LECTURE NOTES

ON

SPACE MECHANICS

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Introduction to Space Mechanics

1 Introduction to Space Mechanics

1.1 Basic concepts

Space Mechanics or Orbital mechanics, also called flight mechanics, is the study of the motions of artificial satellites and space vehicles moving under the influence of forces such as gravity, atmospheric drag, thrust, etc. Orbital mechanics is a modern offshoot of celestial mechanics which is the study of the motions of natural celestial bodies such as the moon and planets. The root of orbital mechanics can be traced back to the 17th century when mathematician Isaac Newton (1642-1727) put forward his laws of motion and formulated his law of universal gravitation. The engineering applications of orbital mechanics include ascent trajectories, reentry and landing, rendezvous computations, and lunar and interplanetary trajectories.

1.2 The solar system

The Solar System is the gravitationally bound system of the Sun and the objects that orbit it, either directly or indirectly. Of the objects that orbit the Sun directly, the largest are the eight planets, with the remainder being smaller objects, such as the five dwarf planets and small Solar System bodies. Of the objects that orbit the Sun indirectly—the moons—two are larger than the smallest planet, Mercury.

What's in Our Solar System?

Our Solar System consists of a central star (the Sun), the nine planets orbiting the sun, moons, asteroids, comets, meteors, interplanetary gas, dust, and all the "space" in between them.

1.2.1 Classify the planets of our solar system.

They are classified mainly into two groups, viz. Inner and Outer Planets.

Inner plants are

i. Mercury
ii. Venus
iii. Earth
iv. Mars

Outer plants are

i. Jupiter
ii. Saturn
iii. Uranus
iv. Neptune

They can be classified based on their size

a) Small rocky planets (Mercury, Venus, Earth, Mars, and Pluto)
b) Gas giants (Jupiter, Saturn, Uranus, and Neptune)

What are the difference between inner and outer planets?

How the temperature is determined of a start?

A star's temperature determines its "color." The coldest stars are red. The hottest stars are blue.

What are the difference between starts and planets?

The difference between star and planets are distinguished based on their various characteristics. The differences are listed in table below.

Parameters for Comparison	Stars	Planets
Meaning	Stars are the astronomical objects, that emit their own light, produced due to thermonuclear fusion, occurring at its core.	Planets refers to the celestial object that has a fixed path (orbit), in which it moves around the star.
Light	They have their own light.	They do not have their own light.
Position	Their position remain unchanged.	They change position.
Size	Big	Small
Shape	Dot shaped	Sphere-shaped
Temperature	High	Low

Number	There is only one star in the solar system.	There are eight planets in our solar system.
Twinkle	Stars twinkle.	Planets do not twinkle.
Matter	Hydrogen, Helium and other light elements.	Solid, liquid or gases, or a combination thereon.

1.2.2 What are the Characteristics of Small Rocky Planets

i. They are made up mostly of rock and metal.

ii. They are very heavy.

iii. They have no rings and few moons (if any).

iv. They have a diameter of less than 13,000 km.

1.2.3 What are the Characteristics of Gas Giants

- They are made up mostly of gases (primarily hydrogen & helium).
- They are very light for their size.
- They have rings and many moons.
- They have a diameter of more than 48,000 km

Write short notes about the Sun

The sun's energy comes from nuclear fusion (where hydrogen is converted to helium) within its core. This energy is released from the sun in the form of heat and light.

Diameter (km): 1392000

Rotation period (days): 25.380

Polar inclination : 7° 15'

Earth masses: 332946

Density : 1.409

Escape velocity: 617.5 (km/sec)

Albedo : Luminous

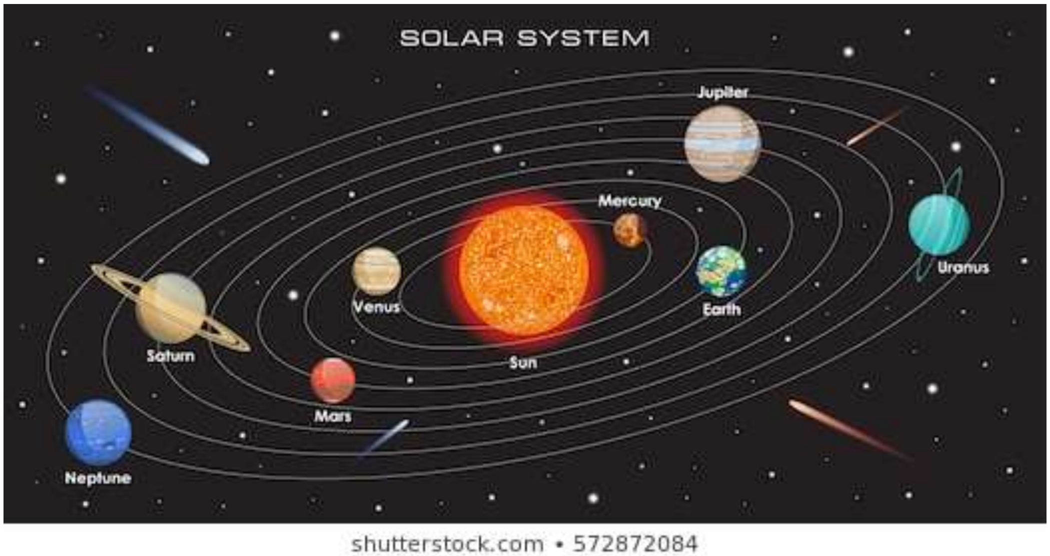


Figure 1 Solar Systems

Write short notes about Mercury

Mercury has a revolution period of 88 days. Mercury has extreme temperature fluctuations, ranging from 800°F (daytime) to -270°F (nighttime). Even though it is the closest planet to the sun, Scientists believe there is ICE on Mercury! The ice is protected from the sun's heat by crater shadows.

Write short notes about Venus

Venus is the brightest object in the sky after the sun and moon because its atmosphere reflects sunlight so well. People often mistake it for a star. Its maximum surface temperature may reach 900°F. Venus has no moons and takes 225 days to complete an orbit.

Name	Sun	Mercury	Venus	Earth	Moon	Mars	Jupiter	Saturn
Diameter (km)	1392000	4880	12100	12756	3476	6790	142200	119300
Rotation period	25.380 Days	58.65 days	243.01 (Retro)	23.9345 hours	27.322 days	24.6229 hours	9.841 hours	10.233 hours
Polar inclination	7° 15'	0°	-2°	23.45°	1° 21'	23.98°	3.08°	29°
Earth masses	332946	0.055	0.815	1	0.0203	0.1074	317.89	95.17

Density	1.409	5.5	5.25	5.517	3.342	3.94	1.33	0.706
Escape velocity (km/sec)	617.5	4.3	10.36	11.18	2.38	5.03	60.22	36.25
Albedo	Luminous	0.06	0.76	0.36	0.07	0.16	0.73	0.76
No of Satellites		0	0	1	-	2	79	62

Uranus	Neptune	Pluto
47100	49500	2300
15.5 hrs (Retro)	15.8 hours	6.3874 hours
97.92°	28.8°	?
14.54	17.23	0.0017
1.19	1.66	0.6-1.7?
21.22	23.6	5.3?
0.93	0.84	0.14
27	14	5

1.2.4 Meteorite vs. Meteoroid

- Meteoroid = while in space a meteorite is called a meteoroid
- Meteorite = a small rock or rocky grain that strikes Earth's surface

So the difference is just based on where the rock is when you are describing it

Meteor

- Sometimes called a "Shooting Star"

- When a meteorite enters Earth's atmosphere, friction causes them to burn up, producing a streak of light

Where do they come from? How big are they?

- Pieces of rock that broke off other objects
- Sizes range from as small as a pebble or as big as a huge boulder

Are they dangerous?

- Most meteoroids disintegrate before reaching the earth by burning up in Earth's atmosphere
- Some leave a trail that lasts several minutes
- Meteoroids that reach the earth are called meteorites. Large ones can cause damage
- 49,000 years ago
- Meteorite about 150 feet in diameter
- Weighed 650 pounds
- Energy = 2.5 million tons of dynamite
- 4000 feet wide, 650 feet deep
- Still visible today

What's a "Meteor Shower"?

- Usual rate = six meteors per hour
- During a Meteor Shower = rate may be as high as 60 meteors per hour
- Occur when Earth passes through the tail or debris of a comet
- Presides (mid-August)
- Leonids (mid-November)

1.2.5 Comets

- Bodies in space made up of ice, dust, small gritty particles
- Sometimes called "dirty snowballs"
- When close to the sun, ice vaporizes, producing a spectacular streak of gas, referred to as a "tail"
- Many in a regular orbit around the sun

Where do comets come from?

- Many ordinate in a region called the Oort cloud which is located beyond the orbit of the dwarf planet Pluto
- Others originate in the Kuiper Belt beyond the orbit of Neptune
- This region is filled with billions of comets

Name few Famous Comets

- Comet Hale-Bopp
- Halley's Comet
- Comet Kohoutek

What do you mean by Asteroids?

- An irregularly shaped rocky object in space (like a space potato)
- May be the shattered remains of objects left over from the time when the planets were formed

How big are asteroids?

- Larger than meteoroids
- (In fact, the main difference between meteoroids and asteroids is their size.)
- Size ranges from 10 feet across to bigger than a mountain
- Approx. 150,000 asteroids in the Solar System
- Most are in a band that orbit the sun between Mars and Jupiter (Asteroid Belt)

Q: What is the difference between a meteoroid, meteororite, and a meteor?

Q: What is the difference between an asteroid and a meteoroid?

Q: Which is larger, asteroid or meteoroid?

Q: Why is it important to study smaller bodies in our Solar System such as comets or asteroids?

Q: Why do planets and moons with atmospheres have less impact craters than those without atmospheres?

Q: Discus what could happen if the Earth experienced another large asteroid impact. How

would it affect life on Earth?

Q: Where is the Asteroid Belt?

Q: What is the Torino Scale?

A system used to rate the hazard level of an object moving toward Earth

Bright streaks of light that result when rocky bodies burn up in the atmosphere are called ___________.

Frozen bodies made of ice, rock, and dust, sometimes called "dirty snowballs" are called _____________.

Small, rocky bodies that revolve around the sun are called ______________.

1.3 Reference frames and coordinate systems

1.3.1 Introduction

To develop an understanding and a basic description of any dynamical system, a physical model of that system must be constructed which is consistent with observations. The fundamentals of orbital mechanics, as we know them today, have evolved over centuries and have continued to require improvements in the dynamical models, coordinate systems and systems of time. The underlying theory for planetary motion has evolved from spheres rolling on spheres to precision numerical integration of the equations of motion based on general relativity. Time has evolved from using the motion of the Sun to describe the fundamental unit of time to the current use of atomic clocks to define the second. As observational accuracy has increased, models have generally increased in complexity to describe finer and finer detail.

To apply the laws of motion to a dynamical system or orbital mechanics problem, appropriate coordinate and time systems must first be selected. Most practical problems involve numerous reference frames and the transformations between them. For example, the equations of motion of a satellite of Mars are normally integrated in a system where the equator of the Earth at the beginning of year 2000 is the fundamental plane. But to include the Mars non-spherical gravitational forces requires the satellite position in the current Mars equatorial system. Planetary ephemerides are usually referred to the ecliptic, so inclusion of solar or Jovian gravitational forces require transformations between the ecliptic and the equator. The correct development of these transformations is tedious and a prime candidate for implementation errors.

Likewise, there are usually numerous time systems in a problem. Spacecraft events might be time tagged by an on board clock or tagged with the universal time that the telemetry is received at the tracking station. In the latter case, tracking station clocks must be synchronized and the time required for the telemetry signal to travel from the s/c to the tracking station must be calculated using the s/c orbit. Depending on the precision desired, this time difference might require special and general relativistic corrections. The independent variable for the equations of motion is called ephemeris time or dynamical time which is offset from universal time. By international agreement, atomic time is the basis of time and is obtained by averaging and correcting numerous atomic clocks around the world. Finally, the location of the zero or prime meridian and the equator are defined by averaging observations of specified Earth "fixed" stations. The understanding of these and other coordinate systems and time systems is fundamental to practicing orbital mechanics.

In this chapter only first order effects will be discussed. This note will also limit coverage to the classical mechanics approach, i.e. special and general relativistic effects might be mentioned but will not be included in any mathematical developments. Calculation for precise orbital mechanics and spacecraft tracking must however include many of these neglected effects. The definitive reference for precise definitions of models and transformations is the Explanatory_Supplement_to the_Astronomical_Almanac.

The first issue that must be addressed in any dynamics problem is to define the relevant coordinate systems. To specify the complete motion of a spacecraft, a coordinate system fixed in the spacecraft at the center of mass is usually selected to specify orientation and a coordinate system fixed in some celestial body is used to specify the trajectory of the center of mass of the spacecraft. The interest here is primarily in the latter system.

Coordinate systems are defined by specifying

1. location of the *origin*,
2. orientation of the *fundamental plane*, and
3. orientation of the *fundamental direction or line* in the fundamental plane.

The **origin** is the (0,0,0) point in a rectangular coordinate system. The **fundamental plane** passes through the origin and is specified by the orientation of the positive normal vector, usually the z-axis. The **fundamental direction** is a directed line in the fundamental plane, usually specifying the +x-axis. The origin, fundamental plane and fundamental line are defined either relative to some previously defined coordinate system or in operational terms. The definitions are usually specified in a seemingly clear statement like: "The origin is the center of mass of the Earth, the fundamental plane (x-y) is the Earth equator and the x-axis points to the vernal equinox." Left as details are subtle issues like the fact that the center of mass of the Earth "moves" within the Earth, that the Earth is not a rigid body and the spin axis moves both in space and in the body, and that the vernal equinox is not a fixed direction. Some of these details are handled by specifying the epoch at which the orientation is defined, i.e. Earth mean equator of 2000.0 is frequently used. Further, it must be recognized that there is no fundamental inertial system to which all motion can be referred. Any system fixed in a planet, the Sun, or at the center of mass of the solar system is undergoing acceleration due to gravitational attraction from bodies inside and outside the solar system. The extent to which these accelerations are included in the dynamical model depends on accuracy requirements and is a decision left to the analyst.

Like many other fields, conventions and definitions are often abused in the literature and this abuse will continue in this text. So "the equator" is jargon for the more precise statement "the plane through the center of mass with positive normal along the spin axis." Likewise, angles should always be defined as an angular rotation about a specified axis or as the angle between two vectors. The angle between a vector and a plane (e.g. latitude) is to be interpreted as the complement of the angle between the vector and the positive normal to the plane. The angle between two planes is defined as the angle between the positive normals to each plane. The more precise definitions often offer computational convenience. For example, after checking the orthogonality of the direction cosines of the positive unit normal (usually +z axis) and the direction cosines of the fundamental direction in the plane (usually +x), the direction cosines of the +y axis can be obtained by a vector cross product. Thus, the entire transformation or rotation matrix is defined by orthogonal x and z unit vectors.

Common origins for coordinate systems of interest in astrodynamics include:

1. **Topocentric**: at an observer fixed to the surface of a planet,
2. **Heliocentric, Geocentric, Areocentric, Selenocentric**, etc.: at the center of mass of the Sun, Earth, Mars, Moon, etc.
3. **Barycentric**: at the center of mass of a system of bodies, i.e. the solar system, Earth-Moon system, etc.

Astronomical observations were traditionally referred to topocentric coordinates since the local vertical and the direction of the spin axis could be readily measured at the site. For dynamics problems, topocentric coordinates might be used for calculating the trajectory of a baseball or a launch vehicle. For the former case, the rotation of the Earth and the variation in gravity with altitude can be ignored because these effects are small compared the errors introduced by the uncertainty in the aerodynamic forces acting on a spinning, rough sphere. For the latter case, these effects cannot be ignored; but, gravitational attraction of the Sun and Moon might be ignored for approximate launch trajectory calculations. The decision is left to the analyst and is usually base on "back of the envelope" calculations of the order of magnitude of the effect compared to the desired accuracy.

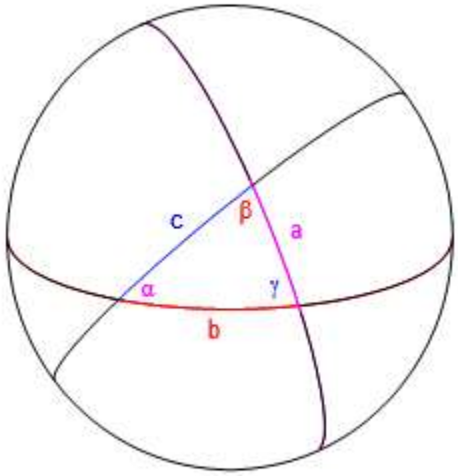
Heliocentric, areocentric, etc. coordinates are traditionally used for calculating and specifying the orbits of both natural and artificial satellites when the major gravitational attraction is due to the body at the origin. During calculation of lunar or interplanetary trajectories, the origin is shifted from one massive body to another as the relative gravitational importance changes; however, the fundamental plane is often kept as the Earth equator at some epoch. Often in what follows only Earth geocentric systems are discussed, but the definitions and descriptions generally apply to planets and moons. Geocentric systems are either terrestrial or celestial. **Terrestrial** systems are fixed to the rotating Earth and can be topocentric, geocentric, or geodetic. **Celestial** systems have either the equator or the ecliptic as the fundamental plane and the vernal equinox as the fundamental direction.

1.3.2 Spherical trigonometry

Transformations of position and velocity vectors between coordinate systems are represented in matrix notation and developed by vector outer and inner products as mentioned above. However, the understanding of the basic concepts of spherical trigonometry is also a necessity when dealing with orbital mechanics. It is convenient to introduce the concept of the celestial sphere**.** The **celestial sphere** is a spherical surface of infinite radius. The location of the center of the celestial sphere is therefore unimportant. For example, one can think of the center as being simultaneously at the center of the Earth and Sun and observer. Any unit vector or direction can thus be represented as a point on the sphere and vice versa. For example, the Earth to Sun line and Sun to

Earth lines could be represented by two points 180 degrees apart. Two distinct points on the sphere can be connected by a **great circle** formed by the intersection on the sphere of the plane formed by the two points and the center of the sphere. If the points are not coincident or 180° apart, the great circle is unique.

The **distance** or **length** between two points on the surface is the central angle subtended by the points, which is also the shorter arc length on the great circle connecting the points. Three points, not on the same great circle, form the **vertices** of a **spherical triangle**. The three **sides** are the great circle arcs connecting each pair of vertices ($0<a,b,c<\pi$ in Figure). The length of a side of a spherical triangle is often referred to as simply the "**side**." With each vertex is associated an "**angle**" ($0 < \alpha, \beta, \gamma < \pi$) that is, the angle between the planes that form the adjacent sides. A spherical triangle has the following propertie



$$\pi < \alpha + \beta + \gamma < 3\pi$$

$$0 < a + b + c < 2\pi$$

$$a + b > c, \text{ etc.}$$

Exercise 1-2. Draw a spherical triangle where both a+b+c is nearly zero and α+β+γ is nearly π. Draw a spherical triangle where both a+b+c is nearly 2π and α+β+γ is nearly 3π. Check using the latter triangle.

The solid angle subtended by the triangle is α+β+γ-π steradian, so if the sphere has radius R, the area of the spherical triangle is given by

$$\text{Area} = R^2(\alpha + \beta + \gamma - \pi)$$

A **right spherical triangle** has either a side or an angle of 90^o and equations can be reduced to two rules and **Napier's Circle**. Consider the latter case and wolog assume $\gamma = 90^o$. Napier's Circle, shown in Figure, is created by putting the side opposite to the 90° angle at the top and proceeding around the triangle in the same direction to fill in the four remaining parts of the circle. The upper three parts are subtracted from 90°. Now consider any three parts of the triangle. The three parts will either be (1) "adjacent" parts, e.g. b, α and Figure. Napier's circle.c in which case α would be called the "middle" part, or (2) two parts will be opposite the third part, e.g. b, α and β and β would be called the "opposite" part. **Napier's Rules of Circular Parts** are then:

1. The sine of the middle part equals the product of the tangents of the adjacent parts.
2. The sine of the middle part equals the product of the cosines of the opposite parts.

As stated above, the first equation is used when the three parts of interest in the triangle are adjacent, e.g. a, β and c are related by cos(β)=tan(a)cot(c), which can be verified using equation. The second equation is used when one of the parts is opposite the other two, e.g.with b, α, and β: cos(β)=cos(b)sin(α), which can be verified using equation . Note that the quadrant is not always determined from the basic equation. Since all parts are less than π, quadrant can not be determined from sine but can be determined from tangent or cosine. Therefore, care must be exercised in determining the quadrant.

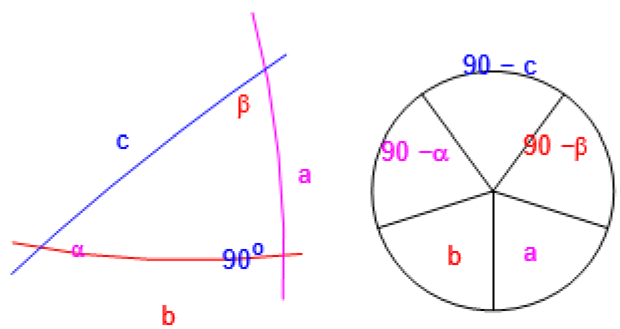


Fig. Napier's Circle

1.4 The celestial sphere

The two conventional **celestial coordinate** system , projected onto the celestial sphere, are shown in Figure below. The two great circles or fundamental planes of interest are the equator of the Earth and the **ecliptic**, i.e. the Earth-Moon barycenter-Sun orbital plane (often called the Earth- Sun plane). The line of intersection where the Sun, moving along the ecliptic, passes from the southern to the northern hemisphere, as seen by a geocentric observer, is called the **first point of Aries** or **the vernal equinox** and is denoted γ. The vernal equinox is the fundamental direction forcelestial systems. The positive direction is from the center of the Earth to the center of the Sun at the time of the vernal equinox. This convention is one of the few remaining concepts from Ptolemy. The angle between the equator and the ecliptic is known as the **obliquity** (ε). The obliquity for the Earth is approximately 23.45o and changes about 0.013o per century. The two intersections of the ecliptic and the equator on the celestial sphere are known as the equinoctial points. When the Sun appears to move southward through the node, it is the autumnal equinox.

The vernal equinox occurs within a day of March 21 and the autumnal occurs within a day of September 21. At either equinox, the length of the day and night are equal at all points on the Earth and the Sun rises (sets) due east (west). When the Sun has maximum northerly declination it is summer solstice in the northern hemisphere and winter solstice in the southern hemisphere, and conversely. At summer solstice in the northern hemisphere, the longest day occurs and the Sun rises and sets at maximum northerly azimuth. Nevertheless, due to the eccentricity of the orbit of the Earth, neither the earliest sunrise nor latest sunset occurs at summer solstice. A fact that, when properly phrased, has won small wagers from non-celestial mechanicians.

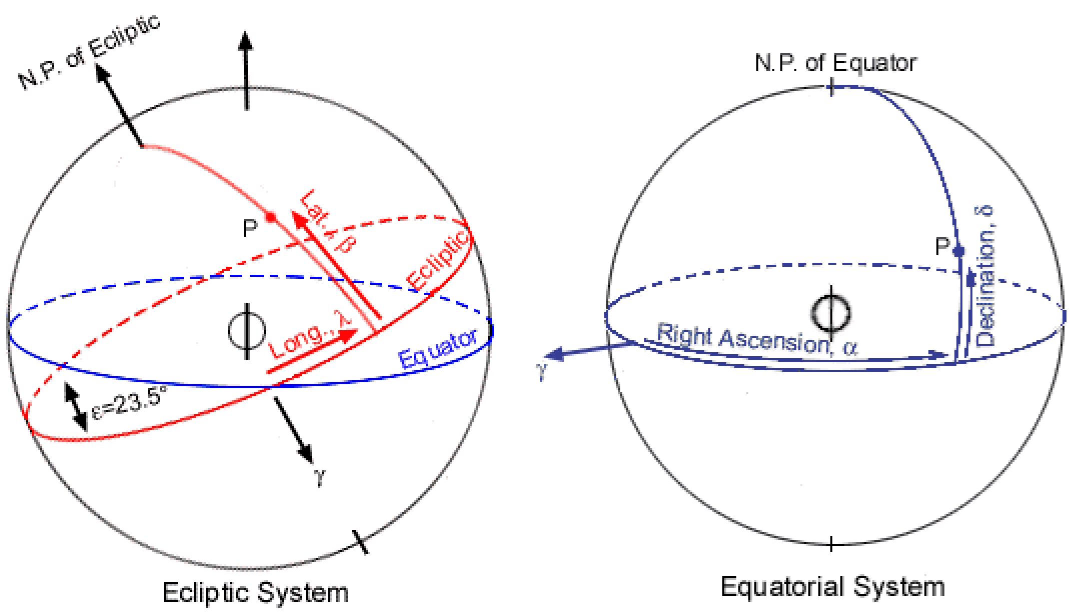


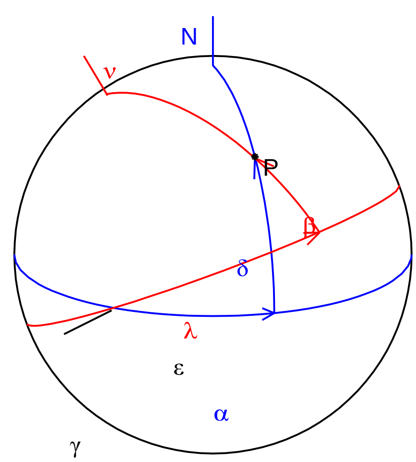
Figure 1-3. Celestial coordinate systems.

It must be recognized that neither the ecliptic nor the equator are fixed planes. Variations in the vernal equinox due to the motion of these planes are termed precession and nutation. **Precession** is the secular component that produces a westward change in the direction of γ that is linear with time. **Nutation** is the quasi-periodic residual that averages to zero over many years. The **mean** equator or ecliptic refers to the position that includes only precession. The **true** equator or ecliptic refers to the position that includes both precession and nutation. The Earth equator is not fixed in space primarily due to lunar and solar gravitational torques applied to the non-spherical Earth. The **luni-solar precession** causes the mean pole of the Earth to move about

50" of arc per year and **luni-solar nutation** has an amplitude of about 9" of arc over an 18.6 year cycle. The cycle of 18.6 years is how long it takes for the orbital planes of the Earth-Moon and the Earth-Sun to return to the same relative configuration. Variations in the ecliptic are primarily due to planetary gravitational forces producing changes in the orbit of Earth-Moon barycenter about the Sun. If the equator was fixed, the **planetary precession** of the ecliptic would cause γ to move along the equator about 12" of arc per century and the obliquity would decrease by 47" per century. To eliminate the need to consider precession and nutation in dynamics problems, the coordinate system is usually specified at some epoch, i.e. **mean equator** and **mean equinox** of 2000.0, otherwise, known as **J2000**. In this case, Earth based observations must be corrected for precession and nutation. Transformations between the J2000 coordinates and the **true** or **apparent** systems are then required .

Another plane that is use in the celestial system is the **invariant plane**. The positive normal to the invariant plane is along the total angular momentum (i.e. rotational plus orbital) of the solar system. In Newtonian mechanics, only gravitational attraction from the distant stars and unmodeled masses can cause this plane to change orientation.

Consider some point P in the geocentric reference system of Figure . The position of point P is projected onto each fundamental plane. In the equatorial system the angle from γ to this projection is call **right ascension** $(0 \le \alpha < 2\pi)$ and the angle between the point P and the equator is called the **declination** $(-\pi/2 \le \delta \le \pi/2)$. In the ecliptic system the corresponding angles are the **celestial longitude** $(0 \le \lambda < 2\pi)$ and **celestial latitude** $(-\pi/2 \le \beta \le \pi/2)$. The "celestial" qualifier is to assure no confusion with traditional terrestrial longitude and latitude. When the context is clear, the qualifier is often omitted. "Celestial" is also sometimes replaced with "ecliptic." The rotation matrix from the ecliptic system to the equatorial system is a single



Transforming between celestial coordinate systems rotation about the x axis by the **obliquity** ε. As the following example illustrates, solving spherical trigonometry problems often involves drawing numerous spherical triangle combinations until the proper combination of knowns and unknowns appears.

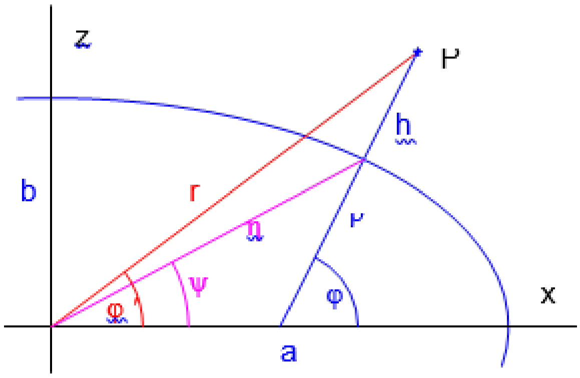
1.4.1 Terrestrial coordinate systems

Astrodynamics problems are generally framed in either the ecliptic or equatorial celestial coordinate system. The locations of observers, receivers, transmitters, and observation targets are usually specified in one of the terrestrial coordinate systems. A **terrestrial coordinate system** is "fixed" in the rotating Earth and is either geocentric or topocentric. Transformations between terrestrial and celestial coordinates are an essential part of orbital mechanics problems involving Earth based observations. These transformations are defined by the **physical ephemeris**, that is, the definition of the pole location and the rotational orientation of the Earth. Precise definitions must include elastic deviations in the solid Earth, plate tectonics , motion of the spin axis in the Earth, and numerous other effects. The largest of these effects is polar motion which produces deviations between the instantaneous and mean spin axis of order 10 meters. Pole location is determined by numerous observation stations and published by international agreement. Irregularities in the rotational rate of the Earth can change the length of the day by a few milliseconds over time scales of interest for orbital mechanics and astronomy problems. One millisecond ≈ 0.46 meters in longitude at theequator. Rotational variations are also monitored and included in the definition of universal time to be discussed later. Specific effect to be included depend on the desired accuracy and the choice is left to the analyst.

The **fundamental terrestrial coordinate** system has the origin at the center of mass and the equator as the fundamental plane. The intersection of the reference meridian with the equator is the fundamental direction. The **origin, the equator, and reference meridian are defined operationally by measurements made at a number of "fixed" stations on the surface**. In the past, the prime meridian was the Greenwich meridian and was defined by the center of a plaque at Greenwich. The phrase "reference meridian" is used to clearly distinguish the fundamental difference in definitions. Nevertheless, the reference meridian is often referred to as the Greenwich meridian, and that practice will be used herein. For remote solid planets, prime meridians are still defined by easily observed sharp surface features. An observer's **local meridian** is defined by the plane through the observer that also contains the spin axis of the Earth. An observer's longitude (λ) is the angle between the reference meridian and the local meridian, more precisely referred to as "terrestrial longitude." Since the spin axis moves in the Earth, an observers true longitude deviates from the mean longitude.

Latitude is specified as either geodetic latitude (φ) or geocentric latitude (φ') Geocentric latitude, often called latitude, is the angle between the equator and the observer. In the **geocentric system**, the location of a point is specified by the **radius** from the center of the Earth, **geocentric latitude** and **longitude**. To satisfy the right hand rule convention for rotations about the pole, longitude should be measured east; but, is often measured west. To be safe, always specify the convention. For example, 75^{o}W = 285^{o}E longitude. **Colatitude** is the angle between the position vector and the normal to the equator and is unambiguous, but latitude is sometimes specified by using a sign convention e.g. -37.5°=37.5°S. Also note that geocentric latitude is often denoted by φ, i.e. the "prime" is omitted when the meaning is clear.

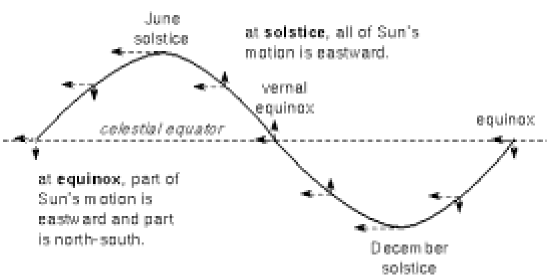
Geodetic coordinates are generally limited to points near the surface of the Earth. **Geodetic latitude** is the angle between the local vertical and the equator. The **local vertical** is determined by the local "gravity" force which is the combination of gravity and a centrifugal contribution due to rotation. An equipotential surface for the two terms is nearly an ellipsoid of revolution. Hence it is convenient to define a **reference ellipsoid** (spheroid) for the mean equipotential surface of the Earth which is approximately the **mean sea level.** This ellipsoid, which is symmetric about the equator and has rotational symmetry about the pole, is defined by the **equatorial radius** (a) and the **flattening** (f). The polar radius is given by b =a(1-f). Reference values are a=6378137m and 1/f=298.25722. Figure shows a cross section of the reference ellipsoid with greatly exaggerated flattening. For the figure, it is assumed that the cross section contains the x-axis, so the equation of the elliptical cross-section is

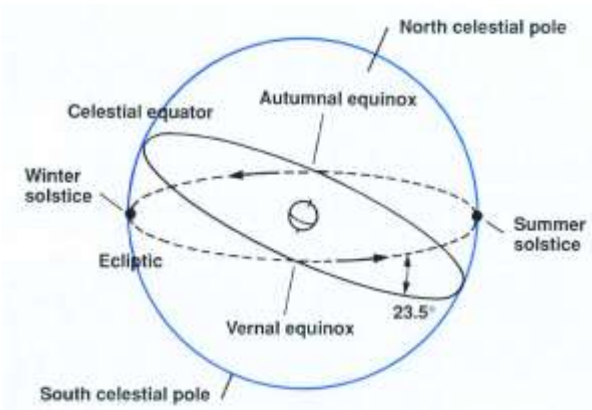


$$f(x,z) = \frac{x^2}{a^2} + \frac{z^2}{b^2} - 1 = 0$$

1.5 The ecliptic

The **ecliptic** is the mean plane of the apparent path in the Earth's sky that the Sun follows over the course of one year; it is the basis of the ecliptic coordinate system. This plane of reference is coplanar with Earth's orbit around the Sun (and hence the Sun's apparent path around Earth). The ecliptic is not normally noticeable from Earth's surface because the planet's rotation carries the observer through the daily cycles of sunrise and sunset, which obscure the Sun's apparent motion against the background of stars during the year

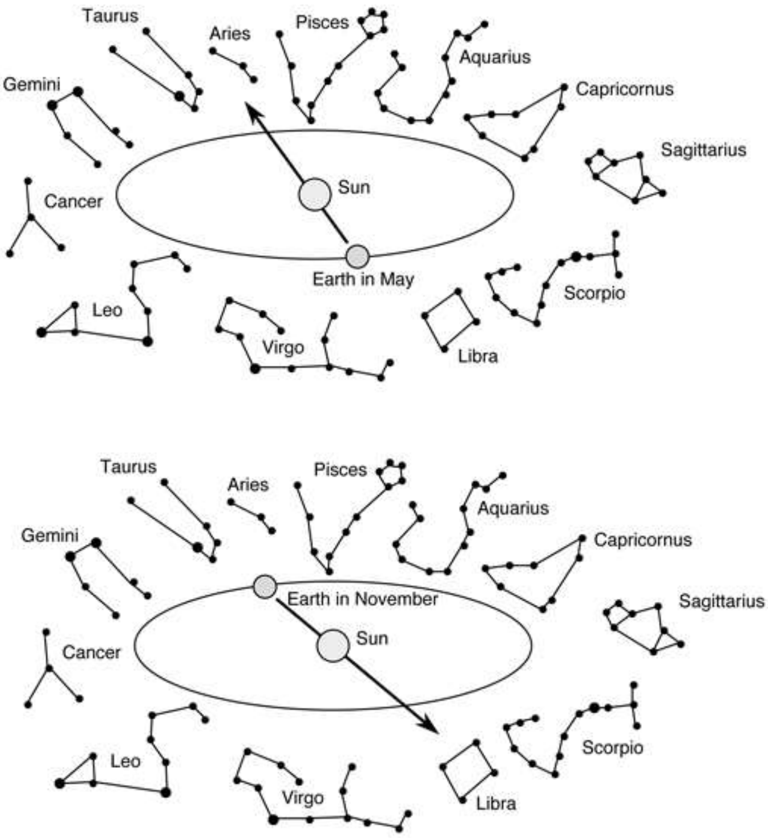




1.6 Motion of vernal equinox

Now that you have your bearings, let's take a look at the position and motion of the closest star to us, the Sun. Every day the Sun rises in an easterly direction, reaches maximum height when it crosses the meridian at local noon, and sets in a westerly direction and it takes the Sun on average 24 hours to go from noon position to noon position the next day. The ``noon position" is when the Sun is on the meridian on a given day. Our clocks are based on this **solar day**. The exact position on the horizon of the rising and setting Sun varies throughout the year (remember though, the celestial equator always intercepts the horizon at exactly East and exactly West). Also, the time of the sunrise and sunset changes throughout the year, very dramatically so if you live near the poles, so the solar day is measured from ``noon to noon".

The Sun appears to drift eastward with respect to the stars (or lag behind the stars) over a year's time. It makes one full circuit of 360 degrees in 365.24 days (very close to 1 degree or twice its diameter per day). This drift eastward is now known to be caused by the motion of the Earth around the Sun in its orbit.



As the Earth moves around the Sun, the Sun **appears** to drift among the zodiac constellations along the path called the **ecliptic.** The ecliptic is the projection of the Earth's orbit onto the sky.

The apparent yearly path of the Sun through the stars is called the **ecliptic**. This circular path is tilted 23.5 degrees with respect to the celestial equator because the Earth's rotation axis is tilted by 23.5 degrees with respect to its orbital plane. Be sure to keep distinct in your mind the difference between the slow drift of the Sun along the ecliptic during the year and the fast motion of the rising and setting Sun during a day.

1.7 Sidereal time

Sidereal time is a timekeeping system that astronomers use to locate celestial objects. Using sidereal time, it is possible to easily point a telescope to the proper coordinates in the night sky. Briefly, sidereal time is a "time scale that is based on Earth's rate of rotation measured relative to the fixed stars".

1.8 Solar Time

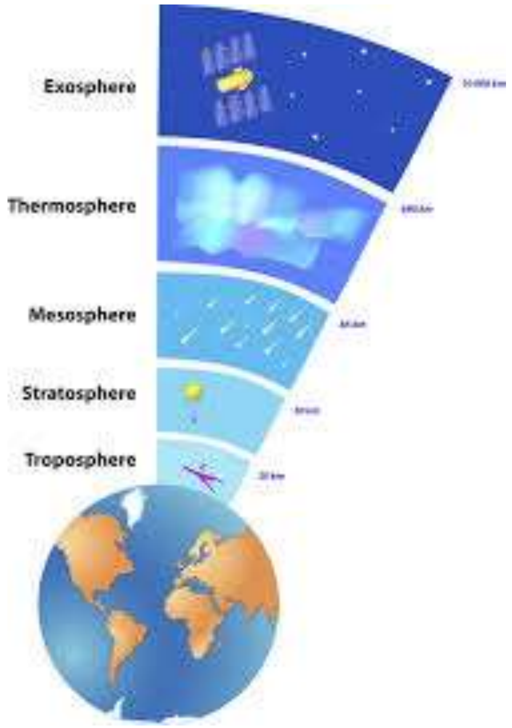
Solar time is a calculation of the passage of **time** based on the position of the **Sun** in the sky. The fundamental unit of **solar time** is the day. Two types of **solar time** are apparent **solar time** (sundial **time**) and mean **solar time** (clock **time**).

1.9 Standard Time

Standard time is the synchronization of clocks within a geographical area or region to a single time standard, rather than using solar time or a locally chosen meridian to establish a local mean time standard.

1.10 The earth's atmosphere

The **atmosphere of Earth** is the layer of gases, commonly known as **air**, that surrounds the planet Earth and is retained by Earth's gravity. The atmosphere of Earth protects life on Earth by creating pressure allowing for liquid water to exist on the Earth's surface, absorbing ultraviolet solar radiation, warming the surface through heat retention (greenhouse effect), and reducing temperature extremes between day and night (the diurnal temperature variation).



- Exosphere
- Thermosphere
- Mesosphere
- Stratosphere
- Troposphere

1.11 Greenwich mean sidereal time

(**GMST**) is the angle between the Greenwich meridian and the mean vernal equinox and would be sensitive to the same variations in rotation as UT1.

1.12 The many body problem

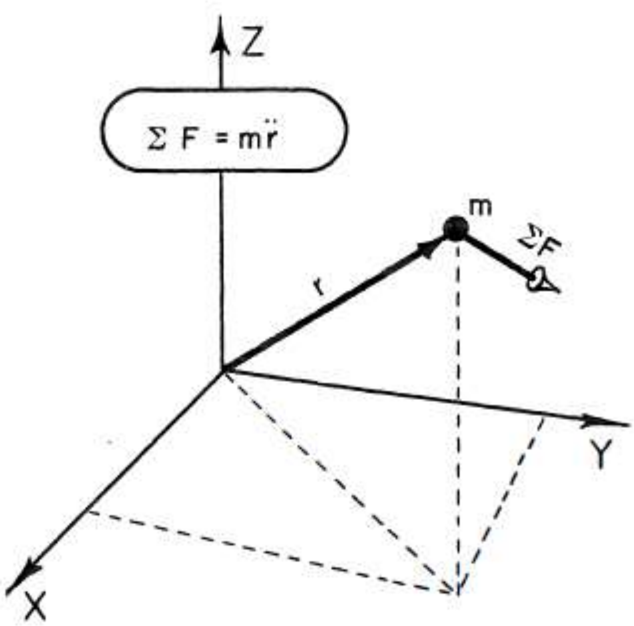
1.12.1 KEPLER'S LAWS

1. First Law-The orbit of each planet is an ellipse, with the sun at a focus .
2. Second Law-The line joining the planet to the sun sweeps out equal areas in equal times.
3. Third Law-The square of the period o f a planet is proportional to the cube of its mean distance from the sun.

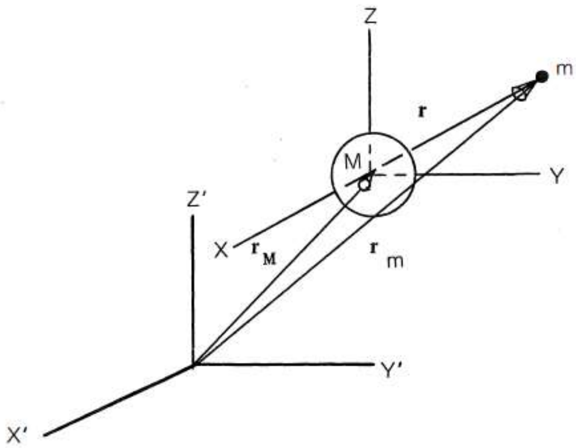
1.12.2 Newton's Laws of Motion.

1. First Law-Every body continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it.
2. Second Law-The rate of change of momentum is proportional to the force impressed and is in the same direction as that force.
3. Third Law-To every action there is always opposed an equal reaction.

The second law can be expressed mathematically as follows:



Newton's Law of Motion



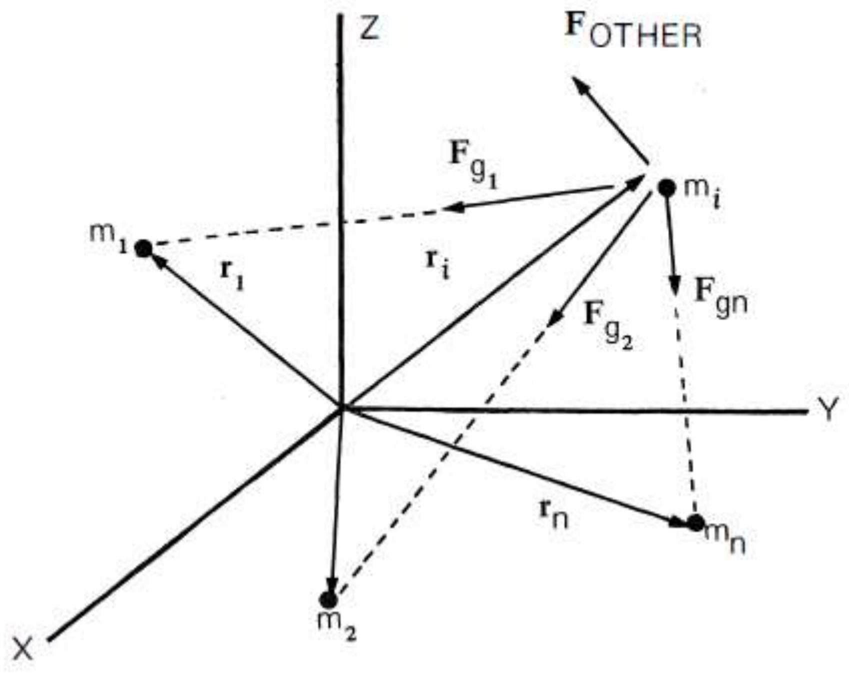
Relative Motion of Two Bodies

Newton's Law of Universal Gravitation. Besides enunciating his three laws of motion in the Principia, Newton fonnulated the law of gravity by stating that any two bodies attract one another with a force proportional to the product of their masses and inversely proportional to the square of the distance between them. We can express this law mathematically in vector notation as :

$$\mathbf{F}_g = -\frac{GMm}{r^2}\frac{\mathbf{r}}{r}$$

where F 9 is the force on mass m due to mass M and r is the vector from M to m. The universal gravitational constant, G, has the value 6.67Ox 1 0-8 dyne cm^2 Igm^2 •

In this section we shall examine in some detail the motion of a body (i.e., an earth satellite , a lunar or interplanetary probe , or a planet). At any given time in its journey, the body is being acted upon by several gravitational masses and may even be experiencing other forces such as drag, thrust and solar radiation pressure . For this examination we shall assume a "system" of nobodies (m}. m2• m3 . . . f'T"h) one of which is the body whose motion we wish to study-call it the jth body , m j. The vector sum o f all gravitational forces and other external forces acting on m . will be used to determine the equation of motion. To determine the gravitational forces we shall apply Newton's law of universal gravitation. In addition, the jth body may be a rocket expelling mass (i.e., propellants) to produce thrust; the motion may be in an atmosphere where drag effects are present; solar radiation may impart some pressure on the body; etc. All of these effects must be considered in the general equation of motion. An important force, not yet mentioned is due to the non-spherical shape of the planets. The earth is flattened at the poles and bulged at the equator; the moon is elliptical about the poles and about the equator. Newton's law of universal gravitation applies only if the bodies are spherical and if the mass is evenly distributed in spherical shells. Thus, variations are introduced to the gravitational forces due primarily to the shape of the bodies. The magnitude of this force for a near-earth satellite is on the order of 1 0-3 g's. Although small, this force is responsible for several important effects not predictable from the studies of Kepler and Newton. The first step in our analysis will b e t o choose a "suitable" coordinate system in which to express the motion. This is not a simple task since any coordinate system we choose has a fair degree of uncertainty as to its inertial qualities. Without losing generality let us assume a "suitable" coordinate system (X, Y, Z) in which the positions of the n masses are known r1 r2, rn' This system is illustrated in Figure



The N-Body Problem

$$\mathbf{F}_g = -Gm_i \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_j}{r_{ji}^{3}} (\mathbf{r}_{ji})$$

1.13 Lagrange-Jacobi identity

1.14 The circular restricted three body problem

1.15 Libration points

A Lagrange point is a location in space where the combined gravitational forces of two large bodies, such as Earth and the sun or Earth and the moon, equal the centrifugal force felt by a much smaller third body. The interaction of the forces creates a point of equilibrium where a spacecraft may be "parked" to make observations.

There are five Lagrange points around major bodies such as a planet or a star. Three of them lie along the line connecting the two large bodies. In the Earth-sun system, for example, the first point, L1, lies between Earth and the sun at about 1 million miles from Earth. L1 gets an uninterrupted view of the sun, and is currently occupied by the Solar and Heliospheric Observatory (SOHO) and the Deep Space Climate Observatory.

L2 also lies a million miles from Earth, but in the opposite direction of the sun. At this point, with the Earth, moon and sun behind it, a spacecraft can get a clear view of deep space. NASA's Wilkinson Microwave Anisotropy Probe (WMAP) is currently at this spot measuring the cosmic background radiation left over from the Big Bang. The James Webb Space Telescope will move into this region in 2018.

The third Lagrange point, L3, lies behind the sun, opposite Earth's orbit. For now, science has not found a use for this spot, although science fiction has. “NASA is unlikely to find any use for the L3 point since it remains hidden behind the sun at all times,” NASA wrote on a web page about Lagrange points. “The idea of a hidden 'Planet-X' at the L3 point has been a popular topic in science fiction writing. The instability of Planet X's orbit (on a time scale of 150 years) didn't stop Hollywood from turning out classics like 'The Man from Planet X.'”

L1, L2 and L3 are all unstable points with precarious equilibrium. If a spacecraft at L3 drifted toward or away from Earth, it would fall irreversibly toward the sun or Earth, "like a barely balanced cart atop a steep hill," according to astronomer Neil DeGrasse Tyson. Spacecraft must make slight adjustments to maintain their orbits.

Points L4 and L5, however, are stable, "like a ball in a large bowl," according to the European Space Agency. These points lie along Earth's orbit at 60 degrees ahead of and behind Earth, forming the apex of two equilateral triangles that have the large masses (Earth and the sun, for example) as their vertices.

Because of the stability of these points, dust and asteroids tend to accumulate in these regions. Asteroids that surround the L4 and L5 points are called Trojans in honor of the asteroids Agamemnon, Achilles and Hector (all characters in the story of the siege of Troy) that are between Jupiter and the Sun. NASA states that there have been thousands of these types of asteroids found in our solar system, including Earth’s only known Trojan asteroid, 2010 TK7.

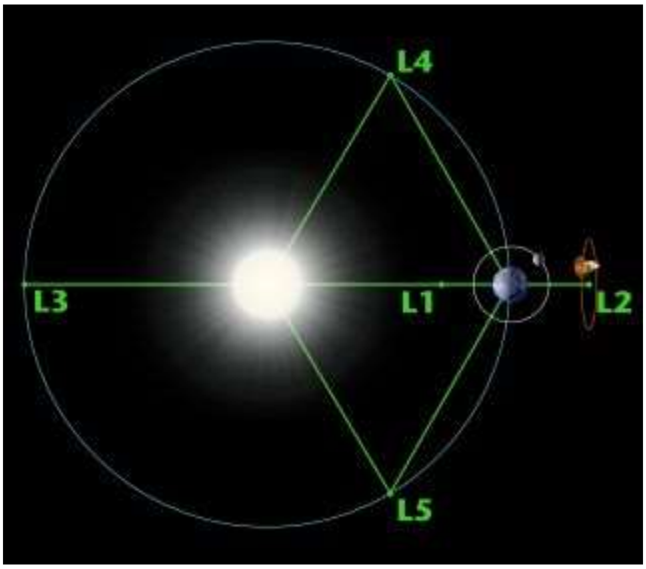


Diagram of the Lagrange points associated with the sun-Earth system

1.16 Relative Motion in the N-body problem

UNIT-II

Two Body Problem

2 Unit-II Two Body Problem

In the previous lecture, we discussed a variety of conclusions we could make about the motion of an arbitrary collection of particles, subject only to a few restrictions. Today, we will consider a much simpler, very well-known problem in physics - an isolated system of two particles which interact through a central potential. This model is often referred to simply as the two-body problem. In the case of only two particles, our equations of motion reduce simply to

$$m1\ddot{r}_1 = F_{21} \; ; \; m_2\ddot{r}_2 = F_{12}$$

2.1 Equations of motion

2.1.1 Newton's Laws of Motion and Universal Gravitation

Newton's laws of motion describe the relationship between the motion of a particle and the forces acting on it.

The first law states that if no forces are acting, a body at rest will remain at rest, and a body in motion will remain in motion in a straight line. Thus, if no forces are acting, the velocity (both magnitude and direction) will remain constant.

The second law tells us that if a force is applied there will be a change in velocity, i.e. an acceleration, proportional to the magnitude of the force and in the direction in which the force is applied. This law may be summarized by the equation

$$F = ma$$

Where F is the force, m is the mass of the particle, and a is the acceleration.

The third law states that if body 1 exerts a force on body 2, then body 2 will exert a force of equal strength, but opposite in direction, on body 1. This law is commonly stated, "for every action there is an equal and opposite reaction".

In his law of universal gravitation, Newton states that two particles having masses m_1 and m_2 and separated by a distance r are attracted to each other with equal and opposite forces directed along the line joining the particles. The common magnitude F of the two forces is

$$F = G\left(\frac{m_1 m_2}{r^2}\right)$$

Where G is an universal constant, called the constant of gravitation, and has the value

6.67259×10^{-11} N-m^2/kg^2 (3.4389×10^{-8} lb-ft^2/slug^2).

Where F is the force, m is the mass of the particle, and a is the acceleration.

Let's now look at the force that the Earth exerts on an object. If the object has a mass m, and the Earth has mass M, and the object's distance from the center of the Earth is r, then the force that the Earth exerts on the object is GmM /r^2 . If we drop the object, the Earth's gravity will cause it to accelerate toward the center of the Earth. By Newton's second law (F = ma), this acceleration g must equal (GmM /r^2)/m, or

$$g = \frac{GM}{r^2}$$

At the surface of the Earth this acceleration has the valve 9.80665 m/s^2 (32.174 ft/s^2).

Many of the upcoming computations will be somewhat simplified if we express the product GM as a constant, which for Earth has the value 3.986005×10^{14} m^3/s^2 (1.408×10^{16} ft^3/s^2). The product GM is often represented by the Greek letter μ.

Mathematical Constants		
π	3.141592653589793	
e	2.718281828459045	
Physical Constants		
Speed of light (c)	299,792,458	m/s
Constant of gravitation (G)	6.67259×10^{-11}	Nm^2/kg^2
Universal gas constant (R)	8,314.4621	J/kmol-K
Stefan-Boltzmann constant (σ)	5.670373×10^{-8}	W/m^2-K^4
Acceleration of gravity (g)	9.80665	m/s^2
Standard atmosphere, sea level	101,325	Pa
Astronomical Constants		

Astronomical unit (AU)	149,597,870	km
Light year (ly)	9.460530×10^{12}	km
Parsec (pc)	3.261633	ly
Sidereal year	365.256366	days
Mass of Sun	1.9891×10^{30}	kg
Radius of Sun	696,000	km
Mass of Earth	5.9737×10^{24}	kg
Equatorial radius of Earth	6,378.137	km
Earth oblateness	1/298.257	
Obliquity of the ecliptic, epoch 2000	23.4392911	degrees
Mean lunar distance	384,403	km
Radius of Moon	1,738	km
Mass of Moon	7.348×10^{22}	kg
Luminosity of Sun	3.839×10^{26}	W
Solar constant, at 1 AU	1,366	W/m^2
Solar maxima	1990 + 11n	(date)
Spaceflight Constants		
GM (Sun)	$1.32712438 \times 10^{20}$	m^3/s^2
GM (Earth)	3.986005×10^{14}	m^3/s^2
GM (Moon)	4.902794×10^{12}	m^3/s^2
GM (Mars)	4.282831×10^{13}	m^3/s^2
J_2 (Earth)	0.00108263	
J_2 (Moon)	0.0002027	
J_2 (Mars)	0.00196045	

2.2 General characteristics of motion for different orbits

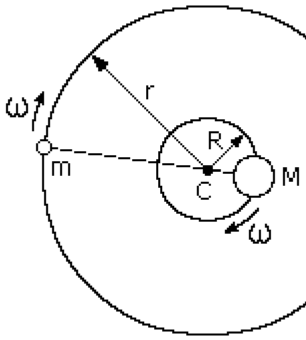
Through a lifelong study of the motions of bodies in the solar system, Johannes Kepler (1571-1630) was able to derive three basic laws known as Kepler's laws of planetary motion. Using the data compiled by his mentor Tycho Brahe (1546-1601), Kepler found the following regularities after years of laborious calculations:

1. All planets move in elliptical orbits with the sun at one focus.
2. A line joining any planet to the sun sweeps out equal areas in equal times.
3. The square of the period of any planet about the sun is proportional to the cube of the planet's mean distance from the sun.

These laws can be deduced from Newton's laws of motion and law of universal gravitation. Indeed, Newton used Kepler's work as basic information in the formulation of his gravitational theory.

As Kepler pointed out, all planets move in elliptical orbits, however, we can learn much about planetary motion by considering the special case of circular orbits. We shall neglect the forces between planets, considering only a planet's interaction with the sun. These considerations apply equally well to the motion of a satellite about a planet.

Let's examine the case of two bodies of masses M and m moving in circular orbits under the influence of each other's gravitational attraction. The center of mass of this system of two bodies lies along the line joining them at a point C such that mr = MR. The large body of mass M moves in an orbit of constant radius R and the small body of mass m in an orbit of constant radius r, both having the same angular velocity ω. For this to happen, the gravitational force acting on each body must provide the necessary centripetal acceleration. Since these gravitational forces are a simple action-reaction pair, the centripetal forces must be equal but opposite in direction. That is, $m\omega^2 r$ must equal $M\omega^2 R$. The specific requirement, then, is that the gravitational force acting on either body must equal the centripetal force needed to keep it



moving in its circular orbit, that is

$$\frac{GMm}{(R+r)^2} = m\omega^2 r$$

If one body has a much greater mass than the other, as is the case of the sun and a planet or the Earth and a satellite, its distance from the center of mass is much smaller than that of the other body. If we assume that m is negligible compared to M, then R is negligible compared to r. Thus, equation above then becomes

$$GM = \omega^2 r^3$$

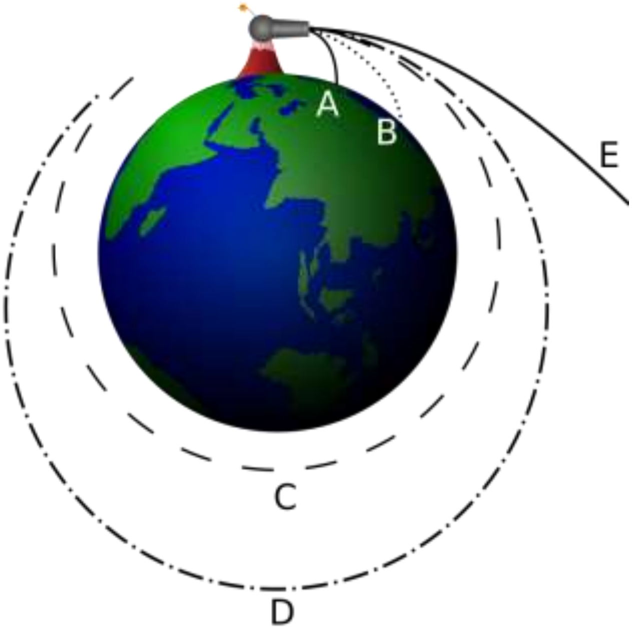
If we express the angular velocity in terms of the period of revolution, $\omega = 2\pi/P$, we obtain

$$GM = \frac{4\pi^2 r^3}{P^2}.$$

$$P^2 = \frac{4\pi^2 r^3}{GM}$$

Where P is the period of revolution. This is a basic equation of planetary and satellite motion. It also holds for elliptical orbits if we define r to be the semi-major axis (a) of the orbit.

A significant consequence of this equation is that it predicts Kepler's third law of planetary motion, that is $P^2 \sim r^3$.



PROBLEM 1.1

A spacecraft's engine ejects mass at a rate of 30 kg/s with an exhaust velocity of 3,100 m/s. The pressure at the nozzle exit is 5 kPa and the exit area is 0.7 m^2. What is the thrust of the engine in a vacuum?

SOLUTION,

Given: q = 30 kg/s
Ve = 3,100 m/s
Ae = 0.7 m2
Pe = 5 kPa = 5,000 N/m2
Pa = 0

Equation

$F = q \times Ve + (Pe - Pa) \times Ae$
$F = 30 \times 3{,}100 + (5{,}000 - 0) \times 0.7$
$F = 96{,}500$ N

In celestial mechanics where we are dealing with planetary or stellar sized bodies, it is often the case that the mass of the secondary body is significant in relation to the mass of the primary, as with the Moon and Earth. In this case the size of the secondary cannot be ignored. The distance R is no longer negligible compared to r and, therefore, must be carried through the derivation. Equation becomes

$$P^2 = \frac{4\pi^2 r(R+r)^2}{GM}$$

More commonly the equation is written in the equivalent form

$$P^2 = \frac{4\pi^2 a^3}{G(M+m)}$$

Where a is the semi-major axis. The semi-major axis used in astronomy is always the primary-to-secondary distance, or the geocentric semi-major axis. For example, the Moon's mean geocentric distance from Earth (a) is 384,403 kilometers. On the other hand, the Moon's distance from the barycentre (r) is 379,732 km, with Earth's counter-orbit (R) taking up the difference of 4,671 km.

Kepler's second law of planetary motion must, of course, hold true for circular orbits. In such orbits both ω and r are constant so that equal areas are swept out in equal times by the line joining a planet and the sun. For elliptical orbits, however, both ω and r will vary with time. Let's now consider this case.

Figure 4.5 shows a particle revolving around C along some arbitrary path. The area swept out by the radius vector in a short time interval Δt is shown shaded. This area, neglecting the small triangular region at the end, is one-half the base times the height or approximately r(rω Δt)/2. This expression becomes more exact as Δt approaches zero, i.e. the small triangle goes to zero more rapidly than the large one. The rate at which area is being swept out instantaneously is therefore

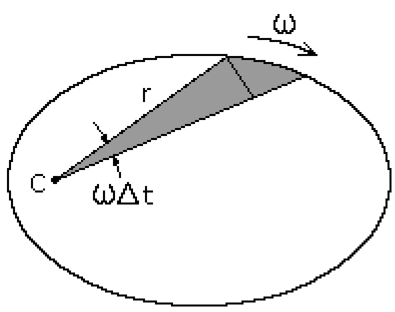


Figure 4.5

$$\lim_{t \to 0} \left[\frac{r(r\omega\Delta t)}{2}\right] = \frac{\omega r^2}{2} \qquad (4.10)$$

For any given body moving under the influence of a central force, the value ωr2 is constant.

Let's now consider two points P1 and P2 in an orbit with radii r1 and r2, and velocities v1 and v2. Since the velocity is always tangent to the path, it can be seen that if γ is the angle between r and v, then

$$v \sin\gamma = \omega r \qquad (4.11)$$

where vsinγ is the transverse component of v. Multiplying through by r, we have

(4.12) $$r v \sin\gamma = \omega r^2 = \text{Constant}$$

or, for two points P1 and P2 on the orbital path

(4.13) $$r_1 v_1 \sin\gamma_1 = r_2 v_2 \sin\gamma_2$$

Note that at periapsis and apoapsis, γ = 90 degrees. Thus, letting P1 and P2 be these two points we get

(4.14) $$R_p V_p = R_a V_a$$

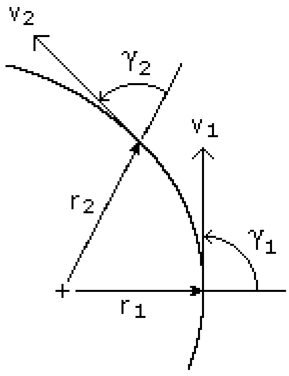


Figure 4.6

Let's now look at the energy of the above particle at points P1 and P2. Conservation of energy states that the sum of the kinetic energy and the potential energy of a particle remains constant. The kinetic energy T of a particle is given by mv2/2 while the potential energy of gravity V is calculated by the equation -GMm/r. Applying conservation of energy we have

$$T_1 + V_1 = T_2 + V_2, \quad \text{or}$$

$$\frac{m v_1^2}{2} - \frac{GMm}{r_1} = \frac{m v_2^2}{2} - \frac{GMm}{r_2}, \quad \text{or}$$

(4.15) $$v_2^2 - v_1^2 = 2GM\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

From equations (4.14) and (4.15) we obtain

(4.16) $$Vp = \sqrt{\frac{2GMR_a}{R_p(R_a + R_p)}}, \quad \text{and}$$

(4.17) $$Va = \sqrt{\frac{2GMR_p}{R_a(R_a + R_p)}}$$

Rearranging terms we get

(4.18) $$R_a = \frac{R_p}{\left(\frac{2GM}{R_p V_p^2} - 1\right)}, \quad \text{and}$$

(4.19) $$R_p = \frac{R_a}{\left(\frac{2GM}{R_a V_a^2} - 1\right)}$$

The eccentricity e of an orbit is given by

(4.20) $$e = \frac{R_p V_p^2}{GM} - 1$$

If the semi-major axis a and the eccentricity e of an orbit are known, then the periapsis and apoapsis distances can be calculated by

(4.21) $R_p = a(1-e)$, and

(4.22) $R_a = a(1+e)$

also note, $R_p + R_a = 2a$

2.3 Relations between position and time for different orbits

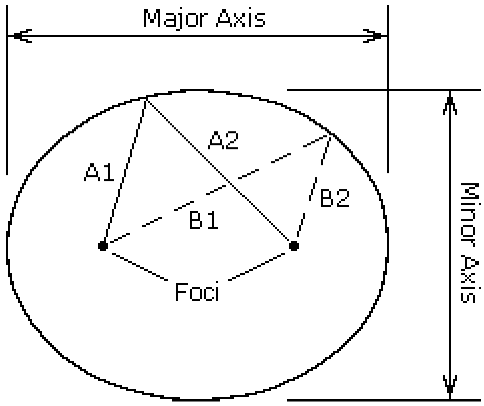
2.4 Expansions in elliptic motion

2.5 Orbital Elements

To mathematically describe an orbit one must define six quantities, called orbital elements. They are

1. Semi-Major Axis, a
2. Eccentricity, e
3. Inclination, i
4. Argument of Periapsis, ω
5. Time of Periapsis Passage, T
6. Longitude of Ascending Node, Ω

An orbiting satellite follows an oval shaped path known as an ellipse with the body being orbited, called the primary, located at one of two points called foci. An ellipse is defined to be a curve with the following property: for each point on an ellipse, the sum of its distances from two fixed points, called foci, is constant (see Figure 4.2). The longest and shortest lines that can be drawn through the center of an ellipse are called the major axis and minor axis, respectively. The semi-major axis is one-half of the major axis and represents a satellite's mean distance from its primary. Eccentricity is the distance between the foci divided



A1 + A2 = B1 + B2

Figure 4.2

by the length of the major axis and is a number between zero and one. An eccentricity of zero indicates a circle.

Inclination is the angular distance between a satellite's orbital plane and the equator of its primary (or the ecliptic plane in the case of heliocentric, or sun centered, orbits). An inclination of zero degrees indicates an orbit about the primary's equator in the same direction as the primary's rotation, a direction called prograde (or direct). An inclination of 90 degrees indicates a polar orbit. An inclination of 180 degrees indicates a retrograde equatorial orbit. A retrograde orbit is one in which a satellite moves in a direction opposite to the rotation of its primary.

Periapsis is the point in an orbit closest to the primary. The opposite of periapsis, the farthest point in an orbit, is called apoapsis. Periapsis and apoapsis are usually modified to apply to the body being orbited, such as perihelion and aphelion for the Sun, perigee and apogee for Earth, perijove and apojove for Jupiter, perilune and apolune for the Moon, etc. The argument of periapsis is the angular distance between the ascending node and the point of periapsis (see Figure 4.3). The time of periapsis passage is the time in which a satellite moves through its point of periapsis.

Nodes are the points where an orbit crosses a plane, such as a satellite crossing the Earth's equatorial plane. If the satellite crosses the plane going from south to north, the node is the ascending node; if moving from north to south, it is the descending node. The longitude of the ascending node is the node's celestial longitude. Celestial longitude is analogous to longitude on Earth and is measured in degrees counter-clockwise from zero with zero longitude being in the direction of the vernal equinox.

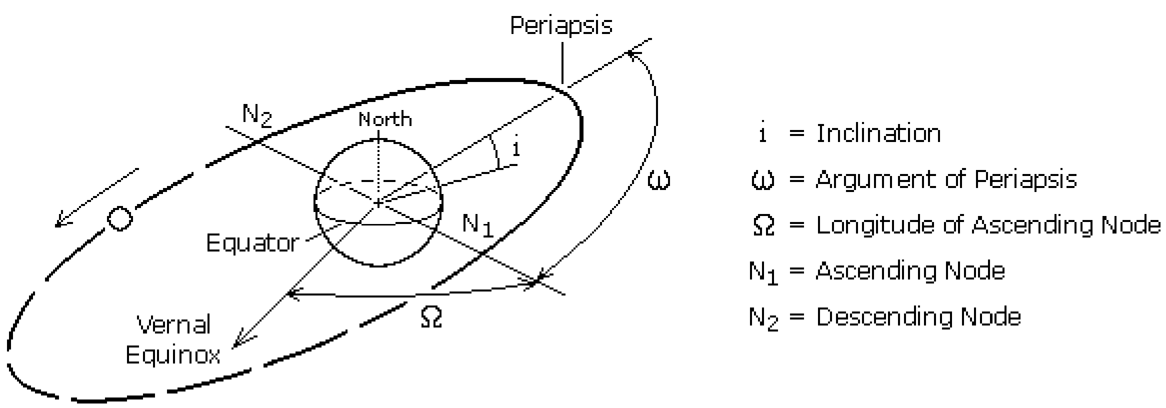


Figure 4.3

In general, three observations of an object in orbit are required to calculate the six orbital elements. Two other quantities often used to describe orbits are period and true anomaly. Period, P, is the length of time required for a satellite to complete one orbit. True anomaly, ν, is the angular distance of a point in an orbit past the point of periapsis, measured in degrees.

2.5.1 Types Of Orbits

For a spacecraft to achieve Earth orbit, it must be launched to an elevation above the Earth's atmosphere and accelerated to orbital velocity. The most energy efficient orbit, that is one that requires the least amount of propellant, is a direct low inclination orbit. To achieve such an orbit, a spacecraft is launched in an eastward direction from a site near the Earth's equator. The advantage being that the rotational speed of the Earth contributes to the spacecraft's final orbital speed. At the United States' launch site in Cape Canaveral (28.5 degrees north latitude) a due east launch results in a "free ride" of 1,471 km/h (914 mph). Launching a spacecraft in a direction other than east, or from a site far from the equator, results in an orbit of higher inclination. High inclination orbits are less able to take advantage of the initial speed provided by the Earth's rotation, thus the launch vehicle must provide a greater part, or all, of the energy required to attain orbital velocity. Although high inclination orbits are less energy efficient, they do have advantages over equatorial orbits for certain applications. Below we describe several types of orbits and the advantages of each:

- Geosynchronous orbits (GEO) are circular orbits around the Earth having a period of 24 hours. A geosynchronous orbit with an inclination of zero degrees is called a geostationary orbit. A spacecraft in a geostationary orbit appears to hang motionless above one position on the Earth's equator. For this reason, they are ideal for some types of communication and meteorological satellites. A spacecraft in an inclined geosynchronous orbit will appear to follow a regular figure-8 pattern in the sky once every orbit. To attain geosynchronous orbit, a spacecraft is first launched into an elliptical orbit with an apogee of 35,786 km (22,236 miles) called a geosynchronous transfer orbit (GTO). The orbit is then circularized by firing the spacecraft's engine at apogee.
- Polar orbits (PO) are orbits with an inclination of 90 degrees. Polar orbits are useful for satellites that carry out mapping and/or surveillance operations because as the

planet rotates the spacecraft has access to virtually every point on the planet's surface.

- Walking orbits: An orbiting satellite is subjected to a great many gravitational influences. First, planets are not perfectly spherical and they have slightly uneven mass distribution. These fluctuations have an effect on a spacecraft's trajectory. Also, the sun, moon, and planets contribute a gravitational influence on an orbiting satellite. With proper planning it is possible to design an orbit which takes advantage of these influences to induce a precession in the satellite's orbital plane. The resulting orbit is called a walking orbit, or precessing orbit.
- Sun synchronous orbits (SSO) are walking orbits whose orbital plane precesses with the same period as the planet's solar orbit period. In such an orbit, a satellite crosses periapsis at about the same local time every orbit. This is useful if a satellite is carrying instruments which depend on a certain angle of solar illumination on the planet's surface. In order to maintain an exact synchronous timing, it may be necessary to conduct occasional propulsive maneuvers to adjust the orbit.
- Molniya orbits are highly eccentric Earth orbits with periods of approximately 12 hours (2 revolutions per day). The orbital inclination is chosen so the rate of change of perigee is zero, thus both apogee and perigee can be maintained over fixed latitudes. This condition occurs at inclinations of 63.4 degrees and 116.6 degrees. For these orbits the argument of perigee is typically placed in the southern hemisphere, so the satellite remains above the northern hemisphere near apogee for approximately 11 hours per orbit. This orientation can provide good ground coverage at high northern latitudes.

2.6 Relation between orbital elements and position and velocity

2.7 Launch vehicle ascent trajectories

2.7.1 Launch of a Space Vehicle

The launch of a satellite or space vehicle consists of a period of powered flight during which the vehicle is lifted above the Earth's atmosphere and accelerated to orbital velocity by a rocket, or launch vehicle. Powered flight concludes at burnout of the rocket's last stage at which time the vehicle begins its free flight. During free flight the space vehicle is assumed to be subjected only to the gravitational pull of the Earth. If the vehicle moves far from the Earth, its trajectory may be affected by the gravitational influence of the sun, moon, or another planet.

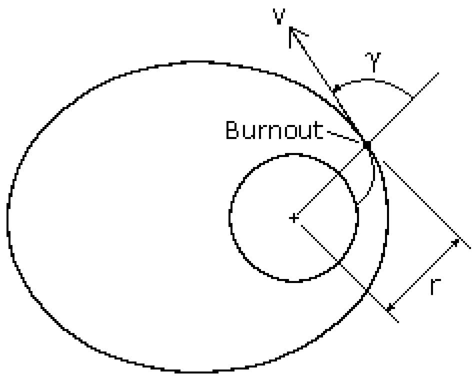


Figure 4.7

A space vehicle's orbit may be determined from the position and the velocity of the vehicle at the beginning of its free flight. A vehicle's position and velocity can be described by the variables r, v, and γ, where r is the vehicle's distance from the center of the Earth, v is its velocity, and γ is the angle between the position and the velocity vectors, called the zenith angle (see Figure 4.7). If we let r1, v1, and γ1 be the initial (launch) values of r, v, and γ, then we may consider these as given quantities. If we let point P2 represent the perigee, then equation (4.13) becomes

$$v_2 = V_p = \frac{r_1 v_1 \sin\gamma_1}{R_p} \qquad (4.23)$$

Substituting equation (4.23) into (4.15), we can obtain an equation for the perigee radius Rp.

$$\frac{r_1^2 v_1^2 \sin^2\gamma_1}{R_p^2} - v_1^2 = 2GM\left(\frac{1}{R_p} - \frac{1}{r_1}\right) \qquad (4.24)$$

Multiplying through by -Rp2/(r12v12) and rearranging, we get

$$\left(\frac{R_p}{r_1}\right)^2 (1-C) + \left(\frac{R_p}{r_1}\right) C - \sin^2\gamma_1 = 0 \qquad (4.25)$$

$$\text{where } C = \frac{2GM}{r_1 v_1^2}$$

Note that this is a simple quadratic equation in the ratio (Rp/r1) and that 2GM /(r1 × v12) is a nondimensional parameter of the orbit.

Solving for (Rp/r1) gives

$$(4.26) \qquad \left(\frac{R_p}{r_1}\right)_{1,2} = \frac{-C \pm \sqrt{C^2 - 4(1-C)(-\sin^2\gamma_1)}}{2(1-C)}$$

Like any quadratic, the above equation yields two answers. The smaller of the two answers corresponds to Rp, the periapsis radius. The other root corresponds to the apoapsis radius, Ra. Please note that in practice spacecraft launches are usually terminated at either perigee or apogee, i.e. γ = 90. This condition results in the minimum use of propellant.

Equation (4.26) gives the values of Rp and Ra from which the eccentricity of the orbit can be calculated, however, it may be simpler to calculate the eccentricity e directly from the equation

$$(4.27) \qquad e = \sqrt{\left(\frac{r_1 v_1^2}{GM} - 1\right)^2 \sin^2\gamma_1 + \cos^2\gamma_1}$$

To pin down a satellite's orbit in space, we need to know the angle ν, the true anomaly, from the periapsis point to the launch point. This angle is given by

$$(4.28) \qquad \tan\nu = \frac{\left(\frac{r_1 v_1^2}{GM}\right)\sin\gamma_1\cos\gamma_1}{\left(\frac{r_1 v_1^2}{GM}\right)\sin^2\gamma_1 - 1}$$

In most calculations, the complement of the zenith angle is used, denoted by ϕ. This angle is called the flight-path angle, and is positive when the velocity vector is directed away from the primary as shown in Figure 4.8. When flight-path angle is used, equations (4.26) through (4.28) are rewritten as follows:

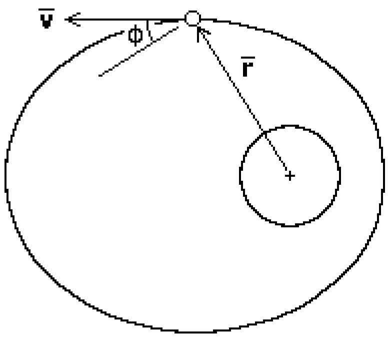


Figure 4.8

$$(4.29) \qquad \left(\frac{R_p}{r}\right)_{1,2} = \frac{-C \pm \sqrt{C^2 - 4(1-C)(-\cos^2\phi)}}{2(1-C)}$$

$$\text{where } C = \frac{2GM}{r v^2}$$

$$(4.30) \qquad e = \sqrt{\left(\frac{r v^2}{GM} - 1\right)^2 \cos^2\phi + \sin^2\phi}$$

$$(4.31) \qquad \tan\nu = \frac{\left(\frac{r v^2}{GM}\right)\cos\phi\sin\phi}{\left(\frac{r v^2}{GM}\right)\cos^2\phi - 1}$$

The semi-major axis is, of course, equal to (Rp+Ra)/2, though it may be easier to calculate it directly as follows:

$$a = \frac{1}{\left(\frac{2}{r} - \frac{v^2}{GM}\right)} \qquad (4.32)$$

If e is solved for directly using equation (4.27) or (4.30), and a is solved for using equation (4.32), Rp and Ra can be solved for simply using equations (4.21) and (4.22).

Orbit Tilt, Rotation and Orientation

Above we determined the size and shape of the orbit, but to determine the orientation of the orbit in space, we must know the latitude and longitude and the heading of the space vehicle at burnout.

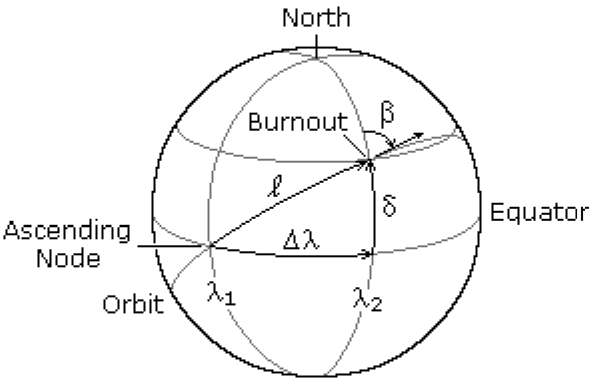


Figure 4.9

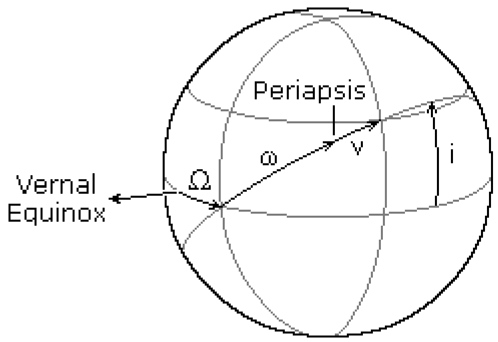


Figure 4.10

Figure 4.9 above illustrates the location of a space vehicle at engine burnout, or orbit insertion. β is the azimuth heading measured in degrees clockwise from north, δ is the geocentric latitude (or declination) of the burnout point, $\Delta\lambda$ is the angular distance between the ascending node and the burnout point measured in the equatorial plane, and ℓ is the angular distance between the ascending node and the burnout point measured in the orbital plane. λ_1 and λ_2 are the geographical longitudes of the ascending node and the burnout point at the instant of engine burnout. Figure 4.10 pictures the orbital elements, where i is the inclination, Ω is the longitude at the ascending node, ω is the argument of periapsis, and ν is the true anomaly.

If β, δ, and λ_2 are given, the other values can be calculated from the following relationships:

(4.33) $\cos i = \cos\delta \sin\beta$

(4.34) $\tan \ell = \dfrac{\tan\delta}{\cos\beta}$

(4.35) $\tan\Delta\lambda = \sin\delta \tan\beta$

(4.36) $\omega = \ell - \nu$

(4.37) $\lambda_1 = \lambda_2 - \Delta\lambda$

In equation (4.36), the value of ν is found using equation (4.28) or (4.31). If ν is positive, periapsis is west of the burnout point (as shown in Figure 4.10); if ν is negative, periapsis is east of the burnout point.

The longitude of the ascending node, Ω, is measured in celestial longitude, while $\lambda 1$ is geographical longitude. The celestial longitude of the ascending node is equal to the local apparent sidereal time, in degrees, at longitude $\lambda 1$ at the time of engine burnout. Sidereal time is defined as the hour angle of the vernal equinox at a specific locality and time; it has the same value as the right ascension of any celestial body that is crossing the local meridian at that same instant. At the moment when the vernal equinox crosses the local meridian, the local apparent sidereal time is 00:00.

2.8 General aspects of satellite injection

Placing a satellite into geosynchronous orbit requires an enormous amount of energy. The launch process can be divided into two phases: the launch phase and the orbit injection phase.

The Launch Phase

During the launch phase, the launch vehicle places the satellite into the transfer orbit--an eliptical orbit that has at its farthest point from earth (apogee) the geosynchronous elevation of 22,238 miles and at its nearest point (perigee) an elevation of usually not less than 100 miles as shown below in **Figure**.

The Orbit Injection Phase

The energy required to move the satellite from the elliptical transfer orbit into the geosynchronous orbit is supplied by the satellite's apogee kick motor (AKM). This is known as the orbit injection phase.

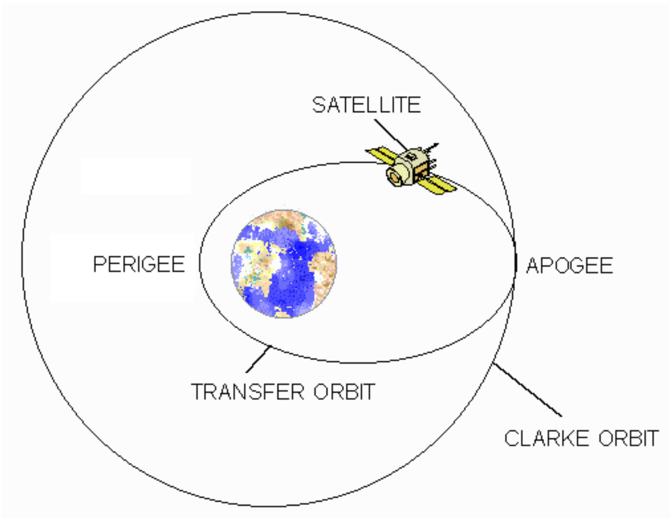


Figure 2The Elliptical Transfer Orbit

2.8.1 Satellite Launch Procedure

The four orbit stages involved in the satellite launch procedure are as follows:

1. Circular low earth orbit
2. Hohmann elliptical transfer orbit
3. Intermediate drift orbit
4. Circular Geostationary orbit

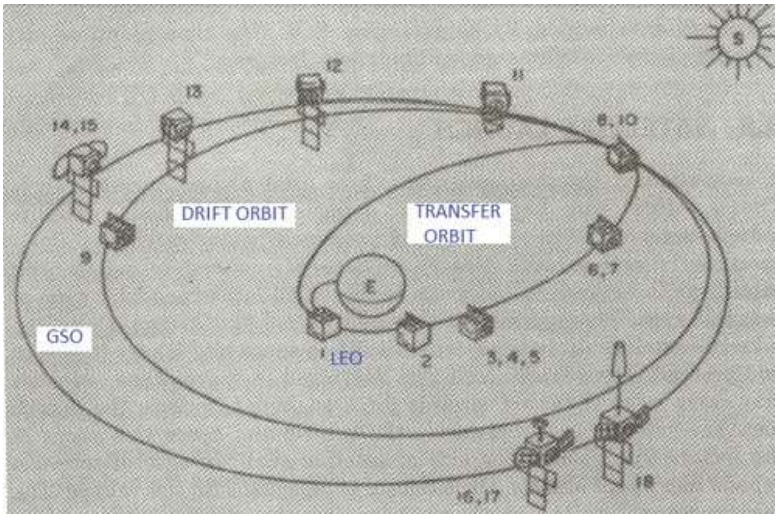


Figure depicts typical satellite launch procedure followed by space companies such as ISRO. Following are the major steps involved in the launch process.

- Step-1: The launch vehicle takes the satellite into low earth orbit. The satellite is injected into desired 3-axes stabilized mode to achieve gyro condition using commands issued by launch vehicle to carry pyro firing.
- Step-2: After satellite reaches apogee AKM is fired for long duration to take satellite to intermediate orbit. This intermediate orbit is referred as transfer orbit. AKM is the short form of Apogee Kick Motor which contains liquid fuel.
- Step-3: The second apogee motor firing is carried out so that satellite attains needed angular velocity and acceleration for Geo-synchronization. This helps satellite to be in LOS from central earth stations. If required it is tracked through other countries earth stations.
- Step-4: Further stabilization and attitude control is achieved using control of momentum/reaction wheels. Antennas and transponders are turned on which brings satellite into stabilized geostationary orbit. Examples of geostationary satellites are INTELSAT, COMSAT, INSAT etc.

Once the satellite is placed in the parking space(i.e. designated orbit), following activities need to be performed as part of maintenance.

- Orbit maintenance
- Attitude maintenance
- Thermal management
- Power management
- battery maintenance
- Payload operations
- Software requirement

Note: Some of these operations are routine in nature whereas some are scheduled as and when required.

2.9 Dependence of orbital parameters on in-plane injection parameters

IN-PLANE ORBIT CHANGES

Due to small errors in burnout altitude , speed, and flight-path angle, the exact orbit desired may not be achieved. Usually this is not serious, but , if a rendezvous is contemplated or if, for some other reason, a very precise orbit is required, it may be necessary to make small corrections in the orbit. This may be done by applying small speed changes or Δv, as they are called , at appropriate points in the orbit . we shall consider both small in-plane corrections to an orbit and . large changes from one circular orbit to a new one of different size.

$a_{tx} = \frac{r_A + r_B}{2}$	semi-major axis of transfer ellipse
$V_{iA} = \sqrt{\frac{GM}{r_A}}$	initial velocity at point A
$V_{fB} = \sqrt{\frac{GM}{r_B}}$	final velocity at point B

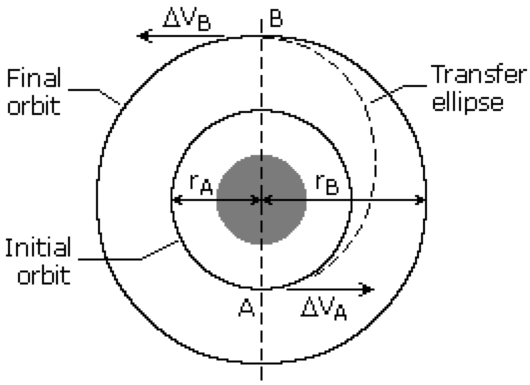


Figure 4.11

2.10 Launch vehicle performances

2.11 Orbit deviations due to injection errors

UNIT-III

Perturbed Satellite Orbit

3 Unit-III Perturbed Satellite Orbit

3.1 Special and general perturbations

3.1.1 Orbit Perturbations

The orbital elements discussed at the beginning of this section provide an excellent reference for describing orbits, however there are other forces acting on a satellite that perturb it away from the nominal orbit. These perturbations, or variations in the orbital elements, can be classified based on how they affect the Keplerian elements. Secular variations represent a linear variation in the element, short-period variations are periodic in the element with a period less than the orbital period, and long-period variations are those with a period greater than the orbital period. Because secular variations have long-term effects on orbit prediction (the orbital elements affected continue to increase or decrease), they will be discussed here for Earth-orbiting satellites. Precise orbit determination requires that the periodic variations be included as well.

3.1.2 Third-Body Perturbations

The gravitational forces of the Sun and the Moon cause periodic variations in all of the orbital elements, but only the longitude of the ascending node, argument of perigee, and mean anomaly experience secular variations. These secular variations arise from a gyroscopic precession of the orbit about the ecliptic pole. The secular variation in mean anomaly is much smaller than the mean motion and has little effect on the orbit, however the secular variations in longitude of the ascending node and argument of perigee are important, especially for high-altitude orbits. For nearly circular orbits the equations for the secular rates of change resulting from the Sun and Moon are

Longitude of the ascending node:

(4.46) $$\Omega_{moon} = -0.00338 \cos(i)/n$$

(4.47) $$\Omega_{sun} = -0.00154 \cos(i)/n$$

Argument of perigee:

(4.48) $\omega_{moon} = 0.00169(4-5\sin^2 i)/n$

(4.49) $\omega_{sun} = 0.00077(4-5\sin^2 i)/n$

where i is the orbit inclination, n is the number of orbit revolutions per day, and Ω and ω are in degrees per day. These equations are only approximate; they neglect the variation caused by the changing orientation of the orbital plane with respect to both the Moon's orbital plane and the ecliptic plane.

3.1.3 Perturbations due to Non-spherical Earth

When developing the two-body equations of motion, we assumed the Earth was a spherically symmetrical, homogeneous mass. In fact, the Earth is neither homogeneous nor spherical. The most dominant features are a bulge at the equator, a slight pear shape, and flattening at the poles. For a potential function of the Earth, we can find a satellite's acceleration by taking the gradient of the potential function. The most widely used form of the geopotential function depends on latitude and geopotential coefficients, Jn, called the zonal coefficients.

The potential generated by the non-spherical Earth causes periodic variations in all the orbital elements. The dominant effects, however, are secular variations in longitude of the ascending node and argument of perigee because of the Earth's oblateness, represented by the J2 term in the geopotential expansion. The rates of change of Ω and ω due to J2 are

(4.50) $$\Omega_{J_2} = -1.5nJ_2(R_E/a)^2(\cos i)(1-e^2)^{-2}$$
$$\approx -2.06474\times10^{14}a^{-7/2}(\cos i)(1-e^2)^{-2}$$

(4.51) $$\omega_{J_2} = 0.75nJ_2(R_E/a)^2(4-5\sin^2 i)(1-e^2)^{-2}$$
$$\approx 1.03237\times10^{14}a^{-7/2}(4-5\sin^2 i)(1-e^2)^{-2}$$

where n is the mean motion in degrees/day, J2 has the value 0.00108263, RE is the Earth's equatorial radius, a is the semi-major axis in kilometers, i is the inclination, e is the eccentricity, and Ω and ω are in degrees/day. For satellites in GEO and below, the J2 perturbations dominate; for satellites above GEO the Sun and Moon perturbations dominate.

Molniya orbits are designed so that the perturbations in argument of perigee are zero. This conditions occurs when the term 4-5sin2i is equal to zero or, that is, when the inclination is either 63.4 or 116.6 degrees.

Perturbations from Atmospheric Drag

Drag is the resistance offered by a gas or liquid to a body moving through it. A spacecraft is subjected to drag forces when moving through a planet's atmosphere. This drag is greatest during launch and reentry, however, even a space vehicle in low Earth orbit experiences some drag as it moves through the Earth's thin upper atmosphere. In time, the action of drag on a space vehicle will cause it to spiral back into the atmosphere, eventually to disintegrate or burn up. If a space vehicle comes within 120 to 160 km of the Earth's surface, atmospheric drag will bring it down in a few days, with final disintegration occurring at an altitude of about 80 km. Above approximately 600 km, on the other hand, drag is so weak that orbits usually last more than 10 years - beyond a satellite's operational lifetime. The deterioration of a spacecraft's orbit due to drag is called decay.

The drag force FD on a body acts in the opposite direction of the velocity vector and is given by the equation

$$D = \frac{1}{2} C_D \rho v^2 A \qquad (4.52)$$

where CD is the drag coefficient, ρ is the air density, v is the body's velocity, and A is the area of the body normal to the flow. The drag coefficient is dependent on the geometric form of the body and is generally determined by experiment. Earth orbiting satellites typically have very high drag coefficients in the range of about 2 to 4. Air density is given by the appendix Atmosphere Properties.

The region above 90 km is the Earth's thermosphere where the absorption of extreme ultraviolet radiation from the Sun results in a very rapid increase in temperature with altitude. At approximately 200-250 km this temperature approaches a limiting value, the average value of which ranges between about 700 and 1,400 K over a typical solar cycle. Solar activity also has a significant affect on atmospheric density, with high solar activity resulting in high density. Below about 150 km the density is not strongly affected by solar activity; however, at satellite altitudes in the range of 500 to 800 km, the density variations between solar maximum and solar minimum are approximately two orders of magnitude. The large variations imply that satellites will decay more rapidly during periods of solar maxima and much more slowly during solar minima.

For circular orbits we can approximate the changes in semi-major axis, period, and velocity per revolution using the following equations:

$$\Delta a_{rev} = \frac{-2\pi C_D A \rho a^2}{m} \qquad (4.53)$$

$$\Delta P_{rev} = \frac{-6\pi^2 C_D A \rho a^2}{mV} \qquad (4.54)$$

$$\Delta V_{rev} = \frac{\pi C_D A \rho a V}{m} \qquad (4.55)$$

where a is the semi-major axis, P is the orbit period, and V, A and m are the satellite's velocity, area, and mass respectively. The term m/(CDA), called the ballistic coefficient, is given as a constant for most satellites. Drag effects are strongest for satellites with low ballistic coefficients, this is, light vehicles with large frontal areas.

A rough estimate of a satellite's lifetime, L, due to drag can be computed from

$$L \approx \frac{-H}{\Delta a_{rev}} \qquad (4.56)$$

Where H is the atmospheric density scale height. A substantially more accurate estimate (although still very approximate) can be obtained by integrating equation (4.53), taking into account the changes in atmospheric density with both altitude and solar activity.

Perturbations from Solar Radiation

Solar radiation pressure causes periodic variations in all of the orbital elements. The magnitude of the acceleration in m/s2 arising from solar radiation pressure is

$$a_R = \frac{-4.5\times10^{-8} A}{m} \qquad (4.57)$$

where A is the cross-sectional area of the satellite exposed to the Sun and m is the mass of the satellite in kilograms. For satellites below 800 km altitude, acceleration from atmospheric drag is greater than that from solar radiation pressure; above 800 km, acceleration from solar radiation pressure is greater.

3.2 Cowell's Method

Cowell's method is perhaps the simplest of the special perturbation methods; mathematically, for mutually interacting bodies, Newtonian forces on body from the other bodies are simply summed thus,

$$\ddot{\mathbf{r}}_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{G m_j (\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3}$$

where

- $\ddot{\mathbf{r}}_i$ is the acceleration vector of body i
- G is the gravitational constant
- m_j is the mass of body j
- $\mathbf{r}_i$ and $\mathbf{r}_j$ are the position vectors of objects i and j
- r_{ij} is the distance from object i to object j

with all vectors being referred to the barycenter of the system. This equation is resolved into components in , , and these are integrated numerically to form the new velocity and position vectors as the simulation moves forward in time. The advantage of Cowell's method is ease of application and programming. A disadvantage is that when perturbations become large in magnitude (as when an object makes a close approach to another) the errors of the method also become large. Another disadvantage is that in systems with a dominant central body, such as the Sun, it is necessary to carry many significant digits in the arithmetic because of the large difference in the forces of the central body and the perturbing bodies.

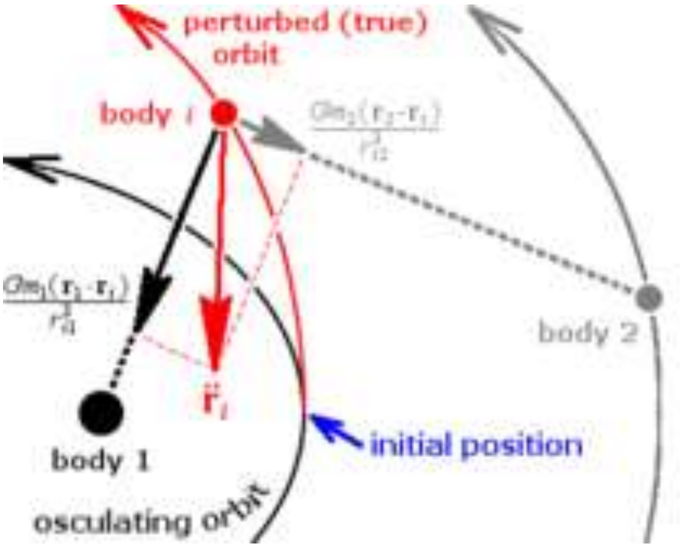


Figure 3Cowell's method

Cowell's method. Forces from all perturbing bodies (black and gray) are summed to form the total force on body *I* (red), and this is numerically integrated starting from the initial position (the *epoch of osculation*).

3.3 Encke's method

Encke's method begins with the osculating orbit as a reference and integrates numerically to solve for the variation from the reference as a function of time. Its advantages are that perturbations are generally small in magnitude, so the integration can proceed in larger steps (with resulting lesser errors), and the method is much less affected by extreme perturbations than Cowell's method. Its disadvantage is complexity; it cannot be used indefinitely without occasionally updating the osculating orbit and continuing from there, a process known as *rectification.*

Lettingbe the radius vector of the osculating orbit, the radius vector of the perturbed orbit, and δr the variation from the osculating orbit.

$\delta\mathbf{r} = \mathbf{r} - \boldsymbol{\rho}$, and the equation of motion of $\delta\mathbf{r}$ is simply

$$\ddot{\delta\mathbf{r}} = \ddot{\mathbf{r}} - \ddot{\boldsymbol{\rho}}.$$

$\ddot{\mathbf{r}}$ and $\ddot{\boldsymbol{\rho}}$ are just the equations of motion of $\mathbf{r}$ and $\boldsymbol{\rho}$,

$$\ddot{\mathbf{r}} = \mathbf{a}_{\text{per}} - \frac{\mu}{r^3}\mathbf{r}$$ for the perturbed orbit and

$$\ddot{\boldsymbol{\rho}} = -\frac{\mu}{\rho^3}\boldsymbol{\rho}$$ for the unperturbed orbit,

Where is the gravitational parameter with and the masses of the central body and the perturbed body, is the perturbing acceleration, and and are the magnitudes of and

$$\ddot{\delta\mathbf{r}} = \mathbf{a}_{\text{per}} + \mu\left(\frac{\boldsymbol{\rho}}{\rho^3} - \frac{\mathbf{r}}{r^3}\right)$$

which, in theory, could be integrated twice to find δr. Since the osculating orbit is easily calculated by two-body methods, and δr are accounted for and can be solved. In practice, the quantity in the brackets, $\frac{\boldsymbol{\rho}}{\rho^3} - \frac{\mathbf{r}}{r^3}$ is the difference of two nearly equal vectors, and further manipulation is necessary to avoid the need for extra significant digits.

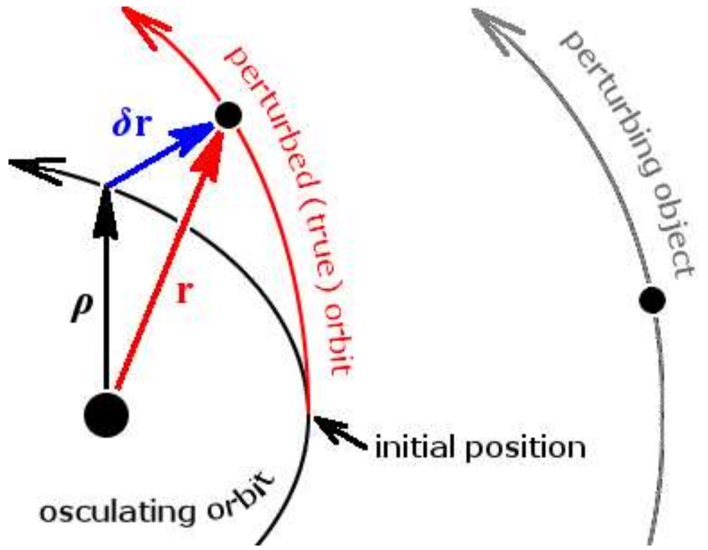


Figure 4Encke's method

Encke's method. Greatly exaggerated here, the small difference δr (blue) between the osculating, unperturbed orbit (black) and the perturbed orbit (red), is numerically integrated starting from the initial position (the epoch of osculation).

3.4 Method of variations of orbital elements

Due to perturbative forces acting on a near earth satellite, the associated classical Keplerian orbital elements vary with time. These variations are divided into secular, short period and long period. The mathematical equations expressing these variations are presented without derivation along with numerical examples. A discussion of the practical applications of these variations to trajectory generation and orbit determination is included.

1. Third-Body Perturbations
2. Perturbations due to Non-spherical Earth
3. Perturbations from Atmospheric Drag
4. Perturbations from Solar Radiation

3.5 General perturbations approach

In methods of **general perturbations**, general differential equations, either of motion or of change in the orbital elements, are solved analytically, usually by series expansions. The result is usually expressed in terms of algebraic and trigonometric functions of the orbital elements of the body in question and the perturbing bodies. This can be applied generally to many different sets of conditions, and is not specific to any particular set of gravitating objects. Historically, general perturbations were investigated first. The classical methods are known as *variation of the elements*, *variation of parameters* or *variation of the constants of integration*. In these methods, it is considered that the body is always moving in a conic section, however the conic section is constantly changing due to the perturbations. If all perturbations were to cease at any particular instant, the body would continue in this (now unchanging) conic section indefinitely; this conic is known as the osculating orbit and its orbital elements at any particular time are what are sought by the methods of general perturbations.

General perturbations takes advantage of the fact that in many problems of celestial mechanics, the two-body orbit changes rather slowly due to the perturbations; the two-body orbit is a good first approximation. General perturbations is applicable only if the perturbing forces are about one order of magnitude smaller, or less, than the gravitational force of the primary body. In the

Solar System, this is usually the case; Jupiter, the second largest body, has a mass of about 1/1000 that of the Sun.

General perturbation methods are preferred for some types of problems, as the source of certain observed motions are readily found. This is not necessarily so for special perturbations; the motions would be predicted with similar accuracy, but no information on the configurations of the perturbing bodies (for instance, an orbital resonance) which caused them would be available.

3.6 Two-dimensional interplanetary trajectories

3.7 Fast interplanetary trajectories

3.8 Three dimensional interplanetary trajectories

3.9 Launch of interplanetary spacecraft

3.10 Trajectory about the target planet

Unit-IV

Ballistic Missile Trajectories

4 Unit-IV Ballistic Missile Trajectories

A ballistic missile trajectory is composed of three parts-the powered flight portion which lasts from launch to thrust cutoff or burnout, the free-flight portion which constitutes most of the trajectory, and the re-entry portion which begins at some ill-defined point where atmospheric drag becomes a significant force in determining the missile's path and lasts until impact.

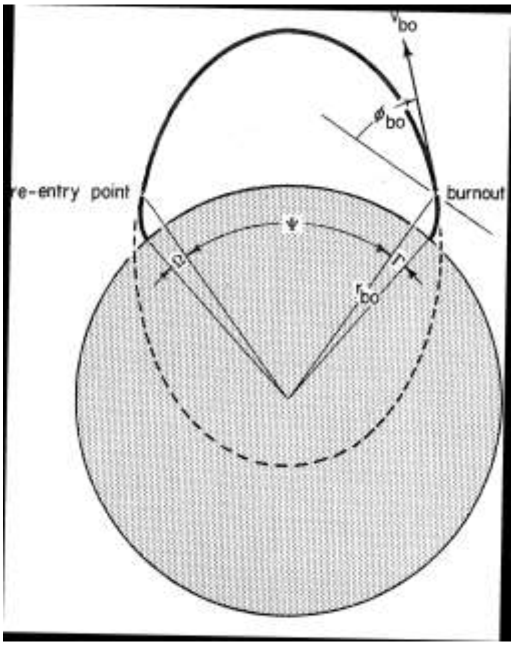
Since energy is continuously being added to the missile during powered flight , we cannot use 2-body mechanics to determine its path from launch to burnout . The path of the missile during this critical part of the flight is determined by the guidance and navigation system. This is the topic of an entire course and will not be covered here; During free-flight the trajectory is part of a conic orbit-almost always an ellipse.

Re-entry involves the dissipation of energy by friction with the atmosphere. It will not be discussed in this text. We will begin by assuming that the Earth does not rotate and that the altitude at which re-entry starts is the same as the burnout altitude. This latter assumption insures that the free-flight trajectory is symmetrical and will allow us to derive a fairly simple expression for the free-flight range of a missile in terms of its burnout conditions. We will then answer a more practical question-"given rbo'vbo' and a desired free-flight range , what flight-path angle at burnout is required? "

4.1 The boost phase

4.1.1 Terminology

The terminology of orbital mechanics, such terms as "*height at burnout*," "*height of apogee*," "*flight-path angle at burnout*," etc., need not be redefined. There are , however, a few new and unfamiliar terms which you must learn before we embark on any derivations.



r - powered flight range angle

R p - ground range of powered flight

Rff - ground range of freeflight

Rre ground range of re-en try

Rt - total ground range

'It - free-flight range angle

n - re-entry range angle

A - total range angle

$$\Lambda = \Gamma + \Psi + \Omega$$

$$R_t = R_p + R_{ff} + R_{re}$$

4.2 The ballistic phase

The Free-Flight Range Equation. Since the free-flight trajectory of a missile is a conic section , the general equation of a conic can be applied to the burnout point.

$$r_{bo} = \frac{p}{1 + e\cos\nu_{bo}}$$

Solving for cos vbo' we get

$$\cos\nu_{bo} = \frac{p - r_{bo}}{e r_{bo}}$$

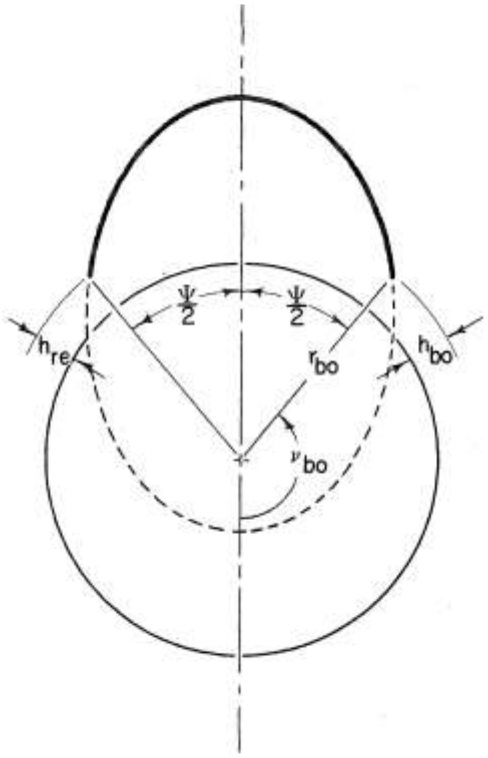


Figure 5Symmetrical traj ectory

Since the free-flight trajectory is assumed to be symmetrical (hbo = hre), half the free-flight range angle , 'l', lies on each side of the major axis, and

$$\cos \frac{\Psi}{2} = - \cos \nu_{bo}$$

above can, therefore, be written as

$$\cos \frac{\Psi}{2} = \frac{r_{bo} - p}{e r_{bo}}$$

We now have an expression for *the free-flight range angle* in terms of p, e, and rbo' Since p = h2 /11 and. h = rv cos cp, we can use the definition of Q to obtain

$$p = \frac{r^2 v^2 \cos^2 \phi}{\mu} = rQ \cos^2 \phi$$

Now, since $p = a(1 - e^2)$,

$$e^2 = 1 - \frac{p}{a}$$

Substituting $p = rQ\cos^2\phi$ and $a = \frac{r}{2-Q}$, we get

$$e^2 = 1 + Q(Q-2)\cos^2\phi$$

If we now substitute equations, we have one form of the *free-flight range equation*:

$$\cos\frac{\Psi}{2} = \frac{1 - Q_{bo}\cos^2\phi_{bo}}{\sqrt{1 + Q_{bo}(Q_{bo}-2)\cos^2\phi_{bo}}}.$$

From this equation we can calculate the free-flight range angle resulting frbm any given combination of burnout conditions, rbo' vbo and CPbO"

While this will prove to be a very valuable equation, it is not particularly useful in solving the typical ballistic missile problem which can be stated thus: Given a particular launch point and target, the total range angle, A, can be calculated as we shall see later in this chapter. If we know how far the missile will travel during powered flight and re-entry, the required free-flight range angle, 'l1, also becomes known. If we now specify rbo and vbo for the missile, what should the flight-path
angle, ¢ro, be in order that the missile will hit the target?

4.3 Trajectory geometry

The Flight-Path Angle Equation. In Figure 6 . 2- 3 we have drawn the local horizontal at the burnout point and also the tangent and normal at the burnout point . The line from the burnout point to the secondary focus, F', is called rbo. The angle between the local horizontal and the tangent (direction of voo') is the flight-path angle , ¢bo. Since rbo is perpendicular to the local horizontal, and the normal is perpendicular to the tangent , the angle between rbo and the normal is also ¢bo. Now, it can be proven (although we won't do it) that the angle between rOO and rbo is bisected by the normal. This fact gives rise to many interesting applications for the ellipse . It means, for example, that, if the ellipse represented the surface of a mirror , light emanating from one focu s would be reflected to the other focus since the angle of reflection equals the angle of incidence. If the ceiling of a room were made in the shape of an ellipsoid, a person standing at a particular point in the room corresponding to one focus could be heard clearly by a person standing at the other focus even though he were whispering. This is, in fact, the basis for the so-called "whispering gallery." What all this means for our derivation is simply that the angle

between roo and rbo is 2¢bo'

Let us concentrate on the triangle formed by F, F' and the burnout point. We know two of the angles in this triangle and the third can be determined from the fact that the angles of a triangle sum to 1 800. If we divide the triangle into two right triangles by the dashed line , d, shown in Figure 6.2-4, we can express d as

$$d = r_{bo} \sin \frac{\Psi}{2}$$

and also as

$$d = r'_{bo} \sin \left[180^{o} - \left(2\phi_{bo} + \frac{\Psi}{2} \right) \right]$$

Combining these two equations and noting that sin (1 800 - x) = sin

x, we get

$$\sin \left(2\phi_{bo} + \frac{\Psi}{2} \right) = \frac{r_{bo}}{r'_{bo}} \sin \frac{\Psi}{2}$$

Since rbo = a(2 - 000) from equation (6.2-3) and r60 + rbo = 2a ,

$$\sin \left(2\phi_{bo} + \frac{\Psi}{2} \right) = \frac{2 - Q_{bo}}{Q_{bo}} \sin \frac{\Psi}{2}$$

This is called the *flight-path angle equation* and it points out some interesting and important facts about ballistic missile trajectories.

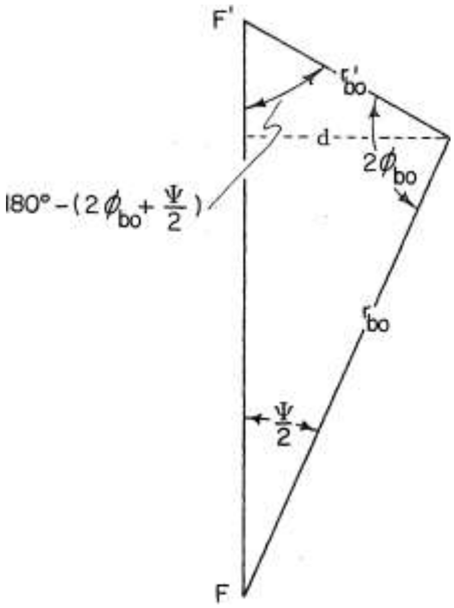


Figure 6Ellipse geometry

Suppose we want a missile to travel a free-flight range of 900 and it has Qbo= .9. Substituting these values into equation (6 .2- 1 6) gives us

$$\sin(2\phi_{bo} + 45^{o}) = \frac{2 - .9}{.9} \sin 45^{o} = .866$$

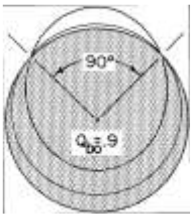
But there are two angles whose sine equals .866, so

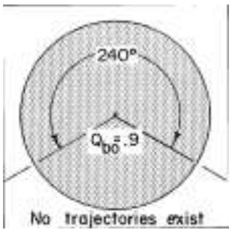
$$2\phi_{bo} + 45^{o} = 60^{o} \text{ and } 120^{o}$$

$$\phi_{bo} = 7.5^{o} \text{ and } 37.5^{o}$$

There are two trajectories to the target which result from the same values of rbo and vbo· The trajectory corresponding to the larger value of flight-path angle is called the high trajectory ; the trajectory associated with the smaller flight-path angle is the low trajectory . The fact that there are two trajectories to the target should not surprise you since even very short-range ballistic trajectories exhibit this property . A familiar illustration of this result is the behavior of water discharged from a garden hose . With constant water pressure and nozzle setting, the speed of the water leaving the nozzle is fixed . If a target well within the maximum range of the hose is selected, the target can be hit by a flat or lofted trajectory. The nature of the high and low trajectory depends primarily on the value of Qbo. If Qbo is less than · 1 there will be a limit to how large 'l' may be in order that the value of the right side of equation (6 . 2- 1 6) does not excee d 1 . This implies that there is a maximum range for a missile with Qbo less than 1 . This maximum range will always be less than 1 800 for Qbo less than 1 . Provide d that 'l' is attainable , there will be both a high and a low trajectory to the target. If Qbo is exactly 1 , one of the trajectorie s to the target will be the circular orbit connecting the burnout and re-entry points . This would not be a very practical missile trajectory, but it does represent the borderline case where ranges of 1 8 00 and more are just attainable . If Qbo is greater than 1 , equation (6 . 2- 1 6) will always yield one positive and one negative value for ¢bo' regardless of range . A negative ¢bo is not practical since the trajectory would penetrate the earth, so only the high trajectory can be realized for Qbo greater than 1 .

The real significance of Qbo greater than 1 is that ranges in excess of 1 800 are possible. An illustration of such a trajectory would be a missile directed at the North American continent from Asia via the south pole . While such a trajectory would avoid detection by our northern radar "fences ," it would be costly in terms of payload delivered and accuracy attainable . Nevertheless, the shock value of such a surprise attack in terms of what it might do towards creating chaos among our defensive forces should not be overlooked by military planners. Since both the high and low traj ectories result from the same rbo and vbo' they both have the same energy. Becau se a = - P/2&, the maj or axis of the high and low traj ectories are the same length. Table 6.2-1 shows which trajectories are possible for various combinations of Qbo and 'l'. Figure 6.2-5 should be helpful in visualizing each case.





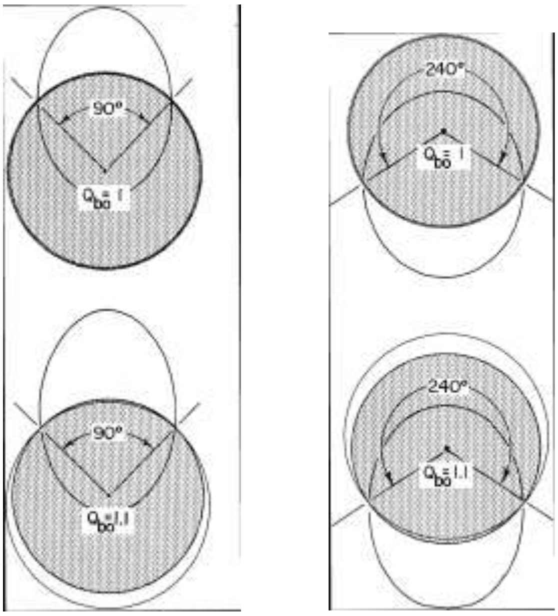


Figure 7Example ballistic missile traj e ctories

Significance of Obo

	$\Psi < 180^0$	$\Psi > 180^0$
$Q_{bo} < 1$	both high and low if $\Psi <$ max. range	impossible
$Q_{bo} = 1$	both high and low ($\phi_{bo} = 0^0$ for low)	high has $\phi_{bo} = 0^0$ one low traj. skims earth
$Q_{bo} > 1$	high only – low traj. hits earth	high only – low traj. hits earth

EXAMPLE PROBLEM. During the test firing of a ballistic missile, the following measurements were made: $h_{bo} = 1/5$ DU, $v_{bo} = 2/3$ DU/TU, $h_{apogee} = 0.5$ DU. Assuming a symmetrical trajectory, what was the free-flight range of the missile during this test in nautical miles?

Before we can use the free-flight range equation to find Ψ, we must find ϕ_{bo} and Q_{bo}

$$\mathcal{E} = \frac{v_{bo}^2}{2} - \frac{\mu}{r_{bo}} = \frac{4}{18} - \frac{1}{1.2} = -\frac{11}{18}\ DU^2/TU^2$$

$$v_a = \sqrt{2\left(\frac{\mu}{r_a} + \mathcal{E}\right)} = \sqrt{2\left(\frac{1}{1.5} - \frac{11}{18}\right)} = \frac{1}{3}\ DU/TU$$

$$h = r_a v_a = 1.5\left(\frac{1}{3}\right) = \frac{1}{2}\ DU^2/TU$$

$$h = r_{bo} v_{bo} \cos\phi_{bo}$$

$$\frac{1}{2} = \frac{6}{5}\left(\frac{2}{3}\right)\cos\phi_{bo}, \quad \therefore \cos\phi_{bo} = 0.625$$

$$Q_{bo} = \frac{v_{bo}^2 r_{bo}}{\mu} = 0.5333$$

Using equation (6.2-12), $\cos\frac{\Psi}{2} = +\frac{19}{20}$.

$$\Psi = 36^{o}24' \text{ or } 36.4^{o}$$

$$R_{ff} = (36.4\,\text{Deg.})\left(60\frac{nmi}{Deg}\right) = \underline{\underline{2{,}184\,nmi}}$$

Using equation (6.2-12), $\cos\frac{\Psi}{2} = +\frac{19}{20}$.

$$\Psi = 36^{o}24' \text{ or } 36.4^{o}$$

$$R_{ff} = (36.4\,\text{Deg.})\left(60\frac{nmi}{Deg}\right) = \underline{\underline{2{,}184\,nmi}}$$

EXAMPLE PROBLEM. A missile's coordinates at burnout are: 30°N, 60°E. Re-entry is planned for 30°S, 60°W. Burnout velocity and altitude are 1.0817 DU/TU and .025 DU respectively. Ψ is less than 180°.

What must the flight-path angle be at burnout?

Before we can use the flight-path angle equation to find ϕ_{bo}, we must find Q_{bo} and Ψ.

$$Q_{bo} = \frac{(1.0817)^2(1.025)}{1} = 1.2$$

From spherical trigonometry,

$$\cos\Psi = \cos 60^\circ \cos 120^\circ + \sin 60^\circ \sin 120^\circ \cos 120^\circ = -.625$$

$$\therefore \Psi = 128^\circ 41'$$

From the flight-path angle equation,

$$\sin\left(2\phi_{bo} + \frac{128^\circ 41'}{2}\right) = \frac{2-1.2}{1.2} \sin\left(\frac{128^\circ 41'}{2}\right) = .6$$

$$\therefore 2\phi_{bo} + 64^\circ 20.5' = 143^\circ 04'$$

or $\phi_{bo} = \underline{\underline{39.36^\circ}}$

4.4 Optimal flights

As will be clear from the foregoing discussion, and as can be seen from Figs. 13.5 and 13.6, there is an optimum injection flight path angle, such that the angular range is maximized for given V_i, r_i and r_e, or, equivalently, the injection velocity is minimized for given Σ, r_i and r_e. We will derive the conditions for these optimal flights using the graphical method discussed in the foregoing section. The geometry of an optimal flight is depicted in Fig. 13.8. We have seen that for a given range the injection velocity is a minimum if the two circles, with P_i and P_e as center and $2a - r_i$ and $2a - r_e$ as radius, touch. The point of contact, F_{opt}, will lie on the chord joining P_i and P_e. So, for optimal flights

$$a_{min} = \tfrac{1}{4}(c + r_i + r_e), \tag{13.2-28}$$

where c is the length of the chord P_iP_e, given by

$$c^2 = r_i^2 + r_e^2 - 2r_ir_e \cos \Sigma. \tag{13.2-29}$$

According to Eq. (13.2-23a), the semi-major axis is a function of k_i and r_i. Substitution of Eq. (13.2-23b) and Eq. (13.2-29) into Eq. (13.2-28) yields an equation for the minimum injection velocity ratio, $k_{i_{min}}$, the solution of which is

$$k_{i_{min}} = \frac{1 - \rho_i - 2\sin^2 \Sigma/2 + \sqrt{(1-\rho_i)^2 + 4\rho_i \sin^2 \Sigma/2}}{\cos^2 \Sigma/2}. \tag{13.2-30}$$

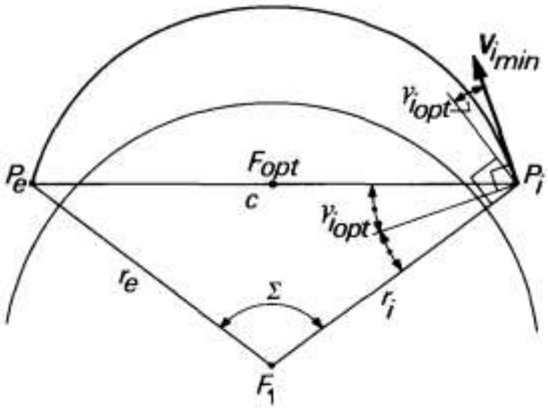


Figure 8Optimal Flight Geometry

This relation is depicted in Fig. 13.9a. The corresponding optimal flight path angle is found by realizing that the angle $F_{opt}P_iF_1$ equals twice the flight path angle. Using a property of plane triangles on the triangle $F_1P_iP_e$ we find

$$\tan^2 \gamma_{i_{opt}} = \frac{(r_i + r_e - c)(-r_i + r_e + c)}{(r_i + r_e + c)(r_i - r_e + c)} = \frac{r_e^2 - (r_i - c)^2}{(r_i + c)^2 - r_e^2}. \tag{13.2-31}$$

Substitution of Eq. (13.2-29) for c into Eq. (13.2-31) gives

$$\tan^2 \gamma_{i_{opt}} = \frac{1 - \rho_i - 2\sin^2 \Sigma/2 + \sqrt{(1-\rho_i)^2 + 4\rho_i \sin^2 \Sigma/2}}{-1 + \rho_i + 2\sin^2 \Sigma/2 + \sqrt{(1-\rho_i)^2 + 4\rho_i \sin^2 \Sigma/2}}. \tag{13.2-32}$$

This relation is depicted in Fig. 13.9b. Therefore, for a given range, Eqs. (13.2-30) and (13.2-32) can be used to determine the optimal injection conditions. Optimal in the sense that for a given value of injection altitude, the injection velocity is minimized. This, of course, also means that the total energy of the trajectory is minimized. Therefore, these trajectories are also called *minimum energy trajectories.* Note, however, that these trajectories are not necessarily optimal in the sense that for a given rocket vehicle the payload is maximized, because we assumed the injection altitude fixed. For the determination of a maximum payload trajectory for a particular rocket, one has to know the characteristics of the powered trajectory, in particular the relations between altitude, velocity and flight path angle. Maximum payload trajectories and minimum injection velocity trajectories will not differ much in general.

For a given injection velocity and radius, the maximum angular range is found by solving Eq. (13.2-30) for Σ. We then find the maximum range that can be reached for a given value of k_i

$$\tan^2 \frac{\Sigma_{max}}{2} = \frac{k_i}{2} \cdot \frac{k_i - 2(1-\rho_i)}{2 - k_i(1+\rho_i)}. \tag{13.2-33}$$

Substitution of this value of Σ into Eq. (13.2-32) yields the optimal injection flight path angle as a function of k_i

$$\tan^2 \gamma_{i_{opt}} = \frac{k_i}{2} \cdot \frac{2 - k_i(1+\rho_i)}{k_i - 2(1-\rho_i)}. \tag{13.2-34}$$

It follows from Eqs. (13.2-33) and (13.2-34) that for a given injection velocity

$$\tan \frac{\Sigma_{max}}{2} \tan \gamma_{i_{opt}} = \frac{k_i}{2}. \tag{13.2-35}$$

In case re-entry and injection altitudes are the same, the equations for the minimum injection velocity at given range, and maximum range for given injection velocity and the optimal flight path angle, simplify to

$$k_{i_{min}} = \frac{2\sin\Sigma/2}{1+\sin\Sigma/2}, \quad (13.2\text{-}36a)$$

$$\tan\gamma_{i_{opt}} = \left[\frac{1-\sin\Sigma/2}{1+\sin\Sigma/2}\right]^{1/2}, \quad (13.2\text{-}36b)$$

$$\tan\frac{\Sigma_{max}}{2} = \frac{k_i}{2\sqrt{1-k_i}}, \quad (13.2\text{-}36c)$$

$$\tan\gamma_{i_{opt}} = \sqrt{1-k_i}. \quad (13.2\text{-}36d)$$

In this case, a simpler expression for the optimum injection flight path angle can be obtained directly. As the sides P_iF_1 and P_eF_1 in the triangle $P_iF_1P_e$ are now equal, and the angle $P_eP_iF_1$ equals $2\gamma_{i_{opt}}$, we find

$$\gamma_{i_{opt}} = \frac{\pi}{4} - \frac{\Sigma}{4}. \quad (13.2\text{-}36e)$$

As the reader may verify himself, this result corresponds with Eq. (13.2-36b). We directly see that for very small ranges, i.e. $\Sigma \rightarrow 0$, the optimum injection flight path angle approaches 45°, the well-known 'flat Earth' result. By solving Eq. (13.2-36d) for k_i and substitution of the result into Eq. (13.2-12b) we find another interesting property of the optimal trajectory with equal injection and re-entry altitudes, viz.

$$e = \tan\gamma_{i_{opt}}. \quad (13.2\text{-}37)$$

The Maximum Range Trajectory . Suppose we plot the free-flight range angle , 'IT, versus the flight-path angle , <Pbo' for a fixed value of Qbo less than 1 . We get a curve like that shown in F igure 6 . 2-6. As the flight-path angle is varied from 0° to 90° the range first increases then reaches a maximum and decrease s to zero again . Notice

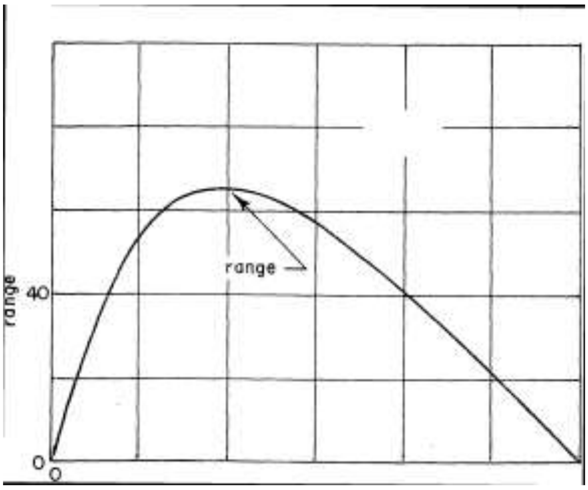


Figure 9Range versus <Pbo

that for every range except the maximum there are two values of <Pbo corresponding to a high and a low trajectory. At rruximum range there is only one path to the targe t.

There are at least two ways that we could derive expressions for the maximllm range condition. One way is to derive an expre ssion for a'l' / a<p and set it equal to zero. A simpler method is to see under what conditions the flight- path angle equation yields a single solution . If the right side of equation (6.2-16) equals exactly 1, we get only a single answer for <Pbo' This must , then, be the maximum range condition.

$$\sin\left(2\phi_{bo} + \frac{\Psi}{2}\right) = \frac{2 - Q_{bo}}{Q_{bo}} \sin\frac{\Psi}{2} = 1$$

from which $2\phi_{bo} + \frac{\Psi}{2} = 90^{o}$

and

$$\boxed{\phi_{bo} = \frac{1}{4}\left(180^{o} - \Psi\right)}$$

for maximum range conditions only. We can easily Cind the maximum range angle attainable with a given Q boo From equation (6.2-1 7),

$$\sin\frac{\Psi}{2} = \frac{Q_{bo}}{2 - Q_{bo}}$$

for maximum range conditions. If we solve this equation for Qbo' we get

$$Q_{bo} = \frac{2\sin(\Psi/2)}{1+\sin(\Psi/2)}$$

for maximum range conditions. This latter form of the equation is useful for determining the lowest value of Qbo that will attain a given range angle

4.5 Time of flight

Time of Free-Flight. The time-of-flight methods developed in Chapter 4 are applicable to the free-flight portion of a ballistic missile traj e ctory , but , due to the symmetry of the case where hre = hbo' the equations are considerably simplified. From the symmetry of Figure 6.2-7 you can see that the time-of-flight from burnout to re-entry is just twice the time-of-flight from burnout (point 1) to apogee (point 2). By inspection, the eccentric anomaly of p oint 2 is 1T radians or 1 800. The value of El can be computed from equation (4.2-8), noting that

VI .1 800 - 'Jl12 .

$$\cos E_1 = \frac{e - \cos\frac{\Psi}{2}}{1 - e\cos\frac{\Psi}{2}}$$

If we now substitute into equation (4.2-9) on page 1 86 we get the time of free-flight

$$t_{ff} = 2\sqrt{\frac{a^3}{\mu}}$$

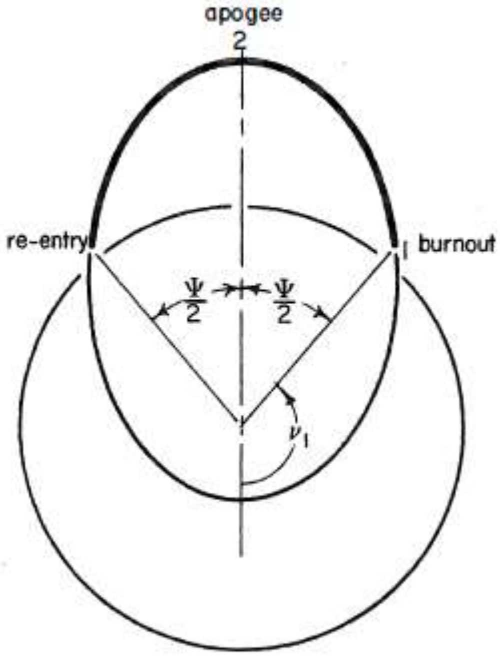


Figure 10Time of free-flight

The semi-maj or axis, a, and the eccentricity, e, can be obtained from equations (6 . 2-3) and (6. 2- 1 1). Figure 6 . 2-9 is an excellent chart for making rapid time-of-flight calculations for the ballistic missile . In fact , since five variables have been plotted on the figure, most ballistic missile problems can be solved completely using just this chart. The free-flight time is read from the chart as the ratio tff!IP cs ' where Wcs is the period of a fictitious circular satellite orbiting at the burnout altitude . Values for TI'cs may be calculated from

$$\mathbb{P}_{cs} = 2\pi\sqrt{\frac{r_{bo}^3}{\mu}}$$

or they may be read directly from Figure 6 . 2-8.

EXAMPLE PROBLEM. A ballistic missile was ob served to have a burnout speed and altitude of 24,300 ft/sec and 2 5 8 nm respectively. What must be the maximum free-flight range capability of this missile?

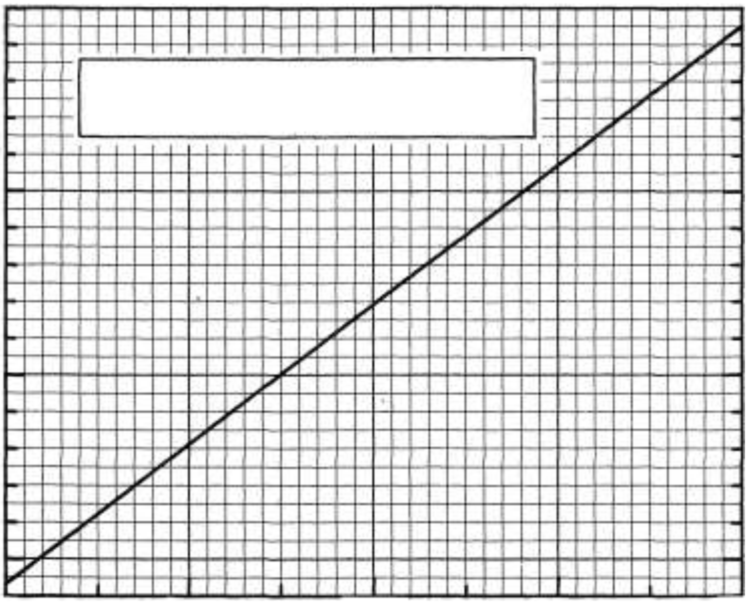


Figure 11Circular satellite period vs altitude

In canonical units

$$Q_{bo} = \left(\frac{2.43\times10^4}{2.59\times10^4}\right)^2\left(1+\frac{258}{3444}\right)$$

$$= (0.884)(1.075) = 0.95$$

$$Q_{bo} = \left(\frac{2.43\times10^4}{2.59\times10^4}\right)^2 \left(1+\frac{258}{3444}\right)$$

$$= (0.884)(1.075) = 0.95$$

From Figure 6.2-9 it is rapidly found that

$$\Psi_{max} = 129^o$$

and $$R_{ff} = (1)\left(\frac{129}{57.3}\right) = \underline{2.25\ DU} = \underline{7{,}750\ nm}$$

EXAMPLE PROBLEM. It is desired to maximize the payload of a new ballistic missile for a free-flight range of 8 ,000 nm. The design burnout altitude has been fixed at 3 44 nm. What should b e the design burnout speed?

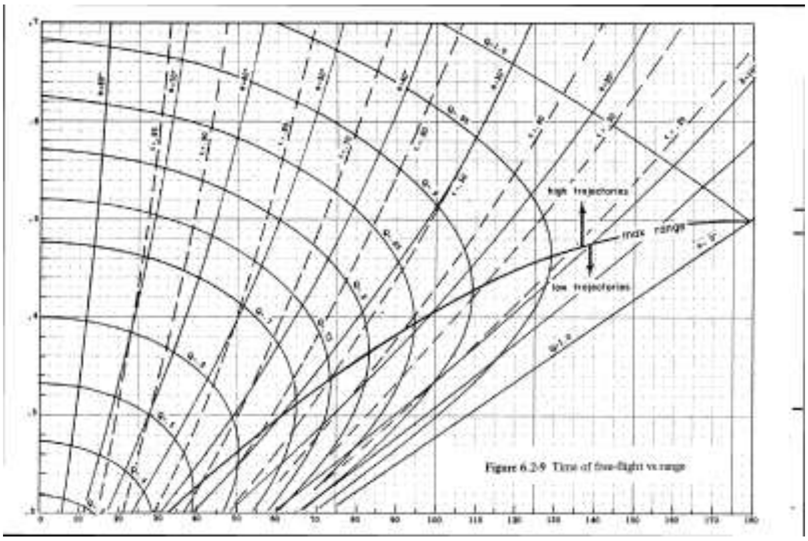


Figure 12Free - flight range 'I' in degrees

For a given amount of propellant, range may b e sacrificed to increase payload and vice-versa. For a fixed burnout altitude , payload may be maximized by minimizing the burnout speed (minimum QbJ.

$$\Psi_{max} = \frac{8{,}000\,nm}{3{,}444\,nm} = 2.32 \text{ rads} = 133.3^{o}$$

From equation (6.2-20)

$$Q_{bo\ min} = \frac{2 \sin 66.7^{o}}{1 + \sin 66.7^{o}} = 0.957$$

From equation (6.2-1)

$$v_{bo} = \sqrt{\frac{.957}{1.1}} = 0.933 \text{ DU/TU} = \underline{24{,}200 \text{ ft/sec}}$$

4.6 Re-entry phase

Re-entry is an extensive subject in itself and falls beyond the scope of this book. We will therefore confine ourselves to some general remarks on the re-entry phase.

The re-entry phase starts at a hypothetical '*re-entry point*'. From this point on downwards, the aerodynamic forces cannot be neglected anymore with respect to the gravitational forces. The re-entry altitude may be determined by first estimating a re-entry altitude and subsequent determination of the

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4.7 The position of the impact point

4.8 Influence coefficients

For classical, non-maneuverable re-entry vehicles the position of the impact point is practically uniquely determined by position and velocity at deployment. Therefore, the guidance system of the rocket vehicle has to generate the commands necessary to ensure correct position and velocity at the time of release of the re-entry vehicle. To this end various guidance schemes are being used. The more advanced guidance laws, e.g. *Hybrid Explicit Guidance*, compute throughout the powered flight, using current position and velocity data, the position of the impact point if the re-entry vehicle were released at that time, as well as the velocity needed to go directly from the current position to the target. When the computed impact point coincides with the target a re-entry vehicle is deployed. Another widely used guidance scheme is *Delta guidance*. In this case a *reference trajectory* is stored in the onboard guidance computer of the rocket vehicle. During powered flight the current state is continuously compared with the reference or nominal state and whenever a deviation from nominal is detected the guidance and control unit directs the missile back to its nominal flight path. To this nominal trajectory correspond nominal shut-down conditions. However, the state at shut-down of an actual missile will never exactly equal the nominal state as a result of guidance system errors. These shut-down errors, expressed in the perturbations Δr_i, $\Delta\alpha_i$, $\Delta\delta_i$, ΔV_i, $\Delta\gamma_i$ and $\Delta\psi_i$, result in impact errors, expressed by a *down-range error*, ΔR_D within the nominal trajectory plane and a *cross-range error*, ΔR_C, normal to the nominal trajectory plane. In this section, we will determine the relations between the impact errors and the shut-down errors. In general, these relations have the following form

$$\Delta R_D = \Delta R_D(\Delta r_i, \Delta\alpha_i, \Delta\delta_i, \Delta V_i, \Delta\gamma_i, \Delta\psi_i), \quad (13.4\text{-}14a)$$

$$\Delta R_C = \Delta R_C(\Delta r_i, \Delta\alpha_i, \Delta\delta_i, \Delta V_i, \Delta\gamma_i, \Delta\psi_i). \quad (13.4\text{-}14b)$$

Assuming the cut-off errors to be small, these equations may be linearized, resulting in

$$\Delta R_D = R_{D_r}\,\Delta r_i + R_{D_\alpha}\,\Delta\alpha_i + \ldots + R_{D_\psi}\,\Delta\psi_i, \quad (13.4\text{-}15a)$$

$$\Delta R_C = R_{C_r}\,\Delta r_i + R_{C_\alpha}\,\Delta\alpha_i + \ldots + R_{C_\psi}\,\Delta\psi_i, \quad (13.4\text{-}15b)$$

where

$$R_{D_r} = \frac{\partial \Delta R_D}{\partial \Delta r_i}, \qquad R_{D_\alpha} = \frac{\partial \Delta R_D}{\partial \Delta \alpha_i}, \quad \text{etc.}$$

The coefficients R_{D_r}, $R_{D_\alpha}, \ldots, R_{D_\psi}$ are the *ballistic down-range influence coefficients*, while R_{C_r}, $R_{C_\alpha}, \ldots, R_{C_\psi}$ are the *ballistic cross-range influence coefficients*. They describe the relation between cut-off errors and impact errors, at least for small cut-off errors. For instance, R_{D_r} is the down-range error as a result of a shut-down radius error of unity.

To simplify our analysis, without affecting the results greatly, we will assume a spherical non-rotating Earth and a ballistic trajectory from launch till impact. So the boost phase is approximated by an impulsive shot, while the re-entry phase is neglected. Firstly, we observe that, with the assumptions made, the plane of the trajectory is uniquely determined by α_i, δ_i and ψ_i. This means that errors in the in-plane injection conditions r_i, γ_i and V_i do not cause any cross-range error. Secondly, a change in injection azimuth causes the trajectory plane to rotate about the injection radius vector and consequently only results in a cross-range error. Thus, we have

$$R_{C_r} = R_{C_\gamma} = R_{C_V} = R_{D_\psi} = 0,$$

and the expressions (13.4-15) simplify to

$$\Delta R_D = R_{D_r}\,\Delta r_i + R_{D_\alpha}\,\Delta\alpha_i + R_{D_\delta}\,\Delta\delta_i + R_{D_V}\,\Delta V_i + R_{D_\gamma}\,\Delta\gamma_i, \qquad (13.4\text{-}16a)$$

$$\Delta R_C = R_{C_\alpha}\,\Delta\alpha_i + R_{C_\delta}\,\Delta\delta_i + R_{C_\psi}\,\Delta\psi_i. \qquad (13.4\text{-}16b)$$

As an error in the in-plane injection state causes a down-range error,

$\Delta R_D = R_0 \Delta\Sigma$, we can write for the in-plane down-range influence coefficients

$$R_{D_r} = R_0 \frac{\partial\Sigma}{\partial r_i}, \qquad R_{D_V} = R_0 \frac{\partial\Sigma}{\partial V_i}, \qquad R_{D_\gamma} = R_0 \frac{\partial\Sigma}{\partial\gamma_i}. \qquad (13.4\text{-}17)$$

Now, $\tan\frac{\Sigma}{2}$ is given in Eq. (13.2-20), and as

$$d\Sigma = \frac{2}{1+\tan^2\Sigma/2}\, d\left(\tan\frac{\Sigma}{2}\right), \qquad (13.4\text{-}18a)$$

we can derive the partial derivatives of range angle with respect to r_i, V_i and γ_i, noting that

$$\frac{\partial}{\partial r_i} = \frac{1}{r_i}\left[\rho_i \frac{\partial}{\partial\rho_i} + k_i \frac{\partial}{\partial k_i}\right], \qquad (13.4\text{-}18b)$$

and

$$\frac{\partial}{\partial V_i} = \frac{2\sqrt{k_i}}{\sqrt{\mu/r_i}} \frac{\partial}{\partial k_i}. \qquad (13.4\text{-}18c)$$

Finally, differentiation of Eq. (13.2-20), using Eqs. (13.4-17) and (13.4-18), and setting $\rho_i = 1$ and $r_i = R_0$ leads to

$$R_{D_r} = \cot\gamma_i + 2k_i \frac{\sin\gamma_i \cos\gamma_i}{\cos^2\gamma_i(1-k_i)^2 + \sin^2\gamma_i}, \qquad (13.4\text{-}19a)$$

$$R_{D_V} = 4\sqrt{\frac{R_0^3}{\mu}}\, k_i \frac{\sin\gamma_i \cos\gamma_i}{\cos^2\gamma_i(1-k_i)^2 + \sin^2\gamma_i}, \qquad (13.4\text{-}19b)$$

$$R_{D_\gamma} = 2R_0 k_i \frac{\cos^2\gamma_i(1-k_i) - \sin^2\gamma_i}{\cos^2\gamma_i(1-k_i)^2 + \sin^2\gamma_i}. \qquad (13.4\text{-}19c)$$

These relations are plotted in Fig. 13.13. The down-range error coefficients along an optimal trajectory are indicated by a dashed line. In Fig. 13.13c this dashed line coincides with the line $R_{D_\gamma} = 0$.

The error coefficients with respect to α_i, δ_i and ψ_i could be obtained by differentiating Eqs. (13.4-4), thus obtaining equations for the partial derivatives of α_f and δ_f with respect to α_i, δ_i and ψ_i. Then, using spherical geometry, the down-range and cross-range errors can be expessed in $\Delta\alpha_f$ and $\Delta\delta_f$, thus leading to the influence coefficients.

Unit-V

Low Thrust Trajectories

5 Unit-V Low Thrust Trajectories

5.1 Equations of Motion

The **Tsiolkovsky rocket equation**, **classical rocket equation**, or **ideal rocket equation** is a mathematical equation that describes the motion of vehicles that follow the basic principle of a rocket: a device that can apply acceleration to itself using thrust by expelling part of its mass with high velocity can thereby move due to the conservation of momentum.

The equation relates the delta-v (the maximum change of velocity of the rocket if no other external forces act) to the effective exhaust velocity and the initial and final mass of a rocket, or other reaction engine.

For any such maneuver (or journey involving a sequence of such maneuvers):

$$\Delta v = v_e \ln \frac{m_0}{m_f} = I_{sp} g_0 \ln \frac{m_0}{m_f}$$

where:

Δv is delta-v – the maximum change of velocity of the vehicle (with no external forces acting).

is the initial total mass, including propellant, also known as wet mass.

is the final total mass without propellant, also known as dry mass.

is the effective exhaust velocity, where:

is the specific impulse in dimension of time.

is standard gravity.

5.2 Constant radial thrust acceleration

5.3 Constant tangential thrust (Characteristics of the motion)

5.4 Linearization of the equations of motion

5.5 Performance analysis

For satellite mission analyses, we have to know the performances of *available* launch vehicles. For preliminary studies, we mostly make us performance diagrams in which the payload capability of a launch vebic plotted versus the so-called *launch vehick* characiertstic velocity, Va.„17.5). To explain how to use these diagrams, we first recall that all exii rocket stages are high-thrust chemical. systems. Once in a parking orbit can assume for orbit transfer maneuvers that the thrust acts for so sbc time that during motor operation only the vehicle's velocity vector is altered, but not its position vector. We thus use the concept of an impuisive *shot* (Section 1123) and consider an orbit transfer maneuver as a sequence of instantaneous velocity changes. The mass of propellant required for each velocity change can be computed from *Tsioikousky's equation,* Eq. (11.2-4). We define a mission *characteristic velocity,* V:1 , as the arithmetic sum of all velocity changes required to perform a specified mission, starting from a 185 km circular parking orbit, plus the velocity in this parking orbit relative to a non-rotating geocentric reference frame (V. = 7.797 km/s). This mission characteristic velocity thus is a measure of the energy required to fly a given mission. With the theory of-orbital maneuvers, we can compute this mission characteristic velocity for each final orbit and each transfer trajectory. As this subject is not treated in this book, we will only discuss some results on the basis of Fig. 17.4. For a full treatment be referred to the standard work of Ehricke [21, and to Gobietz and Doll Pa

The dashed line in Fig. 17.4 shows the initial incremental velocity required at the parking orbit altitude to perform a coplanar *Hohmann transfer* to an apogee altitude corresponding to a desired final orbit altitude. Also indicated is the transfer time to execute this Hohmann transfer. To circularize the orbit at the *pore of the transfer trajectory, an incremental velocity A V2 has to be applied. The sum of lift and A.V2 is also shown in Fig. 17.4. We note that for increasing final orbit altitudes, , first increases, until from an altitude of about 96,000 km upwards it decreases again. One can prove that V:„, for this type of transfer trajectories takes a maximum value at r2/ri im 15.58, where r1 is the radius of the circular parking orbit and r2 is the radius of the final circular orbit. For r2/ri >331, we find that is larger than the escape velocity in the parking orbit.