SPACE MECHANICS

B. Tech VII semester (Autonomous IARE R-16)

BY

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UNIT- I

INTRODUCTION TO SPACE MECHANICS

- **Basic Concepts**
- **The Solar System**
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- **The Celestial Sphere**
- **The Ecliptic And Motion Of Vernal Equinox**
- **Sidereal Time, Solar Time**
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- Expansions in elliptic motion
- Orbital Elements
- Relation between orbital elements and position and velocity
- Launch vehicle ascent trajectories
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- Equations of Motion
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- Constant tangential thrust (Characteristics of the motion)
- Linearization of the equations of motion
- Performance analysis

UNIT –I
INTRODUCTION TO SPACE MECHANICS

Solar System

- What’s in Our Solar System?
- Classify the planets of our solar system.
- Outer plants are
- Small rocky planets (Mercury, Venus, Earth, Mars, and Pluto)
- Gas giants (Jupiter, Saturn, Uranus, and Neptune)
- What are the difference between starts and planets?
- Characteristics of Small Rocky Planets
- Characteristics of Gas Giants
- the Sun
- The natural Satellites (moon)
- Detail about each planets
- Meteorite vs. Meteoroid
- Comets
- Kepler’s Law
- Bode’s Law

What’s in Our Solar System?

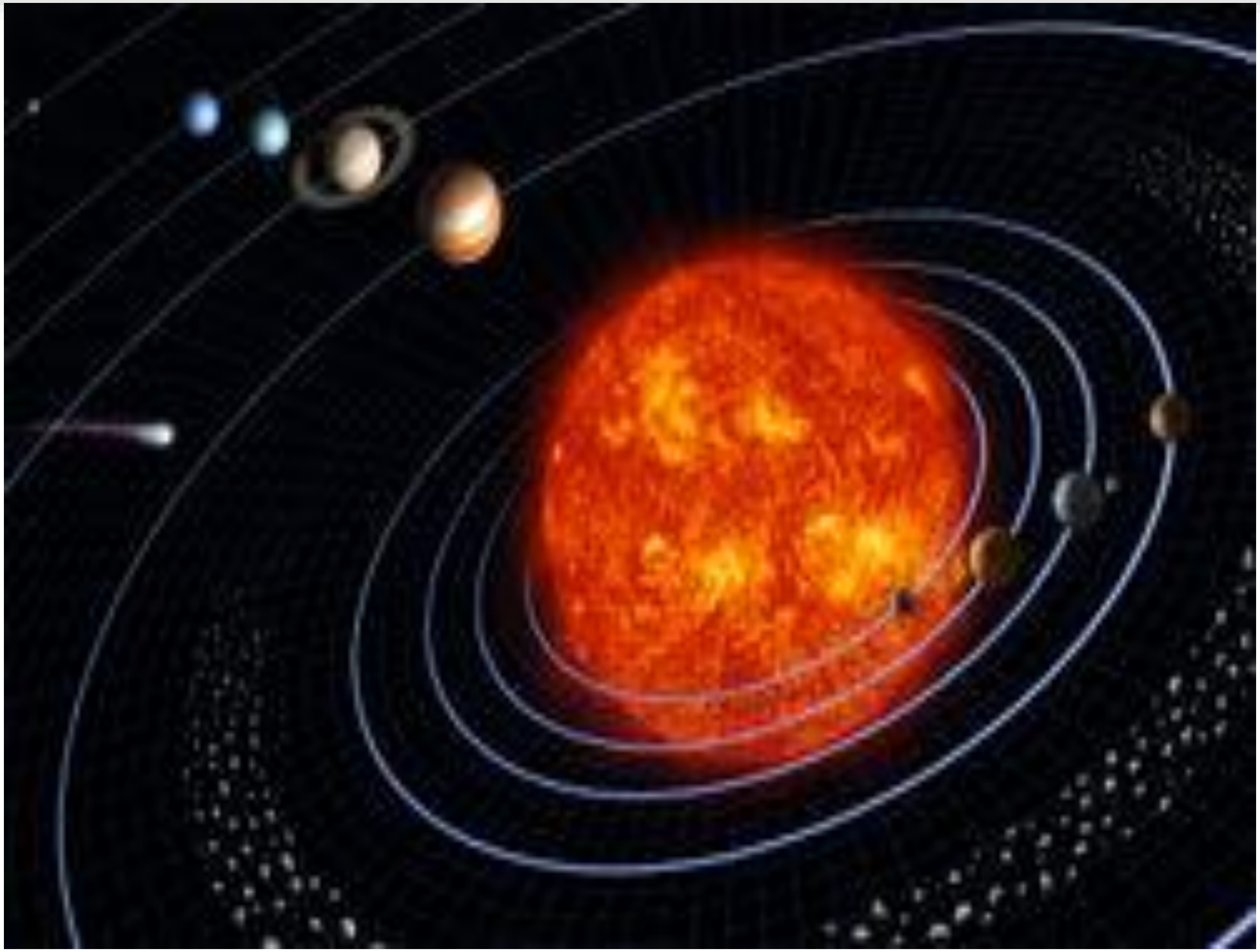
The Solar System

- Our Solar System consists of a central star (the Sun), the Eight planets orbiting the sun, moons, asteroids, comets, meteors, interplanetary gas, dust, and all the “space” in between them.

- The nine planets of the Solar System are named for Greek and Roman Gods and Goddesses.

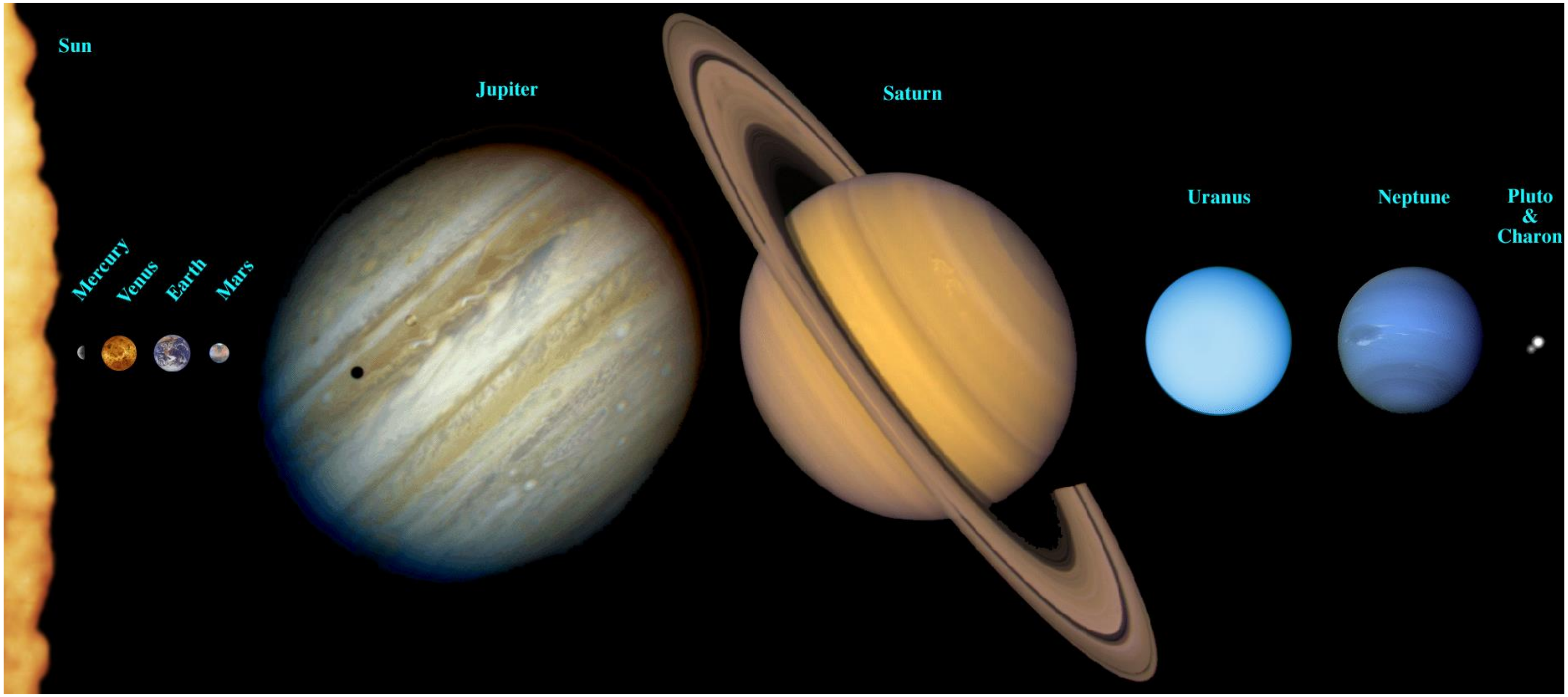
Inner and Outer Planets

- Inner Planets:
  - Mercury
  - Venus
  - Earth
  - Mars
- Outer Planets
  - Jupiter
  - Saturn
  - Uranus
  - Neptune
  - Pluto



Solar System A Pictorial View

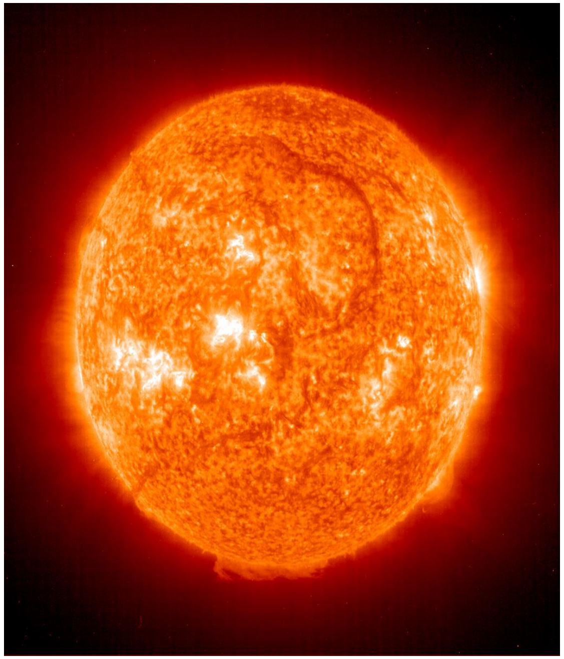
The Relative Size of the Planets



The Relative Size of the Planets in Solar System

The Sun

- The sun's energy comes from nuclear fusion (where hydrogen is converted to helium) within its core. This energy is released from the sun in the form of heat and light.
- Remember: Stars produce light. Planets reflect light.
- A star's temperature determines its "color." The coldest stars are red. The hottest stars are blue.



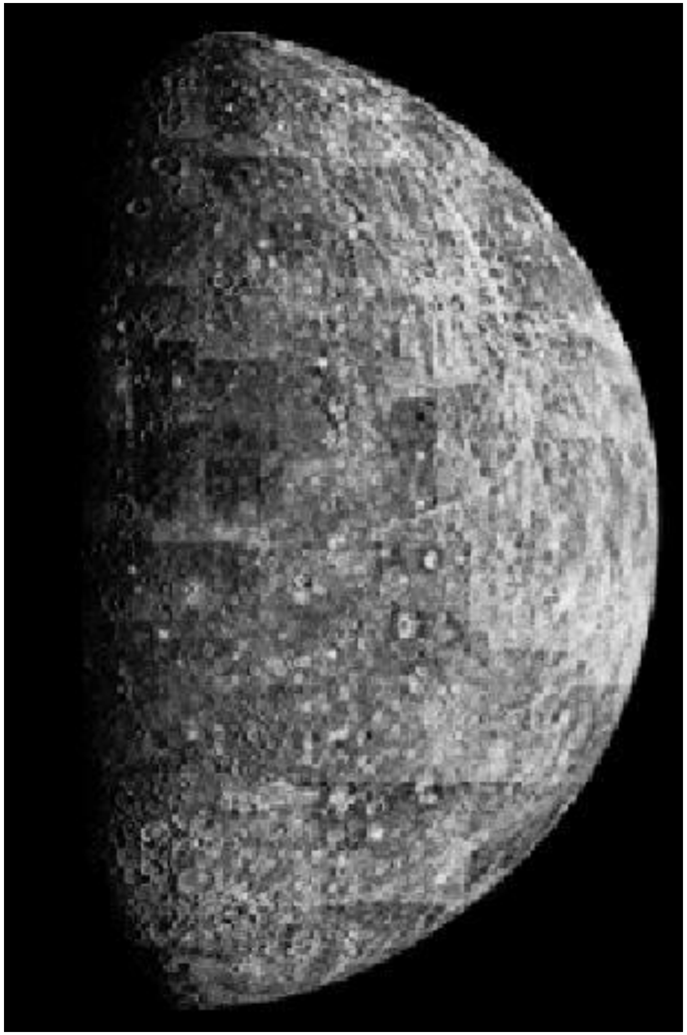
The 8 Planets of the Solar System

- **Planets are categorized according to <u>composition</u> and <u>size</u>. There are two main categories of planets:**
  - **<u>small rocky planets</u> (Mercury, Venus, Earth, Mars, and Pluto)**
  - **<u>gas giants</u> (Jupiter, Saturn, Uranus, and Neptune)**

Characteristics of Small Rocky Planets

- They are made up mostly of rock and metal.
- They are very heavy.
- They move slowly in space.
- They have no rings and few moons (if any).
- They have a diameter of less than 13,000 km.

Mercury



- Mercury has a revolution period of 88 days. Mercury has extreme temperature fluctuations, ranging from 800°F (daytime) to -270°F (night time).
- Even though it is the closest planet to the sun, Scientists believe there is ICE on Mercury! The ice is protected from the sun's heat by crater shadows.

Venus

- Venus is the brightest object in the sky after the sun and moon because its atmosphere reflects sunlight so well. People often mistake it for a star.
- Its maximum surface temperature may reach 900°F.
- Venus has no moons and takes 225 days to complete an orbit.

Earth

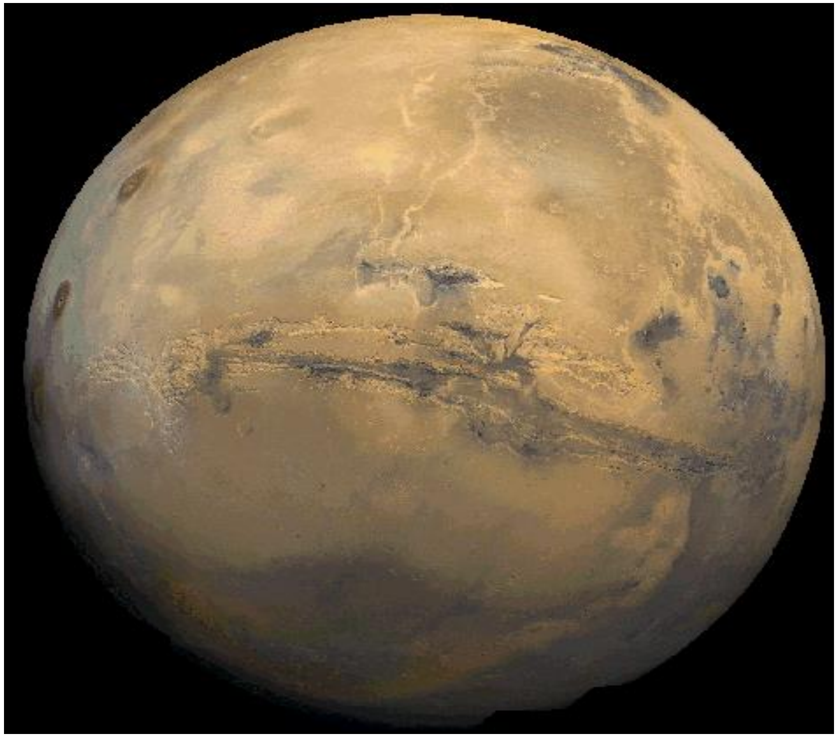
- Earth is the <u>only</u> planet known to support living organisms.
- Earth's surface is composed of 71% water.
  - Water is necessary for life on Earth.
  - The oceans help maintain Earth's stable temperatures.
- Earth has one moon and an oxygen rich atmosphere.

Earth’s Moon

- It takes the moon approximately 29 days to complete one rotation. The same side of the moon always faces us.
- The moon’s surface is covered in dust and rocky debris from meteor impacts. It has no water or atmosphere.
- The moon reflects light from the sun onto the earth’s surface.

Mars

- Like Earth, Mars has ice caps at its poles.
- Mars has the largest volcano in our solar system: Olympus Mons. Olympus Mons is approximately 15 miles high.
- Mars appears red because of iron oxide, or rust, in its soil.
- Mars has two moons and takes about two years to complete an orbit.



Pluto

- Pluto has only one moon and takes about 249 years to orbit the sun.
- Part of Pluto's orbit passes inside that of Neptune, so at times Neptune is the planet farthest from the sun.
- Pluto was located and named in 1930, but today Pluto is no longer considered a planet.

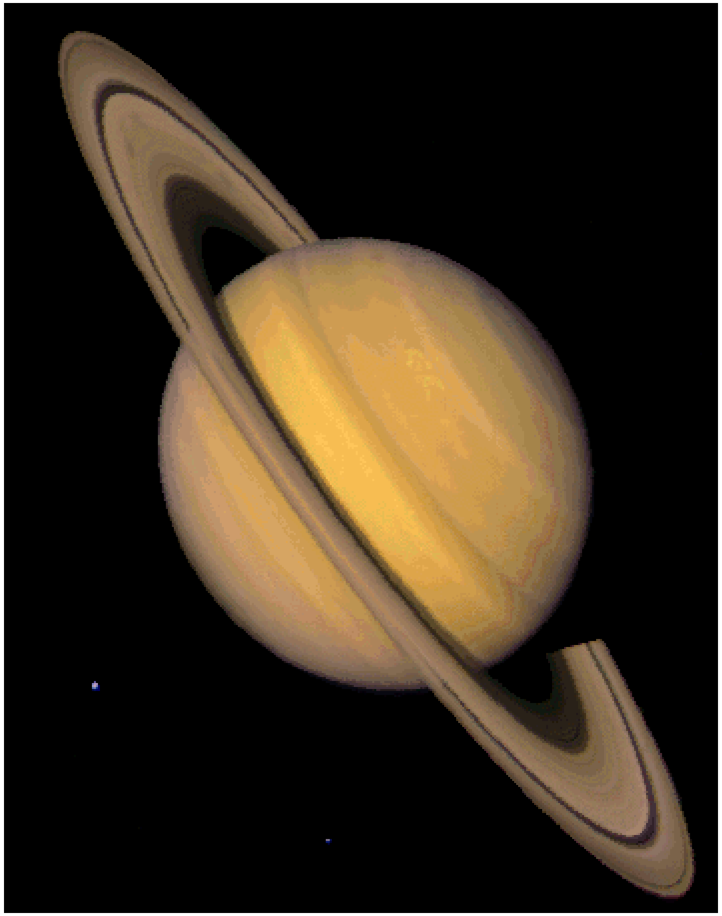
Characteristics of Gas Giants

- **They are made up mostly of gases (primarily hydrogen & helium).**
- **They are very light for their size.**
- **They move quickly in space.**
- **They have rings and many moons.**
- **They have a diameter of less than 48,000 km**

Jupiter

- Jupiter is the largest and most massive planet.
- It's diameter is 11 times bigger than that of the Earth's.
- It takes about 12 years for Jupiter to orbit the sun.
- Jupiter has 16 known moons.

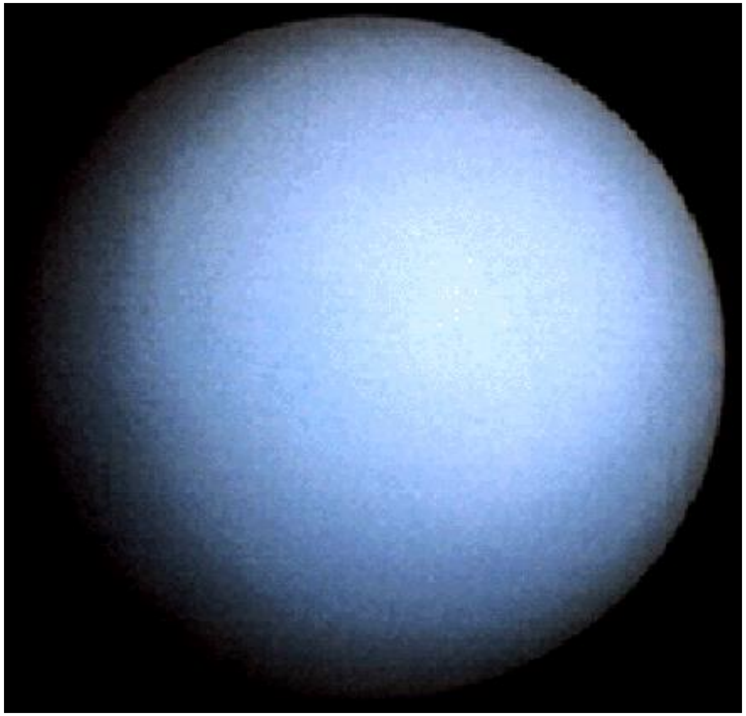
Saturn



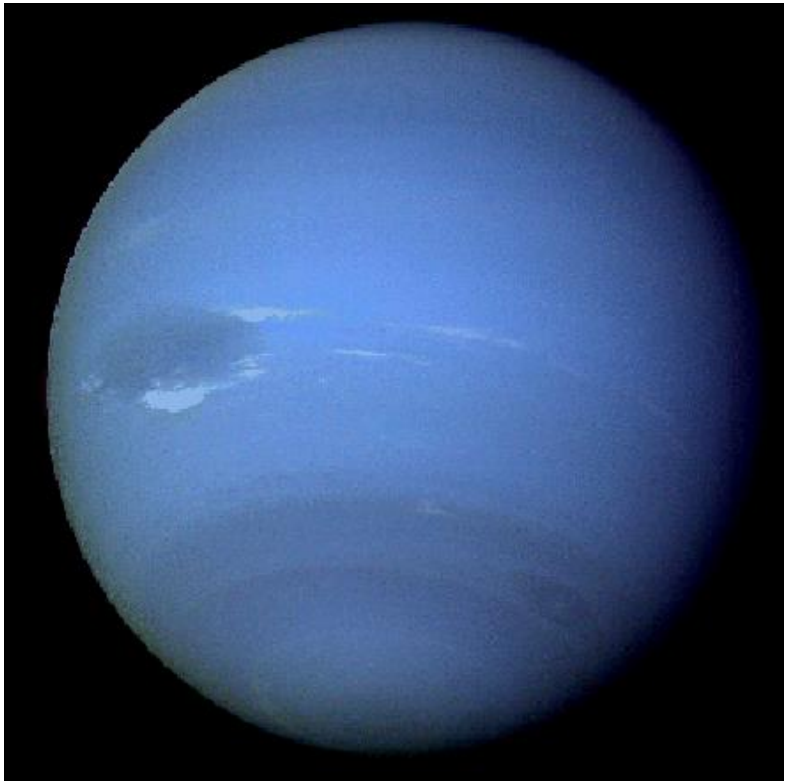
- Saturn is composed almost entirely of hydrogen and helium.
- Saturn has many rings made of ice. Saturn's rings are very wide. They extend outward to about 260,000 miles from the surface but are less than 1 mile thick.
- Saturn has 18 known moons, some of which orbit inside the rings!
- It takes Saturn about 30 years to orbit the sun.

Uranus

- Uranus is blue in color due to methane gas in its atmosphere.
- Uranus has 11 dark rings surrounding it.
- Uranus has 21 known moons and takes 84 years to complete one orbit.

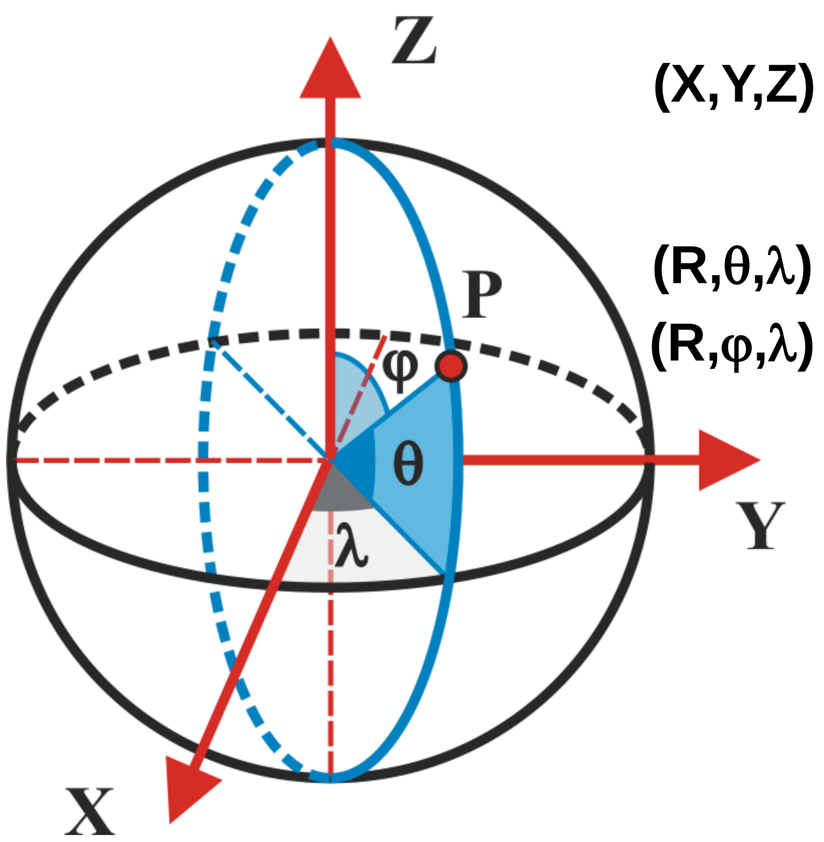


Neptune



- Neptune has the fastest winds in the solar system: up to 2,000 km/hr.
- Neptune is also blue in color due to methane gas in its atmosphere.
- Neptune takes 165 years to orbit the sun and has 8 moons.

COORDINATE SYSTEMS



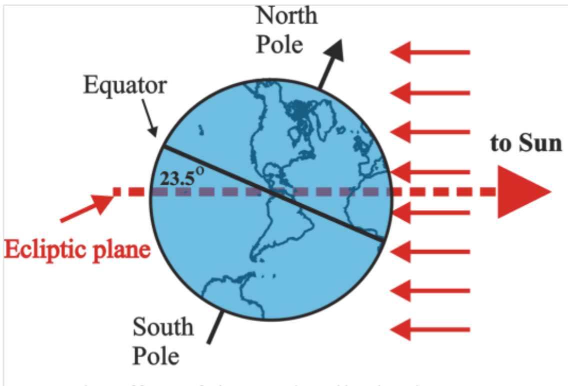
Origin?

- Center of Earth
- Sun or a Star
- Center of a planetary body

Reference Axes

- Axis of rotation or revolution
- Earth spin axis
  - Equatorial Plane
- Plane of the Earth's orbit around the Sun
  - Ecliptic Plane
- Need to pick two axes and then 3rd one is determined

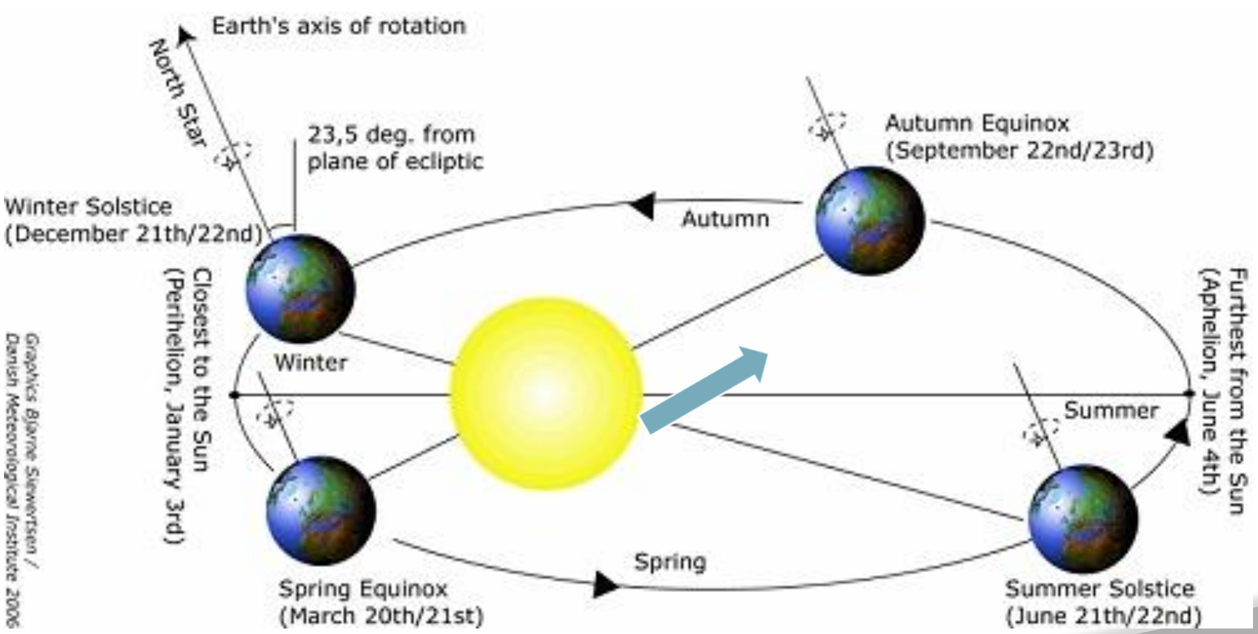
Ecliptic and Equatorial Planes



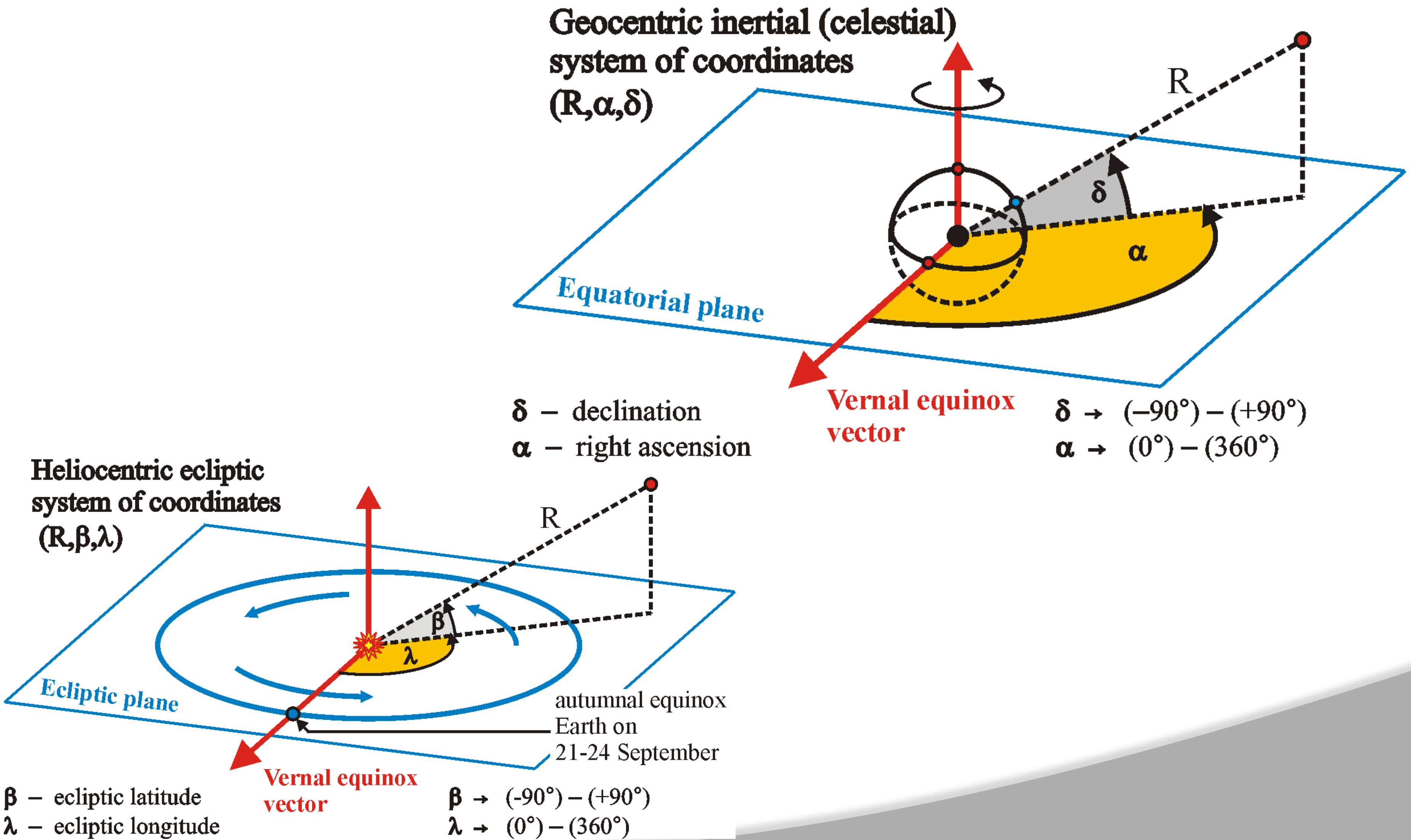
Obliquity of the Ecliptic = 23.44 °

<u>Vernal Equinox vector</u>

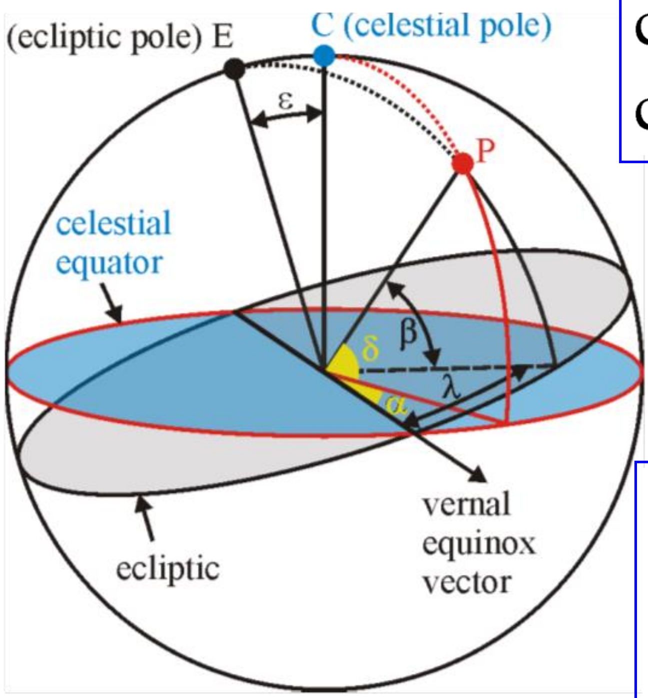
- Earth to Sun on March 21st
- Planes intersect @ Equinox



Inertial Coordinates



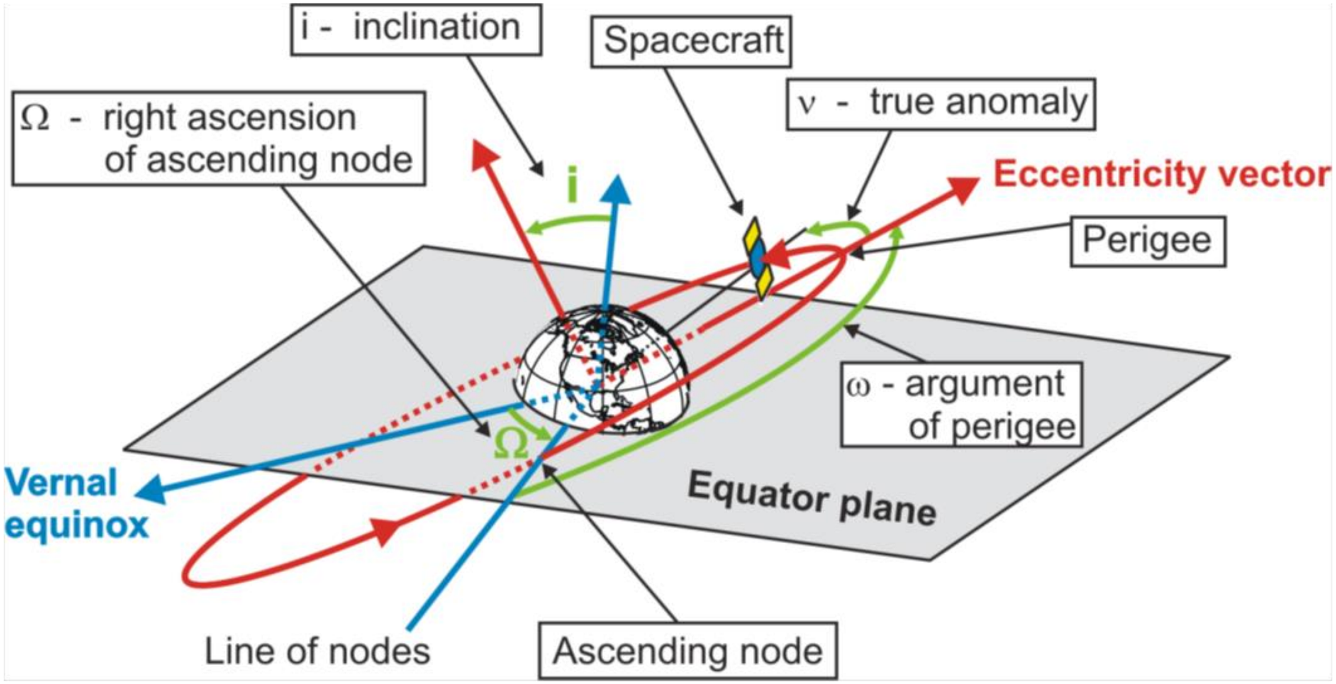
Relationship between Coordinate Frames



$$\sin\beta = \sin\delta\,\cos\varepsilon - \cos\delta\,\sin\varepsilon\,\sin\alpha$$
$$\cos\beta\,\cos\lambda = \cos\delta\,\cos\alpha$$
$$\cos\beta\,\sin\lambda = \sin\delta\,\sin\varepsilon + \cos\delta\,\cos\varepsilon\,\sin\alpha$$

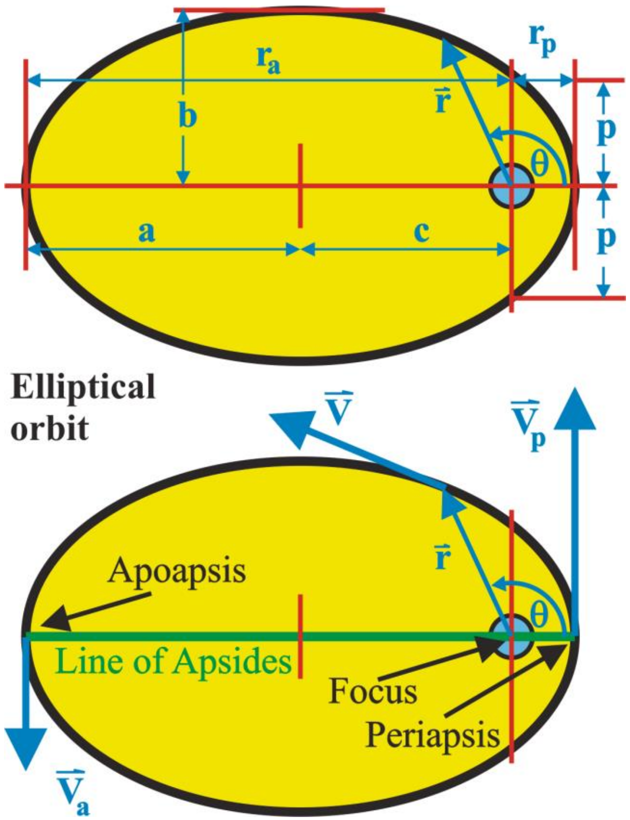
$$\sin\delta = \sin\beta\,\cos\varepsilon + \cos\beta\,\sin\varepsilon\,\sin\lambda$$
$$\cos\delta\,\cos\alpha = \cos\beta\,\cos\lambda$$
$$\cos\delta\,\sin\alpha = -\sin\beta\,\sin\varepsilon + \cos\beta\,\cos\varepsilon\,\sin\lambda$$

Orbital Elements



a - semi-major axis
e - eccentricity
i - inclination
Ω - right ascension of ascending node
ω - argument of perigee
ν - true anomaly

Properties of Orbits



e = 0	→	circle
e < 1	→	ellipse
e = 1	→	parabola
e > 1	→	hyperbola

- **a** is the semi-major axis;
- **b** is the semi-minor axis;
- $\mathbf{r_{MAX} = r_a}$, $\mathbf{r_{MIN} = r_p}$ are the maximum and minimum radius-vectors;
- **c** is the distance between the focus and the center of the ellipse;
- **e = c/a** is eccentricity

$$\left(\frac{b}{a}\right)^2 = 1 - e^2$$

2p is the latus rectum
(*latus* = *side* and *rectum* = *straight*)
p — semi-latus rectum or semi parameter

- **A = ab** is the area of the ellipse

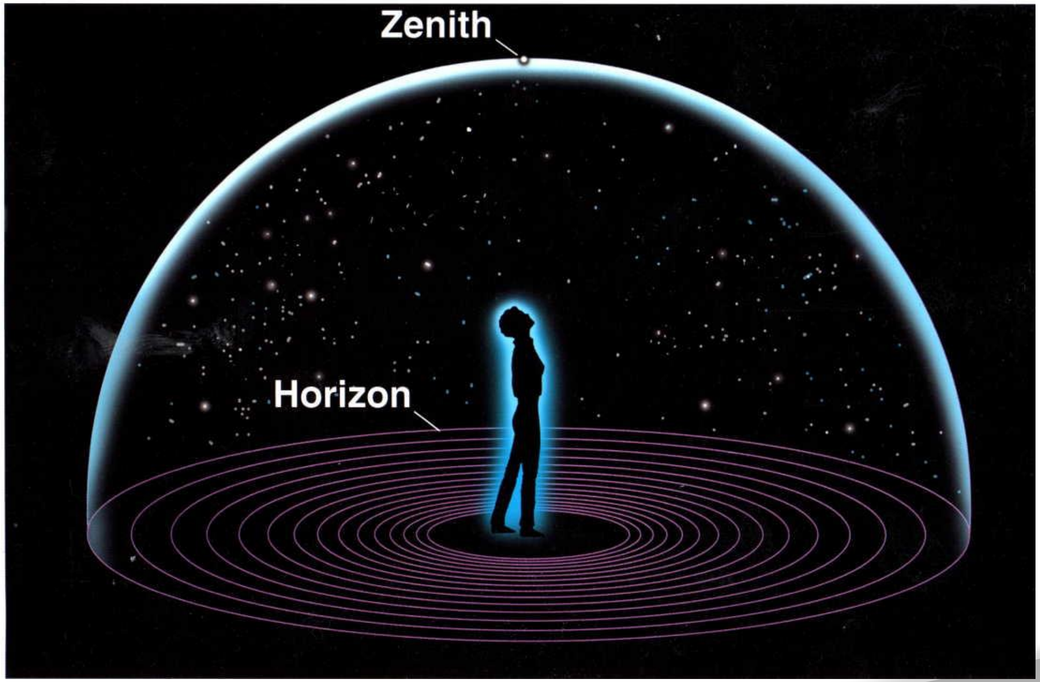
$$\frac{p}{r} = 1 + e\cos\theta$$

Why do we need all this?

- **Launch into desired orbit**
  - Launch window, inclination
  - Ground coverage (ground track/swath)
  - LEO/GEO
  - Purpose of mission?
- **Orbital Manoeuvers**
  - Feasible trajectories
  - Minimize propulsion required
  - Station keeping
  - Tracking, Prediction
- **Interplanetary Transfers**
  - Hyperbolic orbits
  - Changing reference frames
  - Orbital insertion
- **Rendezvous/Proximity Operations**
  - Relative motion
  - Orbital dynamics

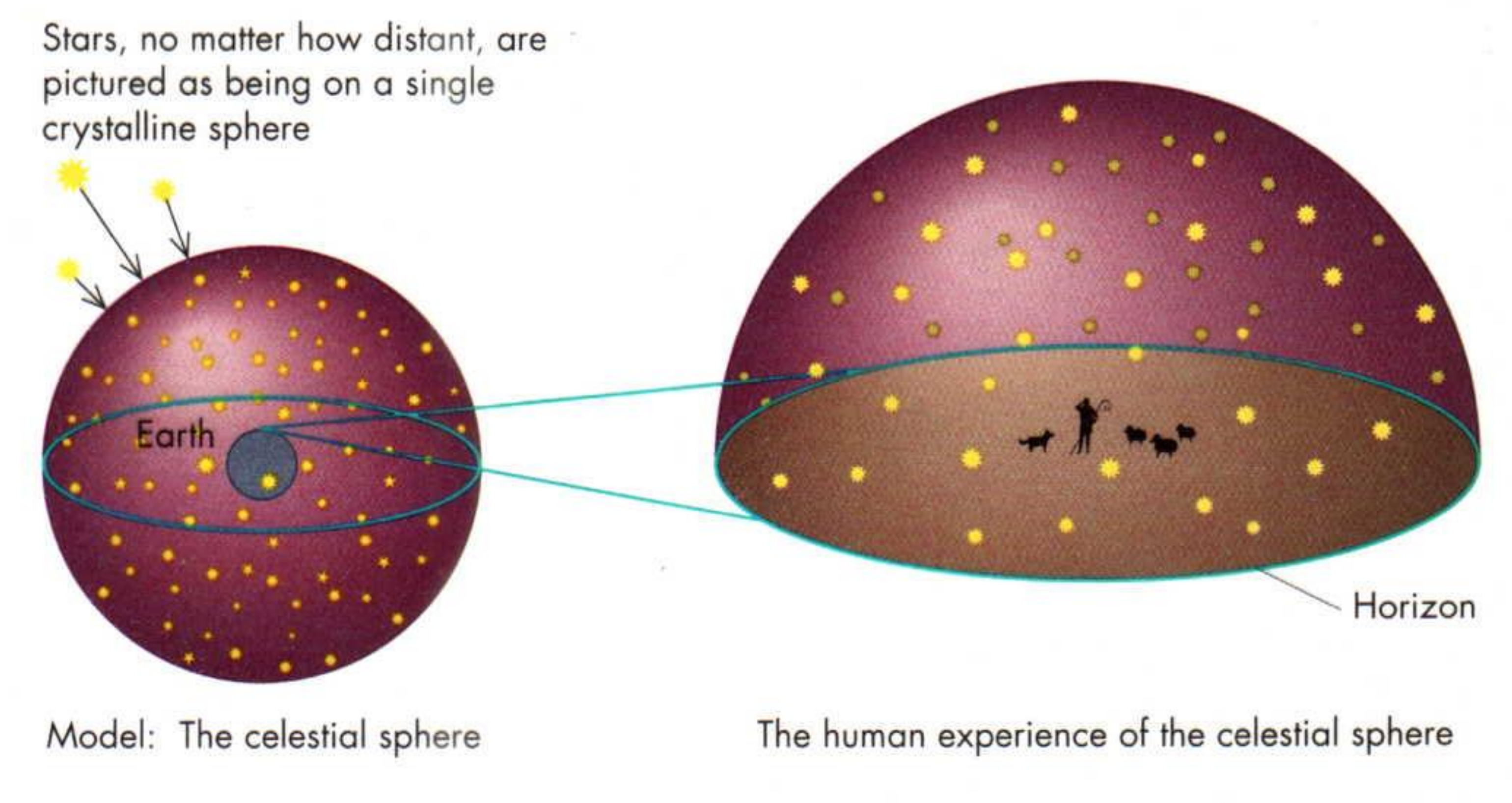
The Sky from Here

From the ground, the sky looks like a big dome above us. Both the “zenith” and horizon are locally defined.



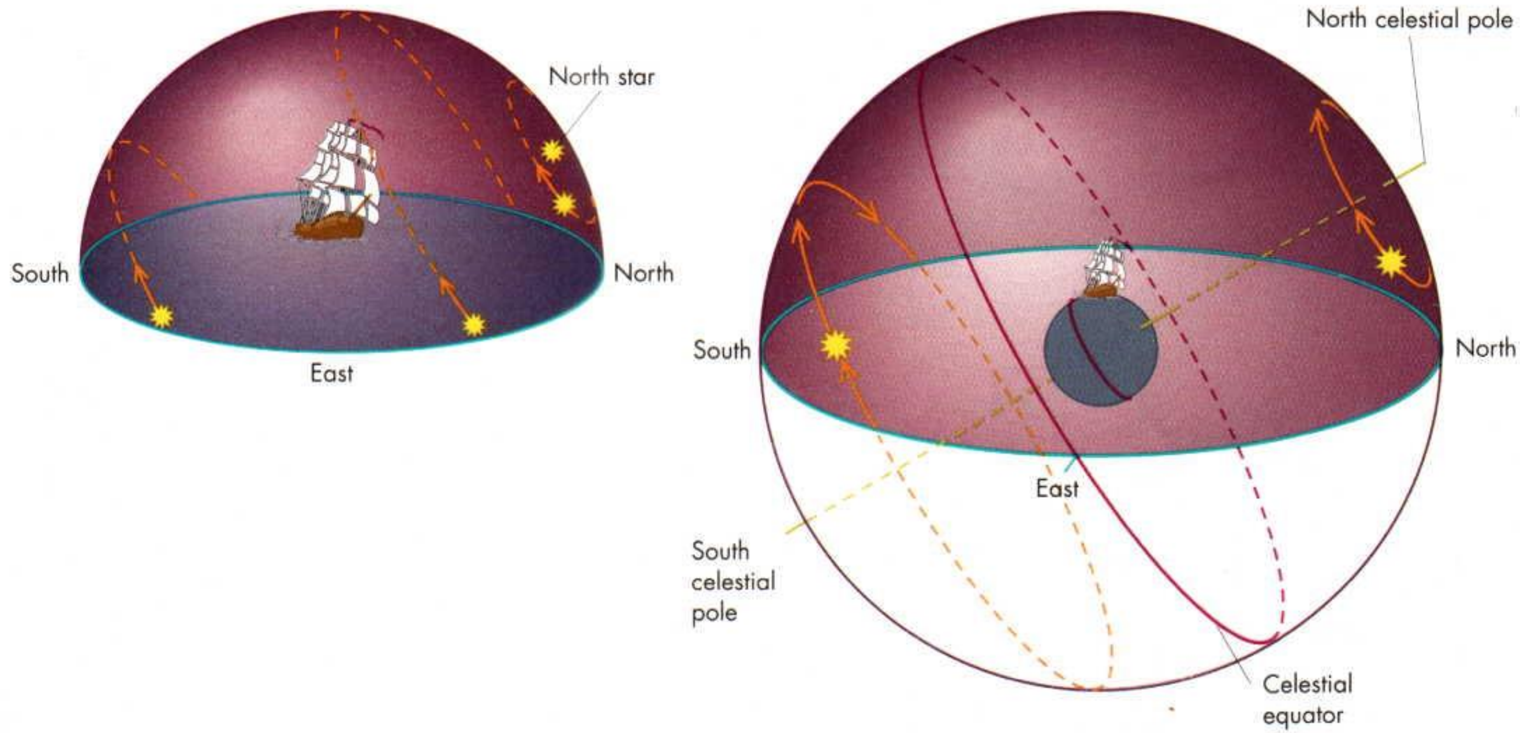
The Celestial Sphere

It is impossible to tell how far away anything is, or whether there is any depth to the "celestial sphere".



Celestial Equator and Pole

We project the Earth into the sky, and its rotation appears reflected there. The “diurnal” (daily) motion of the sky is just due to the spinning Earth.

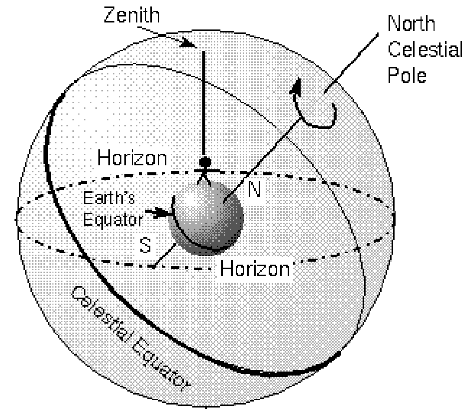


Some stars never set from a given latitude (circumpolar).

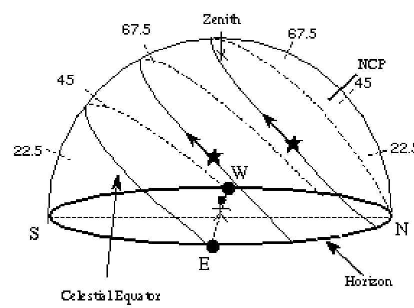
The size of the circumpolar region grows as you approach the poles.

You can never see stars in the opposite circumpolar hemisphere.

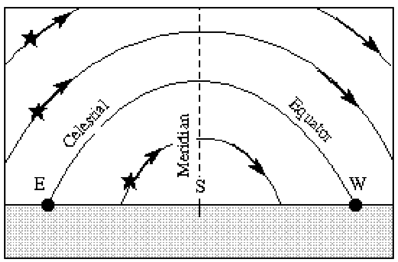
Stars may rise in the East, South-East, or North-East (so might the Sun).



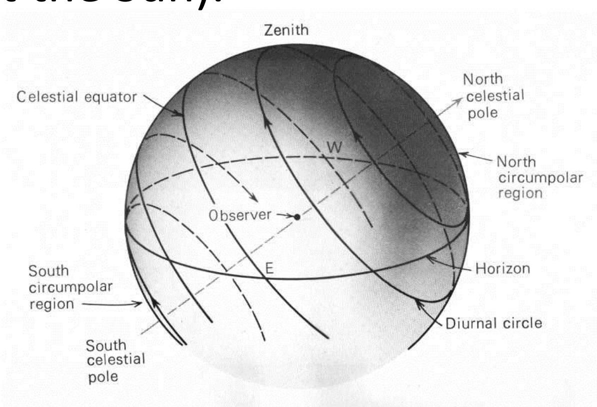
The celestial sphere for an observer in Seattle.
The angle between the zenith and the NCP = the angle between the celestial equator and the horizon.
That angle = 90° – observer's latitude.



Stars motion at Seattle. Stars rotate parallel to the Celestial Equator, so they move at an angle with respect to the horizon here. Altitudes of 1/4, 1/2, and 3/4 the way up to the zenith are marked.

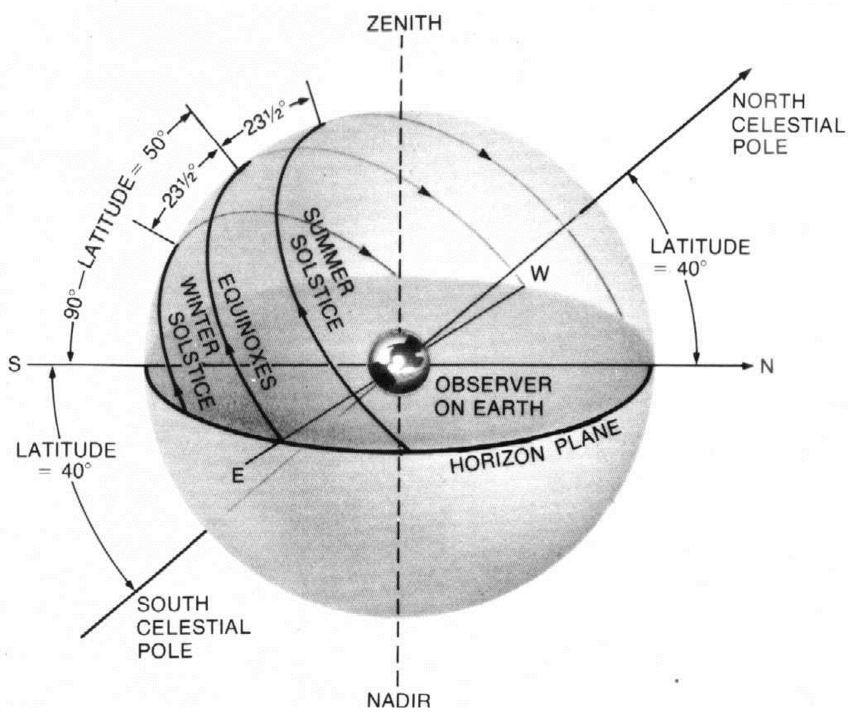


Your view from Seattle. Stars rise in the East half of the sky, reach maximum altitude when crossing the meridian (due South) and set in the West half of the sky. The Celestial Equator goes through due East and due West.



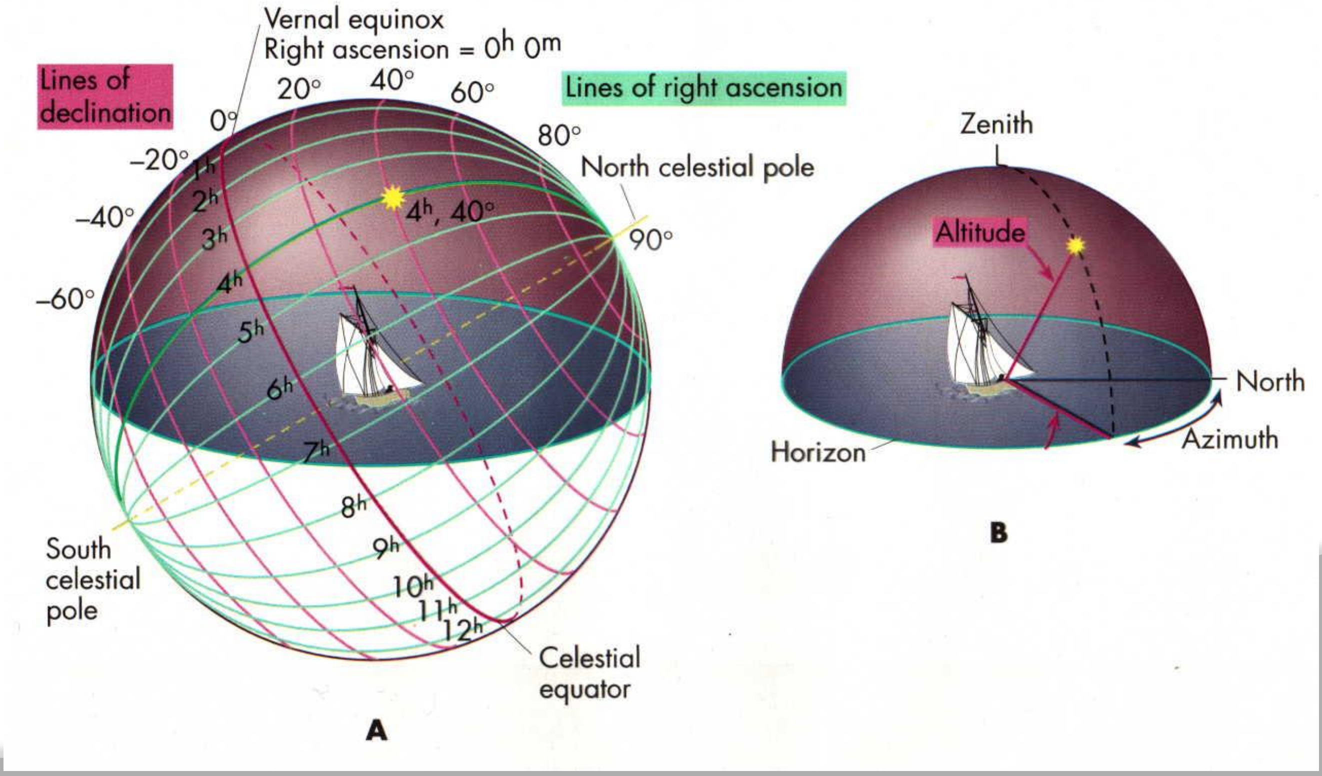
Path of the Sun

- ❖ The altitude of the pole depends on your latitude.
- ❖ The Sun may never pass overhead.
- ❖ The altitude of the Sun depends on the season.

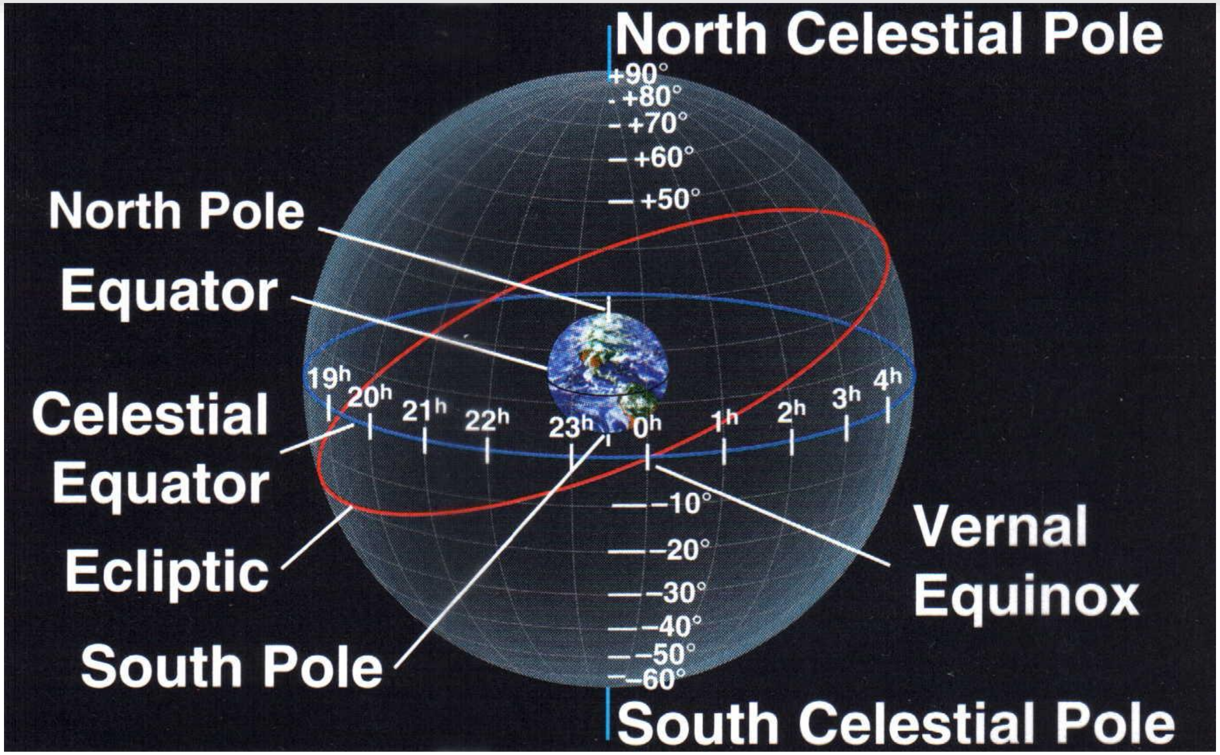


Celestial Coordinates

To "map" a given point in the sky, you can specify how high it is, and in what direction (altitude and azimuth). Or you can project latitude and longitude into the sky, but since the Earth rotates, we must use "right ascension" which is fixed on the stars.



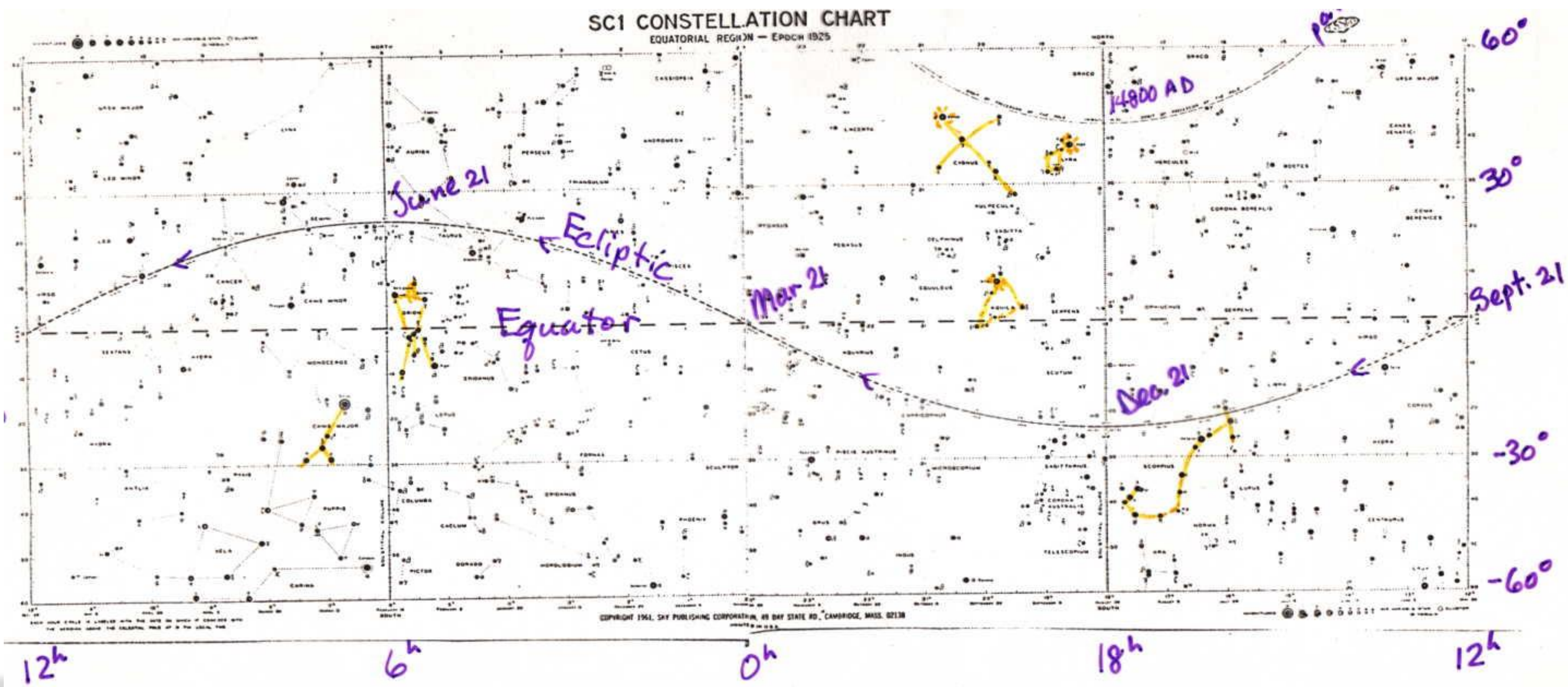
The Ecliptic Plane



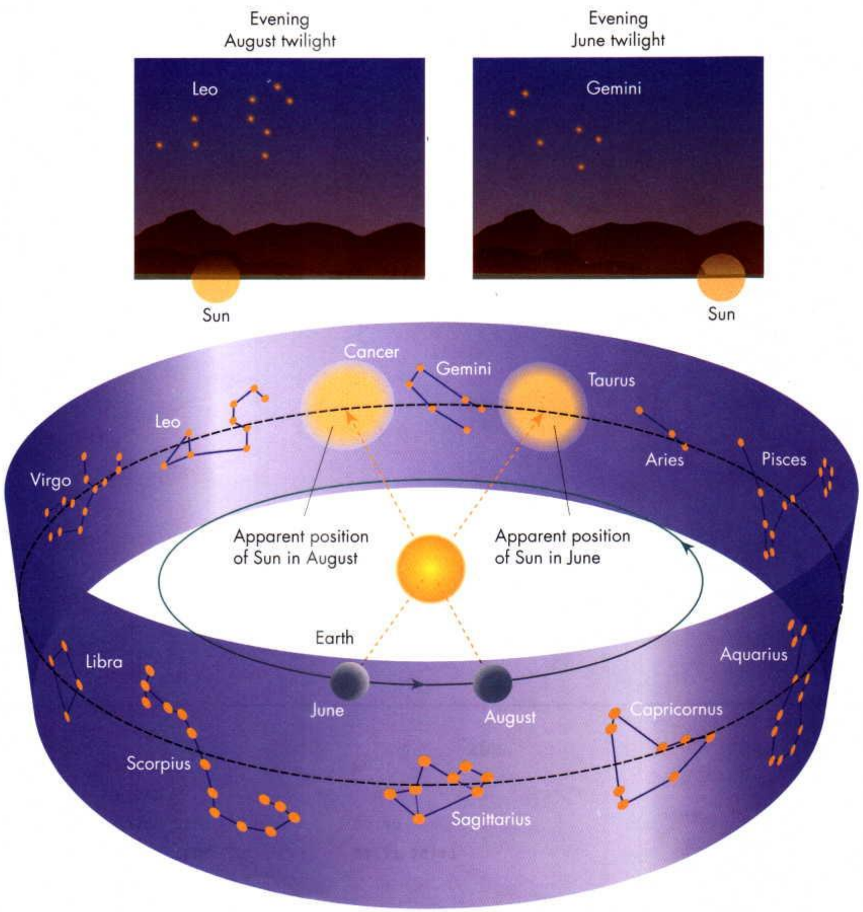
The projection of the Sun's path on the celestial sphere, or equivalently the projection of the plane of the Earth's orbit, is called the "ecliptic". It has a 23.5 degree tilt to the equator.

Chart of the Sky

Note how the Sun appears to go North and South as the year progresses. The zero point of Right Ascension occurs at the Spring crossing of the Equator (vernal equinox). The solstices occur at the maximum N/S excursions.



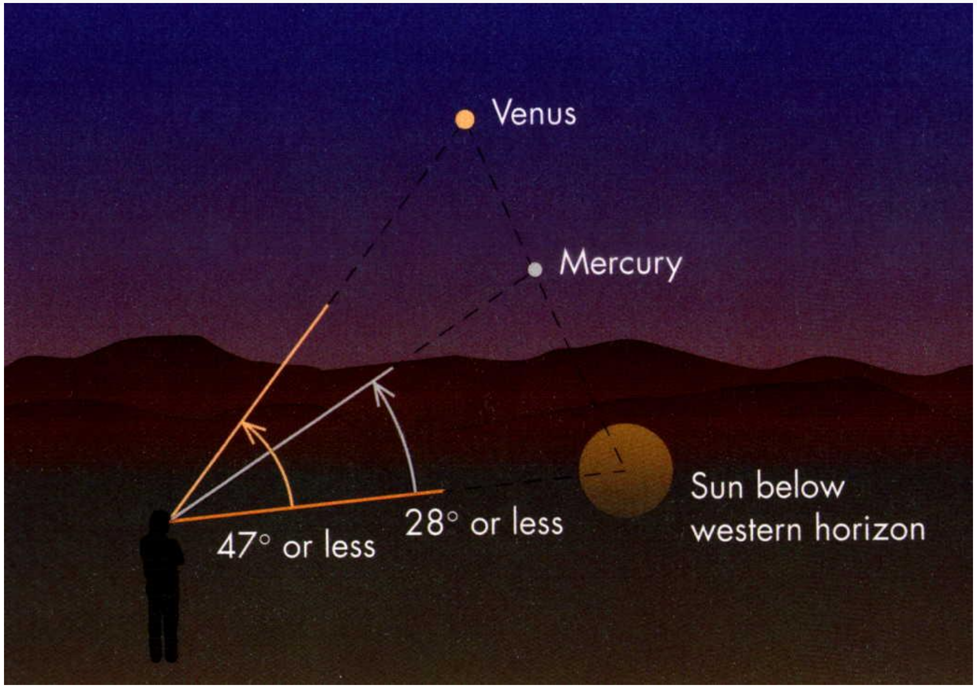
The Seasonal Stars



Constellations along the ecliptic are called the "Zodiac". The visible ones change through the year because the Earth orbits the Sun. The constellations themselves are arbitrary groupings of stars in the sky.

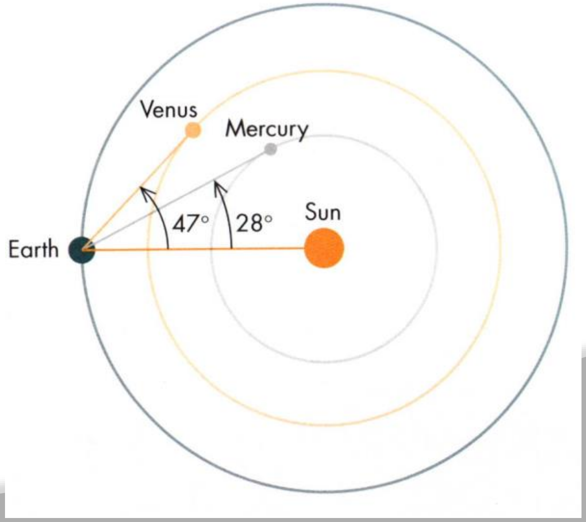
The stars up at night in the summer are up during the daytime in the winter.

Morning and Evening "Stars"

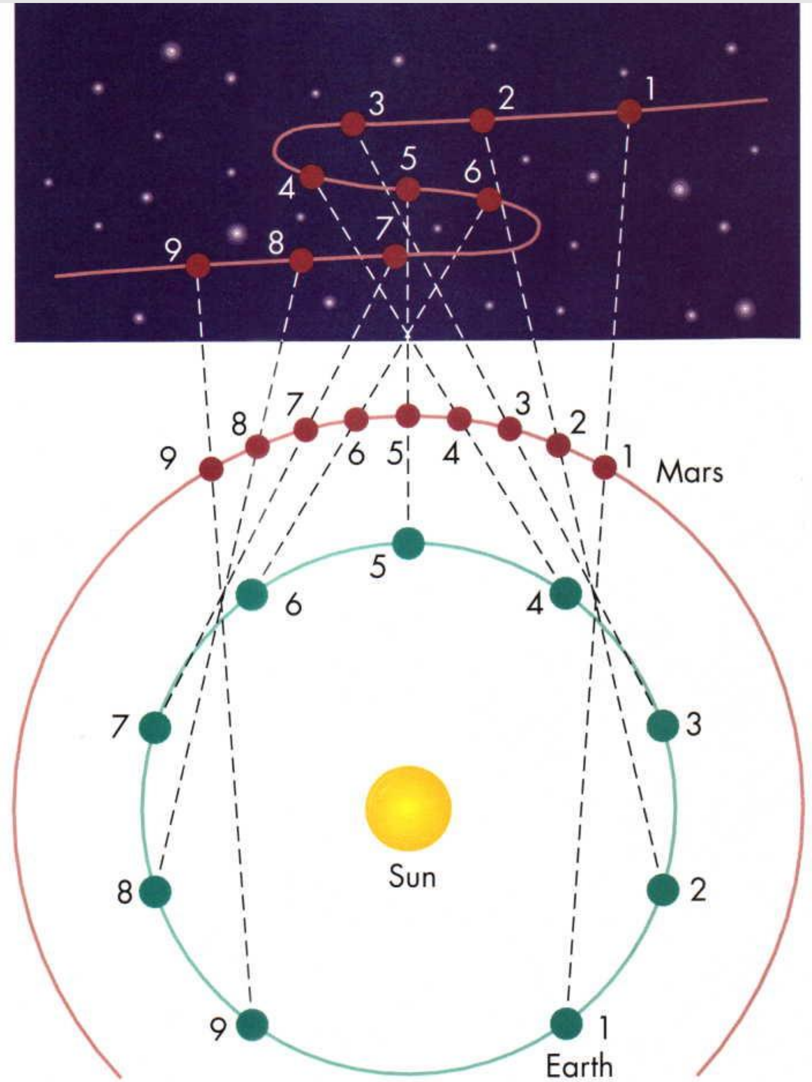


We see Mercury and Venus follow the Sun around in the sky. They may go down after, or come up before it. If they go down after, we see them in the evening.

This is because they have orbits inward of ours. That means they can only be seen to a certain maximum angle away from the Sun.



Retrograde Motion



The outer planets appear to make strange reversals in their motion against the stars. This is due to the fact that the Earth moves around the Sun faster than they do, causing us to overtake them periodically, during which time they appear to move “backwards” in the sky. This caused a lot of headaches for those trying to explain the apparent motion of the planets. The “S” shape is due to the fact that the orbital planes aren’t quite aligned.

Astro Quiz

What is the nightly path of the North Star as seen from the Earth's equator?

1) It rises far north of east, and sets far north of west.

2) It makes a circle around the sky, very low to the horizon.

3) It sits on the horizon in one place all night (so would always be hard to see).

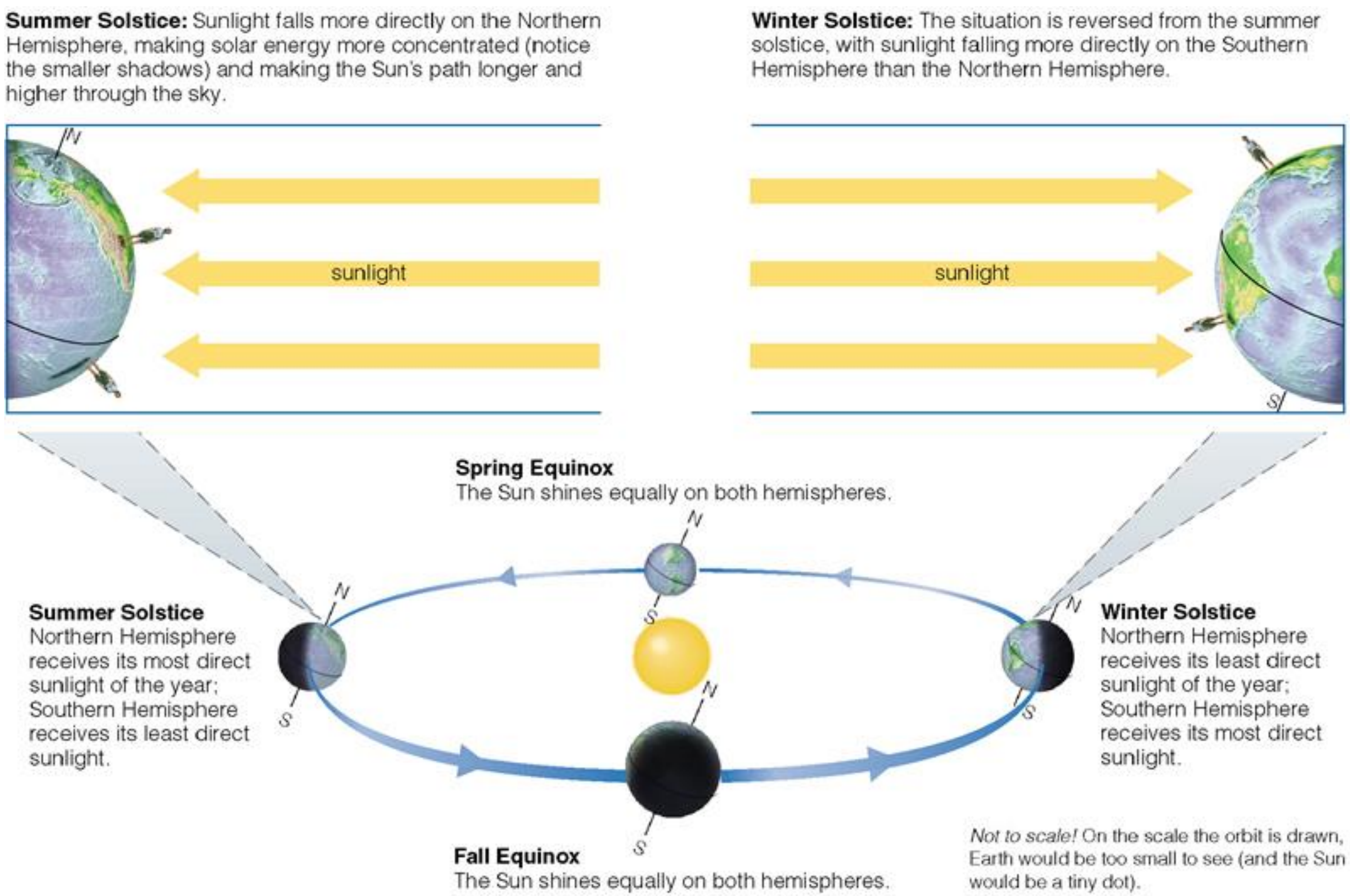
The Four Seasons

Is the changing seasons caused by the change in the distance between the Sun and the Earth?

No. If it is, then

- The northern and southern hemisphere should have the same season, not opposite season like we have.
- We should experience real seasonal changes in Hawaii also.

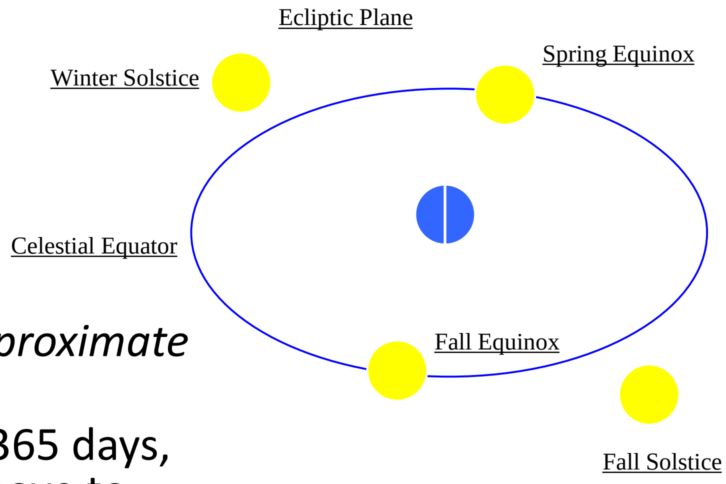
The Effect of the Tilt of Earth's Rotation Axis



Solstices and Equinoxes

- Equinox: An equinox is one of two opposite points on the celestial sphere where the celestial equator and ecliptic intersect.
- Solstice: A solstice is either of the two times of the year when the sun is at its greatest distance from the equator.

  - Spring Equinox ~ March 21
  - Summer Solstice ~ June 21
  - Fall Equinox ~ September 22
  - Winter Solstice ~ December 21



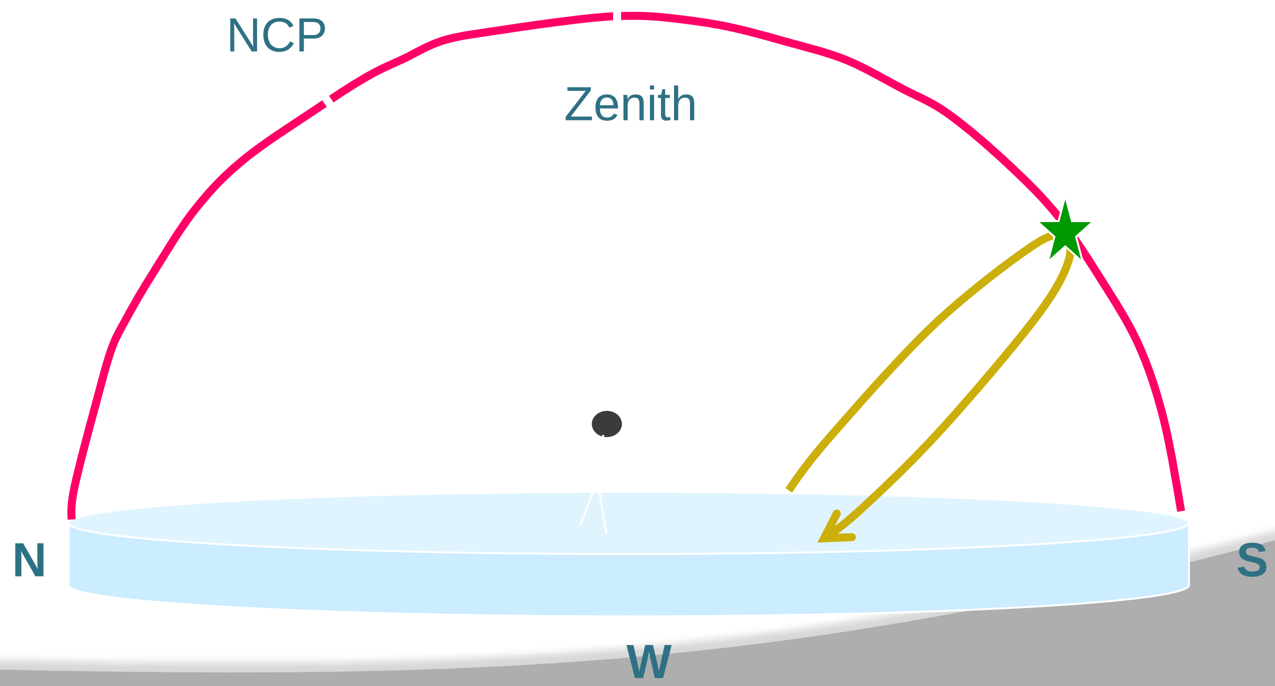
- *The dates of the equinoxes and solstices are only approximate dates.*
  - The actual length of a year is about 365 ¼ days (365 days, 5 hours, 49 minutes), not exactly 365 days. We have to add an extra day to a year every four years to keep the seasons synchronized with the seasons (leap year). Over a longer period of time, we need to skip a leap year to compensate the extra minutes we add in every leap year to keep the calendar in sync.

Time

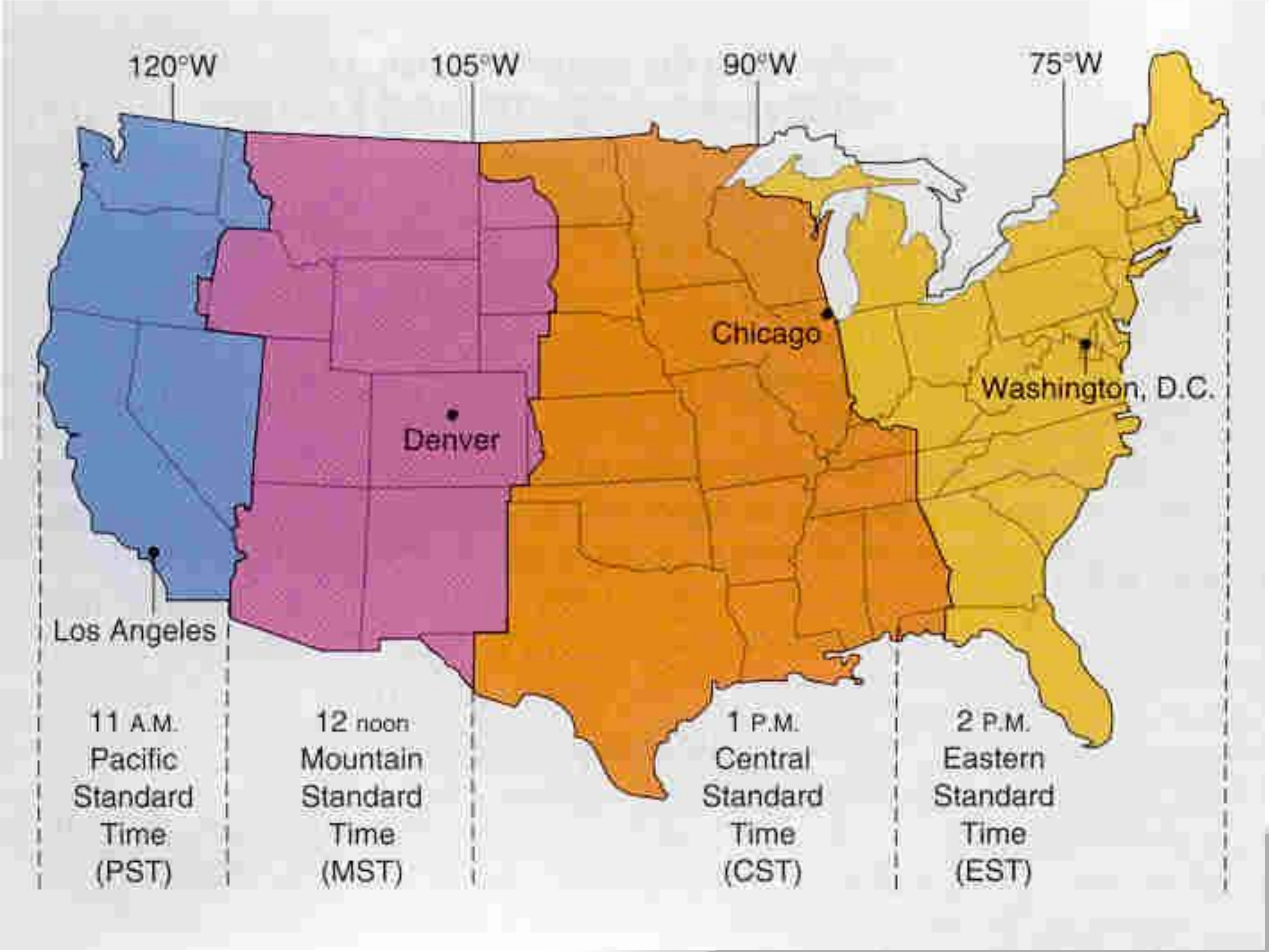
- What Time Is It?
- Before 1884, almost every town in the world kept its own local time. There were no national or international
- conventions which set how time should be measured, or when the day would begin and end, or what length an hour might be.
- However, with the vast expansion of the railway and communications networks during the 1850s and 1860s, there need for a worldwide, international time standard became imperative.

Local Time

- Meridian (North-South Line Through Zenith)
- Meridian Transit
- Local Noon = Solar Meridian Transit



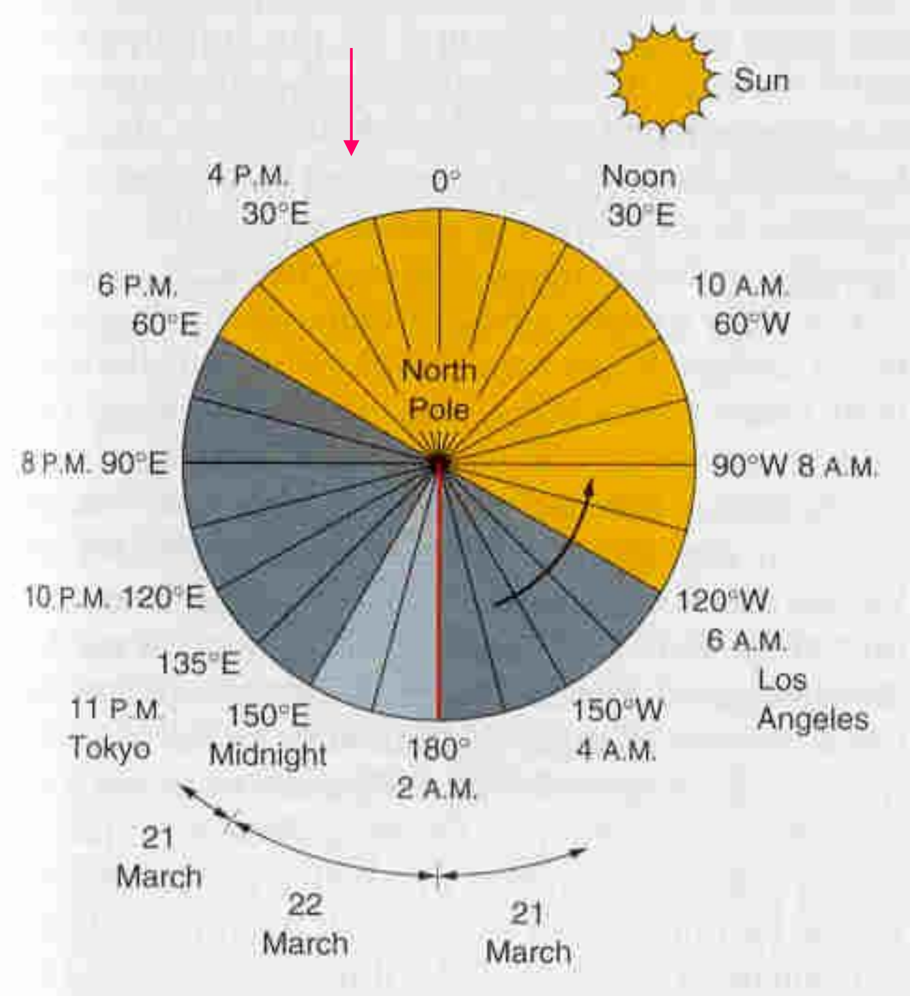
U.S.A. Time Zones



World Time Zones

There are 24 time zones around the world.

They start from Greenwich England (Prime Meridian) and proceed westward.



Astronomical Clocks

- Earth's Rotation on its Axis
  - Successive meridian transits of the sun
  - 1 solar day (clock time)
  - 24 hours (86400 seconds)

Astronomical Clocks

- Earth's Rotation on its Axis
  - Successive meridian transits of the sun
  - 1 solar day (clock time)
  - 24 hours (86400 seconds)
- Earth's Orbit Around the Sun
  - Sun's Path on the Sky Returns to the Same Constellation
  - 1 solar year
  - 365.2422 days

Astronomical Clocks

- Earth's Rotation on its Axis
  - Successive meridian transits of the sun
  - 1 solar day
  - 24 hours (86400 seconds)
- Earth's Orbit Around the Sun
  - Sun's Path on the Sky Returns to the Same Constellation
  - 1 solar year
  - 365.2422 days
- Moon's Orbit Around the Earth
  - 1 month (1 moonth)
  - 29.5 days

Days of the Week

- **The 7 heavenly bodies visible with the unaided eye are each honored with**

In ancient times, the word "planets" was from the Greek for "wanderers" and referred to objects in the sky that were not fixed like the stars. Some of these associations are clearer in English, especially if we compare with names of Norse or Old English gods/goddesses, while others are clearer from comparing French/Spanish with the Roman gods and goddesses.

	Sun	Moon	Mars	Mercury	Jupiter	Venus	Saturn
Roman	Sol	Luna	Mars	Mercury	Jupiter	Venus	Saturn
Norse			Tiw	Woden	Thor	Freya	
Greek	Apollo	Selene			Zeus	Aphrodite	
English	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
French	dimanche	lundi	mardi	mercredi	jeudi	vendredi	samedi
Spanish	domingo	lunes	martes	miercoles	jueves	viernes	sabado
Italian	Domenica	Lunedi	Martedi	Mercoledi	Giovedi	Venerdi	Sabato
German	Sonntag	Montag	Dienstag	Mittwoch	Donnerstag	Freitag	Samstag

Day Length

- Solar Day
  - Observe Successive Meridian Transits of the Sun
  - 24 hours (86,400 seconds)
  - Clock Time
- Sidereal Day
  - Observe Successive Meridian Transits of a Star
  - 23 hours 56 minutes (86,160 seconds)
  - Sky Time

Julian Calendar

- 45 B.C.
- Ten months (Mar, …, Sept, Oct, Nov, Dec???)
  - Add July (Julius Caesar)
  - Add August (Augustus Caesar)

- Three Years of 365 days
- One Year with 366 days (Leap Year)

- This simulates a Calendar with 365.25 day years averaged over four years.

4 Minute Minutes

- Actual Length of Year (365.2422 days)

  365.25 - 365.2422 = 0.0078 days (11 minutes/year)

- After 1500 years

  11 minutes/year X 1500 = 11 days

Gregorian Reformation

- By 1582 A.D. Pope Gregory had had enough.

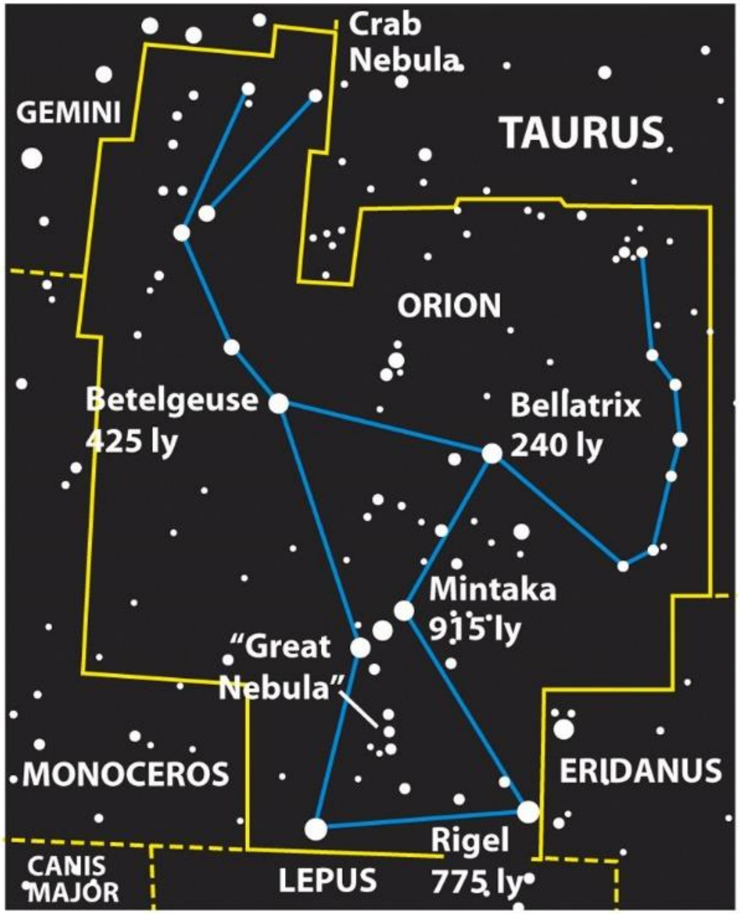
- Proclamation
  - October 4th would be October 15th
  - Century Years Divisible by 400 are NOT Leap Years
  - Average Year 265.2425

365.2424 - 365.2422 = 0.00027 days (23 seconds)

Time needed to accumulate 1 day error (3850 years)

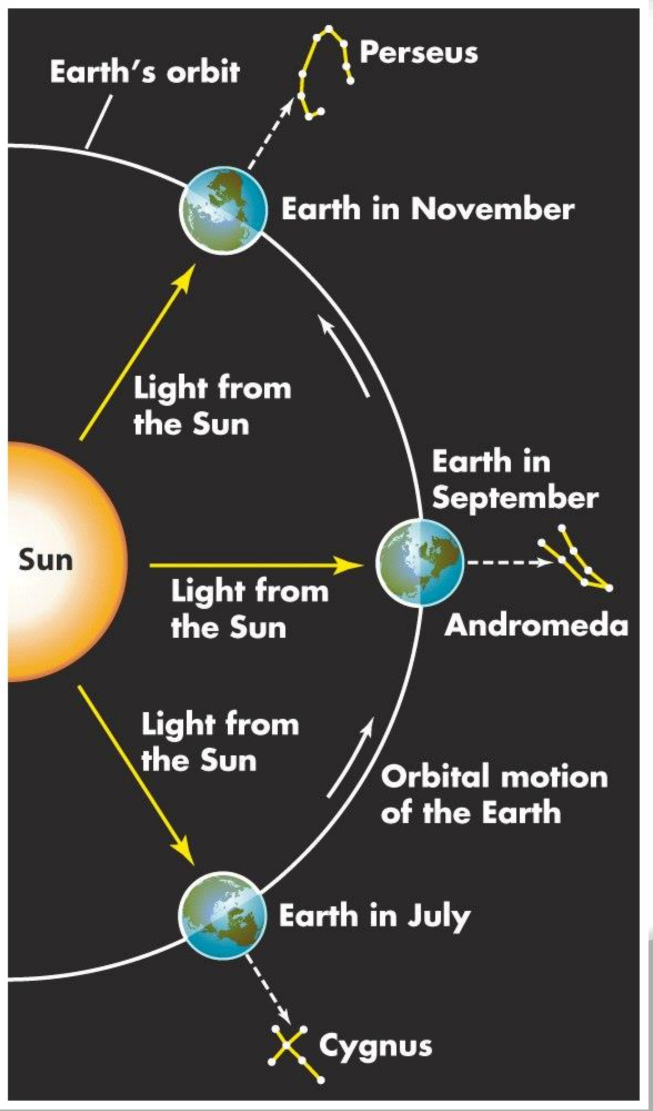
Modern Constellations

- On modern star charts, the entire sky is divided into 88 regions. Each is a constellation
- Most stars in a constellation are nowhere near one another
- They only *appear* to be close together because they are in nearly the same direction as seen from Earth

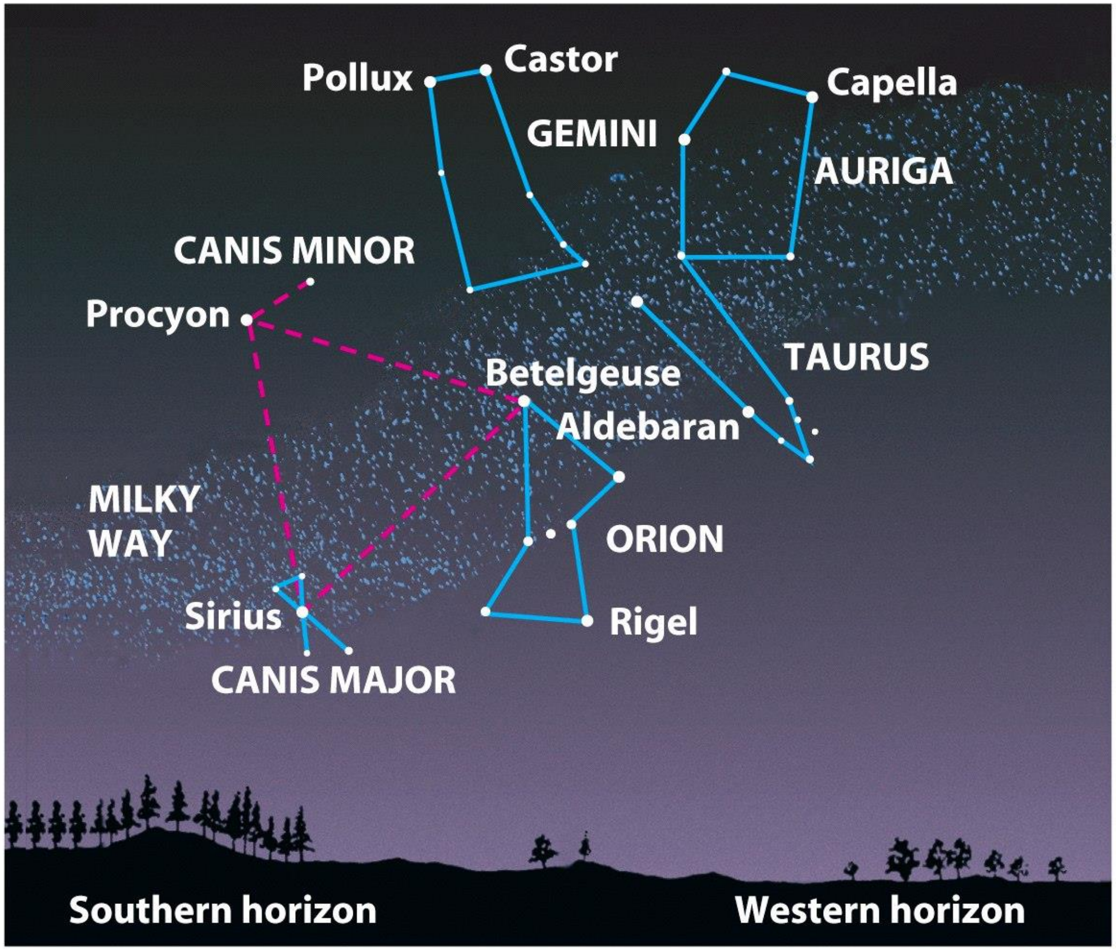


Annual Motion

- The stars also appear to slowly shift in position throughout the year
- This is due to the orbit of the earth around the sun
- If you follow a particular star on successive evenings, you will find that it rises approximately 4 minutes earlier each night, or 2 hours earlier each month

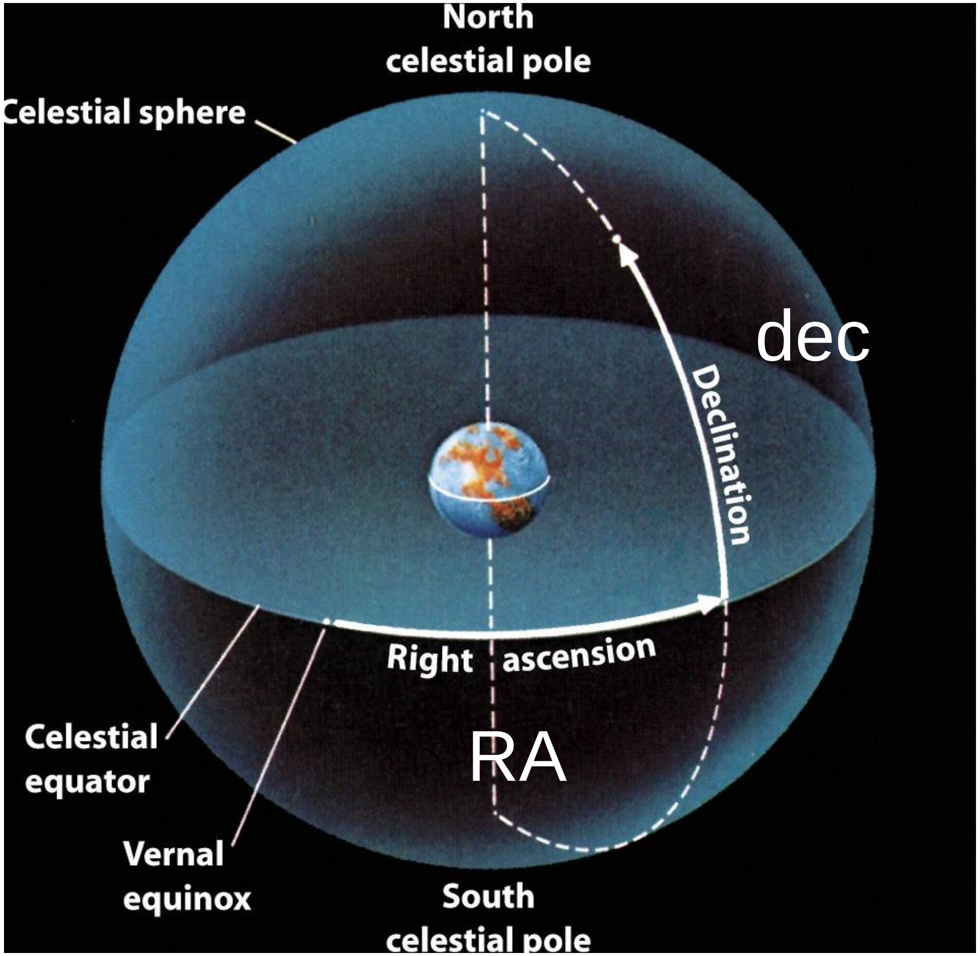


Popular Constellations



Popular Constellations visible in naked Eye

Celestial Equator

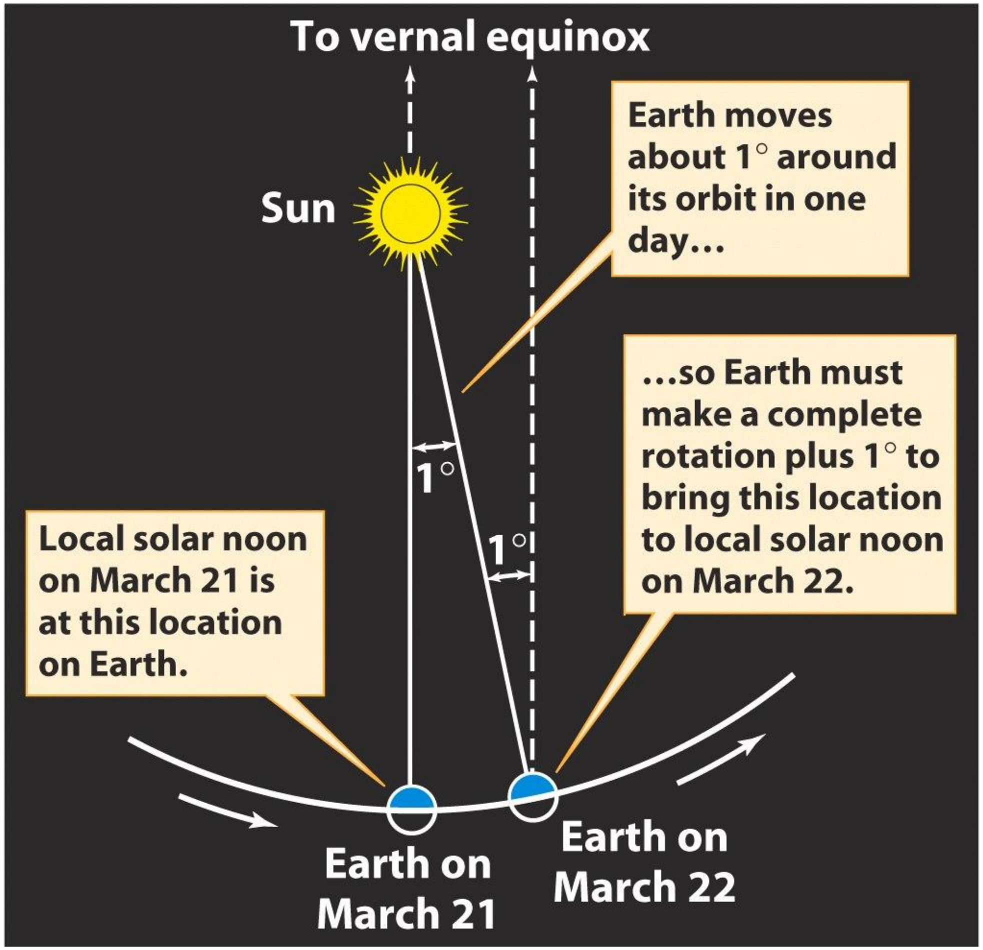


- **Celestial equator** divides the sky into northern and southern hemispheres
- **Celestial poles** are where the Earth's axis of rotation would intersect the celestial sphere
- Polaris is less than 1° away from the north celestial pole, which is why it is called the **North Star** or the Pole Star.
- Point in the sky directly overhead an observer anywhere on Earth is called observer's **zenith.**

Two More Types of Angle and Time

- Hour angle (HA) of an object is the angle between the meridian on which the object is situated and the (observer's) celestial meridian.
- Local Sidereal Time (LST) is the Right Ascension of an observer's celestial meridian.
- LST = RA + HA

Vernal Equinox



Occurrence of Vernal Equinox

Calendars

- Caesar introduced the 365.25 days calendar and thus the Leap Year (an extra day, February 29, every year divisible by 4) .
- However, this is $11^m\ 14^s$ longer than the tropical year. This accumulates to 3 days in 4 centuries error.
- To correct, October 4 was followed by October 15, in 1562 and the century rule was invoked (Gregorian calendar).

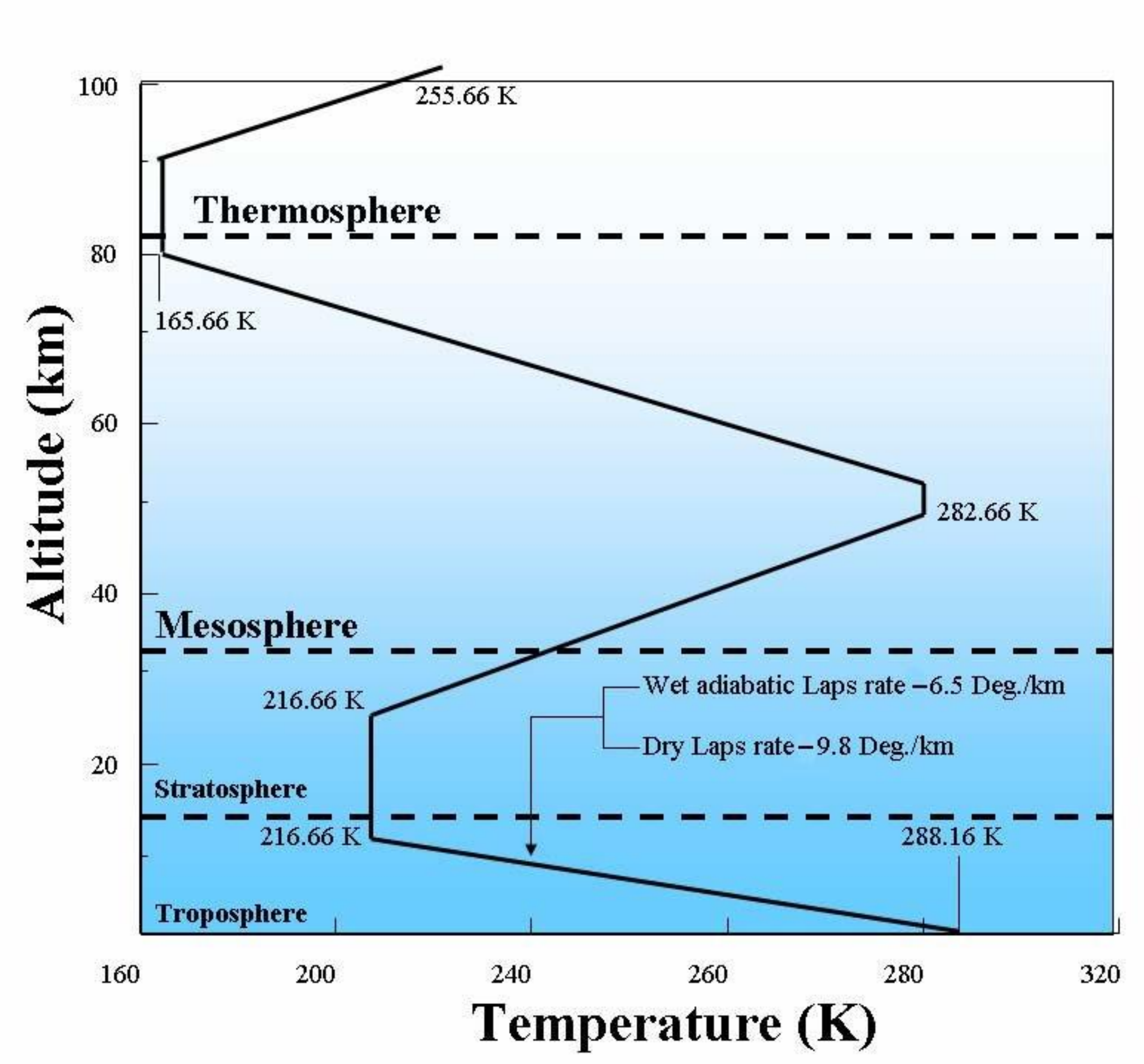
Standard Atmosphere

The "U.S. **Standard Atmosphere** 1976" is an **atmospheric** model of how the pressure, temperature, density, and viscosity of the Earth's **atmosphere** changes with altitude. It is defined as having a temperature of 288.15 K at the sea level 0 km geo-potential height and 1013.25 hPa. The **atmosphere** are divided in.

Standard Atmosphere

Layer	Level Name	Geopotential Altitude above MSL
0	Troposphere	-610
1	Tropopause	11,000
2	Stratosphere	20,000
3	Stratosphere	32,000
4	Stratopause	47,000
5	Mesosphere	51,000
6	Mesosphere	71,000
7	Mesopause	84,852

Standard Atmosphere



Equations of Motion

Newton's Two-Body Equations of Motion

$$\text{Force} = \text{Mass} \times \text{Acceleration}$$

$$\frac{Gm_1m_2}{r^2}\frac{(\mathbf{r}_2 - \mathbf{r}_1)}{r} = m_1\frac{d^2\mathbf{r}_1}{dt^2}$$

$$\frac{Gm_2m_1}{r^2}\frac{(\mathbf{r}_1 - \mathbf{r}_2)}{r} = m_2\frac{d^2\mathbf{r}_2}{dt^2}$$

$$\Longrightarrow$$

$$\frac{d^2}{dt^2}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2) = 0$$

$$-\frac{G(m_1 + m_2)}{r^2}\frac{(\mathbf{r}_2 - \mathbf{r}_1)}{r} = \frac{d^2}{dt^2}(\mathbf{r}_2 - \mathbf{r}_1)$$

Conservation of Total Linear Momentum

$$\frac{d^2\mathbf{r}_{cm}}{dt^2} = 0 \quad \Longrightarrow \quad \mathbf{r}_{cm} = \mathbf{c}_1 t + \mathbf{c}_2 \quad \text{where} \quad \mathbf{r}_{cm} \stackrel{\text{def}}{=} \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

Two-Body Equation of Relative Motion

$$\boxed{\frac{d^2\mathbf{r}}{dt^2} + \frac{\mu}{r^3}\mathbf{r} = 0} \quad \text{or} \quad \frac{d\mathbf{v}}{dt} = -\frac{\mu}{r^3}\mathbf{r} \quad \text{where}$$

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$$

$$r = |\mathbf{r}| = |\mathbf{r}_2 - \mathbf{r}_1|$$

$$\mu = G(m_1 + m_2)$$

Equations of Motion

Kepler's Second Law 1609 **Conservation of Angular Momentum**

$$\mathbf{r} \times \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = 0 \quad \Longrightarrow \quad \boxed{\mathbf{h} = \mathbf{r} \times \mathbf{v}} = \text{Constant}$$

Motion takes place in a plane and angular momentum is conserved

In polar coordinates

$$\mathbf{r} = r\,\mathbf{i}_r \qquad \frac{d\mathbf{r}}{dt} = \mathbf{v} = \frac{dr}{dt}\,\mathbf{i}_r + r\frac{d\theta}{dt}\,\mathbf{i}_\theta = v_r\,\mathbf{i}_r + v_\theta\,\mathbf{i}_\theta$$

so that the angular momentum of m_2 with respect to m_1 is

$$m_2\,r\,v_\theta = m_2\,r^2\frac{d\theta}{dt} \stackrel{\text{def}}{=} m_2\,h = \text{Constant}$$

- **Rectilinear Motion:** For $\mathbf{r} \parallel \mathbf{v}$, then $\boxed{h = 0}$.

As Kepler expressed it, the radius vector sweeps out equal areas in equal time since

$$\boxed{\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{h}{2}} = \text{Constant}$$

Kepler's Law is a direct consequence of radial acceleration!

Equations of Motion

Conservation of Energy $F = G\frac{Mm}{r^2} = m\frac{v^2}{r} = mr\omega^2$

Since $\omega = \frac{2\pi}{P}$

$$\frac{\mu}{r^2} = \frac{4\pi^2 r}{P^2} \quad \text{or} \quad P^2 = \frac{4\pi^2 r^3}{\mu}$$

Other expressions and terminology are used

Mean Motion $n = \frac{2\pi}{P} = \sqrt{\frac{\mu}{a^3}}$ or $\mu = n^2 a^3$ or $\frac{a^3}{P^2} = \text{Constant}$

Conservation of Energy

An orbit is a continually changing balance between potential and kinetic energy

Potential Energy $-G\frac{Mm}{r}$ Kinetic Energy $\frac{1}{2}mv^2$

Using $\epsilon = \frac{{v_a}^2}{2} - \frac{\mu}{r_a} = \frac{{v_p}^2}{2} - \frac{\mu}{r_p}$

$$\frac{v^2}{\mu} = \frac{2}{r} - \frac{1}{a}$$

For any Kepler orbit (elliptic, parabolic, hyperbolic or radial), this is the *Vis Viva* equation

The Hill stability in the General 3-body problem

$$M = m_0 + m_1 + m_2, \quad M^* = m_0 m_1 + m_0 m_2 + m_1 m_2$$

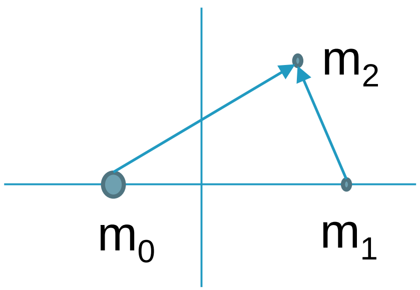
$$a = -\frac{GM^*}{2h}$$ generalized semimajor axis

$$p = -\frac{MC^2}{GM^{*2}}$$ generalized semi-latus rectum

$$\rho^2 = \frac{1}{M^*}\left(m_1 m_2 r_{12}^2 + m_1 m_3 r_{13}^2 + m_2 m_3 r_{23}^2\right)$$ ρ : mean quadratic distance

$$\frac{1}{\nu} = \frac{1}{M^*}\left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}}\right)$$ ν : mean harmonic distance

$$\ldots \Rightarrow \frac{\rho}{\nu} \geq \frac{\rho}{2a} + \frac{p}{2\rho} \overset{h<0}{\Rightarrow} \frac{\rho}{\nu} \geq \sqrt{\frac{p}{a}}$$



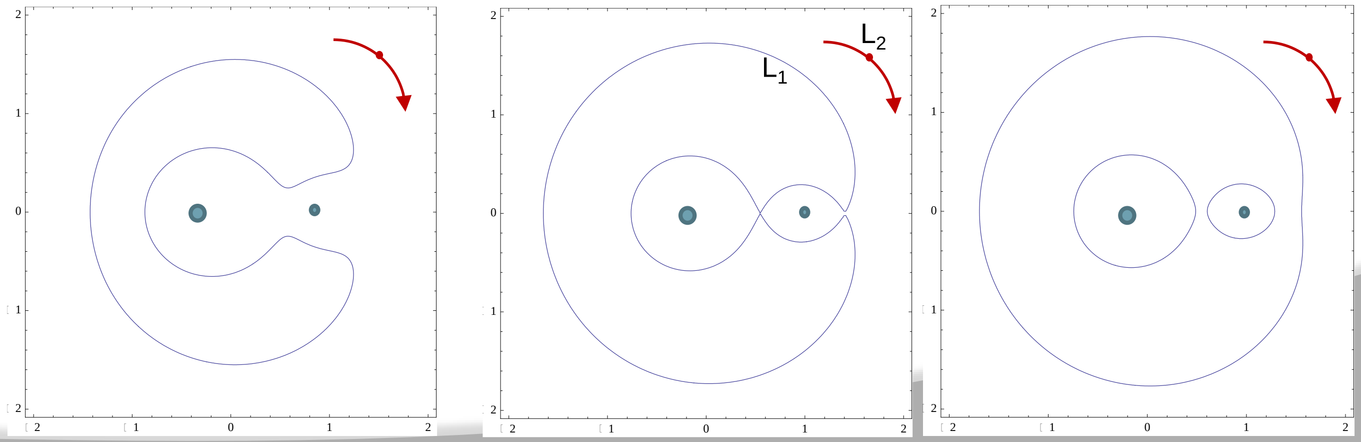
$$\frac{\rho}{\nu} = f(m_i, r_{ij}) = (\text{scale invariant}) = f(m_i, x_2, y_2)$$

$$r_{01} = 1 \qquad \left(x_0 = -\frac{m_1}{m_1 + m_0}, x_1 = \frac{m_0}{m_1 + m_0}, \quad y_0 = y_1 = 0 \right)$$

$$\frac{p}{a} = -\frac{2M}{G^2 M *^3} c^2 h = const.$$

$$\boxed{\frac{\rho}{\nu} = f(m_i, x_2, y_2) \quad \geq \quad const(m_i, c, h)}$$

Possible triangles of the three body if



The Hill stability in the General 3-body problem

Critical values $\left(\frac{p}{a}\right)_k = \left(\frac{\rho}{\nu}(L_k)\right) = f(m_i)$

For the planetary case $m_1 << m_0$, $m_2 << m_0$

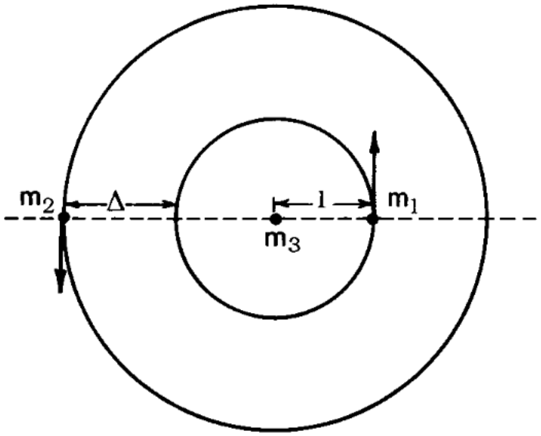
Condition for Hill stability

$$\frac{p}{a} > 1 + 3^{4/3} \frac{m_1 m_2}{m_0^{2/3}(m_1 + m_2)^{4/3}} + ...$$

Hill Stability In 2-planet Systems

Gladman's reformulation

$$\frac{p}{a} = \frac{\mu_1 + \mu_2/\delta^2}{(\mu_1+\mu_2)^3}\left(\mu_1\gamma_1 + \mu_2\gamma_2\delta\right)^2$$



$$\mu_i = \frac{m_i}{m_0} \ll 1, \quad i = 1, 2, \qquad \gamma_i = \sqrt{1-e_i^2}, \qquad \delta = \sqrt{1+\Delta}$$

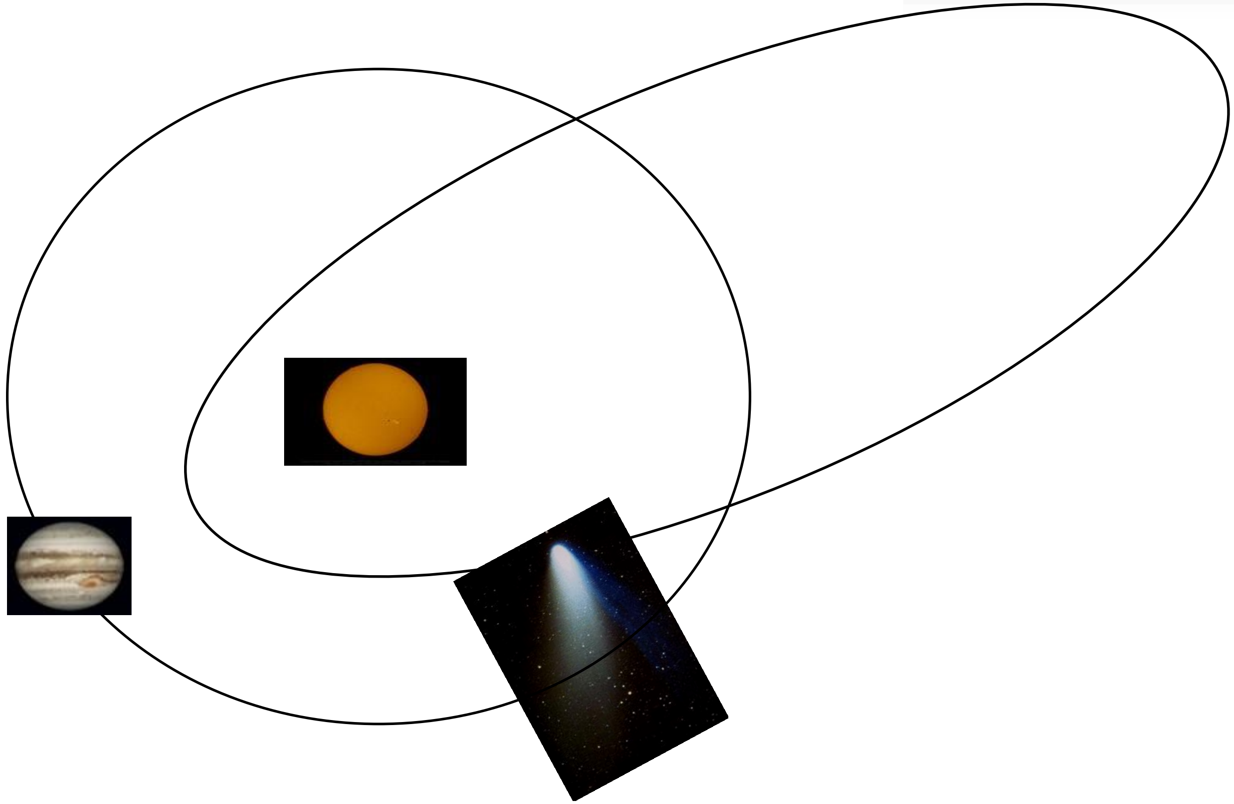
$$\Delta = a_2(1-e_2) - a_1(1+e_1) \quad (a_1 = 1)$$

Minimum separation of elliptic orbits :
inner planet at apocenter (distance from m_0 is d=1)
outer planet at pericenter)

$$\frac{\mu_1 + \mu_2/\delta^2}{(\mu_1+\mu_2)^3}\left(\mu_1\gamma_1 + \mu_2\gamma_2\delta\right)^2 > 1 + 3^{4/3}\frac{\mu_1\mu_2}{(\mu_1+\mu_2)^{4/3}} + \ldots$$

$$f(a_2/a_1, e_1, e_2, \mu_1, \mu_2) > 0$$

The Restricted Three-Body Problem



If two of the bodies in the problem move in circular, coplanar orbits about their common centre of mass and the mass of the third body is too small to affect the motion of the other two bodies, the problem of the motion of the third body is called the *circular, restricted, three -body problem.*

A) **We consider the motion of a small particle of negligible mass moving under the two masses m_1 and m_2.**

B) **We assume that two masses have circular orbit around their common mass center. Then the two masses keep the constant distance and have the same angular velocity.**

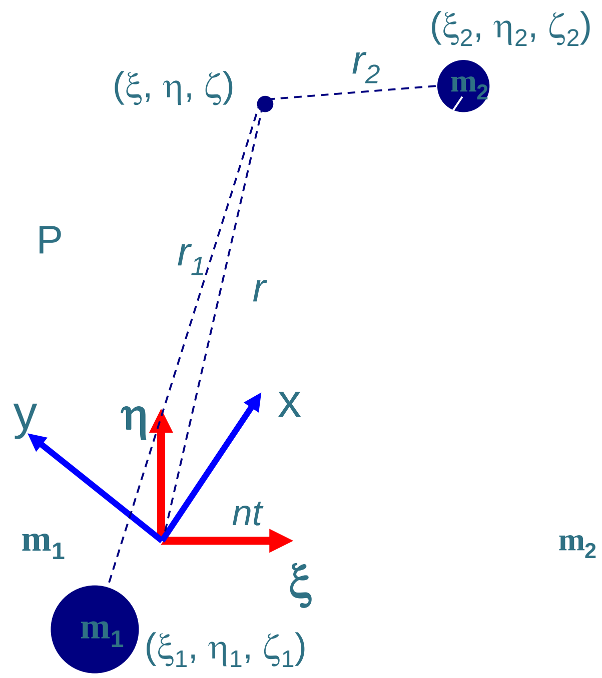
C) **Consider the geometry in the right-hand figure.**

D) **Let the unit of mass be chosen such that**

$$\mu = G(m_1 + m_2) = 1$$

then in this system of units the two masses

The unit of length is chosen such that the constant separation of the two masses is unity. It then follows that the common mean motion, *n*, of the two masses is also unity.



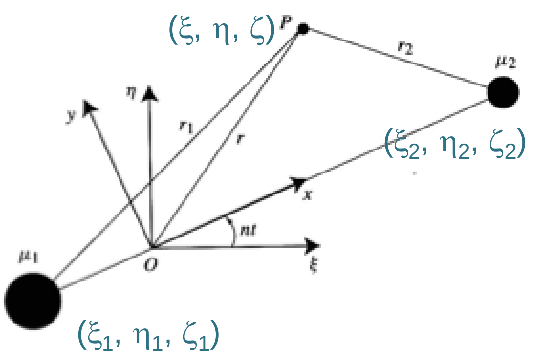
Let the coordinates of the particle in the *inertial or sidereal system* (ξ, η, ζ).

The equation of motion of the particles are

$$\begin{cases} \ddot{x} = m_1 \dfrac{x_1 - x}{r_1^3} + m_2 \dfrac{x_2 - x}{r_2^3} \\ \ddot{h} = m_1 \dfrac{h_1 - h}{r_1^3} + m_2 \dfrac{h_2 - h}{r_2^3} \\ \ddot{z} = m_1 \dfrac{z_1 - z}{r_1^3} + m_2 \dfrac{z_2 - z}{r_2^3} \end{cases}$$

where

$$\begin{cases} r_1^2 = (x_1 - x)^2 + (h_1 - h)^2 + (z_1 - z)^2 \\ r_2^2 = (x_2 - x)^2 + (h_2 - h)^2 + (z_2 - z)^2 \end{cases}$$



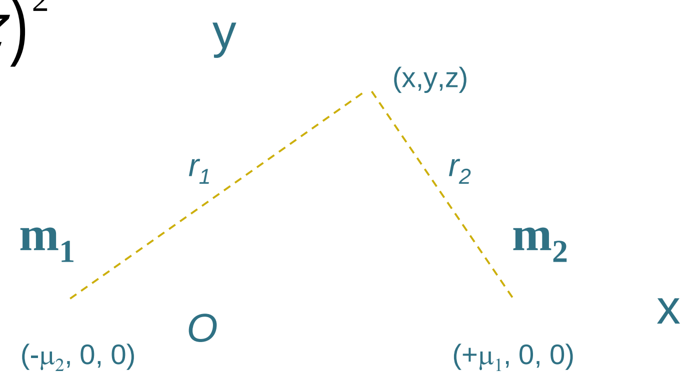
We assume that the circular orbits of two masses, which imply the distance to these two is kept constant. In this condition, we can consider that the motion of the particle in a rotating reference frame in which the locations of the two masses are also fixed.

Consider a new, rotating coordinate system that has the same origin as the ξ, η, ζ system but which is rotating at a uniform rate n in the positive direction.

The direction of the x-axis is chosen such that the two masses always lie along it with coordinates (x1,y1,z1)=(-μ2,0,0) and (x2,y2,z2)=(+μ1,0,0). Hence from Eq. (3.6~7) we have

$$\begin{cases} r_1^2 = (\xi_1 - \xi)^2 + (\eta_1 - \eta)^2 + (\zeta_1 - \zeta)^2 \\ r_2^2 = (\xi_2 - \xi)^2 + (\eta_2 - \eta)^2 + (\zeta_2 - \zeta)^2 \end{cases}$$

$$\begin{cases} r_1^2 = (x + \mu_2)^2 + y^2 + z^2 \\ r_2^2 = (x - \mu_1)^2 + y^2 + z^2 \end{cases}$$



where (x,y,z) are the coordinates of the particle with respect to the *rotating, or synodic system*.

The coordinates (x,y,z) are related to the coordinates in the sidereal system by the following rotation

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} \cos nt & -\sin nt & 0 \\ \sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

sidereal coordinate 항성의 synodic coordinate 삭망

If we now differentiate each component in Eq. (3.10) twice we get

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \end{pmatrix} = n \begin{pmatrix} -\sin nt & -\cos nt & 0 \\ \cos nt & -\sin nt & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \cos nt & -\sin nt & 0 \\ \sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

$$= \begin{pmatrix} \cos nt & -\sin nt & 0 \\ \sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{x} - ny \\ \dot{y} + nx \\ \dot{z} \end{pmatrix}$$

-----(skip)-----

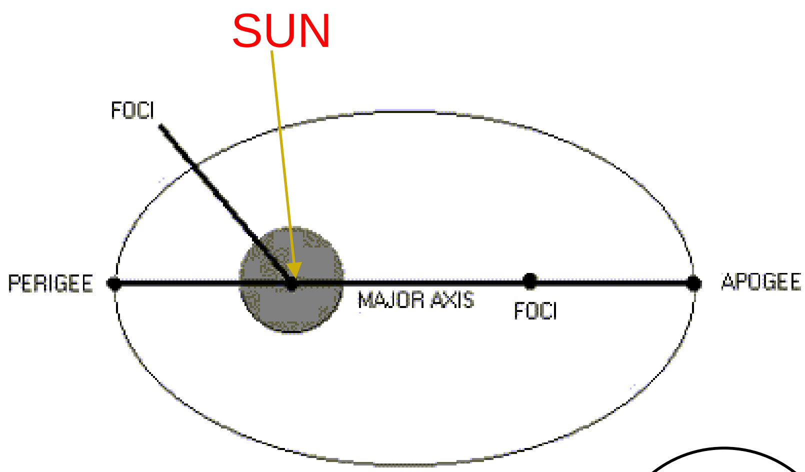
$$\begin{pmatrix} \ddot{\xi} \\ \ddot{\eta} \\ \ddot{\zeta} \end{pmatrix} = \begin{pmatrix} \cos nt & -\sin nt & 0 \\ \sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \ddot{x} - 2n\dot{y} - n^2 x \\ \ddot{y} + 2n\dot{x} - n^2 y \\ \ddot{z} \end{pmatrix}$$

Centrifugal acceleration

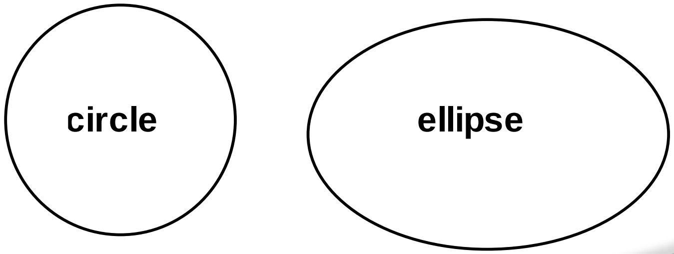
Corioli’s acceleration

Kepler's Laws of Planetary Motions

1. Planets move in elliptical orbits with the sun at one of the foci



Foci – 2 points that are equidistant from center on major axis

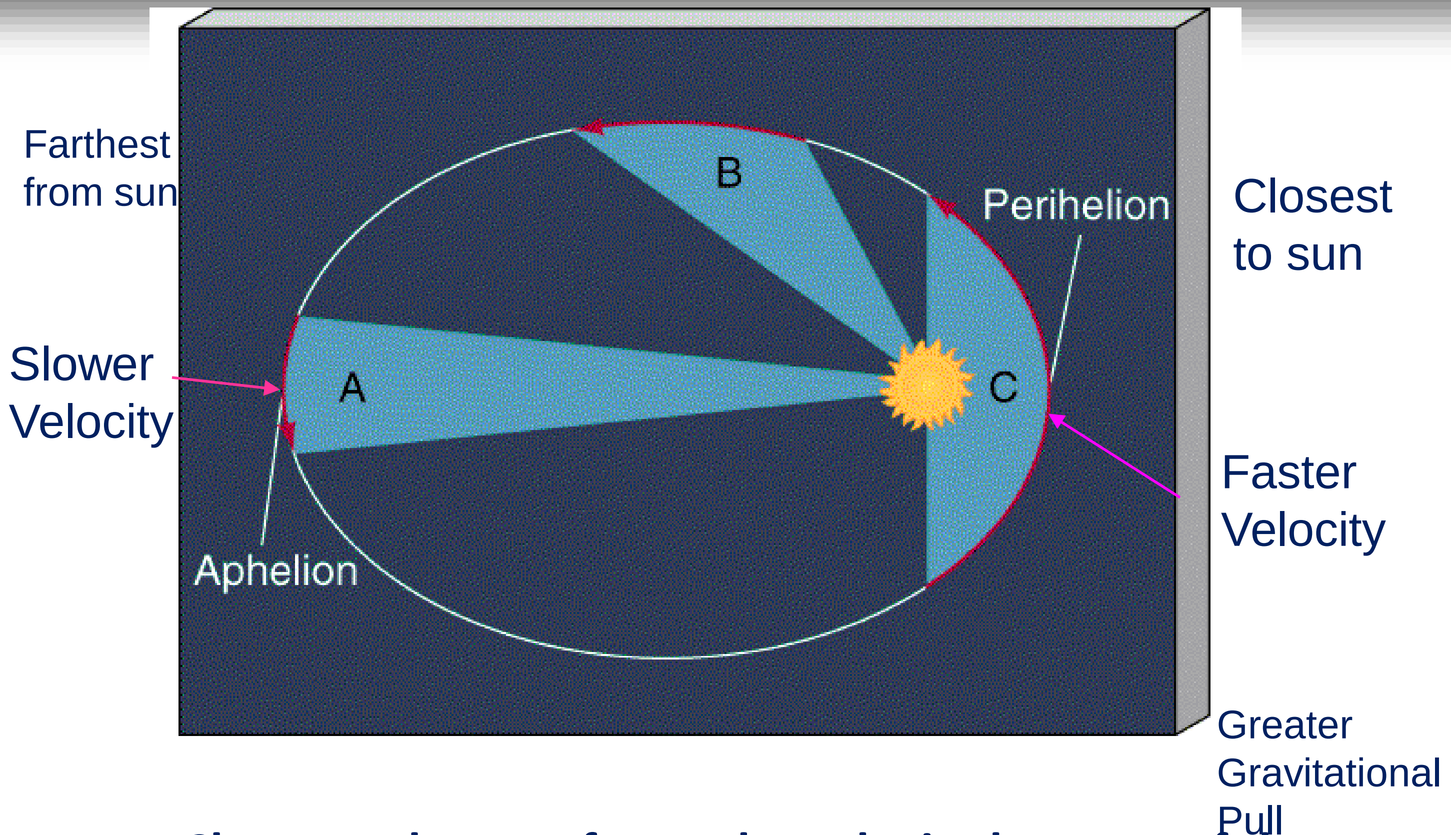


Law of Areas

Line from the sun to a planet sweeps with equal areas in equal time. A planet will move through equal area of space in an equal amount of time

- Perihelion – closest to sun
- Ahelion – farthest from sun
  a. When a planet is in perihelion its orbital velocity increases
  b. When a planet is in aphelion its orbital velocity decreases

UNIT - II
THE TWO BODY PROBLEM



Closer to the sun, faster the velocity because of the gravitational pull.

Law of Periods

Law of Periods - The farther a planet is from the focus, the longer the period of revolution.

Ex: Earth is closer to the sun than Jupiter, therefore the Earth has a shorter period of revolution. (ESRT)

Newton's Law of Gravity

- The force of attraction between any two objects depends on their masses and the distance between them.
- Thus, the closer the objects, the __greater_______ the gravitational pull
- Thus, the bigger the object the _greater__ the gravitational pull

Sir Issac Newton (1642 – 1727)

Although Kepler discovered what is now known the Three *Laws of Planetary Motion*, he could not explain *why* they were true. That did not come until years later from Issac Newton formulated the laws of motion that are the basis of mechanics—that are still valid today!

Sir Issac Newton

Newton formulated what is now known as his 2nd Law of Motion:

$$F_{net} = ma$$

Sir Issac Newton

This enabled him to formulate how objects are influenced (or attracted) in a gravitational field:

$$W = mg$$

Sir Issac Newton

He was also the first to identify the acceleration on objects forced to move in circles as:

$$a_c = \frac{v^2}{r}$$

Sir Issac Newton

And therefore the net force:

$$F_c = ma_c$$

$$= \frac{mv^2}{r}$$

Sir Issac Newton

And finally what is perhaps the greatest intellectual discovery of all time—the

Law of Universal Gravitation:

$$F_{net} = ma$$

$$W = mg$$

$$F_c = \frac{mv^2}{r}$$

$$F_g = \frac{GMm}{r^2}$$

Sir Issac Newton

And finally what is perhaps the greatest intellectual discovery of all time—the

Law of Universal Gravitation:

$$F_{net} = ma$$

$$W = mg$$

$$F_c = \frac{mv^2}{r}$$

$$F_g = \frac{GMm}{r^2}$$

This simple algebraic expression Mm/r^2 says how *everything in the universe* is related to *everything else*—a far-reaching statement indeed!

Although the orbits of the planets are ellipses, they are *very* close to circles. The gravitational pull of the sun provides the force that causes

$F_g = F_c$

the planet to go in its nearly circular orbit.

$$F_c = F_g$$

The gravitational pull of the Sun provides the centripetal force of the satellite.

$$F_c = F_g$$

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

Gravity provides the centripetal force of the satellite.

$$F_c = F_g$$

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$v^2 = \frac{GM}{r}$$

Since $v = \frac{2\pi r}{T}$

We can square both sides:

$$v^2 = \left(\frac{2\pi r}{T}\right)^2$$

$$= \frac{4\pi^2 r^2}{T^2}$$

Equating equivalent expressions for v^2:

$$\frac{GM}{r} = \frac{4\pi^2 r^2}{T^2}$$

$$\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$$

The ratio of two measurable quantities—*radius* and *period*—equals a constant.

$$\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$$

$$\frac{r^3}{T^2} = k$$

$$\frac{r^3}{T^2} = k_{Sun}$$

The ratio of two measurable quantities—*radius* and *period*—equals a constant.

$$\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$$

$$= k_{Sun}$$

If the distance of the planets to the sun are expressed in convenient units like astronomical units (1AU = the distance from the earth to the sun) and the period *T* is expressed in earth years, then the constant *k* equals 1!

But the same analysis for the planets orbiting the sun applies to moons orbiting Jupiter and can be extended to pairs of stars orbiting their common center of mass. This is how astronomers determine the mass of distant planets and stars.

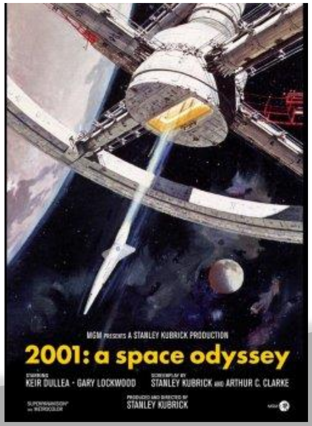
$$M_{Sun} = \frac{4\pi^2}{G}\left(\frac{r_{Sun}^3}{T_{Sun}^2}\right)$$

$$M_J = \frac{4\pi^2}{G}\left(\frac{r_J^3}{T_J^2}\right)$$

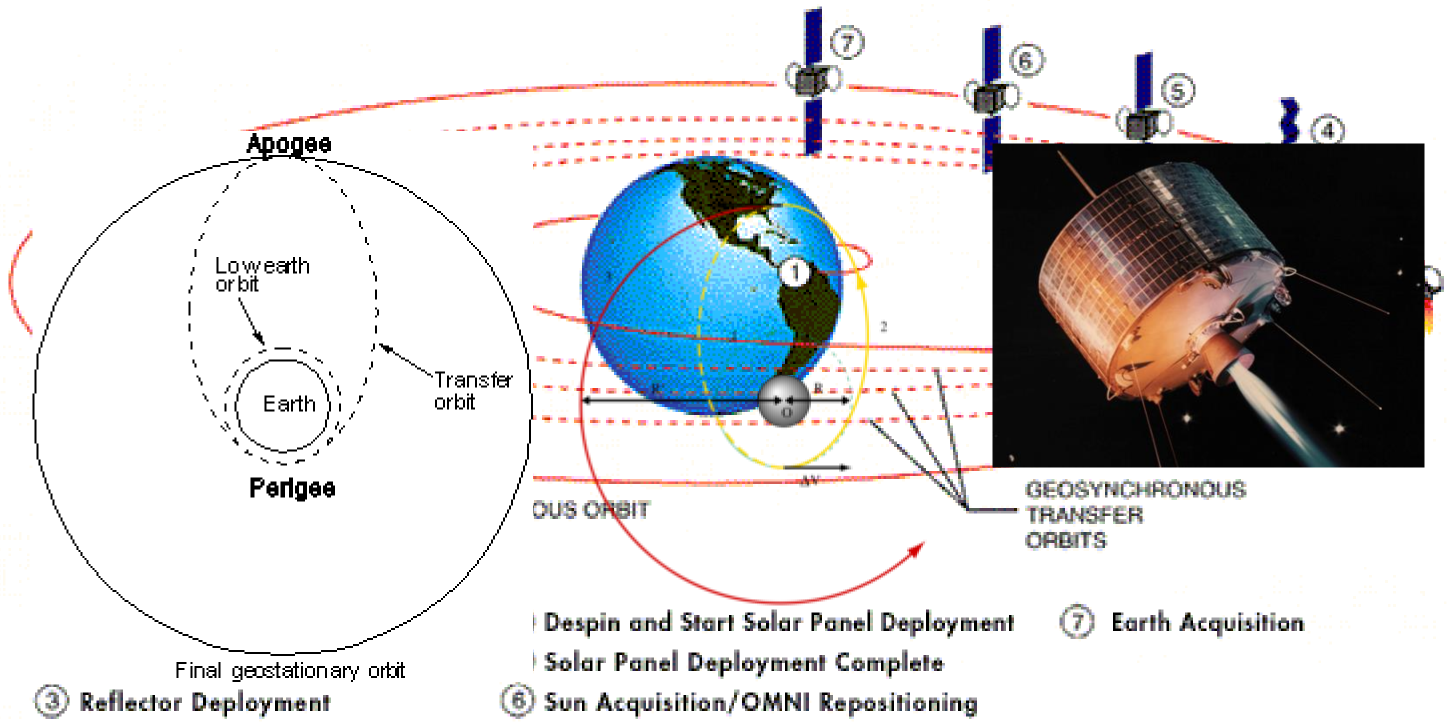
$$m_1 + m_2 = \frac{4\pi^2}{G}\left(\frac{r_{Stars}^3}{T_{Stars}^2}\right)$$

Geosynchronous Orbits

In 1945, British journalist Arthur C. Clarke who later become one of the most famous science fiction novelists of all time proposed that the new invention TV might be someday broadcast from satellites in so-called *geosynchronous orbits* (literally meaning earth-synchronized) from outer space—22,300 miles from the earth's surface. At this distance the orbital period of a satellite equals the rotational period—24 hours for us here on earth. Satellites in this position always appear above the earth in the same point in the sky. Dubbed unfeasible by some and impossible by others, he was largely ignored because of the great distances involved.



Inserting Satellites in Geosynchronous Orbits



How to find *g* at a distance greater than the earth's surface:

$$F_{net} = F_g$$

$$mg = \frac{GMm}{r^2}$$

$$g = \frac{GM}{r^2}$$

Finding the Tangential Speed of the Satellite:

$$F_g = F_c$$

$$mg = \frac{mv^2}{r}$$

$$v = \sqrt{rg}$$

At the surface of the earth *r* is about 6378.137 km

$$v = \sqrt{(6,400,000)(9.8)}$$

$$= 8000 \text{ m/s}$$

$$= 8 \text{ km/s}$$

Equating Equivalent Expressions for the Tangential Velocity and Solving for *r*:

$$v = \sqrt{rg} = \frac{2\pi r}{T}$$

$$v^2 = rg = \frac{4\pi^2 r^2}{T^2}$$

$$r = g\left(\frac{T^2}{4\pi^2}\right)$$

Substitute the value of *g* at the geosynchronous orbit and solve for *r*:

$$r = \left(\frac{GM}{r^2}\right)\left(\frac{T^2}{4\pi^2}\right)$$

$$r^3 = \frac{GM}{4\pi^2}\left(T^2\right)$$

$$r = \sqrt[3]{\frac{GM}{4\pi^2}\left(T^2\right)}$$

Substituting the known values for the universal gravitational constant, *G*, the mass of the earth, *M*, and the number of seconds in a year, *T*, the distance is:

$$r = \sqrt[3]{\frac{GM}{4\pi^2}\left(T^2\right)}$$

$r =$ 22,300 miles above the earth's surface

The slope of the graph of log of R vs. T is the functional relationship between the variables

$$\frac{r^3}{T^2} = k$$

$$r^3 = kT^2$$

$$\log\left(r^3\right) = \log\left(kT^2\right)$$

$$3\log r = 2\log T + \log k$$

$$\log r = \frac{2}{3}\log T + \frac{1}{3}\log k$$

$$y = mx + b$$

$$\therefore m = \frac{2}{3}$$

UNIT -III
PERTURBED SATELLITE ORBIT

- **Orbital perturbations**
- **Launches and launch vehicles**
- **Placing a satellite in a geo-stationary orbit**
- **Orbital effects**
- **Examples**

Important note: Slides present summary of the results. Detailed derivations are given in notes.

Sources of orbital perturbations

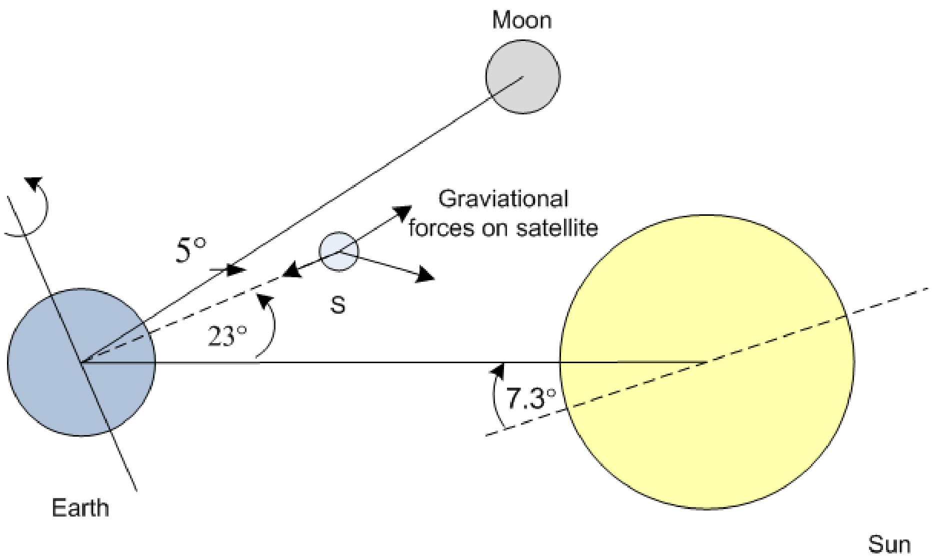
- Orbital perturbation – difference between real orbit and Keplerian orbit obtained from two body equations of motion
- Major sources of orbital perturbations
  - Perturbations due to non-ideal Earth
  - Third body perturbations
  - Atmospheric drag
  - Solar radiation and solar wind
- Importance of perturbation source depends on the satellite altitude
- Modeling of the real satellite orbit
  - Find "osculating orbit" at some time
  - Assume orbit elements vary linearly with time
  - Use measured data to determine rate of change for orbital parameters

Effects of non-ideal Earth

- Earth is not a sphere
  - Equatorial radius: ~ 6,378 km
  - Polar radius: ~ 6,356 km (about 22km smaller)
  - Equatorial radius not constant (small variations ~ 100m)
- Earth mass distribution
  - Earth mass distribution not uniform
  - Regions of mass concentrations: mascons
- Non-ideal Earth causes non-ideal gravitational field
- For LEO and MEO satellites this effect is not very significant
- GEO satellites are impacted the most
  - GEO satellites drift towards mascons in the “east-west” direction
  - Longitudes of two stable equilibrium points: 75°E and 252°E (or 108°W)
  - Longitudes of two unstable equilibrium points: 162°E and 348°E (or 12°W)

Third-body perturbation

- Motion of the satellite is not a "two-body" problem
- Satellite experiences gravitational pull from Sun and Moon as well
- The orbital relationship is complex and time dependent
- Gravitational forces from Sun and Moon tend to move satellite out of the orbit
- Under these condition the orbit will precess and its inclination will change (up to 1°/year)
- In practice:
  - Some of these effects are planed
  - Most of these effects are corrected using the on-board fuel
  - For GEO stationary satellite, the goal is to keep its apparent position within 0.05 degree box



Orbital position of a satellite the Earth, the Sun and the Moon

Note: Orbital parameters are continuously measured and published as TLE data

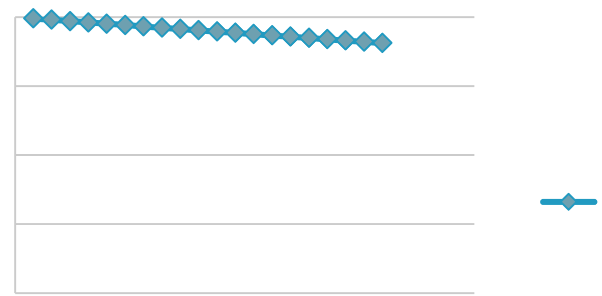
Atmospheric drag

- **Significant for satellites below 1000km**
- **Drag reduces the velocity of the satellite**
  - **Semi-major axis is reduced**
  - **Eccentricity is reduced**
- **Approximate equation for change of semi-major axis**

$$a = a_0 \left[1 + \frac{n_0'}{n_0} \left(t - t_0 \right) \right]^{-2/3}$$

where
a_0 – semi-major axis at t0
n_0 – mean motion (revs/day)
n'_0 – first derivative of mean motion (rev/day^2)

Example: Consider ISS. From TLE data
Mean motion: 15.72137770 rev/day
Derivative of MM/2: 0.00011353 rev/day^2



Note: change is relatively small and it is long term. In practice it can be easily resolved through satellite maneuvering.

Note: Derivative of mean motion is provided in TLE data

Propagators

- **Determine position of a satellite over time**
- **Three types:**
  - **Analytic**
  - **Semi-analytic**
  - **Numerical**
- **Analytic – approximate motion of the satellite through closed form equations**
  - **TwoBody – considers only forces of gravity from Earth, which is modeled as a point mass**
  - **J2 Perturbation – models Earths oblateness, solar and lunar gravitational forces**
  - **J4 Perturbation – second order improvements of J2 Perturbation**
  - **SGP4 - Simplified General Perturbations**

- Semi-analytic – combination of analytical and numerical methods
  - LOP – Long-term Orbital Predictor. Uses same elements as analytic propagators. Accurate over many months.
  - SGP4 for non LEO satellites
- Numerical – Solve complex sets of differential equations. Accuracy is obtained at the expense of computational speed
  - Astrogator – used for trajectory and maneuver planning and includes targeting capabilities
  - HPOP – High Precision Orbital Propagator. Uses the same orbital elements as the analytic propagators.

Note: Propagators are included in commercial software packages

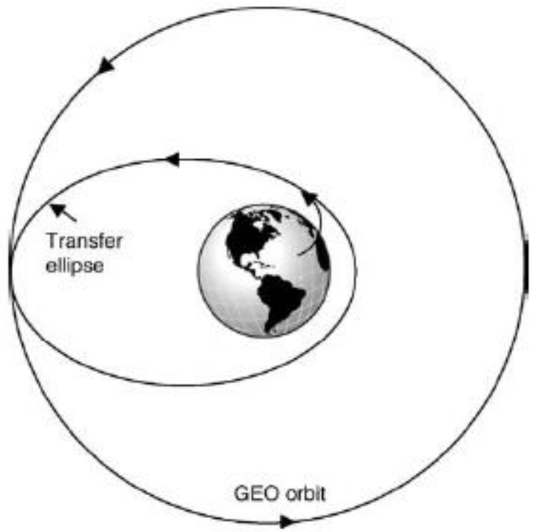
Satellite launch

- Two categories of satellite launches
  - Expandable launch vehicles
    - Ariane (EU)
    - Atlas, Delta (US)
    - Soyuz (Russia)
  - Reusable launch vehicles
    - Space Shuttle (STS) – up to 2011
    - Dragon, Falcon 9 (Space X)
    - Buran? Orion?
- Launches are usually done in several stages
  - Launch vehicle is used to place satellite in one of the transfer LEO orbits
  - Satellite is maneuvered from a transfer orbit into the final orbit
- Due to Earth rotation – the launch is easiest from equator
  - Velocity boost from Earth rotation ~ 0.47km/sec
  - LEO orbits require velocities ~ 7.5km/sec
  - Launching from equator ~ 6% fuel savings
- Launches from sites that are not on the equator place satellites in inclined orbits
  - If the satellite is to be placed in GEO stationary orbit, the correction of the orbit inclination needs to be performed

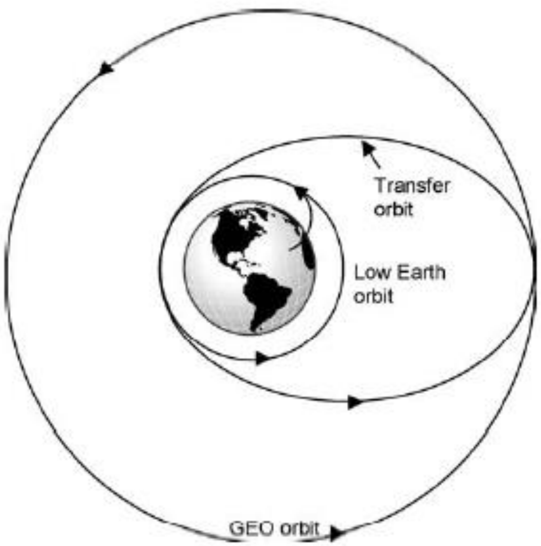
Satellite Launch Site	Latitude	Longitude
Alcantara, Brazil	2.3S	44.4E
Cape Canaveral, Florida, USA	28.5N	81.0W
Edwards Air Force Base, California, USA	35N	118W
Cape Canaveral, Florida, USA	28.5N	81.0W
Jiuquan, China	40.6N	99.9E
Kagoshima, Japan	31.2N	131.3E
Kourou, French Guiana	5.2N	52.8W
Kapustin Yar, Russia	48.4N	45.8E
Palmachim Air Force Base, Israel	31.5N	34.5E
Plesetsk, Russia	62.8N	40.1E
Shar Centre, Srihrikota, India	13.9N	80.4E
Svobdny, Russia	51.37N	128.3E
Taiyuan, China	37.5N	112.6E
Tanagashima, Japan	30.4N	131.0E
Torrejon, Spain	40.488N	3.457E
Tyuratam, Kazakhstan	45.6N	63.4E
Wallops Island, Virginia, USA	37.8N	75.5W
Woomera, Australia	31.1S	136.8E
Western Test Range, Vandenberg, California, USA	34.4N	120.35W
Xichang, China	28.25N	102E

Some of the most frequently used launch sites

Two typical GEO satellite launch approaches



Case 1: single elliptical orbit

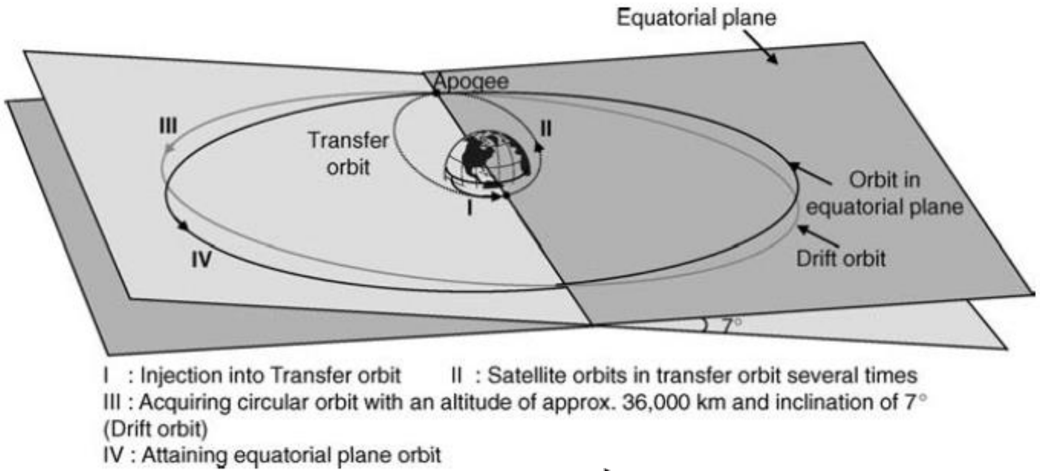


Case 2: two transfer orbits

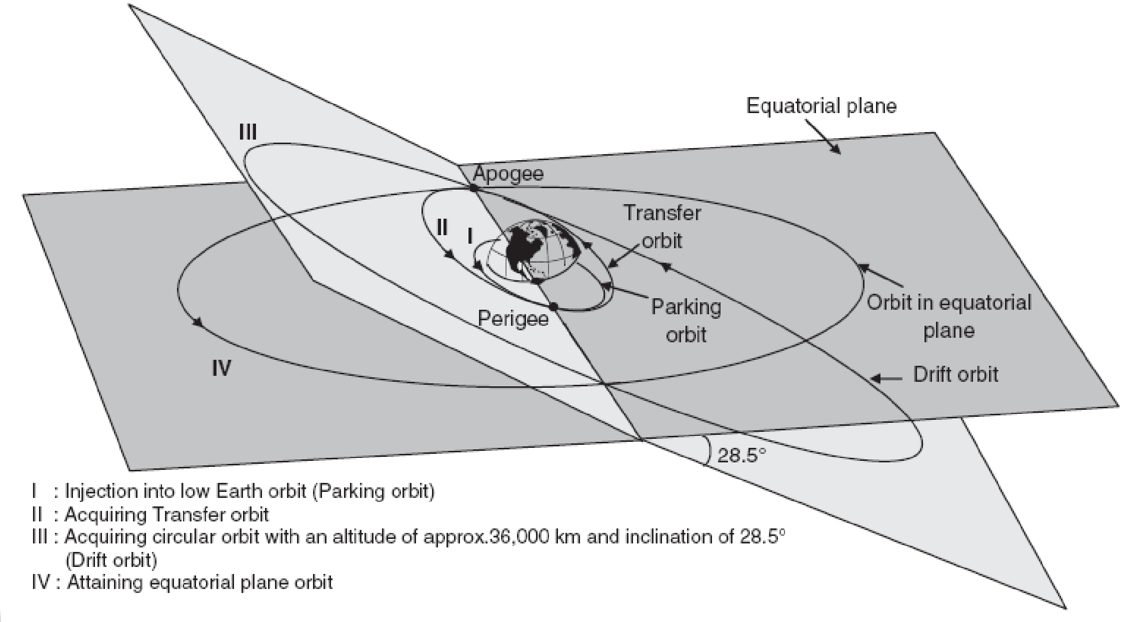
- **Case 1: Launch vehicle puts the satellite into an elliptical transfer orbit**
  - **The orbit has high eccentricity with apogee at the geostationary (or geosynchronous) orbit**
  - **When the satellite passes through apogee, additional velocity is given to the satellite by Apogee Kick Motor (AKM) and the satellite changes orbits**
  - **If the orbit is not GEO stationary, additional adjustments are required to correct for the orbit inclination**
- **Case 2: Lunch vehicle put the satellite into circular LEO orbit**
  - **Two maneuvers are required**
  - **Perigee maneuver to transform circular orbit into elliptical orbit, and**
  - **Apogee maneuver to transform elliptical orbit into GEO stationary orbit**

Note: other launch approaches are possible

Example of GEO launches: French Guyana and KSC



Launch from French Guyana



Launch from Cape Carnival, FL

Orbital Effects

- Motion of the satellite has significant impact on its performance
- Some of the orbital effects to take into account
  - Doppler shift
  - Solar eclipse
  - Sun transit outage

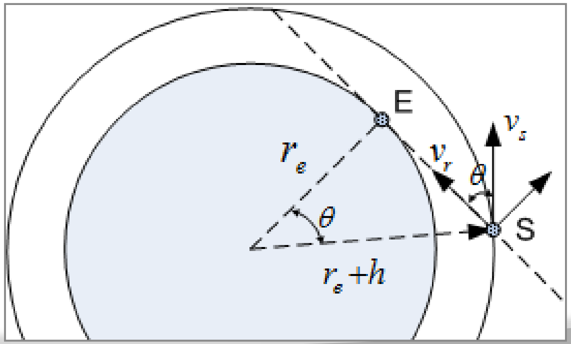
Doppler shift

- **Important in case of LEO and MEO satellites**
- **Negligible for GEO satellites**
- **Caused by relative motion between the satellite and earth station**
- **The magnitude of Doppler shift is given by:**

$$\frac{f_R - f_T}{f_T} = \frac{\Delta f}{f_T} = \frac{V_T}{c}$$

$$\Delta f = V_T f_T / c = V_T / \lambda$$

Where: f_R – received frequency, f_T – transmit frequency, V_T – velocity between satellite and Earth station, λ – wavelength, c – speed of light



Geometry used for calculation of maximum velocity

Example. Consider LEO satellite in circular polar orbit with altitude of 1000km. Transmit frequency is 2.65GHz.
The velocity of satellite in orbit

$$T^2 = 4\pi^2 (r_e + h)^3 / \mu \rightarrow 6306.94\,\text{sec}$$

$$v_s = 2\pi (r_e + h)/T \rightarrow 7.350\,\text{km/sec}$$

Largest component of velocity between satellite and Earth station occurs when the satellite is coming out of horizon directly over the station

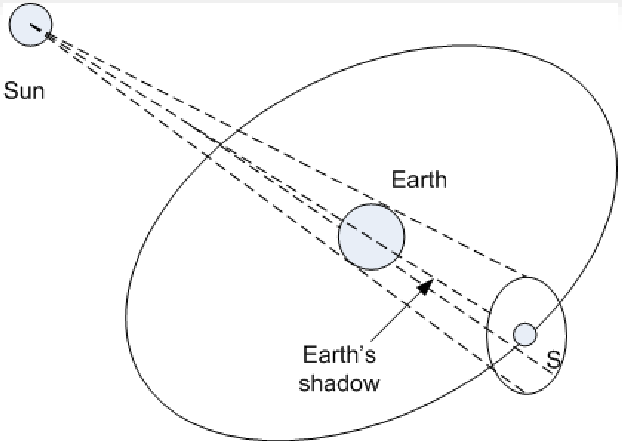
$$v_{r\max} = v_s \cos(\theta) = v_s \frac{r_e}{r_e + h} \rightarrow 6.354\,\text{km/sec}$$

Maximum Doppler shift

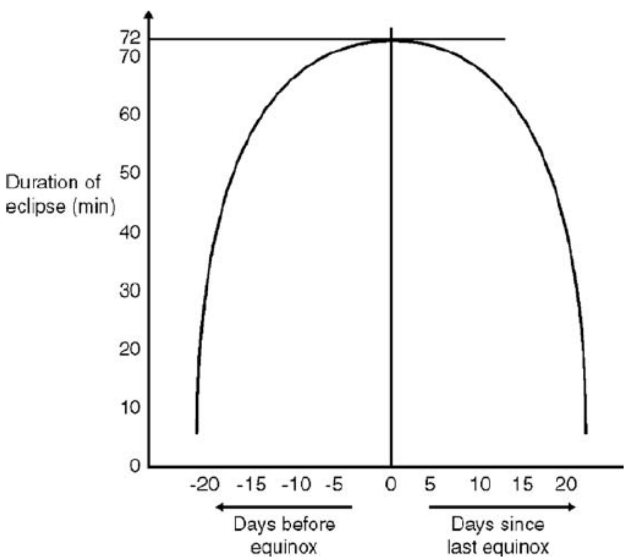
$$\Delta f_{\max} = v_{r\max} f_T / c \rightarrow 56.13\,\text{kHz}$$

Note: the shift becomes larger with higher frequencies.

Solar eclipse



Geometry of solar eclipse

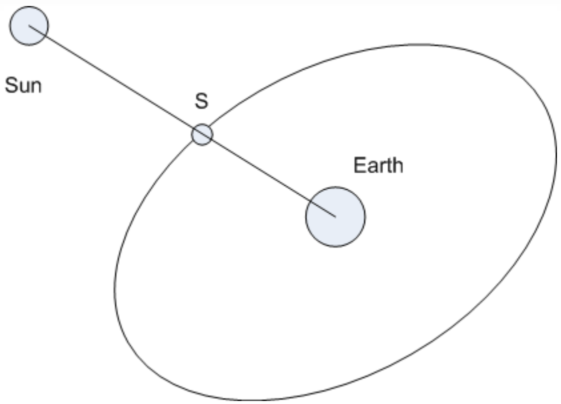


Duration of solar eclipse of a geo stationary satellite relative to equinox time

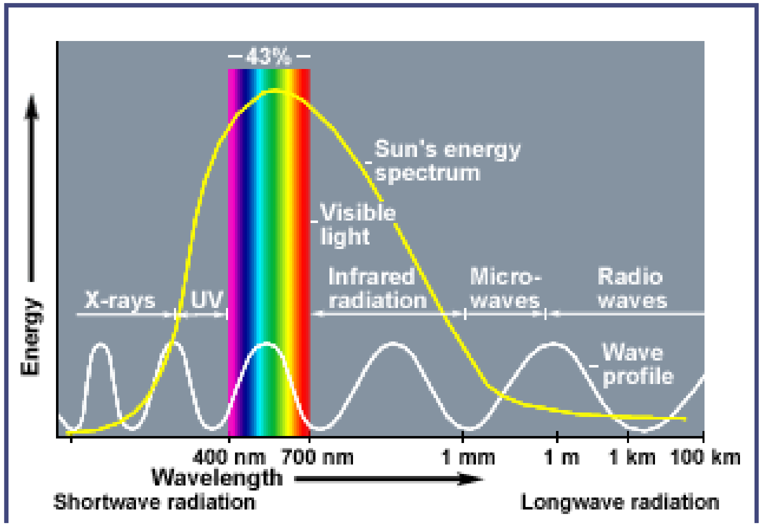
- **Eclipse – sunlight fails to reach satellite due to obstruction from either Earth or Moon**
- **More common type of eclipse – eclipse due to satellite coming in the shadow of the earth**
- **For geostationary satellite the solar eclipse occurs for about 82 days every year**
- **The eclipse times vary from day to day**
- **The longest eclipses are on equinox days (days when the sun passes through equatorial plane) – Nominally: March 21st and September 23st**
- **When the satellite passes through eclipse**
  - **It loses its source of power (relies on battery power)**
  - **It may need to shut off some of its transponders**
  - **Experiences cooling down in temperature**
  - **Transients due to switching of the equipment and temperature variations may cause equipment failure**

Note: for non geostationary satellites orbital study necessary to determine eclipse times

Sun transit outage



Geometry of sun transit eclipse



Radiation spectrum of the Sun

- Sun transit outage - occurs when the satellite passes in front of the Sun
- Sun is a source of electromagnetic radiation
- Sun – noise source with equivalent temperature of 6000K – 11000K
- In most cases this noise overcomes link design margins
- During sun transit outage – communication lost
- Duration of the outage as much as 10 min/day
- For geo-stationary satellites outages occur 0.02% of the time
- Duration problematic since it usually occurs in a traffic intensive part of the day
- Satellite providers try to re-route the traffic through other satellites

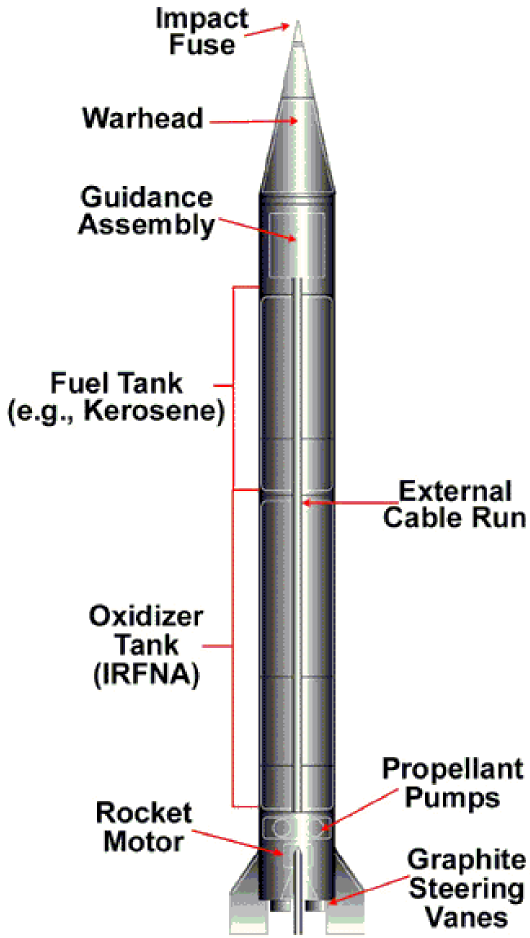
UNIT –IV
BALLISTIC MISSILE

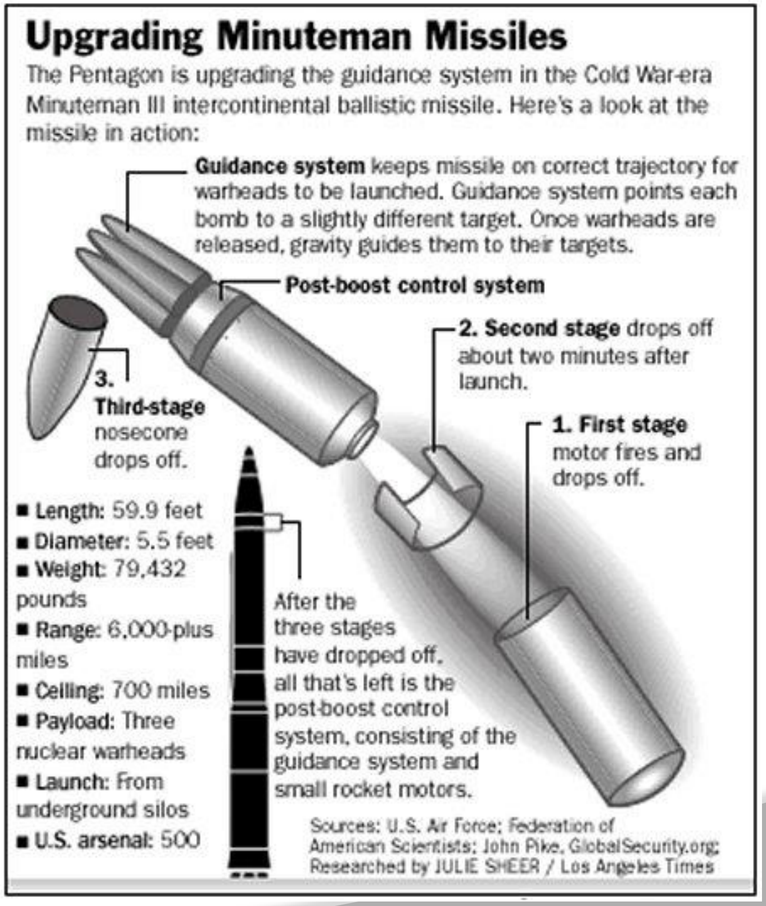
Ballistic Missile

- A ballistic missile is a missile which follows a sub-orbital flight path to a predetermined target.
- A ballistic missile is a missile that has ballistic trajectory over most of its flight path.
- A ballistic missile can deliver one or more warheads to a predetermined target.
- It governed by the laws of orbital mechanics and ballistics (during a part of its entire phase).

- Ballistic missiles are used for transportation of a payload from one point on the Earth (launch site) to another point on the surface of the Earth (impact point or target).
- They are accelerated to a high velocity during a relatively short period.
- Then a re-entry vehicle, containing the warhead, is released and this vehicle then simply coasts in a ballistic or free-fall trajectory to the final impact point.
- To date, ballistic missiles have been propelled by chemical rocket engines of various types.

Ballistic Missile Components





Categorization of Ballistic Missile

United States

Classes	Range
Intercontinental Ballistic Missile (ICBM)	over 5500 kilometers
Intermediate Range Ballistic Missile (IRBM)	3000 to 5500 kilometers
Medium Range Ballistic Missile (MRBM)	1000 to 3000 kilometers
Short Range Ballistic Missile (SRBM)	up to 1000 kilometers

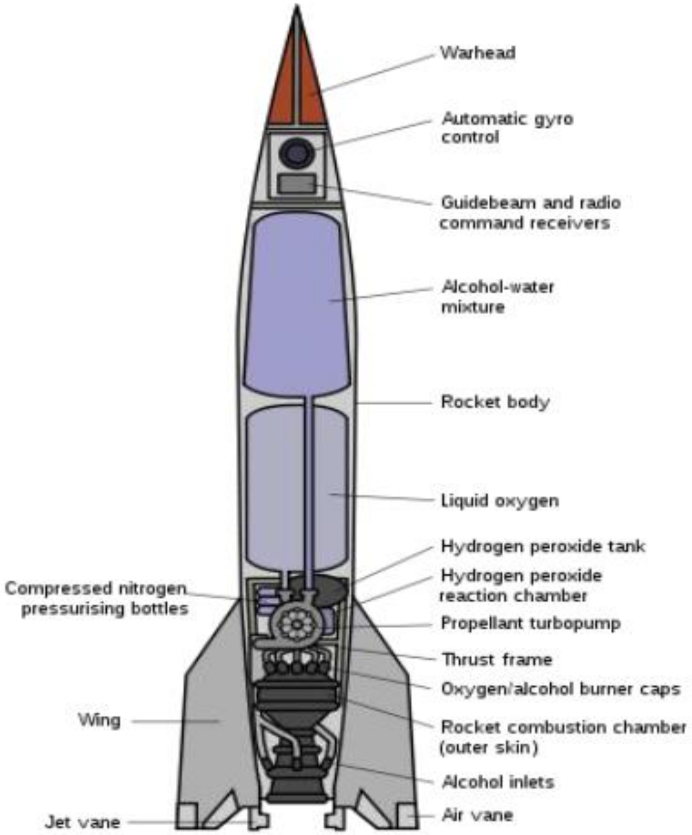
Soviet and Russian Military

Classes	Range
Strategic	over 1000 kilometers
Operational Strategic	500 to 1000 kilometers
Operational	300 to 500 kilometers
Operational Tactical	50 to 300 kilometers
Tactical	up to 50 kilometers

General Category of Ballistic Missile

Classes	Range
Tactical Ballistic Missile (a) Battlefield Range Ballistic Missile (BRBM)	Between 150 to 300 kms Less than 200 kms
Theatre Ballistic Missile (TBM) (a) Short Range Ballistic Missile (SRBM) (b) Medium Range Ballistic Missile (MRBM)	Between 300 to 3500 kms 1000 kms or less Between 1000 to 3500 kms
Intermediate Range Ballistic Missile (IRBM) or Long Range Ballistic Missile (LRBM)	Between 3500 to 5500 kms
Intercontinental Ballistic Missile (ICBM)	More than 5500 kms

Indian Ballistic Missile



V-2, the first ballistic missile

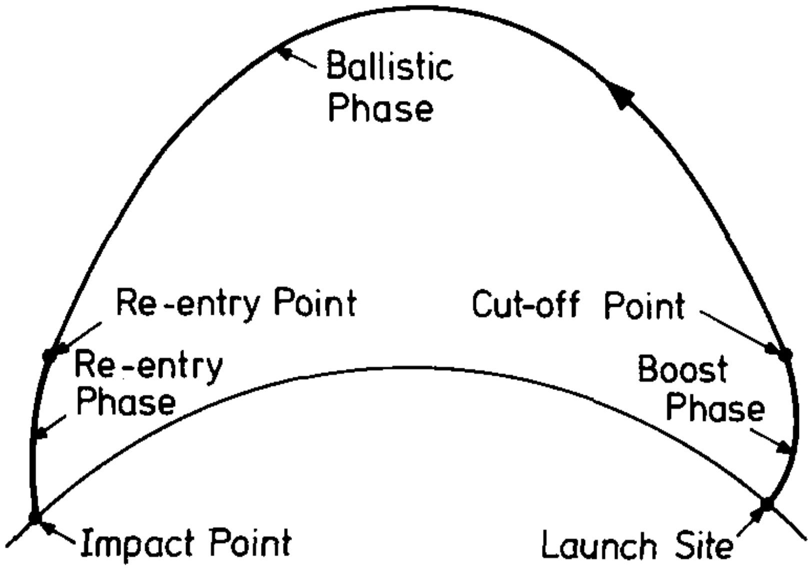
India's Agni-II MRBM

Ballistic Missile

Trident II, a submarine launched ballistic missile (US Navy)

Ballistic Missile Trajectory

The trajectory of a ballistic missile differs from a satellite orbit in only one respect – it intersects the surface of the Earth. Otherwise, it follows a conic orbit during the free-flight portion of its trajectory.



Ballistic Missile Trajectory

- A ballistic missile trajectory is composed of three parts:

(1) the **powered flight portion** which lasts from launch to thrust cutoff or burnout (3–5 minutes, 150–400 km altitude, 7 km/s burnout speed)

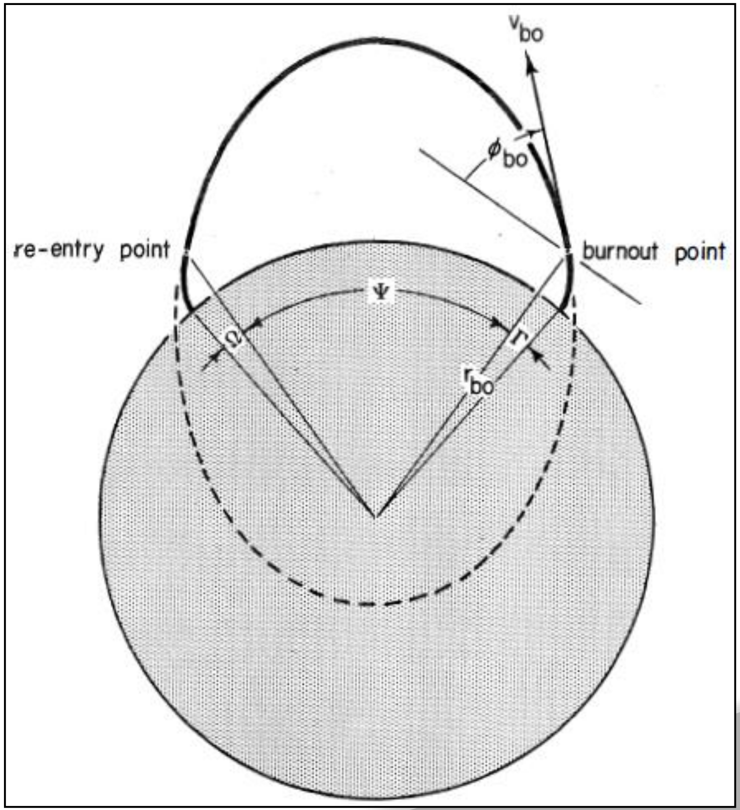
(2) the **free-flight portion** which constitutes most of the trajectory (approx. 25 minutes, apogee altitude approx. 1200 km, semi-major axis between 3186–6372 km)

(3) the **re-entry portion** which begins at some ill-defined point (altitude of 100 km) where atmospheric drag becomes a significant force in determining the missile's path and lasts until impact (2 minutes to impact at a speed of up to 4 km/s)

Ballistic Missile Trajectory

- Powered flight (guidance and navigation system)
- During free-flight, trajectory is part of a conic orbit – almost always an ellipse
- Re-entry involves the dissipation of energy by friction with the atmosphere

$$\Lambda = \Gamma + \Psi + \Omega$$



Non-dimensional Parameter

- Here we need to define a non-dimensional parameter *Q* as

$$Q = \left(\frac{v}{v_{cs}} \right)^2 = \frac{v^2 r}{\mu}$$

- Note that the value of *Q* is not constant for a satellite but varies from point to point in the orbit. (*Q* = 1, 2 or > 2)

- From the energy equation $\xi = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$ we can prove

$$a = \frac{r}{2-Q} \quad or \quad Q = 2 - \frac{r}{a}$$

Free-flight Range Equation

- Objective is to get a simple expression for the free-flight range (Ψ) of a missile in terms of its **burnout conditions**.

- Initial assumption that the Earth does not rotate and that the altitude at which re-entry starts is the same as the burnout altitude (symmetrical free-flight trajectory).

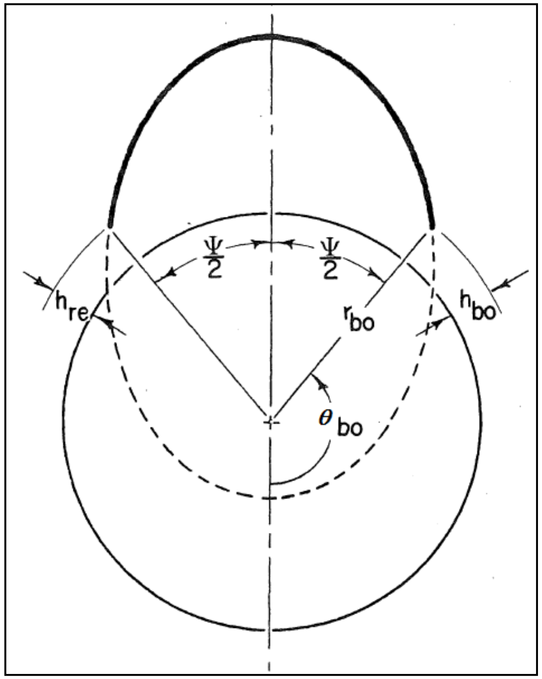
- Since the free-flight trajectory of a missile is a conic section, the general equation of a conic can be applied to the burnout point.

$$r_{bo} = \frac{p}{1 + e\cos\theta_{bo}}$$

- Solving for $\cos\theta_{bo}$ we get

$$\cos\theta_{bo} = \frac{p - r_{bo}}{r_{bo}\, e}$$

Free-flight Range Equation



Since $h_{bo} = h_{re}$, half the free-flight range angle (Y) lies on each side of the major axis, and

$$\cos\frac{\Psi}{2} = -\cos\theta_{bo}$$

And equation can be written as

$$\cos\frac{\Psi}{2} = \frac{r_{bo} - p}{r_{bo}\, e} \qquad (6)$$

Equation (6) is an expression for the free-flight range angle in terms of *p*, *e*, and r_{bo}.

Free-flight Range Equation

- Since $p = h^2/\mu$ and $h = rv\cos\phi$, we can use the definition of parameter *Q* to obtain

$$p = \frac{r^2 v^2 \cos^2\phi}{\mu} = rQ\cos^2\phi$$

- Now, since

$$p = a\left(1 - e^2\right), \;\Rightarrow e^2 = 1 - \frac{p}{a}$$

- From above equations, we get

$$e^2 = 1 + Q\cos^2\phi\,(Q - 2)$$

Free-flight Range Equation

- Now substituting equations (7) and (9) into equation (6) we have one form of the **free-flight range equation**:

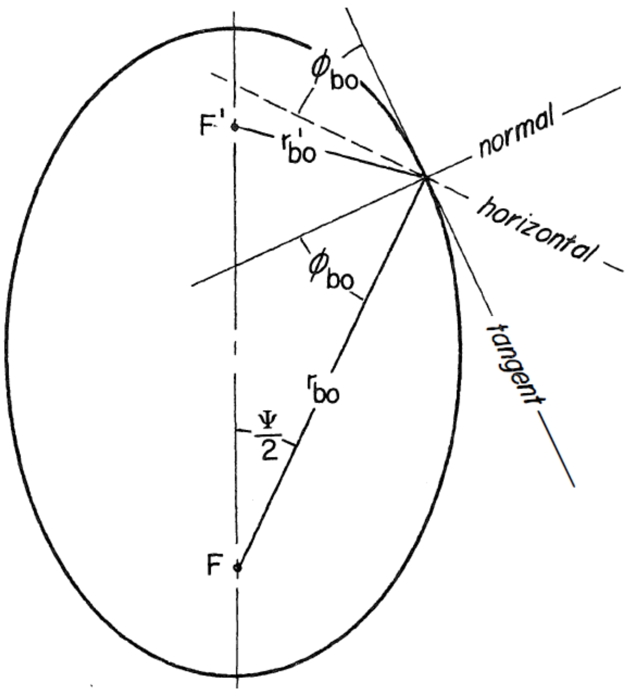
$$\cos\frac{\Psi}{2} = \frac{1 - Q_{bo}\cos^2\phi_{bo}}{\sqrt{1 + Q_{bo}\cos^2\phi_{bo}\left(Q_{bo} - 2\right)}}$$

- Given a particular launch point and target, the total range angle, Λ, can be calculated. If we know how far the missile will travel during powered flight and re-entry, the required free-flight range angle, Ψ, also becomes known.

Flight Path Angle

Flight path Angle Equation

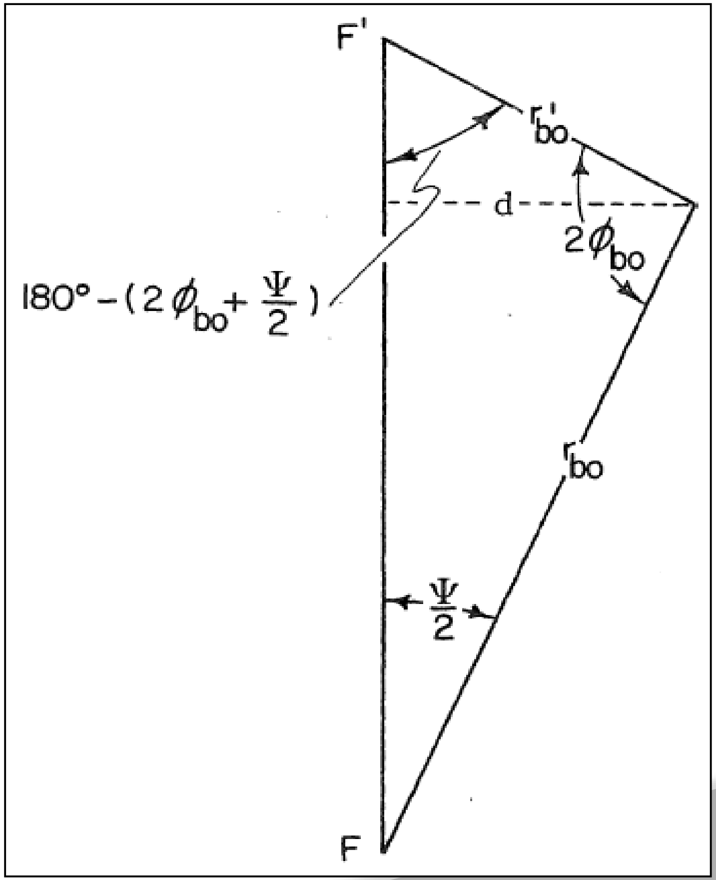
- If we now specify r_{bo} and v_{bo} for the missile, what should the flight-path angle, ϕ_{bo}, be in order that the missile will hit the target?
- So we have to derive an expression for ϕ_{bo} in terms of r_{bo}, v_{bo} and Y.



- So we need to consider a geometry shown here to derive an expression for **<u>flight-path angle equation</u>**.

Flight Path Angle Equation

Let us concentrate on the triangle formed by F, F' and the burnout point. Let us divide the triangle into two right triangles by the dashed line, *d*, as shown in the right hand side diagram.



Flight path Angle Equation

From previous diagram, we can express *d* as $d = r_{bo} \sin \frac{\Psi}{2}$

$$d = r'_{bo} \sin \left[180^\circ - \left(2\phi_{bo} + \frac{\Psi}{2} \right) \right]$$

and also as

Combining the two equations we get

$$\sin\left(2\phi_{bo} + \frac{\Psi}{2} \right) = \frac{r_{bo}}{r'_{bo}} \sin \frac{\Psi}{2}$$

Flight Path Angle Equation

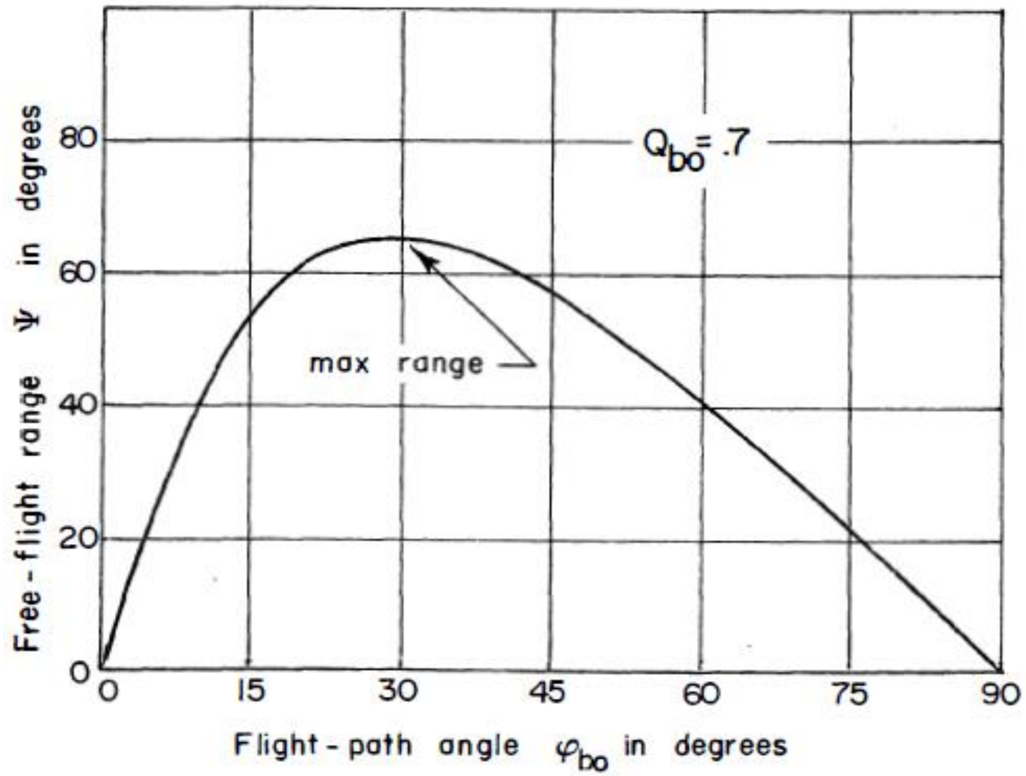
Since $r_{bo} + r'_{bo} = 2a$ and from equation we can write the equation as

$$r_{bo} = a\left(2 - Q_{bo}\right)$$

Following Equation is called the flight path angle equation. Low trajectory and High trajectory.

$$\sin\left(2\phi_{bo} + \frac{\Psi}{2}\right) = \frac{2 - Q_{bo}}{Q_{bo}} \sin\frac{\Psi}{2}$$

Maximum Range Trajectory



Range vs flight-path angle

Maximum Range Trajectory

To derive expressions for the maximum range condition, a simpler method is to see under what conditions the flight-path angle equation yields a single solution.

If the right side of equation (14) equals exactly 1, we get only a single answer for ϕ_{bo}. This must, then, be the maximum range condition.

$$\sin\left(2\phi_{bo} + \frac{\Psi}{2}\right) = \frac{2 - Q_{bo}}{Q_{bo}} \sin\frac{\Psi}{2} = 1$$

$$\Rightarrow 2\phi_{bo} + \frac{\Psi}{2} = 90^\circ \quad \Rightarrow \phi_{bo} = \frac{1}{4}\left(180^\circ - \Psi\right)$$

for maximum range conditions only.

Maximum Range Trajectory

- maximum range angle attainable with a given Q_{bo}.

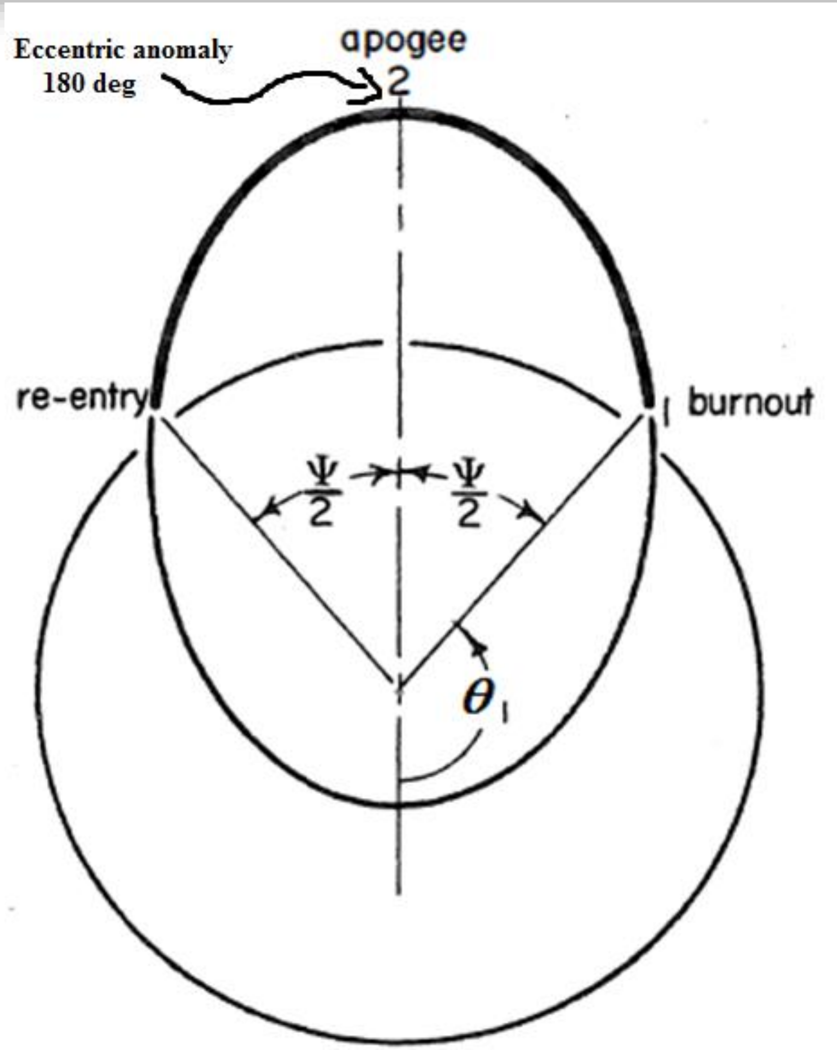
$$\sin\frac{\Psi}{2} = \frac{Q_{bo}}{2-Q_{bo}}$$

for maximum range conditions.

- Solving for Q_{bo}, we get $$Q_{bo} = \frac{2\sin(\Psi/2)}{1+\sin(\Psi/2)}$$

for maximum range conditions.

Time of Free-flight



Ballistic Missile Free Flight Path

Time Of Free-flight

- The value of eccentric anomaly can be computed by taking $\theta_1 = 180^\circ - \Psi/2$ as

$$\cos E_1 = \frac{e - \cos(\Psi/2)}{1 - e\cos(\Psi/2)}$$

- And the time of free-flight can be obtained from

$$t_{ff} = 2\sqrt{\frac{a^3}{\mu}}\left(\pi - E_1 + e\sin E_1\right)$$

Effect of Launching Errors on Range

- Variations in the speed, position, and launch direction of the missile at thrust cutoff will produce errors at the impact point.

- These errors are of two types – errors in the intended plane which cause either a long or a short hit, and out-of-plane errors which cause the missile to hit to the right or left of the target.

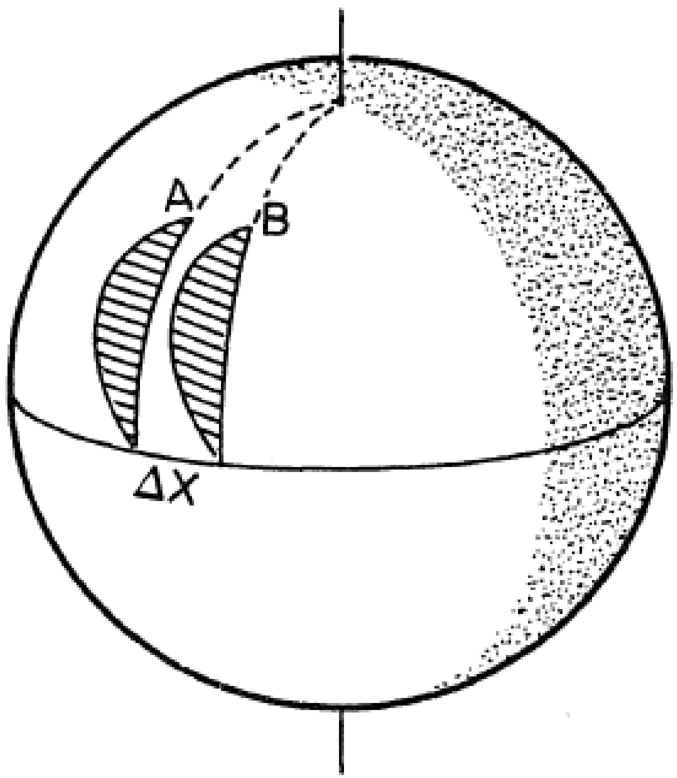
- We will refer to errors in the intended plane as "**down-range**" errors, and out-of-plane errors as "**cross-range**" errors.

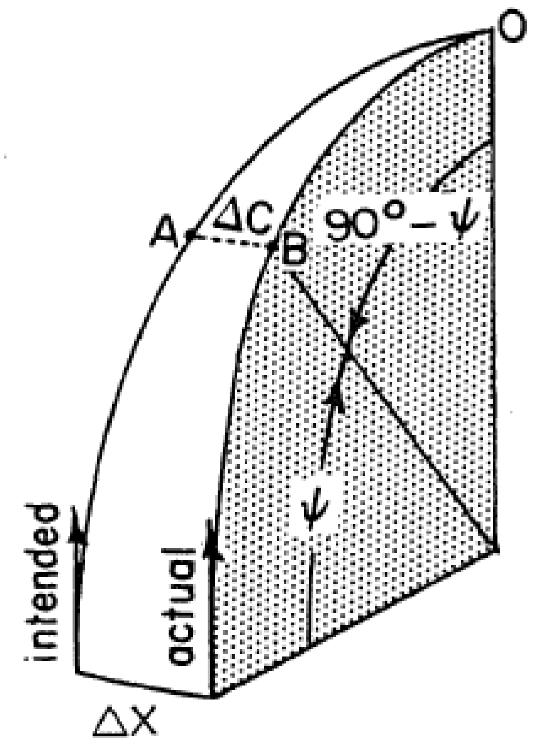
Cross-Range and Down-Range Errors

- There are two possible sources of **cross-range error**:
  - Lateral displacement of the burnout point.
  - Incorrect Launch Azimuth.

- And the sources of **down-range error** are:
  - Down-range displacement of the burnout point.
  - Errors in burnout flight-path angle.
  - Incorrect burnout height.
  - Incorrect speed at burnout.

Effect of Lateral Displacement of Burnout Point





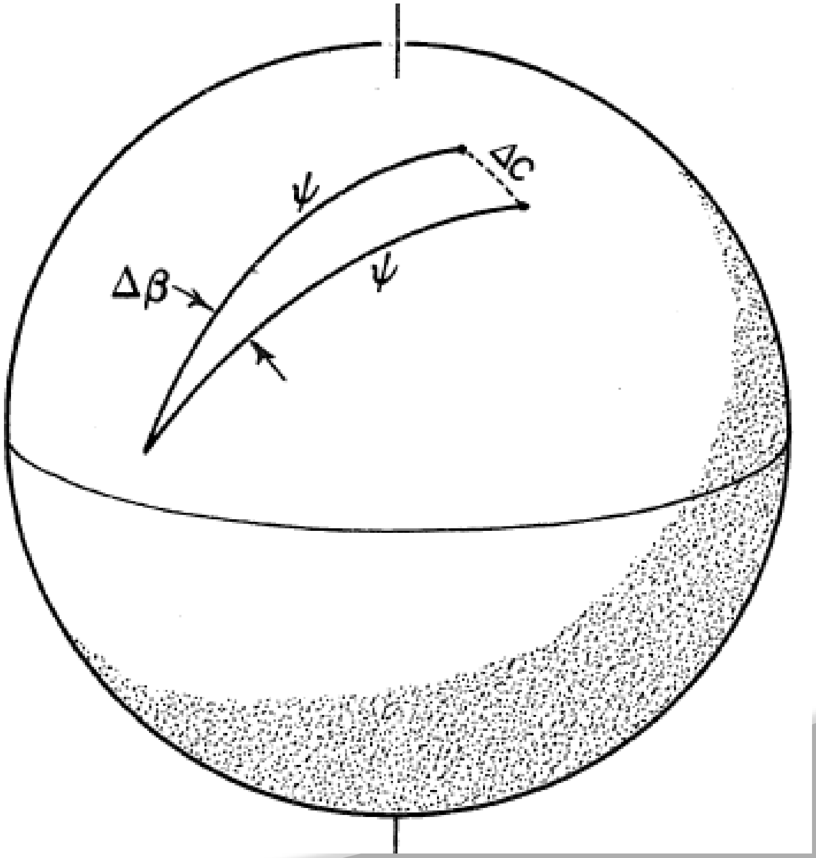
$$\cos \Delta C = \sin^2 \Psi + \cos^2 \Psi \cos \Delta X \quad \Rightarrow \quad \boxed{\Delta C \approx \Delta X \ \cos \Psi}$$

Cross-range Error due to Incorrect Launch Azimuth

➢ If the actual launch azimuth differs from the intended launch azimuth by an amount, *Δb*, a cross-range error, *ΔC*, will result.

$$\cos \Delta C = \cos^2 \Psi + \sin^2 \Psi \cos \Delta \beta$$

$$\Delta C \approx \Delta \beta \sin \Psi$$



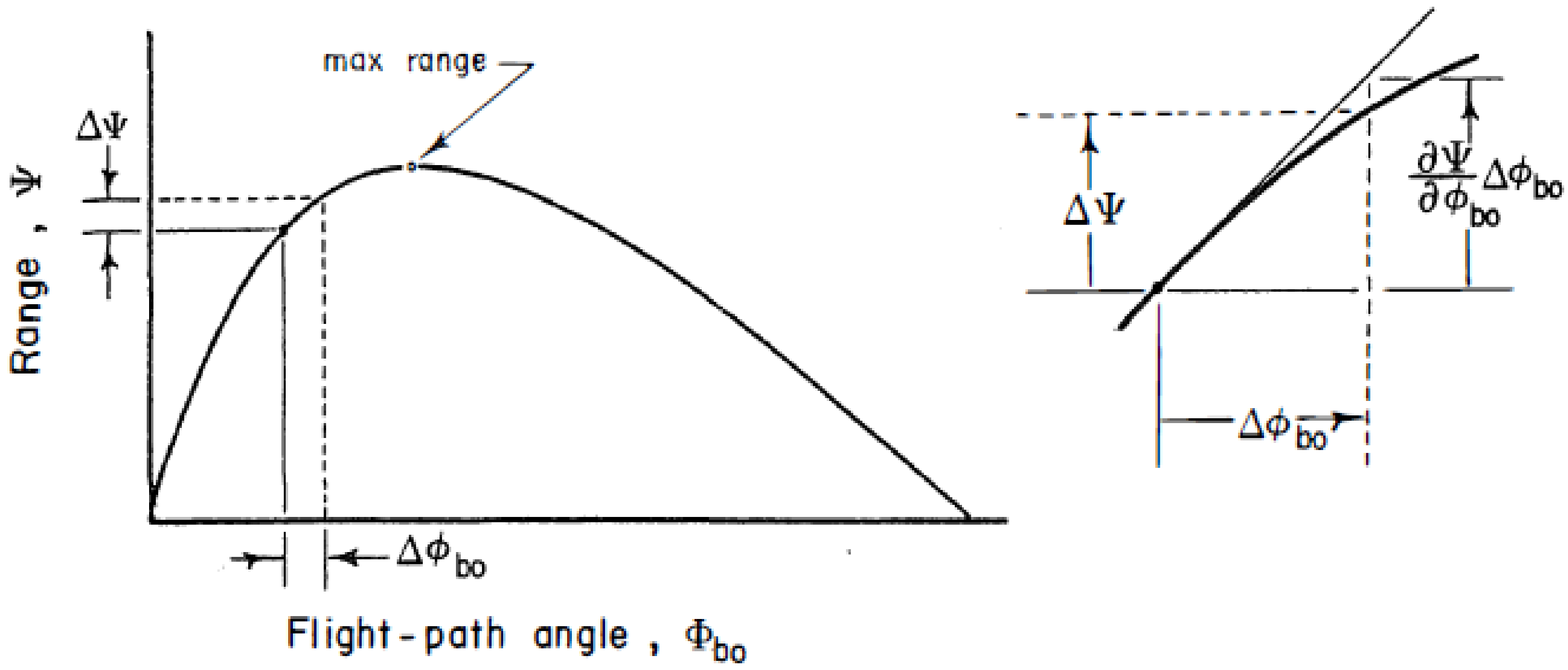
Effect of Down-Range Displacement of the Burnout Point

- An error in down-range position at thrust
  - cutoff produces an equal error at impact.

- If the actual burnout point is 1 nm farther down-range than was intended,
  - the missile will overshoot the target by exactly 1 nm.

Burnout Flight-path Angle Errors on Range

In the above graph DY will represent a down-range error causing the missile to undershoot or overshoot the target.

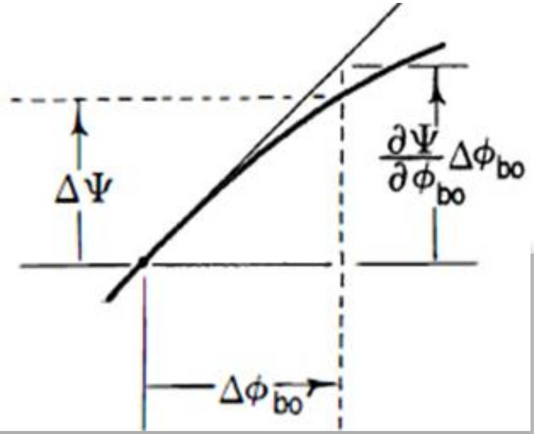


Burnout Flight-path Angle Errors on Range

A good approximate value for DY for very small values of is given by $\Delta\phi_{bo}$

$$\Delta\Psi \approx \frac{\partial\Psi}{\partial\phi_{bo}}\Delta\phi_{bo}$$

where $\frac{\partial\Psi}{\partial\phi_{bo}}$ is the slope of the curve at the point corresponding to the intended trajectory.



Effect of burnout flight-path angle errors on range

- The expression for $\frac{\partial \Psi}{\partial \phi_{bo}}$ may be obtained by implicit partial differentiation of the free-flight range equation.

- The free-flight range equation can be converted into an alternate form for the simple differentiation.

- Recall the free-flight range equation

$$\cos\frac{\Psi}{2} = \frac{1 - Q_{bo}\cos^2\phi_{bo}}{\sqrt{1 + Q_{bo}\cos^2\phi_{bo}\left(Q_{bo} - 2\right)}}$$

Effect of burnout flight-path angle errors on range

- Let us consider the numerator of equation (10) as *a* and denominator as *b*.

Then $$\cos\frac{\Psi}{2}=\frac{\alpha}{\beta}\quad and\quad \cot\frac{\Psi}{2}=\frac{\alpha}{\sqrt{\beta^2-\alpha^2}}$$

Substituting for *a* and *b* we get $$\cot\frac{\Psi}{2}=\frac{1-Q_{bo}\cos^2\phi_{bo}}{Q_{bo}\cos\phi_{bo}\sqrt{1-\cos^2\phi_{bo}}}$$

But $\sqrt{1-\cos^2\phi_{bo}}=\sin\phi_{bo}$, therefore,

$$\cot\frac{\Psi}{2}=\frac{1-Q_{bo}\cos^2\phi_{bo}}{Q_{bo}\cos\phi_{bo}\sin\phi_{bo}}$$

Burnout Flight-path Angle Errors On Range

Effect of burnout flight-path angle errors on range

- Since $\sin 2x = 2\cos x \sin x$, we can further

simplify to obtain

$$\cot\frac{\Psi}{2} = \frac{2}{Q_{bo}}\csc 2\phi_{bo} - \cot\phi_{bo}$$

Now express the above equation in terms of r_{bo}, v_{bo} and ϕ_{bo},

$$\cot\frac{\Psi}{2} = \frac{2\mu}{v_{bo}^2\, r_{bo}}\csc 2\phi_{bo} - \cot\phi_{bo}$$

Now we can differentiate equation (26) implicitly with respect to ϕ_{bo} considering r_{bo}, v_{bo} as constants.

Effect of burnout flight-path angle errors on range

$$\Rightarrow \frac{\partial \Psi}{\partial \phi_{bo}} = \frac{2\sin\left(\Psi + 2\phi_{bo}\right)}{\sin 2\phi_{bo}} - 2$$

- This partial derivative is called an influence coefficient since it influences the size of the range error resulting from a particular burnout error.

- Therefore the free-flight range error due to burnout flight-path angle error is given by

$$\Delta\Psi = \left(\frac{2\sin\left(\Psi + 2\phi_{bo}\right)}{\sin 2\phi_{bo}} - 2\right)\Delta\phi_{bo}$$

Incorrect Burnout Height

Down-Range Errors caused by Incorrect Burnout Height

- Again a good approximate value for DY for very small values of Δr_{bo} is given by

$$\Delta\Psi \approx \frac{\partial\Psi}{\partial r_{bo}}\Delta r_{bo}$$

- Again differentiating the equation (26) implicitly

with respect to r_{bo}, and solving for $\frac{\partial\Psi}{\partial r_{bo}}$, we get

$$\frac{\partial\Psi}{\partial r_{bo}} = \frac{4\mu}{v_{bo}^2 r_{bo}^2}\frac{\sin^2\frac{\Psi}{2}}{\sin 2\phi_{bo}}$$

Incorrect Speed at Burnout

Down-Range Errors caused by Incorrect Speed at Burnout

- A good approximate value for DY for very small values of is given by

$$\Delta\Psi \approx \frac{\partial\Psi}{\partial v_{bo}} \Delta v_{bo}$$

Δv_{bo}

- Again differentiating the equation (26) implicitly

with respect to v_{bo}, and solving for $\frac{\partial\Psi}{\partial v_{bo}}$, we get

$$\frac{\partial\Psi}{\partial v_{bo}} = \frac{8\mu}{v_{bo}^3 \, r_{bo}} \frac{\sin^2 \frac{\Psi}{2}}{\sin 2\phi_{bo}}$$

Total Down-Range Error

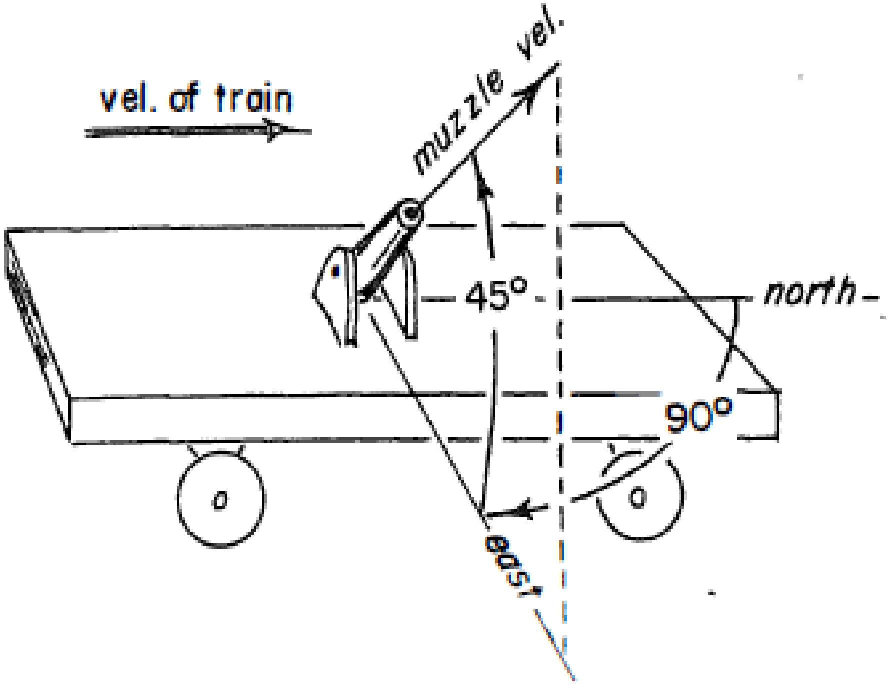
Total down range-error is given by

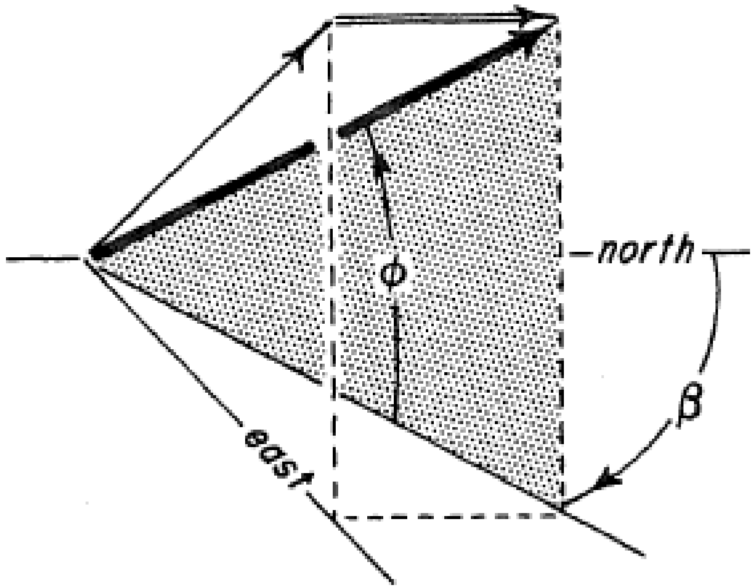
$$\Delta\Psi_{TOTAL} = \frac{\partial\Psi}{\partial\phi_{bo}}\Delta\phi_{bo} + \frac{\partial\Psi}{\partial r_{bo}}\Delta r_{bo} + \frac{\partial\Psi}{\partial v_{bo}}\Delta v_{bo}$$

Effect of Earth Rotation

- The Earth rotates once on its axis in 23 hrs 56 min producing a surface velocity at the equator of approx 0.465 km/sec (or 1524 ft/sec). The rotation is from west to east.
- The free-flight portion of a ballistic missile trajectory is inertial in character. That is, it remains fixed in the XYZ inertial frame while the Earth runs under it.
- Relative to this inertial XYZ frame, both the launch point and the target are in motion.
- Thus we need to compensate for motion of the launch site and the motion of the target due to earth rotation.

Initial Velocity of the Missile





$$V_o = 1524 \cos L_0 \text{ (ft/sec)}$$

Compensating for Initial Velocity of Missile

- The north, east, and up components of the true velocity *v* can be obtained as

$$v_N = v_e \cos\phi_e \cos\beta_e$$

$$v_E = v_e \cos\phi_e \sin\beta_e + v_0$$

$$v_Z = v_e \sin\phi_e$$

- Now the true velocity, flight-path angle, and azimuth can then be found from

$$v = \sqrt{v_N^2 + v_E^2 + v_Z^2}$$

$$\sin\phi = v_Z / v \quad ; \quad \tan\beta = v_E / v_N$$

Compensating for Movement of the Target

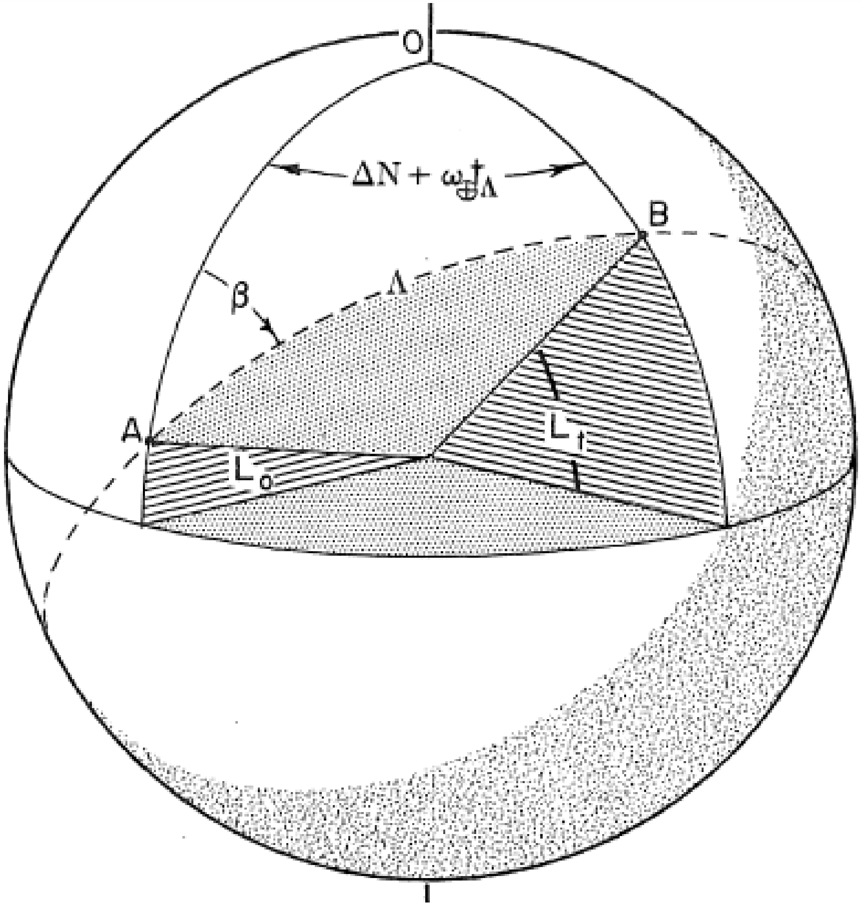


Figure: *Launch site and aiming point at the instant of launch*

Compensating for Movement of the Target

The angle formed at O is just the difference in longitude between the launch point and the aiming point, $\Delta N + \omega_{\oplus} t_{\Lambda}$, where Δ*N* is the difference in longitude between launch point and target.

By considering the launch azimuth, *b* in the spherical triangle,

$$\Rightarrow \cos\Lambda = \sin L_0 \sin L_t + \cos L_0 \cos L_t \cos\left(\Delta N + \omega_{\oplus} t_{\Lambda}\right)$$

$$\sin L_t = \sin L_0 \cos\Lambda + \cos L_0 \sin\Lambda \cos\beta$$

$$\Rightarrow \cos\beta = \frac{\sin L_t - \sin L_0 \cos\Lambda}{\cos L_0 \sin\Lambda}$$

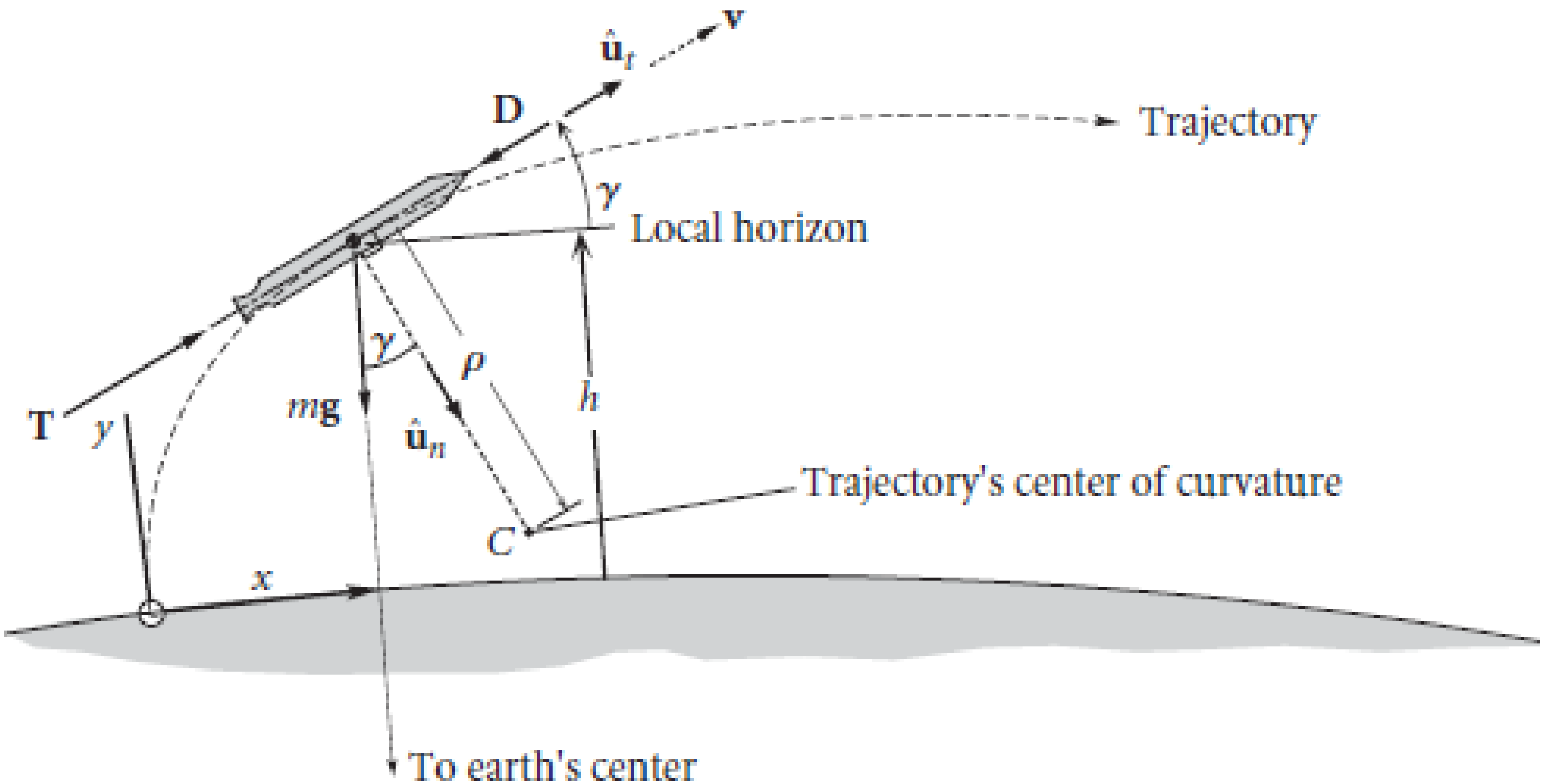
UNIT –V
LOW-THRUST TRAJECTORIES

LOW-THRUST TRAJECTORIES

- EQUATIONS OF MOTION
- CONSTANT RADIAL THRUST ACCELERATION
- CONSTANT TANGENTIAL THRUST

  (Characteristics of the motion)
- LINEARIZATION OF THE EQUATIONS OF MOTION
- PERFORMANCE ANALYSIS

EQUATIONS OF MOTION



Launch Vehicle Boost Trajectory. Γ Is The Flight Path Angle

CONSTANT RADIAL THRUST ACCELERATION

The **normal force** is a **force** perpendicular to the ground that opposes the downward **force** of the weight of the **object**.

The normal acceleration is *an =v2/* (where is the radius of curvature). It was that for flight over a flat surface, *v/=−dγ/dt*, in which case the normal acceleration can be expressed in terms of the flight path angle as

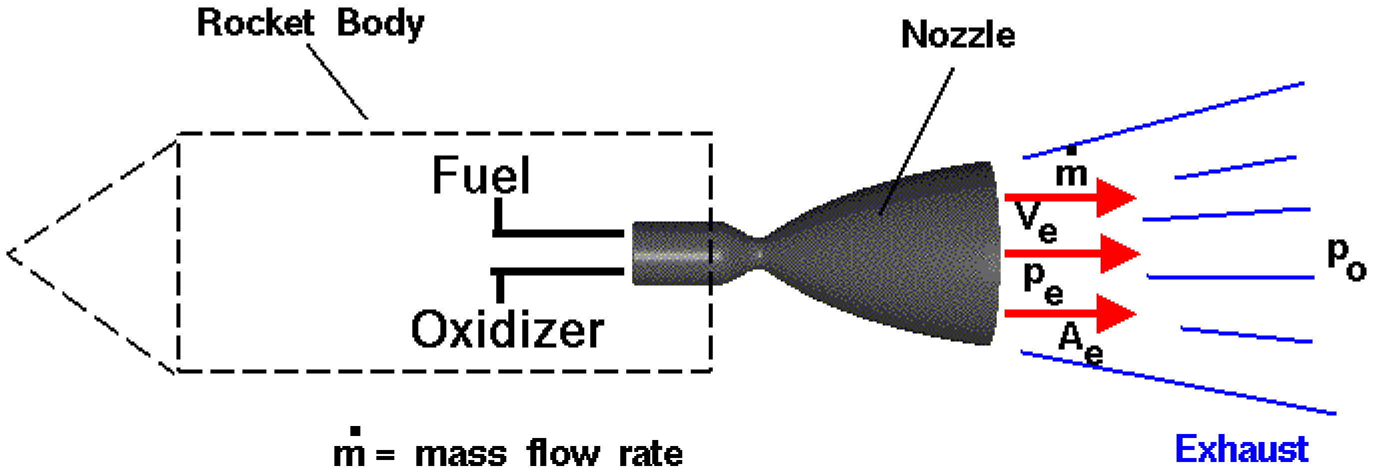
$$a_n = -v\frac{d\gamma}{dt}$$

Normal Thrust Component

With origin at the earth's center to show that a term must be added to this expression, so that it becomes

$$a_n = -v\frac{d\gamma}{dt} + \frac{v^2}{R_E + h}\cos\gamma$$

Characteristics of the Motion



$\dot{m}$ = mass flow rate
p = pressure
V = Velocity
A = Area

Thrust = $F = \dot{m} V_e + (p_e - p_0) A_e$

LINEARIZATION OF THE EQUATIONS OF MOTION

In the direction of $\hat{\mathbf{u}}_t$ Newton's second law requires

$T - D - mg \sin \gamma = ma_t$

Whereas in the $\hat{\mathbf{u}}_n$ direction

$mg \cos \gamma = ma_n$

After combining Equations these expressions may be written as

$$\frac{dv}{dt} = \frac{T}{m} - \frac{D}{m} - g \sin \gamma$$

Downrange Distance and Altitude

$$\frac{dv}{dt} = \frac{T}{m} - \frac{D}{m} - g \sin \gamma$$

The equations for downrange distance *x* and altitude *h*

$$\frac{dx}{dt} = \frac{R_E}{R_E + h} v \cos \gamma \qquad \frac{dh}{dt} = v \sin \gamma$$

Thrust and Specific Impulse

PERFORMANCE ANALYSIS

$$T = -I_{sp} g_0 \frac{dm}{dt} \Rightarrow \frac{dm}{dt} = -\frac{T}{I_{sp} g_0}$$

If the thrust and specific impulse are constant, then the integral of this expression over the burn time *t* is

$$\Delta m = -\frac{T}{I_{sp} g_0} \Delta t$$

$$\Delta t = \frac{I_{sp} g_0}{T}(m_0 - m_f) = \frac{I_{sp} g_0}{T} m_0 \left(1 - \frac{m_f}{m_0}\right)$$


Thank
You