FINITE ELEMENT MODELLING

Course code:AME014

VI- semester

Regulation: IARE R-16

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COs	COURSE OUTCOMES
CO1	Describe the concept of FEM and difference between the FEM with other methods and problems based on 1-D bar elements and shape functions.
CO2	Derive elemental properties and shape functions for truss and beam elements and related problems.
CO3	Understand the concept deriving the elemental matrix and solving the basic problems of CST and axi-symmetric solids
CO4	Explore the concept of steady state heat transfer in fin and composite slab
CO5	Understand the concept of consistent and lumped mass models and solve the dynamic analysis of all types of elements.

UNIT-I
INTRODUCTION TO FEM

Introduction

Introduction to FEM:

- Stiffness equations for a axial bar element in local co-ordinates using Potential Energy approach and Virtual energy principle.
- Finite element analysis of uniform, stepped and tapered bars subjected to mechanical and thermal loads.
- Assembly of Global stiffness matrix and load vector.
- Quadratic shape functions.
- Properties of stiffness matrix

Axially Loaded Bar – Governing Equations and Boundary Conditions

- Differential Equation

$$\frac{d}{dx}\left[EA(x)\frac{du}{dx}\right] + f(x) = 0 \qquad 0 < x < L$$

- Boundary Condition Types
  - Prescribed displacement (essential BC)
  - Prescribed force/derivative of displacement (natural BC)

Axially Loaded Bar –Boundary Conditions

Examples

- Fixed end

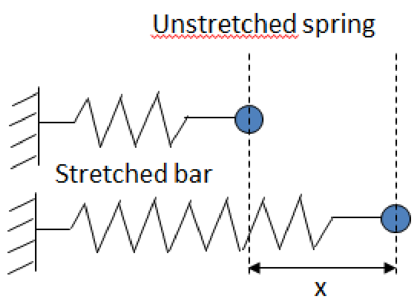
- Simple support

- Free end

Potential Energy

Elastic Potential Energy (PE)

- Spring case



$$PE = 0$$

$$PE = \frac{1}{2} kx^2$$

- Axially loaded bar

Unreformed: $PE = 0$

deformed:

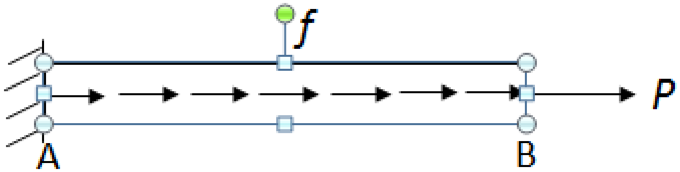
$$PE = \frac{1}{2}\int_0^L \sigma\varepsilon A dx$$

- Elastic body

$$PE = \frac{1}{2}\int_V \boldsymbol{\sigma}^T \boldsymbol{\varepsilon} dv$$

Potential Energy

- Work Potential (WE)



f: distributed force over a line
P: point force
u: displacement

- Total Potential Energy

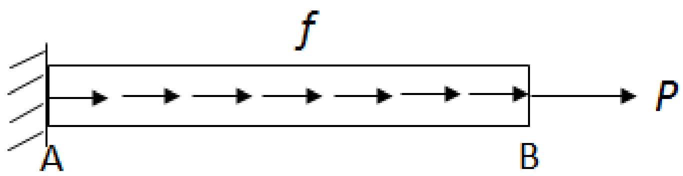
$$\Pi = \frac{1}{2}\int_0^L \sigma\varepsilon A dx - \int_0^L u \cdot f dx - P \cdot u_B$$

- **Principle of Minimum Potential Energy**

For conservative systems, of all the kinematically admissible displacement fields, those corresponding to equilibrium extremize the total potential energy. If the extremum condition is a minimum, the equilibrium state is stable.

Potential Energy + Rayleigh-Ritz Approach

- Example:

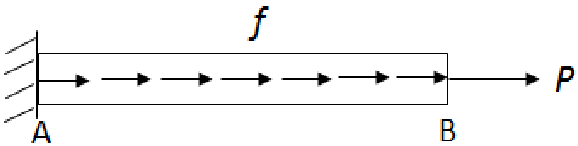


- Step 1: assume a displacement field $u = \sum_i a_i \phi_i(x) \quad i = 1 \text{ to } n$
  - f is shape function / basis function
  - *n* is the order of approximation

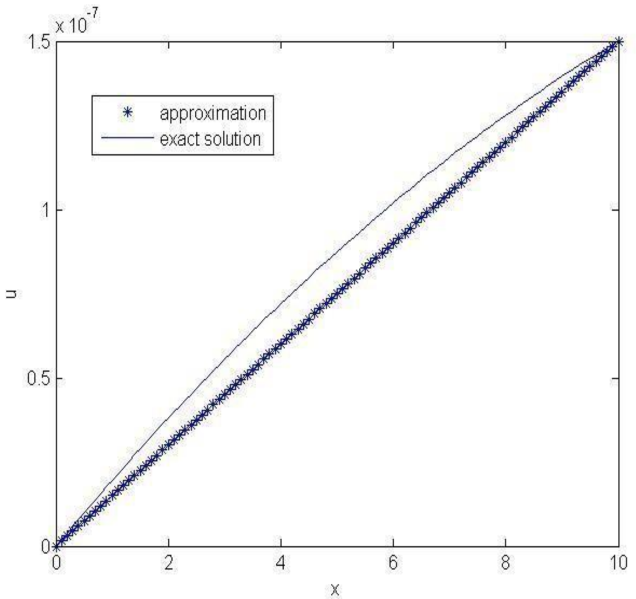
- Step 2: calculate total potential energy

Potential Energy + Rayleigh-Ritz Approach

- Example:

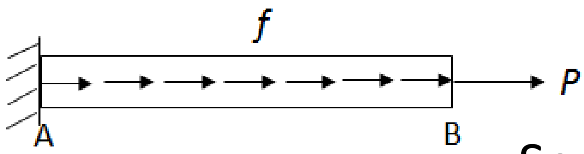


- Step 3: select a_i so that the total potential energy is minimum



Galerkin's Method

- Example:



$$\begin{cases} \dfrac{d}{dx}\left[EA(x)\dfrac{du}{dx}\right] + f(x) = 0 \\ u(x=0) = 0 \\ EA(x)\dfrac{du}{dx}\Big|_{x=L} = P \end{cases}$$

Seek an approximation $\tilde{u}$ so

$$\int_V w_i \left(\frac{d}{dx}\left[EA(x)\frac{d\tilde{u}}{dx}\right] + f(x) \right) dV = 0$$

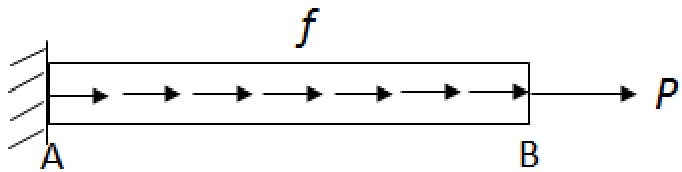
$$\tilde{u}(x=0) = 0$$

$$EA(x)\frac{d\tilde{u}}{dx}\Big|_{x=L} = P$$

- In the Galerkin's method, the weight function is chosen to be the same as the shape function.

Galerkin's Method

- Example:



$$\int_V w_i \left(\frac{d}{dx}\left[EA(x)\frac{d\tilde{u}}{dx} \right] + f(x) \right) dV = 0$$

$$\Rightarrow \quad -\underbrace{\int_0^L EA(x)\frac{d\tilde{u}}{dx}\frac{dw_i}{dx}dx}_{①} + \underbrace{\int_0^L w_i\, f dx}_{②} + \underbrace{w_i EA(x)\frac{d\tilde{u}}{dx}\Bigg|_0^L}_{③} = 0$$

FEM Formulation of Axially Loaded Bar – Governing Equations

- Differential Equation

$$\frac{d}{dx}\left[EA(x)\frac{du}{dx}\right] + f(x) = 0 \qquad 0 < x < L$$

- Weighted-Integral Formulation

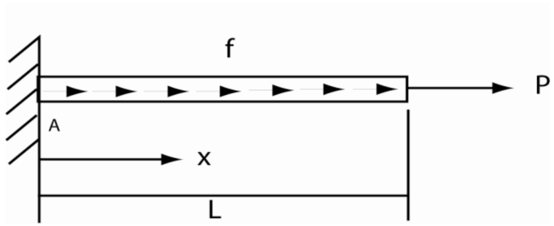
$$\int_0^L w\left(\frac{d}{dx}\left[EA(x)\frac{du}{dx}\right] + f(x)\right)dx = 0$$

- Weak Form

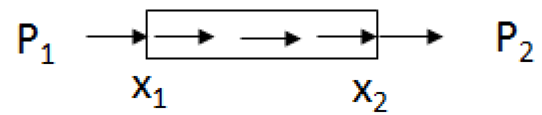
$$0 = \int_0^L \left[\frac{dw}{dx}\left(EA(x)\frac{du}{dx}\right) - wf(x)\right]dx - \left.w\left(EA(x)\frac{du}{dx}\right)\right|_0^L$$

Approximation Methods – Finite Element Method

- Example:



- Step 1: Discretization



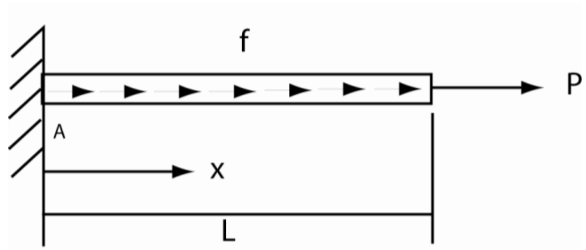
- Step 2: Weak form of one element

$$\int_{x_1}^{x_2}\left[\frac{dw}{dx}\left(EA(x)\frac{du}{dx}\right)-w(x)\,f(x)\right]dx - w(x)\left(EA(x)\frac{du}{dx}\right)\Bigg|_{x_1}^{x_2}=0$$

$$\int_{x_1}^{x_2}\left[\frac{dw}{dx}\left(EA(x)\frac{du}{dx}\right)-w(x)\,f(x)\right]dx - w\left(x_2\right)P_2 - w\left(x_1\right)P_1=0$$

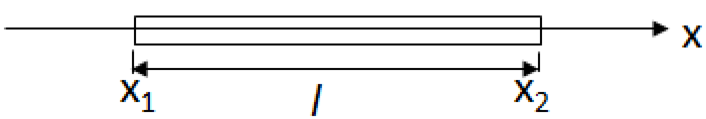
Approximation Methods – Finite Element Metho

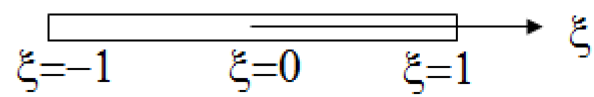
- Example (*cont*):



- Step 3: Choosing shape functions- linear shape functions

$$u = \phi_1 u_1 + \phi_2 u_2$$





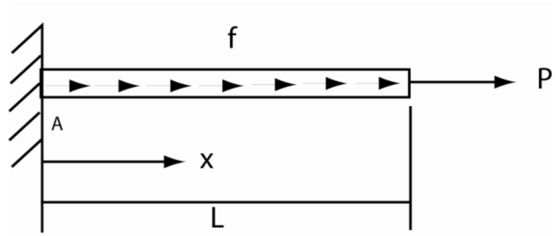
$$\phi_1 = \frac{x_2 - x}{l}; \quad \phi_2 = \frac{x - x_1}{l}$$

$$\phi_1 = \frac{1-\xi}{2}; \quad \phi_2 = \frac{1+\xi}{2}$$

$$\xi = \frac{2}{l}(x - x_1) - 1; \quad x = \frac{(\xi + 1)l}{2} + x_1$$

Approximation Methods – Finite Element Metho

- Example (*cont*):



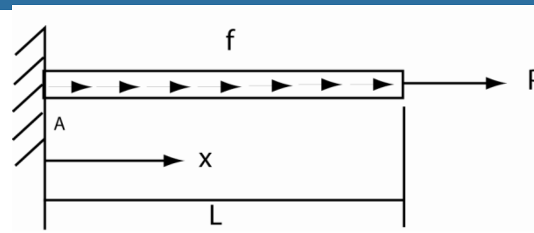
- Step 4: Forming element equation

- Let $w = \phi_1$, weak form becomes $\int_{x_1}^{x_2} -\frac{1}{l}\left(EA \cdot \frac{u_2 - u_1}{l}\right) dx - \int_{x_1}^{x_2} \phi_1 f\, dx - \phi_1 P_2 - \phi_1 P_1 = 0 \longrightarrow \frac{EA}{l} u_1 - \frac{EA}{l} u_2 = \int_{x_1}^{x_2} \phi_1 f\, dx + P_1$

- Let $w = \phi_2$, weak form becomes $\int_{x_1}^{x_2} \frac{1}{l}\left(EA \cdot \frac{u_2 - u_1}{l}\right) dx - \int_{x_1}^{x_2} \phi_2 f\, dx - \phi_2 P_2 - \phi_2 P_1 = 0 \longrightarrow -\frac{EA}{l} u_1 + \frac{EA}{l} u_2 = \int_{x_1}^{x_2} \phi_2 f\, dx + P_2$

$$\frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} \int_{x_1}^{x_2} \phi_1 f dx \\ \int_{x_1}^{x_2} \phi_2 f dx \end{Bmatrix} + \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} + \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

- Example (*cont*):



- Step 5: Assembling to form system equation

Approach 1:

Element 1:

$$\frac{E^I A^I}{l^I}\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}u_1^I\\ u_2^I\\ 0\\ 0\end{Bmatrix}=\begin{Bmatrix}f_1^I\\ f_2^I\\ 0\\ 0\end{Bmatrix}+\begin{Bmatrix}P_1^I\\ P_2^I\\ 0\\ 0\end{Bmatrix}$$

Element 2:

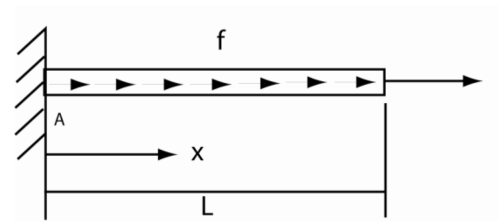
$$\frac{E^{II} A^{II}}{l^{II}}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}0\\ u_1^{II}\\ u_2^{II}\\ 0\end{Bmatrix}=\begin{Bmatrix}0\\ f_1^{II}\\ f_2^{II}\\ 0\end{Bmatrix}+\begin{Bmatrix}0\\ P_1^{II}\\ P_2^{II}\\ 0\end{Bmatrix}$$

Element 3:

$$\frac{E^{III} A^{III}}{l^{III}}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ 0\\ u_1^{III}\\ u_2^{III}\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ f_1^{III}\\ f_2^{III}\end{Bmatrix}+\begin{Bmatrix}0\\ 0\\ P_1^{III}\\ P_2^{III}\end{Bmatrix}$$

Approximation Methods – Finite Element Metho

- Example (*cont*):



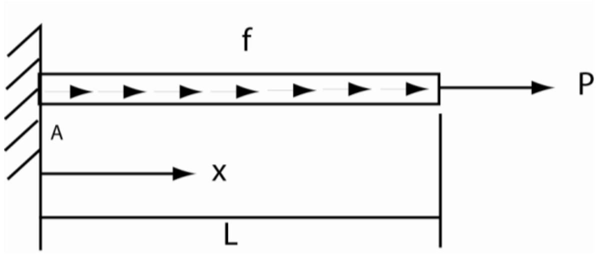
- Step 5: Assembling to form system equation

Assembled System:

$$\begin{bmatrix} \frac{E^I A^I}{l^I} & -\frac{E^I A^I}{l^I} & 0 & 0 \\ -\frac{E^I A^I}{l^I} & \frac{E^I A^I}{l^I}+\frac{E^{II} A^{II}}{l^{II}} & -\frac{E^{II} A^{II}}{l^{II}} & 0 \\ 0 & -\frac{E^{II} A^{II}}{l^{II}} & \frac{E^{II} A^{II}}{l^{II}}+\frac{E^{III} A^{III}}{l^{III}} & -\frac{E^{III} A^{III}}{l^{III}} \\ 0 & 0 & -\frac{E^{III} A^{III}}{l^{III}} & \frac{E^{III} A^{III}}{l^{III}} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} + \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \begin{Bmatrix} f_1^I \\ f_2^I + f_1^{II} \\ f_2^{II} + f_1^{III} \\ f_2^{III} \end{Bmatrix} + \begin{Bmatrix} P_1^I \\ P_2^I + P_1^{II} \\ P_2^{II} + P_1^{III} \\ P_2^{III} \end{Bmatrix}$$

Approximation Methods

- Example (*cont*):



- Step 5: Assembling to form system equation

Approach 2: Element connectivity table

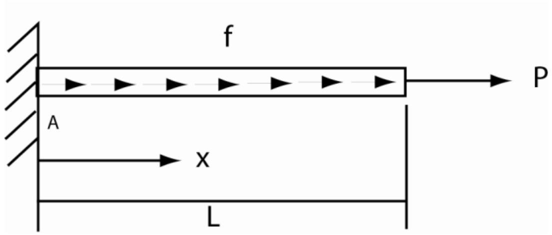
$$k_{ij}^{e} \rightarrow K_{IJ}$$

	Element 1	Element 2	Element 3
1	1	2	3
2	2	3	4

local node (i,j) — global node index (I,J)

Approximation Methods

- Example (*cont*):

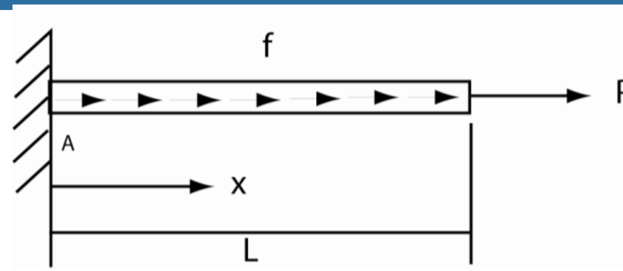


- Step 6: Imposing boundary conditions and forming condense

Condensed system:

$$\begin{pmatrix} \frac{E^I A^I}{l^I} + \frac{E^{II} A^{II}}{l^{II}} & -\frac{E^{II} A^{II}}{l^{II}} & 0 \\ -\frac{E^{II} A^{II}}{l^{II}} & \frac{E^{II} A^{II}}{l^{II}} + \frac{E^{III} A^{III}}{l^{III}} & -\frac{E^{III} A^{III}}{l^{III}} \\ 0 & -\frac{E^{III} A^{III}}{l^{III}} & \frac{E^{III} A^{III}}{l^{III}} \end{pmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_2 \\ f_3 \\ f_4 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ P \end{Bmatrix}$$

- Example (*cont*):



- Step 7: solution

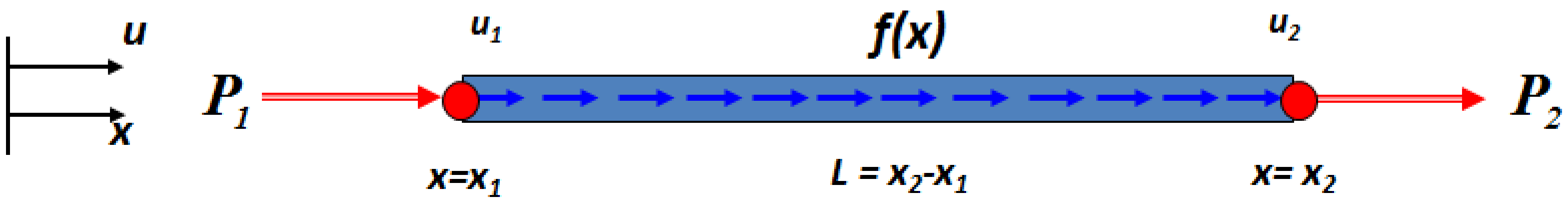
- Step 8: post calculation

$$u = u_1\phi_1 + u_2\phi_2 \quad \Rightarrow \quad \varepsilon = \frac{du}{dx} = u_1\frac{d\phi_1}{dx} + u_2\frac{d\phi_2}{dx} \quad \Rightarrow \quad \sigma = E\varepsilon = Eu_1\frac{d\phi_1}{dx} + Eu_2\frac{d\phi_2}{dx}$$

Summary - Major Steps in FEM

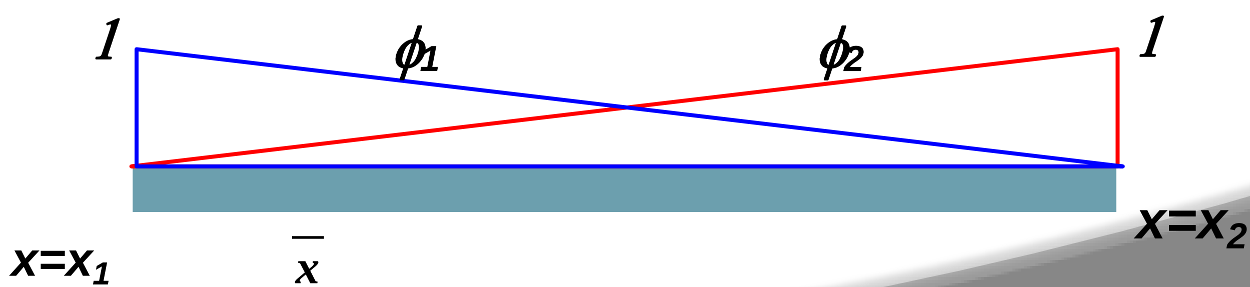
- Discretization
- Derivation of element equation
  - weak form
  - construct form of approximation solution over oneelement
  - derive finite element model
- Assembling – putting elements together
- Imposing boundary conditions
- Solving equations
- Post computation

Linear Formulation for Bar Element

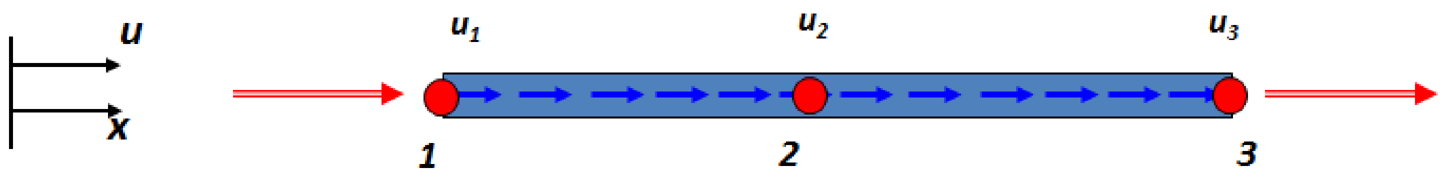


$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

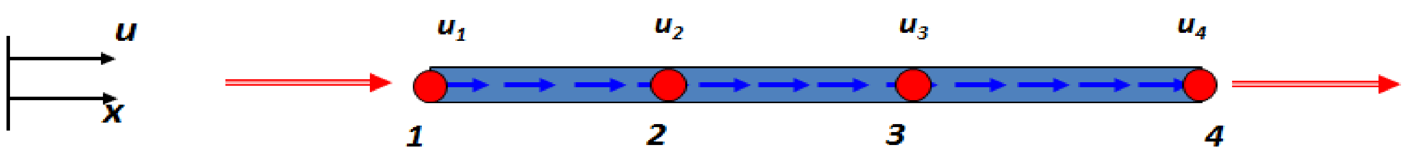
$$\text{where} \quad K_{ij} = \int_{x_1}^{x_2} EA \left(\frac{d\phi_i}{dx} \frac{d\phi_j}{dx} \right) dx = K_{ji} \quad , \quad f_i = \int_{x_1}^{x_2} (\phi_i f)\, dx$$



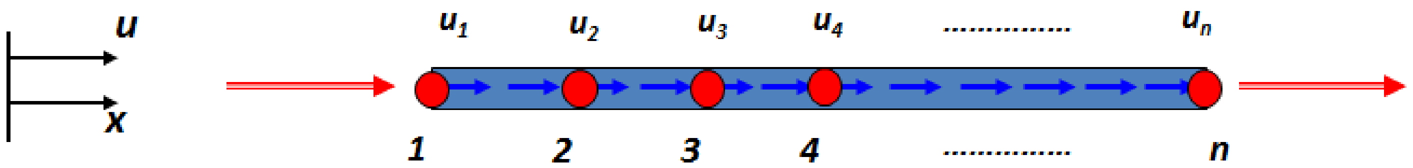
Higher Order Formulation for Bar Element



$$u(x) = u_1\phi_1(x) + u_2\phi_2(x) + u_3\phi_3(x)$$

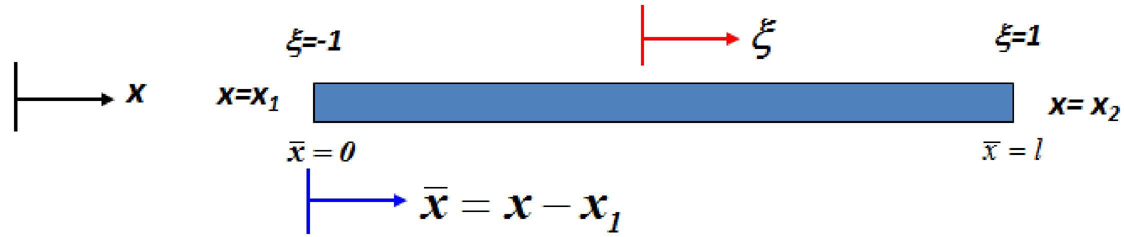


$$u(x) = u_1\phi_1(x) + u_2\phi_2(x) + u_3\phi_3(x) + u_4\phi_4(x)$$

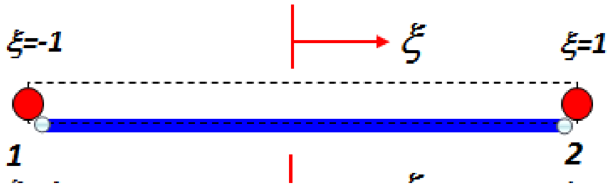


$$u(x) = u_1\phi_1(x) + u_2\phi_2(x) + u_3\phi_3(x) + u_4\phi_4(x) + \bullet\bullet\bullet\bullet\bullet\bullet + u_n\phi_n(x)$$

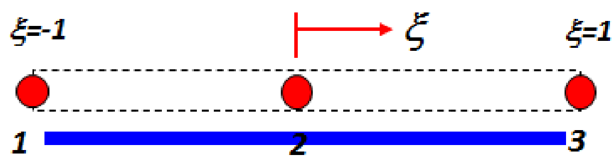
Natural Coordinates and Interpolation Functions



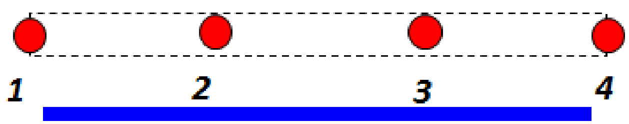
Natural (or Normal) Coordinate: $\xi = \dfrac{x - \dfrac{x_1 + x_2}{2}}{l/2}$



$$\phi_1 = -\frac{\xi - 1}{2}, \quad \phi_2 = \frac{\xi + 1}{2}$$



$$\phi_1 = \frac{\xi(\xi - 1)}{2}, \quad \phi_2 = -(\xi + 1)(\xi - 1), \quad \phi_3 = \frac{(\xi + 1)\xi}{2}$$



$$\phi_1 = -\frac{9}{16}\left(\xi + \frac{1}{3}\right)\left(\xi - \frac{1}{3}\right)(\xi - 1), \quad \phi_2 = \frac{27}{16}(\xi + 1)\left(\xi - \frac{1}{3}\right)(\xi - 1)$$

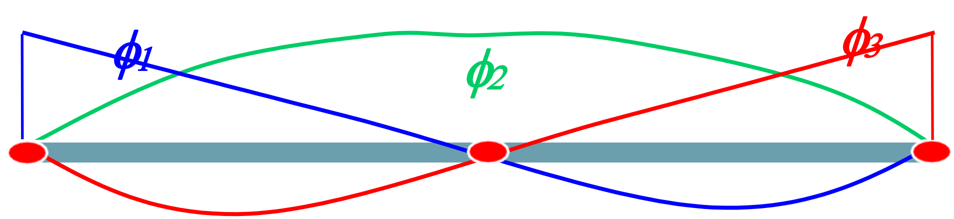
$$\phi_3 = -\frac{27}{16}(\xi + 1)\left(\xi + \frac{1}{3}\right)(\xi - 1), \quad \phi_4 = \frac{9}{16}(\xi + 1)\left(\xi + \frac{1}{3}\right)\left(\xi - \frac{1}{3}\right)$$

Quadratic Formulation for Bar Element

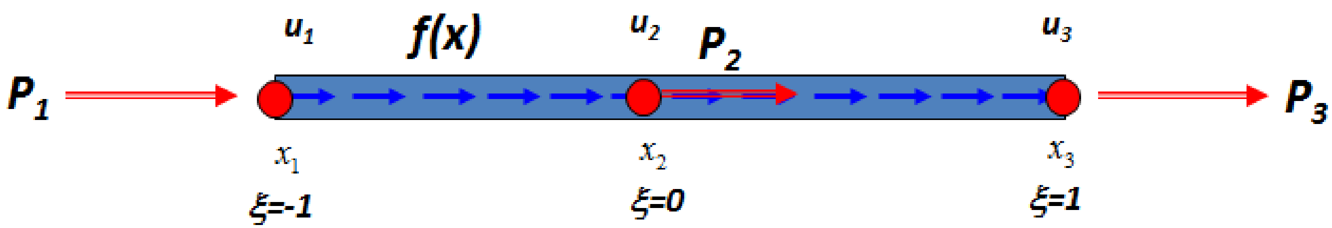
$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\text{where} \quad K_{ij} = \int_{x_1}^{x_2} EA \left(\frac{d\phi_i}{dx} \frac{d\phi_j}{dx} \right) dx = \int_{-1}^{1} EA \left(\frac{d\phi_i}{d\xi} \frac{d\phi_j}{d\xi} \right) \frac{2}{l} d\xi = K_{ji}$$

$$\text{and} \quad f_i = \int_{x_1}^{x_2} (\varphi_i f) dx = \int_{-1}^{1} (\varphi_i f) \frac{l}{2} d\xi, \quad i, j = 1, 2, 3$$



Quadratic Formulation for Bar Element



$$u(\xi) = u_1\phi_1(\xi) + u_2\phi_2(\xi) + u_3\phi_3(\xi) = u_1\frac{\xi(\xi-1)}{2} - u_2(\xi+1)(\xi-1) + u_3\frac{(\xi+1)\xi}{2}$$

$$\phi_1 = \frac{\xi(\xi-1)}{2}, \quad \phi_2 = -(\xi+1)(\xi-1), \quad \phi_3 = \frac{(\xi+1)\xi}{2}$$

$$\xi = \frac{x - \frac{x_1 + x_2}{2}}{l/2} \qquad \frac{l}{2}\, d\xi = dx \qquad \frac{d\xi}{dx} = \frac{2}{l}$$

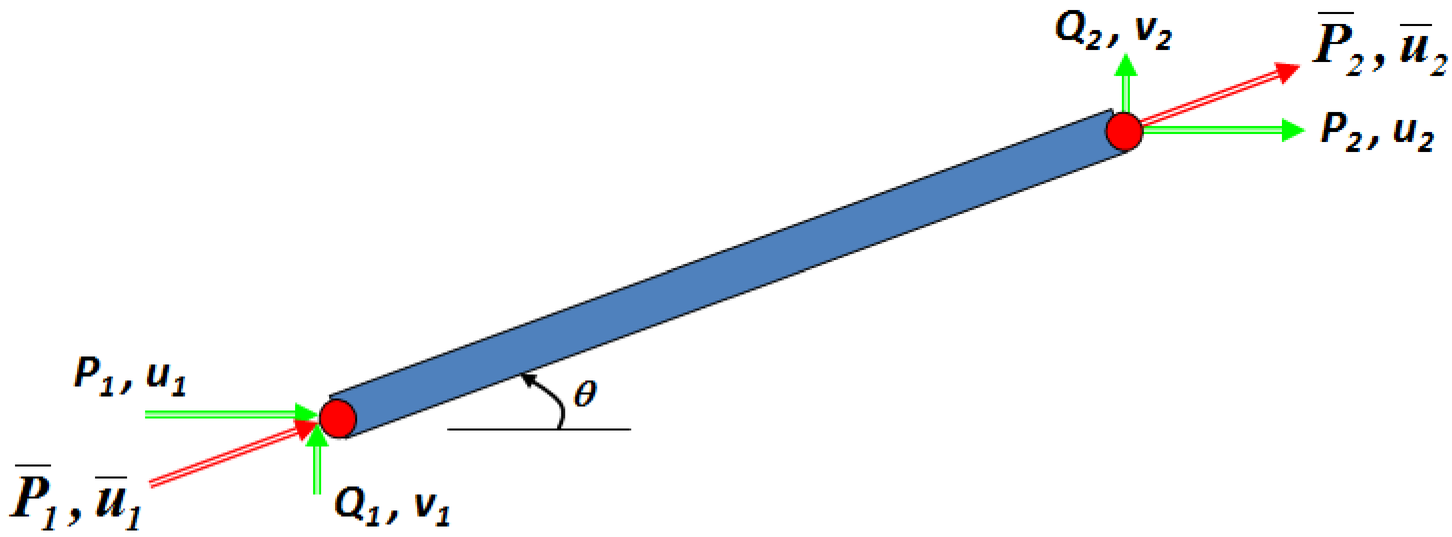
$$\frac{d\phi_1}{dx} = \frac{2}{l}\frac{d\phi_1}{d\xi} = \frac{2\xi - 1}{l}, \quad \frac{d\phi_2}{dx} = \frac{2}{l}\frac{d\phi_2}{d\xi} = -\frac{4\xi}{l}, \quad \frac{d\phi_3}{dx} = \frac{2}{l}\frac{d\phi_3}{d\xi} = \frac{2\xi + 1}{l}$$

UNIT-II
ANALYSIS OF TRUSSES AND BEAMS

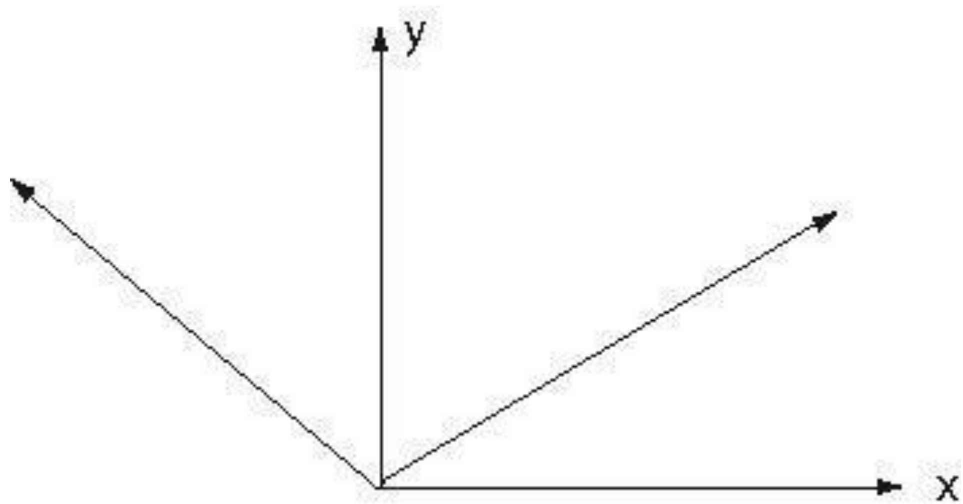
Finite Element Analysis of Trusses:

- Stiffness equations for a truss bar element oriented in 2D plane
- Finite Element Analysis of Trusses
- Plane Truss and Space Truss elements
- Methods of assembly

Arbitrarily Oriented 1-D Bar Element on 2-D Plane



Relationship Between Local Coordinates and Global Coordinates

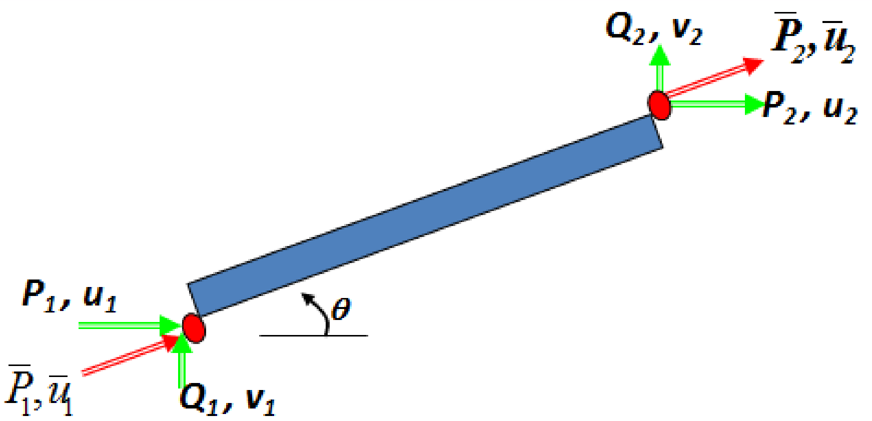


$$\begin{Bmatrix} u_1 \\ v_1 = 0 \\ u_2 \\ v_2 = 0 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

Relationship Between Local Coordinates and Global Coordinates

$$\begin{Bmatrix} P_1 \\ 0 \\ P_2 \\ 0 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} P_1 \\ Q_1 \\ P_2 \\ Q_2 \end{Bmatrix}$$

Stiffness Matrix of 1-D Bar Element on 2-D Plane

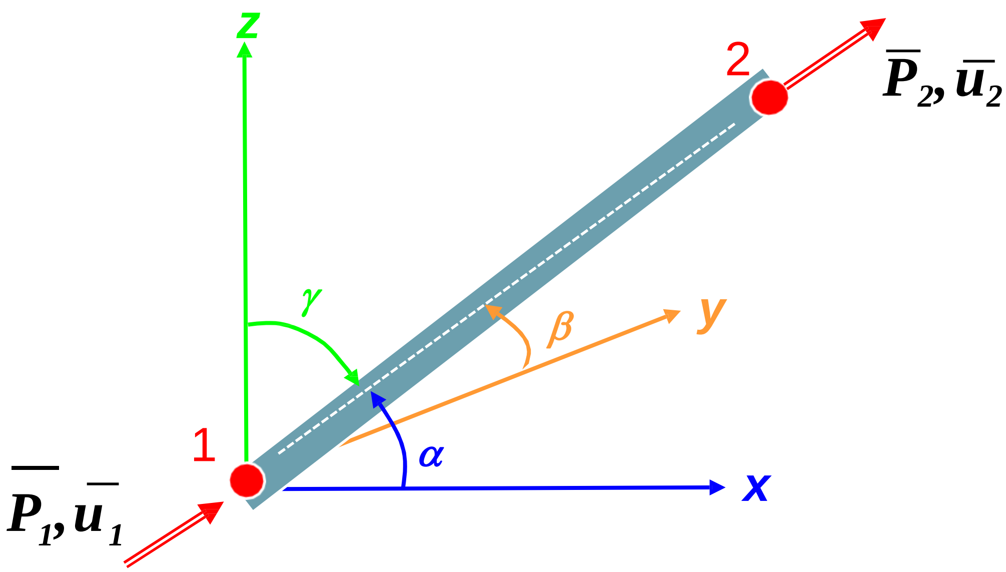


$$
\left[\bar{K}_{ij}\right] = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
\begin{Bmatrix} P_1 \\ Q_1 \\ P_2 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \left[\bar{K}_{ij}\right] \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}
$$

$$
\begin{Bmatrix} P_1 \\ Q_1 \\ P_2 \\ Q_2 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta & -\cos^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\sin\theta\cos\theta & \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}
$$

Arbitrarily Oriented 1-D Bar Element in 3-D Space

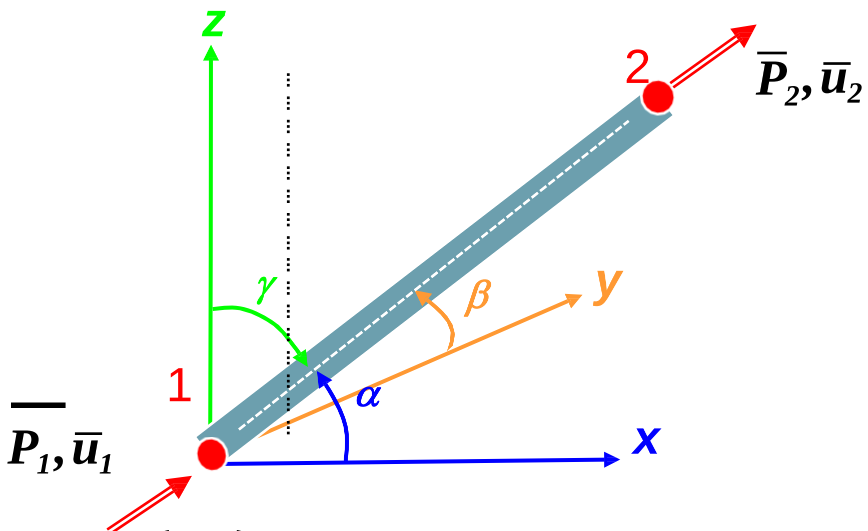


α , β, γ are the Direction Cosines of the bar in the x-y-z coordinate system

$$\begin{Bmatrix} \overline{u}_1 \\ \overline{v}_1 = 0 \\ \overline{w}_1 = 0 \\ \overline{u}_2 \\ \overline{v}_2 = 0 \\ \overline{w}_2 = 0 \end{Bmatrix} = \begin{bmatrix} \alpha_x & \beta_x & \gamma_x & 0 & 0 & 0 \\ \alpha_y & \beta_y & \gamma_y & 0 & 0 & 0 \\ \alpha_z & \beta_z & \gamma_z & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_x & \beta_x & \gamma_x \\ 0 & 0 & 0 & \alpha_y & \beta_y & \gamma_y \\ 0 & 0 & 0 & \alpha_z & \beta_z & \gamma_z \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix}$$

$$\begin{Bmatrix} \overline{P}_1 \\ \overline{Q}_1 = 0 \\ \overline{R}_1 = 0 \\ \overline{P}_2 \\ \overline{Q}_2 = 0 \\ \overline{R}_2 = 0 \end{Bmatrix} = \begin{bmatrix} \alpha_x & \beta_x & \gamma_x & 0 & 0 & 0 \\ \alpha_y & \beta_y & \gamma_y & 0 & 0 & 0 \\ \alpha_z & \beta_z & \gamma_z & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_x & \beta_x & \gamma_x \\ 0 & 0 & 0 & \alpha_y & \beta_y & \gamma_y \\ 0 & 0 & 0 & \alpha_z & \beta_z & \gamma_z \end{bmatrix} \begin{Bmatrix} P_1 \\ Q_1 \\ R_1 \\ P_2 \\ Q_2 \\ R_2 \end{Bmatrix}$$

Stiffness Matrix of 1-D Bar Element in 3-D Space



$$\begin{Bmatrix} \bar{P}_1 \\ \bar{Q}_1 = 0 \\ \bar{R}_1 = 0 \\ \bar{P}_2 \\ \bar{Q}_2 = 0 \\ \bar{R}_2 = 0 \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 = 0 \\ \bar{w}_1 = 0 \\ \bar{u}_2 \\ \bar{v}_2 = 0 \\ \bar{w}_2 = 0 \end{Bmatrix}$$

$$\begin{Bmatrix} P_1 \\ Q_1 \\ R_1 \\ P_2 \\ Q_2 \\ R_2 \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} \alpha_x^2 & \alpha_x\beta_x & \alpha_x\gamma_x & -\alpha_x^2 & -\alpha_x\beta_x & -\alpha_x\gamma_x \\ \alpha_x\beta_x & \beta_x^2 & \beta_x\gamma_x & -\alpha_x\beta_x & -\beta_x^2 & -\beta_x\gamma_x \\ \alpha_x\gamma_x & \beta_x\gamma_x & \gamma_x^2 & -\alpha_x\gamma_x & -\beta_x\gamma_x & -\gamma_x^2 \\ -\alpha_x^2 & -\alpha_x\beta_x & -\alpha_x\gamma_x & \alpha_x^2 & \alpha_x\beta_x & \alpha_x\gamma_x \\ -\alpha_x\beta_x & -\beta_x^2 & -\beta_x\gamma_x & \alpha_x\beta_x & \beta_x^2 & \beta_x\gamma_x \\ -\alpha_x\gamma_x & -\beta_x\gamma_x & -\gamma_x^2 & \alpha_x\gamma_x & \beta_x\gamma_x & \gamma_x^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix}$$

Matrix Assembly of Multiple Bar Elements

- **Element I**

$$\begin{Bmatrix} P_1 \\ Q_1 \\ P_2 \\ Q_2 \end{Bmatrix} = \frac{AE}{L} \left[\begin{array}{cc:cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hdashline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

- **Element II**

$$\begin{Bmatrix} P_2 \\ Q_2 \\ P_3 \\ Q_3 \end{Bmatrix} = \frac{AE}{4L} \left[\begin{array}{cc:cc} 1 & -\sqrt{3} & -1 & \sqrt{3} \\ -\sqrt{3} & 3 & \sqrt{3} & -3 \\ -1 & \sqrt{3} & 1 & -\sqrt{3} \\ \sqrt{3} & -3 & -\sqrt{3} & 3 \end{array} \right] \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

- **Element III**

$$\begin{Bmatrix} P_1 \\ Q_1 \\ P_3 \\ Q_3 \end{Bmatrix} = \frac{AE}{4L} \left[\begin{array}{cc:cc} 1 & \sqrt{3} & -1 & -\sqrt{3} \\ \sqrt{3} & 3 & -\sqrt{3} & -3 \\ -1 & -\sqrt{3} & 1 & \sqrt{3} \\ -\sqrt{3} & -3 & \sqrt{3} & 3 \end{array} \right] \begin{Bmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{Bmatrix}$$

Matrix Assembly of Multiple Bar Elements

- Element I

$$\begin{Bmatrix} P_1 \\ Q_1 \\ P_2 \\ Q_2 \\ P_3 \\ Q_3 \end{Bmatrix} = \frac{AE}{4L} \left[\begin{array}{cc|cc|cc} 4 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -4 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

- Element II

$$\begin{Bmatrix} P_1 \\ Q_1 \\ P_2 \\ Q_2 \\ P_3 \\ Q_3 \end{Bmatrix} = \frac{AE}{4L} \left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & -\sqrt{3} & -1 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 3 & \sqrt{3} & -3 \\ \hline 0 & 0 & -1 & \sqrt{3} & 1 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & -3 & -\sqrt{3} & 3 \end{array}\right] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

- Element III

$$\begin{Bmatrix} P_1 \\ Q_1 \\ P_2 \\ Q_2 \\ P_3 \\ Q_3 \end{Bmatrix} = \frac{AE}{4L} \left[\begin{array}{cc|cc|cc} 1 & \sqrt{3} & 0 & 0 & -1 & -\sqrt{3} \\ \sqrt{3} & 3 & 0 & 0 & -\sqrt{3} & -3 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -1 & -\sqrt{3} & 0 & 0 & 1 & \sqrt{3} \\ -\sqrt{3} & -3 & 0 & 0 & \sqrt{3} & 3 \end{array}\right] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

Matrix Assembly of Multiple Bar Elements

$$
\begin{Bmatrix} R_1 \\ S_1 \\ R_2 \\ S_2 \\ R_3 \\ S_3 \end{Bmatrix} = \frac{AE}{4L}
\left[\begin{array}{cc|cc|cc}
4+1 & 0+\sqrt{3} & -4 & 0 & -1 & -\sqrt{3} \\
0+\sqrt{3} & 0+3 & 0 & 0 & -\sqrt{3} & -3 \\
\hline
-4 & 0 & 4+1 & 0-\sqrt{3} & -1 & \sqrt{3} \\
0 & 0 & 0-\sqrt{3} & 0+3 & \sqrt{3} & -3 \\
\hline
-1 & -\sqrt{3} & -1 & \sqrt{3} & 1+1 & \sqrt{3}-\sqrt{3} \\
-\sqrt{3} & -3 & \sqrt{3} & -3 & \sqrt{3}-\sqrt{3} & 3+3
\end{array}\right]
\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}
$$

Apply known boundary conditions

$$
\begin{Bmatrix} R_1 = ? \\ S_1 = 0 \\ R_2 = F \\ S_2 = ? \\ R_3 = ? \\ S_3 = ? \end{Bmatrix} = \frac{AE}{4L}
\left[\begin{array}{cc|cc|cc}
5 & \sqrt{3} & -4 & 0 & -1 & -\sqrt{3} \\
\sqrt{3} & 3 & 0 & 0 & -\sqrt{3} & -3 \\
\hline
-4 & 0 & 5 & -\sqrt{3} & -1 & \sqrt{3} \\
0 & 0 & -\sqrt{3} & 3 & \sqrt{3} & -3 \\
\hline
-1 & -\sqrt{3} & -1 & \sqrt{3} & 2 & 0 \\
-\sqrt{3} & -3 & \sqrt{3} & -3 & 0 & 6
\end{array}\right]
\begin{Bmatrix} u_1 = 0 \\ v_1 = ? \\ u_2 = ? \\ v_2 = 0 \\ u_3 = 0 \\ v_3 = 0 \end{Bmatrix}
$$

Solution Procedures

$$
\begin{Bmatrix} R_2 = F \\ S_1 = 0 \\ R_1 = ? \\ S_2 = ? \\ R_3 = ? \\ S_3 = ? \end{Bmatrix} = \frac{AE}{4L} \begin{bmatrix} -4 & 0 & 5 & -\sqrt{3} & -1 & \sqrt{3} \\ \sqrt{3} & 3 & 0 & 0 & -\sqrt{3} & -3 \\ 5 & \sqrt{3} & -4 & 0 & -1 & -\sqrt{3} \\ 0 & 0 & -\sqrt{3} & 3 & \sqrt{3} & -3 \\ -1 & -\sqrt{3} & -1 & \sqrt{3} & 2 & 0 \\ -\sqrt{3} & -3 & \sqrt{3} & -3 & 0 & 6 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ v_1 = ? \\ u_2 = ? \\ v_2 = 0 \\ u_3 = 0 \\ v_3 = 0 \end{Bmatrix}
$$

→ $u_2 = 4FL/5AE, \; v_1 = 0$

$$
\begin{Bmatrix} R_2 = F \\ S_1 = 0 \\ R_1 = ? \\ S_2 = ? \\ R_3 = ? \\ S_3 = ? \end{Bmatrix} = \frac{AE}{4L} \begin{bmatrix} -4 & 0 & 5 & -\sqrt{3} & -1 & \sqrt{3} \\ \sqrt{3} & 3 & 0 & 0 & -\sqrt{3} & -3 \\ 5 & \sqrt{3} & -4 & 0 & -1 & -\sqrt{3} \\ 0 & 0 & -\sqrt{3} & 3 & \sqrt{3} & -3 \\ -1 & -\sqrt{3} & -1 & \sqrt{3} & 2 & 0 \\ -\sqrt{3} & -3 & \sqrt{3} & -3 & 0 & 6 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 = \dfrac{4FL}{5AE} \\ v_2 = 0 \\ u_3 = 0 \\ v_3 = 0 \end{Bmatrix}
$$

Recovery of axial forces

Element I

$$\begin{Bmatrix} P_1 \\ Q_1 \\ P_2 \\ Q_2 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 = -\frac{4FL}{5AE} \\ v_2 = 0 \end{Bmatrix} = F\begin{Bmatrix} -4/5 \\ 0 \\ 4/5 \\ 0 \end{Bmatrix}$$

Element II

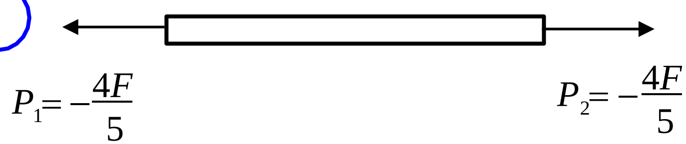
$$\begin{Bmatrix} P_2 \\ Q_2 \\ P_3 \\ Q_3 \end{Bmatrix} = \frac{AE}{4L}\begin{bmatrix} 1 & -\sqrt{3} & -1 & \sqrt{3} \\ -\sqrt{3} & 3 & \sqrt{3} & -3 \\ -1 & \sqrt{3} & 1 & -\sqrt{3} \\ \sqrt{3} & -3 & -\sqrt{3} & 3 \end{bmatrix}\begin{Bmatrix} u_2 = -\frac{4FL}{5AE} \\ v_2 = 0 \\ u_3 = 0 \\ v_3 = 0 \end{Bmatrix} = F\begin{Bmatrix} 1/5 \\ -\sqrt{3}/5 \\ -1/5 \\ \sqrt{3}/5 \end{Bmatrix}$$

Element III

$$\begin{Bmatrix} P_1 \\ Q_1 \\ P_3 \\ Q_3 \end{Bmatrix} = \frac{AE}{4L}\begin{bmatrix} 1 & \sqrt{3} & -1 & -\sqrt{3} \\ \sqrt{3} & 3 & -\sqrt{3} & -3 \\ -1 & -\sqrt{3} & 1 & \sqrt{3} \\ -\sqrt{3} & -3 & \sqrt{3} & 3 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_3 = 0 \\ v_3 = 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

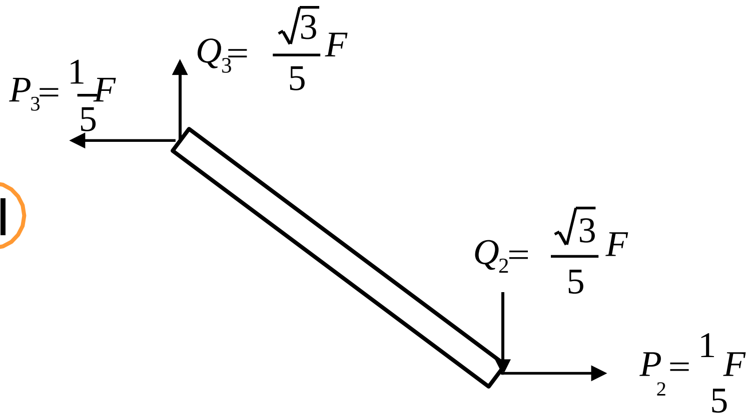
Stresses inside members

Element I



$$\sigma = \frac{4F}{5A}$$

Element II



Element III

Finite Element Analysis of Beams

- Hermite shape functions
- Element stiffness matrix
- Load vector
- Problems

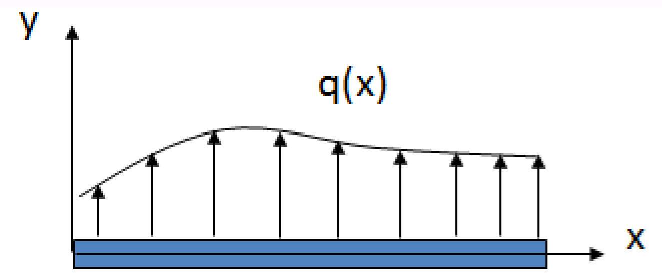
Review

Pure bending problems

- Normal strain: $\varepsilon_x = -\frac{y}{\rho}$
- Normal stress: $\sigma_x = -\frac{Ey}{\rho}$
- Normal stress with bending moment: $\int -\sigma_x y\,dA = M$
- Moment-curvature relationship: $\frac{1}{\rho} = \frac{M}{EI}$ → $M = EI\frac{1}{\rho} \approx EI\frac{d^2y}{dx^2}$
- Flexure formula: $\sigma_x = -\frac{My}{I}$ $\quad I = \int y^2 dA$

Bending Beam

Review



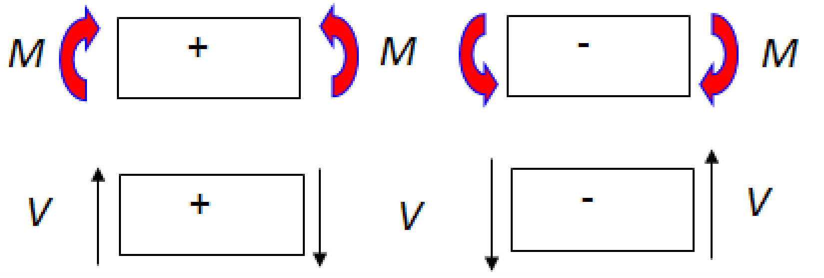
Relationship between shear force, bending moment and transverse load:

Deflection

$$\frac{dV}{dx} = q \qquad \frac{dM}{dx} = V$$

$$EI\frac{d^4y}{dx^4} = q$$

Sign convention:



Governing Equation and Boundary Condition

- Governing Equation

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v(x)}{dx^2}\right)-q(x)=0, \qquad 0<x<L$$

- Boundary Conditions

$$v=?\ \&\ \frac{dv}{dx}=?\ \&\ EI\frac{d^2v}{dx^2}=?\&\ \frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right)=?, \qquad \text{at } x=0$$

$$v=?\ \&\ \frac{dv}{dx}=?\ \&\ EI\frac{d^2v}{dx^2}=?\&\ \frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right)=?, \qquad \text{at } x=L$$

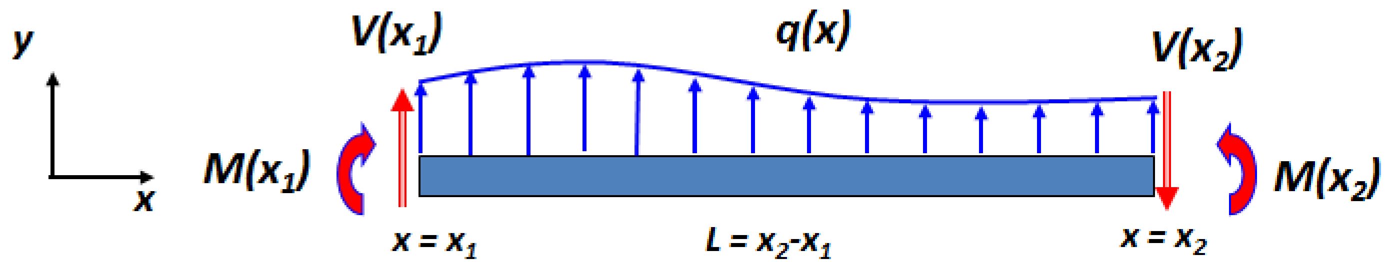
Essential BCs – if v or $\frac{dv}{dx}$ ***is specified at the boundary.***

Natural BCs – if v or is specified at the boundary.

$$EI\frac{d^2v}{dx^2} \qquad \frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right)$$

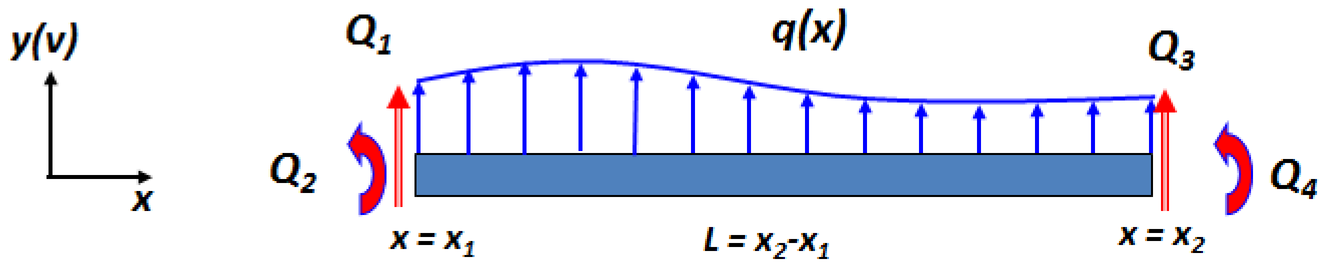
Weak Form from Integration-by-Parts ----- (2nd time)

$$0 = \int_{x_1}^{x_2} \left[\frac{d^2 w}{dx^2} \left(EI \frac{d^2 v}{dx^2} \right) - wq \right] dx + w \frac{d}{dx} \left(EI \frac{d^2 v}{dx^2} \right) \Bigg|_{x_1}^{x_2} - \frac{dw}{dx} \left(EI \frac{d^2 v}{dx^2} \right) \Bigg|_{x_1}^{x_2}$$



$$0 = \int_{x_1}^{x_2} \left[\frac{d^2 w}{dx^2} \left(EI \frac{d^2 v}{dx^2} \right) - wq \right] dx + \left[wV - \frac{dw}{dx} M \right] \Bigg|_{x_1}^{x_2}$$

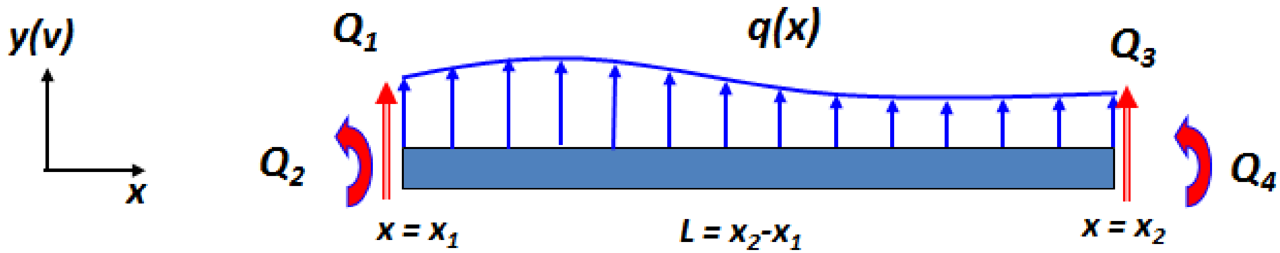
$$0 = \int_{x_1}^{x_2} \left[\frac{d^2 w}{dx^2} \left(EI \frac{d^2 v}{dx^2} \right) - wq \right] dx + \left[wV - \frac{dw}{dx} M \right]_{x_1}^{x^2}$$



$$Q_1 = V(x_1), \quad Q_2 = -M(x_1), \quad Q_3 = -V(x_2), \quad Q_4 = M(x_2)$$

$$\int_{x_1}^{x_2} \left[\frac{d^2 w}{dx^2} \left(EI \frac{d^2 v}{dx^2} \right) - wq \right] dx = w(x_1)Q_1 + w(x_2)Q_3 + \left.\frac{dw}{dx}\right|_1 Q_2 + \left.\frac{dw}{dx}\right|_2 Q_4$$

Ritz Method for Approximation



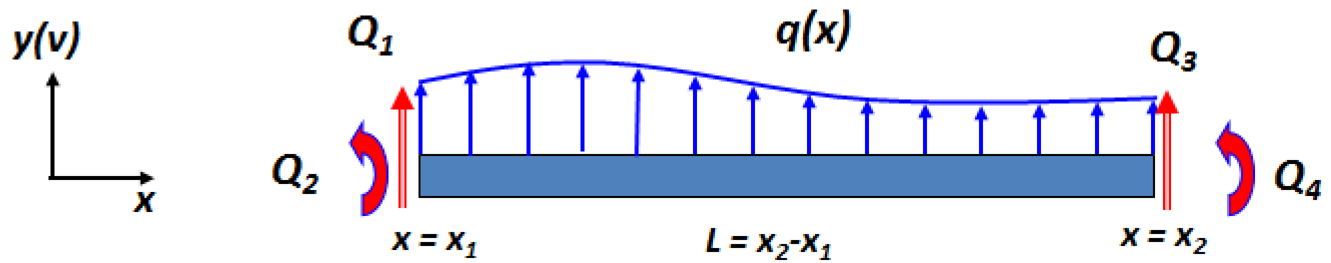
$$Let \quad v(x)=\sum_{j=1}^{n} u_j\phi_j(x) \quad \text{and } n=4$$

$$u_1=v(x_1);\quad u_2=\left.\frac{dv}{dx}\right|_{x=x_1};\quad u_3=v(x_2);\quad u_4=\left.\frac{dv}{dx}\right|_{x=x_2};$$

$$\int_{x_1}^{x_2}\left[\frac{d^2w}{dx^2}EI\left(\sum_{j=1}^{4}u_j\frac{d^2\phi_j}{dx^2}\right)-wq\right]dx = w(x_1)Q_1 + w(x_2)Q_3 + \left.\frac{dw}{dx}\right|_1 Q_2 + \left.\frac{dw}{dx}\right|_2 Q_4$$

- **Let w(x)= f_i (x), i = 1, 2, 3, 4**

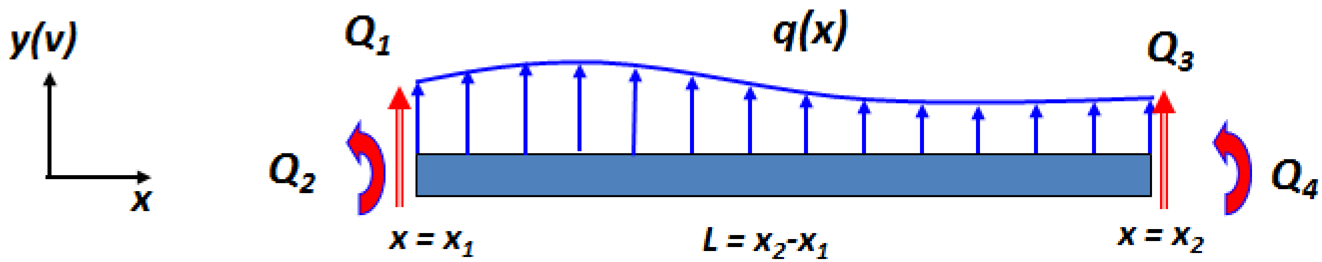
$$\int_{x_1}^{x_2}\left[\frac{d^2\phi_i}{dx^2}EI\left(\sum_{j=1}^{4}u_j\frac{d^2\phi_j}{dx^2}\right)-\phi_i q\right]dx=\phi_i(x_1)Q_1+\phi_i(x_2)Q_3+\left.\frac{d\phi_i}{dx}\right|_1 Q_2+\left.\frac{d\phi_i}{dx}\right|_2 Q_4$$



$$\left[\left(\phi_i\right)\Big|_{x_1} Q_1 + \left(-\frac{d\phi_i}{dx}\right)\Big|_{x_1} Q_2 + \left(\phi_i\right)\Big|_{x_2} Q_3 + \left(-\frac{d\phi_i}{dx}\right)\Big|_{x_2} Q_4 \right] = \sum_{j=1}^{4} K_{ij} u_j - q_i$$

where $K_{ij} = \int_{x_1}^{x_2} EI \left(\frac{d^2\phi_i}{dx^2} \frac{d^2\phi_j}{dx^2} \right) dx$ and $q_i = \int_{x_1}^{x_2} \phi_i q \, dx$

Ritz Method for Approximation



$$\begin{bmatrix} \phi_1|_{x_1} & -\left(\frac{d\phi_1}{dx}\right)\Big|_{x_1} & \phi_1|_{x_2} & -\left(\frac{d\phi_1}{dx}\right)\Big|_{x_2} \\ \phi_2|_{x_1} & -\left(\frac{d\phi_2}{dx}\right)\Big|_{x_1} & \phi_2|_{x_2} & -\left(\frac{d\phi_2}{dx}\right)\Big|_{x_2} \\ \phi_3|_{x_1} & -\left(\frac{d\phi_3}{dx}\right)\Big|_{x_1} & \phi_3|_{x_2} & -\left(\frac{d\phi_3}{dx}\right)\Big|_{x_2} \\ \phi_4|_{x_1} & -\left(\frac{d\phi_4}{dx}\right)\Big|_{x_1} & \phi_4|_{x_2} & -\left(\frac{d\phi_4}{dx}\right)\Big|_{x_2} \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} - \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

Selection of Shape Function

$$\begin{bmatrix} \phi_1\big|_{x_1} & \left.\frac{d\phi_1}{dx}\right|_{x_1} & \phi_1\big|_{x_2} & \left.\frac{d\phi_1}{dx}\right|_{x_2} \\ \phi_2\big|_{x_1} & \left.\frac{d\phi_2}{dx}\right|_{x_1} & \phi_2\big|_{x_2} & \left.\frac{d\phi_2}{dx}\right|_{x_2} \\ \phi_3\big|_{x_1} & \left.\frac{d\phi_3}{dx}\right|_{x_1} & \phi_3\big|_{x_2} & \left.\frac{d\phi_3}{dx}\right|_{x_2} \\ \phi_4\big|_{x_1} & \left.\frac{d\phi_4}{dx}\right|_{x_1} & \phi_4\big|_{x_2} & \left.\frac{d\phi_4}{dx}\right|_{x_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ Interpolation Properties

→

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{12} & K_{22} & K_{23} & K_{24} \\ K_{13} & K_{23} & K_{33} & K_{34} \\ K_{14} & K_{24} & K_{34} & K_{44} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} - \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

Derivation of Shape Function for Beam Element Local Coordinates

$$v(\xi) = \tilde{u}_1\phi_1 + \tilde{u}_2\phi_2 + \tilde{u}_3\phi_3 + \tilde{u}_4\phi_4$$

and $$\frac{dv(\xi)}{d\xi} = \tilde{u}_1\frac{d\phi_1}{d\xi} + \tilde{u}_2\frac{d\phi_2}{d\xi} + \tilde{u}_3\frac{d\phi_3}{d\xi} + \tilde{u}_4\frac{d\phi_4}{d\xi}$$

where $$\tilde{u}_1 = v_1 \quad \tilde{u}_2 = \frac{dv_1}{d\xi} \quad \tilde{u}_3 = v_2 \quad \tilde{u}_4 = \frac{dv_2}{d\xi}$$

Let $$\phi_i = a_i + b_i\xi + c_i\xi^2 + d_i\xi^3$$

Find coefficients to satisfy theinterpolation properties.

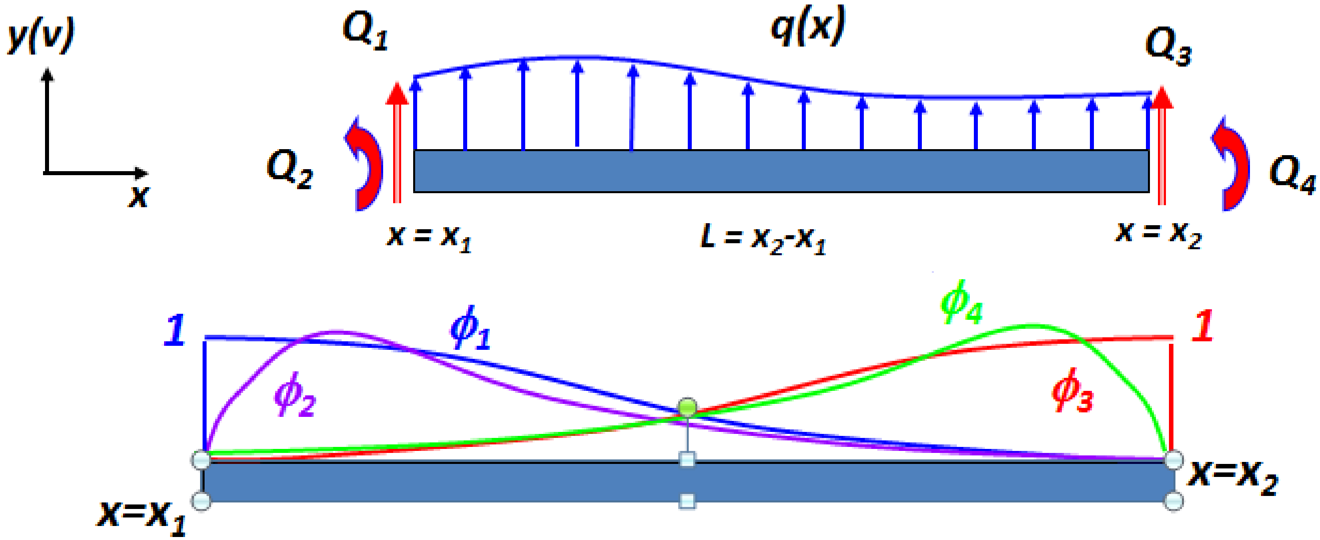
Derivation of Shape Function for Beam Element

In the global coordinates:

$$v(x) = v_1\phi_1(x) + \frac{l}{2}\frac{dv_1}{dx}\phi_2(x) + v_2\phi_3(x) + \frac{l}{2}\frac{dv_2}{dx}\phi_4(x)$$

$$\begin{Bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{Bmatrix} = \begin{Bmatrix} 1-3\left(\frac{x-x_1}{x_2-x_1}\right)^2+2\left(\frac{x-x_1}{x_2-x_1}\right)^3 \\ \frac{2}{l}(x-x_1)\left(1-\frac{x-x_1}{x_2-x_1}\right)^2 \\ 3\left(\frac{x-x_1}{x_2-x_1}\right)^2-2\left(\frac{x-x_1}{x_2-x_1}\right)^3 \\ \frac{2}{l}(x-x_1)\left[\left(\frac{x-x_1}{x_2-x_1}\right)^2-\frac{x-x_1}{x_2-x_1}\right]\end{Bmatrix}$$

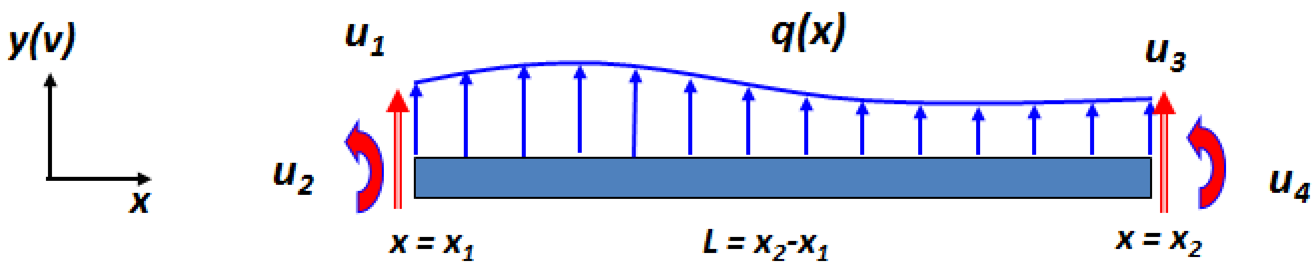
Element Equations of 4th Order 1-D Model



$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} + \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{12} & K_{22} & K_{23} & K_{24} \\ K_{13} & K_{23} & K_{33} & K_{34} \\ K_{14} & K_{24} & K_{34} & K_{44} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$where \quad K_{ij} = \int_{x_1}^{x_2} EI \left(\frac{d^2\phi_i}{dx^2} \frac{d^2\phi_j}{dx^2} \right) dx = K_{ji} \quad and \quad q_i = \int_{x_1}^{x_2} \phi_i q \, dx$$

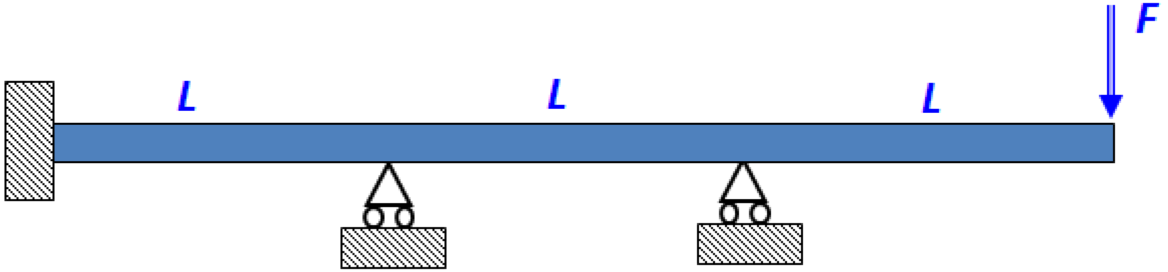
Element Equations of 4th Order 1-D Model



$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} + \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \frac{2EI}{L^3} \begin{bmatrix} 6 & 3L & -6 & 3L \\ 3L & 2L^2 & -3L & L^2 \\ -6 & -3L & 6 & -3L \\ 3L & L^2 & -3L & 2L^2 \end{bmatrix} \begin{Bmatrix} u_1 = v_1 \\ u_2 = \theta_1 \\ u_3 = v_2 \\ u_4 = \theta_2 \end{Bmatrix}$$

$$where \quad q_i = \int_{x_1}^{x_2} \phi_i q \, dx$$

Example 1.



Governing equation:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) - q(x) = 0 \qquad 0 < x < L$$

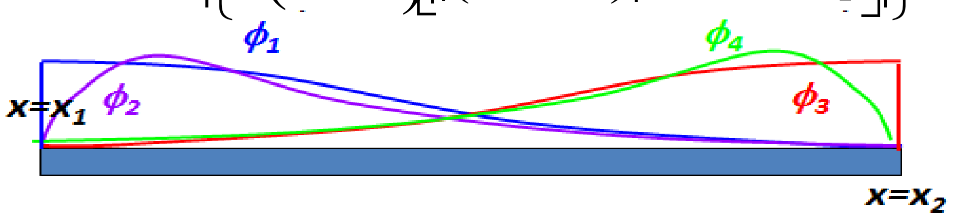
Weak form for one element

where $\displaystyle\int_{x_1}^{x_2}\left(EI\frac{d^2w}{dx^2}\frac{d^2v}{dx^2} - wq\right)dx - w(x_1)Q_1 - \left.\frac{dw}{dx}\right|_{x_1} Q_2 - w(x_2)Q_3 - \left.\frac{dw}{dx}\right|_{x_2} Q_4 = 0$

$$Q_1 = V(x_1) \qquad Q_2 = -M(x_1) \qquad Q_3 = -V(x_2) \qquad Q_4 = M(x_2)$$

- Approximation function: $v(x) = v_1\phi_1(x) + \frac{l}{2}\frac{dv_1}{dx}\phi_2(x) + v_2\phi_3(x) + \frac{l}{2}\frac{dv_2}{dx}\phi_4(x)$

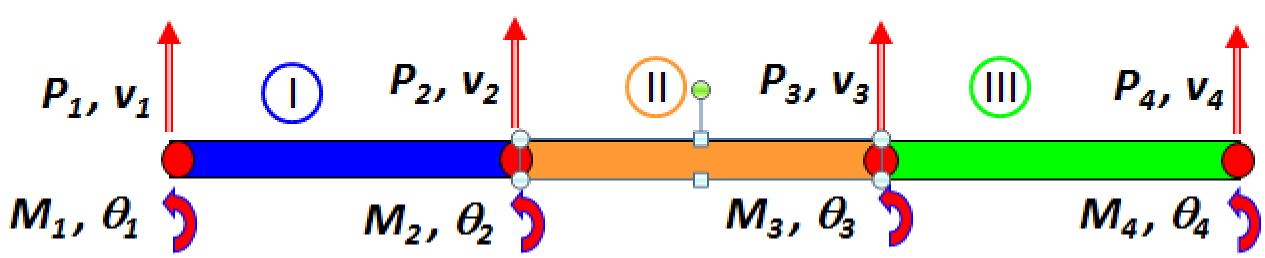
$$\begin{Bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{Bmatrix} = \begin{Bmatrix} 1-3\left(\frac{x-x_1}{x_2-x_1}\right)^2+2\left(\frac{x-x_1}{x_2-x_1}\right)^3 \\ \frac{2}{l}(x-x_1)\left(1-\frac{x-x_1}{x_2-x_1}\right)^2 \\ 3\left(\frac{x-x_1}{x_2-x_1}\right)^2-2\left(\frac{x-x_1}{x_2-x_1}\right)^3 \\ \frac{2}{l}(x-x_1)\left[\left(\frac{x-x_1}{x_2-x_1}\right)^2-\frac{x-x_1}{x_2-x_1}\right] \end{Bmatrix}$$



Finite element model:

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \frac{2EI}{L^3}\begin{bmatrix} 6 & 3L & -6 & 3L \\ 3L & 2L^2 & -3L & L^2 \\ -6 & -3L & 6 & -3L \\ 3L & L^2 & -3L & 2L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

Discretization:



Matrix Assembly of Multiple Beam Elements

Element I

$$\begin{Bmatrix} Q_1^I \\ Q_2^I \\ Q_3^I \\ Q_4^I \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \frac{2EI}{L^3}\begin{bmatrix} 6 & 3L & -6 & 3L & 0 & 0 & 0 & 0 \\ 3L & 2L^2 & -3L & L^2 & 0 & 0 & 0 & 0 \\ -6 & -3L & 6 & -3L & 0 & 0 & 0 & 0 \\ 3L & L^2 & -3L & 2L^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ v_4 \\ \theta_4 \end{Bmatrix}$$

Element II

$$\begin{Bmatrix} 0 \\ 0 \\ Q_1^{II} \\ Q_2^{II} \\ Q_3^{II} \\ Q_4^{II} \\ 0 \\ 0 \end{Bmatrix} = \frac{2EI}{L^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 3L & -6 & 3L & 0 & 0 \\ 0 & 0 & 3L & 2L^2 & -3L & L^2 & 0 & 0 \\ 0 & 0 & -6 & -3L & 6 & -3L & 0 & 0 \\ 0 & 0 & 3L & L^2 & -3L & 2L^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ v_4 \\ \theta_4 \end{Bmatrix}$$

Element III

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ Q_1^{III} \\ Q_2^{III} \\ Q_3^{III} \\ Q_4^{III} \end{Bmatrix} = \frac{2EI}{L^3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 3L & -6 & 3L \\ 0 & 0 & 0 & 0 & 3L & 2L^2 & -3L & L^2 \\ 0 & 0 & 0 & 0 & -6 & -3L & 6 & -3L \\ 0 & 0 & 0 & 0 & 3L & L^2 & -3L & 2L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ v_4 \\ \theta_4 \end{Bmatrix}$$

$$\begin{Bmatrix} P_1 \\ M_1 \\ P_2 \\ M_2 \\ P_3 \\ M_3 \\ P_4 \\ M_4 \end{Bmatrix} = \frac{2EI}{L^3} \begin{bmatrix} 6 & 3L & -6 & 3L & 0 & 0 & 0 & 0 \\ 3L & 2L^2 & -3L & L^2 & 0 & 0 & 0 & 0 \\ -6 & -3L & 6+6 & -3L+3L & -6 & 3L & 0 & 0 \\ 3L & L^2 & -3L+3L & 2L^2+2L^2 & -3L & L^2 & 0 & 0 \\ 0 & 0 & -6 & -3L & 6+6 & -3L+3L & -6 & 3L \\ 0 & 0 & 3L & L^2 & -3L+3L & 2L^2+2L^2 & -3L & L^2 \\ 0 & 0 & 0 & 0 & -6 & -3L & 6 & -3L \\ 0 & 0 & 0 & 0 & 3L & L^2 & -3L & 2L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ v_4 \\ \theta_4 \end{Bmatrix}$$

Apply known boundary conditions

$$
\begin{Bmatrix} P_1 = ? \\ M_1 = ? \\ P_2 = ? \\ M_2 = 0 \\ P_3 = ? \\ M_3 = 0 \\ P_4 = -F \\ M_4 = 0 \end{Bmatrix}
= \frac{2EI}{L^3}
\begin{bmatrix}
6 & 3L & -6 & 3L & 0 & 0 & 0 & 0 \\
3L & 2L^2 & -3L & L^2 & 0 & 0 & 0 & 0 \\
-6 & -3L & 12 & 0 & -6 & 3L & 0 & 0 \\
3L & L^2 & 0 & 4L^2 & -3L & L^2 & 0 & 0 \\
0 & 0 & -6 & -3L & 12 & 0 & -6 & 3L \\
0 & 0 & 3L & L^2 & 0 & 4L^2 & -3L & L^2 \\
0 & 0 & 0 & 0 & -6 & -3L & 6 & -3L \\
0 & 0 & 0 & 0 & 3L & L^2 & -3L & 2L^2
\end{bmatrix}
\begin{Bmatrix} v_1 = 0 \\ \theta_1 = 0 \\ v_2 = 0 \\ \theta_2 = ? \\ v_3 = 0 \\ \theta_3 = ? \\ v_4 = ? \\ \theta_4 = ? \end{Bmatrix}
$$

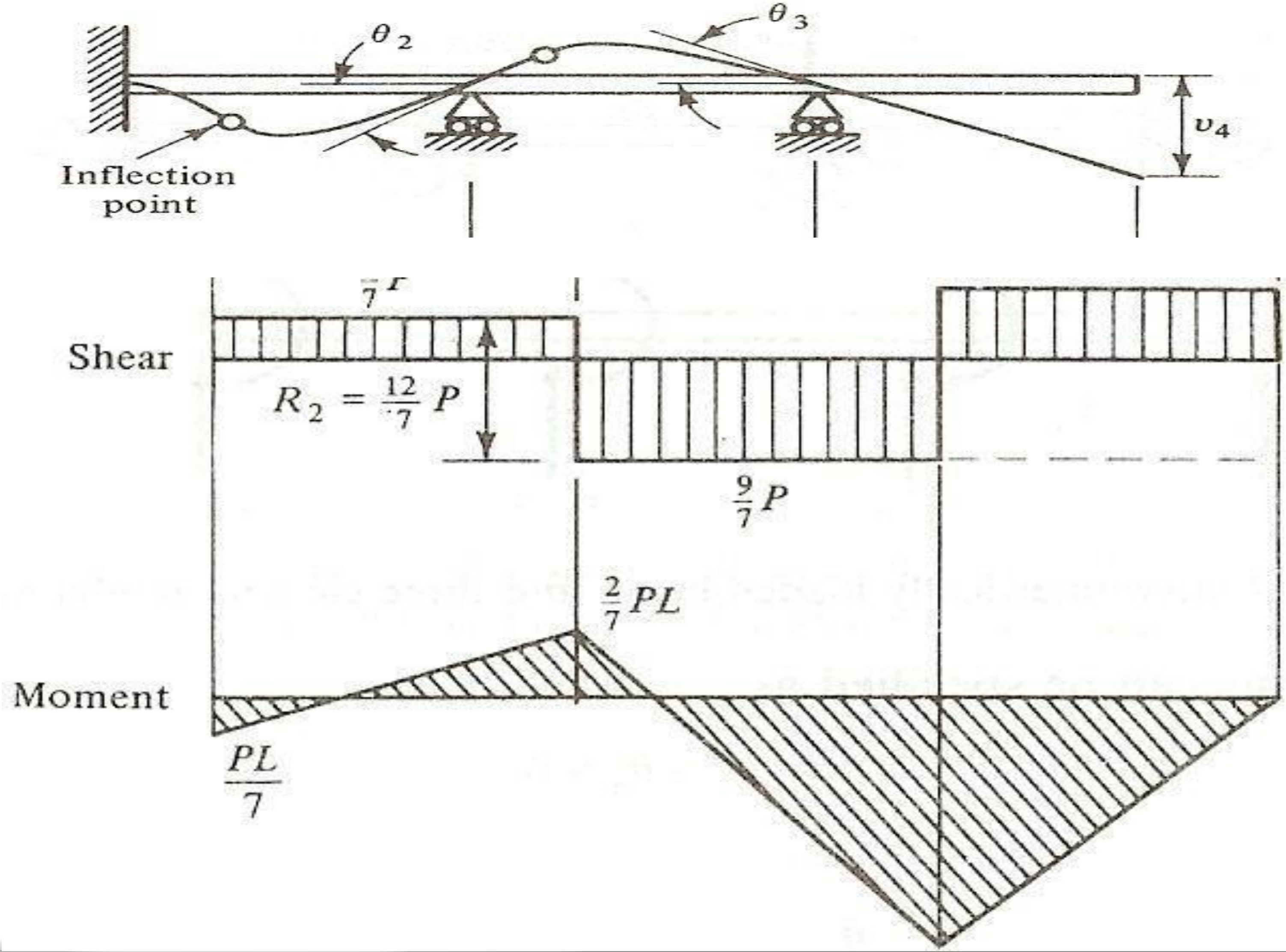
$$
\begin{Bmatrix} M_2 = 0 \\ M_3 = 0 \\ P_4 = -F \\ M_4 = 0 \\ P_1 = ? \\ M_1 = ? \\ P_2 = ? \\ P_3 = ? \end{Bmatrix}
= \frac{2EI}{L^3}
\begin{bmatrix}
3L & L^2 & 0 & 4L^2 & -3L & L^2 & 0 & 0 \\
0 & 0 & 3L & L^2 & 0 & 4L^2 & -3L & L^2 \\
0 & 0 & 0 & 0 & -6 & -3L & 6 & -3L \\
0 & 0 & 0 & 0 & 3L & L^2 & -3L & 2L^2 \\
6 & 3L & -6 & 3L & 0 & 0 & 0 & 0 \\
3L & 2L^2 & -3L & L^2 & 0 & 0 & 0 & 0 \\
-6 & -3L & 12 & 0 & -6 & 3L & 0 & 0 \\
0 & 0 & -6 & -3L & 12 & 0 & -6 & 3L
\end{bmatrix}
\begin{Bmatrix} v_1 = 0 \\ \theta_1 = 0 \\ v_2 = 0 \\ \theta_2 = ? \\ v_3 = 0 \\ \theta_3 = ? \\ v_4 = ? \\ \theta_4 = ? \end{Bmatrix}
$$

Solution Procedures

$$
\begin{Bmatrix} M_2 = 0 \\ M_3 = 0 \\ P_4 = -F \\ M_4 = 0 \\ P_1 = ? \\ M_1 = ? \\ P_2 = ? \\ P_3 = ? \end{Bmatrix} = \frac{2EI}{L^3}
\left[\begin{array}{cccc:c:c:cc}
3L & L^2 & 0 & -3L & 4L^2 & L^2 & 0 & 0 \\
0 & 0 & 3L & 0 & L^2 & 4L^2 & -3L & L^2 \\
0 & 0 & 0 & -6 & 0 & -3L & 6 & -3L \\
0 & 0 & 0 & 3L & 0 & L^2 & -3L & 2L^2 \\
6 & 3L & -6 & 0 & 3L & 0 & 0 & 0 \\
3L & 2L^2 & -3L & 0 & L^2 & 0 & 0 & 0 \\
-6 & -3L & 12 & -6 & 0 & 3L & 0 & 0 \\
0 & 0 & -6 & -12 & 3L & 0 & -6 & 3L
\end{array}\right]
\begin{Bmatrix} v_1 = 0 \\ \theta_1 = 0 \\ v_2 = 0 \\ v_3 = 0 \\ \theta_2 = ? \\ \theta_3 = ? \\ v_4 = ? \\ \theta_4 = ? \end{Bmatrix}
$$

$$
\begin{Bmatrix} M_2 = 0 \\ M_3 = 0 \\ P_4 = -F \\ M_4 = 0 \end{Bmatrix} = \frac{2EI}{L^3}
\begin{bmatrix} 4L^2 & L^2 & 0 & 0 \\ L^2 & 4L^2 & -3L & L^2 \\ 0 & -3L & 6 & -3L \\ 0 & L^2 & -3L & 2L^2 \end{bmatrix}
\begin{Bmatrix} \theta_2 = ? \\ \theta_3 = ? \\ v_4 = ? \\ \theta_4 = ? \end{Bmatrix}
\qquad
\begin{Bmatrix} P_1 = ? \\ M_1 = ? \\ P_2 = ? \\ P_3 = ? \end{Bmatrix} = \frac{2EI}{L^3}
\begin{bmatrix} 3L & 0 & 0 & 0 \\ L^2 & 0 & 0 & 0 \\ 0 & 3L & 0 & 0 \\ -3L & 0 & -6 & 3L \end{bmatrix}
\begin{Bmatrix} \theta_2 \\ \theta_3 \\ v_4 \\ \theta_4 \end{Bmatrix}
$$

Shear Resultant & Bending Moment Diagram



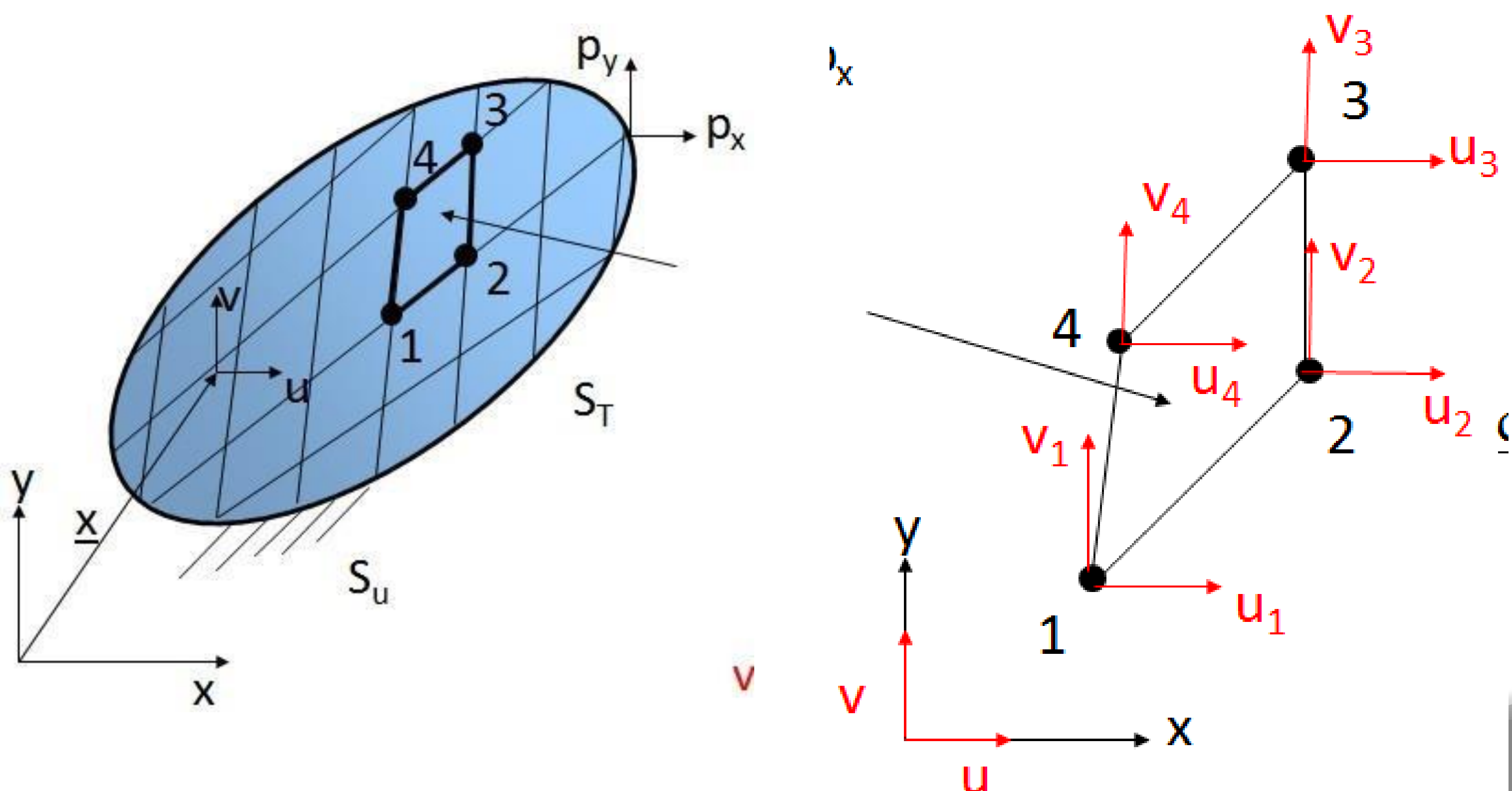
UNIT-III
2D ANALYSISENTS

Introduction

- Computation of shape functions for constant straintriangle
- Properties of the shape functions
- Computation of strain-displacement matrix
- Computation of element stiffness matrix
- Computation of nodal loads due to body forces
- Computation of nodal loads due to traction
- Recommendations for use
- Example problems

Finite element formulation for 2D

- Divide the body into connected to each other through special points (“nodes”)



$$\underline{d} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$u(x,y) \approx N_1(x,y)u_1 + N_2(x,y)u_2 + N_3(x,y)u_3 + N_4(x,y)u_4$$

$$v(x,y) \approx N_1(x,y)v_1 + N_2(x,y)v_2 + N_3(x,y)v_3 + N_4(x,y)v_4$$

$$\underline{u} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\underline{u} = \underline{N}\,\underline{d}$$

TASK 2: APPROXIMATE THE STRAIN and STRESS WITHIN EACH ELEMENT

Approximation of the strain in element 'e'

$$\varepsilon_x = \frac{\partial u(x,y)}{\partial x} \approx \frac{\partial N_1(x,y)}{\partial x} u_1 + \frac{\partial N_2(x,y)}{\partial x} u_2 + \frac{\partial N_3(x,y)}{\partial x} u_3 + \frac{\partial N_4(x,y)}{\partial x} u_4$$

$$\varepsilon_y = \frac{\partial v(x,y)}{\partial y} \approx \frac{\partial N_1(x,y)}{\partial y} v_1 + \frac{\partial N_2(x,y)}{\partial y} v_2 + \frac{\partial N_3(x,y)}{\partial y} v_3 + \frac{\partial N_4(x,y)}{\partial y} v_4$$

$$\gamma_{xy} = \frac{\partial u(x,y)}{\partial y} + \frac{\partial v(x,y)}{\partial x} \approx \frac{\partial N_1(x,y)}{\partial y} u_1 + \frac{\partial N_1(x,y)}{\partial x} v_1 + \ldots\ldots$$

$$\underline{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$= \underbrace{\begin{bmatrix} \dfrac{\partial N_1(x,y)}{\partial x} & 0 & \dfrac{\partial N_2(x,y)}{\partial x} & 0 & \dfrac{\partial N_3(x,y)}{\partial x} & 0 & \dfrac{\partial N_4(x,y)}{\partial x} & 0 \\ 0 & \dfrac{\partial N_1(x,y)}{\partial y} & 0 & \dfrac{\partial N_2(x,y)}{\partial y} & 0 & \dfrac{\partial N_3(x,y)}{\partial y} & 0 & \dfrac{\partial N_4(x,y)}{\partial y} \\ \dfrac{\partial N_1(x,y)}{\partial y} & \dfrac{\partial N_1(x,y)}{\partial x} & \dfrac{\partial N_2(x,y)}{\partial y} & \dfrac{\partial N_2(x,y)}{\partial x} & \dfrac{\partial N_3(x,y)}{\partial y} & \dfrac{\partial N_3(x,y)}{\partial x} & \dfrac{\partial N_4(x,y)}{\partial y} & \dfrac{\partial N_4(x,y)}{\partial x} \end{bmatrix}}_{\underline{B}} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\underline{\varepsilon} = \underline{B}\ \underline{d}$$

Summary: For each element

- Displacement approximation in terms of shape functions

$$\underline{u} = \underline{N}\,\underline{d}$$

- Strain approximation in terms of strain-displacement

$$\underline{\varepsilon} = \underline{B}\,\underline{d}$$

- Stress approximation $\sigma = \underline{DBd}$

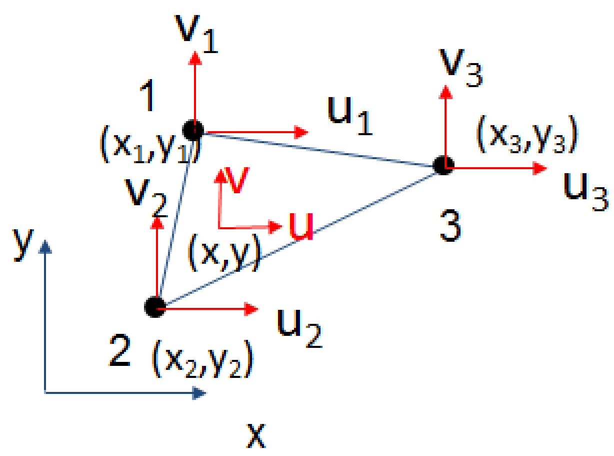
- Element stiffness matrix matrix

$$\underline{k} = \int_{V^e} \underline{B}^T \underline{D}\,\underline{B}\, dV$$

- Element nodal load vector

$$\underline{f} = \underbrace{\int_{V^e} \underline{N}^T \underline{X}\, dV}_{\underline{f}_b} + \underbrace{\int_{S_T^e} \underline{N}^T \underline{T}_S\, dS}_{\underline{f}_S}$$

- **<u>Constant Strain Triangle (CST) :</u>** Simplest 2D finite element



- 3 nodes per element
- 2 dofs per node (each node can move in x- and y- directions)
- Hence 6 dofs per element

- The displacement approximation in terms of shape functionsis

$$u\ (x,y) \approx N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v(x,y) \approx N_1 v_1 + N_2 v_2 + N_3 v_3$$

$$\underline{u} = \begin{Bmatrix} u\ (x,y) \\ v\ (x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\underline{u}_{2\times1} = \underline{N}_{2\times6}\, \underline{d}_{6\times1}$$

$$\underline{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

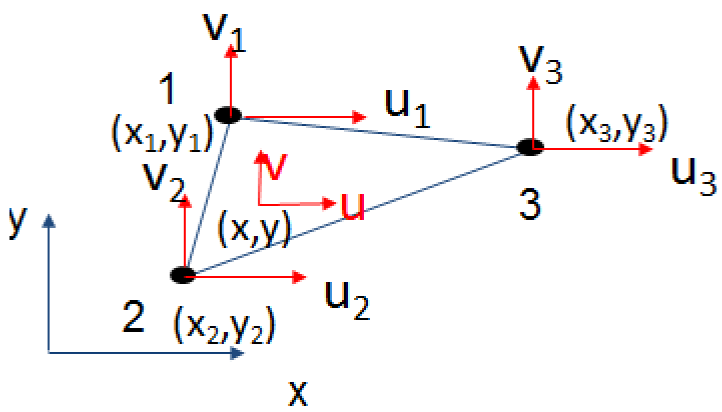
- Formula for the shape functions are

$$N_1 = \frac{a_1 + b_1 x + c_1 y}{2A}$$

$$N_2 = \frac{a_2 + b_2 x + c_2 y}{2A}$$

$$N_3 = \frac{a_3 + b_3 x + c_3 y}{2A}$$

where



$$A = area\,of\ \ triangle = \frac{1}{2}\det\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

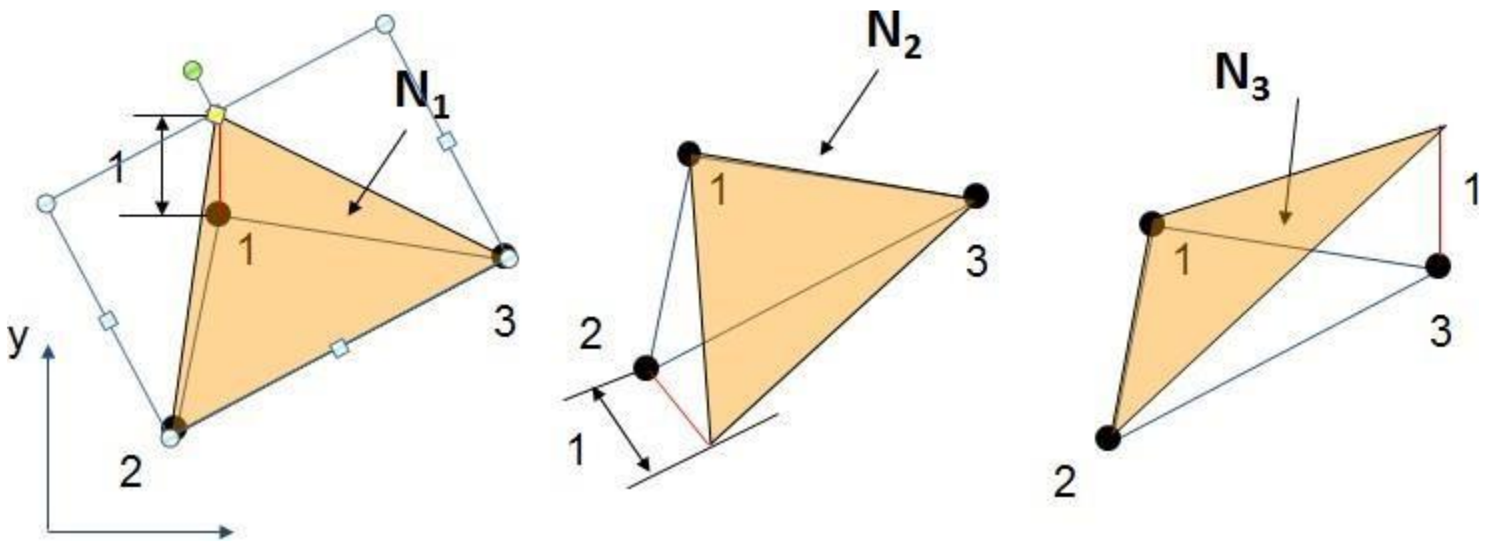
$$a_1 = x_2 y_3 - x_3 y_2 \qquad b_1 = y_2 - y_3 \qquad c_1 = x_3 - x_2$$

$$a_2 = x_3 y_1 - x_1 y_3 \qquad b_2 = y_3 - y_1 \qquad c_2 = x_1 - x_3$$

$$a_3 = x_1 y_2 - x_2 y_1 \qquad b_3 = y_1 - y_2 \qquad c_3 = x_2 - x_1$$

Properties of the shape functions

- The shape functions N_1, N_2 and N_3 are linear functions of x and y



$$N_i = \begin{cases} 1 & at\ node\ 'i' \\ 0 & at\ other\ nodes \end{cases}$$

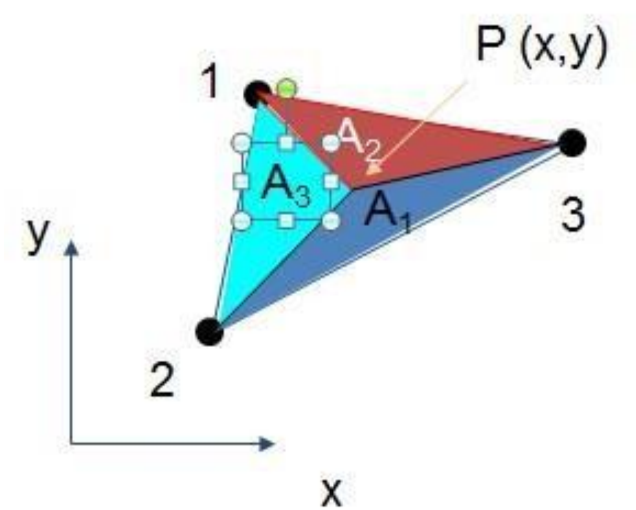
- At every point in the domain

$$\sum_{i=1}^{3} N_i = 1$$

$$\sum_{i=1}^{3} N_i x_i = x$$

$$\sum_{i=1}^{3} N_i y_i = y$$

- Geometric interpretation of the shape functions, at any point P(x,y) that the shape functions are evaluated



$$N_1 = \frac{A_1}{A}$$

$$N_2 = \frac{A_2}{A}$$

$$N_3 = \frac{A_3}{A}$$

- Approximation of the strains

$$\underline{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \approx \underline{B\,d}$$

$$\underline{B} = \begin{bmatrix} \dfrac{\partial N_1(x,y)}{\partial x} & 0 & \dfrac{\partial N_2(x,y)}{\partial x} & 0 & \dfrac{\partial N_3(x,y)}{\partial x} & 0 \\ 0 & \dfrac{\partial N_1(x,y)}{\partial y} & 0 & \dfrac{\partial N_2(x,y)}{\partial y} & 0 & \dfrac{\partial N_3(x,y)}{\partial y} \\ \dfrac{\partial N_1(x,y)}{\partial y} & \dfrac{\partial N_1(x,y)}{\partial x} & \dfrac{\partial N_2(x,y)}{\partial y} & \dfrac{\partial N_2(x,y)}{\partial x} & \dfrac{\partial N_3(x,y)}{\partial y} & \dfrac{\partial N_3(x,y)}{\partial x} \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

- Inside each element, all components of strain are constant: hence the name Constant Strain Triangle.

- Element stresses (constant inside each element).

$$o = \underline{DB}\ \underline{d}$$

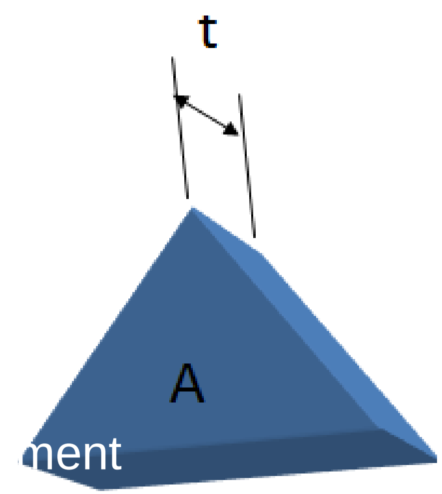
IMPORTANT NOTE:

- The displacement field is continuous across element boundaries
- The strains and stresses are NOT continuous across element boundaries

Element stiffness matrix

$$\underline{k} = \int_{V^e} \underline{B}^T \underline{D}\, \underline{B}\, dV$$

Since $\underline{B}$ is constant



$$\underline{k} = \underline{B}^T \underline{D}\, \underline{B} \int_{V^e} dV = \underline{B}^T \underline{D}\underline{B}\, At$$

t=thickness of the element

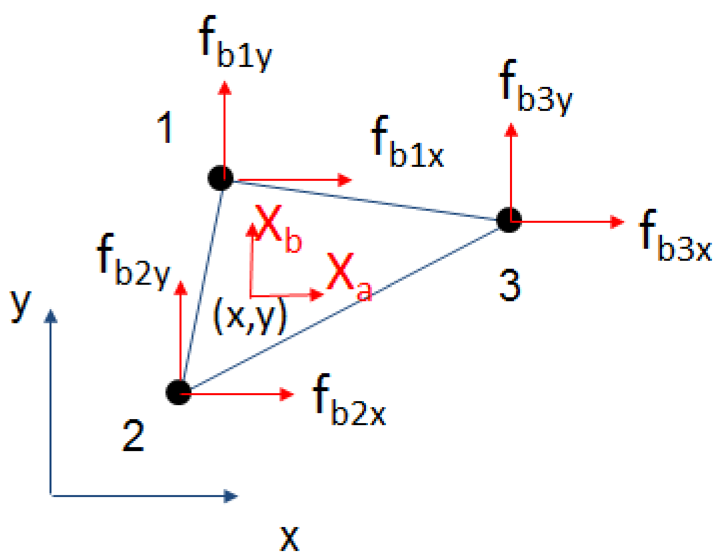
A=surface area of the element

Element nodal load vector

$$\underline{f} = \underbrace{\int_{V^e} \underline{\mathrm{N}}^T \underline{X}\, dV}_{\underline{f}_b} + \underbrace{\int_{S_T^e} \underline{\mathrm{N}}^T T_S\, dS}_{\underline{f}_S}$$

Element nodal load vector due to body forces

$$\underline{f}_b = \int_{V^e} \underline{N}^T \underline{X} \; dV = t\int_{A^e} \underline{N}^T \underline{X} \; dA$$



$$\underline{f}_b = \begin{Bmatrix} f_{b1x} \\ f_{b1y} \\ f_{b2x} \\ f_{b2y} \\ f_{b3x} \\ f_{b3y} \end{Bmatrix} = \begin{Bmatrix} t\int_{A^e} N_1 X_a \; dA \\ t\int_{A^e} N_1 X_b \; dA \\ t\int_{A^e} N_2 X_a \; dA \\ t\int_{A^e} N_2 X_b \; dA \\ t\int_{A^e} N_3 X_a \; dA \\ t\int_{A^e} N_3 X_b \; dA \end{Bmatrix}$$

EXAMPLE:

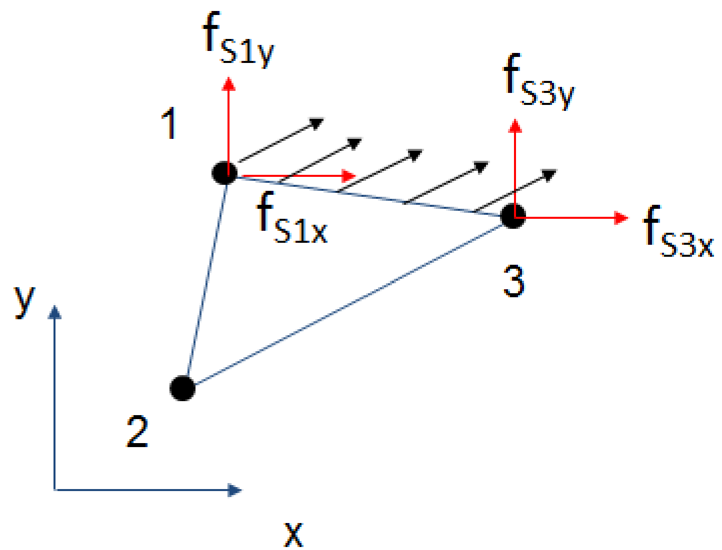
If X_a=1 and X_b=0

$$f_b = \begin{Bmatrix} f_{b1x} \\ f_{b1y} \\ f_{b2x} \\ f_{b2y} \\ f_{b3x} \\ f_{b3y} \end{Bmatrix} = \begin{Bmatrix} t\int_{A^e} N_1 X_a \, dA \\ t\int_{A^e} N_1 X_b \, dA \\ t\int_{A^e} N_2 X_a \, dA \\ t\int_{A^e} N_2 X_b \, dA \\ t\int_{A^e} N_3 X_a \, dA \\ t\int_{A^e} N_3 X_b \, dA \end{Bmatrix} = \begin{Bmatrix} t\int_{A^e} N_1 dA \\ 0 \\ t\int_{A^e} N_2 dA \\ 0 \\ t\int_{A^e} N_3 dA \\ 0 \end{Bmatrix} = \begin{Bmatrix} \frac{tA}{3} \\ 0 \\ \frac{tA}{3} \\ 0 \\ \frac{tA}{3} \\ 0 \end{Bmatrix}$$

Element nodal load vector due to traction

$$\underline{f}_S = \int_{S_T^e} \underline{\mathbf{N}}^T \underline{T}_S \, dS$$

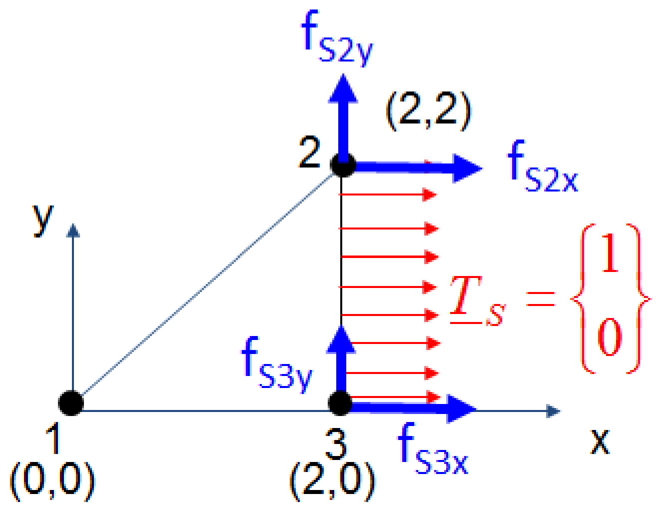
EXAMPLE:



$$\underline{f}_S = t \int_{l_{1-3}^e} \underline{\mathbf{N}}^T \Big|_{along\ 1-3} T_S \, dS$$

Element nodal load vector due to traction

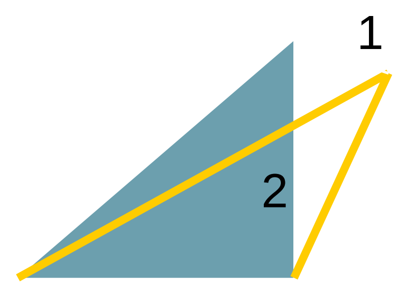
Example



$$\underline{f}_S = t\int_{l_{2-3}^e} \underline{N}^T\Big|_{along\ 2-3} \underline{T}_S \, dS$$

$$f_{S_{2x}} = t\int_{l_{2-3}^e} N_2\Big|_{along\ 2-3} (1)\, dy$$

$$= t\left(\frac{1}{2}\right)\times 2\times 1 = t$$



Similarly, compute

$$f_{S_{y}} = 0$$

$$f_{S_{x}} = t$$

$$f_{S_{3y}} = 0$$

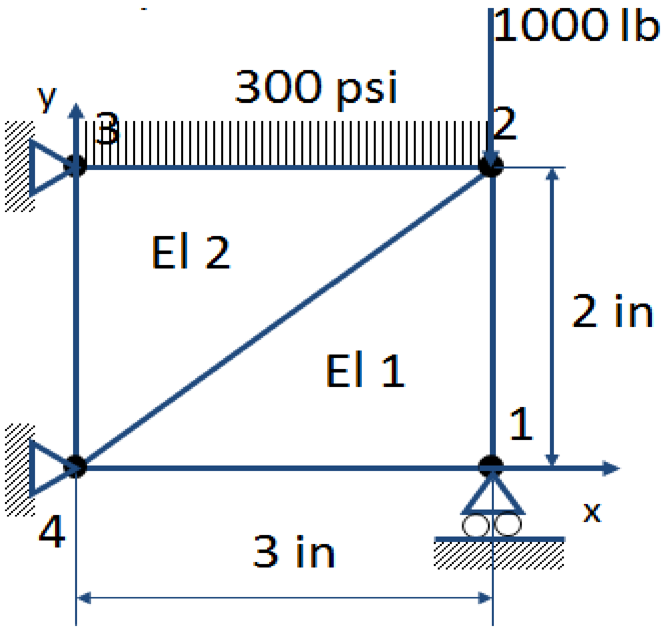
2

3

Recommendations for use of CST

- Use in areas where strain gradients are small
- Use in mesh transition areas (fine mesh to coarse mesh)
- Avoid CST in critical areas of structures (e.g., stress concentrations, edges of holes, corners)
- In general CSTs are not recommended for general analysis purposes as a very large number of these elements are required for reasonable accuracy.

Example



Thickness (t) = 0.5 in

E= 30×10^6 psi

n=0.25

(a) Compute the unknown nodal displacements.

(b) Compute the stresses in the two elements.

Realize that this is a plane stress problem and therefore we needto use

$$\underline{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} 3.2 & 0.8 & 0 \\ 0.8 & 3.2 & 0 \\ 0 & 0 & 1.2 \end{bmatrix} \times 10^{7} \; psi$$

<u>Step 1</u>: Node-element connectivity chart

ELEMENT	Node 1	Node 2	Node 3	Area (sqin)
1	1	2	4	3
2	3	4	2	3

Node	x	y
1	3	0
2	3	2
3	0	2
4	0	0

Nodal coordinates

Step 2: Compute strain-displacement matrices for the elements

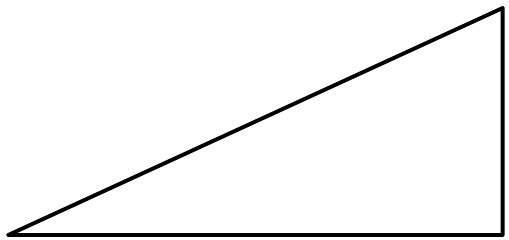
Recall

$$\underline{B} = \frac{1}{2A}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

with

$$b_1 = y_2 - y_3 \quad b_2 = y_3 - y_1 \quad b_3 = y_1 - y_2$$
$$c_1 = x_3 - x_2 \quad c_2 = x_1 - x_3 \quad c_3 = x_2 - x_1$$

For Element #1:



(local numbers within brackets)

$$y_1 = 0; y_2 = 2; y_3 = 0$$
$$x_1 = 3; x_2 = 3; x_3 = 0$$

Hence

$$b_1 = 2 \quad b_2 = 0 \quad b_3 = -2$$
$$c_1 = -3 \quad c_2 = 3 \quad c_3 = 0$$

Therefore

$$\underline{B}^{(1)} = \frac{1}{6}\begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & -3 & 0 & 3 & 0 & 0 \\ -3 & 2 & 3 & 0 & 0 & -2 \end{bmatrix}$$

For Element #2:

$$\underline{B}^{(2)} = \frac{1}{6}\begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 3 & -2 & -3 & 0 & 0 & 2 \end{bmatrix}$$

Step 3: Compute element stiffness matrices

$$\underline{k}^{(1)} = At\underline{B}^{(1)^T}\underline{D}\,\underline{B}^{(1)} = (3)(0.5)\underline{B}^{(1)^T}\underline{D}\,\underline{B}^{(1)}$$

$$= \begin{bmatrix} 0.9833 & -0.5 & -0.45 & 0.2 & -0.5333 & 0.3 \\ & 1.4 & 0.3 & -1.2 & 0.2 & -0.2 \\ & & 0.45 & 0 & 0 & -0.3 \\ & & & 1.2 & -0.2 & 0 \\ & & & & 0.5333 & 0 \\ & & & & & 0.2 \end{bmatrix} \times 10^7$$

u_1	v_1	u_2	v_2	u_4	v_4

$$\underline{k}^{(2)} = At\underline{B}^{(2)^T}\underline{D}\,\underline{B}^{(2)} = (3)(0.5)\underline{B}^{(2)^T}\underline{D}\,\underline{B}^{(2)}$$

$$= \begin{bmatrix} 0.9833 & -0.5 & -0.45 & 0.2 & -0.5333 & 0.3 \\ & 1.4 & 0.3 & -1.2 & 0.2 & -0.2 \\ & & 0.45 & 0 & 0 & -0.3 \\ & & & 1.2 & -0.2 & 0 \\ & & & & 0.5333 & 0 \\ & & & & & 0.2 \end{bmatrix} \times 10^7$$

$$\begin{matrix} u_3 & v_3 & u_4 & v_4 & u_2 & v_2 \end{matrix}$$

Step 4: Assemble the global stiffness matrix corresponding to the *nonzero* degrees of freedom

- $u_3 = v_3 = u_4 = v_4 = v_1 = 0$

- Hence we need to calculate only a small (3x3) stiffness matrix

- $$\underline{K} = \begin{bmatrix} 0.983 & -0.45 & 0.2 \\ 0.45 & 0.983 & 0 \\ 0.2 & 0 & 1.4 \end{bmatrix} \times 10^7 \quad \begin{matrix} u_1 \\ u_2 \\ v_2 \end{matrix}$$

$$\begin{matrix} u_1 & u_2 & v_2 \end{matrix}$$

$$\underline{f} = \begin{Bmatrix} f_{1x} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ f_{2y} \end{Bmatrix}$$

$$f_{2y} = -1000 + f_{S_{2y}}$$

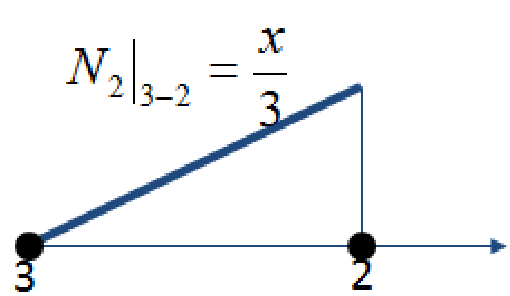
The consistent nodal load due to traction on the edge 3-2

$$f_{S}^{2y} = \int_{x=0}^{3} N_3\big|_{3-2} (-300) t\, dx$$

$$= (-300)(0.5) \int_{x=0}^{3} N_3\big|_{3-2} dx$$

$$= -150 \int_{x=0}^{3} \frac{x}{3} dx$$

$$= -50 \left[\frac{x^2}{2} \right]_0^3 = -50 \left(\frac{9}{2} \right) = -225\, lb$$



Hence

$$f_{2y} = -1000 + f_{S_{2y}}$$

$$= -1225\ lb$$

<u>Step 6</u>: Solve the system equations to obtain the unknown nodal loads

$$\underline{Kd} = \underline{f}$$

$$10^7 \times \begin{bmatrix} 0.983 & -0.45 & 0.2 \\ -0.45 & 0.983 & 0 \\ 0.2 & 0 & 1.4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -1225 \end{Bmatrix}$$

Solve to get

$$\begin{Bmatrix} u_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0.2337 \times 10^{-4}\ in \\ 0.1069 \times 10^{-4}\ in \\ -0.9084 \times 10^{-4}\ in \end{Bmatrix}$$

Step 7: Compute the stresses in the elements

In Element #1

$$\sigma^{(1)} = \underline{\mathrm{D}}\underline{\mathrm{B}}^{(1)}\,\underline{\mathrm{d}}^{(1)}$$

With

$$\underline{\mathrm{d}}^{(1)T} = \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_4 & v_4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2337\times10^{-4} & 0 & 0.1069\times10^{-4} & -0.9084\times10^{-4} & 0 & 0 \end{bmatrix}$$

Calculate

$$\sigma^{(1)} = \begin{Bmatrix} -114.1 \\ -1391.1 \\ -76.1 \end{Bmatrix} psi$$

In Element #2

$$\underline{\sigma}^{(2)} = \underline{DB}^{(2)}\,\underline{d}^{(2)}$$

With

$$\underline{d}^{(2)T} = \begin{bmatrix} u_3 & v_3 & u_4 & v_4 & u_2 & v_2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0.1069\times10^{-4} & -0.9084\times10^{-4} \end{bmatrix}$$

Calculate

$$\sigma^{(2)} = \begin{bmatrix} 114.1 \\ 28.52 \\ -363.35 \end{bmatrix} psi$$

Notice that the stresses are constant in each element

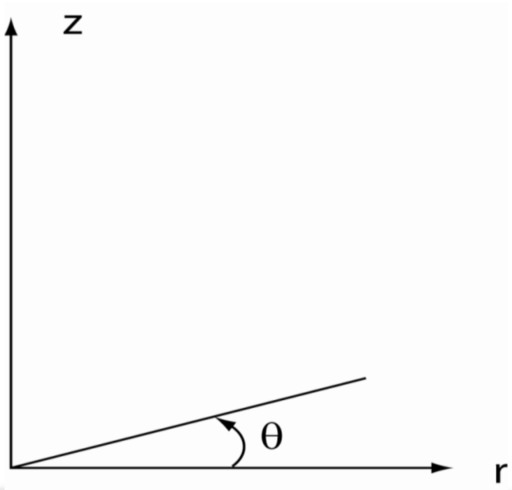
Axi-symmetric Problems

Definition:

A problem in which geometry, loadings, boundary conditions and materials are symmetric about one axis.

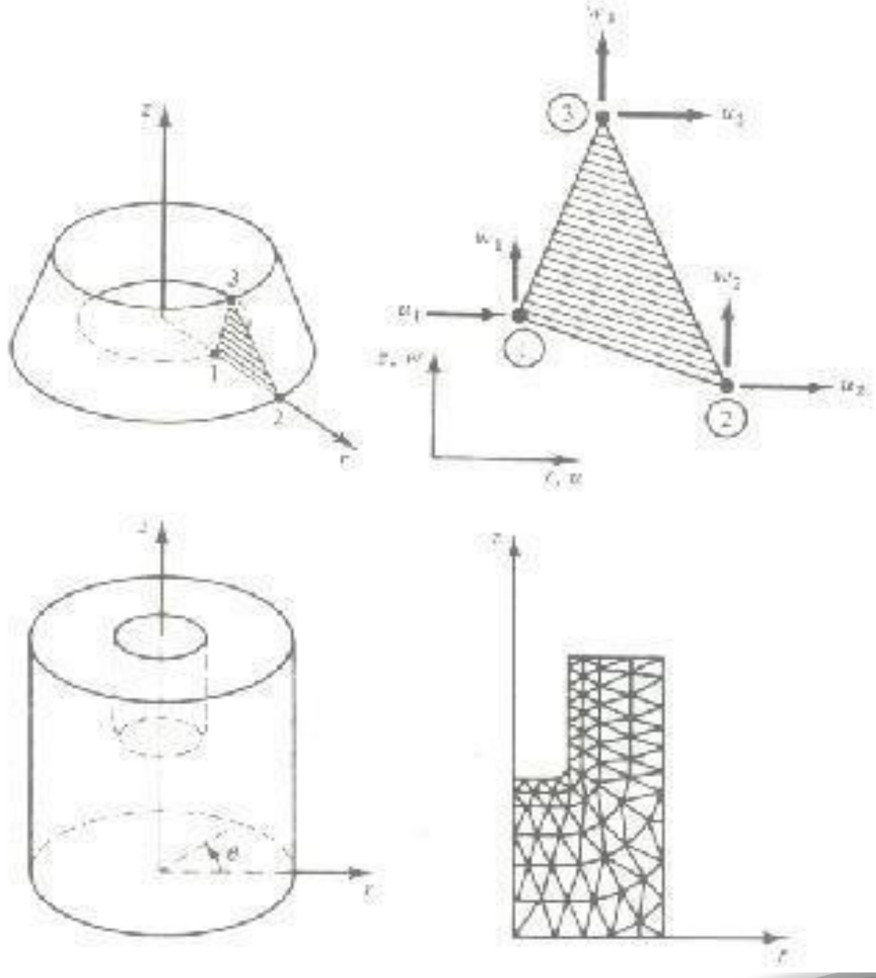
Axi-symmetric Analysis

$$x = r\cos\theta; \quad y = r\sin\theta; \quad z = z$$



- quantities depend on *r* and *z* only
- 3-D problem
- 2-D problem

Axi-symmetric Analysis



Axi-symmetric Analysis – Single-Variable Problem

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r a_{11}\frac{\partial u(r,z)}{\partial r}\right)-\frac{\partial}{\partial z}\left(a_{22}\frac{\partial u(r,z)}{\partial z}\right)+a_{00}u-f(r,z)=0$$

Weak form:

$$0=\int_{\Omega_e}\left[\frac{\partial w}{\partial r}\left(a_{11}\frac{\partial u}{\partial r}\right)+\frac{\partial w}{\partial z}\left(a_{22}\frac{\partial u}{\partial z}\right)+a_{00}wu-wf(r,z)\right]r\,dr\,dz$$

$$-\oint_{\Gamma_e} w q_n\, ds$$

where $q_n = a_{11}\dfrac{\partial u(r,z)}{\partial r}n_r + a_{22}\dfrac{\partial u(r,z)}{\partial z}n_z$

Finite Element Model – Single-Variable Problem

$$u = \sum_j u_j \phi_j \quad \text{where} \quad \phi_j(r, z) = \phi_j(x, y)$$

Ritz method: $w = \phi_i$

Weak form → $$\sum_{j=1}^{n} K_{ij}^e u_j^e = f_i^e + Q_i^e$$

where $$K_{ij}^e = \int_{\Omega_e} \left(a_{11} \frac{\partial \phi_i}{\partial r} \frac{\partial \phi_j}{\partial r} + a_{22} \frac{\partial \phi_i}{\partial z} \frac{\partial \phi_j}{\partial z} + a_{00} \phi_i \phi_j \right) r dr dz$$

$$f_i^e = \int_{\Omega_e} \phi_i f r dr dz$$

$$Q_i^e = \int_{\Gamma_e} \phi_i q_n ds$$

Heat Transfer:

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T(r,z)}{\partial r}\right)-\frac{\partial}{\partial z}\left(k\frac{\partial T(r,z)}{\partial z}\right)-f(r,z)=0$$

Weak form

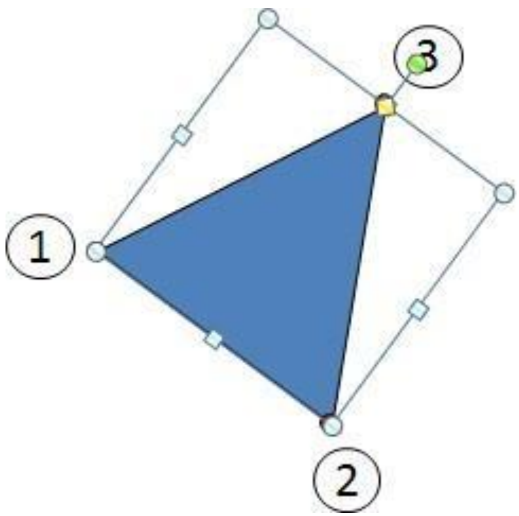
$$0=\int_{\Omega_e}\left[\left(\frac{\partial w}{\partial r}\left(k\frac{\partial T}{\partial r}\right)+\frac{\partial w}{\partial z}\left(k\frac{\partial T}{\partial z}\right)-wf(r,z)\right)\right]rdrdz$$

$$-\int_{\Gamma_e} wq_n ds$$

where

$$q_n=k\frac{\partial T(r,z)}{\partial r}n_r+k\frac{\partial T(r,z)}{\partial z}n_z$$

3-Node Axi-symmetric Element



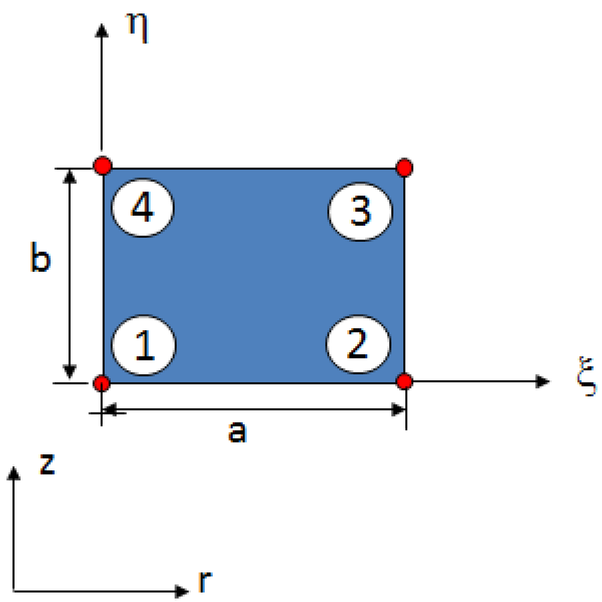
$$T(r,z) = T_1\phi_1 + T_2\phi_2 + T_3\phi_3$$

$$\phi_1 = \frac{\begin{pmatrix}1 & r & z\end{pmatrix}}{2A^e}\begin{Bmatrix} r_2 z_3 - r_3 z_2 \\ z_2 - z_3 \\ r_3 - r_2 \end{Bmatrix}$$

$$\phi_2 = \frac{\begin{pmatrix}1 & r & z\end{pmatrix}}{2A_e}\begin{Bmatrix} r_3 z_1 - r_1 z_3 \\ z_3 - z_1 \\ r_1 - r_3 \end{Bmatrix}$$

$$\phi_3 = \frac{\begin{pmatrix}1 & r & z\end{pmatrix}}{2A_e}\begin{Bmatrix} r_1 z_2 - r_2 z_1 \\ z_1 - z_2 \\ r_2 - r_1 \end{Bmatrix}$$

4-Node Axi-symmetric Element

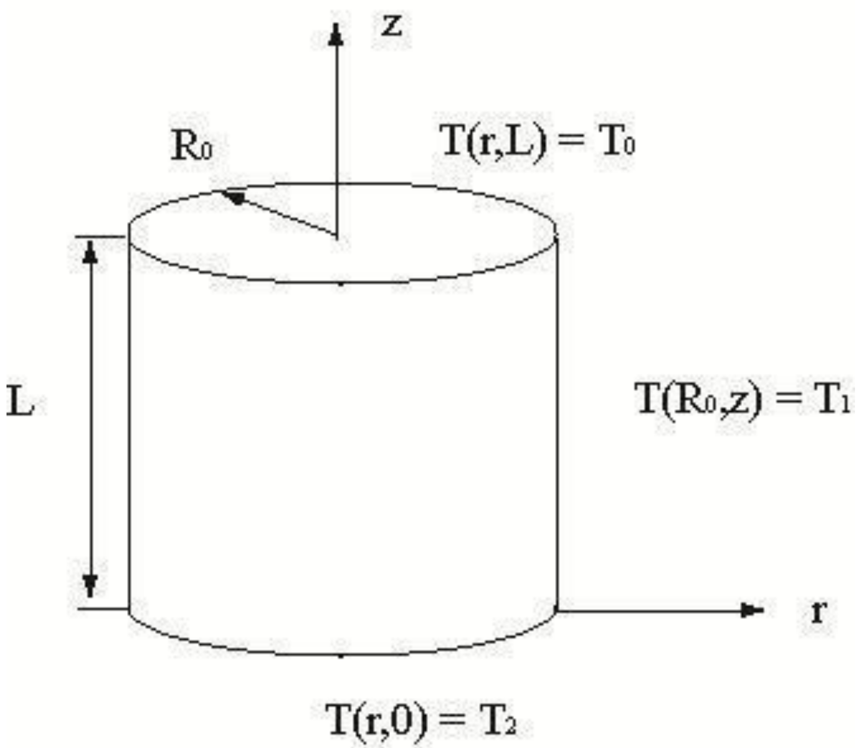


$$T(r,z) = T_1\phi_1 + T_2\phi_2 + T_3\phi_3 + T_4\phi_4$$

$$\phi_1 = \left(1 - \frac{\xi}{a}\right)\left(1 - \frac{\eta}{b}\right) \qquad \phi_2 = \frac{\xi}{a}\left(1 - \frac{\eta}{b}\right)$$

$$\phi_3 = \frac{\xi\eta}{ab} \qquad \phi_4 = \left(1 - \frac{\xi}{a}\right)\frac{\eta}{b}$$

Single-Variable Problem – Example



Step 1: Discretization

Step 2: Element equation

$$K_{ij}^{e} = \int_{\Omega_e} \left(\kappa \frac{\partial \phi_i}{\partial r} \frac{\partial \phi_j}{\partial r} + \kappa \frac{\partial \phi_i}{\partial z} \frac{\partial \phi_j}{\partial z} \right) r dr dz$$

$$f_i^{e} = \int_{\Omega_e} \phi_i \, f r dr dz \qquad Q_i^{e} = \int_{\Gamma_e} \phi_i q_n ds$$

Review of CST Element

Constant Strain Triangle (CST) - easiest and simplest finite element

- Displacement field in terms of generalized coordinates

$$u = \beta_1 + \beta_2 x + \beta_3 y$$
$$v = \beta_4 + \beta_5 x + \beta_6 y$$

- Resulting strain field is

$$\varepsilon_x = \beta_2 \qquad \varepsilon_y = \beta_6 \qquad \gamma_{xy} = \beta_3 + \beta_5$$

- Strains do not vary within the element. Hence, the name constant strain triangle (CST)
  - •Other elements are not so lucky.
  - •Can also be called linear triangle because displacement field is linear in x and y - sides remain straight.

Constant Strain Triangle

- The strain field from the shape functions looks like:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

  - Where, x_i and y_i are nodal coordinates (i=1, 2, 3)
  - $x_{ij} = x_i - x_j$ and $y_{ij}=y_i - y_j$
  - 2A is twice the area of the triangle, $2A = x_{21}y_{31}-x_{31}y_{21}$

- Node numbering is arbitrary except that the sequence 123 must go clockwise around the element if A is to be positive.

Constant Strain Triangle

- Stiffness matrix for element k $=B^T E B\, tA$

- The CST gives good results in regions of the FE model where there is little strain gradient
  - Otherwise it does not work well.

Changes the shape functions and results in quadratic displacement distributions and linear strain distributions within the element.

Linear Strain Triangle

$$u = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2 + \beta_5 xy + \beta_6 y^2$$

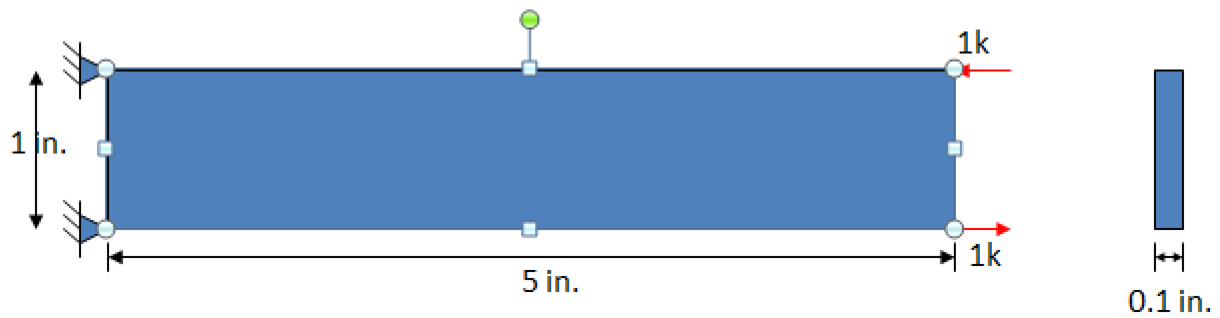
$$v = \beta_7 + \beta_8 x + \beta_9 y + \beta_{10} x^2 + \beta_{11} xy + \beta_{12} y^2$$

$$\varepsilon_x = \beta_2 + 2\beta_4 x + \beta_5 y$$

$$\varepsilon_y = \beta_9 + \beta_{11} x + 2\beta_{12} y$$

$$\gamma_{xy} = (\beta_3 + \beta_8) + (\beta_5 + 2\beta_{10})x + (2\beta_6 + \beta_{11})y$$

Consider the problem we were looking at:



$$I = 0.1 \times 1^3 / 12 = 0.008333 in^4$$

$$o = \frac{M \times c}{I} = \frac{1 \times 0.5}{0.008333} = 60\ ksi$$

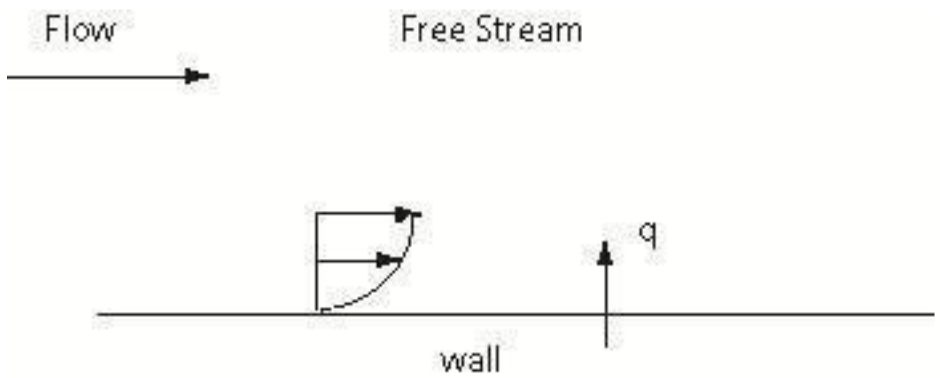
$$\varepsilon = \frac{\sigma}{E} = 0.00207$$

$$\delta = \frac{ML^2}{2EI} = \frac{25}{2 \times 29000 \times 0.008333} = 0.0517 in.$$

UNIT-IV
STEADY STATE HEAT TRANSFER ANALYSIS

CLOs	Course Learning Outcomes
CLO 1	Understand the concepts of steady state heat transfer analysis for one dimensional slab, fin and thin plate.
CLO 2	Derive the stiffness matrix for fin element.
CLO 3	Solve the steady state heat transfer problems for fin and composite slab.

Thermal Convection



Newton’s Law of Cooling

$$q = h(T_s - T_\infty)$$

h: convective heat transfer coefficient $(W \quad m^2 \cdot C^o)$

Thermal Conduction in 1-D

Boundary conditions:

- Dirichlet BC
- Natural BC
- Mixed BC

Weak Formulation of 1-D Heat Conduction (Steady State Analysis)

Governing Equation of 1-D Heat Conduction

$$-\frac{d}{dx}\left(\kappa(x)A(x)\frac{dT(x)}{dx}\right) - AQ(x) = 0 \qquad \mathbf{0<x<L}$$

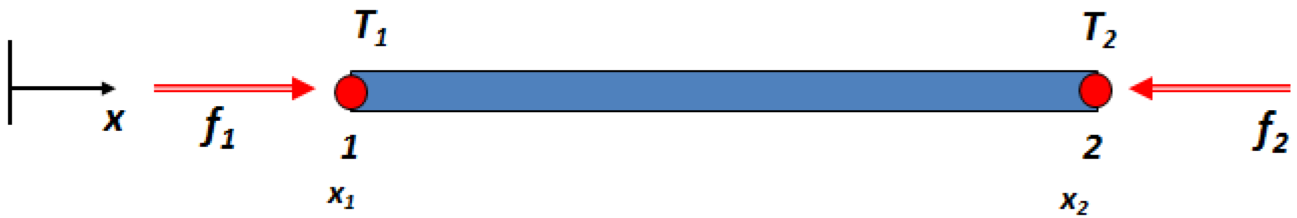
Weighted Integral Formulation

$$0 = \int_0^L w(x)\left[-\frac{d}{dx}\left(\kappa(x)A(x)\frac{dT(x)}{dx}\right) - AQ(x)\right]dx$$

Weak Form from Integration-by-Parts

$$0 = \int_0^L\left[\frac{dw}{dx}\left(\kappa A\frac{dT}{dx}\right) - wAQ\right]dx - w\left(\kappa A\frac{dT}{dx}\right)\Bigg|_0^L$$

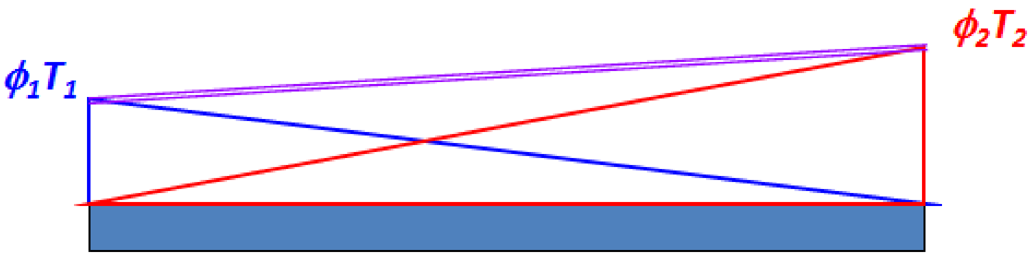
Formulation for 1-D Linear Element



$$f_1(x) = -\kappa A \left.\frac{\partial T}{\partial x}\right|_1, \qquad f_2(x) = \kappa A \left.\frac{\partial T}{\partial x}\right|_2$$

$$T(\mathrm{x}) = T_1\phi_1(\mathrm{x}) + T_2\phi_2(\mathrm{x})$$

$$\phi_1(x) = \frac{x_2 - x}{l}, \qquad \phi_2(x) = \frac{x - x_1}{l}$$



Formulation for 1-D Linear Element

- Let w(x)= $f_i(x)$, i = 1, 2

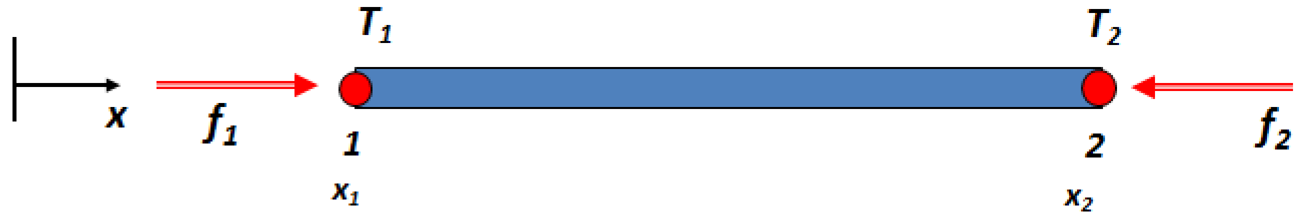
$$0 = \sum_{j=1}^{2} T_j \left[\int_{x_1}^{x_2} \kappa A \left(\frac{d\phi_i}{dx} \frac{d\phi_j}{dx} \right) dx \right] - \int_{x_1}^{x_2} (\phi_i A Q) dx - [\phi_i(x_2) f_2 + \phi_i(x_1) f_1]$$

$$= \sum_{j=1}^{2} K_{ij} T_j - Q_i - [\phi_i(x_2) f_2 + \phi_i(x_1) f_1]$$

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$\text{where } K_{ij} = \int_{x_1}^{x_2} \kappa A \left(\frac{d\phi_i}{dx} \frac{d\phi_j}{dx} \right) dx, \quad Q_i = \int_{x_1}^{x_2} (\phi_i A Q) dx, \quad f_1 = -\kappa A \left. \frac{dT}{dx} \right|_{x_1}, \quad f_2 = \kappa A \left. \frac{dT}{dx} \right|_{x_2}$$

Element Equations of 1-D Linear Element

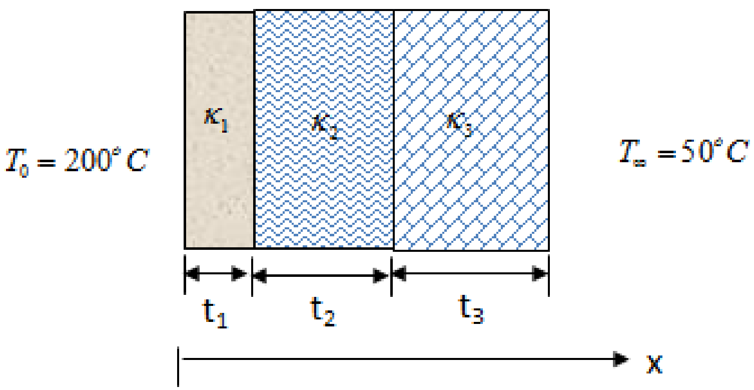


$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \frac{\kappa A}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$where\ Q_i = \int_{x_1}^{x_2} (\phi_i A Q) dx,\ \ f_1 = -\kappa A \left.\frac{dT}{dx}\right|_{x=x_1},\ \ f_2 = \kappa A \left.\frac{dT}{dx}\right|_{x=x_2}$$

1-D Heat Conduction - Example

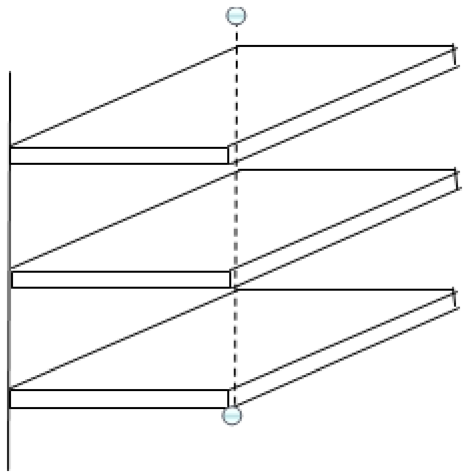
A composite wall consists of three materials, as shown in the figure below. The inside wall temperature is 200°C and the outside air temperature is 50°Cwith a convection coefficient of h = 10 W(m².K). Find the temperature alongthe composite wall.

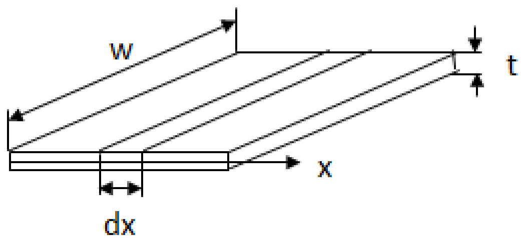


$$\kappa_1 = 70W/(m \cdot K), \quad \kappa_2 = 40W/(m \cdot K), \quad \kappa_3 = 20W/(m \cdot K)$$

$$t_1 = 2cm, \quad t_2 = 2.5cm, \quad t_3 = 4cm$$

Thermal Conduction and Convection- Fin





Objective: to enhance heat transfer

Governing equation for 1-D heat transfer in thin fin

$$\frac{d}{dx}\left(\kappa A_c \frac{dT}{dx}\right) + A_c Q = 0$$

$$Q_{loss} = \frac{2h(T - T_\infty)\cdot dx \cdot w + 2h(T - T_\infty)\cdot dx \cdot t}{A_c \cdot dx} = \frac{2h(T - T_\infty)\cdot(w + t)}{A_c}$$

$$\frac{d}{dx}\left(\kappa A_c \frac{dT}{dx}\right) - Ph\left(T - T_\infty\right) + A_c Q = 0$$

where $P = 2\left(w + t\right)$

Fin (*Steady State Analysis*)

Governing Equation of 1-D Heat Conduction

$$-\frac{d}{dx}\left(\kappa(x)A(x)\frac{dT(x)}{dx}\right)+Ph\left(T-T_{\infty}\right)-AQ=0 \qquad \mathbf{\mathit{0<x<L}}$$

Weighted Integral Formulation

$$0=\int_0^L w(x)\left[-\frac{d}{dx}\left(\kappa(x)A(x)\frac{dT(x)}{dx}\right)+Ph(T-T_{\infty})-AQ(x)\right]dx$$

Weak Form from Integration-by-Parts

$$0=\int_0^L\left[\frac{dw}{dx}\left(\kappa A\frac{dT}{dx}\right)+wPh(T-T_{\infty})-wAQ\right]dx-w\left(\kappa A\frac{dT}{dx}\right)\Bigg|_0^L$$

Formulation for 1-D Linear Element

Let w(x)= f_i (x), i = 1, 2

$$0 = \sum_{j=1} T_j \left[\int_{x_1}^{x_2} \left(\kappa A \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} + Ph\phi_i\phi_j \right) dx \right] - \int_{x_1}^{x_2} \phi_i \left(AQ + PhT_\infty \right) dx$$

$$- \left[\phi_i(x_2) f_2 + \phi_i(x_1) f_1 \right]$$

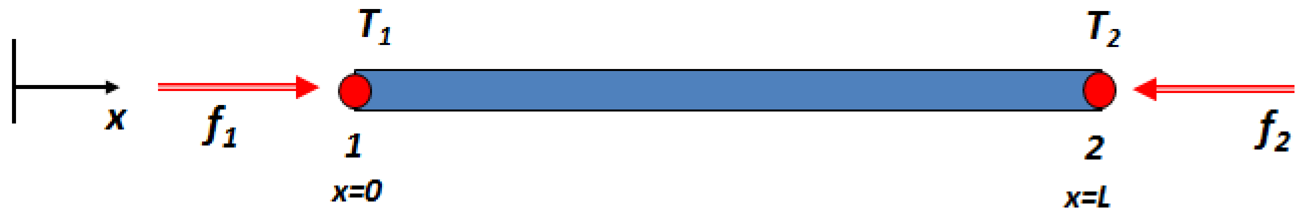
$$= \sum_{j=1}^{2} K_{ij} T_j - Q_i - \left[\phi_i(x_2) f_2 + \phi_i(x_1) f_1 \right]$$

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$where \;\; K_{ij} = \int_{x_1}^{x_2} \left(\kappa A \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} + Ph\phi_i\phi_j \right) dx, \quad Q_i = \int_{x_1}^{x_2} \phi_i \left(AQ + PhT_\infty \right) dx,$$

$$f_1 = -\kappa A \left. \frac{dT}{dx} \right|_{x=x_1}, \quad f_2 = \kappa A \left. \frac{dT}{dx} \right|_{x=x_2}$$

Element Equations of 1-D Linear Element



$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \left\{ \frac{\kappa A}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{Phl}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right\} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$\text{where} \quad Q_i = \int_{x_1}^{x_2} \phi_i (AQ + PhT_\infty) dx, \quad f_1 = -\kappa A \left. \frac{dT}{dx} \right|_{x=x_1}, \quad f_2 = \kappa A \left. \frac{dT}{dx} \right|_{x=x_2}$$

Time-Dependent Problems

In general $u(x,t)$

Two approaches:

$$u(x,t)=\sum u_j \phi_j(x,t)$$

$$u(x,t)=\sum u_j(t)\phi_j(x)$$

Model Problem I – Transient Heat Conduction

$$c\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) + f(x,t)$$

Weak form:

$$0 = \int_{x_1}^{x_2}\left(a\frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + cw\frac{\partial u}{\partial t} - wf\right)dx - Q_1 w(x_1) - Q_2 w(x_2)$$

$$Q_1 = -\left[a\frac{du}{dx}\right]_{x_1}; \qquad Q_2 = \left[a\frac{du}{dx}\right]_{x_2}$$

Transient Heat Conduction

let: $u(x,t) = \sum_{j=1}^{n} u_j(t)\phi_j(x)$ and $w = \phi_i(x)$

$$0 = \int_{x_1}^{x_2} \left(a\frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + cw\frac{\partial u}{\partial t} - wf \right) dx - Q_1 w(x_1) - Q_2 w(x_2)$$

$$[K]\{u\} + [M]\{\dot{u}\} = \{F\}$$

$$K_{ij} = \int_{x_1}^{x_2} a\frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x} dx$$

$$M_{ij} = \int_{x_1}^{x_2} c\phi_i\phi_j dx$$

$$F_i = \int_{x_1}^{x_2} \phi_i f dx + Q_i$$

Time Approximation – First Order ODE

$$a\frac{du}{dt} + bu = f(t) \qquad 0 < t < T \qquad u(0) = u_0$$

Forward difference approximation – explicit

$$u_{k+1} = u_k + \frac{\Delta t}{a}\left[f_k - bu_k \right]$$

Backward difference approximation - implicit

$$u_{k+1} = u_k + \frac{\Delta t}{a + b\Delta t}\left[f_k - bu_k \right]$$

Stability Requirement

$$\Delta t \leq \Delta t_{cri} = \frac{2}{(1-2\alpha)\lambda_{\max}}$$

where $$([K] - \lambda[M])\{u\} = \{Q\}$$

Note: One must use the same discretization for solvingthe eigenvalue problem.

Transient Heat Conduction - Example

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad 0 < x < 1$$

$$u(0,t) = 0 \qquad \frac{\partial u}{\partial t}(1,t) = 0 \qquad t > 0$$

$$u(x,0) = 1.0$$

UNIT-V
DYNAMIC ANALYSIS

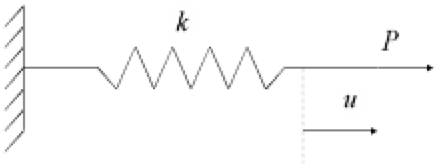
DYNAMIC EQUATIONS

For many structural system, the approximation of linear structural behavior is made in order to convert the physical equilibrium statement, to the following set of second order linear differential equation

$$M\ddot{u}(t)_a + C\dot{u}(t)_a + Ku(t)_a = P(t)$$

Stiffness and flexibility stiffness matrix

Consider a uniform elastic spring subjected to a load P. This Structure obeys Hook's law. If a force P is applied to a spring fixed. At one end, to produce a displacement then the linear force displacement is u.



P = ku

K = fP

Stiffness and flexibility stiffeness matrix

- K is called the stiffeness of the spring
- f is called the flexibility of spring

Suppose the uniform elastic spring has nodal points and 2 at its ends, and that the forces at these points are P_1 and P_2 with corresponding displacements u_1 and u_2.

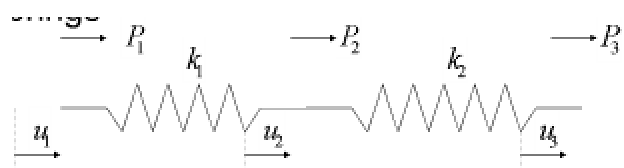
- From equilibrium consid

$$P_1 = k(u_1 - u_2)$$
$$P_2 = -P_1 = k(u_2 - u_1)$$

- It is convenient to show the above in matrix form as follows

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

- Simple system consisting of just two springs



The system is in equilibrium

$$P_1+P_2+P_3 = 0$$

$$P_1 = k_1(u_1-u_2)$$

$$P_2 = k_2(u_3-u_2)$$

$$P_2 = -k_1u_1+(k_1+k_2)-k_2u_3$$

actual distributed mass of the element.

- The element mass matrix is defined as

$$[M] = \int_v \rho [N]^T [N] \, dV$$

Dynamic equations

- The force equilibrium of a multi degree of freedom lumped mass system

$$P(t)_i + P(t)_D + P(t)_S = P(t)$$

  - Vector of inertia forces acting on the node masses $P(t)_i$
  - Vector of viscous damping or energy dissipation forces $P(t)_D$
  - A vector of internal forces carried by the structure $P(t)_S$
  - Vector of externally applied loads $P(t)$

- For many structural systems the approximation of linear structural behavior is made in order to convert the physical equlibrium statement, to the following set of second order linear differential equation

$$M\ddot{u}(t)_a + C\dot{u}(t)_a + Ku(t)_a = P(t)$$

Vibration analysis

when loads are suddenly applied or when the loads are of a variable nature, the mass and acceleration effects come into the picture. If a solid such as an engineering structure is deformed elastically and suddenly released. It tends to vibrate about its equilibrium position. This periodic motion due to the restoring strain energy is called free vibration. The number of cycles per unit time is called frequency. The maximum displacement from the equilibrium position is the amplitude.

- Equation for damped forced vibration

$$M\ddot{u}(t) + C\dot{u}(t) + Ku(t) = P(t)$$

- If there is no damping the equation become

$$M\ddot{u}(t) + Ku(t) = P(t)$$

- Free undamped vibration equation $M\ddot{u}(t) + Ku(t) = 0$

- The free undamped vibration equation is linear and homogeneous. Its general solution is a linear combination of exponentials. Under matrix definiteness conditions the exponentials can be expressed as a combination of trignometric functions: sines and cosines of argument ωt.

- A compact representation of such functions is obtained by using the exponential form $e^{j\omega t}$

$$u(t) = \sum v_i \, e^{j\omega t}$$

Replace $u(t) = \ v_i \, e^{j\omega t}$

$$M\ddot{u}(t) + Ku(t) = 0$$

The Vibration Eigen problem

- The time dependence to the exponential is segregated

$$(-\omega^2 M + K) v e^{j\omega t} = 0$$

- Since is not identically zero, it can be dropped leaving the algebraic condition

$$(-\omega^2 M + K) v = 0$$

- Because v cannot be the null vector this equation is an algebraic Eigen value problem in ω^2.The Eigen values $\lambda_i = {\omega_i}^2$ are the roots of the characteristic polynomial be index by I

$$\det(K - {\omega_i}^2 M) = 0$$

- Dropping the index I this Eigen problem is usually written as

$$Kv = \omega^2 \mathrm{Mv}$$


Thank you