SIGNALS AND SYSTEMS

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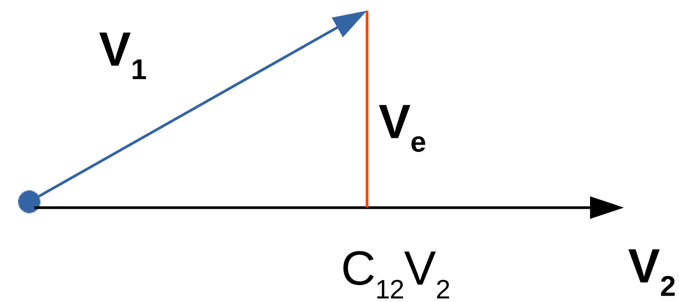
Module – I
Signal Analysis

Analogy between Vectors and Signals

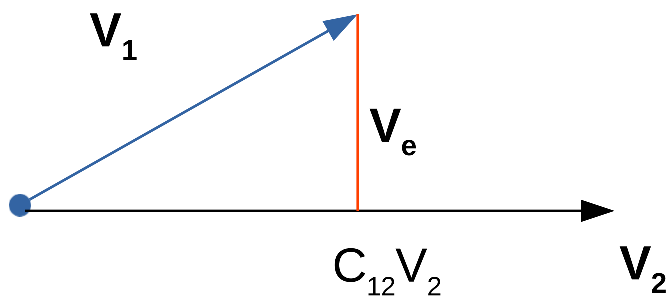
- There is a perfect analogy between vectors and signals which gives better understanding of signal analysis.
- A vector contains magnitude and direction.
- We shall denote all vectors by boldface type and their magnitudes by lightface type.
- For example, **A** is a certain vector with magnitude A.

Analogy between Vectors and Signals

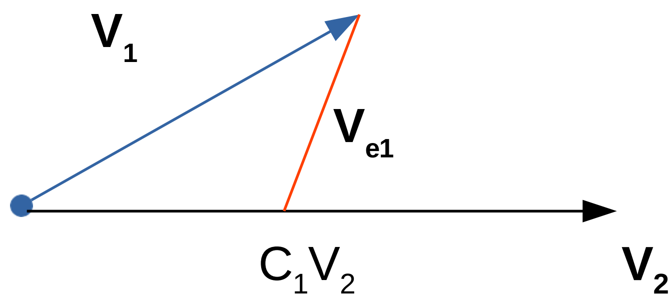
- Consider two vectors $\mathbf{V_1}$ and $\mathbf{V_2}$ as shown in Figure. Let
- the component of $\mathbf{V_1}$ along $\mathbf{V_2}$ be given by $C_{12}V_2$.
- Geometrically the component of a vector $\mathbf{V_1}$ along the vector $\mathbf{V_2}$ is obtained by drawing a perpendicular from the end of $\mathbf{V_1}$ on the vector $\mathbf{V_1}$.
- $\mathbf{V_1} = C_{12}\mathbf{V_2} + \mathbf{V_e}$



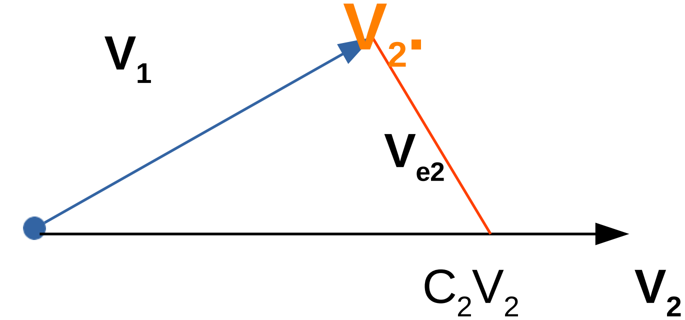
Analogy between Vectors and Signals



Minimum error of V_e is present when it is dropped perpendicular on V_2.



$$V_1 = C_1 V_2 + V_{e1}$$



$$V_1 = C_2 V_2 + V_{e2}$$

Analogy between Vectors and Signals

- If C_{12} is zero, then the vector has no component along the other vector and hence the two vectors are mutually perpendicular.
- Such vectors are known as orthogonal vectors.
- **Orthogonal vectors** are thus **independent** vectors.

Analogy between Vectors and Signals

- **A.B =** AB cosθ
- **A.B = B.A**
- Component **of A** along **B =** A**cos**θ= $\frac{A.B}{B}$
- Component **of B** along **A =** B **cos**θ= $\frac{A.}{\underline{B}}$
- Component **of V_1** along **V_2=** $\frac{V_1 . V_2 A}{V_2} = . \quad C_{12} V_2$

Analogy between Vectors and Signals

$$C_{12} = \frac{V_1 \cdot V_2}{V_2^2} = \frac{V_1 \cdot V_2}{V_2 \cdot V_2}$$

- If $\mathbf{V_1}$ and $\mathbf{V_2}$ are orthogonal then $\mathbf{V_1 . V_2 = 0}$ and $C_{12}=0$

Analogy between Vectors and Signals

- The concept of vector comparison and orthogonality can be extended to signals.
- Let us consider two signals, $f_1(t)$ and $f_2(t)$ and approximate $f_1(t)$ in terms of $f_2(t)$ over a certain interval $(t_1<t<t_2)$
- **$f_1(t) \sim= C_{12}\, f_2(t)$** for $(t_1<t<t_2)$
- If a error function is defined between actual and approximated function is minimum over the interval $(t_1<t<t_2)$
- **$f_e(t)=f_1(t) - C_{12}\, f_2(t)$**

Analogy between Vectors and Signals

- Possible criterion for minimizing the error $f_e(t)$ over the taken interval is to minimize the average value of $f_e(t)$ over this,to minimize

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_1(t) - C_{12} f_2(t)]\, dt$$

- This criterion is inadequate because there can be large positive and negative errors present that may cancel one another in this process of averaging and error becomes zero.

Analogy between Vectors and Signals

- This can be corrected if we choose square of the error instead of error itself.

$$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_e(t)]^2 dt$$

$$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_1(t) - C_{12} f_2(t)]^2 dt$$

Analogy between Vectors and Signals

- To find value of C_{12} which will minimize ε, we must have

$$\frac{d\varepsilon}{dC_{12}} = 0$$

That is

$$\frac{d}{dC_{12}}\left[\frac{1}{t_2 - t_1}\int_{t_1}^{t_2}[f_1(t) - C_{12} f_2(t)]^2\, dt\right] = 0$$

Analogy between Vectors and Signals

- Changing the order of integration and differentiation, we get

$$\frac{1}{t_2-t_1}\left[\int_{t_1}^{t_2} \frac{d}{dC_{12}} f_1^2(t)\,dt - 2\int_{t_1}^{t_2} f_1(t) f_2(t)\,dt + 2C_{12}\int_{t_1}^{t_2} f_2^2(t)\,dt\right] = 0$$

The first integral is obviously zero and hence

$$C_{12} = \frac{\int_{t_1}^{t_2} f_1(t) f_2(t)\,dt}{\int_{t_1}^{t_2} f_2^2(t)\,dt}$$

Analogy between Vectors and Signals

- By analogy with vectors, $f_1(t)$ has a component of waveform $f_2(t)$ and this component has a magnitude C_{12}.
- If **C_{12} disappears**, then the signal $f_1(t)$ contains no component of signal $f_2(t)$, so the **two functions are orthogonal** over the interval(t_1,t_2).
- **Condition for orthogonality**

$$\int_{t_1}^{t_2} f_1(t)\, f_2(t)\, dt = 0$$

Analogy between Vectors and Signals

- It can be shown that the functions sin $n\omega_0 t$ and sin $m\omega_0 t$ are orthogonal over any interval $(t_0, t_0 + 2\pi/\omega_0)$ for integral values of 'm' and 'n'.

- Consider Integral I:

$$I = \int_{t_0}^{t_0 + 2\pi/\omega_0} \sin n\omega_0 t \sin m\omega_0 t \, dt$$

$$I = \int_{t_0}^{t_0 + 2\pi/\omega_0} \frac{1}{2} [\cos (n-m)\omega_0 t - \cos (n+m)\omega_0 t] \, dt$$

- Since ‘n’ and ‘m’ are integers, (n-m) and (n+m) are also integers

- In that case the integral **I** is zero.

- Hence, the two functions are orthogonal.
- Similarly, it can be shown that **sin $n\omega_0 t$** and **cos $m\omega_0 t$** are orthogonal functions and **cos $n\omega_0 t$, cos $m\omega_0 t$** are also mutually orthogonal.

Analogy between Vectors and Signals

Graphical Evaluation of a Component of one Function in the other

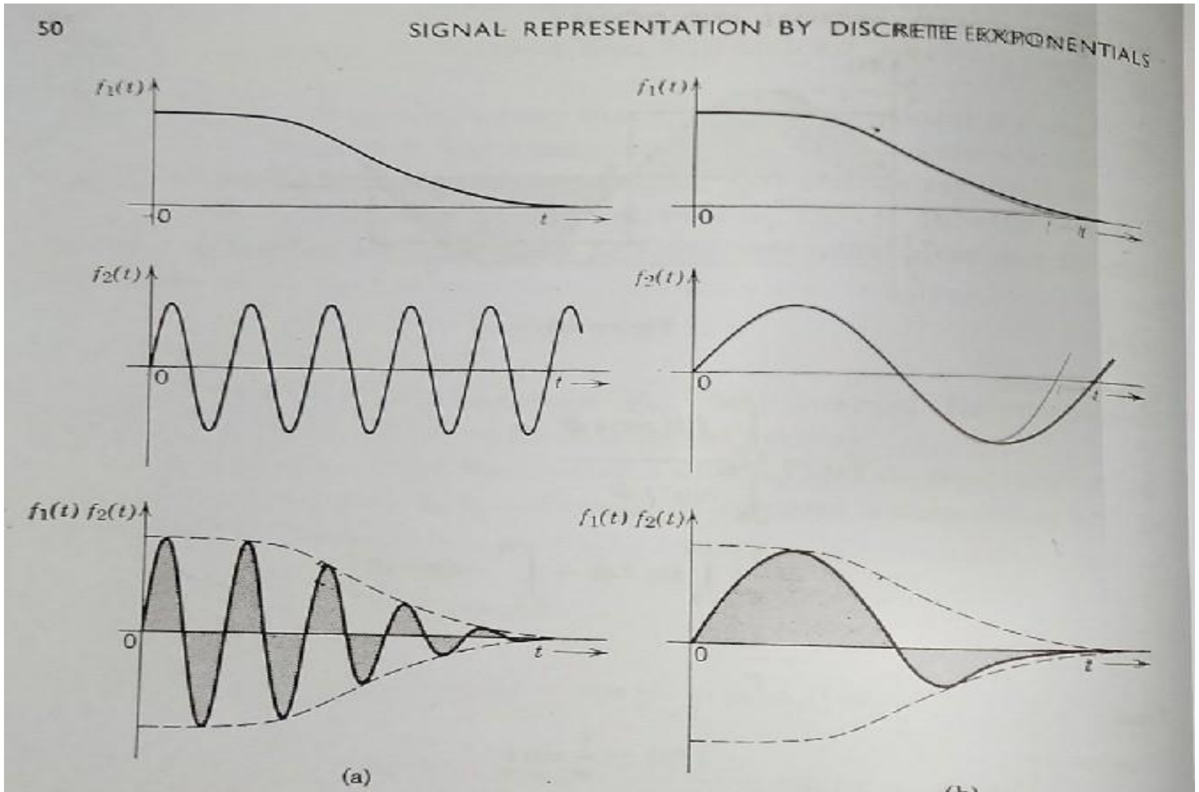
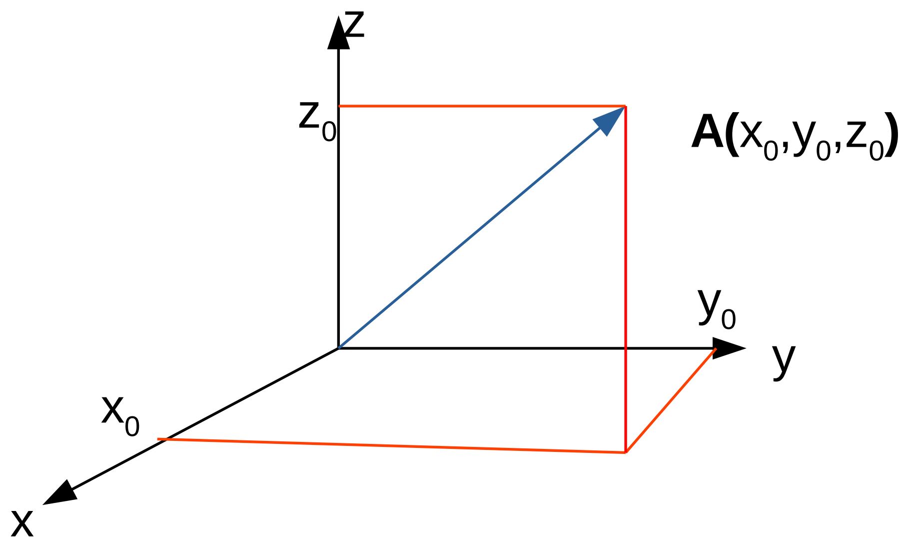


Figure 3.4 Graphical-evaluation of the component of waveform $f_2(t)$ in a signal $f_1(t)$

Orthogonal Vector Space

- Analogy can be extended further to 3-dimensional space.



Analogy between Vectors and Signals

Orthogonal Vector Space

- Component of **A** along the x axis = $\mathbf{A.a}_x$
- Component of **A** along the y axis = $\mathbf{A.a}_y$
- Component of **A** along the z axis = $\mathbf{A.a}_z$

$$\mathbf{A}= x_0a_x+y_0a_y+z_0a_z$$

$a_x.a_y=a_y.a_z=a_z.a_x=0$

$a_x.a_x=a_y.a_y=a_z.a_z=1$

Orthogonal Vector Space

$a_m . a_n = 0 \quad m \neq n$

$= 1 \quad m = n$

Considering n mutually perpendicular coordinates

$$\mathbf{A} = C_1 x_1 + C_2 x_2 + C_3 x_3 + \ldots\ldots + C_n x_n$$

$x_m . x_n = 0 \quad m \neq n$

$= 1 \quad m = n$

Orthogonal Vector Space

Component $C_r = \mathbf{A}.x_r$ For an orthogonal vector space,

$\mathbf{A}.x_r = C_r x_r . x_r = C_r k_r$

$$C_r = \frac{A . x_r}{k_r}$$

$$x_m . x_n = 0 \quad m \neq n$$
$$= k_m \quad m = n$$

Orthogonal Vector Space

If vector space is complete, any vector **F** can be expressed as

$$\mathbf{F} = C_1x_1+C_2x_2+C_3x_3+.....+C_rx_r+.....$$

$$C_r = \frac{F\,.\,x_r}{k_r} = .\,\frac{F\,.x_r}{x_r.\,x_r}$$

Orthogonal Signal Space

Let us consider a set of *n* functions $g_1(t), g_2(t), \ldots, g_n(t)$ which are Orthogonal to one another over an interval t_1 to t_2

$$\int_{t_1}^{t_2} g_j(t)\, g_k(t)\, dt = 0 \qquad j \neq k$$

And let

$$\int_{t_1}^{t_2} g_j^2(t)\, dt = K_j$$

Orthogonal Signal Space

Let an arbitrary function f(t) be approximated over an interval (t_1,t_2) by a linear combination of these n mutually orthogonal Functions.

$$f(t) \approx C_1 g_1(t) + C_2 g_2(t) + \ldots\ldots. + C_k g_k(t) + \ldots\ldots C_n g_n(t)$$

$$f(t) = \sum_{r=1}^{n} C_r g_r(t)$$

Orthogonal Signal Space

$$f_e(t) = f(t) - \sum_{r=1}^{n} C_r g_r(t)$$

$$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[f(t) - \sum_{r=1}^{n} C_r g_r(t) \right]^2 dt$$

$$\frac{\delta \varepsilon}{\delta C_1} = \frac{\delta \varepsilon}{\delta C_2} = \ldots = \frac{\delta \varepsilon}{\delta C_j} = \ldots\ldots = \frac{\delta \varepsilon}{\delta C_n} = 0$$

Orthogonal Signal Space

$$\frac{\delta \varepsilon}{\delta C_j} = 0$$

$$\frac{\delta}{\delta C_j}\left[\int_{t_1}^{t_2} [f(t) - \sum_{r=1}^{n} C_r g_r(t)]^2 dt\right] = 0$$

$$\frac{\delta}{\delta C_j}\int_{t_1}^{t_2} [f^2(t)]\,dt = \frac{\delta}{\delta C_j}\int_{t_1}^{t_2} [C_r^2(t)\, g_r^2(t)]\,dt = \frac{\delta}{\delta C_j}\int_{t_1}^{t_2} [C_r f(t)\, g_r(t)]\,dt = 0$$

Orthogonal Signal Space

This leaves only two non zero terms

$$\frac{\delta}{\delta \varepsilon} \int_{t_1}^{t_2} [-2\, C_j\, f(t)\, g_j(t) + C_j^2\, g_j^2(t)]\, dt = 0$$

Changing the order of integration and differentiation

$$2 \int_{t_1}^{t_2} f(t)\, g_j(t)\, dt = 2\, C_j \int_{t_1}^{t_2} g_j^2(t)\, dt$$

Orthogonal Signal Space

Therefore,

$$C_j = \frac{\int_{t_1}^{t_2} f(t)\, g_j(t)\, dt}{\int_{t_1}^{t_2} g_j^2(t)\, dt} = \frac{1}{K_j}\int_{t_1}^{t_2} f(t)\, g_j(t)\, dt$$

Orthogonal Signal Space

- Given a set of n functions $g_1(t), g_2(t), \ldots\ldots. g_n(t)$ mutually orthogonal over the interval (t_1, t_2), it is possible to approximate an arbitrary function f(t) over the interval by a linear combination of these n functions.

$$f(t) \approx C_1 g_1(t) + C_2 g_2(t) + \ldots\ldots + C_k g_k(t) + \ldots\ldots C_n g_n(t)$$

$$f(t) = \sum_{r=1}^{n} C_r g_r(t)$$

Orthogonal Signal Space

For best approximation we have to choose $C_1, C_2, \ldots\ldots C_n$ such that it will minimize **Mean of the square of the error** over the **interval**

Evaluation of Mean Square Error

- Let it be to consider to find the value of **'ε'** when optimum values of coefficients $C_1, C_2, \ldots, C_n$ are chosen as to give

$$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[f(t) - \sum_{r=1}^{n} C_r g_r(t) \right]^2 dt$$

$$\varepsilon = \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} f^2(t)\, dt + \sum_{r=1}^{n} C_r^2 \int_{t_1}^{t_2} g_r^2(t)\, dt - 2 \sum_{r=1}^{n} C_r \int_{t_1}^{t_2} f(t) g_r(t)\, dt \right]$$

Evaluation of Mean Square Error

- But from previous approximation,

$$\int_{t_1}^{t_2} f(t)\, g_r(t)\, dt = C_r \int_{t_1}^{t_2} g_r^2(t)\, dt = C_r K_r$$

- Substituting this in above equation

$$\varepsilon = \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} f^2(t)\, dt + \sum_{r=1}^{n} C_r^2 K_r - 2 \sum_{r=1}^{n} C_r^2 K_r\right]$$

Evaluation of Mean Square Error

So, the error **ε**

$$\varepsilon = \frac{1}{t_2 - t_1} [\int_{-t_1}^{t_2} f^2(t)\, dt \quad \sum_{r=1}^{n} C_r^2 K_r]$$

This implies mean square error can be evaluated by

$$\varepsilon = \frac{1}{t_2 - t_1} [\int_{t_1}^{t_2} f^2(t)\, dt - (C_1^2 K_1 + C_2^2 K_2 + ... + C_n^2 K_n)]$$

Representation of a Function by a Complete Set of Mutually Orthogonal Signals

- From above equation it is evident that if we increase n, if we approximate f(t) by a larger number of orthogonal functions, the error will be smaller.

- But by its very definition, **ε is a positive quantity**; i.e., in the limit as the number of terms is made infinity, the

sum $\sum_{r=1}^{n} C_r^2 K_r$ may converge to integral $\int_{t_1}^{t_2} f^2(t)\, dt$

Representation of a Function by a Complete Set of Mutually Orthogonal Signals

When integral and summation converge then **'ε'** vanishes.

$$\int_{t_1}^{t_2} f^2(t)\,dt = \sum_{r=1}^{n} C_r^2 K_r$$

Under these conditions f(t) is represented by the infinite series:

$$f(t) = C_1 g_1(t) + C_2 g_2(t) + \ldots.\ C_r g_r(t) + \ldots\ .$$

Representation of a Function by a Complete Set of Mutually Orthogonal Signals

- The infinite series on the right-hand side of above equation converges to f(t) such that the mean square of the error is zero.
- The series is said to converge in the mean.
- Note that f(t) is now exact.
- And should there be no other x(t) having orthogonality with any $g_r(t)$.

Representation of a Function by a Complete Set of Mutually Orthogonal Signals

- Let us now summarize the results. For a set $\{g_r(t)\}$, (r=1,2,....) mutually orthogonal over the interval (t_1, t_2),

$$\int_{t_1}^{t_2} g_m(t)\, g_n(t)\, dt \quad \begin{cases} = 0 & \text{if } m \neq n \\ = K_m & \text{if } m = n \end{cases}$$

Representation of a Function by a Complete Set of Mutually Orthogonal Signals

- If this function set is complete, then any function f(t), can be expressed as

$$f(t) = C_1 g_1(t) + C_2 g_2(t) + \dots C_r g_r(t) + \dots$$

where

$$C_r = \frac{\int_{t_1}^{t_2} f(t)\, g_r(t)\, dt}{K_r} = \frac{\int_{t_1}^{t_2} f(t)\, g_r(t)\, dt}{\int_{t_1}^{t_2} g_r^2(t)\, dt}$$

Representation of a Function by a Complete Set of Mutually Orthogonal Signals

- This draws an analogy between vectors and signals.
  - Any vector can be expressed as a sum of its components along ***'n' mutually orthogonal vectors*,** provided these vectors form a complete set.
  - Similarly, any function f(t) can be expressed as a sum of its components along mutually orthogonal functions, provided these functions form a closed or complete set.

Analogy between Vectors and Signals

$$A \cdot B \sim \int_{t_1}^{t_2} f_A(t)\, f_B(t)\, dt$$

$$A \cdot A = A^2 \sim \int_{t_1}^{t_2} f_A^2(t)\, dt$$

If a vector is expressed in terms of its mutually orthogonal Components, the square of the length is given by the sum of the squares of the lengths of the component vectors.

- **Representation of f(t) by a set of infinite mutually orthogonal functions is called generalized Fourier Series Representation of f(t)**.

Orthogonality in Complex Functions

- Let us consider two signals, $f_1(t)$ and $f_2(t)$ as complex functions of real variable t, over a certain interval $(t_1<t<t_2)$

$f_1(t) \sim= C_{12} f_2(t)$ for $(t_1<t<t_2)$

$$C_{12} = \frac{\int_{t_1}^{t_2} f_1(t) f_2^*(t)\, dt}{\int_{t_1}^{t_2} f_2(t) f_2^*(t)\, dt}$$

Orthogonality in Complex Functions

Condition for orthogonality

$$\int_{t_1}^{t_2} f_1(t) f_2^*(t)\, dt = \int_{t_1}^{t_2} f_1^*(t)\, f_2(t) dt = 0$$

Orthogonality in Complex Functions

For a set of complete functions $\{g_r(t)\}$, $(r=1,2,\ldots)$ mutually orthogonal over the interval (t_1,t_2):

$$\int_{t_1}^{t_2} g_m(t)\, g_n^*(t)\,dt = 0 \qquad m \neq n$$

$$\int_{t_1}^{t_2} g_m(t)\, g_n^*(t)\,dt = K_m \qquad m = n$$

Orthogonality in Complex Functions

If this set of functions is complete, then any function *f(t)* can be expressed as

$f(t) \approx C_1 g_1(t) + C_2 g_2(t) + \ldots\ldots + C_r g_r(t) + \ldots..$

$$C_r = \frac{1}{K_r} \int_{t_1}^{t_2} f(t)\, g_r^*(t)\, dt$$

Orthogonality in Complex Functions

- If this set of functions is real, then $g_r{*}(t)=g(t)$ and all the results

  for complex functions reduce to those obtained for real functions as shown the analysis of real functions.

Summary

i) With two functions

$$C_{12} = \frac{V_1 . V_2}{V_2^2} = \frac{V_1 . V_2}{V_2 . V_2}$$

$V_1.V_2$**=0** and $C_{12}=0$

$$C_{12} = \frac{\int_{t_1}^{t_2} f_1(t) f_2(t)\, dt}{\int_{t_1}^{t_2} f_2^2(t)\, dt}$$

$$\int_{t_1}^{t_2} f_1(t) f_2(t)\, dt = 0$$

Summary

ii) With n dimensional functions

A = $C_1x_1+C_2x_2+C_3x_3+.....+C_nx_n$

$$C_r = \frac{A.x_r}{k_r}$$

$f(t) \approx C_1g_1(t)+C_2g_2(t)+.....C_ng_n(t)$

$$f(t) = \sum_{r=1}^{n} C_r g_r(t)$$

$$C_j = \frac{\int_{t_1}^{t_2} f(t)g_j(t)\,dt}{\int_{t_1}^{t_2} g_j^2(t)\,dt} = \frac{1}{K_j}\int_{t_1}^{t_2} f(t)g_j(t)\,dt$$

Summary

iii) For a complete set of mutually orthogonal functions

F = $C_1x_1+C_2x_2+C_3x_3+.....+C_rx_r+.....$

$$C_r = \frac{F.x_r}{k_r} = \frac{F.x_r}{x_r.x_r}$$

$$f(t) = C_1 g_1(t) + C_2 g_2(t) + ... C_r g_r(t) + ...$$

$$C_r = \frac{\int_{t_1}^{t_2} f(t)\, g_r(t)\, dt}{K_r} = \frac{\int_{t_1}^{t_2} f(t)\, g_r(t)\, dt}{\int_{t_1}^{t_2} g_r^2(t)\, dt}$$

Summary

iv) For Complex functions

$$f(t) \approx C_1 g_1(t) + C_2 g_2(t) + \ldots\ldots + C_r g_r(t) + \ldots\ldots$$

$$C_r = \frac{1}{K_r} \int_{t_1}^{t_2} f(t)\, g_r^*(t)\, dt$$

$$C_{12} = \frac{\int_{t_1}^{t_2} f_1(t)\, f_2^*(t)\, dt}{\int_{t_1}^{t_2} f_2(t)\, f_2^*(t)\, dt}$$

Signals

- A **signal** is a function representing a **physical quantity or variable**, and typically it contains information about the behavior or nature
  of the phenomenon.
- Signals are represented by real- or complex-valued functions of one or more independent variables.

- They may be one-dimensional, that is, functions of only one independent variable, or multidimensional.

Classification of Signals

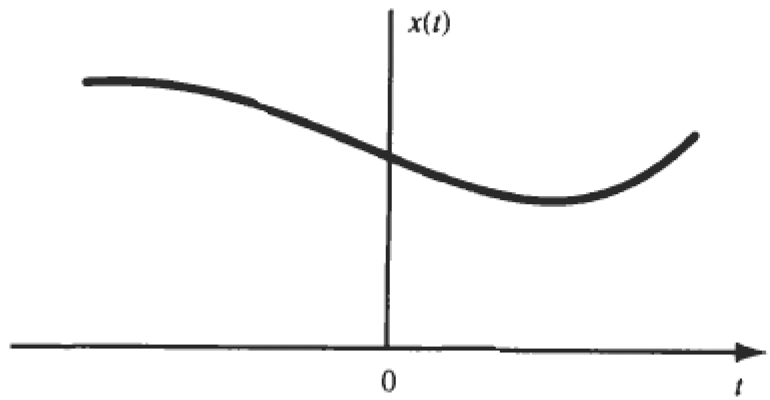
Signals can be classified into:

1. Continuous-time and Discrete-time signals
2. Analog and Digital Signals
3. Real and Complex Signals
4. Deterministic and Random Signals
5. Even and Odd signals
6. Periodic and Non-periodic signals
7. Energy and Power signals

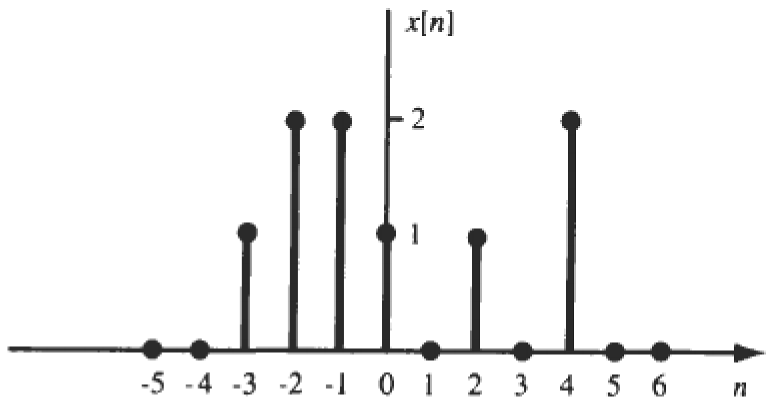
Continuous-time and Discrete-time signals

- A signal **x(t)** is a **continuous-time** signal if **t** is a continuous variable.
- If t is a discrete variable-that is, x(t) is defined at discrete times- then x(t) is a discrete-time signal.
- Since a **discrete-time** signal is defined at discrete times, a discrete-time signal is often identified as a sequence of numbers, denoted by **$\{x_n\}$ or x[n]**, where n = integer.

Continuous-time and Discrete-time signals



Continuous Time Signal



Discrete Time Signal

Continuous-time and Discrete-time signals

$$x[n] = x_n = \begin{cases} \left(\frac{1}{2}\right)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$\{x_n\} = \left\{1, \frac{1}{2}, \frac{1}{4}, \ldots, \left(\frac{1}{2}\right)^n, \ldots\right\}$$

$$\{x_n\} = \{\ldots, 0, 0, 1, 2, 2, \underset{\uparrow}{1}, 0, 1, 0, 2, 0, 0, \ldots\}$$

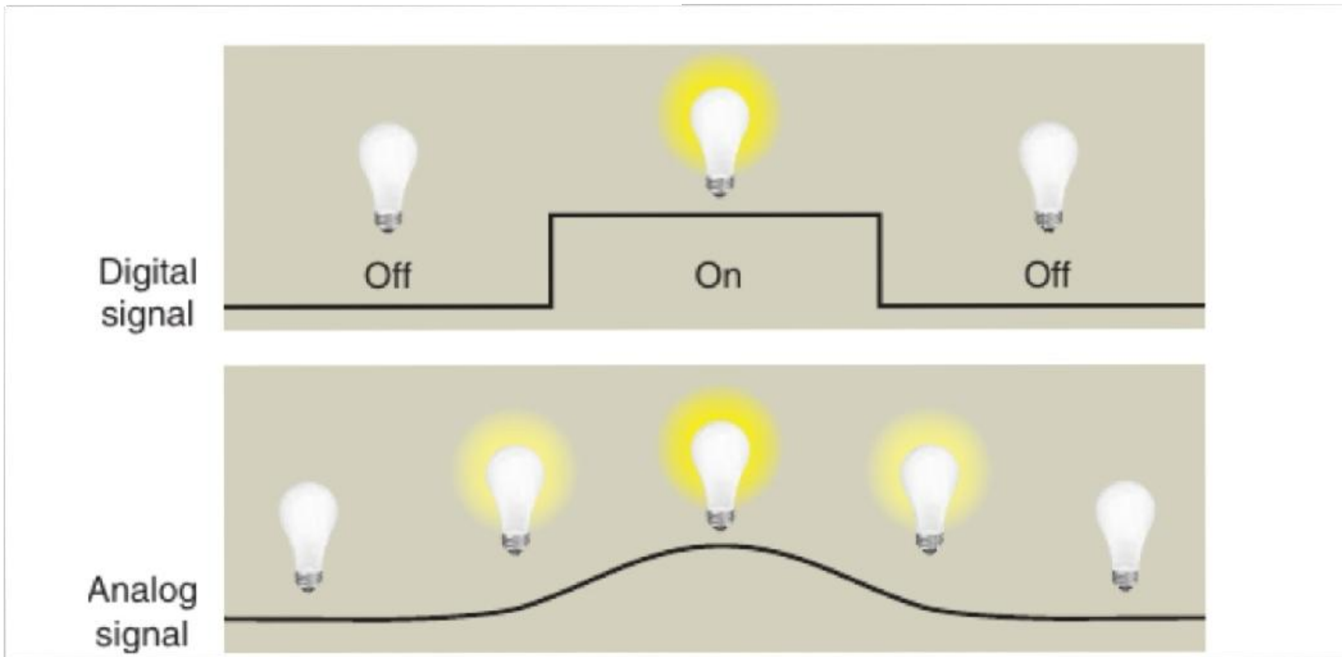
$$\{x_n\} = \{1, 2, 2, \underset{\uparrow}{1}, 0, 1, 0, 2\}$$

Representation of discrete signals

Analog and Digital Signals

- If a continuous-time signal x(t) can take on any value in the continuous interval (a, b), where a may be -∞ and b may be +∞ , then the continuous-time signal x(t) is called an analog signal.
- If a discrete-time signal x[n] can take on only a finite number of distinct values, then we call this signal a digital signal.

Analog and Digital Signals



Real and Complex Signals

A signal x(t) is a real signal if its value is a real number, and a signal x(t) is a complex signal if its value is a complex number.

A general complex signal x(t) is a function of the form

$$x(t) = x_1(t) + jx_2(t)$$

Deterministic and Random Signals

- Deterministic signals are those signals whose values are completely specified for any given time. Thus, a deterministic signal can be modeled by a known function of time t.

- Random signals are those signals that take random values at any given time and must be characterized statistically.

Even and Odd Signals

$$x(-t) = x(t)$$

$$x[-n] = x[n]$$

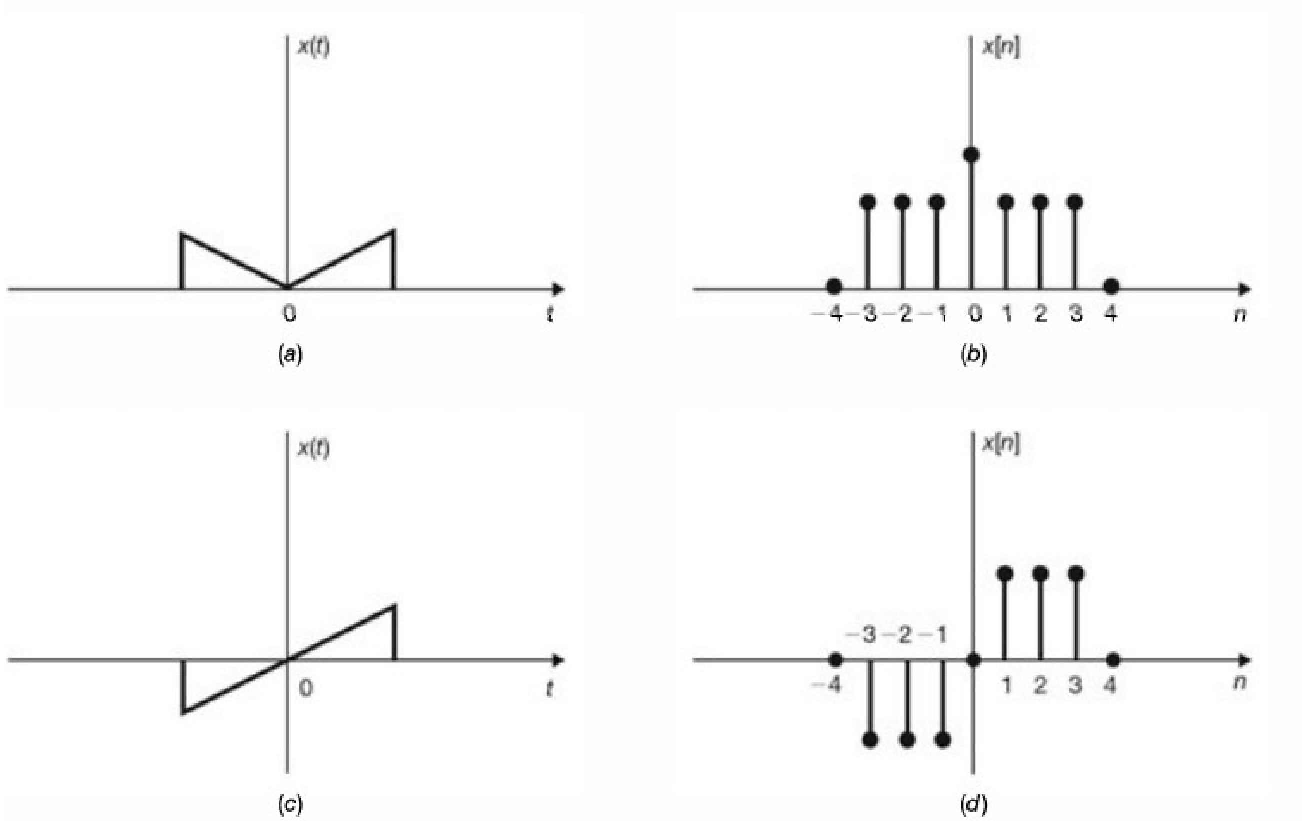
Even Signal

$$x(-t) = -x(t)$$

$$x[-n] = -x[n]$$

Odd Signal

Even and Odd Signals



Even and Odd Signals

- Any signal can be split into even and odd parts
- $x(t) = x_e(t) + x_o(t)$

$$x[n] = x_e[n] + x_o[n]$$

Even and Odd Signals

- $x_e(t) = 1/2 \{x(t) + x(-t)\}$ even part of $x(t)$
- $x_e[n] = 1/2 \{x[n] + x[-n]\}$ even part of $x[n]$
- $x_o(t) = 1/2 \{x(t) - x(-t)\}$ odd part of $x(t)$
- $x_o[n] = 1/2 \{x[n] - x[-n]\}$ odd part of $x[n]$

Periodic and Non-Periodic Signals

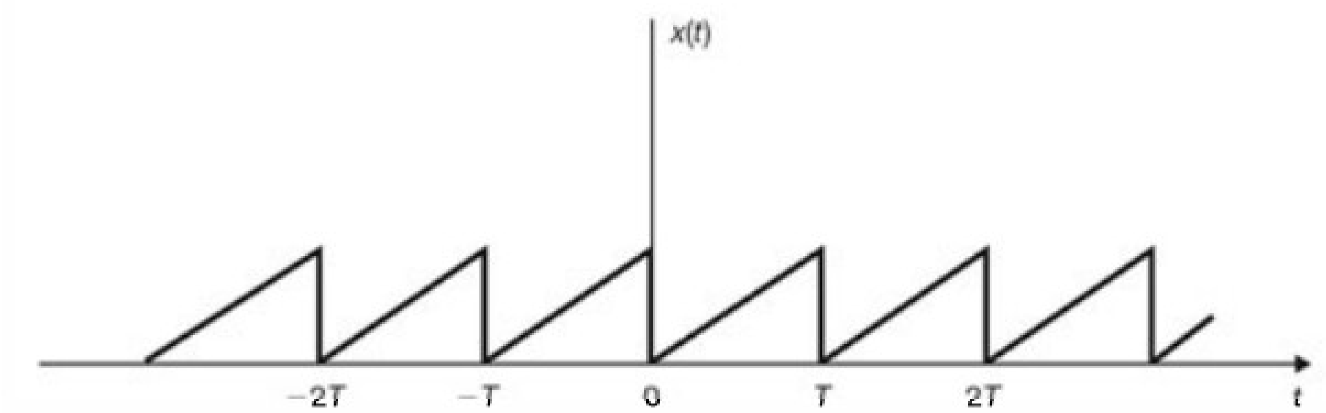
- A continuous-time signal x(t) is said to be periodic with period T if there is a positive nonzero value of T for which

  x(t + T) = x(t)   all t

  x(t + mT) = x(t) for m an integer

- The fundamental period T_0 of x(t) is the smallest positive value of
- T.
- This definition does not work for a constant signal x(t) (known as a dc signal).
  a constant signal x(t) the fundamental period is undefined since x(t) is periodic for any choice of T.

Periodic and Non-Periodic Signals



Continuous Periodic Signal

- Any continuous-time signal which is not periodic is called a
nonperiodic signal.

Periodic and Non-Periodic Signals

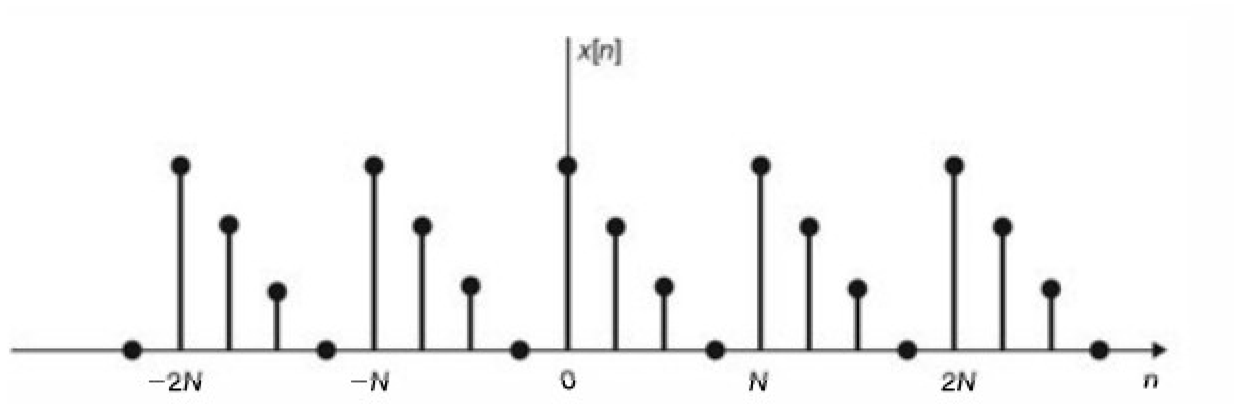
For a discrete-time signal,

$$x[n + N] = x[n] \text{ all } n$$

$$x[n + mN] = x[n] \text{ for } m \text{ an integer}$$

The fundamental period N_0 of x[n] is the smallest positive integer N.

Periodic and Non-Periodic Signals



Periodic Sequence

Periodic and Non-Periodic Signals

- Note that a sequence obtained by uniform sampling of a periodic continuous-time signal may not be periodic.
- Note also that the sum of two continuous-time periodic signals may not be periodic but that the sum of two periodic sequences is always periodic.

Energy and Power Signals

Consider v(t) to be the voltage across a resistor R producing a current i(t). The instantaneous power p(t) per ohm is defined as

$$p(t) = \frac{v(t)i(t)}{R} = i^2(t)$$

Total energy is

$$E = \int_{-\infty}^{\infty} i^2(t)\, dt$$

Average power is

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} i^2(t)dt$$

Energy and Power Signals

For an arbitrary continuous-time signal x(t), the normalized energy content E of x(t) is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Normalized Average power is

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

Energy and Power Signals

- Similarly, for a discrete-time signal x[n], the normalized energy content E of x[n] is defined as

$$E=\sum_{n=-\infty}^{n=\infty}|x[n]|^2$$

- The normalized average power P of x[n] is defined as

$$P=\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{n=N}|x[n]|^2$$

Energy and Power Signals

- Similarly, for a discrete-time signal x[n], the normalized energy content E of x[n] is defined as

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- The normalized average power P of x[n] is defined as

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Energy and Power Signals

- A signal with finite energy has zero power. ***(ENERGY SIGNAL)***
- A signal with finite power has infinite energy. ***(POWER SIGNAL)***
- A signal cannot both be an energy signal and a power signal.
- There are signals, that are neither energy nor power signals.
- A periodic signal is a power signal if its energy content per period is finite, and then the average power of this signal need only be calculated over a period. **Not all periodic signals are power signals.**

Operations on Signals

- Sometime a given mathematical function may completely describe a signal .
- Different operations are required for different purposes of arbitrary signals.
- The operations on signals can be Time Shifting

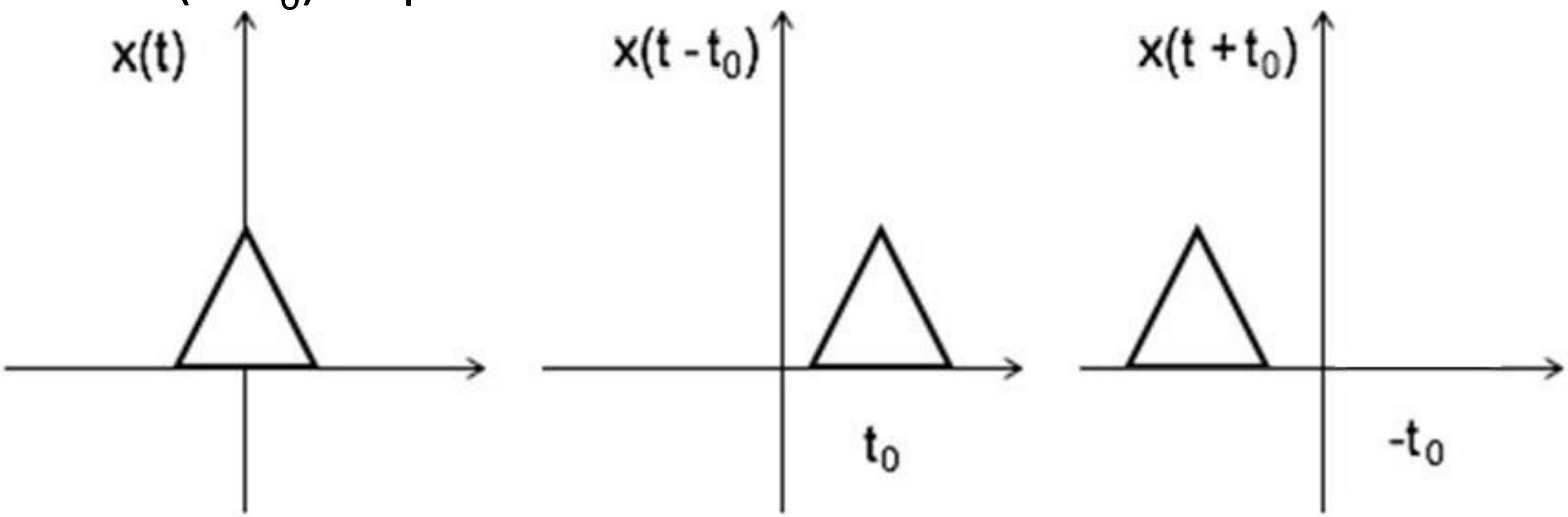
Time Scaling

Time Inversion or Time Folding

Time Shifting

$x(t \pm t_0)$ is time shifted version of the signal $x(t)$. $x(t + t_0) \rightarrow$ negative shift

$x(t - t_0) \rightarrow$ positive shift

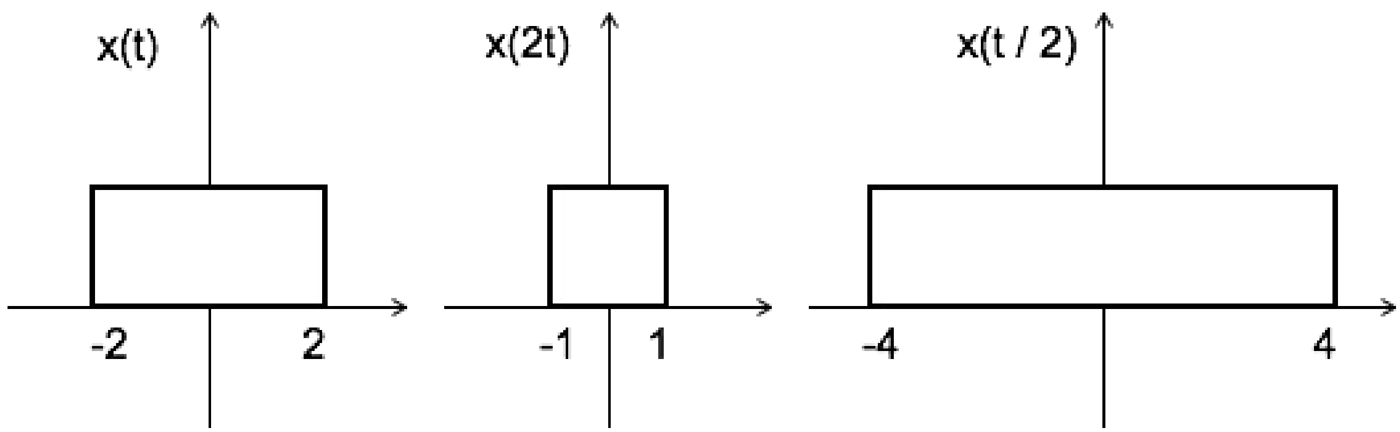


Time Scaling

x(At) is time scaled version of the signal x(t). where A is always positive.

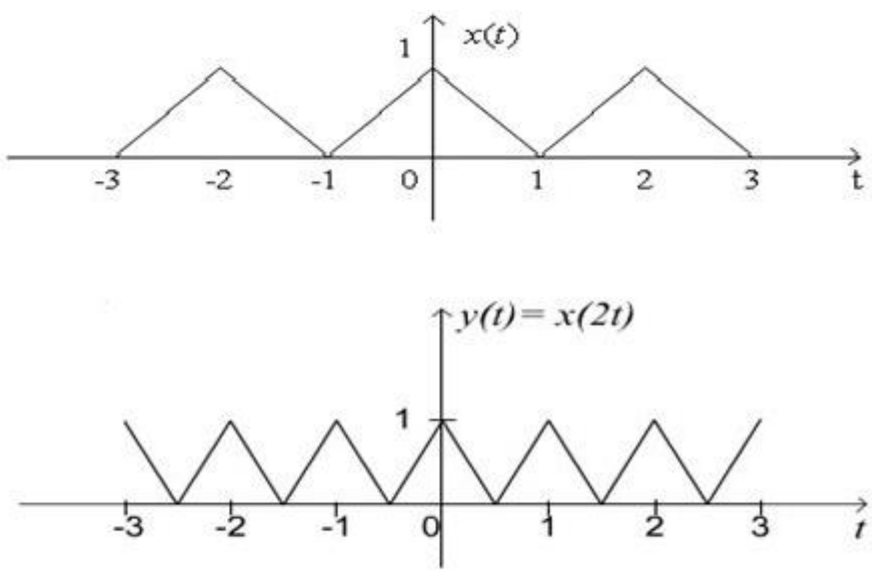
|A| > 1 → Compression of the signal

|A| < 1 → Expansion



Time Scaling

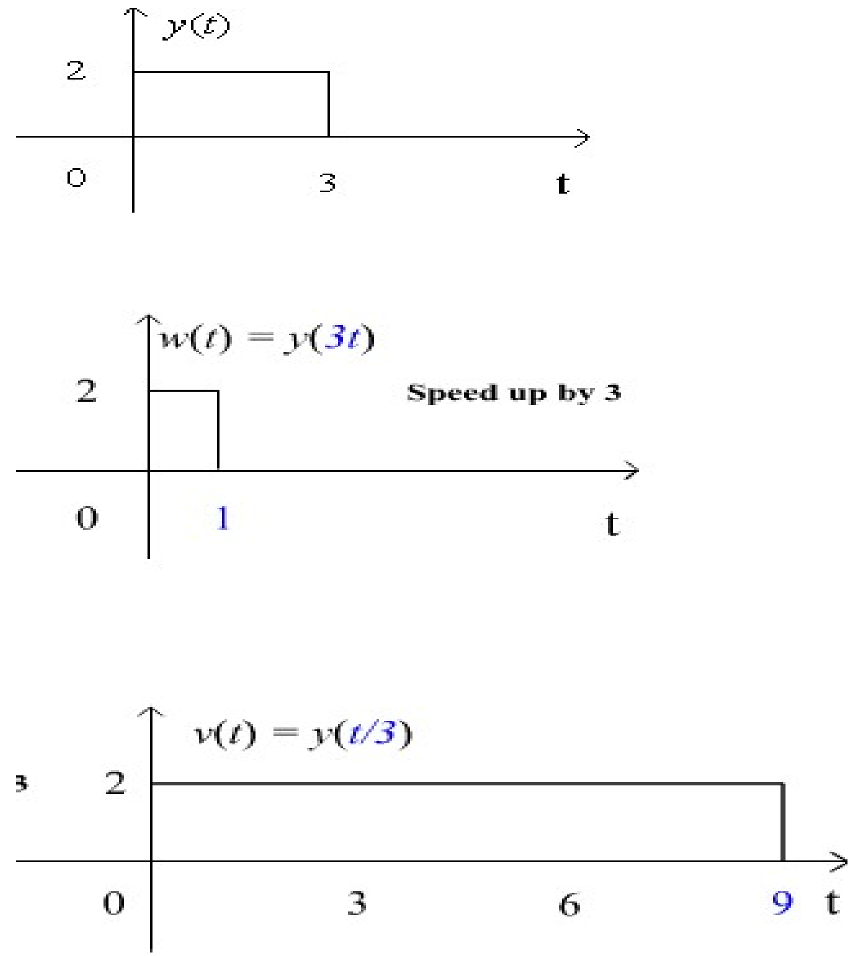
Example: Given $x(t)$ and we are to find $y(t) = x(2t)$



The period of $x(t)$ is 2 and the period of $y(t)$ is 1,

Time Scaling

•Given *y*(*t*),
find *w*(*t*) = *y*(3*t*)
and v(*t*) = *y*(*t*/3

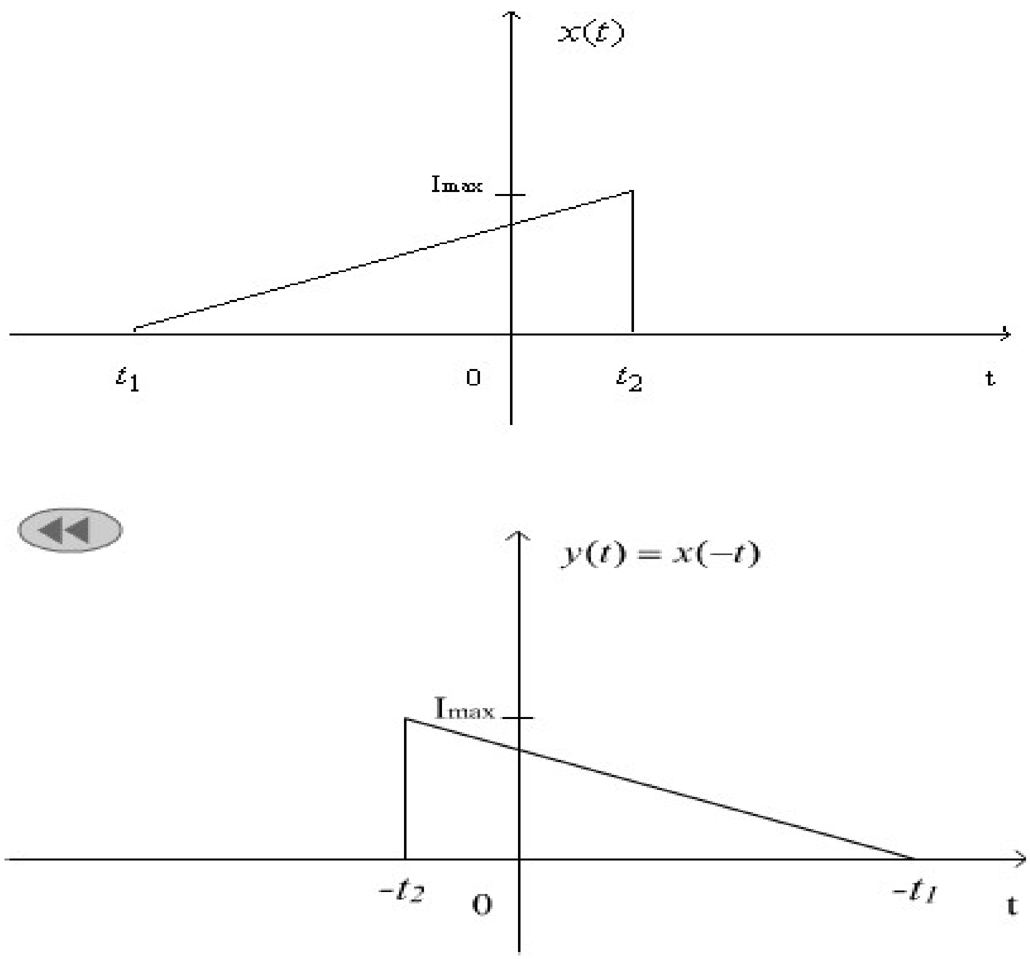


Time Reversal (Or) Time Folding

- Time reversal is also called time folding
- In Time reversal signal is reversed with respect to time i.e.

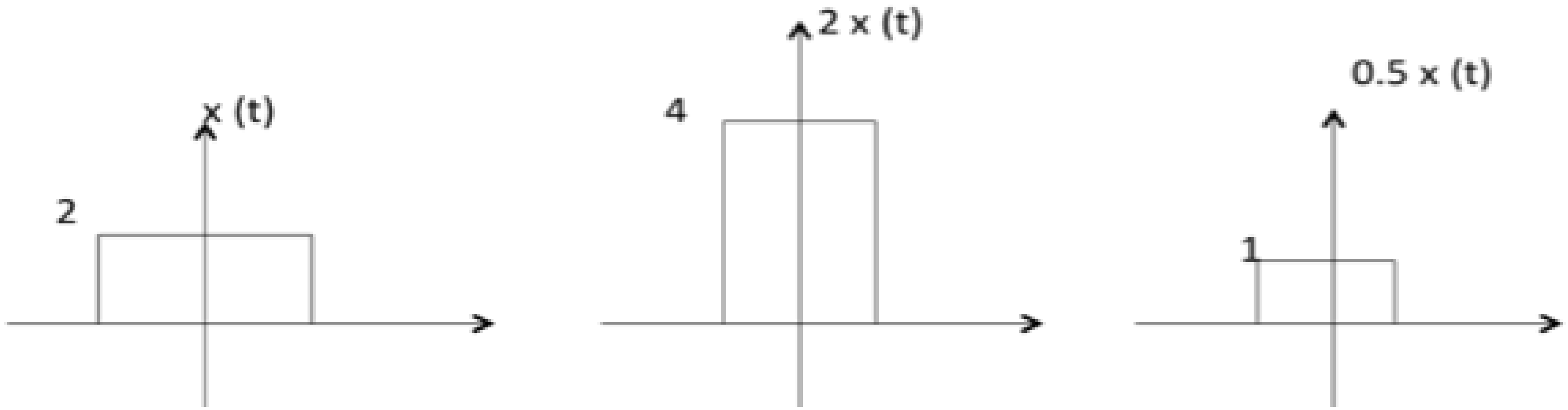
$y(t) = x(-t)$ is obtained for the givenfunction

Time Reversal (Or) Time

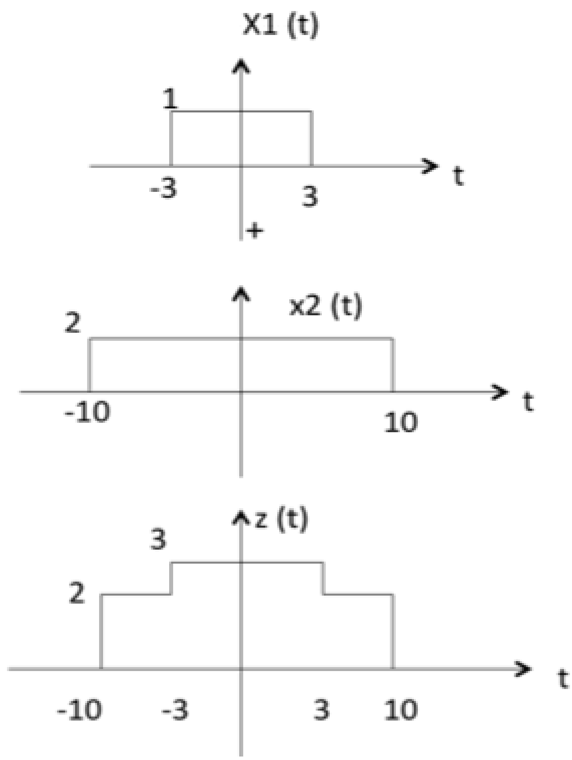


Amplitude Scaling

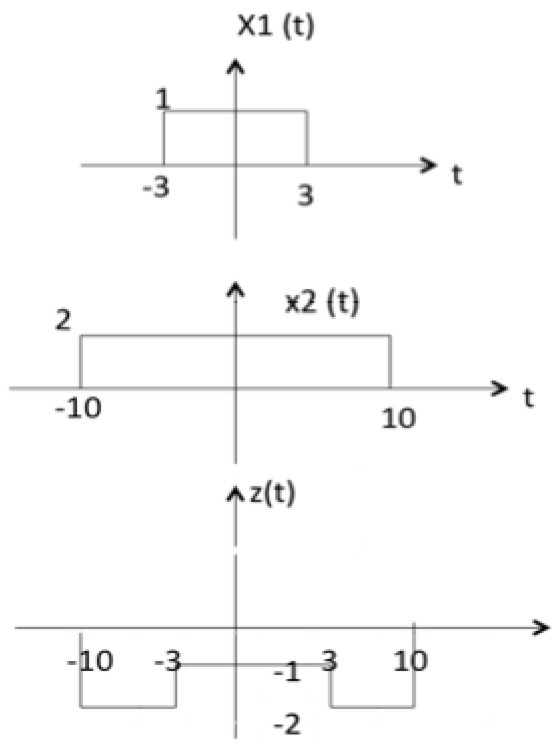
C x(t) is a amplitude scaled version of x(t) whose amplitude is scaled by a factor C.



Addition

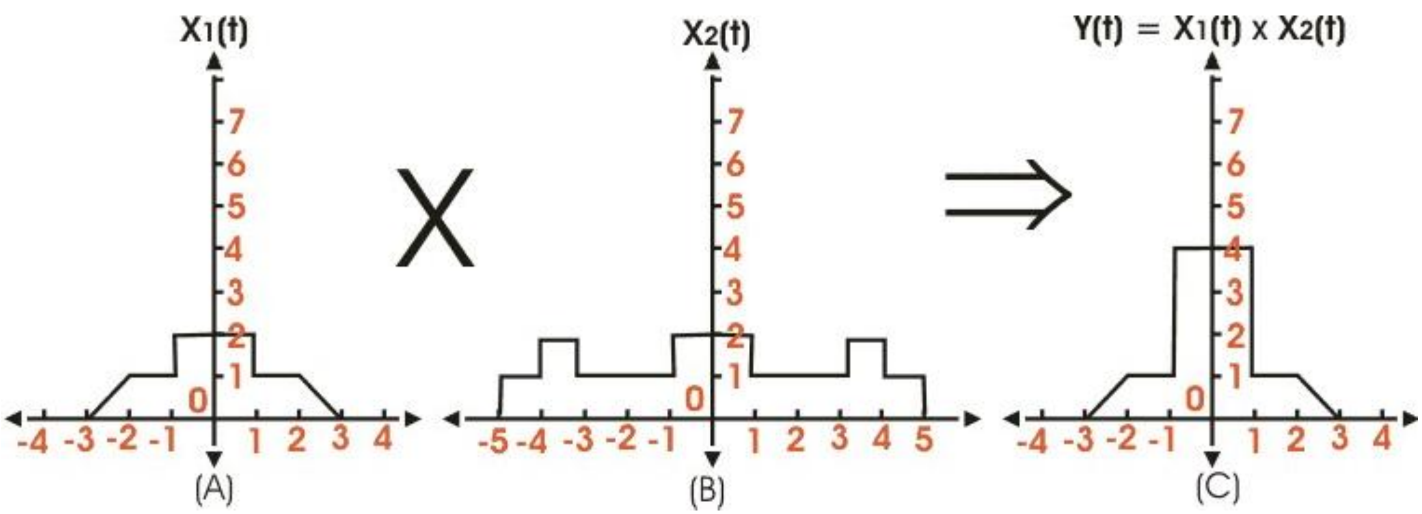


Subraction



Multiplication

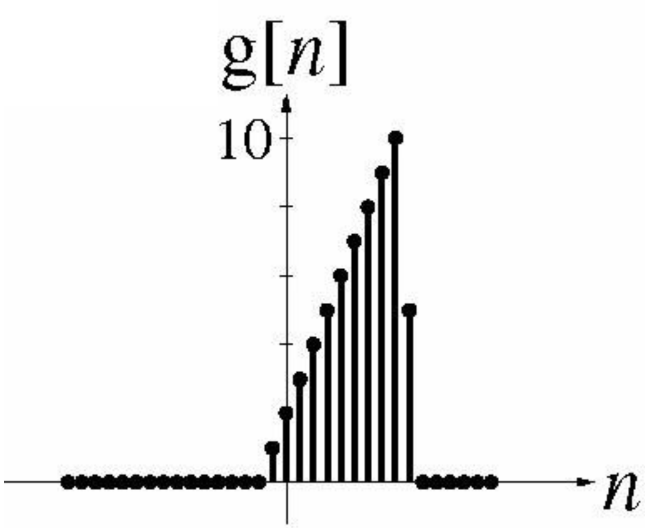
Here multiplication of amplitude of two or more signals at each instance of time or any other independent variables is done which are common between the signals.



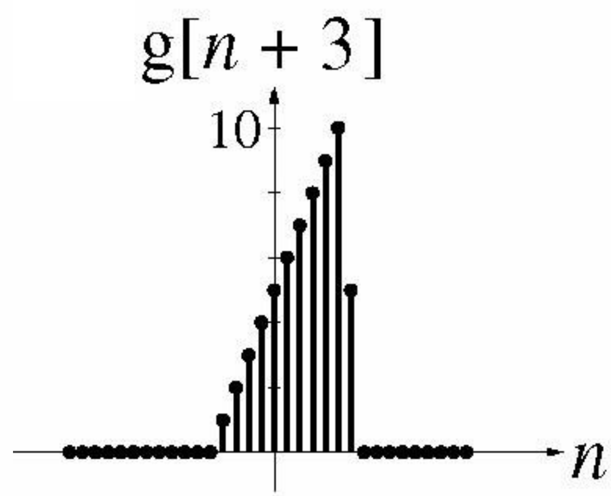
Operations on Signals

Time Shifting for discrete sequences

Time shifting $\quad n \rightarrow n + n_0$, n_0 an integer



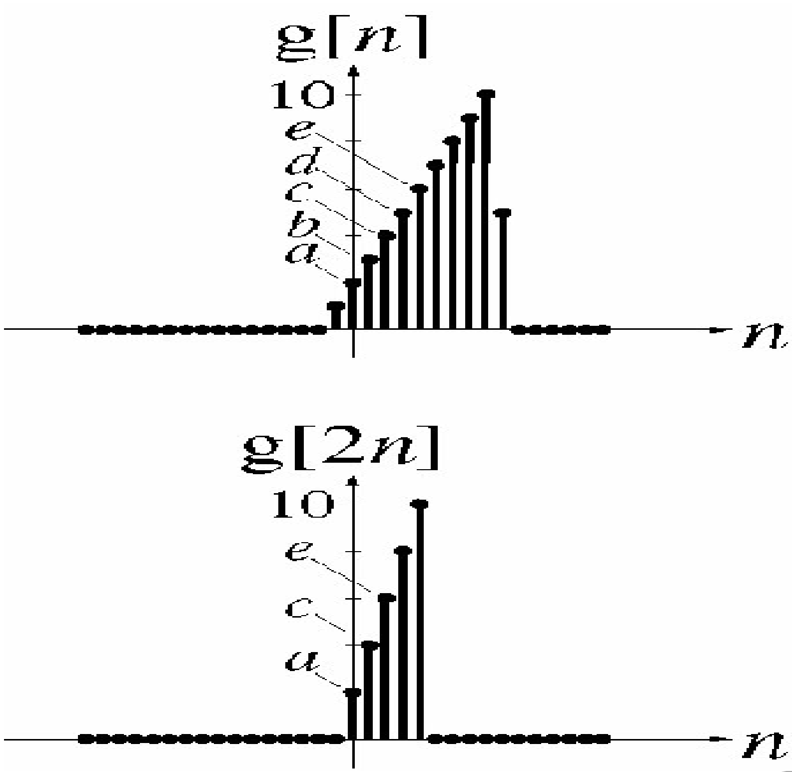
n	$g[n]$
–1	1
0	2
1	3
2	4
3	5
4	6
5	7
6	8
7	9
8	10
9	5
10	0



n	$n+3$	$g[n+3]$
–4	–1	1
–3	0	2
–2	1	3
–1	2	4
0	3	5
1	4	6
2	5	7
3	6	8
4	7	9
5	8	10
6	9	5
7	10	0

Scaling for discrete sequences

$n \to Kn$ $\quad$ K an integer > 1



n	$2n$	$g[2n]$
0	0	2
1	2	4
2	4	6
3	6	8
4	8	10

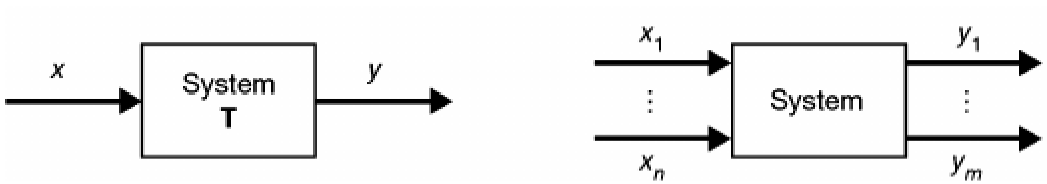
Systems and Classification

- A **system** is a **mathematical model** of a physical process that relates the input (or excitation) signal to the output(or response)
- signal.

Let x and y be the input and output signals, respectively, of a system.

Then the system is viewed as a transformation (or mapping) of x into y.

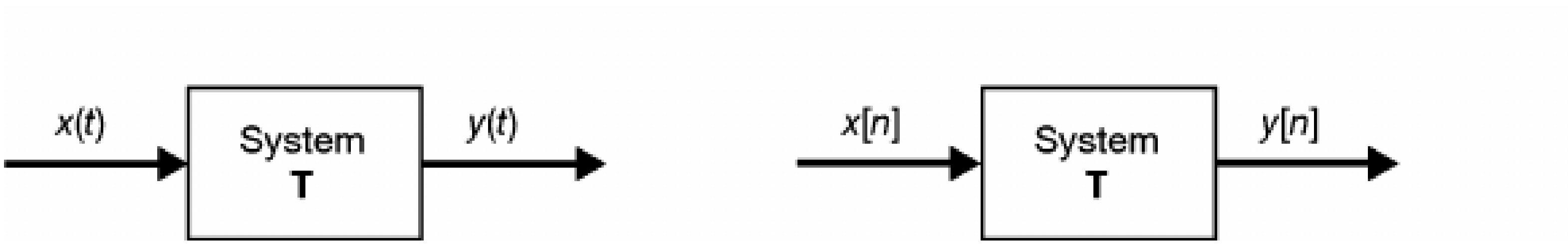
y=**T**x



Deterministic and Stochastic Systems

- If the **input and output signals** x and y are **deterministic** signals, then the system is called a deterministic system.
- If the **input and output signals** x and y are **random** signals, then the system is called a stochastic system.

Continuous-Time and Discrete-Time Systems



- A **continuous** time system is characterized by **differential** equation.
- A **discrete** time system is often expressed by **difference** equation

Systems with Memory and without Memory

- A system is said to be memoryless if the output at any time depends on only the input at that same time.

- Otherwise, the system is said to have memory.
- An example of a memoryless system is a resistor R with the input x(t) taken as the current and the voltage taken as the output y(t).

$$y = R\,x(t)$$

Systems with Memory and without Memory

- An example of a system with memory is a capacitor C with the current as the input x(t) and the voltage as the output y(t); then

$$y = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\, d\tau$$

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Causal and Non-Causal Sytems

- A system is called **causal** if its output at the present time depends on only the **present and/or past** values of the input.
- Thus, in a causal system, it is not possible to obtain an output before an input is applied to the system.

- A system is called noncausal (or anticipative) if its output at the present time depends on future values of the input.

Causal and Non-Causal Sytems

Examples of non-causal Systems

$$y(t) = x(t+1)$$

$$y[n] = x[-n]$$

- Note that all memoryless systems are causal, but not vice versa.

Linear Systems and Nonlinear Systems

- A system is said to be linear if it possesses additivity and homogenity.
- **T{**x_1+x_2**} =** y_1+y_2 **(Additivity)**
- **T{**ax**} =** ay **(Homogeneity** or **Scaling)**
  - **T{**$a_1x_1+a_2x_2$**} =** $a_1y_1+a_2y_2$ **(Superposition)**

Linear Systems and Nonlinear Systems

- Consequence of homogeneity is that for a linear system that
  zero input yields zero output.

  Examples of non linear systems

$$y = x^2 \qquad\qquad y = \cos x$$

Time In-Variant and Time Varying Systems

- A system is called time-invariant if a time shift (delay or advance) in the input signal causes the same time shift in the output signal.

$$\mathbf{T}\{x(t - \tau)\} = y(t - \tau)$$

$$\mathbf{T}\{x[n - k]\} = y[n - k]$$

- To check a system for time-invariance, we can compare the shifted output with the output produced by the shifted input.

Linear Time-Invariant Systems

- If the system is **linear** and also **time-invariant**, then it is called a **linear time-invariant (LTI)** system.

Stable Systems

- A system is bounded-input/bounded-output (BIBO) stable

if for any bounded input ‘x’ defined by

$$|x| \leqslant k_1$$

the corresponding output y is also bounded defined by

$$|y| \leqslant k_2$$

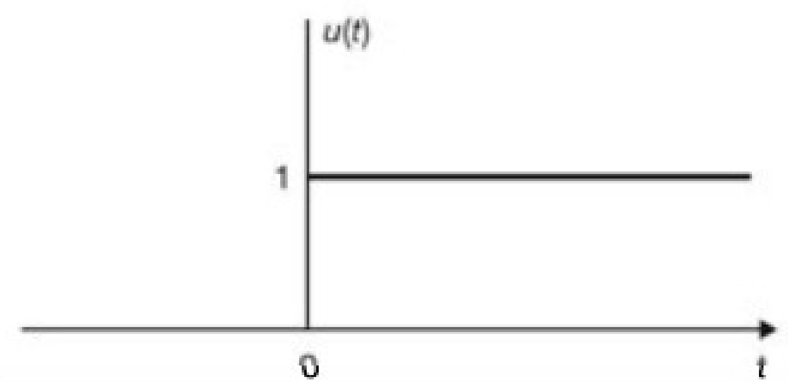
where k_1 and k_2 are finite real constants

An unstable system is one in which not all bounded inputs lead to bounded output.

Unit Step Signal

- The unit step function u(t), also known as the Heaviside unit function, is defined as

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

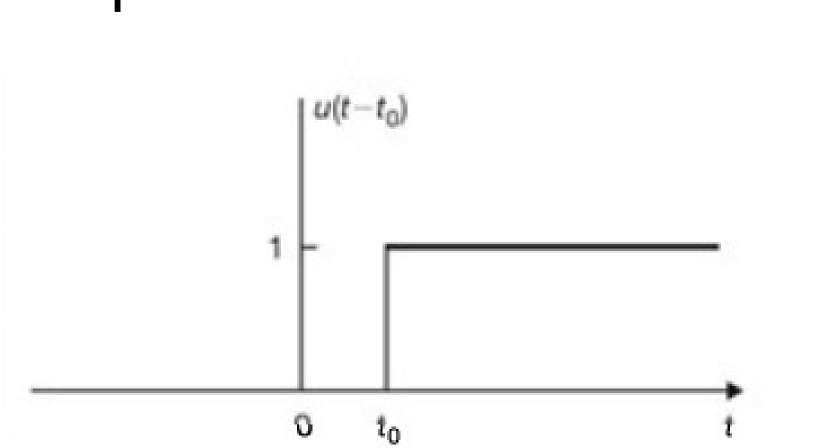


Note that it is discontinuous at t = 0 and that the value at t = 0 is undefined.

Unit Step Signal

- Time shifted version of unit step signal

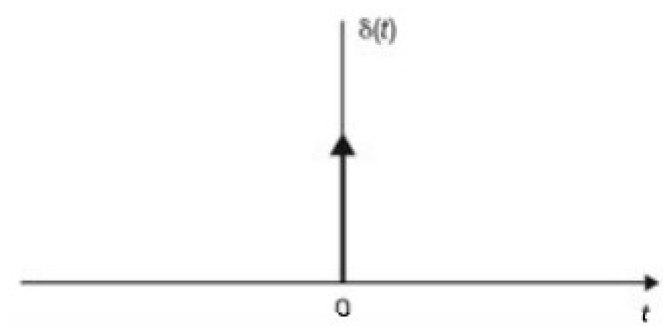
$$u(t-t_0)=\begin{cases}1 & t>t_0\\ 0 & t<t_0\end{cases}$$



Unit Impulse Function

- The unit impulse function, δ(t), also known as the Dirac delta function, is defined as:

$$\delta(t) = 0 \text{ for } t \neq 0;$$
$$= \text{undefined for } t = 0$$



Unit Impulse Function

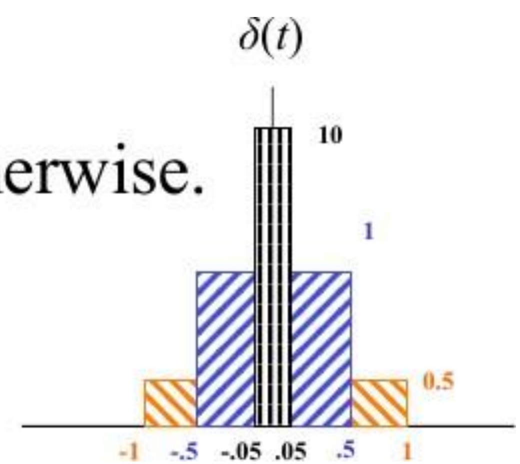
$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)dt = f(\tau)$$

$$\therefore \int_{-\infty}^{\infty} \delta(t)dt = 1$$

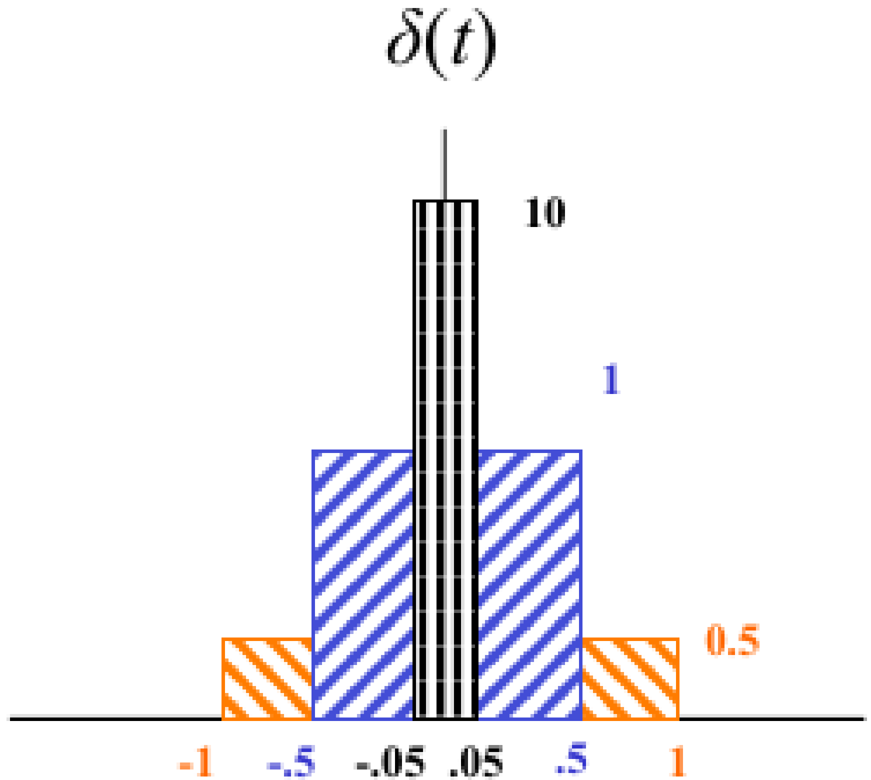
Unit Impulse

A consequence of the delta function is that it can be approximated by a narrow pulse as the width of the pulse approaches zero while the area under the curve = 1

$$\lim_{\varepsilon \to 0} \delta(t) \approx 1/\varepsilon \text{ for } -\varepsilon/2 < t < \varepsilon/2; = 0 \text{ otherwise.}$$

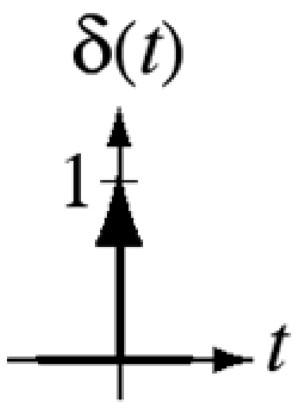


Unit Impulse

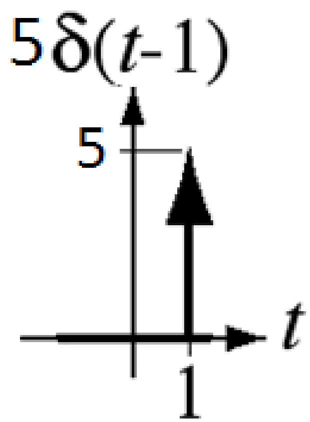


Unit Impulse Function

- The area under an impulse is called its strength or weight. It is represented graphically by a vertical arrow. An impulse with a strength of one is called a unit impulse.



Representation of Unit Impulse



Shifted Impulse of Amplitude5

Standard Signals

Unit Impulse Function

The Sampling Property

$$\int_{-\infty}^{\infty} g(t)\delta(t-t_0)dt = g(t_0)$$

The Scaling Property

$$\delta(a(t-t_0)) = \frac{1}{|a|}\delta(t-t_0)$$

Unit Impulse Function

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$$

- $\delta(t) \longrightarrow u(t)$
- $u(t) \longrightarrow tu(t)$ — 1^{st} order
- $tu(t) \longrightarrow \frac{t^2}{2!}u(t)$ — 2^{nd} order

•

•

•

- $\longrightarrow \frac{t^n}{n!}u(t)$ — n^{th} order

Uses of Impulse Function

Modeling of electrical, mechanical, physical phenomenon:

– point charge,

– impulsive force,

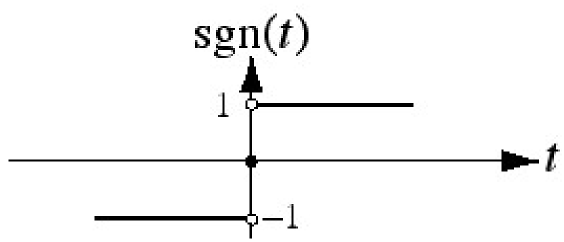
– point mass

– point light

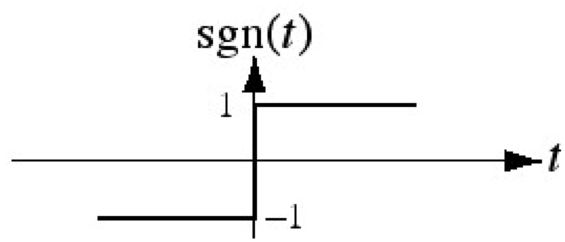
Signum Function

$$\operatorname{sgn}(t) = \begin{cases} 1 & , \ t > 0 \\ 0 & , \ t = 0 \\ -1 & , \ t < 0 \end{cases} = 2\,\mathrm{u}(t) - 1$$

Precise Graph



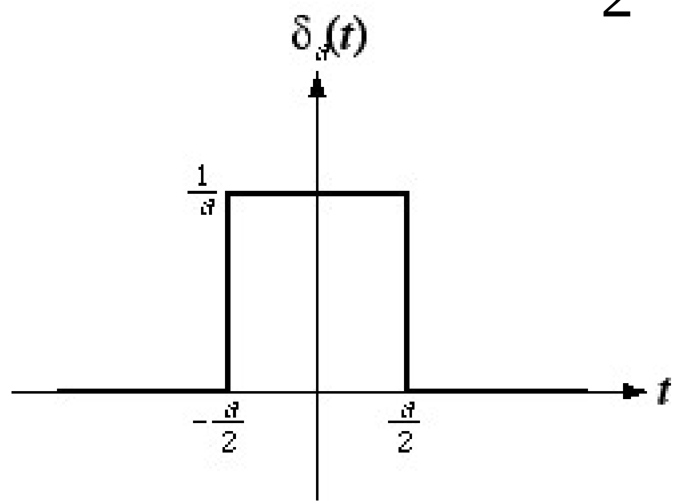
Commonly-Used Graph



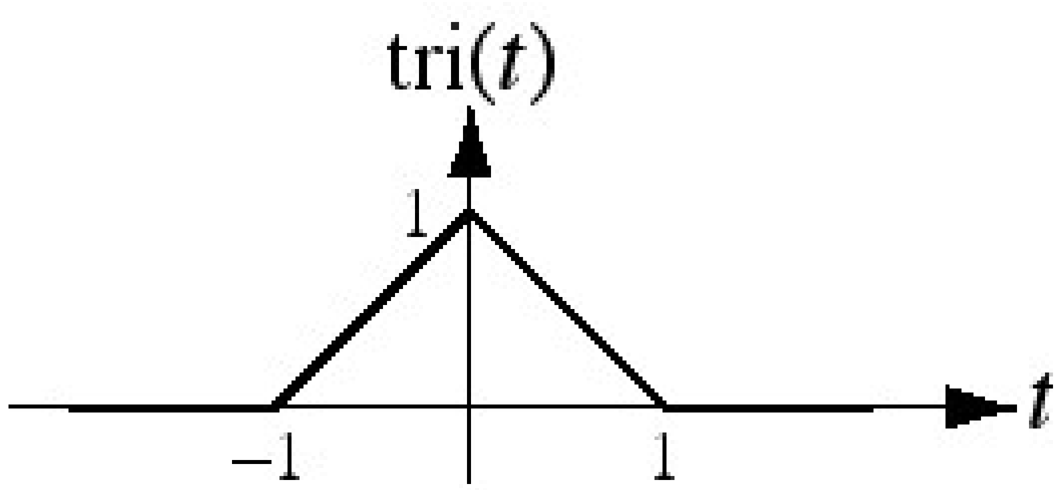
The signum function, is closely related to the unit-step function.

Rectangular Pulse or Gate Function

Rectangular pulse, $$\delta_a(t) = \begin{cases} 1/a & , \quad |t| < a/2 \\ 0 & , \quad |t| > a/2 \end{cases}$$

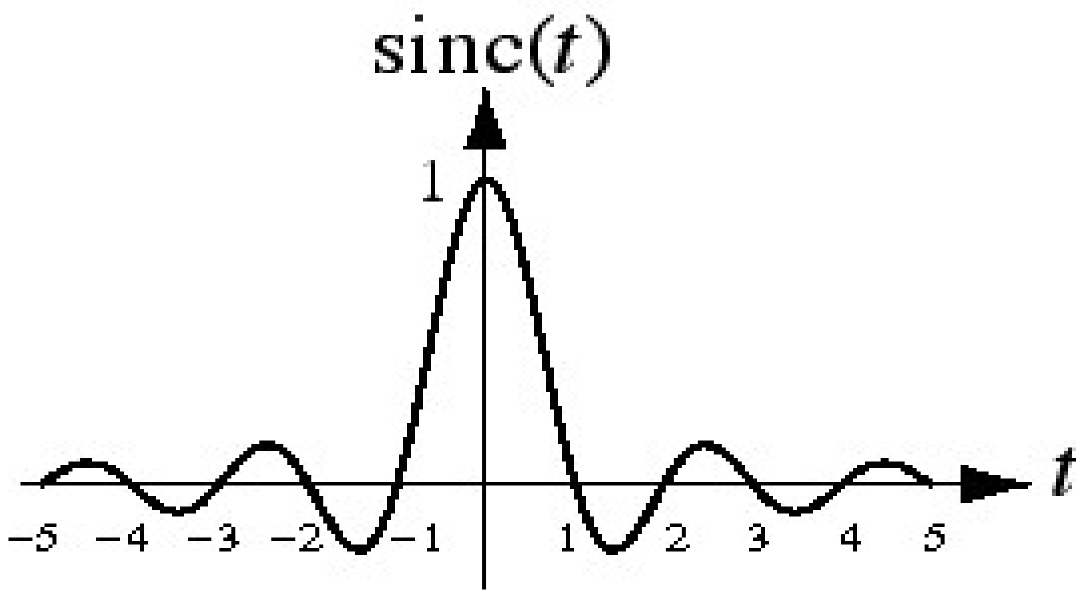


Unit Triangular function



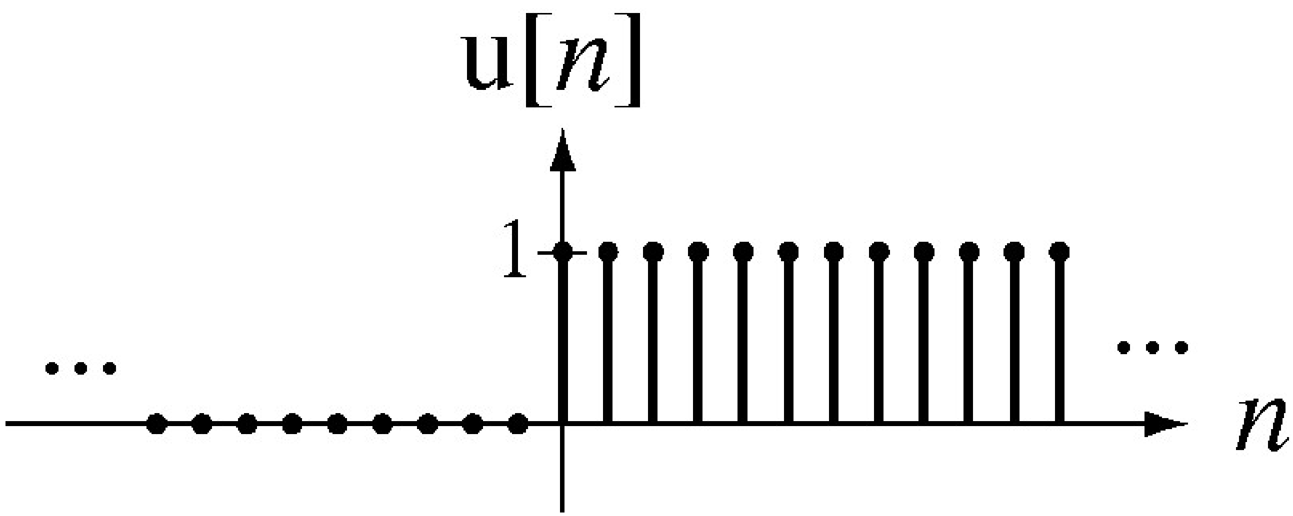
Sinc function

$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$



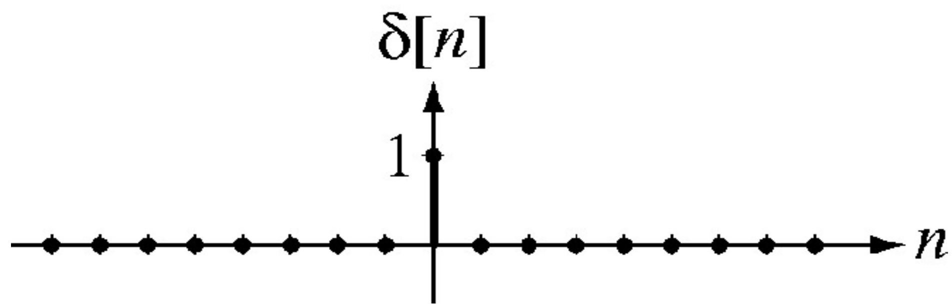
Discrete unit Step function

$$u[n] = \begin{cases} 1 \ , & n \geq 0 \\ 0 \ , & n < 0 \end{cases}$$



Discrete unit impulse function

$$\delta[n] = \begin{cases} 1 \ , & n = 0 \\ 0 \ , & n \neq 0 \end{cases}$$



FOURIER SERIES

Introduction to Fourier Series

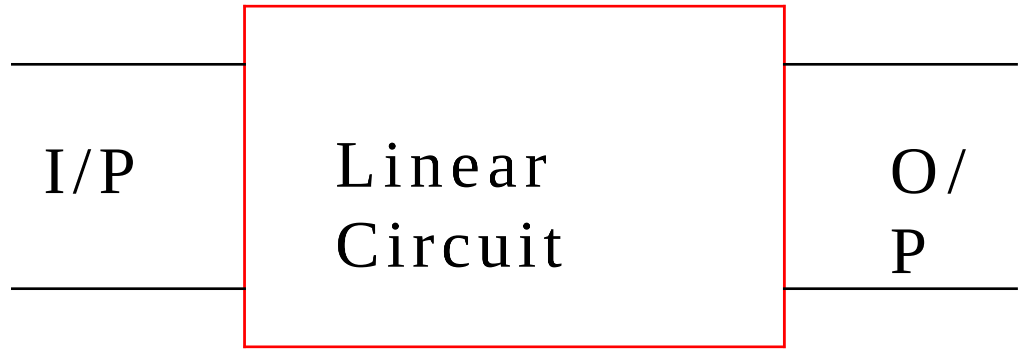
- Fourier Series is a representation of signals as a linear combination of a set of basic signals(sinusoidal or exponential).

- Representation of continuous-time and discrete-time periodic signals is referred as Fourier Series.

- Representation of aperiodic, finite energy signals is done through Fourier Transform.

- Used for analyzing, designing and understanding signals and **LTI** systems

Introduction to Fourier Series



Sinusoidal Inputs → **O K**

Nonsinusoidal Inputs

Nonsinusoidal Inputs → Sinusoidal Inputs

Fourier Series

Introduction to Fourier Series

Perception of Fourier Series

- Trigonometric sums – Babylonians - predict Astronomical events
- Year **1748 –** L **Euler –** examined motion of string – normal modes – discarded trigonometric series
- Year **1753 –** D **Bernoulli –** linear combinations of normal modes.
- Year **1759 – J. L Lagrange –** criticized use of trigonometric series for vibrating strings.

Perception of Fourier Series

- After a half century later **Fourier** developed his ideas on Trigonometric series.

Joseph Fourier 1768 to 1830

Introduction to Fourier Series

Perception of Fourier Series

- Year **1807 –** Fourier represented a series for temperature distribution through a body.

- Any **periodic signal** could be represented by such a **series**.

- For **aperiodic signals** *weighted integrals* of sinusoids that are *not at all* harmonically related.

- **Lagrange** rejected this trigonometric series saying discontinuities can never be represented in sinusoidal.

Introduction to Fourier Series

Perception of Fourier Series

- Year **1807 –** Fourier represented a series for temperature distribution through a body.

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  not at all harmonically related.

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Introduction to Fourier Series

Application areas of Fourier Series

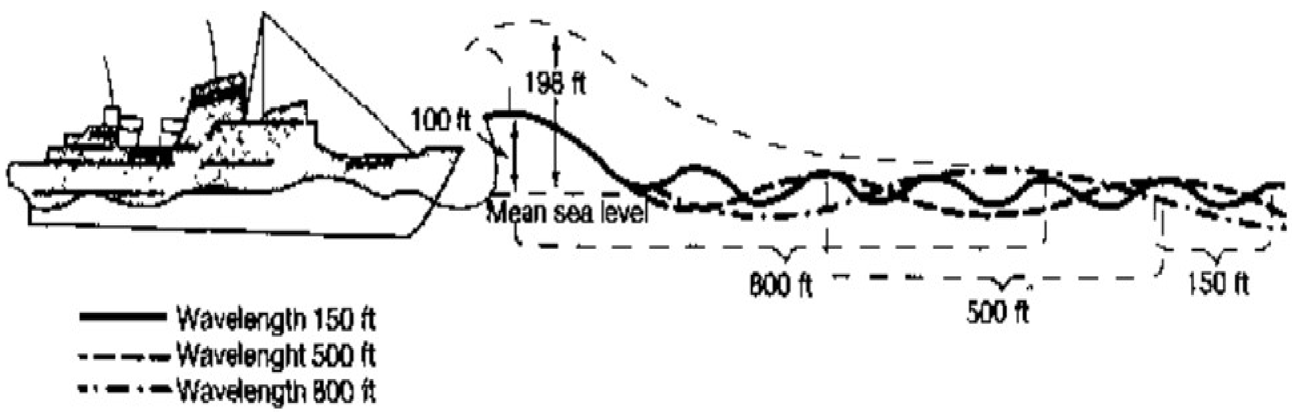
- In Theory of Integration, point-set topology and eigen function expansion.

- Sinusoidal signals arise naturally in describing the motion of the planets and periodic behaviour of the earth's climate.

- Alternating current sources generate voltages and currents used for describing **LTI** systems.

Application areas of Fourier Series

- Waves in ocean – linear combination of sinusoidal waves of diff. wavelengths (**or**) periods.



Introduction to Fourier Series

Application areas of Fourier Series

- Radio signals are sinusoidal in nature.
- **Discrete-time concepts and methods** – numerical analysis.
- Predicting motion of a heavenly body, given a sequence of observations.

- Mid **1960s –** FFT was introduced – reduced the time of computation
- With this tool many interesting but previously impractical ideas with discrete time Fourier series and transform have come practical.

Periodic Signals

Linear Combinations of harmonically Related Complex Exponentials

- A periodic signal with period of T , x(t) = x(t + T) for all t ,

$$x(t) = \cos\omega_0 t \qquad x(t) = e^{j\omega_0 t}$$

Both these signals are periodic with fundamental frequency ω_0 and fundamental period **$T = 2\pi / \omega_0$** .

Fourier series Representation – CT Periodic Signals

Linear Combinations of harmonically Related Complex Exponentials

- The set of harmonically related complex exponentials

$$\varphi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t}$$

$$k = 0, \pm 1, \pm 2, \ldots.$$

- Each of these signals is periodic with period of T

Fourier series Representation – CT Periodic Signals

Linear Combinations of harmonically Related Complex Exponentials

- Thus, a linear combination of harmonically related complex exponentials of the form

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

is also periodic with period of T

k = 0 , x (t) is a constant.

k = + 1 and k = − 1 , both have fundamental frequency equal to ω_0 and are collectively

referred to as the **fundamental components or the first harmonic components.**

k = + 2 and k = − 2 , the components are referred to as the **second harmonic** components. k = + N and k = − N , the components are referred to as the **Nth harmonic** components.

Fourier series Representation – CT Periodic Signals

Linear Combinations of harmonically Related Complex Exponentials

- If x (t) is real, that is, x (t) = x * (t)

$$x(t) = x^*(t) = \sum_{k=-\infty}^{+\infty} a^*_k \, e^{-jk\omega_0 t}$$

Replacing k by − k in the summation, we have

$$x(t) = \sum_{k=-\infty}^{+\infty} a^*_{-k} \, e^{jk\omega_0 t}$$

Fourier series Representation – CT Periodic Signals

Linear Combinations of harmonically Related Complex Exponentials

By comparison with first equation

$a_k = a^*_{-k}$, or equivalently $a^*_k = a_{-k}$

To derive the alternative forms of the Fourier series, we rewrite the summation

$$x(t) = a_0 + \sum_{k=1}^{+\infty}\left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk(2\pi/T)t}\right]$$

Fourier series Representation – CT Periodic Signals

Linear Combinations of harmonically Related Complex Exponentials

Substituting a^*_k for a_{-k}, we have

$$x(t) = a_0 + \sum_{k=1}^{+\infty} \left[a_k e^{jk\omega_0 t} + a^*_k \, e^{-jk(2\pi/T)t} \right].$$

Since the two terms inside the summation are complex conjugate of each other, this can be expressed as

$$x(t) = a_0 + \sum_{k=1}^{+\infty} 2\,\mathrm{Re}\left\{ a_k e^{jk\omega_0 t} \right\}$$

Fourier series Representation – CT Periodic Signals

Linear Combinations of harmonically Related Complex Exponentials

If a $_k$ is expressed in polar from as

$$a_k = A_k e^{j\theta_k}$$

$$x(t) = a_0 + \sum_{k=1}^{+\infty} 2\,\mathrm{Re}\left\{A_k e^{j(k\omega_0 t + \theta_k)}\right\}$$

Fourier series Representation – CT Periodic Signals

Linear Combinations of harmonically Related Complex Exponentials

$$x(t) = a_0 + 2\sum_{k=1}^{+\infty} A_k \cos(k\omega_0 t + \theta_k)$$

It is **one** commonly encountered **form** for the Fourier series of real periodic signals in continuous time.

Fourier series Representation – CT Periodic Signals

Linear Combinations of harmonically Related Complex Exponentials

Another form is obtained by writing a_k in rectangular form as

$$a_k = B_k + jC_k$$

$$x(t) = a_0 + 2\sum_{k=1}^{+\infty}\left[B_k \cos k\omega_0 t - C_k \sin k\omega_0 t\right]$$

Fourier series Representation – CT Periodic Signals

Linear Combinations of harmonically Related Complex Exponentials

For real periodic functions, the Fourier series in terms of complex exponential has the following *three* equivalent forms:

$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$
$x(t) = a_0 + 2\sum_{k=1}^{+\infty} A_k \cos(k\omega_0 t + \theta_k)$
$x(t) = a_0 + 2\sum_{k=1}^{+\infty} \left[B_k \cos k\omega_0 t - C_k \sin k\omega_0 t\right]$

Fourier series Representation – CT Periodic Signals

Convergence of Fourier Series – Dirichlet Conditions

The **Dirichlet conditions** for the periodic signal x are as follows:

- 1)Over a single period, x is **absolutely integrable**(i.e., $\int_T |x(t)|dt < \infty$)

  2)Over a single period, x has a **finite number of maxima and minima** (i.e., x is of bounded variation).

  3)Over any finite interval, x has a **finite number of discontinuities** each of which is **finite** .

Fourier series Representation – CT Periodic Signals

Convergence of Fourier Series – Dirichlet Conditions

If a periodic signal x satisfies the Dirichlet conditions , then:

1.The Fourier series converges pointwise everywhere to x , except at the points of discontinuity of x .

2.At each point $t = t_a$ of discontinuity of x , the Fourier series x converges to

$$\tilde{x}(t_a) = \frac{1}{2}\left[x(t_a^-) + x(t_a^+)\right],$$

where $x(t_a^-)$ and $x(t_a^+)$ denote the values of the signal x on the left- and −right-hand sides of the discontinuity, respectively.

Fourier series Representation – CT Periodic Signals

Convergence of Fourier Series – Dirichlet Conditions

- Since most signals tend to satisfy the Dirichlet conditions and the

  above convergence result specifies the value of the Fourier series

  at every point, this result is often very useful in practice.

Fourier series Representation – CT Periodic Signals

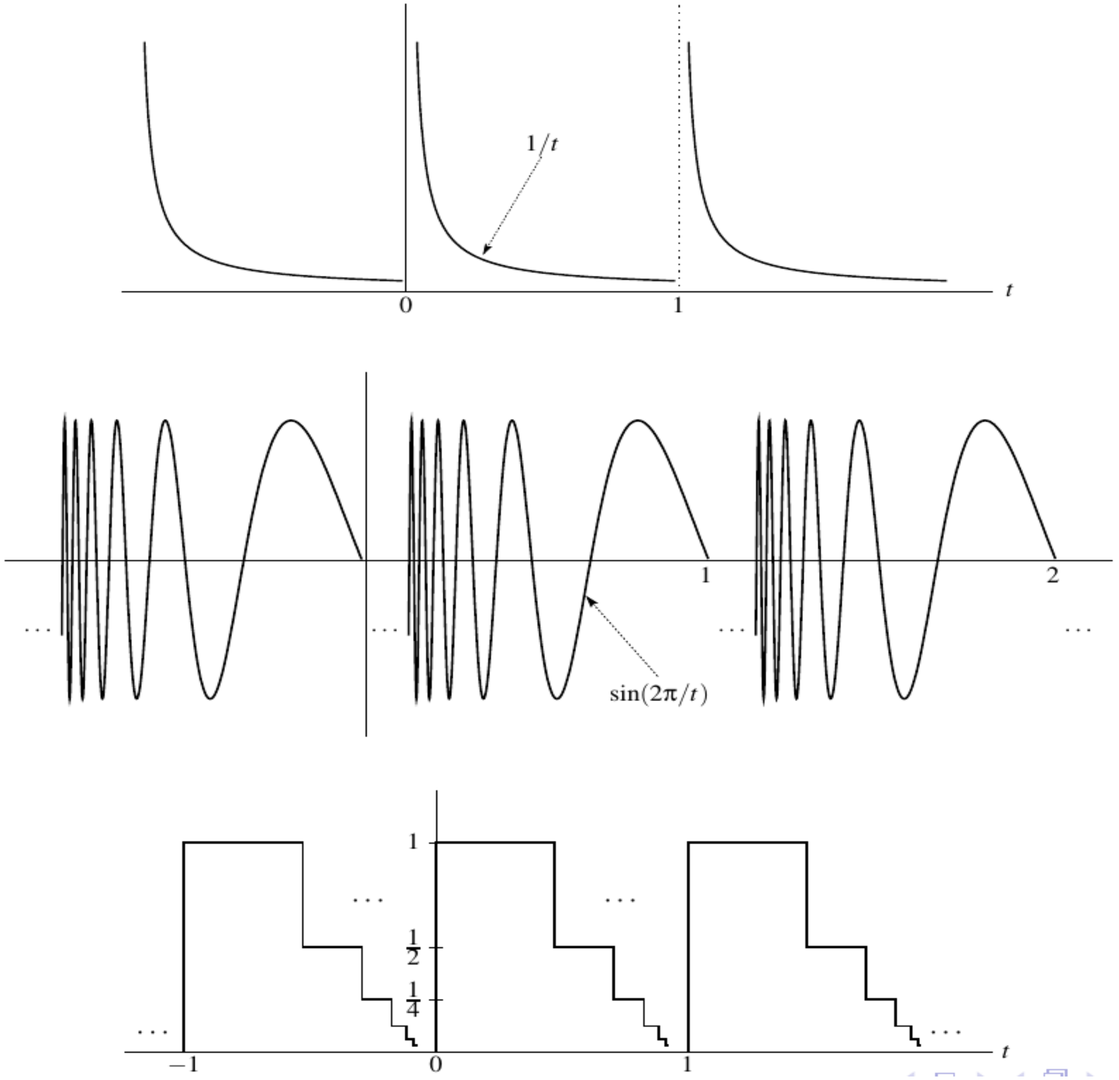
Convergence of Fourier Series – Dirichlet Conditions

- Since most signals tend to satisfy the Dirichlet conditions and the

  above convergence result specifies the value of the Fourier series

  at every point, this result is often very useful in practice.

Examples of Functions Violating Dirichlet Conditions



Periodic Signals

Gibbs Phenomenon

- In practice, we frequently encounter signals with discontinuities.
- When a signal x has discontinuities, the Fourier series representation of does not converge uniformly (i.e., at the same rate everywhere).
- The rate of convergence is much slower at points in the vicinity of a discontinuity.

Periodic Signals

Gibbs Phenomenon

- Furthermore, in the vicinity of a discontinuity, the truncated Fourier series x_N exhibits ripples, where the peak amplitude of the ripples does not seem to decrease with increasing N .

- As it turns out, as N increases, the ripples get compressed towards discontinuity, but, for any finite N , the peak amplitude of the ripples remains approximately constant.

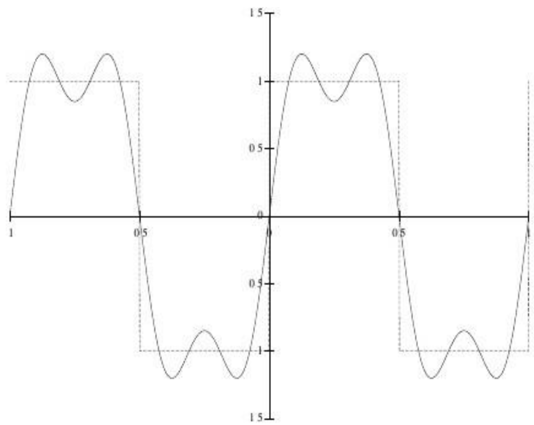
Periodic Signals

Gibbs Phenomenon

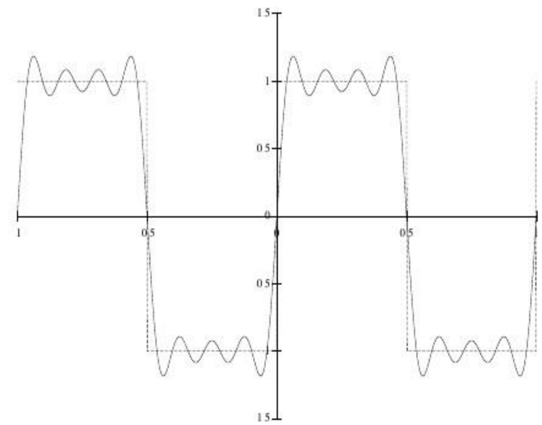
- This behavior is known as **Gibbs phenomenon.**

- The above behavior is one of the weaknesses of Fourier series (i.e., Fourier series converge very slowly near discontinuities).

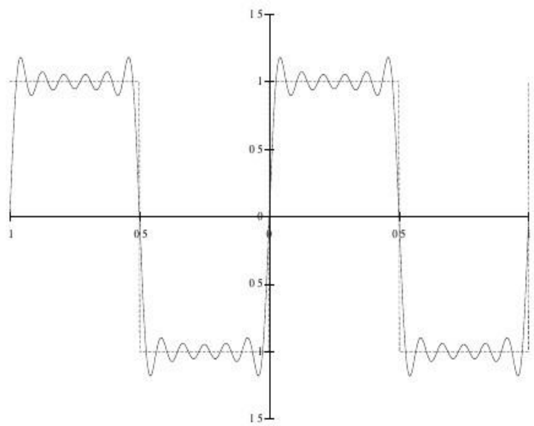
Gibbs Phenomenon



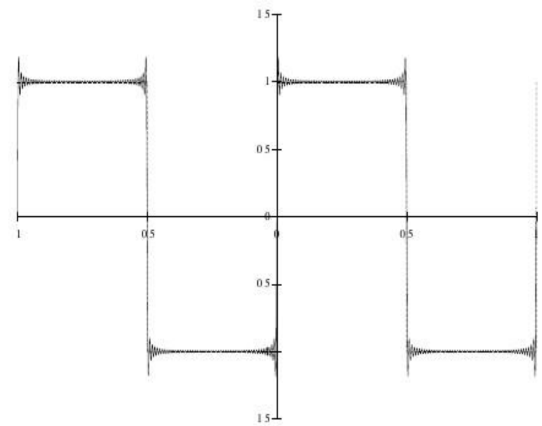
Fourier series truncated after the 3rd harmonic components



Fourier series truncated after the 7th harmonic components



Fourier series truncated after the 11th harmonic components



Fourier series truncated after the 101th harmonic components

Fourier series Representation – CT Periodic Signals

Determination of the Fourier Series Representation of a Continuous-Time Periodic Signal

Multiply both side of $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ by $e^{-jn\omega_0 t}$

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

Integrating both sides from 0 to T = 2 π / ω_0, we have

$$\int_0^T x(t)e^{-jn\omega_0 t}\,dt = \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{jk\omega_0 t} e^{-jn\omega_0 t}\,dt \right] = \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{j(k-n)\omega_0 t}\,dt \right]$$

Fourier series Representation – CT Periodic Signals

Determination of the Fourier Series Representation of a Continuous-Time Periodic Signal

$$\int_0^T e^{j(k-n)\omega_0 t}dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$$

For k=n

$$a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt$$

Fourier series Representation – CT Periodic Signals

Determination of the Fourier Series Representation of a Continuous-Time Periodic Signal

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

→ Synthesis Equation

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

→ Analysis Equation

Fourier series Representation – CT Periodic Signals

Determination of the Fourier Series Representation of a Continuous-Time Periodic Signal

- The set of coefficient { a_k } are often called the Fourier series coefficients (or) the spectral coefficients of x(t).
- The coefficient a_0 is the dc or constant component and is given with k = 0 , that is

$$a_0 = \frac{1}{T}\int_T x(t)dt$$

Fourier series Representation – CT Periodic Signals

Determination of the Fourier Series Representation of a Continuous-Time Periodic Signal

Example: consider the signal $x(t) = \sin \omega_0 t$.

$$\sin \omega_0 t = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}.$$

Comparing the right-hand side of this equation with synthesis equation

$$a_1 = \frac{1}{2j}, \qquad a_{-1} = -\frac{1}{2j}$$

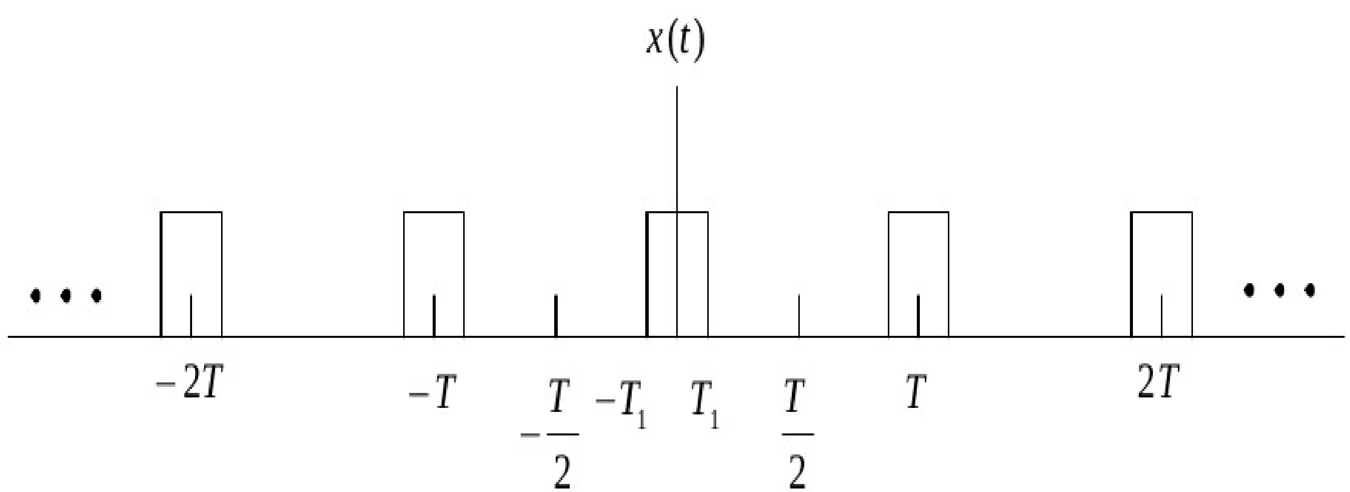
$$a_k = 0, \qquad k \neq +1 \text{ or } -1$$

Determination of the Fourier Series Representation of a Continuous-Time Periodic Signal

Example: The periodic square wave, sketched in the figure below and define over one period is

The signal has a fundamental period T and fundamental frequency $\omega_0 = 2\pi / T$.

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$



Fourier series Representation – CT Periodic Signals

Determination of the Fourier Series Representation of a Continuous-Time Periodic Signal

- To determine the Fourier series coefficients for x(t) , we use analysis equation.

- Because of the symmetry of x(t) about t = 0 , we choose − T / 2 ≤ t ≤ T / 2 as the interval over which the integration is performed, although any other interval of length T is valid the thus lead to the same result.

For k=0

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} x(t)dt = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T},$$

Determination of the Fourier Series Representation of a Continuous-Time Periodic Signal

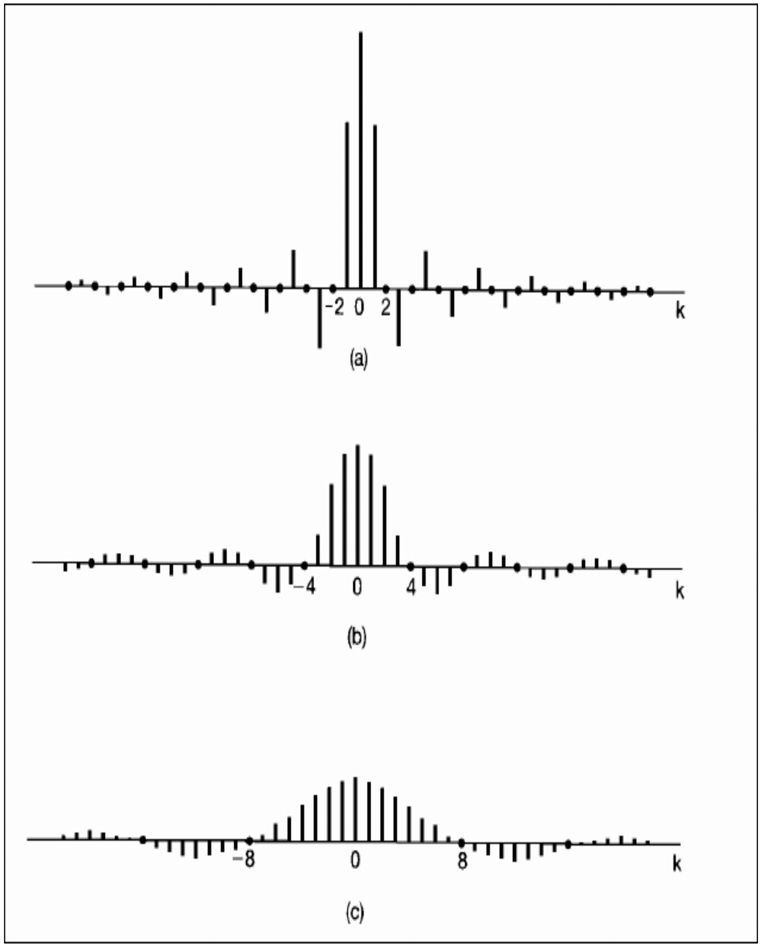
For k ≠ 0 , we ol

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt = -\frac{1}{jk\omega_0 T} e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}$$

$$= \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right]$$

$$= \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}$$

Determination of the Fourier Series Representation of a Continuous-Time Periodic Signal



Periodic Signals

Convergence of the Fourier Series

If a periodic signal x (t) is approximated by a linear combination of finite number of harmonically related complex exponentials

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}.$$

Let $e_N(t)$ denote the approximation error

The criterion used to measure quantitatively the **approximation error** is the **energy** in the error over one period:

Periodic Signals

Convergence of the Fourier Series

$$E_N = \int_T |e_N(t)|^2 dt .$$

The particular choice for the coefficients that minimize the energy in the error is

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt .$$

The limit of E_N as N -> ∞ is zero.

Fourier series Representation – CT Periodic Signals

Convergence of the Fourier Series

One class of periodic signals that are representable through Fourier series is those signals which have finite energy over a period,

$$\int_T |x(t)|^2 dt < \infty ,$$

When this condition is satisfied, we can guarantee that the coefficients obtained from are finite. We define

$$e(t) = x(t) - \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} ,$$

then

$$\int_T |e(t)|^2 dt = 0$$

Fourier series Representation – CT Periodic Signals

Convergence of the Fourier Series

- The **convergence guaranteed** when x(t) has **finite energy** over a period is very useful.

- In this case, we may say that x(t) and its Fourier series representation are **indistinguishable**.

FOURIE TRANSFORM:

- A periodic signal can be represented as linear combination of complex exponentials which are harmonically related.
- An aperiodic signal can be represented as linear combination of complex exponentials, which are infinitesimally close in frequency. So the representation take the form of an integral rather than a sum
- In the Fourier series representation, as the period increases the fundamental frequency decreases and the harmonically related components become closer in frequency. As the period becomes infinite, the frequency components form a continuum and the Fourier series becomes an integral.

FOURIE TRANSFORM:

- The main drawback of Fourier series is, it is only applicable to periodic signals. There are some naturally produced signals such as nonperiodic or aperiodic, which we cannot represent using Fourier series.
- To overcome this shortcoming, Fourier developed a mathematical model to transform signals between time *or* spatial domain to frequency domain & vice versa, which is called 'Fourier transform.
- Fourier transform has many applications in physics and engineering such as analysis of LTI systems, RADAR, astronomy, signal processing etc.

Deriving FOURIE TRANSFORM from FOURIER SERIES:

Consider a periodic signal f*t* with period T. The complex Fourier series representation of f*t* is given as

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T_0}kt} \ldots\ldots(1)$$

Deriving FOURIE TRANSFORM from FOURIER SERIES:

Let $\frac{1}{T_0} = \Delta f$, then equation 1 becomes

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k\Delta f t} \ldots\ldots\ldots\ldots (2)$$

but you know that

$$a_k = \frac{1}{T_0} \int_{t_0}^{t_0+T} f(t) e^{-jk\omega_0 t} dt$$

Substitute in equation 2.

$$2 \Rightarrow f(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} \int_{t_0}^{t_0+T} f(t) e^{-jk\omega_0 t} dt\, e^{j2\pi k\Delta f t}$$

Deriving FOURIE TRANSFORM from FOURIER SERIES:

Let $t_0 = \frac{T}{2}$

$$= \sum_{k=-\infty}^{\infty} \left[\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t) e^{-j2\pi k\Delta ft} \, dt\right] e^{j2\pi k\Delta ft} \, . \Delta f$$

In the limit as $T \rightarrow \infty$, Δf approaches differential df, $k\Delta f$ becomes a continuous variable f, and summation becomes integration

$$f(t) = \lim_{T\rightarrow\infty} \left\{\sum_{k=-\infty}^{\infty} \left[\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t) e^{-j2\pi k\Delta ft} \, dt\right] e^{j2\pi k\Delta ft} \, . \Delta f\right\}$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t) e^{-j2\pi ft} \, dt\right] e^{j2\pi ft} \, df$$

Deriving FOURIE TRANSFORM from FOURIER SERIES:

$$f(t) = \int_{-\infty}^{\infty} F[\omega] e^{j\omega t} d\omega$$

Where $F[\omega] = [\int_{-\infty}^{\infty} f(t) e^{-j2\pi ft} dt]$

Fourier transform of a signal

$$f(t) = F[\omega] = [\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt]$$

Inverse Fourier Transform is

$$f(t) = \int_{-\infty}^{\infty} F[\omega] e^{j\omega t} d\omega$$

FOURIE TRANSFORM :

FT of Impulse Function

$$FT[\omega(t)] = [\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt]$$

$$= e^{-j\omega t} \mid t = 0$$

$$= e^{0} = 1$$

$$\therefore \delta(\omega) = 1$$

FOURIE TRANSFORM :

FT of Unit Step Function:

$U(\omega) = \pi\delta(\omega) + 1/j\omega$

FT of Exponentials

$$e^{-at}u(t) \overset{\text{F.T}}{\longleftrightarrow} 1/(a + j\omega)$$

$$e^{-at}u(t) \overset{\text{F.T}}{\longleftrightarrow} 1/(a + j\omega)$$

$$e^{-a|t|} \overset{\text{F.T}}{\longleftrightarrow} \frac{2a}{a^2+\omega^2}$$

$$e^{j\omega_0 t} \overset{\text{F.T}}{\longleftrightarrow} \delta(\omega - \omega_0)$$

FT of Signum Function

$$sgn(t) \overset{\text{F.T}}{\longleftrightarrow} \frac{2}{j\omega}$$

FOURIE TRANSFORM :

Conditions for Existence of Fourier Transform

Any function ft can be represented by using Fourier transform only when the function satisfies Dirichlet's conditions. i.e.

- The function ft has finite number of maxima and minima.
- There must be finite number of discontinuities in the signal ft,in the given interval of time.
- It must be absolutely integrable in the given interval of time i.e.

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$$

DTFT:

The discrete-time Fourier transform *DTFT* or the Fourier transform of a discrete–time sequence x[n] is a representation of the sequence in terms of the complex exponential sequence *ejωn* .

The DTFT sequence x[n] is given by

$$X(\omega) = \Sigma_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \; \ldots\ldots (1)$$

Here, X*ω* is a complex function of real frequency variable ω and it can be written as

$$X(\omega) = X_{re}(\omega) + jX_{img}(\omega)$$

Inverse Discrete-Time Fourier Transform IDTFT:

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}\,d\omega \text{.........} (2)$$

Convergence Condition:

The infinite series in equation 1 may be converges or may not. x*n* is absolutely summable

$$\text{when} \sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

DTFT:

Where Xreω, Ximgω are real and imaginary parts of Xω respectively.

$$X_{re}(\omega) = |X(\omega)| \cos \theta(\omega)$$

$$X_{img}(\omega) = |X(\omega)| \sin \theta(\omega)$$

$$|X(\omega)|^2 = |X_{re}(\omega)|^2 + |X_{im}(\omega)|^2$$

And Xω can also be represented as $X(\omega) = |X(\omega)| e^{j\theta(\omega)}$

Where $\theta(\omega) = argX(\omega)$

$|X(\omega)|$, $\theta(\omega)$ are called magnitude and phase spectrums of Xω.

Linearity Property

$$\text{If } x(t) \xleftrightarrow{\text{F.T}} X(\omega)$$

$$\& \; y(t) \xleftrightarrow{\text{F.T}} Y(\omega)$$

Then linearity property states that

$$ax(t) + by(t) \xleftrightarrow{\text{F.T}} aX(\omega) + bY(\omega)$$

If

$$x(t) \overset{F}{\leftrightarrow} X(j\omega)$$

Then

$$x(t-t_0) \overset{F}{\leftrightarrow} e^{-j\omega t_0} X(j\omega)$$

Proof

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

Now replacing t by t-t_0

$$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega(t-t_0)}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(e^{-j\omega t_0} X(j\omega)\right)e^{j\omega t}\,d\omega$$

Recognising this as

$$F\{x(t-t_0)\} = e^{-j\omega t_0} X(j\omega)$$

A signal which is shifted in time does not have its Fourier transform magnitude altered, only a shift in phase.

Frequency Shifting Property

$$\text{If } x(t) \stackrel{\text{F.T}}{\longleftrightarrow} X(\omega)$$

Then frequency shifting property states that

$$e^{j\omega_0 t}.\, x(t) \stackrel{\text{F.T}}{\longleftrightarrow} X(\omega - \omega_0)$$

Time Reversal Property

$$\textbf{If } x(t) \stackrel{\textbf{F.T}}{\longleftrightarrow} X(\omega)$$

Then Time reversal property states that

$$x(-t) \stackrel{\textbf{F.T}}{\longleftrightarrow} X(-\omega)$$

Time Scaling Property

$$\text{If } x(t) \overset{\text{F.T}}{\longleftrightarrow} X(\omega)$$

Then Time scaling property states that

$$x(at) \frac{1}{|a|} X \frac{\omega}{a}$$

Differentiation and Integration Properties

$$If\ x(t) \stackrel{F.T}{\longleftrightarrow} X(\omega)$$

Then Differentiation property states that

$$\frac{dx(t)}{dt} \stackrel{F.T}{\longleftrightarrow} j\omega.X(\omega)$$

$$\frac{d^n x(t)}{dt^n} \stackrel{F.T}{\longleftrightarrow} (j\omega)^n.X(\omega)$$

and integration property states that

$$\int x(t)\,dt \stackrel{F.T}{\longleftrightarrow} \frac{1}{j\omega} X(\omega)$$

$$\iiint \ldots \int x(t)\,dt \stackrel{F.T}{\longleftrightarrow} \frac{1}{(j\omega)^n} X(\omega)$$

Multiplication and Convolution Properties

$$\text{If } x(t) \overset{\text{F.T}}{\longleftrightarrow} X(\omega)$$

$$\& \ y(t) \overset{\text{F.T}}{\longleftrightarrow} Y(\omega)$$

Then multiplication property states that

$$x(t).\,y(t) \overset{\text{F.T}}{\longleftrightarrow} X(\omega) * Y(\omega)$$

and convolution property states that

$$x(t) * y(t) \overset{\text{F.T}}{\longleftrightarrow} \frac{1}{2\pi} X(\omega).\,Y(\omega)$$

Differentiation in frequency domain

$$\mathcal{F}[tx(t)] = j\frac{d}{d\omega}X(j\omega)$$

$$\frac{d}{d\omega}X(j\omega) \quad : \quad \frac{d}{d\omega}[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt] = \int_{-\infty}^{\infty} x(t)\frac{d}{d\omega}e^{-j\omega t}dt$$

$$: \quad \int_{-\infty}^{\infty} x(t)(-jt)e^{-j\omega t}dt$$

$$\mathcal{F}[-jtx(t)] = \frac{d}{d\omega}X(j\omega)$$

$$\frac{d}{d\omega}X(j\omega) = \int_{-\infty}^{\infty} tx(t)e^{-j\omega t}dt = \mathcal{F}[tx(t)]$$

$$\mathcal{F}[t^n x(t)] = j^n\frac{d^n}{d\omega^n}X(j\omega)$$

Complex Conjugation

$$\text{if } \mathcal{F}[x(t)] = X(j\omega), \quad \text{then } \mathcal{F}[x^*(t)] = X^*(-j\omega)$$

Proof: Taking the complex conjugate of the inverse Fourier transform, we get

$$x^*(t) = [\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega]^* = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t}d\omega$$

Replacing ω by $-\omega'$ we get the desired result:

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(-\omega')e^{j\omega' t}d\omega' = \mathcal{F}^{-1}[X^*(-\omega)]$$

Parseval's equation

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(j\omega) d\omega = \text{Total energy in } x(t)$$

$$S_X(j\omega) \triangleq |X(j\omega)|^2$$

Symmetry (or Duality)

$$\text{if} \quad \mathcal{F}[x(t)] = X(j\omega), \quad \text{then} \quad \mathcal{F}[X(t)] = 2\pi \, x(-j\omega)$$

$$\text{if} \quad \mathcal{F}[x(t)] = X(f), \quad \text{then} \quad \mathcal{F}[X(t)] = x(-f)$$

$$x(t) = \mathcal{F}^{-1}[X(j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$x(-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{-j\omega t'}d\omega$$

Module - 3

- Signal transmission through Linear system

Linear System

Linear system , it satisfies principle superposition.

The response of linear system to weighted sum of input signals is equal to the same weighted sum of output signals.

$$x_i(t) \rightarrow y_i(t) = T[\, x_i(t)\,]$$

$$x(t) = \sum_{i=1}^{N} a_i x_i(t) \ \ where\ a_i\ is\ any\ arbitary\ constant$$

$$y(t) = T[x(t)\,] = T\left[\sum_{i=1}^{N} a_i x_i(t)\right] = \sum_{i=1}^{N} a_i T[x_i(t)]$$

$$y(t) = \sum_{i=1}^{N} a_i y_i(t)$$

Classification of Linear systems

Lumped and distributed systems

Time invariant and variant systems

Classification of Linear systems : Lumped systems

Lumped systems:

Consisting of Lumped elements which are connected particular way.

The energy in the system considered to be as stored of dissipated in distinct isolated elements.

Disturbance initiated at any point propagated instantaneously at every point in the system.

Dimensions of elements is very small compare to signal wave length.

Obeys ohm law and Kirchhoff laws only and system are expressed by ordinary differential equations.

Classification of Linear systems : Distributed systems

Elements are distributed over a long distances.

Dimensions of the circuits are small compared to the wave length of signals to be transmitted.

system takes finite amount of time for disturbance at one point to be propagated to the other point.

Expressed with partial differential equations.

Example are transmission lines , optical fiber , wave guides, antennas, semiconductor devices , beams etc.,

Classification of Linear systems : Linear time invariant system and variant system

LTI system , it satisfies linear and time invariant properties.

A system is Time invariant , if a time shift of input signal leads to an identical time shift in the output signal.

$$y(t) = T\,[\,x(t)]$$

$$if\ input\ delated\ or\ advanced\ by\ t_0\ seconds$$

$$y_1(t) = T\,[\,x(t \mp t_0)]$$

$$y_1(t) = y\,(t \mp t_0)$$

$$= y(t, t_0)\ \ time\ invariant\ other\ wise\ variant$$

Representation of Arbitrary signal

Let us consider an arbitrary signal

$$\widehat{x(t)} is an\ approxximation\ of\ x(t)\ and$$

$$it\ can\ be\ expressed\ as\ linear combination\ of\ shifted\ impulses$$

$$\widehat{x(t)} = \cdots \ldots + x(-2\Delta)\delta_{\Delta}(t+2\Delta) + x(-\Delta)\delta_{\Delta}(t+\Delta) + x(0)\delta_{\Delta}(t) + x(\Delta)\delta_{\Delta}(t-\Delta) + x(2\Delta)\delta_{\Delta}(t-2\Delta) + \cdots \ldots..$$

$$\widehat{x(t)} = \sum_{k=-\infty}^{\infty} x(\mathrm{k}\Delta)\delta_{\Delta}(t-k\Delta)\,\Delta$$

$$x(t) = \lim_{\Delta\to 0} \widehat{x(t)}$$

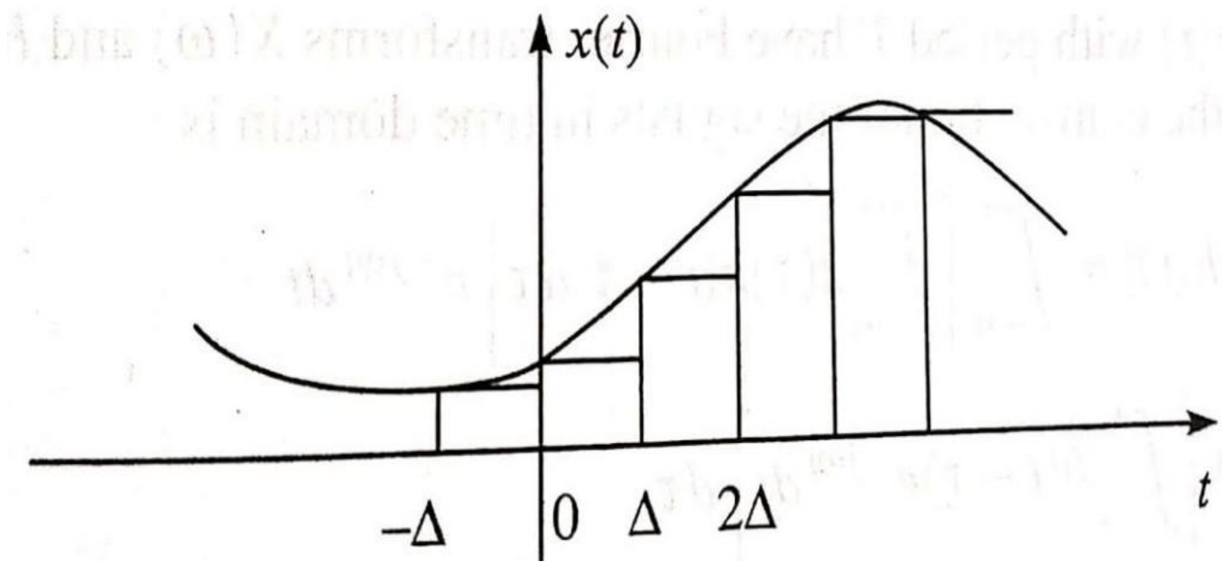
As $\Delta \to 0, \delta_{\Delta}(t) \to \delta(\mathrm{t})$, summation becomes integration $\mathrm{k}\Delta \to \tau, \Delta \to \mathrm{d}\tau$

Representation of Arbitrary signal

$$\delta_\Delta(t) = \frac{1}{\Delta} \quad 0 < t < \Delta \;\; other\,wise\; 0$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

A continuous time signal can be expressed as integral of weighted shifted impulses.



Impulse response of LTI system

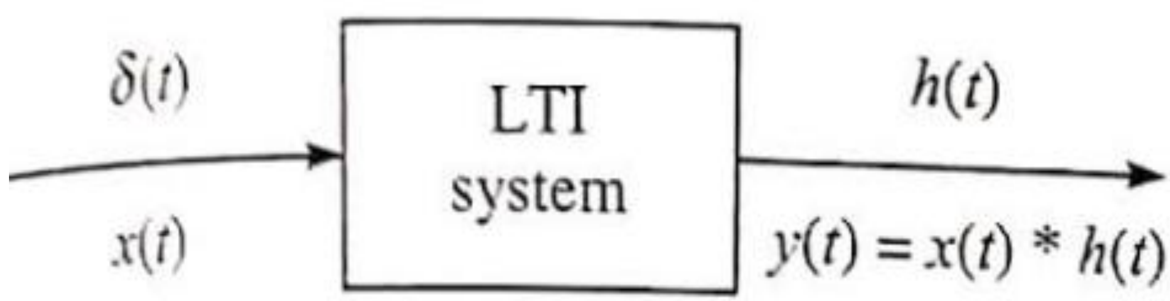
$$y(t)\ is\ a\ reponse\ of\ x(t)$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

$$y(t) = \ T\ [\,x(t)]$$

$$y(t) = \ T\ [\,x(t) = T\left[\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)T[\delta(t-\tau)]d\tau$$



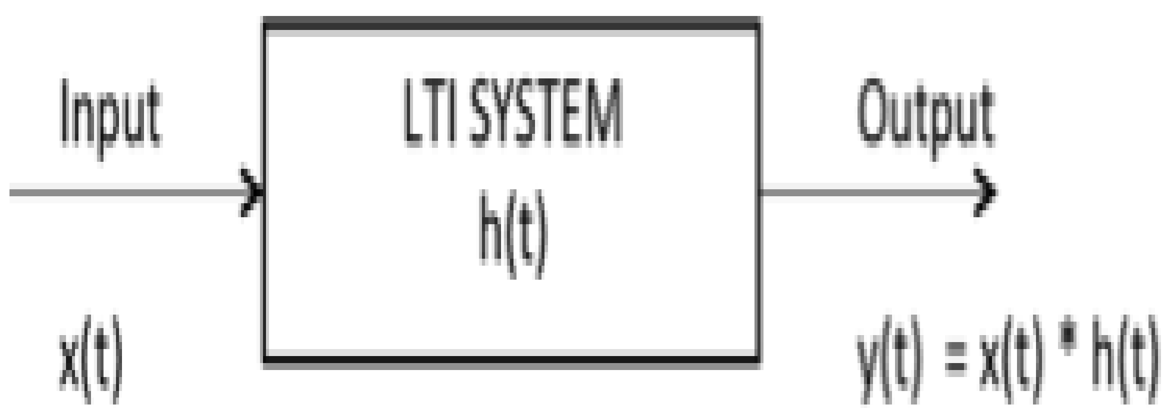
Impulse response of LTI system

$$h(t-\tau) = T\,[\,\delta(t-\tau) this\ satisfies\ time\ invariat\ property$$

$$h(t) = T\,[\,\delta(t)]\ this\ shows\ impuse\ response\ of\ LTI\ system$$

Impulse response of LTI system due to impulse input applies at t=o is h(t).

This is known as convolution integral and it gives relationship among input signal, output signal and impulse response of system.LTI system completely characterized by impulse response



Frequency response of LTI System

Consider LTI system with impulse response h(t)

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$y(t)\overleftrightarrow{Fourier\ transform}\,Y(\omega)$$

$$x(t)\overleftrightarrow{Fourier\ transform}\,X(\omega)$$

$$h(t)\overleftrightarrow{Fourier\ transform}\,H(\omega)$$

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}\,dt$$

$$Y(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)e^{-j\omega t}d\tau\,dt$$

Frequency response of LTI System

$$t-\tau = \lambda, dt = d\lambda$$

$$Y(\omega) = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}\,d\tau \int_{-\infty}^{\infty} h(\lambda)e^{-j\omega\lambda}\,d\lambda$$

$$Y(\omega) = H(\omega)X(\omega)$$

$|H(\omega)| = magnetude\ response\ of\ LTI\ sytem$ and it symmetric

$\angle H(\omega) = phase\ response\ of\ LTI\ system$ and it is anti symmetric

$H(\omega)$

Response to Eigen functions

If input to the system is an exponential function

$$x(t) = e^{j\omega t}$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{j\omega(t-\tau)}d\tau$$

$$y(t) = e^{j\omega t} H(\omega) = x(t) H(\omega)$$

Output is a complex exponential of the same frequency as input multiplied by the complex constant $H(\omega)$.

Properties of LTI System

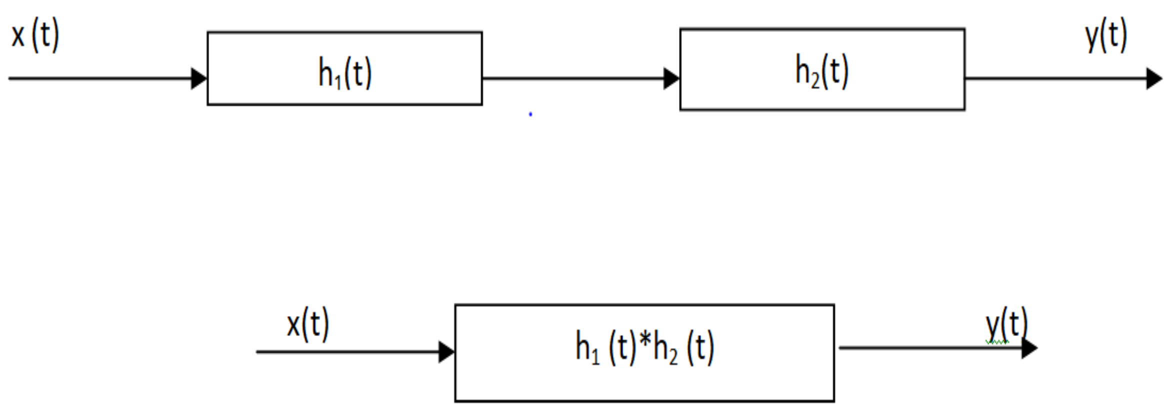
Commutative Property

$$y(t) = x(t) * h(t) = h(t) * x(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Properties of LTI System

Associate Property : cascading of two or more LTI system will results to single system with impulse response equal to the convolution of the impulse response of the cascading systems



$$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_2(t) * h_1(t)\}$$

$$h(t) = h_2(t) * h_1(t)$$

Distributive Property: This property gives that addition of two or more LTI system subjected to same input will results single system with impulse response equal to the sum of impulse response of two or more individual systems.

$$x(t) * \{ h_1(t) + h_2(t)\} = x(t) * h_1(t) + x(t) * h_2(t)$$

Properties of LTI System

Static and Dynamic system:

A system is static or memory less if its output at any time depends only on the value of its input at that instant of time

For LTI systems, this property can hold if its impulse response is itself an impulse.

convolution property, the output depends on the previous samples of the input, therefore an LTI system has memory and hence it is dynamic system

Properties of LTI System

Causality :A continuous time LTI system is said to causal if and only if

$$h(t) = \begin{cases} non\ zero & for\ t \geq 0 \\ 0 & for\ t < 0 \end{cases}$$

Properties of LTI System

Stability: continuous time system is BIBO stable if and only if the impulse response is absolutely Integrable.

Consider LTI system with impulse response h(t) . the output y(t) is

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

If the input $x(t)$ is bounded that is $|x(t)| \leq M_x < \infty$

$$|y(t)| = \left|\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right|$$

$$|y(t)| = \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|d\tau$$

Properties of LTI System

$$|y(t)| = M_x \int_{-\infty}^{\infty} |h(\tau)| d\tau$$

For bounded output, the impulse response is absolutely Intergrable

that is $\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$

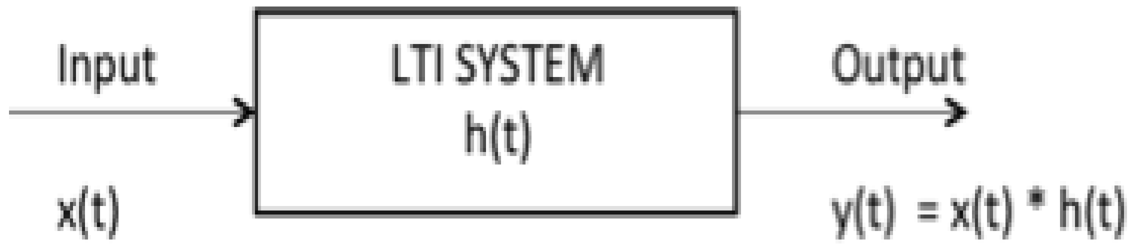
The above equation gives necessary and sufficient condition for BIBO stability.

Properties of LTI System

Inverse systems :A system T said to be invertible if and only if there exits an inverse system T^{-1} for such that T T^{-1} is an identical system

Transfer function of LTI system

Transfer function of LTI system defined as the ratio of Fourier transform of the output signal to Fourier transform of the input signal.



$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

h(t) = IFT of $H(\omega)$.

Transfer function of LTI system

Input and output relationship of continuous time causal LTI system described by linear constant coefficient differential equations with zero initial conditions is given by

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

where a_k andb_k any arbitary constants and N >M

N refer to highest derivative of y(t)

Transfer function of LTI system

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Apply Fourier Transform to above equation

$$\sum_{k=0}^{N} a_k \,(j\omega)^k \,Y(\omega) = \sum_{k=0}^{M} b_k \,(j\omega)^k \,X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k \,(j\,\omega)^k}{\sum_{k=0}^{N} a_k \,(j\,\omega)^k} =$$

Distortion less Transmission Through LTI System

Distortion less transmission through the LTI system requires that the response be exact replica of input signal.

The replica may have different magnetude and delayed in time.

$$any\ \ arbitary\ input\ x(t), if\ output\ y(t) = \ k\ x(t - t_0)$$

$$Y(\omega) = \ kX(\omega)\ e^{-j\omega t_0}$$

$$H(\omega) = k\ e^{-j\omega t_0}$$

$$|H(\omega)| = k, \qquad \angle H(\omega) = n\pi - \omega t_0$$

Magnetude response of system $|H(\omega)|$ must be constant over entire frequency range.

Phase response of the system $\angle H(\omega)$ must be linear with frequency

Signal Band Width

Signal Band width:

It is the range of significant frequency components present in the signal.

For any practical signals, the energy content decreases with frequency, only some of frequency components of signals have significant amplitude within a certain frequency band; outside this band have negligible amplitude. $\frac{1}{\sqrt{2}}$

The amplitude of significant frequency components within the times of maximum signal amplitude.

System Band Width

The band width of system is defined as the interval of frequencies over which the magnitude spectrum of remains within $\frac{1}{\sqrt{2}}$ times (3dB) its value at the mid band.

$\omega_1 = lower\ 3dB\ frequency = lower\ cutoff\ frequency =$
$lower frequency\ at\ which\ magnetude\ of H(\omega)\ \frac{1}{\sqrt{2}}$

Times of its value at the mid band.

$\omega_2 = upper\ cutoff\ frequency = Upper\ 3dB\ frequency$

$= highest\ frequency\ at\ which\ magnetude\ of H(\omega)\ \frac{1}{\sqrt{2}}\ times\ its\ mid\ band\ value$

System band width = $Upper\ 3dB\ frequency - lower\ 3dB\ frequency$

System Band Width

For distortion less transmission, a system should have infinite bandwidth. But due to physical limitations it is impossible to design an ideal filters having infinite bandwidth.

For satisfactory distortion less transmission, an LTI system should have high bandwidth compared to the signal bandwidth

Filter characteristics of linear system

LTI system acts as filter depending on the transfer function of system.

The system modifies the spectral density function of input signal according to transfer function.

system act as some kind of filter to various frequency components.

Some frequency components are boosted in strength, some are attenuated, and some may remain unaffected.

each frequency component suffers a different amount of phase shift in the process of transmission.

Types of filters

LTI system may be classified into five types of filter

Low pass filter

High pass filter

Band pass filter

Band reject filter

All pass filter.

Types of Ideal filters

Pass Band : Passes all frequency components in its pass band without distortion .

Stop Band : completely blocks frequency components outside of pass band. There is discontinuity between pass band and stop band in frequency spectrum.

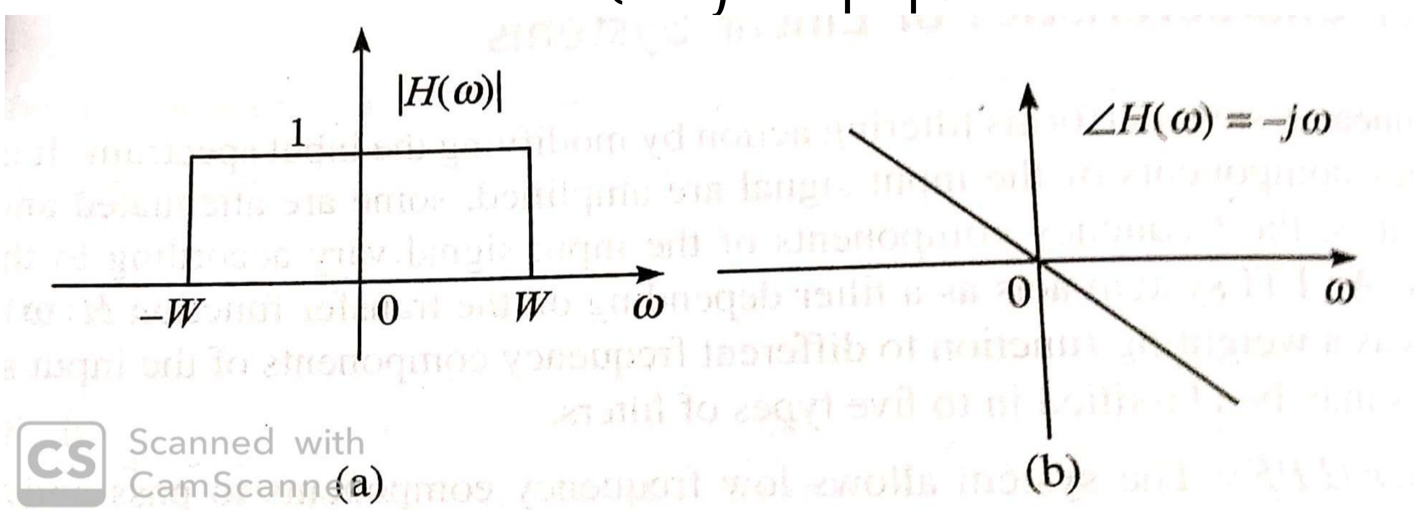
Transition band : For Practical filters, The range of frequencies over which there is a gradual Transition between pass band and stop band.

Types of Ideal filters : Ideal Low Pass Filter

An ideal low pass filter transmits all frequency components below the certain frequency ω_c rad /sec called cutoff frequency, without distortion. The signal above these frequencies is filtered completely.

Transfer function of Ideal LPF

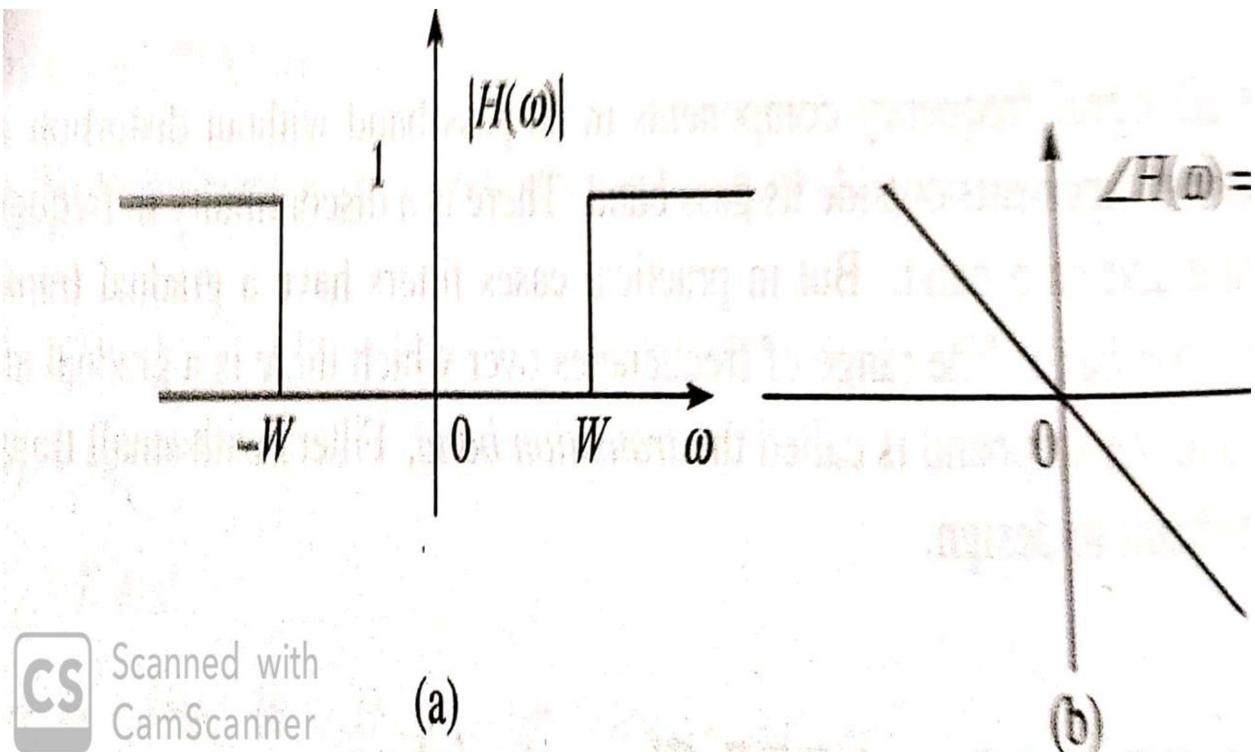
$$H(\omega) = \begin{cases} e^{-j\omega_0 t} & for\ |\omega| < W \\ 0 & for\ |\omega| > W \end{cases}$$



Types of Ideal filters : Ideal High Pass Filter

An ideal high pass filter transmits all frequency components above the certain frequency W rad/sec called cutoff frequency, without distortion. The signal below these frequencies is filtered completely.

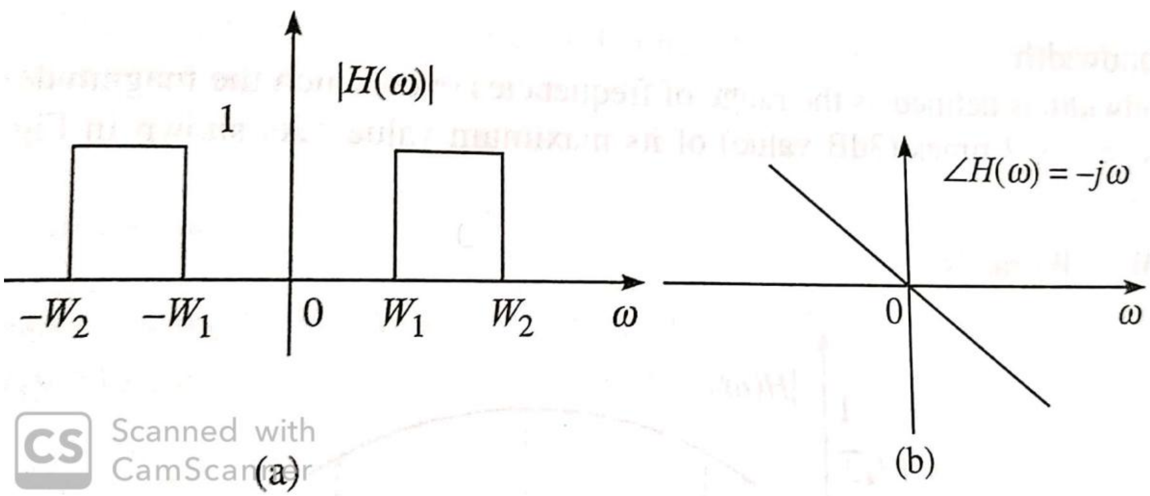
$$H(\omega) = \begin{cases} e^{-j\omega t_0} & for \ |\omega| > W \\ 0 & for \ |\omega| < W \end{cases}$$



Types of Ideal filters : Ideal Band Pass Filter

An ideal band pass filter transmits all frequency components within certain frequency band W_1 to W_2 rad /sec, without distortion. The signal with frequency outside this band is stopped completely.

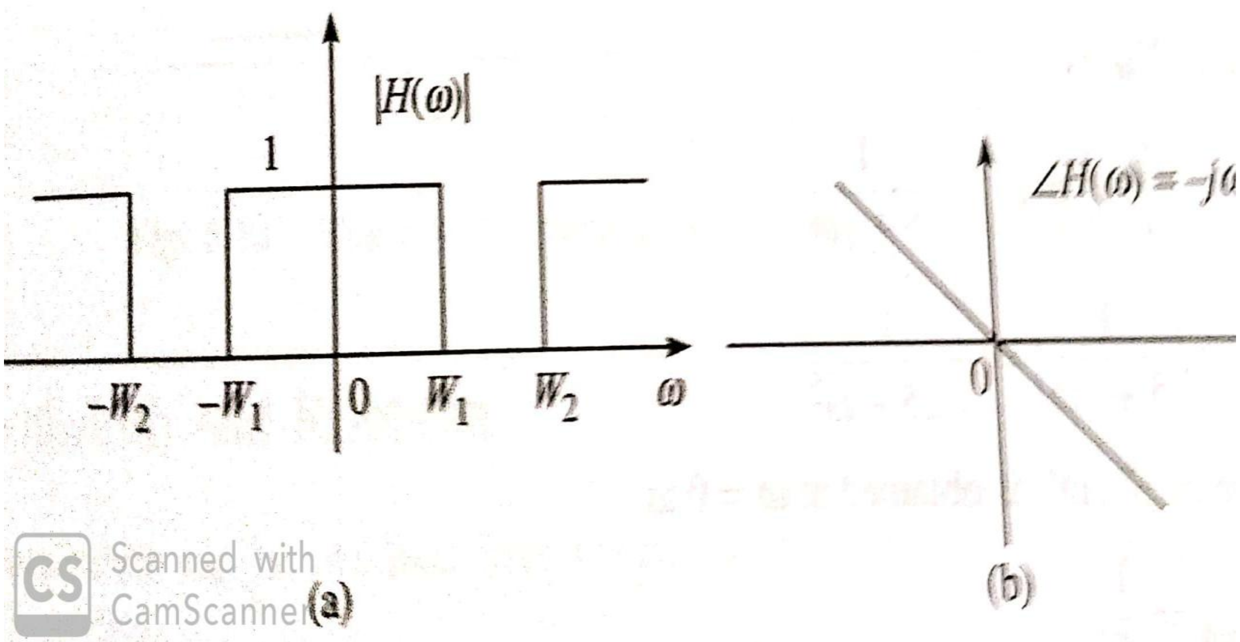
$$H(\omega) = \begin{cases} e^{-j\omega t_0} & for\ W1 < |\omega| < W2 \\ 0 & other\ wise \end{cases}$$



Types of Ideal filters : Ideal Band Reject Filter

An ideal band reject filter rejects all frequency components within certain frequency band W_1 to rad W_2/sec. The signal outside this band is transmitted without distortion.

$$H(\omega) = \begin{cases} 0 & for W_1 < |\omega| < W_2 \\ e^{-j\omega t_0} & other\ wise \end{cases}$$



Causality and Physical Reliability: Paley Wiener criterion

For physically realizable systems, that cannot have response before the input signal applied.

In time domain approach the impulse response of physically realizable systems must be causal.

Frequency domain, The necessary and sufficient condition for magnetude response to be physically realizable is known as the Paley – Wiener criterion

$$\int_{-\infty}^{\infty} \frac{\left|ln|H(\omega)|\right| d\omega}{1+\omega^2} < \infty$$

This condition known as the Paley – Wiener criterion

To satisfy the the Paley – Wiener criterion, the function H (ω) must be square integral .

All causal system satisfy the Paley –Wiener criterion.

Ideal filters are not physically realizable. But it possible to construct physically realizable filters close to the filter characteristics.

Where ε an arbitrary small value

$$H(\omega) = \begin{cases} e^{-j\omega t_0} \; for \; |\omega| < W \\ \varepsilon \quad for \; |\omega| > W \end{cases}$$

Band width and Rise time

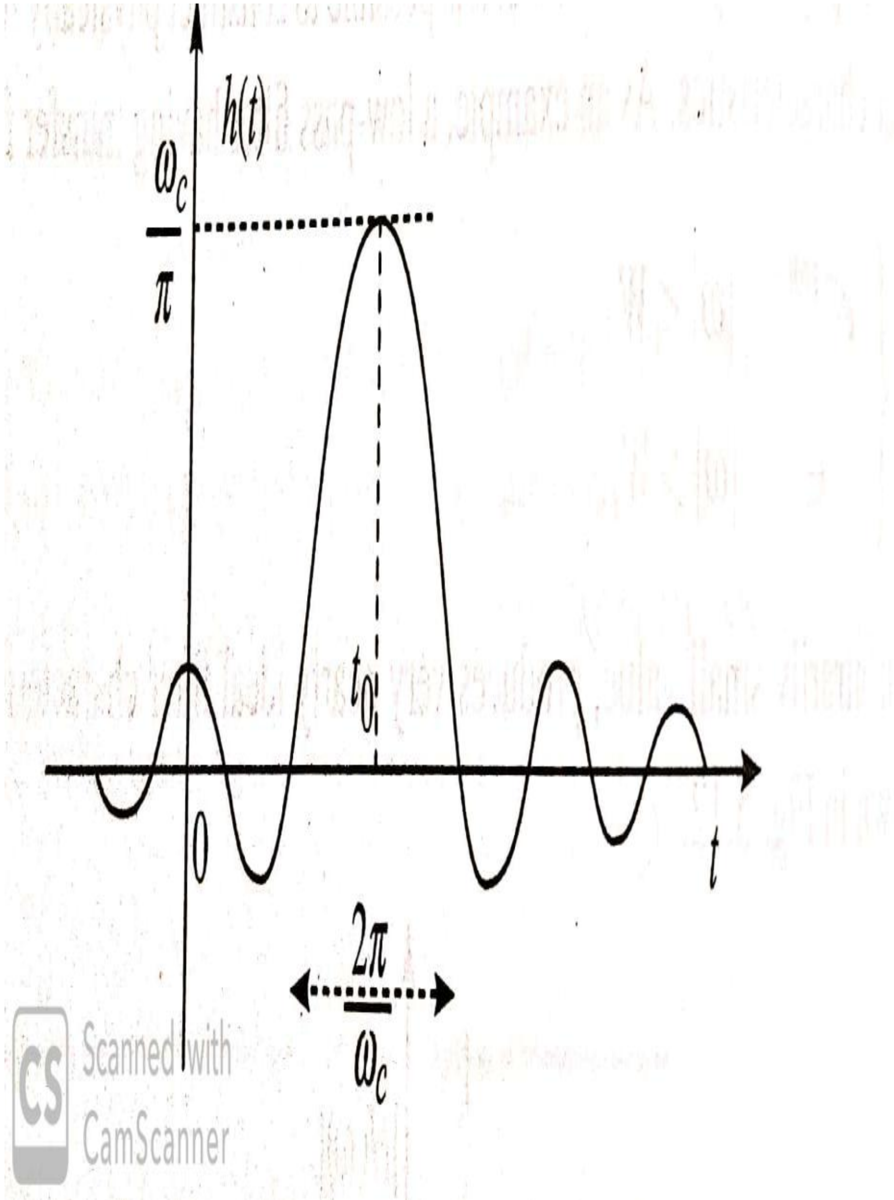
The Rise time (t_r) of output response is defined as the time the response take to reach from 10 % to 90% of the final value of signal.

$$\left.\frac{dy(t)}{dt}\right|_{t_0} = \frac{1}{t_r}$$

System band Width can be derived from output response

Consider LPF with transfer function $H(\omega) = \begin{cases} e^{-j\omega t_o} \; for \; |\omega| < \omega_c \\ 0 \; for \; |\omega| > \omega_c \end{cases}$

Rise time and Band width



$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega) e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(t-t_0)}\, d\omega = \frac{1}{\pi}\frac{sin\omega_c(t-t_0)}{(t-t_0)}$$

$$h(t) = \frac{\omega_c sinc\omega_c(t-t_0)}{\pi}$$

Rise time and band width

$$y(t) = h(t) * \delta(t) = \int_{-\infty}^{\infty} h(\tau)d\tau$$

$$\frac{dy(t)}{dt} = \frac{\omega_c}{\pi} sinc\omega_c(t\text{-}t_0)$$

$$\left.\frac{dy(t)}{dt}\right|_{t_0} = \frac{\omega_c}{\pi} = \frac{1}{t_r}$$

$$t_r = \frac{\pi}{\omega_c}$$

Band width of LPF is ω_c rad/sec

The convolution integral

The process of expressing the output signal in terns of the superposition of weighted and shifted impulse responses is called convolution.

The mathematical tool for evaluating the convolution of continuous time signal is called convolution integral. For discrete time signal is called convolution sum.

Characterizing input – output relationship of LTI systems.

Play important role in time and frequency domain analysis.

The convolution integral

Let $x_1(t)$ *and* $x_2(t)$ *be two* continuous time signals. Then convolution *of* $x_1(t)$ *and* $x_2(t)$ can be expressed as

$$\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\,d\tau$$

where* τ *is dummy variable

The output of any continuous time LTI system is the convolution of the input x(t) with impulse response h(t) of the system.

The convolution Integral

Case 1

If the input signal is causal $x(t) = \begin{cases} non\ zero\ value & t \geq 0 \\ 0 & for\ other\ wise \end{cases}$

$$y(t) = \int_{0}^{\infty} x(\tau)h(t-\tau)d\tau$$

Case 2

If LTI system is causal $h(t) = \begin{cases} non\ zero\ value & t \geq 0 \\ 0 & for\ other\ wise \end{cases}$

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

Case 3

If both input signal and system are causal

$$y(t) = \int_{0}^{\infty} x(\tau)h(t-\tau)d\tau$$

Properties of convolution integral :

Commutative Property:

$$let\ x_1(t) and\ x_2(t) are\ the\ continuous\ time\ signals$$

$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$t - \tau = \lambda$$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_2(\lambda)\, x_2(t-\lambda)\, d\lambda = x_2(t) * x_1(t)$$

Properties of convolution integral :

Distributive Property:

$$x_1(t) * [\, x_2(t) + x_3(t)] = x_1(t) * \ x_2(t) + \ x_1(t) * \ x_3(t)$$

Associate Property:

$$x_1(t) * [\, x_2(t) * x_3(t)] = [x_1(t) * \ x_2(t)] * \ x_3(t)$$
$$= \ x_1(t) * \ x_2(t) * \ x_3(t)$$

Shifting property:

$$x_1 \ (t) * \ x_1 \ (t - t_0) = x(t - t_0)$$

$$x_1 \ (t - t_1) * \ x_1 \ (t - t_2) = x(t - t_1 - t_2)$$

Properties of convolution integral

Convolution with impulse function

$$x(t) * \delta(t) = x(t)$$

$$x(t) * \delta(t - t_0) = x(t - t_0)$$

Convolution with unit step function

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\ d\tau$$

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau) * \delta(\tau)\ d\tau = \int_{-\infty}^{t} x(\tau)\ d\tau$$

Properties of convolution integral

Width Property:

Let us consider finite duration of two signals $x_1(t)\ and\ x_2(t)$ are T_1 and T_2 respectively **then duration of y(t) =** $x_1(t) *\ x_2(t)$ is equal to the sum of duration of $x_1(t)\ and\ x_2(t)$.

Area under finite signals $x_1(t)\ \ and\ \ x_2(t)$ are A_1 and A_2 respectively then the area under y (t) is product of both areas.

A = area under y (t) = area under $x_1(t)$ and area under $x_2(t)$ = A_1 A_2

Convolution property of Fourier Transform

$$x(t) \leftrightarrow X(\omega), y(t) \leftrightarrow Y(\omega)$$

$$Fourier\ Transform\ of\ x(t) * y(t) = X(\omega)Y(\omega)$$

Convolution in Frequency Domain

$$Fourier\ Transform\ of\ X(\omega) * Y(\omega) = 2\pi[x(t)y(t)]$$

Method of Graphical Convolution

Increase the time t along positive axis . Multiply the signals and integrate over the period of two signals to obtain convolution at t.

Increase the time shift step by step and obtain convolution using step 4.

Draw the convolution x (t) with the values obtained in steps 4 and 5 as function of t.

Increase the time t along positive axis . Multiply the signals and integrate over the period of two signals to obtain convolution at t.

Increase the time shift step by step and obtain convolution using step 4.

Draw the convolution x (t) with the values obtained in steps 4 and 5 as function of t.

MODULE-IV

LAPLACE TRANSFORM:

➢A Laplace transform of function f (t) in a time domain, where t is the real number greater than or equal to zero, is given as F(s), where there

$$F(s) = \int_0^{\infty} f(t)e^{-st}\,dt$$

It is the complex number in frequency domain .i.e. s = σ+jω
The above equation is considered as **unilateral Laplace transform equation**
When the limits are extended to the entire real axis then the **Bilateral Laplace transform** can be defined as

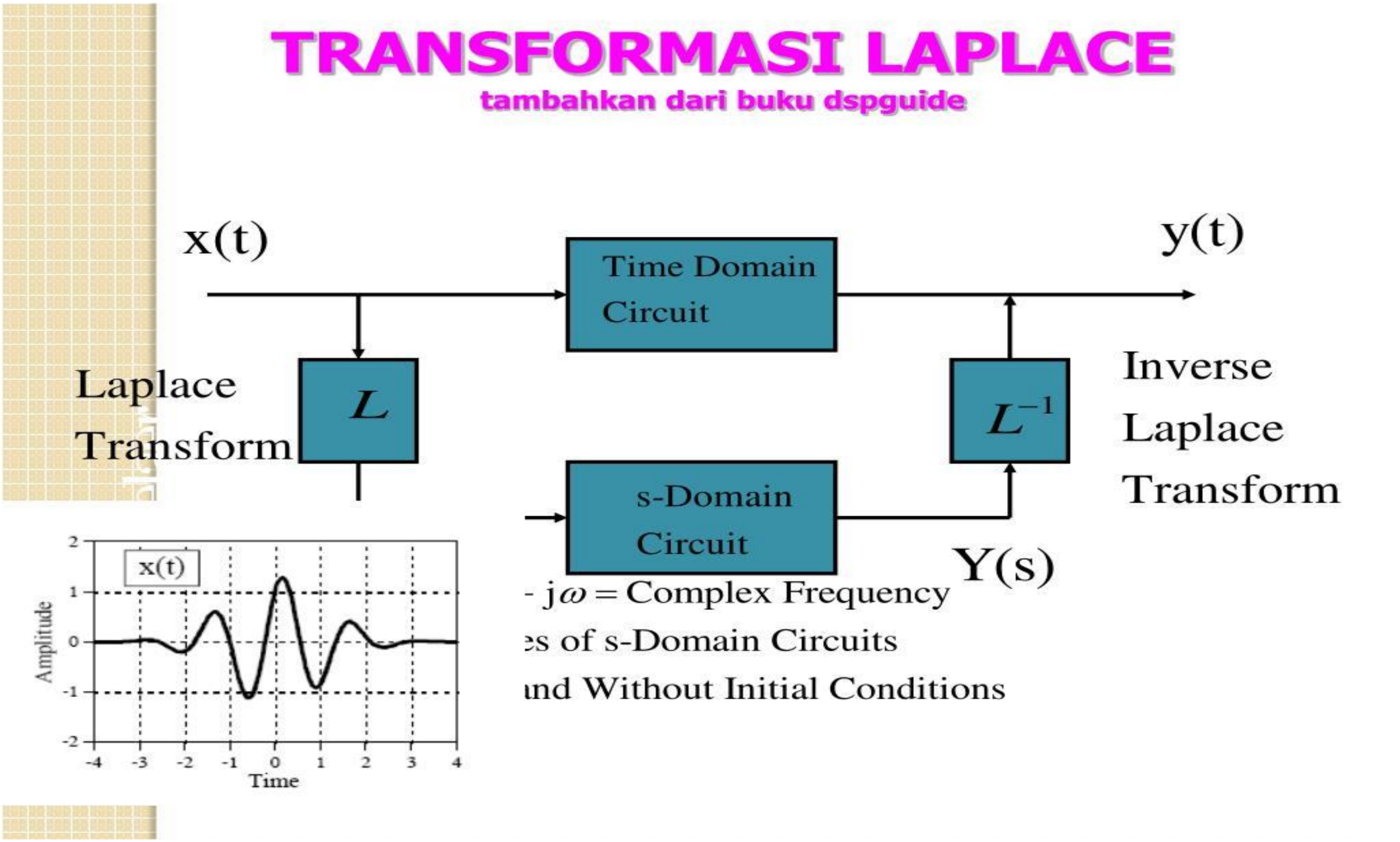
$$F(s) = \int_{\infty}^{-\infty} f(t)e^{-st}\,dt$$

LAPLACE TRANSFORM:

The techniques of Laplace transform are not only used in circuit analysis, but also in

- Proportional-Integral-Derivative (PID) controllers
- DC motor speed control systems
- DC motor position control systems
- Second order systems of differential equations (under damped, over damped and critically damped)

LAPLACE TRANSFORM:



Definition

From $\mathcal{L}\{f(t)\} = F(s)$, the value $f(t)$ is called the inverse Laplace transform of $F(s)$. In symbol,

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

where $\mathcal{L}^{-1}$ is called the inverse Laplace transform operator.

To find the inverse transform, express $F(s)$ into partial fractions which will, then, be recognizable as one of the following standard forms.

$$f(t) = \mathcal{L}^{-1}\{F(s)\}(t) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s)\, ds,$$

REGION OF CONVERGENCE OF LAPLACE TRANSFORM:

Conditions For Applicability of Laplace Transform

Laplace transforms are called integral transforms so there are necessary conditions for convergence of these transforms.

$$F(s) = \int_{\infty}^{-\infty} f(t)e^{-\sigma t}dt < \infty:$$

i.e. f must be locally integral for the interval [0, ∞) and depending on whether σ is positive or negative, e^(-σt) may be decaying or growing. For bilateral Laplace transforms rather than a single value the integral converges over a certain range of values known as Region of Convergence.

PROPERTIES OF LAPLACE TRANSFORM:

1.LINEARITY:

$$f_1(t) \xrightarrow{L.T.} F_1(s) \;\; with\ ROC = R_1$$

$$f_2(t) \xrightarrow{L.T.} F_2(s) \;\; with\ ROC = R_2$$

$$af_1(t) + bf_2(t) \xrightarrow{L.T.} aF_1(s) + bF_2(s):\ ROC = R_1 \cap R_2$$

$$\mathfrak{L}\{a \cdot f(t) + b \cdot g(t)\} = \int_{0^-}^{\infty} (a \cdot f(t) + b \cdot g(t)) * e^{-st} dt$$

$$= a \underbrace{\int_{0^-}^{\infty} f(t) * e^{-st} dt}_{F(s)} + b \underbrace{\int_{0^-}^{\infty} g(t) * e^{-st} dt}_{G(s)}$$

PROPERTIES OF LAPLACE TRANSFORM:

First Derivative Property :

The first derivative in time is used in deriving the Laplace transform for capacitor and inductor impedance. The general formula

$$u(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t)$$

Transformed to the Laplace domain using (???)

$$\mathfrak{L}\left\{\tfrac{\mathrm{d}}{\mathrm{d}t} f(t)\right\} = \int_{0^-}^{\infty} e^{-st} \frac{\mathrm{d}f(t)}{\mathrm{d}t} \mathrm{d}t = \int_{0^-}^{\infty} \underbrace{e^{-st}}_{u(t)} \underbrace{\frac{\mathrm{d}f(t)}{\mathrm{d}t}}_{v'(t)} \mathrm{d}t \Rightarrow$$

Recall integration by parts, based on the product rule, from your favorite calculus class

$$\begin{cases} \int_a^b u(t)\, v'(t)\, \mathrm{d}t = [u(t)\, v(t)]_a^b - \int_a^b u'(t)\, v(t)\, \mathrm{d}t \\ u(t) = \int_{0^-}^{t} f(\tau)\mathrm{d}\tau \Rightarrow u'(t) = f(t) \\ v'(t) = e^{-st} \Rightarrow v(t) = -\frac{1}{s} e^{-st} \end{cases}$$

PROPERTIES OF LAPLACE TRANSFORM:

Second Derivative Property :

The second derivative in time is found using the Laplace transform for the first derivative. The general formula

$$u(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t)$$

Introduce $g(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t)$

$$\begin{cases} u(t) = \frac{\mathrm{d}}{\mathrm{d}t} g(t) \\ g(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t) \end{cases}$$

Integration Property:

Determine the Laplace transform of the integral

$$u(t) = \int_{0^-}^{t} f(\tau)\mathrm{d}\tau$$

Apply the Laplace transform definition

$$\mathfrak{L}\left\{\int_{0^-}^{t} f(\tau)\tau\right\} = \int_{0^-}^{\infty} \underbrace{\left(\int_{0^-}^{t} f(\tau)\mathrm{d}\tau\right)}_{u(t)} \underbrace{e^{-st}}_{v'(t)} \mathrm{d}t \Rightarrow$$

$$\left.\begin{aligned}
\mathfrak{L}\left\{\int_{0^-}^{t} f(\tau)\tau\right\} &= \int_{0^-}^{\infty} u(t)\, v'(t)\, \mathrm{d}t \\
\int_{a}^{b} u(t)\, v'(t)\, \mathrm{d}t &= [u(t)\, v(t)]_a^b - \int_{a}^{b} u'(t)\, v(t)\, \mathrm{d}t \\
u(t) &= \int_{0^-}^{t} f(\tau)\mathrm{d}\tau \Rightarrow u'(t) = f(t) \\
v'(t) &= e^{-st} \Rightarrow v(t) = -\tfrac{1}{s}e^{-st}
\end{aligned}\right\}$$

PROPERTIES OF LAPLACE TRANSFORM:

Time Scaling:

$$x(at) \longleftrightarrow \frac{1}{|a|}X\left(\frac{s}{a}\right) \quad x(at) \longleftrightarrow \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

Proof.

$$\mathcal{L}\{x(at)\} = \int_0^\infty x(at)e^{-st}dt = \int_0^\infty x(\tau)e^{-s\frac{\tau}{a}}\frac{d\tau}{|a|} = \int_0^\infty x(\tau)e^{-\frac{s}{a}\tau}\frac{d\tau}{|a|} = \frac{1}{|a|}X\left(\frac{s}{a}\right)$$

PROPERTIES OF LAPLACE TRANSFORM:

Time shift:

$$\mathcal{L}\{x(t-t_0)u(t-t_0)\} = X(s)e^{-st_0}$$

Proof.

$$\mathcal{L}\{x(t-t_0)u(t-t_0)\} = \int_{t_0}^{\infty} x(t-t_0)e^{-st}dt = \int_{0}^{\infty} x(\tau)e^{-s(\tau+t_0)}d\tau = e^{-st_0}\int_{0}^{\infty} x(\tau)e^{-s\tau}d\tau = e^{-st_0}X(s)$$

Frequency shift:

$$\mathcal{L}\{x(t-t_0)u(t-t_0)\} = X(s)e^{-st_0}$$

Proof.

$$\mathcal{L}\{x(t-t_0)u(t-t_0)\} = \int_{t_0}^{\infty} x(t-t_0)e^{-st}dt = \int_0^{\infty} x(\tau)e^{-s(\tau+t_0)}d\tau = e^{-st_0}\int_0^{\infty} x(\tau)e^{-s\tau}d\tau = e^{-st_0}X(s)$$

PROPERTIES OF LAPLACE TRANSFORM:

Differentiation in the s-domain:

$$\mathcal{L}\{x(t-t_0)u(t-t_0)\} = X(s)e^{-st_0}$$

Proof.

$$\mathcal{L}\{x(t-t_0)u(t-t_0)\} = \int_{t_0}^{\infty} x(t-t_0)e^{-st}dt = \int_{0}^{\infty} x(\tau)e^{-s(\tau+t_0)}d\tau = e^{-st_0}\int_{0}^{\infty} x(\tau)e^{-s\tau}d\tau = e^{-st_0}X(s)$$

PROPERTIES OF LAPLACE TRANSFORM:

Initial value theorem:

$$x(0) = \lim_{s \to \infty} sX(s)$$

Proof. Consider $\dfrac{dx(t)}{dt} \longleftrightarrow sX(s) - x(0)$

$$\int_0^\infty \frac{dx(t)}{dt} e^{-st} dt = sX(s) - x(0)$$

Take $\lim\limits_{s \to \infty}$ on both side,

$$\underbrace{\lim_{s \to \infty} \int_0^\infty \frac{dx(t)}{dt} e^{-st} dt}_{0} = \lim_{s \to \infty} sX(s) - x(0)$$

$$\underbrace{x(0)}_{\text{t-domain}} = \underbrace{\lim_{s \to \infty} sX(s)}_{\text{s-domain}}$$

PROPERTIES OF LAPLACE TRANSFORM:

Final value theorem:

$$x(\infty) = \lim_{s \to 0} sX(s)$$

Proof. Take $\lim_{s \to 0}$ on both side,

$$\underbrace{\lim_{s \to 0} \int_0^{\infty} \frac{dx(t)}{dt} e^{-st} dt}_{\int_0^{\infty} \mathrm{d}x(t)t = x(\infty) - x(0)} = \lim_{s \to \infty} sX(s) - x(0)$$

$$x(\infty) = \lim_{s \to \infty} sX(s)$$

$-END-$

Relation between FOURIER and LAPLACE TRANSFORM:

The (unilateral) Laplace transform of a function g:

$$\{\mathcal{L}^* g\}(s) = \int_0^\infty e^{-st}\, dg(t).$$

The function *g* is assumed to be of bounded variation. If *g* is the ant derivative of *f*:

$$g(x) = \int_0^x f(t)\, dt$$

- The **Z-transform** converts a discrete-time signal, which is a sequence of real or complex numbers, into a complex frequency-domain representation.
- The Z-transform can be defined as either a *one-sided* or two-sided transform.

Bilateral Z-transform

The *bilateral* or *two-sided* Z-transform of a discrete-time signal x [n] is the formal power series X (z) defined as

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Z-TRANSFORM

Unilateral Z-transform

Alternatively, in cases where x [n] is defined only for n ≥ 0 , the *single-sided* or *unilateral* Z-transform is defined as

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=0}^{\infty} x[n] z^{-n}.$$

In signal processing, this definition can be used to evaluate the Z-transform of the unit impulse response of a discrete-time causal system.

Z-TRANSFORM:

Inverse Z-transform

$$x[n] = Z^{-1}\{X(z)\} = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

where *C* is a counterclockwise closed path encircling the origin and entirely in the region of convergence (ROC).

This contour can be used when the ROC includes the unit circle, which is always guaranteed when X (z)hen all the poles are inside the unit circle.

Region of convergence:

The region of convergence (ROC) is the set of points in the complex plane for which the Z-transform summation converges.

$$\mathrm{ROC} = \left\{ z : \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| < \infty \right\}$$

PROPERTIES OF ROC:

- ROC of z-transform is indicated with circle in z-plane.
- ROC does not contain any poles.
- If x(n) is a finite duration causal sequence or right sided sequence, then the ROC is entire z-plane except at $z = 0$.
- If x(n) is a finite duration anti-causal sequence or left sided sequence, then the ROC is entire z-plane except at $z = \infty$.
- If x(n) is a infinite duration causal sequence, ROC is exterior of the circle with radius a. i.e. $|z| > a$.
- If x(n) is a infinite duration anti-causal sequence, ROC is interior of the circle with radius a. i.e. $|z| < a$.
- If x(n) is a finite duration two sided sequence, then the ROC is entire z-plane except at $z = 0$ & $z = \infty$.

PROPERTIES OF Z-TRANSFORM:

LINEARITY:

$$a_1 x_1[n] + a_2 x_2[n]$$

$$a_1 X_1(z) + a_2 X_2(z)$$

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} (a_1 x_1(n) + a_2 x_2(n)) z^{-n} \\ &= a_1 \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n) z^{-n} \\ &= a_1 X_1(z) + a_2 X_2(z) \end{aligned}$$

PROPERTIES OF Z-TRANSFORM:

TIME EXPANSION:

$$x_K[n] = \begin{cases} x[r], & n = Kr \\ 0, & n \notin K\mathbb{Z} \end{cases}$$

$X(z^K)$

$$\begin{aligned} X_K(z) &= \sum_{n=-\infty}^{\infty} x_K(n) z^{-n} \\ &= \sum_{r=-\infty}^{\infty} x(r) z^{-rK} \\ &= \sum_{r=-\infty}^{\infty} x(r)(z^K)^{-r} \\ &= X(z^K) \end{aligned}$$

PROPERTIES OF Z-TRANSFORM:

TIME SHIFTING:

$$\mathcal{Z}[x[n-n_0]] = z^{-n_0}X(z),$$

$$\mathcal{Z}[x[n-n_0]] = \sum_{n=-\infty}^{\infty} x[n-n_0]z^{-n}$$

Define $m = n - n_0$

we have and $n = m + n_0$

$$\sum_{m=-\infty}^{\infty} x[m]z^{-m}z^{-n_0} = z^{-n_0}X(z)$$

$$\sum_{m=-\infty}^{\infty} x[m]z^{-m}z^{-n_0} = z^{-n_0}X(z)$$

PROPERTIES OF Z-TRANSFORM:

CONVOLUTION:

$$\mathcal{Z}[x[n] * y[n]] = X(z)Y(z),$$

The ROC of the convolution could be larger than the intersection of and , due to the possible pole-zero cancellation caused by the convolution

PROPERTIES OF Z-TRANSFORM:

Time Reversal :

$$\mathcal{Z}[x[-n]] = X(1/z) \quad ROC = 1/R_x$$

$$\mathcal{Z}[x[-n]] = \sum_{n=-\infty}^{\infty} x[-n]z^{-n} = \sum_{m=-\infty}^{\infty} x[m](\frac{1}{z})^{-m} = X(1/z)$$

PROPERTIES OF Z-TRANSFORM:

Differentiation in z-Domain :

$$\mathcal{Z}[nx[n]] = -z\,\frac{d}{dz}X(z), \qquad ROC = R_x$$

$$\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} x[n]\frac{d}{dz}(z^{-n}) = \sum_{n=-\infty}^{\infty} (-n)x[n]z^{-n-1} = \frac{-1}{z}\sum_{n=-\infty}^{\infty} nx[n]z^{-n}$$

Conjugation

$$\mathcal{Z}[x^*[n]] = X^*(z^*), \qquad ROC = R_x$$

$$X^*(z) = [\sum_{n=-\infty}^{\infty} x[n]z^{-n}]^* = \sum_{n=-\infty}^{\infty} x^*[n](z^*)^{-n}$$

PROPERTIES OF Z-TRANSFORM:

Time reversal:

$$\mathcal{Z}[x[-n]] = X(1/z) \quad ROC = 1/R_x$$

$$\mathcal{Z}[x[-n]] = \sum_{n=-\infty}^{\infty} x[-n]z^{-n} = \sum_{m=-\infty}^{\infty} x[m](\frac{1}{z})^{-m} = X(1/z)$$

PROPERTIES OF Z-TRANSFORM:

Time reversal:

$$\mathcal{Z}[x[-n]] = X(1/z) \quad ROC = 1/R_x$$

$$\mathcal{Z}[x[-n]] = \sum_{n=-\infty}^{\infty} x[-n]z^{-n} = \sum_{m=-\infty}^{\infty} x[m](\frac{1}{z})^{-m} = X(1/z)$$

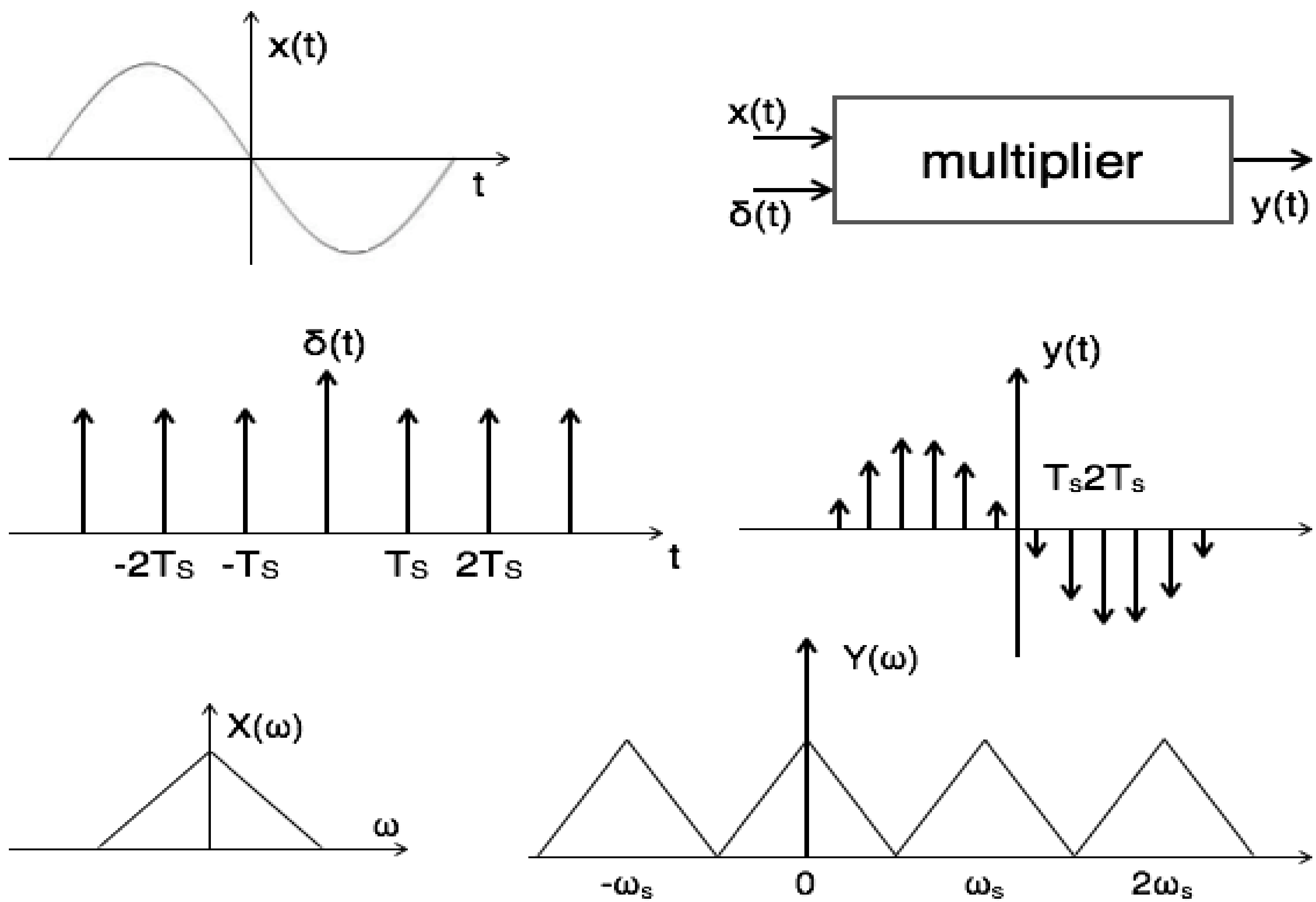
MODULE-V

Graphical and analytical proof for Band Limited Signals:

Sampling theorem: A continuous time signal can be represented in its samples and can be recovered back when sampling frequency f_s is greater than or equal to the twice the highest frequency component of message signal. i. e. $fs \geq 2fm$

Proof: Consider a continuous time signal x(t). The spectrum of x(t) is a band limited to f_m Hz i.e. the spectrum of x(t) is zero for $|\omega|>\omega_m$.Sampling of input signal x(t) can be obtained by multiplying x(t) with an impulse train δ(t) of period T_s. The output of multiplier is a discrete signal called sampled signal which is represented with y(t) in the following diagrams:

Graphical and analytical proof for Band Limited Signals:



Graphical and analytical proof for Band Limited Signals:

Here, you can observe that the sampled signal takes the period of impulse. The process of sampling can be explained by the following mathematical expression

Sampled signal $y(t) = x(t).\, \delta(t) \ldots\ldots (1)$

The trigonometric Fourier series representation of δ (t) is given by

$$\delta(t) = a_0 + \Sigma_{n=1}^{\infty}(a_n \cos n\omega_s t + b_n \sin n\omega_s t) \ldots\ldots (2)$$

Where $a_0 = \frac{1}{T_s}\int_{\frac{-T}{2}}^{\frac{T}{2}} \delta(t)dt = \frac{1}{T_s}\delta(0) = \frac{1}{T_s}$

$$a_n = \frac{2}{T_s}\int_{\frac{-T}{2}}^{\frac{T}{2}} \delta(t)\cos n\omega_s\, dt = \frac{2}{T_2}\delta(0)\cos n\omega_s 0 = \frac{2}{T}$$

$$b_n = \frac{2}{T_s}\int_{\frac{-T}{2}}^{\frac{T}{2}} \delta(t)\sin n\omega_s t\, dt = \frac{2}{T_s}\delta(0)\sin n\omega_s 0 = 0$$

Graphical and analytical proof for Band Limited Signals:

Substitute above values in equation 2.

$$\therefore\ \delta(t) = \frac{1}{T_s} + \Sigma_{n=1}^{\infty}\left(\frac{2}{T_s}\cos n\omega_s t + 0\right)$$

Substitute δ(t) in equation 1.

$$\rightarrow y(t) = x(t).\,\delta(t)$$

$$= x(t)\left[\frac{1}{T_s} + \Sigma_{n=1}^{\infty}\left(\frac{2}{T_s}\cos n\omega_s t\right)\right]$$

$$= \frac{1}{T_s}\left[x(t) + 2\Sigma_{n=1}^{\infty}(\cos n\omega_s t)x(t)\right]$$

$$y(t) = \frac{1}{T_s}\left[x(t) + 2\cos\omega_s t.\,x(t) + 2\cos 2\omega_s t.\,x(t) + 2\cos 3\omega_s t.\,x(t)\ldots\ldots\right]$$

Take Fourier transform on both sides.

Graphical and analytical proof for Band Limited Signals:

$$Y(\omega) = \frac{1}{T_s}[X(\omega) + X(\omega - \omega_s) + X(\omega + \omega_s) + X(\omega - 2\omega_s) + X(\omega + 2\omega_s) + \dots]$$

$$\therefore\ Y(\omega) = \frac{1}{T_s}\Sigma_{n=-\infty}^{\infty} X(\omega - n\omega_s) \qquad where\ n = 0, \pm 1, \pm 2, \dots$$

To reconstruct x(t), you must recover input signal spectrum X(ω) from sampled signal spectrum Y(ω), which is possible when there is no overlapping between the cycles of Y(ω).

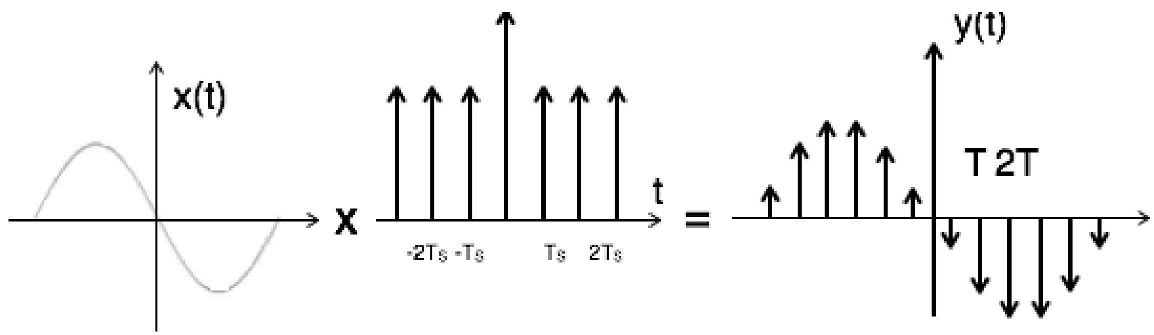
There are three types of sampling techniques:

- Impulse sampling.
- Natural sampling.
- Flat Top sampling.

Graphical and analytical proof for Band Limited Signals:

Impulse Sampling

Impulse sampling can be performed by multiplying input signal x(t) with impulse train of period 'T'. Here, the amplitude of impulse changes with respect to amplitude of input signal x(t). The output of sampler is given by



$y(t) = x(t) \times$ impulse train

$= x(t) \times \Sigma_{n=-\infty}^{\infty} \delta(t - nT)$

$y(t) = y_\delta(t) = \Sigma_{n=-\infty}^{\infty} x(nt)\delta(t - nT) \ldots \ldots 1$

Graphical and analytical proof for Band Limited Signals:

To get the spectrum of sampled signal, consider Fourier transform of equation 1 on both sides

$$Y(\omega) = \frac{1}{T}\Sigma_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

This is called ideal sampling or impulse sampling. You cannot use this practically because pulse width cannot be zero and the generation of impulse train is not possible practically.

Natural Sampling:

Natural sampling is similar to impulse sampling, except the impulse train is replaced by pulse train of period T. i.e. you multiply input signal x(t) to pulse train

Graphical and analytical proof for Band Limited Signals:

Substitute p(t) in equation 1

$$y(t) = x(t) \times p(t)$$

$$= x(t) \times \frac{1}{T} \Sigma_{n=-\infty}^{\infty} P(n\omega_s)\, e^{jn\omega_s t}$$

$$y(t) = \frac{1}{T} \Sigma_{n=-\infty}^{\infty} P(n\omega_s)\, x(t)\, e^{jn\omega_s t}$$

To get the spectrum of sampled signal, consider the Fourier transform on both sides.

$$F.T\,[y(t)] = F.T[\frac{1}{T} \Sigma_{n=-\infty}^{\infty} P(n\omega_s)\, x(t)\, e^{jn\omega_s t}]$$

$$= \frac{1}{T} \Sigma_{n=-\infty}^{\infty} P(n\omega_s)\, F.T\,[x(t)\, e^{jn\omega_s t}]$$

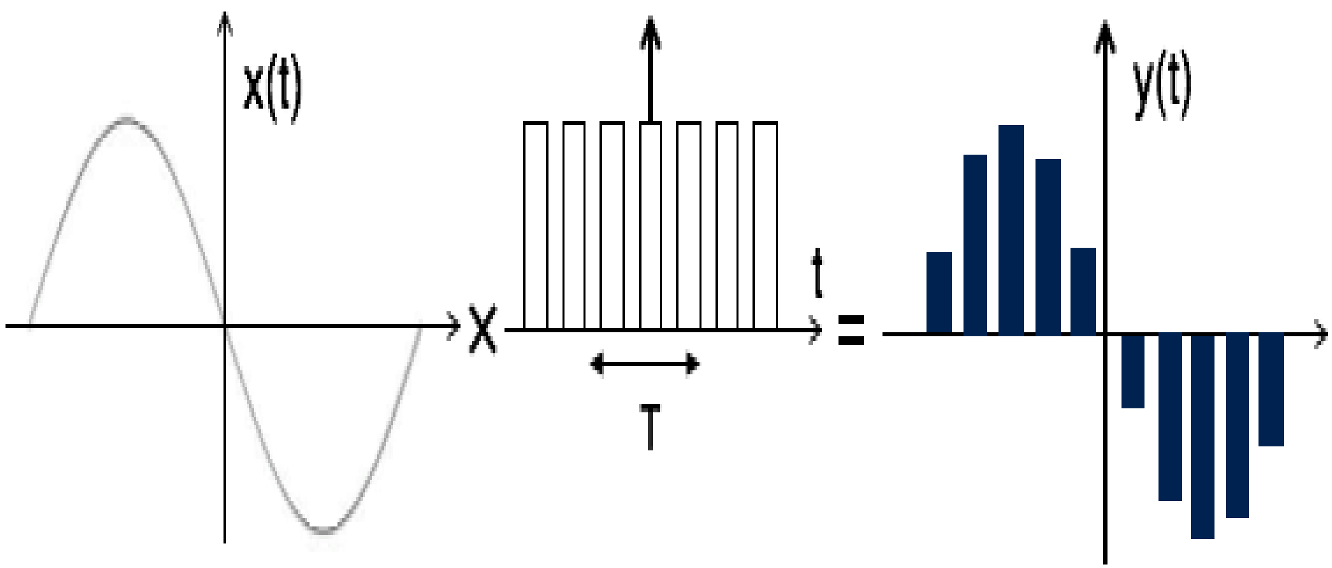
According to frequency shifting property

$$F.T\,[x(t)\, e^{jn\omega_s t}] = X[\omega - n\omega_s]$$

$$\therefore\ Y[\omega] = \frac{1}{T} \Sigma_{n=-\infty}^{\infty} P(n\omega_s)\, X[\omega - n\omega_s]$$

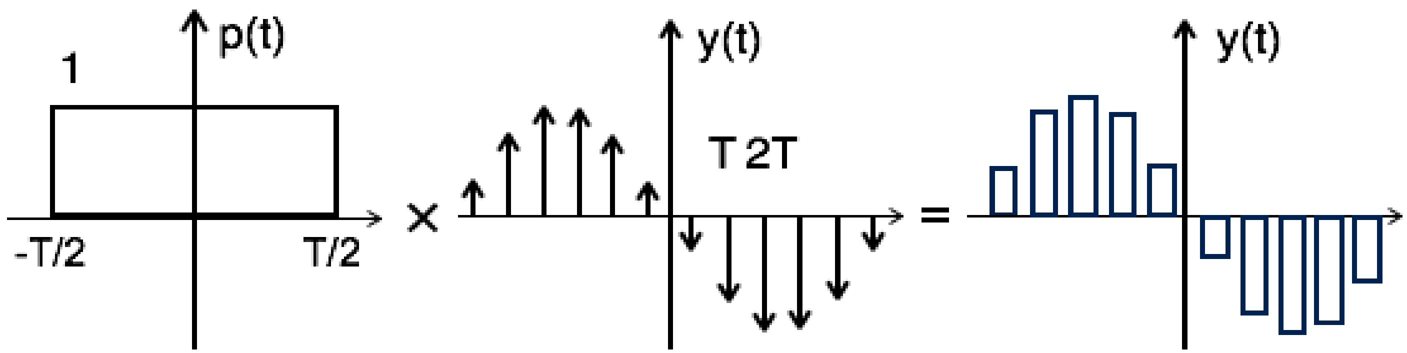
Graphical and analytical proof for Band Limited Signals:

Flat Top Sampling: During transmission, noise is introduced at top of the transmission pulse which can be easily removed if the pulse is in the form of flat top. Here, the top of the samples are flat i.e. they have constant amplitude. Hence, it is called as flat top sampling or practical sampling. Flat top sampling makes use of sample and hold circuit.



Graphical and analytical proof for Band Limited Signals:

i.e. $y(t) = p(t) \times y_\delta(t) \ldots\ldots (1)$



To get the sampled spectrum, consider Fourier transform on both sides for equation 1

$$Y[\omega] = F.T\,[P(t) \times y_\delta(t)]$$

By the knowledge of convolution property,

$$Y[\omega] = P(\omega)\,Y_\delta(\omega)$$

Here $P(\omega) = TSa(\frac{\omega T}{2}) = 2\sin\omega T/\omega$

Graphical and analytical proof for Band Limited Signals:

Nyquist Rate:

It is the minimum sampling rate at which signal can be converted into samples and can be recovered back without distortion.

Nyquist rate $f_N = 2f_m$ hz

Nyquist interval = $1/f_N = 1/2fm$ seconds.

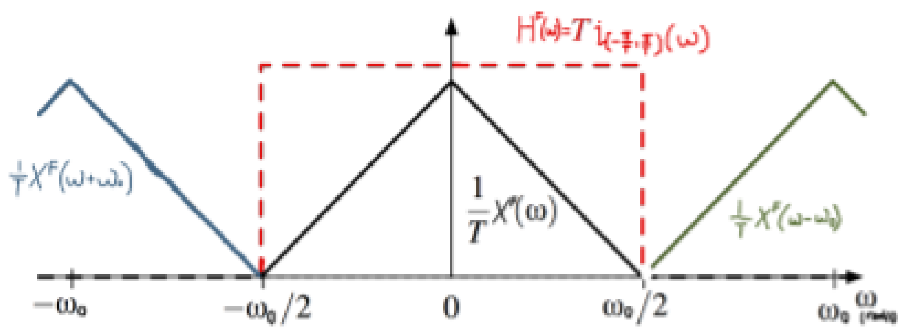
Reconstruction of signal from its samples:

Assume that the Nyquist requirement ω0 > 2ωm is satisfied.
We consider two reconstruction schemes:

- ideal reconstruction (with ideal band limited interpolation),
- reconstruction with zero-order hold.

Ideal Reconstruction: Shannon interpolation formula

$$X_p(t) = \ldots + \frac{1}{T}X^{\mathrm{F}}(\omega+\omega_0) + \frac{1}{T}X^{\mathrm{F}}(\omega) + \frac{1}{T}X^{\mathrm{F}}(\omega-\omega_0) + \ldots$$



Reconstruction of signal from its samples:

Our ideal reconstruction filter has the frequency response:

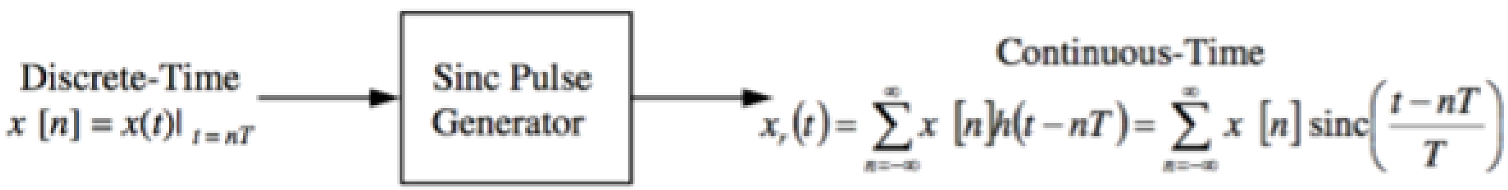
and, consequently, the impulse response

$$h(t) = \text{sinc}\left(\frac{t}{T}\right).$$

Now, the reconstructed signal is

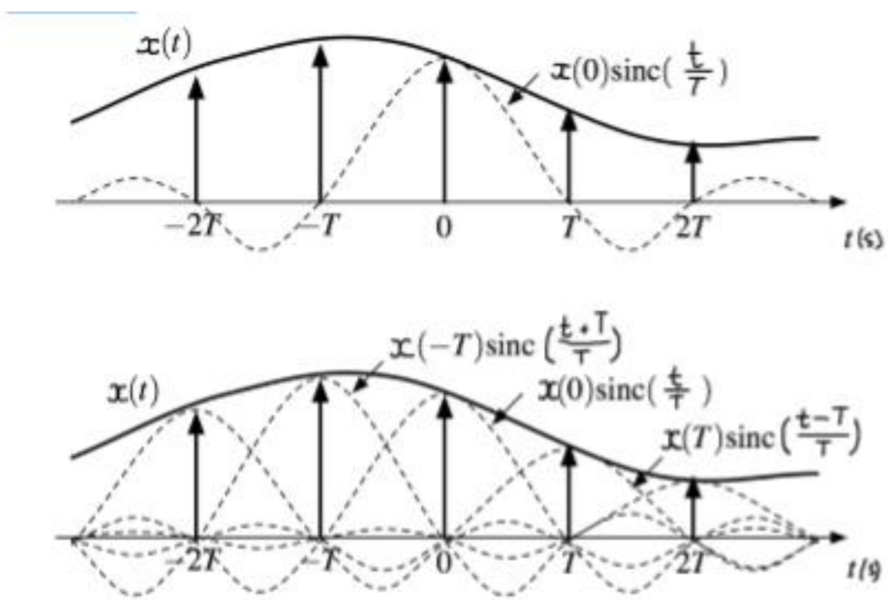
$$x(t) = \underbrace{x_p(t)}_{\text{impulse-sampled signal}} \star h(t) = \sum_{n=-\infty}^{+\infty} x(nT)\, \underbrace{\delta(t - nT) \star h(t)}_{h(t-nT),\ \text{see (3)}} = \sum_{n=-\infty}^{+\infty} x(nT)\, \text{sinc}\left(\frac{t - nT}{T}\right)$$

which is the Shannon interpolation (reconstruction) formula. The actual reconstruction system mixes continuous and discrete time.



Reconstruction of signal from its samples:

The reconstructed signal xr(t) is a train of sinc pulses scaled by the samples x[n]. • This system is difficult to implement because each sinc pulse extends over a long (theoretically infinite) time interval.



A general reconstruction filter

For the development of the theory, it is handy to consider the impulse-sampled signal xP(t) and its CTFT.

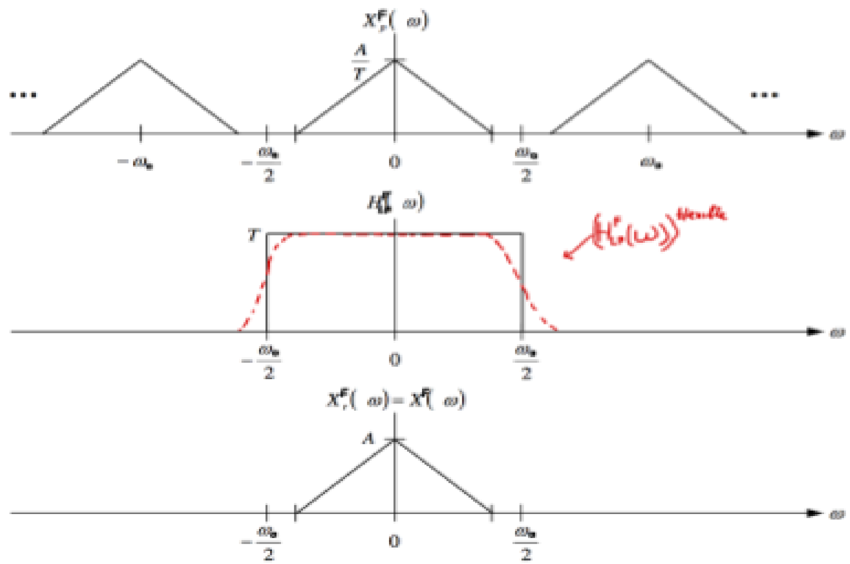


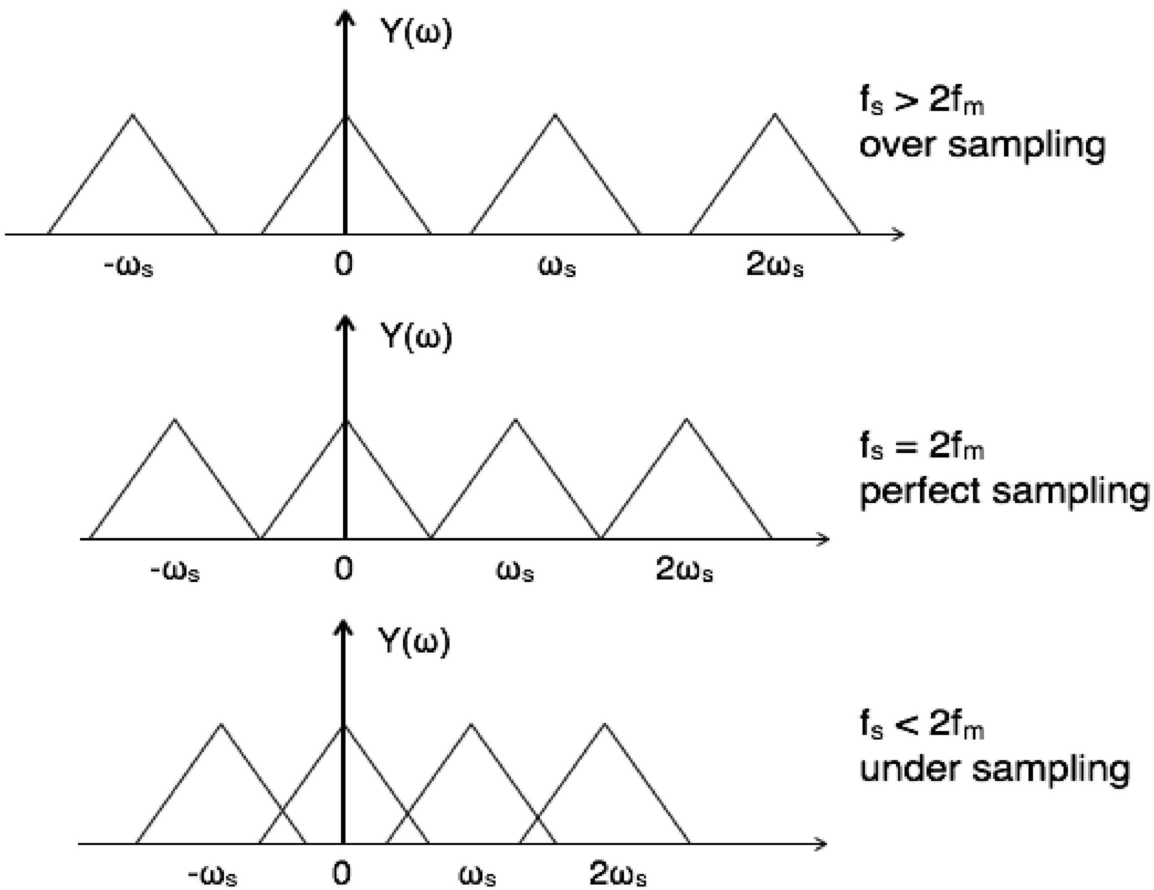
Figure : Reconstruction in the frequency domain is low pass filtering

Here, the reconstructed signal is $x_r(t)$, with CTFT

$$X_r^F(\omega) = H_{LP}^F(\omega)\, X_P^F(\omega) \overset{\text{sampling th.}}{=} H_{LP}^F(\omega)\, \frac{1}{T} \sum_{k=-\infty}^{+\infty} X^F\Big(\omega - \underbrace{\frac{2\pi k}{T}}_{k\omega_0}\Big).$$

Effect of under sampling – Aliasing :

Possibility of sampled frequency spectrum with different conditions is given by the following diagrams



Aliasing Effect:

The overlapped region in case of under sampling represents aliasing effect, which can be removed by

- considering $f_s > 2f_m$
- By using anti aliasing filters .

Samplings of Band Pass Signals:

In case of band pass signals, the spectrum of band pass signal X[ω] = 0 for the frequencies outside the range $f_1 \leq f \leq f_2$. The frequency f_1 is always greater than zero. Plus, there is no aliasing effect when $f_s > 2f_2$. But it has two disadvantages:

Samplings of Band Pass Signals:

The sampling rate is large in proportion with f_2. This has practical limitations.

The sampled signal spectrum has spectral gaps.

To overcome this, the band pass theorem states that the input signal x(t) can be converted into its samples and can be recovered back without distortion when sampling frequency $f_s < 2f_2$.

Also,

$$f_s = \frac{1}{T} = \frac{2f_2}{m}$$

Samplings of Band Pass Signals:

Where m is the largest integer < $\frac{f_2}{B}$

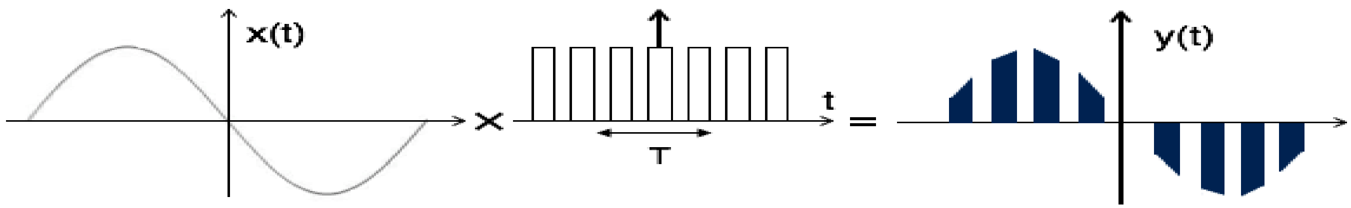
and B is the bandwidth of the signal. If f_2=KB, then

$$f_s = \frac{1}{T} = \frac{2KB}{m}$$

For band pass signals of bandwidth $2f_m$ and the minimum sampling rate f_s= 2 B = $4f_m$,

the spectrum of sampled signal is given by $Y[\omega] = \frac{1}{T}\Sigma_{n=-\infty}^{\infty} X[\omega - 2nB]$

Samplings of Band Pass Signals:



The output of sampler is

$$y(t) = x(t) \times \textbf{pulse train}$$

$$= x(t) \times p(t)$$

$$= x(t) \times \Sigma_{n=-\infty}^{\infty} P(t - nT) \ldots \ldots (1)$$

The exponential Fourier series representation of p(t) can be given as

$$p(t) = \Sigma_{n=-\infty}^{\infty} F_n e^{jn\omega_s t} \ldots \ldots (2)$$

$$= \Sigma_{n=-\infty}^{\infty} F_n e^{j2\pi n f_s t}$$

Where $F_n = \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} p(t) e^{-jn\omega_s t} dt$

$$= \frac{1}{TP}(n\omega_s)$$

Substitute F_n value in equation 2

$$\therefore p(t) = \Sigma_{n=-\infty}^{\infty} \frac{1}{T} P(n\omega_s) e^{jn\omega_s t}$$

$$= \frac{1}{T} \Sigma_{n=-\infty}^{\infty} P(n\omega_s) e^{jn\omega_s t}$$

Correlation:

Cross Correlation and Auto Correlation of Functions:

Correlation

Correlation is a measure of similarity between two signals. The general formula for correlation is

$$\int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)dt$$

There are two types of correlation:

- Auto correlation
- Cross correlation

Auto Correlation Function:

It is defined as correlation of a signal with itself. Auto correlation function is a measure of similarity between a signal & its time delayed version. It is represented with R(τ).

Consider a signals x(t). The auto correlation function of x(t) with its time delayed version is given by

$$R_{11}(\tau) = R(\tau) = \int_{-\infty}^{\infty} x(t)x(t-\tau)dt \qquad [\text{+ve shift}]$$

$$= \int_{-\infty}^{\infty} x(t)x(t+\tau)dt \qquad [\text{-ve shift}]$$

Auto Correlation Function:

Where τ = searching or scanning or delay parameter.

If the signal is complex then auto correlation function is given by

$$R_{11}(\tau) = R(\tau) = \int_{-\infty}^{\infty} x(t)x*(t-\tau)dt \qquad \text{[+ve shift]}$$

$$= \int_{-\infty}^{\infty} x(t+\tau)x*(t)dt \qquad \text{[-ve shift]}$$

Cross Correlation Function:

Cross correlation is the measure of similarity between two different signals.

Consider two signals $x_1(t)$ and $x_2(t)$. The cross correlation of these two signals *R*12(*τ*)R12(τ) is given by

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t) x_2(t-\tau)\, dt \qquad \text{[+ve shift]}$$

$$= \int_{-\infty}^{\infty} x_1(t+\tau) x_2(t)\, dt \qquad \text{[-ve shift]}$$

Cross Correlation Function:

signals are complex then

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)x_2^*(t-\tau)\,dt \qquad \text{[+ve shift]}$$

$$= \int_{-\infty}^{\infty} x_1(t+\tau)x_2^*(t)\,dt \qquad \text{[-ve shift]}$$

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t)x_1^*(t-\tau)\,dt \qquad \text{[+ve shift]}$$

$$= \int_{-\infty}^{\infty} x_2(t+\tau)x_1^*(t)\,dt \qquad \text{[-ve shift]}$$

Properties of Cross Correlation Function:

Auto correlation exhibits conjugate symmetry i.e. R (τ) = R*(-τ)

Proof: The autocorrelation of an energy signal $x(t)$ is given by

$$R(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)\,dt$$

Taking the complex conjugate, we have

$$R^*(\tau) = \int_{-\infty}^{\infty} x^*(t)\,x(t-\tau)\,dt$$

$$\therefore \quad R^*(-\tau) = \int_{-\infty}^{\infty} x^*(t)\,x(t+\tau)\,dt = R(\tau)$$

$$\therefore \quad R(\tau) = R^*(-\tau)$$

Properties of Cross Correlation Function:

Auto correlation function of energy signal at origin i.e. at τ =0 is equal to total energy of that signal, which is given as:

Proof: We have

$$R(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)\,dt$$

Putting $\tau = 0$ gives

$$R(0) = \int_{-\infty}^{\infty} x(t)x^*(t)\,dt = \int_{-\infty}^{\infty} |x(t)|^2\,dt = E$$

Properties of Cross Correlation Function:

Auto correlation function is maximum at τ =0 i.e $|R(\tau)| \leq R(0)\ \forall\ \tau$

Proof: Consider the functions $x(t)$ and $x(t+\tau)$. $[x(t) \pm x(t+\tau)]^2$ is always greater than or equal to zero since it is squared, i.e.

$$x^2(t) + x^2(t+\tau) \pm 2x(t)\,x(t+\tau) \geq 0$$

or

$$x^2(t) + x^2(t+\tau) \geq \pm 2x(t)\,x(t+\tau)$$

Integrating both the sides, we get

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt + \int_{-\infty}^{\infty} |x(t+\tau)|^2\,dt \geq 2\int_{-\infty}^{\infty} x(t)\,x(t+\tau)\,dt$$

$$\therefore \quad E + E \geq 2R(\tau) \quad \text{[If } x(t) \text{ is real valued function]}$$

$$\therefore \quad E \geq R(\tau)$$

or

$$R(0) \geq |R(\tau)| \quad (\text{Since } R(0) = E)$$

Properties of Cross Correlation Function:

Auto correlation function and energy spectral densities are Fourier transform pairs. i.e.

$$F.T[R(\tau)]=S_{XX}(\omega)$$

$$S_{XX}(\omega)= \int R(\tau)e^{-j\omega\tau}d\tau \text{ where } -\infty < \tau<\infty$$

$$R(\tau)=x(\tau)*x(-\tau)$$

Properties of Cross Correlation Function

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•Auto correlation exhibits conjugate symmetry i.e. $R_{12}(\tau)=R^*_{21}(-\tau)$.

•Cross correlation is not commutative like convolution i.e.

$$R_{12}(\tau)\neq R_{21}(-\tau)$$

•If $R_{12}(0) = 0$ means, if $\int x_1(t)x^*_2(t)dt=0$ over interval$(-\infty,\infty)$, then the two signals are said to be orthogonal.

•Cross correlation function corresponds to the multiplication of spectrums of one signal to the complex conjugate of spectrum of another signal. i.e.

$$R_{12}(\tau)\leftarrow\rightarrow X_1(\omega)X^*_2(\omega)$$

This also called as correlation theorem

Energy Density Spectrum:

Energy spectral density describes how the energy of a signal or a time series is distributed with frequency. Here, the term energy is used in the generalized sense of signal processing; Energy density spectrum can be calculated using the formula:

$$E = \int_{-\infty}^{\infty} |x(f)|^2 df$$

Properties of ESD: The following are the properties of ESD.

1. The total area under the energy density spectrum is equal to the total energy of the signal.

   i.e. $$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\omega)\, d\omega = \int_{-\infty}^{\infty} \psi(f)\, df$$

2. If $x(t)$ is the input to an LTI system with impulse response $h(t)$, then the input and output ESD functions are related as:

   $$\psi_y(\omega) = |H(\omega)|^2\, \psi_x(\omega)$$

   or $$\psi_y(f) = |H(f)|^2\, \psi_x(f)$$

3. The autocorrelation function $R(\tau)$ and ESD $\psi(\omega)$ form a Fourier transform pair, i.e.

   $$R(\tau) \longleftrightarrow \psi(\omega)$$

   or $$R(\tau) \longleftrightarrow \psi(f)$$

Power Density Spectrum:

The above definition of energy spectral density is suitable for transients (pulse-like signals) whose energy is concentrated around one time window; then the Fourier transforms of the signals generally exist. For continuous signals over all time, such as stationary processes, one must rather define the *power spectral density* (PSD); this describes how power of a signal or time series is distributed over frequency, as in the simple example given previously. Here, power can be the actual physical power, or more often, for convenience with abstract signals, is simply identified with the squared value of the signal.
Power density spectrum can be calculated by using the formula:

$$P = \Sigma_{n=-\infty}^{\infty} |C_n|^2$$

Power Density Spectrum:

The spectrum of a real valued process (or even a complex process using the above definition) is real and an even function of frequency:

$$S_{xx}(-\omega) = S_{xx}(\omega).$$

If the process is continuous and purely in deterministic, the auto covariance function can be reconstructed by using the Inverse Fourier transform

•The PSD can be used to compute the variance (net power) of a process by integrating over frequency:

$$\mathrm{Var}(X_n) = \frac{1}{\pi}\int_0^{\infty} S_{xx}(\omega)\, d\omega.$$

Relation between Autocorrelation Function and Energy/Power Spectral Density Function:

Relation between Autocorrelation Function and Energy Spectral Density Function

The autocorrelation function $R(\tau)$ and energy spectral density function $\psi(\omega)$ form a Fourier transform pair, i.e.

$$R(\tau) \longleftrightarrow \psi(\omega)$$

Proof: The autocorrelation of a function $x(t)$ is given as:

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

Replacing $x^*(t-\tau)$ by its inverse transform, we have

$$R(\tau) = \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega(t-\tau)}\, d\omega\right]^* dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t) \left[\int_{-\infty}^{\infty} X^*(\omega)\, e^{-j\omega(t-\tau)}\, d\omega\right] dt$$

Interchanging the order of integration, we have

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega) \left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt\right] e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\, X(\omega)\, e^{j\omega\tau}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\omega)\, e^{j\omega\tau}\, d\omega \qquad [\text{since } |X(\omega)|^2 = \psi(\omega)]$$

$$= F^{-1}[\psi(\omega)]$$

$$\therefore \quad \psi(\omega) = F[R(\tau)]$$

This proves that $R(\tau)$ and $\psi(\omega)$ form a Fourier transform pair.

$$R(\tau) \longleftrightarrow \psi(\omega)$$

Relation between Autocorrelation Function and Energy/Power Spectral Density Function:

Relation between Autocorrelation Function and Power Spectral Density Function

The autocorrelation function $R(\tau)$ and the power spectral density (PSD), $S(\omega)$ of a power signal form a Fourier transform pair, i.e.

$$R(\tau) \longleftrightarrow S(\omega)$$

Proof: The autocorrelation function of a power (periodic) signal $x(t)$ in terms of Fourier series coefficients is given as:

$$R(\tau) = \sum_{n=-\infty}^{\infty} C_n C_{-n} e^{jn\omega_0 \tau}$$

where C_n and C_{-n} are the exponential Fourier series coefficients.

$$\therefore \quad R(\tau) = \sum_{n=-\infty}^{\infty} |C_n|^2 e^{jn\omega_0 \tau}$$

Taking Fourier transform on both sides, we have

$$\mathrm{F}[R(\tau)] = \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} |C_n|^2 e^{jn\omega_0 \tau} \right) e^{-j\omega\tau} \, d\tau$$

Interchanging the order of integration and summation, we get

$$\mathrm{F}[R(\tau)] = \sum_{n=-\infty}^{\infty} |C_n|^2 \int_{-\infty}^{\infty} e^{-j\tau(\omega - n\omega_0)} \, d\tau$$

$$= 2\pi \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(\omega - n\omega_0) = \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(f - nf_0)$$

The RHS is the PSD $S(\omega)$ or $S(f)$ of the periodic function $x(t)$.

$$\therefore \quad \mathrm{F}[R(\tau)] = S(\omega) \quad [\text{or } S(f)]$$

and

$$\mathrm{F}^{-1}[S(\omega)] \quad \{\text{or } \mathrm{F}^{-1}[S(f)]\} = R(\tau)$$

i.e.

$$R(\tau) \longleftrightarrow S(\omega) \quad [\text{or } S(f)]$$

Relation between Autocorrelation Function and Energy/Power Spectral Density Function:

Relation between Convolution and Correlation:

The convolution of $x_1(t)$ and $x_2(-t)$ is given by

$$x_1(t) * x_2(-t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(\tau - t)\, d\tau$$

Replacing the dummy variable τ in the above integral by another variable n, we have

$$x_1(t) * x_2(-t) = \int_{-\infty}^{\infty} x_1(n)\, x_2(n - t)\, dn$$

Changing the variable from t to τ, we get

$$x_1(\tau) * x_2(-\tau) = \int_{-\infty}^{\infty} x_1(n)\, x_2(n - \tau)\, dn = R_{12}(\tau)$$

Hence $\quad R_{12}(\tau) = x_1(t) * x_2(-t)\big|_{t=\tau}$

Similarly, $\quad R_{21}(\tau) = x_2(t) * x_1(-t)\big|_{t=\tau}$