LECTURE NOTES

ON

NETWORK ANALYSIS

B. Tech III Semester (IARE-R18)

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SYLLABUS

MODULE-I NETWORK THEOREMS (DC AND AC)

Network Theorems: Tellegen's, superposition, reciprocity, Thevenin's, Norton's, maximum power transfer, Milliman's and compensation theorems for DC and AC excitations, numerical problems.

MODULE-II SOLUTION OF FIRST AND SECOND ORDER NETWORKS

Transient response: Initial conditions, transient response of RL, RC and RLC series and parallel circuits with DC and AC excitations, differential equation and Laplace transform approach.

MODULE-III LOCUS DIAGRAMS AND NETWORKS FUNCTIONS

Locus diagrams: Locus diagrams of RL, RC, RLC circuits.

Network Functions: The concept of complex frequency, physical interpretation, transform impedance, series and parallel combination of elements, terminal ports, network functions for one port and two port networks, poles and zeros of network functions, significance of poles and zeros, properties of driving point functions and transfer functions, necessary conditions for driving point functions and transfer functions, time domain response from pole-zero plot.

MODULE-IV TWO PORTNETWORK PARAMETERS

Two port network parameters: Z, Y, ABCD, hybrid and inverse hybrid parameters, conditions for symmetry and reciprocity, inter relationships of different parameters, interconnection (series, parallel and cascade) of two port networks, image parameters.

MODULE-V FILTERS

Filters: Classification of filters, filter networks, classification of pass band and stop band, characteristic impedance in the pass and stop bands, constant-k low pass filter, high pass filter, m-derived T-section, band pass filter and band elimination filter.

Text Books:

1. A Chakrabarthy, "Electric Circuits", Dhanpat Rai & Sons, 6th Edition, 2010.
2. A Sudhakar, Shyammohan S Palli, "Circuits and Networks", Tata McGraw-Hill, 4th Edition, 2010
3. M E Van Valkenberg, "Network Analysis", PHI, 3rd Edition, 2014.
4. Rudrapratap, "Getting Started with MATLAB: A Quick Introduction for Scientists and Engineers", Oxford University Press, 1st Edition, 1999.

Reference Books:

1. John Bird, "Electrical Circuit Theory and technology", Newnes, 2nd Edition, 2003
2. C L Wadhwa, "Electrical Circuit Analysis including Passive Network Synthesis", New Age International, 2nd Edition, 2009.
3. David A Bell, "Electric Circuits", Oxford University Press, 7th Edition, 2009.

UNIT – I

NETWORK THEOREMS (DC AND AC)

1. INTRODUCTION:

In electric network analysis, the fundamental rules are Ohm's Law and Kirchhoff's Laws. While these humble laws may be applied to analyze just about any circuit configuration (even if we have to resort to complex algebra to handle multiple unknowns), there are some "shortcut" methods of analysis to make the math easier for the average human.

As with any theorem of geometry or algebra, these network theorems are derived from fundamental rules. In this chapter, I'm not going to delve into the formal proofs of any of these theorems. If you doubt their validity, you can always empirically test them by setting up example circuits and calculating values using the "old" (simultaneous equation) methods versus the "new" theorems, to see if the answers coincide.

Network theorems are also can be termed as network reduction techniques. Each and every theorem got its importance of solving network. Let us see some important theorems with DC and AC excitation with detailed procedures.

1. 1 TELLEGEN'S THEOREM:

Dc Excitation:

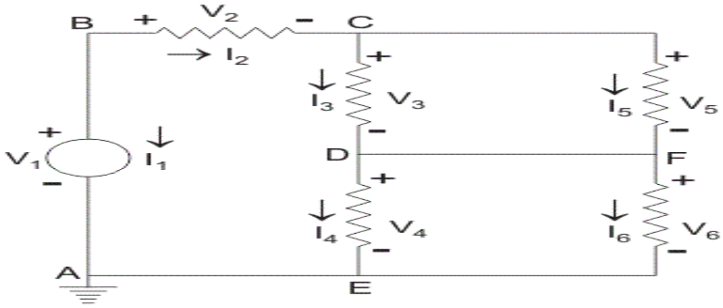
Tellegen's theorem states algebraic sum of all delivered power must be equal to sum of all received powers.

According to Tellegen's theorem, the summation of instantaneous powers for the n number of branches in an electrical network is zero. Are you confused? Let's explain. Suppose n number of branches in an electrical network have i1, i2, i3…. in respective instantaneous currents through them. These currents satisfy Kirchhoff's Current Law. Again, suppose these branches have instantaneous voltages across them are v1, v2, v3, ... vn respectively. If these voltages across these elements satisfy Kirchhoff Voltage Law then,

$$\sum_{k=1}^{n} v_k \cdot i_k = 0$$

v_k is the instantaneous voltage across the k^{th} branch and i_k is the instantaneous current flowing through this branch. **Tellegen's theorem** is applicable mainly in general class of lumped networks that consist of linear, non-linear, active, passive, time variant and time variant elements.

This theorem can easily be explained by the following example.



In the network shown, arbitrary reference directions have been selected for all of the branch currents, and the corresponding branch voltages have been indicated, with positive reference direction at the tail of the current arrow. For this network, we will assume a set of branch voltages satisfy the Kirchhoff voltage law and a set of branch current satisfy Kirchhoff current law at each node.

We will then show that these arbitrary assumed voltages and currents satisfy the equation.

$$\sum_{k=1}^{n} v_k \cdot i_k = 0$$

And it is the condition of **Tellegen's theorem**. In the network shown in the figure, let v_1, v_2 and v_3 be 7, 2 and 3 volts respectively. Applying Kirchhoff Voltage Law around loop ABCDEA. We see that v_4 = 2 volt is required. Around loop CDFC, v_5 is required to be 3 volt and around loop DFED, v_6 is required to be 2. We next apply Kirchhoff's Current Law successively to nodes B, C and D. At node B let i_i = 5 A, then it is required that i_2 = - 5 A. At node C let i_3 = 3 A and then i_5 is required to be - 8. At node D assume i_4 to be 4 then i_6 is required to be - 9. Carrying out the operation of equation,

We get,

$$7 \times 5 + 2 \times (-5) + 3 \times 3 + 2 \times 4 + 3 \times (-8) + 2 \times (-9) = 0$$

Hence **Tellegen's theorem** is verified.

1.2 SUPER-POSITION THEOREM:

DC: " In an any linear , bi-lateral network consisting number of sources , response in any element(resistor) is given as sum of the individual Reponses due to individual sources, while other sources are non-operative**"**

AC: " In an any linear , bi-lateral network consisting number of sources , response in any element(impedance) is given as sum of the individual Reponses due to individual sources, while other sources are non-operative**"**

Procedure of Superposition Theorem:

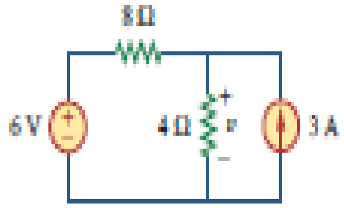
Follow these steps in order to find the response in a particular branch using superposition theorem.

Step 1 − Find the response in a particular branch by considering one independent source and eliminating the remaining independent sources present in the network.

Step 2 − Repeat Step 1 for all independent sources present in the network.

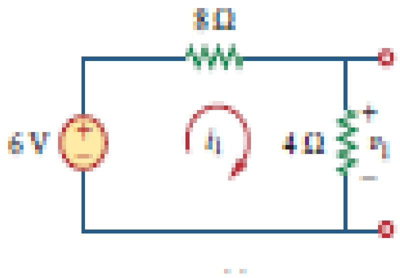
Step 3 − Add all the responses in order to get the overall response in a particular branch when all independent sources are present in the network.

Eg:



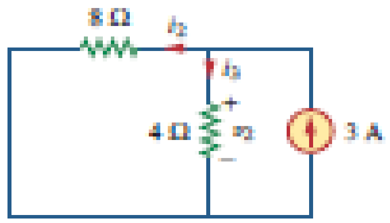
Let V = 6v, I = 3A, R1 = 8 ohms and R2 = 4 ohms

Let us find current through 4 ohms using V source, while I is zero. Then equivalent circuit is



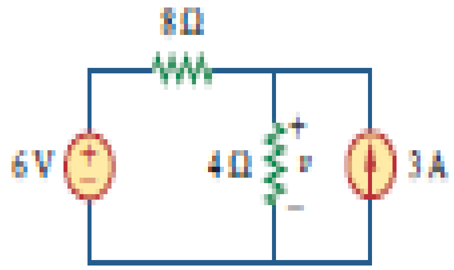
Let i_1 is the current through 4 ohms, i1 = V / (R1+R2)

Let us find current through 4 ohms using I source, while V is zero. Then equivalent circuit is



Let i_2 is the current through 4 ohms, i_2 = I. R1 / (R_1+R_2)

Hence total current through 4 ohms is = I_1+I_2(as both currents are in same direction or otherwise I1-I2)

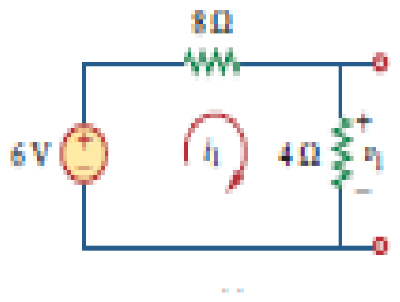


Eg:

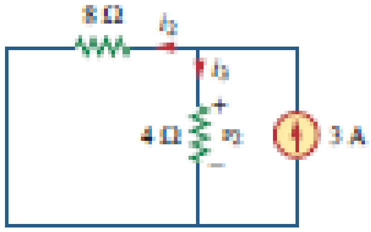
Let V = 6v, I = 3A, Z1 = 8 ohms and Z_2 = 4 ohms

Let us find current through 4 ohms using V source , while I is zero. Then equivalent circuit is

Let i_1 is the current through 4 ohms, i_1 = V / (Z1+Z2)



Let us find current through 4 ohms using I source, while V is zero. Then equivalent circuit is

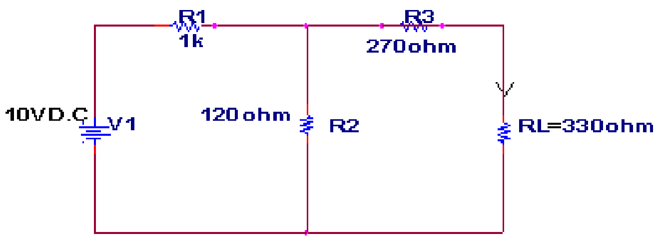


Let i2 is the current through 4 ohms, i2 = I. Z1 / (Z1+Z2)

Hence total current through 4 ohms is = I_1+I_2 (as both currents are in same direction or otherwise I_1-I_2).

1.3 RECIPROCITY THEOREM:

DC & AC: " In any linear bi-lateral network ratio of voltage in one mesh to current in other mesh is same even if their positions are inter-changed".



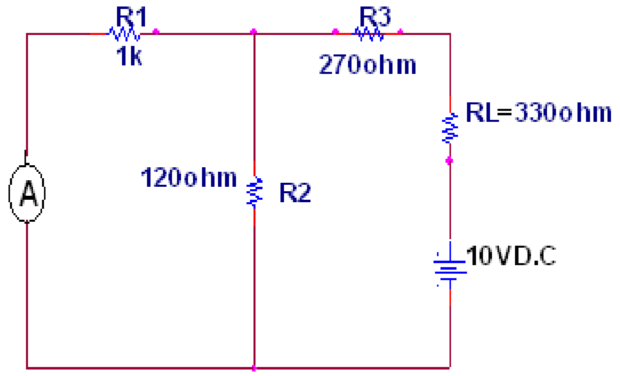
Eg:
Find the total resistance of the circuit, Rt = R1+ [R2(R3+Rl)] / R2+R3+RL.

Hence source current, I = V1 / Rt.

Current through RL is I1 = I. R2 / (R2+R3+RL)

Take the ratio of , V1 / I1 ---1

Draw the circuit by inter changing position of V1 and I1



Find the total resistance of the circuit, Rt = (R3+RL) + [R2(Rl)] / R2+R1.

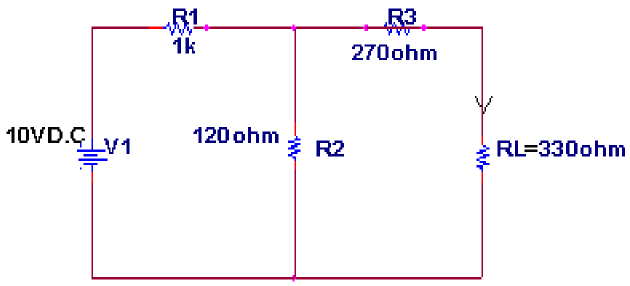
Hence source current, I = V1 / Rt.

Current through RL is I1 = I. R2 / (R2+R1)

Take the ratio of , V1 / I1 ---2

If ratio 1 = ratio 2, then circuit is said to be satisfy reciprocity.

Eg: **With AC source**



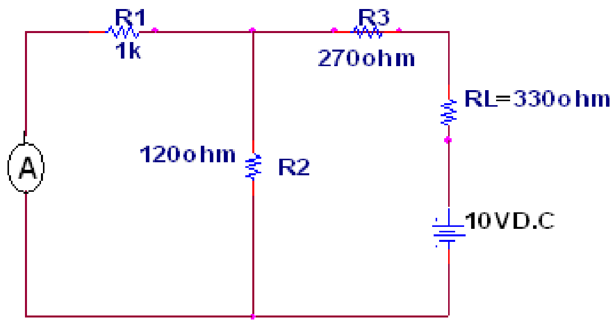
Find the total impedance of the circuit, Zt = Z1+ [Z2(Z3+ZL)] / Z2+ Z 3+ Z L.

Hence source current, I = V1 / Z t.

Current through ZL is I1 = I. Z 2 / (Z2+ Z3+ ZL)

Take the ratio of , V1 / I1 ---1

Draw the circuit by inter changing position of V1 and I1



Find the total impedanceof the circuit, Zt = (Z3+ ZL) + [Z2(ZL)] / Z2+ Z1.

Hence source current, I = V1 / Z t.

Current through ZL is I1 = I. Z2 / (Z2+ Z1)

Take the ratio of , V1 / I1 ---2

If ratio 1 = ratio 2, then circuit is said to be satisfy reciprocity.

*** Here only magnitudes are compared

1.4 THEVENIN'S THEOREM:

DC: " An complex network consisting of number voltage and current sources and be replaced by simple series circuit consisting of equivalent voltage source in series with equivalent resistance, where equivalent voltage is called as open circuit voltage and equivalent resistance is called as Thevenin's resistance calculated across open circuit terminals while all energy sources are non-operative"

AC: " An complex network consisting of number voltage and current sources and be replaced by simple series circuit consisting of equivalent voltage source in series with equivalent impedance, where equivalent voltage is called as open circuit voltage and equivalent impedance is called as Thevenin's impedance calculated across open circuit terminals while all energy sources are non-operative"

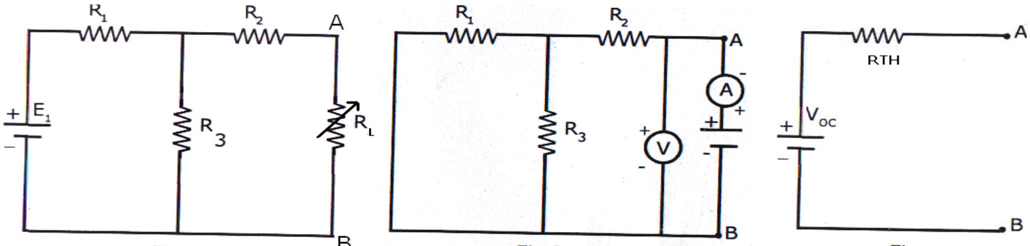


Fig 1 Fig 2 Fig3

Eg:
Here we need to find current through RL using Thevenin's theorem.

Open circuit the AB terminals to find the Thevenin's voltage.

Thevenin’s voltage , Vth = E1. R3 / (R1+R3) ----1 from figure .1

Thevenin’s resistance, Rth = (R1.R3)/ (R1+R3) + R2 ----2 from figure 2.

Now draw the thevenin’s equivalent circuit as shown in figure 3 with calculated values.

Eg: **With AC excitation**

Here we need to find current through ZL using thevenin’s theorem.

Open circuit the AB terminals to find the Thevenin’s voltage.

Thevenin’s voltage , Vth = E1. R3 / (R1+R3) ----1 from figure .1

Thevenin’s impedance, Zth = (Z1. Z3)/ (Z1+ Z3) + Z2 ----2 from figure 2.

Now draw the thevenin’s equivalent circuit as shown in figure 3 with calculated values.

1.5 NORTON’S THEOREM:

DC: “ An complex network consisting of number voltage and current sources and be replaced by simple parallel circuit consisting of equivalent current source in parallel with equivalent resistance, where equivalent current source is called as short circuit current and equivalent resistance is called as Norton’s resistance calculated across open circuit terminals while all energy sources are non-operative”

AC: “An complex network consisting of number voltage and current sources and be replaced by simple parallel circuit consisting of equivalent current source in parallel with equivalent impedance, where equivalent current source is called as short circuit current and equivalent impedance is called as Norton’s impedance calculated across open circuit terminals while all energy sources are non-operative”

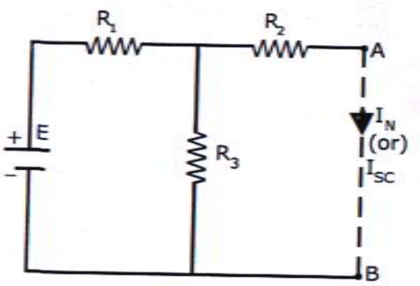


Fig 1

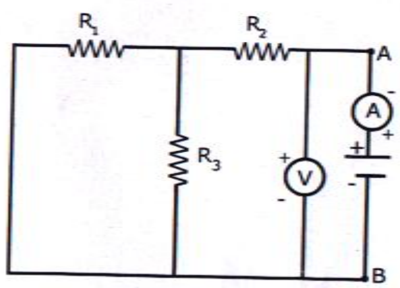


Fig 2

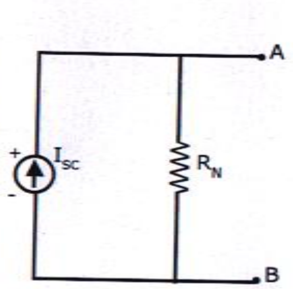


Fig 3

Here we need to find current through RL using Norton’s theorem.

Short circuit the AB terminals to find the Norton’s current.

Total resistance of circuit is, Rt = (R2.R3) / (R2+R3) + R1

Source current, I = E / Rt

Norton’s current , IN = I. R3 / (R2+R3) ----1 from figure .1

Norton’s resistance, RN = (R1.R3)/ (R1+R3) + R2 ----2 from figure 2.

Now draw the Norton's equivalent circuit as shown in figure 3 with calculated values.

Eg: **With AC excitation**

Here we need to find current through ZL using Norton's theorem.

Short circuit the AB terminals to find the Norton's current.

Total impedance of circuit is, Zt = (Z2. Z3) / (Z2+Z3) + Z1

Source current, I = E / Zt

Norton's current, IN = I. Z3 / (Z2+Z3) ----1 from figure .1

Norton's impedance, ZN = (Z1. Z3)/ (Z1+Z3) + Z2 ----2 from figure 2.

Now draw the Norton's equivalent circuit as shown in figure 3 with calculated values.

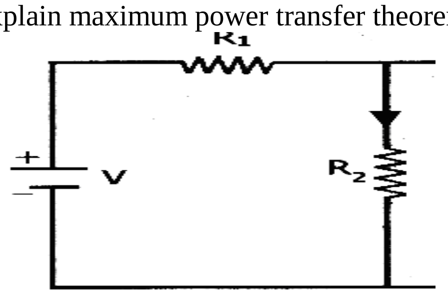
*** These two theorems are useful in determining the load value for which maximum power transfer can be happened.

1.6 MAXIMUM POWER TRANSFER THEOREM:

DC: **"** In linear bi-lateral network maximum power can be transferred from source to load if load resistance is equal to source or thevenin's or internal resistances".

AC: **"** In linear bi-lateral network maximum power can be transferred from source to load if load impedance is equal to complex conjugate of source or thevenin's or internal impedances"

Eg: For the below circuit explain maximum power transfer theorem.



Let I be the source current, $I = V / (R1+R2)$

Power absorbed by load resistor is, $PL = I^2 .R2$

$$= [V / (R1+R2)]^2 .R2.$$

To say that load resistor absorbed maximum power , $dPL / dR2 = 0$.

When we solve above condition we get, R2 = R1.

Hence maximum power absorbed by load resistor is, PLmax = V^2 / 4R2.

Eg: **AC excitation**

Let I be the source current, I = V / (Z1+ Z 2)

Power absorbed by load impedance is, PL = I^2 . Z2

= **[** V / (Z1+ Z2)]^2 . Z2.

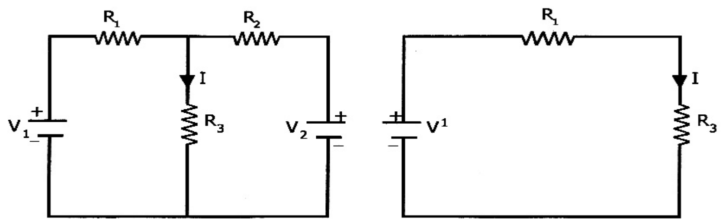
To say that load resistor absorbed maximum power , dPL / d Z2 = 0.

When we solve above condition we get, Z2 = Z1*.

Hence maximum power absorbed by load resistor is, PLmax = V^2 / 4 Z2.(magnitude)

1.7 MILLIMAN'S THEOREM:

DC: " An complex network consisting of number of parallel branches , where each parallel branch consists of voltage source with series resistance, can be replaced with equivalent circuit consisting of one voltage source in series with equivalent resistance"



Where equivalent voltage source value is , V' = $\dfrac{V1G1+V2G2+\text{------}+VnGn}{G1+G2+\text{----------------}Gn}$

Equivalent resistance is , R' = 1 / (G1+G2+------------------Gn)

AC: " An complex network consisting of number of parallel branches , where each parallel branch consists of voltage source with series impedance, can be replaced with equivalent circuit consisting of one voltage source in series with equivalent impedance"

Where equivalent voltage source value is , V' = $\dfrac{V1Y1+V2Y2+\text{------}+VnYn}{Y1+Y2+\text{----------------}Yn}$

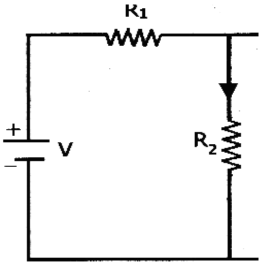
Equivalent resistance is , Z' = 1 / (Y1+Y2+------------------Yn)

*** It is also useful in designing load value for which it absorbs maximum power.

1.8 . COMPENSATION THEOREM:

DC &AC: "compensation theorem states that any element in the network can be replaced with

Voltage source whose value is product of current through that element and its value”
It is useful in finding change in current when sudden change in resistance value.



For the above circuit source current is given as, I = V / (R1+R2)

Element R2 can be replaced with voltage source of ,V’ = I.R2

Let us assume there is change in R2 by ΔR, now source current is I’= V / (R1+R2+ ΔR)

Hence actual change in current from original circuit to present circuit is = I – I’.

This can be find using compensation theorem as, making voltage source non-operative and replacing ΔR with voltage source of I’. ΔR.

Then change in current is given as = I’. ΔR / (R1+R2)

Eg: **AC excitation**

For the above circuit source current is given as, I = V / (Z1+Z2)

Element R2 can be replaced with voltage source of ,V’ = I.Z2

Let us assume there is change in R2 by ΔR, now source current is I’= V / (Z1+Z2+ ΔZ)

Hence actual change in current from original circuit to present circuit is = I – I’.

This can be find using compensation theorem as, making voltage source non-operative and replacing ΔR with voltage source of I’. ΔZ.

Then change in current is given as = I’. Z/ (Z1+Z2)

EXAMPLES:

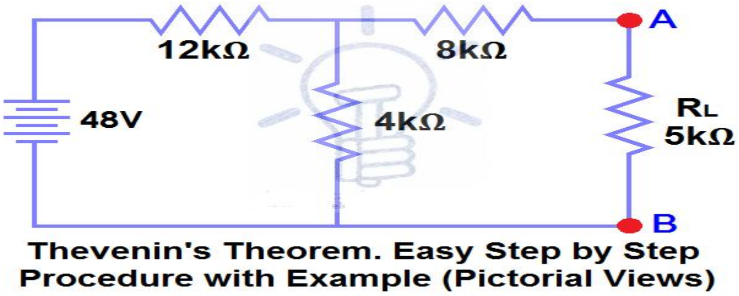
Steps to Analyze Electric Circuit through Thevenin’s Theorem

1. Open the load resistor.
2. Calculate / measure the open circuit voltage. This is the **Thevenin Voltage (V_{TH})**.
3. Open current sources and short voltage sources.
4. Calculate /measure the Open Circuit Resistance. This is the **Thevenin Resistance (R_{TH})**.
5. Now, redraw the circuit with measured **open circuit Voltage (V_{TH})** in Step (2) as voltage source and measured **open circuit resistance (R_{TH})** in step (4) as a series resistance and connect the load resistor which we had removed in Step (1). This is the **equivalent Thevenin circuit** of that **linear electric network** or **complex circuit** which had to be ***simplified and analyzed by Thevenin’s Theorem***. You have done.
6. Now find the Total current flowing through load resistor by using the Ohm’s Law: $I_T = V_{TH}/ (R_{TH}+ R_L)$.

Solved Example by Thevenin's Theorem:

Example:

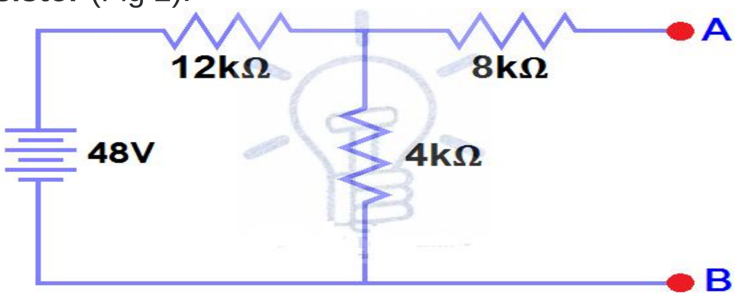
Find V_{TH}, R_{TH} and the load current flowing through and load voltage across the load resistor in fig (1) *by using Thevenin's Theorem*.



Solution:

Step 1:

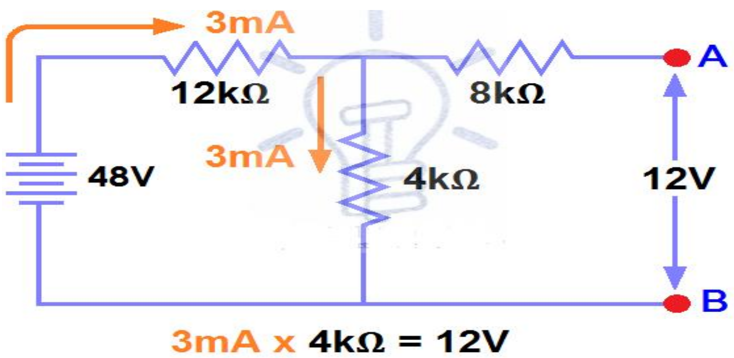
Open the **5kΩ load resistor** (Fig 2).



Step 2:

Calculate / measure the open circuit voltage. This is the **Thevenin Voltage (V_{TH})**. Fig (3).
We have already removed the load resistor from figure 1, so the circuit became an **open circuit** as shown in fig 2. Now we have to calculate the Thevenin's Voltage. Since **3mA** current flows in both **12kΩ** and **4kΩ resistors** as this is a series circuit because current will not flow in the **8kΩ** resistor as it is open.
So **12V (3mA x 4kΩ)** will appear across the **4kΩ resistor**. We also know that current is not flowing through the 8kΩ resistor as it is open circuit, but the 8kΩ resistor is in parallel with **4k resistor**. So the same voltage i.e. **12V will appear across the 8kΩ resistor** as well as 4kΩ resistor. Therefore 12V will appear across the AB terminals. So,

$$V_{TH} = 12V$$



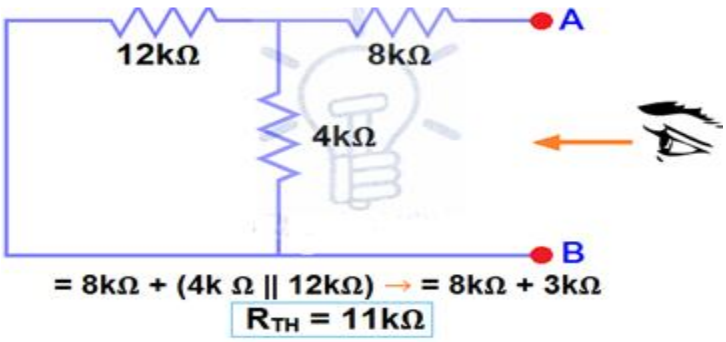
Step 3:Open current sources and **short voltage sources** as shown below. Fig (4)

calculate / **measure the open circuit resistance**. This is the Thevenin Resistance (R_{TH})
We have removed the **48V DC source** to **zero** as equivalent i.e. 48V DC source has been replaced with a short in step 3 (as shown in figure 3). We can see that 8kΩ resistor is in series with a parallel connection of 4kΩ resistor and 12k Ω resistor. i.e.:

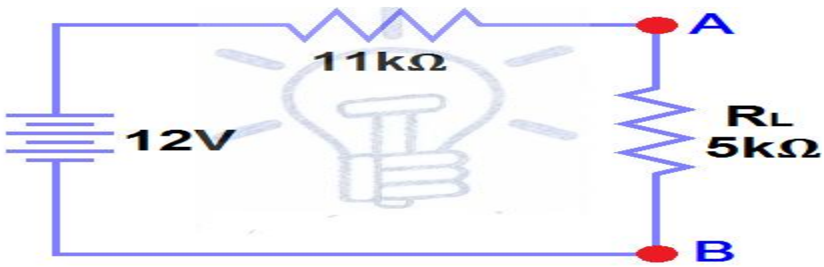
8kΩ + (4k Ω || 12kΩ) ….. (|| = in parallel with)

$$R_{TH} = 8k\Omega + [(4k\Omega \times 12k\Omega) / (4k\Omega + 12k\Omega)]$$
$$R_{TH} = 8k\Omega + 3k\Omega$$

$$R_{TH} = 11k\Omega$$



Step 5.
Connect the R_{TH}in series with Voltage Source V_{TH} and re-connect the load resistor. This is shown in fig (6) i.e. Thevenin circuit with load resistor. This the Thevenin's equivalent circuit.



now apply the last step i.e Ohm's law . **Calculate the total load current & load voltage** as shown in fig 6.

$$I_L = V_{TH} / (R_{TH} + R_L)$$
$$= 12V / (11k\Omega + 5k\Omega) \rightarrow = 12/16k\Omega$$
$$I_L = 0.75mA$$
And
$$V_L = I_L x R_L$$
$$V_L = 0.75mA \times 5k\Omega$$
$$V_L = 3.75V$$

UNIT – II

SOLUTION OF FIRST AND SECOND ORDER NETWORKS

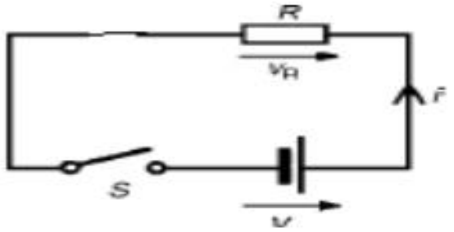
TRANSIENT RESPONSE FOR DC CIRCUITS

INTRODUCTION:

For higher order differential equation, the number of arbitrary constants equals the order of the equation. If these unknowns are to be evaluated for particular solution, other conditions in network must be known. A set of simultaneous equations must be formed containing general solution and some other equations to match number of unknown with equations.

We assume that at reference time t=0, network condition is changed by switching action. Assume that switch operates in zero time. The network conditions at this instant are called initial conditions in network.

2. 1 Resistor :



$$V = i \times R \longrightarrow (1)$$

Equation 1 is linear and also time dependent. This indicates that current through resistor changes if applied voltage changes instantaneously. Thus in resistor, change in current is instantaneous as there is no storage of energy in it.

2.2. Inductor:

If dc current flows through inductor, dil/dt becomes zero as dc current is constant with respect to time. Hence voltage across inductor, VL becomes zero. Thus, as for as dc quantities are considered, in steady stake, inductor acts as short circuit.

$$V_L = L \bullet \frac{diL}{dt} \longrightarrow (2)$$

$$iL = \frac{1}{L}\int V_L dt$$

In above eqn. The limits of integration is from $-\infty$ to $t(0^-)$.

Assuming that switching takes place at t=0, we can split limits into two intervals as $-\infty$ to $t(0^-)$.

$$i_L = \frac{1}{L}\int_{-\infty}^{t} V_L dt$$

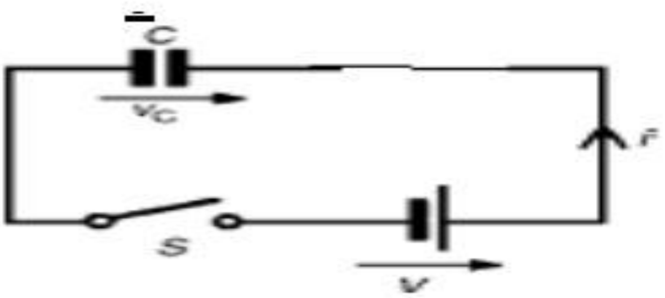
$$i_L = \frac{1}{L}\int_{-\infty}^{0^-} V_L dt + \frac{1}{L}\int_{0^-}^{t} V_L dt$$

$$i_L = i_L(0^-) + \frac{1}{L}\int_{0^-}^{t} V_L dt$$

at $t = 0^t$ we can write $i_L(0^+) = i_L(0^-) + \frac{1}{L}\int V_L dt$

$i_L(0^+) = i_L(0^-)$

2.3. capacitor:



$$i_C = C\frac{dV_C}{dt}$$

If dc voltage is applied to capacitor, dVC / dt becomes zero as dc voltage is constant with respect to time.

Hence the current through capacitor iC becomes zero, Thus as far as dc quantities are considered capacitor acts as open circuit.

$$V_C = \frac{1}{C}\int i_C dt$$

$$V_C = \frac{1}{C}\int_{-\infty}^{t} i_C dt$$

Splitting limits of integration

$$V_C = \frac{1}{C}\int_{-\infty}^{0-} i_C dt + \frac{1}{C}\int_{0^-}^{t} i_C dt$$

At $t(0^+)$, equation is given by

$$V_C(0^+) = V_C(0^-) + \frac{1}{C}\int_{0-}^{0+} i_C dt$$

$$V_C(0^+) = V_C(0^-)$$

Thus voltage across capacitor can not change instantaneously.

2.4Initial Condition for (DC steady state solution)

• Initial condition: response of a circuit before a switch is first activated.
– Since power equals energy per unit time, finite power requires continuous change in energy.
• Primary variables: capacitor voltages and inductor currents-> energy storage elements

$$W_L(t) = \frac{1}{2}Li_L^2(t) \quad W_C(t) = \frac{1}{2}Cv_C^2(t)$$

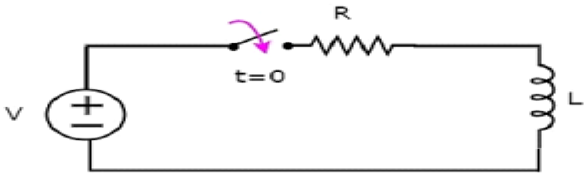
- ❖ Capacitor voltages and inductor currents cannot change instantaneously but should be continuous. -> continuity of capacitor voltages and inductor currents

- ❖ The value of an inductor current or a capacitor voltage just prior to the closing (or opening) of a switch is equal to the value just after the switch has been closed (or opened).

$$v_C(t = 0^-) = v_C(t = 0^+)$$
$$i_L(t = 0^-) = i_L(t = 0^+)$$

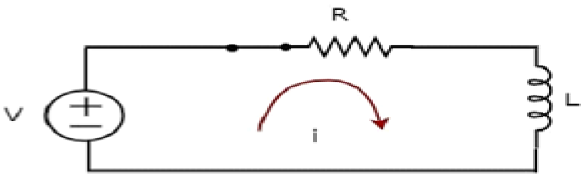
2.5 TRANSIENT RESPONSE OF RL CIRCUITS WITH DC EXCITATIO:

Consider the following **series RL circuit** diagram.



In the above circuit, the **switch** was kept **open** up to t = 0 and it was closed at t = 0. So, the DC voltage source having V volts is not connected to the series RL circuit up to this instant. Therefore, there is **no initial current** flows through inductor.

The circuit diagram, when the **switch** is in **closed** position is shown in the following figure.



Now, the current ***i*** flows in the entire circuit, since the DC voltage source having ***V*** volts is connected to the series RL circuit.

Now, apply **KVL** around the loop.

$$V = Ri + L\frac{di}{dt}$$

$\frac{di}{dt} + (\frac{R}{L})i = \frac{V}{L}$ **Equation 1**

The above equation is a first order differential equation and it is in the form of

$\frac{dy}{dt} + Py = Q$ **Equation 2**

By **comparing** Equation 1 and Equation 2, we will get the following relations.

$$x = t$$

$$y = i$$

$$P = \frac{R}{L}$$

$$Q = \frac{V}{L}$$

The **solution** of Equation 2 will be

$ye^{\int pdx} = \int Qe^{\int pdx}dx + k$ **Equation 3**

Where, **k** is the constant.

Substitute, the values of x, y, P & Q in Equation 3.

$$ie^{\int(\frac{R}{L})dt} = \int(\frac{V}{L})(e^{\int(\frac{R}{L})dt})dt + k$$

$$\Rightarrow ie^{(\frac{R}{L})t} = \frac{V}{L}\int e^{(\frac{R}{L})t}dt + k$$

$$\Rightarrow ie^{(\frac{R}{L})t} = \frac{V}{L}\{\frac{e^{(\frac{R}{L})t}}{\frac{R}{L}}\} + k$$

$$\Rightarrow i = \frac{V}{R} + ke^{-(\frac{R}{L})t}$$ **Equation 4**

We know that there is no initial current in the circuit. Hence, substitute, $t = 0$ and $i = 0$ in Equation 4 in order to find the value of the constant ***k***.

$$0 = \frac{V}{R} + ke^{-(\frac{R}{L})(0)}$$

$$0 = \frac{V}{R} + k(1)$$

$$k = -\frac{V}{R}$$

Substitute, the value of k in Equation 4.

$$i = \frac{V}{R} + (-\frac{V}{R})e^{-(\frac{R}{L})t}$$

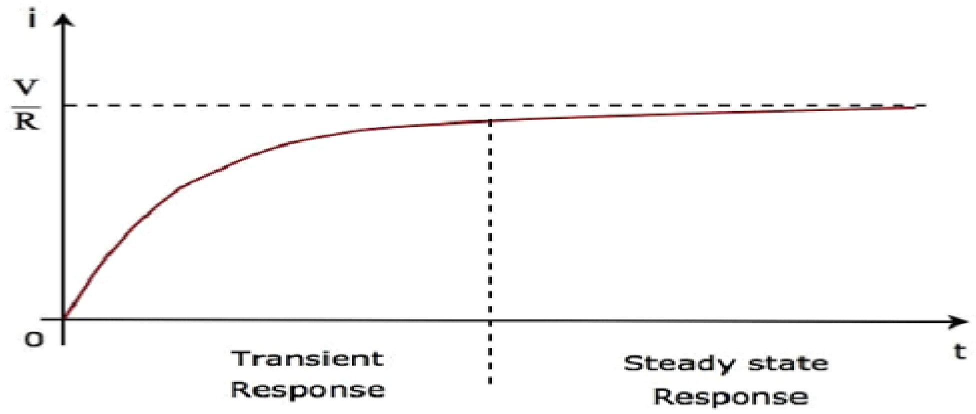
$$i = \frac{V}{R} - \frac{V}{R}e^{-(\frac{R}{L})t}$$

Therefore, the **current** flowing through the circuit is

$$i = -\frac{V}{R}e^{-(\frac{R}{L})t} + \frac{V}{R}$$ **Equation 5**

So, the response of the series RL circuit, when it is excited by a DC voltage source, has the following two terms.

- The first term $-\frac{V}{R}e^{-(\frac{R}{L})t}$ corresponds with the **transient response**.
- The second term $\frac{V}{R}$ corresponds with the **steady state response**. These two responses are shown in the following figure.



We can re-write the Equation 5 as follows −

$$i = \frac{V}{R}(1 - e^{-(\frac{R}{L})t})$$

$$\Rightarrow i = \frac{V}{R}(1 - e^{-(\frac{t}{\tau})})$$ **Equation 6**

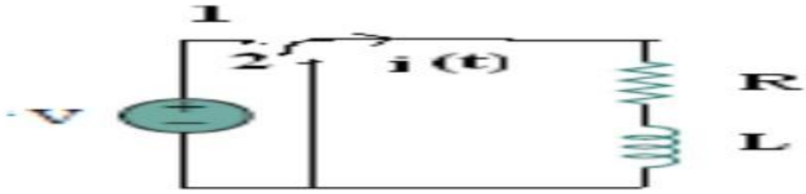
Where, τ is the **time constant** and its value is equal to $\frac{L}{R}$.

Both Equation 5 and Equation 6 are same. But, we can easily understand the above waveform of current flowing through the circuit from Equation 6 by substituting a few values of **t** like 0, τ, 2τ, 5τ, etc.

In the above waveform of current flowing through the circuit, the transient response will present up to five time constants from zero, whereas the steady state response will present from five time constants onwards.

Actual time (t) in sec	Growth of current in inductor (Eq.10.15)
$t = 0$	$i(0) = 0$
$t = \tau\left(= \frac{L}{R}\right)$	$i(\tau) = 0.632 \times \frac{V_s}{R}$
$t = 2\tau$	$i(2\tau) = 0.865 \times \frac{V_s}{R}$
$t = 3\tau$	$i(3\tau) = 0.950 \times \frac{V_s}{R}$
$t = 4\tau$	$i(4\tau) = 0.982 \times \frac{V_s}{R}$
$t = 5\tau$	$i(5\tau) = 0.993 \times \frac{V_s}{R}$

2.5.1 Current decay in source free series RL circuit: -



t = 0- , , switch k is kept at position ‘a’ for very long time. Thus, the network is in steady state. Initial current through inductor is given as,

$$i_L(0^-) = I_0 = \frac{V}{R} = i_L(0^+) \text{ ------------------------- } 1$$

Because current through inductor can not change instantaneously
Assume that at t = 0 switch k is moved to position 'b'.
Applying KVL,

$$L\frac{di}{dt} + iR = 0 \text{ ---------------------------- } 2$$

$$\therefore L\frac{di}{dt} = -iR$$

Rearranging the terms in above equation by separating variables

$$\frac{di}{i} = -\frac{R}{L}dt$$

Integrating both sides with respect to corresponding variables

$$\therefore \ln\left[\ \right] = -\frac{R}{L} \text{ --------------------------- } 3$$

Where k’ is constant of integration.
To find- k’:
Form equation 1, at t=0, i=I0
Substituting the values in equation 3
Where k’ is constant of integration.

To find- k':Form equation 1, at t=0, i=I0
Substituting the values in eq

$$\ln(i) = -\frac{R}{L}t + \ln(I_0)$$

$$\ln(i) - \ln(I_0) = -\frac{R}{L}t$$

$$\frac{i}{i_0} = e^{-\frac{R}{L}t}$$

$$\therefore i = I_0 \cdot e^{-\frac{R}{L}t} \quad \text{-----------------------5}$$

fig. shows variation of current i with respect to time

From the graph, H is clear that current is exponentially decaying. At point P on graph. The current value is (0.363) times its maximum value. The characteristics of decay are determined by values R and L which are two parameters of network.

The voltage across inductor is given by

$$V_L = L\frac{di}{dt} = L\frac{d}{dt}\left[I_0 \cdot e^{-\frac{R}{L}t}\right] = L \cdot I_0\left(-\frac{R}{L}\right) \cdot e^{-\frac{R}{L}t}$$

$$\therefore V_L = -I_0 \cdot \mathrm{R}e^{-\frac{R}{L}t}$$

$$\text{But} \quad I_0 \cdot R = V$$

$$\therefore V_L = -V \cdot e^{-\frac{R}{L}t} \quad \text{Volts}$$

Voltage, v_c and current i are reduced to 36.8 % of their initial value

2.6. TRANSIENT RESPONSE OF RC CIRCUIT WITH DC EXCITATION:

Let us consider a simple series $R-C$ circuit shown in fig. 10.17(a) is connected through a switch 'S' to a constant voltage source V_S.

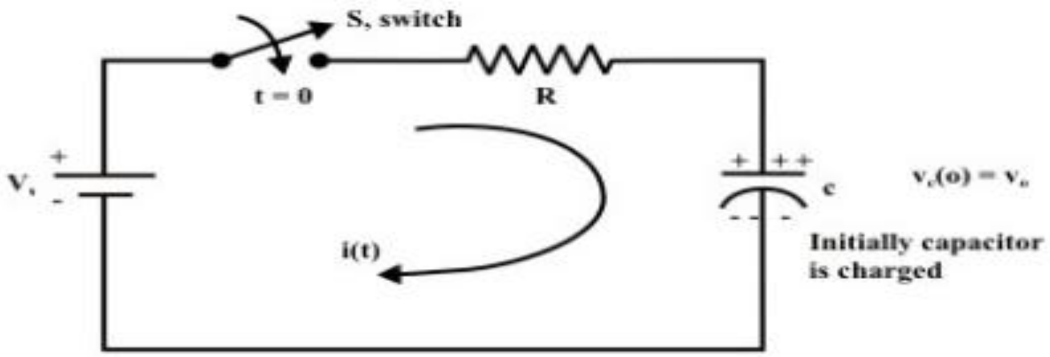


Fig. 10.17(a): Charging of a RC circuit

The switch 'S' is closed at time '$t=0$' (see fig. 10.7(a)). It is assumed that the capacitor is initially charged with a voltage $v_c(0)=v_0$ and the current flowing through the circuit at any instant of time 't' after closing the switch is $i(t)$.

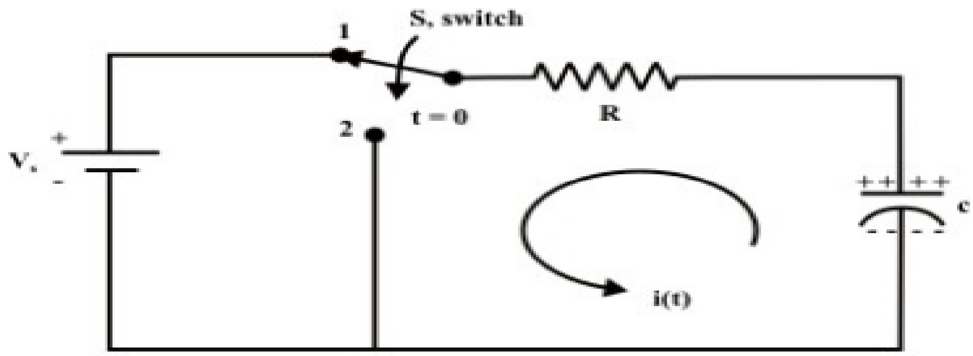


Fig. 10.17(b): Discharging of a RC circuit

The KVL equation around the loop can be written as

$$V_S = R\,i(t) + v_c(t) \Rightarrow V_S = R\,C\frac{dv_c(t)}{dt} + v_c(t)$$

The solution of the above first-order differential equation (10.41) due to forcing function V_s is given by

$v_c(t) = v_{cn}(t)$ (natural response/transient response) + $v_{cf}(t)$ (steady-state response)

$$= A_1 e^{\alpha t} + A$$

The constants A_1, and A are computed using the initial and boundary conditions. The value of α is obtained from the characteristic equation given by (see in detail in Appendix)

$$RC\alpha + 1 = 0 \Rightarrow \alpha = -\frac{1}{RC}$$

$$v_c(0)=v_0=A_1 e^{-\frac{1}{RC}\times 0}+A \Rightarrow A_1=v_0-A=v_0-V_S$$

The values of A_1 , A, and Eq. (10.43) together will give us the final expression for capacitor voltage as

$$v_c(t)=(v_0-V_S)e^{-\frac{1}{RC}t}+V_S \Rightarrow v_c(t)=V_S\left(1-e^{-\frac{1}{RC}t}\right)+v_0 e^{-\frac{1}{RC}t}$$

Thus,

$$v_c(t)=\begin{cases} v_0 & t<0 \\ v_c(t)=V_S\left(1-e^{-\frac{1}{RC}t}\right)+v_0 e^{-\frac{1}{RC}t} & t>0 \end{cases}$$

Special Case: Assume initial voltage across the capacitor at time '$t=0$' is zero i.e., $v_c(0)=v_0=0$. The voltage expression for capacitor at any instant of time can be written from Eq.(10.44) with $v_c(0)=v_0=0$.

Voltage across the capacitance $v_c(t)=V_S\left(1-e^{-\frac{1}{RC}t}\right)$

Voltage across the resistance $v_R(t)=V_S-v_c(t)=V_S\, e^{-\frac{1}{RC}t}$

Charging current through the capacitor $i(t)=\frac{v_R}{R}=\frac{V_S}{R}e^{-\frac{1}{RC}t}$

Charge accumulated on either plate of capacitor at any instant of time is given by

$$q(t)=Cv_c(t)=C\,V_S\left(1-e^{-\frac{1}{RC}t}\right)=Q\left(1-e^{-\frac{1}{RC}t}\right)$$

Charging current through the capacitor $i(t)=\frac{v_R}{R}=\frac{V_S}{R}e^{-\frac{1}{RC}t}$

Charge accumulated on either plate of capacitor at any instant of time is given by

$$q(t)=Cv_c(t)=C\,V_S\left(1-e^{-\frac{1}{RC}t}\right)=Q\left(1-e^{-\frac{1}{RC}t}\right)$$

where Q is the final charge accumulated in the capacitor at steady state (i.e., $t\to\infty$). Once the voltage across the capacitor $v_c(t)$ is known, the other quantities (like, $v_R(t)$, $i(t)$, *and* $q(t)$) can easily be computed using the above expressions. Fig. 10.19(a) shows growth of capacitor voltage $v_c(t)$ for different choices of circuit parameters (assumed that the capacitor is initially not charged). A sketch for $q(t)$ *and* $i(t)$ is shown

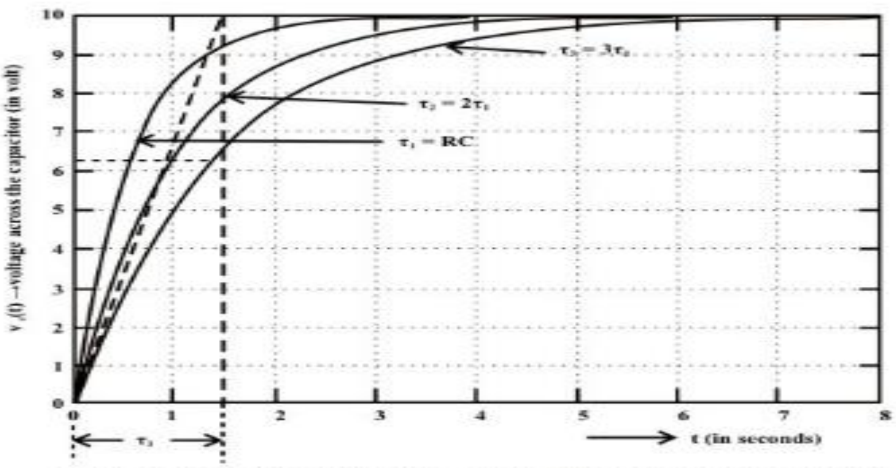


Fig. 10.19(a): Growth of capacitor voltage (assumed initial capacitor voltage is zero)

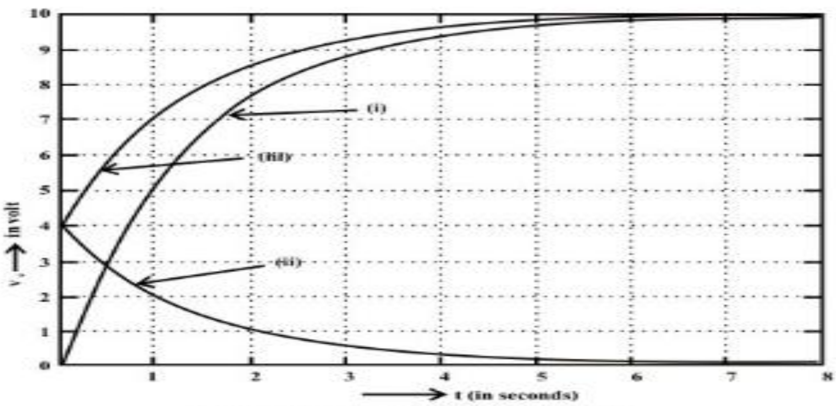


Fig. 10.18: Voltage across the capacitor due to
(i) the forcing function V_s acting alone
(ii) discharge of capacitor initial voltage v_o
(iii) Combine effect of (i) and (ii)

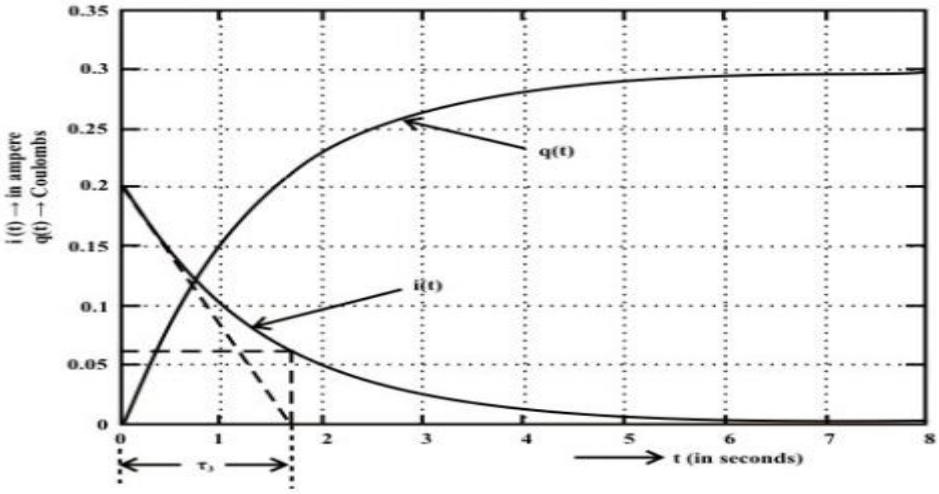


Fig. 10.19(b): System response due to the forcing function V_s (assumed capacitor initial voltage $v_c = 0$)

Discharging of a capacitor or Fall of a capacitor voltage in dc circuits

Fig. (b) shows that the switch 'S' is closed at position '1' for sufficiently long time and the circuit has reached in steady-state condition. At '$t = 0$' the switch' S' is opened and kept in position '2' and remains there. Our job is to find the expressions for (i) voltage across the capacitor (v_c) (ii) voltage across the resistance (v_R) (iii) current $(i(t))$ through the capacitor (discharging current) (iv) discharge of charge $(q(t))$ through the circuit.

Solution: For $t < 0$, the switch 'S' in position 1. The capacitor acts like an open circuit to dc, but the voltage across the capacitor is same as the supply voltage V_S. Since, the capacitor voltage cannot change instantaneously, this implies that

$v_c(0^-) = v_c(0^-) = V_s$

When the switch is closed in position '2', the current $i(t)$ will flow through the circuit until capacitor is completely discharged through the resistance R. In other words, the discharging cycle will start at $t = 0$. Now applying KVL around the loop, we get

$$R\,C\frac{d\,v_c(t)}{dt} + v_c(t) = 0 \qquad (10.49)$$

The solution of input free differential equation (10.49) is given by

$$v_c(t) = A_1\,e^{\alpha t} \qquad (10.50)$$

where the value of α is obtained from the characteristic equation and it is equal to $\alpha = -\dfrac{1}{RC}$. The constant A_1 is obtained using the initial condition of the circuit in Eq.(10.50). Note, at '$t = 0$'(when the switch is just closed in position '2') the voltage across the capacitor $v_c(t) = V_S$. Using this condition in Eq.(10.50), we get

$v_c(0) = V_s = A_1\,e^{-\frac{1}{RC}\times 0} \Rightarrow A_1 = V_s$

Now the following expressions are written as

Voltage across the capacitance $v_c(t) = V_S\,e^{-\frac{1}{RC}t}$ (10.51)

Voltage across the resistance $v_R(t) = -v_c(t) = -V_S\,e^{-\frac{1}{RC}t}$ (10.52)

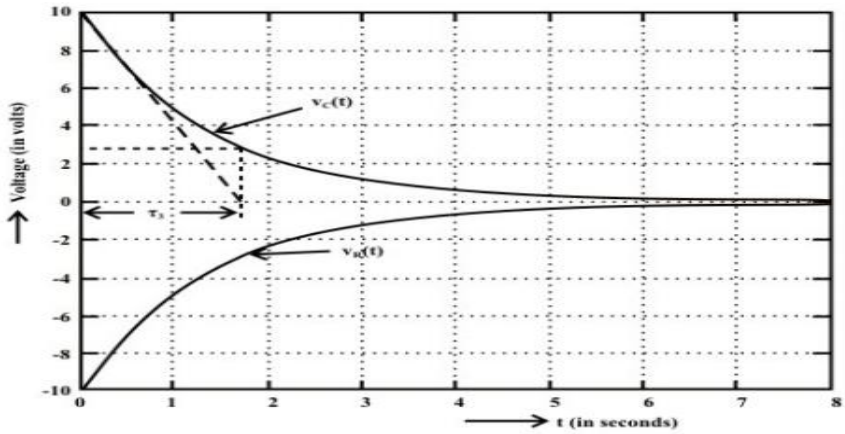
Charging current through the capacitor $i(t) = \dfrac{v_R}{R} = -\dfrac{V_S}{R}e^{-\frac{1}{RC}t}$ (10.53)

An inspection of the above exponential terms of equations from (10.51) to (10.53) reveals that the time constant of RC circuit is given by

An inspection of the above exponential terms of equations from (10.51) to (10.53) reveals that the time constant of RC circuit is given by

$$\tau = RC \text{ (sec.)}$$

This means that at time $t = \tau$, the capacitor's voltage v_c drops to 36.8% of its initial value (see fig. 10.20(a)). For all practical purposes, the dc transient is considered to end after a time span of 5τ. At such time steady state condition is said to be reached. Plots of above equations as a function of time are depicted in fig. 10.20(a) and fig. 10.20(b) respectively.



(a): Discharge of capacitor voltage with time in R-C circuit

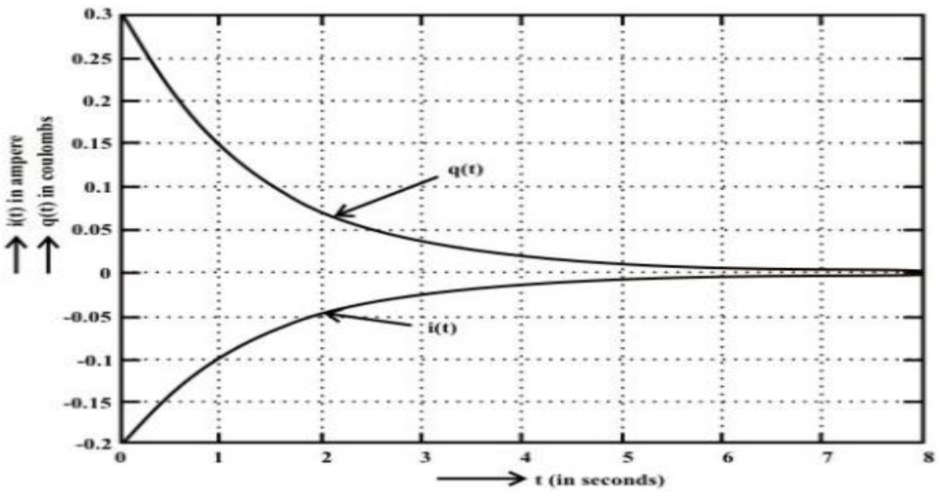


Fig. (b): System response due to capacitor discharge

2.7 TRANSIENT RESPONSE OF RLC CIRCUITS WITH DC EXCITATION:

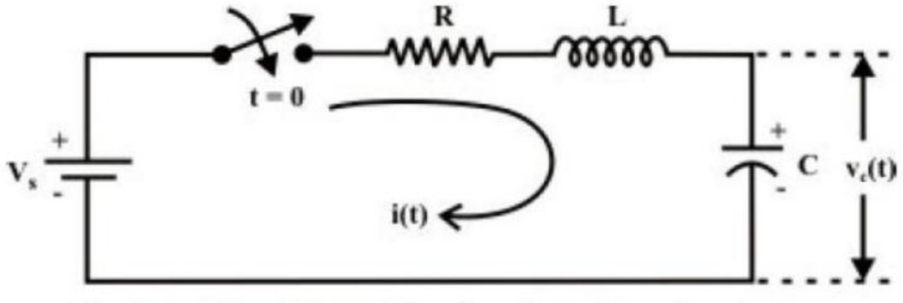
In the preceding lesson, our discussion focused extensively on dc circuits having resistances with either inductor () or capacitor () (i.e., single storage element) but not both. Dynamic response of such first order system has been studied and discussed in detail. The presence of resistance, inductance, and capacitance in the dc circuit introduces at least a second order differential equation or by two simultaneous coupled linear first order differential equations. We shall see in next section that the complexity of analysis of second order circuits increases significantly when compared with that encountered with first order circuits. Initial conditions for the circuit variables and their derivatives play an important role and this is very crucial to analyze a second order dynamic system.

Response of a series R-L-C circuit

Consider a series RL circuit as shown in fig.11.1, and it is excited with a dc voltage source C−−sV.

Applying around the closed path for,

$$L\frac{di(t)}{dt}+Ri(t)+v_c(t)=V_s$$



The current through the capacitor can be written as Substituting the current ‘’expression in eq.(11.1) and rearranging the terms,

$$i(t)=C\frac{dv_c(t)}{dt}$$

$$LC\frac{d^2v_c(t)}{dt^2}+RC\frac{dv_c(t)}{dt}+v_c(t)=V_s$$

The above equation is a 2nd-order linear differential equation and the parameters associated with the differential equation are constant with time. The complete solution of the above differential equation has two components; the transient response and the steady state response. Mathematically, one can write the complete solution as

$$v_c(t) = v_{cn}(t) + v_{cf}(t) = \left(A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}\right) + A$$

$$LC\frac{d^2v_c(t)}{dt^2} + RC\frac{dv_c(t)}{dt} + v_c(t) = 0 \Rightarrow \frac{d^2v_c(t)}{dt^2} + \frac{R}{L}\frac{dv_c(t)}{dt} + \frac{1}{LC}v_c(t) = 0$$

$$a\,\frac{d^2v_c(t)}{dt^2} + b\,\frac{dv_c(t)}{dt} + c\,v_c(t) = 0 \;\; (\text{where } a=1,\, b=\frac{R}{L} \; and \; c=\frac{1}{LC})$$

Since the system is linear, the nature of steady state response is same as that of forcing function (input voltage) and it is given by a constant value. Now, the first part of the total response is completely dies out with time while and it is defined as a transient or natural response of the system. The natural or transient response (see Appendix in Lesson-10) of second order differential equation can be obtained from the homogeneous equation (i.e., from force free system) that is expressed by

$$\alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} = 0 \quad \Rightarrow a\,\alpha^2 + b\,\alpha + c = 0 \;\; (\text{where } a=1,\, b=\frac{R}{L} \; and \; c=\frac{1}{LC})$$

and solving the roots of this equation (11.5) on that associated with transient part of the complete solution (eq.11.3) and they are given below.

$$\alpha_1 = \left(-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right) = \left(-\frac{b}{2a} + \frac{1}{a}\sqrt{\left(\frac{b}{2}\right)^2 - ac}\right);$$

$$\alpha_2 = \left(-\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right) = \left(-\frac{b}{2a} - \frac{1}{a}\sqrt{\left(\frac{b}{2}\right)^2 - ac}\right)$$

$$\text{where, } b = \frac{R}{L} \;\; and \;\; c = \frac{1}{LC}.$$

The roots of the characteristic equation are classified in three groups depending upon the values of the parameters ,,Rand of the circuit Case-A (over damped response): That the roots are distinct with negative real parts. Under this situation, the natural or transient part of the complete solution is written as

$$v_{cn}(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}$$

and each term of the above expression decays exponentially and ultimately reduces to zero as and it is termed as over damped response of input free system. A system that is over damped responds slowly to any change in excitation. It may be noted that the exponential term t→∞11tAeαtakes longer time to decay its value to zero than the term21tAeα. One can introduce a factor ξ that provides an information about the speed of system response and it is defined by damping ratio

$$(\xi)=\frac{Actual\ \ damping}{critical\ \ damping}=\frac{b}{2\sqrt{ac}}=\frac{R/L}{2/\sqrt{LC}}>1$$

Case-B (critically damped response): When $\left(\frac{R}{2L}\right)^2-\frac{1}{LC}=0$, this implies that the roots of eq.(11.5) are same with negative real parts. Under this situation, the form of the natural or transient part of the complete solution is written as

$$v_{cn}(t)=\left(A_1 t+A_2\right)e^{\alpha t} \quad (\text{where } \alpha=-\frac{R}{2L}) \tag{11.9}$$

where the natural or transient response is a sum of two terms: a negative exponential and a negative exponential multiplied by a linear term. The expression (11.9) that arises from the natural solution of second order differential equation having the roots of characteristic equation are same value can be verified following the procedure given below.

The roots of this characteristic equation (11.5) are same $\alpha=\alpha_1=\alpha_2=\frac{R}{2L}$ when $\left(\frac{R}{2L}\right)^2-\frac{1}{LC}=0\Rightarrow\left(\frac{R}{2L}\right)^2=\frac{1}{LC}$ and the corresponding homogeneous equation (11.4) can be rewritten as

$$\frac{d^2v_c(t)}{dt^2}+2\frac{R}{2L}\frac{dv_c(t)}{dt}+\frac{1}{LC}v_c(t)=0$$

$$or\ \frac{d^2v_c(t)}{dt^2}+2\alpha\frac{dv_c(t)}{dt}+\alpha^2 v_c(t)=0$$

$$or\ \frac{d}{dt}\left(\frac{dv_c(t)}{dt}+\alpha v_c(t)\right)+\alpha\left(\frac{dv_c(t)}{dt}+\alpha v_c(t)\right)=0$$

$$or\ \frac{df}{dt}+\alpha f=0 \quad \text{where} \quad f=\frac{dv_c(t)}{dt}+\alpha v_c(t)$$

The solution of the above first order differential equation is well known and it is given by

$f = A_1\, e^{\alpha t}$

Using the value of f in the expression $f = \frac{dv_c(t)}{dt} + \alpha v_c(t)$ we can get,

$$\frac{dv_c(t)}{dt} + \alpha v_c(t) = A_1\, e^{-\alpha t} \Rightarrow e^{\alpha t}\frac{dv_c(t)}{dt} + e^{\alpha t}\,\alpha v_c(t) = A_1 \quad \Rightarrow \frac{d}{dt}\left(e^{\alpha t}\, v_c(t)\right) = A_1$$

Integrating the above equation in both sides yields,

$v_{cn}(t) = \left(A_1 t + A_2\right) e^{\alpha t}$

In fact, the term $A_2\, e^{\alpha t}$ (with $\alpha = -\frac{R}{2L}$) decays exponentially with the time and tends to zero as $t \to \infty$. On the other hand, the value of the term $A_1 t\, e^{\alpha t}$ (with $\alpha = -\frac{R}{2L}$) in equation (11.9) first increases from its zero value to a maximum value $A_1 \frac{2L}{R} e^{-1}$ at a time $t = -\frac{1}{\alpha} = -\left(-\frac{2L}{R}\right) = \frac{2L}{R}$ and then decays with time, finally reaches to zero. One can easily verify above statements by adopting the concept of maximization problem of a single valued function. The second order system results the speediest response possible without any overshoot while the roots of characteristic equation (11.5) of system having the same negative real parts. The response of such a second order system is defined as a critically damped system's response. In this case damping ratio

$$(\xi) = \frac{Actual\ damping}{critical\ damping} = \frac{b}{2\sqrt{ac}} = \frac{R/L}{2/\sqrt{LC}} = 1 \qquad (11.10)$$

Case-C (underdamped response): When $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} < 0$, this implies that the roots of eq.(11.5) are complex conjugates and they are expressed as $\alpha_1 = \left(-\frac{R}{2L} + j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}\right) = \beta + j\gamma$; $\alpha_2 = \left(-\frac{R}{2L} - j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}\right) = \beta - j\gamma$. The

form of the natural or transient part of the complete solution is written as

$$\begin{aligned} v_{cn}(t) &= A_1 e^{\alpha_1 t} + A_2\, e^{\alpha_2 t} = A_1 e^{(\beta + j\gamma)} + A_2\, e^{(\beta - j\gamma)} \\ &= e^{\beta t}\left[\left(A_1 + A_2\right)\cos(\gamma t) + j\left(A_1 - A_2\right)\sin(\gamma t)\right] \qquad (11.11) \\ &= e^{\beta t}\left[B_1 \cos(\gamma t) + B_2 \sin(\gamma t)\right] \text{ where } B_1 = A_1 + A_2\ ;\ B_2 = j\left(A_1 - A_2\right) \end{aligned}$$

For real system, the response $v_{cn}(t)$ must also be real. This is possible only if A_1 *and* A_2 conjugates. The equation (11.11) further can be simplified in the following form:

$$e^{\beta t}\, K \sin(\gamma t + \theta) \qquad (11.12)$$

form of the natural or transient part of the complete solution is written as

$$\begin{aligned} v_{cn}(t) &= A_1 e^{\alpha_1 t} + A_2\, e^{\alpha_2 t} = A_1 e^{(\beta + j\gamma)} + A_2\, e^{(\beta - j\gamma)} \\ &= e^{\beta t}\left[\left(A_1 + A_2\right)\cos(\gamma t) + j\left(A_1 - A_2\right)\sin(\gamma t)\right] \qquad (11.11) \\ &= e^{\beta t}\left[B_1 \cos(\gamma t) + B_2 \sin(\gamma t)\right] \text{ where } B_1 = A_1 + A_2\ ;\ B_2 = j\left(A_1 - A_2\right) \end{aligned}$$

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$$e^{\beta t}\, K \sin(\gamma t + \theta) \qquad (11.12)$$

form of the natural or transient part of the complete solution is written as

$$v_{cn}(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} = A_1 e^{(\beta + j\gamma)} + A_2 e^{(\beta - j\gamma)}$$

$$= e^{\beta t}\left[\left(A_1 + A_2\right)\cos(\gamma t) + j\left(A_1 - A_2\right)\sin(\gamma t)\right] \quad (11.11)$$

$$= e^{\beta t}\left[B_1 \cos(\gamma t) + B_2 \sin(\gamma t)\right] \text{ where } B_1 = A_1 + A_2 \text{ ; } B_2 = j\left(A_1 - A_2\right)$$

For real system, the response $v_{cn}(t)$ must also be real. This is possible only if A_1 *and* A_2 conjugates. The equation (11.11) further can be simplified in the following form:

$$e^{\beta t} K \sin(\gamma t + \theta) \quad (11.12)$$

where $\beta -$ real part of the root , $\gamma -$ complex part of the root, $K = \sqrt{B_1^2 + B_2^2}$ *and* $\theta = \tan^{-1}\left(\frac{B_1}{B_2}\right)$. Truly speaking the value of K *and* θ can be calculated using the initial conditions of the circuit. The system response exhibits oscillation around the steady state value when the roots of characteristic equation are complex and results an under-damped system's response. This oscillation will die down with time if the roots are with negative real parts. In this case the damping ratio

$$(\xi) = \frac{Actual\ damping}{critical\ damping} = \frac{b}{2\sqrt{ac}} = \frac{R/L}{2/\sqrt{LC}} < 1 \quad (11.13)$$

Finally, the response of a second order system when excited with a dc voltage source is presented in fig.L.11.2 for different cases, i.e., (i) under-damped (ii) over-damped (iii) critically damped system response.

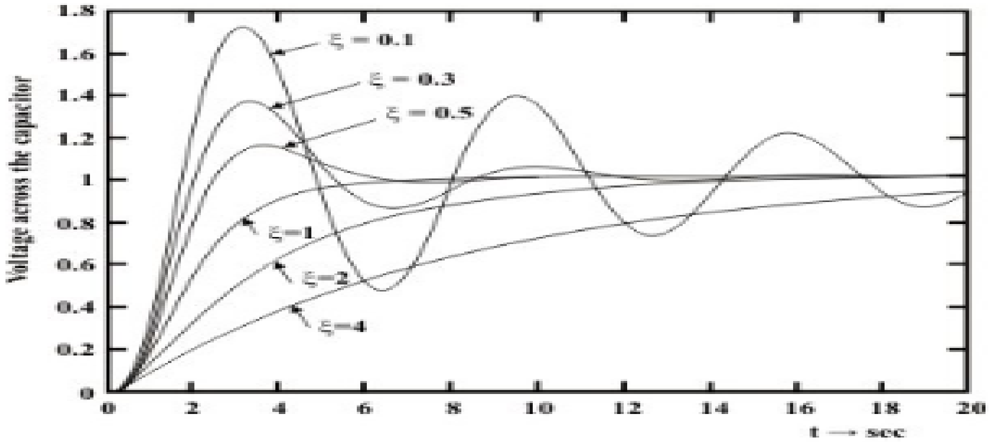
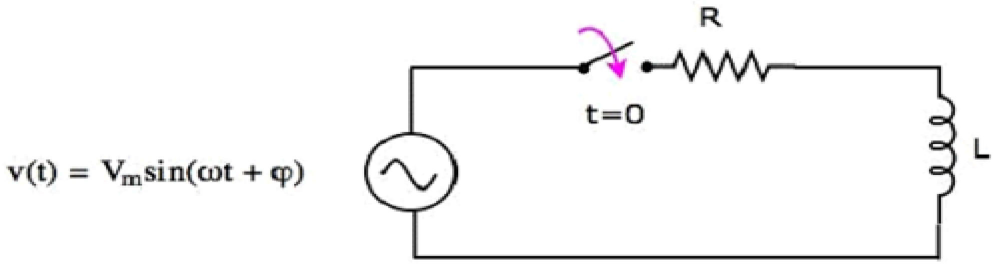


Fig. .2: System response for series R-L-C circuit:
(a) underdamped
(b) critically damped
(c) overdamped system

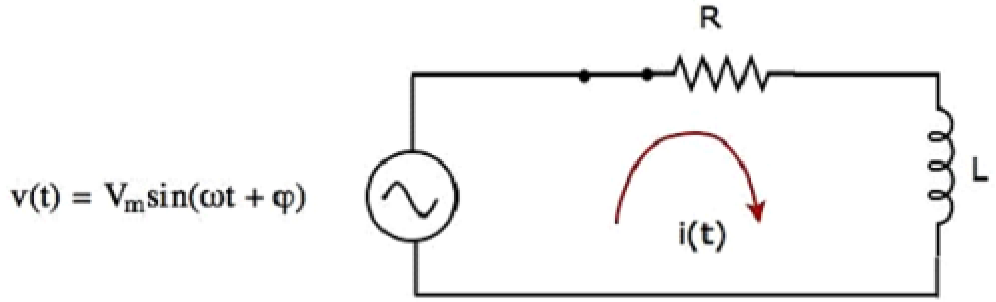
2.8. RESPONSE OF SERIES RL CIRCUIT (AC EXCITATION):

Consider the following **series RL circuit** diagram.



In the above circuit, the **switch** was kept **open** up to $t = 0$ and it was closed at $t = 0$. So, the AC voltage source having a peak voltage of V_m volts is not connected to the series RL circuit up to this instant. Therefore, there is **no initial current** flows through the inductor.

The circuit diagram, when the **switch** is in **closed** position, is shown in the following figure.



Now, the current $i(t)$ flows in the entire circuit, since the AC voltage source having a peak voltage of V_m volts is connected to the series RL circuit.

We know that the current $i(t)$ flowing through the above circuit will have two terms, one that represents the transient part and other term represents the steady state.

Mathematically, it can be represented as

$$i(t) = i_{Tr}(t) + i_{ss}(t)$$ **Equation 1**

Where,

- $i_{Tr}(t)$ is the transient response of the current flowing through the circuit.
- $i_{ss}(t)$ is the steady state response of the current flowing through the circuit.

In the previous chapter, we got the transient response of the current flowing through the series RL circuit. It is in the form of $Ke^{-\left(\frac{t}{\tau}\right)}$

Substitute $i_{Tr}(t) = Ke^{-\left(\frac{t}{\tau}\right)}$ in Equation 1.

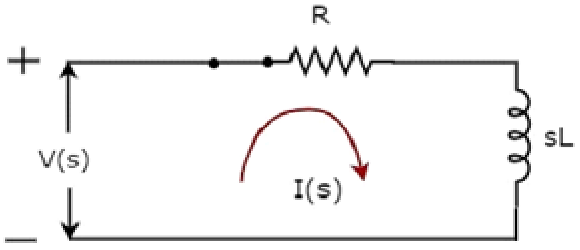
$$i(t) = Ke^{-\left(\frac{t}{\tau}\right)} + i_{ss}(t)$$ **Equation 2**

Calculation of Steady State Current

If a sinusoidal signal is applied as an input to a Linear electric circuit, then it produces a steady state output, which is also a **sinusoidal signal**. Both the input and output sinusoidal signals will be having the same frequency, but different amplitudes and phase angles.

We can calculate the steady state response of an electric circuit, when it is excited by a sinusoidal voltage source using **Laplace Transform approach**.

The s-domain circuit diagram, when the **switch** is in **closed** position, is shown in the following figure.



In the above circuit, all the quantities and parameters are represented in **s-domain**. These are the Laplace transforms of time-domain quantities and parameters.

The **Transfer function** of the above circuit is

$$H(s) = \frac{I(s)}{V(s)}$$

$$\Rightarrow H(s) = \frac{1}{Z(s)}$$

$$\Rightarrow H(s) = \frac{1}{R + sL}$$

Substitute $s = j\omega$ in the above equation.

$$H(j\omega) = \frac{1}{R + j\omega L}$$

Magnitude of $H(j\omega)$ is

$$|H(j\omega)| = \frac{1}{\sqrt{R^2 + \omega^2 L^2}}$$

Phase angle of $H(j\omega)$ is

$$\angle H(j\omega) = -tan^{-1}\left(\frac{\omega L}{R}\right)$$

We will get the **steady state current** $i_{ss}(t)$ by doing the following two steps –

- Multiply the peak voltage of input sinusoidal voltage and the magnitude of $H(j\omega)$.
- Add the phase angles of input sinusoidal voltage and $H(j\omega)$.

The **steady state current** $i_{ss}(t)$ will be

$$i_{ss}(t)=\frac{V_m}{\sqrt{R^2+\omega^2L^2}}sin(\omega t+\varphi-tan^{-1}(\frac{\omega L}{R}))$$

Substitute the value of $i_{ss}(t)$ in Equation 2.

$i(t)=Ke^{-(\frac{t}{\tau})}+\frac{V_m}{\sqrt{R^2+\omega^2L^2}}sin(\omega t+\varphi-tan^{-1}(\frac{\omega L}{R}))$ **Equation 3**

We know that there is no initial current in the circuit. Hence, substitute $t = 0$ & $i(t) = 0$ in Equation 3 in order to find the value of constant, K.

$$0=Ke^{-(\frac{0}{\tau})}+\frac{V_m}{\sqrt{R^2+\omega^2L^2}}sin(\omega(0)+\varphi-tan^{-1}(\frac{\omega L}{R}))$$

$$\Rightarrow 0=K+\frac{V_m}{\sqrt{R^2+\omega^2L^2}}sin(\varphi-tan^{-1}(\frac{\omega L}{R}))$$

$$\Rightarrow K=-\frac{V_m}{\sqrt{R^2+\omega^2L^2}}sin(\varphi-tan^{-1}(\frac{\omega L}{R}))$$

Substitute the value of K in Equation 3.

$i(t)=-\frac{V_m}{\sqrt{R^2+\omega^2L^2}}sin(\varphi-tan^{-1}(\frac{\omega L}{R}))e^{-(\frac{t}{\tau})}$ **Equation 4**

$+\frac{V_m}{\sqrt{R^2+\omega^2L^2}}sin(\omega t+\varphi-tan^{-1}(\frac{\omega L}{R}))$

Equation 4 represents the current flowing through the series RL circuit, when it is excited by a sinusoidal voltage source. It is having two terms. The first and second terms represent the transient response and steady state response of the current respectively.

We can **neglect the first term** of Equation 4 because its value will be very much less than one. So, the resultant current flowing through the circuit will be

$$i(t)=\frac{V_m}{\sqrt{R^2+\omega^2L^2}}sin(\omega t+\varphi-tan^{-1}(\frac{\omega L}{R}))$$

It contains only the **steady state term**. Hence, we can find only the steady state response of AC circuits and neglect transient response of it.

2.9.Transient Response of a series R-L-C circuit

Consider a series RL circuit as shown in fig.11.1, and it is excited with a dc voltage source C−−sV. Applying around the closed path for

$$L\frac{di(t)}{dt}+R\,i(t)+v_c(t)=V_s$$

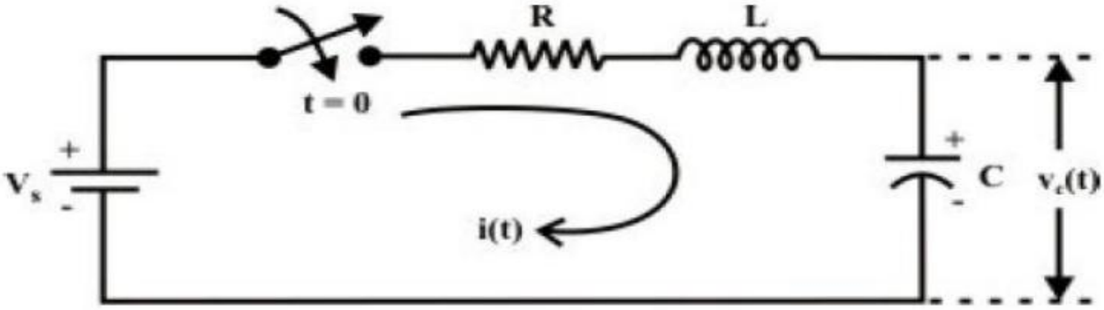


Fig. 11.1: A Simple R-L-C circuit excited with a dc voltage source

The current through the capacitor can be written as

$$i(t)=C\frac{dv_c(t)}{dt}$$

Substituting the current ' $i(t)$ 'expression in eq.(11.1) and rearranging the terms,

$$LC\frac{d^2v_c(t)}{dt^2}+RC\frac{dv_c(t)}{dt}+v_c(t)=V_s \qquad (11.2)$$

The above equation is a 2[nd]-order linear differential equation and the parameters associated with the differential equation are constant with time. The complete solution of the above differential equation has two components; the transient response $v_{cn}(t)$ and the steady state response $v_{cf}(t)$. Mathematically, one can write the complete solution as

$$v_c(t)=v_{cn}(t)+v_{cf}(t)=\left(A_1 e^{\alpha_1 t}+A_2 e^{\alpha_2 t}\right)+A \qquad (11.3)$$

Since the system is linear, the nature of steady state response is same as that of forcing function (input voltage) and it is given by a constant value A. Now, the first part $v_{cn}(t)$ of the total response is completely dies out with time while $R>0$ and it is defined as a transient or natural response of the system. The natural or transient response (see Appendix in Lesson-10) of second order differential equation can be obtained from the homogeneous equation (i.e., from force free system) that is expressed by

$$LC\frac{d^2v_c(t)}{dt^2}+RC\frac{dv_c(t)}{dt}+v_c(t)=0 \Rightarrow \frac{d^2v_c(t)}{dt^2}+\frac{R}{L}\frac{dv_c(t)}{dt}+\frac{1}{LC}v_c(t)=0$$

$$a\,\frac{d^2v_c(t)}{dt^2}+b\,\frac{dv_c(t)}{dt}+c\,v_c(t)=0 \;\;(\text{where } a=1,\ b=\frac{R}{L} \text{ and } c=\frac{1}{LC}) \qquad (11.4)$$

The characteristic equation of the above homogeneous differential equation (using the

$$a\,\frac{d^2v_c(t)}{dt^2}+b\,\frac{dv_c(t)}{dt}+c\;v_c(t)=0\;\;(\text{where } a=1,\; b=\frac{R}{L}\; and\; c=\frac{1}{LC}) \qquad (11.4)$$

The characteristic equation of the above homogeneous differential equation (using the operator $\alpha=\frac{d}{dt}$, $\alpha^2=\frac{d^2}{dt^2}$ and $v_c(t)\neq 0$) is given by

$$\alpha^2+\frac{R}{L}\alpha+\frac{1}{LC}=0 \;\;\Rightarrow a\,\alpha^2+b\,\alpha+c=0\;\;(\text{where } a=1,\; b=\frac{R}{L}\; and\; c=\frac{1}{LC}) \qquad (11.5)$$

and solving the roots of this equation (11.5) one can find the constants α_1 *and* α_2 of the exponential terms that associated with transient part of the complete solution (eq.11.3) and they are given below.

$$\alpha_1=\left(-\frac{R}{2L}+\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}\right)=\left(-\frac{b}{2a}+\frac{1}{a}\sqrt{\left(\frac{b}{2}\right)^2-ac}\right);$$
$$\alpha_2=\left(-\frac{R}{2L}-\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}\right)=\left(-\frac{b}{2a}-\frac{1}{a}\sqrt{\left(\frac{b}{2}\right)^2-ac}\right) \qquad (11.6)$$

where, $b=\frac{R}{L}$ *and* $c=\frac{1}{LC}$.

The roots of the characteristic equation (11.5) are classified in three groups depending upon the values of the parameters R, L, and C of the circuit.

Case-A (overdamped response): When $\left(\frac{R}{2L}\right)^2-\frac{1}{LC}>0$, this implies that the roots are distinct with negative real parts. Under this situation, the natural or transient part of the complete solution is written as

$$v_{cn}(t)=A_1\,e^{\alpha_1 t}+A_2\,e^{\alpha_2 t} \qquad (11.7)$$

and each term of the above expression decays exponentially and ultimately reduces to zero as $t\to\infty$ and it is termed as overdamped response of input free system. A system that is overdamped responds slowly to any change in excitation. It may be noted that the exponential term $A_1\,e^{\alpha_1 t}$ takes longer time to decay its value to zero than the term $A_1\,e^{\alpha_2 t}$. One can introduce a factor ξ that provides an information about the speed of system response and it is defined by damping ratio

$$(\xi)=\frac{Actual\;\;damping}{critical\;\;damping}=\frac{b}{2\sqrt{ac}}=\frac{R/L}{2/\sqrt{LC}}>1 \qquad (11.8)$$

Case-B (critically damped response): When $\left(\frac{R}{2L}\right)^2-\frac{1}{LC}=0$, this implies that the roots of eq.(11.5) are same with negative real parts. Under this situation, the form of the natural or transient part of the complete solution is written as

$$v_{cn}(t)=\left(A_1 t+A_2\right)e^{\alpha t} \qquad (\text{where } \alpha=-\frac{R}{2L}) \qquad (11.9)$$

where the natural or transient response is a sum of two terms: a negative exponential and a negative exponential multiplied by a linear term. The expression (11.9) that arises from the natural solution of second order differential equation having the roots of characteristic equation are same value can be verified following the procedure given below.

The roots of this characteristic equation (11.5) are same $\alpha = \alpha_1 = \alpha_2 = \frac{R}{2L}$ when $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} = 0 \Rightarrow \left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$ and the corresponding homogeneous equation (11.4) can be rewritten as

$$\frac{d^2 v_c(t)}{dt^2} + 2\frac{R}{2L}\frac{dv_c(t)}{dt} + \frac{1}{LC} v_c(t) = 0$$

$$or\ \frac{d^2 v_c(t)}{dt^2} + 2\alpha \frac{dv_c(t)}{dt} + \alpha^2 v_c(t) = 0$$

$$or\ \frac{d}{dt}\left(\frac{dv_c(t)}{dt} + \alpha v_c(t)\right) + \alpha\left(\frac{dv_c(t)}{dt} + \alpha v_c(t)\right) = 0$$

$$or\ \frac{df}{dt} + \alpha f = 0 \quad \text{where} \quad f = \frac{dv_c(t)}{dt} + \alpha v_c(t)$$

The solution of the above first order differential equation is well known and it is given by

$f = A_1\, e^{\alpha t}$

Using the value of f in the expression $f = \frac{dv_c(t)}{dt} + \alpha v_c(t)$ we can get,

$$\frac{dv_c(t)}{dt} + \alpha v_c(t) = A_1\, e^{-\alpha t} \Rightarrow e^{\alpha t}\frac{dv_c(t)}{dt} + e^{\alpha t}\alpha v_c(t) = A_1 \quad \Rightarrow \frac{d}{dt}\left(e^{\alpha t} v_c(t)\right) = A_1$$

Integrating the above equation in both sides yields,

$v_{cn}(t) = \left(A_1 t + A_2\right) e^{\alpha t}$

In fact, the term $A_2\, e^{\alpha t}$ (with $\alpha = -\frac{R}{2L}$) decays exponentially with the time and tends to zero as $t \to \infty$. On the other hand, the value of the term $A_1 t\, e^{\alpha t}$ (with $\alpha = -\frac{R}{2L}$) in equation (11.9) first increases from its zero value to a maximum value $A_1 \frac{2L}{R} e^{-1}$ at a time $t = -\frac{1}{\alpha} = -\left(-\frac{2L}{R}\right) = \frac{2L}{R}$ and then decays with time, finally reaches to zero. One can easily verify above statements by adopting the concept of maximization problem of a single valued function. The second order system results the speediest response possible without any overshoot while the roots of characteristic equation (11.5) of system having the same negative real parts. The response of such a second order system is defined as a critically damped system's response. In this case damping ratio

$$(\xi) = \frac{Actual\ damping}{critical\ damping} = \frac{b}{2\sqrt{ac}} = \frac{R/L}{2/\sqrt{LC}} = 1 \tag{11.10}$$

Case-C (underdamped response): When $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} < 0$, this implies that the roots of eq.(11.5) are complex conjugates and they are expressed as $\alpha_1 = \left(-\frac{R}{2L} + j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \right) = \beta + j\gamma$; $\alpha_2 = \left(-\frac{R}{2L} - j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \right) = \beta - j\gamma$. The form of the natural or transient part of the complete solution is written as

$$\begin{aligned} v_{cn}(t) &= A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} = A_1 e^{(\beta + j\gamma)} + A_2 e^{(\beta - j\gamma)} \\ &= e^{\beta t} \left[(A_1 + A_2)\cos(\gamma t) + j(A_1 - A_2)\sin(\gamma t) \right] \qquad (11.11) \\ &= e^{\beta t} \left[B_1 \cos(\gamma t) + B_2 \sin(\gamma t) \right] \text{ where } B_1 = A_1 + A_2 \; ; \; B_2 = j(A_1 - A_2) \end{aligned}$$

For real system, the response $v_{cn}(t)$ must also be real. This is possible only if A_1 *and* A_2 conjugates. The equation (11.11) further can be simplified in the following form:

$$e^{\beta t} K \sin(\gamma t + \theta) \qquad (11.12)$$

where $\beta =$ real part of the root , $\gamma =$ complex part of the root, $K = \sqrt{B_1^{\,2} + B_2^{\,2}}$ *and* $\theta = \tan^{-1}\left(\frac{B_1}{B_2}\right)$. Truly speaking the value of K *and* θ can be calculated using the initial conditions of the circuit. The system response exhibits oscillation around the steady state value when the roots of characteristic equation are complex and results an under-damped system's response. This oscillation will die down with time if the roots are with negative real parts. In this case the damping ratio

$$(\xi) = \frac{Actual\ damping}{critical\ damping} = \frac{b}{2\sqrt{ac}} = \frac{R/L}{2/\sqrt{LC}} < 1 \qquad (11.13)$$

Finally, the response of a second order system when excited with a dc voltage source is presented in fig.L.11.2 for different cases, i.e., (i) under-damped (ii) over-damped (iii) critically damped system response.

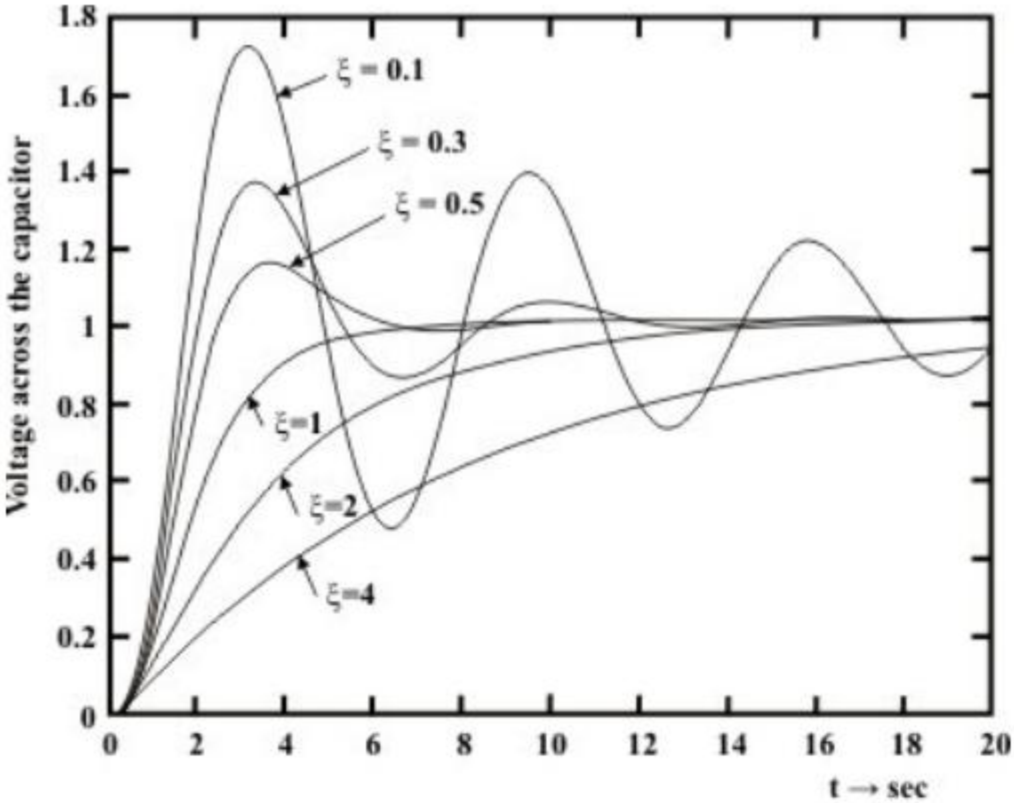
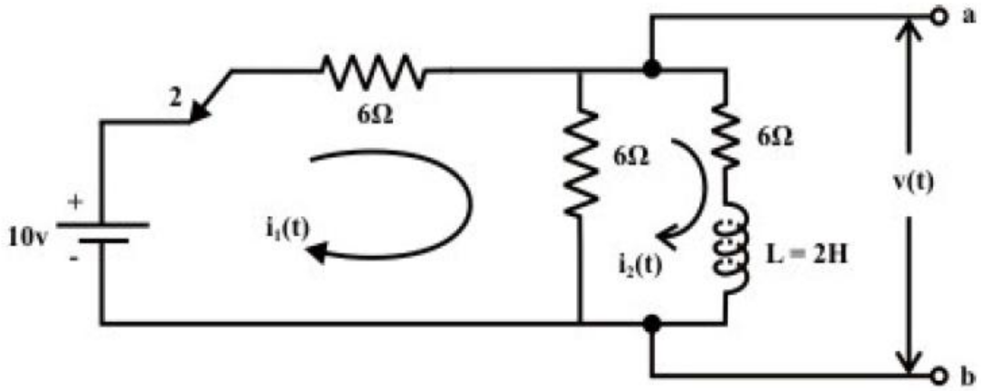


Fig. 11.2: System response for series R-L-C circuit:
(a) underdamped
(b) critically damped
(c) overdamped system

NUMERICAL PROBLEMS:

1. **Assign the loop currents in clockwise directions and redrawn the circuit as shown in . The voltage across the terminals ‘’ and ‘’ can be obtained by solving the following loop equations.**



Solution

Loop-1:

$$10-6i_1(t)-6(i_1(t)-i_2(t))=0 \Rightarrow 10=12i_1(t)-6i_2(t) \Rightarrow i_1(t)=\frac{1}{12}(10+6i_2(t))$$

Loop-2:

$$-6i_2(t)-L\frac{di_2(t)}{dt}-6(i_2(t)-i_1(t))=0 \Rightarrow -6i_1(t)+12i_2(t)+2\frac{di_2(t)}{dt}=0$$

Using the value of $i_1(t)$ in equation (10.37) , we get

$$9i_2(t)+2\times\frac{di_2(t)}{dt}=5$$

To solve the above first order differential equation we must know inductor's initial and final conditions and their values are already known (see, $\Rightarrow i_2(0^-)=i_2(0^+)=3A$ and $i_2(t=\infty)=\frac{5}{3+6}=0.555\ amp.$). The solution of differential equation (10.38) provides an expression of current $i_2(t)$ and this, in turn, will give us the expression of $i_1(t)$. The voltage across the terminals 'a' and 'b' is given by

$$v_{ab}=10-6\times i_1(t)=6i_2(t)+2\frac{di_2(t)}{dt}=\left(3.339-7.335\times e^{-\frac{9}{2}t}\right)V$$

where, $i_2(t)$ and $i_1(t)$can be obtained

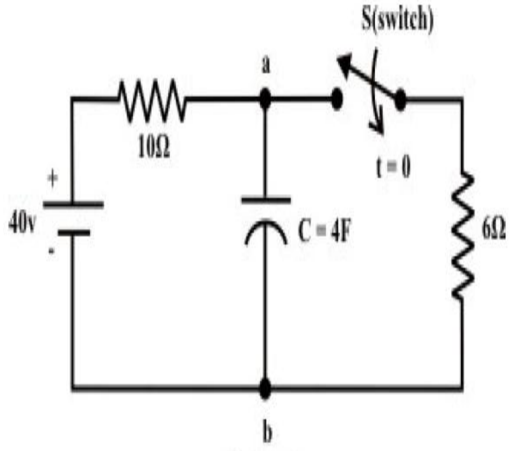
$$i_2(t)=\left(2.445\times e^{-\frac{9}{2}t}+0.555\right) \text{ and } i_1(t)=\frac{1}{12}(10+6i_2(t))=\left(1.11+1.2225e^{-\frac{9}{2}t}\right)$$

2.The switch ‘ *S* ’ shown is kept open for a long time and then it is closed at time ‘ *t* = 0 ’. Find

Find (i) $v_c(0^-)$ (ii) $v_c(0^+)$ (iii) $i_c(0^-)$ (iv) $i_c(0^+)$ (v)

$\left.\frac{dv_c(t)}{dy}\right|_{t=0^+}$ (vi) find the time constants of the circuit before and after the switch is closed

(iv) $v_c(\infty)$



Solution: As we know the voltage across the capacitor $v_c(t)$ cannot change instantaneously due to the principle of conservation of charge. Therefore, the voltage across the capacitor just before the switch is closed $v_c(0^-)$= voltage across the capacitor just after the switch is closed $v_c(0^+)$ = 40 V (note the terminal 'a' is positively charged. It may be noted that the capacitor current before the switch 'S' is closed is $i_c(0^-)=0\,A$. On the other hand, at $t=0$, the current through 10Ω resistor is zero but the current through capacitor can be computed as

$i_c(0^+)=\frac{v_c(0)}{6}=\frac{40}{6}=6.66\ A$ (note, voltage across the capacitor cannot change instantaneously at instant of switching). The rate of change of capacitor voltage at time '$t=0$' is expressed as

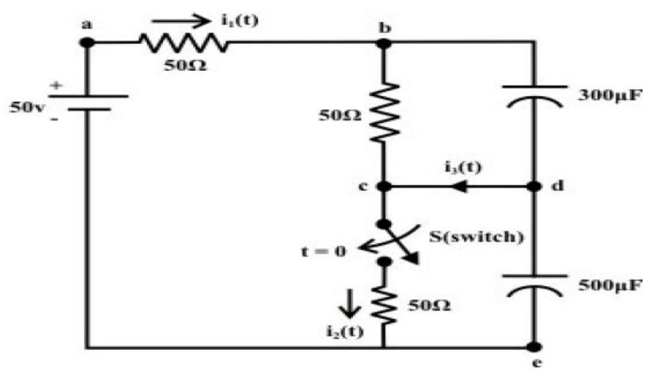
$$C\left.\frac{dv_c(t)}{dt}\right|_{t=0^+}=i_c(0) \Rightarrow \frac{dv_c(0^+)}{dt}=\frac{i_c(0^+)}{C}=\frac{6.66}{4}=1.665\ volt/\sec.$$

Time constant of the circuit before the switch was closed = $\tau = RC = 10\times 4 = 40\sec$. Time constant of the circuit after the switch is closed is $\tau = R_{Th}C = \frac{10\times 6}{10+6}\times 4 = 15\sec$. (replace the part of the circuit than contains only independent sources and resistive elements by an equivalent Thevenin's voltage source. In this case, we need only to find the Thevenin resistance R_{Th}).

Note: When the switch is kept in closed position, initially the capacitor will be in discharge state and subsequently its voltage will decrease with the increase in time. Finally, at steady state the capacitor is charged with a voltage $v_c(t=\infty)=\frac{40}{10+6}\times 6=15\ V$ (theoretically, time required to reach the capacitor voltage at steady value is $5\tau=5\times 15=75$ sec.).

The circuit shown in Fig.below has been established for a long time

The switch is closed at time $t=0$. Find the current (i) $i_1(0^+)$, $i_2(0^+)$, $i_3(0^+)$, *and* $\left.\frac{dv_{de}}{dt}\right|_{t=0^+}$ (ii) at steady state the voltage across the capacitors, $i_1(\infty)$, $i_2(\infty)$ *and* $i_3(\infty)$.



Solution: (i) At $t=0^-$ no current flowing through the circuit, so the voltage at points 'b' and 'd' are both equal to $50\,volt$. When the switch 'S' closes the capacitor voltage remains constant and does not change its voltage instantaneously. The current $i_1(0^+)$ through $a-b$ branch must then equal to zero, since voltage at terminal 'b' is equal to $v_b(0^+)=50\ volt.$, current through $b-c$ is also zero. One can immediately find

out the current through $c-e$ equal to $i_2(0^+)=\frac{50}{50}=1\ A$. Appling KCL at point '$c$', $i_3(0^+)=1\,A$ which is the only current flow at $t=0^+$ around the loop '$d-c-e-d$'. Note the capacitor across '$d-e$' branch acts as a voltage source, the change of capacitor voltage $\left.\frac{dv_{de}}{dt}\right|_{t=0^+}=\frac{1}{500\times 10^{-6}}\ i_3(0^+)=2\ k\,volt/\text{sec}.$

(ii) at steady state the voltage across each capacitor is given

$=\frac{50}{150}\times 50=16.666\,volt.$

At steady state current delivered by the source to the different branches are given by

$i_1(\infty)=\frac{50}{150}=0.333\,A;\ i_2(\infty)=0.333\,A\ \ and\ \ i_3(\infty)=0\,A$

Transient Analysis of Electric Circuits Using Laplace Transform

In electrical engineering, a **transient response** or **natural response** is the electrical response of a system to a change from equilibrium.

The condition prevailing in an electric circuit between two steady-state conditions is known as the *transient state*; it lasts for a very short time. The currents and voltages during the transient state are called *transients*.

In general, transient phenomena occur whenever

i. a circuit is suddenly connected or disconnected to/from the supply,

ii. there is a sudden change in the applied voltage from one finite value to another,

iii. a circuit is short-circuited.

We consider the transient analysis for the following circuits subject to step input, impulse input and sinusoidal input:

1. *RL* Series Circuit,
2. *RC* Series Circuit,
3. *RLC* Series Circuit, and
4. *RLC* Parallel Circuit.

Steps for Circuit Analysis Using Laplace Transform Method

1. All circuit elements are transformed from time-domain to Laplace domain with initial conditions.
2. Excitation function is transformed into Laplace domain.
3. The circuit is solved using different circuit analysis techniques, such as, mesh analysis, node analysis, etc.
4. Time domain solution is obtained by taking inverse Laplace transform of the solution.

Convolution Theorem

If $f_1(t)$ and $f_2(t)$ are two functions of time which are zero for $t < 0$, and if their Laplace transforms are $F_1(s)$ and $F_2(s)$, respectively, then the convolution theorem states that the Laplace transform of the convolution of $f_1(t)$ and $f_2(t)$ is given by the product $F_1(s)\,F_2(s)$.

Application of Convolution Theorem

The convolution theorem is used to find the response of a linear system to any arbitrary excitation if the impulse response of the system is known.

We know that the transfer function is defined as the ratio of response transform to excitation transform with zero initial conditions. Thus,

$$\text{Transfer Function} = \left.\frac{\text{Laplace transform of Response}}{\text{Laplace transform of Excitation}}\right|_{\text{all initial conditions reduced to zero}}$$

or $H(s) = \left.\frac{Y(s)}{W(s)}\right|_{IC=0}$

Thus, $Y(s) = H(s)W(s)$

Laplace Transform Table

There is always a table that is available to the engineer that contains information on the Laplace transforms. An example of Laplace transform table has been made below. We will come to know about the Laplace transform of various common functions from the following table .

$$\mathcal{L}\,[\delta(t)] = 1$$

$$\mathcal{L}\,[u(t)] = \frac{1}{s}$$

$$\mathcal{L}\,[t] = \frac{1}{s^2}$$

$$\mathcal{L}\,[t^2] = \frac{2}{s^3}$$

$$\mathcal{L}\,[t^3] = \frac{6}{s^4}$$

$$\mathcal{L}\,[t^n] = \frac{n!}{s^{n+1}}$$

$$\pounds\left[e^{-\alpha t}\right]=\frac{1}{s+\alpha}$$

$$\pounds\left[e^{\alpha t}\right]=\frac{1}{s-\alpha}$$

$$\pounds\left[te^{-\alpha t}\right]=\frac{1}{(s+\alpha)^2}$$

$$\pounds\left[te^{\alpha t}\right]=\frac{1}{(s-\alpha)^2}$$

$$\pounds\left[t^n e^{-\alpha t}\right]=\frac{n!}{(s+\alpha)^{n+1}}$$

$$\pounds\left[\cos\omega t\right]=\frac{s}{s^2+\omega^2}$$

$$\pounds\left[\sin\omega t\right]=\frac{\omega}{s^2+\omega^2}$$

$$\pounds\left[e^{-\alpha t}\sin\omega t\right]=\frac{\omega}{(s+\alpha)^2+\omega^2}$$

$$\pounds\left[e^{-\alpha t}\cos\omega t\right]=\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$$

$$\pounds\left[\sinh\alpha t\right]=\frac{\alpha}{s^2-\alpha^2}$$

$$\pounds\left[\cosh\alpha t\right]=\frac{s}{s^2-\alpha^2}$$

2.10. TRANSIENT ANALYSIS OF A SERIES RL CIRCUITS USING LAPLACE TRANSFORM:

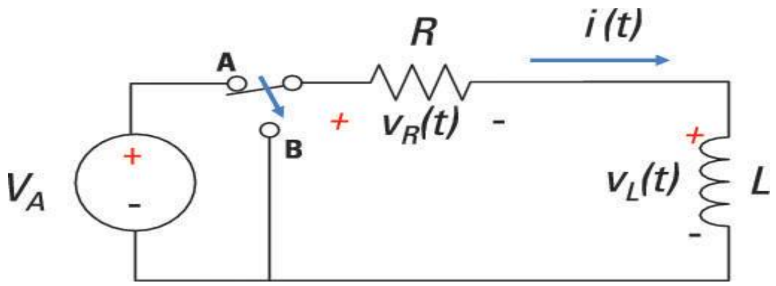
Using the Laplace transform as part of your circuit analysis provides you with a prediction of circuit response. Analyze the poles of the Laplace transform to get a general idea of output behavior. Real poles, for instance, indicate exponential output behavior.

Follow these basic steps to analyze a circuit using Laplace techniques:

1. Develop the differential equation in the time-domain using Kirchhoff's laws and element equations.
2. Apply the Laplace transformation of the differential equation to put the equation in the *s*-domain.
3. Algebraically solve for the solution, or response transform.
4. Apply the inverse Laplace transformation to produce the solution to the original differential equation described in the time-domain.

To get comfortable with this process, you simply need to practice applying it to different types of circuits such as an RC (resistor-capacitor) circuit, an RL (resistor-inductor) circuit, and an RLC (resistor-inductor-capacitor) circuit.

Here is an RL circuit that has a switch that's been in Position A for a long time. The switch moves to Position B at time t = 0.



For this circuit, you have the following KVL equation:

$v_R(t) + v_L(t) = 0$

Next, formulate the element equation (or *i*-*v* characteristic) for each device. Using Ohm's law to describe the voltage across the resistor, you have the following relationship:

$v_R(t) = i_L(t)R$

The inductor's element equation is

$$v_L(t) + L\frac{di_L(t)}{dt}$$

Substituting the element equations, $v_R(t)$ and $v_L(t)$, into the KVL equation gives you the desired first-order differential equation:

$$L\frac{di_L(t)}{dt} + i_L(t)R = 0$$

On to Step 2: Apply the Laplace transform to the differential equation:

$$\mathcal{L}\left[L\frac{di_L(t)}{dt}+i_L(t)R\right]=0$$

$$\mathcal{L}\left[L\frac{di_L(t)}{dt}\right]+\mathcal{L}\left[i_L(t)R\right]=0$$

The preceding equation uses the linearity property which says you can take the Laplace transform of each term. For the first term on the left side of the equation, you use the differentiation property:

$$\mathcal{L}\left[L\frac{di_L(t)}{dt}\right]=L\left[sI_L(s)-I_0\right]$$

This equation uses $I_L(s) = \mathcal{L}[i_L(t)]$, and I_0 is the initial current flowing through the inductor.

The Laplace transform of the differential equation becomes

$I_L(s)R + L[sI_L(s) - I_0] = 0$

Solve for $I_L(s)$:

$$I_L(s)=\frac{I_0}{s+\frac{R}{L}}$$

For a given initial condition, this equation provides the solution $i_L(t)$ to the original first-order differential equation. You simply perform an inverse Laplace transform of $I_L(s)$ — or look for the appropriate transform pair in this table to get back to the time-domain.

Signal Description	***Time-Domain Waveform, f(t)***	***s-Domain Waveform, F(s)***
Step	$u(t)$	$\frac{1}{s}$
Exponential	$\left[e^{-\alpha t}\right]u(t)$	$\frac{1}{s+\alpha}$
Impulse	$\delta(t)$	1
Ramp, *r(t)*	$tu(t)$	$\frac{1}{s^2}$
Sine	$\left[\sin\beta t\right]u(t)$	$\frac{\beta}{s^2+\beta^2}$
Cosine	$\left[\cos\beta t\right]u(t)$	$\frac{s}{s^2+\beta^2}$
Damped Pairs		
Damped ramp	$te^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^2}$
Damped sine	$\left[e^{-\alpha t}\sin\beta t\right]u(t)$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$
Damped cosine	$\left[e^{-\alpha t}\cos\beta t\right]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$

The preceding equation has an exponential form for the Laplace transform pair. You wind up with the following solution:

$$I_L(s) = \frac{I_0}{s+\frac{R}{L}} \leftrightarrow i_L(t) = I_0 e^{-\left(\frac{R}{L}\right)t}$$

The result shows as time *t* approaches infinity, the initial inductor current eventually dies out to zero after a long period of time — about 5 time constants (*L/R*)

2.11.TRANSIENT RESPONSE OF SERIES RC CIRCUIT USING LAPLACE TRANSFORMS:

Using the Laplace transform as part of your circuit analysis provides you with a prediction of circuit response. Analyze the poles of the Laplace transform to get a general idea of output behavior. Real poles, for instance, indicate exponential output behavior.

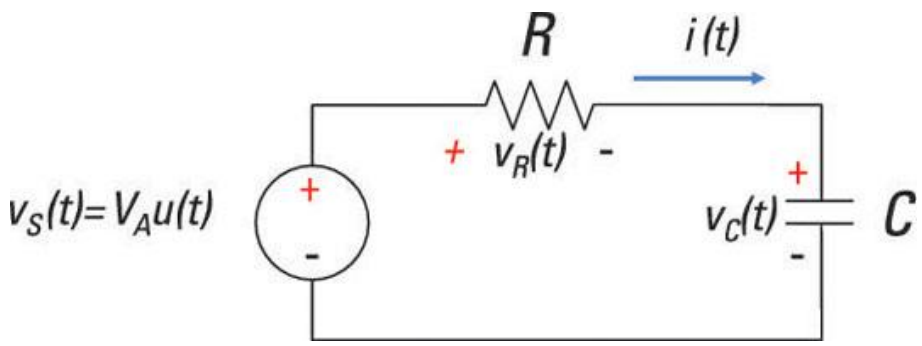
Follow these basic steps to analyze a circuit using Laplace techniques:

1. Develop the differential equation in the time-domain using Kirchhoff's laws and element equations.
2. Apply the Laplace transformation of the differential equation to put the equation in the *s*-domain.
3. Algebraically solve for the solution, or response transform.
4. Apply the inverse Laplace transformation to produce the solution to the original differential equation described in the time-domain.

To get comfortable with this process, you simply need to practice applying it to different types of circuits such as an RC (resistor-capacitor) circuit, an RL (resistor-inductor) circuit, and an RLC (resistor-inductor-capacitor) circuit.

Consider the simple first-order RC series circuit shown here. To set up the differential equation for this series circuit, you can use Kirchhoff's voltage law (KVL), which says the sum of the voltage rises and drops around a loop is zero. This circuit has the following KVL equation around the loop:

$-v_S(t) + v_r(t) + v_c(t) = 0$



Next, formulate the element equation (or *i-v* characteristic) for each device. The element equation for the source is

$v_S(t) = V_A u(t)$

Use Ohm's law to describe the voltage across the resistor:

$v_R(t) = i(t)R$

The capacitor's element equation is given as

$$i(t) = C\frac{dv_c(t)}{dt}$$

Substituting this expression for *i(t)* into $v_R(t)$ gives you the following expression:

$$v_R(T) = i(t)R = RC\frac{dv_c(t)}{dt}$$

Substituting $v_R(t)$, $v_C(t)$, and $v_S(t)$ into the KVL equation leads to

$$-v_s(t) + v_R(t) + v_c(t) = 0$$

$$-V_A u(t) + RC\frac{dv_c(t)}{dt} + v_c(t) = 0$$

Now rearrange the equation to get the desired first-order differential equation:

$$RC\frac{dv_c(t)}{dt} + v_c(t) = V_A u(t)$$

Now you're ready to apply the Laplace transformation of the differential equation in the *s*-domain. The result is

$$\mathcal{L}\left[RC\frac{dv_c(t)}{dt} + v_c(t)\right] = \mathcal{L}\left[V_A u(t)\right]$$

$$\mathcal{L}\left[RC\frac{dv_c(t)}{dt}\right] + \mathcal{L}\left[v_c(t)\right] = \mathcal{L}\left[V_A u(t)\right]$$

On the left, the linearity property was used to take the Laplace transform of each term. For the first term on the left side of the equation, you use the differentiation property, which gives you

$$\mathcal{L}\left[RC\frac{dv_c(t)}{dt}\right] = RC\left[sV_c(s) - V_0\right]$$

This equation uses $V_C(s) = \mathcal{L}[v_C(t)]$, and V_0 is the initial voltage across the capacitor.

Using the following table, the Laplace transform of a step function provides you with this:

$$\mathcal{L}\left[V_A u(t)\right] = \frac{V_A}{s}$$

Signal Description	***Time-Domain Waveform, f(t)***	***s-Domain Waveform, F(s)***
Step	$u(t)$	$\frac{1}{s}$
Exponential	$\left[e^{-\alpha t}\right]u(t)$	$\frac{1}{s+\alpha}$
Impulse	$\delta(t)$	1
Ramp, *r(t)*	$tu(t)$	$\frac{1}{s^2}$
Sine	$\left[\sin\beta t\right]u(t)$	$\frac{\beta}{s^2+\beta^2}$
Cosine	$\left[\cos\beta t\right]u(t)$	$\frac{s}{s^2+\beta^2}$
Damped Pairs		
Damped ramp	$te^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^2}$
Damped sine	$\left[e^{-\alpha t}\sin\beta t\right]u(t)$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$
Damped cosine	$\left[e^{-\alpha t}\cos\beta t\right]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$

Based on the preceding expressions for the Laplace transforms, the differential equation becomes the following:

$$RC\left[sV_c(s)-V_0\right]+V_c(s)=\frac{V_A}{s}$$

Next, rearrange the equation:

$$\left[s+\frac{1}{RC}\right]V_c(s)=\frac{V_A}{RC}\left(\frac{1}{s}\right)+V_0$$

Solve for the output $V_c(s)$ to get the following transform solution:

$$V_c(s)=\frac{V_A}{RC}\left[\frac{1}{s\left(s+\frac{1}{RC}\right)}\right]+\frac{V_0}{s+\frac{1}{RC}}$$

By performing an inverse Laplace transform of $V_C(s)$ for a given initial condition, this equation leads to the solution $v_C(t)$ of the original first-order differential equation.

On to Step 3 of the process. To get the time-domain solution $v_C(t)$, you need to do a partial fraction expansion for the first term on the right side of the preceding equation:

2.13. TRANSIENT ANALYSIS OF A SERIES RLC CIRCUIT USING LAPLACE TRANSFORMS:

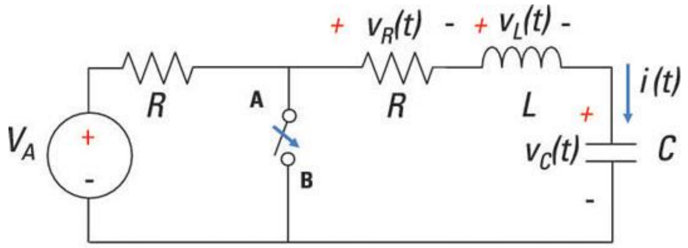
Using the Laplace transform as part of your circuit analysis provides you with a prediction of circuit response. Analyze the poles of the Laplace transform to get a general idea of output behavior. Real poles, for instance, indicate exponential output behavior.

Follow these basic steps to analyze a circuit using Laplace techniques:

1. Develop the differential equation in the time-domain using Kirchhoff's laws and element equations.
2. Apply the Laplace transformation of the differential equation to put the equation in the *s*-domain.
3. Algebraically solve for the solution, or response transform.
4. Apply the inverse Laplace transformation to produce the solution to the original differential equation described in the time-domain.

To get comfortable with this process, you simply need to practice applying it to different types of circuits such as an RC (resistor-capacitor) circuit, an RL (resistor-inductor) circuit, and an RLC (resistor-inductor-capacitor) circuit.

Here you can see an RLC circuit in which the switch has been open for a long time. The switch is closed at time $t = 0$.



$R = 800\ \Omega$ $\quad C = \frac{1}{4.1} \cdot 10^{-5}\ \text{F} = 2.439\ \mu\text{F}$

$L = 1\ \text{H}$ $\quad V_A = 5\ \text{V}$

In this circuit, you have the following KVL equation:

$v_R(t) + v_L(t) + v(t) = 0$

Next, formulate the element equation (or i-v characteristic) for each device. Ohm's law describes the voltage across the resistor (noting that $i(t) = i_L(t)$ because the circuit is connected in series, where $I(s) = I_L(s)$ are the Laplace transforms):

$v_R(t) = i(t)R$

The inductor's element equation is given by

$$v_L(t) = L\frac{di_L(t)}{dt}$$

And the capacitor's element equation is

$$v_c(t) = \frac{1}{C}\int_0^t i(\tau) + v_c(0)$$

Here, $v_C(0) = V_0$ is the initial condition, and it's equal to 5 volts.

Substituting the element equations, $v_R(t)$, $v_C(t)$, and $v_L(t)$, into the KVL equation gives you the following equation (with a fancy name: the *integro-differential equation*):

$$L\frac{di_L(t)}{dt} + i_L(t)R + \frac{1}{C}\int_0^t i(\tau) + v_c(0) = 0$$

The next step is to apply the Laplace transform to the preceding equation to find an *I(s)* that satisfies the integro-differential equation for a given set of initial conditions:

$$\mathcal{L}\left[L\frac{di_L(t)}{dt} + i(t)R + \frac{1}{C}\int_0^t i(\tau)d\tau + V_0\right] = 0$$
$$\mathcal{L}\left[L\frac{di_L(t)}{dt}\right] + \mathcal{L}\left[i(t)R\right] + \mathcal{L}\left[\frac{1}{C}\int_0^t i(\tau)d\tau + V_0\right] = 0$$

The preceding equation uses the linearity property allowing you to take the Laplace transform of each term. For the first term on the left side of the equation, you use the differentiation property to get the following transform:

$$\mathcal{L}\left[L\frac{di(t)}{dt}\right] = L\left[sI(s) - I_0\right]$$

This equation uses $I_L(s) = \mathcal{L}[i(t)]$, and I_0 is the initial current flowing through the inductor. Because the switch is open for a long time, the initial condition I_0 is equal to zero.

For the second term of the KVL equation dealing with resistor *R*, the Laplace transform is simply

$\mathcal{L}[i(t)R] = I(s)R$

For the third term in the KVL expression dealing with capacitor *C*, you have

$$\mathcal{L}\left[\frac{1}{C}\int_0^t i(\tau)d\tau + V_0\right] = \frac{I(s)}{sC} + \frac{V_0}{s}$$

The Laplace transform of the integro-differential equation becomes

$$L\left[sI(s) - I_0\right] + I(s)R + \frac{I(s)}{sC} + \frac{V_0}{s} = 0$$

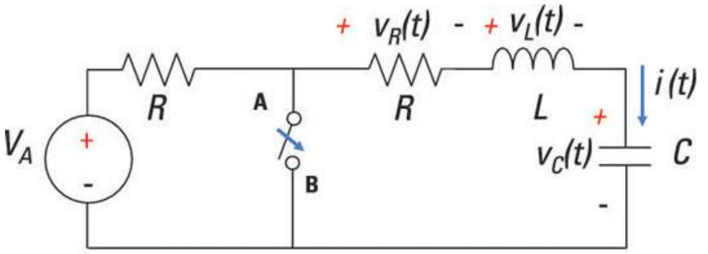
Rearrange the equation and solve for I(s):

$$I(s) = \frac{sI_0 - \frac{V_0}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

To get the time-domain solution *i(t)*, use the following table, and notice that the preceding equation has the form of a damping sinusoid.

Signal Description	***Time-Domain Waveform, f(t)***	***s-Domain Waveform, F(s)***
Step	$u(t)$	$\frac{1}{s}$
Exponential	$\left[e^{-\alpha t}\right]u(t)$	$\frac{1}{s+\alpha}$
Impulse	$\delta(t)$	1
Ramp, *r(t)*	$tu(t)$	$\frac{1}{s^2}$
Sine	$[\sin\beta t]u(t)$	$\frac{\beta}{s^2+\beta^2}$
Cosine	$[\cos\beta t]u(t)$	$\frac{s}{s^2+\beta^2}$
Damped Pairs		
Damped ramp	$te^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^2}$
Damped sine	$\left[e^{-\alpha t}\sin\beta t\right]u(t)$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$
Damped cosine	$\left[e^{-\alpha t}\cos\beta t\right]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$

Now, you plug in $I_0 = 0$ and some numbers from this figure:



$R = 800\ \Omega$ $\qquad C = \frac{1}{4.1} \cdot 10^{-5}\ \text{F} = 2.439\ \mu\text{F}$

$L = 1\ \text{H}$ $\qquad V_A = 5\ \text{V}$

Now you've got this equation:

$$I(s) = -\frac{5}{s^2 + 800s + 401 \cdot 10^5}$$

$$= -\frac{5}{500}\left[\frac{500}{(s+400)^2 + (500)^2}\right]$$

You wind up with the following solution:

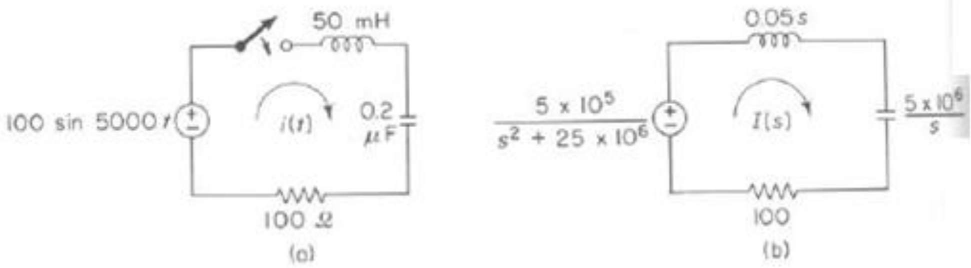
$i(t) = [-0.01e^{-400t} \sin 500t]u(t)$

For this RLC circuit, you have a damping sinusoid. The oscillations will die out after a long period of time. For this example, the time constant is 1/400 and will die out after 5/400 = 1/80 seconds.

Numerical Problems on RLC Circuits:

1.

The relaxed series RLC circuit of Fig. 6-23a is excited at $t = 0$ by the sinusoidal source shown. Solve for the current $i(t)$ for $t > 0$.



Solution Although the mathematics will eventually reveal the type of response, a preliminary calculation should prove interesting. We will first calculate $R/2L$ and $1/\sqrt{LC}$.

$$\frac{R}{2L} = \frac{100}{2 \times 0.05} = 10^3 \qquad (6\text{-}125a)$$

$$\frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.05 \times 0.2 \times 10^{-6}}} = 10^4 \qquad (6\text{-}125b)$$

Since $R/2L < 1/\sqrt{LC}$, the circuit is underdamped and oscillatory. We have

$$\alpha = 10^3 \text{ nepers}^\dagger \qquad (6\text{-}126)$$

$$\omega_0 = 10^4 \text{ rad/s} \qquad (6\text{-}127)$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 9.95 \times 10^3 \text{ rad/s} \qquad (6\text{-}128)$$

As a result of the relatively small amount of damping, the damped resonant frequency differs from the undamped resonant frequency by only 0.5%. As a matter of interest, the damped repetition frequency is $f_d = \omega_d/2\pi = 1548$ Hz. Notice that the natural damped frequency is about twice the frequency of the excitation. Again, we point out that these preliminary calculations are not absolutely necessary as the results will "fall out" of the math that follows.

The transformed circuit is shown in Fig. 6-23b. Using the impedance concept, we have

$$\begin{aligned} Z(s) &= 0.05s + 100 + \frac{5 \times 10^6}{s} \\ &= \frac{0.05s^2 + 100s + 5 \times 10^6}{s} \\ &= \frac{s^2 + 2000s + 10^8}{20s} \end{aligned} \qquad (6\text{-}129)$$

The current is

$$I(s) = \frac{E(s)}{Z(s)} = \frac{10^7 s}{(s^2 + 25 \times 10^6)(s^2 + 2000s + 10^8)} \qquad (6\text{-}130)$$

The poles due to the quadratic with three terms are

$$\begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} = -10^3 \pm j9.95 \times 10^3 \qquad (6\text{-}131)$$

which agrees with our preliminary calculations.

We obtain the final desired result by finding the inverse transform of $I(s)$. Since one quadratic has imaginary roots and the other has complex roots, we may invert the function by applying the special formula of Section 5-7 individually to the two quadratic factors. The reader is invited to show that the result is

$$i(t) = 0.133\epsilon^{-1000t} \sin(9.95 \times 10^3 t - 99.51°) + 0.132 \sin(5000t + 82.41°) \quad (6\text{-}132)$$

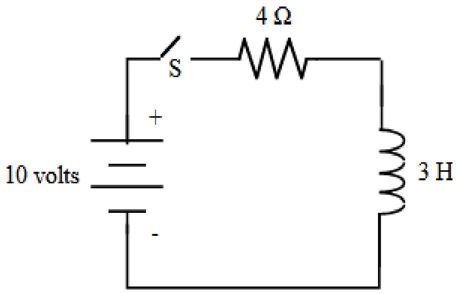
The response is seen to consist of a damped sinusoidal term whose frequency is the natural damped resonant frequency of the circuit, and an undamped sinusoid whose frequency is that of the excitation. The former term is transient in nature, whereas the latter term is the steady-state response. After the transient disappears, the steady-state or forced response is

$$i_{ss}(t) = 0.132 \sin(5000t + 82.41°) \quad (6\text{-}133)$$

SOLVED PROBLEMS

The switch S closes at T=0. The complete response for i(t) for t>0 is

(a) $2.5 + 6e^{-0.75t}$
(b) $2.5 - 2.5e^{-0.75t}$
(c) $2.5 + 2.5e^{-1.33t}$
(d) 0
(e) $2.5 - 2.5e^{-1.33t}$



We use KVL to write the differential equation for the circuit using the correct expression for the impedance of the inductor.

$$-10 + 4i + 3\frac{di}{dt} = 0$$

Re-writing this in conventional form with the sources on the right hand side of the equation

$$3\frac{di}{dt} + 4i = 10$$

The dc (or homogeneous) solution is obtained by setting the derivatives equal to zero or, in this case

$4i = 10$
giving i=2.5 amps.

The transient solution is always an exponential in form. Substituting $i(t)=Ae^{kt}$ into the differential equation and setting the source (the right hand side of the equation) equal to zero we obtain

differential equation and setting the source (the right hand side of the equation) equal to zero we obtain

$$3\frac{di}{dt} + 4i = 0$$

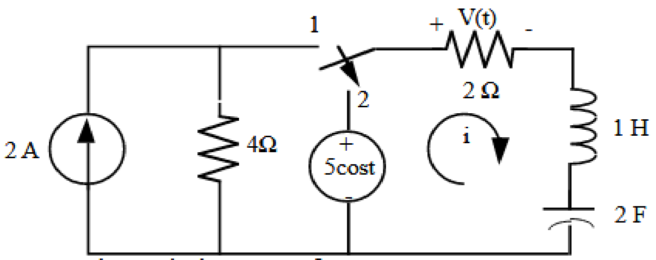
$3kAe^{kt} + 4Ae^{kt} = 0$
$3k+4 = 0$
$k=-4/3$

The total solution is then $i(t) = i_{transient} + i_{homogeneous} = Ae^{-1.33t} + 2.5$
The coefficient A is solved for by using the boundary condition that $i(0^+)=i(0^-)=0$.
This requires that $i(0^+) = A + 2.5 = 0$, or that $A=-2.5$.

Then $i(t) = 2.5 - 2.5e^{-1.33t}$ and the correct answer is (e).

The form of the transient part of V(t) for t>0 is

(a) $(A_1+A_2)e^{-0.5t}$
(b) $A_1\cos(0.5t) + A_2\sin(0.5t)$
(c) $A_1e^{-t}\cos(0.5t) + A_2e^{-t}\sin(0.5t)$
(d) $A_1e^{-1.71t} + A_2e^{-0.29t}$
(e) 0



At t=0 the switch moves from 1 to 2.

Solution:
For t>0 the equation of the circuit is

$$\frac{di}{dt} + 2i + \frac{1}{2}\int_0^t i(\alpha)d\alpha + V_C(0^+) = 5\cos(t)$$

where $V_c(0^+)$, the initial voltage on the capacitor, is zero.
Differentiating the above equation to remove the integral

$$\frac{d^2i}{dt^2} + 2\frac{di}{dt} + \frac{1}{2}\,i(t) = -5\sin(t)$$

The left hand side of this equation describes the transient response. For the transient

$$\frac{d^2i}{dt^2} + 2\frac{di}{dt} + \frac{1}{2}\,i(t) = 0$$

If we let $i(t)=Ae^{mt}$ we get

$$m^2Ae^{mt} + 2mAe^{mt} + \frac{1}{2}Ae^{mt} = 0$$

which reduces to the characteristic equation

$$m^2 + 2m + 0.5 = 0$$

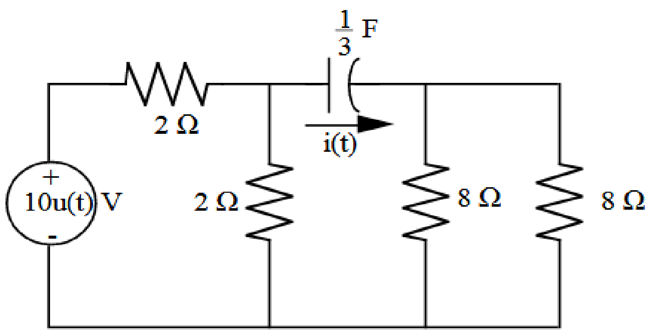
This equation can be solved using the quadratic formula

$$m = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(0.5)}}{2} = -1.71 \text{ and } -0.29$$

The only answer with these exponents is (d).

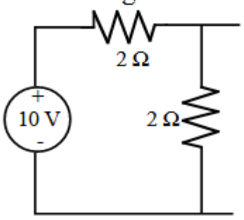
i(t) is

(a) $e^{-0.6t}$
(b) $e^{-1.67t}$
(c) $-e^{-1.67t}$
(d) $-e^{-0.6t}$
(e) $2-e^{-1.67t}$



Solution:
There are many ways to solve this problem but, perhaps, the easiest way is to Thevenize the left hand side of the circuit (the voltage source and the two 2Ω resistors) and replace the right hand side of the circuit (the two 8Ω resistors in parallel) by its equivalent resistance.

Thevenizing the left hand side of the circuit



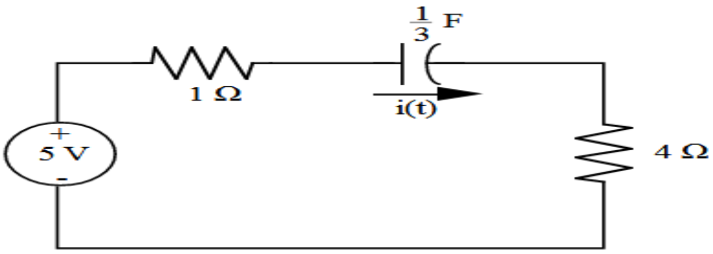
$$V_T = \frac{2}{2+2}10 = 5 \text{ volts}$$

and

$$R_T = \frac{2\times 2}{2+2} = 1\ \Omega$$

NOTE: This is a good technique to use to get rid of current sources in problems.

The 8 Ω resistors in parallel can be replaced by a 4 Ω resistor. Redrawing the original circuit and replaceing the left hand side by its Thevenin equivalent and replacing the two 8Ω resistors by a single 4Ω resistance, we get the following circuit



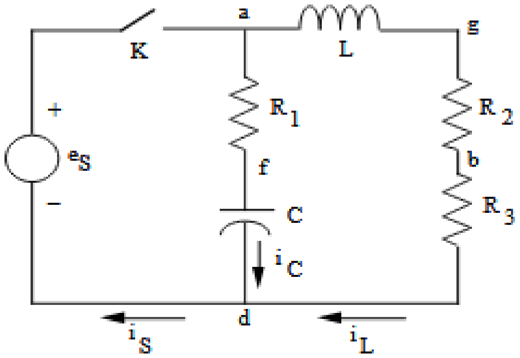
Since $V_C(0^+)=V_C(0^-)=0$

$$i_C(0^+) = \frac{5 \text{ volts}}{1\Omega + 4\Omega} = 1 \text{ amp}$$

The time constant for the circuit can be directly computed as

$$\tau = RC = (5\ \Omega)(\tfrac{1}{3}\text{ F}) = 1.67 \text{ seconds}$$

The solutions should be of the form



You are given that $e_S(t)=E+E_1\sin(500t)+E_2\sin(1000t)$, L=10 millihenries, C=200 microfarads, R_1=10 ohms, R_2=5.0 ohms and R_3=5.0 ohms in the above circuit.

For questions 11–14 assume that switch K is closed at t=0 and answer the questions for the instant immediately after the switch is closed, i.e. for time $t=0^+$.

11. If E=30V, E_1=40V and E_2=20V, the current i_C is most nearly
(A) 0.0 amperes
(B) 1.5 amperes
(C) 2.8 amperes
(D) 3.0 amperes
(E) 6.0 amperes

This problem is most easily solved by recalling the initial conditions for capacitors which require that V(0-) = V(0+). The initial voltage on the capacitor is 0 volts so the voltage across the capacitor immediately after the switch is closed must also be 0 volts. The applied voltage $e_S(t=0+) \approx 30$ since $\sin(0+) \approx 0$. At t=0+ e_S appears enitrely across R_1 and the resulting current (which is equal to i_C since R_1 and C are in series) must be given by

$$i_C = \frac{e_S}{R_1} = \frac{30 \text{ volts}}{10\Omega} = 3.0 \text{ Amperes}$$

The correct answer is (D).

In an RL circuit of Fig. 2.6, the switch closes at $t = 0$. Find the complete current response, if $R = 10\ \Omega$, $L = 0.01$ H, and $v_s = 120\sqrt{2}\sin(1000t + 15^\circ)$ V.

1) The time constant of the circuit is

$$\tau = \frac{L}{R} = \frac{0{,}01}{10} = 10^{-3} = 1\,ms$$

and the natural response is

$$i_n = Ae^{-1000t}.$$

2) The steady-state current is calculated by phasor analysis. The impedance of the circuit is $Z(j\omega) = R + j\omega L = 10 + j10 = 10\sqrt{2}\angle 45^\circ\,\Omega$, the voltage source phasor is $\underline{V}_{sm} = 100\sqrt{2}e^{j15^\circ}$. Thus, the current phasor will be

$$\underline{I}_f = \frac{\underline{V}_{sm}}{\underline{Z}} = \frac{100\sqrt{2}\angle 15^\circ}{10\sqrt{2}\angle 45^\circ} = 10\angle -30^\circ A$$

and the current versus time is

$$i_f = 10\sin(1000t - 30^\circ)\,A$$

3)The initial condition is zero, i.e., $i(0_+) = i(0_-) = 0$.

4) Non-dependent initial conditions are needed.

5) The integration constant can now be found $A = i(0) - i_f(0) = 0 - 10\sin(-30^\circ) = 5$ and the complete response is

$$i(t) = 10\sin(1000t - 30^\circ) + 5e^{-1000t} A.$$

UNIT- III
LOCUS DIAGRAMS AND NETWORKS FUNCTIONS

LOCUS DIAGRAMS

3.1.INTRODUCTION:

Locus diagrams are the graphical representations of the way in which the response of electrical circuits vary, when one or more parameters are continuously changing. They help us to study the way in which

a. Current / power factor vary, when voltage is kept constant,
b. Voltage / power factor vary, when current is kept constant, when one of the parameters of the circuit (whether series or parallel) is varied.

The Locus diagrams yield such important information as I_{max}, I_{min} , V_{max} ,V_{min} & the power factor`s at which they occur. In some parallel circuits, they will also indicate whether or not, a condition for response is possible.

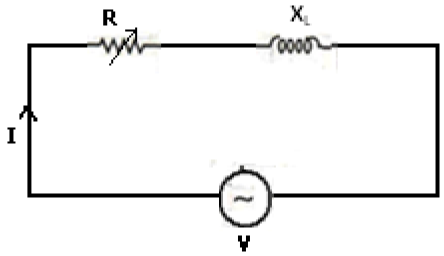
3.2. RL Series Circuit:

Consider an R – X_L series circuit as shown below, across which a constant voltage is applied. By varying R or X_L, a wide range of currents and potential differences can be obtained.

`R ` can be varied by the rheostatic adjustment and X_L can be varied by using a variable inductor or by applying a variable frequency source.

When the variations are uniform and lie between 0 and infinity, the resulting locus diagrams are circles

Case 1: when `R` is varied



When R = 0 , the current is maximum and is given by $I_{max} = \frac{V}{X_1}$ and lags V by 90^0

∴ Power factor is zero

When R = infinity, the current is minimum and is given by $I_{min} = 0$, $\emptyset = 0$ and power factor = 1

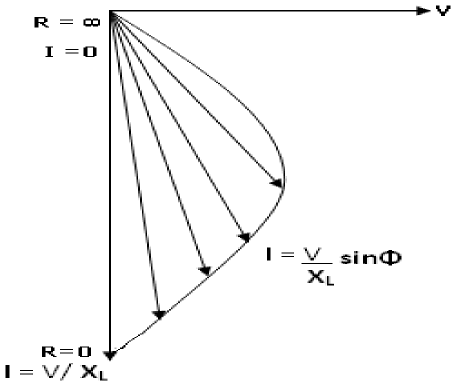
For any other values of`R`, the current lags the voltage by an angle $\emptyset = \tan^{-1}\frac{X_L}{R}$

∴ The general expression for current is

$$I = \frac{V}{\sqrt{R^2+{X_L}^2}} = \frac{V}{Z}\frac{X_L}{X_L} = \frac{V}{X_L}\frac{X_L}{Z} = \frac{V}{X_L}\sin\emptyset$$

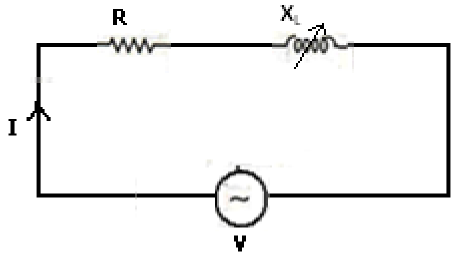
The equation $I = \frac{V}{X_L}\sin\emptyset$ is the equation of a circle in the polar form, where $\frac{V}{X_L}$ is the diameter of the circle.

The Locus diagram of current i.e the way in which the current varies in the circuit, as `R` is varied from zero to infinity is shown in below which is a semi -circle.



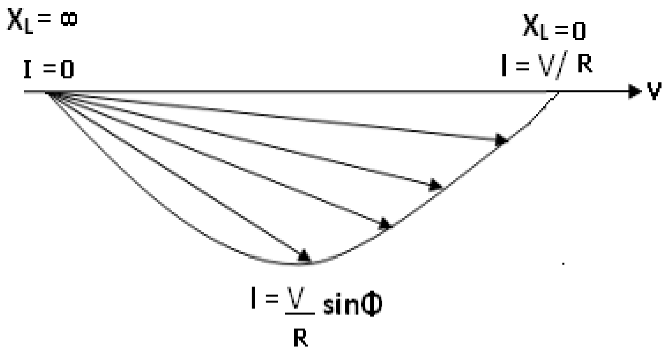
∴ Locus of current in a series RL circuit is a semi circuit with radius = $\frac{V}{2X_L}$ & whose center is given by $(0, \frac{V}{2X_L})$

Case 2: When X_L is varied



When $X_L = 0$, current is maximum and is given by $\frac{V}{R}$ and is in phase with V. The power factor is unity.
When X_L = to infinity, the current is zero, the power factor is zero and $\emptyset = 90^0$
For any other value of `R`, the current lags the voltage by an angle $\emptyset = \frac{X_L}{R}$
∴ The general expression for current is I = $\frac{V}{\sqrt{R^2+X_L{}^2}} = \frac{V}{R}\frac{R}{R} = \frac{V}{R}\frac{R}{Z} = \frac{V}{R}\cos\ \emptyset$

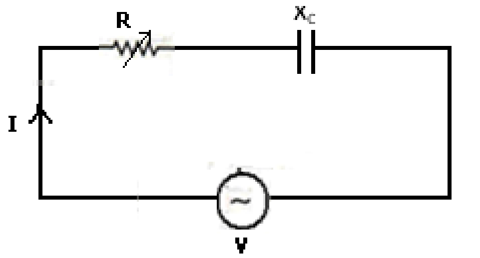
The equation of a circle in the polar form where $\frac{V}{R}$ is the diameter of the circle



∴ The Locus of current in a series RL circuit is a semi circuit whose radius is $\frac{V}{2R}$ and whose center is $\left(\frac{V}{2R}, 0\right)$

3.3. RC Series Circuit:

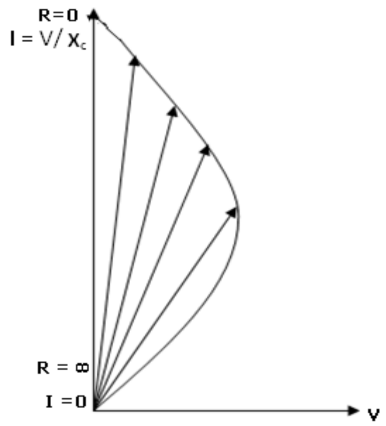
Case 1: when `R` is varied



When R = 0 current is maximum and is given by $I_{max} = \frac{V}{X_c}$, which leads the voltage by 90^0 .Power factor is zero.
When R = ∞, the current is zero. The power factor is unity & $\emptyset = 0$
For any other value of R the current leads the voltage by an angle $\emptyset = tan^{-1}\frac{X_c}{R}$
∴ The general expression for current is

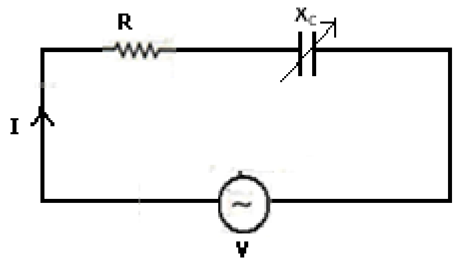
$$I = \frac{V}{\sqrt{R^2+X_L{}^2}} = \frac{V}{Z}\frac{X_c}{X_c} = \frac{V}{X_c}\ \frac{X_c}{Z} = \frac{V}{X_c}\sin\emptyset$$

∴ $\frac{V}{X_c}\sin\emptyset$ is the equation of a circle in the polar form, where $\frac{V}{X_c}$ is the diameter of the circle.



∴ Locus is a semi – circle where radius is $\frac{R}{2X_c}$ & center is$\left(0,\ \frac{V}{2X_c}\right)$.

Case 2: Where X_c is varied



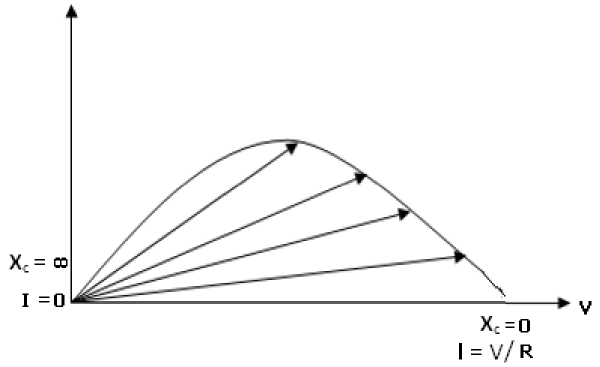
When X_c = 0 , current is maximum & is given by $I_{max} = \frac{V}{R}$, which is in phase with V. Power factor is unity and $\emptyset = 0$

When $X_c = \infty$, the current is zero. Power factor is 0 & $\emptyset = 90^0$, for any other value of X_c , the current leads the voltage by an angle $\emptyset = tan^{-1}\frac{X_c}{R}$

The general equation for the current is

$$I = \frac{V}{Z} = \frac{V}{Z}\frac{R}{R} = \frac{V}{R}X\frac{R}{Z} = \frac{V}{R}\ cos\emptyset$$

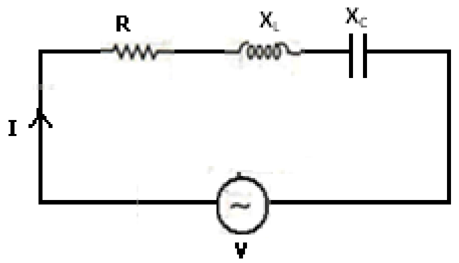
The equation $I = \frac{V}{R}\ cos\emptyset$ is the equation of the circle in polar form, where $\frac{V}{R}$ is the diameter of the circle.



∴

The locus is a circle of radius$\left(\frac{V}{2R}\ ,0\right)$.

3.4. RLC series circuit:



The figure represents an $R - X_L - X_c$ series circuit across which, a constant voltage source is applied `I` is the current flowing through the circuit. The characteristics of this circuit can be studied by varying any one of the parameters, R, X_L, X_c & f .

Case1: when R is varied and the other three parameters are constant, the locus diagram of current are similar to those of

a) $R - X_L$ series circuit, if $X_L > X_c$

b) $R - X_c$ Series circuit if $X_c > X_L$

The only difference would be, the resulting reactance is either $X_L - X_c$ or $X_c - X_L$

Case2: When X_L is varied

When $X_c = 0$ the circuit behaves as an R-X_c series circuit & the current is given by

$$I = \frac{V}{\sqrt{R^2+X_L{}^2}} \quad \& \quad \emptyset = tan^{-1}\frac{X_c}{R}$$

When $X_L = X_c$, the circuit behaves as a pure resistance, circuit the current is maximum or is given by $I_{max} = \frac{V}{R}$ & $\emptyset = 0$ The power factor is unity

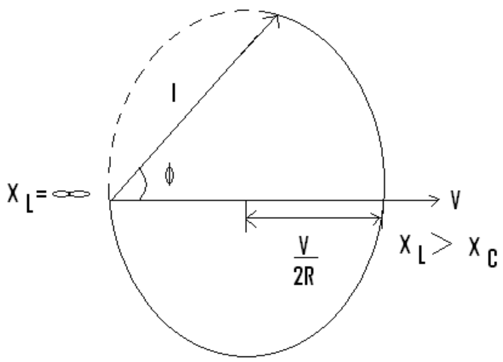
Where $X_L > X_C$, The circuit behaves as an $R - X_L$ series circuit & the current is given by

$$I = \frac{V}{\sqrt{R^2+(X_L-X_C)^2}} \;\&\; \emptyset = tan^{-1}\frac{X_L-X_c}{R}\; (lagging)$$

When $X_L = \infty$, I = 0

For any value of X_L Lying between X_C & ∞, the locus of current is a semi circle of radius = $\frac{V}{2R}$.

The complete locus diagram of current as X_L varies from zero to infinity is as shown below.



Case3 : When X_C is varied

When $X_C = 0$ the circuit behaves as an R-X_L series circuit & the circuit is given by

$$I = \frac{V}{\sqrt{R^2+(X_L-X_C)^2}} \;\&\; \emptyset = tan^{-1}\frac{X_L}{R}\; (lagging)$$

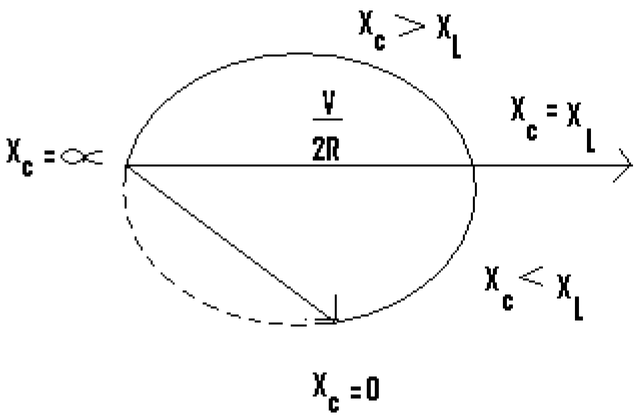
When $X_C = X_L$, the circuit behaves as a pure resistance circuit. The current is maximum and is given by $I_{max} = \frac{V}{R}$ & $\emptyset = 0$. The power factor is unity

When $X_C > X_L$, the circuit behaves as an $R - X_C$ series circuit and the current is given by

$$I = \frac{V}{\sqrt{R^2+(X_C-X_L)^2}} \;\&\; \emptyset = tan^{-1}\frac{X_C-X_L}{R}\; (leading)$$

For any value of X_C lying between X_L & ∞, the locus of current is a semi circle of radius $\frac{V}{2R}$.

The complete locus diagram of current as X_C varies from o to ∞ is as shown below.



Case 4: When ‘f’ is varied

When f=0, $X_c = \infty$, hence I=0.

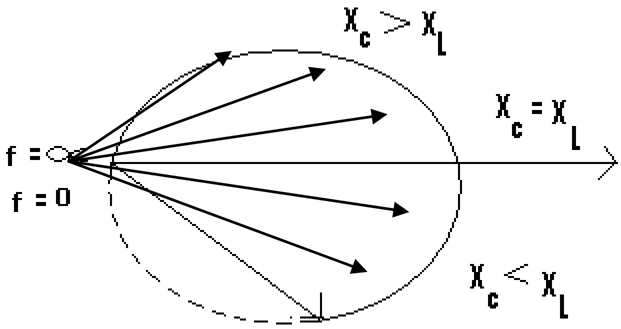
For values of ‘f’, for which $X_c > X_L$, the circuit behaves as an R-X_c series circuit and the locus is a semi circle in upper half of X-Y plane with V/2R as radius.

For values of ‘f’, which $X_c = X_L$, the current is maximum and is equal to $I_{max} = V/R$, $\Phi=0$, P.F=1.

For values of ‘f’, for which $X_c < X_L$ the circuit behaves as an R-X_L series circuit and the locus is a semi circle in lower half of X-Y plane with V/2R as radius.

For $f = \infty$, $X_L = \infty$, $X_c = 0$ and hence I=0.

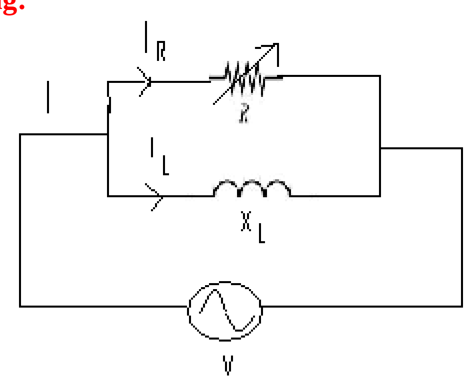
Therefore the complete locus diagram of current as f varies from 0 to ∞ is as shown in figure bellow



3.5. Locus Diagrams of parallel circuits:

When a constant voltage, constant frequency source is applied across a parallel circuit and any one parameters in one of the parallel branches is verified, current varies only in that branch and the total current locus is get by adding the variable current locus with the constant current flowing in the other branch.

Case 1: R & X_L in parallel R Varying:

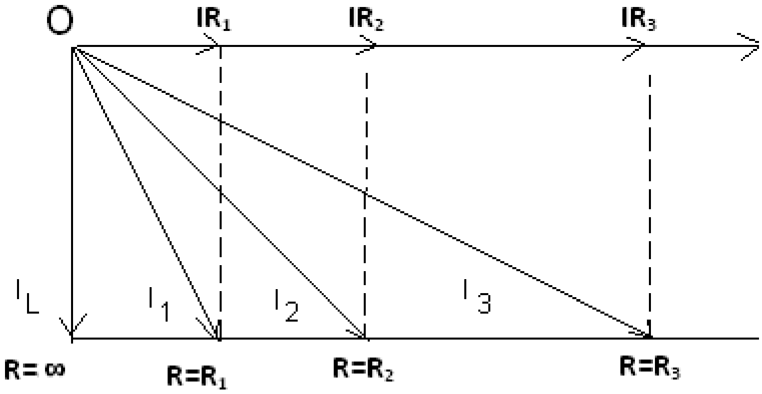


Consider a parallel circuit as shown below, across which a constant voltage, constant frequency source is applied.

$$\vec{I} = \vec{I_L} + \vec{I_R} \quad \vec{I} = \vec{I_L} + \vec{I_R}$$

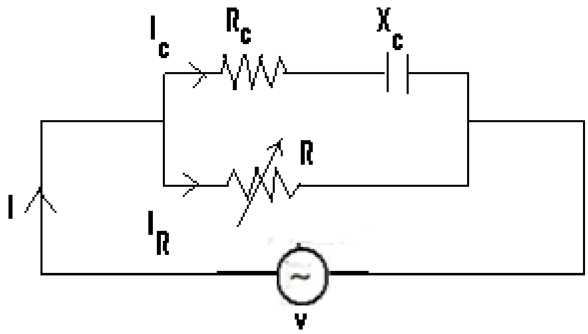
As X_L Is constant I_L is constant
As R is variable I_R is Variable
When R = ∞, I_R = 0 and I = I_L which lags V by 90^0

For any other values of R = R_1, the current I_L remains constant, but $I_{R1} = \frac{V}{R_1}$ and is in phase with V.



For other values of R=R_2, R_3.. etc., I_{R2},I_{R3} etc., and I_1 ,I_2 etc., can be found and plotted.

Case 2: R-X_C in parallel with R & 'R' varying.



Consider a parallel circuit consisting of R_C-X_C branch in parallel with 'R' as shown.

$$I = \vec{I_C} + \vec{I_R}$$

As R_C & X_C are constants, I_C remains constant & is given by

$$I_C = \frac{V}{\sqrt{(R_C{}^2)+(X_C{}^2)}} \quad \& \ \emptyset_C = tan^{-1}\frac{X_C}{R} \ (leading)$$

As R is variable I_R is also variable.
When R = ∞ I_R = 0, hence I = I_C

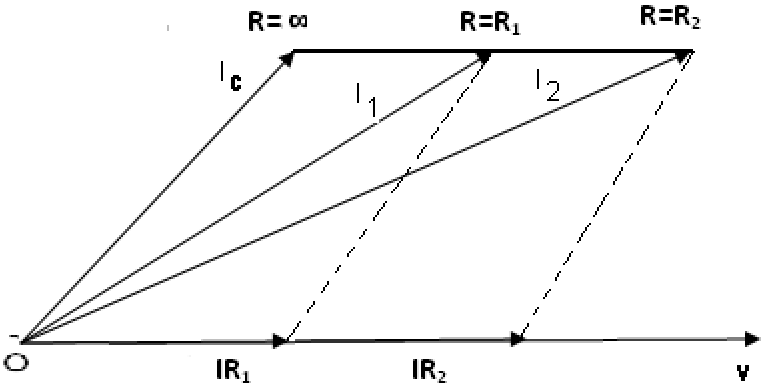
For any other values of R = R_1, I_C remains constant, but $I_{R1} = \frac{V}{R_1}$ & is in phase with V

The total current is given by Type equation here.

$$\vec{I} = \vec{I_C} + \vec{I_R}$$

Similarly for other values of $R_2, R_3, etc., I_{R_2}, I_{R_3} etc., \& I_2 \ I_3 etc.$, can be plotted

The locus of the total current is as shown below.



Problem: A 230 volts, 50 H source is connected to a series circuit consisting of a resistance of 30 ohms and an inductance which varies between 0.03 henries and 0.15 henries. Draw the Locus Diagram of current.

Diameter of circle = $\frac{V}{R} = \frac{230}{30}$ = 7.67 amps

X_{min} = 2X 3.14 X 50 X 0.03 = 9.42 ohms

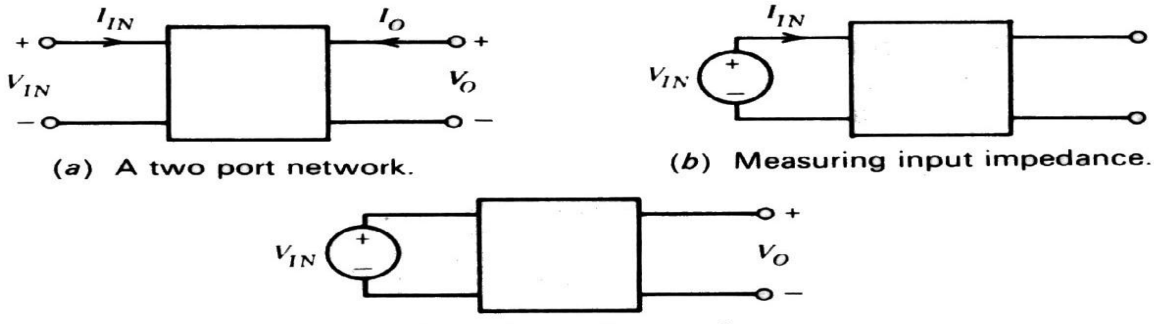
$I_{max} = \frac{230}{\sqrt{(30)^2+(9.42)}}$

X_{max} = 2X3.14x50x0.15 = 47.1ohms

$I_{min} = \frac{230}{\sqrt{(30)^2+(47.1)^2}}$ = 4.52 am

3.6 Network Functions:

A network function is the Laplace transform of an impulse response. Its format is a ratio of two polynomials of the complex frequencies. Consider the general two-port network shown in Figure 2.2a. The terminal voltages and currents of the two-port can be related by two classes of network functions, namely, the driving point functions and the transfer functions.



(a) A two port network. (b) Measuring input impedance. (c) Measuring voltage gain.

The driving point functions relate the voltage at a port to the current at the same port. Thus, these functions are a property of a single port. For the input port the driving point impedance function $Z_{IN}(s)$ is defined as:

This function can be measured by observing the current I_{IN} when the input port is driven by a voltage source V_{IN} (Figure 2.2b). The driving point admittance function $Y_{IN}(s)$ is the reciprocal of the impedance function, and is given by:

$$Y_{IN}(s) = \frac{I_{IN}(s)}{V_{IN}(s)}$$

The output port driving point functions are denned in a similar way. The transfer functions of the two-port relate the voltage (or current) at one port to the voltage (or current) at the other port. The possible forms of transfer functions are:

1. The voltage transfer function, which is a ratio of one voltage to another voltage.
2. The current transfer function, which is a ratio of one current to another current.
3. The transfer impedance function, which is the ratio of a voltage to a current.
4. The transfer admittance function, which is the ratio of a current to a voltage.

The voltage transfer functions are defined with the output port an open circuit, as:

$$\text{voltage gain} = \frac{V_O(s)}{V_{IN}(s)}$$

$$\text{voltage loss (attenuation)} = \frac{V_{IN}(s)}{V_O(s)}$$

To evaluate the voltage gain, for example, the output voltage V_O is measured with the input port driven by a voltage source V_{IN} (Figure 2.2c). The other three types of transfer functions can be defined in a similar manner. Of the four types of transfer functions, the voltage transfer function is the one most often specified in the design of filters.

The functions defined above, when realized using resistors, inductors, capacitors, and active devices, can be shown to be the ***ratios of polynomials in s*** with real coefficients. This is so because the network functions are obtained by solving simple algebraic node equations, which involve at most the terms R, sL, sC and their reciprocals. The active device, if one exists, the solution still involves only the addition and multiplication of simple terms, which can only lead to a ratio of polynomials in s. In addition, all the coefficients of the numerator and denominator polynomials will be real. Thus, the general form of a network function is:

$$H(s) = \frac{a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_0}{b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \cdots + b_0}$$

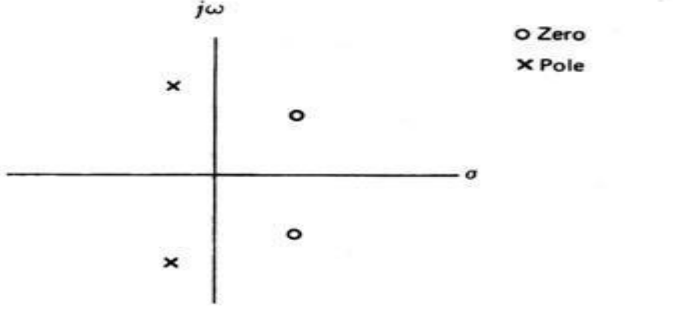
$$\text{where} \quad a_n \neq 0 \qquad b_m \neq 0$$

and all the coefficients a_i and b_i are real. If the numerator and denominator polynomials are factored, an alternate form of H(s) is obtained:

$$H(s) = \frac{a_n(s - z_1)(s - z_2)\cdots(s - z_n)}{b_m(s - p_1)(s - p_2)\cdots(s - p_m)}$$

I

n this expression $z_1, z_2, ..., z_n$ are called the zeros of H(s), because H(s) = 0 when $s = z_i$. The roots of the denominator $p_1, p_2, ..., p_m$ are called the poles of H(s). It can be seen that H(s) = ∞ at the poles, $s = p_i$.The poles and zeros can be plotted on the complex s plane ($s = \sigma + j\omega$), which has the real p art σ for the abscissa, and the imaginary part jω for the ordinate below



Poles and zeros plotted in the complex *s* plane.

3.7. Properties of all Network Functions:

We have already seen that ***network functions are ratios of polynomials in s with real coefficients.*** A consequence of this property is that ***complex poles (and zeros) must occur in conjugate pairs.*** To demonstrate this fact consider a complex root at (s = -a – jb) which leads to the factor (s + a + jb) in the network function. The jb term will make some of the coefficients complex in the polynomial, unless the conjugate of the complex root at (s = -a + jb) is also present in the polynomial. The product of a complex factor and its conjugate is

$$(s + a + jb)(s + a - jb) = s^2 + 2as + a^2 + b^2$$

Further important properties of network functions are obtained by restricting ***the networks to be stable***, by which we mean that a bounded input excitation to the network must yield a bounded response. Put differently, the output of a stable network cannot be made to increase indefinitely by the application of a bounded input excitation. Passive networks are stable by their very nature, since they do not contain energy sources that might inject additional energy into the network. Active networks, however, do contain energy sources that could join forces with the input excitation to make the output increase indefinitely. Such unstable networks, however, have no use in the world of practical filters and are therefore precluded from all our future discussions.
A convenient way of determining the stability of the general network function H(s)

is by considering its response to an impulse function, which is obtained by taking the inverse Laplace transform of the partial fraction expansion of the function.

- If the network function has a simple pole on the real axis, the impulse response due to it (for t >= 0) will have the form:

$$h(t) = \mathscr{L}^{-1} \frac{K_1}{s - p_1} = K_1 e^{p_1 t}$$

For p_1 positive, the impulse response is seen to increase exponentially with time, corresponding to an unstable circuit. Thus, ***H(s) cannot have poles on the positive real axis.***

- Suppose H(s) has a pair of complex conjugate poles at s = a +/- jb. The contribution to the impulse response due to this pair of poles is

$$h(t) = \mathscr{L}^{-1}\left(\frac{K_1}{s-a-jb} + \frac{K_1}{s-a+jb}\right) = \mathscr{L}^{-1}\,\frac{2K_1(s-a)}{(s-a)^2+b^2}$$

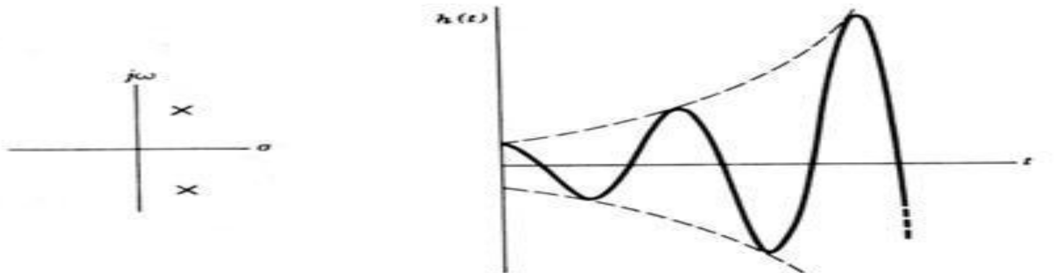
$$= 2K_1 e^{at}\cos bt$$

Now if a is positive, corresponding to poles in the right half s plane, the response is seen to be an exponentially increasing sinusoid (Figure 2.4b). Therefore, ***H(s) cannot have poles in the right half s plane.*** An additional restriction on the poles of H(s) is that ***any poles on the imaginary axis must be simple.***

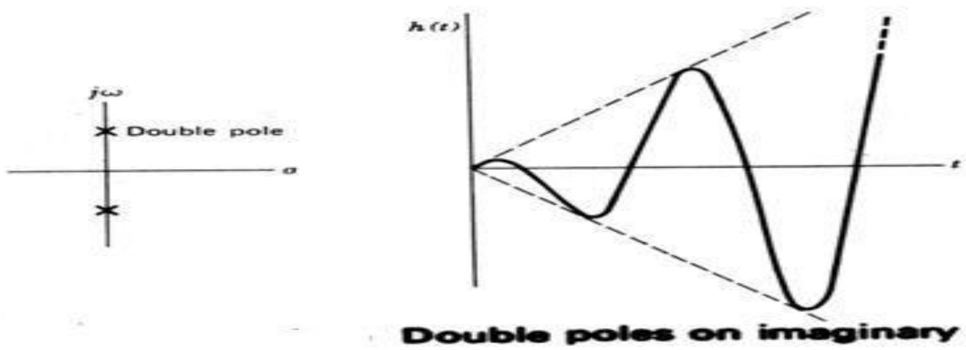
Similarly, it can be shown that higher order poles on the jω axis will also cause the network to be unstable. From the above discussion we see that H(s) has the following factored form:

$$H(s) = \frac{N(s)}{\prod_i (s+a_i)\prod_k (s^2+c_k s+d_k)}$$

Where N(s) is the numerator polynomial and the constants associated with the denominator a_i, c_k, and d_k are real and nonnegative. The $(s + a_i)$ terms represent poles on the negative real axis and the second order terms represent complex conjugate poles in the left half s plane. It is easy to see that the product of these factors can only lead to a polynomial, all of whose coefficients are real and positive; moreover, none of the coefficients may be zero unless all the even or all the odd terms are missing.



Complex poles in the right half s plane.



Double poles on imaginary axis.

In summary, the network functions of all passive networks and all stable active Must be rational functions in s with real coefficients.

- May not have poles in the right half s plane.
- May not have multiple poles on the jω axis.

Example: Check to see whether the following are stable network functions:

$$\text{(a)}\ \frac{s}{s^2 - 3s + 4} \qquad \text{(b)}\ \frac{s - 1}{s^2 + 4}$$

The first function cannot be realized by a stable network because one of the coefficients in the denominator polynomial is negative. It can easily be verified that the poles are in the right half s plane.

The second function is stable. The poles are on the jω axis (at s = +/- 2j) and are simple. Note that the function has a zero in the right half s plane; however, this does not violate any of the requirements on network functions.

3.8.Properties of Driving Point (Positive Real) Functions:

These conditions are required to satisfy to be positive realness

- Y(s) must be a rational function in s with real coefficients, i.e., the coefficients of the numerator and denominator polynomials is real and positive.
- The poles and zeros of Y(s) have either negative or zero real parts, i.e., Y(s) not have poles or zeros in the right half s plane.
- Poles of Y(s) on the imaginary axis must be simple and their residues must be real and positive, i.e., Y(s) not has multiple poles or zeros on the jω axis. The same statement applies to the poles of l/Y(s).
- The degrees of the numerator and denominator polynomials in Y(s) differ at most by 1. Thus the number of finite poles and finite zeros of Y(s) differ at most by 1.
- The terms of lowest degree in the numerator and denominator polynomials of Y(s) differ in degree at most by 1. So Y(s) has neither multiple poles nor zeros at the origin.
- There be no missing terms in numerator and denominator polynomials unless all even or all odd terms are missing.

Test for necessary and sufficient conditions:

- ➢ Y(s) must be real when s is real.
- ➢ If Y(s) = p(s)/q(s), then p(s) + q(s) must be Hurwitz. This requires that:

i. the continued fraction expansion of the Hurwitz test give only real and positive coefficients, and

ii. the continued fraction expansion not end prematurely.

- ➢ In order that Re [Y(jω)] >= 0 for all ω, it is necessary and sufficient that

$$A(\omega^2) = m_1m_2 - n_1n_2 \Big|_{s=j\omega}$$

have no real positive roots of odd multiplicity. This may be determined by factoring $A(\omega^2)$ or by the use of Sturm's theorem.

Example The function

$$Y_1(s) = 5\frac{s^2+2s+1}{s^3+2s^2+2s+40} = 5\frac{(s+1)(s+1)}{(s+4)(s^2-2s+10)}$$

is not positive real because it has two poles in the right half plane.

$$Y_2(s) = \frac{s^3+5s}{s^4+2s^2+1} = \frac{s(s^2+5)}{(s^2+1)^2}$$

is not positive real because of the multiple poles on the imaginary axis.

3.9. System Poles and Zeros:

The transfer function provides a basis for determining important system response characteristics without solving the complete differential equation. As defined, the transfer function is a rational function in the complex variable $s = \sigma + j\omega$, that is

$$H(s) = \frac{a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_0}{b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \cdots + b_0}$$

$$\text{where} \quad a_n \neq 0 \quad b_m \neq 0$$

It is often convenient to factor the polynomials in the numerator and denominator, and to write the transfer function in terms of those factors

$$H(s) = \frac{a_n(s-z_1)(s-z_2)\cdots(s-z_n)}{b_m(s-p_1)(s-p_2)\cdots(s-p_m)}$$

Where the numerator and denominator polynomials, $N(s)$ and $D(s)$, have real coefficients defined by the system's differential equation and $K = b_m/a_n$. As written in Eq. (2) the z_i's are the roots of the equation

$$N(s) = 0$$

and are defined to be the system zeros, and the p_i's are the roots of the equation

$$D(s) = 0,$$

and are defined to be the system *poles*. In Eq. (2) the factors in the numerator and denominator are written so that when $s = z_i$ the numerator $N(s) = 0$ and the transfer function vanishes, that is

$$\lim_{s \to z_i} H(s) = 0$$

and similarly when $s = p_i$ the denominator polynomial $D(s) = 0$ and the value of the transfer function becomes unbounded,

$$\lim_{s \to p_i} H(s) = \infty$$

All of the coefficients of polynomials $N(s)$ and $D(s)$ are real, therefore the poles and zeros must be either purely real, or appear in complex conjugate pairs. In general for the poles, either $p_i = \sigma_i$, or else p_i, p_i+1 = $\sigma_i + j\omega_i$. The existence of a single complex pole without a corresponding conjugate pole would generate complex coefficients in the polynomial $D(s)$. Similarly, the system zeros are either real or appear in complex conjugate pairs.

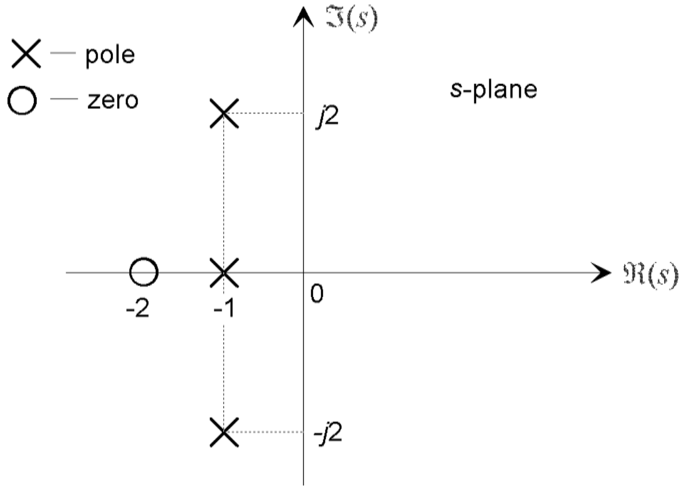


Figure 1: The pole-zero plot for a typical third-order system with one real pole and a complex conjugate pole pair, and a single real zero.

3.10.Pole-Zero Plot:

A system is characterized by its poles and zeros in the sense that they allow reconstruction of the input/output differential equation. In general, the poles and zeros of a transfer function may be complex, and the system dynamics may be represented graphically by plotting their locations on the complex *s*-plane, whose axes represent the real and imaginary parts of the complex variable *s*. Such plots are known as *pole-zero plots*. It is usual to mark a zero location by a circle (◦) and a pole location a cross (×). The location of the poles and zeros provide qualitative insights into the response characteristics of a system.

System stability:

The stability of a linear system may be determined directly from its transfer function. An *n*th order linear system is asymptotically stable only if all of the components in the homogeneous response from a finite set of initial conditions decay to zero as time increases, or

$$\lim^{n} ce^{p_i}$$

UNIT-IV

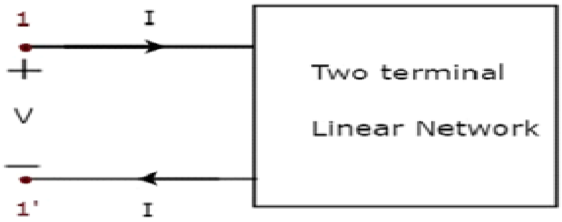
TWO PORT NETWORK PARAMETERS

TWO PORT NETWORK PARAMETERS

4.1. Introduction:

In general, it is easy to analyze any electrical network, if it is represented with an equivalent model, which gives the relation between input and output variables. For this, we can use **two port network** representations. As the name suggests, two port networks contain two ports. Among which, one port is used as an input port and the other port is used as an output port. The first and second ports are called as port1 and port2 respectively.

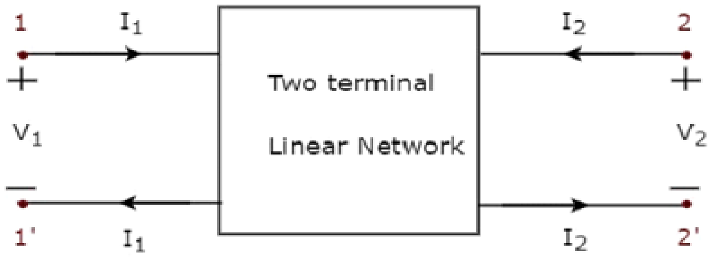
One port network: it is a two terminal electrical network in which, current enters through one terminal and leaves through another terminal. Resistors, inductors and capacitors are the examples of one port network because each one has two terminals. One port network representation is shown in the following figure.



Here, the pair of terminals, 1 & 1' represents a port. In this case, we are having only one port since it is a one port network.

Similarly,

Two port network: it is a pair of two terminal electrical network in which, current enters through one terminal and leaves through another terminal of each port. Two port network representation is shown in the following figure.



Here, one pair of terminals, 1 & 1' represents one port, which is called as **port1** and the other pair of terminals, 2 & 2' represents another port, which is called as **port2**.

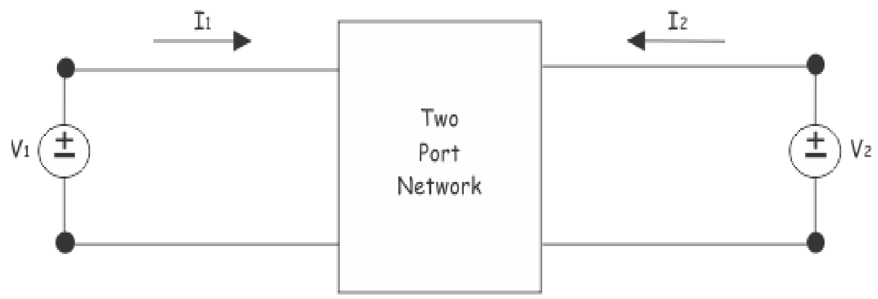
There are **four variables** V_1, V_2, I_1 and I_2 in a two port network as shown in the figure. Out of which, we can choose two variables as independent and another two variables as dependent. So, we will get six possible pairs of equations. These equations represent the dependent variables in terms of independent variables. The coefficients of independent variables are called as **parameters**. So, each pair of equations will give a set of four parameters.

4.2. IMPEDANCE PARAMETERS (OR) Z PARAMETERS:

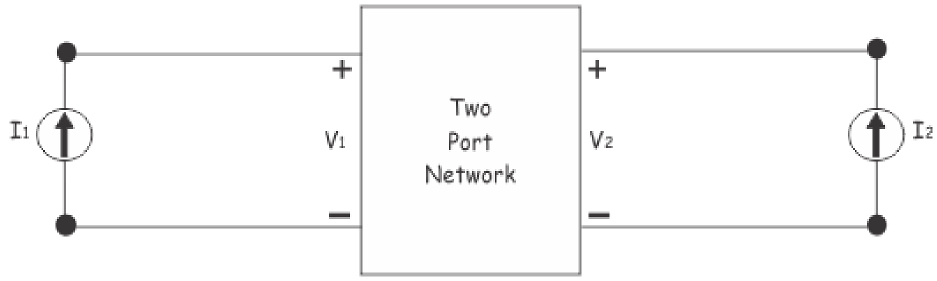
We will get the following set of two equations by considering the variables V_1 & V_2 as dependent and I_1 & I_2 as independent. The coefficients of independent variables, I_1 and I_2 are called as Z parameters.
Z parameters are also known as impedance parameters. When we use Z parameter for analyzing two part network, the voltages are represented as the function of currents. So

$$V_1 = f_1(I_1, I_2)\ and\ V_2 = f_2(I_1, I_2)$$

the input and output of a two port network can either be voltage or current. If the network is voltage driven, that can be represented as shown below.



If the network is driven by current, that can be represented as shown below.



From, both of the figures above, it is clear that, there are only four variables. One pair of voltage variables V_1 and V_2 and one pair of current variables I_1 and I_2. Thus, there are only four ratio of voltage to current, and those are,

$$\frac{V_1}{I_1}, \frac{V_1}{I_2}, \frac{V_2}{I_1}\ and\ \frac{V_2}{I_2}$$

These four rations are considered as parameters of the network. We all know, This is why these parameters are called either **impedance parameter** or **Z parameter**. The values of **Z parameter**s of a two port network, can be evaluated by making once

$$Impedance(Z) = \frac{Voltage(V)}{Current(I)}$$

This is why these parameters are called either **impedance parameter** or **Z parameter**. The values of these **Z parameter**s of a two port network, can be evaluated by making once and another once

$$I_1 = 0$$

$$I_2 = 0$$

The Z parameters are,

$$Z_{22} = Output\ impedance\ keeping\ input\ open = \frac{V_2}{I_2}\Big|_{I_1=0}$$

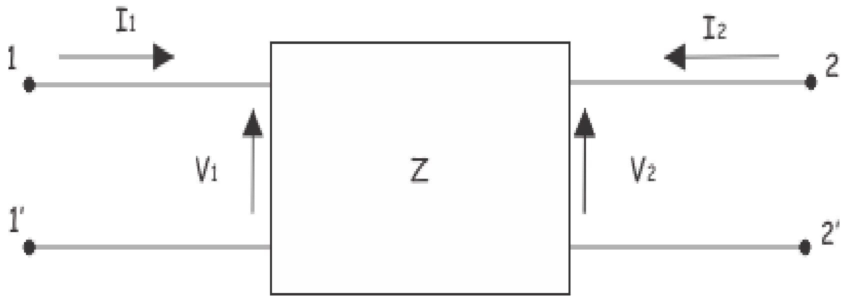
$$Z_{12} = Reverse\ transfer\ impedance\ keeping\ input\ open = \frac{V_1}{I_2}\Big|_{I_1=0}$$

$$Z_{21} = Forward\ transfer\ impedance\ keeping\ output\ open = \frac{V_2}{I_1}\Big|_{I_2=0}$$

$$Z_{11} = Input\ impedance\ keeping\ output\ open = \frac{V_1}{I_1}\Big|_{I_2=0}$$

The voltages are represented as

$$V_1 = Z_{11}I_1 + Z_{12}I_2\ and\ V_2 = Z_{21}I_1 + Z_{22}I_2$$



$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

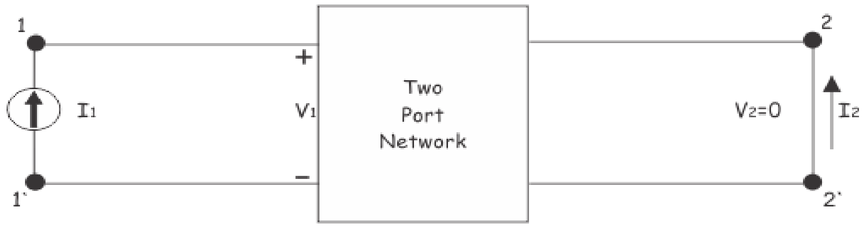
Admittance parameters or short circuit parameters(Y):We can represent current in terms of voltage bt **admittance parameters** of a two port network. Then we will represent the current voltage relations as,

$$I_1 = Y_{11}V_1 + Y_{12}V_2 \cdots\cdots\cdots\cdot(iii)$$
$$I_2 = Y_{21}V_1 + Y_{22}V_2 \cdots\cdots\cdots\cdot(iv)$$

This can also be represented in matrix form as,

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Here, Y_{11}, Y_{12}, Y_{21} and Y_{22} are **admittance parameter**. Sometimes these are called as **Y parameters**. We can determine the values of the parameters of a particular two port network by making short-circuited output port and input port alternatively as follows. First let us apply current source of I_1 at input port keeping the output port short circuited as shown below.



Now, the ratio of input current I_1 to input voltage V_1 while output voltage $V_2 = 0$, is

$$\left.\frac{I_1}{V_1}\right|_{V_2 = 0} = Y_{11}$$

This is called short circuit input **admittance**. The ratio of output current I_2 to input voltage V_1 while output voltage $V_2 = 0$, is

$$\left.\frac{I_2}{V_1}\right|_{V_2 = 0} = Y_{21}$$

This is referred as short circuit transfer **admittance** from input port to output port. Now, let us short circuit the input port of the network and apply current I_2 at output port, as shown below.

In this case,

$$\left.\frac{I_2}{V_2}\right|_{V_1=0} = Y_{22}$$

This is called short circuit output **admittance**.

$$\left.\frac{I_1}{V_2}\right|_{V_1=0} = Y_{12}$$

This is called short circuit transfer admittance from input port to output port. So finally,

$$\left.\frac{I_1}{V_1}\right|_{V_2=0} = Y_{11} = short\ circuit\ input\ admittance$$

$$\left.\frac{I_1}{V_2}\right|_{V_1=0} = Y_{12} = short\ circuit\ transfer\ admittance\ from\ output\ port\ to\ input\ port$$

$$\left.\frac{I_2}{V_1}\right|_{V_2=0} = Y_{21} = short\ circuit\ transfer\ admittance\ from\ intput\ port\ to\ output\ port$$

$$\left.\frac{I_2}{V_2}\right|_{V_1=0} = Y_{22} = short\ circuit\ output\ admittance$$

Hybrid Parameters or h Parameters:

Hybrid parameters are also referred as **h parameters**. These are referred as hybrid because, here Z parameters, Y parameters, voltage ratio, current ratio, all are used to represent the relation between voltage and current in a two port network. The relations of voltages and current in **hybrid parameters** are represented as,

$$V_1 = h_{11}I_1 + h_{12}V_2$$
$$I_2 = h_{21}I_1 + h_{22}V_2$$

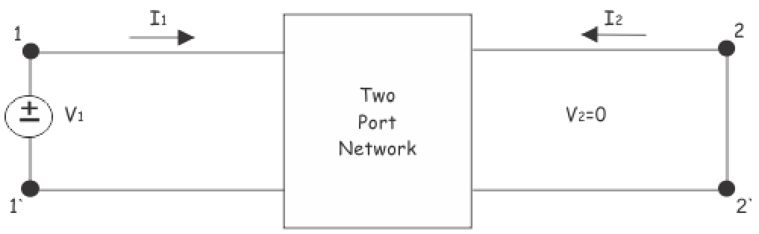
This can be represented in matrix form as,

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

Hybrid parameters or **h parameters** are very much useful in analyzing electronics circuit where, transistors like elements are connected. In those circuits, sometimes it is difficult to measure Z parameters and Y parameters but h parameters can be measured in much easier way.

Determining h Parameters

Let us short circuit the output port of a two port network as shown below,



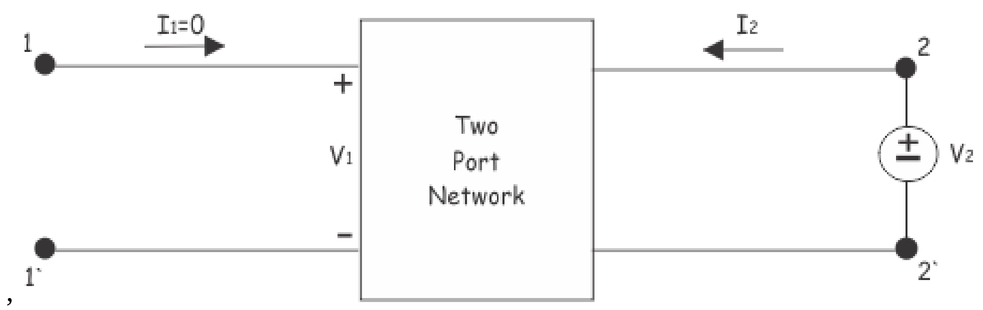
Now, ratio of input voltage to input current, at short circuited output port, is

$$\left.\frac{V_1}{I_1}\right|_{V_2=0} = h_{11}$$

this is referred as short circuit input impedance. Now, the ratio of the output current to input current at short circuited output port, is

$$\left.\frac{I_2}{I_1}\right|_{V_2=0} = h_{21}$$

This is called short circuit current gain of the network. Now, let us open circuit the port 1. At that condition, there will be no input current (I_1=0) but open circuit voltage V_1 appears across the port 1, as shown below



$$\left.\frac{V_1}{V_2}\right|_{I_1=0} = h_{12} = open\ circuit\ reverse\ voltage\ gain$$

This is referred as reverse voltage gain because, this is the ratio of input voltage to output voltage of the network, but voltage gain is defined as ratio of output voltage to input voltage of a network. Now,

$$\left.\frac{I_2}{V_2}\right|_{I_1 = 0} = h_{21}$$

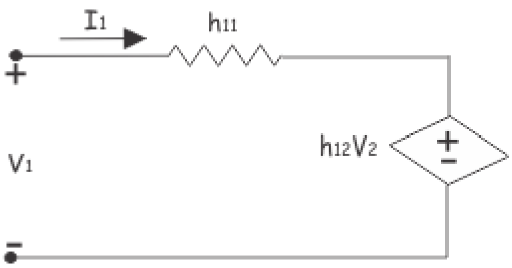
It is referred as open circuit output admittance.

h Parameter Equivalent Network of Two Port Network

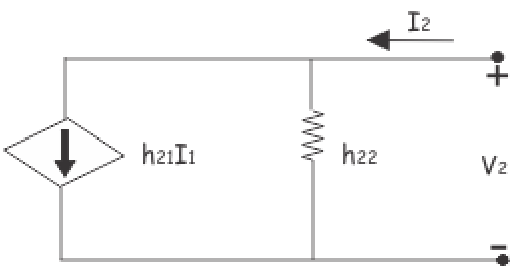
To draw **h parameter** equivalent network of a two port network, first we have to write the equation of voltages and currents using h parameters. These are,

$$V_1 = h_{11}I_1 + h_{12}V_2 \cdots\cdots\cdots\cdots\cdots (i)$$
$$I_2 = h_{21}I_1 + h_{22}V_2 \cdots\cdots\cdots\cdots\cdots (ii)$$

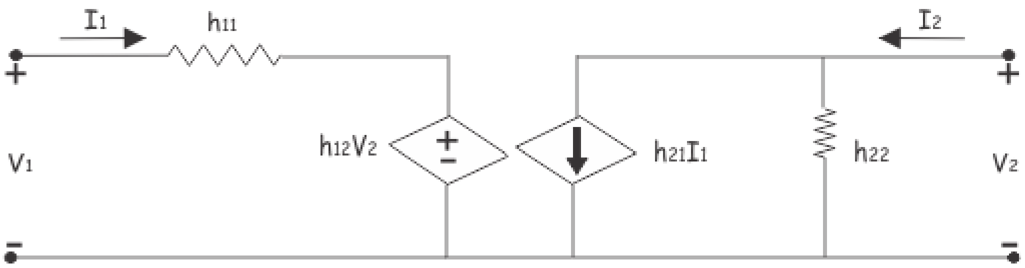
clearly, the equation (i) can be represented as circuit based on Kirchhoff Voltage Law.



Clearly, the equation (ii) can be represented as circuit based on Kirchhoff Current Law.



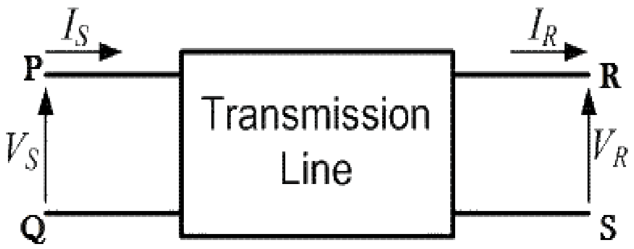
Combining these two parts of the network, we get,



The h parameters equivalent network of a two port network

ABCD Parameters of Transmission Line parameters:

A major section of power system engineering deals in the transmission of electrical power from one particular place (eg. generating station) to another like substations or distribution units with maximum efficiency. So it's of substantial importance for power system engineers to be thorough with its mathematical modeling. Thus the entire transmission system can be simplified to a **two port network** for the sake of easier calculations.The circuit of a 2 port network is shown in the diagram below. As the name suggests, a 2 port network consists of an input port PQ and an output port RS. In any 4 terminal network, (i.e. linear, passive, bilateral network) the input voltage and input current can be expressed in terms of output voltage and output current. Each port has 2 terminals to connect itself to the external circuit. Thus it is essentially a 2 port or a 4 terminal circuit, having



$$\begin{aligned} \textit{Supply end voltage} &= V_S \\ \textit{and Supply end current} &= I_S \end{aligned}$$

Given to the input port PQ.

$$\begin{aligned} \textit{And there is the Receiving end voltage} &= V_R \\ \textit{and Receiving end current} &= I_R \end{aligned}$$

Now the **ABCD parameters** or the transmission line parameters provide the link between the supply and receiving end voltages and currents, considering the circuit elements to be linear in nature.

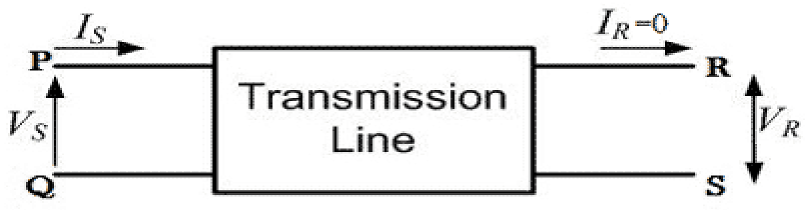
Thus the relation between the sending and receiving end specifications are given using **ABCD parameters** by the equations below.

$$V_S = AV_R + BI_R \quad \cdots\cdots\cdots\cdots\cdots\cdots (1)$$

$$I_S = CV_R + DI_R \quad \cdots\cdots\cdots\cdots\cdots\cdots (2)$$

Now in order to determine the ABCD parameters of transmission line let us impose the required circuit conditions in different cases.

ABCD Parameters, When Receiving End is Open Circuited



The receiving end is open circuited meaning receiving end current $I_R = 0$. Applying this condition to equation (1) we get,

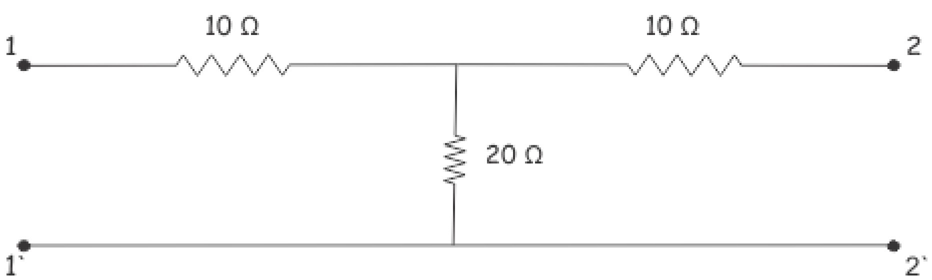
$$V_S = A\,V_R + B\,0 \Rightarrow V_S = A\,V_R + 0$$
$$A = \left.\frac{V_S}{V_R}\right|_{I_R = 0}$$

Thus its implies that on applying short circuit condition to ABCD parameters, we get parameter B as the ratio of sending end voltage to the short circuit receiving end current. Since dimension wise B is a ratio of voltage to current, its unit is Ω. Thus B is the short circuit resistance and is given by $B = V_S/I_R$ Ω. Applying the same short circuit condition i.e $V_R = 0$ to equation (2) we get

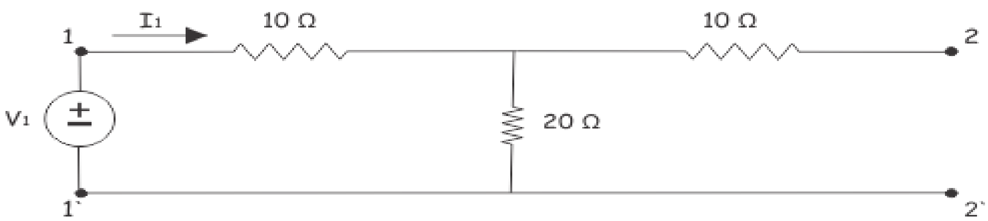
$$I_S = C\,0 + D\,I_R \Rightarrow I_S = 0 + D\,I_R$$
$$D = \left.\frac{I_S}{I_R}\right|_{V_R = 0}$$

Thus its implies that on applying short circuit condition to ABCD parameters, we get parameter D as the ratio of sending end current to the short circuit receiving end current. Since dimension wise D is a ratio of current to current, it's a dimension less parameter.

Find the z parameters for network shown in figure



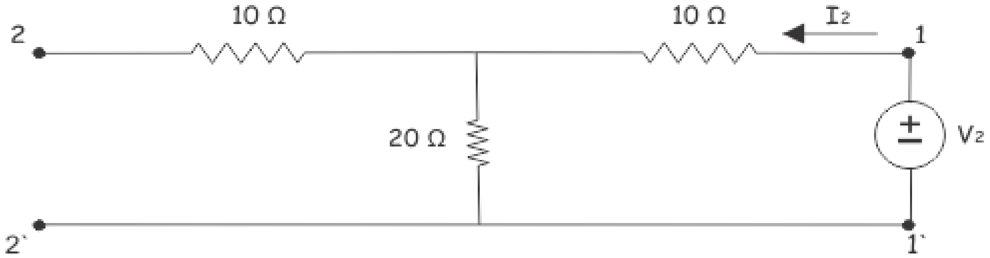
Let us put a voltage source V_1 at input,



$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2 = 0} = \frac{(10 + 20)I_1}{I_1} = 30\ \Omega$$

$$Z_{21} = \frac{V_2}{I_1}\Big|_{I_2 = 0} = \frac{20I_1}{I_1} = 20\ \Omega$$

Now, let us connect one voltage source V_2 at output port and leave the input port as open as shown, below



$$Z_{22} = \frac{V_2}{I_2}\Big|_{I_1 = 0} = \frac{(10+20)I_2}{I_2} = 30\ \Omega$$

$$Z_{12} = \frac{V_1}{I_2}\Big|_{I_1 = 0} = \frac{20I_2}{I_2} = 20\ \Omega$$

Now, $Z_{11} = Z_{22}\ and\ Z_{12} = Z_{21}$

Therefore the above network is symmetrical, reciprocal network

Parameter	Symmetry	Reciprocity
Z	$Z_{11} = Z_{22}$	$Z_{12} = Z_{21}$
Y	$Y_{11} = Y_{22}$	$Y_{12} = Y_{21}$
h	$\Delta_h = 1$	$h_{12} = -h_{21}$
G	$\Delta_G = 1$	$G_{12} = -G_{21}$
T	$A = D$	$\Delta_T = 1$

S. No.	Name	Function: Express	Function: In terms of	Matrix form
1.	Open circuit Impedance or [Z] Parameter	V_1, V_2	I_1, I_2	$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$
2.	Short circuit admittance or [Y] Parameter	I_1, I_2	V_1, V_2	$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$
3.	ABCD or Transmission Parameter	V_1, I_1	V_2, I_2	$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$
4.	Inverse Transmission Parameter	V_2, I_2	V_1, I_1	$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} V_1 \\ -I_1 \end{bmatrix}$
5.	Hybrid or [h] Parameter	V_1, I_2	I_1, V_2	$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$
6.	Inverse hybrid or [g] Parameter	I_1, V_2	V_1, I_2	$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$

Procedure of two port parameter conversions

In the previous chapter, we discussed about six types of two-port network parameters. Now, let us convert one set of two-port network parameters into other set of two port network parameters. This conversion is known as two port network parameters conversion or simply, **two-port parameters conversion**.

Sometimes, it is easy to find one set of parameters of a given electrical network easily. In those situations, we can convert these parameters into the required set of parameters instead of calculating these parameters directly with more difficulty.

Now, let us discuss about some of the two port parameter conversions

- **Step 1** − Write the equations of a two port network in terms of desired parameters.
- **Step 2** − Write the equations of a two port network in terms of given parameters.
- **Step 3** − Re-arrange the equations of Step2 in such a way that they should be similar to the equations of Step1.
- **Step 4** − By equating the similar equations of Step1 and Step3, we will get the desired parameters in terms of given parameters. We can represent these parameters in matrix form.

Z parameters to T parameters

Here, we have to represent T parameters in terms of Z parameters. So, in this case T parameters are the desired parameters and Z parameters are the given parameters.

Step 1 − We know that, the following set of two equations, which represents a two port network in terms of **T parameters**.

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

Step 2 − We know that the following set of two equations, which represents a two port network in terms of **Z parameters**.

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$

$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

Step 3 − We can modify the above equation as

$$\Rightarrow V_2 - Z_{22}I_2 = Z_{21}I_1$$

$$\Rightarrow I_1 = \left(\frac{1}{Z_{21}}\right)V_2 - \left(\frac{Z_{22}}{Z_{21}}\right)I_2$$

Step 4 − The above equation is in the form of $I_1 = CV_2 - DI_2$. Here,

$$C = \frac{1}{Z_{21}}$$

$$D = \frac{Z_{22}}{Z_{21}}$$

Step 5 − Substitute I_1 value of Step 3 in V_1 equation of Step 2.

$$V_1 = Z_{11}\{(\frac{1}{Z_{12}})V_2 - (\frac{Z_{22}}{Z_{21}})I_2\} + Z_{12}I_2$$

$$\Rightarrow V_1 = (\frac{Z_{11}}{Z_{21}})V_2 - (\frac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_{21}})I_2$$

Step 6 − The above equation is in the form of $V_1 = AV_2 - BI_2$. Here,

$$A = \frac{Z_{11}}{Z_{21}}$$

$$B = \frac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_{21}}$$

Step 3 − We can modify it as

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}^{-1} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$ **Equation 4**

Step 4 − By equating Equation 3 and Equation 4, we will get

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}^{-1}$$

$$\Rightarrow \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \frac{\begin{bmatrix} Y_{22} & -Y_{12} \\ -Y_{21} & Y_{11} \end{bmatrix}}{\Delta Y}$$

Where,

$$\Delta Y = Y_{11}Y_{22} - Y_{12}Y_{21}$$

So, just by doing the **inverse of Y parameters matrix**, we will get the Z parameters matrix.

Y parameters to T parameters

Here, we have to represent T parameters in terms of Y parameters. So, in this case, T parameters are the desired parameters and Y parameters are the given parameters.

Step 1 − We know that, the following set of two equations, which represents a two port network in terms of **T parameters**.

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

Step 2 − We know that the following set of two equations of two port network regarding Y parameters.

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$

$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

Step 3 − We can modify the above equation as

$$\Rightarrow I_2 - Y_{22}V_2 = Y_{21}V_1$$

$$\Rightarrow V_1 = (\frac{-Y_{22}}{Y_{21}})V_2 - (\frac{-1}{Y_{21}})I_2$$

Step 4 − The above equation is in the form of $V_1 = AV_2 - BI_2$. Here,

$$A = \frac{-Y_{22}}{Y_{21}}$$

$$B = \frac{-1}{Y_{21}}$$

Step 5 − Substitute V_1 value of Step 3 in I_1 equation of Step 2.

$$I_1 = Y_{11}\{(\frac{-Y_{22}}{Y_{21}})V_2 - (\frac{-1}{Y_{21}})I_2\} + Y_{12}V_2$$

$$\Rightarrow I_1 = (\frac{Y_{12}Y_{21} - Y_{11}Y_{22}}{Y_{21}})V_2 - (\frac{-Y_{11}}{Y_{21}})I_2$$

Step 6 − The above equation is in the form of $I_1 = CV_2 - DI_2$. Here,

$$C = \frac{Y_{12}Y_{21} - Y_{11}Y_{22}}{Y_{21}}$$

$$D = \frac{-Y_{11}}{Y_{21}}$$

Step 7 − Therefore, the **T parameters matrix** is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{-Y_{22}}{Y_{21}} & \frac{-1}{Y_{21}} \\ \frac{Y_{12}Y_{21} - Y_{11}Y_{22}}{Y_{21}} & \frac{-Y_{11}}{Y_{21}} \end{bmatrix}$$

T parameters to h-parameters

Here, we have to represent h-parameters in terms of T parameters. So, in this case hparameters are the desired parameters and T parameters are the given parameters.

Step 1 − We know that, the following **h-parameters** of a two port network.

$$h_{11} = \frac{V_1}{I_1}, \; when \; V_2 = 0$$

$$h_{12} = \frac{V_1}{V_2}, \; when \; I_1 = 0$$

$$h_{21} = \frac{I_2}{I_1}, \; when \; V_2 = 0$$

$$h_{22} = \frac{I_2}{V_2}, \; when \; I_1 = 0$$

Step 2 − We know that the following set of two equations of two port network regarding **T parameters**.

$$V_1 = AV_2 - BI_2 \qquad \textbf{Equation 5}$$

$$I_1 = CV_2 - DI_2 \qquad \textbf{Equation 6}$$

Step 3 − Substitute $V_2 = 0$ in the above equations in order to find the two h-parameters, h_{11} and h_{21}.

$$\Rightarrow V_1 = -BI_2$$

$$\Rightarrow I_1 = -DI_2$$

Substitute, V_1 and I_1 values in h-parameter, h_{11}.

$$h_{11} = \frac{-BI_2}{-DI_2}$$

Substitute I_1 value in h-parameter h_{21}.

$$h_{21} = \frac{I_2}{-DI_2}$$

$$\Rightarrow h_{21} = -\frac{1}{D}$$

Step 4 − Substitute $I_1 = 0$ in the second equation of step 2 in order to find the h-parameter h_{22}.

$$0 = CV_2 - DI_2$$

$$\Rightarrow CV_2 = DI_2$$

$$\Rightarrow \frac{I_2}{V_2} = \frac{C}{D}$$

$$\Rightarrow h_{22} = \frac{C}{D}$$

Step 5 − Substitute $I_2 = (\frac{C}{D})V_2$ in the first equation of step 2 in order to find the h-parameter, h_{12}.

$$V_1 = AV_2 - B(\frac{C}{D})V_2$$

$$\Rightarrow V_1 = (\frac{AD - BC}{D})V_2$$

$$\Rightarrow \frac{V_1}{V_2} = \frac{AD - BC}{D}$$

$$\Rightarrow h_{12} = \frac{AD - BC}{D}$$

Step 6 − Therefore, the h-parameters matrix is

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} \frac{B}{D} & \frac{AD-BC}{D} \\ -\frac{1}{D} & \frac{C}{D} \end{bmatrix}$$

h-parameters to Z parameters

Here, we have to represent Z parameters in terms of h-parameters. So, in this case Z parameters are the desired parameters and h-parameters are the given parameters.

Step 1 − We know that, the following set of two equations of two port network regarding **Z parameters**.

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$

$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

Step 2 − We know that, the following set of two equations of two-port network regarding **h-parameters**.

$$V_1 = h_{11}I_1 + h_{12}V_2$$

$$I_2 = h_{21}I_1 + h_{22}V_2$$

Step 3 − We can modify the above equation as

$$\Rightarrow I_2 - h_{21}I_1 = h_{22}V_2$$

$$\Rightarrow V_2 = \frac{I_2 - h_{21}I_1}{h_{22}}$$

$$\Rightarrow V_2 = (\frac{-h_{21}}{h_{22}})I_1 + (\frac{1}{h_{22}})I_2$$

The above equation is in the form of $V_2 = Z_{21}I_1 + Z_{22}I_2. Here,$

$$Z_{21} = \frac{-h_{21}}{h_{22}}$$

$$Z_{22} = \frac{1}{h_{22}}$$

Step 4 − Substitute V_2 value in first equation of step 2.

$$V_1 = h_{11}I_1 + h_{21}\{(\frac{-h_{21}}{h_{22}})I_1 + (\frac{1}{h_{22}})I_2\}$$

$$\Rightarrow V_1 = (\frac{h_{11}h_{22} - h_{12}h_{21}}{h_{22}})I_1 + (\frac{h_{12}}{h_{22}})I_2$$

The above equation is in the form of $V_1 = Z_{11}I_1 + Z_{12}I_2$. Here,

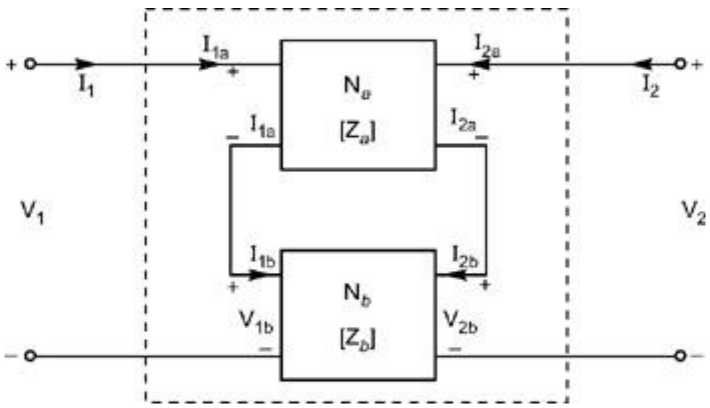
$$Z_{11} = \frac{h_{11}h_{22} - h_{12}h_{21}}{h_{22}}$$

$$Z_{12} = \frac{h_{12}}{h_{22}}$$

Interconnections of two-port networks

Two-port networks may be interconnected in various configurations, such as series, parallel, cascade, series-parallel, and parallel-series connections. For each configuration a certain set of parameters may be more useful than others to describe the network.

series connection



Series connection of two two-port networks For network N $_a$,

$$\begin{bmatrix} V_{1a} \\ V_{2a} \end{bmatrix} = \begin{bmatrix} Z_{11a} & Z_{12a} \\ Z_{21a} & Z_{22a} \end{bmatrix} \begin{bmatrix} I_{1a} \\ I_{2a} \end{bmatrix}$$

$$(10.63) V_{2a} = Z_{21a} I_{1a} + Z_{22a} I_{2a}.$$

$$(10.63) V_{2a} = Z_{21a} I_{1a} + Z_{22a} I_{2a}.$$

$$\begin{bmatrix} V_{1b} \\ V_{2b} \end{bmatrix} = \begin{bmatrix} Z_{11b} & Z_{12b} \\ Z_{21b} & Z_{22}b \end{bmatrix} \begin{bmatrix} I_{1b} \\ I_{2b} \end{bmatrix}$$

$$(10.65) V_{2b} = Z_{21b} I_{1b} + Z_{22b} I_{2b}$$

$$(10.65) V_{2b} = Z_{21b} I_{1b} + Z_{22b} I_{2b}$$

The condition for series connection is

$$(10.67) V_2 = V_{2a} + V_{2b}$$

Putting the values of V $_{1a}$ and V $_{1b}$ from Equation (10.62) and Equation (10.64), Putting the values of V $_{2a}$ and V $_{2b}$ from Equation (10.63) and Equation (10.65) into Equation (10.67), we get

$$(10.69) V_2 = (Z_{21a} + Z_{21b}) I_1 + (Z_{22a} + Z_{22b}) I_2$$

$$\begin{aligned} V_1 &= Z_{11a} I_{1a} + Z_{12a} I_{2a} + Z_{11b} I_{1b} + Z_{12b} I_{2b} \\ (10.68) \qquad &= Z_{11a} I_1 + Z_{12a} I_2 + Z_{11b} I_1 + Z_{12b} I_2 \ [I_{1a} = I_{1b} = I_1, \ I_{2a} = I_{2b} = I_2] \\ V_1 &= (Z_{11a} + Z_{11b}) I_1 + (Z_{12a} + Z_{12b}) I_2. \end{aligned}$$

$$\begin{aligned} V_1 &= Z_{11} I_1 + Z_{12} I_2 \\ V_2 &= Z_{21} I_1 + Z_{22} I_2, \end{aligned}$$

The Z-parameters of the series-connected combined network can be written as

$$
\begin{aligned}
V_1 &= Z_{11}I_1 + Z_{12}I_2 \\
V_2 &= Z_{21}I_1 + Z_{22}I_2,
\end{aligned}
$$

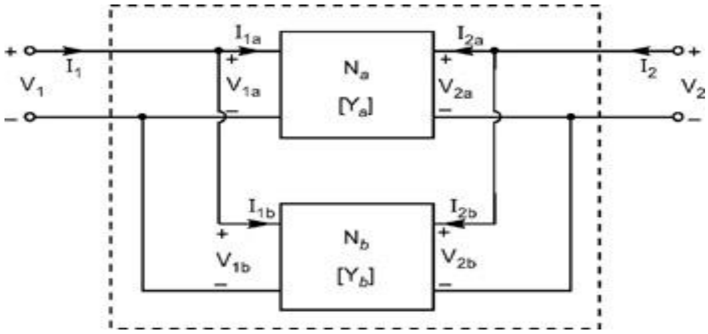
$$
\begin{aligned}
Z_11 &= Z_{11a} + Z_{11b} \\
Z_12 &= Z_{12a} + Z_{12b} \\
Z_21 &= Z_{21a} + Z_{21b} \\
Z_22 &= Z_{22a} + Z_{22b}
\end{aligned}
$$

$$[Z] = [Z_a] + [Z_b].$$

The overall Z-parameter matrix for series connected two-port networks is simply the sum of Z-parameter matrices of each individual two-port network connected in series.

Parallel Connection

Parallel connection of two two-port networks N_a and N_b. The resultant of two admittances connected in parallel is $Y_1 + Y_2$. So in parallel connection, the parameters are Y-parameters.



Parallel connections for two two-port networks For network N_a

for a network,

$$
\begin{bmatrix} I_{1a} \\ I_{2a} \end{bmatrix} = \begin{bmatrix} Y_{11a} & Y_{12a} \\ Y_{21a} & Y_{22a} \end{bmatrix} \begin{bmatrix} V_{1a} \\ V_{2a} \end{bmatrix}
$$

$$(10.71) I_{2a} = Y_{21a}V_{1a} + Y_{22a}V_{2a}.$$

$$(10.71) I_{2a} = Y_{21a}V_{1a} + Y_{22a}V_{2a}.$$

for b network,

$$
\begin{bmatrix} I_{1b} \\ I_{2b} \end{bmatrix} = \begin{bmatrix} Y_{11b} & Y_{12b} \\ Y_{22b} & Y_{22b} \end{bmatrix} \begin{bmatrix} V_{1b} \\ V_{2b} \end{bmatrix}
$$

$$(10.72) I_{1b} = Y_{11b}V_{1b} + Y_{12b}V_{2b}$$

$$(10.73) I_{2b} = Y_{21b}V_{1b} + Y_{22b}V_{2b}$$

$$V_{1a} = V_{1b} = V_1,\ V_{2a} = V_{2b} = V_2\ [\mathit{Same\ voltage}]$$

$$(10.74) I_1 = I_{1a} + I_{1b}$$

$$(10.75) I_2 = I_{2a} + I_{2b}$$

$$\begin{aligned} I_1 &= Y_{11a}V_{1a} + Y_{12a}V_{2a} + Y_{11b}V_{1b} + Y_{12b}V_{2b} \\ &= Y_{11a}V_1 + Y_{12a}V_2 + Y_{11b}V_1 + Y_{12b}V_2 \end{aligned}$$

$$(10.76) I_1 = (Y_{11a} + Y_{11b})V_1 + (Y_{12a} + Y_{12b})V_2$$

$$(10.77) I_2 = (Y_{21a} + Y_{21b})V_1 + (Y_{22a} + Y_{22b})V_2$$

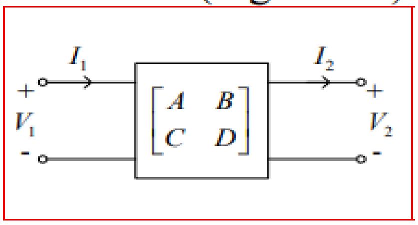
the Y-parameters of the parallel connected combined network can be written as

$$\begin{aligned} I_1 &= Y_{11}V_1 + Y_{12}V_2 \\ I_2 &= Y_{21}V_1 + Y_{22}V_2 \end{aligned}$$

Cascade connection of twoport networks:

There is another set of network parameters particularly suited for cascading two-port networks. This set is called the *ABCD* matrix or, equivalently, the transmission matrix.

Consider this two-port network (Fig. 4.11a):



Unlike in the definition used for Z and Y parameters, notice that I_2 is directed away from the port. This is an important point and we'll discover the reason for it shortly.

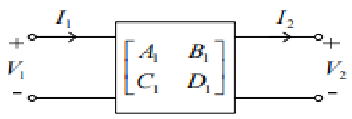
It is easy to show that

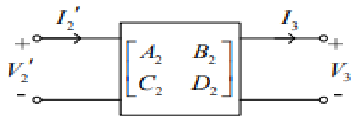
$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0}, \quad B = \left.\frac{V_1}{I_2}\right|_{V_2=0}$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0}, \quad D = \left.\frac{I_1}{I_2}\right|_{V_2=0}$$

Note that not all of these parameters have the same units.

To see this, consider the following two-port networks:





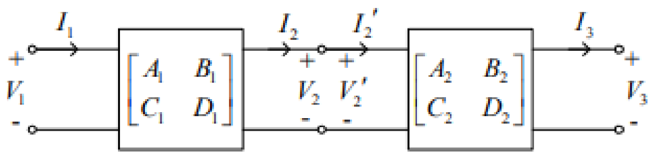
In matrix form

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \cdot \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \qquad (2)$$

and

$$\begin{bmatrix} V_2' \\ I_2' \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \cdot \begin{bmatrix} V_3 \\ I_3 \end{bmatrix} \qquad (3)$$

When these two-ports are cascaded,



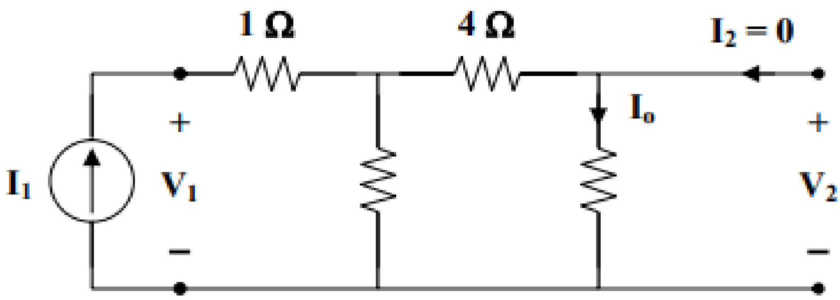
it is apparent that $V_2' = V_2$ and $I_2' = I_2$. (The latter is the reason for assuming I_2 out of the port.) Consequently, substituting (3) into (2) yields

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \cdot \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \cdot \begin{bmatrix} V_3 \\ I_3 \end{bmatrix} \qquad (4)$$

We can consider the matrix-matrix product in this equation as describing the cascade of the two networks.

Solution 1.

To get $\mathbf{z}_{11}$ and $\mathbf{z}_{21}$, consider the circuit in Fig. (a).



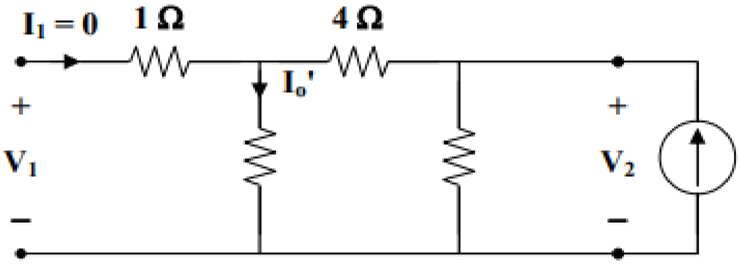
(a)

$$\mathbf{z}_{11} = \frac{\mathbf{V}_1}{\mathbf{I}_1} = 1 + 6 \,||\, (4+2) = 4\,\Omega$$

$$\mathbf{I}_o = \frac{1}{2}\mathbf{I}_1, \qquad \mathbf{V}_2 = 2\mathbf{I}_o = \mathbf{I}_1$$

$$\mathbf{z}_{21} = \frac{\mathbf{V}_2}{\mathbf{I}_1} = 1\,\Omega$$

To get $\mathbf{z}_{22}$ and $\mathbf{z}_{12}$, consider the circuit in Fig. (b).



(b)

$$\mathbf{z}_{22} = \frac{\mathbf{V}_2}{\mathbf{I}_2} = 2 \,||\, (4+6) = 1.667\,\Omega$$

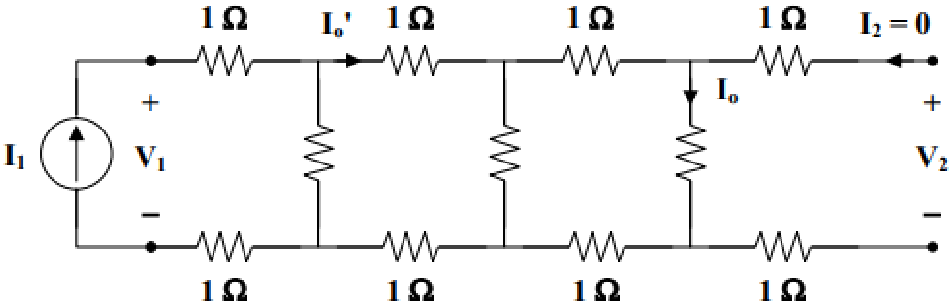
$$\mathbf{I}_o' = \frac{2}{2+10}\mathbf{I}_2 = \frac{1}{6}\mathbf{I}_2, \qquad \mathbf{V}_1 = 6\mathbf{I}_o' = \mathbf{I}_2$$

$$\mathbf{z}_{12} = \frac{\mathbf{V}_1}{\mathbf{I}_2} = 1\,\Omega$$

Hence, $$[\mathbf{z}] = \begin{bmatrix} 4 & 1 \\ 1 & 1.667 \end{bmatrix}\Omega$$

Solution 2.

Consider the circuit in Fig. (a) to get $\mathbf{z}_{11}$ and $\mathbf{z}_{21}$.



(a)

$$\mathbf{z}_{11} = \frac{\mathbf{V}_1}{\mathbf{I}_1} = 2 + 1 \,\|\, [\,2 + 1 \,\|\, (2+1)\,]$$

$$\mathbf{z}_{11} = 2 + 1 \,\|\left(2 + \frac{3}{4}\right) = 2 + \frac{(1)(11/4)}{1 + 11/4} = 2 + \frac{11}{15} = 2.733$$

$$\mathbf{I}_o = \frac{1}{1+3}\mathbf{I}_o' = \frac{1}{4}\mathbf{I}_o'$$

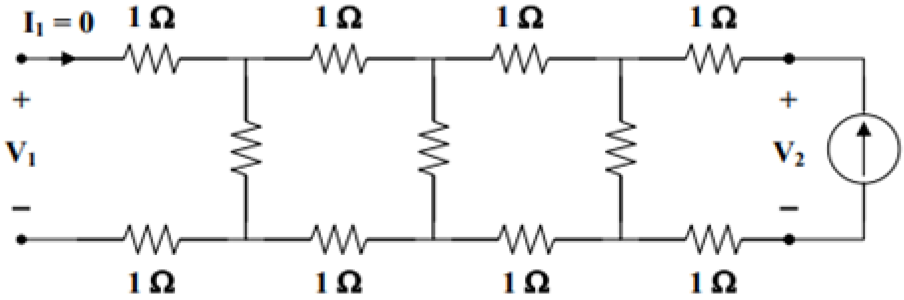
$$\mathbf{I}_o' = \frac{1}{1+11/4}\mathbf{I}_1 = \frac{4}{15}\mathbf{I}_1$$

$$\mathbf{I}_o = \frac{1}{4}\cdot\frac{4}{15}\mathbf{I}_1 = \frac{1}{15}\mathbf{I}_1$$

$$\mathbf{V}_2 = \mathbf{I}_o = \frac{1}{15}\mathbf{I}_1$$

$$\mathbf{z}_{21} = \frac{\mathbf{V}_2}{\mathbf{I}_1} = \frac{1}{15} = \mathbf{z}_{12} = 0.06667$$

To get $\mathbf{z}_{22}$, consider the circuit in Fig. (b).



(b)

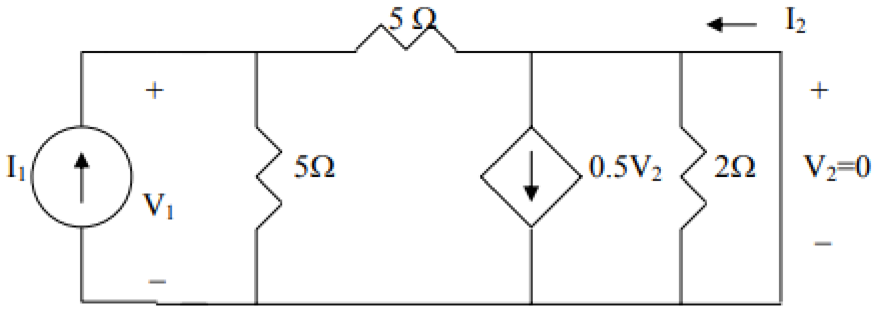
$$\mathbf{z}_{22} = \frac{\mathbf{V}_2}{\mathbf{I}_2} = 2 + 1 \,||\, (2 + 1 \,||\, 3) = \mathbf{z}_{11} = 2.733$$

Thus,

$$[\mathbf{z}] = \begin{bmatrix} \mathbf{2.733} & \mathbf{0.06667} \\ \mathbf{0.06667} & \mathbf{2.733} \end{bmatrix} \mathbf{\Omega}$$

Solution 2

To obtain y_{11} and y_{21}, consider the circuit below.

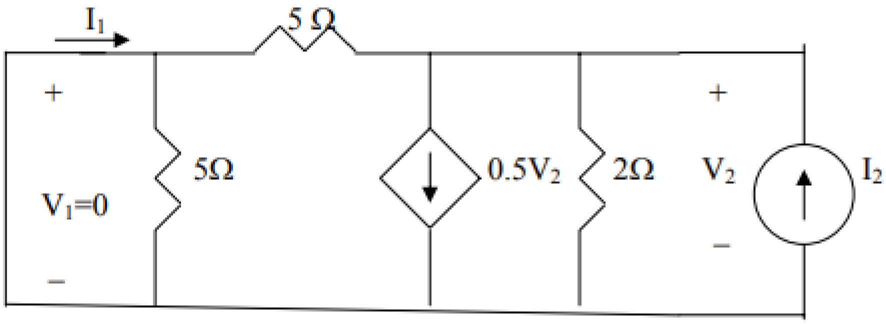


The 2-Ω resistor is short-circuited.

$$V_1 = 5\frac{I_1}{2} \quad \longrightarrow \quad y_{11} = \frac{I_1}{V_1} = \frac{2}{5} = 0.4$$

$$I_2 = \frac{1}{2} I_1 \quad \longrightarrow \quad y_{21} = \frac{I_2}{V_1} = \frac{\frac{1}{2} I_1}{2.5 I_1} = 0.2$$

To obtain y_{12} and y_{22}, consider the circuit below.



At the top node, KCL gives

$$I_2 = 0.5V_2 + \frac{V_2}{2} + \frac{V_2}{5} = 1.2V_2 \longrightarrow y_{22} = \frac{I_2}{V_2} = 1.2$$

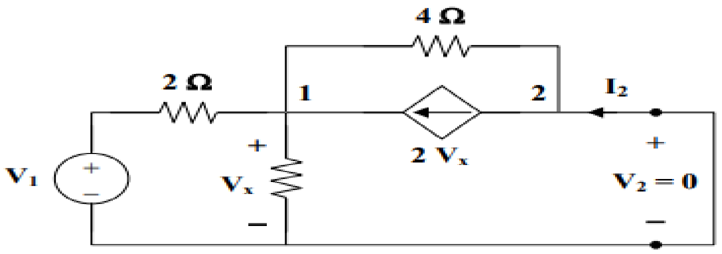
$$I_1 = -\frac{V_2}{5} = -0.2V_2 \longrightarrow y_{12} = \frac{I_1}{V_2} = -0.2$$

Hence,

$$[y] = \underline{\begin{bmatrix} 0.4 & -0.2 \\ 0.2 & 1.2 \end{bmatrix} S}$$

Find the y parameters for the network shown in figure below?

To get $\mathbf{y}_{11}$ and $\mathbf{y}_{21}$, consider the circuit in Fig. (a).



(a)

At node 1,

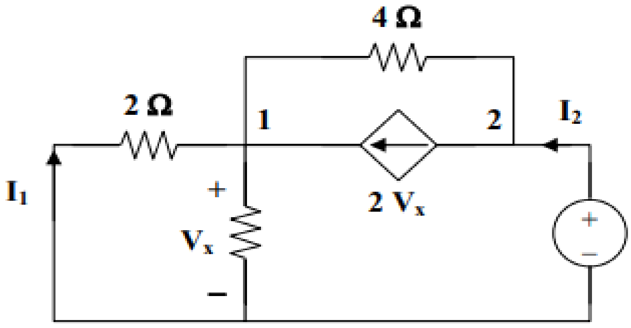
$$\frac{\mathbf{V}_1 - \mathbf{V}_x}{2} + 2\mathbf{V}_x = \frac{\mathbf{V}_x}{1} + \frac{\mathbf{V}_x}{4} \longrightarrow 2\mathbf{V}_1 = -\mathbf{V}_x \qquad (1)$$

But $$\mathbf{I}_1 = \frac{\mathbf{V}_1 - \mathbf{V}_x}{2} = \frac{\mathbf{V}_1 + 2\mathbf{V}_1}{2} = 1.5\mathbf{V}_1 \longrightarrow \mathbf{y}_{11} = \frac{\mathbf{I}_1}{\mathbf{V}_1} = 1.5$$

Also, $$\mathbf{I}_2 + \frac{\mathbf{V}_x}{4} = 2\mathbf{V}_x \longrightarrow \mathbf{I}_2 = 1.75\mathbf{V}_x = -3.5\mathbf{V}_1$$

$$\mathbf{y}_{21} = \frac{\mathbf{I}_2}{\mathbf{V}_1} = -3.5$$

To get $\mathbf{y}_{22}$ and $\mathbf{y}_{12}$, consider the circuit in Fig.(b).



(b)

At node 2,

$$\mathbf{I}_2 = 2\mathbf{V}_x + \frac{\mathbf{V}_2 - \mathbf{V}_x}{4} \qquad (2)$$

At node 1,

$$2\mathbf{V}_x + \frac{\mathbf{V}_2 - \mathbf{V}_x}{4} = \frac{\mathbf{V}_x}{2} + \frac{\mathbf{V}_x}{1} = \frac{3}{2}\mathbf{V}_x \longrightarrow \mathbf{V}_2 = -\mathbf{V}_x \qquad (3)$$

Substituting (3) into (2) gives

$$\mathbf{I}_2 = 2\mathbf{V}_x - \frac{1}{2}\mathbf{V}_x = 1.5\mathbf{V}_x = -1.5\mathbf{V}_2$$

Substituting (3) into (2) gives

$$\mathbf{I}_2 = 2\mathbf{V}_x - \frac{1}{2}\mathbf{V}_x = 1.5\mathbf{V}_x = -1.5\mathbf{V}_2$$

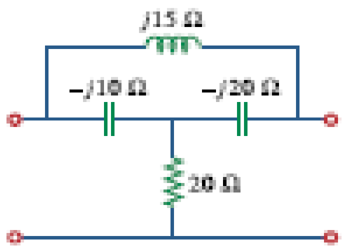
$$\mathbf{y}_{22} = \frac{\mathbf{I}_2}{\mathbf{V}_2} = -1.5$$

$$\mathbf{I}_1 = \frac{-\mathbf{V}_x}{2} = \frac{\mathbf{V}_2}{2} \longrightarrow \mathbf{y}_{12} = \frac{\mathbf{I}_1}{\mathbf{V}_2} = 0.5$$

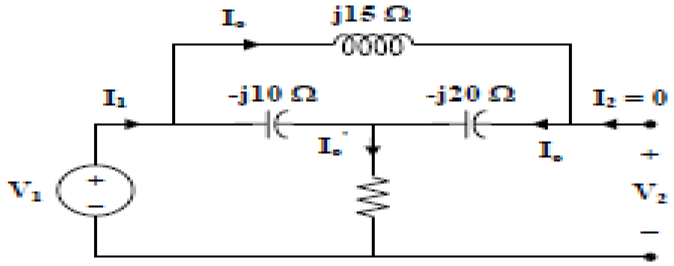
Thus,

$$[\mathbf{y}] = \begin{bmatrix} \mathbf{1.5} & \mathbf{0.5} \\ \mathbf{-3.5} & \mathbf{-1.5} \end{bmatrix} \mathbf{S}$$

Determine the transmission parameters of the circuit in Fig. below



To determine $\mathbf{A}$ and $\mathbf{C}$, consider the circuit in Fig.(a).



(a)

$$\mathbf{V}_1 = [\,20 + (\text{-j}10) \,||\, (\text{j}15 - \text{j}20)\,]\mathbf{I}_1$$

$$\mathbf{V}_1 = \left[\,20 + \frac{(\text{-j}10)(\text{-j}5)}{\text{-j}15}\right]\mathbf{I}_1 = \left[\,20 - \text{j}\frac{10}{3}\right]\mathbf{I}_1$$

$$\mathbf{I}_o' = \mathbf{I}_1$$

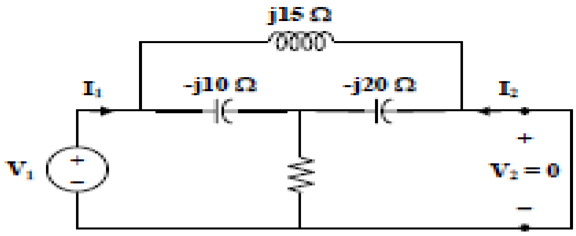
$$\mathbf{I}_o = \left(\frac{\text{-j}10}{\text{-j}10 - \text{j}5}\right)\mathbf{I}_1 = \left(\frac{2}{3}\right)\mathbf{I}_1$$

$$\mathbf{V}_2 = (\text{-j}20)\mathbf{I}_o + 20\mathbf{I}_o' = -\text{j}\frac{40}{3}\mathbf{I}_1 + 20\mathbf{I}_1 = \left(20 - \text{j}\frac{40}{3}\right)\mathbf{I}_1$$

$$\mathbf{A} = \frac{\mathbf{V}_1}{\mathbf{V}_2} = \frac{(20 - \text{j}10/3)\mathbf{I}_1}{\left(20 - \text{j}\frac{40}{3}\right)\mathbf{I}_1} = 0.7692 + \text{j}0.3461$$

$$\mathbf{C} = \frac{\mathbf{I}_1}{\mathbf{V}_2} = \frac{1}{20 - \text{j}\frac{40}{3}} = 0.03461 + \text{j}0.023$$

To find $\mathbf{B}$ and $\mathbf{D}$, consider the circuit in Fig. (b).



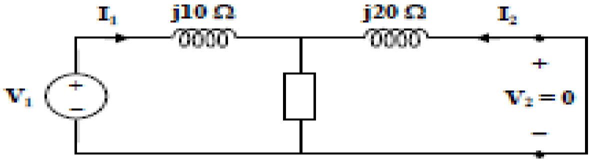
(b)

We may transform the Δ subnetwork to a T as shown in Fig. (c).

$$\mathbf{Z}_1 = \frac{(\text{j}15)(\text{-j}10)}{\text{j}15 - \text{j}10 - \text{j}20} = \text{j}10$$

$$\mathbf{Z}_2 = \frac{(\text{-j}10)(\text{-j}20)}{\text{-j}15} = \text{-j}\frac{40}{3}$$

$$\mathbf{Z}_3 = \frac{(\text{j}15)(\text{-j}20)}{\text{-j}15} = \text{j}20$$



(c)

$$-\mathbf{I}_2 = \frac{20 - j40/3}{20 - j40/3 + j20}\mathbf{I}_1 = \frac{3 - j2}{3 + j}\mathbf{I}_1$$

$$\mathbf{D} = \frac{-\mathbf{I}_1}{\mathbf{I}_2} = \frac{3 + j}{3 - j2} = 0.5385 + j0.6923$$

$$\mathbf{V}_1 = \left[j10 + \frac{(j20)(20 - j40/3)}{20 - j40/3 + j20} \right]\mathbf{I}_1$$

$$\mathbf{V}_1 = [\, j10 + 2(9 + j7)\,]\mathbf{I}_1 = j\mathbf{I}_1\,(24 - j18)$$

$$\mathbf{B} = \frac{-\mathbf{V}_1}{\mathbf{I}_2} = \frac{-j\mathbf{I}_1\,(24 - j18)}{\dfrac{-(3-j2)}{3+j}\mathbf{I}_1} = \frac{6}{13}(-15 + j55)$$

$$\mathbf{B} = -6.923 + j25.385\ \Omega$$

$$[\mathbf{T}] = \underline{\begin{bmatrix} 0.7692 + j0.3461 & -6.923 + j25.385\ \Omega \\ 0.03461 + j0.023\ \text{S} & 0.5385 + j0.6923 \end{bmatrix}}$$

Given the transmission parameters

$$[\mathbf{T}] = \begin{bmatrix} 3 & 20 \\ 1 & 7 \end{bmatrix}$$

obtain the other five two-port parameters.

Chapter 19, Solution 57.

$$\Delta_T = (3)(7) - (20)(1) = 1$$

$$[\mathbf{z}] = \begin{bmatrix} \frac{A}{C} & \frac{\Delta_T}{C} \\ \frac{1}{C} & \frac{D}{C} \end{bmatrix} = \underline{\begin{bmatrix} 3 & 1 \\ 1 & 7 \end{bmatrix}\Omega}$$

$$[\mathbf{y}] = \begin{bmatrix} \frac{D}{B} & \frac{-\Delta_T}{B} \\ \frac{-1}{B} & \frac{A}{B} \end{bmatrix} = \underline{\begin{bmatrix} \frac{7}{20} & \frac{-1}{20} \\ \frac{-1}{20} & \frac{3}{20} \end{bmatrix}\text{S}}$$

$$[\mathbf{h}] = \begin{bmatrix} \frac{B}{D} & \frac{\Delta_T}{D} \\ \frac{-1}{D} & \frac{C}{D} \end{bmatrix} = \underline{\begin{bmatrix} \frac{20}{7}\,\Omega & \frac{1}{7} \\ \frac{-1}{7} & \frac{1}{7}\,\text{S} \end{bmatrix}}$$

$$[\mathbf{g}] = \begin{bmatrix} \frac{C}{A} & \frac{-\Delta_T}{A} \\ \frac{1}{A} & \frac{B}{A} \end{bmatrix} = \underline{\begin{bmatrix} \frac{1}{3}\,\text{S} & \frac{-1}{3} \\ \frac{1}{3} & \frac{20}{3}\,\Omega \end{bmatrix}}$$

$$[\mathbf{t}] = \begin{bmatrix} \frac{D}{\Delta_T} & \frac{B}{\Delta_T} \\ \frac{C}{\Delta_T} & \frac{A}{\Delta_T} \end{bmatrix} = \underline{\begin{bmatrix} 7 & 20\,\Omega \\ 1\,\text{S} & 3 \end{bmatrix}}$$

❖ Find the transmission parameters for z parameters of the network are Z=[25 20;24 30]

$$[z]=\begin{bmatrix}25 & 20\\ 24 & 30\end{bmatrix}, \quad \Delta z = 25x30-20x24=270$$

$$A=\frac{z_{11}}{z_{21}}=\frac{25}{24}, \quad B=\frac{\Delta z}{z_{21}}=\frac{270}{24}$$

$$C=\frac{1}{z_{21}}=\frac{1}{24}, D=\frac{z_{22}}{z_{21}}=\frac{30}{24}$$

$t\rightarrow\infty$

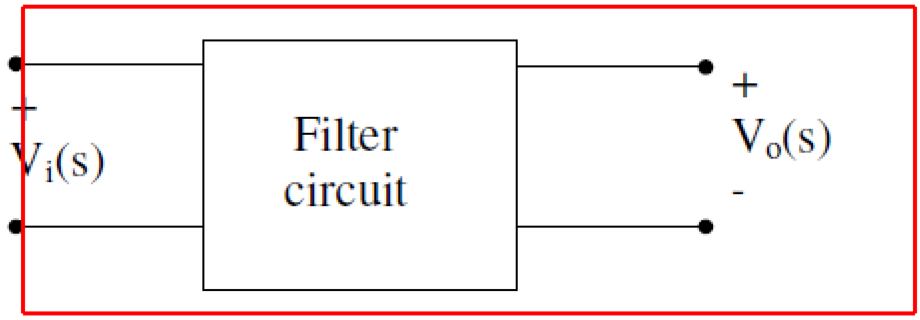
UNIT- V
FILTERS

5. Introduction :

Filters are essential building blocks in many systems, particularly in communication and instrumentation systems. A filter passes one band of frequencies while rejecting another. Typically implemented in one of three technologies: passive RLC filters, active RC filters and switched capacitor filters. Crystal and SAW filters are normally used at very high frequencies. Passive filters work well at high frequencies, however, at low frequencies the required inductors are large, bulky and non-ideal.

Furthermore, inductors are difficult to fabricate in monolithic from and are incompatible with many modern assembly systems. Active RC filters utilize op-amps together with resistors and capacitors and are fabricated using discrete, thick film and thin-film technologies. The performance of these filters is limited by the performance of the op-amps (e.g., frequency response, bandwidth, noise, offsets, etc.). Switched-capacitor filters are monolithic filters which typically offer the best performance in the term of cost. Fabricated using capacitors, switched and op-amps. Generally poorer performance compared to passive LC or active RC filters.

Filters are generally linear circuits that can be represented as a two-port network:



The filter transfer function is given as follows:

$$T(j\omega) = T(s) = \frac{V_o(s)}{V_i(s)}$$

The magnitude of the transmission is often expressed in dB in terms of gain function:

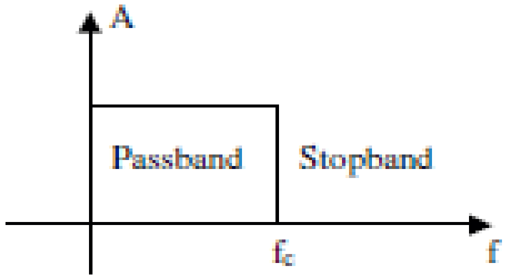
$$G(\omega)dB=20\log(|T(j\omega)|$$

Or, alternatively, in terms of the attenuation function:

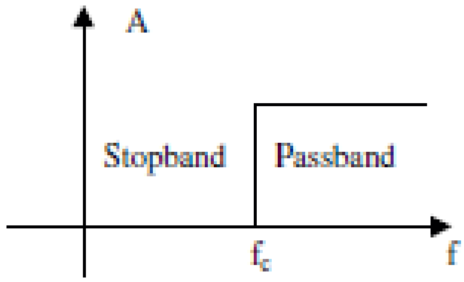
$$A(\omega)dB=-20\log(|T(j\omega)|$$

5.1.Classification Of Filters:

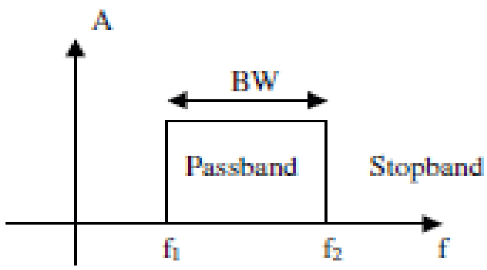
A filter shapes the frequency spectrum of the input signal, according to the magnitude of the transfer function. The phase characteristics of the signal are also modified as it passes through the filter. Filters can be classified into a number of categories based on which frequency bands are passes through and which frequency bands are stopped. Figures below show ideal responses of various filters.



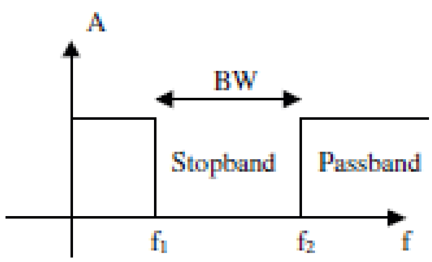
LOW-PASS FILTER RESPONSE



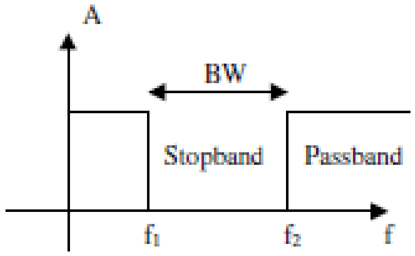
HIGH-PASS FILTER RESPONSE



BANDPASS FILTER RESPONSE



BANDSTOP FILTER RESPONSE



BANDSTOP FILTER RESPONSE

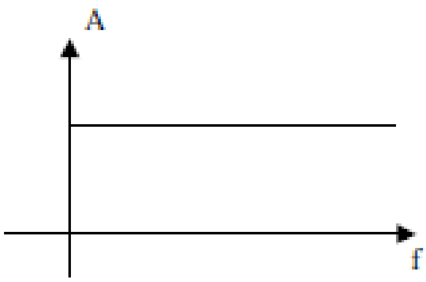
Center frequency

$$f_0 = \sqrt{f_1 f_2}$$

Quality factor Q (how fast the roll-off is)

$$Q = \frac{f_0}{BW}$$

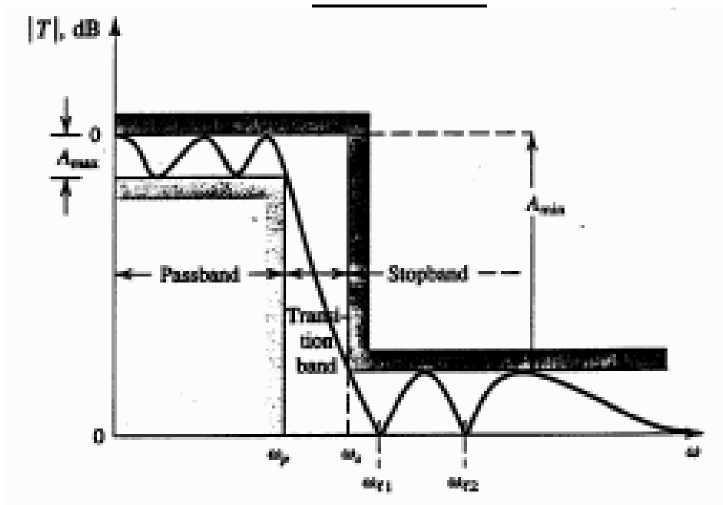
Wideband filter: $Q < 1$
Narrowband filter: $Q > 1$



ALLPASS FILTER RESPONSE

5.2.1. Classification of Pass band and Stop band:

Ideal filters could not be realized using electrical circuits, therefore the actual response of the filter is not a brick wall response as shown above but increases or decreases with a roll-off factor. Realistic transmission characteristics for a low pass filter are shown below.

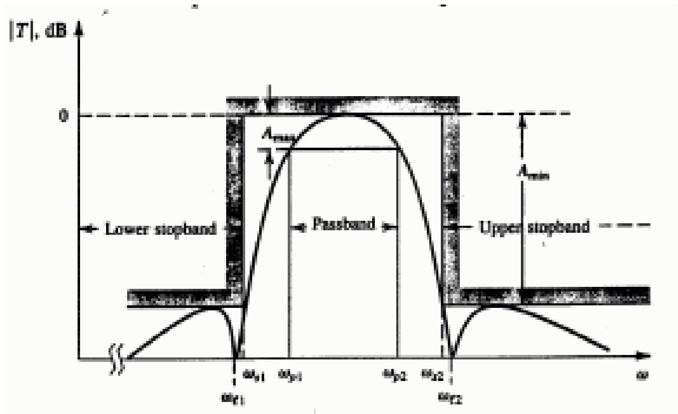


Transmission of a low pass filter is specified by four parameters:

- *Pass band edge*, ωp
- Maximum allowed variation in *pass band transmission,* Amax
- *Stop band edge,* ωs
- Minimum required *stop band attenuation,* Amin

The ratio ws/wp is usually used to measure the sharpness of the filter response and is called the selectivity *factor*. The more tightly one specifies a filter (i.e., lower Amax, higher Amin, ws/wp Closer to unity) the resulting filter must be of higher order and thus more complex and expensive. Amax is commonly referred as the *pass band ripple.* The process of obtaining a transfer function that meets given specifications is known as filter approximation. Filter approximation is usually performed using computer programs or filter design tables. In simple cases, filter approximation can be performed using closed form expressions.

Figure below shows transmission specifications for a band pass filter.



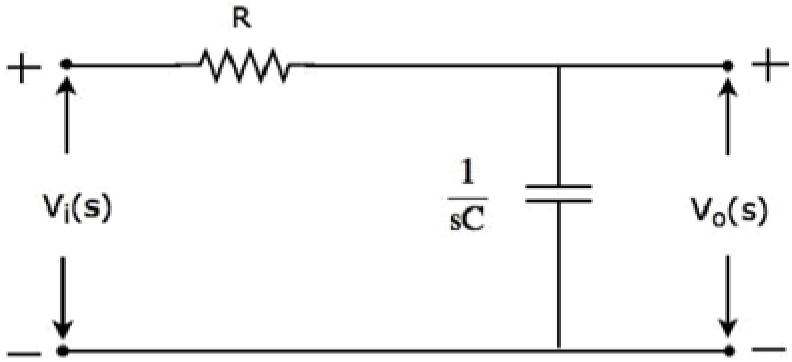
In circuit theory, a filter is an electrical network that alters the amplitude and/or phase characteristics of a signal with respect to frequency. Ideally, a filter will not add new frequencies to the input signal, nor will it change the component frequencies of that signal, but it will change the relative amplitudes of the various frequency components and/or their phase relationships.

Filters are often used in electronic systems to emphasize signals in certain frequency ranges and reject signals in other frequency ranges

5.2.2. Low-Pass:

Low pass filter as the name suggests, it allows (passes) only **low frequency** components. That means, it rejects (blocks) all other high frequency components.

The s-domain **circuit diagram** (network) of Low Pass Filter is shown in the following figure.



It consists of two passive elements resistor and capacitor, which are connected in **series**. Input voltage is applied across this entire combination and the output is considered as the voltage across capacitor.

Here, $V_i(s)$ and $V_o(s)$ are the Laplace transforms of input voltage, $v_i(t)$ and output voltage, $v_o(t)$ respectively.

The **transfer function** of the above network is

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}}$$

$$\Rightarrow H(s) = \frac{1}{1 + sCR}$$

Substitute, $s = j\omega$ in the above equation.

$$H(j\omega) = \frac{1}{1 + j\omega CR}$$

Magnitude of transfer function is

$$|H(j\omega)| = \frac{1}{\sqrt{(1 + (\omega CR)^2}}$$

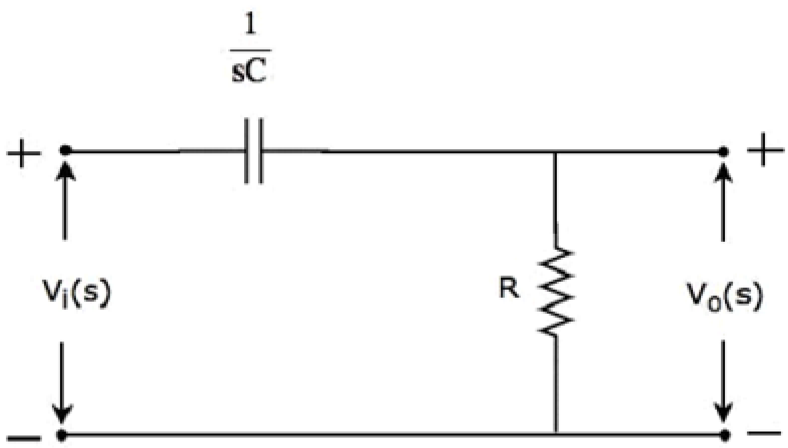
- At $\omega = 0$, the magnitude of transfer function is equal to 1.
- At $\omega = 1/CR$, the magnitude of transfer function is equal to 0.707.
- At $\omega = \infty$, the magnitude of transfer function is equal to 0.

Therefore, the magnitude of transfer function of **Low pass filter** will vary from 1 to 0 as ω varies from 0 to ∞.

5.2.3.High-Pass :

High pass filter as the name suggests, it allows (passes) only **high frequency** components. That means, it rejects (blocks) all low frequency components

The s-domain **circuit diagram** (network) of High pass filter is shown in the following figure.



It consists of two passive elements capacitor and resistor, which are connected in **series**. Input voltage is applied across this entire combination and the output is considered as the voltage across resistor

Here, $V_i(s)$ and $V_o(s)$ are the Laplace transforms of input voltage, $v_i(t)$ and output voltage, $v_o(t)$ respectively.

The **transfer function** of the above network is

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{R}{R + \frac{1}{sC}}$$

$$\Rightarrow H(s) = \frac{sCR}{1 + sCR}$$

Substitute, $s = j\omega$ in the above equation.

$$H(j\omega) = \frac{j\omega CR}{1 + j\omega CR}$$

Magnitude of transfer function is

$$|H(j\omega)| = \frac{\omega CR}{\sqrt{(1 + (\omega CR)^2}}$$

- At $\omega = 0$, the magnitude of transfer function is equal to 0.
- At $\omega = 1/CR$, the magnitude of transfer function is equal to 0.707.
- At $\omega = \infty$, the magnitude of transfer function is equal to 1.

Therefore, the magnitude of transfer function of **High pass filter** will vary from 0 to 1 as ω varies from 0 to ∞.

5.2.4.Band Pass :

Band pass filter as the name suggests, it **allows** (passes) only **one band** of frequencies. In general, this frequency band lies in between low frequency range and high frequency range. That means, this filter rejects (blocks) both low and high frequency components

The s-domain **circuit diagram** (network) of Band pass filter is shown in the following

It consists of three passive elements inductor, capacitor and resistor, which are connected in **series**. Input voltage is applied across this entire combination and the output is considered as the voltage across resistor.

Here, $V_i(s)$ and $V_o(s)$ are the Laplace transforms of input voltage, $v_i(t)$ and output voltage, $v_o(t)$ respectively.

The **transfer function** of the above network is

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{R}{R + \frac{1}{sC} + sL}$$

$$\Rightarrow H(s) = \frac{sCR}{s^2LC + sCR + 1}$$

Substitute $s = j\omega$ in the above equation.

$$H(j\omega) = \frac{j\omega CR}{1 - \omega^2LC + j\omega CR}$$

Magnitude of transfer function is

$$|H(j\omega)| = \frac{\omega CR}{\sqrt{(1 - \omega^2LC)^2 + (\omega CR)^2}}$$

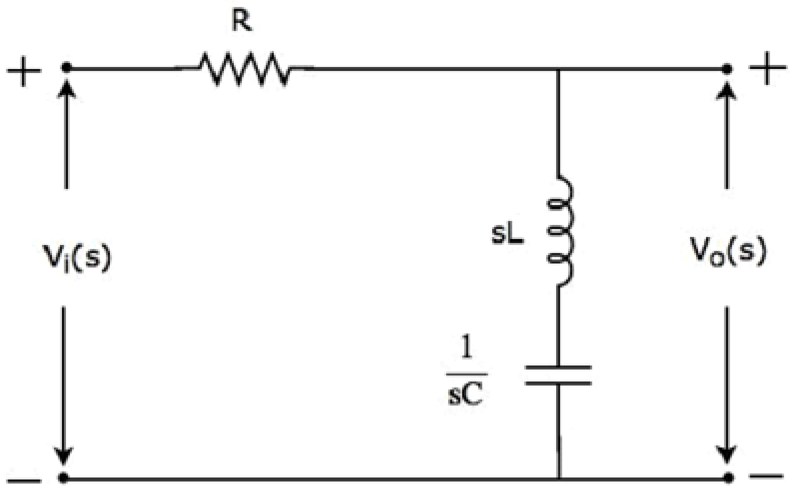
- At $\omega = 0$, the magnitude of transfer function is equal to 0.
- At $\omega = 1/LC$, the magnitude of transfer function is equal to 1.
- At $\omega = \infty$, the magnitude of transfer function is equal to 0.

Therefore, the magnitude of transfer function of **Band pass filter** will vary from 0 to 1 & 1 to 0 as ω varies from 0 to ∞.

5.2.5.Band elimination Filter:

Band stop filter as the name suggests, it rejects (blocks) only one band of frequencies. In general, this frequency band lies in between low frequency range and high frequency range. That means, this filter allows (passes) both low and high frequency components.

The s-domain (network) of **circuit diagram**and stop filter is shown in the following figure.



It consists of three passive elements resistor, inductor and capacitor, which are connected in **series**. Input voltage is applied across this entire combination and the output is considered as the voltage across the combination of inductor and capacitor.

Here, $V_i(s)$ and $V_o(s)$ are the Laplace transforms of input voltage, $v_i(t)$ and output voltage, $v_o(t)$ respectively.

The **transfer function** of the above network is

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{sL + \frac{1}{sC}}{R + sL + \frac{1}{sC}}$$

$$\Rightarrow H(s) = \frac{s^2LC + 1}{s^2LC + sCR + 1}$$

Substitute, $s = j\omega$ in the above equation.

$$H(j\omega) = \frac{1 - \omega^2LC}{1 - \omega^2LC + j\omega CR}$$

Magnitude of transfer function is

$$|H(j\omega)| = \frac{1 - \omega^2LC}{\sqrt{(1 - \omega^2LC)^2 + (\omega CR)^2}}$$

At $\omega = 0$, the magnitude of transfer function is equal to 1.

At $\omega = 1/\sqrt{LC}$ the magnitude of transfer function is equal to 0.

At $\omega = \infty$, the magnitude of transfer function is equal to 1.

Therefore, the magnitude of transfer function of **Band stop filter** will vary from 1 to 0 & 0 to 1 as ω varies from 0 to ∞.

5.2.6.Active Filters:

Active filters use amplifying elements, especially op amps, with resistors and capacitors in their feedback

loops, to synthesize the desired filter characteristics. Active filters can have high input impedance, low

output impedance, and virtually any arbitrary gain. They are also usually easier to design than passive

filters

Active Filters contain active components such as operational amplifiers, transistors or FET's within their circuit design. They draw their power from an external power source and use it to boost or amplify the output signal.

5.3. Constant – K Low Pass Filter

A network, either *T* or \[\pi\], is said to be of the constant-*k* type if Z_1 and Z_2 of the network satisfy the relation

$$Z_1 Z_2 = k^2$$

where Z_1 and Z_2 are impedance in the T and [pi] sections as shown in Fig.17.8. Equation 17.20 states that Z_1 and Z_2 are inverse if their product is a constant, independent of frequency. *k* is a real constant, that is the resistance. *k* is often termed as design impedance or nominal impedance of the constant *k*-filter.

The constant *k, T* or \[\pi\] type filter is also known as the prototype because other more complex networks can be derived, where $Z_1 = j\omega_L$ and $Z_2 = 1/j\omega_C$. Hence Z_1Z_2= \[{L \over C}={k^2}\] which is independent of frequencyThe pass band can be determined graphically. The reactance's of Z_1 and $4Z_2$ will vary with frequency as drawn in Fig.30.2. The cut-off frequency at the intersection of the curves Z_1 and $4Z_2$ is indicated as f_c. On the X-axis as $Z_1 = -4Z_2$ at cut-off frequency, the pass band lies between the frequencies at which $Z_1 = 0$, and $Z_1=-4Z_2$.

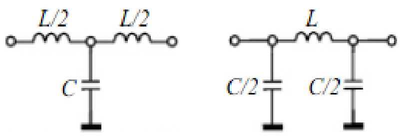
All the frequencies above f_c lie in a stop or attenuation band

The characteristic impedance of a \[\pi\]-network is given by

THE LOW PASS CONSTANT-*K* FILTER

The constant-*k* *LPF* can have the configurations from *Figure* The cutoff frequency is given by:

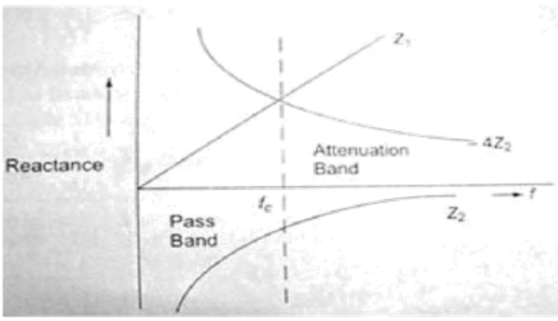
$$\omega_t = \frac{2}{\sqrt{LC}}$$



Figure

Generally the filter works on a constant load (R_s). To design the filter, R_s and ω_t are given. The matching can not be done at any frequency therefore we have to choose the frequency at which the filter will match. Most of the times, *LPF* matches in d.c. ($\omega = 0$). The elements of the filter are given by:

$$L = \frac{2R_s}{\omega_t} \qquad C = \frac{2}{\omega_t R_s}$$

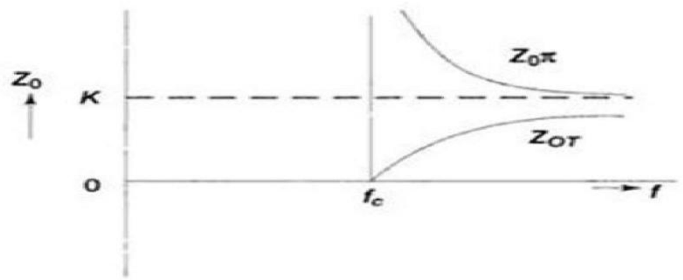


$$Z_{0\pi} = \frac{Z_1 Z_2}{Z_{0T}} = \frac{k}{\sqrt{1-\left(\frac{f}{f_c}\right)^2}} \qquad \text{...(30.5)}$$

5.4.Constant K-High Pass Filter:

Constant K-high pass filter can be obtained by changing the positions of series and shunt arms of the networks shown in Fig.30.1. The prototype high pass filters are shown in Fig.30.5, where Z_1 =-j/ω_C and $Z_2 = j\omega L$.

gain, it can be observed that the product of Z_1 and Z_2 is independent of frequency, and the filter design obtained will be of the constant k type.The plot of characteristic impedance with respect to frequency is shown



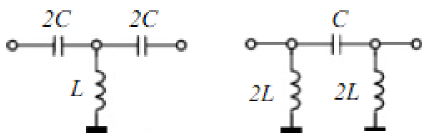
THE HIGH PASS CONSTANT-*K* FILTER

The possible configurations of the constant-*k* *HPF* are shown
The cutoff frequency is given by:

$$\omega_t = \frac{1}{2\sqrt{LC}}$$

If we are interesting in matching at very high frequency ($\omega \to \infty$), then L and C are given by:

$$L = \frac{R_s}{2\omega_t} \qquad C = \frac{1}{2R_s\omega_t}$$



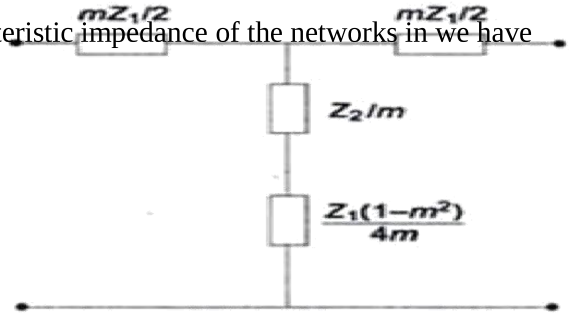
Figure

(a) (b)

5.5.m-Derived T-Section:

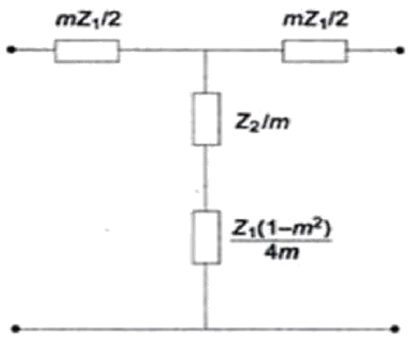
It is clear from previous chapter Figs 30.3 & 30.7 that the attenuation is not sharp in the stop band for k-type filters. The characteristic impedance, Z_0 is a function of frequency and varies widely in the transmission band. Attenuation can be increased in the stop band by using ladder section, i.e. by connecting two or more identical sections. In order to join the filter sections, it would be necessary that their characteristic impedance be equal to each other at all frequencies. If their characteristic impedances match at all frequencies, they would also have the same pass band. However, cascading is not a proper solution from a practical point of view. This is because practical elements have a certain resistance, which gives rise to attenuation in the pass band also.

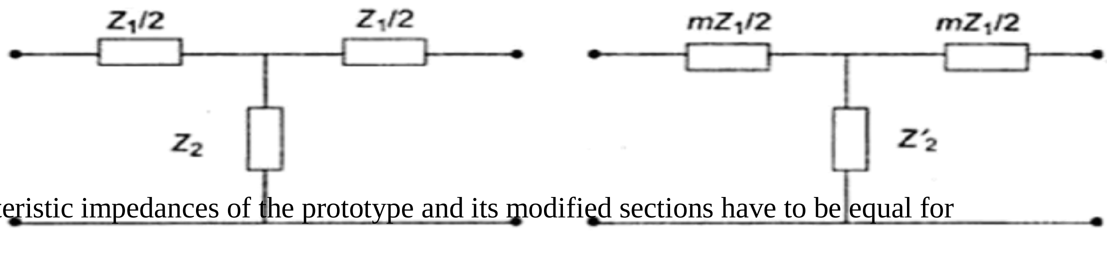
Therefore, any attempt to increase attenuation in stop band by cascading also results in an increase of _a‘ in the pass band. If the constant k section is regarded as the prototype, it is possible to design a filter to have rapid attenuation in the stop band, and the same characteristic impedance as the prototype at all frequencies. Such a filter is called m-derived filter. Suppose a prototype T-network shown in Fig.31.1 (a) has the series arm modified as shown in Fig.31.1 (b), where m is a constant. Equating the characteristic impedance of the networks in we have



where $Z_{0T'}$ is the characteristic impedance of the modified (m-derived) T-network.

Thus m-derived section can be obtained from the prototype by modifying its series and shunt arms. The same technique can be applied to \[\pi\] section network. Suppose a prototype p-network shown in Fig.31.3 (a) has the shunt arm modified as shown in Fig.31.3 (b).





The characteristic impedances of the prototype and its modified sections have to be equal for matching.

(a) (b)

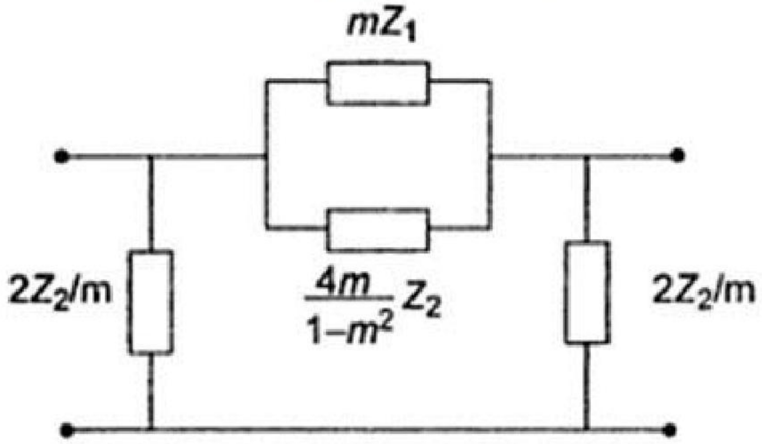
$$\therefore \quad \sqrt{\frac{Z_1 Z_2}{1+\frac{Z_1}{4Z_2}}} = \sqrt{\frac{Z_1^{'}\frac{Z_2}{m}}{1+\frac{Z_1^{'}}{4.Z_2/m}}}$$

The characteristic impedance of the modified (m-derived) \[\pi\]-network

Or $$Z_1' = \frac{Z_1Z_2}{\frac{Z_1}{4m} + \frac{Z_2}{m} - \frac{mZ_1}{4}}$$

$$= \frac{Z_1Z_2}{\frac{Z_2}{m} + \frac{Z_1}{4m}\left(1-m^2\right)}$$

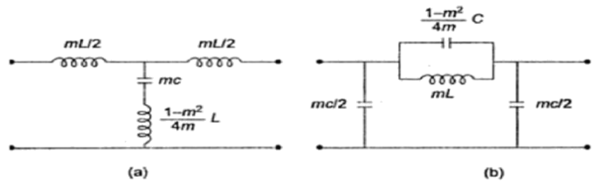
$$Z_1' = \frac{Z_1Z_2\frac{4m^2}{\left(1-m^2\right)}}{\frac{Z_2 4m_2}{m\left(1-m_2\right)} + Z_1 m} = \frac{mZ_1\frac{Z_2 4m}{\left(1-m^2\right)}}{mZ_1 + \frac{Z_2 4m}{\left(1-m^2\right)}} \quad \ldots\ldots\ldots\ldots(31.2)$$



The series arm of the m-derived \[\pi\] section is a parallel combination of mZ_1 and $4mZ_2/1\text{-}m^2$

5.5.1.m-Derived Low Pass Filter

In Fig.31.5, both m-derived low pass T and \[\pi\] filter sections are shown. For the T-section shown Fig.31.5(a), the shunt arm is to be chosen so that it is resonant at some frequency f_x above cut-off frequency f_c its impedance will be minimum or zero. Therefore, the output is zero and will correspond to infinite attenuation at this particular frequency



$$m\omega_r L = \frac{1}{\left(\frac{1-m^2}{4M}\right)\omega_r C}$$

$$\omega_r^2 = \frac{4}{LC\left(1-m^2\right)}$$

$$f_r = \frac{1}{\pi\sqrt{LC\left(1-m^2\right)}}$$

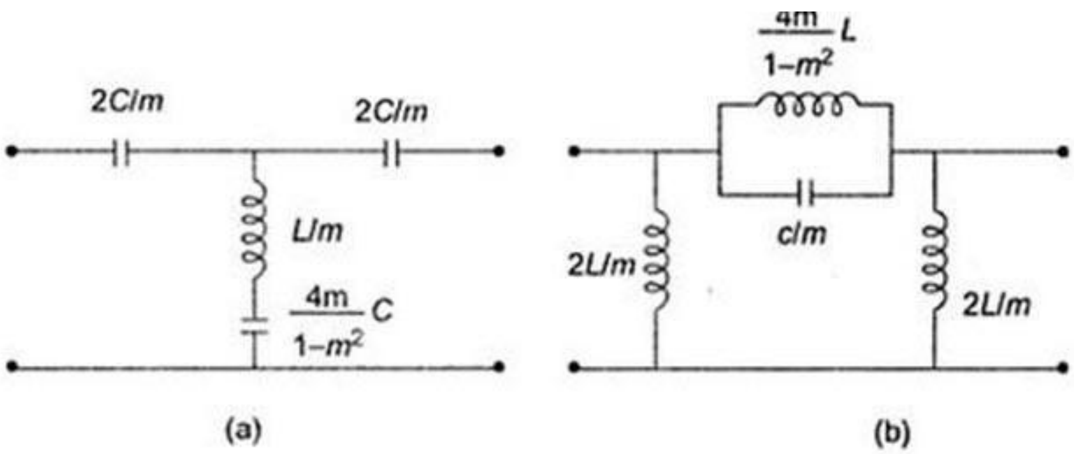
$$\therefore \qquad \alpha = 2\cosh^{-1}\frac{m\frac{f}{f_c}}{\sqrt{1-\left(\frac{f}{f_\alpha}\right)^2}}$$

And $$\beta = 2\sin^{-1}\sqrt{\left|\frac{Z_1}{4Z_1}\right|} = 2\sin^{-1}\frac{m\frac{f}{f_c}}{\sqrt{1-\left(\frac{f}{f_c}\right)^2(1=m)^2}}$$

5.5.2. m-derived High Pass Filter:

If the shunt arm in T-section is series resonant, it offers minimum or zero impedance. Therefore, the output is zero and, thus, at resonance frequency, or the frequency corresponds to infinite attenuation.

$$\omega_r \frac{L}{m} = \frac{1}{\omega_r \frac{4m}{1-m^2} C}$$

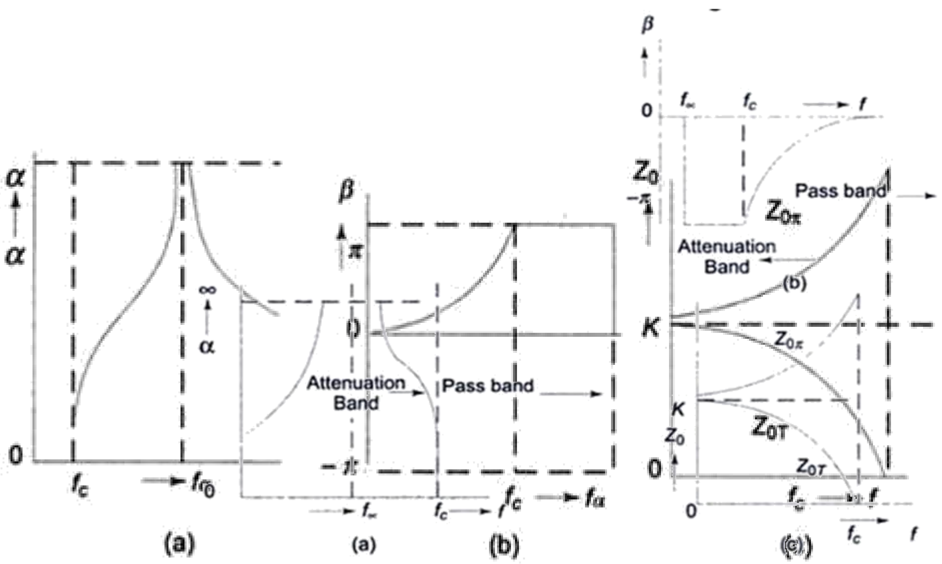


The m-derived \[\pi\]-section, the resonant circuit is constituted by the series arm inductance and capacitance

$$\frac{4m}{1-m^2} \omega_r L = \frac{1}{\frac{\omega_r}{m} C}$$

$$\omega_r^2 = \omega_\alpha^2 = \frac{1-m^2}{4LC}$$

$$\omega_\alpha = \frac{\sqrt{1-m^2}}{2\sqrt{LC}} \; or \; f_\alpha = \frac{\sqrt{1-m^2}}{4\pi\sqrt{LC}}$$



5.5.3..Band Pass Filter:

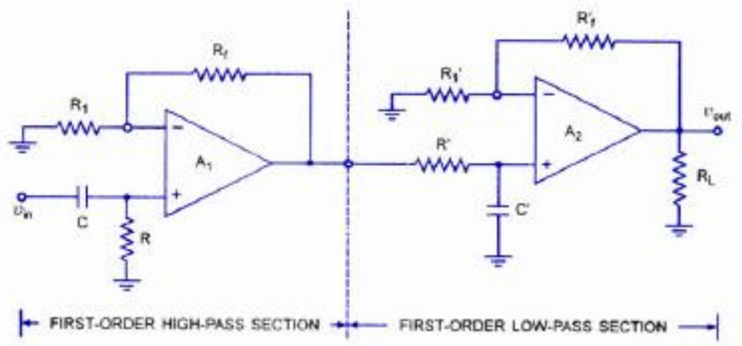
A band-pass filter is a circuit which is designed to pass signals only in a certain band of frequencies while attenuating all signals outside this band. The parameters of importance in a band pass filter are the high and low cut-off frequencies (fH and fl), the bandwidth (BW), the centre frequency fc, centre-frequency gain, and the selectivity or Q.

There are basically two types of band pass filters viz wide band pass and narrow band pass filters. Unfortunately, there is no set dividing line between the two. However, a band pass filter is defined as a wide band pass if its figure of merit or quality factor Q is less than 10 while the band pass filters with Q > 10 are called the narrow band pass filters. Thus Q is a measure of selectivity, meaning the higher the value of Q the more selective is the filter, or the narrower is the bandwidth (BW). The relationship between Q, 3-db bandwidth, and the centre frequency fc is given by an equation

For a wide band pass filter the centre frequency can be defined as where fH and fL are respectively the high and low cut-off frequencies in Hz. In a narrow band pass filter, the output voltage peaks at the centre frequency fc.

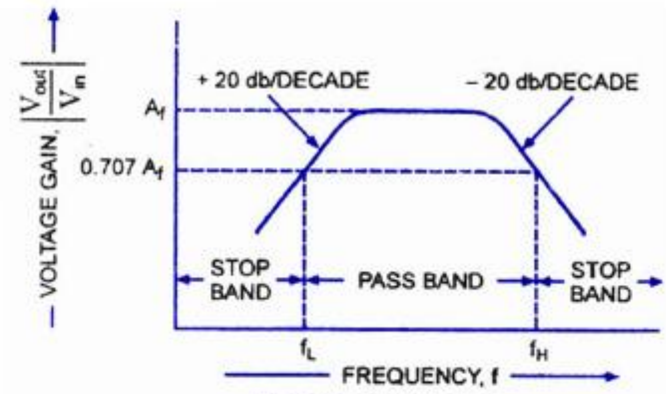
Wide Band Pass Filter:

A wide band pass filter can be formed by simply cascading high-pass and low-pass sections and is generally the choice for simplicity of design and performance though such a circuit can be realized by a number of possible circuits. To form a ± 20 db/ decade band pass filter, a first-order high-pass and a first-order low-pass sections are cascaded; for a ± 40 db/decade band pass filter, second-order high- pass filter and a second-order low-pass filter are connected in series, and so on. It means that, the order of the band pass filter is governed by the order of the high-pass and low-pass filters it consists of.



Circuit Diagram

Wide Band Pass Filter



Frequency Response

Frequency Response wide bandpass filter

A ± 20 db/decade wide band pass filter composed of a first-order high-pass filter and a first-order low-pass filter, is illustrated in fig. (a). Its frequency response is illustrated in fig. (b).

Narrow Band pass Filter:

A narrow bandpass filter employing multiple feedback is depicted in figure. This filter employs only one op-amp, as shown in the figure. In comparison to all the filters discussed so far, this filter has some unique features that are given below.

1. It has two feedback paths, and this is the reason that it is called a multiple-feedback filter.
2. The op-amp is used in the inverting mode.

The frequency response of a narrow bandpass filter is shown in fig(b).

Generally, the narrow bandpass filter is designed for specific values of centre frequency fc and Q or fc and BW. The circuit components are determined from the following relationships.

For simplification of design calculations each ofC1 and C2 may be taken equal to C.

R1 = Q/2Π fcCAf
R2 =Q/2Π fc C(2Q2-Af)
and R3 = Q / Π fc C

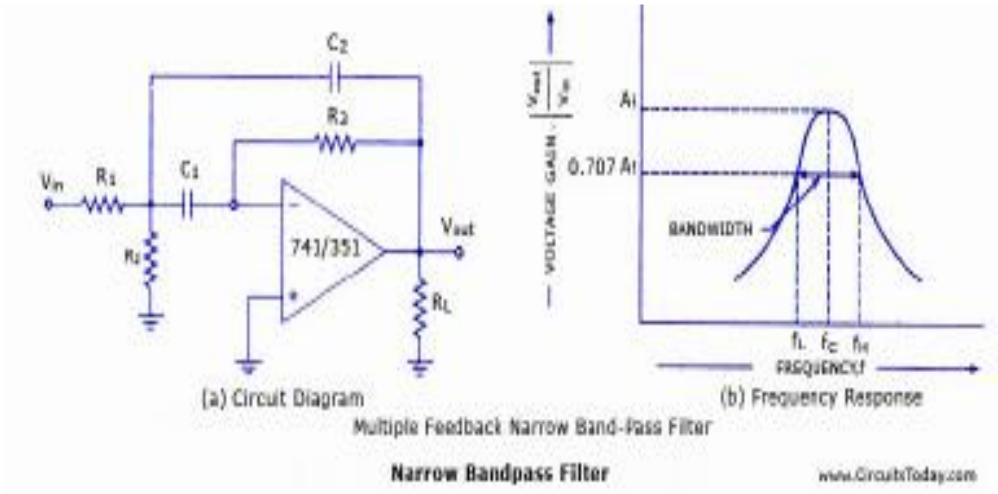
where Af, is the gain at centre frequency and is given as

Af = R3 / 2R1

The gain Af however must satisfy the condition Af < 2 Q2.

The centre frequency fc of the multiple feedback filter can be changed to a new frequency fc‘ without changing, the gain or bandwidth. This is achieved simply by changing R2 to R’2 so that

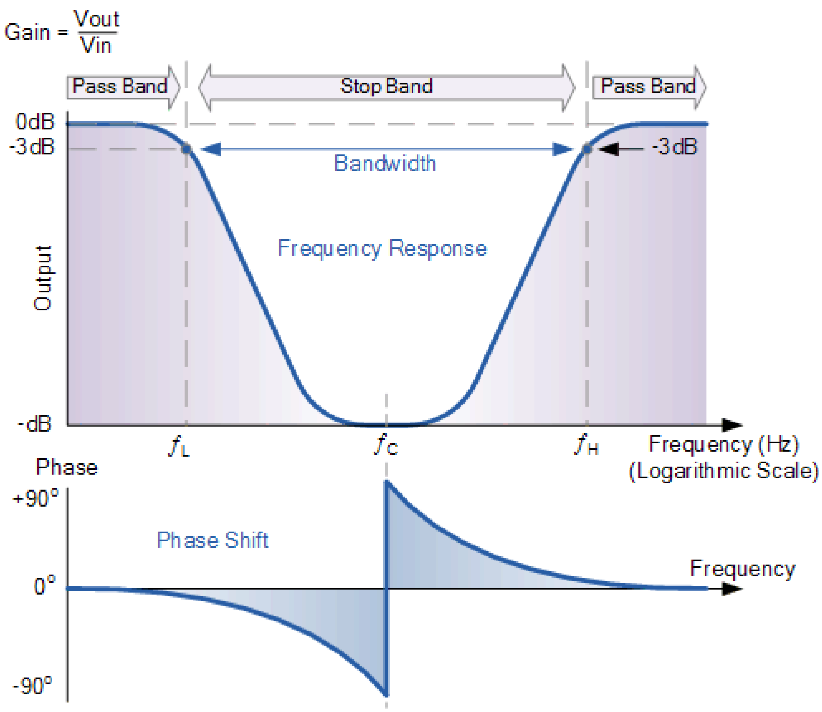
R’2 = R2 [fc/f’c]2



Band Stop or Band Elimination Filter:

By combining a basic RC low-pass filter with a RC high-pass filter we can form a simple band-pass filter that will pass a range or band of frequencies either side of two cut-off frequency points. But we can also combine these low and high pass filter sections to produce another kind of RC filter network called a band stop filter that can block or at least severely attenuate a band of frequencies within these two cut-off frequency points.

Band Stop Filter Response



The Band Stop Filter, (BSF) is another type of frequency selective circuit that functions in exactly the opposite way to the Band Pass Filter we looked at before. The band stop filter, also known as a band reject filter, passes all frequencies with the exception of those within a specified stop band which are greatly attenuated.

If this stop band is very narrow and highly attenuated over a few hertz, then the band stop filter is more commonly referred to as a notch filter, as its frequency response shows that of a deep notch with high selectivity (a steep-side curve) rather than a flattened wider band.

Also, just like the band pass filter, the band stop (band reject or notch) filter is a second-order (two-pole) filter having two cut-off frequencies, commonly known as the -3dB or half-power points producing a wide stop band bandwidth between these two -3dB points.

Then the function of a band stop filter is too pass all those frequencies from zero (DC) up to its first (lower) cut-off frequency point f_L, and pass all those frequencies above its second (upper) cut-off frequency f_H, but block or reject all those frequencies in-between. Then the filters bandwidth, BW is defined as: $(f_H - f_L)$.

So for a wide-band band stop filter, the filters actual stop band lies between its lower and upper -3dB points as it attenuates, or rejects any frequency between these two cut-off frequencies. The frequency response curve of an ideal band stop filter is therefore given as:

Band Stop Filter Response:

We can see from the amplitude and phase curves above for the band pass circuit, that the quantities f_L, f_H and f_C are the same as those used to describe the behaviour of the band-pass filter. This is because the band stop filter is simply an inverted or complimented form of the standard band-pass filter. In fact the definitions used for bandwidth, pass band, stop band and center frequency are the same as before, and we can use the same formulas to calculate bandwidth, BW, center frequency, f_C, and quality factor, Q.

The ideal band stop filter would have infinite attenuation in its stop band and zero attenuation in either pass band. The transition between the two pass bands and the stop band would be vertical (brick wall). There are several ways we can design a "Band Stop Filter", and they all accomplish the same purpose.

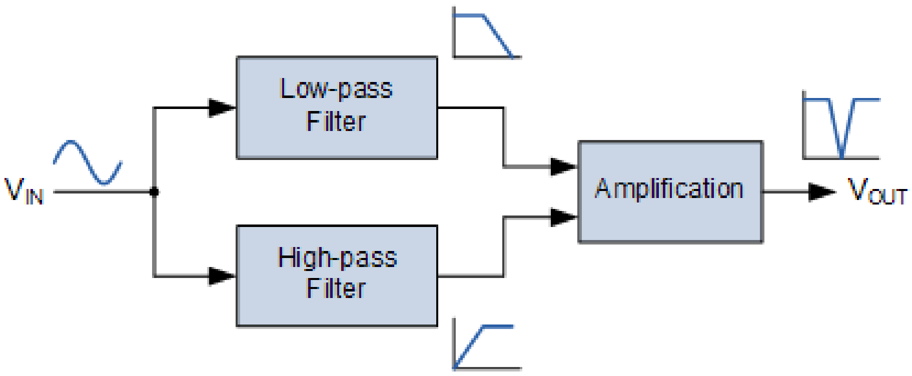
Generally band-pass filters are constructed by combining a low pass filter (LPF) in series with a high pass filter (HPF). Band stop filters are created by combining together the low pass and high pass filter sections in a "parallel" type configuration as shown.

Typical Band Stop Filter Configuration:

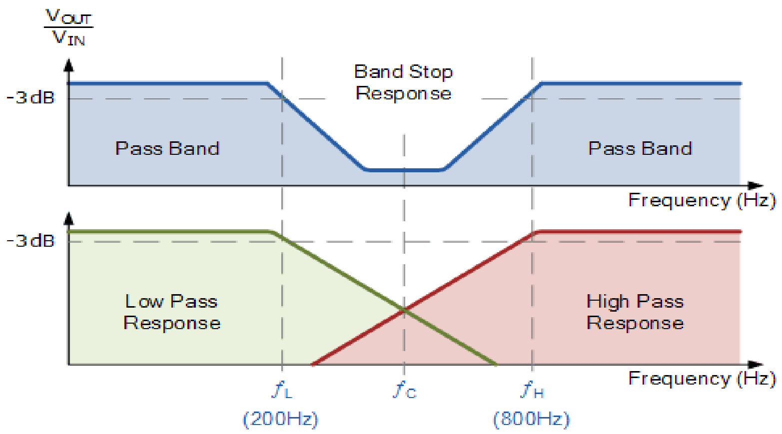
The summing of the high pass and low pass filters means that their frequency responses do not overlap, unlike the band-pass filter. This is due to the fact that their start and ending frequencies are at different frequency points. For example, suppose we have a first-order low-pass filter with a cut-off frequency, f_L of 200Hz connected in parallel with a first-order high-pass filter with a cut-off frequency, f_H of 800Hz. As the two filters are effectively connected in parallel, the input signal is applied to both filters simultaneously as shown above.

All of the input frequencies below 200Hz would be passed unattenuated to the output by the low-pass filter. Likewise, all input frequencies above 800Hz would be passed unattenuated to the output by the high-pass filter. However, and input signal frequencies in-between these two frequency cut-off points of 200Hz and 800Hz, that is f_L to $f_H$would be rejected by either filter forming a notch in the filters output response.

In other words a signal with a frequency of 200Hz or less and 800Hz and above would pass unaffected but a signal frequency of say 500Hz would be rejected as it is too high to be passed by the low-pass filter and too low to be passed by the high-pass filter. We can show the effect of this frequency characteristic below.



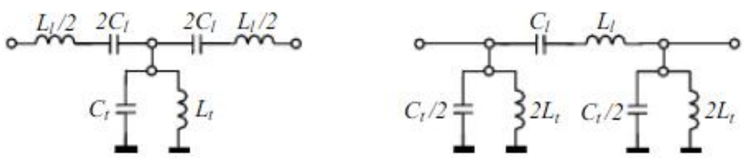
Band Stop Filter Characteristics:



The transformation of this filter characteristic can be easily implemented using a single low pass and high pass filter circuits isolated from each other by non-inverting voltage follower, (Av = 1). The output from these two filter circuits is then summed using a third operational amplifier connected as a voltage summer (adder) as shown.

THE BAND PASS CONSTANT-K FILTER

The constant-*k* *BPF* configurations are shown in *Figure*



Figure

If the cutoff frequencies (ω_L and ω_H) and the load (R_s) are given, then the elements of the filter are given by:

$$\begin{cases} L_l = \dfrac{2R_s}{\omega_s - \omega_i} \\ C_l = \dfrac{\omega_s - \omega_i}{2R_s\omega_0^2} \end{cases} \qquad \begin{cases} L_t = \dfrac{(\omega_s - \omega_i)R}{2\omega_0^2} \\ C_t = \dfrac{2}{(\omega_s - \omega_i)R} \end{cases}$$

5.5.7.Band Stop Filter Circuit:

The use of operational amplifiers within the band stop filter design also allows us to introduce voltage gain into the basic filter circuit. The two non-inverting voltage followers can easily be converted into a basic non-inverting amplifier with a gain of $Av = 1 + Rf/Rin$by the addition of input and feedback resistors, as seen in our non-inverting op-amp tutorial.

Also if we require a band stop filter to have its -3dB cut-off points at say, 1kHz and 10kHz and a stop band gain of -10dB in between, we can easily design a low-pass filter and a high-pass filter with these requirements and simply cascade them together to form our wide-band band-pass filter design.

Now we understand the principle behind a Band Stop Filter, let us design one using the previous cut-off frequency values.