INSTITUTE OF AERONAUTICAL ENGINEERING

(Autonomous)
Dundigal, Hyderabad -500 043

MECHANICAL ENGINEERING

COURSE LECTURE NOTES

Course Title	**MATERIALS AND MECHANICS OF SOLIDS**
Course Code	AMEB11
Programme	B. Tech.
Semester	IV ME
Chief Coordinator	Dr. K Vishwanath Allamraju, Professor
Course Faculty	Dr. K Vishwanath Allamraju, Professor Mr. A. Somaiah, Assistant Professor

COURSE OBJECTIVES:

I. Determination of mechanical properties of different materials.
II. Establish the constitutive relations in metals using destructive methods.
III. Understand the behavior of members during twisting and transverse loading.
IV. Familiarize with standard test procedures.
V. Deriving slope and deflection for different types of beams.

COURSE OUTCOMES:

I. Describe the different types of crystal structures.
II. Discuss the phase transformations and equilibrium diagram.
III. Ability to apply the principles of elasticity and plasticity.
IV. Able to draw shear force and bending moment diagrams for various loads.
V. Determination of slope and deflection of various types of beams.

COURSE LEARNING OUTCOMES:

CLO CODE	DESCRIPTION
AMEB11.01	Understand the concepts crystallography, crystal structures, unit cells, crystallographic planes, directions and miller indices.
AMEB11.02	Discuss the crystal imperfections and Frank Reed source of dislocation.
AMEB11.03	Demonstrate the concept of Bauschinger‟s effect, twinning, strain hardening and seasons cracking.
AMEB11.04	Knowledge of yield point phenomenon, cold/hot working, recovery, re-crystallization, grain growth and strengthening of metals.
AMEB11.05	Discuss the constitution of alloys and phase diagrams, constitution of alloys, solid solutions, substitutional and interstitial.
AMEB11.06	Demonstrate the phase diagrams, isomorphous, eutectic, peritectic, eutectoid and peritectoid reactions.
AMEB11.07	Construction of iron –Iron carbide equilibrium diagram.

AMEB11.08	Classification of steel and cast-Iron microstructure, properties and application.
AMEB11.09	Discuss Hooke’s law, stresses and strains
AMEB11.10	Derive relationship between elastic constants.
AMEB11.11	Describe the concept of poisson’s ratio, linear and lateral strains.
AMEB11.12	Construct the Mohr’s circle to solve principal stresses and strains.
AMEB11.13	Understand the beams and types transverse loading on beams, shear force and bend moment diagrams.
AMEB11.14	Discuss types of beam supports, simply supported and over-hanging beams, cantilevers.
AMEB11.15	Understand theory of bending of beams, bending stress distribution and neutral axis.
AMEB11.16	Understand the shear stress distribution, point and distributed loads.
AMEB11.17	Understand moment of inertia about an axis and polar moment of inertia.
AMEB11.18	Derive the deflection of a beam using double integration Method.
AMEB11.19	Computation of slopes and deflection in beams.
AMEB11.20	Discuss Maxwell‟s reciprocal theorems.

SYLLABUS:

<table>
<tr><td>Module-I</td><td>FUNDAMENTALS OF MATERIAL SCIENCE</td></tr>
<tr><td colspan="2">Basic Crystallography Crystal structure BCC, FCC and HCP structure, unit cell, crystallographic planes and directions, miller indices. Crystal imperfections, point, line, planar and volume defects, grain size, ASTM grain size number. Frank Reed source of dislocation Elastic & plastic modes of deformation, slip & twinning, strain hardening, seasons cracking, Bauschinger‟s effect, yield point phenomenon, cold/hot working, recovery, re-crystallization, and grain growth, strengthening of metals.</td></tr>
<tr><td>Module-II</td><td>ALLOYS AND PHASE DIAGRAMS</td></tr>
<tr><td colspan="2">Constitution of alloys and phase diagrams; constitution of alloys, solid solutions, substitutional and interstitial. phase diagrams, isomorphous, eutectic, peritectic, eutectoid and peritectoid reactions. iron –
iron carbide equilibrium diagram. classification of steel and cast-iron microstructure, properties and application.</td></tr>
<tr><td>Module-III</td><td>SIMPLE STRESSES AND STRAINS, PRINCIPAL STRESSES</td></tr>
<tr><td colspan="2">Hooke‟s law, stress and strain- tension, compression and shear stresses elastic constants and their relations
Volumetric, linear and shear strains principal stresses and principal planes, Mohr‟s circle.</td></tr>
<tr><td>Module-IV</td><td>SHEAR FORCE AND BENDING MOMENT DIAGRAMS, FLEXURAL STRESSES, SHEAR STRESSES</td></tr>
<tr><td colspan="2">Beams and types transverse loading on beams shear force and bend moment diagrams types of beam supports, simply supported and over-hanging beams, cantilevers. theory of bending of beams, bending stress distribution and neutral axis, shear stress distribution, point and distributed loads.</td></tr>
<tr><td>Module-V</td><td>SLOPE AND DEFLECTION</td></tr>
</table>

Moment of inertia about an axis and polar moment of inertia, deflection of a beam using double integration method, computation of slopes and deflection in beams, Maxwell‟s reciprocal theorems.

Text Books:

1. Sidney H Avner, “Introduction to Physical Metallurgy”, McGraw-Hill Education, 2nd Edition, 2008.
2. Donald R Askeland, Thomson, “Essentials of Material Science and Engineering”, Thomson Press, 1st
3. Edition, 2005.
4. R. S. Kurmi, Gupta, “Strength of Materials”, S Chand & Co, New Delhi, 1st Edition, 2013.
5. Egor P. Popov, “Solid Mechanics” Pearson, 2nd Edition, 2002.

Reference Books:

1. Jindal, “Strength of Materials”, Pearson Education, 1st Edition, 2012.
2. Vazirani, Ratwani, “Analysis of Structures”, Khanna Publishers, 19th Edition, 2014.
3. S. Ramamrutam, “Strength of Materials”, Dhanpat Rai Publishing Company, 18th Edition, 2014.
4. R K. Rajput, “Strength of Materials”, S.Chand & Co New Delhi, 4th Edition, 2007.

MODULE-I

FUNDAMENTALS OF MATERIAL SCIENCE

1.0 What is Materials Science?

Materials make modern life possible—from the polymers in the chair you're sitting on, the metal ball-point pen you're using, and the concrete that made the building you live or work in to the materials that make up streets and highways and the car you drive. All these items are products of materials science and technology (MST). Briefly defined, materials science is the study of "stuff." Materials science is the study of solid matter, inorganic and organic. Figures 1.1, 1.2, 1.3, and 1.4 depict how these materials are classified.

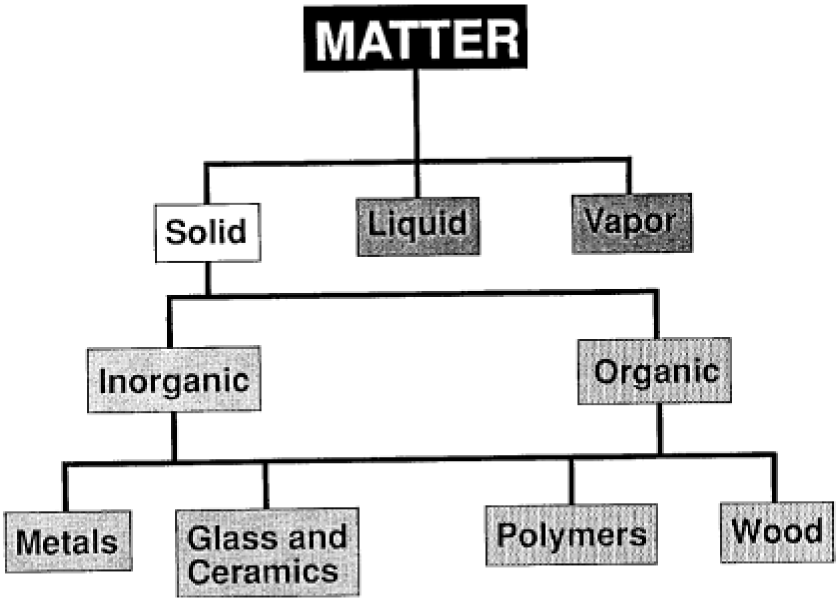


Figure 1.1. Physical Classification of Materials by State

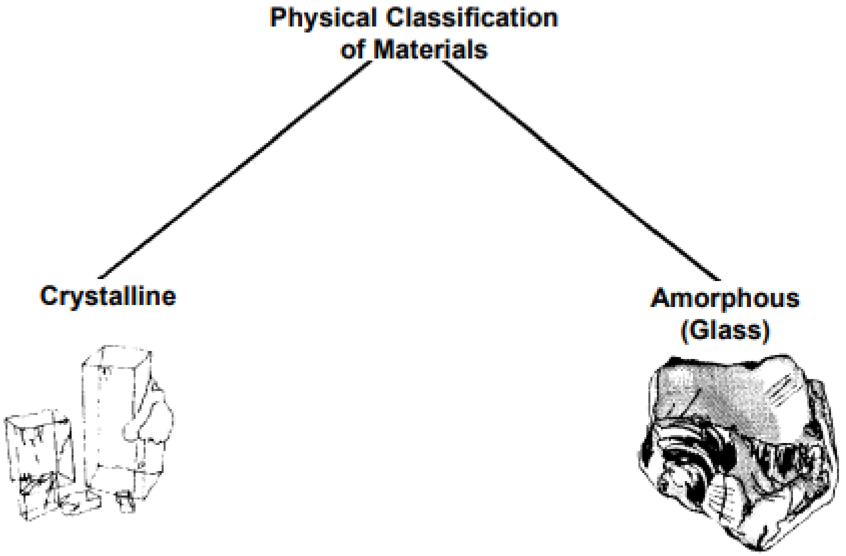


Figure 1.2. Physical Classification of Materials by Morphological Structure

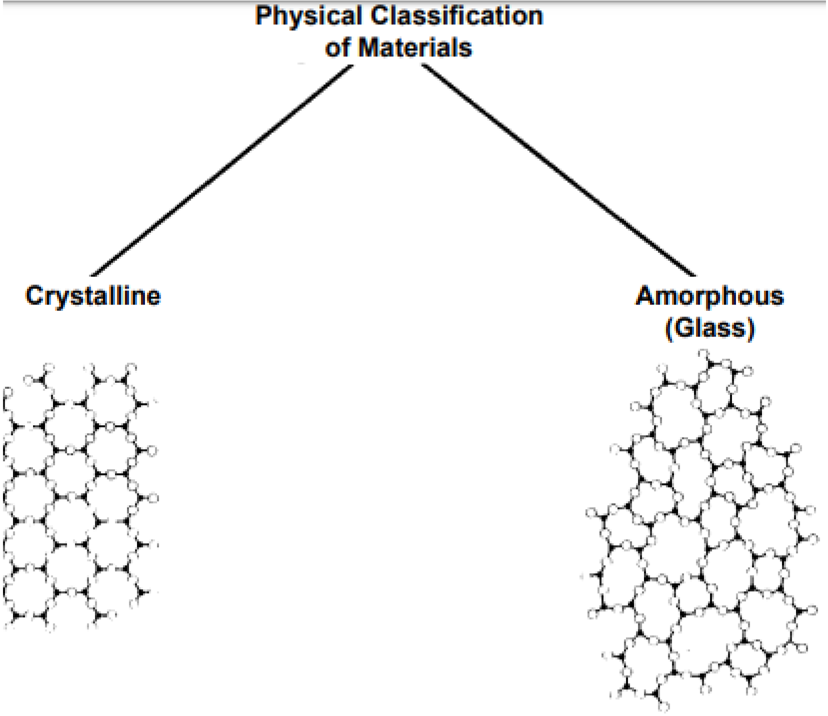


Figure 1.3. Physical Classification of Materials by Atomic Structure

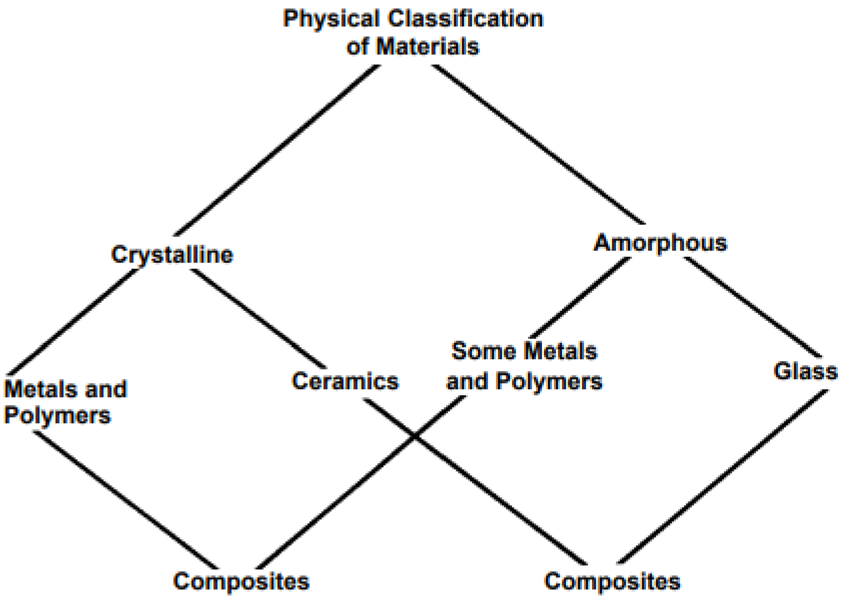


Figure 1.4. Interrelationships Between Classes of Materials

Materials science and technology is a multidisciplinary approach to science that involves designing, choosing, and using three major classes of materials—metals, ceramics, and polymers (plastics). Wood also could be used. Another class of materials used in MST is composites, which are made of a combination of materials (such as in particle board or fiberglass). Materials science combines many areas of science. Figure 1.5 illustrates how materials science draws from chemistry, physics, and engineering to make better, more useful, and more economical and efficient "stuff." Because of the interdisciplinary nature of materials science, it can be used both as an introductory course to interest students in science and engineering and also as an additional course to expand the horizons of students already taking science and mathematics courses.

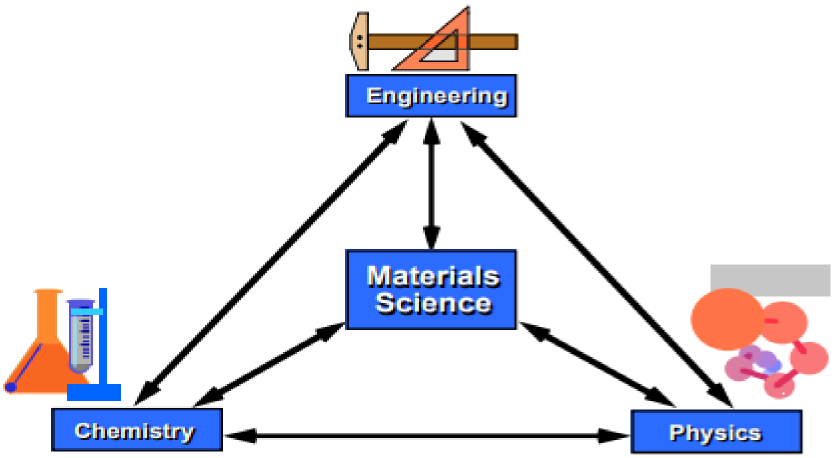
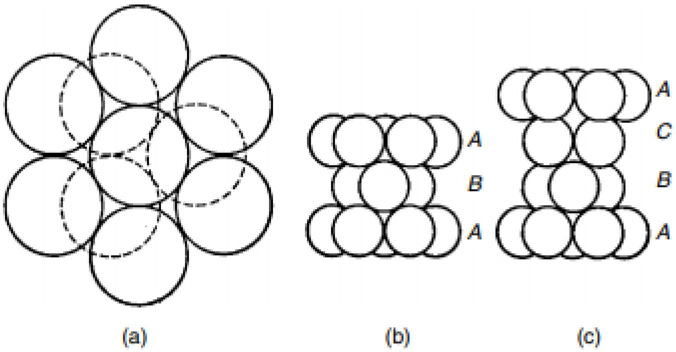


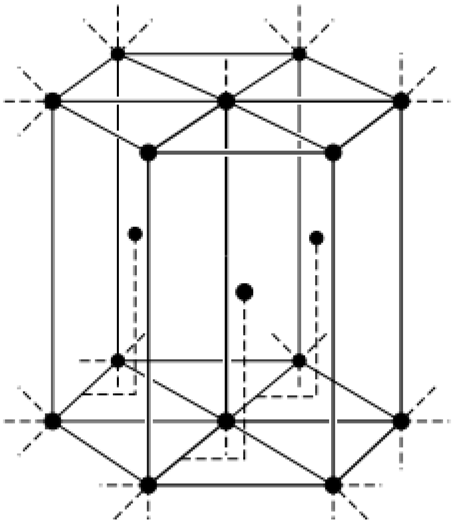
Figure 1.5. Materials Science and Technology—A Multidisciplinary Approach

1.1 STRUCTURE OF METALS AND ALLOYS

Since the electrons in a metallic lattice are in a “gas,” we must use the core electrons and nuclei to determine the structure in metals. This will be true of most solids we will describe, regardless of the type of bonding, since the electrons occupy such a small volume compared to the nucleus. For ease of visualization, we consider the atomic cores to be hard spheres. Because the electrons are delocalized, there is little in the way of electronic hindrance to restrict the number of neighbors a metallic atom may have. As a result, the atoms tend to pack in a close-packed arrangement, or one in which the maximum number of nearest neighbors (atoms directly in contact) is satisfied. Refer to Figure 1.16. The most hard spheres one can place in the plane around a central sphere is six, regardless of the size of the spheres (remember that all of the spheres are the same size). You can then place three spheres in contact with the central sphere both above and below the plane containing the central sphere. This results in a total of 12 nearest-neighbor spheres in contact with the central sphere in the close-packed structure. Closer inspection of Figure 1.16a shows that there are two different ways to place the three nearest neighbors above the original plane of hard spheres. They can be directly aligned with the layer below in an ABA type of structure, or they can be rotated so that the top layer does not align core centers with the bottom layer, resulting in an ABC structure. This leads to two different types of close-packed structures. The ABAB... structure (Figure 1.16b) is called hexagonal close-packed (HCP) and the ABCABC... structure is called face-centered cubic (FCC). Remember that both

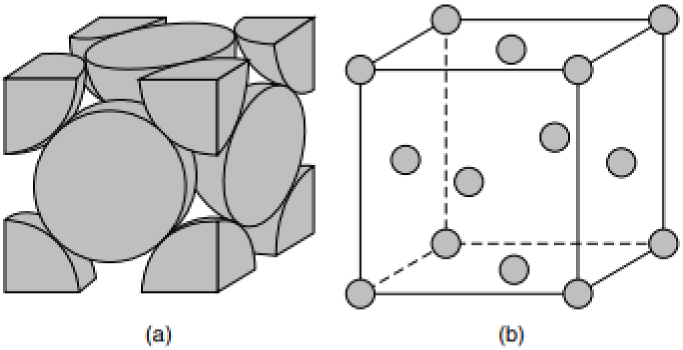


Close-packing of spheres. (a) Top view, (b) side view of ABA structure, (c) side view of ABC structure



The extended unit cell of the hexagonal close-packed (HCP) structure.

of these close-packed arrangements have a coordination number (number of nearest neighbors surrounding an atom) of 12: 6 in plane, 3 above, and 3 below.∗ Keep in mind that for close-packed structures, the atoms touch each other in all directions, and all nearest neighbors are equivalent. Let us first examine the HCP structure. Figure 1.17 is a section of the HCP lattice, from which you should be able to see both hexagons formed at the top and bottom of what is called the unit cell. You should also be able to identify the ABA layered structure in the HCP unit cell of Figure 1.17 through comparison with Figure 1.16. Let us count the number of atoms in the HCP unit cell. The three atoms in the center of the cell are completely enclosed. The atoms on the faces, however, are shared with adjacent cells in the lattice, which extends to infinity. The center atoms on each face are shared with one other HCP unit cell, either above (for the top face) or below (for the bottom face), so they contribute only half of an atom each to the HCP unit cell under consideration. This leaves the six corner atoms on each face (12 total) unaccounted for. These corner atoms are at the intersection of a total of six HCP unit cells (you should convince yourself of this!), so each corner atom contributes only one-sixth of an atom to our isolated HCP unit cell.



The face-centered cubic (FCC) structure showing (a) atoms touching and (b) atoms as small spheres.

Counting the atoms in the FCC structure is performed in a similar manner, except that visualizing the FCC structure takes a little bit of imagination and is virtually impossible to show on a two-dimensional page. Take the ABC close-packed structure shown in Figure 1.16c, and pick three atoms along a diagonal. These three atoms form the diagonal on the face of the FCC unit cell, which is shown in Figure 1.18. There is a trade-off in doing this: It is now difficult to see the close-packed layers in the FCC structure, but it is much easier to see the cubic structure (note that all the edges of the faces have the same length), and it is easier to count the total number of atoms in the FCC cell. In a manner similar to counting atoms in the HCP cell, we see that there are zero atoms completely enclosed by the FCC unit cell, six face atoms that are each shared with an adjacent unit cell, and eight corner atoms at the intersection of eight unit cells to give 6 × (1/2) = 3 face atoms 8 × (1/8) = 1 corner atom 4 total atoms

Crystal Structures Our description of atomic packing leads naturally into crystal structures. While some of the simpler structures are used by metals, these structures can be employed by heteronuclear structures, as well. We have already discussed FCC and HCP, but there are 12 other types of crystal structures, for a total of 14 space lattices or Bravais lattices. These 14 space lattices belong to more general classifications called crystal systems, of which there are seven.

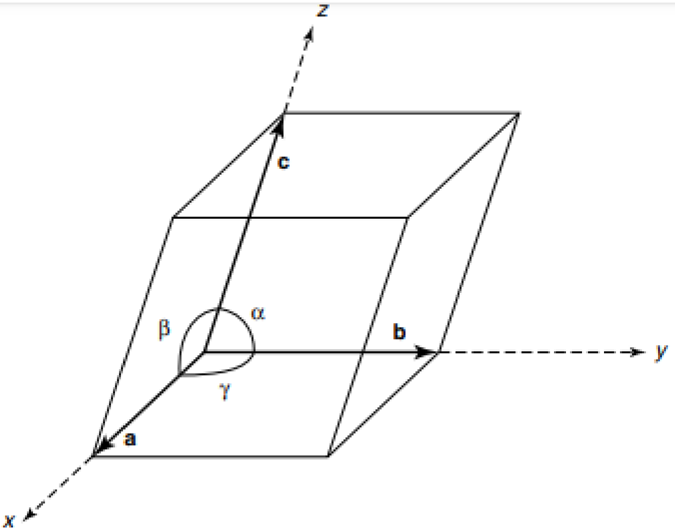


Figure 1.19 Definition of a coordinate system for crystal structures.

Before describing each of the space lattices, we need to define a coordinate system. The easiest coordinate system to use depends upon which crystal system we are looking at. In other words, the coordinate axes are not necessarily orthogonal and are defined by the unit cell. This may seem a bit confusing, but it simplifies the description of cell parameters for those systems that do not have crystal faces at right angles to one another. Refer to Figure 1.19. For each crystal system, we will define the space lattice in terms of three axes, x, y, and z, with interaxial angles α, β, γ . Note that the interaxial angle α is defined by the angle formed between axes z and y, and also note that angles β and γ are defined similarly. Only in special circumstances are α, β, γ equal to 90◦ . The distance along the y axis from the origin to the edge of the unit cell is called the lattice translation vector, b. Lattice translation vectors a and c are defined similarly along the axes x and z, respectively. The magnitudes (lengths) of the lattice translation vectors are called the lattice parameters, a, b, and c. We will now examine each of the seven crystal systems in detail.

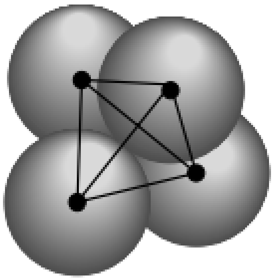
1.2 Crystal Systems.

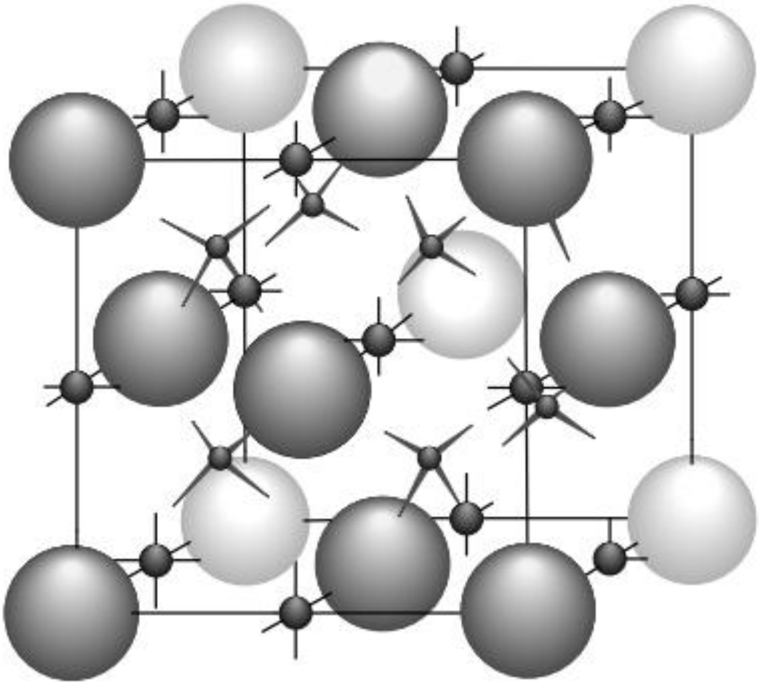
The cubic crystal system is composed of three space lattices, or unit cells, one of which we have already studied: simple cubic (SC), bodycentered cubic (BCC), and face-centered cubic (FCC). The conditions for a crystal to be considered part of the cubic system are that the lattice parameters be the same (so there is really only one lattice parameter, a) and that the interaxial angles all be 90◦ . The simple cubic structure, sometimes called the rock salt structure because it is the structure of rock salt (NaCl), is not a close-packed structure (see Figure 1.20). In fact, it contains about 48% void space; and as a result, it is not a very dense structure. The large space in the center of the SC structure is called an interstitial site, which is a vacant position between atoms that can be occupied by a small impurity atom or alloying element. In this case, the interstitial site is surrounded by eight atoms. All eight atoms in SC are equivalent and are located at the intersection of eight adjacent unit cells, so that there are 8 × (1/8) = 1 total atoms in the SC unit cell.

Body-centered cubic (BCC) is the unit cell of many metals and, like SC, is not a close-packed structure. The number of atoms in the BCC unit cell are calculated as follows:

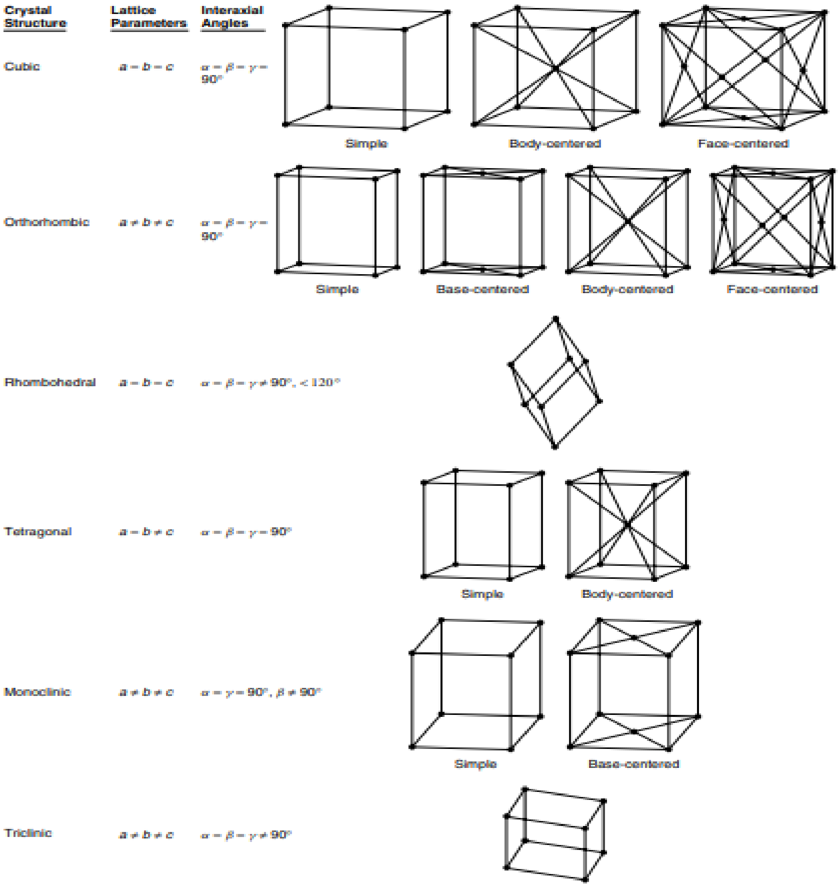
$1 \times 1 \quad = 1$ center atom
$8 \times (1/8) = 1$ corner atom
2 total atoms

Finally, face-centered cubic (FCC) has already been described (Figure 1.18). Even though FCC is a close-packed structure, there are interstitial sites, just as in SC. There are actually two different types of interstitial sites in FCC, depending on how many atoms surround the interstitial site. A group of four atoms forms a tetrahedral interstice, as shown in Figure 1.21. A group of six atoms arranged in an octahedron (an eight-sided geometric figure), creates an octahedral interstice (Figure 1.22). Figure 1.23 shows the locations of these interstitial sites within the FCC lattice. Note that there are eight total tetrahedral interstitial sites in FCC and there are four total octahedral interstitial sites in FCC (prove it!), which are counted in much the same way as we previously counted the total number of atoms in a unit cell. We will see later on that these interstitial sites play an important role in determining solubility of impurities and phase stability of alloys. Interstitial sites are the result of packing of the spheres. Recall from Figure 1.18 that the spheres touch along the face diagonal in FCC. Similarly, the spheres touch along the body diagonal in BCC and along an edge in SC. We should, then, be able to calculate the lattice parameter, a, or the length of a face edge, from a knowledge of the sphere radius. In SC, it should be evident that the side of a unit cell is simply 2r. Application of a little geometry should prove to you that in FCC, $a = 4r/\sqrt{2}$. The relationship between a and r for BCC is derived in Example Problem 1.4; other geometric relationships, including cell volume for cubic structures, are listed in Table 1.8. Finally, atomic radii for the elements can be found in Table 1.9. The radius of an atom is not an exactly defined quantity, and it can vary depending upon the bonding environment.





Location of interstitial sites in FCC.



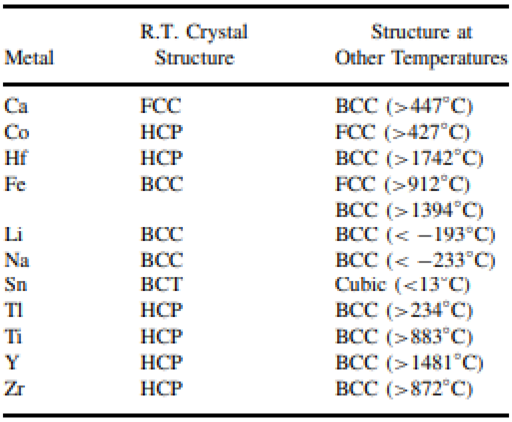
Crystal Structure	Lattice Parameters	Interaxial Angles
Cubic	$a = b = c$	$\alpha = \beta = \gamma = 90°$
Orthorhombic	$a \neq b \neq c$	$\alpha = \beta = \gamma = 90°$
Rhombohedral	$a = b = c$	$\alpha = \beta = \gamma \neq 90°, < 120°$
Tetragonal	$a = b \neq c$	$\alpha = \beta = \gamma = 90°$
Monoclinic	$a \neq b \neq c$	$\alpha = \gamma = 90°, \beta \neq 90°$
Triclinic	$a \neq b \neq c$	$\alpha = \beta = \gamma \neq 90°$

Summary of the 14 Bravais space lattices.

Orthorhombic crystals are similar to both tetragonal and cubic crystals because their coordinate axes are still orthogonal, but now all the lattice parameters are unequal. There are four types of orthorhombic space lattices: simple orthorhombic, face-centered orthorhombic, body-centered orthorhombic, and a type we have not yet encountered, base-centered orthorhombic. The first three types are similar to those we have seen for the cubic and tetragonal systems. The base-centered orthorhombic space lattice has a lattice point (atom) at each corner, as well as a lattice point only on the top and bottom faces (called basal faces). All four orthorhombic space lattices are shown in Figure 1.20. There is only one space lattice in the rhombohedral crystal system. This crystal is sometimes called hexagonal R or trigonal R, so don't confuse it with the other two similarly-named crystal systems. The rhombohedral crystal has uniform lattice parameters in all directions and has equivalent interaxial angles, but the angles are nonorthogonal and are less than 120◦ . The crystal descriptions become increasingly more complex as we move to the monoclinic system. Here all lattice parameters are different, and only two of the interaxial angles are orthogonal. The third angle is not 90◦ . There are two types of monoclinic space lattices: simple monoclinic and base-centered monoclinic. The triclinic crystal, of which there is only one type, has three different lattice parameters, and none of its interaxial angles are orthogonal, though they are all equal. Finally, we revisit the hexagonal system in order to provide some additional details. The lattice parameter and interaxial angle conditions shown in Figure 1.20 for the hexagonal cell refer to what is called the primitive cell for the hexagonal crystal, which can be seen in the front quadrant of the extended cell in Figure 1.17. The primitive hexagonal cell has lattice points only at its corners and has one atom in the center of the primitive cell, for a basis of two atoms. A basis is a unit assembly of atoms identical in composition, arrangement, and orientation that is placed in a regular manner on the lattice to form a space lattice. You should be able to recognize that there are three equivalent primitive cells in the extended HCP structure. The HCP extended cell, which is more often used to represent the hexagonal structure, contains a total of six atoms, as we calculated earlier. In the extended structure, the ratio of the height of 38 THE STRUCTURE OF MATERIALS Table 1.10 Axial Ratios for Some HCP Metals Metal c/a Be, Y 1.57 Hf, Os, Ru, Ti 1.58 Sc, Zr 1.59 Tc, Tl 1.60 La 1.61 Co, Re 1.62 Mg 1.63 Zn 1.85 Cd 1.89 Ideal (sphere packing) 1.633 the cell to its base, c/a, is called the axial ratio. Table 1.10 lists typical values of the axial ratio for some common HCP crystals. A table of crystal structures for the elements can be found in Table 1.11 (excluding the Lanthanide and Actinide series). Some elements can have multiple crystal structures, depending on temperature and pressure. This phenomenon is called allotropy and is very common in elemental metals (see Table 1.12). It is not unusual for closepacked crystals to transform from one stacking sequence to the other, simply through a shift in one of the layers of atoms. Other common allotropes include carbon (graphite at ambient conditions, diamond at high pressures and temperature), pure iron (BCC at room temperature, FCC at 912◦ C and back to BCC at 1394◦ C), and titanium (HCP to BCC at 882◦ C).

1.3 Crystal Locations, Planes, and Directions.

In order to calculate such important quantities as cell volumes and densities, we need to be able to specify locations and directions within the crystal. Cell coordinates specify a position in the lattice and are indicated by the variables u, v, w, separated by commas with no brackets: u distance along the lattice translation vector a v distance along the lattice translation vector b w distance along the lattice translation vector c



Metal	R.T. Crystal Structure	Structure at Other Temperatures
Ca	FCC	BCC (>447°C)
Co	HCP	FCC (>427°C)
Hf	HCP	BCC (>1742°C)
Fe	BCC	FCC (>912°C)
		BCC (>1394°C)
Li	BCC	BCC (< −193°C)
Na	BCC	BCC (< −233°C)
Sn	BCT	Cubic (<13°C)
Tl	HCP	BCC (>234°C)
Ti	HCP	BCC (>883°C)
Y	HCP	BCC (>1481°C)
Zr	HCP	BCC (>872°C)

For example, the center atom in the BCC space lattice (see Figure 1.20) has cell coordinates of 1/2, 1/2, 1/2. Any two points are equivalent if the fractional portions of their coordinates are equal:

$$1/2, 1/2, 1/2 \equiv -1/2, -1/2, -1/2 \text{ (center)}$$
$$0, 0, 0 \equiv 1, 0, 1 \text{ (corner)}$$

A *cell direction* is designated by the vector **r**, which is a combination of the lattice translation vectors **a**, **b**, and **c**:

$$\mathbf{r} = u\mathbf{a} + v\mathbf{b} + w\mathbf{c} \tag{1.27}$$

A direction can also be specified with the cell coordinates in square brackets, with commas and fractions removed:

$$[1\ 1\ 1] \equiv [1/2\ 1/2\ 1/2] \equiv 1/2\mathbf{a} + 1/2\mathbf{b} + 1/2\mathbf{c}$$

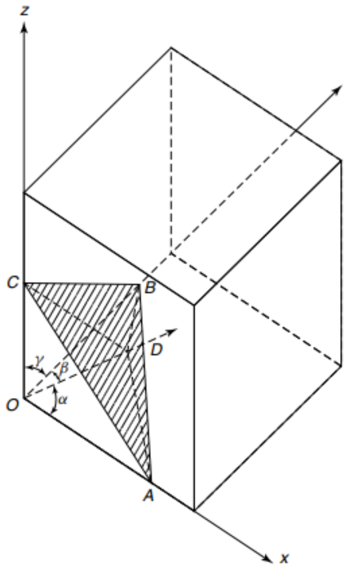
Negative directions are indicated by an overbar [1-1] and are called the "one negative one one" direction. All directions are relative to the origin where the three lattice translation vectors originate (see Figure 1.19).

The *cell volume*, V, can be calculated using the lattice translation vectors:

$$V = |\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}| \tag{1.28}$$

Mathematically, this is a triple scalar product and can be used to calculate the volume of any cell, with only a knowledge of the lattice translation vectors. If the lattice parameters and interaxial angles are known, the following expression for V can be derived from the vector expression:

$$V = abc(1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma)^{1/2} \tag{1.29}$$



Definition of Miller indices for an arbitrary plane

Any planes that have common factors are parallel. For example, a (222) and a (111) plane are parallel, as are (442) and (221) planes. As with cell directions, a minus sign (in this case, indicating a negative intercept) is designated by an overbar. The (221) plane has intercepts at 1/2, −1/2, and 1 along the x, y, and z axes, respectively. Some important planes in the cubic crystal system are shown in Figure 1.25. In a manner similar to that used to calculate the density of a unit cell, we can calculate the density of atoms on a plane, or planar density. The perpendicular intersection of a plane and sphere is a circle, so the radius of the atoms will be helpful in calculating the area they occupy on the plane. Refer back to Example Problem 1.4 when we calculated the lattice parameter for a BCC metal. The section shown along the body diagonal is actually the (110) plane. The body-centered atom is entirely enclosed by this plane, and the corner atoms are located at the confluence of four adjacent planes, so each contributes 1/4 of an atom to the (110) plane. So, there are a total of two atoms on the (110) plane. If we know the lattice parameter or atomic radius, we can calculate the area of the plane, Ap, the area occupied by the atoms, Ac, and the corresponding

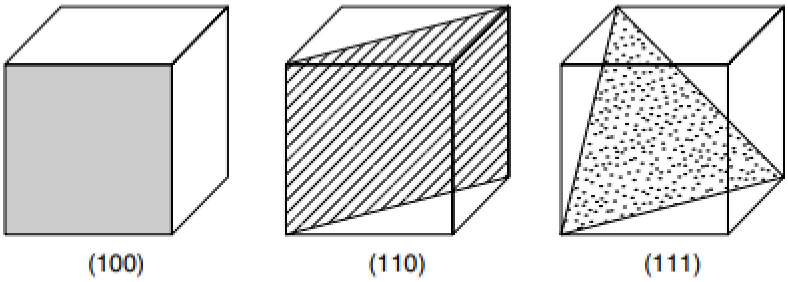


Figure 1.25 Some important planes in the cubic space lattice.

Interplanar Spacings. To this point, we have concentrated on planes in an isolated cell. A crystal lattice, of course, is composed of many individual unit cells, with the planes extending in all directions. So, a real crystal lattice has many (111) planes, for example, all of which are parallel to one another. There is a uniform distance between like planes in a lattice, which we call the interplanar spacing and designate with d, the perpendicular distance between adjacent planes in a set. Note that even though the (111) and (222) planes are parallel to one another, they are not the same plane, since their planar densities may be much different depending on the lattice (for example, compare these two planes in simple cubic). What we are calculating here is the perpendicular distance between the same plane in adjacent cells.

1.4 Crystal structures

Primitive lattice Cell

An ideal crystal is constructed by the infinite repetition of identical structural units in space. In the simplest crystals the structural unit is a single atom, as in copper, silver, gold, iron, aluminium, and the alkali metals.The structure of all crystals can be described in terms of a lattice, with a group of atoms attached to every lattice point. The group of atoms is called the basis; when repeated in space it forms the crystal structure. The basis consists of a primitive cell, containing one single lattice point. Arranging one cell at each lattice point willfill up the entire crystal.

Simple Crystal StructuresThere are several types of crystal structures. The simplest one is the simple cubic lattice (sc). Two other cubic lattices are the body-centered (bcc) and the face-centered (fcc) cubic lattice.

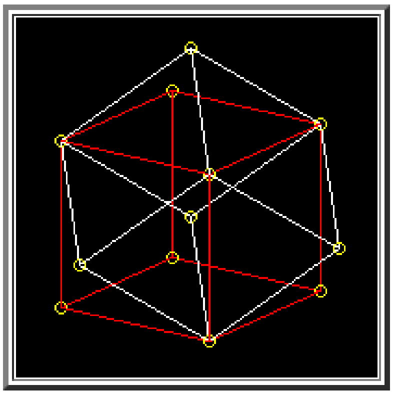


Figure 1.2.1. Body-centered cubic (bcc) lattice

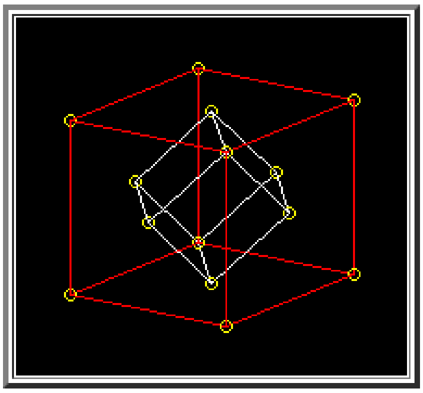


Figure 1.2.2. Face-centered cubic (fcc) lattice

1.5 Diamond and zinkblend lattice structures

The diamond lattice structure is very common in semiconductor materials, Si, Ge. GaAs and GaP has a zinkblende lattice structure which is similar to the diamond lattice structure. The difference between the face-centered lattice structure and the diamond lattice structure is four atoms (see pictures). In the GaAs these four atoms are Ga-atoms and the rest are As-atoms.

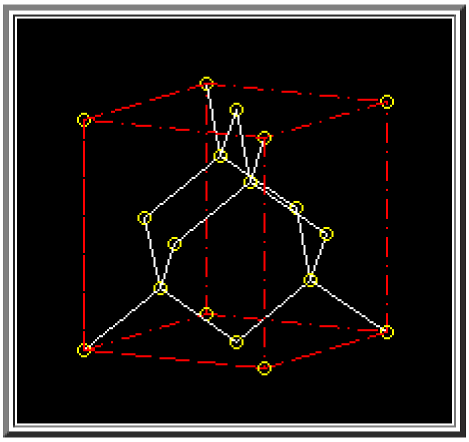
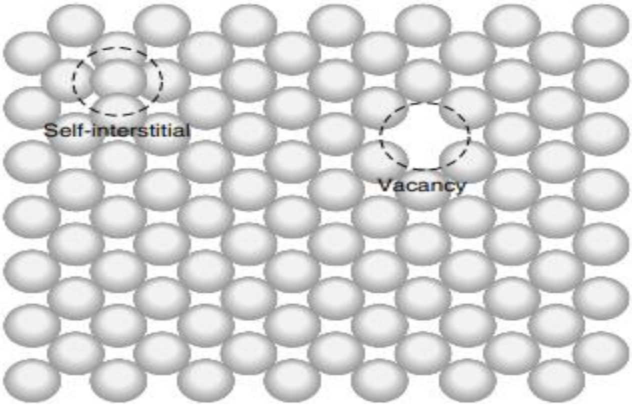


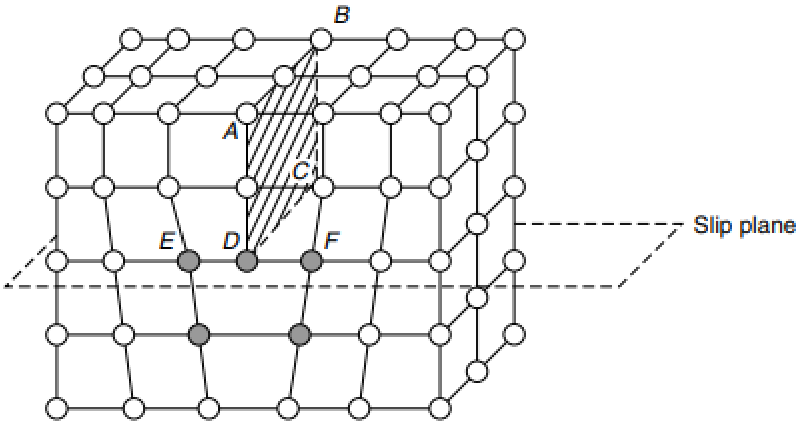
Figure 1.3.1. Diamond lattice structure

1.6 Point Defects

Now that the most important aspects of perfect crystals have been described, it is time to recognize that things are not always perfect, even in the world of space lattices. This is not necessarily a bad thing. As we will see, many important materials phenomena that are based on defective structures can be exploited for very important uses. These defects, also known as imperfections, are grouped according to spatial extent Point defects have zero dimension; line defects, also known as dislocations, are onedimensional; and planar defects such as surface defects and grain boundary defects have two dimensions. These defects may occur individually or in combination. Let us first examine what happens to a crystal when we remove, add, or displace an atom in the lattice. We will then describe how a different atom, called an impurity (regardless of whether or not it is beneficial), can fit into an established lattice. As shown by Eq. (1.36), point defects have equilibrium concentrations that are determined by temperature, pressure, and composition. This is not true of all types of dimensional defects that we will study.



Line Defects and Dislocations We now move on to defects that have some spacial extent, even if only in one dimension. As we continue to increase the geometric complexity of these defects, you may find it more difficult to visualize them. As with crystal structures, three-dimensional models may help you with visualization, and do not limit yourself to one representation of a specific defect—look for multiple views of the same thing. The first type of one-dimensional defect, or line defect, is called a dislocation. A dislocation is a linear disturbance of the atomic arrangement in a crystal caused by the displacement of one group of atoms from an adjacent group. There are three types of dislocations: edge dislocations, screw dislocations, and a combination of these two, termed mixed dislocations. An edge dislocation occurs when a single atomic plane does not extend completely through the lattice. The termination of this half-plane of atoms creates a defect line (dislocation line) in the lattice (line DC in Figure 1.31). The edge dislocation is designated by a perpendicular sign, either ⊥ if the plane is above the dislocation line or if the plane is below the dislocation line. Edge dislocations can be quantified using a vector called the Burger's vector, b, which represents the relative atomic displacement in the lattice due to the dislocation (see Figure 1.32). The Burger's vector is determined as follows: ž Define a positive direction along the dislocation line. This is usually done into the crystal.



MODULE-II

ALLOYS AND PHASE DIAGRAMS

2.0 Introduction

Many of the engineering materials possess mixtures of phases, e.g. steel, paints, and composites. The mixture of two or more phases may permit interaction between different phases, and results in properties usually are different from the properties of individual phases. Different components can be combined into a single material by means of solutions or mixtures. A solution (liquid or solid) is phase with more than one component; a mixture is a material with more than one phase. Solute does not change the structural pattern of the solvent, and the composition of any solution can be varied. In mixtures, there are different phases, each with its own atomic arrangement. It is possible to have a mixture of two different solutions!

A pure substance, under equilibrium conditions, may exist as either of a phase namely vapor, liquid or solid, depending upon the conditions of temperature and pressure. A phase can be defined as a homogeneous portion of a system that has uniform physical and chemical characteristics i.e. it is a physically distinct from other phases, chemically homogeneous and mechanically separable portion of a system. In other words, a phase is a structurally homogeneous portion of matter. When two phases are present in a system, it is not necessary that there be a difference in both physical and chemical properties; a disparity in one or the other set of properties is sufficient.

There is only one vapor phase no matter how many constituents make it up. For pure substance there is only one liquid phase, however there may be more than one solid phase because of differences in crystal structure. A liquid solution is also a single phase, even as a liquid mixture (e.g. oil and water) forms two phases as there is no mixing at the molecular level. In the solid state, different chemical compositions and/or crystal structures are possible so a solid may consist of several phases. For the same composition, different crystal structures represent different phases. A solid solution has atoms mixed at atomic level thus it represents a single phase. A single-phase system is termed as homogeneous, and systems composed of two or more phases are termed as mixtures or heterogeneous. Most of the alloy systems and composites are heterogeneous.

It is important to understand the existence of phases under various practical conditions which may dictate the microstructure of an alloy, thus the mechanical properties and

usefulness of it. Phase diagrams provide a convenient way of representing which state of aggregation (phase or phases) is stable for a particular set of conditions. In addition, phase diagrams provide valuable information about melting, casting, crystallization, and other phenomena.

2.1 Useful terminology:-

Component – is either pure metal and/or compounds of which an alloy is composed. The components of a system may be elements, ions or compounds. They refer to the independent chemical species that comprise the system.

System – it can either refer to a specific body of material under consideration or it may relate to the series of possible alloys consisting of the same components but without regard to alloy composition.

Solid solution – it consists of atoms of at least two different types where solute atoms occupy either substitutional or interstitial positions in the solvent lattice and the crystal structure of the solvent is maintained.

Solubility limit – for almost all alloy systems, at a specific temperature, a maximum of solute atoms can dissolve in solvent phase to form a solid solution. The limit is known as solubility limit. In general, solubility limit changes with temperature. If solute available is more than the solubility limit that may lead to formation of different phase, either a solid solution or compound. Equilibrium Phase Diagrams, Particle strengthening by precipitation and precipitation reactions

2.2 Equilibrium Phase Diagrams

A diagram that depicts existence of different phases of a system under equilibrium is termed as phase diagram. It is also known as equilibrium or constitutional diagram. Equilibrium phase diagrams represent the relationships between temperature and the compositions and the quantities of phases at equilibrium. In general practice it is sufficient to consider only solid and liquid phases, thus pressure is assumed to be constant (1 atm.) in most applications. These diagrams do not indicate the dynamics when one phase transforms into another. However, it depicts information related to microstructure and phase structure of a particular system in a convenient and concise manner. Important information, useful for the scientists and engineers who are involved with materials development, selection, and application in product design, obtainable from a phase diagram can be summarized as follows:

- To show phases are present at different compositions and temperatures under slow cooling (equilibrium) conditions.
- To indicate equilibrium solid solubility of one element/compound in another.
- To indicate temperature at which an alloy starts to solidify and the range of solidification.
- To indicate the temperature at which different phases start to melt.
- Amount of each phase in a two-phase mixture can be obtained.

A phase diagram is actually a collection of solubility limit curves. The phase fields in equilibrium diagrams depend on the particular systems being depicted. Set of solub

Phase equilibrium – it refers to the set of conditions where more than one phase may exist. It can be reflected by constancy with time in the phase characteristics of a system. In most metallurgical and materials systems, phase equilibrium involves just solid phases. However the state of equilibrium is never completely achieved because of very slow rate of approach of equilibrium in solid systems. This leads to non -equilibrium or meta-stable state, which may persist indefinitely and of course, has more practical significance than equilibrium phases. An equilibrium state of solid system can be reflected in terms of characteristics of the microstructure, phases present and their compositions, relative phase amounts and their spatial arrangement or distribution.

Variables of a system – these include two external variables namely temperature and pressure along with internal variable such as composition (C) and number of phases (P). Number of independent variables among these gives the degrees of freedom (F) or variance. All these are related for a chosen system as follows:

$$P + F = C + 2$$

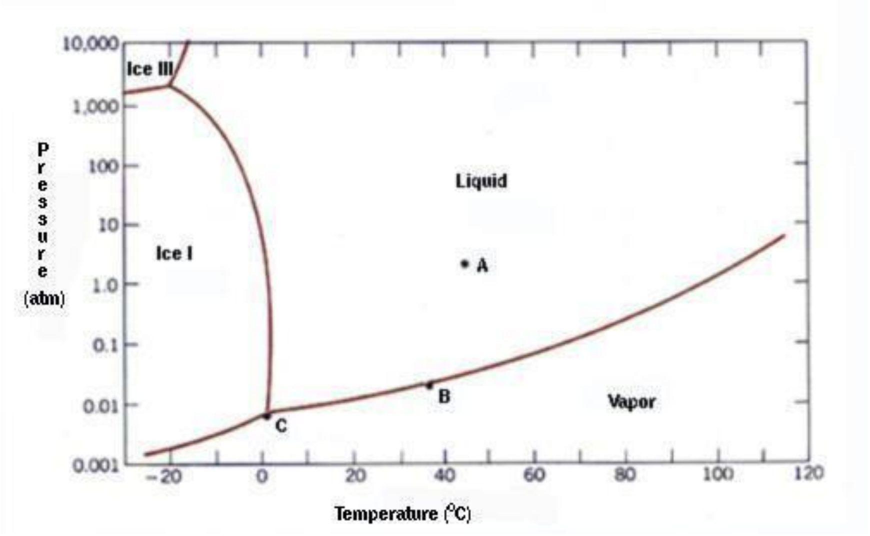
which is known as Gibbs Phase rule. The degrees of freedom cannot be less than zero so that we have an upper limit to the number of phases that can exist in equilibrium for a given system. For practical purpose, in metallurgical and materials field, pressure can be considered as a constant, and thus the condensed phase rule is given as follows:

$P + F = C + 1$

ility curves that represents locus of temperatures above which all compositions are liquid are called liquidus, while solidus represents set of solubility curves that denotes the locus of temperatures below which all compositions are solid. Every phase diagram for two or more components must show a liquidus and a solidus, and an intervening freezing range, except for pure system, as melting of a phase occurs over a range of temperature. Whether the components are metals or nonmetals, there are certain locations on the phase diagram where the liquidus and solidus meet. For a pure component, a contact point lies at the edge of the diagram. The liquidus and solidus also meet at the other invariant positions on the diagram. Each invariant point represents an invariant reaction that can occur only under a particular set of conditions between particular phases, so is the name for it!

Phase diagrams are classified based on the number of components in the system. Single component systems have unary diagrams, two-component systems have binary diagrams, three-component systems are represented by ternary diagrams, and so on. When more than two components are present, phase diagrams become extremely complicated and difficult to represent. This chapter deals mostly with binary phase diagrams.

Unary diagrams: In these systems there is no composition change (C=1), thus only variables are temperature and pressure. Thus in region of single phase two variables (temperature and pressure) can be varied independently. If two phases coexist then, according to Phase rule, either temperature or pressure can be varied independently, but not both. At triple points, three phases can coexist at a particular set of temperature and pressure. At these points, neither temperature nor the pressure can be changed without disrupting the equilibrium i.e. one of the phases may disappear. Figure-1 depicts phase diagram for water.



Binary diagrams: These diagrams constitutes two components, e.g.: two metals (Cu and Ni), or a metal and a compound (Fe and Fe3C), or two compounds (Al2O3 and Si2O3), etc. In most engineering applications, as mentioned before, condensed phase rule is applicable. It is assumed that the same is applicable for all binary diagrams, thus the presentation of binary diagrams becomes less complicated. Thus binary diagrams are usually drawn showing variations in temperature and composition only. It is also to be noted that all binary systems consist only one liquid phase i.e. a component is completely soluble in the other component when both are in liquid state.

Hence, binary systems are classified according to their solid solubility. If both the components are completely soluble in each other, the system is called isomorphous system. E.g.: Cu-Ni, Ag-Au, Ge-Si, Al2O3-Cr2O3. Extent solid solubility for a system of two metallic components can be predicted based on Hume-Ruthery conditions, summarized in the following:

- Crystal structure of each element of solid solution must be the same.
- Size of atoms of each two elements must not differ by more than 15%.

- Elements should not form compounds with each other i.e. there should be no appreciable difference in the electro-negativities of the two elements.
- Elements should have the same valence.

All the Hume- Rothery rules are not always applicable for all pairs of elements which show complete solid solubility.

In systems other than isomorphous systems i.e. in case of limited solid solubility, there exist solid state miscibility gaps; number of invariant reactions can take place; intermediate phases may exist over a range of composition (intermediate solid solutions) or only at relatively fixed composition (compound). These intermediate phases may undergo polymorphic transformations, and some may melt at a fixed temperature (congruent transformations, in which one phase changes to another of the same composition at definite temperature). A solid solution based on a pure component and extending to certain finite compositions into a binary phase diagram is called a terminal solid solution, and the line representing the solubility limit of a terminal solid solution w.r.t a two-phase solid region is called a solvus line (figure-4).

Isomorphous system: Figure -2 depicts a typical phase diagram for an isomorphous system made of two metallic elements A and B. As cited earlier, any phase diagram can be considered as a map. A set of coordinates – a temperature and a composition – is associated with each point in the diagram. If the alloy composition and temperature specified, then the phase diagram allows determination of the phase or phases that will present under equilibrium conditions. There are only two phases in the phase diagram, the liquid and the solid phases. These single-phases regions are separated by a two-phase region where both liquid and solid co-exist. The area in the figure-2 above the line marked liquidus (A'bB') corresponds to the region of stability of the liquid phase, and the area below the solidus line (A'dB') represents the stable region for the solid phase.

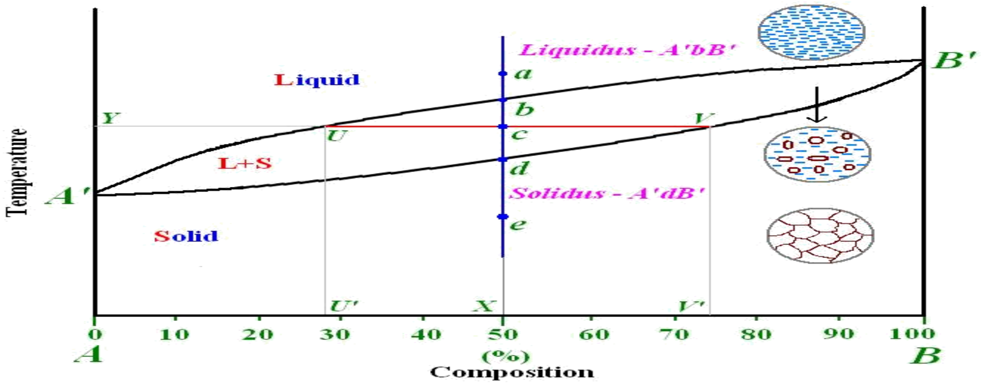


Figure-2: *Phase diagram for typical isomorphous binary system*.

For the interpretation of the phase diagram, let's consider the vertical line ***ae*** drawn corresponding to composition of 50%***A*** +50%***B*** and assume that the system is undergoing equilibrium cooling. The point ***a*** on the line ***ae*** signifies that for that particular temperature and composition, only liquid phase is stable. This is true up to the point ***b*** which lies on the liquidus line, representing the starting of solidification. Completion of solidification of the alloy is represented by the point, ***d***. Point ***e*** corresponds to single- phase solid region up to the room temperature. Point ***c*** lies in the two-phase region made of both liquid and solid phases. Corresponding micro-structural changes are also shown in *figure-2*. As shown in figure-2, above liquidus only a liquid phase exists, and below the solidus single solid phase exists as completely solidified grains. Between these two lines, system consist both solid crystals spread in liquid phase. It is customary to use ***L*** to represent liquid phase(s) and Greek alphabets (***α***, ***β***, ***γ***) for representing solid phases.

Between two extremes of the horizontal axis of the diagram, cooling curves for different alloys are shown in *figure-3* as a function of time and temperature. Cooling curves shown in *figure-3* represent ***A***, ***U'***, ***X***, ***V'*** and ***B*** correspondingly in *figure-2*. Change in slope of the cooling curve is caused by heat of fusion. In fact these changes in slope are nothing but points on either solidus or liquidus of a phase diagram. An experimental procedure where repeated cooling/heating of an alloy at different compositions, and corresponding changes in slope of cooling curves will be used to construct the phase diagram.

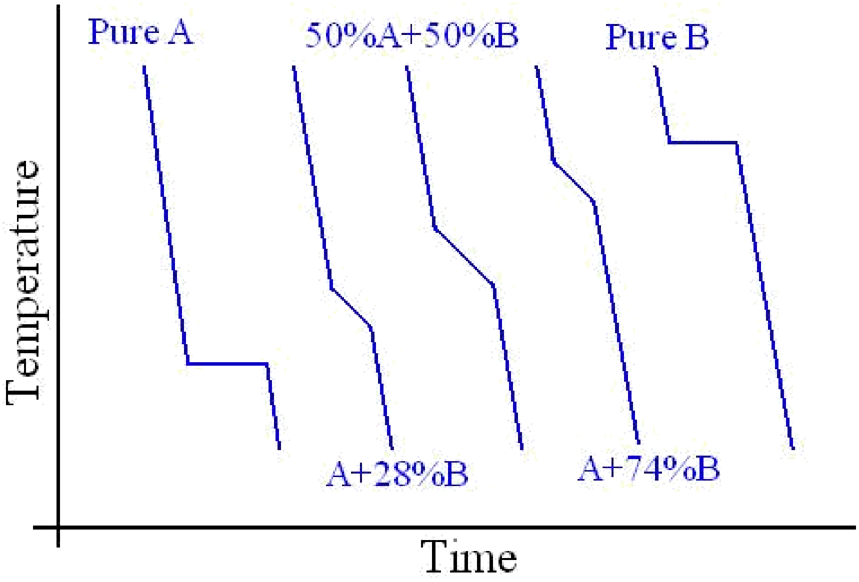


Figure-3: *Cooling curves for isomorphous binary system.*

Another important aspect of interpreting phase diagrams along with phases present is finding the relative amount of phases present and their individual composition.

Procedure to find equilibrium concentrations of phases:

- A *tie-line* or *isotherm* (***UV***) is drawn across two-phase region to intersect the boundaries of the region.
- Perpendiculars are dropped from these intersections to the composition axis, represented by ***U’*** and ***V’*** in *figure-2*, from which each of each phase is read. ***U’*** represents composition of liquid phase and ***V’*** represents composition of solid phase as intersection ***U*** meets liquidus line and ***V*** meets solidus line.

Procedure to find equilibrium relative amounts of phases (*lever rule*):

- A tie-line is constructed across the two phase region at the temperature of the alloy to intersect the region boundaries.
- The relative amount of a phase is computed by taking the length of tie line from overall composition to the phase boundary for the other phase, and dividing by the total tie-line length. From *figure-2*, relative amounts of liquid and solid phases is given respectively by

$$C_L = \frac{cV}{\overline{UV}}, \quad C_S = \frac{Uc}{\overline{UV}}$$

, and it is to be noted that $C_L + C_S = 1$.

<u>Eutectic system</u>: Many binary systems have components which have limited solid solubility, e.g.: Cu-Ag, Pb-Sn. The regions of limited solid solubility at each end of a phase diagram are called *terminal solid solutions* as they appear at ends of the diagram.

Many of the binary systems with limited solubility are of eutectic type, which consists of specific alloy composition known as *eutectic composition* that solidifies at a lower temperature than all other compositions. This low temperature which corresponds to the lowest temperature at which the liquid can exist when cooled under equilibrium conditions is known as *eutectic temperature*. The corresponding point on the phase diagram is called *eutectic point*. When the liquid of eutectic composition is cooled, at or below eutectic temperature this liquid transforms simultaneously into two solid phases (two terminal solid solutions, represented by $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$). This transformation is known as *eutectic reaction* and is written symbolically as:

Liquid (***L***) ↔ solid solution-1 (***α***) + solid solution-2 (***β***)

This eutectic reaction is called invariant reaction as it occurs under equilibrium conditions at a specific temperature and specific composition which can not be varied. Thus, this reaction is represented by a thermal horizontal arrest in the cooling curve of an alloy of eutectic composition. A typical eutectic type phase diagram is shown in *figure-4* along with a cooling curve.

As shown in *figure-4*, there exist three single phase regions, namely liquid (***L***), ***α*** and ***β*** phases. There also exist three two phase regions: ***L+α***, ***L+β*** and ***α+β***. These three two phase regions are separated by horizontal line corresponding to the eutectic temperature. Below the eutectic temperature, the material is fully solid for all compositions. Compositions and relative amount of the phases can be determined using tie-lines and lever rule. Compositions that are on left-hand-side of the eutectic composition are known as *hypo-eutectic compositions* while compositions on right-hand-side of the eutectic composition are called *hyper-eutectic compositions*. Development of micro -structure and respective cooling curves for eutectic alloys are shown in *figure-5, 6, 7 and 8* for different compositions. The phase that forms during cooling but before reaching eutectic temperature is called *pro-eutectic phase*.

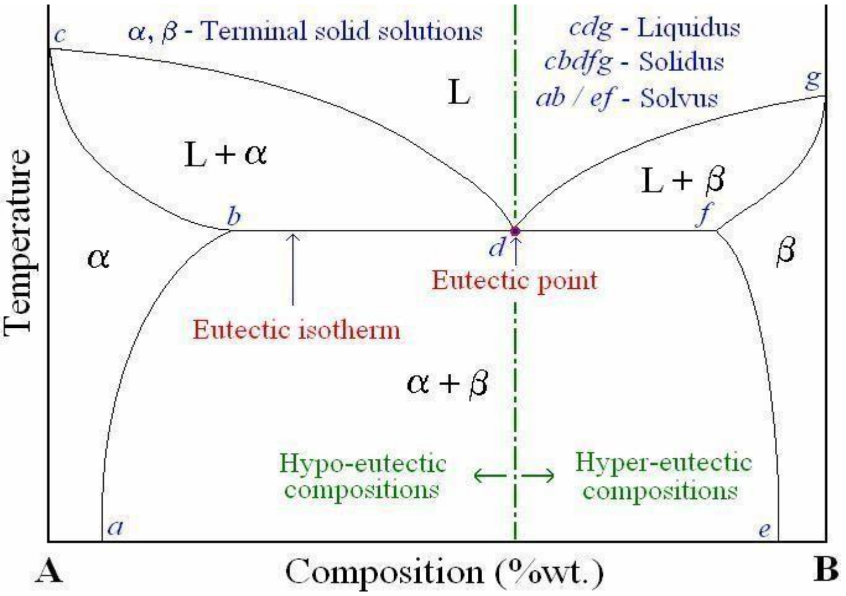


Figure-4: *Typical phase diagram for a binary eutectic system.*

In many systems, solidification in the solid + liquid region may lead to formation of layered (cored) grains, even at very slow cooling rates. This is as a result of very slow or no-diffusion in solid state compared with very high diffusion rates in liquids. The composition of the liquid phase evolves by diffusion, following the equilibrium values that can be derived from the tie-line method. However, new layers that solidify on top of the grains have the equilibrium composition at that temperature but once they are solid their composition does not change.

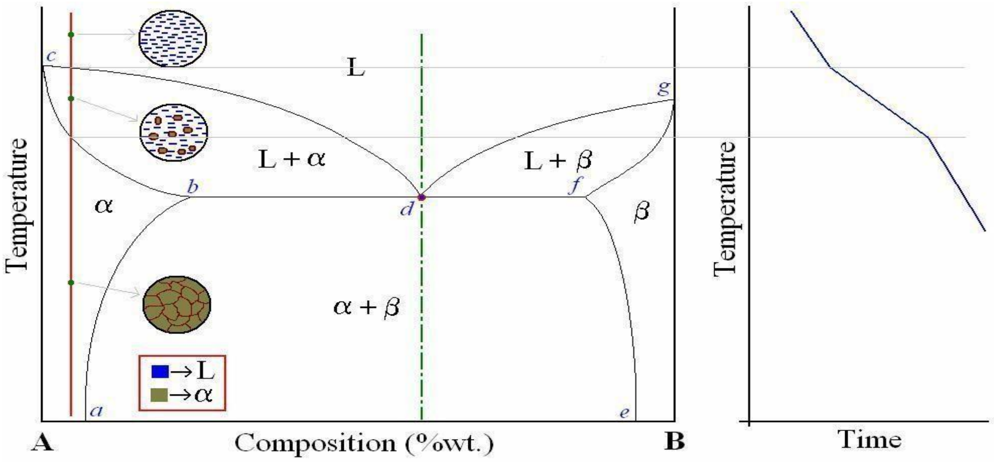


Figure-5: *Cooling curve and micro-structure development for eutectic alloy that passes mainly through terminal solid solution.*

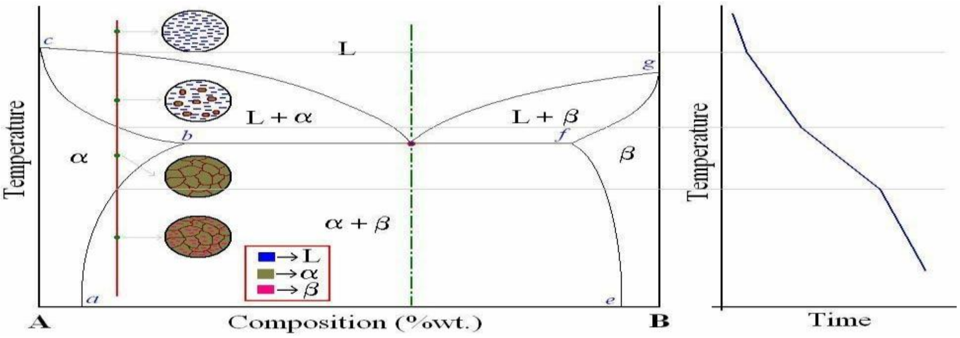


Figure-6: *Cooling curve and micro-structure development for eutectic alloy that passes through terminal solid solution without formation of eutectic solid.*

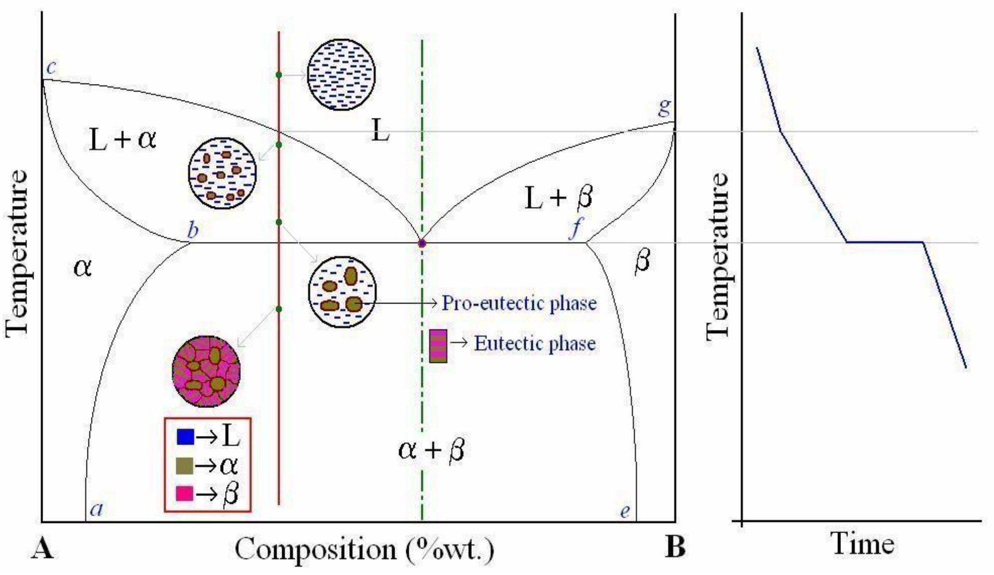


Figure-7: *Cooling curve and micro-structure development for eutectic alloy that passes through hypo-eutectic region.*

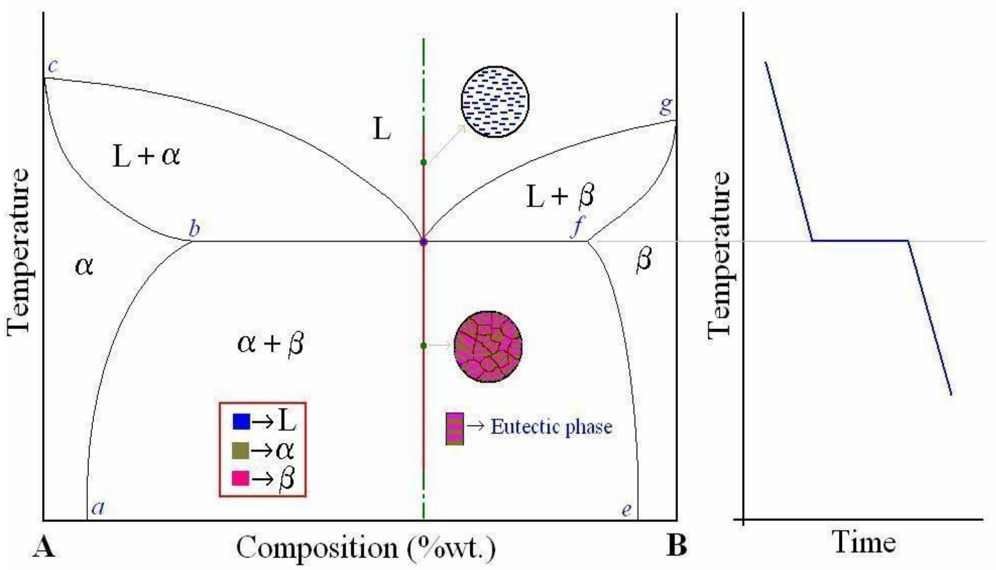


Figure-8: *Cooling curve and micro-structure development for eutectic alloy that passes through eutectic-point.*

2. 3 Invariant reactions:

The *eutectic reaction,* in which a liquid transforms into two solid phases, is just one of the possible three-phase invariant reactions that can occur in binary systems those are not isomorphous. Schematically it can be shown as in *figure-9.* It represents that a liquid phase, ***L***, transforms into two different solids phases (***α*** and ***β***) upon cooling during the eutectic reaction.

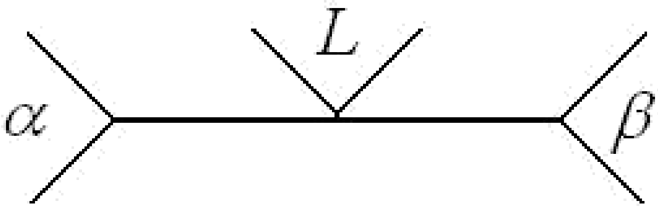


Figure-9: *Schematic of eutectic invariant reaction.*

In the solid state analog of a eutectic reaction, called a *eutectoid reaction,* one solid phase having eutectoid composition transforms into two different solid phases. Another set of invariant reactions that occur often in binary systems are - *peritectic reaction* where a solid phase reacts with a liquid phase to produce a new solid phase, and in *peritectoid reaction,* two solid phases react to form a new solid phase. Peritectic reaction is commonly present as part of more-complicated binary diagrams, particularly if the melting points of the two components are quite different. Peritectic and peritectoid reactions do not give rise to micro-constituents as the eutectic and eutectoid reactions do. Another invariant reaction that involves liquid phase is *monotectic reaction* in which a liquid phase transforms into a solid phase and a liquid phase of different composition. Over a certain range of compositions the two liquids are immiscible like oil and water and so constitute individual phases, thus monotectic reaction can said to be associated with miscibility gaps in the liquid state. Example system for monotectic reaction: Cu-Pb at 954 C and 36%Pb. Analog to monotectic reaction in solid state is *monotectoid reaction* in which a solid phase transforms to produce two solid phases of different compositions. Another notable invariant reaction that is associated with liquid immiscibility is *syntectic reaction* in which two liquid phases react to form a solid phase. All the invariant reactions are summarized in the *table-1* showing both symbolic reaction and schematic part of phase diagram.

Table-1: *Summary of invariant reactions in binary systems.*

Reaction	**Symbolic equation**	**Schematic presentation**	**Example**
Eutectic	$\boldsymbol{L \leftrightarrow \alpha + \beta}$		Fe-C, 4.27% C, 1147 C
Eutectoid	$\boldsymbol{\alpha \leftrightarrow \beta + \gamma}$		Fe-C, 0.80% C, 723 C
Peritectic	$\boldsymbol{L + \alpha \leftrightarrow \beta}$		Fe-C, 0.16%C, 1495 C
Peritectoid	$\boldsymbol{\alpha + \beta \leftrightarrow \gamma}$		
Monotectic	$\boldsymbol{L1 \leftrightarrow L2 + \alpha}$		Fe-C, 0.51%C, 1495 C

Intermediate phases: An intermediate phase may occur over a composition range (intermediate solid solution) or at a relatively fixed composition (compound) inside the phase diagram and are separated from other two phases in a binary diagram by two phase regions. Many phase diagrams contain intermediate phases whose occurrence cannot be readily predicted from the nature of the pure components. Intermediate solid solutions often have higher electrical resistivities and hardnesses than either of the two components. Intermediate compounds form relatively at a fixed composition when there exists a stoichiometric relationship between the components, for example: Mg2Ni and MgNi2 in Mg- Ni system. These are called inter-metallic compounds, and differ from other chemical compounds in that the bonding is primarily metallic rather than ionic or covalent, as would be found with compounds in certain metal-nonmetal or ceramic systems.

Some metal-nonmetal compounds, Fe3C, are metallic in nature, whereas in others, MgO and Mg2Si, bonding is mainly covalent. When using the lever rules, inter-metallic compounds are treated like any other phase, except they appear not as a wide region but as a vertical line.

Number of phase transformations may takes place for each system. Phase transformations in which there are no compositional alternations are said to be congruent transformations, and during incongruent transformations at least one of the phases will experience a change in composition. Examples for (1) congruent transformations: allotropic transformations, and melting of pure materials (2) incongruent transformations: all invariant reactions, and also melting of alloy that belongs to an isomorphous system. Intermediate phases are sometimes classified on the basis of whether they melt congruently or incongruently. $MgNi_2$, for example,

melts congruently whereas Mg_2Ni melts incongruently since it undergoes peritectic decomposition

2.4 The iron – carbon system, phase transformations

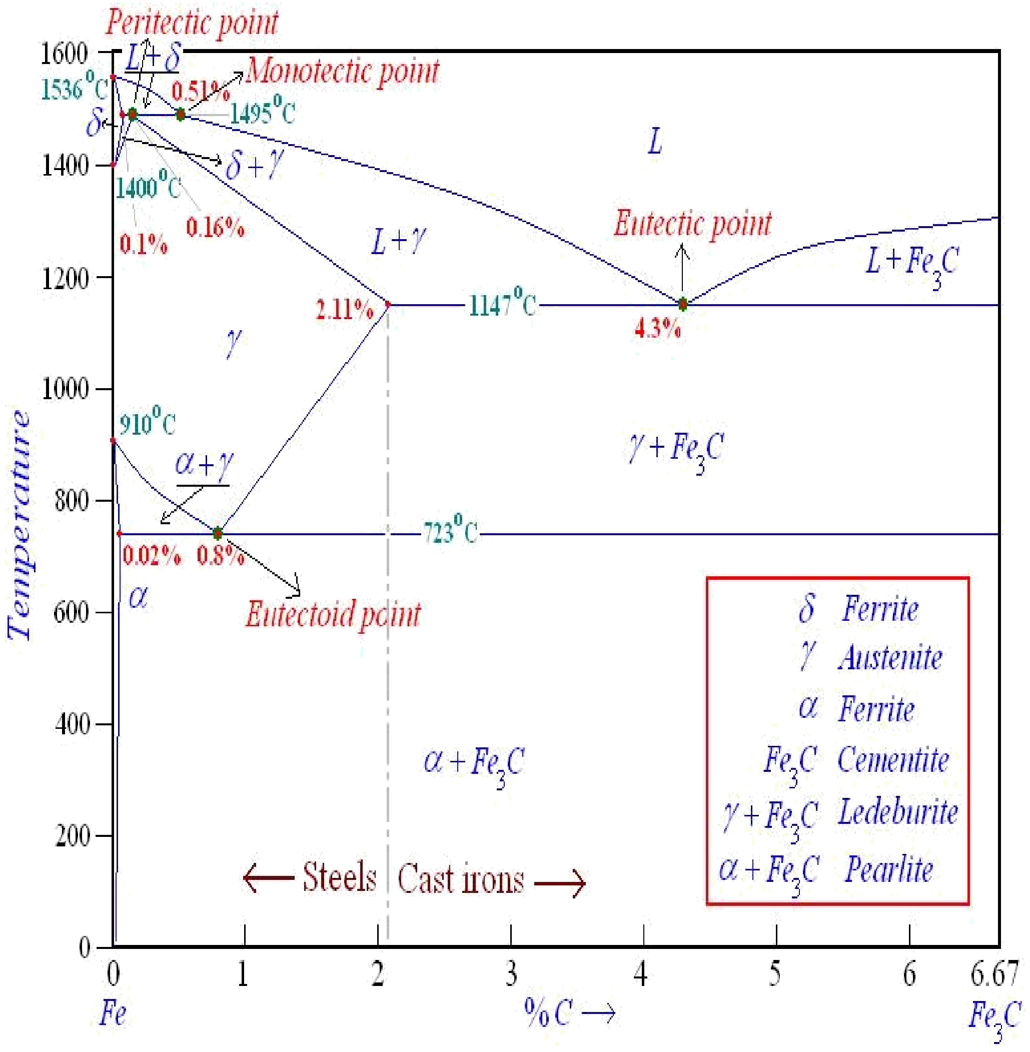
A study of iron-carbon system is useful and important in many respects. This is because

(1) steels constitute greatest amount of metallic materials used by man (2) solid state transformations that occur in steels are varied and interesting. These are similar to those occur in many other systems and helps explain the properties.

Iron-carbon phase diagram shown in figure-16 is not a complete diagram. Part of the diagram after 6.67 wt% C is ignored as it has little commercial significance. The 6.67%C

represents the composition where an inter-metallic compound, cementite (Fe_3C), with solubility limits forms. In addition, phase diagram is not true equilibrium diagram because cementite is not an equilibrium phase. However, in ordinary steels decomposition of cementite into graphite never observed because nucleation of cementite is much easier than that of graphite. Thus cementite can be treated as an equilibrium phase for practical purposes.

Iron – Iron carbide phase diagram.



The Fe -Fe3C is characterized by five individual phases and four invariant reactions. Five phases that exist in the diagram are: α–ferrite (BCC) Fe-C solid solution, γ-austenite (FCC) Fe-C solid solution, δ-ferrite (BCC) Fe-C solid solution, Fe3C (iron carbide) or cementite - an inter-metallic compound and liquid Fe-C solution. Four invariant reactions that cause transformations in the system are namely eutectoid, eutectic, monotectic and peritectic.

As depicted by left axes, pure iron upon heating exhibits two allotropic changes. One involves α–ferrite of BCC crystal structure transforming to FCC austenite, γ-iron, at 910C.

Carbon present in solid iron as interstitial impurity, and forms solid solution with ferrites

/ austenite as depicted by three single fields represented by α, γ and δ. Carbon dissolves least in α–ferrite in which maximum amount of carbon soluble is 0.02% at 723 C. This limited solubility is attributed to shape and size of interstitial position in BCC α– ferrite. However, carbon present greatly influences the mechanical properties of α– ferrite. α–ferrite can be used as magnetic material below 768 C . Solubility of carbon in γ-iron reaches its maximum, 2.11%, at a temperature of 1147 C . Higher solubility of carbon in austenite is attributed to FCC structure and corresponding interstitial sites. Phase transformations involving austenite plays very significant role in heat treatment of different steels. Austenite itself is non-magnetic. Carbon solubility in δ-ferrite is maximum (0.1%) at 1495 C. As this ferrite exists only at elevated temperatures, it is of no commercial importance. Cementite, Fe3C an inter-metallic compound forms when amount of carbon present exceeds its solubility limit at respective temperatures. Out of these four solid phases, cementite is hardest and brittle that is used in different forms to increase the strength of steels. α–ferrite, on the other hand, is softest and act as matrix of a composite material. By combining these two phases in a solution, a material's properties can be varied over a large range.

For technological convenience, based on %C dissolved in it, a Fe-C solution is classified as: commercial pure irons with less than 0.008%C; steels having %C between 0.008- 2.11; while cast irons have carbon in the range of 2.11%-6.67%. Thus commercial pure iron is composed of exclusively α–ferrite at room temperature. Most of the steels and cast irons contain both α–ferrite and cementite. However, commercial cast irons are not simple alloys of iron and carbon as they contain large quantities of other elements such as silicon, thus better consider them as ternary alloys. The presence of Si promotes the formation of graphite instead of cementite. Thus cast irons may contain carbon in form of both graphite and cementite, while steels will have carbon only in combined from as cementite.

As shown in figure-16, and mentioned earlier, Fe-C system constitutes four invariant reactions:

Product phase of eutectic reaction is called ledeburite, while product from eutectoid reaction is called pearlite. During cooling to room temperature, ledeburite transforms into pearlite and

cementite. At room temperature, thus after equilibrium cooling, Fe- C diagram consists of either α–ferrite, pearlite and/or cementite. Pearlite is actually not a single phase, but a micro - constituent having alternate thin layers of α–ferrite (~88%) and
Fe3C, cementite (~12%). Steels with less than 0.8%C (mild steels up to 0.3%C, medium carbon steels with C between 0.3%-0.8% i.e. hypo-eutectoid Fe-C alloys) i.e. consists pro-eutectoid α–ferrite in addition to pearlite, while steels with carbon higher than 0.8% (high-carbon steels i.e. hyper-eutectoid Fe-C alloys) consists of pearlite and pro- eutectoid cementite. Phase transformations involving austenite i.e. processes those involve eutectoid reaction are of great importance in heat treatment of steels.

In practice, steels are almost always cooled from the austenitic region to room temperature. During the cooling upon crossing the boundary of the single phase γ-iron, first pro-eutectoid phase (either α–ferrite or cementite) forms up to eutectoid temperature. With further cooling below the eutectoid temperature, remaining austenite decomposes to eutectoid product called pearlite, mixture of thin layers of α–ferrite and cementite. Though pearlite is not a phase, nevertheless, a constituent because it has a definite appearance under the microscope and can be clearly identified in a structure composed of several constituents. The decomposition of austenite to form pearlite occurs by nucleation and growth. Nucleation, usually, occurs heterogeneously and rarely homogeneously at grain boundaries. When it is not homogeneous, nucleation of pearlite occurs both at grain boundaries and in the grains of austenite. When austenite forms pearlite at a constant temperature, the spacing between adjacent lamellae of cementite is very nearly constant. For a given colony of pearlite, all cementite plates have a common orientation in space, and it is also true for the ferrite plates. Growth of pearlite colonies occurs not only by the nucleation of additional lamellae but also through an advance at the ends of the lamellae. Pearlite growth also involves the nucleation of new colonies at the interfaces between established colonies and the parent austenite. The thickness ratio of the ferrite and cementite layers in pearlite is approximately 8 to 1. However, the absolute layer thickness depends on the temperature at which the isothermal transformation is allowed to occur.

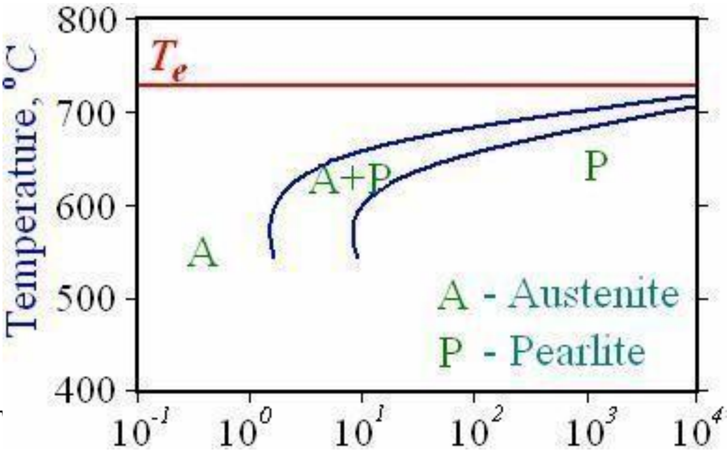
The temperature at which austenite is transformed has a strong effect on the inter- lamellar spacing of pearlite. The lower the reaction temperature, the smaller will be inter- lamellar spacing. For example, pearlite spacing is in order of 10-3 mm when it formed at 700 C, while spacing is in order of 10-4 mm when formed at 600 C . The spacing of the pearlite lamellae has a practical significance because the hardness of the resulting structure depends upon it; the smaller the spacing, the harder the metal. The growth rate of pearlite is also a strong function of temperature. At temperatures just below the eutectoid, the growth rate increases rapidly with decreasing temperature, reaching a maximum at 600 C, and then decreases again at lower temperatures.

Additions of alloying elements to Fe-C system bring changes (alternations to positions of phase boundaries and shapes of fields) depends on that particular element and its concentration. Almost all alloying elements causes the eutectoid concentration to decrease, and most of the alloying elements (e.g.: Ti, Mo, Si, W, Cr) causes the eutectoid temperature to increase while some other (e.g.: Ni, Mn) reduces the eutectoid temperature. Thus alloying additions alters the relative amount of pearlite and pro- eutectoid phase that form.

Fe-C alloys with more than 2.11% C are called cast irons. Phase transformations in cast irons involve formation of pro-eutectic phase on crossing the liquidus. During the further cooling, liquid of eutectic composition decomposes in to mixture of austenite and cementite, known as ledeburite. On further cooling through eutectoid temperature, austenite decomposes to pearlite. The room temperature microstructure of cast irons thus consists of pearlite and cementite. Because of presence of cementite, which is hard, brittle and white in color, product is called white cast iron. However, depending on cooling rate and other alloying elements, carbon in cast iron may be present as graphite or cementite. Gray cast iron contains graphite in form of flakes. These flakes are sharp and act as stress risers. Brittleness arising because of flake shape can be avoided by producing graphite in spherical nodules, as in malleable cast iron and SG (spheroidal graphite) cast iron. Malleable cast iron is produced by heat treating white cast iron (Si < 1%) for prolonged periods at about 900 C
and then cooling it very slowly. The cementite decomposes and temper carbon appears approximately as spherical particles. SG iron is produced by adding inoculants to molten iron. In these Si content must be about 2.5%, and no subsequent heat treatment is required.

Transformation rate effects and TTT diagrams, Microstructure and Property Changes in Fe-C Alloys

Solid state transformations, which are very important in steels, are known to be dependent on time at a particular temperature, as shown in figure-14(b). Isothermal transformation diagram, also known as TTT diagram, measures the rate of transformation at a constant temperature i.e. it shows time relationships for the phases during isothermal transformation. Information regarding the time to start the transformation and the time required to complete the transformation can be obtained from set of TTT diagrams. One such set of diagram for reaction of austenite to pearlite in steel is shown in figure-17. The diagram is not complete in the sense that the transformations of austenite that occur at temperatures below about 550 C are not shown.



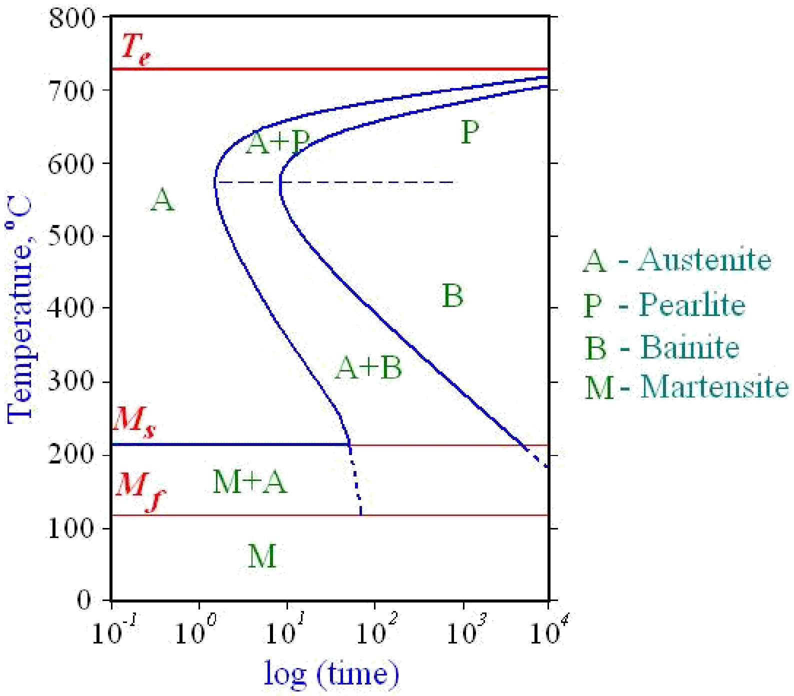
2.5 Partial TTT

As mentioned in previous section, thickness of layers in pearlite depends on the temperature at which the transformation occurred. If the transformation took place at a temperature that is just below the eutectoid temperature, relatively thick layers of **α**– ferrite and cementite are produced in what is called *coarse pearlite*. This is because of high diffusion rates of carbon atoms. Thus with decreasing transformation temperature, sluggish movement of carbon results in thinner layers **α**–ferrite and cementite i.e. *fine pearlite* is produced.

At transformation temperatures below 550 C, austenite results in different product known as bainite. Bainite also consists of **α**–ferrite and cementite phases i.e. transformation is again diffusion controlled but morphologically it consists of very small particles of cementite within or between fine ferrite plates. Bainite forms needles or plates, depending on the temperature of the transformation; the microstructural details of bainite are so fine that their resolution is only possible using electron microscope. It differs from pearlite in the sense that different mechanism is involved in formation ob bainite which does not have alternating layers of **α**–ferrite and cementite. In addition, because of equal growth rates in all directions pearlite tends to form spherical colonies, whereas bainite grows as plates and has a characteristic acicular (needlelike) appearance. *Upper bainite*, formed at the upper end of the temperature range (550 C-350 C), is characterized by relatively coarse, irregular shaped cementite particles in ***α***–ferrite plates. If the transformation is taking place at lower temperatures (350 C-250 C), the ***α***–ferrite plates assume a more regular needlelike shape, and the transformation product is called *lower bainite*. At the same time carbide particles become smaller in size and appear as cross-striations making an angle of about 55 to the axis of the ***α***–ferrite plate. Upper bainite has large rod-like cementite regions, whereas lower bainite has much finer cementite particles as a result of sluggish diffusion of carbon atoms at lower temperatures. Lower bainite is considerably harder than upper bainite. Another characteristic of bainite is that as it has crystallographic orientation that is similar to that found in simple ferrite nucleating from austenite, it is believed that bainite is nucleated by the formation of ferrite. This is in contrast to pearlite which is believed to be nucleated by formation of cementite.

Basically, bainite is a transformation product that is not as close to equilibrium as pearlite. The most puzzling feature of the bainite reaction is its dual nature. In a number of respects, it reveals properties that are typical of a nucleation and growth type of transformation such as occurs in the formation pearlite and also a mixture of ***α***–ferrite and cementite though of quite different morphology (no alternate layers), but at the same time it differs from the Martensite as bainite formation is athermal and diffusion controlled though its microstructure is characterized by acicular (needlelike) appearance.

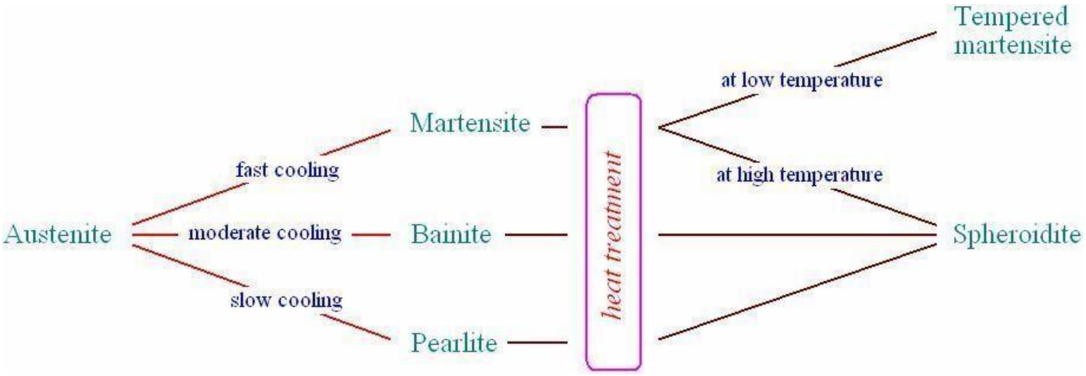
reaction product is bainite only. Thus bainite transformation is favored at a high degree of supercooling, transformations are competitive with each other.



Complete TTT (isothermal transformation) diagram for eutectoid steel.

As explained in earlier section, martensitic transformation can dominate the proceedings if steel is cooled rapid enough so that diffusion of carbon can be arrested. Transformation of austenite to Martensite is diffusion-less, time independent and the extent of transformation depends on the transformation temperature. Martensite is a *meta-stable* phase and decomposes into ferrite and pearlite but this is extremely slow (and not noticeable) at room temperature. Alloying additions retard the formation rate of pearlite and bainite, thus rendering the martensitic transformation more competitive. Start of the transformation is designated by M_S, while the completion is designated by Mf in a transformation diagram. Martensite forms in steels possesses a body centered tetragonal crystal structure with carbon atoms occupying one of the three interstitial sites available. This is the reason for characteristic structure of steel Martensite instead of general BCC. Tetragonal distortion caused by carbon atoms increases with increasing carbon content and so is the hardness of Martensite. Austenite is slightly denser than Martensite, and therefore, during the phase transformation upon quenching, there is a net volume increase. If relatively large pieces are rapidly quenched, they may crack as a result of internal stresses, especially when carbon content is more than about 0.5%.

Mechanically, Martensite is extremely hard, thus its applicability is limited by brittleness associated with it. Characteristics of steel Martensite render it unusable for structural applications in the as-quenched form. However, structure and thus the properties can be altered by *tempering*, heat treatment observed below eutectoid temperature to permit diffusion of carbon atoms for a reasonable period of time. During tempering, carbide particles attain spherical shape and are distributed in ferrite phase – structure called s*pheroidite*. Spheroidite is the softest yet toughest structure that steel may have. At lower tempering temperature, a structure called *tempered Martensite* forms with similar microstructure as that of spheroidite except that cementite particles are much, much smaller. The tempering heat treatment is also applicable to pearlitic and bainitic structures. This mainly results in improved machinability. The mechanism of tempering appears to be first the precipitation of fine particles of hexagonal ε-carbide of composition about $Fe_{2.4}C$ from Martensite, decreasing its tetragonality. At higher temperatures or with increasing tempering times, precipitation of cementite begins and is accompanied by dissolution of the unstable ε-carbide. Eventually the Martensite loses its tetragonality and becomes BCC ferrite, the cementite coalesces into spheres. A schematic of possible transformations involving austenite decomposition are shown in *figure-19*.

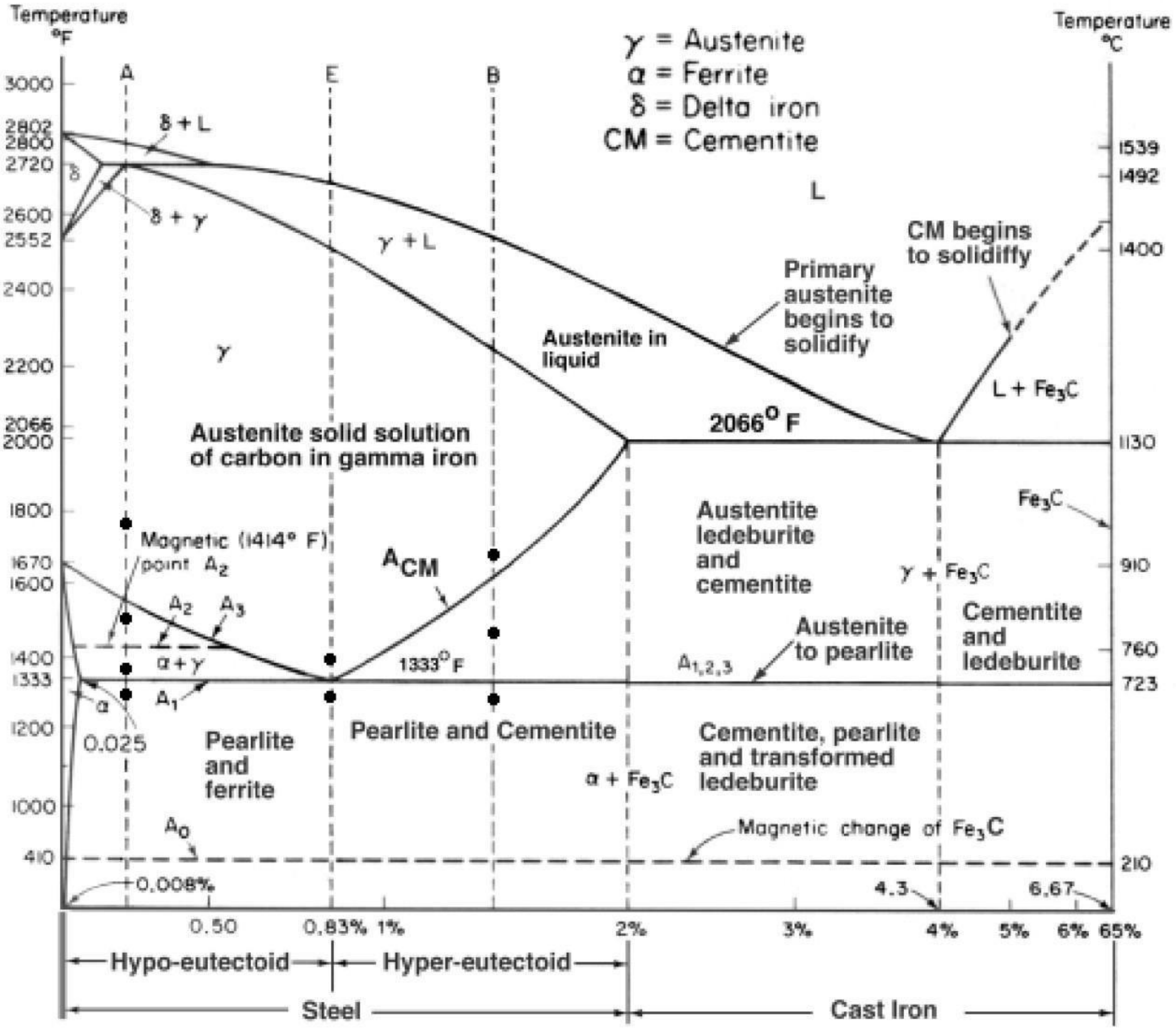


2.6 Possible transformation involving austenite decomposition.

Tempering of some steels may result in a reduction of toughness what is known as *temper embrittlement* . This may be avoided by (1) compositional control, and/or (2) tempering above 575 *or* below 375, followed by quenching to room temperature. The effect is greatest in Martensite structures, less severe in bainitic structures and least severe in pearlite structures. It appears to be associated with the segregation of solute atoms to the grain boundaries lowering the boundary strength. Impurities responsible for temper brittleness are: P, Sn, Sb and As. Si reduces the risk of embrittlement by carbide formation. Mo has a stabilizing effect on carbides and is also used to minimize the risk of temper brittleness in low alloy steels.

TTT diagrams are less of practical importance since an alloy has to be cooled rapidly and then kept at a temperature to allow for respective transformation to take place. However, most industrial heat treatments involve continuous cooling of a specimen to room temperature. Hence, Continuous Cooling Transformation (CCT) diagrams are generally more appropriate for engineering applications as components are cooled (air cooled, furnace cooled, quenched etc.) from a processing temperature as this is more economic than transferring to a separate furnace for an isothermal treatment. CCT diagrams measure the extent of transformation as a function of time for a continuously decreasing temperature. For continuous cooling, the time required for a reaction to begin and end is delayed, thus the isothermal curves are shifted to longer times and lower temperatures.

Both TTT and CCT diagrams are, in a sense, phase diagrams with added parameter in form of time. Each is experimentally determined for an alloy of specified composition. These diagrams allow prediction of the microstructure after some time period for constant temperature and continuous cooling heat treatments, respectively. Normally, bainite will not form during continuous cooling because all the austenite will have transformed to pearlite by the time the bainite transformation has become possible. Thus, as shown in *figure-20,* region representing austenite-pearlite transformation terminates just below the nose.



Fe-Fe3C Phase Diagram

Figure 1 shows the equilibrium diagram for combinations of carbon in a solid solution of iron. The diagram shows iron and carbons combined to form Fe-Fe3C at the 6.67%C end of the diagram. The left side of the diagram is pure iron combined with carbon, resulting in steel alloys. Three significant regions can be made relative to the steel portion of the diagram. They are the eutectoid E, the hypoeutectoid A, and the hypereutectoid B. The right side of the pure iron line is carbon in combination with various forms of iron called alpha iron (ferrite), gamma iron (austenite), and delta iron. The black dots mark clickable sections of the diagram.

Allotropic changes take place when there is a change in crystal lattice structure. From 2802º-2552ºF the delta iron has a body-centered cubic lattice structure. At 2552ºF, the lattice changes from a body-centered cubic to a face-centered cubic lattice type. At

1400ºF, the curve shows a plateau but this does not signify an allotropic change. It is called the Curie temperature, where the metal changes its magnetic properties.

Two very important phase changes take place at 0.83%C and at 4.3% C. At 0.83%C, the transformation is eutectoid, called pearlite.

gamma (austenite) --> alpha + Fe3C (cementite)

At 4.3% C and 2066ºF, the transformation is eutectic, called ledeburite.
L(liquid) --> gamma (austenite) + Fe3C (cementite)

Steels

Steels are alloys of iron and carbon plus other alloying elements. In steels, carbon present in atomic form, and occupies interstitial sites of Fe microstructure. Alloying additions are necessary for many reasons including: improving properties, improving corrosion resistance, etc. Arguably steels are well known and most used materials than any other materials.

Mechanical properties of steels are very sensitive to carbon content. Hence, it is practical to classify steels based on their carbon content. Thus steels are basically three kinds: low-carbon steels (% wt of C < 0.3), medium carbon steels (0.3 <% wt of C < 0.6) and high-carbon steels (% wt of C > 0.6). The other parameter available for classification of steels is amount of alloying additions, and based on this steels are two kinds: (plain) carbon steels and alloy-steels.

Low carbon steels: These are arguably produced in the greatest quantities than other alloys. Carbon present in these alloys is limited, and is not enough to strengthen these materials by heat treatment; hence these alloys are strengthened by cold work. Their microstructure consists of ferrite and pearlite, and these alloys are thus relatively soft, ductile combined with high toughness. Hence these materials are easily machinable and weldable. Typical applications of these alloys include: structural shapes, tin cans, automobile body components, buildings, etc.

A special group of ferrous alloys with noticeable amount of alloying additions are known as HSLA (high-strength low-alloy) steels. Common alloying elements are: Cu, V, Ni, W, Cr, Mo, etc. These alloys can be strengthened by heat treatment, and yet the same time they are ductile, formable. Typical applications of these HSLA steels include: support columns, bridges, pressure vessels.

Medium carbon steels: These are stronger than low carbon steels. However these are of less ductile than low carbon steels. These alloys can be heat treated to improve their strength. Usual heat treatment cycle consists of austenitizing, quenching, and tempering at suitable conditions to acquire required hardness. They are often used in tempered condition. As hardenability of these alloys is low, only thin sections can be heat treated using very high quench rates. Ni, Cr and Mo alloying additions improve their

hardenability. Typical applications include: railway tracks & wheels, gears, other machine parts which may require good combination of strength and toughness.

High carbon steels: These are strongest and hardest of carbon steels, and of course their ductility is very limited. These are heat treatable, and mostly used in hardened and tempered conditions. They possess very high wear resistance, and capable of holding sharp edges. Thus these are used for tool application such as knives, razors, hacksaw blades, etc. With addition of alloying element like Cr, V, Mo, W which forms hard carbides by reacting with carbon present, wear resistance of high carbon steels can be improved considerably.

Stainless steels: The name comes from their high resistance to corrosion i.e. they are rust-less (stain-less). Steels are made highly corrosion resistant by addition of special alloying elements, especially a minimum of 12% Cr along with Ni and Mo. Stainless steels are mainly three kinds: ferritic & hardenable Cr steels, austenitic and precipitation hardenable (martensitic, semi- austenitic) steels. This classification is based on prominent constituent of the microstructure. Typical applications include cutlery, razor blades, surgical knives, etc.

Ferritic stainless steels are principally Fe-Cr-C alloys with 12-14% Cr. They also contain small additions of Mo, V, Nb, and Ni.

Austenitic stainless steels usually contain 18% Cr and 8% Ni in addition to other minor alloying elements. Ni stabilizes the austenitic phase assisted by C and N. Other alloying additions include Ti, Nb, Mo (prevent weld decay), Mn and Cu (helps in stabilizing austenite).

By alloying additions, for martensitic steels Ms is made to be above the room temperature. These alloys are heat treatable. Major alloying elements are: Cr, Mn and Mo.

Ferritic and austenitic steels are hardened and strengthened by cold work because they are not heat treatable. On the other hand martensitic steels are heat treatable. Austenitic steels are most corrosion resistant, and they are produced in large quantities. Austenitic steels are non-magnetic as against ferritic and martensitic steels, which are magnetic.

2.7 HEAT TREATMENT

Thermal processing of metals and alloys

Apart from mechanical processing, metals are very often subjected to thermal processing for various reasons, like: to refine grain structure/size, to minimize residual stresses, to

impart phase changes, to develop special phases over external surfaces, etc. Metals and alloys develop requisite properties by thermal processing either through grain refinement of phase changes. Thermal processing is also known as heat treatment. Heat treatment originated as an ancient art in man’s attempts to improve the performance of materials in their practical applications. In present day metallurgical practice, heat treatment has become very important for obvious reasons. There has been tremendous progress over centuries in the systematic understanding of materials structure and structure-property relationships that eliminated the empiricism in thermal processing. Properly designed and implemented thermal processing can result in optimum modifications in the composition and distribution of phases, corresponding changes in physical, chemical and mechanical properties at substantial levels. However, most of the thermal processes are aimed to improving mechanical characteristics of materials. Thus it is possible to extend the service performance of materials considerably within constraints of available resources.

All metals can be subjected to thermal processing. But the effect of it may differ from one metal to another. Metals are subjected to heat treatment for one or more of the following purposes: improvement in ductility; relieving internal stresses; grain size refinement; increase of strength; improvement in machinability, toughness; etc.

Heat treatment of materials involves number of factors – temperature up to which material is heated, length of time that the material is held at the elevated temperature, rate of cooling, and the surrounding atmosphere under the thermal treatment. All these factors depend on material, pre-processing of the material’s chemical composition, size and shape of the object, final properties desired, material’s melting point/liquidus, etc.

Thermal processes may be broadly classified into two categories based on cooling rates from elevated temperatures – annealing and quenching & tempering. Annealing involved cooling the material from elevated temperatures slowly, while quenching means very fast cooling of the material using cooling medium like water/oil bath. Quenching is done to retain the phases of elevated temperatures at room temperature.

Annealing processes

The term annealing was used by craftsmen who discovered the beneficial effects of heating the material at elevated temperatures followed by slow cooling of it to room temperature. Annealing can be defined as a heat treatment process in which the material is taken to a high temperature, kept there for some time and then cooled. High temperatures allow diffusion processes to occur fast. The time at the high temperature (soaking time) must be long enough to allow the desired transformation to occur. Cooling is done slowly to avoid the distortion (warping) of the metal piece, or even cracking, caused by stresses induced by differential contraction due to thermal

inhomogeneities. Benefits of annealing are:

- relieve stresses
- increase softness, ductility and toughness
- produce a specific microstructure

Depending on the specific purpose, annealing is classified into various types: process annealing, stress relief, full annealing and normalizing.

Process annealing is primarily applied to cold worked metals to negate the effects of cold work. During this heat treatment, material becomes soft and thus its ductility will be increased considerably. It is commonly sandwiched between two cold work operations. During this, recovery and recrystallization are allowed whereas grain growth was restricted.

Stress relief operation removes the stresses that might have been generated during plastic deformation, non-uniform cooling, or phase transformation. Unless removed, these stresses may cause distortion of components. Temperature used is normally low such that effects resulting from cold working are not affected.

Full annealing is normally used for products that are to be machined subsequently, such as transmission gear blanks. After heating and keeping at an elevated temperature, components are cooled in furnace to effect very slow cooling rates. Typically, the product receives additional heat treatments after machining to restore hardness and strength.

Normalizing is used to refine the grains and produce a more uniform and desirable size distribution. It involves heating the component to attain single phase (e.g.: austenite in steels), then cooling in open air atmosphere.

Quenching and Tempering processes

Quenching is heat treatment process where material is cooled at a rapid rate from elevated temperature to produce Martensite phase. This process is also known as hardening. Rapid cooling rates are accomplished by immersing the components in a quench bath that usually contains quench media in form of either water or oil, accompanied by stirring mechanism.

Quenching process is almost always followed by tempering heat treatment. Tempering is the process of heating martensitic steel at a temperature below the eutectoid transformation temperature to make it softer and more ductile. During the tempering process, Martensite transforms to a structure containing iron carbide particles in a matrix of ferrite.

Martempering is a modified quenching procedure used to minimize distortion and cracking that may develop during uneven cooling of the heat-treated material. It involves cooling the austenized steel to temperature just above Ms temperature, holding it there until temperature is uniform, followed by cooling at a moderate rate to room temperature before austenite-to-bainite transformation begins. The final structure of martempered steel is tempered Martensite.

Austempering is different from martempering in the sense that it involves austenite- to-bainite transformation. Thus, the structure of austempered steel is bainite. Advantages of austempering are – improved ductility; decreased distortion and disadvantages are – need for special molten bath; process can be applied to limited number of steels.

Case Hardening

In case hardening, the surface of the steel is made hard and wear resistant, but the core remains soft and tough. Such a combination of properties is desired in applications such as gears.

Induction hardening

Here, an alternating current of high frequency passes through an induction coil enclosing the steel part to be heat treated. The induced emf heats the steel. The depth up to which the heat penetrates and raises the temperature above the elevated temperature is inversely proportional to the square root of the ac frequency. In induction hardening, the heating time is usually a few seconds. Immediately after heating, water jets are activated to quench the surface. Martensite is produced at the surface, making it hard and wear resistant. The microstructure of the core remains unaltered. Induction hardening is suitable for mass production of articles of uniform cross-section.

Flame hardening

For large work pieces and complicated cross-sections induction heating is not easy to apply. In such cases, flame hardening is done by means of an oxyacetylene torch. Heating should be done rapidly by the torch and the surface quenched, before appreciable heat transfer to the core occurs

Laser hardening

In this case, a laser beam can be used for surface hardening. As laser beams are of high intensity, a lens is used to reduce the intensity by producing a defocused spot of size ranging from 0.5 to 25 mm. Proper control of energy input is necessary to avoid melting. Laser hardening has the advantage of precise control over the area to be hardened, an ability to harden reentrant surfaces, very high speed of hardening and no separate quenching step. The disadvantage is that the hardening is shallower than in induction and flame hardening

Carburizing

Carburizing is the most widely used method of surface hardening. Here, the surface layers of low carbon steel are enriched with carbon up to 0.8-1.0%. The source of carbon may be a solid medium, a liquid or a gas. In all cases, the carbon enters the steel at the surface and diffuses intothe steel as a function of time at an elevated temperature. Carburizing is done at 920-950C.

This fully austenitic state is essential. If carburizing is done in the ferritic region, the carbon, with very limited solubility in ferrite, tends to form massive cementite particles near the surface, making the subsequent heat treatment difficult. For this reason, carburizing is always done in the austenitic state, even though longer times are required due to the diffusion rate of carbon in austenite being less that in ferrite at such temperatures.

Cyaniding

Cyaniding is done in a liquid bath of NaCN, with the concentration varying between 30 and 97%. The temperature used for cyaniding is lower than that for carburizing and is in the range of 800-870 C. The time of cyaniding is 1-3 hr to produce a case depth of 0.25 mm or less.

Nitriding

Nitriding is carried out in the ferritic region. No phase change occurs after nitriding. The part to be nitrided should posses the required core properties prior to nitriding. During nitriding, pure ammonia decomposes to yield nitrogen which enters the steel. The solubility of nitrogen in ferrite is small. Most of the nitrogen, that enters the steel, forms hard nitrides (e.g., Fe3 N). The temperature of nitriding is 500-590 C. The time for a case depth of 0.02 mm is about 2 hr. In addition to providing outstanding wear resistance, the nitride layer increases the resistance of carbon steel to corrosion in moist atmospheres

Cast irons

Though ferrous alloys with more than 2.14 wt.% C are designated as cast irons, commercially cast irons contain about 3.0-4.5% C along with some alloying additions. Alloys with this carbon content melt at lower temperatures than steels i.e. they are responsive to casting. Hence casting is the most used fabrication technique for these alloys.

Hard and brittle constituent presented in these alloys, cementite is a meta-stable phase, and can readily decompose to form α-ferrite and graphite. In this way disadvantages of brittle phase can easily be overcome. Tendency of cast irons to form graphite is usually controlled by their composition and cooling rate. Based on the form of carbon present, cast irons are categorized as gray, white, nodular and malleable cast irons.

Gray cast iron: These alloys consists carbon in form graphite flakes, which are surrounded by either ferrite or pearlite. Because of presence of graphite, fractured surface of these alloys look grayish, and so is the name for them. Alloying addition of Si (1- 3wt.%) is responsible for decomposition of cementite, and also high fluidity. Thus castings of intricate shapes can be easily made. Due to graphite flakes, gray cast irons are weak and brittle. However they possess good damping properties, and thus typical applications include: base structures, bed for heavy machines, etc. they also show high resistance to wear.

White cast iron: When Si content is low (< 1%) in combination with faster cooling rates, there is no time left for cementite to get decomposed, thus most of the brittle cementite retains. Because of presence of cementite, fractured surface appear white, hence the name. They are very brittle and extremely difficult to machine. Hence their use is limited to wear resistant applications such as rollers in rolling mills. Usually white cast iron is heat treated to produce malleable iron.

Nodular (or ductile) cast iron: Alloying additions are of prime importance in producing these materials. Small additions of Mg / Ce to the gray cast iron melt before casting can result in graphite to form nodules or sphere-like particles. Matrix surrounding these particles can be either ferrite or pearlite depending on the heat treatment. These are stronger and ductile than gray cast irons. Typical applications include: pump bodies, crank shafts, automotive components, etc.

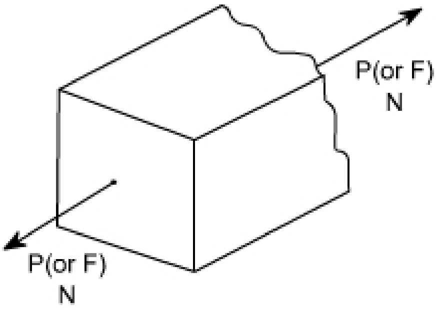
Malleable cast iron: These formed after heat treating white cast iron. Heat treatments involve heating the material up to 800-900 C, and keep it for long hours, before cooling it to room temperature. High temperature incubation causes cementite to decompose and form ferrite and graphite. Thus these materials are stronger with appreciable amount of ductility. Typical applications include: railroad, connecting rods, marine and other heavy- duty services.

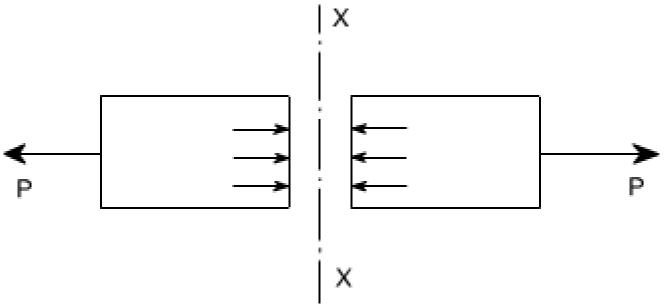
MODULE- III

SIMPLE STRESSES AND STRAINS, PRINCIPAL STRESSES

3.1 Stress

Stress is the internal resistance offered by the body to the external load applied to it per unit cross sectional area. Stresses are normal to the plane to which they act and are tensile or compressive in nature.





As we know that in mechanics of deformable solids, externally applied forces acts on a body and body suffers a deformation. From equilibrium point of view, this action should be opposed or reacted by internal forces which are set up within the particles of material due to cohesion. These internal forces give rise to a concept of stress. Consider a rectangular rod subjected to axial pull P. Let us imagine that the same rectangular bar is assumed to be cut into two halves at section XX. The each portion of this rectangular bar is in equilibrium under the action of load P and the internal forces acting at the section XX has been shown. Now stress is defined as the force intensity or force per unit area. Where A is the area of the $X-X$ section

Here we are using an assumption that the total force or total load carried by the rectangular bar is uniformly distributed over its cross – section. But the stress distributions may be for from uniform, with local regions of high stress known as stress concentrations. If the force carried by a component is not uniformly distributed over its cross – sectional area, A, we must consider a small area, ‘δA’ which carries a small load ‘δP’, of the total force ‘P', Then definition of stress is

As a particular stress generally holds true only at a point, therefore it is defined mathematically as

$$\boxed{\sigma = \lim_{\delta A \to 0} \frac{\delta F}{\delta A}}$$

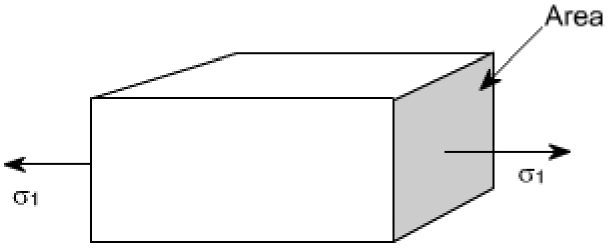
Units :

The basic units of stress in S.I units i.e. (International system) are N / m^2 (or Pa) MPa = 10^6 Pa

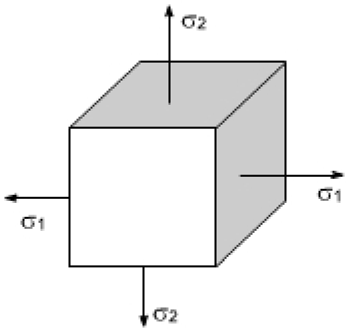
Sometimes N / mm^2 units are also used, because this is an equivalent to MPa. While US customary unit is pound per square inch psi.

3.2 TYPES OF STRESSES : Only two basic stresses exists : (1) normal stress and (2) shear stress. Other stresses either are similar to these basic stresses or are a combination of this e.g. bending stress is a combination tensile, compressive and shear stresses. Torsional stress, as encountered in twisting of a shaft is a shearing stress. Let us define the normal stresses and shear stresses in the following sections.

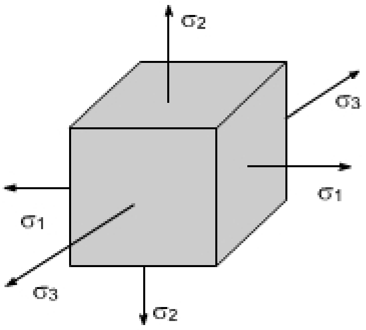
Normal stresses : We have defined stress as force per unit area. If the stresses are normal to the areas concerned, then these are termed as normal stresses. The normal stresses are generally denoted by a Greek letter (σ)



This is also known as uniaxial state of stress, because the stresses acts only in one direction however, such a state rarely exists, therefore we have biaxial and triaxial state of stresses where either the two mutually perpendicular normal stresses acts or three mutually perpendicular normal stresses acts as shown in the figures below :



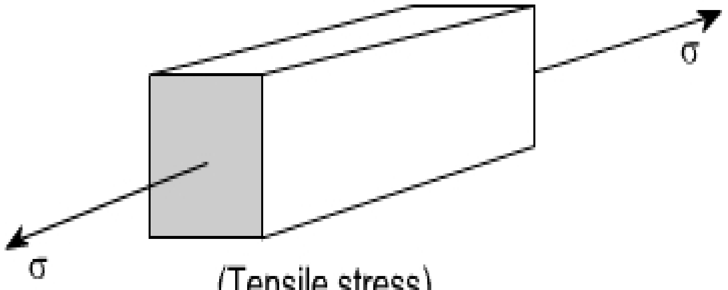
(Biaxial state of stress)



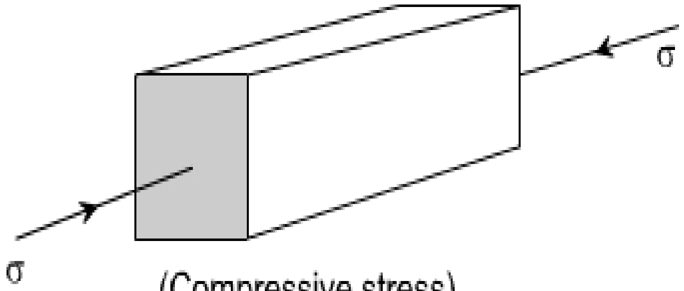
(Triaxial state of stress)

Tensile or compressive Stresses:

The normal stresses can be either tensile or compressive whether the stresses acts out of the area or into the area

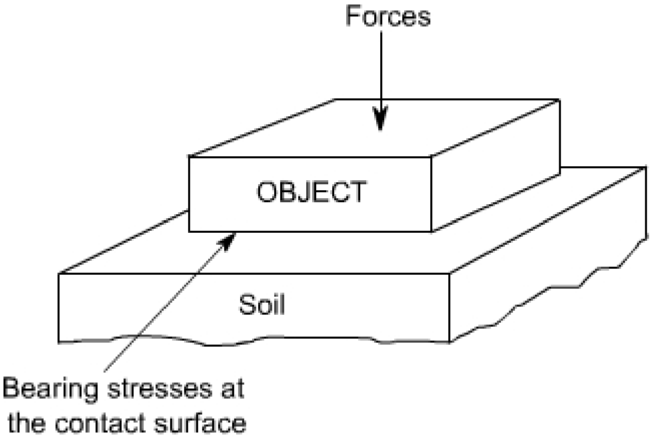


(Tensile stress)



(Compressive stress)

Bearing Stress: When one object presses against another, it is referred to a bearing stress (They are in fact the compressive stresses).



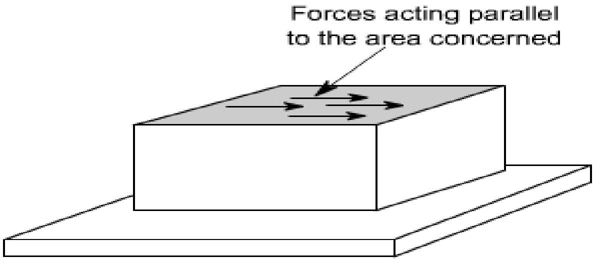
<u>Sign convections for Normal stress</u>

Direct stresses or normal stresses

- tensile +ve
- compressive –ve

<u>Shear Stresses:</u>

Let us consider now the situation, where the cross – sectional area of a block of material is subject to a distribution of forces which are parallel, rather than normal, to the area concerned. Such forces are associated with a shearing of the material, and are referred to as shear forces. The resulting stress is known as shear stress.



The resulting force intensities are known as shear stresses, the mean shear stress being equal to

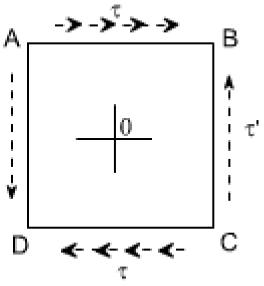
$$\tau = \frac{P}{A}$$

Where P is the total force and A the area over which it acts. As we know that the particular stress generally holds good only at a point therefore we can define shear stress at a point as

$$\tau = \lim_{\delta A \to 0} \frac{\delta F}{\delta A}$$

3.3 Complementary shear stresses:

The existence of shear stresses on any two sides of the element induces complementary shear stresses on the other two sides of the element to maintain equilibrium.



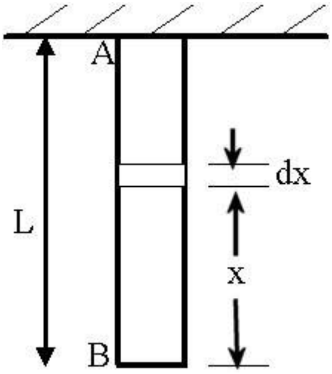
Sign convections for shear stresses:

- tending to turn the element C.W +ve.

- tending to turn the element C.C.W – ve.

Deformation of a Body due to Self Weight

Consider a bar AB hanging freely under its own weight as shown in the figure.



Let

L= length of the bar

A= cross-sectional area of the bar

E= Young's modulus of the bar material w= specific weight of the bar material

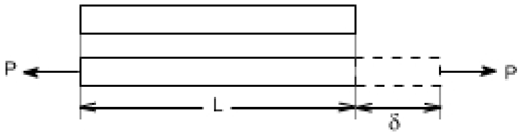
$$\boxed{\delta L = \frac{WL}{2E}}$$

Then deformation due to the self-weight of the bar is

Members in Uni – axial state of stress

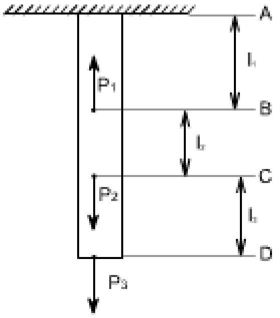
Introduction: [For members subjected to uniaxial state of stress]

For a prismatic bar loaded in tension by an axial force P, the elongation of the bar can be determined as



$$\delta = \frac{PL}{AE} \qquad \text{——— (1)}$$

Suppose the bar is loaded at one or more intermediate positions, then equation (1) can be readily adapted to handle this situation, i.e. we can determine the axial force in each part of the bar i.e. parts AB, BC, CD, and calculate the elongation or shortening of each part separately, finally, these changes in lengths can be added algebraically to obtain the total charge in length of the entire bar.



When either the axial force or the cross – sectional area varies continuosly along the axis of the bar, then equation (1) is no longer suitable. Instead, the elongation can be found by considering a deferential element of a bar and then the equation (1) becomes

$$d\delta = \frac{P_x \, dx}{E.A_x}$$

$$\delta = \int_0^l \frac{P_x \, dx}{E.A_x}$$

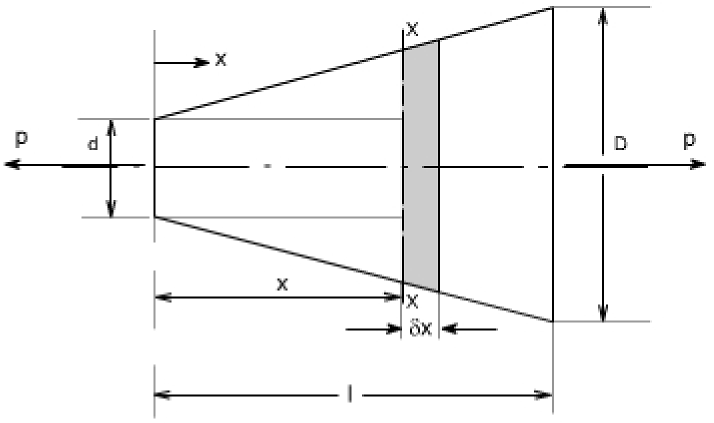
i.e. the axial force $P_x$and area of the cross – section A_x must be expressed as functions of x. If the expressions for $P_x$and A_x are not too complicated, the integral can be evaluated analytically, otherwise Numerical methods or techniques can be used to evaluate these integrals.

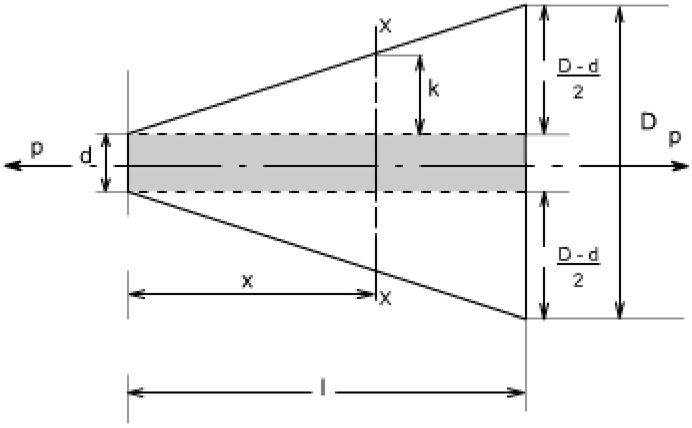
3.4 Principle of Superposition

The principle of superposition states that when there are numbers of loads are acting together on an elastic material, the resultant strain will be the sum of individual strains caused by each load acting separately.

Numerical Problems on stress, shear stress in axially loaded members.

Example 1: Now let us for example take a case when the bar tapers uniformly from d at x = 0 to D at x = l





In order to compute the value of diameter of a bar at a chosen location let us determine the value of dimension k, from similar triangles

$$\frac{(D - d)/2}{l} = \frac{k}{x}$$

$$\text{Thus, } k = \frac{(D - d)x}{2l}$$

therefore, the diameter 'y' at the

X-section is or = d + 2k

$$y = d + \frac{(D - d)x}{l}$$

Hence the cross –section area at section X- X will be

$$A_x \text{ or } a = \frac{\pi}{4} y^2$$

$$= \frac{\pi}{4}\left[d + (D - d)\frac{x}{l}\right]^2$$

hence the total extension of the bar will be given by expression

$$= \frac{P}{E}\int_0^l \frac{\delta x}{a}$$

subsitituting the value of 'a 'to get the total extention of the bar

$$= \frac{\pi P}{4E}\int_0^l \frac{\delta x}{\left[d + (D - d)\frac{x}{l}\right]^2}$$

after carrying out the intergration we get

$$= -\frac{4.P.l}{\pi E}\left[\frac{1}{D} - \frac{1}{d}\right]$$

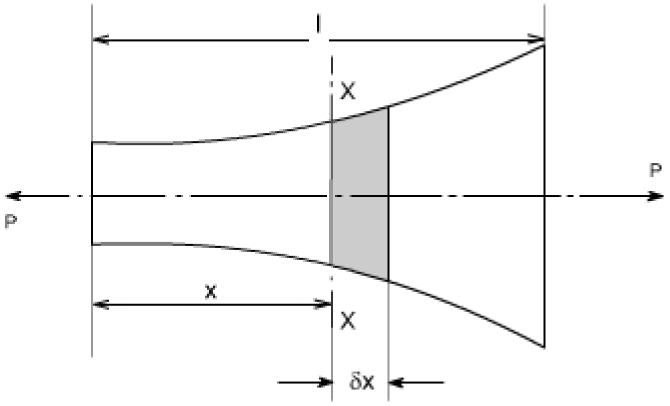
$$= \frac{4.P.l}{\pi E D.d}$$

hence the total strain in the bar = $\boldsymbol{\frac{4.P.l}{\pi E\ D.d}}$

An interesting problem is to determine the shape of a bar which would have a uniform stress in it under the action of its own weight and a load P.

Example 2: stresses in Non – Uniform bars

Consider a bar of varying cross section subjected to a tensile force P as shown below.



Let

a = cross sectional area of the bar at a chosen section XX then

Stress < = p / a

If E = Young's modulus of bar then the strain at the section XX can be calculated

< = < / E

Then the extension of the short element < x. =<< .original length = < / E. <^{x}

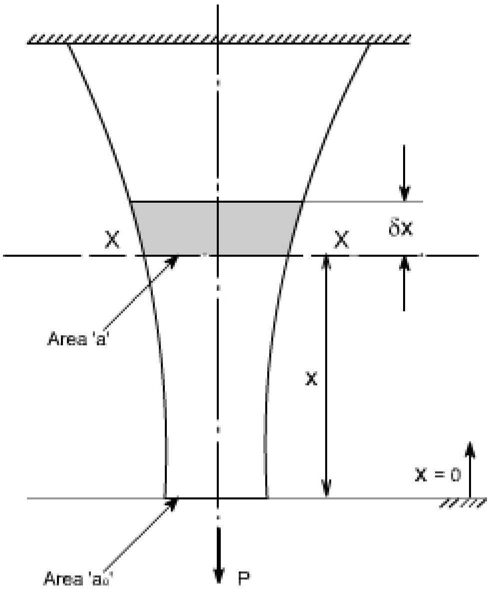
$$= \frac{P}{E.}\frac{\delta x}{a}$$

Thus, the extension for the entire bar is

$$\delta = \int_0^l \frac{P}{E}\frac{\delta x}{a}$$

$$\text{or total extension} = \frac{P}{E}\int_0^l \frac{\delta x}{a}$$

let us consider such a bar as shown in the figure below:



The weight of the bar being supported under section XX is

$$= \int_0^x \rho\, g\, a\, dx$$

where ρ is density of the bar.

thus the stress at XX is

$$\sigma = \frac{P + \int_0^x \rho\, g\, a\, dx}{a}$$

$$\text{or } \sigma.a = P + \int_0^x \rho.g.a\, dx$$

Differentiating the above equation with respect to x we get

$$\sigma.\frac{da}{dx} = \rho.g.a$$

$$\frac{da}{a} = \frac{\rho.g}{\sigma}.dx$$

intergrating the above equation we get

$$\int \frac{da}{a} = \int \frac{\rho.g}{\sigma} dx$$

$$\log_e{}^a = \frac{\rho.g.x}{\sigma} + \text{constant}$$

In order to determine the constant of integration

let us apply the boundary conditions

at $x = 0;\ a = a_0$

thus, constant $= \log_e{}^{a_0}$

or

$$\log_e{}^a = \frac{\rho.g.x}{\sigma} + \log_e{}^{a_0}$$

$$\log_e\left(\frac{a}{a_0}\right) = \frac{\rho.g.x}{\sigma}$$

$$\text{or } \boxed{e^{\frac{\rho.g.x}{\sigma}} = \frac{a}{a_0}}$$

also at $x = 0$

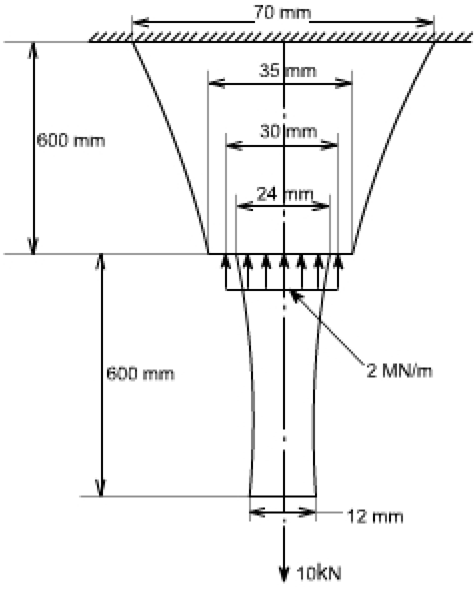
$$\sigma = \frac{P}{a_0}$$

Thus,

$$\frac{a}{a_0} = e^{\frac{\rho.g.x\, a_0}{P}}$$

Example 1: Calculate the overall change in length of the tapered rod as shown in figure below. It carries a tensile load of 10kN at the free end and at the step change in section a compressive load of 2 MN/m evenly distributed around a circle of 30 mm diameter take the value of E = 208 GN / m^2.

This problem may be solved using the procedure as discussed earlier in this section



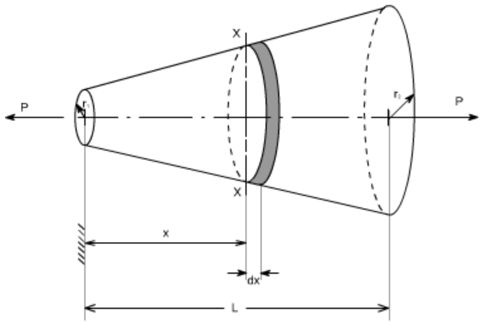
Example 2: A round bar, of length L, tapers uniformly from radius r_1 at ne end to radius $r_2$at the other. Show that the extension produced by a tensile axial load P

$$\frac{PL}{2\pi E r_1^2}$$

is

If $r_2 = 2r_1$, compare this extension with that of a uniform cylindrical bar having a radius equal to the mean radius of the tapered bar.

Solution:



consider the above figure let r$_1$ be the radius at the smaller end. Then at a X crosssection XX located at a distance x from the smaller end, the value of radius is equal to

$$= r_1 + \frac{x}{L}(r_2 - r_1)$$

$$= r_1(1 + kx)$$

$$\text{where } k = \left(\frac{r_2 - r_1}{L}\right).\frac{1}{r_1}$$

$$\text{stress at section XX} = \frac{\text{load}}{\text{area}}$$

$$= \frac{P}{\pi r_1^2 (1 + kx)^2}$$

$$\text{hence strain at this section} = \frac{\text{stress}}{E}$$

$$= \frac{P}{E.\pi r_1^2 (1 + kx)^2}$$

Thus, for a small length dx of the bar at this section the extention is $\dfrac{P.dx}{E\pi r_1^2 (1 + kx)^2}$

Total extension of the bar can be found by integrating the above expression within the limits from x=0 to x=L

$$\text{Extension} = \int_0^L \frac{P.dx}{E.\pi r_1^2 (1 + k.x)^2}$$

$$= \frac{P}{E.\pi r_1^2} \int_0^L (1 + k.x)^{-2} dx$$

$$= \frac{P}{E.\pi r_1^2} \left[\frac{(1+kx)^{-1}}{-k}\right]_0^L$$

$$= \frac{P}{E.\pi r_1^2} \left[\frac{(1+kx)^{-1}}{-k} - \frac{1}{-k}\right]$$

$$= \frac{P}{E.\pi r_1^2 .k} \left[1 - \frac{1}{1 + kL}\right]$$

$$= \frac{PL}{E.\pi r_1^2 (1 + kL)}$$

$$\text{since } k = \frac{(r_2 - r_1)}{r_1 L}$$

$$\text{Thus, } 1 + kL = {}^{r_2}\!/_{r_1}$$

$$\text{Therefore, the extension} = \frac{PL}{\pi E r_1 r_2}$$

Comparing of extensions

For the case when $r_2 = 2.r_1$, the value of computed extension as above

$$\frac{PL}{2\pi E r_1^2}$$

becomes equal to

The mean radius of taper bar

$= 1 / 2(r_1 + r_2)$

$= 1 / 2(r_1 +2\ r_2)$

$= 3 / 2\ .r_1$

Therefore, the extension of uniform bar

= Orginal length . strain

$$= L.\frac{\sigma}{E}$$

$$= \frac{L}{E}.\frac{P}{\pi\left(3/2\, r_1\right)^2}$$

$$= \frac{4PL}{g\pi E\pi r_1^2}$$

hence the

$$\frac{\text{Extension of uniform}}{\text{Extension of tapered}} = \left(\frac{4PL}{g\pi E\pi r_1^2}\right) \Big/ \frac{PL}{2\pi E r_1^2}$$

$$= \frac{8}{g}$$

3.5 Strain:

When a single force or a system force acts on a body, it undergoes some deformation. This deformation per unit length is known as strain. Mathematically strain may be defined as deformation per unit length.

So,

Strain=Elongation/Original length

Elasticity;

The property of material by virtue of which it returns to its original shape and size upon removal of load is known as elasticity.

Hooks Law

It states that within elastic limit stress is proportional to strain.

Where E = Young’s Modulus

Hooks law holds good equally for tension and compression.

Poisson’s Ratio;

The ratio lateral strain to longitudinal strain produced by a single stress is nown as Poisson’s ratio. Symbol used for poisson’s ratio is *nu* or 1/ *m* .

Modulus of Elasticity (or Young’s Modulus)

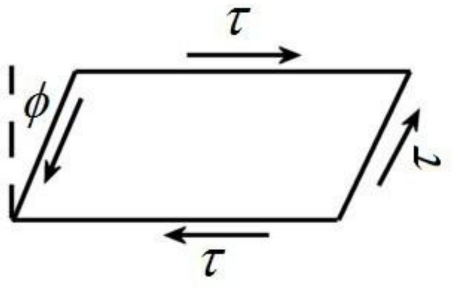
Young’s modulus is defined as the ratio of stress to strain within elastic limit.

Deformation of a body due to load acting on it

We know that young’s modulus E= $\frac{Stress}{Strain}$, So, deformation

Shear Strain

The distortion produced by shear stress on an element or rectangular block is shown in the figure. The shear strain or 'slide' is expressed by angle ϕ and it can be defined as the change in the right angle. It is measured in radians and is dimensionless in nature.



Modulus of Rigidity

For elastic materials it is found that shear stress is proportional to the shear strain within elastic limit. The ratio is called modulus rigidity. It is denoted by the symbol 'G' or 'C'.

Bulk modulus (K): It is defined as the ratio of uniform stress intensity to the volumetric strain. It is denoted by the symbol K.

Relation between elastic constants:

Elastic constants: These are the relations which determine the deformations produced by a given stress system acting on a particular material. These factors are constant within elastic limit, and known as modulus of elasticity E, modulus of rigidity G, Bulk modulus K and Poisson's ratio μ.

Relationship between modulus of elasticity (E) and bulk modulus (K):

$$\boxed{E = 3K(1-2\mu)}$$

Relationship between modulus of elasticity (E) and modulus of rigidity (G):

$$\boxed{E = 2G(1+\mu)}$$

Relation among three elastic constants:

$$\boxed{E = \frac{9KG}{G+3K}}$$

.

3.6 Stress – Strain Relationship

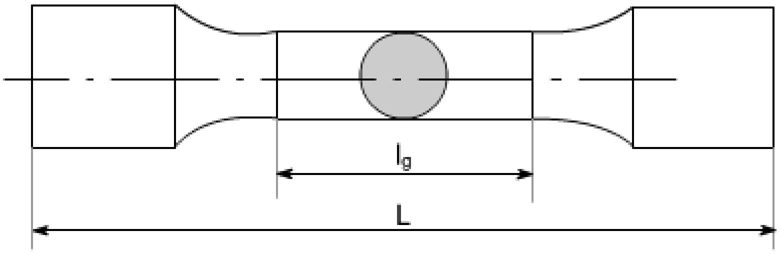
Stress – strain diagram for mild steel

Standard specimen are used for the tension test.

There are two types of standard specimen's which are generally used for this purpose, which have been shown below:

Specimen I:

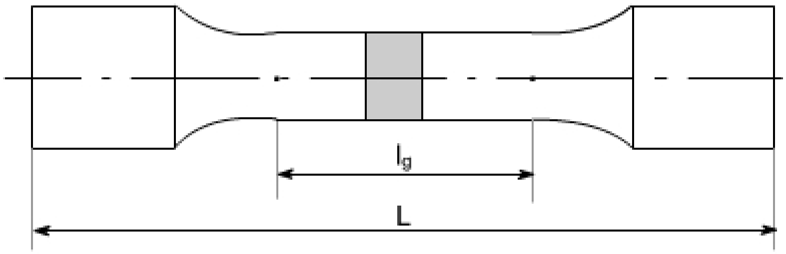
This specimen utilizes a circular X-section.



[specimen with circular X-section]

Specimen II:

This specimen utilizes a rectangular X-section.



[specimen with rectangular X-section]

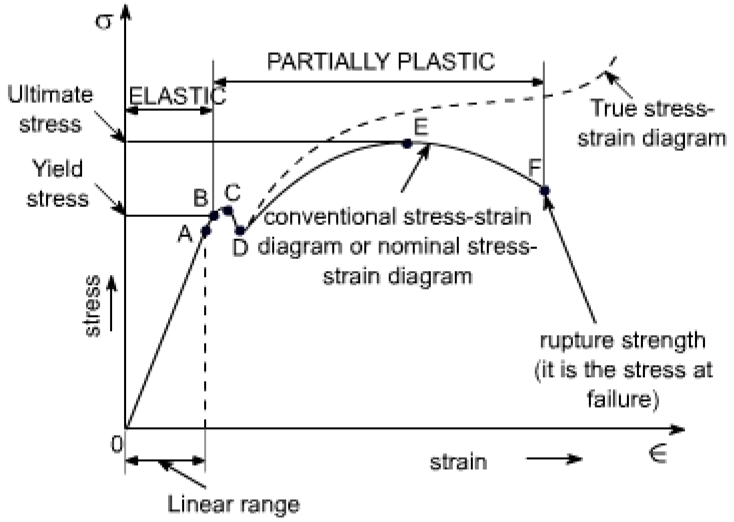
l_g = gauge length i.e. length of the specimen on which we want to determine the mechanical properties.The uniaxial tension test is carried out on tensile testing machine and the following steps are performed to conduct this test.

(i) The ends of the specimen are secured in the grips of the testing machine.

(ii) There is a unit for applying a load to the specimen with a hydraulic or mechanical drive.

(iii) There must be some recording device by which you should be able to measure the final output in the form of Load or stress. So the testing machines are often equipped with the pendulum type lever, pressure gauge and hydraulic capsule and the stress Vs strain diagram is plotted which has the following shape.

A typical tensile test curve for the mild steel has been shown below



<u>**SALIENT POINTS OF THE GRAPH:**</u>

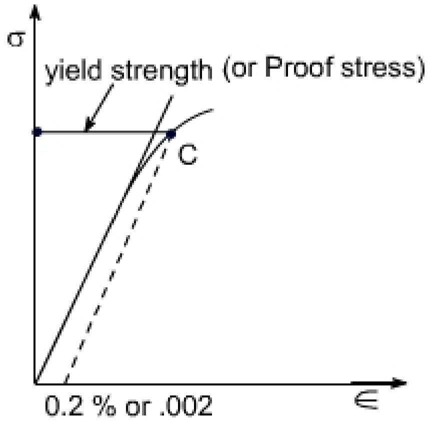
(A) So it is evident form the graph that the strain is proportional to strain or elongation is proportional to the load giving a st.line relationship. This law of proportionality is valid upto a point A.
or we can say that point A is some ultimate point when the linear nature of the graph ceases or there is a deviation from the linear nature. This point is known as **the limit of proportionality or the proportionality limit**.

(B) For a short period beyond the point A, the material may still be elastic in the sense that the deformations are completely recovered when the load is removed. The limiting point B is termed as **Elastic Limit** .

(C) and (D) - Beyond the elastic limit plastic deformation occurs and strains are not totally recoverable. There will be thus permanent deformation or permanent set when load is removed. These two points are termed as upper and lower yield points respectively. The stress at the yield point is called the yield strength.

A study a stress – strain diagrams shows that the yield point is so near the proportional limit that for most purpose the two may be taken as one. However, it is much easier to locate the former. For material which do not posses a well define yield points, In order to find the yield point or yield strength, an offset method is applied.

In this method a line is drawn parallel to the straight line portion of initial stress diagram by off setting this by an amount equal to 0.2% of the strain as shown as below and this happens especially for the low carbon steel.



(E) A further increase in the load will cause marked deformation in the whole volume of the metal. The maximum load which the specimen can with stand without failure is called the load at the ultimate strength.

The highest point ‘E' of the diagram corresponds to the ultimate strength of a material.

s_u = Stress which the specimen can with stand without failure & is known as Ultimate Strength or Tensile Strength.

s_u is equal to load at E divided by the original cross-sectional area of the bar.

(F) Beyond point E, the bar begins to forms neck. The load falling from the maximum until fracture occurs at F. Beyond point E, the cross-sectional area of the specimen begins to reduce rapidly over a relatively small length of bar and the bar is said to form a neck. This necking takes place whilst the load reduces, and fracture of the bar finally occurs at point F.

Nominal stress – Strain OR Conventional Stress – Strain diagrams:

Stresses are usually computed on the basis of the original area of the specimen; such stresses are often referred to as conventional or nominal stresses.

True stress – Strain Diagram:

Since when a material is subjected to a uniaxial load, some contraction or expansion always takes place. Thus, dividing the applied force by the corresponding actual area of the specimen at the same instant gives the so called true stress.

Percentage Elongation: 'd ':

The ductility of a material in tension can be characterized by its elongation and by the reduction in area at the cross section where fracture occurs.

It is the ratio of the extension in length of the specimen after fracture to its initial gauge length, expressed in percentage.

$$\delta = \frac{(l_1 - l_g)}{l_1} \times 100$$

l_I = gauge length of specimen after fracture(or the distance between the gage marks at fracture)

l_g= gauge length before fracture(i.e. initial gauge length)

For 50 mm gage length, steel may here a % elongation d of the order of 10% to 40%.

Ductile and Brittle Materials:

Based on this behaviour, the materials may be classified as ductile or brittle materials

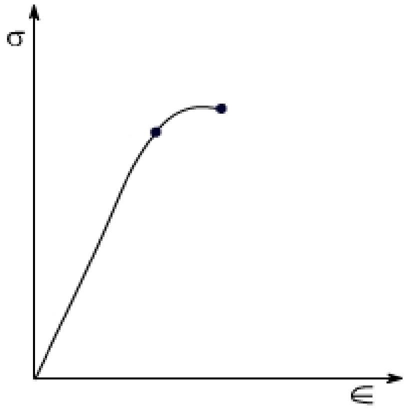
Ductile Materials:

It we just examine the earlier tension curve one can notice that the extension of the materials over the plastic range is considerably in excess of that associated with elastic loading. The Capacity of materials to allow these large deformations or large extensions without failure is termed as ductility. The materials with high ductility are termed as ductile materials.

Brittle Materials:

A brittle material is one which exhibits a relatively small extensions or deformations to fracture, so that the partially plastic region of the tensile test graph is much reduced.

This type of graph is shown by the cast iron or steels with high carbon contents or concrete.



Mechanical Properties of material:

Elasticity: Property of material by virtue of which it can regain its shape after removal of external load

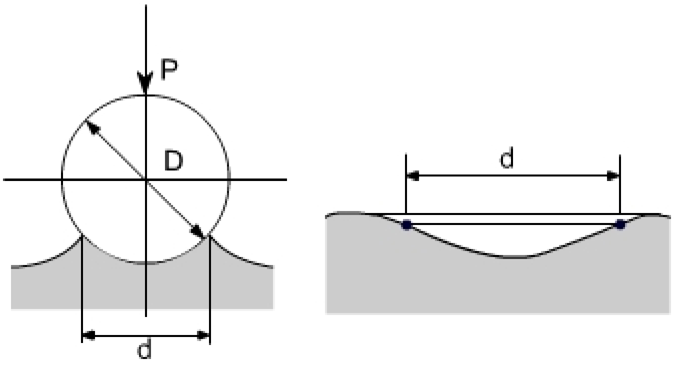
Plasticity: Property of material by virtue of which, it will be in a state of permanent deformation even after removal of external load.

Ductility: Property of material by virtue of which, the material can be drawn into wires.

Hardness: Property of material by virtue of which the material will offer resistance to penetration or indentation.

Ball indentation Tests:

iThis method consists in pressing a hardened steel ball under a constant load P into a specially prepared flat surface on the test specimen as indicated in the figures below :



After removing the load an indentation remains on the surface of the test specimen. If area of the spherical surface in the indentation is denoted as F sq. mm. Brinell Hardness number is defined as :

BHN = P / F

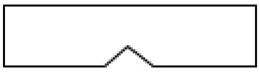
F is expressed in terms of D and d D = ball diameter

d = diametric of indentation and Brinell Hardness number is given by

Then is there is also **Vicker's Hardness Number** in which the ball is of conical shape.

IMPACT STRENGTH

Static tension tests of the unnotched specimen's do not always reveal the susceptibility of metal to brittle fracture. This important factor is determined in impact tests. In impact tests we use the notched specimen's



this specimen is placed on its supports on anvil so that blow of the striker is opposite to the notch the impact strength is defined as the energy A, required to rupture the specimen,

Impact Strength = A / f

Where f = It is the cross – section area of the specimen in cm^2 at fracture & obviously at notch.

The impact strength is a complex characteristic which takes into account both toughness and strength of a material. The main purpose of notched – bar tests is to study the simultaneous effect of stress concentration and high velocity load application

Impact test are of the severest type and facilitate brittle friction. Impact strength values can not be as yet be used for design calculations but these tests as rule provided for in specifications for carbon & alloy steels.Futher, it may be noted that in impact tests fracture may be either brittle or ductile. In the case of brittle fracture, fracture occurs by separation and is not accompanied by noticeable plastic deformation as occurs in the case of ductile fracture.

Impact loads:

Considering a weight falling from a height h, on to a collar attached at the end as shown in the figure.

Let P= equivalent static or gradually applied load which will produce the same extension x as that of the impact load W

Neglecting loss of energy due to impact, we can have:

Loss of potential energy= gain of strain energy of the bar

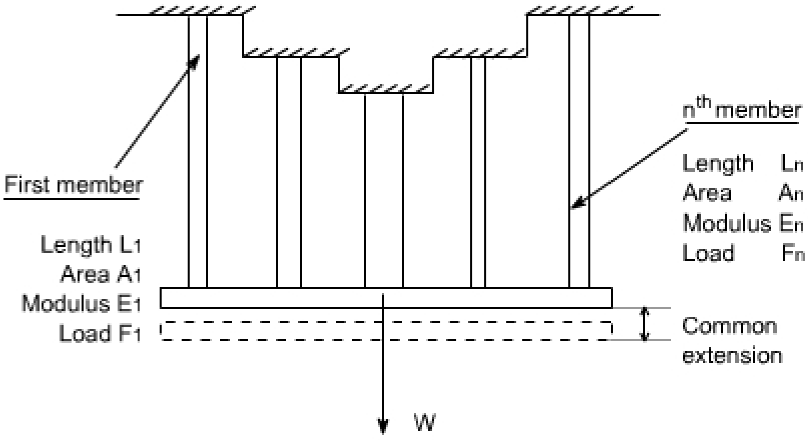
Important Case: for a particular case i.e. for h=0, for a suddenly applied load P=2W,

i.e. the stress produced by a suddenly applied load is twice that of the static stress.

Thermal stresses, Bars subjected to tension and Compression

Compound bar: In certain application it is necessary to use a combination of elements or bars made from different materials, each material performing a different function. In over head electric cables or Transmission Lines for example it is often convenient to carry the current in a set of copper wires surrounding steel wires. The later being designed to support the weight of the cable over large spans. Such a combination of materials is generally termed compound bars.

Consider therefore, a compound bar consisting of n members, each having a different length and cross sectional area and each being of a different material. Let all member have a common extension ‘x' i.e. the load is positioned to produce the same extension in each member.



For the 'n' the members

$$\frac{\text{stress}}{\text{strain}} = E_n = \frac{F_n / A_n}{x_n / L_n}$$

$$= \frac{F_n . L_n}{A_n . x_n}$$

or $$F_n = \frac{E_n . A_n . x_n}{L_n} = \frac{E_n . A_n . x}{L_n} \quad .....(1)$$

Where F_n is the force in the nth member and A_n and L_n are its cross - sectional area and length.

Let W be the total load, the total load carried will be the sum of all loads for all the members.

$$W = \sum \frac{E_n . A_n . x}{L_n}$$

$$= x . \sum \frac{E_n . A_n}{L_n} \qquad \ldots\ldots(2)$$

From equation (1), force in member 1 is given as

$$F_1 = \frac{E_1 . A_1 . x}{L_1}$$

from equation (2)

$$x = \frac{W}{\sum \frac{E_n . A_n}{L_n}}$$

$$\text{Thus, } F_1 = \frac{E_1 . A_1}{L_1} . \frac{W}{\sum \left(\frac{E_n . A_n}{L_n} \right)}$$

Therefore, each member carries a portion of the total load W proportional of EA / L value.

$$F_1 = \frac{\frac{E_1 . A_1}{L_1}}{\sum \frac{E_n . A_n}{L_n}} . W$$

The above expression may be writen as

if the length of each individual member in same then, we may write

$$F_1 = \frac{E_1 . A_1}{\sum E . A} . W$$

Thus, the stress in member '1' may be determined as $< _1 = F_1 / A_1$

Determination of common extension of compound bars: In order to determine the common extension of a compound bar it is convenient to consider it as a single bar of an imaginary material with an equivalent or combined modulus E_c.

Assumption: Here it is necessary to assume that both the extension and original lengths of the individual members of the compound bar are the same, the strains in all members will than be equal.

Total load on compound bar = $F_1 + F_2 + F_3 + \ldots\ldots\ldots + F_n$ where F_1 , F_2 ,….,etc

Dividing throughout by the common strain<< .

$$\frac{\sigma}{\in}(A_1 + A_2 + ... + A_n) = \frac{\sigma_1}{\in}A_1 + \frac{\sigma_2}{\in}A_2 + ... \frac{\sigma_n}{\in}A_n$$

$$\text{i.e } E_c(A_1 + A_2 + ... + A_n) = E_1A_1 + E_2A_2 + ...E_n.A_n$$

$$\text{or } E_c = \frac{E_1A_1 + E_2A_2 + ...E_n.A_n}{A_1 + A_2 + ... + A_n}$$

$$\text{or } E_c = \frac{\sum EA}{\sum A}$$

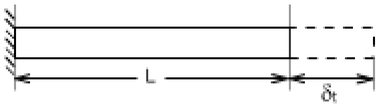
with an external load W applied stress in the equivalent bar may be computed as

$$\text{stress} = \frac{W}{\sum A}$$

$$\text{strain inthe equivalent bar} = \frac{x}{L} = \frac{W}{\sum AE_c}$$

$$\text{hence commen extension } x = \frac{W.L}{E_c\sum A}$$

Compound bars subjected to Temp. Change : Ordinary materials expand when heated and contract when cooled, hence , an increase in temperature produce a positive thermal strain. Thermal strains usually are reversible in a sense that the member returns to its original shape when the temperature return to its original value. However, there here are some materials which do not behave in this manner. These metals differs from ordinary materials in a sence that the strains are related non linearly to temperature and some times are irreversible .when a material is subjected to a change in temp. is a length will change by an amount.



□ = coefficient of linear expansion for the material L = original Length

t = temp. change

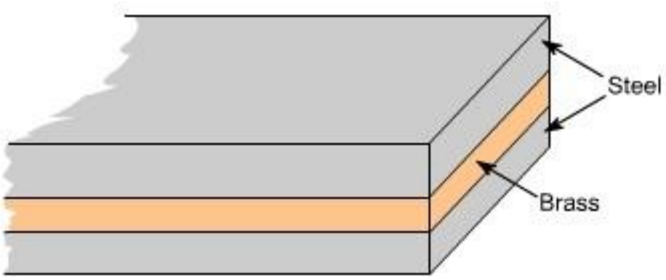
Thus an increase in temperature produces an increase in length and a decrease in temperature results in a decrease in length except in very special cases of materials with zero or negative coefficients of expansion which need not to be considered here.

If however, the free expansion of the material is prevented by some external force, then a stress is set up in the material. They stress is equal in magnitude to that

which would be produced in the bar by initially allowing the bar to its free length and then applying sufficient force to return the bar to its original length.

Consider now a compound bar constructed from two different materials rigidly joined together, for simplicity.

Let us consider that the materials in this case are steel and brass.



If we have both applied stresses and a temp. change, thermal strains may be added to those given by generalized hook's law equation –e.g.

$$\in_x = \frac{1}{E}\left[\sigma_x - \gamma(\sigma_y + \sigma_z)\right] + \alpha\Delta t$$

$$\in_x = \frac{1}{E}\left[\sigma_y - \gamma(\sigma_x + \sigma_z)\right] + \alpha\Delta t$$

$$\in_x = \frac{1}{E}\left[\sigma_z - \gamma(\sigma_x + \sigma_y)\right] + \alpha\Delta t$$

While the normal strains a body are affected by changes in temperatures, shear strains are not. Because if the temp. of any block or element changes, then its size changes not its shape therefore shear strains do not change.

MODULE-IV

SHEAR FORCE AND BENDING MOMENT DIAGRAMS, FLEXURAL STRESSES, SHEAR STRESSES

4.1 Concept of Shear Force and Bending moment in beams:

When the beam is loaded in some arbitrarily manner, the internal forces and moments are developed and the terms shear force and bending moments come into pictures which are helpful to analyze the beams further. Let us define these terms

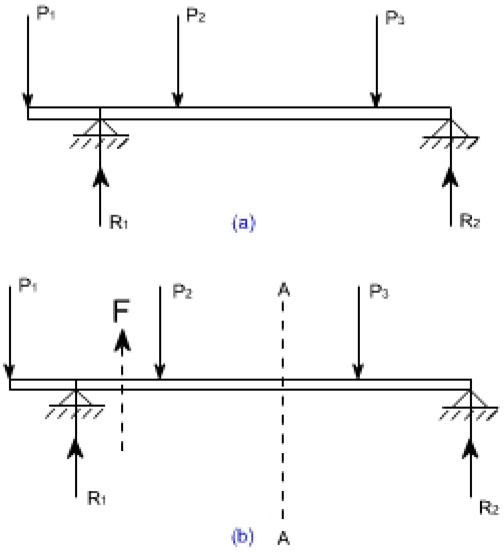


Fig 1

Now let us consider the beam as shown in fig 1(a) which is supporting the loads P_1, P_2, P_3 and is simply supported at two points creating the reactions R_1 and R_2 respectively. Now let us assume that the beam is to divided into or imagined to be cut into two portions at a section AA. Now let us assume that the resultant of loads and reactions to the left of AA is ‘F' vertically upwards, and since the entire beam is to remain in equilibrium, thus the resultant of forces to the right of AA must also be F, acting downwards. This forces ‘F' is as a shear force. The shearing force at any x- section of a beam represents the tendency for the portion of the beam to one side of the section to slide or shear laterally relative to the other portion.

Therefore, now we are in a position to define the shear force ‘F' to as follows:

At any x-section of a beam, the shear force ‘F' is the algebraic sum of all the lateral components of the forces acting on either side of the x-section.

Sign Convention for Shear Force:

The usual sign conventions to be followed for the shear forces have been illustrated in figures 2 and 3.

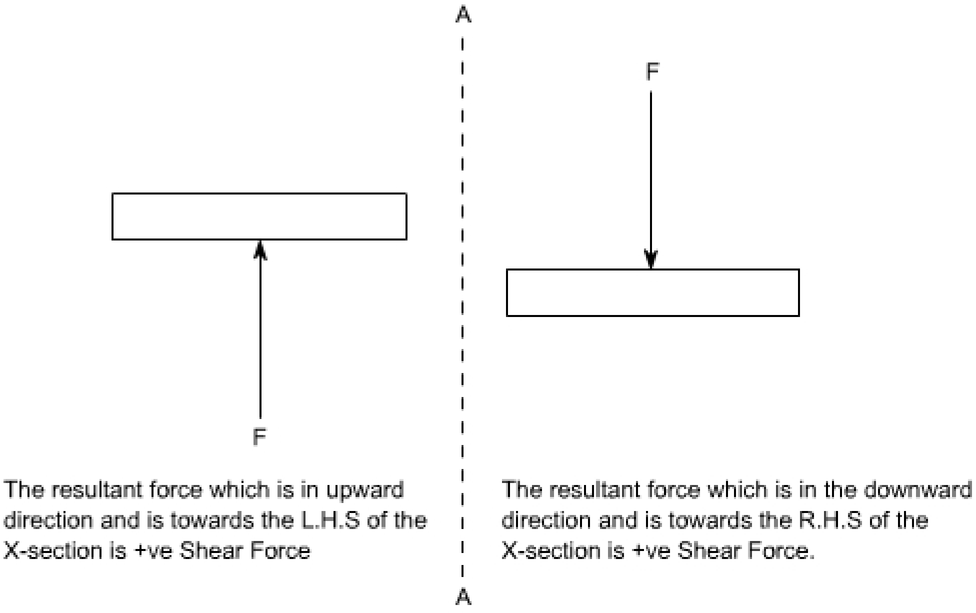


Fig 2: Positive Shear Force

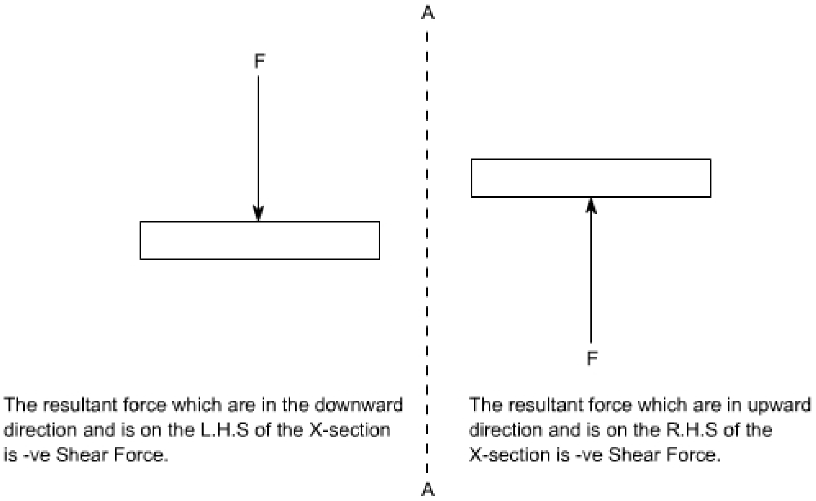


Fig 3: Negative Shear Force

Bending Moment:

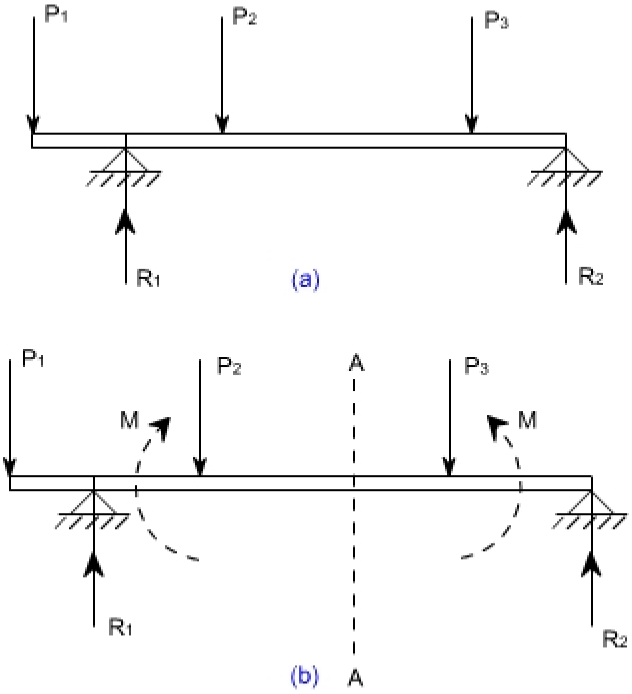


Fig 4

Let us again consider the beam which is simply supported at the two prints, carrying loads P_1, P_2 and P_3 and having the reactions R_1 and R_2 at the supports Fig 4. Now, let us imagine that the beam is cut into two potions at the x-section AA. In a similar manner, as done for the case of shear force, if we say that the resultant moment about the section AA of all the loads and reactions to the left of the x-section at AA is M in C.W direction, then moment of forces to the right of x-section AA must be ‘M' in
C.C.W. Then ‘M' is called as the Bending moment and is abbreviated as B.M. Now one can define the bending moment to be simply as the algebraic sum of the moments about an x-section of all the forces acting on either side of the section

Sign Conventions for the Bending Moment:

For the bending moment, following sign conventions may be adopted as indicated in Fig 5 and Fig 6.

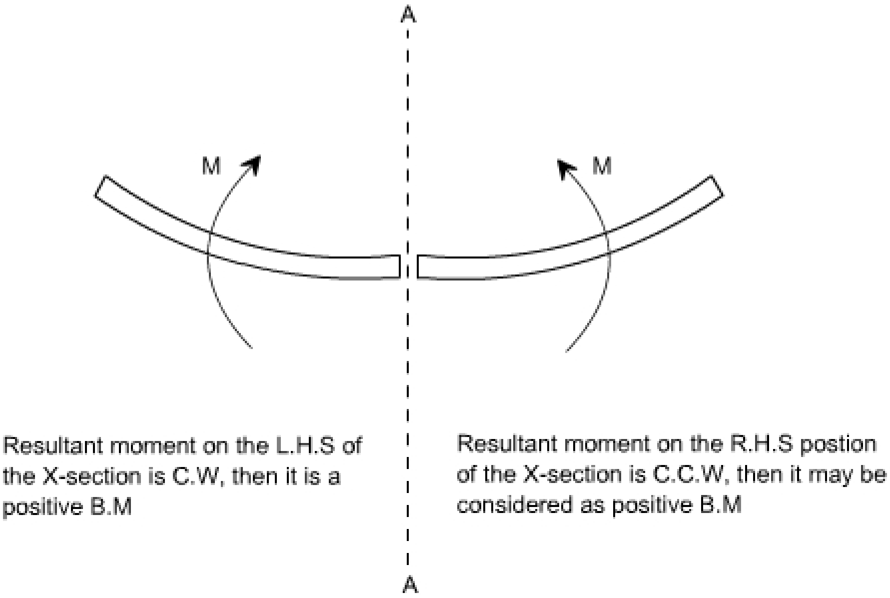


Fig 5: Positive Bending Moment

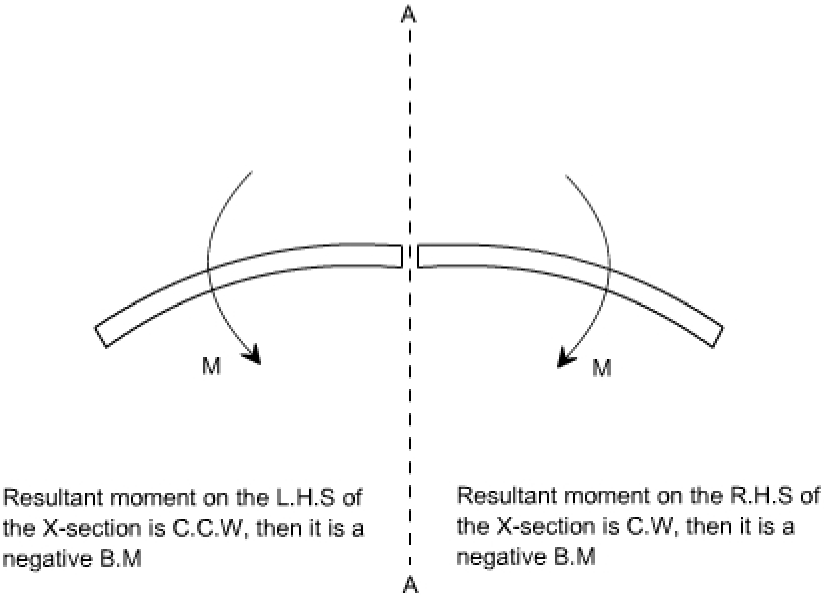


Fig 6: Negative Bending Moment

Some times, the terms ‘Sagging' and Hogging are generally used for the positive and negative bending moments respectively.

Bending Moment and Shear Force Diagrams:

The diagrams which illustrate the variations in B.M and S.F values along the length of the beam for any fixed loading conditions would be helpful to analyze the beam further.

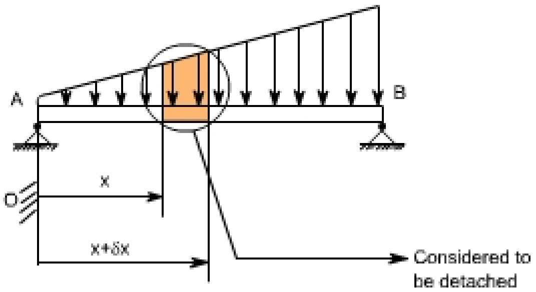
Thus, a shear force diagram is a graphical plot, which depicts how the internal shear force ‘F' varies along the length of beam. If x dentotes the length of the beam, then F is function x i.e. F(x).

Similarly a bending moment diagram is a graphical plot which depicts how the internal bending moment ‘M' varies along the length of the beam. Again M is a function x i.e. M(x).

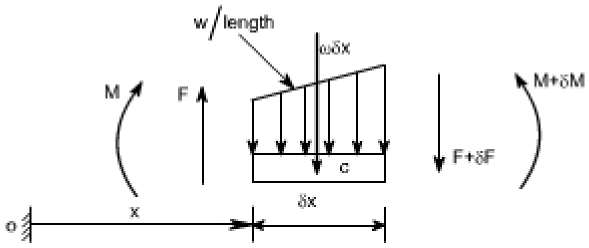
Basic Relationship Between The Rate of Loading, Shear Force and Bending Moment:

The construction of the shear force diagram and bending moment diagrams is greatly simplified if the relationship among load, shear force and bending moment is established.

Let us consider a simply supported beam AB carrying a uniformly distributed load w/length. Let us imagine to cut a short slice of length dx cut out from this loaded beam at distance ‘x' from the origin ‘0'.



Let us detach this portion of the beam and draw its free body diagram.



The forces acting on the free body diagram of the detached portion of this loaded beam are the following

- The shearing force F and F+ δF at the section x and x + δx respectively.

- The bending moment at the sections x and x + δx be M and M + dM respectively.

- Force due to external loading, if ‘w' is the mean rate of loading per unit length then the total loading on this slice of length δx is w. δx, which is approximately acting through the centre ‘c'. If the loading is assumed to be uniformly distributed then it would pass exactly through the centre ‘c'.

This small element must be in equilibrium under the action of these forces and couples.

Now let us take the moments at the point ‘c'. Such that

$$M + F.\frac{\delta x}{2} + (F + \delta F).\frac{\delta x}{2} = M + \delta M$$

$$\Rightarrow F.\frac{\delta x}{2} + (F + \delta F).\frac{\delta x}{2} = \delta M$$

$$\Rightarrow F.\frac{\delta x}{2} + F.\frac{\delta x}{2} + \delta F.\frac{\delta x}{2} = \delta M \text{ [Neglecting the product of } \delta F \text{ and } \delta x \text{ being small quantities]}$$

$$\Rightarrow F.\delta x = \delta M$$

$$\Rightarrow F = \frac{\delta M}{\delta x}$$

Under the limits $\delta x \rightarrow 0$

$$\boxed{F = \frac{dM}{dx}} \qquad \ldots\ldots\ldots\ldots \quad (1)$$

Resolving the forces vertically we get

$$w.\delta x + (F + \delta F) = F$$

$$\Rightarrow w = -\frac{\delta F}{\delta x}$$

Under the limits $\delta x \rightarrow 0$

$$\Rightarrow w = -\frac{dF}{dx} \text{ or } -\frac{d}{dx}\left(\frac{dM}{dx}\right)$$

$$\boxed{w = -\frac{dF}{dx} = -\frac{d^2M}{dx^2}} \qquad \ldots\ldots\ldots\ldots(2)$$

Conclusions: From the above relations,the following important conclusions may be drawn

- From Equation (1), the area of the shear force diagram between any two points, from the basic calculus is the bending moment diagram

$$M = \int F.dx$$

- The slope of bending moment diagram is the shear force, thus

$$F = \frac{dM}{dx}$$

Thus, if F=0; the slope of the bending moment diagram is zero and the bending moment is therefore constant.'

- The maximum or minimum Bending moment occurs where $\frac{dM}{dx} = 0.$

The slope of the shear force diagram is equal to the magnitude of the intensity of the distributed loading at any position along the beam. The –ve sign is as a consequence of our particular choice of sign conventions

Procedure for drawing shear force and bending moment diagram: Preamble:

The advantage of plotting a variation of shear force F and bending moment M in a beam as a function of ‘x' measured from one end of the beam is that it becomes easier to determine the maximum absolute value of shear force and bending moment.

Further, the determination of value of M as a function of ‘x' becomes of paramount importance so as to determine the value of deflection of beam subjected to a given loading.

Construction of shear force and bending moment diagrams:

A shear force diagram can be constructed from the loading diagram of the beam. In order to draw this, first the reactions must be determined always. Then the vertical components of forces and reactions are successively summed from the left end of the beam to preserve the mathematical sign conventions adopted. The shear at a section is simply equal to the sum of all the vertical forces to the left of the section.

When the successive summation process is used, the shear force diagram should end up with the previously calculated shear (reaction at right end of the beam. No shear force acts through the beam just beyond the last vertical force or reaction. If the shear force diagram closes in this fashion, then it gives an important check on mathematical calculations.

The bending moment diagram is obtained by proceeding continuously along the length of beam from the left hand end and summing up the areas of shear force diagrams giving due regard to sign. The process of obtaining the moment diagram from the shear force diagram by summation is exactly the same as that for drawing shear force diagram from load diagram.

It may also be observed that a constant shear force produces a uniform change in the bending moment, resulting in straight line in the moment diagram. If no shear force exists along a certain portion of a beam, then it indicates that there is no change in moment takes place. It may also further observe that dm/dx= F therefore, from the fundamental theorem of calculus the maximum or minimum moment occurs where the shear is zero. In order to check the validity of the bending moment diagram, the terminal conditions for the moment must be satisfied. If the end is free or pinned, the computed sum must be equal to zero. If the end is built in, the moment computed by the summation must be equal to the one calculated initially for the reaction. These conditions must always be satisfied.

Illustrative problems:

In the following sections some illustrative problems have been discussed so as to illustrate the procedure for drawing the shear force and bending moment diagrams

1. A cantilever of length carries a concentrated load 'W' at its free end.

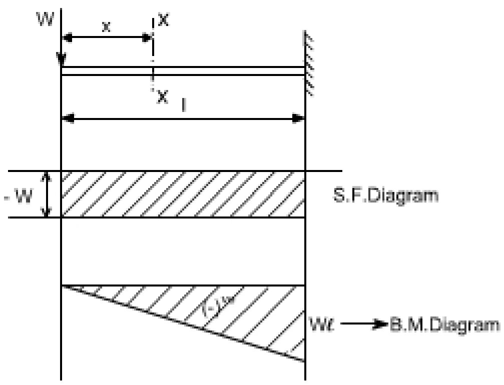
Draw shear force and bending moment.

Solution:

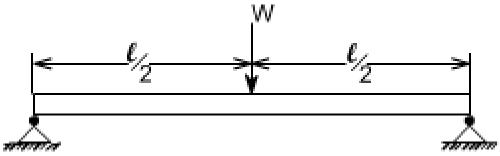
At a section a distance x from free end consider the forces to the left, then F = -W (for all values of x) -ve sign means the shear force to the left of the x-section are in downward direction and therefore negative
Taking moments about the section gives (obviously to the left of the section)

M = -Wx (-ve sign means that the moment on the left hand side of the portion is in the anticlockwise direction and is therefore taken as –ve according to the sign convention)
so that the maximum bending moment occurs at the fixed end i.e. M = -W l

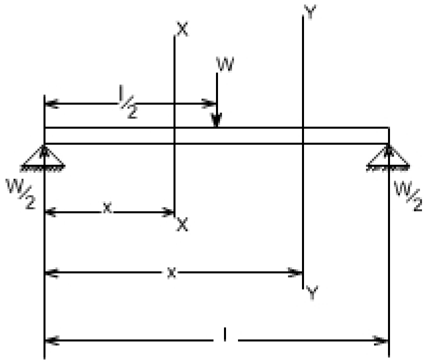
From equilibrium consideration, the fixing moment applied at the fixed end is Wl and the reaction is W. the shear force and bending moment are shown as,



2. Simply supported beam subjected to a central load (i.e. load acting at the mid- way)



By symmetry the reactions at the two supports would be W/2 and W/2. now consider any section X-X from the left end then, the beam is under the action of following forces.

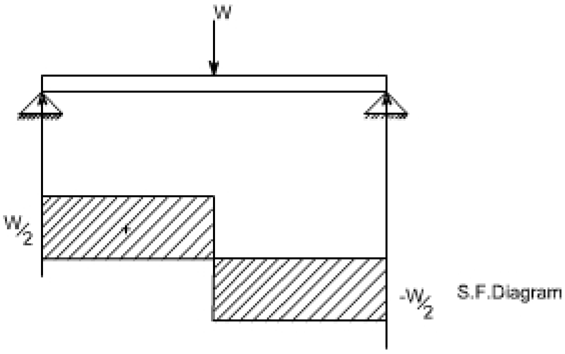


.So the shear force at any X-section would be = W/2 [Which is constant upto x < l/2]

If we consider another section Y-Y which is beyond l/2 then

$$S.F_{Y-Y} = \frac{W}{2} - W = \frac{-W}{2}$$

for all values greater = l/2 Hence S.F diagram can be plotted as,



.For B.M diagram:

If we just take the moments to the left of the cross-section,

$$B.M_{x\text{-}x} = \frac{W}{2}\,x \text{ for } x \text{ lies between } 0 \text{ and } l/2$$

$$B.M_{\text{at } x = \frac{l}{2}} = \frac{W}{2}\,\frac{l}{2} \text{ i.e B.M at } x = 0$$

$$= \frac{Wl}{4}$$

$$B.M_{Y\text{-}Y} = \frac{W}{2}\,x - W\left(x - \frac{l}{2}\right)$$

Again

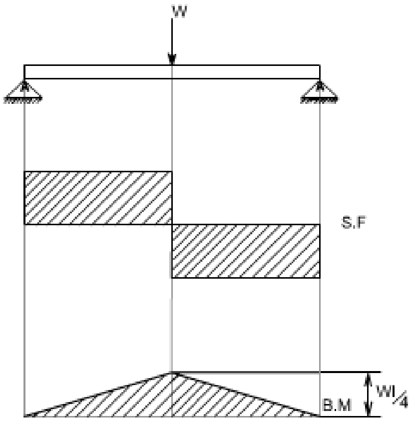
$$= \frac{W}{2}\,x - Wx + \frac{Wl}{2}$$

$$= -\frac{W}{2}\,x + \frac{Wl}{2}$$

$$B.M_{\text{at } x = l} = -\frac{Wl}{2} + \frac{Wl}{2}$$

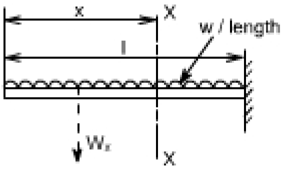
$$= 0$$

Which when plotted will give a straight relation i.e.



It may be observed that at the point of application of load there is an abrupt change in the shear force, at this point the B.M is maximum.

3. A cantilever beam subjected to U.d.L, draw S.F and B.M diagram.



Here the cantilever beam is subjected to a uniformly distributed load whose intensity is given w / length.

Consider any cross-section XX which is at a distance of x from the free end. If we just take the resultant of all the forces on the left of the X-section, then

$S.F_{xx} = -Wx$ for all values of ‘x' --------------**(1)**

$S.F_{xx} = 0$

S.Fxx at x=1 = -Wl

So if we just plot the equation No. (1), then it will give a straight line relation. Bending Moment at X-X is obtained by treating the load to the left of X-X as a concentrated load of the same value acting through the centre of gravity.

Therefore, the bending moment at any cross-section X-X is

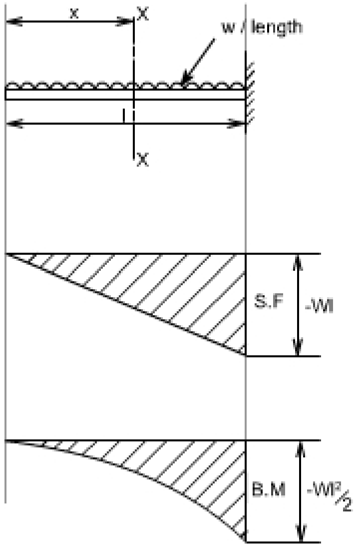
$$B.M_{X-X} = -\,W x\,\frac{x}{2}$$
$$= -\,W\frac{x^2}{2}$$

The above equation is a quadratic in x, when B.M is plotted against x this will produces a parabolic variation.

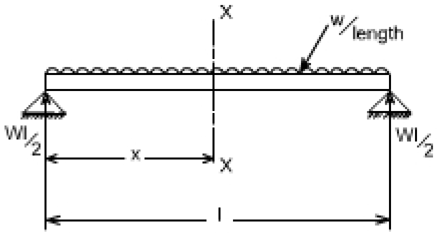
The extreme values of this would be at x = 0 and x = l

$$B.M_{\text{at } x = l} = -\frac{Wl^2}{2}$$
$$= \frac{Wl}{2} - Wx$$

Hence S.F and B.M diagram can be plotted as follows:



4. Simply supported beam subjected to a uniformly distributed load [U.D.L].



The total load carried by the span would be

= intensity of loading x length

= w x l

By symmetry the reactions at the end supports are each wl/2

If x is the distance of the section considered from the left hand end of the beam.

S.F at any X-section X-X is

$$= \frac{Wl}{2} - Wx$$

$$= W\left(\frac{l}{2} - x\right)$$

Giving a straight relation, having a slope equal to the rate of loading or intensity of the loading.

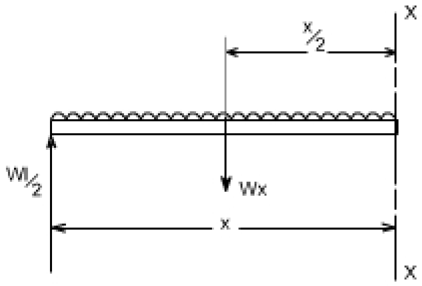
$$S.F_{\text{at } x=0} = \frac{wl}{2} - wx$$

so at

$$S.F_{\text{at } x=\frac{l}{2}} = 0 \text{ hence the S.F is zero at the centre}$$

$$S.F_{\text{at } x=l} = -\frac{Wl}{2}$$

The bending moment at the section x is found by treating the distributed load as acting at its centre of gravity, which at a distance of x/2 from the section



$$B.M_{x\text{-}x} = \frac{Wl}{2}x - Wx.\frac{x}{2}$$

so the

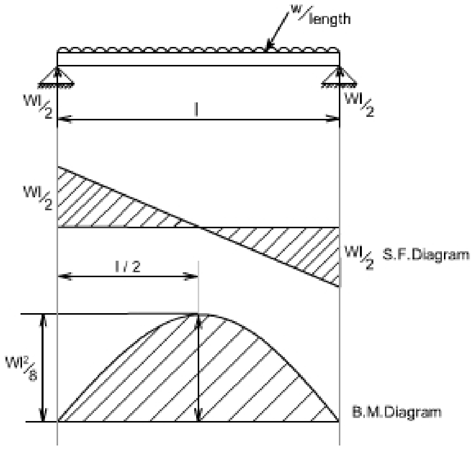
$$= W.\frac{x}{2}(l - 2) \quad \text{........(2)}$$

$$B.M_{\text{at } x=0} = 0$$

$$B.M_{\text{at } x=l} = 0$$

$$B.M\Big|_{\text{at } x=l} = -\frac{Wl^2}{8}$$

So the equation (2) when plotted against x gives rise to a parabolic curve and the shear force and bending moment can be drawn in the following way will appear as follows:



4.2 Loading restrictions:

As we are aware of the fact internal reactions developed on any cross-section of a beam may consists of a resultant normal force, a resultant shear force and a resultant couple. In order to ensure that the bending effects alone are investigated, we shall put a constraint on the loading such that the resultant normal and the resultant shear forces are zero on any cross-section perpendicular to the longitudinal axis of the member,

That means F = 0

since $\frac{dM}{dX} = F = 0$ or M = constant.

Thus, the zero shear force means that the bending moment is constant or the bending is same at every cross-section of the beam. Such a situation may be visualized or envisaged when the beam or some portion of the beam, as been loaded only by pure couples at its ends. It must be recalled that the couples are assumed to be loaded in the plane of symmetry.

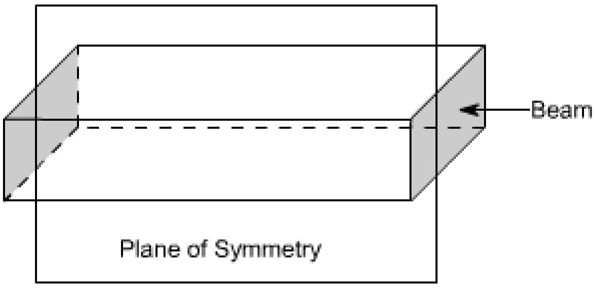


Fig (1)

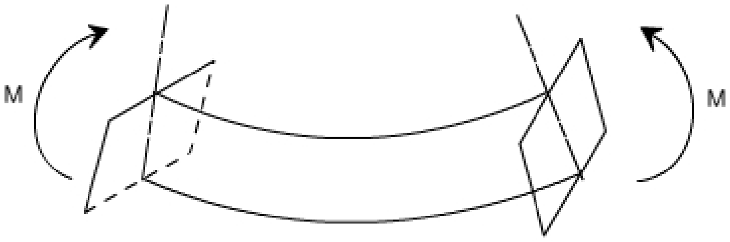
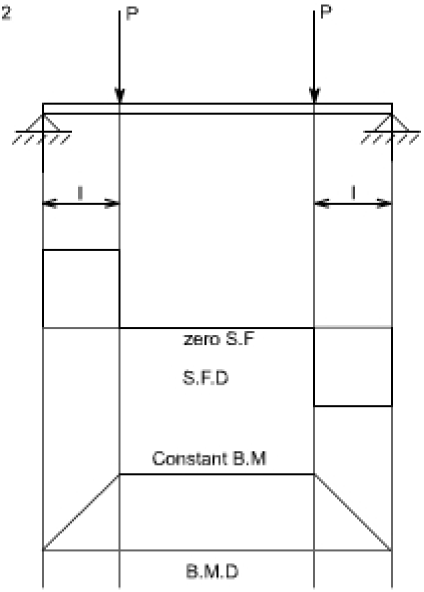


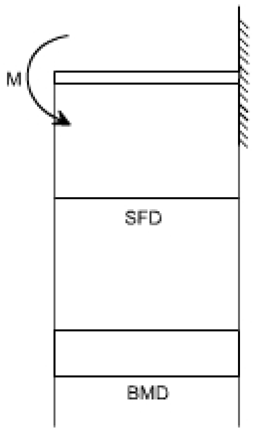
Fig (2)

When a member is loaded in such a fashion it is said to be in **pure bending.** The examples of pure bending have been indicated in EX 1and EX 2 as shown below :

EX .2

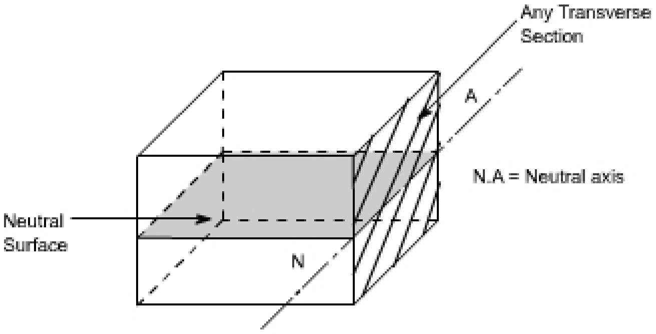


EX. 1



When a beam is subjected to pure bending are loaded by the couples at the ends, certain cross-section gets deformed and we shall have to make out the conclusion that,

1. Plane sections originally perpendicular to longitudinal axis of the beam remain plane and perpendicular to the longitudinal axis even after bending , i.e. the cross-section A'E', B'F' (refer Fig 1(a)) do not get warped or curved.
2. In the deformed section, the planes of this cross-section have a common intersection i.e. any time originally parallel to the longitudinal axis of the beam becomes an arc of circle.



We know that when a beam is under bending the fibres at the top will be lengthened while at the bottom will be shortened provided the bending moment M acts at the ends. In between these there are some fibres which remain unchanged in length that is they are not strained, that is they do not carry any stress. The plane containing such fibres is called neutral surface.

The line of intersection between the neutral surface and the transverse exploratory section is called the neutral axisNeutral axis **(N A)** .

4.3 Bending Stresses in Beams or Derivation of Elastic Flexural formula :

In order to compute the value of bending stresses developed in a loaded beam, let us consider the two cross-sections of a beam**HE** and **GF** , originally parallel as shown in fig 1(a).when the beam is to bend it is assumed that these sections remain parallel i.e.**H'E'** and **G'F'** , the final position of the sections, are still straight lines, they then subtend some angle < .

Consider now fiber AB in the material, at adistance y from the N.A, when the beam bends this will stretch to A'B'

Therefore,

$$\text{strain in fibre } AB = \frac{\text{change in length}}{\text{orginal length}}$$

$$= \frac{A'B' - AB}{AB} \qquad \text{But } AB = CD \text{ and } CD = C'D'$$

refer to fig1(a) and fig1(b)

$$\therefore \text{strain} = \frac{A'B' - C'D'}{C'D'}$$

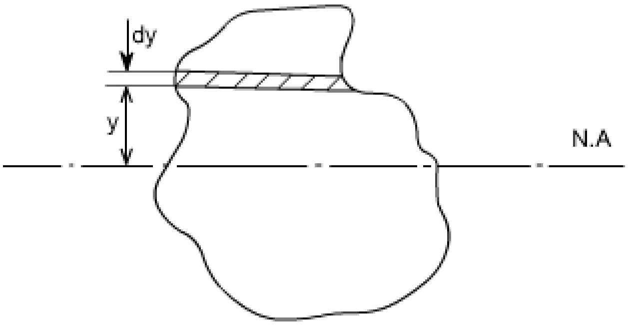
Since CD and C'D' are on the neutral axis and it is assumed that the Stress on the neutral axis zero. Therefore, there won't be any strain on the neutral axis

$$= \frac{(R+y)\theta - R\theta}{R\theta} = \frac{R\theta + y\theta - R\theta}{R\theta} = \frac{y}{R}$$

However $\frac{\text{stress}}{\text{strain}} = E$ where E = Young's Modulus of elasticity

Therefore, equating the two strains as obtained from the two relations i.e,

$$\frac{\sigma}{E} = \frac{y}{R} \text{ or } \frac{\sigma}{y} = \frac{E}{R} \qquad \ldots\ldots\ldots\ldots(1)$$



Consider any arbitrary a cross-section of beam, as shown above now the strain on a fibre at a distance ‘y' from the N.A, is given by the expression

$$\sigma = \frac{E}{R} y$$

if the shaded strip is of area 'dA'
then the force on the strip is

$$F = \sigma\, \delta A = \frac{E}{R} y\, \delta A$$

Moment about the neutral axis would be $= F.y = \frac{E}{R} y^2 \delta A$

The toatl moment for the whole cross-section is therefore equal to

$$M = \sum \frac{E}{R} y^2\, \delta A = \frac{E}{R} \sum y^2 \delta A$$

Now the term $\sum y^2 \delta A$ is the property of the material and is called as a second moment of area of the cross-section and is denoted by a symbol I.

Therefore

$$M = \frac{E}{R} I \qquad \ldots\ldots\ldots(2)$$

combining equation 1 and 2 we get

$$\boxed{\frac{\sigma}{y} = \frac{M}{T} = \frac{E}{R}}$$

This equation is known as the Bending Theory Equation. The above proof has involved the assumption of pure bending without any shear force being present.

Therefore this termed as the pure bending equation. This equation gives distribution of stresses which are normal to cross-section i.e. in x-direction.

Section Modulus:

From simple bending theory equation, the maximum stress obtained in any cross-section is given as

$$\sigma_{max}{}^{m} = \frac{M}{I} y_{max}{}^{m}$$

For any given allowable stress the maximum moment which can be accepted by a particular shape of cross-section is therefore

$$M = \frac{I}{y_{max}{}^{m}} \sigma_{max}{}^{m}$$

For ready comparison of the strength of various beam cross-section this relationship is some times written in the form

$$M = Z \sigma_{max}{}^{m} \text{ where } Z = \frac{I}{y_{max}{}^{m}}$$

Is termed as section modulus

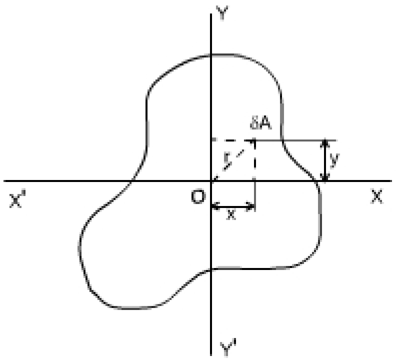
The higher value of Z for a particular cross-section, the higher the bending moment which it can withstand for a given maximum stress.

Theorems to determine second moment of area: There are two theorems which are helpful to determine the value of second moment of area, which is required to be used while solving the simple bending theory equation.

Second Moment of Area :

Taking an analogy from the mass moment of inertia, the second moment of area is defined as the summation of areas times the distance squared from a fixed axis. (This property arised while we were driving bending theory equation). This is also known as the moment of inertia. An alternative name given to this is second moment of area, because the first moment being the sum of areas times their distance from a given axis and the second moment being the square of the distance $\int y^2 \, dA$ or

.



Consider any cross-section having small element of area d A then by the definition I_x(Mass Moment of Inertia about x-axis) = $\int y^2 \, dA$ and I_y(Mass Moment of Inertia about y-axis) = $\int x^2 \, dA$

Now the moment of inertia about an axis through 'O' and perpendicular to the plane of figure is called the polar moment of inertia. (The polar moment of inertia is also the area moment of inertia).

i.e,

J = polar moment of inertia

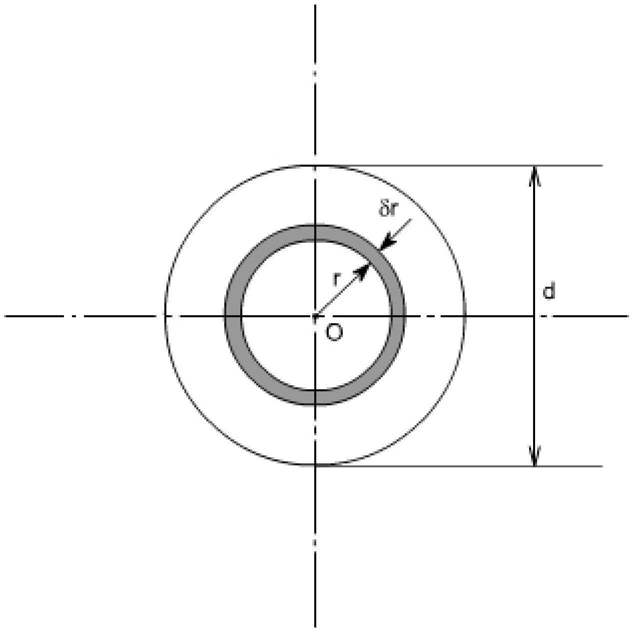
$$
\begin{aligned}
&= \int r^2 dA \\
&= \int (x^2 + y^2) dA \\
&= \int x^2 dA + \int y^2 dA \\
&= I_X + I_Y \\
\text{or } J &= I_X + I_Y \qquad \ldots\ldots\ldots\ (1)
\end{aligned}
$$

The relation (1) is known as the **perpendicular axis theorem** and may be stated as follows:

The sum of the Moment of Inertia about any two axes in the plane is equal to the moment of inertia about an axis perpendicular to the plane, the three axes being concurrent, i.e, the three axes exist together.

CIRCULAR SECTION :

For a circular x-section, the polar moment of inertia may be computed in the following manner



Consider any circular strip of thickness < r located at a radius 'r'. Than the area of the circular strip would be dA = 2< r. < r

$$J = \int r^2 dA$$

Taking the limits of intergration from 0 to d/2

$$J = \int_0^{\frac{d}{2}} r^2 2\pi r \delta r$$

$$= 2\pi \int_0^{\frac{d}{2}} r^3 \delta r$$

$$J = 2\pi \left[\frac{r^4}{4} \right]_0^{\frac{d}{2}} = \frac{\pi d^4}{32}$$

however,by perpendicular axistheorem

$$J = I_x + I_y$$

But for the circular cross-section ,the I_x and I_y are both equal being moment of inertia about a diameter

$$I_{dia} = \frac{1}{2} J$$

$$I_{dia} = \frac{\pi d^4}{64}$$

for a hollow circular section of diameter D and d,
the values of J and I are defined as

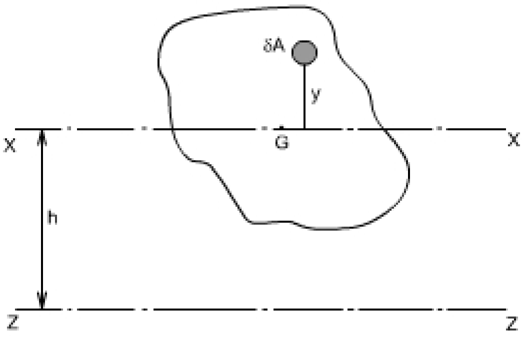
$$J = \frac{\pi \left(D^4 - d^4 \right)}{32}$$

$$I = \frac{\pi \left(D^4 - d^4 \right)}{64}$$

Thus

Parallel Axis Theorem:

The moment of inertia about any axis is equal to the moment of inertia about a parallel axis through the centroid plus the area times the square of the distance between the axes.



If ‘ZZ' is any axis in the plane of cross-section and ‘XX' is a parallel axis through the centroid G, of the cross-section, then

$$I_z = \int (y+h)^2 \, dA \text{ by definition (moment of inertia about an axis ZZ)}$$

$$= \int \left(+ 2yh + h^2 \right) dA$$

$$= \int y^2 dA + h^2 \int dA + 2h \int y dA$$

$$\text{Since } \int y dA = 0$$

$$= \int y^2 dA + h^2 \int dA$$

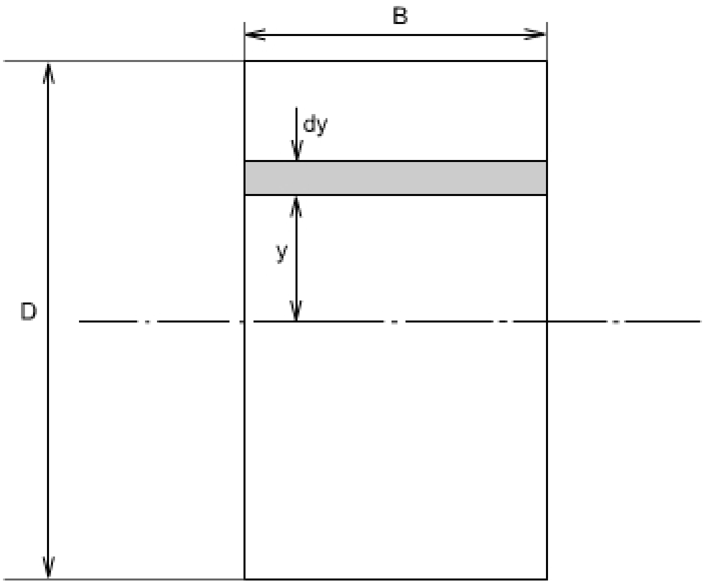
$$= \int y^2 dA + h^2 A$$

$$I_z = I_x + Ah^2 \qquad I_x = I_G \text{ (since cross-section axes also pass through G)}$$

$$\text{Where } A = \text{Total area of the section}$$

Rectangular Section:

For a rectangular x-section of the beam, the second moment of area may be computed as below :



Consider the rectangular beam cross-section as shown above and an element of area **dA** , thickness **dy** , breadth **B** located at a distance **y** from the neutral axis, which by symmetry passes through the centre of section. The second moment of area **I** as defined earlier would be

$$I_{N.A} = \int y^2 \, dA$$

Thus, for the rectangular section the second moment of area about the neutral axis i.e., an axis through the centre is given by

$$I_{N.A} = \int_{\frac{-D}{2}}^{\frac{D}{2}} y^2 (B \, dy)$$

$$= B \int_{\frac{-D}{2}}^{\frac{D}{2}} y^2 \, dy$$

$$= B \left[\frac{y^3}{3} \right]_{\frac{D}{2}}^{\frac{-D}{2}}$$

$$= \frac{B}{3} \left[\frac{D^3}{8} - \left(\frac{-D^3}{8} \right) \right]$$

$$= \frac{B}{3} \left[\frac{D^3}{8} + \frac{D^3}{8} \right]$$

$$I_{N.A} = \frac{BD^3}{12}$$

MODULE – V

SLOPE AND DEFLECTION

5.1 Introduction:

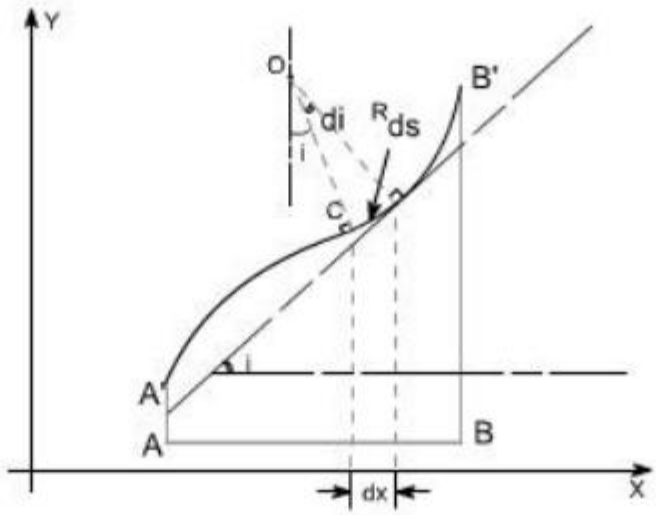
In all practical engineering applications, when we use the different components, normally we have to operate them within the certain limits i.e. the constraints are placed on the performance and behavior of the omponents. For instance we say that the particular component is supposed to operate within this value of stress and the deflection of the component should not exceed beyond a particular value.

In some problems the maximum stress however, may not be a strict or severe condition but there may be the
deflection which is the more rigid condition under operation. It is obvious therefore to study the methods by which we can predict the deflection of members under lateral loads or transverse loads, since it is this form of loading which will generally produce the greatest deflection of beams.

Assumption: The following assumptions are undertaken in order to derive a differential equation of elastic curve for the loaded beam

1. Stress is proportional to strain i.e. hooks law applies. Thus, the equation is valid only for beams that are not
stressed beyond the elastic limit.
2. The curvature is always small.
3. Any deflection resulting from the shear deformation of the material or shear stresses is neglected.

It can be shown that the deflections due to shear deformations are usually small and hence can be ignored.



Consider a beam AB which is initially straight and horizontal when unloaded. If under the action of loads the beam deflect to a position A'B' under load or infact we say that the axis of the beam bends to a shape A'B'. It is customary to call A'B' the curved axis of the beam as the elastic line or deflection curve.

In the case of a beam bent by transverse loads acting in a plane of symmetry, the bending moment M varies along the length of the beam and we represent the variation of bending moment in B.M diagram. Futher, it is assumed that the simple bending theory equation holds good.

$$\boxed{\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}}$$

If we look at the elastic line or the deflection curve, this is obvious that the curvature at every point is different; hence the slope is different at different points.

To express the deflected shape of the beam in rectangular co-ordinates let us take two axes x and y, x-axis coincide with the original straight axis of the beam and the y – axis shows the deflection.

Futher,let us consider an element ds of the deflected beam. At the ends of this element let us construct the normal which intersect at point O denoting the angle between these two normal, but for the deflected shape of the beam the slope i at any point C is defined,

$$\tan i = \frac{dy}{dx} \quad \ldots\ldots(1) \quad \text{or} \quad i = \frac{dy}{dx} \quad \text{Assuming } \tan i = i$$

Futher

$$ds = R\,di$$

however,

$$ds = dx \text{ [usually for small curvature]}$$

Hence

$$ds = dx = R\,di$$

$$\text{or } \boxed{\frac{di}{dx} = \frac{1}{R}}$$

substituting the value of i, one get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{R} \text{ or } \frac{d^2y}{dx^2} = \frac{1}{R}$$

From the simple bending theory

$$\frac{M}{I} = \frac{E}{R} \text{ or } M = \frac{EI}{R}$$

so the basic differential equation governing the deflection of beams is

$$\boxed{M = EI\frac{d^2y}{dx^2}}$$

This is the differential equation of the elastic line for a beam subjected to bending in the plane of symmetry. Its

solution y = f(x) defines the shape of the elastic line or the deflection curve as it is frequently called.

Relationship between shear force, bending moment and deflection: The relationship among shear force,bending moment and deflection of the beam may be obtained as Differentiating the equation as derived

$$\frac{dM}{dx} = EI\frac{d^3y}{dx^3} \quad \text{Recalling } \frac{dM}{dx} = F$$

Thus,

$$F = EI\frac{d^3y}{dx^3}$$

Therefore, the above expression represents the shear force whereas rate of intensity of loading can also be found out by differentiating the expression for shear force

$$\text{i.e } w = -\frac{dF}{dx}$$

$$w = -EI\frac{d^4y}{dx^4}$$

Therefore if 'y' is the deflection of the loaded beam, then the following important relations can be arrived at

$$\boxed{\text{slope} = \frac{dy}{dx}}$$

$$\boxed{B.M = EI\frac{d^2y}{dx^2}}$$

$$\boxed{\text{Shear force} = EI\frac{d^3y}{dx^3}}$$

$$\boxed{\text{load distribution} = EI\frac{d^4y}{dx^4}}$$

Methods for finding the deflection: The deflection of the loaded beam can be obtained various methods.The one of the method for finding the deflection of the beam is the direct integration method, i.e. the method using the differential equation which we have derived.

Direct integration method: The governing differential equation is defined as

$$M = EI\frac{d^2y}{dx^2} \quad \text{or} \quad \frac{M}{EI} = \frac{d^2y}{dx^2}$$

on integrating one get,

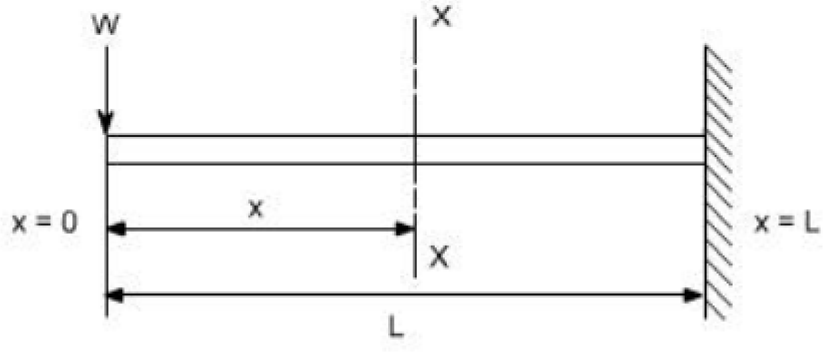
$$\frac{dy}{dx} = \int\frac{M}{EI}dx + A \text{ - - - - this equation gives the slope of the loaded beam.}$$

Integrate once again to get the deflection.

$$y = \int\int\frac{M}{EI}dx + Ax + B$$

Where A and B are constants of integration to be evaluated from the known conditions of slope and deflections for the particular value of x.

Illustrative examples : let us consider few illustrative examples to have a familiarty with the direct integration method Case 1: Cantilever Beam with Concentrated Load at the end:- A cantilever beam is subjected to a concentrated load W at the free end, it is required to determine the deflection of the beam



In order to solve this problem, consider any X-section X-X located at a distance x from the left end or the reference, and write down the expressions for the shear force abd the bending moment

$S.F|_{x-x} = -W$

$B.M|_{x-x} = -W.x$

Therefore $M|_{x-x} = -W.x$

the governing equation $\frac{M}{EI} = \frac{d^2y}{dx^2}$

substituting the value of M interms of x then integrating the equation one get

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{Wx}{EI}$$

$$\int \frac{d^2y}{dx^2} = \int -\frac{Wx}{EI}\,dx$$

$$\frac{dy}{dx} = -\frac{Wx^2}{2EI} + A$$

Integrating once more,

$$\int \frac{dy}{dx} = \int -\frac{Wx^2}{2EI}\,dx + \int A\,dx$$

$$y = -\frac{Wx^3}{6EI} + Ax + B$$

The constants A and B are required to be found out by utilizing the boundary conditions as defined below

i.e at x= L ; y= 0 -------------------- (1)

at x = L ; dy/dx = 0 -------------------- (2)

Utilizing the second condition, the value of constant A is obtained as

$$A = \frac{Wl^2}{2EI}$$

While employing the first condition yields

$$y = -\frac{WL^3}{6EI} + AL + B$$

$$B = \frac{WL^3}{6EI} - AL$$

$$= \frac{WL^3}{6EI} - \frac{WL^3}{2EI}$$

$$= \frac{WL^3 - 3WL^3}{6EI} = -\frac{2WL^3}{6EI}$$

$$B = -\frac{WL^3}{3EI}$$

Substituting the values of A and B we get

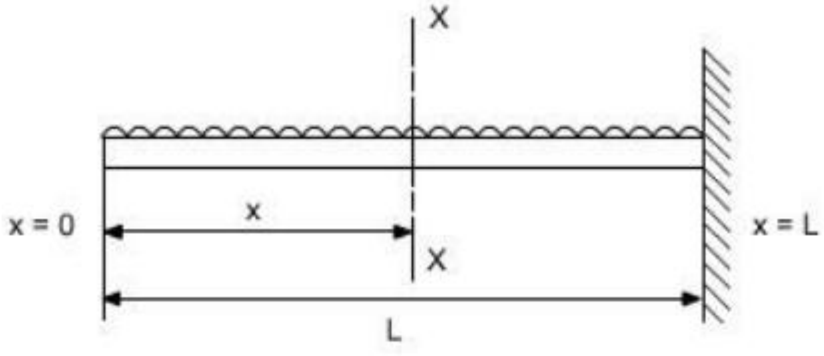
$$y = \frac{1}{EI}\left[-\frac{Wx^3}{6EI} + \frac{WL^2x}{2EI} - \frac{WL^3}{3EI}\right]$$

The slope as well as the deflection would be maximum at the free end hence putting x=0 we get,

$$\boxed{y_{max} = -\frac{WL^3}{3EI}}$$

$$\boxed{(\text{Slope})_{max}m = +\frac{WL^2}{2EI}}$$

Case 2: A Cantilever with Uniformly distributed Loads:- In this case the cantilever beam is subjected to U.d.l with rate of intensity varying w / length.The same procedure can also be adopted in this case



$$S.F\big|_{x-x} = -w$$

$$B.M\big|_{x-x} = -w.x.\frac{x}{2} = w\left(\frac{x^2}{2}\right)$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{wx^2}{2EI}$$

$$\int\frac{d^2y}{dx^2} = \int -\frac{wx^2}{2EI}dx$$

$$\frac{dy}{dx} = -\frac{wx^3}{6EI} + A$$

$$\int\frac{dy}{dx} = \int -\frac{wx^3}{6EI}dx + \int A\,dx$$

$$y = -\frac{wx^4}{24EI} + Ax + B$$

Boundary conditions relevant to the problem are as follows:

1. At $x = L$; $y = 0$
2. At $x = L$; $dy/dx = 0$

The second boundary conditions yields

$$A = +\frac{wx^3}{6EI}$$

whereas the first boundary conditions yields

$$B = \frac{wL^4}{24EI} - \frac{wL^4}{6EI}$$

$$B = -\frac{wL^4}{8EI}$$

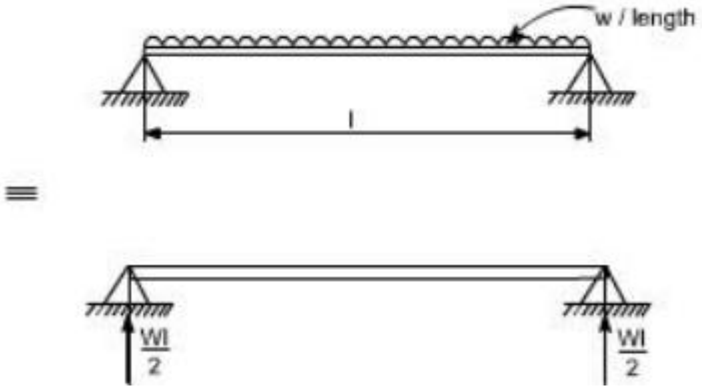
Thus, $y = \frac{1}{EI}\left[-\frac{wx^4}{24} + \frac{wL^3 x}{6} - \frac{wL^4}{8}\right]$

So y_{maxm} will be at $x = 0$

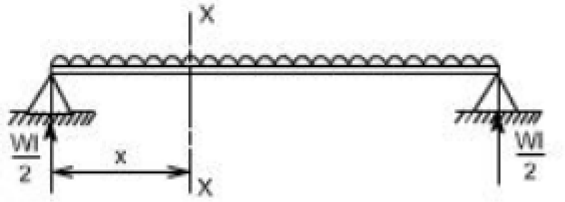
$$\boxed{y_{maxm} = -\frac{wL^4}{8EI}}$$

$$\boxed{\left(\frac{dy}{dx}\right)_{maxm} = \frac{wL^3}{6EI}}$$

Case 3: Simply Supported beam with uniformly distributed Loads:- In this case a simply supported beam is subjected to a uniformly distributed load whose rate of intensity varies as w / length.



In order to write down the expression for bending moment consider any cross-section at distance of x metre from left end support.



$$S.F|_{x-x} = w\left(\frac{l}{2}\right) - w.x$$

$$B.M|_{x-x} = w.\left(\frac{l}{2}\right).x - w.x.\left(\frac{x}{2}\right)$$

$$= \frac{wl.x}{2} - \frac{wx^2}{2}$$

The differential equation which gives the elastic curve for the deflected beam is

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{1}{EI}\left[\frac{wl.x}{2} - \frac{wx^2}{2}\right]$$

$$\frac{dy}{dx} = \int \frac{wlx}{2EI}dx - \int \frac{wx^2}{2EI}dx + A$$

$$= \frac{wlx^2}{4EI} - \frac{wx^3}{6EI} + A$$

Integrating, once more one gets

$$y = \frac{wlx^3}{12EI} - \frac{wx^4}{24EI} + A.x + B \qquad \text{-----}(1)$$

Boundary conditions which are relevant in this case are that the deflection at each support must be zero.

i.e. at x = 0; y = 0 : at x = l; y = 0

let us apply these two boundary conditions on equation (1) because the boundary conditions are on y, This yields B = 0.

$$0 = \frac{wl^4}{12EI} - \frac{wl^4}{24EI} + A.l$$

$$A = -\frac{wl^3}{24EI}$$

So the equation which gives the deflection curve is

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

Futher

In this case the maximum deflection will occur at the centre of the beam where x = L/2 [i.e. at the position where the load is being applied].So if we substitute the value of x = L/2

$$\text{Then}\qquad y_{max^m} = \frac{1}{EI}\left[\frac{wL}{12}\left(\frac{L^3}{8}\right) - \frac{w}{24}\left(\frac{L^4}{16}\right) - \frac{wL^3}{24}\left(\frac{L}{2}\right)\right]$$

$$\boxed{y_{max^m} = -\frac{5wL^4}{384EI}}$$

Conclusions

(i) The value of the slope at the position where the deflection is maximum would be zero.

(ii) Thevalue of maximum deflection would be at the centre i.e. at x = L/2.

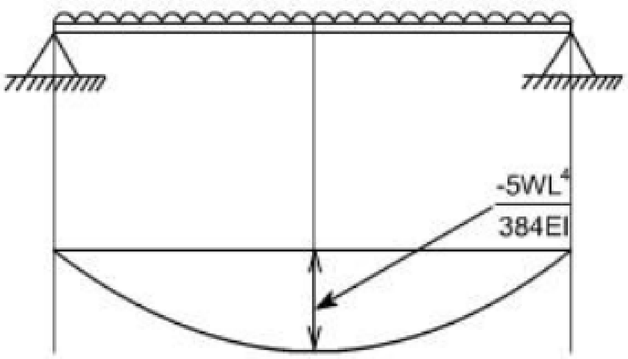
The final equation which is governs the deflection of the loaded beam in this case is

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

By successive differentiation one can find the relations for slope, bending moment, shear force and rate of loading.

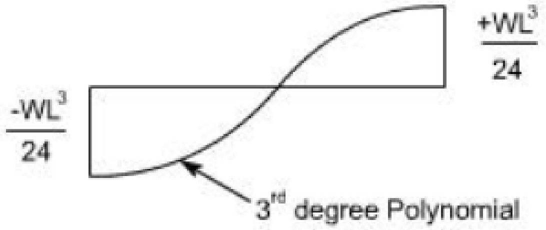
Deflection (y)

$$yEI = \left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

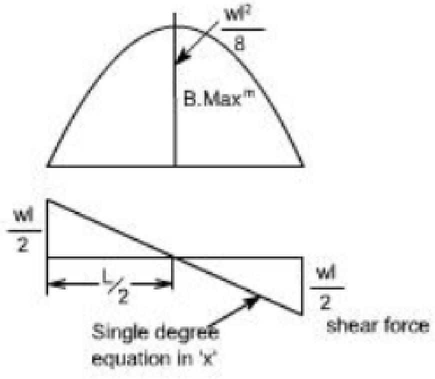


Slope (dy/dx)

$$EI.\frac{dy}{dx} = \left[\frac{3wLx^2}{12} - \frac{4wx^3}{24} - \frac{wL^3}{24}\right]$$



So the bending moment diagram would be



Bending Moment

$$\frac{d^2y}{dx^2} = \frac{1}{EI}\left[\frac{wLx}{2} - \frac{wx^2}{2}\right]$$

Shear Force

Shear force is obtained by taking

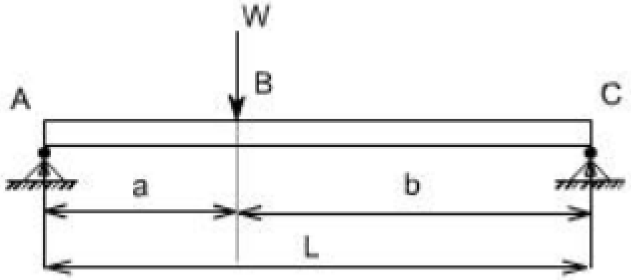
third derivative.

$$EI\frac{d^3y}{dx^3} = \frac{wL}{2} - w.x$$

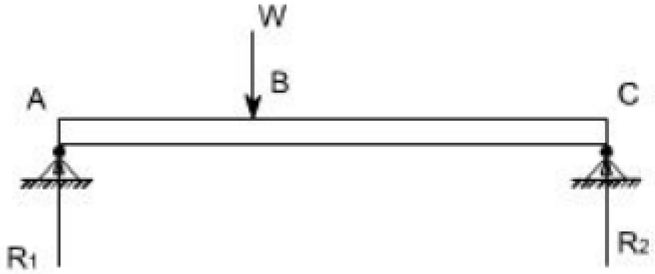
Rate of intensity of loading

$$EI\frac{d^4y}{dx^4} = -w$$

Case 4: The direct integration method may become more involved if the expression for entire beam is not valid for the entire beam.Let us consider a deflection of a simply supported beam which is subjected to a concentrated load W acting at a distance 'a' from the left end.



Let R_1 & R_2 be the reactions then,



B.M for the portion AB

$M|_{AB} = R_1.x \quad 0 \leq x \leq a$

B.M for the portion BC

$M|_{BC} = R_1.x - W(x - a) \quad a \leq x \leq l$

so the differential equation for the two cases would be,

$$EI\frac{d^2y}{dx^2} = R_1\,x$$

$$EI\frac{d^2y}{dx^2} = R_1\,x - W\,(x - a)$$

These two equations can be integrated in the usual way to find ‘y' but this will result in four constants of integration two for each equation. To evaluate the four constants of integration, four independent boundary conditions will be needed since the deflection of each support must be zero, hence the boundary conditions (a) and (b) can be realized.

Further, since the deflection curve is smooth, the deflection equations for the same slope and deflection at the point of application of load i.e. at x = a. Therefore four conditions required to evaluate these constants may be defined as follows:

(a) at x = 0; y = 0 in the portion AB i.e. $0 \le x \le a$

(b) at x = l; y = 0 in the portion BC i.e. $a \le x \le l$

(c) at x = a; dy/dx, the slope is same for both portion

(d) at x = a; y, the deflection is same for both portion

By symmetry, the reaction R1 is obtained as

$$R_1 = \frac{Wb}{a+b}$$

Hence,

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{(a+b)}x \qquad 0 \le x \le a \text{ --------(1)}$$

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{(a+b)}x - W(x-a) \qquad a \le x \le l \text{ ---------(2)}$$

integrating (1) and (2) we get,

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 + k_1 \qquad 0 \le x \le a \text{ --------(3)}$$

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 - \frac{W(x-a)^2}{2} + k_2 \qquad a \le x \le l \text{ --------(4)}$$

Using condition (c) in equation (3) and (4) shows that these constants should be equal, hence letting

$K_1 = K_2 = K$

Hence

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 + k \qquad 0 \le x \le a \text{------}(3)$$

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 - \frac{W(x-a)^2}{2} + k \qquad a \le x \le l \text{-------}(4)$$

Integrating agian equation (3) and (4) we get

$$EIy = \frac{Wb}{6(a+b)}x^3 + kx + k_3 \qquad 0 \le x \le a \text{------}(5)$$

$$EIy = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4 \qquad a \le x \le l \text{------}(6)$$

Utilizing condition (a) in equation (5) yields

$$k_3 = 0$$

Utilizing condition (b) in equation (6) yields

$$0 = \frac{Wb}{6(a+b)}l^3 - \frac{W(l-a)^3}{6} + kl + k_4$$

$$k_4 = -\frac{Wb}{6(a+b)}l^3 + \frac{W(l-a)^3}{6} - kl$$

But $a + b = l$,

Thus,

$$k_4 = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6} - k(a+b)$$

Now lastly k3 is found out using condition (d) in equation (5) and equation (6), the condition (d) is that,

At x = a; y; the deflection is the same for both portion

$$\text{Therefore } y\big|_{\text{from equation 5}} = y\big|_{\text{from equation 6}}$$

or

$$\frac{Wb}{6(a+b)}x^3 + kx + k_3 = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4$$

$$\frac{Wb}{6(a+b)}a^3 + ka + k_3 = \frac{Wb}{6(a+b)}a^3 - \frac{W(a-a)^3}{6} + ka + k_4$$

Thus, $k_4 = 0$;

OR

$$k_4 = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6} - k(a+b) = o$$

$$k(a+b) = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6}$$

$$k = -\frac{Wb(a+b)}{6} + \frac{Wb^3}{6(a+b)}$$

so the deflection equations for each portion of the beam are

$$EIy = \frac{Wb}{6(a+b)}x^3 + kx + k_3$$

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{Wb(a+b)x}{6} + \frac{Wb^3x}{6(a+b)} \qquad \text{- - - -}\mathbf{for\ 0 \le x \le a}\text{- - - - - (7)}$$

and for other portion

$$EIy = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4$$

Substituting the value of 'k' in the above equation

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{W(x-a)^3}{6} - \frac{Wb(a+b)x}{6} + \frac{Wb^3x}{6(a+b)} \qquad \text{For } \mathbf{for\ a \le x \le l}\text{- - - - - (8)}$$

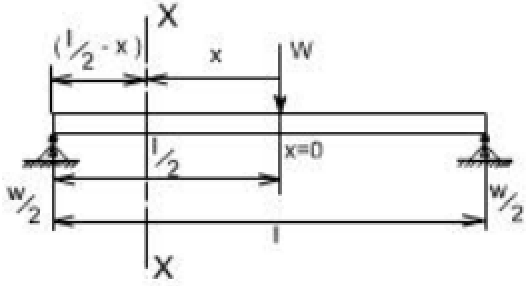
so either of the equation (7) or (8) may be used to find the deflection at x = a hence substituting x = a in either of the equation we get

$$Y\big|_{x=a} = -\frac{Wa^2b^2}{3EI(a+b)}$$

OR if $a = b = l/2$

$$\boxed{Y_{max^m} = -\frac{WL^3}{48EI}}$$

ALTERNATE METHOD: There is also an alternative way to attempt this problem in a more simpler way. Let us considering the origin at the point of application of the load,



$$S.F|_{xx} = \frac{W}{2}$$

$$B.M|_{xx} = \frac{W}{2}\left(\frac{l}{2} - x\right)$$

substituting the value of M in the governing equation for the deflection

$$\frac{d^2y}{dx^2} = \frac{\frac{W}{2}\left(\frac{l}{2} - x\right)}{EI}$$

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{WLx}{4} - \frac{Wx^2}{4}\right] + A$$

$$y = \frac{1}{EI}\left[\frac{WLx^2}{8} - \frac{Wx^2}{12}\right] + Ax + B$$

Boundary conditions relevant for this case are as follows

(i) at x = 0; dy/dx= 0

hence, A = 0

(ii) at x = l/2; y = 0 (because now l / 2 is on the left end or right end support since we have taken the origin at the centre)

Thus,

$$0=\left[\frac{WL^3}{32}-\frac{WL^3}{96}+B\right]$$

$$B=-\frac{WL^3}{48}$$

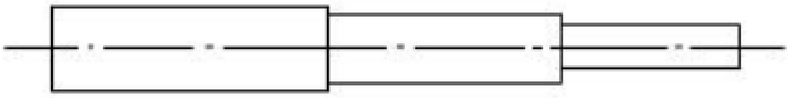
Hence he equation which governs the deflection would be

$$y=\frac{1}{EI}\left[\frac{WLx^2}{8}-\frac{Wx^3}{12}-\frac{WL^3}{48}\right]$$

Hence

$$\boxed{\begin{array}{lll} Y_{max^m}\Big|_{at\,x=0}=-\dfrac{WL^3}{48EI} & \text{At the centre} \\ \left(\dfrac{dy}{dx}\right)_{max^m}\Big|_{at\,x=\pm\frac{L}{2}}=\pm\dfrac{WL^2}{16EI} & \text{At the ends} \end{array}}$$

Hence the integration method may be bit cumbersome in some of the case. Another limitation of the method would be that if the beam is of non uniform cross section,



i.e. it is having different cross-section then this method also fails.

So there are other methods by which we find the deflection like

1. Macaulay's method in which we can write the different equation for bending moment for different sections.
2. Area moment methods
3. Energy principle methods

Introduction: slope

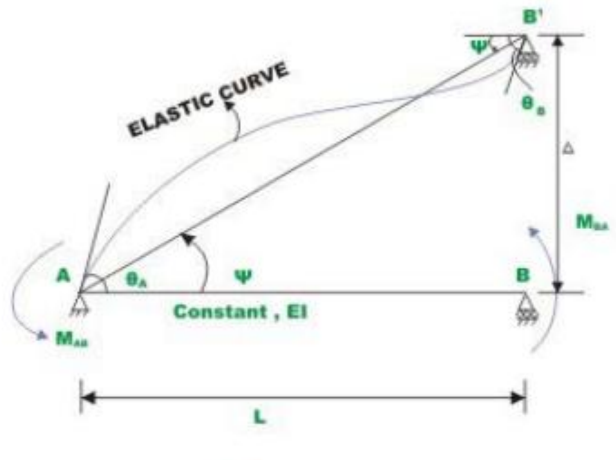
In the last lesson, slope-deflection equations were derived without considering the rotation of the beam axis. In this lesson, slope-deflection equations arederived considering the rotation of beam axis. In statically indeterminate structures, the beam axis rotates due to support yielding and this would in turn induce reactions and stresses in the structure. Hence, in this case the beam end moments are related to rotations, applied loads and beam axes rotation. After deriving the slope-deflection equation in section 15.2, few problems are solved to illustrate the procedure.

Consider a beam AB as shown in Fig.15.1.The support B is at a higher elevation compared to A by an amount Δ . Hence, the member axis has rotated by an amount ψ from the original direction as shown in the figure. Let L be the span of the beam and flexural rigidity of the beam EI , is assumed to be constant for the beam. The chord has rotated in the counterclockwise direction with respect to its original direction. The counterclockwise moment and rotations are assumed to be positive. As stated earlier, the slopes and rotations are derived by

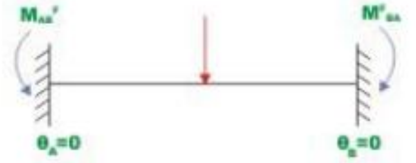
superposing the end moments developed due to

(1) Externally applied moments on beams.

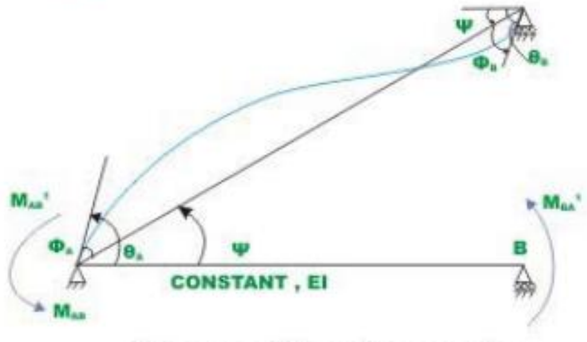
(2) Displacements θ A ,θ B and Δ (settlement)



(a)



b) kinematically determinate retrained structure



c) Beam with end moments

The given beam with initial support settlement may be thought of as superposition of two simple cases as shown in fig, the kinematically determinate beam is shown with the applied load. For this case, the fixed end moments are calculated by force method. Let φ A and φ B be the end rotations of the elastic curve with respect to rotated beam axis AB' (see Fig.15.1c) that are caused by end moments and

$$\phi_B = \theta_B - \psi = \frac{M_{BA}'L}{3EI} - \frac{M_{AB}'L}{6EI}$$

Now solving for M_A' and M_B' in terms of θ_A, θ_B and ψ ,

$$M_{AB}' = \frac{2EI}{L}(2\theta_A + \theta_B - 3\psi)$$

$$M_{BA}' = \frac{2EI}{L}(2\theta_B + \theta_A - 3\psi)$$

Now superposing the fixed end moments due to external load and end moments due to displacements, the end moments in the actual structure is obtained .Thus

$$M_{AB} = M_{AB}^F + M_{AB}'$$

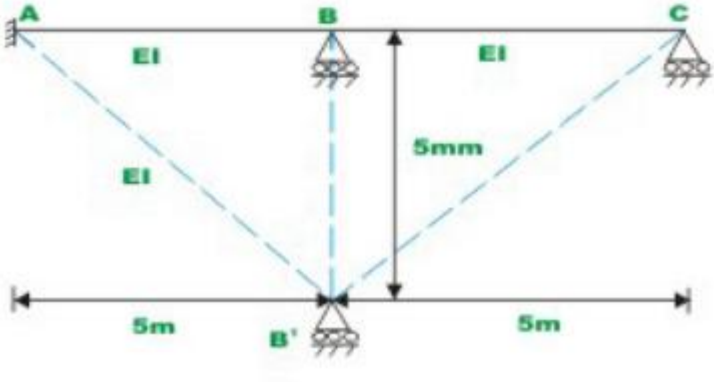
$$M_{BA} = M_{BA}^F + M_{BA}'$$

$$M_{AB} = M_{AB}^F + \frac{2EI}{L}(2\theta_A + \theta_B - 3\psi)$$

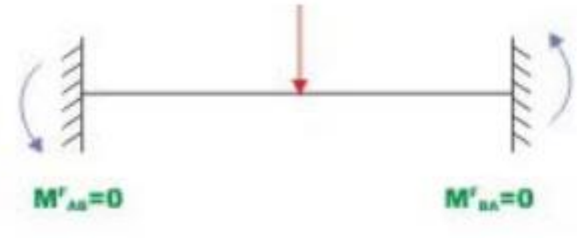
$$M_{BA} = M_{BA}^F + \frac{2EI}{L}(2\theta_B + \theta_A - 3\psi)$$

Example 15.1

Calculate the support moments in the continuous beam having constant flexural rigidity ABC EI throughout ,due to vertical settlement of the support B by 5mm. Assume E =200 GPa and I = .Also plot quantitative elastic curve.



In the continuous beam ABC, two rotations θ B and θ C need to be evaluated. Hence, beam is kinematically indeterminate to second degree. As there is no external load on the beam, the fixed end moments in the restrained beam are zero.



For each span, two slope-deflection equations need to be written. In span AB, B is below A . Hence, the chord AB rotates in clockwise direction. Thus, ψ AB is taken as negative

$$\psi_{AB} = \frac{-5\times10^{-3}}{5} = -1\times10^{-3}$$

Writing slope-deflection equation for span AB ,

$$M_{AB} = \frac{2EI}{L}(2\theta_A + \theta_B - 3\psi_{AB})$$

For span AB , $\theta_A = 0$, Hence,

$$M_{AB} = \frac{2EI}{5}(\theta_B + 3\times10^{-3})$$

$$M_{AB} = 0.4EI\theta_B + .0012EI$$

Similarly, for beam-end moment at end B , in span AB

$$M_{BA} = 0.4EI(2\theta_B + 3\times10^{-3})$$
$$M_{BA} = 0.8EI\theta_B + 0.0012EI$$

In span BC , the support C is above support B , Hence the chord joining ′CB rotates in anticlockwise direction.

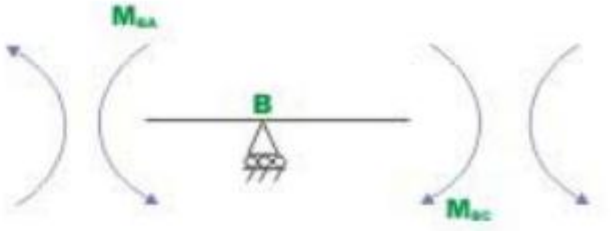
$$\psi_{BC} = \psi_{CB} = 1 \times 10^{-3}$$

Writing slope-deflection equations for span BC,

$$M_{BC} = 0.8EI\theta_B + 0.4EI\theta_C - 1.2 \times 10^{-3} EI$$

$$M_{CB} = 0.8EI\theta_C + 0.4EI\theta_B - 1.2 \times 10^{-3} EI$$

Now, consider the joint equilibrium of support B (see Fig.15.2c)



$$M_{BA} + M_{BC} = 0$$

Substituting the values of M_{BA} and M_{BC} in equation (6),

$$0.8EI\theta_B + 1.2 \times 10^{-3} EI + 0.8EI\theta_B + 0.4EI\theta_C - 1.2 \times 10^{-3} EI = 0$$

Simplifying,

$$1.6\theta_B + 0.4\theta_C = 1.2 \times 10^{-3}$$

Also, the support C is simply supported and hence, $M_{CB} = 0$

$$M_{CB} = 0 = 0.8\theta_C + 0.4\theta_B - 1.2 \times 10^{-3} EI$$

$$0.8\theta_C + 0.4\theta_B = 1.2 \times 10^{-3}$$

We have two unknowns θ B and θ C and there are two equations in θ B andθ C. Solving equations,

$\theta_B = -0.4286 \times 10^{-3}$ radians

$\theta_C = 1.7143 \times 10^{-3}$ radians

Substituting the values of θ B ,θ C and EI in slope-deflection equations,

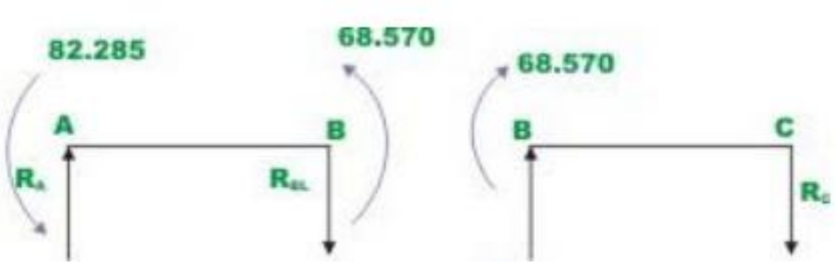
$$M_{AB} = 82.285\,\text{kN.m}$$

$$M_{BA} = 68.570\,\text{kN.m}$$

$$M_{BC} = -68.573\,\text{kN.m}$$

$$M_{CB} = 0\,\text{kN.m}$$

Reactions are obtained from equations of static equilibrium



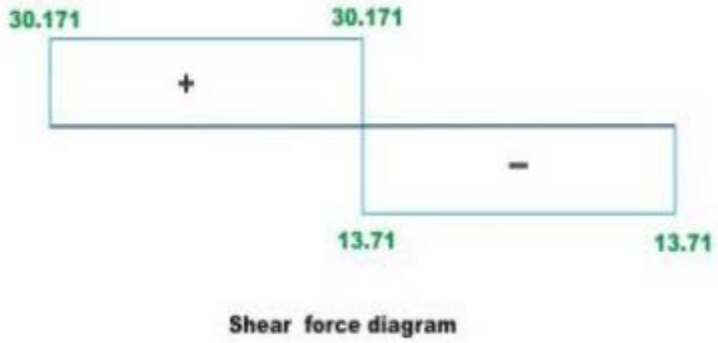
In beam AB ,

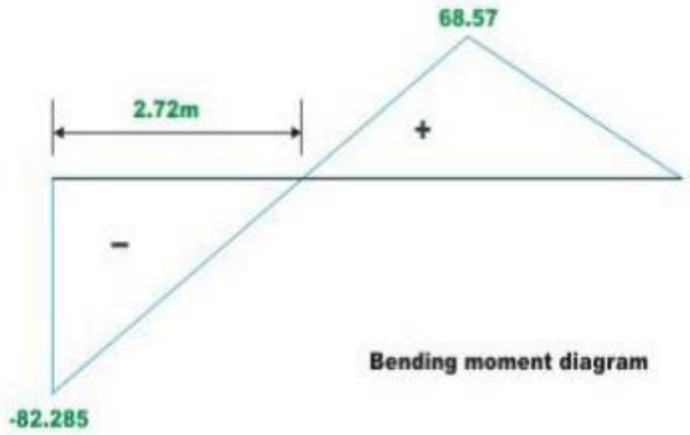
$$\sum M_B = 0,\ R_A = 30.171\,\text{kN}(\uparrow)$$

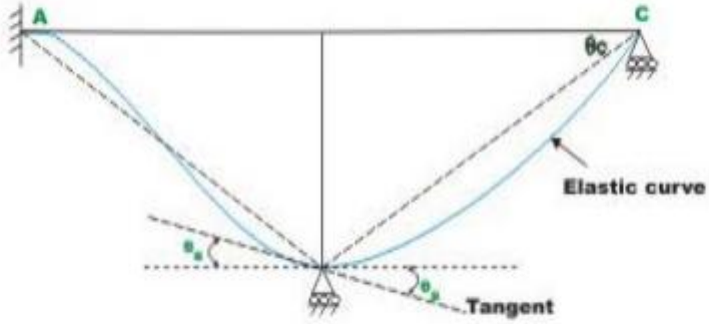
$$R_{BL} = -30.171\,\text{kN}(\downarrow)$$

$$R_{BR} = -13.714\,\text{kN}(\downarrow)$$

$$R_C = 13.714\,\text{kN}(\uparrow)$$







Elastic curve

Summary

slope-deflection equations are derived for the case of beam with yielding supports. Moments developed at the ends are related to rotations and support settlements. The equilibrium equations are written at each support. The continuous beam is solved using slope-deflection equations. The deflected shape of the beam is sketched. The bending moment and shear force diagrams are drawn for the examples solved in this lesson.