ and substitute in equation (15), we obtain

$$-\frac{\sqrt{(G+j\omega C)(R+j\omega L)}}{(R+j\omega L)}(A\sinh\gamma z + B\cosh\gamma z = I \qquad (16)$$

- The equation can be simplified and written as

$$-\sqrt{\frac{(G+j\omega C)}{(R+j\omega L)}}(A\sinh\gamma z + B\cosh\gamma z = I$$

$$-\frac{1}{Z_0}(A\sinh\gamma z + B\cosh\gamma z = I \qquad (17)$$

- Where Z_0 is another constant and can be called as characteristic impedance along the line

$$Z_0 = \sqrt{\frac{(R+j\omega L)}{(G+j\omega C)}}$$

- Form equation (9) and equation (17) the solution of voltage and current wave equation in terms of constants A and B can be re written as

$$V = A\cosh\gamma z + B\sinh\gamma z \qquad (18)$$

$$I = -\frac{1}{Z_0}(A\sinh\gamma z + B\cosh\gamma z) \qquad (19)$$

- Now the constants A and B can be obtained by applying initial conditions of the transmission line at Z=0
- Let V_s and I_s be the source voltage and current respectively. At the source end, Z=0 the voltage V= V_s, current I= I_s, then the equations (18) and (19) can be simplified as

$$V_s = A\cosh\gamma(0) + B\sinh\gamma(0)$$

$$I_s = -\frac{1}{Z_0}(A\sinh\gamma(0) + B\cosh\gamma(0))$$

$$V_s = A$$

$$-I_s Z_0 = B$$

- Substitute A and B values in equation (18) and equation (19), we obtain

$$V = V_s\cosh\gamma z - I_s Z_0\sinh\gamma z \qquad (20)$$

$$I = -\frac{V_s}{Z_0}\sinh\gamma z + I_s\cosh\gamma z \qquad (21)$$

- These equations are called transmission line equations. The can represent voltage and current at any point z from the source voltage and current.

Infinite Transmission Line:
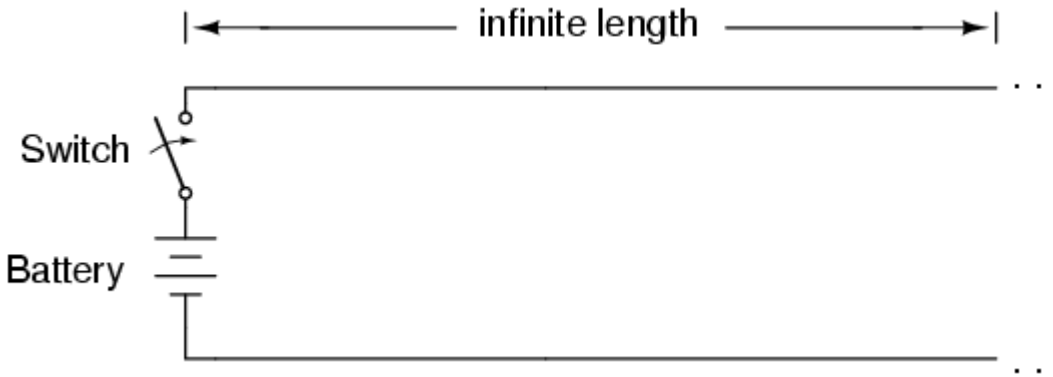
If a line of infinite length is considered then all the power fed into it will be absorbed. The reason is as we move away from the input terminals towards the load, the current and voltage will decrease along the line and become zero at an infinite distance, because the voltage drops across the inductor and current leaks through the capacitor. By considering this hypothetical line of infinite line an important terminal condition is formed.

Let V_s be the sending end voltage and I_s be the sending end current and Z_s be the input impedance which is given by

$$Z_S = \frac{V_S}{I_S}$$

Current at any point distance x from sending end is given by

$$I = ce^{Px} + de^{-Px}$$

The value of c & d can now determined by considering an infinite line.

At the sending end x=0 and I =I_s.

$$I_S = c + d$$

At the receiving end, I=0 and x=∞.

$$0 = c * \infty,$$

$$\infty \neq 0,$$

$$therefore\ c = 0.$$

$$if\ c = 0, then\ I_S = d$$

$$therefore\ I = I_S e^{-Px}$$

The above equation gives current at any point of an infinite line.

And $\quad V = V_s e^{-Px}$

Similarly the above equation gives voltage at any point of an infinite line.

Infinite line is equivalent to a finite line terminated in its Z_o

When a finite length of line is joined with a similar kind of infinite line, their total input impedance is same as that of infinite line itself, because they together make one infinite line however the infinite line alone presents an impedance Z_o at its input PQ because the input impedance of an infinite line is Z_o.

It is therefore concluded that a finite line has an impedance Z_o when it is terminated in Z_o.

Or A finite line terminated by its Z_o behaves as an infinite line.

Let a finite length of 'l' is terminated by its characteristic impedance Z_o and is having voltage and current V_R and I_R at terminating end.

Therefore $\quad\quad\quad\quad\quad\quad Z_o = \dfrac{V_R}{I_R}$

$$Putting\ x = l, V = V_R, I = I_R\ in\ general\ equations, we\ have$$

$$V_R = V_S coshpl - I_S Z_O sinhpl$$

$$I_R = I_S coshpl - \frac{V_S}{Z_O} sinhpl$$

Dividing V_R by I_R we will get the value of Z_O.

$$Z_o = \frac{V_R}{I_R} = \frac{V_S coshpl - I_S Z_O sinhpl}{I_S coshpl - \dfrac{V_S}{Z_O} sinhpl}$$

Multiplying right hand side numerator and denominator by Z_O we get,

$$Z_o = \frac{V_S coshpl - I_S Z_O sinhpl}{I_S coshpl - \dfrac{V_S}{Z_O} sinhpl}\frac{Z_o}{Z_o}$$

$$Z_o\ I_S coshpl - V_S sinhpl = V_S coshpl - I_S Z_O sinhpl$$

$$Z_O I_S(coshpl + sinhpl) = Z_o(coshpl + sinhpl)$$

Therefore

$$V_S = Z_O I_S \quad and \ Z_O = \frac{V_S}{I_S}$$

But $\frac{V_S}{I_S}$ is the input impedance of the line($Z_{in} = Z_o$)

Therefore $Z_{in} = Z_o$

Thus the input impedance of a finite line terminated in its characteristic impedance is the characteristic impedance of the line. Since the input impedance of an infinite line is the characteristic impedance of the line.

Therefore a finite line terminated in its Z_O is equivalent to an infinite line as both will have an input impedance of $Z_{O.}$

CHARACTERISTIC IMPEDANCE AND PROPAGATION CONSTANT:

Equivalent-circuit model of transmission line section:
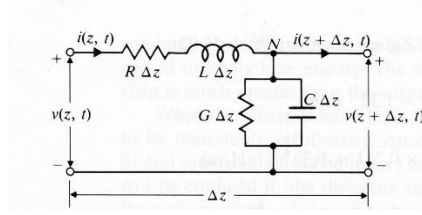
$$R(\Omega/m),\ L(H/m),\ G(S/m),\ C(F/m)$$

Transmission line equations: In higher-frequency range, the transmission line model is utilized to analyze EM power flow.

$$\begin{cases} -\dfrac{v(z+\Delta z,t)-v(z,t)}{\Delta z} = Ri(z,t)+L\dfrac{\partial i(z,t)}{\partial t} \\ -\dfrac{i(z+\Delta z,t)-i(z,t)}{\Delta z} = Gv(z,t)+C\dfrac{\partial v(z,t)}{\partial t} \end{cases} \Rightarrow \begin{cases} -\dfrac{\partial v}{\partial z} = Ri+L\dfrac{\partial i}{\partial t} \\ -\dfrac{\partial i}{\partial z} = Gv+C\dfrac{\partial v}{\partial t} \end{cases}$$

Set $v(z,t)=Re[V(z)e^{j\omega t}]$, $i(z,t)=Re[I(z)e^{j\omega t}]$

$$\Rightarrow \begin{cases} -\dfrac{dV}{dz} = (R+j\omega L)I(z) \\ -\dfrac{dI}{dz} = (G+j\omega C)V(z) \end{cases} \Rightarrow \begin{cases} \dfrac{d^2V(z)}{dz^2} = (R+j\omega L)(G+j\omega C)V(z) = \gamma^2 V(z) \\ \dfrac{d^2I(z)}{dz^2} = (R+j\omega L)(G+j\omega C)I(z) = \gamma^2 I(z) \end{cases}$$

where $\gamma = \alpha + j\beta = \sqrt{(R+j\omega L)(G+j\omega C)} \Rightarrow V(z)=V_0^+ e^{-\gamma z}+V_0^- e^{\gamma z},\ I(z)=I_0^+ e^{-\gamma z}+I_0^- e^{\gamma z}$

Characteristic impedance: $Z_0 = \dfrac{V_0^+}{I_0^+} = -\dfrac{V_0^-}{I_0^-} = \dfrac{R+j\omega L}{\gamma} = \dfrac{\gamma}{G+j\omega C} = \sqrt{\dfrac{R+j\omega L}{G+j\omega C}}$

Note:

1. International Standard Impedance of a Transmission Line is $Z_0 = 50\Omega$.
2. In transmission-line equivalent-circuit model, $G \ne 1/R$.

The ratio of Forward Voltage and Current waves is always Z_0 , and the ratio of the Backward Voltage and Current waves is always - Z_0 . The parameters γ and Z_0 completely define the voltage and current behaviour on a transmission line. These two parameters are related to R, L, G, and C, and the frequency of the signal. In transmission line analysis knowledge of γ and Z_0 is adequate and the explicit knowledge of R, L, G, C is rarely needed.

Eg. A *d-c* generator of voltage and internal resistance is connected to a lossy transmission line characterized by a resistance per unit length *R* and a conductance per unit length *G*. (a) Write the governing voltage and current transmission-line equations. (b) Find the general solutions for *V(z)* and *I(z)*.

(Sol.) (a) $\omega = 0 \Rightarrow \gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{RG}$

$$\frac{d^2 V(z)}{dz^2} = RGV(z), \quad \frac{d^2 I(z)}{dz^2} = RGI(z)$$

(b) $V(z) = V_0^+ e^{-\sqrt{RG}z} + V_0^- e^{\sqrt{RG}z}, \quad I(z) = I_0^+ e^{-\sqrt{RG}z} + I_0^- e^{\sqrt{RG}z}$

Lossless line (*R=G=0*):

$$\gamma = \alpha + j\beta = j\omega\sqrt{LC} \Rightarrow \alpha = 0, \quad \beta = \omega\sqrt{LC}, \quad v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}, \quad Z_0 = \sqrt{\frac{L}{C}} = R_0 + jX_0 \Rightarrow R_0 = \sqrt{\frac{L}{C}}, \quad X_0 = 0$$

Low-loss line (*R<<ωL, G<<ωC*):

$$\gamma = \alpha + j\beta \approx j\omega\sqrt{LC}(1 + \frac{1}{2j\omega}(\frac{R}{L} + \frac{G}{C})] \Rightarrow \alpha =\approx \frac{1}{2}(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}), \beta = \omega\sqrt{LC}, \quad v_p \approx \frac{1}{\sqrt{LC}}$$

$$Z_0 \approx \sqrt{\frac{L}{C}}[1 + \frac{1}{2j\omega}(\frac{R}{L} - \frac{G}{C})]$$

Distortionless line (*R/L=G/C*):

$$\gamma = \alpha + j\beta = \sqrt{\frac{C}{L}}(R + j\omega L) \Rightarrow \alpha = R\sqrt{\frac{C}{L}}, \quad \beta = \omega\sqrt{LC}, \quad v_p = \frac{1}{\sqrt{LC}}, \quad Z_0 = \sqrt{\frac{L}{C}}$$

Large-loss line (*ωL<<R, ωC<< G*):

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta = \sqrt{RG} \cdot (1 + j\frac{\omega L}{R})^{\frac{1}{2}}(1 + \frac{j\omega C}{G})^{\frac{1}{2}} \approx \sqrt{RG}[1 + \frac{j\omega}{2}(\frac{L}{R} + \frac{C}{G})]$$

$$\therefore \alpha \approx \sqrt{RG}, \beta \approx \frac{\omega}{2}(L \cdot \sqrt{\frac{G}{R}} + C \cdot \sqrt{\frac{R}{G}}), \ v_p = \frac{1}{2}(L \cdot \sqrt{\frac{G}{R}} + C \cdot \sqrt{\frac{R}{G}})$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{R}{G}} \cdot (1 + \frac{j\omega L}{R})^{\frac{1}{2}} \cdot (1 + \frac{j\omega C}{G})^{-\frac{1}{2}} = \sqrt{\frac{R}{G}} \cdot [1 + \frac{j\omega}{2}(\frac{L}{R} - \frac{C}{G})]$$

GROUP VELOCITY AND PHASE VELOCITY

PHASE VELOCITY

A single (infinite) wave is described by the expression cos(ωt − kx) or sin[2π λ (x − vt)] or equivalent. The pattern travels with a velocity (actually a speed)

v_p = λ T = fλ = ω/k v_p is what matters with interference.

GROUP VELOCITY

An infinite wave is unrealistic. A real wave has to have beginning and end. The overall shape is called the envelope. Various shapes are possible - abrupt or gentle. v_g = dω /dk is the velocity of the envelope.

Illustration

Consider two waves almost in step.

They have ω1, k1 and ω2, k2 (and λ1, λ2...)

Write the means and differences ω = ω1+ω2/ 2 , k = k1+k2 /2

Δω = ω1−ω2 /2 , Δk = k1−k2/ 2

the original quantities can be expressed in terms of these ω1 = ω + Δω, ω2 = ω − Δω etc

Adding the two waves gives a total wave

$e^{i(\omega 1t - k1x)} + e^{i(\omega 2t - k2x)}$

This can be written

$e^{i(\omega t + \Delta \omega t - kx - \Delta kx)} + e^{i(\omega t - \Delta \omega t - kx + \Delta kx)}$

Take out a common factor:

$e^{i(\omega t - kx)} (e^{i(\Delta \omega t - \Delta kx)} + e^{i(-\Delta \omega t + \Delta kx)})$

Remembering cosθ = $e^{i\theta}$+$e^{-i\theta}$ /2

this is $2\cos(\Delta\omega t - \Delta kx)e^{i(\omega t - kx)}$ The first term is clearly the envelope. It has small wavenumber and frequency and so a long wavelength and period. It travels with velocity $v_g = \Delta\omega/\Delta k$. This generalises $v_g = d\omega/dk$

Finding v_g often v_p is known from measurements or from basic principles. Take the expression for v_p and write ω/k for v_p in it. Turn all the λ and f etc terms into ω and k. Then differentiate with respect to k. This gives an expression involving $d\omega/dk$ from which v_g can be extracted.

As a trivial example, suppose v_p is constant (i.e. independent of wavelength) with value c. Then $\omega = ck$ and

LOSS LESS LINE:

A transmission line is said lo be lossless if the conductors of the line are perfect $(\sigma_c = \infty)$ and the dielectric medium separating them is lossless $(\sigma = 0)$.

$$R = 0 = G$$

This is a necessary condition for a line to be lossless.

Thus for such a line, the following Parameters are

Propagation Constant $\Upsilon = j\beta = j\omega \sqrt{(LC)}$

Attenuation Constant $\alpha = 0$

Phase Constant $\beta = \omega \sqrt{(LC)}$

Velocity $\vartheta = \omega/\beta$

Then $X0 = 0$ $Z0 = R0 = \sqrt{L/C}$

Distortion Less Line:

A transmission line has a distributed inductance on each line and a distributed capacitance between the two conductors. We will consider the line to have zero series resistance and the insulator to have infinite resistance (a zero conductance or perfect insulator).

A signal normally consists of a band of frequencies; wave amplitudes of different frequency components will be attenuated differently in a lossy line as α is *frequency dependent.* This results in distortion.

A distortion less line is one in which the attenuation constant α *is frequency independent* while the phase constant β is linearly dependent on frequency. From the general expression for α and β [in eq. (10)], a distortion less line results if the line parameters are such that

$$R/L = G/C$$

Thus for a distortion less line

$\Upsilon = \sqrt{[RG(1+j\omega L/R)(1+j\omega C/G)]}$

We know that

$\Upsilon = \alpha + j\beta$

then

$\alpha = \sqrt{RG}$

$\beta = \omega \sqrt{(LC)}$

Impedance of the line is

$$ZO \sqrt{\frac{R(1+\frac{j\omega L}{R})}{G(1+\frac{j\omega C}{R})}} = \sqrt{R/G} = \sqrt{L/C}$$

Resistance

$$Ro = \sqrt{R/G} = \sqrt{L/C} \quad XO = 0$$

Phase Velocity:

$$\vartheta = \omega/\beta = 1/\sqrt{LC}$$

The phase velocity is independent of frequency because the phase constant β Linearly depends on frequency. We have shape distortion of signals unless *α and u are* independent of frequency. *u and Z_o remain the same as for lossless lines.*

A lossless line is also a distortion less line, but a distortion less line is not necessarily lossless. Although lossless lines are desirable in power transmission, telephone lines are required to be distortion less.

DISTORTION LESS LINE:

      A transmission line is said to be distortion less if the attenuation constant 'α' is frequency independent, white the phase constant is linearly dependent of frequency. From the general equations of α & β, the distortionless line results, if the line parameters are such that,

$$\frac{R}{L} = \frac{G}{C}$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega c}}$$

$$= \sqrt{\frac{L\left(\frac{R}{L} + j\omega\right)}{C\left(\frac{G}{C} + j\omega\right)}} = \sqrt{\frac{L}{C}}$$

$$\alpha = P = \sqrt{(R + j\omega L)(G + j\omega c)}$$

$$\sqrt{R\left(1 + j\omega\frac{L}{R}\right)G\left(1 + j\omega\frac{C}{G}\right)}$$

$$P = \sqrt{RG}\left(1 + j\omega\frac{C}{G}\right)$$

$$\alpha = \sqrt{RG}, \quad \beta = \omega\sqrt{LC}$$

$$VP = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{2C}} = \frac{1}{\sqrt{2c}} \text{ [phase velocity]}$$

CONCLUSIONS:

 ➢ The phase velocity is independent of frequency because, the phase constant of β, linearly depends upon frequency. We have distortions of signals unless α & V_p are independent of frequency.
 ➢ Both V_P and Z_0 remains the same for loss less line.
 ➢ A loss less line is also a distortion less line, but a distortion less line is not necessarily loss less. Loss less lines are desirable in power transmission telephone lines are to b distortion less.

$$\boxed{V_P = \frac{\omega}{f} = \lambda f}$$

Condition for Minimum Attenuation:

$\propto = \sqrt{RG}, \omega\sqrt{LC}$

From the equation the attenuation of a line is expressed by attenuation constant DC as

$$\propto = \frac{1}{2}\sqrt{(RG + \omega^2 LC) + \sqrt{R^2 + \omega^2 L^2(G^2 + \omega^2 C^2)}}$$

It is observed that $\propto$ depends on the four primary constants in addition to the frequency. The Value of L for minimum attenuation.

Let us assume the three line constants C,G and R including ω are constant and only L may be varied.

Therefore, differentiating above value of $\propto$, W.R.T L and equating it to zero, we get

$$\frac{dx}{dL} = \frac{1}{2}\left[\frac{1}{2}\left\{\frac{2\omega^2 L(G^2+\omega^2 C^2)}{\sqrt{R^2+\omega^2 L^2(G^2+\omega^2 C^2)}} - \omega^2 C\right\}\right] = 0$$

By cross multiplying

$$\frac{\omega^2 L(G^2+\omega^2 C^2)}{\sqrt{R^2+\omega^2 L^2(G^2+\omega^2 C^2)}} - \omega^2 C = 0$$

$$\cancel{\omega^2}\, L\sqrt{\frac{G^2+\omega^2 C^2}{R^2+\omega^2 L^2}} = \cancel{\omega^2} C$$

After cancellation remaining terms are

$$L^2\sqrt{G^2 + \omega^2 C^2} = C\sqrt{R^2 + \omega^2 L^2}$$

Squaring on both sides

$$L^2(G^2 + \omega^2 C^2) = C(R^2 + \omega^2 L^2)$$

$$L^2 G^2 = C^2 R^2$$

$$LG = CR$$

$$\boxed{\frac{G}{C} = \frac{R}{L}}$$

The condition is similar to the distortion less line. Therefore, the value of $\propto$ obtained in above equation is some for line with minimum attenuation. From the final equation if only L is variable, the attenuation is minimum when $L = \frac{CR}{G}$ H/km.

This result is important because, in practice, L is normally less than desired value and hence the attenuation of a line can be reduced by increasing L artificially.

LOADING OF TRANSMISSION LINES: loading is a method of increasing the series inductance of a line by the addition of external inductance its purpose is to improve the performance of the line by reducing attenuation and distortion

TYPES OF LOADING:

1. Continuous loading (iron wire)
2. Lumped loading(copper wire)

1. In this type of iron bar or other magnetic material is wound to the bar to be load. This increasing permeability of surrounding medium and thereby increasing conducting.
    ➢ This is measured at and increase in inductance up to 65mH/km.
    ➢ It is quite expensive. Furthermore eddy current and hysteresis loses in the magnetic material increase the primarily loss,

    The inductance of a line can be increased by the introduction of loading coil of uniform intervals. This method of loading is more convenient.

LOADED LINES

The delay and frequency distortion is introduced by a cable. Let us consider a typical 16 gauge paper insulated cable pair whose line parameters:

$$R = 42.1 \text{ohms/km}$$
$$G = 1.5 \ \mu \text{ mhos/km}$$
$$C = 0.062 \ \mu\text{F/km}$$
$$L = 1 \text{ mH/km.}$$

Therefore, $\frac{R}{L} = \frac{42.1}{1 \ X \ 12^{-3}}$ - 42.1 X 10^3

$\frac{G}{C} = \frac{1.5 \ X \ 10^{-6}}{0.062 \ X \ 10^{-6}} = 0.0242$ X 10^3

Thus, $\frac{R}{L} >> \frac{G}{C}$.

Hence, distortion less and minimum attenuation conditions as expressed by and are not satisfied.

However, both the conditions can be approached in four ways:

(i) Reduce R. this will decrease the attenuation but will require large conductors, which in turn causes an increase in cable size and cost. Reduction of T will also lower $|Z_0|$.

(ii) Increase G. this can be accomplished by lowering conductor insulation or by introducing shunt conductance along the line. This is a poor solution because it increases losses. Increasing G also lowers $|z_0|$.

(iii) Decrease C . This will increase the spacing between conductor, resulting increase of cable size and cost. Decreasing C increases $|z_0|$ although lower α.

(iv)   Increase L. this decreases α and reduces distortion and hence offers the   best approach to achieve distortion less and minimum attenuation condition

LOADING:

The advantage of loading is not so great – on open wire line which have an appreciable inductance of their own and so have much less distortion than cable. As a result the practice of loading open wire line has been abandoned. the loading practice is, therefore normally restricted to cables only. There are three type of loading in practice this day:

(i)   **Continuous loading**. A type of iron or some other magnetic material as numeral is wound round the conductor to be loaded thus increasing the permeability of the surrounding medium and thereby increasing inductance. This method can give an increase in inductance upto about 65 mH per km but, it is quite expensive due to laborious construction. Therefore, continuous loading is used only on ocean cables, where the problem of making water – tight joints at loading points renders lumped loading difficult. Furthermore, repair of a creak in the cable would normally result in alteration in the loading coil spacing for that section resulting in irregularity.

(ii)   **Patch loading**. This type of loading employs sections of continuously loaded cable separated by section of unloaded cable in the way the advantage of continuous loading is obtained but the cost is greatly reduced. In fact, in submarine cable, it has been found unnecessary to use continuous loading over the entire cable to obtain the required reduction in attenuation and desired result without continuous loading over the entire length of the cable .the typical length for the section is normally quarter kilometer.

(iii)   **Lumped loading**. the inductance of a line can also be increased by the introduction of loading coil or uniform intervals. This is called lumped loading. As already explained in Art 5.6 that a lumped loaded line behaves as a low pass filter. This method of loading is more convenient than continuous loading provided that a limited frequency ranged upto f_c the loading coils have a certain resistance and thus, increasing total effective inductance increases R.

Problem 1:

An air line has characteristic impedance of 70Ω and a phase constant of 3Rafm at 100MHz calculate the inductance per meter and capacitance per meter of the line .

Solution:
 Air line is nothing but loss less line.
For loss less line,

$$R = 0, G = 0$$

$$Z_0 = \sqrt{\frac{L}{c}} = \beta = j\omega \sqrt{LC}$$

$$P = j\omega \sqrt{LC}$$

$$V_p = \frac{1}{\sqrt{LC}}$$

$$\frac{z_0}{\beta} = \frac{\sqrt{\frac{L}{c}}}{j\omega \sqrt{LC}} = \frac{1}{j\omega c}$$

$$C = \frac{\beta}{z_0}$$

$$C = \frac{3}{2\pi X 100 X 10^6 \, (70)} = 68.\text{PF/m}.$$

$$Z_0 = \sqrt{\frac{L}{c}} \ , \ L = Z_0^2 c = (70)^2 (68.2 X 10^{-12})$$

$$= 334.2 \text{nH/m}$$

Problem 2:

1. A distortion less line has $Z_0 = 60\Omega$ X=20mvP/m, VP=0.6C, where c is the speed of line in vacuum. Find R, L, G, C, and λ at 100MHz.

$$\frac{R}{L} = \frac{G}{C}$$

$$Z_0 = \sqrt{\frac{L}{c}}$$

$$\propto = \sqrt{RG}, \quad \beta = \omega\sqrt{LC}$$

$$V_P = \frac{1}{\sqrt{Lc}}$$

$$Rc = GL$$

$$G = \frac{RC}{L}$$

$$\propto = \sqrt{RG} = {}^R\sqrt{\frac{C}{L}}$$

$$\propto = \frac{R}{Z_0}$$

R = $\propto$ $\underline{Z_0}$ = 1.2Ω/m

λ $\quad = \frac{V_P}{f} = 1.8$m

V_p $Z_0 = \frac{1}{C}$

$C = \frac{1}{V_P z_0} = 92.59$pF/m

$G = \frac{\propto^2}{R} = 0.33 = 333\mu$s/m

$\frac{Z_0}{V_P} = L = 333$nH/m.

Problem 3:

Calculate Primary constants for loop kilometer are R = 196Ω, C=0.09mF, L=0.71mH and leakage conductance is negligible. Calculate intrinsic impedance and propagation constant for

$$f = \frac{2000}{2\pi f} \text{ Hz} \qquad\qquad\qquad Z = R + j\omega L$$

$$\omega = 2\pi f$$
$$= 2\pi\left(\frac{2000}{2\pi}\right)$$

$$= 196 + (2000)(0.7mH) \qquad \omega$$
$$= 196 + j(1.4)$$
$$Z = 196.00 \angle 0.40$$

$y = G + j\omega C$
$= j\,(2000)(0.09)mf$
$0.18\,j$
$Y = 0.18 \angle 90$

$Z_0 = \sqrt{\dfrac{z}{y}}$

$Z_0 = \sqrt{1088} \angle -89.6$
$\quad = \sqrt[8]{17} \angle -89.6/2$
$\quad = 32.98 \angle -89.6/2$
$\quad = 32.98 \angle -44.8$

$P = \sqrt{z\,y}$
$\quad = \sqrt{(196.00 \angle 0.40)(0.18 \angle 90)}$
$\quad = 5.93 \angle 45.2$

Problem1:

An open wire transmission line terminated in its characteristic impedance has the following primary constant at 1KHz. R=6 Ω/km, L=2 mH/km, G=0.5 u℧, C=0.005 uF/km. Calculate the phase velocity and attenuation in decibels suffered by a signal in a length of 100 kms.

Problem2:

The primary constants of a cable are R=80 Ω/km, L=2 mH /km and G=0.3 u℧ /km. C=0.01 uF/km. Calculate the secondary constants at a frequency of 1KHz.

Problem3:

A generator of 1V, 1 KHz supplies power to a 100 km long line terminated in Z_o and having the following constants, R = 10.4 Ω/km, L = 0.00367 H/km, G = 0.8x10^{-6} ℧/km and C = 0.00835x10^{-6} F/km. Calculate Z_o, attenuation constant α, phase constant β, wavelength λ and velocity V.

Problem3:

A loss less transmission line has 115 Ω characteristic impedance. The line is terminated in a load impedance of 100-j250 Ω. The maximum voltage measured on the line is 120V. Find the maximum current and minimum voltage on the line.

UNIT – V
UHF TRANSMISSION LINES AND APPLICATIONS

Input Impedance Relations:-

- Let the voltage and current transmission line equations at any point on the line from the source end can be written as

$$V = V_s \cosh \gamma z - I_s Z_0 \sinh \gamma z$$

$$I = -\frac{V_s}{Z_0} \sinh \gamma z + I_s \cosh \gamma z$$

- Suppose, a transmission line, which is terminated with some load impedance Z_R at a distance 'l' from the load and the voltage and current at the terminating end can be written as V_R and I_R respectively.

- So that at z= l, the voltage and current can be written as

$$V_R = V_s \cosh \gamma l - I_s Z_0 \sinh \gamma l$$

$$I_R = -\frac{V_s}{Z_0} \sinh \gamma l + I_s \cosh \gamma l$$

- Now the load impedance from the terminating point can be written as

$$Z_R = \frac{V_R}{I_R} = \frac{V_s \cosh \gamma l - I_s Z_0 \sinh \gamma l}{-\dfrac{V_s}{Z_0} \sinh \gamma l + I_s \cosh \gamma l}$$

- By taking the cross multiplication the above equation can be written as

$$Z_R \left(-\frac{V_s}{Z_0} \sinh \gamma l + I_s \cosh \gamma l \right) = V_s \cosh \gamma l - I_s Z_0 \sinh \gamma l$$

- Take the current I_S on both sides, the above equation can be written as

$$I_s Z_R \left(-\frac{V_s}{I_s Z_0} \sinh \gamma l + \cosh \gamma l \right) = I_s \left(\frac{V_s}{I_s} \cosh \gamma l - Z_0 \sinh \gamma l \right)$$

- Now the ratio of $\dfrac{V_s}{I_s}$ can be written as the source impedance or input impedance (Z_{in} or Z_S). So, the above equation can be modified as

$$Z_R \left(-Z_s \sinh \gamma l + Z_0 \cosh \gamma l \right) = Z_0 \left(Z_s \cosh \gamma l - Z_0 \sinh \gamma l \right)$$

- Modify the above equation by separating Z_S terms,

$$Z_R \left(-Z_s \sinh \gamma l + Z_0 \cosh \gamma l \right) = Z_0 \left(Z_s \cosh \gamma l - Z_0 \sinh \gamma l \right)$$

$$Z_0\left(Z_0 \sinh \gamma l + Z_0 \cosh \gamma l\right) = \left(Z_0 Z_s \cosh \gamma l + Z_s Z_R \sinh \gamma l\right)$$

$$Z_s = Z_0 \frac{\left(Z_0 \sinh \gamma l + Z_R \cosh \gamma l\right)}{\left(Z_0 \cosh \gamma l + Z_R \sinh \gamma l\right)}$$

- Where Z_S is the source impedance, which is also called input impedance. So, the input impedance can be written as

$$Z_{in} = Z_s = Z_0 \frac{\left(Z_0 \sinh \gamma l + Z_R \cosh \gamma l\right)}{\left(Z_0 \cosh \gamma l + Z_R \sinh \gamma l\right)}$$

- The above expression can be written in terms of hyperbolic tan functions as

$$Z_{in} = Z_s = Z_0 \frac{\left(Z_0 \tanh \gamma l + Z_R\right)}{\left(Z_0 + Z_R \tanh \gamma l\right)}$$

- The above expression represents input impedance of a general transmission, which is terminated with load impedance at a distance 'l' from the source end.

- Now, consider a special case, If the line is lossless line (attenuation constant $\alpha=0$) the propagation constant γ can be written as

$$\gamma = \alpha + j\beta = 0 + j\beta$$

- So, the input impedance can be written as

$$Z_{in} = Z_s = Z_0 \frac{\left(Z_0 \tanh j\beta l + Z_R\right)}{\left(Z_0 + Z_R \tanh j\beta l\right)}$$

- The hyperbolic tan function $\tanh j\beta l$ can be written as $\tanh j\beta l = j\tanh \beta l$, therefore the input impedance can be written as

$$Z_{in} = Z_s = Z_0 \frac{\left(jZ_0 \tan \beta l + Z_R\right)}{\left(Z_0 + jZ_R \tan \beta l\right)}$$

- We, now express some special cases from the above general lossless line input impedance relations.

Case i: When the line is terminated with characteristics impedance, i.e. $Z_R=Z_0$,

$$Z_{in} = Z_0 \frac{\left(jZ_0 \tan \beta l + Z_0\right)}{\left(Z_0 + jZ_0 \tan \beta l\right)} = Z_0$$

- So, the line is terminated with characteristic impedance, the input impedance is equal to the characteristic impedance. This condition is called matched load condition, which means that there are no reflections from the load.

Case ii: When the line is terminated with open circuited, i.e. Z_R=infinity (∞),

$$Z_{in} = Z_0 \frac{(jZ_0 \tan \beta l + Z_R)}{(Z_0 + jZ_R \tan \beta l)}$$

- Take Z_R common from the above expression, the equation reduces to

$$Z_{in} = Z_{OC} = Z_0 \frac{\left(j\dfrac{Z_0}{Z_R} \tan \beta l + 1 \right)}{\left(\dfrac{Z_0}{Z_R} + j \tan \beta l \right)}$$

$$Z_{in} = Z_{OC} = Z_0 \frac{1}{j \tan \beta l}$$

$$Z_{in} = Z_{OC} = -jZ_0 \cot \beta l$$

Case iii: When the line is terminated with short circuited, i.e. Z_R=0,

$$Z_{in} = Z_0 \frac{(jZ_0 \tan \beta l + Z_R)}{(Z_0 + jZ_R \tan \beta l)}$$

Substitute ZR= 0 in the above expression, the equation reduces to

$$Z_{in} = Z_{SC} = jZ_0 \tan \beta l$$

Note: If the line is terminated with open circuit or short circuit, the input impedance can be purely imaginary. Depending on the value of length of the line, it may be capacitive or inductive.

- Moreover, from the short circuit and open circuit impedance relations, the characteristic impedance can be written as

$$Z_{OC}Z_{SC} = (jZ_0 \tan \beta l) \frac{Z_0}{j \tan \beta l}$$

$$Z_{OC}Z_{SC} = Z_0^2$$

$$\sqrt{Z_{OC}Z_{SC}} = Z_0$$

- From the above relation, it can be concluded, the characteristic impedance of the line can be measured from open and short circuit Transmission line.

Transmission Line with Load Impedance: -

- In previous lecture, we have derived the expression for input impedance of a transmission line, if the source voltage and source current is known to us.

- In this lecture, we are deriving similar expression, if we know the reinitiating voltage and current, instead of source voltage and current.

- To derive the input impedance expression, consider a basic transmission line, which is terminated at a distance l, with known terminating voltage and current.

- The voltage and current equations, which was derived in our previous lectures with two unknown constants can be rewritten here as

$$V = A\cosh\gamma z + B\sinh\gamma z$$

$$I = -\frac{1}{Z_0}(A\sinh\gamma z + B\cosh\gamma z)$$

- Where, A and B are constants, Z_0 is the characteristic impedance.

- Now the terminating voltage and current at a distance $z=l$ can be written as

$$V_R = A\cosh\gamma l + B\sinh\gamma l$$

$$I_R = -\frac{1}{Z_0}(A\sinh\gamma l + B\cosh\gamma l)$$

- To derive the constants A and B, the above voltage and current equations can be multiplied with $\cosh\gamma l$ and $\sinh\gamma l$, respectively and add the resultant equations. Then, we get the constants A as

$$V_R\cosh\gamma l = A\cosh^2\gamma l + B\sinh\gamma l\cosh\gamma l$$

$$I_R\sinh\gamma l = -\frac{1}{Z_0}(A\sinh^2\gamma l + B\sinh\gamma l\cosh\gamma l)$$

$$V_R\cosh\gamma l + Z_0 I_R\sinh\gamma l = A\cosh^2\gamma l + B\sinh\gamma l\cosh\gamma l - A\sinh^2\gamma l - B\sinh\gamma l\cosh\gamma l$$

$$V_R\cosh\gamma l + Z_0 I_R\sinh\gamma l = A(\cosh^2\gamma l - \sinh^2\gamma l)$$

- Therefore,

$$V_R\cosh\gamma l + Z_0 I_R\sinh\gamma l = A$$

- Similarly, the constant B can be obtained by multiplying voltage equation with $\sinh \gamma l$ and current equation with $\cosh \gamma l$, and adding both the equations, we obtain B as

$$V_R \sinh \gamma l - Z_0 I_R \cosh \gamma l = A \sinh \gamma l \cosh \gamma l + B \cosh^2 \gamma l - A \sinh \gamma l \cosh \gamma l - B \sinh^2 \gamma l$$

$$V_R \cosh \gamma l - Z_0 I_R \sinh \gamma l = B(\cosh^2 \gamma l - \sinh^2 \gamma l)$$

$$V_R \cosh \gamma l - Z_0 I_R \sinh \gamma l = B$$

- Substituting the values of A and B in the basic equation, we obtain voltage and current at any point from the terminating point, which can be written as

$$V = V_R \cosh \gamma(l - z) + \frac{V_R}{Z_0} \sinh \gamma(l - z)$$

$$I = \frac{V_R}{Z_0} \sinh \gamma(l - z) + I_R \cosh \gamma(l - z)$$

- Where, $l - z$ is the distance from the terminating end.

- Now from the above expressions, the input impedance z=0 can be written as

$$V_S = V_R \cosh \gamma(l - 0) + I_R Z_0 \sinh \gamma(l - 0)$$

$$I_S = \frac{V_R}{Z_0} \sinh \gamma(l - 0) + I_R \cosh \gamma(l - 0)$$

$$\frac{V_S}{I_S} = \frac{V_R \cosh \gamma l + I_R Z_0 \sinh \gamma l}{\frac{V_R}{Z_0} \sinh \gamma l + I_R \cosh \gamma l}$$

$$Z_{in} = Z_0 \frac{(Z_0 \tanh \gamma l + Z_R)}{(Z_0 + Z_R \tanh \gamma l)}$$

Illustrative problem

A source with 50 Ω source impedance drives a 50 Ω transmission line that is 1/8 of wavelength long, terminated in a load $Z_L = 50 - j25$ Ω. Calculate:

(i)The reflection coefficient, Γ_L, (ii)   VSWR, (iii) The input impedance seen by the source.

Solution: The given problem can be represented with the following figure

(i) The reflection coefficient,

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$= \frac{(50 - j25) - 50}{(50 - j25) + 50} = 0.242 e^{-j76^0}$$

(ii) VSWR

$$VSWR = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = 1.64$$

(iii)The input impedance seen by the source, Z_{in}

To calculate input impedance, it needs to calculate the electrical length, $\beta l = \dfrac{2\pi}{\lambda} \dfrac{\lambda}{8} = \dfrac{\pi}{4}$

Therefore,

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l}$$

$$= 50 \frac{50 - j25 + j50}{50 + j50 + 25}$$

$$= 30.8 - j3.8 \Omega$$
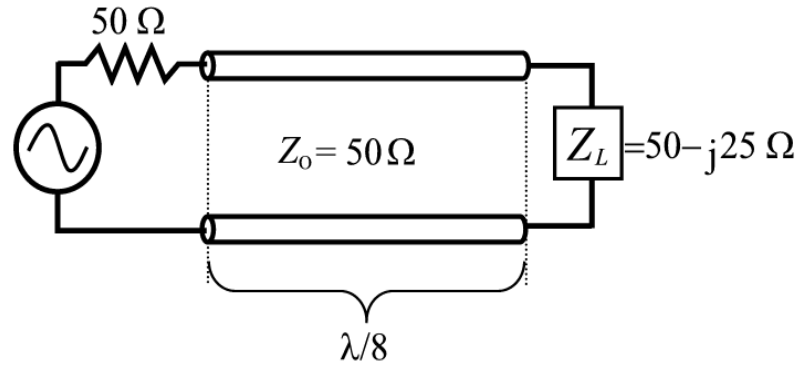
Reflection and Reflction Coefficient: -

- Consider a transmission line terminated in a load of impedance Z_R, which is not equal to the characteristic impedance, the incident wave from the source will be reflected at the termination.
- The phenomenon of wave being reflected from termination is called reflection.

- So, in general, there will be forward and backward waves going in the line. Hence, the general transmission line consists of both incident and reflection waves. These can be written from the solutions of transmission line wave equations as

$$V = ae^{+\gamma z} + be^{-\gamma z}$$

$$I = \frac{1}{Z_0}\left(be^{-\gamma z} - ae^{+\gamma z}\right)$$

- Where, $e^{+\gamma z}, e^{-\gamma z}$ are the reflected and incident waves, respectively.
- Now the reflection coefficient is defined as the ratio of reflected voltage/current to the incident voltage/current. From the above equations, the voltage and current reflection coefficient can be written as,

$$\Gamma = \frac{ae^{+\gamma z}}{be^{-\gamma z}}$$

$$\Gamma = -\frac{ae^{-\gamma z}}{be^{-\gamma z}}$$

- To determine the constants in the reflection coefficient expressions shown above, we have to consider the voltage and current from termination. So, the distance measured from termination in opposite direction can be expressed as x=-z, the expressions can be simplified as

$$V = ae^{-\gamma x} + be^{\gamma x}$$

$$I = \frac{1}{Z_0}\left(b^{\gamma x} - ae^{-\gamma x}\right)$$

- Now at the load point i.e. x=0, the voltage V=V$_R$ and current I=I$_R$. So, the above equations can be simplified as

$$V_R = a + b$$

$$I_R = \frac{1}{Z_0}(b - a)$$

- Solving the above two equations, we obtain

$$V_R + I_R Z_0 = a + b + b - a$$

$$\frac{V_R + I_R Z_0}{2} = b$$

Similarly,

$$V_R - I_R Z_0 = a + b - b + a$$

$$\frac{V_R - I_R Z_0}{2} = a$$

- Substitute these equations on voltage or current reflection coefficient expression $\Gamma = \dfrac{ae^{+\gamma z}}{be^{-\gamma z}}$, we obtain

$$\Gamma_V = \frac{\dfrac{V_R - I_R Z_0}{2} e^{+\gamma z}}{\dfrac{V_R + I_R Z_0}{2} e^{-\gamma z}}$$

$$\Gamma_I = -\frac{\dfrac{V_R - I_R Z_0}{2} e^{-\gamma z}}{\dfrac{V_R + I_R Z_0}{2} e^{-\gamma z}}$$

- The above equations can be simplified as

$$\Gamma_V = \frac{\dfrac{V_R}{I_R} - Z_0 e^{+\gamma z}}{\dfrac{V_R}{I_R} + Z_0 e^{-\gamma z}}$$

$$\Gamma_I = -\frac{\dfrac{V_R}{I_R} - Z_0 e^{+\gamma z}}{\dfrac{V_R}{I_R} + Z_0 e^{-\gamma z}}$$

- Where, $\dfrac{V_R}{I_R}$ is the load impedance Z_R, therefore the voltage/current reflection coefficient becomes

$$\Gamma_V = \frac{Z_R - Z_0 e^{+\gamma z}}{Z_R + Z_0 e^{-\gamma z}}$$

$$\Gamma_I = -\frac{Z_R - Z_0 e^{+\gamma z}}{Z_R + Z_0 e^{-\gamma z}}$$

- From the above expression it can be observed that the reflection coefficient completely depends on load impedance and characteristic impedance.

- For a lossless line, the reflection coefficient can be expressed as

$$\Gamma_V = \frac{Z_R - Z_0 e^{+j\beta z}}{Z_R + Z_0 e^{-j\beta z}}$$

$$\Gamma_I = -\frac{Z_R - Z_0 e^{+j\beta z}}{Z_R + Z_0 e^{-j\beta z}}$$

- Now the magnitude of the voltage/current reflection coefficient can be expressed as

$$\left|\Gamma_{V/I}\right| = \left|\frac{Z_R - Z_0}{Z_R + Z_0}\right|$$

- Now, we will consider some special cases for reflection coefficient

Case i: When the line is terminated with characteristics impedance, i.e. $Z_R = Z_0$,

$$\left|\Gamma\right| = \left|\frac{Z_0 - Z_0}{Z_0 + Z_0}\right| = 0$$

- So, the line is terminated with characteristic impedance, the magnitude of reflection coefficient equal to zero. This condition is called matched load condition, which means that there are no reflections from the load.

Case ii: When the line is terminated with open circuited, i.e. Z_R=infinity (∞),

$$\left|\Gamma\right| = \left|\frac{1 - \dfrac{Z_0}{Z_R}}{1 + \dfrac{Z_0}{Z_R}}\right| = 1$$

Case iii: When the line is terminated with short circuited, i.e. $Z_R = 0$,

$$\left|\Gamma\right| = \left|\frac{Z_R - Z_0}{Z_R + Z_0}\right| = 1$$

- If the line is terminated with open circuit or short circuit, the magnitude of reflection coefficient is equal to one. Which indicate the complete wave is reflected from the terminating point. The Table below shows the behavior of voltage and current at the end of an open circuited or shorted transmission line

Table 1: Reflection properties of open and shorted transmission lines

Transmission Line Termination	Effect on voltage	Effect on current
open circuit	100% reflection no phase shift	100% reflection 180 deg phase shift
short circuit	100% reflection 180 deg phase shift	100% reflection no phase shift

Standing Wave Ratio: -

- When a transmission line (cable) is terminated by an impedance that does not match the characteristic impedance of the transmission line, not all of the power is absorbed by the termination.

- Part of the power is reflected back down the transmission line. So that the forward (or incident) signal mixes with the reverse (or reflected) signal cause a standing wave pattern along the transmission line.

- The standing wave pattern along the transmission line is as shown in **Figure 1**. So, the degree of voltage/current mismatch on the transmission line can be measured by a quantity called standing wave ratio(SWR). This can be measured either voltage/current or power using standing wave ratio meter.
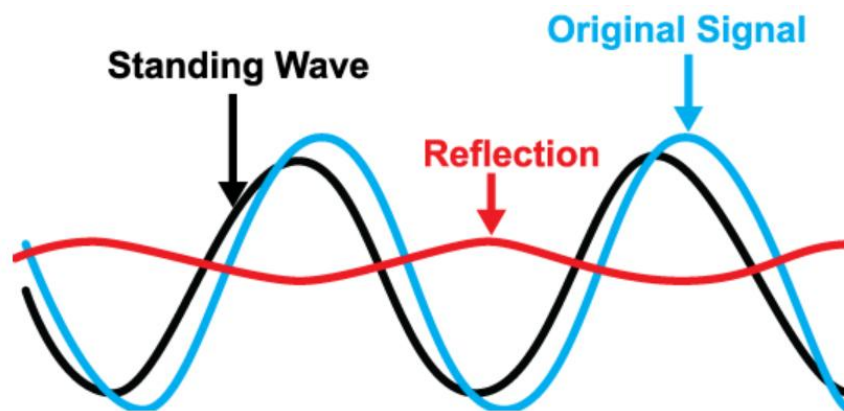

Figure 1. Standing wave pattern representation from incident and reflected wave

- The voltage SWR can be defined as the ratio of maximum voltage to minimum voltage along the line.

$$|VSWR| = \left| \frac{V_{max}}{V_{min}} \right|$$

- Where, V_{max}, V_{min} are the maximum and minimum voltages of the standing waves.

- The maximum voltage occurs when the incident and reflected waves are added and minimum voltage occurs when the incident and reflected waves are subtracted. Therefore, the SWR can be written in terms of incident and reflected voltages as

$$|VSWR| = \left| \frac{V_{max}}{V_{min}} \right| = \frac{|V_i| + |V_r|}{|V_i| - |V_r|}$$

$$|VSWR| = \frac{1 + \left|\dfrac{V_r}{V_i}\right|}{1 - \left|\dfrac{V_r}{V_i}\right|}$$

- But, we know the ratio $\dfrac{V_r}{V_i}$ is the reflection coefficient Γ.

- Now the VSWR in terms of reflection coefficient can be written as

$$|VSWR| = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

- Similarly, the reflection coefficient in terms of SWR can be written as

$$|\Gamma| = \frac{|VSWR| - 1}{|VSWR| + 1}$$

- Now, we will consider some special cases for SWR

Case i: When the reflection coefficient $|\Gamma| = 0$,

$$|VSWR| = \frac{1 + |0|}{1 - |0|} = 1$$

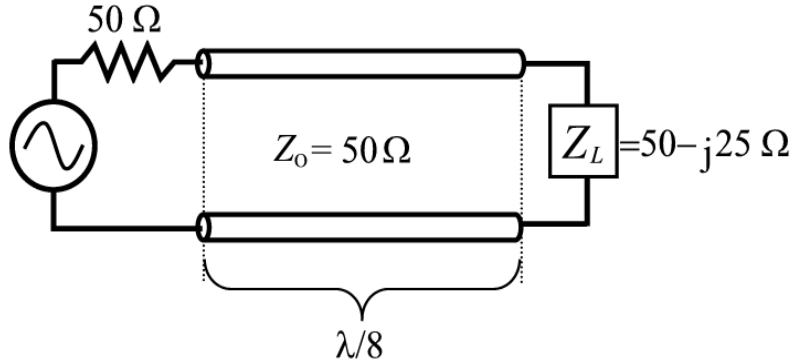**Case ii: When the reflection coefficient $|\Gamma| = \pm 1$,**

$$|VSWR| = \frac{1 + 1}{1 - 1} = \infty$$

Illustrative Problem

A source with 50 Ω source impedance drives a 50 Ω transmission line that is 1/8 of wavelength long, terminated in a load $Z_L = 50 - j25$ Ω. Calculate:

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Solution: The given problem can be represented with the following figure

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Tocalculate input impedance ,it needs to calculate the electrical length, $\beta l = \frac{2\pi}{\lambda} \frac{\lambda}{8} = \frac{\pi}{4}$

Therefore,

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l}$$

$$= 50 \frac{50 - j25 + j50}{50 + j50 + 25}$$

$$= 30.8 - j3.8\Omega$$

UHF Lines as Circuit Elements: -

- The lossless transmission lines at ultra-high frequency considered as circuit elements from the special cases of
  - ✓ A short at the load,
  - ✓ An open at the load.
- We know the short circuited and open circuited input impedance are purely imaginary. Those short circuited and open circuited impedance equations can be rewritten as below.

$$Z_{SC} = jZ_0 \tan \beta l$$

$$Z_{OC} = -jZ_0 \cot \beta l$$

- As the distance l increases from terminating point, the short and open circuit impedance changes and is purely reactive. Depending on the length of the line, the impedance can be either inductive or capacitive.

- Now to study the effect on the length of short circuit and open circuit line, the following special cases with different length are studied.

Case i: When the length of the line is less than quarter wavelength $\left(0 < l < \dfrac{\lambda}{8} \right)$,

$$\text{Let } l = \frac{\lambda}{8}, \quad \text{then } \beta l = \frac{\pi}{4}$$

So that,

$$Z_{SC} = jZ_0 \tan \frac{\pi}{4} = jZ_0$$

$$Z_{OC} = -jZ_0 \cot \frac{\pi}{4} = -jZ_0$$

- Therefore, for first quarter wave length $\left(0 < l < \dfrac{\lambda}{8} \right)$, the short-circuited line acts as inductive and open circuited line acts as capacitive

Case ii: When the length of the line is greater than quarter wavelength, but less than half wave length $\left(\dfrac{\lambda}{4} < l < \dfrac{\lambda}{2} \right)$,

$$\text{Let } l = \frac{\lambda}{3}, \text{ then } \beta l = \frac{2\pi}{3}$$

So that,

$$Z_{SC} = jZ_0 \tan \frac{2\pi}{3} = -j\sqrt{3}Z_0$$

$$Z_{OC} = -jZ_0 \cot \frac{2\pi}{3} = j\sqrt{3}Z_0$$

- Therefore, for length $\left(\dfrac{\lambda}{4} < l < \dfrac{\lambda}{2} \right)$, the short-circuited line acts as capacitive and open circuited line acts as inductive.

Case iii: When the length of the line is equal to quarter wavelength $\left(l = \dfrac{\lambda}{4} \right)$,

$$\text{Let } l = \frac{\lambda}{4}, \quad \text{then } \beta l = \frac{\pi}{2}$$

So that,

$$Z_{SC} = jZ_0 \tan \frac{\pi}{2} = \pm\infty$$

$$Z_{OC} = -jZ_0 \cot \frac{\pi}{2} = 0$$

- Therefore, for quarter wave length $\left(l = \dfrac{\lambda}{4} \right)$, the short-circuited line looks like a parallel resonant circuit (parallel resonant circuit impedance is infinity at resonance frequency) and open circuited line looks like a series resonant circuit (series resonant circuit impedance zero at resonance frequency)

Case iv: When the length of the line is equal to a half wavelength $\left(l = \dfrac{\lambda}{2} \right)$,

$$\text{Let } l = \frac{\lambda}{2}, \quad \text{then } \beta l = \pi$$

So that,

$$Z_{SC} = jZ_0 \tan \pi = 0$$

$$Z_{OC} = -jZ_0 \cot \pi = \pm\infty$$

- Therefore, for half wave length $\left(l = \dfrac{\lambda}{2} \right)$, the short-circuited line looks like a series resonant circuit (series resonant circuit impedance zero at resonance frequency) and open circuited line looks like a series resonant circuit (parallel resonant circuit impedance is infinity at resonance frequency)

Note: -These conditions repeat with multiple one-quarter or one-half wavelengths line.

Impedance Transformation: -

- A transmission line terminated with some impedance, Z_R, other than the characteristic impedance, Z_0, input impedance of a transmission line depends on its length, terminating impedance, which is given by

$$Z_{in} = Z_0 \frac{\left(Z_0 \tanh \gamma l + Z_R \right)}{\left(Z_0 + Z_R \tanh \gamma l \right)}$$

- Where γ is the line propagation constant

- Thus, to match the input impedance to deliver maximum power to the load, a short section of transmission line is added to the main transmission line.

- The important short section transmission lines with different lengths are studied here.

- In many situations, the transmission line will have no appreciable loss along the length of the line. So, the effect of these short section transmission lines, are studied for a lossless line.

- For a lossless line, the attenuation constant is zero and the propagation constant reduces $j\beta$, so that the input impedance relation reduces to

$$Z_{in} = Z_0 \frac{\left(jZ_0 \tan \beta l + Z_R \right)}{\left(Z_0 + jZ_R \tan \beta l \right)}$$

- Now the important short length of transmission lines are considered with different cases

Case (i) **Eight wavelength transmission line** $\left(l = \dfrac{\lambda}{8} \right)$

$$\text{Let } l = \frac{\lambda}{8}, \quad \text{then } \beta l = \frac{\pi}{4}$$

So that,

$$Z_{in} = Z_0 \frac{\left(jZ_0 \tan \dfrac{\pi}{4} + Z_R \right)}{\left(Z_0 + jZ_R \tan \dfrac{\pi}{4} \right)}$$

$$Z_{in} = Z_0 \frac{\left(jZ_0 + Z_R \right)}{\left(Z_0 + jZ_R \right)}$$

Case (ii) **Quarter wavelength transmission line** $\left(l = \dfrac{\lambda}{4} \right)$

$$\text{Let } l = \frac{\lambda}{4}, \quad \text{then } \beta l = \frac{\pi}{2}$$

So that,

$$Z_{in} = Z_0 \frac{\left(jZ_0 \tan \dfrac{\pi}{2} + Z_R \right)}{\left(Z_0 + jZ_R \tan \dfrac{\pi}{2} \right)}$$

$$Z_{in} = \frac{Z_0{}^2}{Z_R}$$

$$Z_0 = \sqrt{Z_{in} Z_R}$$

- Which can be called as quarter-wave impedance transformer.

- In other words, if the transmission line is precisely one-quarter wavelength long, the input impedance is inversely proportional to the load impedance.

- Suppose a 50 Ω line needs to be matched to a 100 Ω load. So that, to eliminate reflections we insert a $\dfrac{\lambda}{4}$ section of transmission line with 70.7Ω characteristic impedance to act as an impedance transformer

Case (iii) Half wavelength transmission line $\left(l = \dfrac{\lambda}{2} \right)$

$$\text{Let } l = \frac{\lambda}{2}, \quad \text{then } \beta l = \pi$$

So that,

$$Z_{in} = Z_0 \frac{\left(jZ_0 \tan \pi + Z_R \right)}{\left(Z_0 + jZ_R \tan \pi \right)}$$

$$Z_{in} = Z_0 \frac{Z_R}{Z_0} = Z_R$$

- A $\dfrac{\lambda}{2}$ length line transfers the load impedance to the source end.

Impedance Matching: -

- Impedance matching is one of the important aspects of high frequency circuit analysis.
- When UHF line is terminated with load impedance which is not equal to characteristics impedance, a mismatch occurs.

- To avoid reflections and power loss from transmission line sections impedance matching techniques can be used. There are various impedance matching techniques which are discussed in the following:

Quarter Wavelength Transformer

- A quarter wave length of line is generally used

  - For matching a resistive load to a transmission line
  - For matching two resistive loads
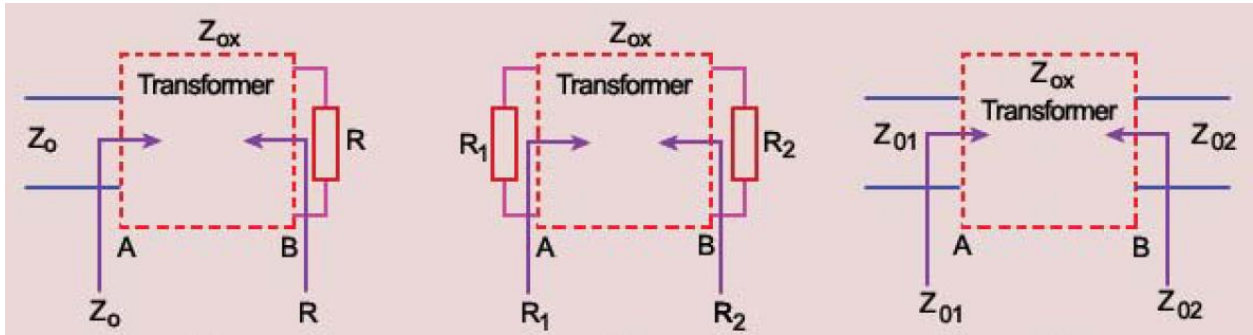 - For matching two transmission lines with unequal characteristic impedances

Figure 1 Quarter wave length of line for different cases

- By the introduction of the quarter wavelength transmission line acting as a transformer and perfectly matched in between the three cases mentioned above. That is the impedance seen towards right at A (**Figure 1**), and impedance seen towards left at B should be same.
- For the transformer we have two parameters to control, the characteristic impedance of the transformer section, and the length of the transformer section.
- Let us assume that the characteristic impedance of the transformer section is Z_{0x}. So, for quarter wave length, the transformer inverts the normalized impedance. Therefore, the impedance seen at A towards right in Figure (a) would be

$$Z_A = \frac{1}{R/Z_{0x}} Z_{0x} = \frac{Z_{0x}^{\ 2}}{R}$$

- For matching at A, Z_A, should be equal to Z_0 .i.e.

$$\frac{Z_{0x}^{\ 2}}{R} = Z_0$$
$$\Rightarrow Z_{0x} = \sqrt{R Z_0}$$

- Two resistive impedances can be matched by a section of a transmission line which is quarter-wavelength long and has characteristic impedance equal to the geometric mean of the two resistances.

- The quarter wavelength transfer is commonly used at the junction of two transmission lines of unequal characteristic impedances.

Disadvantages

- This technique needs special line of characteristics impedance for every pair of resistances to be matched.
- To add quarter wave length, we have to cut the main transmission line
- To avoid the above disadvantages other methods can be used, which are either open or short circuited transmission line attached at some position parallel to main transmission line. Those techniques are called stub matching.

Stub Matching-

The main advantage of this stub matching is

- The length and characteristic impedance remains same
- Since the stub is added in shunt, there is no need to cut the main line
- The susceptance of the stub can be adjusted for perfect matching

The choice of open or shorted stub may depend in practice on a number of factors.

- A short-circuited stub is less prone to leakage of electromagnetic radiation and is somewhat easier to realize.
- Open circuited stub may be more practical for certain types of transmission lines, for example microstrips where one would have to drill the insulating substrate to short circuit the two conductors of the line.
- Moreover, based on the number of parallel stubs connected to the line stub matching techniques can be classified into two types

    ✓ **Single-Stub Matching Technique**
    ✓ **Double Stub Matching Technique**

Single stub matching: -

- In this section, a short-circuited section (stub) of a transmission line connected in parallel to the main transmission line (Figure 2). since the stub is connected in parallel, so it is easy to deal with admittance analysis instead of impedance.

- There are two design parameters for single stub matching:
    ✓ The location of the stub with reference to the load can be represented as d_{stub}
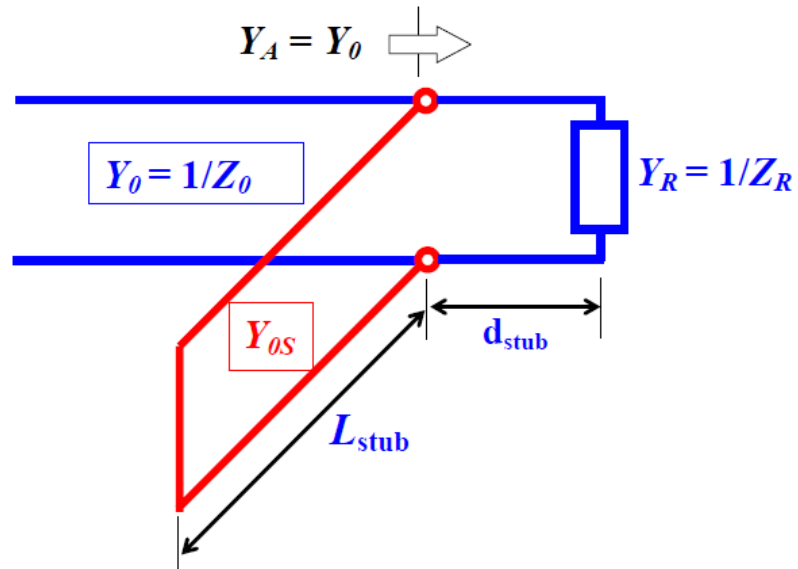   ✓ The length of the stub line Lstub

Figure 2. Stub matching technique

- For proper impedance matching, the admittance at the location of the stub can be written as

$$Y_A = Y_{stub} + Y(d_{stub}) = Y_0 = \frac{1}{Z_0}$$
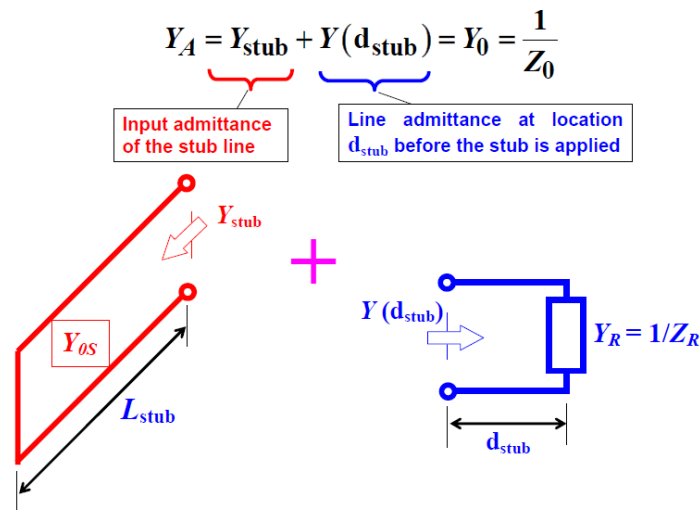
- Which is show in in Figure 3.

Figure 3. Admittance of location and length of stub

Location of the stub:-

- To determine the length and location of the short circuited stubs, consider the input impedance of a lossless transmission line and convert it to admittance.

- Let the input impedance of the loss line can be written as

$$Z(d_{stub}) = Z_{in} = Z_0 \frac{Z_R + jZ_0 \tan \beta l}{Z_0 + jZ_R \tan \beta l}$$

- Converting the impedance into admittance,

$$Y_{in} = Y_0 \frac{Y_R + jY_0 \tan \beta l}{Y_0 + jY_R \tan \beta l}$$

- The normalized admittance can be written as

$$y_{in} = \frac{Y_{in}}{Y_0} = \frac{\dfrac{Y_R}{Y_0} + j \tan \beta l}{1 + j\dfrac{Y_R}{Y_0} \tan \beta l}$$

$$y_{in} = \frac{Y_{in}}{Y_0} = \frac{y_R + j \tan \beta l}{1 + j y_R \tan \beta l}$$

- Separating the real and imaginary parts by rationalizing

$$y_{in} = \left(\frac{y_R + j \tan \beta l}{1 + j y_R \tan \beta l} \right) \left(\frac{1 - j y_R \tan \beta l}{1 - j y_R \tan \beta l} \right)$$

$$y_{in} = \frac{y_R + y_R \tan^2 \beta l - j y_R^2 \tan \beta l + j \tan \beta l}{1 + y_R^2 \tan^2 \beta l}$$

$$= \frac{y_R \left(1 + \tan^2 \beta l\right)}{1 + y_R^2 \tan^2 \beta l} + \frac{j\left(1 - y_R^2\right) \tan \beta l}{1 + y_R^2 \tan^2 \beta l}$$

For no reflection, at a distance $l = d_{stub}$ the real part of the admittance is unity.

That is $\text{Re}[Y_{in}] = Y_0$ or $y_{in} = 1$

So at $l = d_{stub}$, $\dfrac{y_R \left(1 + \tan^2 \beta d_{stub}\right)}{1 + y_R^2 \tan^2 \beta d_{stub}} = 1$

$$y_R \left(1 + \tan^2 \beta d_{stub}\right) = 1 + y_R^2 \tan^2 \beta d_{stub}$$

$$\left(y_R - y_R^2\right) \tan^2 \beta d_{stub} = 1 - y_R$$

$$\tan^2 \beta d_{stub} = \frac{1 - y_R}{\left(y_R - y_R^2\right)} = \frac{1}{y_R}$$

$$\beta d_{stub} = \tan^{-1}\left(\frac{1}{\sqrt{y_R}}\right)$$

$$d_{stub} = \frac{1}{\beta} \tan^{-1}\left(\frac{1}{\sqrt{y_R}}\right)$$

$$d_{stub} = \frac{2\pi}{\lambda} \tan^{-1}\left(\sqrt{\frac{Y_0}{Y_R}}\right)$$

- Therefore, the location of short circuited stub is

$$d_{stub} = \frac{\lambda}{2\pi} \tan^{-1}\left(\sqrt{\frac{Z_0}{Z_R}}\right)$$

At this location the imaginary part of susceptance b_s of y_{in} can be written as

$$\mathrm{Im}[y_{in}] = b_s = \frac{\left(1 - y_R^2\right) \tan \beta l}{1 + y_R^2 \tan^2 \beta l} = \frac{\left(1 - y_R^2\right) \tan \beta d_{stub}}{1 + y_R^2 \tan^2 \beta d_{stub}}$$

$$b_s = \frac{\left(1 - y_R^2\right) \tan \beta l}{1 + y_R^2 \tan^2 \beta l} = \frac{\left(1 - y_R^2\right) \tan \beta d_{stub}}{1 + y_R^2 \tan^2 \beta d_{stub}}$$

$$b_s = \frac{\left(1 - y_R^2\right) \dfrac{1}{\sqrt{y_R}}}{1 + y_R^2 \dfrac{1}{y_R}} = \frac{\left(1 - y_R^2\right)}{\sqrt{y_R (1 + y_R)}} = \frac{1 - y_R}{\sqrt{y_R}}$$

$$i.e. \quad b_s = \frac{1 - \dfrac{Y_R}{Y_0}}{\sqrt{\dfrac{Y_R}{Y_0}}} = \frac{Y_0 - Y_R}{\sqrt{Y_R Y_0}}$$

- Therefore, at length d_{stub} the input admittance is

$$y_{in} = 1 + jb_s$$

- If short circuited stub added parallelly at this point with susceptance equl to $-jb_s$, then the admittance becomes perfectly matched

- That is

$$y_{in} = 1 + jb_s - jb_s$$
$$y_{in} = 1$$

- Therefore, matching is achieved

$$Y_{in} = Y_0$$

Length of the stub: -

- Now to determine the length of the short-circuited stub, which was represented in Figure 2, consider the short-circuited stub impedance and convert in to admittance.

- That is

$$Z_{sc} = jX_s = jZ_0 \tan \beta l$$

$$Y_{sc} = -jB_s = -jY_0 \cot \beta l$$

- The required normalized suspetance of the short circuited stub b_s can be written as

$$b_s = \frac{B_s}{Y_0} = \cot \beta l$$

Equating this with

$$\cot \beta L_{stub} = \frac{Y_0 - Y_R}{\sqrt{Y_R Y_0}}$$

or

$$\tan \beta L_{stub} = \frac{\sqrt{Y_R Y_0}}{Y_0 - Y_R}$$

- Converting into impedance,

$$\tan \beta L_{stub} = \frac{\dfrac{1}{\sqrt{Z_R Z_0}}}{\dfrac{1}{Z_R} - \dfrac{1}{Z_0}}$$

$$= \frac{\sqrt{Z_R Z_0}}{Z_R - Z_0}$$

- Therefore, the length of short circuited stub is

$$L_{stub} = \frac{\lambda}{2\pi} \tan^{-1}\left(\frac{\sqrt{Z_R Z_0}}{Z_R - Z_0} \right) \quad \text{for } Z_R > Z_0$$

$$L_{stub} = \frac{\lambda}{2\pi} \tan^{-1}\left(\frac{\sqrt{Z_R Z_0}}{Z_0 - Z_R} \right) \text{for } Z_0 > Z_R$$

- Any load impedance can be matched to the line by using single stub technique. The drawback of this approach is that if the load is changed, the location of insertion may have to be moved.

Smith Chart-Configuration and Applications:

- For complex transmission line problems, the use of the formulae becomes increasingly difficult and inconvenient. An indispensable graphical method of solution is the use of Smith Chart.
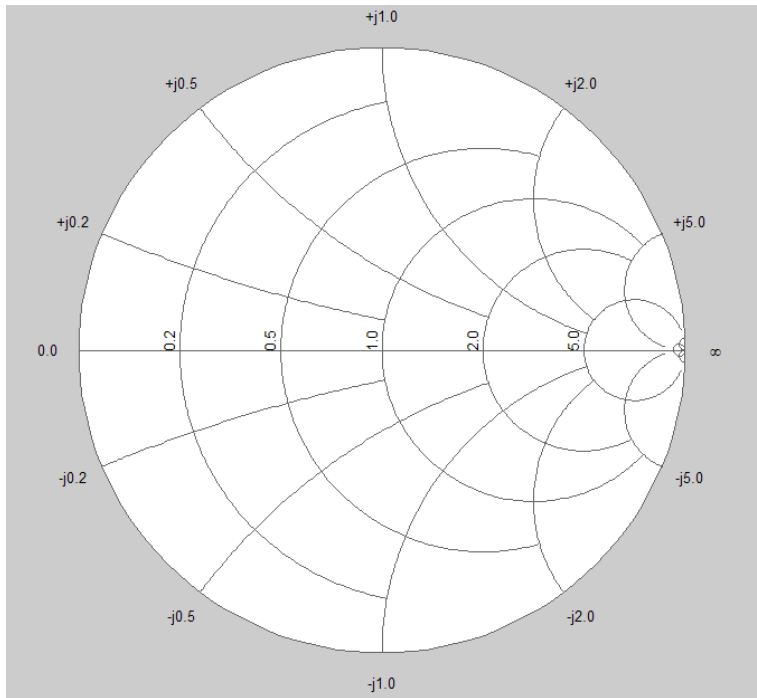
- The Smith Chart is used for analyzing and designing transmission-line circuits.

- Impedances represented by normalized values, Z_0.

- Reflection coefficient is
$$z_L = \frac{1 + K}{1 - K}$$

- Normalized load admittance is $y_L = \dfrac{1}{z_L} = \dfrac{1 - K}{1 + K}$ (dimensionless)

- **Horizontal line:** The horizontal line running through the center of the Smith chart represents either the resistive ir the conductive component. Zero resistance or conductance is located on the left end and infinite resistance or conductance is located on the right end of the line.

- **Circles of constant resistance and conductance:** Circles of constant resistance are drawn on the Smith chart tangent to the right-hand side of the chart and its intersection with the centerline. These circles of constant resistance are used to locate complex impedances and to assist in obtaining solutions to problems involving the Smith chart.

- **Lines of constant reactance:** Lines of constant reactance are shown on the Smith chart with curves that start from a given reactance value on the outer circle and end at the right-hand side of the center line.
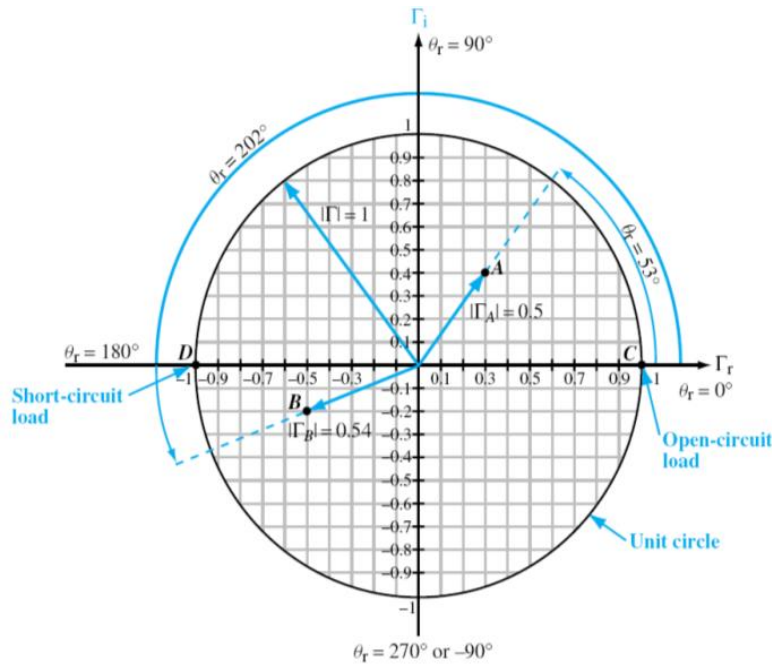
Fig. Smith Chart (location of maxima and minima)

The types of problems for which Smith charts are used include te following:

1) Plotting a complex impedance on a Smith chart

2) Finding VSWR for a given load

3) Finding the admittance for a given impedance

4) Finding the input impedance of a transmission line terminated in a short or open.

5) Finding the input impedance at any distance from a load Z_L.

6) Locating the first maximum and minimum from any load

7) Matching   a transmission line to a load with a single series stub.

8) Matching a transmission line with a single parallel stub

9) Matching a   transmission line to a load with two parallel stubs.

Plotting a Complex Impedance:

- To locate a complex impedance, $Z = R +- jX$ or admittance $Y = G +- jB$ on a Smith chart, normalize the real and imaginary part of the complex impedance.

- Locating the value of the normalized real term on the horizontal line scale locates the resistance circle

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- Locating the normalized value of the imaginary term on the outer circle locates the curve of constant reactance.

- The intersection of the circle and the curve locates the complex impedance on the Smith chart.

Finding the VSWR for a given load:

- Normalize the load and plot its location on the Smith chart.

- Draw a circle with a radius equal to the distance between the 1.0 point and the location of the normalized load and the center of the Smith chart as the center.

- The intersection of the right-hand side of the circle with the horizontal resistance line locates the value of the VSWR.

Finding the Input Impedance at any Distance from the Load:

- The load impedance is first normalized and is located on the Smith chart.

- The VSWR circle is drawn for the load.

- A line is drawn from the 1.0 point through the load to the outer wavelength scale.

- To locate the input impedance on a Smith chart of the transmission line at any given distance from the load, advance in clockwise direction from the located point, a distance in wavelength equal to the distance to the new location on the transmission line.

Power Loss:

- Return Power Loss: When an electromagnetic wave travels down a transmission line and encounters a mismatched load or a discontinuity in the line, part of the incident power is reflected back down the line.

- The return loss is defined as:
$$P_{return} = 10 \log_{10} P_i/P_r$$
$$P_{return} = 20 \log_{10} 1/G$$

- Mismatch Power Loss: The term mismatch loss is used to describe the loss caused by the reflection due to a mismatched line. It is defined as
$$P_{mismatch} = 10 \log_{10} P_i/(P_i - P_r)$$