LECTURE NOTES

ON

OPTIMIZATION TECHNIQUES (AMEB12)

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MODULE-I
DEVELOPMENT OF OR AND ALLOCATION

Operations Research is the science of rational decision-making and the study, design and integration of complex situations and systems with the goal of predicting system behavior and improving or optimizing system performance. The formal activities of operation research were initiated in England during World War II to make decisions regarding the best utilization of war material. After the war the ideas advanced in military operations were adapted to improve efficiency and productivity in the civilian sector. That developed to today's dominant and indispensable decision-making tool, Operations research. It encompasses managerial decision making, mathematical and computer modeling and the use of information technology for informed decision-making.

The concepts and methods of Operations Research are pervasive. Students and graduates advise the public and private sectors on energy policy; design and operation of urban emergency systems; defense; health care; water resource planning; the criminal justice system; transportation issues. They also address a wide variety of design and operational issues in communication and data networks; computer operations; marketing; finance; inventory planning; manufacturing; and many areas designed to improve business productivity and efficiency. The subject impacts biology, the internet, the airline system, international banking and finance. It is a subject of beauty, depth, infinite breadth and applicability.

The Meaning of Operations Research

From the historical and philosophical summary just presented, it should be apparent that the term "operations research" has a number of quite distinct variations of meaning. To some, OR is that certain body of problems, techniques, and solutions that has been accumulated under the name of OR over the past 30 years, and we apply OR when we recognize a problem of that certain genre. To others, it is an activity or process something we do, rather than know-which by its very nature is applied.

Perhaps in time the meaning will stabilize, but at this point it would be premature to exclude any of these interpretations. It would also be counterproductive to attempt to make distinctions between "operations research" and the "systems approach." While these terms are sometimes viewed as distinct, they are often conceptualized in such a manner as to defy separation. Any attempt to draw boundaries between them would in practice be arbitrary.

The Operational Research Society of Great Britain has adopted the following definition:

- Operational research is the application of the methods of science to complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business, government, and defense. The distinctive approach is to develop a scientific model of the system, incorporating measurements of factors such as chance and risk, with which to predict and compare the outcomes of alternative decisions, strategies or controls. The purpose is to help management determine its policy and actions scientifically.
- Operations research is concerned with scientifically deciding how to best design and operate man-machine systems, usually under conditions requiring the allocation of scarce resources.
- Although both of these definitions leave something to be desired, they are about as specific as one would want to be in defining such a broad area. It is noteworthy that both definitions emphasize the motivation for the work; namely, to aid decision makers in dealing with complex real-world problems. Even when the methods seem to become so abstract as to lose real-world relevance, the student may take some comfort in the fact that the ultimate goal is always some useful application.
- Both definitions also mention methodology, describing it only very generally as “scientific.” That term is perhaps a bit too general, inasmuch as the methods of science are so diverse and varied. A more precise description of the OR methodology would indicate its reliance on “models.” Of course, that term would itself require further elaboration, and it is to that task that we now turn our attention.
- Operations Research has been defined so far in various ways and still not been defined in an authoritative way. Some important and interesting opinions about the definition of OR which have been changed according to the development of the subject been given below:
- OR is a scientific method of providing executive departments with a quantitative basis for decision regarding the operations under their control. -Morse and Kimbal(9164)
- OR is a scientific method of providing executive with an analytical and objective basis for decisions. -P.M.S.Blacket(1948)
- OR is the application of scientific methods, techniques and tools to problems involving the

- operations of systems so as to provide these in control of the operations with optimum solutions to the problem." - Churchman, Acoff, Arnoff (1957)

- OR is the art of giving bad answers to problems to which otherwise worse answers are given. -T. L Saaty (1958)

CHARECTERISTIS (FEATURES) OPERATIONS RESEARCH

The main characteristics of OR are as follows:

1) Inter-disciplinary team approach: In OR, the optimum solution is foud by a team of scientists selected from various disciplines such as mathematics, statistics, economics, engineering, physics, etc.
2) Wholistic approach to the system: The most of the problems tackled by OR have the characteristic that OR have the characteristic that OR tries to find the best (optimum) deisions relative to larger possible portion of the total organization.
3) Imperfectness of solutions: by OR techniques, we cannot obtain perfect answers to our problems but, only the quality of the solution is improved from worse to bad answers.
4) Use of scientific research: OR uses techniques of scientific research to reach the optimum solutions.
5) To optimize the total output> OR tries to optimize total return by maximizing the profit and minimizing the cost or loss.

APPLICATIONS OF OPEARATIONS RESEARCH

1) In Agriculture:

-Optimum allocation of land to various crops in accordance with climatic conditions.

- Optimum distribution of water from various resources like canal for irrigation purposes**.**

2) In Finance:

-To maximize the per capita income with minimum recourses.

-To find out the profit plan for the company.

-To determine the best replacement policies.

3) In Industry: allocation of various limited resources such as men, machines, material, money,

4) In marketing: with th help of OR techniques a marketing manager can decide

- Where to distribute the products for sles so that the total cost of transportation is minimum.

- The minimum per MODULE cost sale price.
- Thee size of stock to meet the future demand.
- How to select the best advertising media with respect to time, cost etc.
- How, when , and where to purchase at the minimum possible cost?

5**) In Personnel Management**: A Personnel manager can use OR techniques

- To appoint the most suitable persons on minimum salary.
- To determine the best age of retirement for employees.
- To find out the number of persons to be appointed on full time basis when the work load is seasonal.

6**) In Production management**. A production manager can use OR techniques

- To find out the number and size of the items to be replaced.
- In scheduling and sequencing the production runs by proper allocation of machines.
- In calculating the optimum product mix.
- To select , locate and design the sites for production plants.

8**) In LIC:**

- What should be the premium rates for various modes of policies.
- How best the profits could be distributed in he case of with profit policies.

QUANTITATIVE TECHNIQUES OF OR:

Distribution (Allocation) Models: Distribution models are concerned with allotment of available recourses so as to minimize cost or maximize profits subject to prescribed conditions.

Production/Inventory Models: These models are concerned with determination of the optimal (economic) order quantity and ordering (production) intervals considering the factors such s demand per MODULE ime, cost of placing orders, costs associated with goods held up in he inventory and the cost due to shortage of goods etc.

Waiting Line (Queueing) Models: In queueing models an attempt is made to predict

Markovian Models: These models are applicable in such a situations where the state of the system can be defined by some descriptive measure of numerical value and here the system moves from one state to another on a probability basis. Example is Brand switching problems

Competitive Strategy Models(Game Theory): These models are used to determine the ehaviour of

decission making under competition or conflict.

Network Models: These models are applicable in large projects involving complexities and inter-dependencies od activities. PERT and CPM are used for planning , scheduling and controlling complex project which can be characterized as net-work.

Job Sequencing Models: These models involve the selection of such a sequence of performing a series of jobs to be done on service facilities(machines) that optimize th efficiency measure of performance of the system.

Replacement Models: These models deal with determination of optimum replacement policy in situation that arise when some items or machinery need replacement by a new one.

Simulation Models: Simulation is very powerful technique for solving such complex models which cannot be solved otherwise and thus it is being extensively applied to solve a variety of problems**.**

MODELLING IN OPERATIONS RESEARCH

The essence of the operations research activity lies in the construction and use of models. Although modeling must be learned from individual experimentation, we will attempt here to discuss it in broad, almost philosophical terms. This overview is worth having, and setting a proper orientation in advance may help to avoid misconceptions later.

Definition: A model in the sense used in 0 R is defined as a representation of an actual object or situation. It shows the relationships (direct or indirect) and inter-relationships of action and reaction in terms of cause and effect.

Since a model is an abstraction of reality, it thus appears to be less complete than reality itself. For a model to be complete, it must be a representative of those aspects of reality that are being investigated.

The main objective of a model is to provide means for analyzing the behavior of the system for the purpose of improving its performance. Or, if a system is not in existence, then a model defines the ideal structure of this future system indicating the functional relationships among its elements. The reliability of the solution obtained from a model depends on the validity of the model in representing the real systems. A model permits to ‘examine the behavior of a system without interfering with ongoing operations.

CLASSIFICATION OF MODELS

Models can be Classified According to Following Characteristics:

1. **Classification by Structure:**

i. Iconic models. Iconic models represent the system as it is by scaling it up or down (i.e., by enlarging or reducing the size). In other words, it is an image.

For example, a toy airplane is an iconic model of a real one. Other common examples of it are: photographs, drawings, maps etc. A model of an atom is scaled up so as to make it visible to the naked eye. In a globe, the diameter of the earth is scaled down, but the globe has approximately the same shape as the earth, and the relative sizes of continents, seas, etc., are approximately correct. The iconic model is usually the simplest to conceive and the most specific and concrete. Its function is generally descriptive rather than explanatory. Accordingly, it cannot be easily used to determine or predict what effects many important changes on the actual system.

ii. Analogue models. The models, in which one set of properties is used to represent another set of properties, are called analogue models. After the problem is solved, the solution is reinterpreted in terms of the original system.

For example, graphs are very simple analogues because distance is used to represent the properties such as: time, number, percent, age, weight, and many other properties. Contour lines on a map represent the rise and fall of the heights. In general, analogues are less specific, less concrete but easier to manipulate than are iconic models.

iii. Symbolic (Mathematical) models. The symbolic or mathematical model is one which employs a set of mathematical symbols (i.e., letters, numbers, etc.) to represent the decision variables of the system. These variables are related together by means of a mathematical equation or a set of equations to describe the behavior (or properties) of the system. The solution of the problem is then obtained by applying well-developed mathematical techniques to the model. The symbolic model is usually the easiest to manipulate experimentally and it is most general and abstract. Its function is more often explanatory rather than descriptive.

2. **Classification by Purpose:**

Models can also be classified by purpose of its utility. The purpose of a model may be descriptive predictive or prescriptive.

i. **Descriptive models.** A descriptive model simply describes some aspects of a situation based on observations, survey. Questionnaire results or other available data. The result of an opinion poll represents a descriptive model.

ii. **Predictive models. Such models** can answer 'what if' type of questions, i.e. they can make predictions regarding certain events. For example, based on the survey results, television networks such models attempt to explain and predict the election results before all the votes are actually counted.

iii. **Prescriptive models**. Finally, when a predictive model has been repeatedly successful, it can be used to prescribe a source of action. For example, linear programming is a prescriptive (or normative) model because it prescribes what the managers ought to do.

3. Classification by Nature of Environment

These are mainly of two types:

i. **Deterministic models**. Such models assume conditions of complete certainty and perfect knowledge.

For example, linear programming, transportation and assignment models are deterministic type of models.

ii. **Probabilistic (or Stochastic) models**. These types of models usually handle such situations in which the consequences or payoff of managerial actions cannot be predicted with certainty. However, it is possible to forecast a pattern of events, based on which managerial decisions can be made. For example, insurance companies are willing to insure against risk of fire, accidents, sickness and so on, because the pattern of events have been compiled in the form of probability distributions.

4. Classification by Behavior

Static models. These models do not consider the impact of changes that takes place during the planning horizon, i.e. they are independent of time. Also, in astatic model only one decision is needed for the duration of a given time period.

i. **Dynamic models.** In these models, time is considered as one of the important variables and admits the impact of changes generated by time. Also, in dynamic models, not only one but a series of interdependent' decisions is required during the planning horizon.

5. Classification by Method of Solution

i. **Analytical models**. These models have a specific mathematical structure-and thus can be solved by known analytical or mathematical techniques. For example, general linear programming models as well as the specially structured transportation and assignment models are analytical models. .

ii. **Simulation models.** They also have a mathematical structure but they cannot be solved by purely using the 'tools' and 'techniques' of mathematics. A simulation model is essentially computer-assisted experimentation on a mathematical structure of a real time structure in order to study the system under a variety of assumptions.

Simulation modeling has the advantage of being more flexible than mathematical modeling and hence can be used to represent complex systems which otherwise cannot be formulated mathematically. On the other hand, simulation has the disadvantage of not providing general solutions like those obtained from successful mathematical models.

6. **Classification by use of Digital Computers**: The development of the digital computer has led to the introduction of the following types of modeling in OR.

Analogue and Mathematical models combined. Sometimes analogue models are also expressed in terms of mathematical symbols. Such models may belong to both the types (ii) and (iii) in classification 1 above.

For example, Simulation model is of analogue type but mathematical formulae are also used in it. Managers very frequently use this model to 'simulate' their decisions by summarizing the activities of industry in a scale-down period.

Function models. Such models are grouped on the basis of the function being performed.

For example, a function may serve to acquaint to scientist with such things as-tables, carrying data, a blue-print of layouts, a program representing a sequence of operations (like' in computer programming).

ii **Quantitative models.** Such models are used to measure the observations.

For example, degree of temperature, yardstick, a MODULE of measurement of length value, etc.

Other examples of quantitative models are: (i) transformation models which are useful in converting a measurement of one scale to another (e.g., Centigrade vs. Fahrenheit conversion scale), and (ii) the test models that act as 'standards' against which measurements are compared (e.g., business dealings, a specified standard production control, the quality of a medicine).

Heuristic models:. These models are mainly used to explore alternative strategies (courses of action) that were overlooked previously, whereas mathematical models are used to represent systems possessing well- defined strategies. Heuristic models do not claim to find the best solution to the problem.

PRINCIPLES OF MODELING

Let us now outline general principles useful in guiding to formulate the models within the context of 0 R. The model building and their users both should be consciously aware of the following ten principles:

Do not build up a complicated model when simple one will suffice. Building the strongest possible model is a common guiding principle for mathematicians who are attempting to extend the theory or to develop techniques that have wide applications. However, in the actual practice of building models for specific purposes, the best advice is to “keep it simple”.

Beware of molding the problem to fit the technique. For example, an expert on linear programming techniques may tend to view every problem he encounters as required in a linear programming solutions. In fact, not all optimization problems involve only linear functions. Also, not all OR problems involve optimization. As a matter of fact, not all real-world problems call for operations research! Of course, everyone search reality in his own terms, so the field of OR is not unique in this regard. Being human we rely on the methods we are most comfortable in using and have been most successful within the past. We are certainly not able to use techniques in which we have no competence, and we cannot hope to be competent in all techniques. We must divide OR experts into three main categories:

(i)Technique developers. (ii) Teacher and (iii) Problem solvers.

In particular one should be ready to tolerate the behavior “I have found a cure but I am trying to search a disease to fit it” among technique developers and teachers.

1. **The deduction phase of modeling must be conducted rigorously**. The reason for requiring rigorous deduction is that one wants to be sure that if model conclusions are inconsistent with reality, then the defect lie in the assumptions. One application of this principle is that one must be extremely careful when programming computers. Hidden “bugs” are especially dangerous when they do not prevent the program from running but

simply produce results, which are not consistent with the intention of the model.

2. **Models should be validated prior to implementation.** For example, if a model is constructed to forecast the monthly sales of a particular commodity, it could be tested using historical data to compare the forecasts it would have produced to the actual sales. In case, if the model cannot be validated prior to its implementation, then it can be implemented in phases for validation. For example a new model for inventory control may be implemented for a certain selected

group of items while the older system is retained for the majority of remaining items. If the model proves successful, more items can be placed within its range. It is also worth noting that real things change in time. A highly satisfactory model may very well degrade with age. So periodic re-evaluation is necessary.

5.**A model should never be taken too literally**. For example, suppose that one has to construct an elaborate computer model of Indian economy with many competent researchers spending a great deal of time and money in getting all kinds of complicated interactions and relationships. Under such circumstances, it can be easily believed as if the model duplicates itself the real system. This danger continues to increase as the models become larger and more sophisticated, as they must deal with increasingly complicated problems.

6.**A model should neither be pressed to do, nor criticized for failing to do that for which it was never intended**. One example of this error would be the use of forecasting model to predict so far into the future that the data on which the forecasts are based have no relevance. Another example is the use of certain network methods to describe the activities involved in a complex project. A model should not be stretched beyond its capabilities.

7.**Beware of over-selling a model**. This principle is of particular importance for the OR professional because most non- technical benefactors of an operations researcher's work are not likely to understand his methods. The increased technicality of one's methods also increases the burden of responsibility on the OR professional to distinguish clearly between his role as model manipulator and model interpreter. In those cases where the assumptions can be challenged, it would be dishonest to use the model.

8.**Some of the primary benefits of modeling are associated with the process of developing the model.** It is true in general that a model is never as useful to anyone else as it is to those who are involved in building it up. The model itself never contains the full knowledge and understanding of the real system that the builder must acquire in order to

Successfully model it, and there is no practical way to convey this knowledge and understanding properly. In some cases the sole benefits may occur while the model is being developed. In such cases, the model may have no further value once it is completed, An example of this case might occur when a small group of people attempts to develop a formal plan for some subject. The plan is the final model, but the real problem may be to agree on 'what the objectives ought to be'.

9.**A model cannot be any better than the information that goes into it.** Like a computer program, a model can only manipulate the data provided to it; it cannot recognize and correct for deficiencies in input. Models may condense data or convert it to more useful forms, but they do not have the capacity to generate it. In some situations it is always better to gather more information about the system instead of exerting more efforts on modern constructions.

10.**Models cannot replace decision makers.** The purpose of OR models should not be supposed to provide "Optimal solutions" free from human subjectivity and error. OR models can aid decision makers and thereby permit better decisions to be made. However they do not make the job of decision making easier. Definitely, the role of experience, intuition and judgment in decision- making is undiminished.

GENERAL METHODS FOR SOLVING 'OR' MODLES

In OR, we do not have a single general technique that solves all mathematical models that arise in practice. Instead, the type and complexity of the mathematical model dictate the nature of the solution method. For example, in Section 1.1 the solution of the tickets problem requires simple ranking of the alternatives based on the total purchasing price, whereas the solution of the rectangle problem utilizes differential calculus to determine the maximum area.

The most prominent OR technique is linear programming. It is designed for models with strict linear objective and constraint functions. Other techniques include integer programming (in which the variables assume integer values), dynamic programming (in which the original model can be decomposed into smaller sub. problems), network programming (in which the problem can be modeled as a network), and nonlinear programming (in which the functions of the model are non. linear). The cited techniques are but a partial list of the large number of available OR tools.

A peculiarity of most OR techniques is that solutions are not generally obtained in (formula-like) closed forms.

Instead, they are determined by algorithms. An algorithm provides fixed computational rules that are applied repetitively to the problem with each repetition (called iteration) moving the solution closer to the optimum. Because the computations associated with each iteration are typically tedious and voluminous, it is imperative that these algorithms be executed on the computer. .

Some mathematical models may be so complex that it is impossible to solve them by any of the available optimization algorithms. In such cases, it may be necessary to abandon the search for the optimal solution and simply seek a good solution using heuristics or rules of thumb.

Generally three types of methods are used for solving OR models.

Analytic Method: If the OR model is solved by using all the tools of classical mathematics such as: differential calculus and finite differences available for this task, then such type of solutions are called analytic solutions. Solutions of various inventory models are obtained by adopting the so-called analytic procedure.

Iterative Method: If classical methods fail because of complex- ity of the constraints or of the number of variables, then we are usually forced to adopt an iterative method. Such a procedure starts with a trial solution and a set of rules for improving it. The trial solution is then replaced by the improved solution, and the process is repeated until either no further improvement is possible or the cost of further calculation cannot be justified.

Iterative method can be divided into three groups:

a).After a finite number of repetitions, no further improvement will be possible.

b)Although successive iterations improve the solutions, we are only guaranteed the solution as a limit of an infinite process.

c)Finally we include the trial and error method, which, however, is likely to be lengthy, tedious, and costly even if electronic computers are used.

The Monte-Carlo Method: The basis of so-called Monte- Carlo technique is random sampling of variable's values from a Distribution of that variable. Monte-Carlo refers to the use of sampling methods to estimate the value of non-stochastic variables

Steps of Monte-Carlo method:.

The following are the main steps of Monte-Carlo method:

Step 1. In order to have a general idea of the system, we first draw a flow diagram of the system.

Step 2. Then we take correct sample observations to select some suitable model for the system. In this step we compute the probability distributions for the variables of our interest.

Step 3. We, then, convert the probability distributions to a cumulative distribution function. Step 4. A sequence of random numbers is now selected with the help of random number tables.

Step 5. Next we determine the sequence of values of variables of interest with the sequence of random numbers obtained instep 4.

Step 6. Finally we construct some standard mathematical function to the values obtained in step 5.Step 3. Step 4.Step

"Operations Research (Management Science) is a scientific approach to decision making that seeks to best design and operate a system, usually under conditions requiring the allocation of scarce resources."

A system is an organization of interdependent components that work together to accomplish the goal of the system.

THE METHODOLOGY OF OR

When OR is used to solve a problem of an organization, the following seven step procedure should be followed:

Step 1. Formulate the Problem

OR analyst first defines the organization's problem. Defining the problem includes specifying the organization's objectives and the parts of the organization (or system) that must be studied before the problem can be solved.

Step 2. Observe the System

Next, the analyst collects data to estimate the values of parameters that affect the organization's problem. These estimates are used to develop (in Step 3) and evaluate (in Step4 a mathematical model of the organization's problem. Step 3. Formulate a Mathematical Model of the Problem

The analyst, then, develops a mathematical model (in other words an idealized representation) of the problem. In this class, we describe many mathematical techniques that can be used to model systems.

Step 4. Verify the Model and Use the Model for Prediction

The analyst now tries to determine if the mathematical model developed in Step 3 is an accurate representation of reality. To determine how well the model fits reality, one determines how valid the model is for the current situation.

Step 5. Select a Suitable Alternative

Given a model and a set of alternatives, the analyst chooses the alternative (if there is one) that best meets the organization's objectives.

Sometimes the set of alternatives is subject to certain restrictions and constraints. In many situations, the best alternative may be impossible or too costly to determine.

Step 6. Present the Results and Conclusions of the Study

In this step, the analyst presents the model and the recommendations from Step 5 to the decision making individual or group. In some situations, one might present several alternatives and let the organization choose the decision maker(s) choose the one that best meets her/his/their needs. After presenting the results of the OR study to the decision maker(s), the analyst may find that s/he does not (or they do not) approve of the recommendations. This may result from incorrect definition of the problem on hand or from failure to involve decision maker(s) from the start of the project. In this case, the analyst should return to Step 1, 2, or 3.

Step 7. Implement and Evaluate Recommendation

If the decision maker(s) has accepted the study, the analyst aids in implementing the recommendations. The system must be constantly monitored (and updated dynamically as the environment changes) to ensure that the recommendations are enabling decision maker(s) to meet her/his/their objectives.

LINEARPROGRAMMING

Introduction

Observe the world and predictable patterns will emerge. When air pressure falls, bad weather often follows. High and low tides relate to the setting and rising of the moon. Hair and eye color pass from parent to child. Patterns emerge from systems as small as the atom and as large as the universe. Mathematics allows us to express these patterns as equations, and thereby to predict future changes. Linear algebra, the mathematics of systems of equations, further allows complex interactions within a system to be expressed as a unified entity representing the entire system all at once: a matrix.

Linear algebra provides the methods necessary to analyze unwieldy systems. Insight into information such as the stresses and strains at multiple locations in a building, harmonic frequencies of an aircraft design, and how to maximize and minimize many individual aspects of an entire system allows the development of larger and more complex designs. Of special interest to industry, even after 40 years of intense research, is the ability to optimize.

Linear Programming: Linear programming is a powerful quantitative technique (or operational research technique) designs to solve allocation problem. The term 'linear programming' consists

of the two words 'Linear' and 'Programming'. The word 'Linear' is used to describe the relationship between decision variables, which are directly proportional. For example, if doubling (or tripling) the production of a product will exactly double (or triple) the profit and required resources, then it is linear relationship. The word 'programming' means planning of activities in a manner that achieves some 'optimal' result with available resources. A programme is 'optimal' if it maximizes or minimizes some measure or criterion of effectiveness such as profit, contribution (i.e. sales-variable cost), sales, and cost. Thus, 'Linear Program- ming' indicates the planning of decision variables, which are directly proportional, to achieve the 'optimal' result considering the limitations within which the problem is to be solved.

Decision Variables:

The decision variables refer to the economic or physical quantities, which are competing with one another for sharing the given limited resources. The relationship among these variables must be linear under linear programming. The numerical values of decision variables indicate the solution of the linear programming problem

Objective Function

The objective function of a linear programming problem is a linear function of the decision variable expressing the objective of the decision maker. For example, maximization of profit or contribution, minimization of cost/time.

Constraints

The constraints indicate limitations on the resources, which are to be allocated among various decision variables. These resources may be production capacity, manpower, time, space or machinery. These must be capable of being expressed as linear equation (i.e. =) on inequalities (i.e. > or<; type) in terms of decision variables. Thus, constraints of a linear programming problem are linear equalities or inequalities arising out of practical limitations.

Non-negativity Restriction

Non-negativity restriction indicates that all decision variables must take on values equal to or greater than zero.

Divisibility

Divisibility means that the numerical values of the decision variables are continuous and not limited to integers. In other words, fractional values of the decision variables must be permissible in obtaining optimal solution.

FORMULATION OF LP PROBLEMS

Step Involved in the Formulation of LP Problem

The steps involved in the formation of linear programming problem are as follows:

Step 1: Identify the Decision Variables of interest to the decision maker and express them as X_1, X_2, X_3

Step 2: Ascertain the Objective of the decision maker whether he wants to minimize or to maximize.

Step 3: Ascertain the cost (in case of minimization problem) or the profit (in case of maximization problem) per MODULE of each of the decision variables.

Step 4: Ascertain the constraints representing the maximum availability or minimum commitment or equality and represent them as less than or equal to (≤ type inequality or greater than or equal to (>≥) type inequality or ‘equal to’ (=) type equality respectively.

Step 5: Put non-negativity restriction as under: $X_j > 0$; j = 1, 2… n (non-negativity restriction) Step 6: Now formulate the LP problem as under:

Example-1: (Production allocation problem): A firm manufactures two types of products A and B and sells them at a profit of Rs. 2 on type A and Rs 3 on type B. Each product is processed on two machines G and H. Type A requires one minute of processing time on G and two minutes on H; type B requires one minute on G and one minute on H. the machine G is available for not more than 6 hour 40 minutes while machine H is available for 10 hours during any of working day. Formulate the problem as a linear programming problem.

Formulation: let X_1 be the number of products of type A

X_2 the number products of type B.

After carefully understanding the problem, the given information can be systematically arranged in the form of the following table:

Table-1

	time of products (minutes)		Available time (minutes)
	TypeA (X_1 MODULEs)	Type B (X_2 MODULEs)	
G	1	1	400
H	2	1	600
Profit per MODULE	Rs 2	Rs 3	

Since the profit on type A is Rs 2 per product, and Rs 3 on product B.

The total profit for X_1 MODULEs of typew A, and X_2 MODULEs of type B . Z = 2X1 + 3X2 (objective function)

Considering processing time on machine G for processing X_1 MODULEs of type A, and X_2 MODULEs of type B = $1 X_1 + 1 X_2$

but machine G available time is – 400 minutes

$1 X_1 + 1 X_2 \leq 400$ ------ eqn-1 (first constraint)

Considering processing time on machine H for processing X1 MODULEs of type A, and X2 MODULEs of type B = $2 X_1 + 1 X_2$

but machine H available time is – 600 minutes

$1 X_1 + 1 X_2 \leq 600$ ------ eqn-2 (second constraint)

Since it is not possible to produce negative quantities

$X_1 \geq 0, \quad X_2 \geq 0$ (non negativity restrictions

Hence the allocation problem of the firm ca be finally put in the form

Find X_1 and X_2 such that the profit Z = $2X_1 + 3X_2$ is maximum
subject to the conditions:

$$1 X_1 + 1 X_2 \leq 400 ,$$
$$1 X_1 + 1 X_2 \leq 600$$
$$X_1 \geq 0, \quad X_2 \geq 0$$

Example-2. A company produces two types of Hats. Each hat of first type requires twice as much labour than the second type. If all the hats are of the second type only, the company can produce a total of 500 hats a day. The material limits the daily sales of the first and second type to 150 and 250 hats . Assuming that the profits per hat are Rs8 for type A and Rs 5 for type B, formulate the problem as a linear programming model in order to determine the number of hats of each type so as to maximize the profit.

Formulation: let X_1 be the number of MODULEs of hat type A

X_2 be the number of MODULEs of hat type B.

After carefully understanding the problem, the given information can be systematically arranged in the form of the following table:

Table-1

Machine	time of products (minutes)		Available time (minutes)
	Type A (X_1 MODULEs)	Type B (X_2 MODULEs)	
hat type A	2t	1t	500 t
Hat Type B	2	1	600
Profit pper MODULE	Rs 8	Rs 5	

Since the profit on hat of type A is Rs 8 per MODULE, , and Rs 5 on hat of type B.

The total profit for X1 MODULEs of type A, and X2 MODULEs of type B . $Z = 8X_1 + 5X_2$ (objective function)

Since the company can produce a at the most 500 hats in a day and a type of hat requires twice as mach time as that of hat type B, the production restriction is given by $2*t\ X_1 + 1*t\ X_2 \leq 500\ t$ where t is the labour time per MODULE of second type ie

$2X_1 + X_2 \leq 500$ ----- constraint-1

Considering limitations on the sale of hats

$X_1 \leq 150$

$X_2 \leq 250$

Also , since the company cannot produce negative quantities

$X_1 \geq 0, \quad X_2 \geq 0$ (non negativity restrictions

Hence the allocation problem of the firm ca be finally put in the form

Find X_1 and X_2 such that the profit $Z = 8X_1 + 5X_2$ is maximum subject to the restrictions:

$$2X_1 + X_2 \leq 500$$
$$X_1 \leq 150$$
$$X_2 \leq 250$$
$$X_1 \geq 0, \quad X_2 \geq 0$$

E production plan for medicines A and B. There are sufficient ingredients available to make 20,000 bottles of medicine A and 40,000 bottles of medicine B, but there are only 45,000 bottles into which either of the medicines can be filled. Further it takes three hours to prepare enough material to fill 1000 bottles of medicine A and 1 hour to prepare enough material to fill 1000 bottles of medicine B, and there are 66 hours available for this application. The profit is Rs 8 per bottle for medicine A and Rs 7 per bottle for medicine B. Further, it takes three hours to prepare enough material to fill1000 bottles of medicine A and one hour to prepare enough material to fill 1 000 bottles of medicine B, and there are 66hours available for this operation. The profit is Rs. 8 per bottle for medicine A and Rs. 7 per bottle for medicine B.

i. Formulate this problem as a LP.P. ,
ii. How the manufacturer schedule his production in order to maximize profit.

Formulation.

(i) Suppose the manufacturer produces X_I and X_2 thousand of bottles of medicines A and B, respectively. Since it takes three hours to prepare 1000 bottles of medicine A,

The time required to fill X_I thousand bottles of medicine A will be $3X_1$ hours. Similarly, the time required to prepare X_2thousand bottles of medicine B will be X_2 hours. Therefore, total time required to prepare X_I thousand bottles of medicine A and X_2 thousand bottles of medicine B will be $3X_1 + 1.X_2$.

Since the total time available for this operation is 66 hours,

The constraint becomes $3X_1 + 1.X_2 . \leq 66$

Since there are only 45 thousands bottles available for filling medicines A and B, the constraint becomes $X_1 + X_2 . \leq 45$

Considering ingredient availability, possible to make 20 thousands of medicine A and 40 thousands bottles off medicine B, the constraints becomes

$$X_1 \leq 20$$
$$X_2 \leq 40$$

Number of bottles being non-negative, $X_1 \geq 0$, $X_2 \geq 0$

Considering profit: profit at the rate of Rs 8 per bottle for type A medicine and Rs 7 per bottle for type B medicine,

The total profit on X_1 thousands of medicine A and X_2 thousands of medicine B will become

Profit $Z = 8*1000\, X_1 + 7*1000 X_2 = 8000\, X_1 + 7000 X_2$

Hence the allocation problem of the firm can be finally put in the form:

Find X_1 and X_2 such that the profit $Z = 8000X_1 + 7000X_2$ is maximum
subject to the restrictions:

$$3X_1 + 1.\, X_2 \leq 66$$

Exa ... doll-A and a deluxe version doll-B. each ... , and the company would have time to make a maximum of 2000 per day. The supply of plastic is sufficient to produce 1500 dolls per day (both A and B combined). The deluxe version requires a fncy dress of which there are only 600 per day available. If the company makes a profit of Rs 3 and Rs 5 per doll, respectively on doll A and B, then how many of each doll should be produced per dy to maximize the total profit. Formulate the problem.

Formulation: let X_1 be the number of doll A.

X_2 the number products of doll type B.

After carefully understanding the problem, the given information can be systematically arranged in the form of the following table:

Machine			Availability
	Doll A (X_1 MODULEs)	Type B (X_2 MODULEs)	
Time in hours	t	2t	2000 t
Plastic material	2	1	1500
Dress constraint	-	1	600
Profit pper MODULE	Rs 3	Rs 5	

Since the profit on doll A is Rs 3 per product, and Rs 5 on doll B.

The total profit for X1 MODULEs of doll A, and X2 MODULEs of doll B . $Z = 3X1 + 5X2$ (objective function)

Considering processing time: let the doll A require t hours, so that the doll B require 2t hours. Since the total time to manufacture X_1 MODULEs of doll A, and X_2 MODULEs of doll B . , should not exceed 2000t hours. Therefore $t X_1 + 2 X_2 \leq 2000 t$

Considering plastic material constant for X_1 MODULEs of doll A, and X_2 MODULEs of doll B

$1 X_1 + 1 X_2 \leq 1500$ ------ eqn-2 (second constraint)

Considering fancy dress material constraint $X_2 \leq 600$

Since it is not possible to produce negative quantities of dolls

$X_1 \geq 0, \quad X_2 \geq 0$ (non negativity restrictions

Hence the allocation problem of the firm can be finally put in the form

Find X_1 and X_2 such that the profit $Z = 3X_1 + 5X_2$ is maximum subject to the conditions:

$$X_1 + 2X_2 \leq 2000,$$
$$X_1 + 1X_2 \leq 1500$$
$$X_2 \leq 600$$
$$X_1 \geq 0, \quad X_2 \geq 0$$

C. three kinds of wool are required for it, say: red, green and blue wool. One MODULE length of type A cloth needs 2 meters of red wool and 3 meters of blue wool; one MODULE length of type B cloth needs 3 meters of red wool, 2 meters of green wool and 2 meters of blue wool; and one MODULE of type C cloth needs 5 meters of green wool and 4 meters of blue wool. The firm has only a stock of 8 meters of red wool, 10 meters of green wool and 15 meters of blue wool. It is assumed that the income obtained from one MODULE length of type A cloth is Rs 3.00, of type B cloth is Rs 5.00, and of type C cloth is Rs 4.00. Determine, how the firm should use the available material so as to maximize the income from the finished cloth.

Formulation: after understanding the problem carefully it is convenient to represent the data in following table.

Quality of wool	Availability of wool (in meters)			
	A (X_1 MODULEs)	B (X_2 MODULEs)	C (X_2 MODULEs)	Availability of wool
Red	2	3	0	8
Green	0	2	5	10
Blue	3	2	4	15
Income per MODULE length of cloth	Rs 3	Rs 5	4	

let X_1, X_2, , X_3 be quantity (in meters) produced of type A, B, C respectively.

Since the profit on type –A is Rs 3 per product, and Rs 5 on type-B, and Rs 4 on type-C.

The total profit for X_1 MODULEs of type- A, and X_2 MODULEs of type-B and X_3 MODULEs of type-C .

$Z = 3X1 + 5X2 + 4X_3$ (objective function)

Since 2 meters of red wool are required for each meter of cloth A and 3 meters of this type of cloth are required for each meter of cloth B and zero meter are required for each meter of cloth C.

Thus total quantity of red wool becomes: $2X_1 + 3X_2 + 0X_3$

Considering red wool availability of 8 meters, the constraint becomes

$2X_1 + 3X_2 + 0X_3 \leq 8$ --------- eqn-1

Considering green wool availability the constraint becomes : $0.X_1 + 2X_2 + 5X_3 \leq 10$ ----eqn-2

Considering blue wool availability the constraint becomes : $3.X_1 + 2X_2 + 4X_3 \leq 15$ ----- eqn-3

Since it is not possible to produce negative quantities of product A, B and C

$X_1 \geq 0$, $X_2 \geq 0$ $X_3 \geq 0$ (non negativity restrictions)

Hence the allocation problem of the firm ca be finally put in the form

Find X_1 and X_2 such that the profit $Z = 3X_1 + 5X_2$ is maximum
subject to the conditions:

$2X_1 + 3X_2 + 0X_3 \leq 8$

$2X_2 + 5X_3 \leq 10$

$3X_1 + 2X_2 + 4X_3 \leq 15$

and $X_1 \geq 0$, $X_2 \geq 0$, $X_3 \geq 0$

L P GRAPHICAL SOLUTION

Linear programming graphical Method :

Graphical Method is used for solving linear programming problems that involve only two variables.

Closed Half Plane

A linear inequality in two variables is known as a half plane. The corresponding equality or the line is known as the boundary of the half plane. The half plane along with its boundary is called a closed half plane. Thus, a closed half plane is a linear inequality in two variables, which include the value of the variable for which equality is attained.

Feasible Solution

Any non-negative solution which satisfies all the constraints is known as a feasible solution of the problem.

Feasible Region

The collection of all feasible solutions is known as a feasible region.

Convex Set

A set (or region) is convex if only if for any two points on the set, the line segment joining those points lies entirely in the set. Thus, the collection of feasible solutions in a linear program- ming problem form a convex set. In other words, the feasible region of a linear programming problem is a convex set.

Convex Polygon

A convex polygon is a convex set formed by the intersection of a finite number of closed half planes.

Extreme Points or Vertexes or Corner Points

The extreme points of a convex polygon are the points of intersection of the lines bounding the feasible region. The value of the decision variables, which maximize or minimize the objective function is located on one of the extreme points of the convex polygon. If the maximum or minimum value of a linear function defined over a convex polygon exists, then it must be on one of the extreme points.

Redundant Constraint

Redundant constraint is a constraint, which does not affect the feasible region.

Multiple Solutions

Multiple solutions of a linear programming problem are solutions each of which maximize or minimize the objective function. Under graphical method, the existence of multiple solutions is indicated by a situation under which the objective function line coincides with one of the half planes generated by a constraint. In other words, where both the objective function line and one of constraint lines have the same slope.

Unbounded Solution

An unbounded solution of a linear programming problem is a solution whose objective function is infinite. A linear program- ming problem is said to have unbounded solution if its solution can be made infinitely large without violating any of the constraints in the problem. Since there is no real applied problem, which has infinite returns, hence an unbounded solution always represents a problem that has been incorrectly formulated.

For example, in a maximization problem at least one of the constraints must be an equality' or 'less than or equal to' (<) type. If all of the constraints are 'greater than or equal to' (>) type, then there will be no upper limit on the feasible region. Similarly for minimization problem, at least one of constraints must be an 'equality' or 'a greater than or equal to' type (>) if a solution is to be bounded.

Under graphical method, the feasible solution region extends indefinitely.

Infeasible Problem

A linear programming problem is said to be infeasible if there is no solution that satisfies all the constraints. It represents a state of inconsistency in the set of constraints.

Practical Steps Involved in Solving LPP By Graphical Method

Simple linear programming problems of two decision variables can be easily solved by graphical method. The outlines of graphical procedure are as follows:

Step 1: Consider each inequality constraint as equation.

Step 2: plot each equation on he graph, as each will geometrically represent a straight line.

\step-3. Shade the feasible region. Every point on the line will satisfy the equation of the line. If the inequality-constraint corresponding to that line is '<=', then the region below the line lying

in the first quadrant (due to non-negativity of variables) is shaded. For the equality –constraint with ' >=' sign, the region above he line in the first quadrant is shaded. The points lying in the

common region will satisfy all the constraints simultaneously. The common region thus obtained is called the feasible region.

Step-4. Choose the convenient value of Z (say =0) and plot the objective function line.

Step-5. Pull the objective function line until the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin and passing through at least one carner of he feasible region. In the minimization case, , the line , this line will stop nearest to the origin and passing through at least one corner of the feasible region.

Step-6. Read the coordinates of the extreme point (s) selected in step-5, and find the maximum or minimum (as the case may be) value of Z.

Example-1. Find a geometrical interpretation and a solution as well for the following problem.(Graphical method)

Maximize $Z = 3X_1 + 5X_2$, subjective to restrictions:

$X_1 + 2X_2 \leq 2000$, , $X_1 + X_2 \leq 1500$, , $X_2 \leq 600$ and $X_1 \geq 0$, , $X_2 \geq 0$,

Graphical Solution.

Step-1. To graph the inequality constraints, consider two mutually perpendicular lines OX_1 and OX_2 as axix of coordinates. Since $X_{1,} X_2 \geq 0$ we have to draw in first quadrant only.

To plot the $X_1 + 2X_2 \leq 2000$, consider the equation $X_1 + 2X_2 = 2000$, for $x_1 = 0$, $X_2 = 1000$; $X_2 = 0$, $X_1 = 2000$

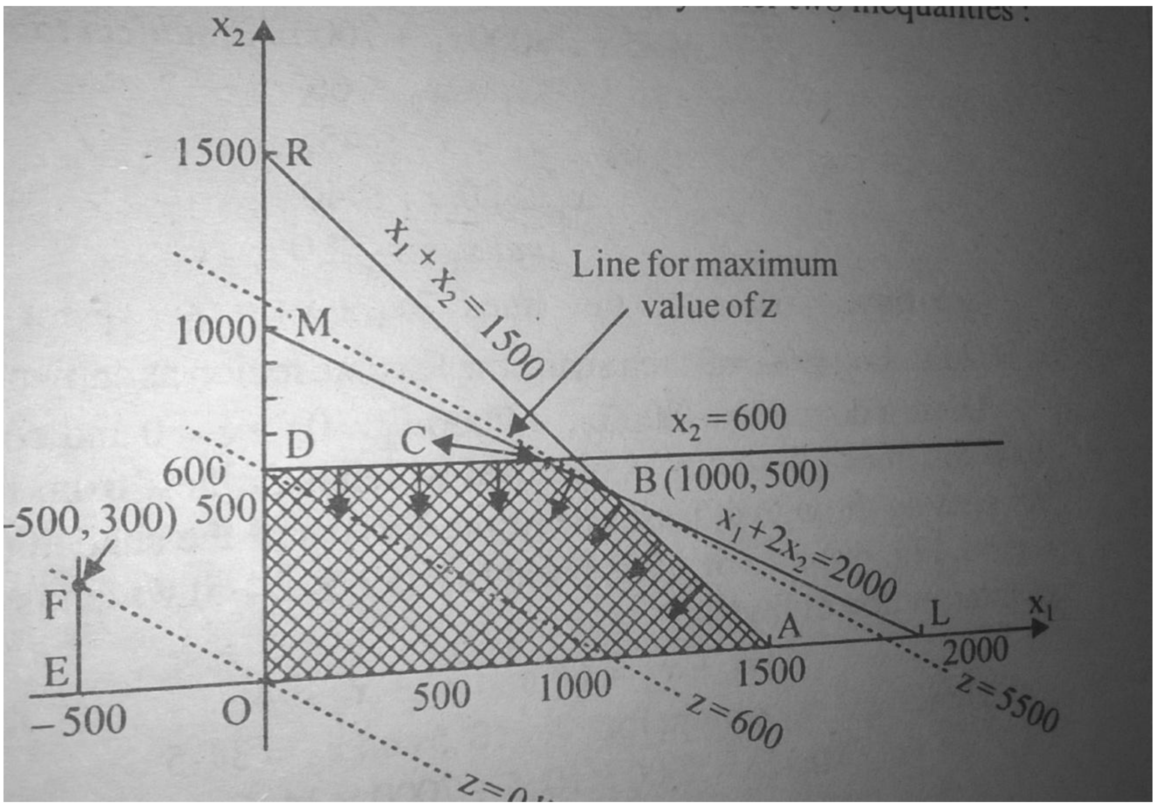
Take appropriate scale and reprensent the points and draw the line of $X_1 + 2X_2 = 2000$, since constraint is of $\leq$ type, consider region toward the origin I as feasible region.

Consuder second constraint $X_1 + X_2 \leq 1500$, , fo

when $X_1 = 0$,	$X_2 = 1500$
when $X_2 = 0$	$X_1 = 1500$

Consider third constraint $X_2 \leq 600$, $X_2 = 600$, when $X_1 = 0$, $X_2 = 600$

Similarly plot the all the straight lines and consider common n region towards origin satisfying all the all the constraints as feasible region.



Step-2. Find the feasible region or solution space by combining all constraints. A common shaded area of OABCD is obtained which is feasible solution to the given LPP. Which satisfies all the constraints?

Step-3. Find the coordinates of the corner points of feasible region O, A, B, C, D

Step-4. Locate the corner point of optimal solution either by calculating the value of Z for each corner point o, A, B, C and D.

Point A (0, 1500) $Z = 3x_1 + 5x_2$ = 0+ 7500) = 7500

At point B (1000, 500), Z =(3*1000 + 5*500) = **5500 maximum**

At point C (800,600), Z= (3*800 + 5*600) = 5400

At point D (0, 600), Z= (3 *0 + 5* 600) = 3000

At point O (0, 0) Z = (3*0 + 5*0) = 0

In this problem, maximum value of Z is attained at the corner point B (1000, 500), which is poin of intersection of lines -

$X_1 + 2X_2 = 2000$, and $X_1 + X_2 \leq 1500$,

Hence the required solution is $X_1 = 1000$, $X_2 = 500$ and maximum value Z = Rs 5500

ANOTHER METHOD TO FIND OPTIMUM POINT:

To find the point or points in the feasible region . for some fixed value of Z, Z $= 3X_1 + 5X_2$ is a straight line and any point on it gives the same value of Z

Considering Z=3000, $3X_1 + 5X_2 = 3000$ when x1 = 0, X2= 600

When x2=0 X1 = 1000

Represent the objective function line also on the feasible region graph. Move th objective function straight line parallely such that it touches the extreme poit on the feasible region.. it touches point C, where x1 = 1000, X2= 500, maximum objective function value =Zmax = 5500.

Example-2.

Solve by graphical method Max $Z = X_1 + X_2$ Subject to

$X_1 + 2X_2 \leq 2000$, , $X_1 + X_2 \leq 1500$, , $X_2 \leq 600$ and $X_1 \geq 0$, , $X_2 \geq 0$,

Considering First constraint as equation $X_1 + 2X_2 = 2000$,

When [$X_1 = 0$, $X_2 = 1000$] and When $X_2 = 0$, $X_1 = 2000$,

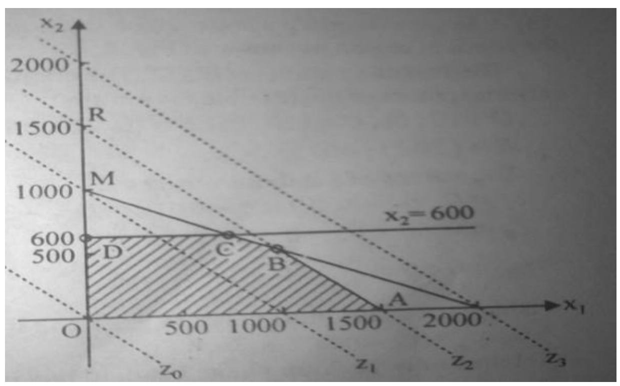
Considering the second constraint as equation , $X_1 + X_2 = 1500$

when [X_1 = 0, X_2 = 1500] & { When X_2 =0 , X_1 =1500]

The feasible region is similar to earlier problem.

It is clear from figure that Z is maximum and it lies along the edge AB of the polygon of the feasible solutions. This indicates that the values of X_1, X_2, which maximize Z are not unique., but any point on the edge AB of OABCD polygon will give the optimum value of Z. the maximum value of Z is always unique, but there will be an infinite number of feasible solutions which give unique value of 'Z'. thus , two corners A and B as well as any other point on the line AB (segment) will give optimal solution of this problem.

It should be noted that if a linear programming problem has mre than one optimum solution, there exists alternative optimum solutions.. and one of the optimum solutions will be corresponding to corner point B. ie

$_1$= 1000, X_2= 500 with max. profit Z 

= 1000+500= 1500

Figure-2

Example-3: Solve the following LP problem graphically.

Max Z = 8000 X_1 + 7000X_2, subject to

3 X_1 + X_2 ≤ 66, X_1 + 2X_2 ≤ 45, X_1 ≤ 20, X_2 ≤ 40

and X_1 ≥ 0, , X_2 ≥ 0

Solution: First plot the lines 3 X_1 + X_2 = 66, X_1 + 2X_2 = 45, X_1 = 20 and X_2 = 40 and shade the fusible region as shown in fig-3.

Draw a dotted line 8000 X_1 + 7000X_2= 0 for z-0 and continue to draw the lines till a point is obtained which is farthest from the origin but passing through at least one of the corners of the shaded (feasible) region.

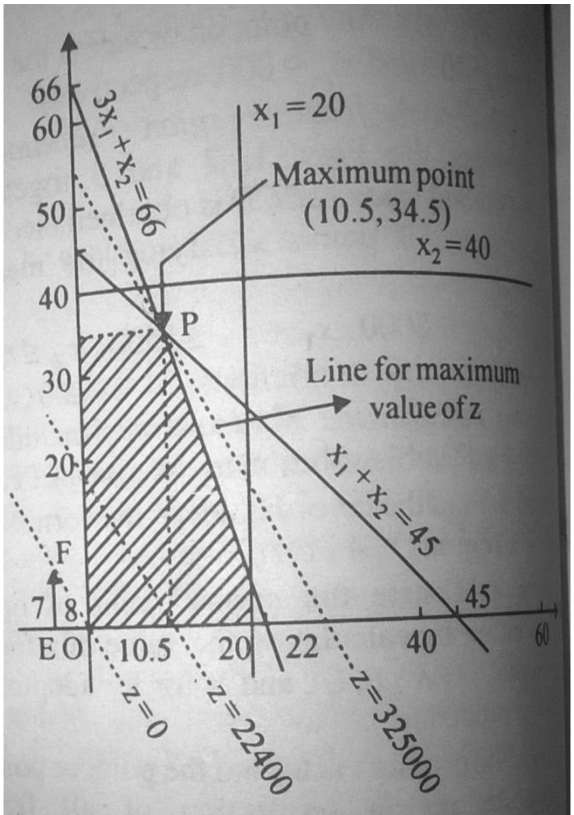


Fig. 6

The figure shows that this point P (10.5, 34.5) which is the intersection of lines

$3\ X_1 + X_2 = 66,$

$X_1 + 2X_2 = 45,$

Hence , Z is maximum for X_1= 10.5 and X_2= 34.5

Max Z = 8000* 100.5 + 7000*34.5 = Rs 325000.

GRAPHICAL METHOD FOR MINIMIZATION PROBLEM:

Consider the problem

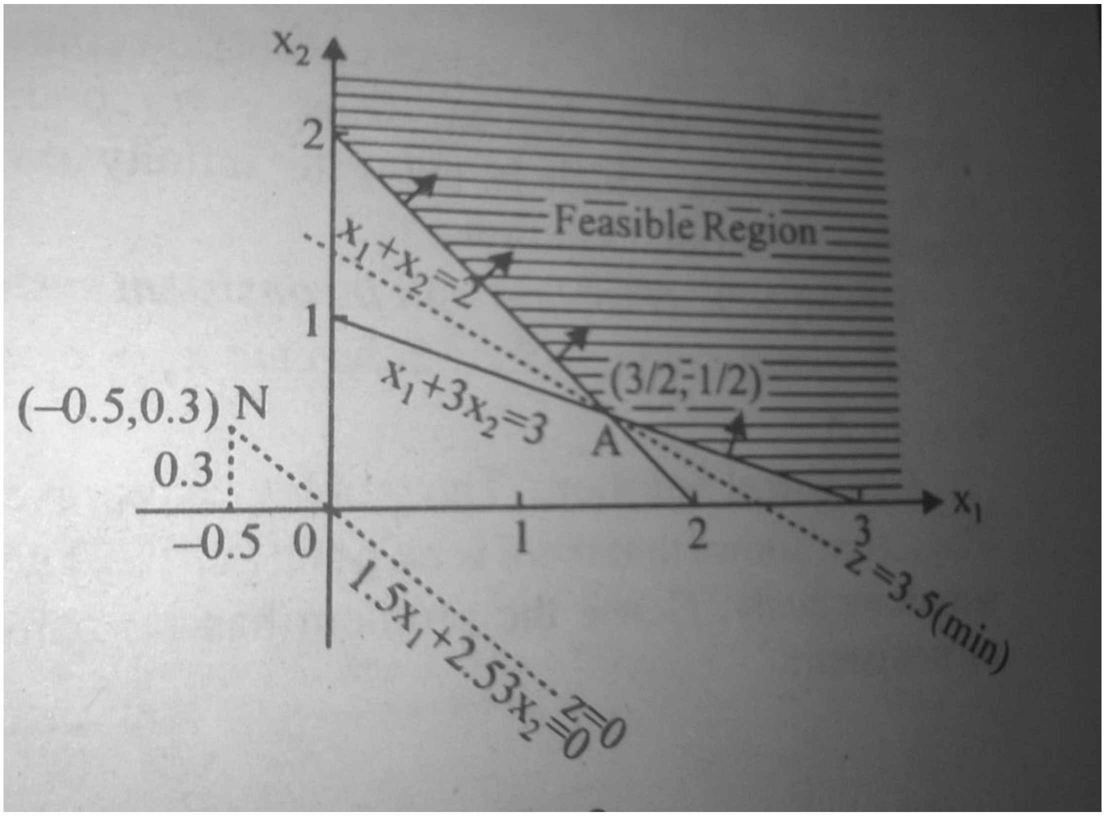
Example-4. Min $Z = 1.5\ X_1 + 2.5\ X_2$ subject to

$X_1 + 3X_2 \geq 3$ $X_1 + X_2 \geq 2$ and $X_1 \geq 0$, , $X_2 \geq 0$

Graphical Solution: The geometrical interpretation of the problem is given in fig-7. The minimum is attained at the point of intersection A of the lines

$X_1 + 3X_2 = 3$ and $X_1 + X_2 = , 2$ .This is the unique point to give the minimum value of Z. now , solving these two equations

$X_1 = 1.5$, $X_2 = 0.5$ and min Z = 3.5



GRAPHICAL SOLUTION IN SPECIAL EXCEPTIONAL CASES

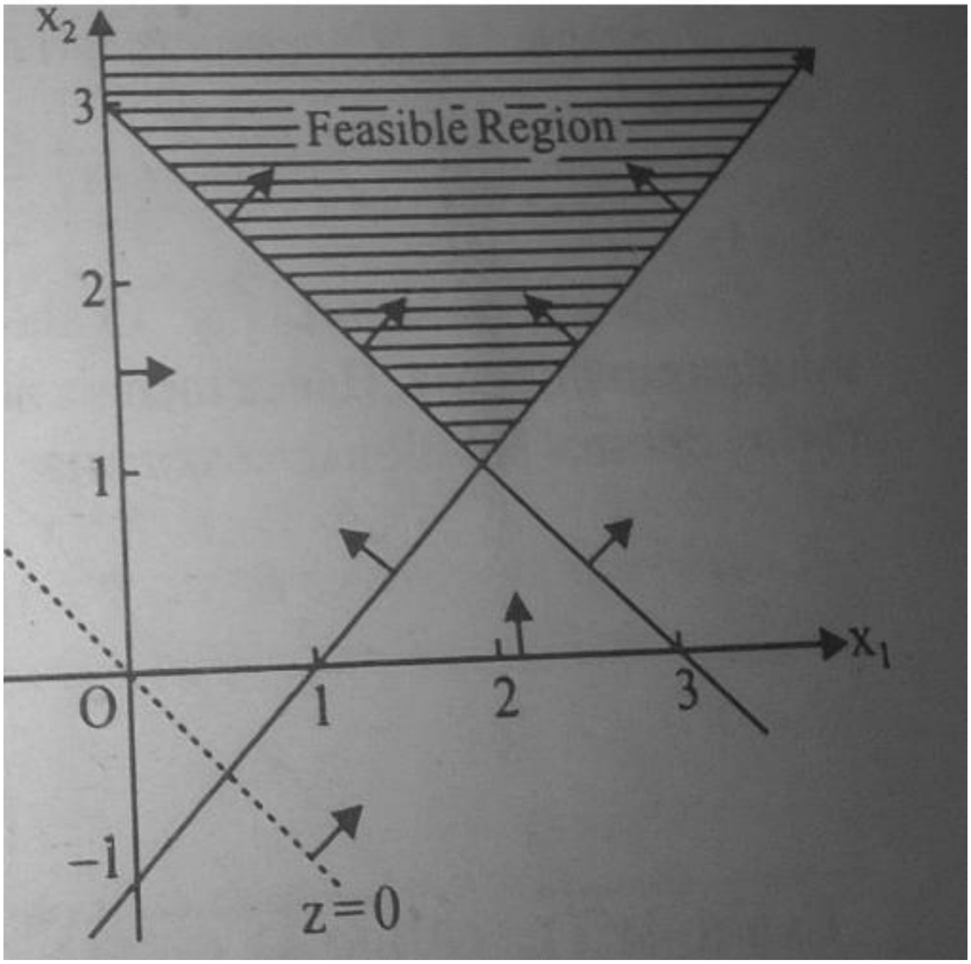
Example -7 .(Problem having unbounded solution)

Max $Z= 3X_1 + 2 X_2$ subject to $X_1 – X_2 \leq 1$, $X_1 + X_2 \geq 3$, and $X_1 \geq 0$, , $X_2 \geq 0$

Graphical Solution: the region of feasible solution is the shaded area as shown in figure-7.

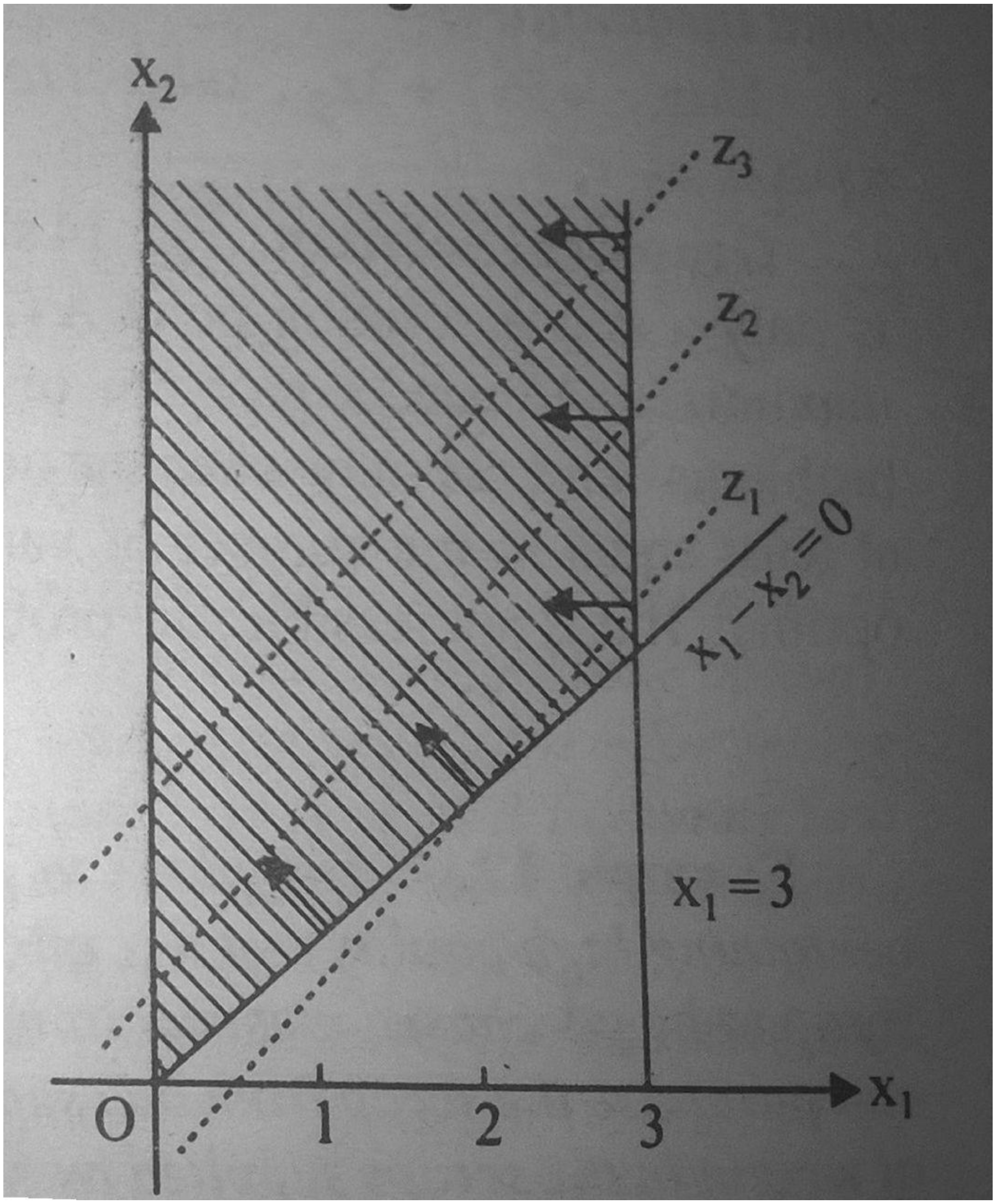
It is clear from the figure that the line representing the objective function can be moved far even parallel to itself in the direction of increasing Z, and still have some points in the region of feasible solutions.

Hence Z can be made arbitrarily large, and the problem has no finite maximum value of Z. such problems are said to have unbounded solutions. Infinite profit in practical problems of LP can not be expected. If LP problem has been formulated by committing some mistake, it may lead to an unbounded solution.



Example-8 Max $Z= -3X_1 + 2 X_2$ subject to $X_1 \leq 3$, $X_1 – X_2 \leq 0$ and X_1 , $X_2 \geq 0$

Graphical Solution: I previous example-7, it has been seen that the both the variables can be made arbitrarily large as Z is increased. In this problem, an unbounded solution does not necessarily imply that all the variables can be made arbitrarily large as Z approaches infinity. Here the variable X1 remains constant as shown in figure.



Example-9. (problem which is not completely normal)

Maximize $Z= -X_1 + 2 X_2$ subject to $- X_1 +X_2 \leq 1$, $- X_1 + 2X_2 \geq 4$ and X_1, $X_2 \geq 0$

Graphical solution: the problem is solved graphically in fig.

Observing graph figure-11., the lne of objective function coincides with the edge of equation-1 of the region of the feasible solutions. Thus every point (X_1, X_2) lying on this edge ($- X_1 + 2X_2 = 4$), which is going infinity on the right gives Z=4 and is therefore an optimal; solution.

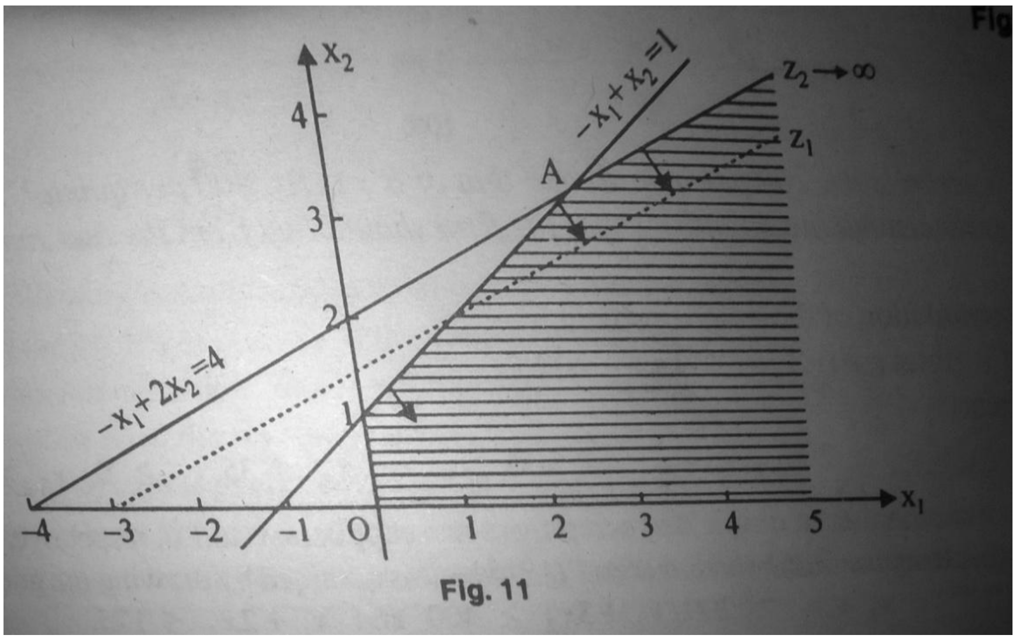
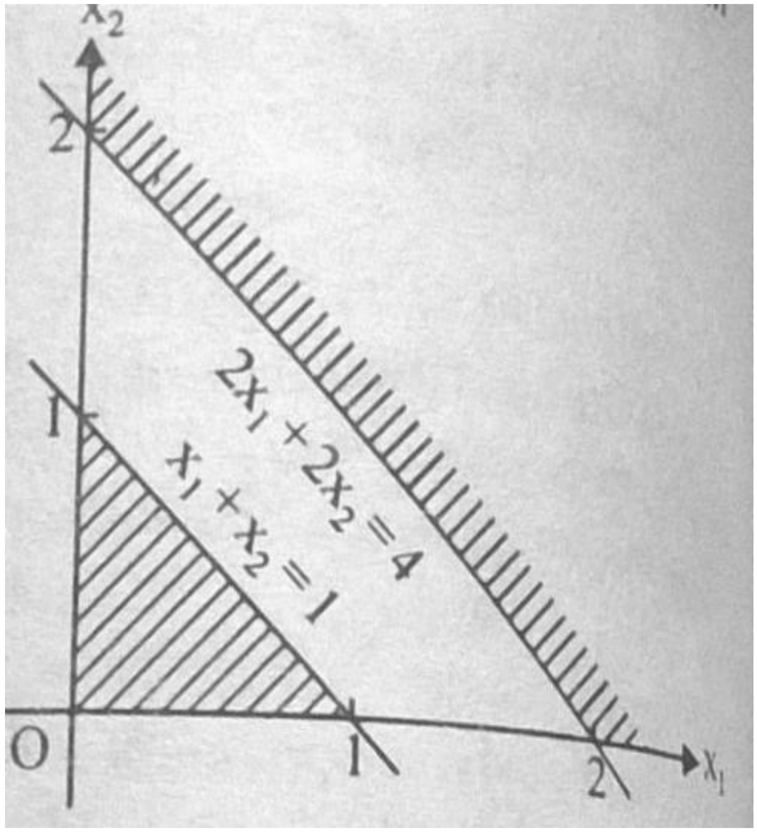


Fig. 11

Example-10. (Problem with inconsistent system of constraints)

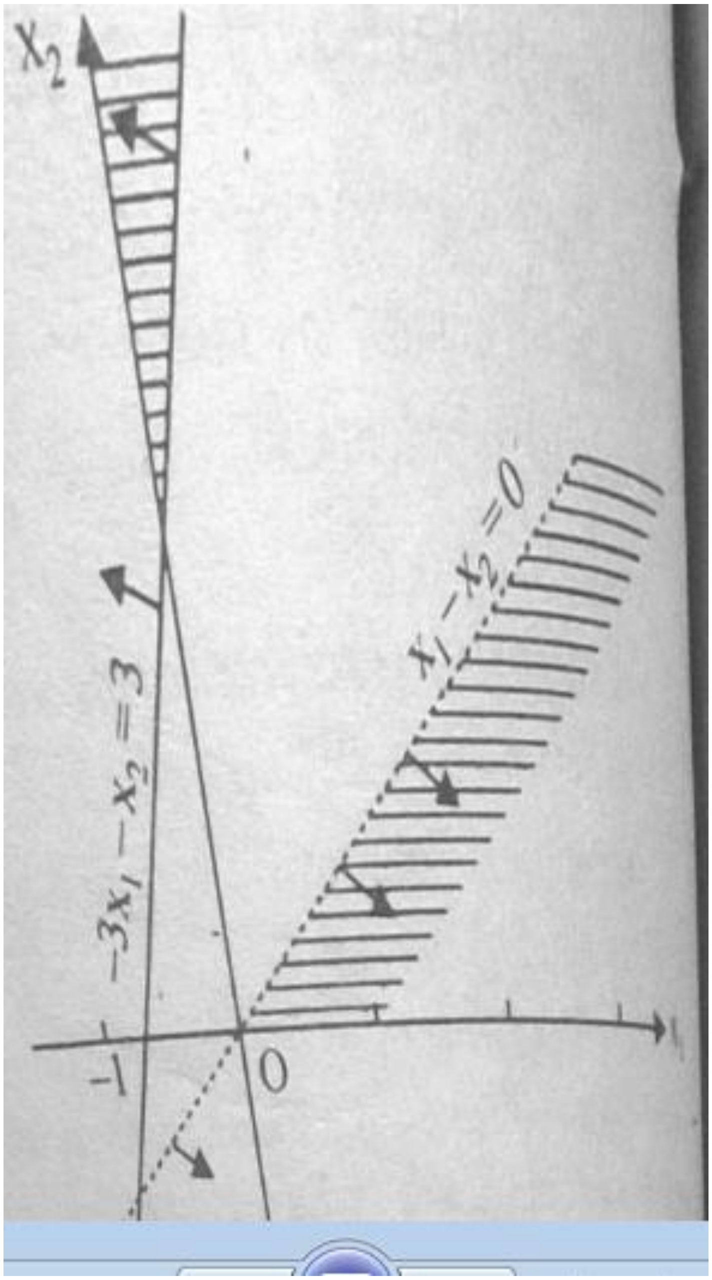
Maximize $Z= 3X_1 - 2 X_2$ subject to $X_1 +X_2 \leq 1$, $2X_1 + 2X_2 \geq 4$ and X_1, $X_2 \geq 0$

Graphical solution: the problem is represented graphically in fig-12. The figure shows that there is no point (X_1, X_2) which satisfies both the constraints simultaneously. Hence the problem has no solution because the constraints are inconsistent.



Example-13. (Constraint can be consistent and yet thee may be no solution)

Maximize $Z = X_1 + X_2$ subject to $X_1 - X_2 \geq 0$, $-3X_1 + X_2 \geq 3$, and X_1 , $X_2 \geq 0$



Graphical solution: the figure shows that there is no region of feasible solutions in this case. Hence there is no feasible solution. So the question of having optimal solution does not arise.

Example-12. (Problem in which constraints are equations rather than inequalities)

Maximize $Z = 5X_1 + 3X_2$ subject to $3X_1 + 5X_2 = 15$, , $5X_1 + 2X_2 = 10$, and X_1 , $X_2 \geq 0$

Graphical solution: fig-14 shows the graphical solution. Since there is only a single solution point A (20/19, 45/19), thee is nothing to be maximized. Hence , a problem of this kind is of no importance. Such problems can arise only when the number of equations in the constraints is at least equal to the number of variables. If the solution is feasible,

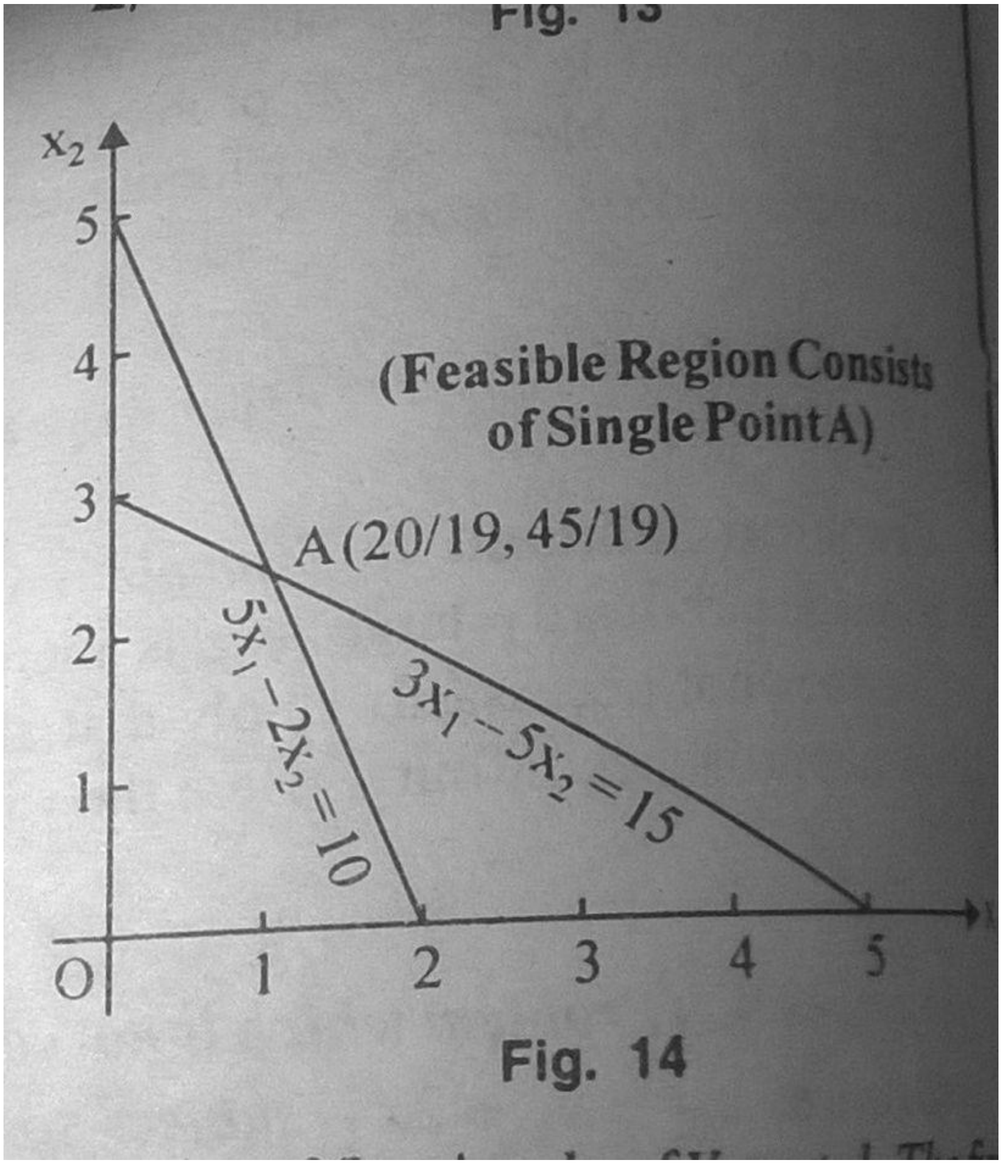


Fig. 14

it is optimal,

INTRODUCTION TO SIMPLEX METHOD

The Simplex Method is a systematic procedure for generating and testing candidate vertex solutions to a linear program.” (Gill,Murray,andWright,p.337). It begins at an arbitrary corner of the solution set. At each iteration, the Simplex Method selects the variable that will produce the largest change towards the minimum (or maximum) solution. That variable replaces one of its compatriots that is most severely restricting it, thus moving the Simplex Method to a different corner of the solution set and closer to the final solution. In addition, the Simplex Method can determine if no solution actually exists.

The Simplex Method solves a linear program of the form described in figure. Here, the coefficients c_j represent the respective weights, or costs, b_i of the variables x_j . The minimized statement is similarly called the cost of the solution. a_{ij} The coefficients of the system of

equations are represented by, and any constant values in the system of equations are combined on the right-hand side of the inequality in the variables

. Combined, these statements represent a linear program, to which we seek a solution of minimum cost.

$$\text{minimize} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum a_{ij} x_j \leq b_i \quad (i = 1,2,\ldots,m < n)$$

$$x_j \geq 0 \quad (j = 1,2,\ldots,n)$$

This linear program involves solution of the set of equations.

If no solution to the set of equations is yet known, slack variables, x_{n+1}, x_{n+2}, ……, x_{n+m}, adding no cost to the solution are introduced. The initial basic feasible solution (BSF) will be the solution of the linear program.

THE SIMPLEX METHOD DEFINITIONS

Objective Function: The function that is either being minimized or maximized. For example, it may represent the cost that you are trying to minimize.

Optimal Solution: A vector *x*, which is both feasible (satisfying the constraints) and optimal (obtaining the largest or smallest objective value).

Constraints: A set of equalities and inequalities that the feasible solution must satisfy.

Feasible Solution: A solution vector, *x,* which satisfies the constraints**.**

Basic Solution: *x* of (*Ax=b*) is a *basic solution* if the *n* components of *x* can be partitioned in to *m* “basic” and *n-m*“ non-basic” variables in such a way that: the*m* columns of *A* corresponding to the basic variables forma nonsingular basis and the value of each “non-basic” variable is 0.

The constraint matrix *A* has *m* rows (constraints) and *n* columns (variables).

Basis: The set of basic variables.

Basic Variables: A variable in the basic solution (value is not 0).

Non-basic Variables: A variable not in the basic solution (value = 0).

Slack Variable: A variable added to the problem to eliminate less than constraints.

Surplus Variable: A variable added to the problem to eliminate greater-than constraints.

Artificial Variable: A variable added to a linear program in phase 1 to aid finding a feasible solution.

Unbounded Solution: For some linear programs it is possible to make the objective arbitrarily small (with out bound).Such an LP is said to have an un bounded solution.

The Simplex Algorithm

Note that in the examples considered at the graphical solution, the unique optimal solution to the LP occurred at a vertex (corner) of the feasible region. In fact it is true that for *any* LP the optimal solution occurs at a vertex of the feasible region. This fact is the key to the simplex algorithm for solving LP's.

Essentially the simplex algorithm starts at one vertex of the feasible region and moves (at each iteration) to another (adjacent) vertex, improving (or leaving unchanged) the objective function as it does so, until it reaches the vertex corresponding to the optimal LP solution.

The simplex algorithm for solving linear programs (LP's) was developed by Dantzig in the late 1940's and since then a number of different versions of the algorithm have been developed. One of these later versions called the *revised simplex* algorithm (sometimes known as the "product form of the inverse" simplex algorithm) forms the basis of most modern computer packages for solving LP's.

Steps

1. Convert the LP to standard form

2. Obtain a basic feasible solution (bfs) from the standard form
3. Determine whether the current bfs is optimal. If it is optimal, stop.
4. If the current bfs is not optimal, determine which non-basic variable should become a basic variable and which basic variable should become a non-basic variable to find a new bfs with a better objective function value
5. Go back to Step3.

Related concepts:

- Standard form: all constraints are equations and all variables are non negative
- BSF : any basic solution where all variables are non-negative
- Non-basic variable: a chosen set of variables where variables equal to0
- Basic variable: the remaining variables that satisfy the system of equations at the standard form

COMPUTATIONAL PROCEDURE OF SIMPLEX METHOD

Example 1.

Maximize $Z = 3x_1 + 2x_2$

Subjective to the constraints, $X_1 + X_2 \leq 4$

$X_1 - X_2 \leq 2$ and $X_1 \geq 0$, $X_2 \geq 0$

Step-1: First observe whether all the right side constants of the constraints are non-negative. If not , it can be changed into positive value on multiplying both sides of the constraints by -1. In this example all 'b_i' (right side constants) are already positive.

Step-2. Next convert the inequality constraints to equation by by introducing the non-negative slack or surplus variable. The coefficients of slack or surplus variables are always taken as zero in the objective function.

In this problem, all inequality constraints being $\leq$, only slack variables S_1, S_2 are needed.

Therefore, given problem now becomes

Maximize $Z = 3x_1 + 2x_2 + 0.\,S_1 + 0.\,S_2$

$X_1 + X_2 + S_1 \leq 4$

$X_1 - X_2 + S_2 \leq 2$

$X_1, X_2, S_1, S_2 \geq 0,$

Step-3. Now, represent the constraint equation matrix form.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} * \begin{bmatrix} X_1 \\ X_2 \\ S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Step-4> Construct the starting simplex table . It shold remembered that the values of non-basic variables are always taken as zero at each iteration. So $X_1 = 0$, $X_2 = 0$

Simplex Table-1. C_j -> 3 2 0 0

Basic variable	C_B	X_B	X_1	X_2	S_1	S_2	Mini RatioX_B/X_K for $X_K > 0$
S_1	**0**	**4**	**1**	**1**	**1**	**0**	$\frac{4}{1} = 4$
S_2	**0**	**2**	**1**	**-1**	**0**	**1**	$\frac{2}{2} = 1$ → mini
	Z= $C_B . X_B$ = 0	**Δ1= - 3**	**Δ2 = -2**	**Δ3= 0**	**Δ4= 0**		

Mini (Entering Vector) key element

The X_B gives the values of basic variables ie $S_1 = 4$, $S_2 = 2$, $X_1 = 0$, $X_2 = 0$, the value of objective function is Zero

$$\Delta j = Z_j - C_j = \sum Cbi - Cj$$

$\Delta 1 = 0 * 1 + 0 * 1 - 3 = -3$, $\Delta 3 = 0 * 1 + 0 * 0 - 0 = 0$

$\Delta 2 = 0 * 1 + 0 * 1 - 2 = -2$, , $\Delta 3 = 0 * 0 + 0 * 1 - 0 = 0$

Optimality Test:

1) If all $\Delta j \geq 0$, the solution under test will be optimal. Alternative optimal solution will exist if any non-basic

Δj *is also zero.*

2) If at least one Δj is negative, the solution under test is not optimal, then proceed to improve the solution in the next step.
3) If corresponding to any negative Δj, all the elements of the column X_j are negative or zero (≤ 0) , the solution under test will be unbalanced.

Applying these rules for testing the optimality of starting basic feasible solution, it is observed that $\Delta 1$ and $\Delta 2$ both are negative. Hence to improve solution proceed to step-6.

Step-6. Decide incoming vector and outgoing vector.

Simplex Table-2.		C_j ->	3	2	0	0	
Basic variable	C_B	X_B	X_1	X_2	S_1	S_2	Mini Ratio X_B/X_K for $X_K > 0_0$
S_1	**0**	**2**	**0**	**2**	**1**	**-1**	**$\frac{2}{2}$ = 1 mini**
X_1	**3**	**2**	**1**	**-1**	**0**	**1**	**$\frac{2}{2}$ = 1 mini** →
	Z=∑ C_B X_B = 6		**Δ_1 = 0**	**Δ_2 = - 5** ↑	**Δ_3 = 0**	**Δ_4 = 3**	

Mini (Entering Vector) key element

The X_B gives the values of basic variables ie $S_1 = 0$, $S_2 = 0$, $X_1 = 3$, $X_2 = 0$, the value of objective function is Z = 6

$$\Delta j = Z_j - C_j = \sum Cbi.Zij - Cj$$

$\Delta 1 = 0 * 0 + 3 * 1 - 3 = 0$, $\Delta 3 = 0 *1 + 3 *0 - 0 = 0$

$\Delta 2 = 0 * 2 + 3* (-1) - 2 = -5$ (minimum) , $\Delta 4 = 0 *(-1) + 3 *1 - 0 = 3$

Simplex Table-3		**C_j ->**	**3**	**2**	**0**	**0**	
Basic variable	**C_B**	**X_B**	**X_2**	**X_B**	**S_1**	**S_2**	**Mini Ratio X_B/X_K for $X_K > 0_0$**
X_2	**2**	**1**	**0**	**1**	**0.5**	**- 0.5**	**$\frac{2}{2}$ = 1**
X_1	**3**	**3**	**1**	**0**	**0.5**	**0.5**	**$\frac{2}{2}$ = 1**

	Z^{1*}= C_B .X_B = 11	**Δ_1 = 0**	**Δ_2 = 0**	**Δ3= 2.5**	**Δ4= 0.5**	

New element = old element – (corresponding key column element * corresponding key row element)/ key element

NE = OE – (KCE*KRE)/KE

Where NE = new Element
OE = Old Element
KCE = Corresponding Key Column Element
KRE = Corresponding Key Row Element
KE = Key Element

Since all $\Delta j \geq 0$, the optimal solution is attained.
Optimum solution is X_1= ,3, X_2= 1, max Z = 2*1+ 3*3 = 11

Example-2: Mini Z = X_1 - $3X_2$ +$3X_3$ Subject to

$3 X_1 – X_2 + 3X_3 \leq 7$,
$-2X_1 + 4 X_2 \leq 12$,
$-4X_1 +3X_2 + 8 X_3 \leq 10$
and $X_1, X_2, X_3 = 0$

This is problem of minimization. Converting the objective function from minimization to maximization, we have

Min Z = Max (-Z) = max (Z^1) where –Z = Z^1

Max (Z^1) = $- X_1 + 3X_2 - 3X_3$

subject to
$3 X_1 – X_2 + 3X_3 \leq 7$, $-2X_1 + 4 X_2 \leq 12$, $-4X_1 +3X_2 + 8 X_3 \leq 10$

and $X_1, X_2, X_3 = 0$

Adding slack variable S_1 , S_2, S_3 to equations 1, 2, 3 respectively

Max ($Z^{1)}$ = $- X_1 + 3X_2 - 3X_3 + o S_1 + 0S_2 + 0 S_3$

$3 X_1 – X_2 + 3X_3 + S_1 = 7$,

$-2X_1 + 4X_2 \quad + S_2 \quad = 12,$

$-4X_1 + 3X_2 + 8X_3 \quad + \quad S_3 = 10$

and $X_1, X_2, X_3, S_1, S_2, S_3 = 0$

Simplex Table-1

C_j →			**- 1**	**3**	**- 2**	**0**	**0**	**0**	
Basic variable	**C_B**	**X_B**	**X_1**	**X_2**	**X_3**	**S_1**	**S_2**	**S_3**	**Mini Ratio X_B/X_K for $X_K > 0_0$**
S_1	**0**	**4**	**1**	**--1**	**1**	**1**	**0**	**0**	---
S_2	**0**	**2**	**0**	**4**	**0**	**0**	**1**	**0**	$\frac{2}{4}$ =0.5 mini
S_3	**0**	**4**	**8**	**3**	**8**	**0**	**0**	**1**	4/3= 1.33
$X_1=X_2,=0, X_3=0$	$Z^1=0$, Z=0		**$\Delta_1=$**	**$\Delta_2 = -3$**	**$\Delta_3 = -3$**	**$\Delta_4 = 2$**	**$\Delta_5 = 0$**	**$\Delta_6 = 0$**	

Simplex Table-2

.	**C_j ->**	**- 1**	**3**	**- 2**	**0**	**0**	**0**	**0**	
Basic variable	**C_B**	**X_B**	**X_1**	**X_2**	**X_3**	**S_1**	**S_2**	**S_3**	**Mini Ratio X_B/X_K for $X_K > 0_0$**
S_1	0	10	5/2	0	3	1	1/4	0	10/2.5
X_2	0	10	-1/2	1	0	0	1/4	0	---
S_3	0	1	5/2	0	8	0	-3/4	0	---
x1=x3=s2=0	**Z^1= 9** **z= -9**		**$\Delta_1=$** - 1/2	**Δ_2 =** 0	**Δ_3 =** 2	**Δ_4 = 0**	**Δ_5 =** -3/4	**Δ_6 =** 0	

Simplex Table-3

.	**C_j ->**	**- 1**	**3**	**- 2**	**0**	**0**	**0**	**0**	
Basic variable	**C_B**	**X_B**	**X_1**	**X_2**	**X_3**	**S_1**	**S_2**	**S_3**	**Mini Ratio X_B/X_K for $X_K > 0_0$**

S1	**0**	**10**	**1**	**0**	**6/5**	**2/5**	**1/10**	**0**	
X2	**0**	**10**	**0**	**1**	**3/5**	**1/5**	**3/10**	**0**	
S3	**0**	**1**	**0**	**0**	**11**	**1**	**- 1/2**	**1**	
x1=s1=x 2=0	**Z^1=11 z= -11**		**$\Delta_{1=}$ 0**	**$\Delta_{2=}$ 0**	**$\Delta_{3=}$ 13/5**	**$\Delta_{4=}$ 1/5**	**$\Delta_{5=}$ 8/10**	**$\Delta_{6=0}$**	

Since all $\Delta j \geq 0$, the optimal solution is attained.

Optimal solution is

X1=4, x2= 5, x3 = 0 min Z = -11

Example-3: Max Z = 3x1 + 2 x2 + 5x3 subjective to

X1+2x2+x3 <= 430
3x1+2 x3 <= 460
X1+ 4x2 <=420 and x1, x2, x3 >=0

Adding slack variables X_4, X_5, X_6 to the equations respectively
X1+2x2+x3 + X_4 = 430
3x1+2 x3 + X_5 = 460
X1+ 4x2+ X_6 = 420 and x1, x2, x3 >=0
Max Z = 3x1 + 2 x2 + 5x3 + 0 X_{4+} 0 X_4 + 0 X_5

simplex table-1

		C^j -->	3	2	5	0	0	0
Basic variable	C_B	X_B	X_1	X_2	X_3	x4	X_5	X_6	Mini Ratio X_B/X_K for $X_K > 0_0$
X_4	0	430	1	2	1	1	0	0	430/1 = 430
X_5 ←	0	460	3	0	2	0	1	0	$\frac{460}{2}$ = 230 → mini
X_6	0	420	1	4	0	0	0	1	4/3= 1.33
$x_1=x_2=x_3=0$	Z^1= C_B .X_B = $\sum Cbi.Zij$ 0		Δ1= -3	Δ2 = - 2	Δ3 = - 5 min ↑	Δ4 = 0	Δ5= 0	Δ6= 0	

				i				

Simplex-Table-2

$C_j \rightarrow$ 3 2 5 0 0 0

Basic variable	**C_B**	**X_B**	**X_1**	**X_2**	**X_3**	**x4**	**X_5**	**X_6**	
X_4	0	200	- 1/2	2	0	1	1/4	0	200/2= 100 mini
X_2	0	230	3/2	1	1	0	- 1/2	0	---
X_6	0	420	1	0	0	0	0	1	420/4
x_1=0, x2 =230 x_3=0 x_4=200 x_6=450	Z= 1150		9/2	- 2 min	0	0	5/2	0	

Simplex-Table-3

$C_j \rightarrow$ 3 2 5 0 0

Basic variable	**C_B**	**X_B**	**X_1**	**X_2**	**X_3**	**x4**	**X_5**	**X_6**	
X_2	2	**100**	-1/4	1	0	1/2	-1/4	0	
X_3	5	230	3/2	0	1	0	1/2	0	
X_6	0	20	2	0	0	- 2	1	1	
x_1=x_4=x_5=0	Z=1350		4	-0	0	1	2	0	All $\Delta j \geq 0$

Since All $\Delta j \geq 0$, the solution is X_1=0, X_2=100 , X_3= 230, Max Z = 2*100 + 5*230 +0 = 1350

Example-4. Solve the LP problem, :Max Z = 3x1 + 5 x2 + 4x3 subjective to

$2X_1+3x_2 \leq 8$
$2x_2+5x_3 \leq 10$
$3X1+ 2x2+4X_3 \leq 15$ and x_1, x_2, x_3 >=0

Adding slock variables X_4, X_5, X_6 to the equations respectively

Example-5. Solve the LP problem, :Min Z = $x2 - 3 x3 - 2 X_5$ subjective to

$3x2 - x3 + 2 X5 \leq 7$
$- 2x_2+4x_3 \leq 12$
$- 2x_2+3X_3+ 8X_5 \leq 10$ and $x_1, x_2, x_3 >=0$
Adding slock variables X_1, X_4, X_6 to the equations respectively6 , the constraint equations becomes

$X_1 + 3x_2 - X_3 + 2 X_5 = 7$
$- 2x_2+4X_3 + X_4 = 12$
$- 2x2+3X_3+ 8X_5 +X_6 = 10$

$X_1 + 3x_2 - X_3 + 0X_4 + 2 X_5 + 0 X_6 = 7$
$0. X_1 - 2x_2+ 4X_3 + X_4 + 0 X_5 + 0 X_6 = 12$
$0. X_1 - 2x2 + 3X_3 + 0. X_4+ 8X_5 + X_6 = 10$

Max $Z^1 = Max(-Z) = - x_2 - 3 x_3 - 2 X_5 + 0 X_1 + 0. X_4 + X_6 = 0 X_1 - x2 + 3 x3 + 0. X_4 - 2 X5 + X_6$ +

ARTIFICIAL VARIABLE TECHNIQUE

1. Two Phase method
2. Big-M method

Two Phase Method: (Two phase Simplex method): Two phase simplex method is used to solve a given problem in which some artificial variables are involved. The solution is obtained in two phases.

Phase-1: In this phase, the simplex method is applied to a specially constructed auxiliary linear programming problem leading to a final simplex table containing a basic feasible solution to the original problem.

Step-1: Assign a cost -1 to each artificial variable and a cost ‘0’ to all other variables (in place of their original cost) in the objective function.

Step-2. Construct auxiliary linear programming problem In whch the new objective function Z^* to be maximum subject to the given set of constraints.

Step-3. Solve the auxiliary problem by simplex method until either of the following three possibilities do arise.

1) Max $Z^* < 0$ and at least one artificial vector appear in the optimum basis at a +ve level. In this case given

problem does not possess any feasible solution.

2) Max Z^* =0 and at least one artificial vector appears in the basis at zero level. In this case proceed to phase-II.

3) Max Z^* = 0 and no artificial vector appears in the optimum basis. In this case also proceed to phase –II.

Phase-II: Now assign the actual costs to the variables in the objective function and a a zero cost to every artificial variable that appears in the basis at the zero level. This new objective function is now maximized by simplex method subject to the given constraints. That is simplex method is applied to the modified simplex table obtained at the end of phase-I, until an optimum basic feasible solution has been attained (if exists). The artificial variables which are non-basic at the end of phase-I are removed.

Problem-1. Use two phase simplex method to solve the problem.

`

Minimize $Z = X_1 - 2X_2 - 3X_3$ subject to the constraints.

$$-2X_1 + X_2 + 3X_3 = 2$$
$$2X_1 + 3X_2 + 4X_3 = 1 \quad \text{and } x_1, x_2, x_3 >= 0$$

Step-1. First convert the objective function into maximization form

Max $Z^1 = -X_1 + 2X_2 + 3X_3$, where $Z^1 = -Z$

Introducing the artificial variables $a1 \geq 0$ $a_2 \geq 0$

The constraints of the given problem become

$$-2X_1 + X_2 + 3X_3 + a_1 = 2$$
$$2X_1 + 3X_2 + 4X_3 + a_2 = 1$$
$$\text{and } x_1, x_2, x_3, a_1, a_2 >= 0$$

Phase-I: Auxiliary LP problem is Max $Z^1 = 0X_1 + 0X_2 + 0X_3 - 1a_1 - a_2$

Subject to the above given

$$-2X_1 + X_2 + 3X_3 + a_1 = 2$$
$$2X_1 + 3X_2 + 4X_3 + a_2 = 1$$

Auxiliary Table-1

		$C_j \rightarrow$	0	0	0	- 1	- 1	
Basic variable	C_B	X_B	X_1	X_2	X_3	A_1	A_2	Mini RatioX_B/X_K for $X_K > 0_0$
a_1	- 1	2	-2	1	3	1	0	2/3
a_2	- 1	1	2	3	4	0	1	1/4 mini →
	Z^{1*} =	– 3	0	- 4	- 7 ↑	0	0	

The auxiliary table-2		$C_j \rightarrow$	0	0	0	-1	-1	
Basic variable	C_B	X_B	X_1	X_2	X_3	X_4	A_1	Mini Ratio X_B/X_K for $X_K > 0_0$
a_1	-1	5/4	-7/2	-5/4	0	1	-3/4	3
X_3	0	1/4	1/2	3/4	1	0	1/4	$\frac{6}{5}$ = 1.2 → mini
	$Z^{1*} = -5/4$		7/2 ↑	5/4 ↑	0	0	-1/4	

Since $\Delta j \geq 0$, an optimum basic feasible solution to the auxiliary LPP has been attained. But at the same time max Z^{1*} is negative and the artificial variable a_1 apperars in the basic solution at a + ve level (ie +5/4). Hence the original problem does not possess any feasible solution.
Here there is no need to enter phase-II.

Problem-2. Use two phase simplex method to solve the problem.

Minimize $Z = 15/2\, X_1 - 3X_2$ subject to the constraints.

$3X_1 - X_2 - X_3 \geq 3$

$X_1 - X_2 \geq 2$ and $x_1, x_2, x_3 >= 0$

Step-1. First convert the objective function into maximization form

Max $Z^1 = -15/2\, X_1 + 3X_2$, where $Z^1 = -Z$

Introducing the surplus variables $X_4 \geq 0$, and $X_5 \geq 0$

Introducing the artificial variables $a_1 , \geq 0$ $a_2 \geq 0$

The constraints of the given problem become

$3X_1 - X_2 - X_3 - X_4 + a_1 , = 3$

$X_1 - X_2 - X_5 + a_2 = 2$ and $x_1, x_2, x_3 , a_1 , a_2 , >= 0$

Phase-1. Assigning cost -1 to the artificial variables a_1 , and a_2 ,

Assigning cost 0 to the all other variables in the objective function

Max $Z^1 = 0\, X_1 + 0\, X_2 + 0\, X_3 + 0\, X_4 , - 1\, a_1 - a_2$

$3X_1 - X_2 - X_3 - X_4 + a_1 , = 3$

$X_1 - X_2 - X_5 + a_2 = 2$ and $x_1, x_2, x_3 , a_1 , a_2 , >= 0$

Auxiliary simplex table--1

Basic variable	C_B	X_B	X_1	X_2	X_3	X_4	X_5	A_1	A_2	Mini Ratio X_B/X_K for $X_K > 0_0$
$C_j \rightarrow$			0	0	0	0	0	- 1	- 1	
a_1	- 1	3	3	-1	-1	-1	0	1	0	3/3 =1 mini →
a_2	- 1	2	1	-1	1	0	-1	0	1	2/1=2
	Z^{1*} = - 5		-4 min	2	0	1	1	0	0	Δj

Since $\Delta j \geq 0$, and no artificial variable appears in the basis, an optimum solution to the auxiliary problem has been attained. No we can proceed to phase-II.

Auxiliary simplex table--2

Basic variable	C_B	X_B	X_1	X_2	X_3	X_4	X_4	A_1	A_2	Mini Ratio X_B/X_K for $X_K > 0_0$
$C_j \rightarrow$			0	0	0	0	0	- 1	-- 1	
X_1	0	1	1	-1/3	-1/3	-1/3	0	1/3	0	--
a_2	- 1	1	0	0	4/3	1/3	-1	1/3	1	¾ mini →
	Z^{1*} = - 1		0	0	- 4/3 min ↑	-1/3	1	2/3	0	

Phase-II: in second phase consider the actual costs associated with original variables, the objective function thus becomes

Max $Z1^* = - 15/2\ X_1 + 3\ X_2 + 0\ X_3 + 0\ X_4 + 0\ X_5$ and apply simplex method in the usual manner.

Simplex table-1								
		→ C_j	- 15/2	3	0	0	0	
Basic variable	C_B	X_B	X_1	X_2	X_3	X_4	X_5	Mini Ratio X_B/ X_K for X_K > 0_0
X_1	-15/2	5/4	1	-1/2	0	-1/4	-1/4	--
X_3	0	¾	0	-1/2	1	-1/4	-3/4	
	Z^{1*} = -75/8		0	3/4	0	15/8	15/8	

Since all $\Delta j \geq 0$, an optimum basic feasible solution to the auxiliary LPP has been attained. Solution is X_1= 5/4, X_3= ¾ minimum Z = 75/8

Big- M –Method (Charne's Penalty Method)

Computational steps of Big-M- method are as stated below:

Step-1. Express the problem in the standard form.

Step-2. Add non-negative artificial variables to the left side of each of the equations corresponding to constraints of type (≥) and '='. When artificial variables are added, it causes violation of the corresponding constraints. This difficulty is removed by introducing a condition which ensures that artificial variables will be zero in the final solution(provided the solution of the problem exists). On the other hand, if the problem does not have a solution, at least one of the artificial variables will appear in the final solution with positive value. This is achieved by assigning very large price (per MODULE penalty) to these variables in the objective function. Such a large price will be designated by –M for maximization (+M for minimization problems).

Step-3. In the last, use the artificial variables for the starting solution and proceed with te usual simplex routine until optimal solution is obtained..

Example-1: Solve big-M method the following Linear Programming problem.
Max Z = $-2x_1 - X_2$ subjective to

$3X_1 + x_2 = 3$
$4X_1 + 3x_2 \geq 6$
$X_1 + 2x2 \leq 4$ and $x_1, x_2 >= 0$

Solution step-1. Introducing slack variables and artificial variables, the system of constraint equation becomes

$3X_1 + X_2 \quad + a_1 = 3$

$4X_1 + 3X_2 - X_3 \quad + a_2 = 6$

$X_1 + 2X_2 + X4 \quad = 4$

and x_1, x >=0

swhich can be written in the matrix form as:

$$\begin{bmatrix} X_1 & X_2 & X_3 & X_4 & A_1 & \\ 3 & 1 & 0 & 0 & 1 & \\ 4 & 3 & -1 & 0 & 0 & \\ 1 & 2 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X2 \\ X3 \\ X4 \\ A1 \\ A2 \end{bmatrix} * \begin{bmatrix} \\ \end{bmatrix} = \begin{matrix} 3 \\ 6 \\ 4 \end{matrix}$$

Step-2. Assigning large negative price -M to the artificial variables a_1, a_2, the objective function becomes

Max $Z = -2x_1 - X_2 + 0X_3 + 0X_4 - MA_1 - MA_2$

Step-3. Construct the simplex table as followings:

Simplex table-1									
		C_j →	-2	-1	0	0	- M	--M	
Basic variable	C_B	X_B	X_1	X_2	X_3	X_4	A_1	A_2	Mini Ratio X_B/X_K for $X_K > 0_0$
a_1	- M	3	3	1	0	0	1	0	3/3 = 1 mini →
a_2	- M	6	4	3	-1	0	0	1	$\frac{6}{4}$ = 1.5
X_4	0	4	1	2	0	1	0	0	4/1 = 4
	Z = -9 M		(2 – 7M) min ↑	(1-4M)/	M	0	0	0	

Simplex Table-2									
- M		C_j →		- 2	- 1	0	0	-M	
Basic variable	C_B	X_B	X_1	X_2	X_3	X_4	A_1	A_2	Mini Ratio X_B/X_K for $X_K > 0_0$
X_1	-2	1	1	1/3	0	1	1	0	3
a_2	-M	2	0	5/3	1	0	0	1	$\frac{6}{5}$ = 1.2 mini →
X_4	0	3	0	5/3	0	0	0	0	9/5 = 1/.8
	Z= (- 2 -2 M)		0min	(1-5M)/3 1-4M	M	0	(-2+ 7M)/3 0	0	

↑

Simplex table-3									
	C_j →		-2	-1	0	0	- M	- M	
Basic variable	C_B	X_B	X_1	X_2	X_3	X_4	A_1	A_2	Mini Ratio X_B/X_K for $X_K > 0_0$
X_1	-2	**3/5**	**1**	**0**	**1/5**	**1**	**3/5**	**-1/5**	
X_2	-1	**6/5**	**0**	**1**	**-3/5**	**0**	**-4/5**	**3/5**	
X_4	0	**1**	**0**	**0**	**1**	**0**	**1**	**-1**	
	Z= -12/5		**0**	**0**	**1/5**	**0**	**M-2/5**	**M- 1/5**	

Since all $\Delta j \geq 0$ *since M is as large as possible*, Δ3, Δ3,, Δ3 are all positive . consequently, the optimal solution is $X_1 =$ 3/5, $X_2 =$ 6/5, max Z = -12/5

Example-2. Solve the following problem by Big-M method.

Max Z = $X_1 + 2X_2 + 3X_3 - X_4$, bject to

$X_1 + 2X_2 + 3X_3 = 15$

$2X_1 + X_2 + 5X_3 = 20$

$X_1 + 2X_2 + X_3 + X_4 = 10$ and $x_1, x_2, x_3, x_4 \geq 0$.

Since the constraints of the given problem are equations, introduce the artificial variables a_1, $a_2 \geq$ **0**

$$\text{Max } Z = X_1 + 2X_2 + 3X_3 - X_4 - MA_1 - MA_2$$

subject to

$$X_1 + 2X_2 + 3X_3 + a_1 = 15$$

$$2X_1 + X_2 + 5X_3 + a_2 = 20$$

$$X_1 + 2X_2 + X_3 + X_4 + a_3 = 10$$

and $x_1, x_2, x_3, x_4, a_1, a_2 \geq 0$.

Simplex Table-1

	$C_j \rightarrow$		1	2	3	-1	- M	- M	
Basic variable	C_B	X_B	X_1	X_2	X_3	X_4	A_1	A_2	Mini Ratio X_B/X_K for $X_K > 0_0$
a_1	-M	15	1	2	3	0	1	0	15/3
a_2	-M	20	2	1	5	0	0	1	$\frac{20}{5}$ = 4 mini
X_4	-1	10	1	2	1	1	0	0	10/1 = 10
	Z= (35M-10)		(-3M-2)	(-3M-2)	(-8M-4)	0	0	0	

Simplex Table-2

	$C_j \rightarrow$		1	2	3	- 1	- M	- M	
Basic variable	C_B	X_B	X_1	X_2	X_3	X_4	A_1	A_2	Mini Ratio X_B/X_K for $X_K > 0_0$
a_1	-M	3	-1/5	7/5 7/5	0	0	1	X	3/(7/5)=15/7
X_3	3	4	2/5	1/5	1	0	0	X	$\frac{4}{1/5}$ = 20
X_4	-1	6	3/5	9/5	0	1	0	X	$\frac{6}{9/5}$ = 30/9
	Z= (-3M+6)		(M-2)/5	(-7M+16)/5	0	0	0	X	

Simplex Table-3

Basic variable	C_B	X_B	X_1	X_2	X_3	X_4	A_1	A_2	Mini Ratio X_B/X_K for $X_K > 0_0$
$C_j \rightarrow$			1	2	3	- 1	- M	- M	
X_2	2	15/7	-1/7	1	0	0	X	X	-
X_3	3	25/7	3/7	0	1	0	X	X	25/3
X_4	-1	15/7	6/7	0	0	1	X	X	$\frac{15}{6}$ min
	Z= 90/7		-6/7	0	0	0	X	X	

Simplex Table-4

Basic variable	C_B	X_B	X_1	X_2	X_3	X_4	A_1	A_2	Mini Ratio X_B/X_K for $X_K > 0_0$
$C_j \rightarrow$			1	2	3	- 1	- M	- M	
X_2	2	5/2	0	1	0	1/6	X	X	
X_1	3	5/2	0	0	1	1/2	X	X	
X_4	1	5/2	1	0	0	7/6	X	X	
	Z= 15		0	0	0	75/36	X	X	

Since all $\Delta j \geq 0$, an optimum basic feasible solution has been obtained . tly, the optimal solution is
$X_1 = X_2 = X_2 = 5/2$, max $Z = 15$

MODULE – II

TRANSPORTATION PROBLEM & ASSIGNMENT PROBLEM

Definition: The Transportation Problem is to transport various amounts of a single homogeneous commodity that are initially stored at various Origins, to different destinations in such a way that the totakl transportation cost is a minimum.

For example, a tyre manufacturing concern has ' m' factories located in 'm' different cities. The total supply potential of manufactured product is absorbed by 'n' retail dealers in 'n' different cities of the country. Then the transportation problem is to determine the transportation schedule that minimizes the total cost of transporting tyres from various factory locations to various retail dealers.

General Mathematical model of Transportation Problem

Let there be 'm' sources of supply, S_1 , S_2 , S_3 .... S_m having a_i (I = 1, 2, 3, ……….,m) MODULEs of supply (or capacity) respectively, to be transported to among 'n' destinations, D_1, D_2, D_3, ………D_n with b_j *(j = 1, 2, 3, ……, n)* MODULEs of demand (or requirement respectively. Let C_{ij} be the cost of shipping one MODULE of the commodity from source I to destination j, the problem is to determine the transportation schedule so as to minimize the total transportation cost satisfying supply and demand conditions.
Let x_{ij} be the quantity transported from source 'i' to destination 'j'
Mathematically , the problem in general may be stated as follows:

Minimize (total cost) $Z = \sum_{i=}^{m} \sum_{j=1}^{j=n} C_{ij} X_{ij}$ ---------- ,objective function

Subject to the constraints
Considering availability quantity

$\sum_{j=1}^{j=n} x_{ij} = a_i$ for i=1,2,3,….., m (Supply constraints) -----1

$\sum_{i=1}^{i=m} x_{ij} = b_j$ for j=1,2,3,….., n (demand constraint)---------2

It is assumed that the total availabities (supplies) $\sum a_i$ satisfy the total requirement (Demand) $\sum b_j$

Considering constraint of total supply = total demand

$\sum a_i = \sum b_j$ (i=1,2,3….,m; j=1,2,3,…..n) ------------3

It may be observed that the constraint equations and the objective functions are all linear in x_{ij} . so it may be looked like a linear transportation problem.

	D_1	D_2	. . . . .	D_n	Supply a_i
S_1	C_{11} $X11$	C_{12} $X12$	……..	C_{1n} $X1n$	a_1
S_2	C_{21} $X21$	C_{22} $X21$	‘’’’’’	C_{2n} $X2n$	a_2
..	. .	. .	. .	. .	
Sm	C_{m1} $Xm1$	C_{m2} $Xm2$	‘’’’’’’j	Cmn Xmn	a_m
Demand b_j	b_1	b_2	…. ..	b_n	$\sum a_i = \sum b_j$

There will be (m+n) constraints one for each source of supply and destination and m*n variables.
It follows hat any feasible solution for a transportation problem must have exactly (m+n-1) non-negative basic variables (or allocations) x_{ij} satisfying total supply = total demand condition.

Step by Step Procedure for solving Transportation Problem.

STEP-1: Understand the given problem and formulate in Standard form.

STEP-2: Check the equality of the total supply & demand (if not, we need to balance it creating dummy row or dummy column according to the problem. assign Zero transportation cost for the dummy cells

how many ship of MODULE should be transported from each demand to each distribution center can be expressed as (X_{ij})

- i ……..for Factory and j …….for distribution center

STEP-3: Establish the Initial Basic Feasible Solution using any one of the following methods

i. The Northwest corner rule
ii. Least Cost Method
iii. Vogel’s Approximation Model

Step-4. Check the initial feasible solution for non-degeneracy.

If it non-degenerate [no of required cell (m+n-1) = occupied cell in IBFS] proceed to optimal solution using MODI method.

If it is degenerate make (occupied cells < no. of required cells), consider one cell as occupied cell with small quantity epsilon) so as to make (occupied cell = Required cells). Then proceed for Optimal solution using

MODI method.

Step-5. Find the set of values Ui, Vj values considering the occupied cell using equation

$$Cij = Ui + Vj$$

Step-6. Find the net cell evaluation of all un-accupied cells using the equation

d_{ij} = Cij – (Ui + Vj) = Cij - Ui – Vj)

Step-7. Test for Optimality.

If all d_{ij} > = 0, then the solution is said to be optimal.

If atleast one d_{ij} < 0 , we say the solution is not optimal.

Step-8. Form a loop at unoccupied cell the net cell evaluation value is the most negative.

Form the closed loop horizontally and vertically keeping two cell in direction and revise the transportation allocation.

Step-9. Repeat steps-5 tp 7 till we get all dij >=0

Example-1. A company has three production facilities S_1, S_{21}, and S_3 with production capacity of 7, 9, and 18 MODULEs (in 100s) per week of a product, respectively. These MODULEs are to be shipped to four ware houses D_1, D_2, D_3 and D_4 with requirement of 5, 6, 7 and 14 MODULEs (in 100s) per week, respectively.The transportation costs (in rupees) per MODULE between factories to warehouses are I the table below.

	Warehouse				Supply/capacity
	D_1	D	D_3	D4	
Factoty -S_1	19	30	50	10	7
Factory -S_2	70	30	40	60	9
Factory-S_3	40	8	70	20	18
Demand / W.H requirements	5	8	7	14	34

Formulate this transportation problem as an lp model to minimize the total transportation cost.

Model formulation : let x_{ij} = number of MODULEs of the product to be transported from factory I (I =1, 2, 3) to warehouse j (j = 1, 2, 3,4)

The transportation problem is stated as an LP model as follows:

Minimize (total transportation coast) Z = $19 x_{11} + 30 x_{12} + 50 x_{13} + 10 x_{14} + 70 x_{21} + 30 x_{22} + 40 x_{23} + 60 x_{24} + 40 x_{31} + 8 x_{32} + 70 x_{33} + 20 x_{34}$

Subject to the constraints

Capacity constraints

$$x_{11} + x_{12} + x_{13} + x_{14} = 7$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 9$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 18$$

considering columns

$$x_{11} + x_{21} + x_{31} = 5$$
$$x_{12} + x_{22} + x_{32} = 8$$
$$x_{13} + x_{23} + x_{33} = 7$$
$$x_{14} + x_{24} + x_{34} = 14$$

and $x_{ij} \geq 0$ for I anfd j

in this LP model , there are m*n = 3* 4 =12 decision variables, x_{ij} and m+n = 7 contraints whre m ae the number of rows and n are the number of columns in a general transportation table.

TRANSPORTATION ALGORITHM:

The algorithm for solving to a transportation problem may be summarized in the following steps:

Step-1. Formulate the problem and arrange the data in the matrix form.: the formulation of the transportation problem is similar to the LP problem formulation. Here the objective function is the total transportation cost and the constraints are the supply and demand available at each source and destination, respectively

Step-2: obtain the Initial Basic Feasible Solution

Under this three methods are discussed to obtain IBF Solution.

North West Corner Method / Rule.
Least cost Method
Vogel's Approximation (or Penalty) Method

Initial Basic feasible Solution is obtained by any of above three methods must satisfy following conditions:

i. The solution must be feasible if it satisfies supply = demand constraint
ii. The number of positive allocations must be equal to m+n-1, where m is the number of rows and n is the number of columns.
   Any solution that satisfies the above condition s is called non-degenerate basic feasible solution
iii. **Test the initial basic feasible solution for Optimality**

iv. In this Modified Distribution (MODI) method is is use to test the optimality of the solution obtained in step-2, if the current solution is optimal, then stop. Otherwise, determine a new improved solution.

v. **Updating the Solution:** Repeat step-3 until an optimal solution is attained.

3.1 METHODS OF FINDING INITIAL BASIC FEASIBLE SOLUTION:

There are several methods available to obtain an initial basic feasible solution. Here we will discuss three following three methods

3.1.1North West Corner Method:

Step-1: Start with the cell at the upper left (north-west) comer of the transportation matrix and allocate as much as possible (min(supply, demand) ie mini $(a_{1,}b_{1)}$

Step-2a:if allocation made in step-1 is equal to the supply available at first source (a1, in first row), them move vertically down to the cell (2,1) in the second row and first column and apply step-1 again , for next allocation.

Step-2b: if the allocation made in step-1 is equal to the demand of the first destination (b_1 in first column), then move horizontally to the cell (1,2) in the first row and second column and apply step-1 again for next allocation.

Step-2c: if $a_1 = b_{1,}$ allocate $x_{11} = a_1$ or b_1 and move diagonally to the cell (2,2)

Step-3: continue the procedure step by step till an allocation is made in the south-east corner cell of the transportation problem.

iii. LEAST COST METHOD (MINIMUM COST METHOD):

This method takes into account the minimum MODULE cost of transportation for obtaining initial basic feasible solution and is as follows:

Step-1. Select the cell with the lowest MODULE cost in the entire transportation table and allocate minimum of (supply , demand, and eliminate (line out) that row or column in which either supply or demand is exhausted. If both row and column is satisfied simultaneously , respective row and column is be eliminated.

Step-2. After adjusting the supply and demand for all uncrossed-out rows and columns repeat the procedure with next lowest MODULE cost among the remaining rows and columns of the transportation table and allocate minimum of supply and demand to this cekk and eliminate (line out) that row and column in which either supply or demand is exhausted.

Step-3. Repeat the procedure until entire available supply at various sources and demand at at various destinations is satisfied. Iii). Vogel's Method

:Begin by computing for each row and column a penalty equal to the difference between the two smallest costs in the row and column. Next find the row or column with the largest penalty. Choose as the first basic variable the variable in this row or column that has the smallest cost. As described in the NWC method, make this variable as large as possible, cross out row or column, and change the supply or demand associated with the basic variable (See Northwest Corner Method for the details!). Now recomputed new penalties (using only cells that do not lie in a crossed out row or column), and repeat the procedure until only one uncrossed cell remains. Set this variable equal to the supply or demand associated with the variable, and cross out the variable's row and column.

		To		
		warehouse-A	Warehouse-B	Warehose-C
From	Factory-D	5	4	3
	Facory-E	8	4	3
	Factory-F	9	7	5

The following example was used to demonstrate the formulation of the transportation model.

Example-1. Bath tubes are produced in three Factories namely Factory-D, Factory-E and Factory-F. The product is shipped to the distributers namely WH-1, WH-2 and WH-3 in different cities in railroad. Each Factory is able to supply the following number of MODULEs , Factory-D 100 MODULEs, Factory-E 300 MODULEs and Factory-F 300 MODULEs .

The distribution requires the following MODULEes ; WH-1 300 MODULEs , WH-2 200 MODULEs and WH-3 200 MODULEs

The transportation cost in Rupees matrix is shown in below table

TRANSPORTATION MATRIX

From \ To	WH-A	WH-B	WH-C	Factory capacity
FACTORY-D	5	4	3	100
FACTORY-E	8	□4	3	300
FACTOFY-C	9	7	5	300
Warehouse requirement	300	200	200	700

Factory-A capacity constraint

Cell representing a possible source-to-destination shipping assignment (factory-B to WH-2)

Cost of shipping 1 unit from Factory-C to warehouse-2

warehouse-3 demand

Total demand and total supply

Find the initial basic feasible solution using north west corner rule, least cost method and Vogel's approximation method

Solution: with the data given , form late the standard transportation problem and is shown as below.

TRANSPORTATION PROBLEM

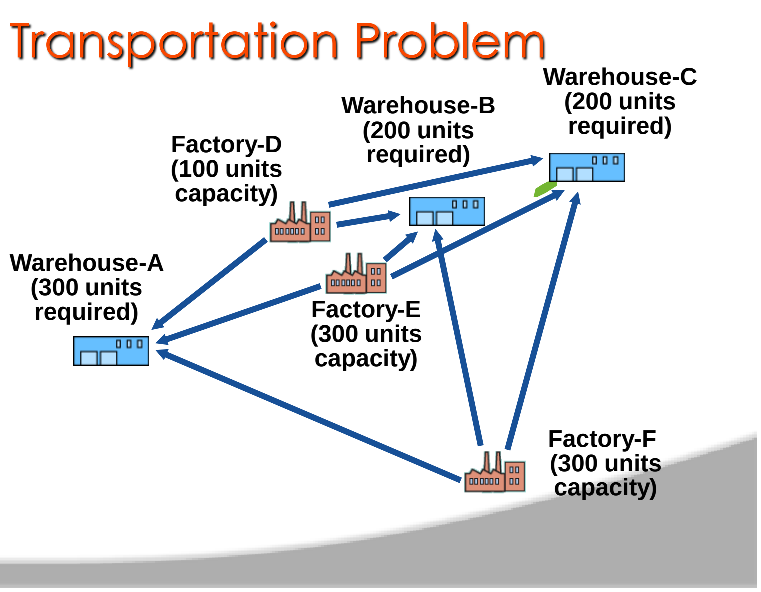


Figure C.1

TRANSPORTATION MATRIX

From \ To	WH-A	WH-B	WH-C	Factory capacity
FACTORY-D	5	4	3	100
FACTORY-E	8	4	3	300
FACTOFY-C	9	7	5	300
Warehouse requirement	300	200	200	700

Factory-A capacity constraint

Cell representing a possible source-to-destination shipping assignment (factory-B to WH-2)

Cost of shipping 1 unit from Factory-C to warehouse-2

warehouse-3 demand

Total demand and total supply

i. Northwest-Corner Rule for finding Initial Basic Feasible Solution.

- Start in the upper left-hand cell (or northwest corner) of the table and allocate units to shipping routes as follows:
  1. Exhaust the supply (factory capacity) of each row before moving down to the next row
  2. Exhaust the (warehouse) requirements of each column before moving to the next column
  3. Check to ensure that all supplies and demands are met

INITIAL BASIC FEASIBLE SOLUTION USING NORTHWEST-CORNER RULE

From \ To	WH-A	WH-B	WH-C	Factory capacity
FACTORY-D	100 ₹ 5	₹ 4	₹ 3	100
FACTORY-E	100 ₹ 8	100 ₹ 4	₹3	300
FACTORY-F	₹ 9	100 ₹7	200 ₹ 5	300
Warehouse requirement	300	200	200	700

Means that the Factory-F is shipping 100 bathtubs from Factory-F to WH-B

Initial basic feasible solution:

Trransport form factory-D to Warehoue—A = 100 MODULEs

Transport form factory-E to Warehoue—A = 200 MODULEs

Transport form factory-E to Warehoue—B = 100 MODULEs

Transport form factory-F to Warehoue-B = 100 MODULEs

Transport form factory-F to Warehoue-C = 200 MODULEss

Total transport cost = 100*5 + 200 * 8 + 100*4 + 100 *7 + 200 * 5 = Rs 4200

ii) Using Least Cost Method to find Initial Basic Feasible Solution:

1. **Evaluate the transportation cost and select the cell with the lowest cost (in case a Tie make an arbitrary selection).**
2. **Depending upon the supply & demand condition , allocate the maximum possible MODULEs (min of capacity, demand)to lowest cost cell.**
3. **Delete the satisfied allocated row or the column (or both).**
4. **Repeat steps 1 and 3 until all MODULEs have been allocated.**

The Lowest-Cost Method

L-6, L-3, L-2, L-1, L-4, L-5

To / From	WH-A	WH-B	WH-C	Factory capacity
FACTORY-D	5	4	100 ($3)	100
FACTORY-E	8	200 ($4)	100 ($3)	300
FACTORY-F	300 (9)	7	5	300
Warehouse requirement	300	200	200	700

STEP-4: **Finally, select left out cell(F,A) and ship 300 units from FACTORY-F to WAREHOUSE-A as this is the only remaining cell to complete the allocations**

Figure C.4

Solution Table:

The Lowest-Cost Method

To / From	WH-A	WH-B	WH-C	Factory capacity
FACTORY -D	5	4	100 (3)	100
FACTORY –E	8	200 (4)	100 (3)	300
FACTORY -F	300 (9)	7	5	300
Warehouse requirement	300	200	200	700

initial basic feasible solution:
assign (transport) form facory-D to warehoue-- A 100 units @ cost of rs 5/-
assign (transport) form facory-E to warehoue--A 200 units @ cost of rs 8/-
assign (transport) form facory-E to warehoue-B 100 units @ cost of rs 4/-

transport form facory-F to warehoue-B 100 units units @ cost rs 7/-
transport form facory-F to warehoue-C 200 units units @ cost rs 5/-

Total Cost = 3(100) + 4(200) + 3(100) + 9 (300)
= Rs4,100

iii) Vogel's approximation method for finding Initial Basic Feasible solution:

1. Calculate penalties for each row and column by taking the difference between the smallest and next smallest MODULE transportation cost .
2. observe the penalty computed row and colum , select the cell with largest penalty and allocate maximum possible quanty. min of capacity, demand).
3. Adjust the supply and demand and cross out the satisfied row or column (or both).
4. Repeat steps 1 and 3 until all MODULEs have been allocated.

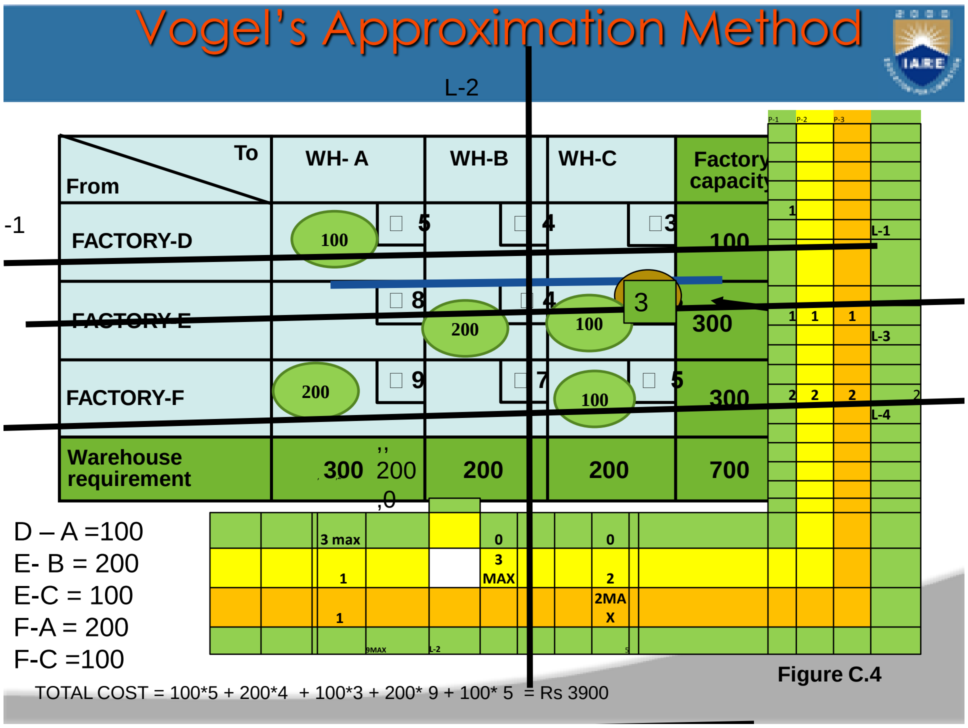


Figure C.4

Initial Basic Feasible Solution using Vogel's Approximation Method

Computed Shipping Cost

Route				
From	To	Tubs Shipped	Cost per Unit	Total Cost
D	A	100	5	Rs 500
E	B	200	4	800
E	C	100	3	300
F	A	200	9	1800
F	C	100	5	500
			Total:	3900

OPTIMALITY SOLUTION USING MODI (Modified Distribution Method)

For given problem initial Basic feasible solution is found by using any of IBFS-methods. Then test the Initial Basic Feasible Solution for optimality .

Step-1. We will find Initial Basic Feasible Solution using Vogels Approximation method.

The Initial Basic Feasible solution is

USING VOGEL'S APPROXIMATION METHOD TO FIND IBFS

From \ To	WH- A	WH-B	WH-C	Factory capacity
FACTORY-D	100 (5)	(4)	(3)	100
FACTORY-E	(8)	200 (4)	100 (3)	300
FACTORY-F	200 (9)	(7)	100 (5)	300
Warehouse requirement	300	200	200	700

D – A =100
E- B = 200
E-C = 100
F-A = 200
F-C =100

TOTAL COST = 100*5 + 200*4 + 100*3 + 200* 9 + 100* 5 = Rs 3900

Figure C.4

The Initial Basic Feasible Solution Using Vogel's Approximation Method

From \ To	WH--A	WH-B	WH-C	Factory capacity	
Factory-D	100 (5)	(4)	(3)	100	$U_1 = - 2$
Factory-E	(8)	200 (4)	100 (3)	300	$U_2 = 0$
Factory-F	200 (9)	($7)	100 ($5)	300	$U_3 = 2$
Warehouse requirement	300	200	200	700	
	$V_1 = 7$	$V_2 = 4\ 8$	$V_3 = 3$		

Total Cost = 5(100) + 4(200) + 3(100) + 9(200) + 5(100)

= Rs 39,000 is this the optimal solution

No. ocupied cells = m+n - 1 = 3+3-1 = 5
The actual cells = 5
Since no. of acupied cells = no. of actual cells, the problem is said to be non-degenerate. Thus Optimal Solution can be attained using MODI method

Figure C.8

FIND THE OPTIMAL SOLUTION USING MODI (Modified distribution Method)

1. **Check IBFS for non-degeneracy.**

no of actulas cells = (m +n-1) = 5 satisfies

2. To calculate set of U_i, V_j values consider row or column which has more no of ocupied cells (Ui or Vj) as zero.

3. For ocupied cells $C_{ij} = U_i + V_j$

considering U2 = 0 , $U_2+V_3 = C_{23}$, U2 + V3 = 3, 0+ V3 = 3-→ V3= 3

$U_2 + V_2 = C_{22}$, 0 + V_2 = 4 -→ V2 = 4

Considering C33 = U3+V3 5 = U3 + 3→ U3 = 2

Considering C31 = U3 +V1 , 9= 2 + v1 -→ V1 = 7

Considering C11 = U1 +V1 , 5 = U1 + 7 -→ U1 = -2

.

Computation of Cell Evaluation for un-ocupied cells dij = $C_{ij} – u_i - v_j$

From \ To	WH-(A) Albuquerque	(B) Boston	(C) Cleveland	Factory capacity	
Factory-D	100 (5)	4	3	100	$U_1 = -2$
Factory-E	8	200 (4)	100 (3)	300	$U_2 = 0$
Factory-F	200 (9)	$7	100 ($5)	300	$U_3 = 2$
Warehouse requirement	300	200	200	700	
	$V_1 = 7$	$V_2 = 4$	$V_3 = 3$		

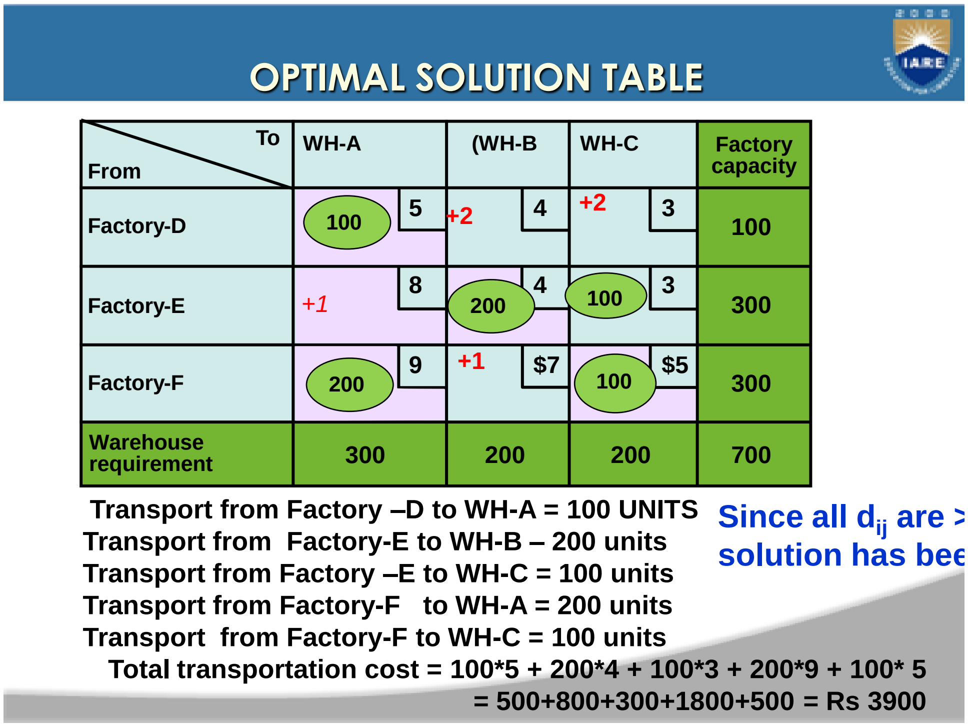
d_{12} = C22 – U2 –v2 , = 4 – (-2 + 4) = 2

d_{23} = C23 – U2 –V3 = 3 – 0 – 3 = 0

d_{21} = C21 – U2 – V1 = 8 – 0 -7 = 1

d_{32} = C32 – U3 – V2 = 7 – 2 – 4 = 1

Since all D_{ij} are >=0, the optimal solution has been achieved/



OPTIMAL SOLUTION TABLE

To / From	WH-A	(WH-B	WH-C	Factory capacity
Factory-D	100 (5)	+2 (4)	+2 (3)	100
Factory-E	+1 (8)	200 (4)	100 (3)	300
Factory-F	200 (9)	+1 ($7)	100 ($5)	300
Warehouse requirement	300	200	200	700

Transport from Factory –D to WH-A = 100 UNITS
Transport from Factory-E to WH-B – 200 units
Transport from Factory –E to WH-C = 100 units
Transport from Factory-F to WH-A = 200 units
Transport from Factory-F to WH-C = 100 units

Since all d_{ij} are >
solution has bee

Total transportation cost = 100*5 + 200*4 + 100*3 + 200*9 + 100* 5
= 500+800+300+1800+500 = Rs 3900

- Example-2: The has three production facilities S1, S2 and S3 with Production Capacity0f 7, 9 and 18 MODULEs (in ooo’s) per week f a product, respectively. These MODULEs are to be shipped to four warehouses D1, D2, D3 and D4 with requirement of 5,6, 7 and 14 MODULEs (in 000^s per week , respectively. The transportation costs (in rupees) per MODULE between factories to ware houses are given in the table below.

		To			
		D_1	D_2	D3	D_4
From	S_1	19	30	50	10
	S_2	70	30	40	60
	S_3	40	8	70	20

INITIAL TRANSPORTATION TABLE

	D1	D2	D3	D4	CAPACITY
S1	19र	30र	50र	10र	7
S2	70र	30र	40र	60र	9
S3	र40	8र	70र	20र	18
DEMAND	5	8	7	14	34

IBFS- using Vogel's approximation

L-2 L-1 L-3

	D1	D2	D3	D4	CAPACITY	PENALTY P-1	PENALTY -P2	P3	P4	
S1	19र (5)	30र	50र	10र (2) 2	7,2,0 7	9	9	9	10	1(
S2	70र	30र	40र (7)	60र (2)	9,2,0 9	10	20	20 MAX	MAX 60	
S3	40र 3	8र (8)	र70	20र (10)	18	12	12	20	20	20
	5	8	7	14	34					
P1	21	MAX 22	14,12,2 10	10						
P2	MAX21		10	10						
P3				10						

IBFS- using Vogel's approximation

L-2 L-1 L-3

	D1	D2	D3	D4	CAPACITY
s1	19₹ (5)	30₹	50₹	10₹ (2)	7
S2	70₹	30₹	40₹ (7)	60₹ (2)	9
s3	40₹ 3	8₹ (8)	₹70 (10)	20₹ 10	18
	5	8	7	14	34

S1-D1 = 5 UNITS
S1-D4 = 2 UNITS
S2-D3 = 7 UNITS
S2-D4 = 2 UNITS
S3- D2 = 8 UNITS
S3 –D4= 10 UNITS

TOTAL TRANSPORT COST = 5*19 + 2 * 10 + 7*40 + 2 * 60 + 8* 8+ 10* 20 = Rs779

APPLYING OPTIMALITY TEST (MODI -METHOD)

	D1	D2	D3	D4	CAPACITY	U_i
s1	19₹ (5)	30₹	50₹	10₹ (2)	7	$U_1 = 10$
S2	70₹	30₹ add	40₹ (7)	60₹ (2) sub	9	$U_2 = 60$
s3	40₹ 3	8₹ (8) sub	₹70	20₹ add (10)	18	$U_3 = 20$
DEMAND	5	8	7	14	34	
V_j	$V_1 = 9$	$V_2 = -12$	V_3 - 20	$V_4 = 0$		

$d_{12} =$ C12 – U1 –v2 , = 19 -10 - (-12)= 21
$d_{13} =$ C13 – U1 –v3 , = 50 – 10 – (-20) = 60 d_{23} = C23 – U2 –V3 = 3 – 0 – 3 =
d_{21} = C21 – U2 – V1 = 70 – 60 – 9 = 1
d_{22} = C22 – U2 – V2 = 30 – 60 – (-12) = - 18
$d_{31} =$ C31 – U3 – V1 = 40 – 20 – 9 = 11
$d_{33} =$ C33 – U3 – V3 = 70 – 20 – (-20) = 70
Since $D_{22} < 0$, ie = -18 , solution is not optimal, form loop starting from most – ve cell

Revised Table , test for optimality computing net evaluation of unoccupied cell

	D1	D2	D3	D4	CAPACITY	U_i
S1	19र (5)	30र	50र	10र (2)	7	U_1 =10
S2	70र	30र (2)	40र (7)	60र	9	U_2 = 42
S3	40र 3	8र (6) sub	र70	20र (12)	18	U_3 =20
DEMAND	5	8	7	14	34	
V_j	V_1 =9	V_2 = - 12	V_3 =- 2	V_4 = 0		

d_{12} = C12 – U1 –v2 , = 19 – 10 - (-12)= 21
d_{13} = C13 – U1 –v3 , = 50 – 10 – (-2) = 38
d_{21} = C21 – U2 – V1 = 70 – 42 – 9 = 19
d_{24} = C24 – U2 – V4 = 60 –42 – 0 = 18
d_{31} = C31 – U3 – V1 = 40 – 20 – 9 = 11
d_{33} = C33 – U3 – V3 = 70 – 20 – (- 2) = 52
Since all Dij >= an optimal solution has been attained

Total Transport cost = 5*19 + 2 * 10 + 2 * 30 + 7 * 40 + 6 * 8 + 12 * 20 = 743

APPLYING OPTIMALITY TEST (MODI -METHOD) Revised Table , test for optimality computing net evaluation of unoccupied cell

	D1	D2	D3	D4	CAPACITY	U_i
S1	19र (5)	30र	50र	10र (2)	7	U_1 =10
S2	70र	30र (2)	40र (7)	60र	9	U_2 = 42
S3	40र 3	8र (6) sub	र70	20र (12)	18	U_3 =20
DEMAND	5	8	7	14	34	
V_j	V_1 =9	V_2 = - 12	V_3 =- 2	V_4 = 0		

d_{12} = C12 – U1 –v2 , = 19 – 10 - (-12)= 21
d_{13} = C13 – U1 –v3 , = 50 – 10 – (-2) = 38
d_{21} = C21 – U2 – V1 = 70 – 42 – 9 = 19
d_{24} = C24 – U2 – V4 = 60 –42 – 0 = 18
d_{31} = C31 – U3 – V1 = 40 – 20 – 9 = 11
d_{33} = C33 – U3 – V3 = 70 – 20 – (- 2) = 52
Since all Dij >= an optimal solution has been attained

Total Transport cost = 5*19 + 2 * 10 + 2 * 30 + 7 * 40 + 6 * 8 + 12 * 20 = 743

THE OPTIMUM SOLTUTION FOR TRANSPORTATION PROBLEM

	D1	D2	D3	D4	CAPACITY	U_i
S1	19₹ (5)	30₹	50₹	10₹ (2)	7	U_1 =10
S2	70₹	30₹ (2)	40₹ (7)	60₹	9	U_2 = 42
S3	40₹	8₹ (6) sub	₹70	20₹ (12)	18	U_3 =20
DEMAND	5	8	7	14	34	
V_j	V_1 =9	V_2 = - 12	V_3 =- 2	V_4 = 0		

Transport from S1 to D1 = 5 UNITS
Transport from S1 TO D4 = 2 UNITS
Transport from S2 TO D2 = 2 UNITS
Transport from S2 TO D3 = 7 UNITS
Transport from S3 TO D2 = 6 UNITS
Transport from S3 TO D4 = 12 UNITS

Total Transport cost = 5*19 + 2 * 10 + 2 * 30 + 7 * 40 + 6 * 8 + 12 * 20 = 743

UNBALANCED TRANSPORTATION PROBLEM:

UNBALANCED TRANSPORTATION PROBLEM

From \ To	WH-A	WH-B	WH-C	Factory capacity
FACTORY-D	₹ 5	₹ 4	₹ 3	250
FACTORY-E	₹ 8	₹ 4	₹3	300
FACTORY-F	₹ 9	₹7	₹ 5	300
Warehouse requirement	300	200	200	850

TOTAL FACTORY CAPACITY = 850 UNITS
TOTAL WAREHOUSE REQUIREMENT = 700 UNITS
TOTALS ARE NOT EQUAL WE HAV E INSERT DUMMY –WAREHOUSE OF REQUIREMENT = 150 UNITS WITH TRANSPORT COST = 0

BALANCING AN UNBALANCED TRANSPORTATION PROBLEM

Excess Supply

If total supply exceeds total demand, we can balance a transportation problem by creating a ***dummy demand point*** that has a demand equal to the amount of excess supply. Since shipments to the dummy demand point are not real shipments, they are assigned a cost of zero. These shipments indicate unused supply capacity.

Unmet Demand

If total supply is less than total demand, actually the problem has no feasible solution. To solve the problem it is sometimes desirable to allow the possibility of leaving some demand unmet. In such a situation, *a penalty is often associated with unmet demand.* This means that a ***dummy supply point*** should be introduced.

UNBLANCED TP PROBLEM
IBFS BY USING NOH WEST CORNER RULE

To / From	WH-A	WH-B	WH-C	WH-Dummy	Factory capacity
FACTORY-D	250 (5र)	(र4)	(र3)	(0)	250
FACTORY-E	50 (8र)	200 (र4)	50 (र3)	(0)	300
FACTORY-F	(र9)	(र7)	150 (र5)	150 (0)	300
Warehouse requirement	300	200	200	150	850

Total Cost = 250(5) + 50(8) + 200(4) + 50(3) + 150(5) + 150(0)
= Rs 3,350

New Factory-E capacity

SPECIAL ISSUES IN MODELING
UNBLANCED TP PROBLEM

To / From	WH-A	WH-B	WH-C	WH-Dummy	Factory capacity
FACTORY-D	($5)	($4)	($3)	(0)	250
FACTORY-E	50 ($8)	($4)	($3)	(0)	300
FACTORY-F	($9)	($7)	($5)	(0)	300
Warehouse requirement	300	200	200	150	850

Total Cost = 250(5) + 50(8) + 200(4) + 50(3) + 150(5) + 150(0)
= Rs 3,350

New Factory-E capacity

CHECK FOR OPTIMALITY TEST:

1). no of occupied cell required = m + n -1 = 3+4-1 = 6

no of available occupied cells = 6

if both are equal, this problem is not degenerate, it has optimal solution

2).find set of Ui, Vj for all row and columns such that

Cij = Ui + Vj

3) then compute net cell evaluation for unoccupied cells (d_{ij})

4) If all (d_{ij}) >=0 , then solution table is optimal

5) If not select most –ve cell and form loop for modified solution and repeat the steps 2 to 4

DEGENERACY IN TRANSPORTATION PROBLEM:

M (number of rows) + N (number of columns) = allocated cells

If a solution does not satisfy this rule it is called degenerate

Degeneracy in transportation problems can occur in two ways

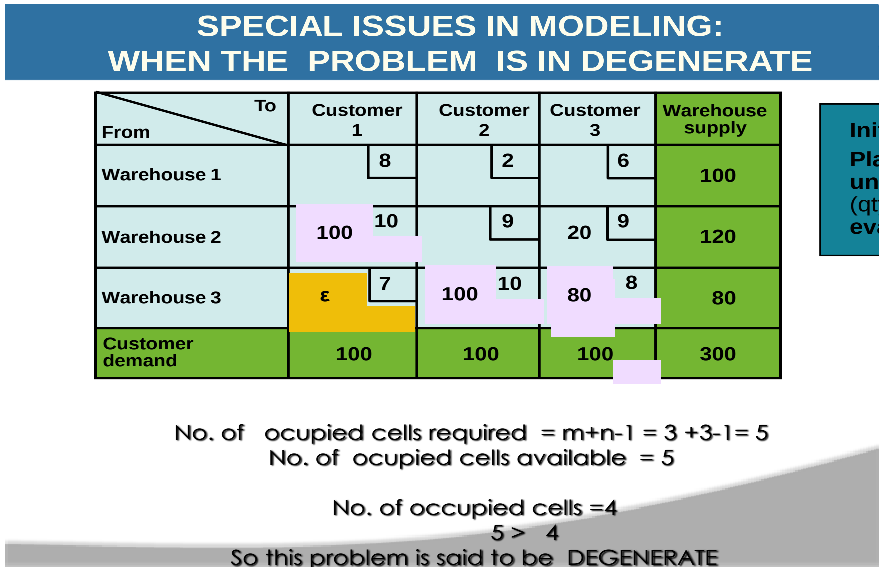
1. Basic feasible solutions may be degenerate from the initial stage onwards.
2. They may become degenerate at any intermediate stage.

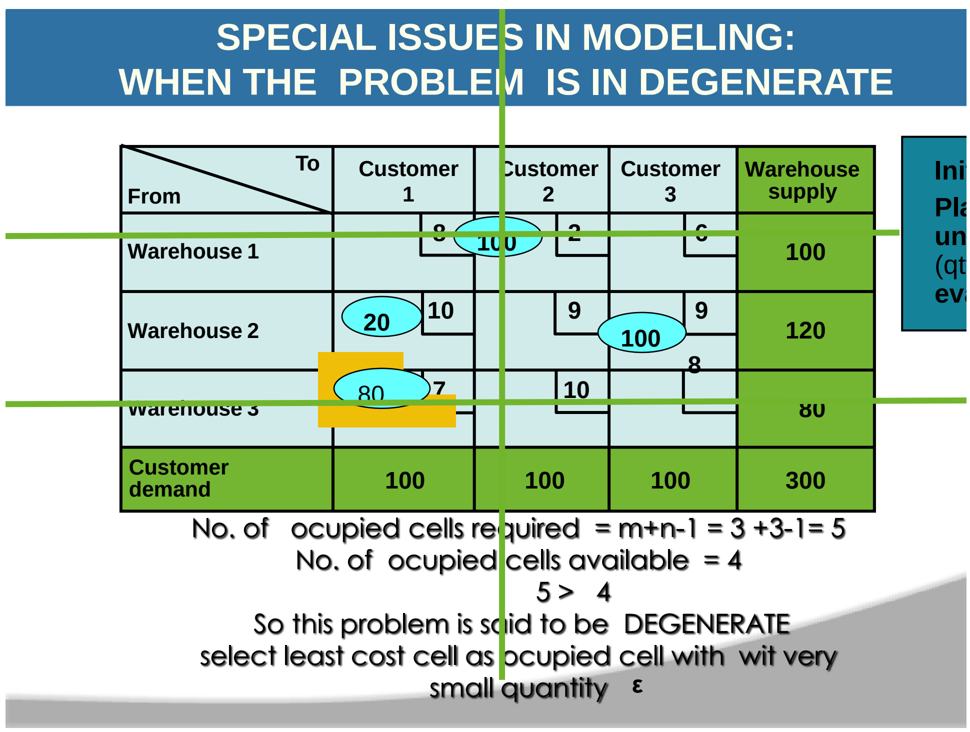
Resolution of Degeneracy During the Initial Stage

To resolve degeneracy, allocate an extremely small amount of goods (close to zero) to *one* or *more* of the empty cells so that a number of occupied cells becomes *m*+*n*-I. The cell containing this extremely small allocation is, of course, considered to be an occupied cell.

Rule: The extremely small quantity usually denoted by the Greek letter ~ (delta) [also sometimes by E (epsilon)] is introduced in the least cost independent cell subject to the

following assumptions. If necessary, two or more □ ‘ s can be introduced in theleast and





When number of occupied cells required = m+n-1 = 3+3-1 = 5
Number of occupied cell available = 4
5> 4, the problem is said to be in degenate.
To resolve degeneracy , we have to consider a least un-occupied cell with very small quantity 'ε' .
Suc that **ε** + quanty = quantity , quantity - **ε** = quantity.
Consider this small cell as a occupied cell calculate set of Ui and Vi , finally compute net cell evaluation for optimality test.

Optimal solution for degeneracy transportation problem

From \ To	Customer 1	Customer 2	Customer 3	Warehouse supply	
Warehouse 1	8	2 (100)	6 (ε)	100	U1= - 3
Warehouse 2	10 (20)	9	9 (100)	120	U2 = 0
Warehouse 3	7 (80)	10	8	80	U3 = - 3
Customer demand	100	100	100	300	
	V1 =10	V2 = 5	V3=9		

Transport warehouse-1 customer -2 = 100 units
Transport warehouse-1 customer -3 = 0 units
Transport warehouse-2 customer -1 = 20 units
Transport warehouse-3 customer -3 = 100 units
Transport warehouse-3 customer -1 = 100 units ε
Total Transportation cost = 100 * 2 + 0 * 6 + 20 * 10 + 100 *9 + 80 *7 = Rs 1860

ASSIGNMENT PROBLEMS

This chapter deals with

- Introduction
- Steps Involved in Solving TP
- Examples

What Is Assignment Problem

Assignment Problem is a special type of linear programming problem where the objective is to minimize the cost or time of completing a number of jobs by a number of persons.

The assignment problem in the general form can be stated as follows:

"Given *n* facilities, *n* jobs and the effectiveness of each facility for each job, the problem is to assign each facility to one and only one job in such a way that the measure of effectiveness is optimized (Maximized or Minimized)."Several problems of management have a structure identical with the assignment problem.

Example I: A manager has four persons (i.e. facilities) available for four separate jobs (i.e. jobs) and the cost of assigning (i.e. effectiveness)each job to each person is given. His objective is to assign each person to one and only one job in such a way that the total cost of assignment is minimized.

Example II: A manager has four operators for four separate jobs and the time of completion of each job by each operator is given. His objective is to assign each operator to one and only one job in such a way that the total time of completion is minimized.

Example III A tourist car operator has four cars in each of the four cities and four customers in four different cities. The distance between different cities is given. His objective is to assign a cell to one and only one customer in such a way that the total distance covered is minimized.

Hungarian Method

Although an assignment problem can be formulated as a linear programming problem, it is solved by a special method known as Hungarian Method because of its special structure. If the time of completion or the costs corresponding to every assignment is written down in a matrix form, it is referred to as a Cost matrix. The Hungarian Method is based on the principle that if a constant is added to every element of a row and/or a column of cost matrix, the optimum solution of the resulting assignment problem is the same as the original problem and vice versa. The original cost matrix can be reduced to another cost matrix by adding constants to the elements of rows and columns where the total cost or the total completion time of an assignment is zero. Since the optimum solution remains unchanged after this reduction, this assignment is also the optimum solution of the original problem. If the objective is to maximize the effectiveness through Assignment, Hungarian Method can be applied to a revised cost matrix obtained from the original matrix.

Balanced Assignment Problem

Balanced Assignment Problem is an assignment problem where the number of facilities is equal to the number of jobs.

Unbalanced Assignment Problem

Unbalanced Assignment problem is an assignment problem where the number of facilities is not equal to the number of jobs. To make unbalanced assignment problem, a balanced one, a dummy facility(s) or a dummy job(s) (as the case may be) is introduced with zero cost or time.

Dummy Job/Facility

A dummy job or facility is an imaginary job/facility with zero cost or time introduced to make an unbalanced assignment problem balanced.

An Infeasible Assignment

An Infeasible Assignment occurs in the cell(i,j) of the assignment cost matrix if *ith* person is unable to perform *jth* job..

It is sometimes possible that a particular person is incapable of doing certain work or a specific job cannot be performed on a particular machine. The solution of the assignment problem should take in to account the restrictions so that the infeasible assignments can be avoided. This can be achieved by assigning a very high cost to the cells where assignments are prohibited.

Practical Steps Involved In Solving Minimization Problems

Step 1: See whether Number of Rows are equal to Number of Column. If yes, problem is balanced one; if not, then add a Dummy Row or Column to make the problem a balanced one by allotting zero value or specific value (if any given) to each cell of the Dummy Row or Column, as the case maybe.

Step 2: Row *Subtraction:* Subtract the minimum element of each row from all elements of that row.

Note: If there is zero in each row, there is no need for row subtraction.

Step 3: Column *Subtraction:* Subtract the minimum element of each column from all elements of that column.

Note: If there is zero in each column, there is no need for column subtraction.

Step 4: Draw minimum number of Horizontal and/or Vertical Lines to cover all zeros. To draw minimum number of lines the following procedure may be followed:

1. Select a row containing exactly one uncovered zero and draw a vertical line through the column containing this zero and repeat the process still no such row is left.
1. Select a column containing exactly one uncovered zero and draw a horizontal line through the row containing the zero and repeat the process still no such column is left.

Step 5: If the total lines covering all zeros are equal to the size of the matrix of the Table, we have got the optimal solution; if not, subtract the minimum uncovered element from all uncovered elements and add this element to all elements at the intersection point of the lines covering zeros.

Step 6: Repeat Steps 4 and 5 till minimum number of lines covering all zeros is equal to the size of the matrix of the Table.

Step 7: *Assignment:* Select a row containing exactly one unmarked zero and surround it

by , 'and draw a vertical line through the column containing this zero. Repeat this process till no such row is left; then select a column containing exactly one unmarked zero and surround it by, and draw a horizontal line through the row containing this zero and repeat this process till no such column is left.

Note: If there is more than one unmarked zero in any row or column, it indicates that an alternative solution exists. In this case, select anyone arbitrarily and pass two lines horizontally and vertically.

Step 8: Add up the value attributable to the allocation, which shall be the minimum value.

Step 9 :*Alternate Solution:* If there are more than one unmarked zero in any row or column, select the other one(i.e., other than the one selected in Step7) and pass through lines horizontally and vertically. Add up the value attributable to the allocation, which shall be the minimum value.

LINE DRAWING PROCEDURE:

when it is not possible to assign one cell on each of the row and column (ie any row or column found with out assigment), we have to draw minimum no of lines covering all zeros

step-1. tick ( ) row that do not have any assinment

Step-2. tick ( ) column having crossed zero (0)

Step-3. observe the ticked column, mark rows having assignment. (assigned zero)

Step-4. Repeat the steps-2 and 3 until the chain of ticking complete.

Step-5. draw lines through all ticked column and un-ticked rows. This wil give minimum number of lines . then go for assignment whre the ror or colum contains only one zero.

Step 5: If the total lines covering all zeros are equal to the size of the matrix of the Table, we have got the optimal solution; if not, subtract the minimum uncovered element from all uncovered elements and add this element to all elements at the intersection point of the lines covering zeros.

Step 6: Repeat Steps 4 and 5 till minimum number of lines covering all zeros is equal to the size of the matrix of the Table.

Step 7: Assignment: Select a row containing exactly one unmarked zero and surround it by ,‘and draw a vertical line through the column containing this zero. Repeat this process till no such row is left; then select a column containing exactly one unmarked zero and surround it by, and draw a horizontal line through the row containing this zero and repeat this process till no such column is left.

Note: If there is more than one unmarked zero in any row or column, it indicates that an alternative solution exists. In this case, select anyone arbitrarily and pass two lines horizontally and vertically.

This will give minimum no. of lines. Then go for Assignment

Example-1. ABC Corporation has four plants each of which can manufacture any one of the four products. Product costs differ from one plant to another as follows

You are required to obtain which product each plant would produce to minimize cost.

PLANT	**PRODUCT**			
	1	**2**	**3**	**4**
A	**33**	**40**	**43**	**32**
B	**45**	**28**	**31**	**23**
C	**42**	**29**	**36**	**29**
D	**27**	**42**	**44**	**38**

SOLUTION

PLANT	PRODUCT			
	1	2	3	4
A	33	40	43	32
B	45	28	31	23
C	42	29	36	29
D	27	42	44	38

Initial Table

Step 1 : Row Deduction: Subtracting the minimum element of each row from all the elements of that row

PLANT	PRODUCT			
	1	2	3	4
A	1	8	11	0
B	22	5	8	0
C	13	0	7	0
D	0	15	17	11

After row operation

Solution

SOLUTION

PLANT	PRODUCT			
	1	2	3	4
A	1	8	11	0
B	22	5	8	0
C	13	0	7	0
D	0	15	17	11

Step 2 : Column Deduction: Subtracting the minimum element of each column from all the elements of that column

PLANT	PRODUCT			
	1	2	3	4
A	1	8	4	0
B	22	5	1	0
C	13	0	0	0
D	0	15	17	11

After column operation

SOLUTION

Step-3: perform assignment considering row / Columns having single zero

PLANT	PRODUCT			
	1	2	3	4
A	1	8	4	[0]
B	22	5	1	~~0~~
C	13	~~0~~	[0]	~~0~~
D	[0]	15	17	11

✔ 1

Here row-c does not have any assignment. To proceed further apply line drawing procedure to revise the table.

Example 1.*(Minimization Problem).*

STEP-3 now test whether it is possible to make an assignment using only zeros. If it is possible , the assignment must be optimal

product

	1	2	3	4
A	1	8	4	[0]
B	22	5	1	~~0~~
C	13	[0]	~~0~~	~~0~~
D	[0]	15	17	11

2 ✔ 3 ✔ 1 ✔

,minimum unlined element = 1

- **Subtract mini element from all unlined elements**
- **add min element to elements of line of intersection**

Assign Machine A to Producto-IV =Rs 32
Assign Machine B to Product- III = 31
Assign Machine C to Product –II = 29
Assign Mchine –D to product -I = 27
total cost = Rs 119 hours

Solution Table

PRODUCT

MACHINE	I	II	III	IV
A	0	7	3	[0]
B	21	4	[0]	0
C	13	[0]	0	1
D	[0]	15	17	12

Example -1 *OPTIMAL SOLLUTION*

STEP-6 now test whether it is possible to make an assignment using only zeros. If it is possible , the assignment must be optimal

PRODUCT (columns I–IV); MACHINE (rows A–D)

	I	II	III	IV
A	33	40	43	**32**
B	45	28	**31**	23
C	42	**29**	36	29
D	**27**	42	44	38

Assign Machine A to Product-IV = Rs 32
Assign Machine B to Product- III = 31
Assign Machine C to Product –II = 29
Assign Mchine –D to product -I = 27
total cost = 119

STEPS INVOLVING MAXIMIZATION TYPE OF ASSIGNMENT PROBLEM

- Step 1: See whether Number of Rows is equal to Number of Columns. If yes, problem is a balanced one; if not, then adds a Dummy Row or Column to make the problem a balanced one by allotting zero value or specific value (if any given) to each cell of the Dummy Row or Column, as the case may be.
- Step 2: Derive Profit Matrix by deducting cost from revenue.
- Step 3: Derive Loss Matrix by deducting all elements from the largest element.
- Step 4: Follow the same Steps 2 to 9 as involved in solving Minimization Problems.

Example-2:

Example 2.*(MiNization Problem).*

A Department head has four subordinates, and four tasks have to be performed. Subordinates differ in efficiency and tasks differ in their intrensic difficulty. Time each man would take to perform each task is given in the effectiveness matrix. How the tasks should be allocated to each person so as to minimize the total man-hours.

SUBORDINATES

TASKS	I	II	III	IV
A	8	26	17	11
B	13	28	4	26
C	38	19	18	15
D	19	26	24	3

Example 2.*(Minimization Problem).*

STEP-1. Subtract the smallest element in each row from every element of that row, we will get the reduced matrix

SUBORDINATES

›KS	I	II	III	IV
A	8	26	17	11
B	13	28	4	26
C	38	19	18	15
D	19	26	24	3

Reduced matrix

	I	II	III	IV
A	0	18	9	3
B	9	24	0	22
C	23	4	3	0
D	16	23	21	0

Example 2.*(Manimization Problem).*

STEP-2. next Subtract the smallest element in each rcolumn from every element of that column, we will get the reduced matrix

SUBORDINATES

ꞋKS

	I	II	III	IV
A	0	18	9	3
B	9	24	0	22
C	23	4	3	0
D	16	23	21	0

Reduced matrix

	I	II	III	IV
A	0	14	9	3
B	9	20	0	22
C	23	0	3	0
D	16	19	21	0

Example 2.*(Manimization Problem).*

STEP-2. next Subtract the smallest element in each rcolumn from every element of that column, we will get the reduced matrix

SUBORDINATES

ꞋKS

	I	II	III	IV
A	0	18	9	3
B	9	24	0	22
C	23	4	3	0
D	16	23	21	0

Reduced matrix

	I	II	III	IV
A	0	14	9	3
B	9	20	0	22
C	23	0	3	0
D	16	19	21	0

Example-3: ASSIGNMENT (minimization)

- Five men are available to do five different jobs. From the past records , the time (in hours) that each man takes to do each job is known and given in the following table.

Job

Man		I	II	III	IV	V
	A	2	9	2	7	1
	B	6	8	7	6	1
	C	4	6	5	3	1
	D	4	2	7	3	1
	E	5	3	9	5	1

Find the assignment of men to jobs that will minimize the total time taken.

Example-3: ASSIGNMENT

- Step-1. Subtracting the smallest element of each row from every element of the corresponding row, we get the reduced matrix

Job

Man		I	II	III	IV	V
	A	2	9	2	7	1
	B	6	8	7	6	1
	C	4	6	5	3	1
	D	4	2	7	3	1
	E	5	3	9	5	1

Reduced Matrix

	I	II	III	IV	V
A	1	8	1	6	0
B	5	7	6	5	0
C	3	5	4	2	0
D	3	1	6	2	0
E	4	2	8	4	0

Example-3: Assignment ...contd.

- Step-2. Subtracting the smallest element of each column from every element of the corresponding column , to get the adjoining reduced matrix reduced matrix

Job

	I	II	III	IV	V
A	1	8	1	6	0
B	5	7	6	5	0
C	3	5	4	2	0
D	3	1	6	2	0
E	4	2	8	4	0

Reduced Matrix

	I	II	III	IV	V
A	0	7	0	4	0
B	4	6	5	3	0
C	2	4	3	0	0
D	2	0	5	0	0
E	3	2	7	2	0

Example-3: Assignment .. Contd.

- Step-3 Perform assignment for the rows/ columns where there is only one zero only. Mini = 2
- Unlined .

	I	II	IV	V
A	0	7	0	0
B	4	6	3	0
C	2	4	0	0
D	2	0	0	0
E	3	2	2	0

Reduced Matrix

	I	II	III	IV	V
A	0	7	0	4	2
B	2	4	3	2	0
C	2	4	3	0	2
D	2	0	5	0	2
E	1	0	5	0	0

Example-3: Solution

Assign man –A to job-III = 2 hours
Assign Man-B to job-V = 1 hour
Assign Man-C to Job –IV = 3 hours
Assign Man –D to Job –II = 2 hours
Assign Man –E to Job-I = 5 hours Total Time = 13 hours

	I	II	III	IV	V
A	2	9	2	7	1
B	6	8	7	6	1
C	4	6	5	3	1
D	4	2	7	3	1
E	5	3	9	5	1

	I	II	III	IV	V
A	0	8	0	5	2
B	2	5	3	3	0
C	1	4	2	0	1
D	1	0	4	0	1
E	0	0	4	0	0

MAXIMAL ASSSIGNMENT PPROBLEM:

Sometimes, the assignment problem deals with the maximization of an objective function rather than to, minimize it. For example, it may be required to assign persons to jobs in such a way that the expected profit is maximum. Such problem may be solved easily by first converting it to a minimization problem and then applying the usual procedure of assignment algorithm. This conversion can be very easily done by subtracting from the highest element, all the elements of the given profit matrix; or equivalently, by placing minus sign before each element of the profit-matrix in order to make it cost-matrix.

Following examples will make the procedure clear.

Example 4. ASSIGNMEMT *(Maximization Problem).*

A company has 5 jobs to be done. The following matrix shows the return in rupees on assigning i th (i = 1,2,3,4,5) machine to the jth job (j =A, B, C, D, E). Assign the five jobs to the five machines so as to maximize the total expected profit

Jobs

Machines	A	B	C	D	E
1	5	11	10	12	4
2	2	4	6	3	5
3	3	12	5	14	6
4	6	14	4	11	7
7	9	8	12	5	5

SOLUTION OF EXAMPLE-4 Contd..

Step 3.. Performing column operation : consider each of the column and subtract smallest element from other elements of that column

	A	B	C	D	E
1	7	1	2	0	8
2	4	2	0	3	1
3	11	2	9	0	8
4	8	0	10	3	7
5	5	3	4	0	7

After column operation

	A	B	C	D	E
1	3	1	2	0	7
2	0	2	0	3	0
3	7	2	9	0	7
4	4	0	10	3	6
5	1	3	4	0	7

SOLUTION OF EXAMPLE-4 (maximization) ..contd.

Step-4. after assigning, row-3 and row-5 does not contain any assignment. At this stage no optimal solution is attained. Draw minimum no. of lines passing through all zeros. Select minimum among unlined element.

3* No change of elemen[t] on the line other than li[ne] intersection

- **1.Subtract this element from all other unlined elements.**
- 2. **add this element to intersection elements.**

JOB

	A	B	C	D	E	
				✔ 3		
1	3	1	2	[0]	7	✔ 4
2	[0]	2	0	3	0	
3	7	2	9	0	7	✔ 1
4	4	0	10	3	[0]	
5	1	3	4	0	7	✔ 2

Optimal solution is attained

	A	B	C	D	E
1	2	[0]	1	0	6
2	0	2	[0]	4	0
3	6	1	8	[0]	6
4	5	0	10	4	[0]
5	[0]	2	3	0	6

OPTIMAL SOLUTION OF EXAMPLE-4

Optimal solution is attained

	A	B	C	D	E
1	2	[0]	1	0	6
2	0	2	[0]	4	0
3	6	1	8	[0]	6
4	5	0	10	4	[0]
5	[0]	2	3	0	6

MACHINE / JOB

	A	B	C	D	E
1	5	11	10	12	4
2	2	4	6	3	5
3	3	12	5	14	6
4	6	14	4	11	7
7	7	8	12	5	1

plant	JOB	PROFIT(Rs)
Assign Machine-1	B	11
Assign Machine -2	C	6
Assign Machine-3	D	14
Assign Machine-4	E	7
Assign Machine-5	A	7
Maximum Profit		Rs 45

UNBALANCED ASSIGNMENT PROBLEM

- If the cost matrix of an assignment problem is not a square matrix (number of sources not equal to number of destinations), then the assignment problem is called an Unbalanced problem,
- In such cases , fictitious rows / columns with '0' costs are added in the matrix so as to form a square matrix. Then the usual assignment algorithm can be applied to the resulting problem.

Example : UNBALANCED ASSIGNMENT PROBLEM SOLUTION ...... Contd.

JOBS

Profit matrix

IINES

	A	B	C	D	E	Dummy
1	25	50	10	60	10	0
2	20	50	15	70	30	0
3	30	65	20	80	30	0
4	35	70	20	90	45	0
5	40	70	30	90	60	0
6	60	90	50	100	60	0

Step-1. introducing one dummy column with profit elements as zeo
Since this problem is maximization type . Convert the problem into cost matrix by subtracting all elements from the highest element (100)

Example : UNBALANCED ASSIGNMENT PROBLEM SOLUTION ...... Contd.

Step-2. . Convert the problem into cost matrix by subtracting all elements from the highest element (100)

cost matrix

IINES

	A	B	C	D	E	Dummy
1	75	50	90	40	90	100
2	80	50	85	30	70	100
3	70	35	80	20	70	100
4	65	30	80	20	55	100
5	60	30	70	10	40	100
6	40	10	50	0	40	100

Example : UNBALANCED ASSIGNMENT PROBLEM SOLUTION ...... Contd.

Step-3: perform column operation

	A	B	C	D	E	Dummy
1	75	50	90	40	90	100
2	80	50	85	30	70	100
3	70	35	80	20	70	100
4	65	30	80	20	55	100
5	60	30	70	10	40	100
6	40	10	50	0	40	100

1.After row operation

	A	B	C	D	E	Dummy
1	35	10	50	0	50	60
2	50	20	55	0	40	70
3	50	15	60	0	50	80
4	45	10	60	0	45	80
5	50	20	60	0	30	90
6	40	10	50	0	40	100

Example : UNBALANCED ASSIGNMENT PROBLEM SOLUTION ...... Contd.

Step-2: perform column operation

	A	B	C	D	E	Dummy
1	35	10	50	0	50	60
2	50	20	55	0	40	70
3	50	15	60	0	50	80
4	45	10	60	0	45	80
5	50	20	60	0	30	90
6	40	10	50	0	40	100

1.After column operation

	A	B	C	D	E	Dummy
1	0	0	0	0	20	0
2	15	10	5	0	10	10
3	15	5	10	0	20	20
4	10	0	10	0	15	20
5	15	10	10	0	0	30
6	5	0	0	0	10	40

Example : UNBALANCED ASSIGNMENT PROBLEM SOLUTION ...... Contd.

Step-3: Perform assignment with single zeros

	A	B	C	D ✔ 2	E	Dummy	
1	[0]	~~0~~	~~0~~	~~0~~	20	~~0~~	
2	15	10	5	[0]	10	10	✔ 3
3	15	5	10	~~0~~	20	20	✔ 1
4	10	[0]	10	~~0~~	15	20	
5	15	10	10	~~0~~	[0]	30	
6	5	~~0~~	[0]	~~0~~	10	40	

Aftter correcting with unlined small element = 5

	A	B	C	D	E	Dummy
1	[0]	~~0~~	~~0~~	5	20	0
2	10	5	~~0~~	[0]	5	5
3	10	~~0~~	5	~~0~~	15	15
4	10	[0]	10	5	15	20
5	15	10	10	5	[0]	30
6	5	~~0~~	[0]	5	10	40

✔ ✔ ✔

Example : UNBALANCED ASSIGNMENT PROBLEM SOLUTION …… Contd.

	A	B	C	D	E	Dummy
1	0	0	0	5	15	0
2	5	5	5	0	10	10
3	15	5	10	0	20	20
4	10	0	10	0	15	20
5	15	10	10	0	0	30
6	5	0	0	0	10	40

Step-5: perform line drawing operation

	A	B	C	D	E	Dummy
1	0	0	0	5	20	0
2	10	5	0	0	5	5
3	10	0	5	0	15	15
4	10	0	10	5	15	20
5	15	10	10	5	0	30
6	5	0	0	5	10	40

RESTRICTION S ON ASSIGNMENT

- Some times technical, legal or other restrictions do not permit the assignment of a particular facity to a particular job. Such a difficulty can be overcome by assigning a very high cost (say , infinite cost) to the corresponding cell, so that the activity will be automatically excluded from the optimal solution.

Example - RESTRICTION S ON ASSIGNMENT

A job shop has purchased 5 new machines of different type. They are 5 available locations in the shop where a machine could be installed. Some of these locations are more desirable than others for particular machines because of their proximity to work centres which would have a heavy work flow to and from these machines. Therefore , the objective is to assign the new machines to the available locations in order to minimize the total cost of material handling. The estimated cost per unit time of materials handling involving each of the machines is given in table for respective locations. Locations 1,2,3, 4, and 5 ae not considered suitable for machines A, B, C, D and E respectively.

Example - RESTRICTION S ON ASSIGNMENT

Since locations 1, 2, 3, 4 and 5 are not suitable for machines A, B, C, D and E respectively, an ‘ X’ is shown in cost matrix some times denoted by “---”

LOCATIONS

MACHINES	1	2	3	4	5
A	X	10	25	25	10
B	1	X	10	15	2
C	8	9	X	20	10
D	14	10	24	X	15
E	10	8	25	27	X

Example - RESTRICTION S ON ASSIGNMENT

Since locations 1, 2, 3, 4 and 5 are not suitable for machines A, B, C, D and E respectively, an extremely large cost (say ∞) should be attached to these locations. The cost matrix is as follows:

LOCATIONS

MACHINES

	1	2	3	4	5
A	∞	10	25	25	10
B	1	∞	10	15	2
C	8	9	∞	20	10
D	14	10	24	∞	15
E	10	8	25	27	∞

Example : UNBALANCED ASSIGNMENT PROBLEM SOLUTION ...... Contd.

Step-1: perform row operation

	1	2	3	4	5
A	∞	10	25	25	10
B	1	∞	10	15	2
C	8	9	∞	20	10
D	14	10	24	∞	15
E	10	8	25	27	∞

→

1.After row operation

	1	2	3	4	5
A	∞	0	15	15	0
B	0	∞	9	14	1
C	0	1	∞	12	2
D	4	0	14	∞	5
E	2	0	15	17	∞

Example : UNBALANCED ASSIGNMENT PROBLEM SOLUTION …… Contd.

1.After column peration

	1	2	3	4	5
A	∞	0	6	3	0
B	0	∞	0	2	1
C	0	1	∞	0	2
D	4	0	5	∞	5
E	2	0	6	5	∞

Step-2: perform column operation

	1	2	3	4	5
A	∞	0	15	15	0
B	0	∞	9	14	1
C	0	1	∞	12	2
D	4	0	14	∞	5
E	2	0	15	17	∞

Example : UNBALANCED ASSIGNMENT PROBLEM SOLUTION …… Contd.

Step-3. perform assignment operation

	1	2 ✔ 2	3	4	5	
A	∞	0	6	3	[0]	
B	0	∞	[0]	2	1	
C	0	1	∞	[0]	2	M/C
D	4	[0]	5	∞	5	✔ 3
E	2	0	6	5	∞	✔ 1

[The left margin shows boxed 0s beside rows A and E.]

Step-4. unlined mini element = 2
OPTIMUM SOLUTION

LOCATION

	1	2	3	4	5
A	∞	2	6	3	[0]
B	0	∞	[0]	2	1
C	0	3	∞	[0]	2
D	2	[0]	3	∞	3
E	[0]	0	4	3	∞

Example : UNBALANCED ASSIGNMENT PROBLEM SOLUTION …… Contd.

Step-3. perform assignment operation

M/C	1	2	3	4	5
A	∞	0	6	3	0
B	0	∞	0	2	1
C	0	1	∞	0	2
D	4	0	5	∞	5
E	2	0	6	5	∞

Step-4. unlined mini element = 2
OPTIMUM SOLUTION

LOCATION

M/C	1	2	3	4	5
A	∞	2	6	3	0
B	0	∞	0	2	1
C	0	3	∞	0	2
D	2	0	3	∞	3
E	0	0	4	3	∞

Example : UNBALANCED ASSIGNMENT PROBLEM SOLUTION …… Contd.

Step-5. OPTIMUM SOLUTION

M/C	1	2	3	4	5
A	∞	10	25	25	10
B	1	∞	10	15	2
C	8	9	∞	20	10
D	14	10	24	∞	15
E	10	8	25	27	∞

Step-4. unlined mini element = 2
OPTIMUM SOLUTION

LOCATION

M/C	1	2	3	4	5
A	∞	2	6	3	0
B	0	∞	0	2	1
C	0	3	∞	0	2
D	2	0	3	∞	3
E	0	0	4	3	∞

Assign Machine A to Location-5 = Rs 10
Assign Machine- B to Location -3 = 10
Assign Machine –C to Location-4 = 20
Assign Machine-D to Location-2 = 10
Assign Machine-E to location -1 = 10
Total cost **= Rs 60**

TRAVELING SALESPERSON PROBLEMS

"Given a number of cities and the costs of traveling from any city to any other city, what is the cheapest round-trip route (tour) that visits each city once and then returns to the starting city?" This problem is called the traveling salesperson problem (TSP), not surprisingly. An itinerary that begins and ends at the same city and visits each city once is called a ***tour***.

Suppose there are N cities.

Let c_{ij}= Distance from city i to city j (for i j) and

Let c_{ii} = M (a very large number relative to actual distances) Also define x_{ij} as a 0-1 variable as follows:

x_{ij}= 1 if s/he goes from city i to city j; x_{ij}= 0otherwise The formulation of the TSP is: min $\sum_i \sum_j c_{ij}x_{ij}$

s.t. $\sum_i x_{ij} = 1$ for all j

$\sum_j x_{ij}=1$ for all i

$u_i - u_j + N x_{ij} \leq N-1$ for i j; $i,j>$

1 All x_{ij}= 0 or 1, All $u_i \geq 0$

The first set of constraints ensures that s/he arrives once at each city. The second set of constraints ensures that s/he leaves each city once. The third set of constraints ensure the following:

Any set of x_{ij}"s containing a sub-tour will be infeasible Any set of x_{ij}"s that forms a tour will be feasible

$u_i - u_j + N x_{ij} \leq N-1$ for i j; $i, j >1$ Assume N=5

Sub-tours: 1-5-2-1, 3-4-3 ???

Choose the sub-tour that does not contain city 1:

$u_3 - u_4 + 5 x_{34} \leq 4$ $u_4 - u_3 + 5 x_{43} \leq 4$ $5 (x_{34} + x_{43}) \leq 8$ This rules out the possibility that $x_{34} = x_{43} = 1$The formulation of an IP whose solution will solve a TSP becomes unwieldy and inefficient for large TSPs.

When using branch and bound methods to solve TSPs with many cities, large amounts of computer time may be required. For this reason, heuristics, which quickly lead to a good (but not necessarily optimal) solution to a TSP, are often used.

MODULE-III:
SEQUENCING

Introduction

Suppose there are n jobs to perform, each of which requires processing on some or all of *m* different machines. The effectiveness *(i.e.* cost, time or mileage ,etc.)can be measured for any given sequence of jobs at each machine, and the most suitable sequence is to be selected (which optimizes the effectiveness measure) among all *(n!)m* theoretically possible sequences.Although,theoretically,itisa1wayspossibletoselect the best sequence by testing each one, but it is practically impossible because of large number of computations.

In particular, if *m* = 5 and *n* = 5, the total number of possible sequences will be (5 !)5 = 25,000,000,000. Hence the effectiveness for each of (5 !)5 sequences is to be computed before selecting the most suitable one. But, this approach is practically impossible to adopt. So easier methods of dealing with such problems are needed. Before proceeding to our actual discussion we should explain what the sequencing problem is. The problem of sequencing may be defined as follows:

Definition: Suppose there are n jobs (1, 2, 3,..., n), each of which has to be processed one at a time at each of m machines A, *B,* C, ... The order of processing each job through machines is given (for example, job is processed through machines A, C, and B in this order). The time that each job must require on each machine is known. The problem is to find a sequence among *(n!) m* number of all possible sequences (or combinations) (or order) for processing the jobs so that the total elapsed time for all the jobs will be minimum. Mathematically, let

Ai = time for job *i* on machine A,

Bi = time for job *i* on machine *B,* etc.

T = time from start of first job to completion of the last job.

Then, the problem is to determine for each machine a sequence of jobs *i1, i2, i3, ...,in-* where *(i1,i2, i3, ..., in)* is the permutation of the integers which will minimize *T.*

Terminology and Notations

The following *terminology* and notations will be used in this chapter.

i. Number of Machines. It means the service facilities through which. a job must pass before it is completed..

For example, a book to be published has to be processed through composing, printing, binding, etc. In this example, the book constitutes the *job* and the different processes constitute the *number of machines.* .

ii. Processing order:.It refers to the order in which various machines are required for completing the job.

iii. Processing Time: It means the time required by each job on each machine. The notation *Tij* will denote the processing time required for ith job b y*jth* machine$(i = 1,2,..., n; j= 1, 2,...,m)$.

iv. Idle Time on a Machine. This is the time for which a machine remains idle during the total elapsed time. The notation *Xij* shall be (used to denote the idle time of machine *j* between the *end* of the (i - 1)th job and the start of the ith job.

v. Total Elapsed Time. This is the time between starting the first job and completing the last job. This also includes *idle time,* if exists. It will be denoted by the symbol *T*.

vi. No Passing Rule. This rule means that P1lssing is not allowed, *i.e.* the same order of jobs is maintained over each machine. If each of the n-jobs is to be processed through two machines *A* and *B* in the order *AB,* then this rule means that each job will go to machine *A* first and then to *B.*

Principal Assumptions

i. No machine can process more than one operation at a time.

ii. Each operation, once started, must be performed till completion.

iii. A job is an entity, *i.e.* even though the job represents a lot of individual parts, no lot may be processed by more than one machine at a time.

iv. Each operation must be completed before any other operation, which it must precede, can begin.

v. Time intervals for processing are independent of the order in which operations are performed.

vi. There is only one of each type of machine.

vii. A job is processed as soon as possible subjec to ordering requirements.

viii. All jobs are known and are ready to start processing before the period under consideration begins.

ix. The time required to transfer jobs between machines is negligible.

- Processingnjobsand2Machines
- Processingnjobsand3Machines
- Processing n jobs and m Machines

Processing n Jobs Through Two Machines

The problem can be described as: (i) only two machines *A* and *B* are involved, *(ii)* each job is pro-cessedintheorder*AB*,and*(iii)*theexactorexpectedprocessingtimesA_1, A_2, ,*A3*,...,*An; B_1, B2, B3, ..., Bn* are known

The problem is to sequence (order) the jobs so as to minimize the total elapsed time *T*.

The solution procedure adopted by *Johnson* (1954) is given below.

Processing Times	Job (i)				
	1	2	3	...	n
A_i	A_1	A_2	A_3	...	A_n
B_i	B_1	B_2	B_3	...	B_n

Solution Procedure

Step 1. Select, the least processing time occurring in the list A_I, A_2, $A_{3,...,}$ Ar and B_1, B_2, $B_{3'}$ ..., *BII'* If there is a tie, either of the smallest processing time should be selected.

Step 2. If the least processing time is *Ar* select *r th* job *first.* If it is *Bs*, do the s th job *last* (as the given order *is AB).*

Step 3. There are now *n* - I jobs left to be ordered. Again repeat steps I and n for the reduced set of processing times obtained by deleting processing times for both the machines corresponding to the job already assigned. .

Continue till all jobs have been ordered. The resulting ordering will minimize the elapsed time T.

Proof. Since passing is not allowed, all *n* jobs must be processed on machine *A* without any *idle time* for it. On the other hand, machine *B is* subject to its remaining idle time at various stages. Let *Yj*be the time for which m machine B remains idle after completing *(i* - l)th job and before starting processing the ith job *(i*=1,2, ..., *n).* Hence, the total elapsed time T is given by

$$T = \sum_{i=1}^{n} B_i + \sum_{i=1}^{n} Y_i$$

Example: There are five jobs each of which must go through the two machines A and B

Processing time (hours)					
Job	1	2	3	4	5
Time for *A*	5	1	9	3	10
Time for *B*	2	6	7	8	4

in the order A,B. processing times are given below:

Determine a sequence of or five jobs that will minimize the elapsed time T. Calculate the total idle time for the machines in this period.

Solution. Apply steps I and II of solution procedure. It is seen that the smallest processing time is one hour for job 2 on the machine A. So list the job 2 at first place as shown above.

2				

Now, the reduced list of processing times becomes

Job	*A*	*B*
1	5	2
3	9	7
4	3	8
5	10	4

Again, the smallest processing time in the reduced list is 2 for job 1 on the machine *B*. So place job 1 last.

2				1

Continuing in the like manner, the next reduced list is obtained

Job	*A*	*B*
3	9	7
4	3	8
5	10	4

leading to sequence

2	4			1

and the list

Job	*A*	*B*
3	9	7
5	10	4

gives rise to sequence

2	4		5	1

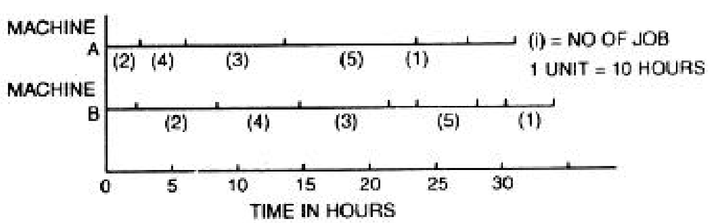
Finally, the optimal sequence is obtained,

2	4	3	5	1

Further, it is also possible to calculate the minimum elapsed time corresponding to the optimal sequencing, using the individual processing time given in the statement of the problem. The details are given in Table

Job sequence	Machine A		Machine B	
	Time in	Time out	Time in	Time out
2	0	1	1	7
4	1	4	7	15
3	4	13	15	22
5	13	23	23	27
1	23	28	28	30

Thus, the minimum time, i.e. the time for starting of job 2 to completion of the last job 1, is 30 hrs only. During this time, the machine A remains idle for 2 hrs (from 28 to 30 hrs) and the machine B remains idle for 3 hrs only (from 0-1,22-23, and 27-28 hrs). The total elapsed time can also be calculated by using Gantt chart as follows:



From the Fig it can be seen that the total elapsed time is 30 hrs, and the idle time of the machine B is 3 hrs. In this problem, it is observed that job may be held in inventory before going to the machine. For example, 4th job will be free on machine A after 4th hour and will start on machine B after 7th hr. Therefore, it will be kept in inventory for 3 hrs. Here it is assumed that the storage space is available and the cost of holding the inventory for each job is either same or negligible. For short duration process problems, it is negligible. Second general assumption is that the order of completion of jobs has no significance, i.e. no job claims the priority.

Processing n Jobs Through Three Machines

The problem can be described as: *(i)* Only three machines *A, B* and C are involved, *(ii)* each job is processed in the prescribed order *ABC, (iii)* transfer of jobs is not permitted, *i.e.* adhere strictly the order over each machine, and *(iv)* exact or expected processing times are given in Table

Job	Machine A	Machine B	Machine C
1	A_1	B_1	C_1
2	A_2	B_2	C_2
3	A_3	B_3	C_3
:	:	:	:
n	A_n	B_n	C_n

Optimal Solution. So far no general procedure is available for obtaining an optimal sequence in this case.

However, the earlier method adopted by Johnson (1954) can be extended to cover the special cases where either one or both of the following conditions hold:

i. The minimum time on machine A the maximum time on machine B.

ii. The minimum time on machine C the maximum time on machine B.

The procedure explained here (without proof) is to replace the problem with an equivalent

$$G_i = A_i + B_i,\ H_i = B_i + C_i.$$

problem, involving n jobs and two fictitious machines denoted by G and H, and corresponding time G_j and H_j are defined by

If this problem with prescribed ordering *GH* is solved, the resulting optimal sequence will also be optimal for the original problem.

Rules for deleting the programs which cannot be optimal.

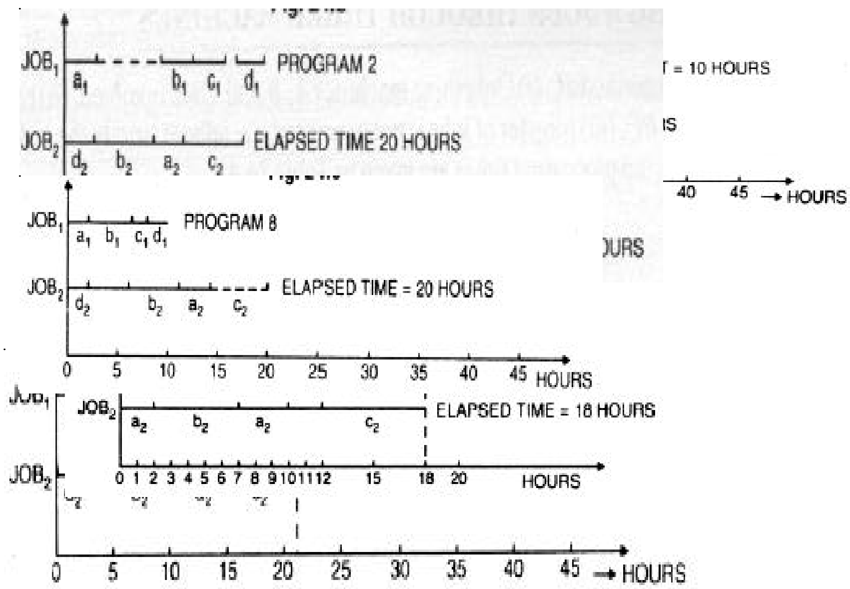
Rule no.	Technological orderings for		Delete programs containing
	Job 1	Job 2	
I	X...Y	Y...	xy
II	X...Y...	...XY	$\bar{x}y$
III	...X...Y	...XY...	$\bar{\bar{x}}y$
IV	...XY...	X...Y...	xy
V	...XY...Z...	...X...YZ...	$\bar{x}yz$
VI	...X...YZ...	...XY...Z	$\bar{x}\bar{y}z$

Here... stands for other machines, if any. Applying the rules to this example, it is observed by taking *A* as X, and D as *Y* (I rule), that delete the programs containing *ad.* Such a program is 16th only. Again by rule taking *A* as X and C as *Y*, all those programs are deleted which contain *lie, i.e.,* the 5th program. Other rules are not applicable to our problem. Thus we have only following five programs.

Program No.				
1	2	4	6	8
$\bar{a}$	a	a	a	a
$\bar{b}$	$\bar{b}$	b	$\bar{b}$	b
$\bar{c}$	$\bar{c}$	$\bar{c}$	c	c
$\bar{d}$	$\bar{d}$	$\bar{d}$	$\bar{d}$	d

Now finally we enumerate all these programs one by one using Gantt Chart as shown below:

'From these charts it is clear that optimum program is 6th and the minimum elapsed time



is 18 hours.

Example. There are five jobs, each of which must go through

Job *i*	Processing Times		
	A_i	B_i	C_i
1	8	5	4
2	10	6	9
3	6	2	8
4	7	3	6
5	11	4	5

machines A, Band C in the order ABC. Processing times are given
in Table

Determine a sequence for five jobs that will minimize the elapsed time T. Solution. Here min *Ai* = 6, max *Bi* = 6, min *Ci*= 4.

Since one of two conditions is satisfied by min *Ai* =max *Bi*, so the procedure adopted in *Example 1* can be followed.

The equivalent problem, involving five jobs and two fictitious machine G and *H*, becomes:

Job i	Processing Times	
	$G_i (= A_i + B_i)$	$H_i (= B_i + C_i)$
1	13	9
2	16	15
3	8	10
4	10	9
5	15	9

This new problem can be solved by the procedure described earlier. Because of ties, possible optimal sequences are :

(*i*) 3 2 1 4 5 (*ii*) 3 2 4 1 5

(*iii*) 3 2 4 5 1 (*iv*) 3 2 5 4 1

(*v*) 3 2 1 5 4 (*vi*) 3 2 5 1 4

It is possible to calculate the minimum elapsed time for first sequence as shown in Table

Job	Machine *A*		Machine *B*		Machine *C*	
	Time in	Time out	Time in	Time out	Time in	Time out
3	0	6	6	8	8	16
2	6	16	16	22	22	31
1	16	24	24	29	31	35
4	24	31	31	34	35	41
5	31	42	42	46	46	51

Thus, any of the sequences from *(i)* to *(vi)* may be used to order the jobs through machines *A, B* and C. and they all will give a minimum elapsed time of 51 hrs. Idle time for machine *A* is 9 hrs, for *B* 31 hrs, for C 19hrs.

Graphical Method

In the two job m-machine problem, there is a graphical procedure, which is rather simple to apply and usually provides good (though not necessarily optimal) results. The following example will make the graphical procedure clear.

Example 3. Use graphical method to minimize the time needed to process the following

Job 1	Sequence of Machines	:	A	B	C	D	E
	Time	:	2	3	4	6	2
Job 2	Sequence of Machines	:	C	A	D	E	B
	Time	:	4	5	3	2	6

jobs onthemachinesshownbelow,i.e.foreachmachinefindthejob,whichshouldbedonefirst. Also calculate the total time needed to complete both thejobs.

Solution.

Step 1. First, draw a set of axes, where the horizontal axis represents processing time on job 1 and the vertical axis represents processing time on job 2

Step 2. Layout the machine time for two jobs on the corresponding axes in the given techno-

logicalorderMachineAtakes2hrsforjob1and5hrsforjob2.ConstructtherectanglePQRS for the machine A. Similarly, other rectangles for machines B, C, D and E are constructed as shown.

Step 3. Make programme by starting from the origin 0 and moving through various states of completion (points) until the point marked

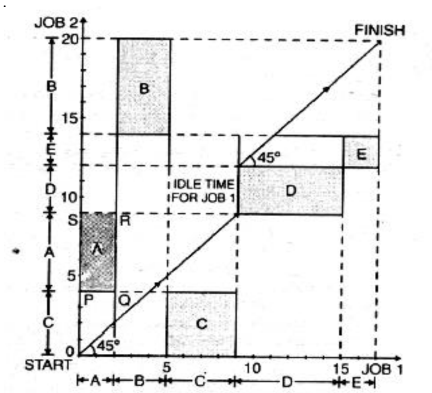
Graphical solution for the 2-job 5-machine sequencing problem. ‘finish’ is obtained. Physical interpretation of the path thus chosen involves the series of segments, which are horizontal or vertical or diagonal making an angle of 45° with the horizontal. Moving to the right means that job 1is proceeding while job 2 is idle, and moving upward means that job 2 is proceeding while job 1 is idle, and moving diagonally means that both the jobs are proceeding simultaneously.

Further, both the jobs can not be processed simultaneously on the same machine

.Graphically, diagonal movement through the blocked-out(shaded) area is not allowed, and similarly for other machines too.

Step4.To find an optimal path. An optimal path (programme) is one that minimizes idle time for job (horizontal movement). Similarly, an optimal path is one that minimizes idle time for job 2 (vertical movement). Choose such a path on which diagonal movement is as much as possible. According to this, choose a good path by inspection as shown by arrows.

Step5. To find the elapsed time. The elapsed time is obtained by adding the idle time for either of the job to the processing time for that job. In this problem, the idle time for the chosen path is seen to be 3 hrs. for the job I, and zero for the job 2. Thus, the total elapsed time, 17 + 3 = 20 hrs is obtained.



REPLACEMENT

In this chapter we will discuss

- Replacement introduction
- Replacement of Items that Deteriorate

Introduction: The Replacement Problem

The replacement problems are concerned with the situation that arise when some items such as men, machines, electric light bulbs, etc. need replacement due to their decreased efficiency, failure or breakdown. Such decreased efficiency or complete breakdown may either be gradual or all of a sudden.

The replacement problem arises because of the following factors:

1. The old item has become in worse condition and work badly or requires expensive maintenance.

2. The old item has failed due to accident or otherwise and does not work at all, or the old item is expected to fail shortly.

3. A better or more efficient design of machine or equipment has become available in the market.

In the case of items whose efficiency go on decreasing according to their age, it requires to spend more money on account of increased operating cost, increased repair cost, increased scrap, etc. So in such cases, the replacement of an old item with new one is the only alternative to prevent such increased expenses.

Thus the problem of replacement is to decide best policy to determine an age at which the replacement is most economical instead of continuing at increased cost. The need for replacement arises in many situations so that different type of decisions may have to be taken. For example,

i. We may decide whether to wait for complete failure of the item (which might cause some loss),or to replace earlier at the expense of higher cost of the item.

ii. The expensive items may be considered individually to decide whether we should replace now or, if not, when it should be reconsidered for replacement.

iii. It may be decided whether we should replace by the same type of item or by different type (latest model) of item. The problem of replacement is encountered in the case of both men and machines. Using probability it is possible to estimate the chance of death (or failure) at various ages.

The main objective of replacement is to direct the organization for maximizing its profit (or minimizing the cost).

Failure Mechanism of Items

The term 'failure' has a wider meaning in *business* than what it has in our daily life. There are *two* kinds of failure.

1. Gradual Failure. The mechanism under this category is progressive. That is, as the life of an item increases, its efficiency deteriorates, causing:

i. Increased expenditure for operating costs,

ii. decreased productivity of the equipment,

iii. Decrease in the value of the equipment, *i.e.,* the resale of saving value decreases.

For example, mechanical items like pistons, bearings, rings etc. Another example is 'Automobile tyres'.

2. Sudden Failure. This type of failure is applicable to those items that do not deteriorate markedly with service but which ultimately fail after some period of using. The period

between installation and failure is not constant for any particular type of equipment but will follow some frequency distribution, which may be progressive, retrogressive or random in nature.

i. Progressive failure: Under this mechanism, probability of failure increases with the increase in the life of an item. For example, *electric light bulbs, automobile tubes,* etc.

ii. Retrogressive failure: Certain items have more probability of failure in the beginning of their life, and as the time passes the chances of failure become less. That is, the ability of the MODULE to survive in the initial period of life increases its expected life. Industrial equipments with this type of distribution of life span are exemplified by aircraft engines.

iii. Random failure: Under this failure, constant probability of failure is associated with items that fail from random causes such as *physical shocks,* not related to age. In such a case, virtually all items fail before aging has any effect. For example, vacuum tubes in air-borne equipment have been shown to fail at a rate independent of the age of the tube.

The replacement situations may be placed into *four* categories:

1. Replacement of capital equipment that becomes worse with time, *e.g.machinestools ,buses in a transport organization, planes,* etc.
2. Group replacement of items that fail completely, *e.g., light bulbs, radio tubes,* etc.
3. Problems of mortality ands taffing.
4. Miscellaneous Problems. Replacement of Items that deteriorate

Costs to be Considered

In general, the costs to be included in considering replacement decisions are all those costs that depend up on the choice or age of machine. In some special problems, certain costs need not be included in the calculations. For example, in considering the optimum decision of replacement for a particular machine, the costs that do not change with the age of the machine need not be considered.

When The Replacement Is Justified?

This question can easily be answered by considering a case of truck owner whose problem is to find the 'best' time at which he should replace the old truck by new one. The truck owner wants to transport goods as cheaply as possible. The associated costs are:

(i) The running costs, and *(ii)* the capital costs of purchasing a truck.

These associated costs can be expressed as average cost per month. Now the truck owner will observe that the average monthly cost will go on decreasing, longer the replacement is postponed. However, there will come an age at which the rate of increase of running costs more than compensates the saving in average capital costs. Thus at this age the replacement is justified.

Replacement Policy for Items Whose Maintenance Cost increases with Time, and Money Value is Constant during a period.

Theorem 22.1.The cost of maintenance of a machine is given, as a function increasing with time and its scrap value is constant.

a) If time is measured continuously, then the average annual cost will be minimized by replacing the machine when the average cost to date becomes equal to the current maintenance cost.

b) if time is measured in discrete MODULEs, then the average annual cost will be minimized by replacing the machine when the next period's maintenance cost becomes greater than the current average cost.

Proof. (a)-When time *'t'* is a continuous variable.

Let R_t= maintenance cost at time t, C = the capital cost of the item, S = the scrap value of the item. Obviously, annual cost of the item at any time $t = Rt + C - S$.

(b) When time 't' is a discrete variable.

Since the time is considered in discrete units, the cost equation (22.1) can be written as

$$F(n) = \frac{P(n)}{n} = \sum_{t=1}^{n} \frac{R_t}{n} + \frac{C-S}{n}.$$

By using finite differences, $F(n)$ will be minimum if the following relationship is satisfied. :

$$\Delta F(n-1) < 0 < \Delta F(n)$$

Now, differencing (22.4) under the summation sign by definition of first difference.

$$\begin{aligned}
\Delta F(n) &= F(n+1) - F(n) \\
&= \left[\sum_{t=1}^{n+1} \frac{R_t}{n+1} + \frac{C-S}{n+1}\right] - \left[\sum_{t=1}^{n} \frac{R_t}{n} + \frac{C-S}{n}\right] \\
&= \left(\frac{R_{n+1}}{n+1} + \sum_{t=1}^{n} \frac{R_t}{n+1}\right) - \sum_{t=1}^{n} \frac{R_t}{n} + (C-S)\left[\frac{1}{n+1} - \frac{1}{n}\right] \\
&= \frac{R_{n+1}}{n+1} + \sum_{t=1}^{n} R_t\left(\frac{1}{n+1} - \frac{1}{n}\right) + (C-S)\left[\frac{1}{n+1} - \frac{1}{n}\right] \\
&= \frac{R_{n+1}}{n+1} - \sum_{t=1}^{n} \frac{R_t}{n(n+1)} - \frac{C-S}{n(n+1)}.
\end{aligned}$$

Since $\Delta F(n) > 0$ for minimum of $F(n)$, so

$$\frac{R_{n+1}}{n+1} > \sum_{t=1}^{n} \frac{R_t}{n(n+1)} + \frac{C-S}{n(n+1)} \quad \text{or} \quad R_{n+1} > \sum_{t=1}^{n} \frac{R_t}{n} + \frac{C-S}{n}$$

or $\quad R_{n+1} > P(n)/n$, by virtue of equation (22.4).

Similarly, it can be shown that $R_n < P(n)/n$, by virtue of $\Delta F(n-1) < 0$

Hence $\quad R_{n+1} > (P(n)/n) > R_n$.

This completes the proof.

$$P(n) = \int_0^n R_t\,dt + C - S.$$

Hence average total cost is given by $\quad F(n) = \frac{P(n)}{n} = \frac{1}{n}\int_0^n R_t\,dt + \frac{C-S}{n}$

Now, we have to find such time n for which $F(n)$ is minimum. Therefore, differentiating $F(n)$ w.r.t. 'n',

$$\frac{dF(n)}{dn} = \frac{1}{n}R_n + \left(-\frac{1}{n^2}\right)\int_0^n R_t\,dt - \frac{C-S}{n^2} = 0, \text{ for minimum of } F(n),$$

which gives $\quad R_n = \frac{1}{n}\int_0^n R_t\,dt + \frac{C-S}{n} = \frac{P(n)}{n}$, by virtue of equation

Hence, maintenance cost at time n = average cost in time n.

Example 1.The cost of a machine is Rs. 6100 and its scrap value is only Rs. 100. The

Year	:	1	2	3	4	5	6	7	8
Maintenance cost in Rs.	:	100	250	400	600	900	1250	1600	2000

maintenance costs are found from experience to be:

When should machine be replaced?

Solution. First, find an average cost per year during the life of the machine as follows:

TotalcostinfirstyearMaintenancecostinfirsyear+lossinpurchaseprice=100+(6100-100)=Rs.6100

:. Average cost in first year =Rs. 6100/1 = **Rs 6100**

Total cost up to two years =Maintenance cost up to two years + loss in purchase price

=(100 + 250) + (6100-100) =Rs. 6350.

:. Average cost per year during first two years =Rs 6350/2 = Rs. 3175.

In a similar fashion, average cost pe year during first threeyears=6750/3=Rs.2250.00, average cost per year during first four years=7350/4 = Rs.1837.50,
Average cost per year during first five years=8250/5=Rs.1650.00, Average cost per year during first six years=9500/6=Rs.1583.33,

Averagecostperyearduringfirstsevenyears=11100/7=Rs.1585.71*(Note)*.

These computations may b summarized in the following tabular form

Replace at the end of year (n)	Maintanence cost (R_n)	Total maintenance cost (ΣR_n)	Difference between Price and Resale price $(C-S)$	Total cost $P(n)$	Average cost $\frac{P(n)}{n}$
(1)	(2)	(3)	(4)	(5) = (3) + (4)	(6)
1	100	100	6000	6100	6100
2	250	350	6000	6350	3175
3	400	750	6000	6750	2250
4	600	1350	6000	7350	1837
5	900	2250	6000	8250	1650
→6	1250	3500	6000	9500	1583
7	1600	5100	6000	11100	1586
8	2000	7100	6000	13100	1638

Hereitisobservedthatthemaintenancecostinthe7thyearbecomesgreaterthantheaverage cost for 6 years *[i.e. R7 >P(6)/6]*. Hence the machine should be replaced at the end of 6th year.

Alternatively, last column of above table shows that the average cost starts increasing in the 7^{th}year, so the machine should be replaced before the beginning of 7^{th}year, i.e.at the end of 6^{th}year.

Example-2: A machine owner finds from his past records that the cost s per year of maintaining a machine whose purchase price is Rs 6000 are given bellow.

Year	1	2	3	4	5	6	7	8
Maintenance cost (Rs)	1000	1200	1400	1800	2300	2800	3400	4000
Resale price	3000	1500	750	375	200	200	200	200

Determine what age is a
replacement due? Solution: here C=
purchase price = Rs 6000
The following table shows the average cost per year during the life of the machine.

Table-1

Replace at the end of year (n)	Maintenance cost (Rn)	Total maintenance cost	Difference between price & Resale price (C- Sa)	Total cost P(n)	Average cost P(n)/n
(1)	(2)	(3)	(4)	(5) = 3+4	(6) = col5/col1
1	1000	1000	6000-3000=3000	4000	4000
2	1200	2200	6000-1500=4500	6700	3350
3	1400	3600	6000-750=5250	8850	2950
4	1800	5400	6000-375=5625	11025	2756
5	2300	7700	6000-200=5800	13500	**2700** min
6	**2800 R6**	10500	6000-200=5800	16300	2717
7	3400	13900	6000-200=5800	19700	2814
8	4000	17900	6000-200=5800	23750	2989

The machine should be replaced at the end of 5th year, because the maintenance cost in the 6th year becomes greater than the average cost for 5 years.

R_6 >= P(5)/5 2800>= 2700

Problem-3: the machine owner has 3 machines of purchase price Rs 6000 each and the cost per year of maintaining each machine is same as problem-2. Two of these machines are two years old and the third is one year old. He is considering a new machine if purchase price Rs8000 with 50% more capacity than one of the old machines. The estimates of maintaining cost and resale price for new machine are as given below.

Year	1	2	3	4	5	6	7	8
Maintenance cost (Rs)	1200	1500	1800	2400	3100	4000	5000	6100
Resale price (Rs)	4000	2000	1000	500	300	300	300	300

Assuming that the loss of flexibility due to fewer machines is of no importance, and he continues to have sufficient work for three of old machines. What should his policy be?

Solution: First compute the average yearly cost during the life of a new machine of larger capacity is computed a shown in table-2

Replace at the end of year (n)	Maintenance cost (R_n)	Total maintenance cost $\sum Rn$	Difference between price & Resale price (C- S_a)	Total cost P(n)	Average cost P(n)/n
(1)	(2)	(3)	(4)	(5) = 3+4	(6) = col5/col1
1	1200	1200	8000-4000=4000	4000	5200
2	1500	2700	8000-2000= 6000	8700	4350
3	1800	4500	8000-1000=7000	11500	3833
4	2400	6900	8000-500= 7500	14400	3600
5	3100	10000	8000-300=7700	17700	**3540** min
6	**4000 R_6**	14000	8000-300= 7700	21700	3616
7	5000	19000	8000-300=7700	26700	3814
8	6000	25000	9000-300=7700	32700	4087

Observation 4000 >= 3540 ie $R_6 > P(5)/5$

Decision: replace new machine at the end of 5th year.

Now decide whether it is economically justified to replace smaller machine by new larger machine.

Since new larger machine has 50% more capacity than that of smaller one, three smaller m/cs will be equivalent to two larger machines.

Consequently the lowest average cost of Rs 3540 as shown in table-2 for new larger m/c is equivalent to (3540 = 1.5 times smaller machine) ie 3540/1.5 =2360 per pay load of smaller machine.

Since the amount of Rs 2360 is less than the minimum average annual cost Rs 2700 (see table -1) for one of the old smaller machines, hence the old smaller m/c will be replaced by a new-larger machine.

Now , decide when the new machine should be purchased.

Assume that for uniformity the replacement will involve two new machines and all the three old machines. The new machines should be purchased when the cost for the next year of running the three old machines exceeds the average yearly cost for two types of machines.

It is seen from the table-1, the total yearly cost for the smaller machine is as follows.

The total cost during first year = Rs 4000

The total cost during second year = 6700-4000 = 2700

The total cost during 3rd year = 8850-6700 = 2150

The total cost during 4th year = 22025-8850 = 2175

The total cost during 5th year = 13500-11025 = 2475

The total cost during 6th year = 16300-13500 = 2800

I 2 3 4

4

Hence the total cost next year for smaller machine aged 2 years & one smaller m/c aged one year

Becomes = 2* 2150 + 1 * 2700 = Rs 7000< 2*3540=7080

Similarly total cost during 2nd year = 2*2175 + 1* 2150 = 6500< 7080 **Similarly total cost during 3rd year = 2*2475 + 1* 2175 = 7125 > 7080** Similarly total cost during 4th year = 2*2800 + 1* 2475 = 8075

But the minimum average cost for two larger machines will be = 2* 3540 = Rs 7080

It has been observed that the cost (Rs6500) for the old m/cs will not exceed the cost (Rs 7080) for the larger new two machines until 3rd year.

Hence all the three small machines should be replaced after two years before any of them reaches the normal replacement age of five years (as seen from table-1).

Example-4.

a) Machine A costs Rs 9000, Annual operating cost is Rs 200 nfor the first year and then increase by Rs 2000 every year, ie, in the fourth year operating cost bcomes Rs 6200. Determine the best age at which to replace the machine . if the optimum replacement policy is followed, what will be the average yearly cost of owning and operating the machine?. (assume that the machine has no resale value when replaced and thar future costs are not discounted).

b) Machine B costs Rs10,000. Annual operating cost is Rs 499 for the first year, and then increased by Rs 800 every year. You have own a machine of type A which is one year old. Should you replace it with B, and if so when.

c) Suppose you are just ready to telace machine A with another machine of the same type, when you hear that machine B will become available in a year. What should you do?

Slution: a) for machine A, the average cost per year can be computed as shown below in the tableFor Machine-A cost =9000, scrap=0

Replace at the end of year (n)	Running cost (Rn)	Total running cost	Difference between price & Resale price (C- S)	Total cost P(n)	Average cost P(n)/n
(1)	(2)	(3)	(4)	(5) = 3+4	(6) = col5/col1

1	200	200	9000 -0 = 9000	9200	9200
2	2200	2400	9000-0 = 9000	11400	5700
3	4200	6600	9000-0 = 9000	15600	**5200 mim**
4	**6200**	12800	9000 -0 = 9000	21800	5450

Thus , the machine A should be replaced at the end of third year and average yearly cost of owning and operating the machine is Rs 5200 at the optimum time (3 years) of replacement.

b)In a similar fashion , prepare a table for machine **B,**

For Machine-B cost =10000, scrap=0

e at the end of year (n)	Running cost (R_n)	Total running cost $\sum Rn$	Difference between price & Resale price (C- S)	Total cost P(n)	Average cost P(n)/n
(1)	(2)	(3)	(4)	(5) = 3+4	(6) = col5/col1
1	Rs 400	400	1000 -0 = 10000	10400	Rs 10400
2	1200	1600	1000 -0 = 10000	11600	5800
3	2000	3600	1000 -0 = 10000	13600	4533
4	2800	6400	1000 -0 = 10000	16400	4100
5	3600	10000	1000 -0 = 10000	20000	**4000 min**
6	**4400**	14400	1000 -0 = 10000	24400	4066

Since the lowest average cost is Rs 4000 for machine B and is less than the lowest average cost of Rs 5200 for machine A, so machine A can be replaced by machine B.

Now decide when the machine B should be purchased. The machine B should be purchased when the cost for next year of running the machine A exceeds the average yearly cost for machine B.

Find the total yearly cost for machine A as follows:

1st year = 11400-9200 = 2200 < 4000

2nd year = 15600-11400 = 4200 > 4000

3rd year = 21800-15600= 6200 > 4000

Thus it has been observed that the cost (rs 2200) foe one year old machine A will not exceed the lowest average cost (Rs 4000) for B until second year hence. Therefore , machine A should be replaced after one year from now before it reaches the normal replacement age of three years.

d) Instead of replacing machine A with another machine of the same type, purchase machine B which is available after a year. The reason is that machine A can be replaced with machine B after one year from now as seen above in (b)

Money Value, Present Worth Factor (PWF), and Discount Rate

Moneyvalue.Sincemoneyhasavalueovertime,weoftenspeak: *'moneyisworth10%peryear'*.This can be explained in the following two alternative ways:

i. *In one way,* spending Rs. 100 today would be equivalent to spending Rs. 110 in a year's time: In other words, if we plan to spend Rs.110 after a year from now,wecouldequivalentlyspendRs. 100 today which would be of worth Rs. 110 next year.

Discount rate (*Depreciation value*). The *present worth factor* of unit amount to be spent after one year is given by $v = (1 + r)^{-1}$, where r is the interest rate. Then, v is called ***discount rate*** (technically known as the ***depreci- ation value***). Following example will explain the effect of considering money value with prescribed rate of interest.

ii. *Alternatively* ,if we borrow Rs.100attherateofinterest10% per year and spend this amount today, then we have to pay Rs. 110 after one year.'Thus we conclude that Rs.100 today will be equivalent to Rs.110 after a year from now. Consequently, $(1.1)^{-1}$ rupee today at the interest rate 10% per year.

One rupee spent two years from now is equivalent to $(1.1)^{-2}$ today.

Similarly, we can say one-rupee spent n years from now is equivalent to $(1.1)^{-n}$ today. This quantity $(1.1)^{-n}$ is called the present worth factor (Pwf) or the present value of one rupee spent 'n' years from now.

More generally, if r is the interest rate, then $(1+r)^{-n}$ is called the present worth factor (Pwf) or present value of one rupee spent in n years time from now onwards. It is also called as Compound amount factor.

Example 7: The cost pattern for two machines A and B, when money value is not considered, is given in the table:

Year	Cost at the beginning of year (in Rs)	
	Machine A	Machine B
1	900	1400
2	600	100
3	**700**	700
total	**2200**	2200

Find the cost pattern for each machine when money is worth 10% per year, and hence find which machine is less costly.

Solution: the total outlay for three years for machine A = 900+600+700 = Rs 2200. The total outlay for three years for machine B – 1400+100+700 = Rs2200

Here we observe that the total outlay for either machine is same for three years when the money value is not taken into account. Hence both the machines will appear to be equally good in this case.

Now considering the money value at rate of 10% per year, the discounted cost pattern for each machine for three years is as shown below table:

Year	Discounted Cost (10% rate) in Rs	
	Machine A	Machine B
1	900 = 900.00	1400 = 1400
2	$600*(100/110)^1$ = 545.45	100*100/110= 90.90
3	**$700*(100/110)^2$ = 578.52**	$700*(100/110)^2$= 578.53
Total outlay	**Rs 2023.97 minimum**	Rs 2069.43

The data shows that the total outlay for machine A is actually Rs 45.46 less than that of machine B. hence machine A will be preferred.

Example 7.Let the value of money be assumed to be 10% per year and suppose that machine A is replaced after every 3 years whereas machine B is replaced after every six

Year :	1	2	3	4	5	6
Machine A :	1,000	200	400	1,000	200	400
Machine B :	1,700	100	200	300	400	500

years. The yearly costs of both the machine are given as under

Determine which machine should be purchased.

Solution. Present worth factor is given by $v = \frac{100}{100 + 10} = \frac{10}{11}$.

$\therefore$ Total discount cost (present worth) of *A* for 3 years = Rs. $\left[1{,}000 + 200 \times \frac{10}{11} + 400 \times \left(\frac{10}{11}\right)^2\right]$

= Rs. 1,512 (nearly).

Again the total discount cost of *B* for six years

$$= \text{Rs.}\left[1{,}700 + 100 \times \frac{10}{11} + 200 \times \left(\frac{10}{11}\right)^2 + 300 \times \left(\frac{10}{11}\right)^3 + 400 \times \left(\frac{10}{11}\right)^4 + 500 \times \left(\frac{10}{11}\right)^5\right]$$

= Rs. 2765.

Average yearly cost of *A* = 1,512/3 = Rs.504

and average yearly cost of *B* = 2,765/6 = Rs. 461.

Although, this shows the apparent advantage with B, but the comparison is unfair because the periods of consideration are different.

So, if we consider 6 years period for machine A also, then the total discount of A will be

$$= 1{,}000 + 200 \times \frac{10}{11} + 400 \times \left(\frac{10}{11}\right)^2 + 1{,}000 \times \left(\frac{10}{11}\right)^3 + 200 \times \left(\frac{10}{11}\right)^4 + 400 \times \left(\frac{10}{11}\right)^5 = \text{Rs. } 2647,$$

which is Rs. 118 less costlier than machine *B* over the same period.

Hence machine *A* should be purchased.

Example 7.Let the value of money be assumed to be 10% per year and suppose that machine A is replaced after every 3 years whereas machine B is replaced after every six years. The yearly costs of both the machines are given as under:

Year :	1	2	3	4	5	6
Machine A :	1,000	200	400	1,000	200	400
Machine B :	1,700	100	200	300	400	500

etermine which machine should be purchased.

Solution. Present worth factor is given by $v = \frac{100}{100+10} = \frac{10}{11}$.

$\therefore$ Total discount cost (present worth) of A for 3 years $= \text{Rs.}\left[1{,}000 + 200 \times \frac{10}{11} + 400 \times \left(\frac{10}{11}\right)^2\right]$

$= \text{Rs. } 1{,}512$ (nearly).

Again the total discount cost of B for six years

$$= \text{Rs.}\left[1{,}700 + 100 \times \frac{10}{11} + 200 \times \left(\frac{10}{11}\right)^2 + 300 \times \left(\frac{10}{11}\right)^3 + 400 \times \left(\frac{10}{11}\right)^4 + 500 \times \left(\frac{10}{11}\right)^5\right]$$

$= \text{Rs. } 2765.$

Average yearly cost of $A = 1{,}512/3 = \text{Rs.}504$

and average yearly cost of $B = 2{,}765/6 = \text{Rs. } 461.$

Although, this shows the apparent advantage with B, but the comparison is unfair because the periods of consideration are different.

So, if we consider 6 years period for machine A also, then the total discount of A will be

$$= 1{,}000 + 200 \times \frac{10}{11} + 400 \times \left(\frac{10}{11}\right)^2 + 1{,}000 \times \left(\frac{10}{11}\right)^3 + 200 \times \left(\frac{10}{11}\right)^4 + 400 \times \left(\frac{10}{11}\right)^5 = \text{Rs. } 2647,$$

which is Rs. 118 less costlier than machine B over the same period.

Hence machine A should be purchased.

Example-8: A manual stamper currently valued at Rs 1000 expected to last two years and Rs 4000 per year to operate. An automatic stamper which can be purchased for Rs 3000 will last for four years and can be operated at an annual cost of rs 3000. If money carries the rate of interest10% per annum. Determine which stamper should be purchased.

Solution: The present worth factor is given by V = (100)/(100+10)= 0.9091.

The two given stampers have different expected lives, so we shall consider a span of four years during which we have to purchase either two manual stampers (the second one being purchased in the third year) or one automatic stamper

	YEAR-1	YEAR-2	YEAR-3	YEAR-4
MANUAL STAMPER	1000+4000	+ 4000	1000+4000	+4000
Automatic stamper	3000+3000	+3000	+3000	+3000

We now $V = 1/(1+r)$

The present worth of investments on the two manual stampers used in 4 years =

$= 1000+4000 + (4000)/(1+r) + (1000+4000)/(1+r)^2 + 4000/(1+r)^3$

$= 1000 + 1000V^2 + 4000 + 4000V + 4000V^2 + 4000V^3$

$= 1000 (!+V^2) + 4000 (1+V+V^2 +V^3)$

$= 1000(1+0.9091^2) + 4000 (1+0.9091 + 0.9091^2 + 0.9091^3) = 15773$

The present worth of investments on the automatic stamper for the next 4 years =

$= (3000+3000)+ 3000V + 3000V^2+ 3000V^3= 6000+3000(V+v^2+V^3) = 3000+10460= 13460$

Since the present worth of future costs for the automatic stamper is less than that of the manual stamper. It will be more profitable to purchase an automatic stamper.

Example-8: A person considering to purchase a machine for his own factory. Relevant data about alternative machines are as follows:

	Machine A	Machine B	Machine-C
Present investment(rs)	10000	12000	15000
Total annual cost (Rs)	2000	1500	1200
Life (years)	10	10	10
Salvage value	500	1000	1200

As an adviser to buyer, you have been asked to select the best machine, considering 12% normal rate of return.
You are given that

i. Single payment present worth factor (pwf) @12% for 10 years = 0.322
ii. Annual series present worth factor (pwf) @12% for 10 years = 5.650

Solution: the present value of total cost of each of the three m/cs for a period of 10 years is given as below:

Machine	Present investment (Rs)	Present value of total annual cost	Present value of salvage value(Rs)	Net cost (Rs)
	(1)	(2)	(3)	(1+2-3)
A	10000	2000*5.65=11300	50080.322=161	21139.00
B	12000	1500*5.65 = 8475	1000*0.322= 322	20153.00 min i
C	15000	1200*5.65= 6780	1200*0.322= 386.40	21393.60

From the above table we conclude that the net cost for m/c-B is the least and hence the machine B should be purchased.

REPLACEMENT POLICY WHEN THE MAINTENANCE COST INCREASES WITH TIME AND MONEY VALUE CHANGES WITH CONSTANT RATE

How to Select the Best Machine?

In the problem of choosing a best machine (or item), the costs that are constant over time for each given machine will still have to be taken into account, although these costs may differ for each machine .Only those costs that are same for the machines under comparison can be excluded. Suppose two machines MI and *M2* are at our choice. The data required for determining the best replacement age of each type of machine is also given from past experience.Thus, a best selection can be done by adopting the following outlined procedure:

Step 1. First find the best replacement age for machine *M*I and *M2*both by using the relationship:

$$R_{n+1} > \frac{F(n)}{\Sigma v^{n-1}} > R_n$$

Supposetheoptimumreplacementageformachines*MI*and*M2*comes out be n_I and n_2'respectively.

Step 2. Compute the fixed annual payment (or weighted average cost) for each machine by using the formula:

$$x = \frac{C + R_1 + R_2 + \ldots + R_n v^{n-1}}{1 + v + v^2 + \ldots + v^{n-1}} = \frac{F(n)}{\Sigma v^{n-1}}$$

Andsubstitutinginthisformula*n*=n 1formachine*M1*and*n* =n*2* for machine *M2*. Let it be *XI* and *X2* for machines *M*I and*M2,* respectively.

Step 3.

i. If $X_1 < X_2$, then choose machine M_I,

ii. If $X_I > X_2$, then choose machine M_2.

iii. If $X_I = X_2$'then both machines are equally good.

problem-10: a manufacturer is offered two machines a and b. a is priced at rs 5000 and running costs are estimated at Rs 800 for each of the first five years, increasing by Rs200 per year in the sixth and subsequent years. Machine B, which has the same capacity as A costs s 2500 but will have running costs of Rs 1200 per year for six years, increasing by Rs 200 per year there after. If money is worth 10% per year, which machine shold be purchased? (assume that machine will eventually be sold for scrap at negligible price.)

Solution: since the money is worth 10% per year

The discount rate is given by V= 1/(1+r)= 1/(1+0.10) = 0.9091

Therefore , the optimum replacement age ,n, must satisfy the relationship.

For Machine A: for the best replacement age ‘n’, tabulate the required calculations of Total maintenance cost machine-A in ntable-1

year (n)	Running cost (R_n)	Pwf (V^{n-1})	(V^{n-1}).R_n	F(n)=(C-S) + $\sum Rn.Vn-1$	$\sum Vn-1$	Weighted average F(n) / $\sum Vn-1$
1	800	1.000	800	5800	1.0000	5800
2	800	0.9091	737	6537	1.9091	3419
3	800	0.8264	661	7198	2.7355	2628
4	800	0.7513	601	7799	3.4868	2234
5	800	0.6830	546	8345	4.1698	19999
6	**1000**	0.6209	621	8966	4.7907	1896
7	1200	0.5645	677	9643	5.3552	1799
8	1400	0.5132	718	10361	5.8684	1764
9	1600	0.4665	746	11107	6.3349	1752
10	**1800**	0.4241	763	11870	6.7590	1755

$$R_{n+1} > \frac{F(n)}{\Sigma v^{n-1}} > R_n$$

R_{10}> F(9)/9
>R_9 1800 >
1752 > 1600

It is observed that R_{10}= 1800 becomes greater than the weighted average cost Rs1752 for nine years. Hence. It would be best to replace machine A after nine years; and also from table1, the equivalent fixed annual payment for machine A is read from last column as x_1 = weighted average for nine years = Rs 1752

Calculations for machine B are shown in table-2

year (n)	Running cost (R_n)	Pwf (V^{n-1})	(V^{n-1}).R_n	F(n)=(C-S) + $\sum Rn.Vn-1$	$\sum Vn-1$	Weighted average F(n) / $\sum Vn-1$
1	1200	1.000	1200	3700	1.0000	3700
2	1200	0.9091	1091	4791	1.9091	2510
3	1200	0.8264	991	5782	2.7355	2114
4	1200	0.7513	902	6684	3.4868	1917
5	1200	0.6830	820	7504	4.1698	1800
6	**1200**	0.6209	745	8249	4.7907	1722
7	1400	0.5645	790	9039	5.3552	1668
8	1600_>	0.5132	821	9860	5.8684	**1680**
9	**1800**	0.4665	840	10700	6.3349	1689

Similarily, best replacement age for machine B can be calculated . Since R_9 > weighted average for 8 years

1800 > 1680

It would be best to rteplace machine after 8 years.

The equivalent fixed annual payment for machine B is read from table-2 as X_2 = weighted average for 8 years = Rs 1680

Since X_1 is grater than x2, it would be better to purchase machine B instead of machine A

GROUP REPLACEMENT OF ITEMS THAT FAIL COMPLETELY

Group replacement is concerned with those items that either work or fail completely. It often happens that a system encounters a ;large number of identical low cost items that are increasingly liable to failure with age. In such cases there is setup cost per replacement that is independent of the number replaced and it may be advantageous to replace all items at fixed intervals. Such a policy is called group replacement policy and is particularly attractive when the value of any individual item is so small that the cost of keeping records of individual ages can not be justified. The classical example of such a policy is used in replacing light bulbs.

Two types of replacement policies are considered

i. Individual replacement: under this policy an item is replaced immediately after its failure.
ii. Group replacement: under this policy decision is taken as to when all the items must be replaced, irrespective of the fact that items have failed and have not failed, with the provision that if any item fails before the optimum time, it may be replaced individually.

Group Replacement Policy Theorem

a. One should group replace at the end of i^{th} period, if the cost of individual replacements for the i^{th} period is greater than the average cost per period through the end of i^{th} period.
b. One should group replace at the end of i^{th} period , if the cost of individual replacements at the end of $(i-1)^{th}$ period is less thanthe average cost per period through the end of i^{th} period.

Problem:11 The following failure rates have been observed for a certain type of light bulbs.

week	1	2	3	4	5
percent failing by end of the week	10	25	50	80	100

There are 1000 bulbs in use and it costs Rs 10 to replace an individual bulbs which has burn out. If all the bulbs were replaced simultaneously it would cost Rs 4 per bulb. It is proposed to replace all bulbs at fixed intervals of time, whether or not they have burn out, and to continue replacing burnt out bulbs as and when they fail. At what intervals all the bulbs should be replaced? At what group replacement price per bulb would on policy of strictly individual replacement become preferable to the adopted policy.

Solution: Let P_i be the probability that a light bulb, which was new when placed in position for use fails during the i^{th} week of its life.

Thus he following frequency distribution is obtained assuming to replace burnt out bulbs as and when they fail.

P_1 = the probability of failure in Ist week = 10/100 = 0.10

P_2 = the probability of failure in IInd week = (25-10)/100 = 0.15

P_3 = the probability of failure in IIIrd week = (50-25)/100 = 0.25

P_4 = the probability of failure in IVth week = (80-50)/100 = 0.30

P_5 = the probability of failure in Vth week = (100-80)/100 = 0.20

Total **=1.00**

Since the sum of all probabilities can never greater than MODULEy, therefore further probabilities P_6, P_7, P_8 and so on, Thus a bulb that has already lasted four weeks is sure to fail during the fifth week.

Furthermore, assume that

i. Bulbs that fail during a week are replaced just before the end of that week.

ii. The actual percentage of failure during a week for a sub population of bulbs with same age is the same as the expected percentage of failures, during the week for that population.

Let N_i = be the number of replacements made at the end of i^{th} week, if all 1000 bulbs are new initially.

Thus $N_0 = N_o$ = 1000

$N_1 = N_0 P_1$ = 1000 * 0.10 = 100

$N_2 = N_0 P_2 + N_1 P_1$ = 1000 * 0.15 + 100 * 0.10 = 160

$N_3 = N_0 P_3 + N_1 P_2 + N_0 P_2 + N_2 P_1$

= 1000 * 0.25 + 100 * 0.15 + 160 * 0.10 = 281

$N_4 = N_0 P_4 + N_1 P_3 + N_2 P_2 + N_3 P_1$

= 1000 * 0.30 + 100 * 0.25 + 160 * 0.15 + 281 * 0.10 = 377

$N_5 = N_0 P_5 + N_1 P_4 + N_2 P_3 + N_3 P_2 N_4 + N_4 P_1$ = 350

$N_6 = N_0 P_6 + N_1 P_5 + N_2 P_4 + N_3 P_3 + N_4 P_2 + N_5 P_1$ = 230

$N_7 = N_0 P_7 + N_1 P_6 + N_2 P_5 + N_3 P_{25} + N_4 P_3 + N_5 P_2 + N_6 P_1$ = 286 and so on

It has been observed that the expected number of bulbs burnt out in a week increases until 4^{th} week and then decreases until 6 th week and again starts increasing. Thus the number will continue to oscillate and ultimately the system settles down to a steady state in which the proportion of bulbs failing in each week is the reciprocal of their average life.

The mean age of bulbs = $1 * P_1 + 2 * P_2 + 3 * P_3 + 4 * P_4 + 5 * P_5$

$= 1 * 0.10 + 2 * 0.15 + 3 * 0.25 + 4 * 0.30 + 5 * 0.20 = 3.35$ weeks

The number of failures in each week in steady state becomes = 1000/3.35 = 299

The cost of replacing bulbs individually only on failure = 10* 299 = 2990 @ 10 per bulb

Since the replacement of all 1000 bulbs simultaneously costs Rs 4 per bulb and replacement of an individual bulb on failure costs Rs10

Therefore the cost of replacement of all bulbs simultaneously is as given in following table:

End of week	Cost of individual replacement)	Total cost of group replacement	Average cost per week
1	100 nos * 10 = 1000	1.000 * 4 + 100*10 = 5000	5000/1 = 5000
2	160 nos* 10 = 1600	5000 + 160 *10 = 6600	6600/2 = 33000
3	281 *10 = 2810	6600 + 281 * 10 = 9410	9410/3 = ***3136.6***
4	377 * 10 = **3770**	9410 + 377 *10 = 13180	13180/4= 3294.80

The cost of individual replacement in the 4th week exceeds the average cost for 3 weeks.

Thus it would be optimal to replace all the bulbs after every 3 weeks other wise average cost will start increasing. Further , since the group replacement at the end of one week costs Rs5000, and the individual replacement after one week cost Rs 2990, the individual replacement is preferred.

Problem-13: A computer contains 10000 resisters. When any one of the resistor fails, it is replaced. The cost of replacing a single resistor is Rs 10 only. If all the rsistors are replaced at the same time, the cost per resistor would be reduced to Rs 3.50. the percent surviving by the end of moth is as follows:

Month	0	1	2	3	4	5	6
% surviving by the end of month	100	97	90	70	30	15	0

Problem-13: A computer contains 10000 resisters. When any one of the resistor fails, it is replaced. The cost of replacing a single resistor is Rs 10 only. If all the rsistors are replaced at the same time, the cost per resistor would be reduced to Rs 3.50. the percent surviving by the end of moth is as follows:

Month	0	1	2	3	4	5	6
% surviving by the end of month	100	97	90	70	30	15	0

What is the optimum plan?

Solution: the probabilities P_t of failure during the month t are

P_1= the prob of falure in Ist month = (100-97) /100 = 0.03

P_2= the prob of falure in 2nd month = (97-90) / 100 = 0.07

P_3 = the prob of falure in 3rd month = (90-70) /100 = 0.20

P_4 = the prob of falure in 4th month = (70-30) /100 = 0.40

P_5 = the prob of falure in 5th month = (30-15) / 100 = 0.15

P_6= the prob of falure in 6th month = (15-0 0 /100 = 0.15

$$\mathbf{1\sum_{t=1}^{6} Pt = 1.0}$$

Let N_t = be the number of replacements made at the end of t^{th} month if all 10000 resistors are new initially.

.

thus $N_0 = N_o$ = 10000

$N_1 = N_0 P_1$ = 10000 * 0.03 = 300

$N_2 = N_0 P_2 + N_1 P_1$ = 10000 * 0.07 + 300 * 0.03 = 709

$N_3 = N_0 P_3 + N_1 P_2 + N_0 P_2 + N_2 P_1$

= 10000 * 0.20 + 300 * 0.07 + 709 * 0.03 = 2042

$N_4 = N_0 P_4 + N_1 P_3 + N_2 P_2 + N_3 P_1$ = 4171

$N_5 = N_0 P_5 + N_1 P_4 + N_2 P_3 + N_3 P_2 \; N_4 + N_4 P_1$ = 2030

$N_6 = N_0 P_{6} + N_1 P_5 + N_2 P_4 + N_3 P_3 + N_4 P_2 + N_5 P_1$ = 2590 and so on

Average life of each resistor = $i = 6 \sum_{i=1} i\, Pi$ =

It has been observed that the expected number of bulbs burnt out in a week increases until 4^{th} week and then decreases until 6 th week and again starts increasing. Thus the number will continue to oscillate and ultimately the system settles down to a steady state in which the proportion of bulbs failing in each week is the reciprocal of their average life.

$= 1 * P_1 + 2 * P_2 + 3 * P_3 + 4 * P_4 + 5 * P_5 + 6 * P_6$

$= 1 * 0.03 + 2 * 0.07 + 3 * 0.20 + 4 * 0.40 + 5 * 0.15 + 6 * 0.15$

$= 0.03 + 0.14 + 0.60 + 1. + 0.75 + 0.90 = 4.02$ months

Average life of each resistor = 4.02 months

Average no. of replacements every month = 10000/4.02 = 2488

Average cost of monthly individual replacements = 2488 * 10 = 24880

Under group replacement we find

End of month	Cost of individual replacement	Total cost of group replacement	Average cost per month
1	300 nos * 10 = 3000	10000 * 3.5 + 300*10 = 38000	38000/1 = 38000
2	709 nos* 10 = 7090	38000 + 709 * 10 = 45090	45090 /2 = 22545
3	2042 *10 = 20420	45090 + 2042 * 10 = 65510	65510 /3 = 21836.66
4	4171 * 10 = 41710	65510 + 4171 * 10 = 107220	107220/ 4 = 26805.00

The cost of individual replacement in the fourth month exceeds the average cost for 3 months. hence it would be optimal to replace all the resistors after every 3 months, otherwise the average cost will start increasing. *** **

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MODULE-IV

THEORY OF GAMES AND INVENTORY

Game Theory covers the following concepts

- Game Theory Introduction
- Minimax criterion of optimality
- Games without saddle point

Introduction

Life is full of struggle and competitions. A great variety of competitive situations is commonly seen in everyday life. For example, candidates fighting an *election* have their conflicting interests, because each candidate is interested to secure more votes than those secured by all others. Besides such pleasurable activities in competitive situations, we come across much more earnest competitive situations, of military battles, advertising and marketing campaigns by competing business firms, etc.

What should be the bid to win a big Government contract in the pace of competition from several contractors? Game must be thought of, in a broad sense, not as a kind of sport but as competitive situation, a kind of conflict in which somebody must *win* and somebody must *lose.*

Game theory is a type of decision theory in which one's choice of action is determined after taking into account all possible alternatives available to an opponent playing the same game, rather than just by the possibilities of several outcomes. The game theory has only been capable of analysing very simple competitive situations. Thus, there has been a great gap between what the theory can handle and most actual competitive situations in industry and elsewhere. So the primary contribution of game theory has been its concepts rather than its formal application to solving real problems.

Game is defined as an activity between two or more persons involving activities by each person according to a set of rules at the end of which each person receives some benefit or satisfaction or suffers loss (negative benefit). The set of rules , at the end of which each person receives some benefit or satisfaction or suffers loss (negative benefit).

Characteristicsof Game Theory

There can be various types of games. They can be classified on the basis of the following characteristics.

i. ***Chance of strategy:*** If in a game, activities are determined by skill, it is said to be *a game of strategy;* if they are determined by chance, it is a *game of chance*. In general, a game may involve game of strategy as well as a game of chance. In this chapter, simplest models of games of strategy will be considered.

ii. ***Number of persons:*** A game is called an n-person game if the number of persons playing is 11.The person means an individual or a group aiming at a particular objective.

iii. ***Number of activities:*** These may *be finite* or *infinite.*

iv. ***Number of alternatives (choices) available to each person*** **in a particular activity** may also be finite or infinite .A *finite* game has a finite number of activities, each involving a finite number of alternatives, otherwise the game is said to be *infinite..*

v. ***Information to the players*** *about the past activities of other players* is completely available, partly available, or not available at all.

vi. ***Payoff:*** A quantitative measure of satisfaction a person gets at the end of each play is called *a payoff.* It is a real-valued function of variables in the game.

Let *vi* be the payoff to the player *Pi,*

1≤i ≤n, in an n person game. If , then the game is said to game. $\sum^{n} v_i = 0$, -sum

In this lesson, we shall discuss rectangular games (also called two- person zero-sum).

Basic Definitions

1. **Competitive Game.** A competitive situation is called a *competitive game* if it has the following four properties:

i. There are finite number *(n)* of competitors (called players) such that *n*~2. In case *n*=2, it is called a two- person game and in case *n*>2, it is referred to as an n- person game.

ii. Each player has a list of finite number of possible activities (the list may not be same for each player).

iii. A play is said to *occur* when each player chooses one of his activities. The choices are assumed to be made simultaneously, *i.e.* no player knows the choice of the other until he has decided on his own.

iv. Every combination of activities determines an outcome (which may be points, money or anything else whatsoever) which results in again of payments(+ ve, - ve or zero) to each player, provided each player is playing uncompromisingly to get as much as possible. Negative gain implies the loss of same amount.

v. Value of game: it is the expected gain of player A , if both players use their best strategies.

We define best strategy on the basis of the minimax criterion of optimality.

Saddle Point: a saddle point is an element of the matrix which is both the smallest element in its row and the largest element in its column,

2. **Zero-sum and Nonzero-sum Games:**

.Competitive games are classified according to the number of players in volved, *i.e.* as a *two person game.. three person game,* etc. Another important distinction is between *zero-sum games* and *nonzero-sum* games. If the players make payments only to each other, *i.e.* the loss of one is the gain of others, and nothing comes from outside, the competitive game is said to be *zero-sum.*

Mathematically, suppose an n-person game is played by *n* players *P1, P2* '..., *P”* whose respective pay-offs at the end of a play of the game are *VI ,V2, ... V”*then, the game will be called zero- sum

$$\text{if} \sum_{i=1}^{n} v_i = 0$$

at each play of the game. A game which is not

zero-sum is called a *nonzero-sum game.* Most of the competitive games are zero-sum games. An example of a nonzero-sum game is the 'poker' game in which a certain part of the pot is removed from the 'house' before the final payoff.

3. **Strategy:** A strategy of a player has been loosely defined as a rule for decision-making in advance of all the plays by which he decides the activities he should adopt. In other words, a strategy for a given player is a set of rules (programmes) that specifies which of the available course of action he should make at each play.

This strategy may be of two kinds:

i. ***Pure Strategy.***: If a player knows exactly what the other player is going to do, a *deterministic* situation is obtained and objective function is to maximize the gain. *Therefore, the pure strategy is a decision rule always to select a particular course of action.* A pure strategy is usually represented by a number with which the course of action is associated.

ii. ***Mixed Strategy.***[Agra92; Kerala (Stat.) 83]: If a player is guessing as to which activity is to be selected by the other on any particular occasion, a *probabilistic* situation is obtained and objective function is to maximize the *expected gain.*

Thus, the mixed strategy is a selection among pure strategies with fixed probabilities.

Mathematically, a mixed strategy for a player with *m* (:2:2) possible courses of action, is denoted by the set S of *m* non-negative real numbers whose sum is MODULEy, representing probabilities with which each course of action is chosen. If Xi*(i* =I, 2, 3, ..., *m)* is the probability of choosing the course *i, then*

$$S = (x_1, x_2, x_3, \ldots, x_m)$$

subject to the conditions $$x_1 + x_2 + x_3 + \ldots + x_m = 1$$

and $$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, \ldots, x_m \geq 0.$$

4. **Two-Person, Zero-Sum *(Rectangular)* Games.** A game with only two players (say, *Player and Player B)* is called a *'two- person, zero-sum game'* if the losses of one player are equivalent to the gains of the other, so that the sum of their net gains is zero.

Two-person, zero-sum games are also called *rectangular games* as these are usually represented by a payoff matrix in rectangular form.

5. Payoff Matrix. Suppose the player *A* has *m* activities and the player *B* has *n* activities. Then a payoff matrix can be formed by adopting the following rules:

i. Row designations for each matrix are activities available to player A.

ii. Column designations for each matrix are activities available to player *B.*

iii. Cell entry *'vij*, is the payment to player *A* in *A's* payoff matrix when *A* chooses the activity *i* and *B* chooses the activity j.

iv. With a 'zero-sum, two person game', the cell entry in the player *B'* s payoff matrix will be negative of the corresponding cell entry 'Vi}, in the player *A's* payoff matrix so that sum of payoff matrices for player *A* and player *B* is ultimately zero.

The player *A*'s payoff matrix

The player *B*'s payoff matrix

		Player *B*					
		1	2	...	*j*	...	*n*
	1	v_{11}	v_{12}	---	v_{1j}	---	v_{1n}
	2	v_{21}	v_{22}	---	v_{2j}	---	v_{2n}
Player *A*	:	:	:		:		:
	i	v_{i1}	v_{i2}	---	v_{ij}	---	v_{in}
	:	:	:		:		:
	m	v_{m1}	v_{m2}	---	v_{mj}	---	v_{mn}

		Player *B*					
		1	2	...	*j*	...	*n*
	1	$-v_{11}$	$-v_{12}$	...	$-v_{1j}$	...	$-v_{1n}$
	2	$-v_{21}$	$-v_{22}$	...	$-v_{2j}$	...	$-v_{2n}$
Player *A*	:	:	:		:		:
	i	$-v_{i1}$	$-v_{i2}$	...	$-v_{ij}$	...	$-v_{in}$
	:	:	:		:		:
	m	$-v_{m1}$	$-v_{m2}$	...	$-v_{mj}$	...	$-v_{mn}$

In order to make the above concepts a clear, consider the coin matching game involving two players only. Each player selects either a head *H* or a tail *T*. If the outcomes match *(H, H* or *T, T), A* wins Re 1from *B* , otherwise *B* wins Re 1from *A* . This game is a two-person zero-sum game. since the winning of one player is taken as losses for the other. Each has his choices between two *pure* strategies *(H* or *T).* This yields the following (2 x 2) payoff matrix to player *A.*

It will be shown later that the optimal solution to such games requires each player to play one pure strategy or a mixture of pure strategies.

Assumptions of the Game:

- each player has available to him a finite number of possible strategies (course of action). The list may not be the same for each player.
- Player act rationally and intelligently.
- List strategist of each player and the amount of gain or loss on individual's choice of strategy is known to each player in advance.

- One player attempts to maximize gains and the other attempts to minimize losses.
- Both players make their decisions individually, prior to the play, without direct communication between them.
- Both players select and announce their strategies simultaneously so that neither player has an advantage resulting from the direct knowledge of the other player's decision.
- The payoff is fixed and determined in advance.

Minimax (Maximin) Criterion And Optimal Strategy The 'minimax criterion of optimality' states that if a player lists the worst possible outcomes of all his potential strategies, he will choose that strategy to be most suitable for him which corresponds to the best of these worst outcomes. Such a strategy is called an optimal strategy.

Problem-1: Consider (two-person, zero-sum) game matrix, which represents payoff to the player A. Find, the optimal strategy, if any.(See Table)

	PLAYER - B			
PLAYER -A		I	II	III
	I	-3	2	6
	II	2	0	2
	III	5	-2	-4

Solution; the player A wishes to obtain the larsest possible 'V_{ij}' by choosing one of his activities (I, II, III),

while the player B is determined to make A's gain the minimum possible by choice of activities from his list (I, II, III). Then the player A is called the maximizing player abd B the minimizing player.

		PLAYER - B				
PLAYER -A		I	II	III	Row minimum	
	I	-3	2	6	-3	maximin= max(-3,0,-)=0
	II	2	**0**	2	0	
	III	5	-2	-4	-4	
	Column maximum	5	0	6		

Minimax value = mini (5,0,6) =0

If the Player A chooses the Ist activity , then it would happen that the player B also chooses his Ist activity. In this case the player B can guarantee a gain of least -3 ie mini(-3, -2, 6) = -3

Similarly, for other choices of player A, ie, II and III activities , B can force the player A to get only o and -4 respectively, by his proper choices from (I, II, III) ie Mini (2.0,2) = 0

Mini (5, -2, -4) =-4

The minimum value in each row guaranteed by the player A is indicated by 'row minimum' as shown in the table.

The best choice for the player A is to to maximize his least gains -3, 0, -4 = 0 and opt II strategy which assures at most the gain '0' ie'

Max (Row minimum of the strategy I, II, III of B) = Max (min (-3,-2, 6), min(2,0,2), min (5, -2, -4))= max(-3, 0, -4)= 0

Mini(column Maximum of the strategy I, II,III of A)= mini(max(-3,2,5), max(-2, 0, -2), max(- 6, 2, -4)= mini(5, 0,2)=0

MaxminiV_{ij}= Mini max $V_{ij\ =\ 0}$ i j j I Saddle point is 'o"

Soluion is:

Player A uses his course of action II throughout Player B uses his course of action II throughout

The value of game for player A =0

The value vof game for player B =-0 This game is fair to ,player A and B The solution to this game is unique. **Rules for determining a saddle Point:**

Step-1: select the minimum element of each row of the payoff matrix and mark them by Spep-2: Select the greatest element of each column of the payoff matrix and mark them by

Step-3: if there appears an element in the payoff matrix marked by □ and ◯ both, the position of that element is a saddle point of the payoff matrix.

Example-2: Consider the following game

	PLAYER - B			
		I	II	III
PLAYER -A	I	3	-4	8
	II	-8	5	-6
	III	6	-7	6

	PLAYER - B					
PLAYER -A		I	I I	II I	Row minimum	
	I	3	- 4	8	-4	Maximin= Maxi(-4,- 8,- 7)= -4
	II	- 8	5	-6	-8	
	III	6	-7	6	-7	
M a	Column maximu	6	5	8		
x m		minimax = mini(6,5,8)=5				

in =-4,

minimax

=5 Maxmin#

minimax

There is no saddle point for this game.

Problem-3. The pay-off matrix for a two person zero-sum game is given below. Find the best strategy for each player and the value of the game.

	PLAYE R - B					
PLAYE R - A		I	II	III	I V	V
	I	-2	0	0	5	3
	II	3	2	1	2	2
	III	-4	-3	0	-2	6
	IV	-4	-3	0	-2	6
	V	5	3	-4	-2	-6

Solution:

	PLAYER - B						
PLAYER - A		I	II	III	IV	V	row mini
	I	-2	0	0	5	3	0
	II	3	2	**1**	2	2	1
	II I	-4	-3	0	-2	6	-4
	I V	-4	-3	0	-2	6	-4
	V	5	3	-4	-2	-6	-6
	Col max	5	3	1	5	6	

maxmin= max(0,1,-4,-6) = 1

Minimax = mni

(5,3,1,5,6) = 1

Maxmin= minimax =

1

The saddle

point is 1

Solution is:

Player A uses his course of action II throughout Player B uses his course of action III throughout The value of game for player A =1

The value vof game for player B =-1 This game is fair to ,player A

The solution to this game is unique.

Problem-4: determine the solution of the following game.

	PLAYER - B			
PLAYER -A		I	II	III
	I	-2	15	-2
	II	-5	-6	-4
	III	-5	20	-8

	PLAYER - B					
PLAYER -A		I	II	III	Row minimum	
	I	-2	15	-2	-2	Maximin= Maxi(-2,-6,-8)= -2
	II	-5	-6	-4	-6	
	III	-5	20	-8	-8	
	col max	-2	20	-2		
		minimax = mini(-2,20,-2)= -2				

The Row minima, corresponding to player A's course of action I,II and III are -2, -6 and -8 respectively

The column maxima, corresponding to B,s course of action are -2, 20, -2.

The largesr of row minimua and the smallest of the column maxima are both equal to -2.

Here the saddle point is -2. However , there are two elements whose value is -2 and hence the game has two saddle points.

maxmin= max(-2, -6, -8) = -2

Minimax = mni (-20, 20, -2)

= -2 Maxmin= minimax = -2

The saddle point is -2

Soluion to such a game is not unique.

Player A uses his course of action I throughout

Player B may uses either of his courses of action I or III throughout, or a mixed strategy which uses any combination of these courses of action.

The value of game for player A = -2 The value of game for player B = +2 This game is fair to

,player B

The solution to this game is not unique.

MIXED STRATEGIS: GAME WITHOUT SADDLE POINT:

In certain cases, there is no pure strategy solution for the game, ie. No saddle point exists. In such cases, to spolve games both players must determine an optimal mixture of strategies to find a saddle (Equilibrium) point.

The optimal strategy mixture for each player may be determined by assignibg to each strategy its probability of being choosen, the strategies , so determined, are called mixed strategies. This is because they are probabilistic combination of the available choices of strategy.

A mixed strategy game can be solved by following methods.

i. Algebric method
ii. Arithmatic method
iii. Matrix method
iv. Graphical method
v. Linear programming method`

THE RULES (PRINCIPLES) OF DOMINANCE

The rules dominance are used to reduce the size of the payoff matrix. These rules can help in deleting certain rows and / or columns of the payoff matrix that are inferior (less attractive) to at least one of the remaining rows and /or columns (strategies), in terms of payoff matrix to both the players. Rows and/or columns once deleted can never be used for determining the optimum strategy for both the players.

For p;layer A, who is assumed to be gainer. If ach element in a row, say R_r, is less than or equal to the corresponding element in another row, say R_s, in the payoff matrix, then the row R_r is said to be (inferior) dominated by row R_s , and therefore, row$R_r$can be deleted from the payoff matrix. In other words , player A will never use the strategy corresponding to row $R_{r,}$, because he will gain less by choosing such a strategy.

For p;layer B, who is assumed to be loser, If ach element in a column, say C_r, is greater than or equal to the corresponding element in another column, C_s, in the payoff matrix, then the column C_r is said to be (inferior) dominated by column C_s , and therefore, column C_r can be deleted from the payoff matrix. In other words , player B will never use the strategy corresponding to column $C_{r,}$, because he will loose more by choosing such strategy.

A strategy say , k can also be dominated if it is inferior (less attractive) to an average of two or more other pure strategies. In this case, if the domination is strict, then strategy k can be deleted

Problem-4: In a game of matchng coins with two players , suppose one player wins Rs 2 when there are two heads and wins nothing when there are two tails; and losses Rs 1 when there are one head and one tail. Determine the payoff matrix , the best strategies for each player and the value of the game.

Solution: The pay of matrix for player a is as follows:

	PLAYER-B				
		H	T	Row min	
PLAYER - A	H prob x	2	-1	-1	Maximin= Maxi(-2,-6,-8)= -2
	T 1-x	-1	0	-1	
Column max		2	0		
		Minimax(2,0) =0			

Here Minimax # maximin, so there is no saddle point. We need to find mixed strategies for each of the player.

Let the player A plays H with probability x ad T with probability 1-x So that sum = 1, x+ (1-x) =1

Then if the player B plays H all the time, A's expected gain will be E(A, H) = 2.x + (-1)(1-x) =

$3x-1$ ------------------------------------- eqn-1

Similarly , if the player B plays T all the time, A's

expectd gain will be $E(A, T) = x(-1) + (1-x).\ 0 = -x$ --

-------- eqn-2

It can be shown mathematically that E(A,

$H) = E(A, T)\ 3x - 1 = -x$

$4x = 1$, $x = ¼$, $1-x = ¾$

Therefore , best strategy for player A is to play H and T with probability ¼ and ¾ respectively. Since it is a mixed strategy, the strategy o player A = [¼, ¾]

The expected gain for player A = $E(A) = (1/4)*2 + (3/4)*(-1) = -1/4$

Let the player B plays H with probability y ad T with

probability 1-y Then if the player B plays H all the

time, B's expected gain will be $E(B, H) = E(B,T)=$

E(B) SAY

$2.y + (-1)(1-y) = 3y - 1$ ---------- eqn-1

Similarly , if the player A plays T all the time, B's

expectd gain will be $E(B, T) = y(-1) + (1-y).\ 0 = -y$ --

-------- eqn-2

It can be shown mathematically that E(A,

$H) = E(A, T)\ 3y - 1 = -y$

$4y = 1$, $y = ¼$, $1-y = ¾$

Therefore , best strategy for player B is to play H and T with probability ¼ and ¾ respectively. Since it is a mixed strategy, the strategy o player B = [¼, ¾]

The expected gain for player B = $E(A) = (1/4)*2 + (3/4)*(-1) = -1/4$

The complete solution of the game is;

- The player A should play H and T with probabilities 1/4,

and ¾ respectively. Thus A's optimal strategy is (¼, ¾)

The player B should play H and T with probabilities ¼

and ¾ respectively Player B's optimal strategy is (1/4, ¾)

The expected value of the game for player A = -1/4

Problem-5:determine the solution of the following game

	PLAYER - B			
PLAYER - A		I	II	III
	I	**0**	-2	7
	II	2	5	6
	III	3	-3	8

Solution:

	PLAYER - B					
PLAYER -A		I	II	III	Row minimum	
	I	**0**	-2	7	-2	Maximin= Maxi(-2,2,-3)= -3
	II	2	5	6	2	
	III	3	-3	8	-3	
col maximum		3	5	8		
		minimax = mini(3,5,8)= 3				

Here Minimax # maximin, so there no saddle point

Applying principle of dominance reduce the payoff matrix to 2*2 form

	PLAYER - B			
PLAYER -A		I	II	III
	I	0	-2	7
	II	2	5	6
	III	3	-3	8

On observing Columns Col-III >= Col-I, col-III is inferior or dominated column. So detete col-III

	PLAYER - B		
PLAYER -A		I	II
	I	0	-2
	II	2	5
	III	3	-3

Observing rows Row-1 is <= Row-II ie Row-1 is inferior to Row-II , Delete Row-1

Here Minimax # maximin, so there is no saddle point. We need to find mixed strategies for each of the player.

Let the player A plays strategy II with probability x and Strategy III with probability 1-x So that sum = 1, x+ (1-x) =1

Then if the player B plays Strategy I all the time, A's expected gain will be E(A, II) = 2.x + 3(1-x) = 3-x ------------------- eqn-1

Similarly , if the player B plays Strategy II all the time, A's expectd gain will be E(A, III) = 5 x + (1-x). (-3) = 8x-3 ---------- eqn-2 B's I strategy

It can be shown mathematically that E(A, I) = E (A, II) B's II strategy 3-x = 8x-3 , x = 6/9

Therefore , best strategy for player A is to play Strategy II and strategy III with probability 2/3 and 1/3 respectively. Since it is a mixed strategy, the strategy of player A = [0, 6/9, 3/9]

The expected gain for player A = E(A) = (6/9)2 + (3/9)3 = 4/3 +1= 7/3

Let the player B plays Strategy I with probability y and strategy II with probability 1-y Then if the player B plays Strategy I all the time, B's expected gain will be

2.y + 5(1-y) = 5- 3y ---------- eqn-1

Similarly , if the player B plays strategyT all the time, B's expectd gain will be E(B, II) = 3 y - 3 (1-y) = 6y-3---------- eqn-2

It can be shown mathematically that E(A, I) = E (A, II) 5-3y = 6y-3

9y =8, y= 8/9, 1-y = 1/9

The expected gain for player B = E(A) = 2* 8/9 + 5 * 1/9 = 21/9 The complete solution of the game is;

A's optimal strategy is [0, 8/9, 1/0]

Player B's optimal strategy is [8/9, 1/9, 0]

The expected value of the game for player A = 19/9

	PLAYER - B		
		I	II
PLAYER - A	II	2	5
	III	3	-3

Problem-5:determine the solution of the following game

	PLAYER - B			
PLAYER -A		I	II	III
	I	**6**	1	**3**
	II	0	9	7
	III	2		4

	PLAYER - B					
PLAYER -A		I	II	III	Row minimum	
	I	**6**	1	**3**	1	Maximin = Maxi(1, 0,2)= 2
	II	0	9	7	0	
	III	2	3	4	2	
colmn max		6	9	7		
		minimax = mini(6,9,7)= 6				

Max Min # minimax, so there is no saddle point.

	PLAYER - B			
PLAYER -A		I	II	III
	I	**6**	1	**3**
	II	0	9	7
	III	2	3	4

This has no saddle point.

Further none of the pure strategies of A is inferior to any of his other pure strategies. However average of A's first & second pure strategies gives us

Row-III <= (average of row –I and row-II) (0 , 9, 7) <= [(6+0)/2, (1+9)/2, (3+7)/2]

[0, 9, 7] <= [3, 5, 5]. , row-III is inferior to other rows, delete Row-III

	PLAYER - B		
PLAYER -A		I	II	III
	I	**6**	1	**3**
	II	0	9	7
	III	2	3	4

Problem :6 Solve the following game and find the value of game:

		PLAYER - B			
PLAYER -A		I	II	III	IV
	II	3	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

Row-1 <= Row-III

(3, 2, 4, 0) <= (4, 2, 4, 0) delete Row-1

		PLAYER - B			
PLAYER -A		I	II	III	IV
	II	3	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

Col –I >= col-III, detete col-I

col-I		col-III
3	➢ =	2
4		4

Column II >= Avg of Col III % col IV

		II		Avg col III &IV
PLAYER-A	II	4	>=	(2+4)/2=3
	III	2		(4+0)/2= 2
	IV	4		(0+8)/2=4

Delete col-II

		III	IV
PLAYER-A	II	2	4
	III	4	0
	IV	0	8

Row-II <= Avg of Row III & row IV

[2, 4] <= [(4+0)/2, (0+8)/2]

[2, 4] <= [2, 4] delete row-II

		III y=2/3	IV 1-y=1/3
PLAYER-A	III x=2/3	4	0
	IV 1-x=1/3	0	8

4x + 0* (1-x) = 0*x + 8 (1-x), 4x+8 x =8, 12x =8 , x= 2/3

4y + 0*(1-y) = 0.y + 8 * (1-y)

4y+ 8y = 8, y = 8/12 = 2/3

Strategy of player A = (0, 0, 2/3, 1/3)

Strategy of player B = (0, 0, 2/3, 1/3)

Value of game for player A = 2/3*4 = 8/3

GRAPHICAL METHODS FOR (2*n and m*2) GAMES

Problem-7: Solve the following (2 X 3) game graphically

	PLAYER - B			
PLAYER -A		y_1 I	y_2 II	y_3 III
	I x_1	1	3	11
	II $1-x_1$	8	5	2

Solution: Max(Row min) = Max (1, 2) =1

Mini (col max) = mini (8, 5, 11) =8

Maxmin # minmax

This game does not have a saddle point.

Thu the player A's expected pay-off corresponding to the player B's pure strategies are given in following table-1

Three expected payoff lines are :

B's Pure strategies	A's Expected payoff $E(x_1)$
I	$E(x_1) = 1x_1 + 8(1-x_1) = -7x_1 + 8$
II	$E(x_1) = 3.x_1 + 5(1-x_1) = -2x_1 + 5$
III	$E(x_1) = 11x_1 + 2(1-x_1) = 9x_1 + 2$

$E(x_1) = -7x_1 + 8$

$E(x_1) = -2x_1 + 5$

$E(x_1) = 9x_1 + 2$ and can be plotted on a graph as follows

First draw two parallel lines one MODULE apart and mark a scale on each. These two lines will represent two strategies available to the player A. then draw lines to represent each of player B's strategies.

For example, to represent the player B's Ist strategy, join mark 1 on scale Iand mark 8 on scale II.

To represent the player B's second strategy, join mark 3 on scale I & mark 5 on scale II and so on.

Since the expected pay-off $E(x_1)$ is a function of x_1 alone, these three expected pay-off lines can be drawn by taking x_1 as x-axis and $E(x_1)$ as y-axis nd $E(x_1)$ as y- axis.

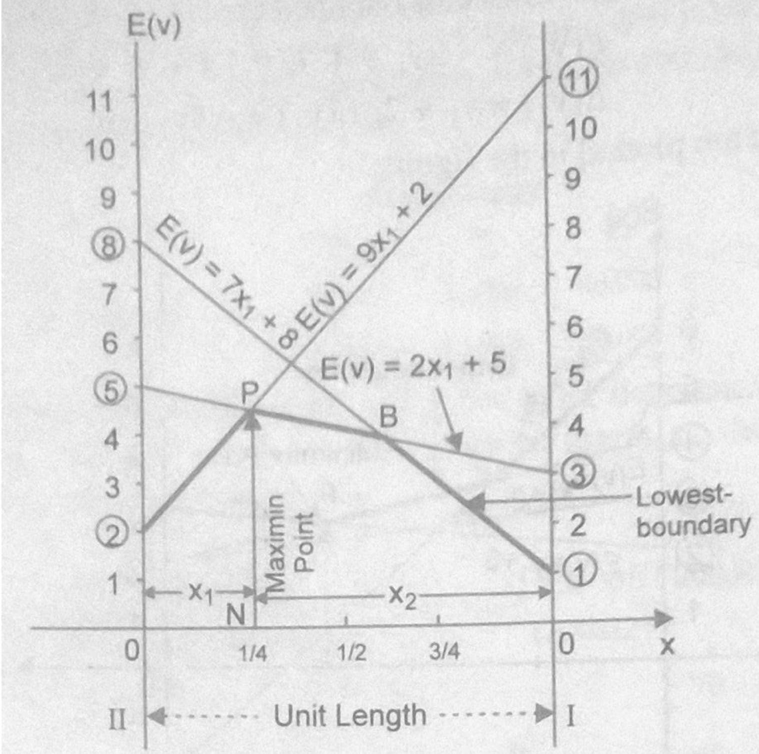


Fig. 2 Graphical representation for solving (2 × n) game.

Points **A, P,B,C on the lowest boundary** represent the lowest possible expected gain to the player A for any value of x_1 between 0 and 1. According to the maximum criterion, the player A chooses the best of these worst outcomes.

Clearly, the highest point P on the lowest boundary will give the largest expected gain PN to A. so the best strategies for the player B are those which pass through the point P. thus the game is reduced to 2 x 2 as shown in table-2.

	PLAYER - B		
		y_2 II	y_3 III
PLAYER - A	I x_1	3	11
	II $1-x_1$	5	2

Considering player-A $3x_1 + 5(1-x_1) = 11x_1 + 2(1-x_1)$,

$$3x_1+5 - 5x_1 = 11x_1 + 2- 2x_1$$

$$3x_1-5x_1 -11x_1 +2x_1 = 2-5$$

$$- \quad 11x_1 = -3 \rightarrow \quad x_1 = 3/11$$

Considering player-B

$$- \quad 3y_2 + 11(1-y_2) = 5y_2 + (1-y_2)\, 2$$

$$3y_2 +11-11y_2 = 5y_2 +2 -2y_2$$

$$3y_2 -11y_2 -5y_2 +2y_2 = 2 -11 \rightarrow \quad -11y_2 = -9$$

$$Y_2 = 9/11$$

$$Y_3 = (1-y_2) = 2/11$$

The solution of the game is obtained as follows:

The player A choose optimal mixed strategy (x1, x2)= (3/11, 9/11)

The player B choose optimal mixed strategy (y1, y2 , y3)= (0, 2/11, 9/11)

The value of game to the player A is V = 49/11

V(A) = 3*3/11 + 5*8/11 = 49/11

Problem-8: solve the game graphically whose payoff matrix for player A is given as follows:

		PLAYER - B	
		y_1 I	y_2 II
PLAYER -A	I x_1	2	4
	II $1-x_1$	2	3
	III	3	2
	IV	-2	6

Solution:

Max(min rows) = max(2,2,2,-2) =2

Min(max colums) = Min(3, 6) = 3

Maxmin # mimax , hence there is no saddle point

The game does not have a saddle point,

Let y_1 and y_2 (= $1-y_1$) be mixed strategies of the player B

Thu the player B's expected pay-off corresponding to the player A's pure strategies are given in following table-1

Four expected payoff lines are :

A's Pure str	B's Expected payoff $E(x_1)$
I	$E(y_1) = 2y_1 + 4(1-y_1) = -2y_1 + 4$
II	$E(y_1) = 2.y_1 + 3(1-y_1) = -y_1 + 3$
III	$E(y_1) = 3y_1 + 2(1-y_1) = y_1 + 2$
IV	$E(y_1) = -2y_1 + 6(1-y_1) = -8y_1 + 6$

$E(y_1) = -2y_1 + 4$

$E(y_1) = -y_1 + 3$

$E(y_1) = y_1 + 2$

$E(y_1) = -8y_1 + 6$ and can be plotted on a graph as follows

First draw two parallel lines one MODULE apart and mark a scale on each. These two lines will represent two strategies available to the player B. then draw lines to represent each of player A's strategies.

For example, to represent the player A's I[st] strategy, join mark 2 on scale Iand mark 4 on scale II.

To represent the player A's second strategy, join mark 2 on scale I & mark 3 on scale II and so on.

Since the expected pay-off $E(y_1)$ is a function of y_1 alone, these four expected pay-off lines can be drawn by taking y_1 as x-axis and $E(y_1)$ as y-axis

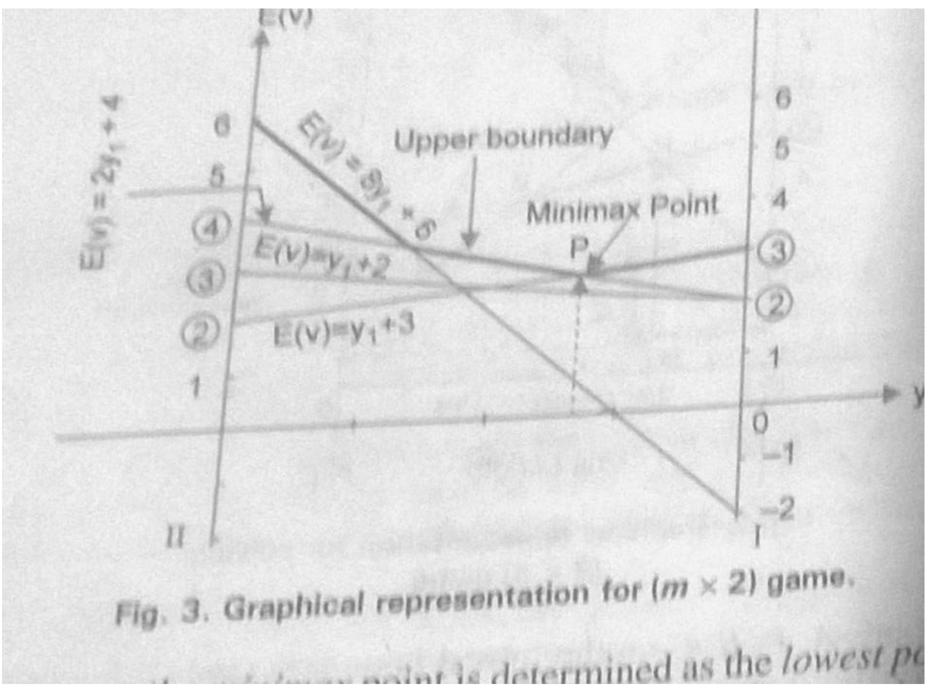


Fig. 3. Graphical representation for (m × 2) game.

In this case, the minima point is determined as the lowest point PointP on the uppermost boundary. Lines intersecting at the minimax point P correspond to the player A's pure strategies I and III. This indicates x2 = 0, x4 =0. Thus the reduced game is given in table-2.

Clearly, the lowest point P on the highest boundary will give the lowest expected gain PN to B. so the best strategies for the player A are those which pass through the point P. thus the game is reduced to 2 x 2 as shown in table-2.

		PLAYER - B	
PLAYER -A		y_1 I	y_2 II
	I x_1	2	4
	III $1-x_1$	3	2

Considering player-A $2x_1 + 3(1-x_1) = 4x_1 + 2(1-x_1)$,

$2x1+3 - 3x_1 = 4x_1 + 2- 2x_1$

2x1-3x1 -4x1 +2 x1 = 2-3

- $3x_1 = -1$ ➔ x1 =1/3 , 1-x1 = 2/3
- Considering player-B 2y1 + 4(1-y1) = 3y1 + (1-y1) 2

2y1 + 4 - 4y1 = 3y1 +2 -2y1

2y1-4y1 -3y1 +2y1 = 2 -4→ -3y1 = -2

Y1 = 2/3

Y2 = (1-y1) = 1/3

Value of game for player A = 2*1/3 + 3 *2/3 = 8/3

The solution of the game is obtained as follows:

The player A choose optimal mixed strategy (x1, x2, x3, x4)= (1/3, 0, 2/3, 0)

The player B choose optimal mixed strategy (y1, y2)= (2/3, 1/3)

The value of game to the player A is V = 8/3

V(A) = 2*1/3 + 3*2/3 = 8/3

------ ** -----

INVENTORY

This chapter will focuses on following concepts.

- Inventory Introduction
- Inventory Model
- Introduction

Definition:

The function of directing the movement of goods through the entire manufacturing cycle from the requisitioning of raw materials to the inventory of finishes goods orderly mannered to meet the objectives of maximum customer-service with minimum investment and efficient (low-cost) plant operation.

What Is Inventory?

In broad sense, inventory may be defined as the stock of goods, commodities or other economic resources that are stored or reserved in order to ensure smooth and efficient running of business affairs.

The inventory or stock of goods may be kept in any of the following forms:

i. Raw material inventory, i.e. raw materials which are kept in stock for using in the production of\goods.

ii. Work-in-process inventory, i.e. semi finished goods or goods in process, which are stored during the production process.

iii. Finished goods inventory, i.e. finished goods awaiting shipment from the factory.

iv. Inventory also includes furniture, machinery, fixtures, etc. The term inventory may be classified in two main categories. **Reasons for carrying Inventory:**

i. To improve customer service

ii. To reduce operating costs

iii. Maintenance of operational capability

iv. To manage irregular supply and demand.

v. To avail quantity discounts.

vi. Avoiding stock outs (shortages) 1- Direct Inventories

The items which playa direct role in the manufacture and become an integral part of finished goods are included in the category of direct inventories. These may be further classified into four main groups:

a. Raw material inventories are provided:

i. for economical bulk purchasing,

ii. to enable production rate changes

iii. to provide production buffer against delays in transportation,

iv. for seasonal fluctuations.

b. Work-in-process inventories are provided:

i. to enable economical lot production,

ii. to cater to the variety of products

iii. for replacement of wastages,

iv. to maintain uniform production even if amount of sales may vary.

c. Finished-goods inventories are provided:

i. for maintaining off-self delivery,

ii. to allow stabilization of the production level

iii. for sales promotion.

d. Spare parts. Indirect Inventories

Indirect inventories include those items, which are necessarily required for manufacturing but do not become the component of finished production, like: oil, grease, lubricants, petrol, and office-material maintenance material, etc.

Types of Inventory Models

Basically, there are five types of inventory models:

i. Fluctuation Inventories: These have to be carried because sales and production times cannot be predicted accurately. In real-life problems, there are fluctuations in the demand and lead- times that affect the production of items. Such types of reserve stocks or safety stocks are called fluctuation inventories.

ii. Anticipation Inventories : .These are built up in advance for the season of large sales, a promotion programe or a plant shutdown period. In fact, anticipation inventories store them en and machine hours for future requirements.

iii. Cycle (lot-size) inventories. In practical situations, it seldom happens that the rate of consumption is the same as the rate of production or purchasing .So the items are procured in larger quantities than they are required. This results in cycle (or lot-size) inventories.

iv. Transportation Inventories : .Such inventories exist because the materials are required to move from one place to another. When the transportation time is long, the items under transport cannot be served to customers. These inventories exist solely because of transportation time.

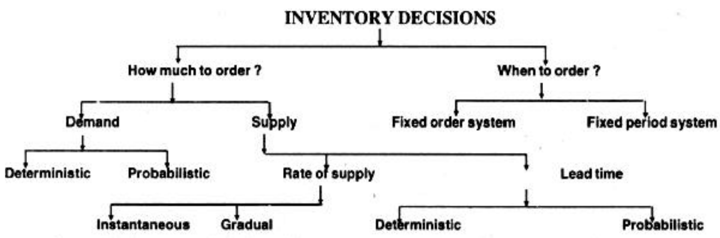
v. Decoupling Inventories: Such inventories are needed for meeting out the demands during the decoupling period of manufacturing or purchasing.

Inventory Decisions

The managers must take two basic decisions in order to accomplish the functions of inventory. The decisions made for every item in the inventory are:

i. How much amount of an item should be ordered when the inventory of that item is to be replenished?

ii. When to replenish the inventory of that item? Inventory decisions may be



Before taking inventory decisions, it is necessary to develop an inventory model.

classified as follows:

Inventory Costs

1. Holding Cost(C_1orCh): The cost associated with carrying or holding the goods in stock is know

as *holding* or *carrying* cost which is usually denoted by C.or *Ch* per MODULE of goods for a MODUL of time. Holding cost is assumed to vary directly with the size of inventory as well as the time for whic the item is held in stock. The following components constitute the holding cost:,

i. *Invested Capital Cost.* This is the interest charge over the capital investment. Since this is the most important component, a careful investigation is required to determine its rate.

ii. *Record Keeping and Administrative Cost.* This signifies the need of keeping funds for maintaining the records and necessary administration.

iii. *Handling Costs:* These include all costs associated with movement of stock such as: *cost of labour over-head cranes, gantries* and *other machinery* required for this purpose.

iv. *Storage Costs:* These involve the rent of storage space or depreciation and interest even if the own space is used.

v. *Depreciation, Deterioration and Obsolescence Costs:* Such costs arise due to the items in stock being out of fashion or the items undergoing chemical changes during storage *(e.g.* rusting in steel).

vi. *Taxes and Insurance Cost:* .All these costs require careful study and generally amounts to I% to 2% of the invested capital.

vii. Purchase Price or Production Costs.: Purchase price per MODULE item is affected by the quantity purchased due to *quantity discounts* or *price-breaks.* Production cost per MODULE item depends upon the length of production runs. For long smooth production runs this cost in lower due to more efficiency of men and machines. So the order quantity must be suitably modified to take the advantage of these price discounts. If *P* is the purchase price of an item and *I* is the stock holding cost per MODULE item expressed as a fraction of stock value (in rupees), then the holding cost C. = *IP*.

viii. *Salvage Costs or Selling Price.* When the demand for an item is affected by its quantity in stock, the decision model of the problem depends upon the profit maximization criterion and includes the revenue (sales tax etc.) from the sale of the item. Generally, salvage costs are combined with the storage costs and not considered independently.

2. Shortage Costs or Stock-out Costs*(C2*or*C,).*The penalty costs that are incurred as a result of running out of stock *(i.e.,* shortage) are known as *shortage* or *stock-out* costs. These are denoted by C2 or *Cs* per MODULE of goods (or a specified period.

These costs arise due to shortage of goods, sales may be lost, and good will may be lost either by a delay in meeting the demand or being quite unable to meet the demand at all. In the case where the unfilled demand for the goods can be satisfied at a latter date (back log case), these costs are usually assumed to vary directly with the shortage quantity and the delaying time both. On the other hand, if the unfilled demand is lost (no backlog case), shortage costs become proportional to shortage quantity only.

3. Set-up Costs *(C3* or *Co),* these include the fixed cost associated with obtaining goods through *placing of an order* or *purchasing* or *manufacturing* or *setting up a machinery* before starting production. So they include costs of purchase, requisition, follow-up, receiving the goods, quality control, etc. These are also called *order costs* or replenishment costs, usually denoted by C3 or *Co* per production run (cycle). They are assumed to be independent of the quantity ordered or produced.

Why Inventory is maintained?

As we are aware of the fact that the inventory is maintained for efficient and smooth running of business affairs. If a manufacturer has no stock of goods at all, on receiving a sale-order he has to place an order for purchase of raw materials, wait for their receipt and then start his production. Thus, the customers will have to wait for a long time for the delivery of the goods and may turn to other suppliers. This results in a heavy loss of business. So it becomes necessary to maintain an inventory because of the following reasons:

i. Inventory helps in smooth and efficient running of business.

ii. Inventory provides service to the customer's immediately or at a short notice. Due to absence of stock, the company may have to pay high prices because of piece-wise purchasing. Maintaining of inventory may earn price discount because of bulk-purchasing.

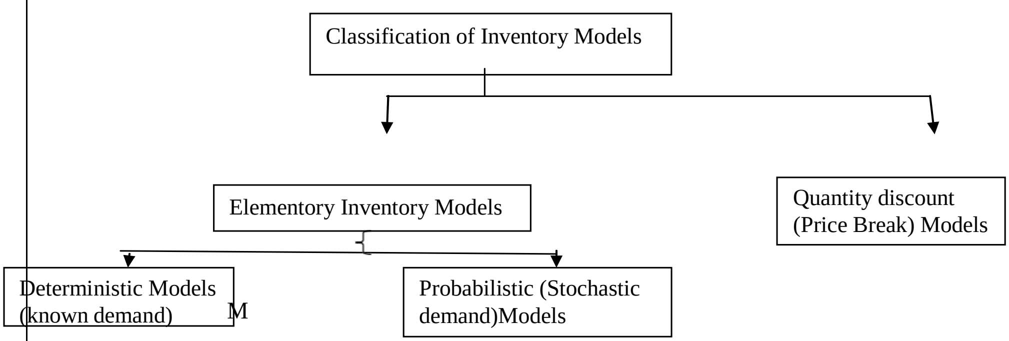
iii. Inventory also acts as a buffer stock when raw materials are received late and so many sale-orders are likely to be rejected.

iv. Inventory also reduces product costs because there is an additional advantage of batching and long smooth running production runs.

v. Inventory helps in maintaining the economy by absorbing some of the fluctuations when the demand for an item fluctuates or is seasonal.

i. Firstly, demand for the item is at a constant rate and is known to the decision maker in advance.

4. Secondly, the lead-time : which is the (elapsed time between the placement of the order and its receipt into inventory) or the time required for acquiring an item is also known. Although above two assumptions are rarely valid for inventory problems in the business world, they do allow us to develop a simple model into which more realistic, complicating factors can be introduced.



INVENTORY CONTROL MODELS WITH OUT SHORTAGES

Model-1(a): EOQ model with constant rate of (uniform) demand' Model -1(b).EOQ model with different rates of demand

Model-1(c): Economic Production Quantity model when supply (Replenishment) is gradual.

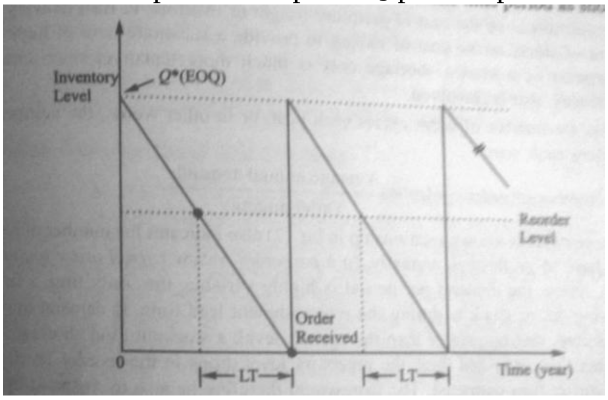
Model-2: Model with Two-price breaks

Model-1(a): EOQ model with constant rate of (uniform) demand'

In this Model we want to derive an Economic lot size formula for the optimum production quantity q per cycle. (ie per production run) of a single product so as to minimize the total average variable cost per MODULE time, where

- Demand is uniform at a rate of R quantity MODULEs per MODULE time.
- Lead time is zero (or known exactly)

- Production rate is infinite ie production is instantaneous.
- Shortages are not allowed (C_2 =0)
- Holding cost is rupees C_1 per quantity MODULE per MODULE time
- Setup cost is rupees C_3 per setup.



Total Inventory cost = total inventory carrying costs + total Ordering cost eqn-1

Since the demand is uniform and known exactly and the supply is instantaneous, the reorder point is that when the inventory falls to zero. Rise and fall in inventory level for a particular item over time reflects the periodic cycles of depletion and replenishment as shown in fig. since the actual consumption of inventory varies constantly, theconcept of average inventory is applicable here with a constant rate of demand.

Total inventory carrying cost = (Average no. of MODULEs in inventory * cost of one MODULE * inventory carrying cost %)

$= (\frac{0+q}{2})*C*I = (\frac{q}{2})*C*I = (\frac{q}{2})*c_1$ ------- Eqn-2

as C * I = C1 -------- eqn-2

where C.I can be written simply as C_1, the holding or carrying cost per MODULE for a MODULE time. The total ordering cost= no. of orders per year * ordering cost per order

=(R/q) * C3 ------------------------- eqn-3

Substituting Eqn-2 and eqn-3 in Equation -1

Total inventory cost = C(q) =( ) C_1 + (−) C_3 --------- Eqn---4

But the total inventory cost C(q) is minimum, when = 0

Or when inventory carrying costs becomes equal to total ordering costs ( ) C_1 = ( ) C_3

$q^{2.}$ = 2. R C_3/ C_1

$q^* = \sqrt{\frac{2C_3R}{C_1}}$

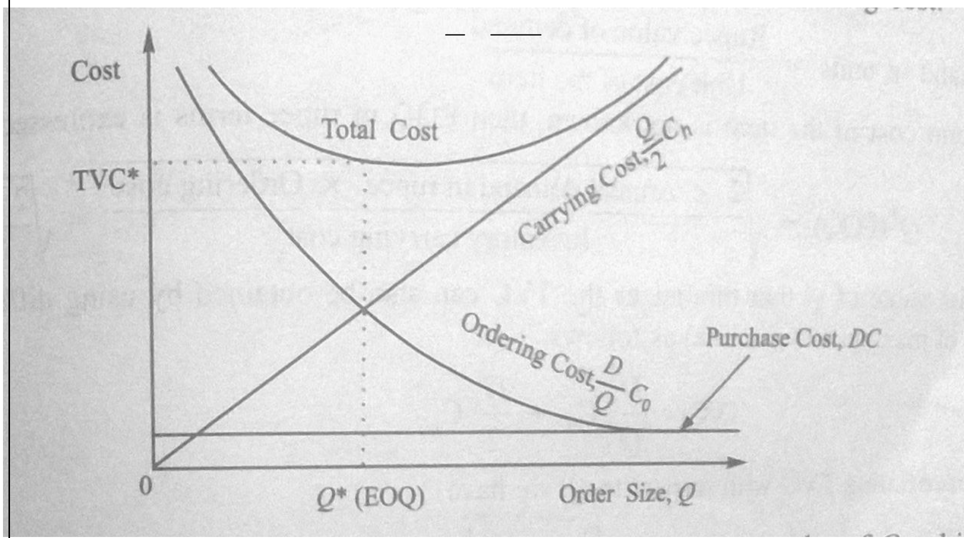
To obtain the minimum of total inventory cost C(q), substitute the value of q^* in the equation---3

$$C_{min} = \sqrt{2C_1C_3R} = \sqrt{2*8000*1.00*0.20*12.50} = Rs200$$

Optimum interval of ordering (t^*) $= \frac{R}{q} = \frac{8000}{1000} = 8 ordersperyear$, where (q=Rt)

Optimum no. of orders (N) = $\frac{\text{Total annual quantity requirement}}{q} = \frac{R}{q} = \frac{8000}{1000} = 8.orders per year$

Number of days supply per optimum order (d) = $\frac{365}{N} = \frac{365}{5} = 45.8$ days



Example: Suppose annual demand equals 8000 MODULEs, ordering cost is Rs 12.50, the carrying cost of average inventory is 20% per year and the cost per MODULE is Rs 1.00. compute economic order qty, C_{mim}, t^*, N^*, number of days supply.

Given R= annual demand = 8000 MODULEs/year C_3 = ordering cost = Rs 12.50 per order

I = 20% , C = 1.00

C_1= C. I = 1 * 20/100 = 0.20

i. EOQ = q = square root[(2*12.5 *8000) / (1.00 * 0.20)] = 1000 MODULEs

ii. Total inventory carrying cost = (q * C * I)/2 = (1000* 1*o.20)/2 = Rs100

iii. Total ordering cost = ($R*C_3$)/q = (8000*12.50)/1000 = Rs100 iv.

v. Optimum no. of orders = R/q = 8000/1000 = 8 orders/year

VI. No. of days supply per optimum order , d = 365/ N = 365/8 = 45.8 days

Model 1(b) : ECONOMIC LOT SIZE WITH DIFFERET RATES OF DEMSND ON DIFFERENT CYCLES…..

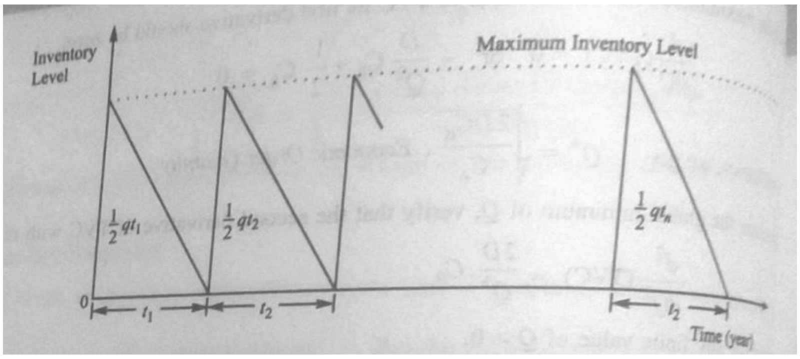


Fig-3 diagram of inventory level with different rates demand in different cycles

If Total demand D is prescribed over time period T, instead odf demand rate being constant for each production cycle. If rate f demand being different in different production cycles. Then derive the optimal lost size formula and the minimum cost.

Let the total demand D be specified as demand during total time period T and q be the stock level to be fixed.

Thus the inventory costs are determined as follows: Carrying costs = Average inventory * C_1*T

Ordering costs =Number of orders * ordering cost pe order= * C_3

No. of production cycles will be given by n = D/q Total period T = $t_1 + t_2 + t_3 + \ldots + t_n$

It is obvious that the fixed quantity q, produced in the beginning of interval t_1, is supposed with uniform rate of demand in interval t_1 only.

Similarly the same quantity q is again produced / procured in the beginning of the next interval t_2 which is supplied with some other different rate of demand during interval t_2 and so on for the remaining intervals t_3, t_4 …., t_n

The graphical representation of this model is shown in figure,

The carrying cost for the period T will be = ((q . t_1) + (q . t_2)+ (q . t_3)+ …….+((q . t_n)

$$= C_1 q\,(t_1 + t_2 + t_3 + + t_n) = C_1 q\, T$$

The set up cost = DC_3 / q

Total inventory cost = carrying cost + Ordering costs C(q) = C_1q T + C_3

For optimization = C_1T - C_3 = 0

$$\sqrt{\frac{2C_3(D/T)}{C_1}}$$

$q\dfrac{d^2c}{dq^2}$ = [- D C_3 (-2)]/ q^3 which is a positive

Therefore the optimum lot size $q^* = \sqrt{\dfrac{2C_3(D/T)}{C_1}}$

$$c_{\min} = \sqrt{2C_1 c_3 (D/T)}$$

Here we observe that fixed demand rate R in the model-1(a) is replaced by Average demand rate (D/T) in this model.

Problem-2: you have to supply your customers 100 MODULEs of certain product every Monday (and only then). You obtain the product from a local supplier at Rs 60 per MODULE. The costs of ordering and transportation from the supplier are Rs 150 per order. The cost of carrying inventory is estimated at 15% per year of the cost of product carried.

a) Find the lot size which will minimize the cost of the system.

b) Determine the optimal cost.

Given Demand rate = R = 100 MODULEs/week

C_3= setup cost or order cost per order

C_1= 15% per year of the cost of the product carried.= Rs (15*60)/ (100*52) per MODULE per week=

= rs 9/52 per MODULE per week (1 year = 52 weeks)

We know C_1= C*I

= 15% * 60 per year

assuming 1 year = 52 weeks

$= \frac{15}{100} * \frac{60}{52}$ per week = Rs $\frac{9}{52}$ per week

$$q^* = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2*150*100}{9/52_1}} = 416 units$$

$$c_{\min} = \sqrt{2C_1 c_3 R} = \sqrt{2*(9/52)150*100} = 72$$

$C^{\min} =$ 60R + $\sqrt{2C_1 c_3 R}$

optimum cost = 60 *100 +72 = Rs 6072

Problem: An air craft company uses rivets at an approximate customer rate of2500 kg per year. Each MODULE costs Rs 30per kg and the company personnel estimates that it costs Rs 130 to place an order and that the carrying cost of inventory is 10% per year. How frequently should orders for revets be placed? Also determine the optimum size of each order.

Solution: R = 2500 kgs per year.

C_1= (cost of each MODULE* Inventory carrying cost)= 30 *10% = = Rs 3 per MODULE per year.

$$q^* = \sqrt{\frac{2C_3R}{C_1}} =$$

$$q^* = \sqrt{\frac{2*2500*130}{30*0.10}} = 466 units$$

$$t^* = \frac{q^*}{R} = 466\frac{466}{2500} - 0.18 year = 2.16 months$$

N; number of orders = R/q = 466/2500 = 0.18 year = 2.16 months.

Number of orders = $n = \frac{R}{q}$ = 2500/466 = 5 orders/year.

.

Model-1c: ECONOMIC LOT SIZE WITH FINITE RATE OF REPLENISHMENT (EOQ production

Model)

Let C_1= holding cost per item per MODULE time

$C_2 = \infty$ ie shortages are not permitted.

R= number of items required per MODULE time (consumption rate)

K= production rate is finite , uniform and greater than R.

T = interval between production cycles.

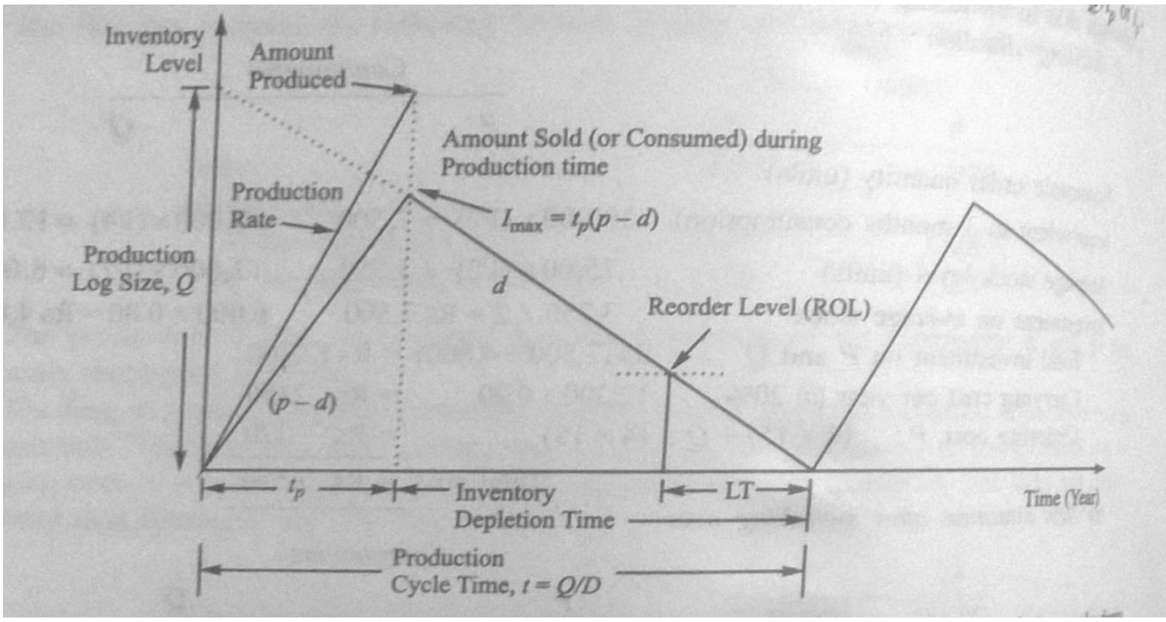
Q= Rt (number of items produced per production run.)

Find the expression for

a) The optimal order quantity.

b) Reorder point

c) Minimum average cost per MODULE time.



The inventory of finished goods does not build up immediately to its maximum point Q. Rather it builds up gradually since goods are being produced faster than they being sold.

K = production rate.

R= consumption rate.

In this model, each production cycle time T consists of two parts t_1& t_2

Where t_1 is the period during which the stock growing at a rate of (K-R) items per MODULE time.

t_2 is the period during which there is no replenishment (or supply or production) but there is only a constant demand at the rate of R.

Further, we assume the q is the stock available at the end of time t_1, which is expected to be consumed during remaining period t_2 at the consumption (or demand) rate R . it is evident from fig

$$t_1 = \frac{Q}{K-R}, \qquad t_2 = \frac{Q}{R}$$

$t_1 + t_2 = t$

$$t_1 = \frac{Q}{K-R}, \qquad t_2 = \frac{Q}{R}$$

$$t = t_1 + t_2 = \frac{Q}{K-R} + \frac{Q}{R} = \frac{QR + QK - QR}{R(K-R)} = \frac{QK}{R(K-R)}$$

we know $t = t_1 + t_2$ and $q=Rt$ =>

$$t = \frac{R}{q}$$

$$t = \frac{QK}{R(K-R)}$$

$$Q = \frac{(K-R)}{K} * q$$

Now holding cost for the time period t = (Area of ONB)C_1 = (ΔMN * OB * C_1)/2

$$= = \frac{1}{2} QtC_1$$

Set up cost for period t = C_3

Therefore the total average cost C(q) = $\frac{1}{2} Qt C_1 + \frac{C_3}{t}$

For optimization $\frac{dc}{dq} = \frac{1}{2} C_1 \frac{(K-R)}{K} - C_3 \frac{R}{q^2} = 0 - - - - - - - - eqn - 1$

$$q^* = \sqrt{\frac{2C_3RK}{C_1(K-R)}}$$

$\frac{d^2c}{dq^2}$ = 0 + 2. D C_3 (-2)]/ q^3 which is a positive

Therefore the optimum lot size $q^* = \sqrt{\frac{2C_3RK)}{C_1(K-R)}}$

$$t^* = \frac{q^*}{R} = \sqrt{\frac{2C_3RK}{C_1R^2(K-R)}} = \sqrt{\frac{2C_3K}{C_1R(K-R)}} = optimum - time\,interval$$

Substituting the alue of q^* in equation-1

$$c_{min} = \sqrt{2C_1(1-\frac{R}{K})C_3R}$$

Here we observe that

- If K=R then C_{min} = 0 which implies there will be no varying cost and no setup cost.
- If K $\rightarrow \infty$ ie production rate is infinite then this problem becomes exactly same as model 1(a).
- Although in this model C_3 is same as model 1(a), 1(b), but the carrying cost is reduced in the ratio $(1-\frac{R}{K})$: 1 for minimum cost.

Problem:4 A contractor has to supply 10000 bearings per day to an automobile manufacturer. He finds that when he starts a production run, he can produce 25000 bearings per day. The cost of holding a bearing in stock for one year is 20 paisa, and setup cost of production run is Rs 18-00. How frequently should production run be made?

Solution: the is a production model ie model-1(c).

$C_{1=}$ carrying cost /MODULE/day=Rs 0.20 per 365 days= Rs 0.00055 per bearing per day

C_3= setup / production run = Rs 180 per production run.

R= no. of items required per day = 10,000 bearings per day.

K= Production rate = 10000 bearings per day (k K> R)

$$t^* = \sqrt{\frac{2C_3K}{RC_1(K-R)}} = \sqrt{\frac{2*180*25000}{10000*0.00055*(25000-10000)}} = 0.3day$$

$$t^* = \sqrt{\frac{2C_3K}{RC_1(K-R)}} = \sqrt{\frac{2*180*25000}{10000*0.00055*(25000-10000)}} = 0.3day$$

PURCHASE INVENTORY MODEL WITH TWO PRICE BREAK:

We consider a purchase situation when wo quantity discounts apply. Such a situation may be represented as follows:

Purchase cost per item	Range of quantity
P_1	$_1 \le q_1 < b_{1...}$
P_2	$b_1 \le q_2 < b_2$
P_3	$b_2 \le q_3$

Where b_1, b_2 are the quantities which determine the price beaks.

Working Rule or Procedure

Step-1: compute $\overset{*}{q}_3$ and compare b_2

i) If $\overset{*}{q}_3 \ge b_2$ then te optimum purchase qty is $\overset{*}{q}_3$

ii) If $\overset{*}{q}_1 = \sqrt{\frac{2c_3R}{P_1.I}} = \sqrt{\frac{2*200*100}{10.00*0.02}} = 447units < b_2$, Then go to step-2

Step-2: compute $\overset{*}{q}_2$

Since $\overset{*}{q}_3 < b_2$ and $\overset{*}{q}_2$ is alsi less than b_2

thus there are only two possibilities when $\overset{*}{q}_3 < b_2$, ie either $\overset{*}{q_2} \geq b_1$ or $\overset{*}{q_2} < b_1$

(i) if $\overset{*}{q_2} < b_2$ but $\overset{*}{q_2} \geq b_1$ then proceed as in the case of one price beak only.

that is compare the costs $C(\overset{*}{q_2})$ and $C(b_2)$ to obtain the optimum purchase quantity. The quantity with lower cost will naturally be the optimum

iii) If $\overset{*}{q_2} <$ (b_2 and b_1 both) then go to step-3.

Step-3: If $\overset{*}{q_2} <$ (b_2 and b_1 both) then compute $\overset{*}{q_1}$ which will satisfy the inequality $\overset{*}{q_1} < b_1$

in this case compare the costs $C(\overset{*}{q_1})$ with $C(b_1)$ and $C(b_2)$ both to determine the optimum purchase quantity.

Problem-1: Find the optimal order quantity for a product for which the price breaks are as follows:

Quantity	$0 \leq q_1 < 500$	$500 \leq q_2 < 750$	$750 \leq q_3$
MODULE cost (Rs)	10.00	9.25	8.75

The monthly demand for a product is 200 MODULEs, the cost of storage is 2% of the MODULE cost and the cost of ordering is Rs 350.

Soln: R = 200 MODULEs / month C_3= Rs 350 I = 0.02

P_1 = Rs 10.00 P_2 = 9.25 P_3 = 8.75

b_1 = 500, b_2= 750

step-1 : compute $\overset{*}{q_3}$ and obtain $\overset{*}{q_3} = \sqrt{\dfrac{2c_3R}{P_3.I}} = \sqrt{\dfrac{2*350*200}{8.75*0.02}} = 894$

since $\overset{*}{q_3} > b_2$ ie 894 > 750, therefore the optimum purchase quantity will be $\overset{*}{q_3} = 894$

Example-2: Find q^*, where R - 200 items/monthC_3= 100/-, I = 0.09 (ie 2% of the MODULE cost) and for

P_1 = Rs 10.00 for $0 \leq q_1 < 500$

P_2= Rs 9.25 for $500 \leq q_2 < 750$

P_3= Rs 8.75 for $500 \leq q_3$

Soln: compute $\overset{*}{q_3}$ and obtain $\overset{*}{q_1} = \sqrt{\dfrac{2c_3R}{P_3.I}} = \sqrt{\dfrac{2*200*100}{8.75*0.02}} = 478$

Since $\overset{*}{q_3} < b_2$

ie (478 < 500) so compute q_2^*

$$q_2^* = \sqrt{\frac{2c_3R}{2.I}} = \sqrt{\frac{2*200*100}{9.75*0.02}} = 465$$

Since $q_2^* < b_1$

ie (465< 500) so compute q_1^*

$$q_1^* = \sqrt{\frac{2c_3R}{p_2.I}} = \sqrt{\frac{2*500*180}{25.000*0.10}} = 268$$

$$\frac{Lossorprofit}{Lossorprofit + \cos tofspare} = \frac{100,00,000}{100,00,000 + 1,00,000} = 0.9901$$

Comput C(q_1^*) = C (447) = 2090.42

C(b_1) = C (500)= = 1937.25

C(b_2) = C)750) = = 1843.29 minimum

Therefore C(750) < C(500) < C (q_1^* = 447)

The optimum purchase quantity q_1^* = 750

Example-3: if b_1=400 (instead of 500)

b_2= 3000 (instead of 750)

Find q^*, where R - 200 items/monthC_3= 100/-, I = 0.09 (ie 2% of the MODULE cost) and for

P_1 = Rs 10.00 for $0 \le q_1 < 400$

P_2 = Rs 9.25 for $400 \le q_2 < 3000$

P_3 = Rs 8.75 for $3000 \le q_3$

Solution: Compute q_3^* and obtain $q_1^* = \sqrt{\frac{2c_3R}{P_3.I}} = \sqrt{\frac{2*200*100}{8.75*0.02}} = 478$ MODULEs…

Since $q_3^* < b_2$

ie (478 < 30000) so compute q_2^*

$$q_2^* = \sqrt{\frac{2c_3R}{2.I}} = \sqrt{\frac{2*200*100}{9.75*0.02}} = 465$$

Since q_2^* < 465which falls with in the range (400 ≤ q_2< 3000 O

There is no need to calculate q_1^*

Rather compare only C(q_2) = C(465) = = Rs1937

ie (465< 500) so compute q_1^*

C (3000)= =Rs 2020.17

Since (q_2^*) < C (3000), the most economical purchase quantity is q_2^* = 465

Example-4:

if b_1 = 1500 (instead of7 50)

P_2= 9.00 (instead of 8.75)

R = 200 items/monthC_3= 100/-, I = 0.02 (ie 2% of the MODULE cost) and for

P_1 = Rs 10.00 for 0 ≤ q_1<500

P_2 = Rs 9.25 for 500 ≤ q_2<1500

P_3 = Rs 9.00 for 1500≤ q

Soln: compute q_3^* and obtain $q_3^* = \sqrt{\frac{2c_3R}{2.I}} = \sqrt{\frac{2*200*100}{9.00*0.02}} = 471\,unit$ which is less than 1500

Since q_3^* < b_2

ie (471 <1500) so compute q_2^*

$$q_2^* 1 = \sqrt{\frac{2c_3R}{2.I}} = \sqrt{\frac{2*200*100}{9.25*0.02}} = 465$$

Since q_2^* < 465which < 500 , so compute q_1^*

$$q_1^* = \sqrt{\frac{2c_3R}{P_1.I}} = \sqrt{\frac{2*200*100}{10.00*0.02}} = 447\,units$$

Compare C(1500), C(500) and C(q_1^*=447)

C(1500) = = Rs 1949.33

C(500)= = Rs 1937.25 minimum

C(q_1^*=447) = =Rs 2090.42

Hence in this situation the optimum purchase quantity is $q^* = 500$

Problem-5:

if $b_1 = 3000$ (instead of 500)
and b2= 5000 (instead of 750)
$P_2 = 9.00$ (instead of 8.75)
R = 200 items/monthC_3= 100/-, I = 0.02 (ie 2% of the MODULE cost) and for

P_1 = Rs 10.00 for $0 \le q_1 < 3000$
P_2 = Rs 9.25 for $3000 \le q_2 < 5000$
P_3 = Rs 8.75 for $1500 \le q_3$

Soln: compute q_3^* and obtain $q_3^* = \sqrt{\frac{2c_3R}{2.I}} = \sqrt{\frac{2*200*100}{8.75*0.02}} = 478 unit$ which is less than 1500
.

Since $q_3^* <$ b_2 , $q_3^* < 500$

ie (471 < 500) so compute q_2^*

$$q_2^* = \sqrt{\frac{2c_3R}{2.I}} = \sqrt{\frac{2*200*100}{9.25*0.02}} = 465 \text{ MODULEs}$$

Since $q_2^* <$ 465which < 3000 , so compute q_1^*

$$q_1^* = \sqrt{\frac{2c_3R}{P_1.I}} = \sqrt{\frac{2*200*100}{10.00*0.02}} = 447 units$$

Compare C(3000), C(5000) and C(q_1^*=447

C(q_1^*=447) = =Rs 2092.42 minimum
C(3000) = = Rs 2135.17
C(5000)= = Rs 2192.50

Hence in this situation the optimum purchase quantity is $q^* = 447$

PURCHASE INVENTORY MODEL WITH NUMBER OF PRICE BEAKS

Problem- : The annual demand for a product is 500 MODULEs. The cost of storage per MODULE per year is 10% of the MODULE cost. The ordering cost is Rs 180 for each order. The MODULE cost depends upon the amount ordered. The range of amount ordered and the MODULE cost price are as follows:

Range of amount ordered	$0 \le q_1 < 500$	$500 \le q_2 < 1500$	$1500 \le q_3 < 3000$,	$3000 \le q_4$
MODULE cost (Rs)	25.00	24.80	24.60	244.40

Find the optimal order quantity?

Obviously this problem has three price beaks only

Soln: compute q_4^* and obtain $q_4^* = \sqrt{\frac{2c_3R}{2.I}} = \sqrt{\frac{2*500*180}{24.40*0.10}} = 272$, q_4^*

Since $q_4^* (= 272) < b_3 (= 3000)$

q_4^* = 272 is not optimal order quantity, so proceed to compute q_3^* as

$$q_3^* = \sqrt{\frac{2c_3R}{2.I}} = \sqrt{\frac{2*500*180}{24.600*0.10}} = 270$$

q_3^* = 270 MODULEs does not lie in the range $1500 \leq q_3 < 3000$, so q_3^* = 270 is not the optimal order qty,

So next compute q_2^*, $q_2^* = \sqrt{\frac{2c_3R}{p_2.I}} = \sqrt{\frac{2*500*180}{24.80*0.10}} = 269$ which is less than b_2, b_1 both .

Therefore as per rule compute

ie (471 < 500) so compute q_1^* and compare the cost C(q_1^*) with C(b_2) and C (b_1)

Since q_2^* < 465which < 3000 , so compute q_1^*

$$q_1^* = \sqrt{\frac{2c_3R}{p_2.I}} = \sqrt{\frac{2*500*180}{25.000*0.10}} = 268$$

Compare C(3000), C(5000) and C(q_1^* = 268)

$$C(q_1^*) = \frac{R}{q_1^*} * C_3 + R p_1 + \frac{1}{2} * I p_1 q_1^*) =$$

$$C(q_1^*) = \frac{R}{q_1^*} * C_3 + R p_1 + \frac{1}{2} * I p_1 q_1^*$$

$$C(q_1^* = 268) = \frac{500}{268} * 180 + 500 * 25 + \frac{1}{2} * 0.10 * 25 * 268 = Rs13{,}170.82 \text{ minimum}$$

$$C(b_1) = \frac{R}{b_1} * C_3 + R p_2 + \frac{1}{2} * I p_2 b_1$$

$$C(500^*) = \frac{500}{500} * 180 + 500 * 24.80 + \frac{1}{2} * 0.10 * 24.80 * 500 = Rs13200.00$$

$$C(b_2) = \frac{R}{b_2^*} * C_3 + R p_3 + \frac{1}{2} * I p_3 b_2$$

$$C(1500) = \frac{500}{1500} * 180 + 500 * 24.60 + \frac{1}{2} * 0.10 * 24.60 * 1500 = 14205.00$$

$$C(3000) = \frac{500}{3000} * 180 + 500 * 24.40 + \frac{1}{2} * 0.10 * 24.40 * 3000 = 30 + 12200 + 3660 = 15890$$

Since $C(q_1^*) < C(b_1) < C(b_2)$

C(268) < C (500) < C(1500)

q_1^* = 268 MODULEs is the optimum order quantity

STOCHASTIC INVENTORY MODELS

Instantaneous demand , no setup cost model

VI (a): discrete case

Find the optimum order level Z which minimizes the total expected cost under the following assumptions:

- t= constant interval between orders (t may be considered as MODULEy, e.g. daily, weekly, monthly etc,)
- Z= the stock (in discrete MODULEs) at the beginning of each day
- d= is the estimated (random) demand at discontinues rate with probability P(d). ie demand d arises at each interval with probability P(d).
- C_1= holding cost per item per 't' time MODULE
- C_2= shortage cost per item per t time nit.
- Lead time is zero.
- Demand is instantaneous.

In this model with instantaneous demand, it is assumed that the total demand is filled-up at the beginning of the period. Thus depending on the amount d demanded, the inventory position just after the demand occurs may be either positive (surplus)or negative (shortage)

News paper problem: a newspaper boy buys papers for Rs 2.60 each and sells them for Rs 3.60 each. He cannot return unused news papers. Daily demand has the following distribution.

No. of customers	23	24	25	26	27	28	29	30	31	32
Probability	0.01	0.03	0.06	0.10	0.20	0.25	0.15	0.10	0.05	0.05

if each day's demand is independent of the previous day's , how many papers should he order each day.

Solution:

Z= no of news papers ordered per day

D = the demand ie demand that could be sold per day, if z $\geq$ d

P(d)= the probability that the demand will be equal to 'd' on a randomly selected day.

c_1 = cost per news paper

c_2 = selling price oer year

if d exceeds ie d>= Z

profit = $c_2 - c_1$ and no paper will be left unsold

demand d < Z

Profit becomes = $(c_2 - c_1)$ d + (Z-d) c_1

No. of customers	23	24	25	26	27	28	29	30	31	32	
Probability	0.01	0.03	0.06	0.10	0.20	0.25	0.15	0.10	0.05	0.05	
cum probability	0.01	0.040.	0.10	0.20	0.40	0.65	0.80	0.90	0.95	0.95	

C_1 = holding cost / paper/day

C_2= shortage cost /paper

$$\sum_{d=23}^{Z} p(d) > \frac{C_2}{c_1 + C_2}$$

But $C_1 = c_1 = 2.60$

$C_2 = c_2 - c_1 = 3.60 - 2.60 = 1.00$

Hence $\frac{C_2}{C_1 + C_2} = \frac{2.60}{1.00 + 2.60} = \frac{2.60}{3.60} = 0.2777$

$\sum_{d=23}^{Z} p(d) >= 0.2777$ gives value of Z= 27 through above table.

Problem-2: Some of the spare parts of Ship costs Rs 1,00,000 each. These spare parts can only be ordered together with the ship. If not ordered at the time , when the ship is constructed, these parts cannot be made available on need. Suppose that a loss of Rs 100, 00,000 is suffered for each spare part is needed, when none is available in the stock. Further , suppose that probabilities that spare parts will be needed as replacement during the life-term of the class of the ship discussed are

Spares required	0	1	2	3	4	5 or more
Probability	0.9488	0..0400	0.0100	0.0010	0.0002	0.0000

How many spares parts should be procured?

Solution:

$C_1 = c_1 = 1,00,000$

$C_2 = c_2$= shortage cost = Rs 100,00,000

Spares required	0	1	2	3	4	5 or more
Probability	0.9488	0..0400	0.0100	0.0010	0.0002	0.0000
cum. prob	0.9488	0.9888	0.9988	09998	1.0000	1.0000

$$\frac{C_2}{C_1+C_2} = \frac{100000}{10,00,000+1,00,00} = \frac{100,00,000}{100,00,000+1,00,000} = \frac{100,00,000}{101,00,000}. = 0.9901$$

$$\sum_{d=23}^{Z} p(d) > \frac{C_2}{C_1+C_2} = \frac{Lossorprofit}{Lossorprofit + \cos tofspare} = \frac{100,00,000}{100,00,000+1,00,000} = 0.9901$$

$\sum_{d=23}^{Z} p(d) > 0.9901$ Which gives Z = 2

Model VI(b): CONTINUOUS CASE:

Instantaneous demand, setup cost zero, stock levels continuous, lead time zero

This model is same as Model VI(a), except that the stock levels are now assumed to be continuous (rather than discrete) quantities. So instead of probability P(d), we shall have f(x)dx, where f(x) is the probability density function.

$\int_{x1}^{x2} f(x)dx$= The probability of an order with in the range x_1 to x_2

now , the cost equation fr this model will be similar to model VI(a). only p(d) is replaced by f(x)dx and the $\sum$ is replaced by the sign of integration $(\int$ $)$. *then the cost equation or this model becomes*:

C(Z) =$C_1\int_0^z (z-x)f(x)dx$ + $C_2\int_x^{\infty} (x-Z)f(x)dx$ ------ eqn-1

Hence we can get optimum value of z satisfying eqation -1 for which the total expected cost C is minimum.

$\int_0^z (z-x)f(x)dx$ = $C_2/(C_1+C_2) = \frac{C2}{C1+C2}$

Problem-3: A backing company sells cake by the kg weight. It makes a profit of Rs 5.00 a kg on each kg sold on the day it is baked. It disposes of all cake not sold on the date of it is baked at a loss of Rs1.20 a kg. If demand is known to be rectangular between 2000 and 3000 kg. determine the optimal daily amount baked.

Soln: let

C_1= the carrying cost unsold cake if not sold on the day ogf baking = Rs 1.20

C_2 = the profit per kg = Rs 5

x = the demand which is continuous with probability density f(x).

$\int_{X1}^{x2} f(x)dx$ = the probability of an order with in the range x_1 to x_2

Z= the stock level

$\int_{x1}^{z} f(x)dx = \frac{C2}{C1+C2}$ -------- eqn-2

x_1= 2000, x_2= 3000

f(x) = $\frac{1}{x2-x1} = \frac{1}{3000-2000} = \frac{1}{1000}$

Substituting the values in the equation-2 we get

$\int_{200o}^{z}(\frac{1}{1000})dx = \frac{5.00}{1.20+5.00}$ = 0.807

[1/1000 x] = 0.807

Z= 2807.

Problem-4 An ice cream company sells one of its types of ice cream by weight. If the product is not sold on the day it is prepared, it can be sold at a loss of 50 paise per kg. but there is an unlimited market for one day old ice-cream . on the other hand, the company makes a profit of Rs 3.20 on every kg of ice-cream sold on the day it is prepared. Past daily orders form a distribution with f(x) = 0.02 – 0.002x, $0 \le x \le 100$. How many Kg of ice-cream should the ccompany prepare every day.

Here C_1 = bear 50 paise for every kg unsold= Rs 0.50

C_2= profit per kg sold= Rs 3.20

f(x) = 0.02 – 0.002 x

directly using the result

$\int_0^z f(x)dx = \frac{C2}{C1+C2}$ -------- eqn-2

$\int_0^z (0.02 - 0.002x)dx = \frac{C2}{C1+C2}$ = 3.20 $\frac{3.20}{0.50+3.20}$---= 0.865----- eqn-2

[0.02x – 0.002x^2] limits 0and Z

Z= 63.2 kgs

Model-VI(c) : Reorder lead time prescribed

It is similar to VI(a), with one important exception, that the re-order lead time is to be taken into consideration . in other words, the time between placing an order and receiving the goods is significant.

n= no. of order cycle periods in the reorder lead time

Z_0 = the tock level at the end of the period preceding placing of order.

Let q_1 q_2 $q_3 q_4$, -, -, -, -, q_{n-1} quantities already ordered and due to be received on the 1st, 2nd, 3rd, ---- and (n-1) th days.

$F(x^1) = f(x_1 + x_2 + x_3 + \ldots + x_n)$ wherex^1 is the demand during lead time.

Our problem is to find the value of quantity q_n in order to minimize the total expected cost of the nth order cycle time. The opimumm value of Z^1 is that value which satisfies the equation.

$$\int_0^z f(x)dx = \frac{C2}{C1+C2}$$

After the optimum value of z^1 or Z^* is obtained , we can easily find the optimum value of q_n by using relationship.

$q_n^* = Z^* - [Z_0 + \sum_{i=1}^{n-1} qi]$

Problem: A shop owner places orders daiy for goods which will be delivered 7 days latr (ie the reorder lead time is 7 days). On certain day, the owner has 10 items in stock. Further more on the 6 previous days, he has already placed orders, for the delivery of 2, 4, 1, 10, 11 and 5 items in that order over each of the next 6 days. To facilitate computations, we shall assume C_1= Rs 0.15 and C_2= Rs 0.95 and the distribution requirement over a 7 day period (x^1) is $f(x^1) = 0.02 - 0.0002\, x^{1.}$

how many items should be ordered for the 7th day hence?

Soln: Substituting the values of C_1 and C_2 and $f(x^1)$ in the result.

$$\int_0^z f(x)dx = \frac{C2}{C1+C2}$$

$\int_0^z (0.02 - 0.0002x)dx = \frac{0.95}{0.15+0.95} = 0.8636$

[0.02x – 0.ooo2. x/2] = 0.8636

$Z^2 - 0.02\ Z + 0.8636 = 0$

$Z^2 - 200\ Z + 8600 = 0$

Solving this quadratic equation we approximately get

$Z^1 = 63$ or 136

Since $Z^1 = Z_0 + \sum_{i=1}^{6} qi + q_7$

we have 63= 10 + (2 + 4+ 1 + 10 +11+ 5) + q_7

q_7 = 63-10-33 = 20 items

the optimum order quantity is 20 items.

UNIFORM DEMAND, NO SETUP COST, MODEL:

Under this probabilistic model, the type of problem s similar to model VI. Except, that with drawals from stocks are continuous (rather than instantaneous) and the rate of with drawl is assumed to be virtually constant.

Model VII(a): continuous version

To find the optimum order level so as to minimize the total expected cost where

- t is the scheduling period which is a prescribed constant
- Z is the stock level to which the stock is raised at the end of every period t.
- f(x) is the probability density function for demand x, whch is known
- C_1 = is the carrying cost per quantity MODULE per MODULE time
- C_2 is the shortage cost per quantity MODULE per MODULE time
- Production is instantaneous
- Lead time is zero
- The demand rate in a period t is constant.

In this case , demand occurs uniformly rather than instantaneously during period t, as shown in figures

$\int_0^z f(x)dx + \int_z^\infty \frac{f(x)dx}{x} = \frac{C2}{C1+C2}$

Cost is minimum for the optimum value of Z given by above equation.

MODULE-V
WAITING LINES AND SIMULATION

In this we will discuss on following concepts

- **Queuing Theory Introduction**
- **MIM/1 Queuing Model**

Introduction

In everyday life, it is seen that a number of people arrive at a cinema ticket window. If the people arrive" too frequently" they will have to wait for getting their tickets or sometimes do without it. Under such circumstances, the only alternative is to form a queue, called the *waiting line,* in order to maintain a proper discipline. Occasionally, it also happens that the person issuing tickets will have to wait, *(i.e.* remains idle), until additional people arrive. Here the arriving people are called the customers and the person issuing the tickets is called a server.

Another example is represented by letters arriving at a typist's desk. Again, the letters represent the *customers* and the typist represents the *server.* A third example is illustrated by a machine breakdown situation. A broken machine represents a *customer* calling for the service of a repair man.These examples show that the term *customer* may be interpreted in various numbers of ways.It is also noticed that a service may be performed either by moving the *server* to the *customer* or the *customer* to the *server.*

Thus, it is concluded that waiting lines are not only the lines of human beings but also the aero planes seeking to land at busy airport, ships to be unloaded, machine parts to be assembled, cars waiting for traffic lights to turn green, customers waiting for attention in a shop or supermarket, calls arriving at a telephone switch-board, jobs waiting for processing by a computer, or anything else that require work done on and for it are also the examples of costly and critical delay situations .Further, it is also observed that arriving MODULEs may form one line and be serviced through only one station (doctor's clinic),may form one line and be served through several stations (as in a barber shop), may form several lines and be served through as many stations *{e.g.* at check out counters of supermarket).
Servers may be in parallel or in series. When in parallel, he arriving customers may form a single queue as shown in *Fig.* Or individual queues in front of each server as is common in big post-offices. Service times may be constant or variable and customers may be served singly or in batches (like passengers boarding a bus).

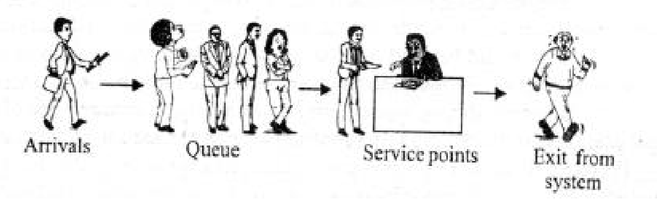


Fig (a). Queuing system with single queue and single service station..

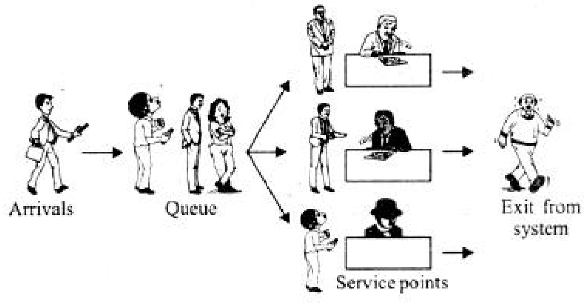


Fig. (b). Queuing system with single queue and several service stations.

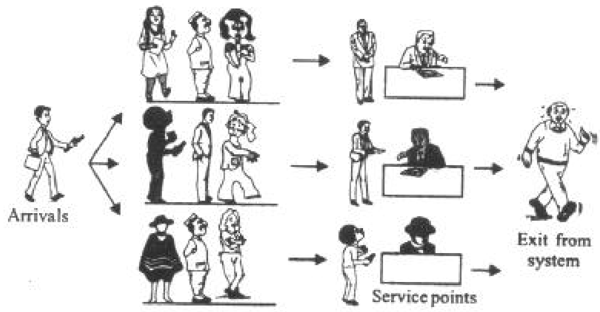


Fig. (c) Queuing system with several queues and several queues

Fig .illustrates how a machine shop may be thought of as a system of queues forming in front of a number of service centers, the arrows between the centers indicating possible routes for jobs processed in the shop. Arrivals at a service centre are either new jobs coming into the system or jobs, partially processed, from some other service centre. Departures from a service centre may be come the arrivals at another service centre or may leave the system entirely, when processing on these items is complete.

Queuing theory is concerned with the statistical description of the behavior of queues with finding, *e.g.,* the probability distribution of the number in the queue from which the mean and variance of queue length and the probability distribution of waiting time for a customer ,or the distribution of a server's busy periods can be found. In operational research problems

Involving queues, Investigators must measure the existing system to make an objective assessment of its characteristics and must determine how changes may be made to the system,what effects of various kinds of changes in the system's characteristics would be, and whether, in the light of the costs incurred in the systems, changes should be made to it. A model of the queuing system under study must be constructed in this kind of analysis and the results of queuing theory are required to obtain the characteristics of the model and to assess the effects of changes, such as the addition of an extra server or a reduction in mean service time.

Perhaps the most important general fact emerging from the theory is that the degree of congestion in a queuing system (measured by mean wait in the queue or mean queue length) is very much dependent on the amount of irregularity in the system. Thus congestion depends not just on mean rates at which customers arrive and are served and may be reduced without altering mean rates by regularizing arrivals or service times, or both where this can be achieved.

Meaning of a Queuing Model

A Queuing Model is a suitable model to represent a service- oriented problem where customers arrive randomly to receive some service, the service time being also a random variable.

Objective of a Queuing Model

The objective of a queuing model is to find out the optimum service rate and the number of servers so that the average cost of being in queuing system and the cost of service are minimized.

The queuing problem is identified by the presence of a group of customers who arrive randomly to receive some service. The customer up on arrival may be served immediately or if willing may have to wait until the server is free.

Application of a Queuing Model

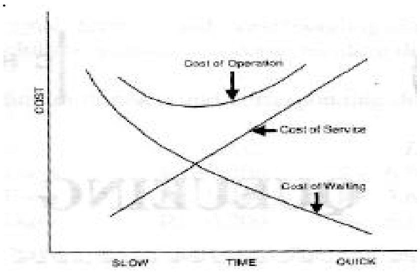
The queuing models are basically relevant to service oriented organizations and suggest ways and means to improve the efficiency of the service. This model can be applied in the field of business (banks, booking counters), industries (servicing of machines), government (railway or post-office counters), transportation (air port, harbor) and everyday life (elevators, restaurants, doctor's clinic).

Relationship between Service and Cost

Improvement of service level is always possible by increasing the number of employees. Apart from increasing the cost an immediate consequence of such a step unutilized or idle time of the servers. In addition, it is unrealistic to assume that a large-scale increase in staff is possible in an organization.

Queuing methodology indicates the optimal usage of existing manpower and other resources to improve the service. It can also indicate the cost implication if the existing service facility has to be improved by adding *more* servers.

The relationship between queuing and service rates can be diagrammatically illustrated using the cost curves as shown in following figure.



At a slow service rate, queues build up and the cost of queuing increases. An ideal service MODULE will minimize the operating cost of the entire system.

Arrival

The statistical pattern of the arrival can be indicated -through -

i. the probability distribution of the number of arrivals in a specific period of time, or

ii. the probability distribution of the time between two successive arrival *(known as inter arrival time)* number of arrivals is a discrete variable whereas the inter arrival times are continuous random and variable. A remarkable resulting this context is that if the number of arrivals follows a 'Poisson Distribution', the corresponding inter arrival time follows an 'Exponential Distribution'. This property is frequently used to derive elegant results on queuing problems.

Service

The time taken by a server to complete service is known as service time. The service time is a statistical variable and can be studied *either* as the number of services completed in a given period of time *or* the time taken to complete the service. The data on actual service time should be analyzed to find out the probability distribution of service time. The number of services completes discrete random variable while the service time is a continuous random variable.

Server

A server is a person or a mechanism through which service is offered. The service may be offered through a single server such as a ticket counter or through several channels such as a train arriving in a station with several platforms. Sometimes the service is to be carried out sequentially through several phases known as multiphase service. In government, the papers move through a number of phases in terms of official hierarchy till they arrive at the appropriate level where a decision can be taken.

Time Spent in the Queueing System

The time spent by a customer in a queuing system is the sum of waiting time before service and the service time. The waiting time of a customer is the time spent by a customer in a queuing system before the service starts. The probability distribution of waiting time depends upon the probability distribution of interarrival time and service time.

Queue Discipline

The queue discipline indicates the order in which members of the queue are selected *for* service. It is most frequently assumed that the customers are served on a first come first serve basis. This is commonly referred to as FIFO (first in, first out) system. Occasionally, a certain group of customers receive priority in service over others even if they arrive late. This is commonly referred to as priority queue. The queue discipline does not always take into account the order of arrival. The server chooses one of the customers to offer service at random. Such a system is known as service in random order (SIRO).
While allotting an item with high demand and limited supply such as a test match ticket or share of a public limited company, SIRO system is the only possible way of offering service when it is not possible to identify the order of arrival.

Kendall's Notation

Kendall's Notation is a system of notation according to which the various characteristics of a queuing model are identified. Kendall (Kendall, 1951) has introduced a set of notations, which have become standard in the literature of queuing models. A general queuing system is denoted by (a/b/c) :(d/e) where

a = probability distribution of the inter arrival time. b = probability distribution of the service time.

c = number of servers in the system.

d = maximum number of customers allowed in the system. e = queue discipline

In addition, the size of the population is important *for* certain types of queuing problem although not explicitly mentioned in the Kendall's notation. Traditionally, the exponential distribu- tion in queuing problems is denoted by M. Thus *(MIMI!):* (¥/ FIFO) indicates a queuing system when the inter arrival times and service times are exponentially distributed having one server in the system with first in first out discipline and the number of customers allowed in the system can be infinite.

State of Queueing System

The transient state of a queuing system is the state where the probability of the number of customers in the system depends upon time. The steady state of a queuing system is the state where the probability of the number of customers in the system is independent of*t*.

Let Pn (t) indicate the probability of having n customers in the system at time *t.* Then if Pn(t) depends upon *t,* the queuing system is said to be in the transient state. After the queuing system has become operative *for* a considerable period of time, the probability Pn(t) may become independent of t. The probabilities are then known as steady state probabilities

Poisson Process

When the number of arrivals in a time interval of length *t* follows a Poisson distribution with parameter *(At),* which is the product of the arrival rate (A-)and the length of the interval *t,* the arrivals are said to follow a poison process.

Relationship Between Poisson Process And Exponential

Probability Distribution

1. If the number of arrivals in a time interval of length *(t)*for How's a Poisson Process, then corresponding inter arrival time follows an 'Exponential Distribution'.
2. If the inter arrival times are independently, identically distributed random variables with an exponential probability distribution, then the number of arrivals in a time interval of length*(t)* follows a Poisson process with arrival rate identical with parameter of the Exponential Distribution.

MIM/1 Queuing Model

The M/M/1 queuing model is a queuing model where the arrivals

follows a Poisson process, service times are exponentially distributed and there is one server.

The assumption of M/M/1 queuing model are as follows:

1. The number of customers arriving in a time interval *t* follows a Poisson Process with parameter A.
2. The interval between any two successive arrivals is exponentially distributed with parameter A.
3. The time taken to complete a single service is exponentially distributed with parameter1.1
4. The number of server is one.
5. Although not explicitly stated both the population and the queue size can be infinity.
6. The order of service is assumed to be FIFO.
7.

If $\frac{\lambda}{\mu} < 1$, the steady state probabilities exist and P_n the number of customers in the system follows a geometric distribution with parameter $\frac{\lambda}{\mu}$ **(also known as traffic intensity)**.

The probabilities are :

$$P_n = P\,(\text{No. of customers in the sytem} = n)$$

$$= \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right); n = 1, 2, \ldots..$$

$$P_0 = 1 - \frac{\lambda}{\mu}$$

The time spent by a customer in the system taking into account both waiting and service time is an **exponential distribution** with parameter the probability distribution of the Waiting time be for service can also be derived in an identical manner. The expected number of customers in the system is given by–

Optimum Value of Service Rate

$$L_s = E(n) = \sum_{n=1}^{\infty} nP_n = \sum_{n=1}^{\infty} n\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

$$= \frac{\lambda}{\mu-\lambda} = \frac{\rho}{1-\rho}$$

The expected number of customers in the queue is given by –

$$L_q = \sum_{n=1}^{\infty}(n-1)P_n = \sum_{n=1}^{\infty} nP_n = \sum_{n=1}^{\infty} P_n$$

$$= \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\rho^2}{1-\rho}$$

Average waiting time of a customer in the system $W_s = \frac{1}{\mu-\lambda}$.

Average waiting time of a customer in the queue $W_q = \frac{\lambda}{\mu(\mu-\lambda)}$.

The study of queuing models helps us to find a cost model, which minimizes the sum of costs of service and of waiting time per MODULE time. Generally, it is assumed that the cost of waiting is directly proportional to the total time that the customers spend in the system, both waiting and in service. The service can usually be ascertained from the various available records.

Assuming a single server model with an arrival rate A and service rate 11and that the service rate 11is controllable, optimum value of service rate 11can be determined as follows:

C_1 = cost per unit increase in μ per unit time.

C_2 = cost per unit time per person of waiting.

$T(\mu)$ = The total cost of average waiting and service per unit time where

$T(\mu) = C_1\mu + C_2L_s$

In an M/M/1 : ∞/FIFO system it follows that –

$$T(\mu) = C_1\mu + \frac{C_2\lambda}{\mu-\lambda}$$

The optimum value of μ is obtained by minimising $T(\mu)$ with respect to μ by calculus methods. This value is given by

$$\mu = \lambda + \left(\frac{C_2\lambda}{C_1}\right)^{1/2}$$

Fundamental Process Component Elements of a Queuing

The fundamental components of a queuing process are listed below:

1. The input process or the arrivals
2. Service mechanism
3. Queue discipline

We now give a brief description of each of the above components:

1. The Input Process: Customers arrive at a service station for service. They do not come at regular intervals but arrivals into the system occur according to some chance mechanism. Often an arrival occurs at random and is independent of what has previously occurred. Customers may arrive for service individually or in groups. Single arrivals are illustrated by customers visiting a bank. On the other hand, families visiting a restaurant, is an example of bulk or group arrival. Arrivals may occur at a constant rate or may be in accordance with some probability distribution such as Poisson distribution or normal distribution etc. Frequently the population of the MODULEs or

customers requiring service may be infinite e.g. passengers waiting across the booking counters but there are situations where the population may be limited such as the number of particular equipment breaking down and requiring service from the factory's maintenance crew. Thus, in connection with the input process, the following information is usually called for and is considered relevant:

i. Arrival distribution
ii. inter-arrival distribution
iii. mean arrival rate;(or the average number of customers arriving in one MODULE of time) and
iv. mean time between arrivals.

2. Service Mechanism: Analogous to the input process, there are probability distribution of service times and number of customers served in an interval. Service time can either be fixed (as in the case of a vending machine) or distributed in accordance with some probability distribution. Service facilities can be anyone of the following types:

i. **Single channel facility or one queue-one service station facility** – This means that there is only one queue in which the customer waits till the service point is ready to take him for servicing. A library counter is an example of this.
ii. **One queue-several service station facilities:** In this case customers wait in a single queue until one of the service stations is ready to take them for servicing. Booking at a service station that has several mechanics each handling one vehicle, illustrates this type of model.
iii. **Several queues-one service stations** - In such a situation, there are several queues and the customer can join anyone of these but the service station is only one.
iv. **Multi-channel facility -** In this model, each of the servers has a different queue. Different cash counters in an Electricity Board Office where the customers can make payment in respect of their electricity bills provides an example of this model. Booking counters at railway station provide another example.
v. **Multi-stage multi-channel facilities -** In this case, customers require several types of service and different service stations are there. Each station provides a specialized service and the customer passes through each of the several stations before leaving the system.

For example, machining of a certain steel item may consist of cutting, turning, knurling, drilling, grinding and packaging etc., each of which is performed by a single server in a series.

In connection with the service mechanism the following information is often obtained from the point of view of the queuing theory:

I. Distribution of number of customers serviced
II. Distribution of time taken to service customers
III. Average number of customers being serviced in one MODULE of time at a service station.
IV. Average time taken to service a customer.

3. Queue Discipline: Queue discipline may refer to many things. One of such things is the order in which the service station selects the next customer from the waiting line to be served. In this context, queue discipline may be like first in first out or last in first out or may be on the basis of certain other criteria. For example, it may be random when a teacher picks up the students for recitation. Sometimes the customer may be given a priority basis for service on the basis of ladies first. Another aspect of queue discipline is whether a customer in a queue can move to a shorter queue in the multi-channel system. Queue discipline also refers to the manner in which the customers form into queue and the manner in which they behave in the queue. For

example, a customer may get impatient and leave the queue.

Conditions For Single Channel Queuing Model

The single channel queuing model can be fitted in situations where the following seven conditions are fulfilled:

1. The number of arrivals per MODULE of time is described by Poisson distribution. The mean arrival rate is denoted by μ.
2. The service time has exponential distribution. The average service rate is denoted by μ.
3. Arrivals are from infinite population.
4. The queue discipline is FIFO, that is, the customers are served on a first come first serve basis.
5. There is only a single service station.
6. The mean arrival rate is less than the mean service rate i.e.< 11. μ.
7. The waiting space available for customers in the queue is infinite.

Limitations of Single Channel Queuing Model

The single channel queuing model referred above is the simplest model, which is based on the above-mentioned assumptions. However, in reality, there are several limitations of this model in Its applications. One obvious limitation is the possibility that the wait That arrival rate is state dependent. That is, potential customers are discouraged from entering the queue if they observe along line at the time they arrive. Another practical limitation of the model is that the arrival process is not stationary. It is quite possible that the service station would experience peak periods and slack periods during which the arrival rate is higher and lower respectively than the overall average.Thesecouldoccuratparticulartimesduringadayoraweekorparticularweeksduringayear.Thereis*not*agreatdealonecandotoa ccount*for* stationary without complicating the mathematics enormously. The population *of* customers served may be finite, the queue discipline may not be first come first serve. In general, the validity *of* these models depends on stringent assumptions that are of ten unrealistic in practice.

Evenwhenthemodelassumptionsarerealistic,thereisanotherlimitation *of* queuing theory that is often over looked. Queuing models give steady state solution, that is, the models tell us what will happen after queuing system has been in operation long enough *to* eliminate the effects *of* starting with an empty queue at the beginning *of* each business day. In some applications, the queuing system never reaches a steady state, so the model solutions are *of* little value.

Model-1: Model (M/M/1): $(\infty/FCFS)$Single channel-poisson arrival with exponential service – infinite population Model.

Problem-1: An airport runway for arrivals only. Arriving aircraft join a single queue fr rthe runway. Exponentially distributed service time with a rate 27 arrivals/hour. arrival rate 20 arrivals / hour. Find

a) Average number of customers in the system.

b) Average no. of customers in the queue.

c) Average time a customer spends in the system.

d) Average time a customer spends in the queue.

$\mu\lambda$

λ = mean arrival rate = 20 arrivals /hour

μ= Mean service rate = 27 nos /hour

a)Average number of customers in the system.= $L_s = \frac{\lambda}{(\mu-\lambda)} = \frac{20}{(27-20)}$= 20/7=2.9 aircrafts

b)Average no. of customers in the queue. $L_q = \frac{\lambda * Ls}{\mu} = \left(\frac{\lambda}{\mu}\right) * \frac{\lambda}{(\mu-\lambda)} = \left(\frac{20}{7}\right) * \frac{20}{(27-20)}$= 2.1 aircrafts

c)Average time a customer spends in the system.=$W_s = \frac{1}{(\mu-\lambda)} = \frac{1}{(27-20)}$= 8.6 min

d)Average time a customer spends in the queue.= $W_q = \frac{\lambda}{\mu} * (W_s) = \frac{\lambda}{\mu} * \frac{1}{(\mu-\lambda)} = \frac{1}{(27-20)}$

$\frac{20}{27} * \frac{1}{(27-20)}$= 0.1058 hour= 6.3 minutes

Problem-2: suppose we are in holiday and the arrival rate increase 25 arrivals /hour. How will the quantities of queuing system change.

λ = mean arrival rate = 25 arrivals /hour

μ= Mean service rate = 27 nos /hour

a) Average number of customers in the system.= L_s= $\frac{\boldsymbol{\lambda}}{(\mu-\lambda)}$= $\frac{25}{(27-25)}$= 12.5 air crafts

b)Average no. of customers in the queue.L_q= $\frac{\boldsymbol{\lambda}* \ \mathbf{Ls}}{\mu}$=$\left(\frac{\lambda}{\mu}\right) * \frac{\boldsymbol{\lambda}}{(\mu-\lambda)} = \left(\frac{25}{27}\right) * \frac{25}{(27-25)}$= 11.6 aircrafts

c)Average time a customer spends in the system.=W_s= $\frac{\mathbf{1}}{(\mu-\lambda)}$= $\frac{\mathbf{1}}{(27-25)}$= 1/5 hour= 30 min

b) Average time a customer spends in the queue.= W_q= $\frac{\lambda}{\mu}$* (W_s) = $\frac{\lambda}{\mu}$*$\frac{\mathbf{1}}{(\mu-\lambda)}$=

$\frac{25}{27} * \frac{1}{(27-25)}$= 0.1058 hour= 2.8 minutes

Problem-3: Suppose we have bad weather and the service rate decreases $\mu = 22\ arrivals\ per\ hour.$ how will the quanties of queuing system changes.

μ= Mean service rate = 22 nos /hour

λ = mean arrival rate = 20 arrivals /hour

a) Average number of customers in the system.= L_s= $\frac{\boldsymbol{\lambda}}{(\mu-\lambda)}$= $\frac{20}{(22-20)}$= 10 air crafts

b)Average no. of customers in the queue.L_q= $\frac{\boldsymbol{\lambda}* \ \mathbf{Ls}}{\mu}$=$\left(\frac{\lambda}{\mu}\right) * \frac{\boldsymbol{\lambda}}{(\mu-\lambda)} = \left(\frac{20}{22}\right) * \frac{20}{(22-20)}$= 9.1 aircrafts

c)Average time a customer spends in the system.=W_s= $\frac{\mathbf{1}}{(\mu-\lambda)}$= $\frac{\mathbf{1}}{(22-20)}$= 1/2 hour= 30 min

b) Average time a customer spends in the queue.= W_q= $\frac{\lambda}{\mu}$* (W_s) = $\frac{\lambda}{\mu}$*$\frac{\mathbf{1}}{(\mu-\lambda)}$=

$\frac{20}{22} * \frac{1}{(22-20)}$= 10/22 hour= 27.27 minutes

e) Problem-4: A self-service store employees one cahier at its counter. None customers arrive at an average every 5 minutes while the cashier can serve 10 customers in 5 minutes. Assuming Poisson distribution for arrival rate and exponential distribution for service rate. Find.

a) Average number of customers in the system.

b) Average no. of customers in the queue.

c) Average time a customer spends in the system.

d) Average time a customer spends in the queue.

Arrival rate = **λ = 9/5 customers per minute= 1.8 customers/minute**

service rate = μ= 10/5 customers per minute= 2.0 customers/minute

a) Average number of customers in the system.= L_s= $\frac{\boldsymbol{\lambda}}{(\mu-\lambda)}$= $\frac{1.8}{(2.0-1.8)}$= 9 customers

b)Avg.no. of customers in the queue.L_q= $\frac{\boldsymbol{\lambda}* \ \mathbf{Ls}}{\mu}$=$\left(\frac{\lambda}{\mu}\right) * \frac{\boldsymbol{\lambda}}{(\mu-\lambda)} = \left(\frac{1.8}{2.0}\right) * \frac{1.8}{(2.0-1.80)}$= 8.1 customers

c)Average time a customer spends in the system.=W_s= $\frac{\mathbf{1}}{(\mu-\lambda)}$= $\frac{\mathbf{1}}{(2.0-1.80)}$= 5 minutes

b) Average time a customer spends in the queue.= W_q= $\frac{\lambda}{\mu}$* (W_s) = $\frac{\lambda}{\mu}$*$\frac{\mathbf{1}}{(\mu-\lambda)}$=

$\frac{1.80}{2.0} * \frac{1}{(2.0-1.8)}$= 4.5 minutes

Problem-5: At a certain petrol pump, customers arrive according to a poisson process with an average time of 5 minutes between arrivals. The service time is exponentially distributed with mean time of 2 minutes.on the basis of this information. Find out

a) What would be the average que length? L_q

b) what would be the average no of customers in the queueing system L_s

c) What is the average time spent by a car in the petrol pump? Ws

d) What is the average waiting time of a car before receiving petrol W_q

e) Solution: ths is an M/M/! queueing model .

Average inter arrival time = 5 minutes

λ = mean arrival rate = 1/5 MODULEs/minutes =(1/5)*60 = 12 MODULEs/hour

average service time = 2 minutes

Mean service rate = $\mu = \frac{1}{2} nos\ per\ minutes = \frac{1}{2} * 60 = 30 \frac{units}{hour}$

Since λ < μ

, the steady state solution exists.

$L_{q=}$ Average queue length = $\frac{\boldsymbol{\lambda} * \mathbf{Ls}}{\mu} = \left(\frac{\lambda}{\mu}\right) * \frac{\boldsymbol{\lambda}}{(\mu-\lambda)} = \left(\frac{12}{30}\right) * \frac{12}{(30-12)}$ = 4/5 customers

Ls = average no of customers in the queue system= = $\frac{\boldsymbol{\lambda}}{(\mu-\lambda)} = \frac{12}{(30-12)}$ = 2/3 customers

Average time spent in the petrol pump = Ws = = $\frac{\mathbf{1}}{(\mu-\lambda)} = \frac{\mathbf{1}}{(30-12)}$ = 1/18 hour= 3.33 minutes

c) Average waiting time of a c$\frac{\lambda}{\mu} * \frac{\mathbf{1}}{(\mu-\lambda)}$= ar before receiving petrol = $W_q = \frac{\lambda}{\mu} * (W_s) = \frac{12}{30} * \frac{\mathbf{1}}{(30-12)}$ = 1/45 hour = 1.33 minutes

Problem: 6 at a certain petrol pump , customers arrive according to a Poisson process with an average time of 55 minutes between arrivals. The service time is exponentially distribution with mean time of 2 minutes. Find

a) The average queue length (Lq)

b) The average no. of customers in the queueing system (L_s)

c) Average time spent a customer in a pertrol pump. (W_s)

d) Average time of a customer before receiving the service (Wq)

e) If the waiting time in queue is 4 min, a second pump will be considered. What should be the arrival rate for scond pump.

Solution: this is an (M/M/1) : $(\infty! FCFS)\ QUEUE\ MODEL$λμ

AVERAGE INTER ARRIVAL TIME = $\frac{1}{\lambda}$= 5 MIN

λ = 1/5 nos per minute

= 12 nos / hour

μ = mean service rate = ½ nos per minutes

= ½ * 60 = 30 nos per hour

As λ < μ , the steady state solution exists.

a) Average length of Queue = $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{12^2}{30(30-12)} = \frac{4}{15}$ nos

b) Average length of system $L_s = \frac{\lambda}{(\mu-\lambda)} = \frac{12}{(30-12)} = \frac{2}{3}$ nos

c) Average time spent in the petrol pump (system)= $W_{s=}$ $W_s = \frac{1}{(\mu-\lambda)} = \frac{1}{(30-12)} = \frac{1}{18} hour = 3.33 \min$

d) Average waiting time of customer in queue ie before receiving the service = $W_{q=}$

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{1}{30(30-12)} = \frac{1}{45} hour = 1.33 \min$$

Or

$$Wq = W_s - \frac{1}{\mu} = 3.33 - 2 = 1.33 \min$$

e) To open second pump Wq

$$W^1_q = 4\text{m} = \frac{\lambda^1}{\mu(\mu-\lambda^1)} = \frac{4}{15} hour$$

$$\frac{1}{15} hour = \frac{\lambda^1}{30(30-\lambda^1)}$$

$$\lambda^1 = 60/3 = 20 nos/hour$$

$$or\ interarrivaltime = 3\min$$

Model-II: Single channel Finite population model with Poisson arrivals and Exponential service times (M:M:1) : (N / FCFS)

In some cases , MODULEs arrive from a limited pool of potential customers. Once a MODULE joins the queue, there is one less MODULE which could arrive , and therefore the probability of an arrival is lowered. When a MODULE is served, it rejoins the pool of potential customers, and the probability of an arrival is there by increased. As a rule of thumb, if the population is less than 40, the equations for finite population should be used. Although the concepts are same as those for infinite populations, some of the terms are different and the equations required for analysis are different.

One distinct difference is that the probability of an arrival depends upon the number of potential customers available to enter the system.

Let the total potential customers population is M,

The no. of customers already in the queuing system = n

Then any arrival must come from M-n number , that is not yet in the system.

1. The probability of an empty system = $P_0 = \dfrac{1}{\sum_{n=0}^{n=M} \frac{M!}{(M-n)!} (\frac{l}{u})^n}$

2. The probability of n customers in the system $P_n = \dfrac{\frac{M!}{(M-n)!}(\frac{\lambda}{\mu})^n}{\sum_{n=0}^{n=M} \frac{M!}{(M-n)!}(\frac{\lambda}{\mu})^n} = P_0 \frac{M!}{(M-n)!}(\frac{\lambda}{\mu})^n$

3) L_s = Expected number of customers in the system. $= \sum_{n-0}^{n=M} n P_n =$

4) Expected Number of customers in the queue= $L_q = L_s - \frac{\lambda}{\mu} = M - \frac{\mu}{\lambda}(1-P_0) - \frac{\lambda}{\mu} =$

$L_q = L_s - \frac{\lambda}{\mu}$

$L_q = M - \frac{\lambda+\mu}{\lambda}(1-P_0)$

$W_s = \frac{L_s}{\lambda} =$

$W_q = W_s - \frac{1}{\mu} = \frac{L_s}{\lambda} - \frac{1}{\mu} = (\frac{L_q}{\lambda} + \frac{\lambda}{\mu}) * \frac{1}{\lambda} - \frac{1}{\mu} = \frac{L_q}{\lambda}$

Problem-5: Amechanic repairs four machines. The mean time between service requirements is 5 hours for each machine and forms an exponential m distribution. The mean repair time is 1 hour and also follows the same distribution pattern. Machine downtime costs Rs 25 per hour and mechanic costs Rs55 per day.

i. Find the expected no. of operating machines
ii. Determine the expected down time cost per day.
iii. Would it be economical to engage two mechanics, each repairing only two machines.

Solution: This situation involves finite population

When mechanic serving 4 machines M = 4

Arrival rate = **λ = 1/5 = 0.2**

Service rate $\mu = \frac{1}{1} = 1$

Let us find the probability of an empty system

$$P_0 = \frac{1}{\sum_{n=0}^{n=M} \frac{M!}{(M-n)!}(\frac{l}{u})^n} = \frac{1}{\sum_{n=0}^{n=4} \frac{4!}{(4-n)!}(\frac{0.2}{1})^n}$$

$$= \;= \frac{1}{1+\frac{4!}{3!}(0.2)^1+\frac{4!}{(4-2)!}(0.2)^2+\frac{4!}{(4-3)!}(0.2)^3+\frac{4!}{(4-4)!}(0.2)^4}$$

$$= \frac{1}{1+4*0.2+4*3(0.2)^2+4*3*2*(0.2)^3+(4*3*2*1)(0.2)^4} = 0.40$$

a) Expected number of broke down machines in the system $L_s = \langle \mid \rangle$

$$L_s = \;= 4-\frac{1}{0.2}(1-0.4) = 4-5*0.6 = 1$$

the expected number of operating machines in the system = 4-1 =3

b) Expected down time cost per day (assuming an 8 hours per day)

= 8 * Expected number of broken down m/cs * Rs 25 per hour= 8*1*25 = Rs 200 per day

c) When there are two mechanics each serving two machines ie M=2

$$P_0 = \frac{1}{\sum_{n=0}^{n=M} \frac{M!}{(M-n)!}(\frac{l}{u})^n} = \frac{1}{\sum_{n=0}^{n=2} \frac{2!}{(2-n)!}(\frac{0.2}{1})^n}$$

$$= \frac{1}{\frac{2!}{2!}+\frac{2!}{(1)!}(0.2)^1+\frac{2!}{}(0.2)^2} = \frac{1}{\frac{2!}{2!}+\frac{2!}{(1)!}(0.2)^1+\frac{2!}{}(0.2)^2}$$

$$= \frac{1}{1+2*0.2+2*0.04} = \frac{1}{1.48} = 0.68$$

It is assumed that each mechanic with his two machines constitute a separate system with no inter play.

$$M - \frac{\mu}{\lambda}(1-P_0)$$

a) Expected number of machines in the system $L_s = \rho = \frac{\lambda}{c.\mu} < 1$ $\quad = 2-\frac{1}{0.2}(1-0.68) = 0.4$

Therefore the expected down time / day = 8 * 0.4* no of mechanics = 8*0.4*2 = 6.4 hr/day

The total cost with two mechanics = Rs 2 * 55 + Rs 6.4 *25 = 110+160= 270 per day

Total cost with one mechanic = Rs 55 +Rs 200 = Rs 255 per day   (**minimum)**

Hence use one mechanic is economical

Uses of two mechanics are not economical.

MODEL-3: M/M/C System : ($(\infty/\ FIFO)$

Let   Arrival rate = **λ   customers per MODULE time**

service rate = μ= customers per MODULE minute

both population size , queue size can be (∞)

no. of servers = C   (C >1)

The steady state probabilities exists if

$$W_q = \frac{2\lambda^2}{\mu(\mu-\lambda)}$$

$$W_q = \frac{5}{10(10-5)} = \frac{1}{10} week$$

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{12^2}{30(30-12)} = \frac{4}{15}$$

$$L_s = \frac{\lambda}{(\mu-\lambda)} = \frac{12}{(30-12)} = \frac{2}{3}$$

$$W_s = \frac{1}{(\mu-\lambda)} = \frac{1}{(30-12)} = \frac{1}{18} hour = 3.33\min$$

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{1}{30(30-12)} = \frac{1}{45} hour = 1.33\min$$

$$Wq = W_s - \frac{1}{\mu} = 3.33 - 2 = 1.33\min$$

$$W^1{}_q = 4\text{m} = \frac{\lambda^1}{\mu(\mu-\lambda^1)} = \frac{4}{15} hour = \frac{1}{15} hour = \frac{\lambda^1}{\mu(\mu-\lambda^1)} = \frac{4}{15} = \frac{1}{30(30-12)} = \frac{1}{45} hour = 1.33\min$$

$$\frac{1}{15} hour = \frac{\lambda^1}{30(30-\lambda^1)}$$

$$\lambda^1 = 60/3 = 20 nos/hour$$

$$or\ \text{int}\ erarrivaltime = 3\min$$

(λ

< c μ)

$$P_0 = \frac{1}{\sum_{n=0}^{c-1} \frac{(\frac{\lambda}{\mu})^n}{n!} + \frac{(\frac{\lambda}{\mu})^c}{c!} * \frac{1}{(1-\rho)}}$$

$$L_q = \frac{\lambda\mu(\frac{\lambda}{\mu})^n}{(c-1)! * (c\mu - \lambda)^2} P_0 + \frac{\lambda}{\mu} - \frac{\lambda}{\mu}$$

$$L_q = \frac{5*10*(\frac{5}{10})^{2c}}{(2-1)! * (2*10-5)^2} * \frac{3}{5} = \frac{1}{30}$$

The probabilities are given by $L_q = P_0 \frac{(\frac{\lambda}{\mu})^c}{c!} * \frac{\rho}{(1-\rho)^2}$

$$\frac{1}{\mu}$$

$$\mathbf{P_n} = \begin{pmatrix} \frac{(\frac{\lambda}{\mu})^n}{n!} P_o & 0 \le n \le c \\ \frac{(\frac{\lambda}{\mu})^n}{c! c^{n-c}} P_0 & n \ge c \end{pmatrix}$$

The expected number of customers in queue $L_q = P_0 \frac{(\frac{\lambda}{\mu})^n}{c!} * \frac{\rho}{(1-\rho)^2}$

1. The expected no. of MODULEs in the system = $L_s = \frac{\lambda\mu(\lambda/\mu)^c}{(c-1)! * (c\mu - \lambda)^2} * P_0 + \frac{\lambda}{\mu}$ ----- eqn-1
2. The expected queue length $L_q = L_s$ - mean number being served = Ls - $\frac{\lambda}{\mu}$ = Ls - $\frac{\lambda}{\mu}$

$$= L_q = \frac{\lambda\mu(\frac{\lambda}{\mu})^n}{(c-1)!*(c\mu-\lambda)^2} P_0 + \frac{\lambda}{\mu} - \frac{\lambda}{\mu}$$ ---- eqn-2

$$L_q = \frac{\lambda\mu(\frac{\lambda}{\mu})^c}{(c-1)!*(c\mu-\lambda)^2} P_0$$

$$P_n = \frac{(\frac{\lambda}{\mu})^n}{n!(c-1)!*(c\mu-\lambda)^2} P_0$$

$$P_n = \frac{(\frac{\lambda}{\mu})^1}{} P_0$$

$$P = (\frac{\lambda}{\mu})^1 P_0 = (\frac{5}{10})^1 * \frac{3}{5} = \frac{3}{10}_0$$

3) Expected waiting time in the queue = W_q = $L_q * \frac{1}{\lambda} = \frac{\mu(\frac{\lambda}{\mu})^c}{(c-1)!*(c\mu-\lambda)^2} P_0$ ---- eqn-3

4) Expected waiting time in the system Ws = $W_q + \frac{1}{\mu} = \frac{\mu(\frac{\lambda}{\mu})^c}{(c-1)!*(c\mu-\lambda)^2} P_0 + \frac{1}{\mu}$

The expected waiting time in the queue = Wq = $\frac{1}{\lambda}$* Lq

the exp[ected waiting time in the system = Ws = Wq + $\frac{1}{\mu}$

the expected no. of customers in the system = Ls = λ W_s

$= \lambda$ (Wq + $\frac{1}{\mu}$)= $\lambda(\frac{Lq}{\lambda} + \frac{1}{\mu}$)=

$$= \mathbf{Lq} + \frac{\lambda}{\mu}$$

Problem: (M/M/C): (∞| $FIFo$) A petroleum company is considering expansion of its one unloading facility at its refiner. Due to random variations in weather, loading delays, ships arriving at the refinery to unload crude oil arrive at a rate of 5 ships per week. The service rate is 10 ships per week. Assume arrivals follows a poisson process and the service time is exponential.

a) Find the average time a ship must wait before beginning to deliver its cargo to the refinery. (Wq)

b) If second birth is rented , what will be the average no. of ships waiting before being unloaded.

c) What would be the average time a ship would wait before being unloaded with two births.

d) What is the average no. of idle births at any specified time.

Solution:

Mean arrival rate λ = 5 ships/ week

Mean service rate = μ = 10 ships per week

a) The expected waiting time (Wq) before ship begins to unload crude oil is Wq= $\frac{\lambda}{\mu(\mu-\lambda)}$

$$= W_q = \frac{5}{10(10-5)} = \frac{1}{10} week$$

b) With a second berth , the queueing model is an M/M/2 system

$$\rho = \frac{\lambda}{C\mu} = \frac{5}{2*10} = 1/4$$

$$P_0 = \frac{1}{\sum_{n=0}^{c-1} \frac{(\frac{\lambda}{\mu})^n}{n!} + \frac{(\frac{\lambda}{\mu})^c}{c!} * \frac{1}{(1-\rho)}}$$

$$= P_0 = \frac{1}{\frac{(5/10)^0}{0!} + \frac{(5/10)^1}{1!} + \frac{(5/10)^{12}}{2!} + \frac{(5/10)^2}{2!}(\frac{10*2}{10*2-5})} = \frac{3}{5}$$

$$\text{Lq} = L_q = \frac{\lambda\mu(\frac{\lambda}{\mu})^n}{(c-1)!*(c\mu-\lambda)^2} P_0 + \frac{\lambda}{\mu} - \frac{\lambda}{\mu}$$

$$= L_q = \frac{5*10*(\frac{5}{10})^{2c}}{(2-1)!*(2*10-5)^2} * \frac{3}{5} = \frac{1}{30}$$

c) $W_q = \frac{L_q}{\lambda} = \frac{1}{30} * \frac{1}{5} = \frac{1}{150} week$

d) If there is no ship in the system then boh the births are idle.

If there is only one ship in the system , one of the births is empty.

$$P_n = \frac{(\frac{\lambda}{\mu})^n}{n!} P_0 \qquad 0 < n < C$$

$$P_1 = P_n = \frac{(\frac{\lambda}{\mu})^1}{} P_0 = (\frac{5}{10})^1 * \frac{3}{5} = \frac{3}{10}$$

Hence the expected number of idle berths is = 2 * P0 + 1 * P1

= 2 * (3/5) + 1* (3/10) = 15/10 = 1.5

Problem-2: A super market has two girls ringing up sales at the counters. If the service time for each customers is exponential with mean 4 minutes, and if people arrive in a Poisson fashion at the rate of 10/hour

a) What is the probability of having to wait for service.
b) What is the expected percentage of idle time for each girl?
c) Find the average length and the average number of MODULEs in the system.

****

SIMULATION

Simulation is the representative model for real situations.

Definition:

- Simulation is a representation of reality through the use of a model or other device which will react in the same manner as reality under a given set of conditions.
- Simulation is the use of system model that has designed the characteristics of reality in order to produce the essence of actual operation

Example: the testing of an air craft model in a wind tunnel from which the performance of the real aircraft is determined for being under fit under real operating conditions. In the laboratories, we often perform a number of experiments on simulated models to predict the behavior of the real system under true environments.

According to Donald G. Malcolm, a simulated model may be defined as one which depicts the working of a large scale system of men, machines, materials and information operating over period of time in a simulated environment of the actual real world conditions.

TYPES OF SIMULATION

- Analog simulation
- Computer simulation (System Simulation)

i. Deterministic models
ii. Stochastic model
iii. Static models
iv. Dynamic models

NEED FOR SIMULATION

- Simulation techniques allow experimentation with a model of real-life system rather than the actual operating system.
- Sometimes there is no sufficient time to allow the actual system to operate extensively.
- The non-technical manager can comprehend simulation more easily than a complex mathematical model.
- The use of simulation enables a manager to provide insights into certain managerial problems.
-

SIMULATION PROCESS

Step-1: First define and identify the problem clearly.
Step-2.Secondly, list the decision variable and decide on rules of the problem.
Step-3. Formulate the suitable model for the given problem.
Step-4. Test the model and compare its behavior with the behavior of real problem situation.
Step-5. Collect and identify the data required to test the model.
Step-6. Execute (run) the simulation model.
Step-7' the results of simulation run are then analyzed. If the simulation run is complete, then choose the best course of action, otherwise , required changes are done in model decision variables, design or parameters and go to step-4.
Step-8. Run the simulation again to find the new solution.
Step-9. Validate the simulation.

LIMITATION OF SIMULATION

- Optimum results cannot
- The another difficulty lies in the quantification of the variables.
- In very large and complex problems, it becomes difficult to make the computer program on account of large number of variables and the involved inter-relationships among them.
- Simulation is comparatively costlier, time consuming method in many situations.

TYPES OF SIMULATION MODELS

- Deterministic models:
In this type of models, the input and output variables cannot be random variables and can be described by exact functional relationships.

- Probabilistic models
In these models, method of random sampling is used. This technique is called 'Monte-Carlo Technique.

- Static Models:
In these types of models, the variable time cannot be taken into account consideration.

- Dynamic Models:These models deal with time varying interaction.

PHASES OF SIMULATION MODEL

Phase-1: Data collection

Data generation involves the sample observation of variables and can be carried with the help of following methods.

- Using random number tables.
- Resorting to mechanical devices.)example: roulettes wheel)
- Using electronic computers

Phase -2. Book-keeping

Book-keeping phase of a simulation model deals with updating the system when new events occur, monitoring and recording the system states as and when they change, and keeping track of quantities of our interest (such as idle time and waiting time) to compute the measure of effectiveness.

GENERATION OF RANDOM NUMBERS

For clear understanding, the following parameters are be defined

- Random variable: it refers to a particular outcome of an experiment.
- Number::it refers to a uniform random variable or numerical value assigned to a random variable following uniform probability density function. (ie., normal, Poisson, exponential, etc).
- Pseudo-random Numbers: Random numbers are called pseudo-random numbers when they are generated by some deterministic process but they qualify the pre-determined statistical test for randomness.

MONTE-CARLO SIMULATION- STEPS INVOLVED

i. First define the problem by
   Identifying the objectives of the problem
   Identifying the main factors having the greatest effect on the objective of the problem

ii. Construct an appropriate model by
   Specifying the variables and parameters of the model
   Formulating the suitable decision rules.
   Identifying the distribution that will be used.
   Specifying the number in which time will change,

Defining the relationship between the variables and parameters.

iii. Prepare the model for experimentation
Defining the starting conditions for the simulation
Specifying the number of runs of simulation.
iv. Using step1 to step 3, test the model by
Defining a coding system that will correlate the factors defined in step1 with random numbers to be generated for simulation.
Selecting a random generator and creating the random numbers to be used in the simulation.
Associating the generated random numbers with the factors as identified in step1 and coded in step4.
v. Summarize and examine the results as obtained in.
vi. Evaluate the results of the simulation
vii. Formulate proposals for advice to management on the course of action to be adopted and modify the model, if required.

APPLICATION OF SIMULATION

- Simulation model can be applied for Solving Inventory problems
- Simulation model can be applied for solving Queuing problems

Simulation model can be applied for solving Inventory problems:

Problem-1: A bakery keeps stock of populr brand f cake. The previous experience shows the daily demand pattern for the item with associated probabilities , is as given below:

Daily demand	**0**	**10**	**20**	**30**	**40**	**50**
Probability	0.01	0.20	0.15	0.50	0.12	0.02

Use following sequence of random numbers to simulate the demand for next 10 days.

Random numbers: 25, 39, 65, 76, 12, 05, 73, 89, 19, 49 .

Estimate the daily average demand for the cakes on the basis of the simulated data.

Solution: Using the daily demand distribution, we first obtain a probability distribution as shown in

table-1.

Daily demand	**Probability**	**Cumulative probability**	**Randon number Interval**
0	0.01	0.01	00
10	0.20	0.21	01 - 20
20	0.15	0.36	21 - 35
30	0.50	0.86	36 - 85
40	0.12	0.98	86 - 97
50	0.02	1.00	98 - 99

Next to conduct the simulation experiment for demand take a sample of 10 random numbers from a given random numbers, which represent the sequence of 10 samples. Each random sample number represents a sample of demand . The simulation calculations for a period of 10 days are given in table-2.

Days	Random number	Simulated demand	
1	40	30	Since R,no 40 falls in the interval of 36-85
2	19	10	
3	87	40	
4	83	30	
5	73	30	
6	84	30	
7	29	20	
8	09	10	
9	02	10	
10	20	10	

Expected demand = total /10 =220/10= 22 MODULEs per day

Problem-2: A company manufacture around 200 mopeds , depending upon the availability of raw material and other conditions. , the daily production has been varying from 196 mopes to 204 mopeds, the probability distribution is as given below,

Production per day .	**196**	**197**	**198**	**199**	**200**	**201**	**202**	**203**	**204**
Probability	0.05	0.09	0.12	0.14	0.20	0.15	0.11	0.08	0.06

The finished mopeds are transported in a specially designed three-storied lorry that can accommodate 200 mopeds. Using the following 15 random numbers 82, 89, 78, 24, 53, 61, 18, 45, 23, 50,77, 27, 54. Simulate the mopeds waiting in the factory. what will be the average number of mopeds waiting in the factory ?. What will be the number of empty spaces waiting in the factory?. What will be the number of empty spaces in the lorry?.
Solution: Using production per day distribution, the daily production is shown in table.

Production / day	**Probability.**	**Cumilative probability**	**Random number interval**
196	0.05	0.05	00 – 04
197	0.09	0.14	05- 13
198	0.12	0.26	14-25
199	0.14	0.40	26-39
200.	0.20	060	40-59
201	0.15	0.75	60 -74
202	0.11	0.86	75-85
203	0.08	0.94	86-93
204	0.06	1.00	94-99

Based on the given 15 random numbers’, simulation experiments of production per day is shown in table

days	Random number	Production per day	No. of moped waiting	Empty space in the lorry
1	82	202	2	-
2	89	203	3	-
3	78	202	2	-
4	24	198	-	2
5	53	200	0	0
6	61	201	1	-
7	18	198.	-	2
8	45	200	00	-
9	04	196	-	4
10	23	198	-	2
11	50	200	0	-
12	77	202	2	-
13	8	199	1-	1
14	54	200	0	1
15	10	197	-	3
	total		10	14

Average number of mopeds waiting in the factory = 1/15(2+3+2+1+2)= 10/15 = 0.66 = 1 mopeds approx
Average number of empty spaces in the lorry = 14/15 =0.86

Problem-3: A book store wishes to carry a particular book in stock. The demand of the book is uncertain and there is a lead time of 2 days for stock replenishment. The probabilities of demand are given below:

Demand (MODULEs/day)	0	1	2	3	4
Probability	0.05	0.10	0.30	0.45	0.10

Each time an order is placed, the store incurs an ordering cost of Rs 10 per order. The store also incurs a carrying cost of Rs 0.50 per book per day. The inventory carrying cost is calculated on the basis of stock at the end of each day. The manager of the book store wishes to compare two options for his investment decision.

a) order 5 books when the present inventory plus any outstanding order falls below 6 books.
b) order 8 books when the present inventory plus any outstanding order falls below 6 books.

Currently (beginning of the 1^{st} day) the store has a stock of 8 books plus 6 books ordered two days of and are expected to arrive the next day.

Carry out simulation run for 10 days to recommend an appropriate option. You may uses random numbers in the sequences, using the first number for day one 89, 34, 78, 63, 61, 81, 39,16, 13,73.

Solution: using the daily demand distribution , we obtain probability distribution as shown in table-1.

Daily demand	**Probability**	**Cumilative probability**	**Random number interval**
0	**0.05**	**0.05**	**00 – 04**
1	**0.10**	**0.15**	**0 5- 14**

2	**0.30**	**0.45**	**15 - 44**
3	**0.45**	**0.90**	**45 - 89**
4	**0.10**	**1.00**	**90- 99**

Case-A: The stock in hand is of 8 books and stock on order is 5 books (Expected next day)

Random number	Daily demand	Opening stock	Receipt	Closing stock in hand	Order quantity	Closing stock
89	3	8	-	8-3 =5	5	5
34	2	5	5	5+5-2=8	-	8
78	3	8	-	8-3=5	5	5
63	3	5	5	5+5-3=7	-	7
61	3	7	-	7+0-3=4	5	4
81	3	4	5	4+5-3=6	-	6
39	2	6	-	6-2=4	5	4
16	2	4	5	4+5-2=7	-	7
13	1	7	-	7-2=6	-	6
73	3	6	-	6-3=3	-	3 TOT;=55

Since 5 books have been ordered four times as shown in table . therefore the totak ordering cost is = 4* Rs10= Rs 40/-
Closing stock of 10 days = 55 books. Therefore ., the holding cost at the rate of Rs 0.5 per book per day is = 55* 0.50 = Rs27.5
Total cost = Rs 40 + 27.50 = 67.50

Case-B: Order 8 books when the present inventory plus any outstanding order falls below 6 books.

The stock in hand is of 8 books and stock on order is 5 books (Expected next day)

Random number	Daily demand	Opening stock	Receipt	Closing stock in hand	Order quantity	Closing stock
89	3	8	-	8-3 =5	8	5
34	2	5	8	5+8-2=11	-	11
78	3	11	-	11-3=8	-	8
63	3	8	-	8-3=5	8	5
61	3	5	8	13-3=10	-	10
81	3	10	-	10-3=7	-	7
39	2	7	-	7-2=5	8	5
16	2	5	8	13-2=11	-	11
13	1	11	-	11-1=10	-	10
73	3	10	-	10-3=7	-	7 TOT;=79

books have been ordered three times, as shown in table, when the inventory of books at the beginning of the day plus outstanding order is less than eight.

Therefore total ordering cost is =no. of orders * ordering cost per order = 3 * 10 = Rs 30.00

The closing stock of 10 days is = 71 books

Therefore , the holding cost @ rs0.50 per book per day is = Rs 71*0.50 = 35.50

The total cost for 10 days = Rs 30.00 + 35.50 = Rs 65.50 this option -case-B is lower than case-A.

Select option case-B

Simulation model can be applied for solving Queuing problems

- Event list: to help determine what happens next:
  - Tracks the future times at which deferent types of events occur.
  - Events usually occur at random times.
- The randomness needed to imitate real life is made possible through the use of random (pseudo-random) numbers (more on this later).

STEPS INVOLVED IN SIMULATION

- The simulations are carried out by the following steps:
  - Determine the input characteristics.
  - Construct a simulation table.
  - For each repetition I, generate a value for each input, evaluate the function, and calculate the value of the response Y_I.
- Simulation examples will be given in queuing, inventory, reliability and net work analysis.
-

APPLICCATION OF SIMULATION FOR QUEUING MODELS

Simulation of Queuing Systems

- A queuing system is described by its calling population, nature of arrivals, service mechanism, system capacity and the queuing discipline.
- In a single-channel queue:
  - The calling population is infinite.
  - Arrivals for service occur one at a time in a random fashion. Once they join the waiting line they are eventually served.

- Arrivals and services are defined by the distribution of the time between arrivals and service times.

Simulation of Queuing Systems

- Assume the only possible service times are 1, 2, 3 and 4 time MODULEs and theyareequally likely to occur, with input generatedas:

Customer	Service Time	Customer	Service Time
1	2	4	2
2	1	5	1
3	3	6	4

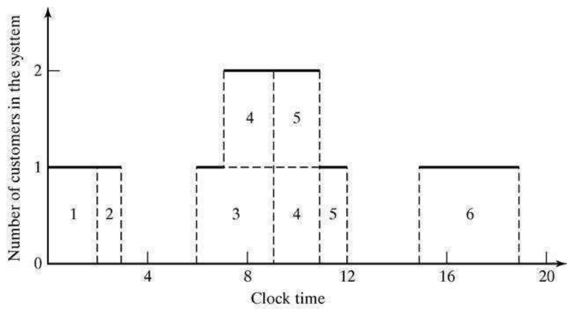
- Resulting simulation table emphasizing clock times:

Customer Number	Arrival Time (clock)	Time Service Begins (Clock)	Service Time (Duration)	Time Service Ends (clock)
1	0	0	2	2
2	2	2	1	3
3	6	6	3	9
4	7	9	2	11
5	9	11	1	12
6	15	15	4	19

Simulation of Queuing Systems

• Another presentation method, by chronological ordering ofevents:

Event Type	Customer Number	Clock Time
Arrival	1	0
Departure	1	2
Arrival	2	2
Departure	2	3
Arrival	3	6
Arrival	4	7
Departure	3	9
Arrival	5	9
Departure	4	11
Departure	5	12
Arrival	6	15
Departure	6	19



Simulation of Queuing Systems

- Grocery store example with only one checkoutcounter:

– Customers arrive at random times from 1 to 8 minutes apart, with equal probability of occurrence:

Time between Arrivals (minutes)	Probability	Cumulative Probability	Random Digit Assignment
1	0.125	0.125	001-125
2	0.125	0.250	126-250
3	0.125	0.375	251-375
4	0.125	0.500	376-500
5	0.125	0.625	501-625
6	0.125	0.750	626-750
7	0.125	0.875	751-875
8	0.125	1.000	876-000

- .
- The demand rate in a period t is constant.

In this case , demand occurs uniformly rather than instantaneously during period t, as shown in figures

Cost is minimum for the optimum value of Z given by above equation.
