LECTURE NOTES

ON

THEORY OF PLATES AND SHELLS

(BSTB03)

M. Tech I year I semester

(IARE- R18)

Mr. Gude Ramakrishna
Associate Professor

DEPARTMENT OF CIVIL ENGINEERING

INSTITUTE OF AERONAUTICAL NGINEERING
(Autonomous) DUNDIGAL,
HYDERABAD - 500 043

UNIT-I

INTRODUCTION

Shells in engineering structures

Thin shells as structural elements occupy a leadership position in engineering and, in particular, in civil, mechanical, architectural, aeronautical, and marine engineering.

Examples of shell structures in civil and architectural engineering are large-span roofs, liquid-retaining structures and water tanks, containment shells of nuclear power plants, and concrete arch domes. In mechanical engineering, shell forms are used in piping systems, turbine disks, and pressure vessels technology. Aircrafts, missiles, rockets, ships, and submarines are examples of the use of shellsin aeronautical and marine engineering. Another application of shell engineering is in the field of biomechanics: shells are found in various biological forms, such as the eye and the skull, and plant and animal shapes. This is only a small list of shell forms in engineering and nature.

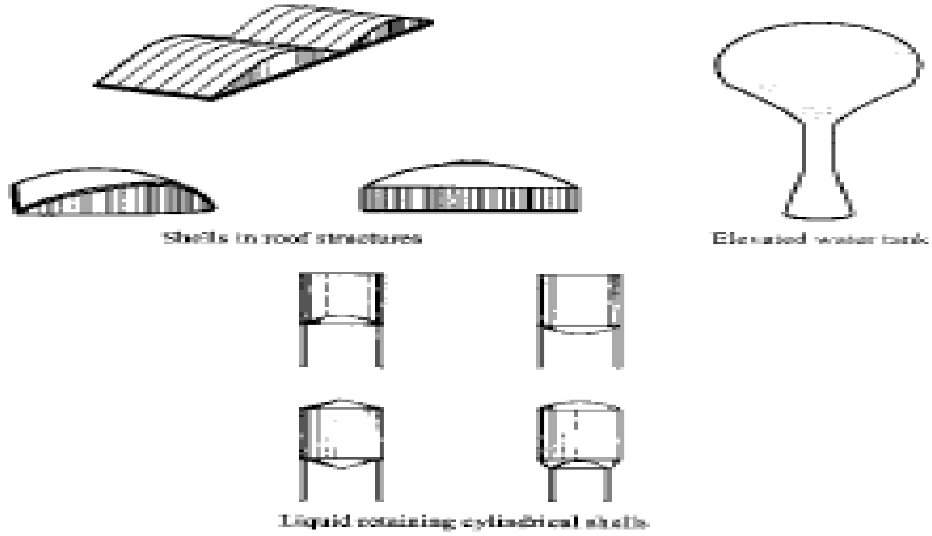
Advantages

1. Efficiency of load-carryingbehavior.
2. High degree of reserved strength and structuralintegrity.
3. High strength : weight ratio. This criterion is commonly used to estimate a structural component efficiency: the larger this ratio, the more optimal is a structure. According to this criterion, shell structures are much superior to other structural systems having the same span and overalldimensions.

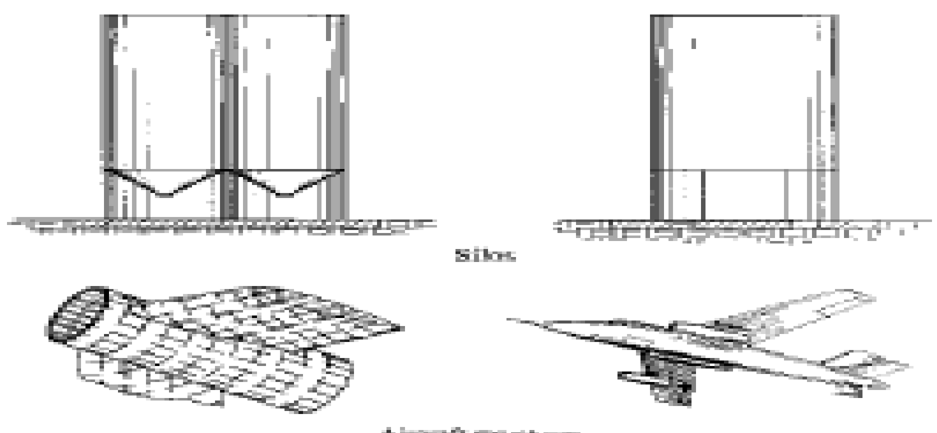
1. Very highstiffness.
2. Containment ofspace.

In addition to these mechanical advantages, shell structures enjoy the unique position of having extremely high aesthetic value in various architectural designs.

Shell structures support applied external forces efficiently by virtue of their geometrical form, i.e., spatial curvatures; as a result, shells are much stronger and stiffer than other structural forms.



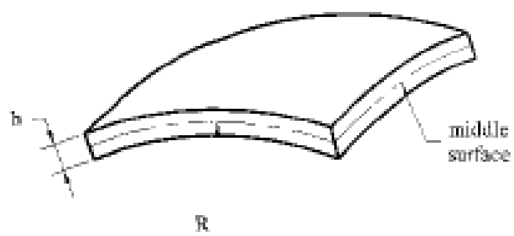
(a)



(b)

GENERAL DEFINITIONS AND FUNDAMENTALS OF SHELLS

We now formulate some definitions and principles in shell theory. The term shell is applied to bodies bounded by two curved surfaces, where the distance between the surfaces is small in comparison with other body dimensions fig shownbelow



The locus of points that lie at equal distances from these two curved surfaces defines the middle surface of the shell. The length of the segment, which is perpendicular to the curved surfaces, is called the thickness of the shell and is denoted by h. The geometry of a shell is entirely defined by specifying the form of the middle surface and thickness of the shell at each point. In this book we consider mainly shells of a constant thickness. Shells have all the characteristics of plates, along with an additional one – curvature. The curvature could be chosen as the primary classifier of a shell because a shell’s behavior under an applied loading is primarily governed by curvature . Depending on the curvature of the surface, shells are divided into cylindrical (noncircular and circular), conical, spherical, ellipsoidal, paraboloidal, toroidal, and hyperbolic paraboloidal shells. Owing to the curvature of the surface, shells are more complicated than flat plates because their bending cannot, in general, be separated from their stretching. On the other hand, a plate may be considered as a special limiting case of a shell that has no curvature; consequently, shells are sometimes referred to as curved plates. This is the basis for the adoption of methods from the theory of plates, discussed in Part I, into the theory of shells. There are two different classes of shells: thick shells and thin shells. A shell is called thin if the maximum value of the ratio h=R (where R is the radius of curvature of the middle surface) can be neglected in comparison with unity. For an engineering accuracy, a shell may be regarded as thin if [1] the following condition issatisfied:

Max (h/ R) <or=1/20

Hence, shells for which this inequality is violated are referred to as thickshells.For a large number of practical applications, the thickness of shells lies in the range

(1/1000) <or=(h/R)<or=1/20

THE LINEAR SHELL THEORIES

The most common shell theories are those based on linear elasticity concepts. Linear shell theories predict adequately stresses and deformations for shells exhibiting small elastic deformations; i.e., deformations for which it is assumed that the equilibrium equation conditions for deformed shell surfaces are the same as if they were not deformed, and Hooke’s lawapplies.

For the purpose of analysis, a shell may be considered as a three-dimensional body, and the methods of the theory of linear elasticity may then be applied. However, a calculation based on these methods will generally be very difficult and complicated. In the theory of shells, an alternative simplified method is therefore employed. According to this method and adapting some hypotheses the 3D problem of shell equilibrium and straining may be reduced to the analysis of its middle surface only, i.e. the given shell, as discussed earlier as a thin plate, may be regarded as some 2D body. In the development of thin shell theories, simplification is accomplished by reducing the shell problems to the study of deformations of the middlesurface.

Shell theories of varying degrees of accuracy were derived, depending on the degree to which the elasticity equations were simplified. The approximations necessary for the development of an adequate theory of shells have been the subject of

considerable discussions among investigators in the field. We present below a brief outline of elastic shell theories in an historical context.

A second class of thin elastic shells, which is commonly referred to as higherorderpproximation, has also been developed. To this grouping it is possible to assign all linear shell theories in which one or another of the Kirchhoff–Love hypotheses are suspended. First, we consider some representative theories in which the thinness assumption is delayed in derivation while the rest of the postulates are retained. In this case, the order of a particular approximate theory will be established by the order of the terms in the thickness coordinate that is retained in the strain and constitutive equations.

The small-defection shell theories discussed above were formulated from the classical linear theory of elasticity. It is known that the equations of these theories, which are based on Hooke's law and the omission of nonlinear terms in both the equations for strain components and equilibrium equations, have a unique solution in every case. In other words, a linear shell theory determines a unique position of equilibrium for every shell with prescribed load and constraints. In reality,however,a solution of physical shell problems is not always unique. A shellunderidentical conditions of loading and constraints may have several possible positions of equilibrium. A theory that takes into account finite or large deformations is referred to asa geometrically nonlinear theory of thin shells. Additionally, a shell may be physically
nonlinear with respect to the stress–strain relations.

The membrane stress condition is an ideal state at which a designer should aim. It should be noted that structural materials are generally far more efficient in an extensional rather in a flexural mode because:

1. Strength properties of all materials can be used completely in tension (or compression), since all fibers over the cross section are equally strained and load-carrying capacity may simultaneously reach the limit for the whole section of thecomponent.
2. The membrane stresses are always less than the correspondingbending stresses for thin shells under the same loadingconditions.

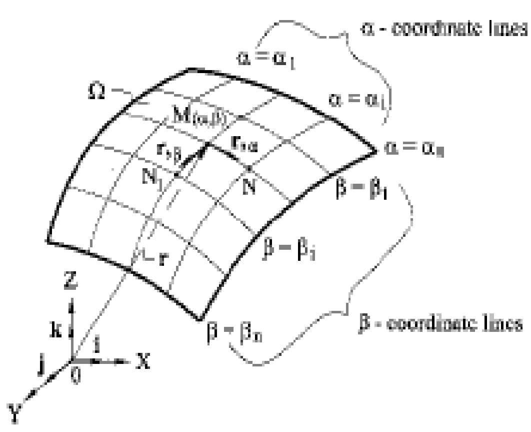
Thus, the momentless or membrane stress conditions determine the basic advantages of shells compared with beams, plates, etc.

The highest efficiency of a shell, as a structural member, is associated with its curvature and thinness. Owing to the shell curvature, the projections of the direct forces on the normal to the middle surface develop an analog of an ‘‘elastic foundation’’ under the shell. So, it can be said that a shell resists an applied transverse loading as a flat plate resting on an elastic foundation. This phenomenon can explain an essential increase in strength and stiffness of a shell compared with a plate. Thus, as a result of the curvature of the surface, a shell acquires a spatial stiffness that gives to ita larger load-carryingcapacity and develops the direct stresses. Owing to its thinness, a shell may balance an applied transverse loading at the expense of the membrane stresses mainly, with bending actions minimized.

As mentioned above, shell thinness demonstrates the high efficiency of shells. It is associated with the shell's low weight and simultaneously its high strength. However, a shell's thinness is, at the same time, a weak point because all the advantages mentioned earlier hold for a tensile state of stress. In this case, a shell material is stretched and its strength properties are used completely. On the other hand, a thinness of shells manifests itself in compression. External forces, as before, are effectively transformed in the constant membrane stresses over the shell thickness. However, the trouble is that the level of the critical stresses at buckling be sufficiently low. This level is just determined by the shell thickness. The thinner the shell, the lower is the level of the critical stresses. The latter can be many factors smaller than the proportional limit of the shell material. In this case, the efficiency of thin shells can be reduced co nsiderably. To avoid the possibility of buckling, a shell structure should be designed in such a way that a dominant part of the structure is intension.

COORDINATE SYSTEM OF THE SURFACE

A surface _ can be defined as a locus of points whose position vector, r, directed from the origin 0 of the global coordinate system, OXYZ, is a function of two independentparameters α and β



If the α- and β -coordinate lines are mutually perpendicular at all points on a surface Ω _ (i.e., the angles between the tangents to these lines are equal to 90), the curvilinear coordinates are said to be orthogonal.

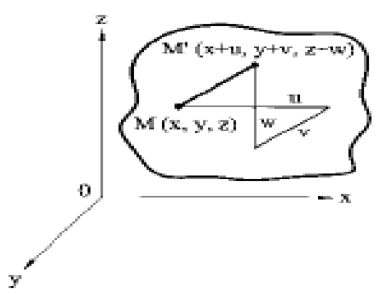
The derivatives of the position vector r with respect to the curvilinear coordinates and are given by the following:

$$\frac{\partial \mathbf{r}}{\partial \alpha} = \mathbf{r}_{,\alpha} \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial \beta} = \mathbf{r}_{,\beta},$$

where we have introduced the comma notation to denote partial derivativeswith respecttoα and β

Strains and displacements

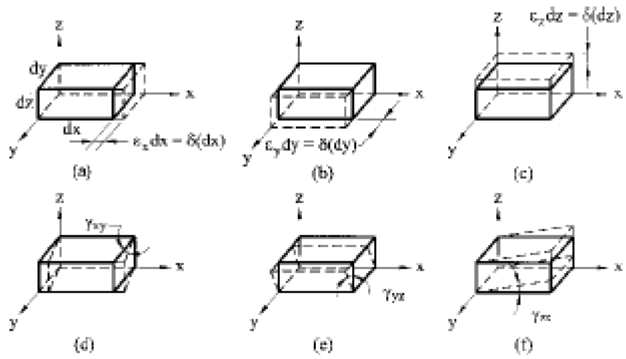
Assume that the elastic body shown in Fig. below is supported in such a way that rigid body displacements (translations and rotations) are prevented. Thus, this body deforms under the action of external forces and each of its points has small elastic displacements. For example, a point M had the coordinates x; y, and z in initial unreformed state. After deformation, this point moved into position M0and its coordinates became the following $x' = x + u,\ y' = y + v,\ z' = z' + w,$where u, v, and w are projections of the displacement vector of point M, vector MM0, on the coordinate axes x, y and z. In the general case, u, v, and w are functions of x, y, andz.



Again, consider an infinitesimal element in the form of parallelepiped enclosing point of interest M. Assuming that a deformation of this parallelepiped is small, we can represent it in the form of the six simplest deformations shown in Fig. a,b,c,d. define the elongation (or contraction) of edges of the parallelepiped in the direction of the coordinate axes and can be definedas

$$\varepsilon_x = \frac{\delta(\mathrm{d}x)}{\mathrm{d}x}, \varepsilon_y = \frac{\delta(\mathrm{d}y)}{\mathrm{d}y}, \varepsilon_z = \frac{\delta(\mathrm{d}z)}{\mathrm{d}z},$$

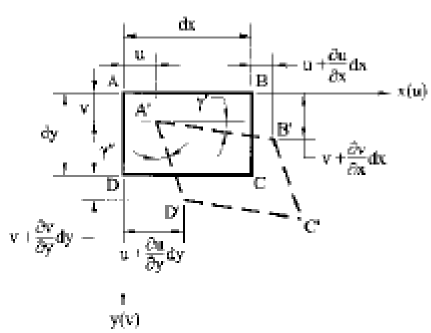
And they are called the normal or linear strains.



the increments delta dx can be expressed by the second term in the Taylor series, i.e.,
i.e., $\delta \mathrm{d}x = (\partial u/\partial x)\mathrm{d}x,$ thus, we can write

$$\varepsilon_x = \frac{\partial u}{\partial x}, \varepsilon_y = \frac{\partial v}{\partial y}, \varepsilon_z = \frac{\partial w}{\partial z}.$$

Since we have confined ourselves to the case of very small deformations, we may omit thequantities $\partial u/\partial x$ and $\partial v/\partial y$ in the denominator of the last expression, as being negligibly small compared with unity. Finally, weobtain



$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}.$$

Similarly, we can obtain _xz and _yz. Thus, the shear strains are given by

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \ \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \ \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}.$$

Similar to the stress tensor (1.1) at a given point, we can define a strain tensor as

$$T_D = \begin{pmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z \end{pmatrix}$$

It is evident that the strain tensor is also symmetric because of

$$\gamma_{xy} = \gamma_{yx}, \ \gamma_{xz} = \gamma_{zx}, \ \gamma_{yz} = \gamma_{zy}$$

Constitutive equations

The constitutive equations relate the stress components to strain components. For the linear elastic range, these equations represent the generalized Hooke's law. In the case of a three-dimensional isotropic body, the constitutive equations are given by

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right], \ \varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right], \ \varepsilon_z = \frac{1}{2}\left[\sigma_z - \nu(\sigma_y + \sigma_x)\right],$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}, \ \gamma_{xz} = \frac{1}{G}\tau_{xz}, \ \gamma_{yz} = \frac{1}{G}\tau_{yz},$$

where E, _, and G are the modulus of elasticity, Poisson's ratio, and the shear modulus, respectively. The following relationship exists between E and G:

$$G = \frac{E}{2(1+\nu)}$$

Equilibrium equations

The stress components introduced previously must satisfy the following differential equations of equilibrium:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0,$$
$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + F_y = 0,$$
$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + F_z = 0,$$

$$\gamma_{xy} = \gamma_{yx}, \ \gamma_{xz} = \gamma_{zx}, \ \gamma_{yz} = \gamma_{zy}$$

whereFx; Fy; and Fz are the body forces (e.g., gravitational, magnetic forces). In deriving these equations, the reciprocity of the shear stresses, Eqs $\gamma_{xy} = \gamma_{yx}, \ \gamma_{xz} = \gamma_{zx}, \ \gamma_{yz} = \gamma_{zy}$, has been used.

Numerical Problem

Design a (‗waist slab‘ type) dog-legged staircase for an office building, given the following data:

• Height between floor = 3.2m;

• Riser = 160 mm, tread = 270mm;

• Width of flight = landing width = 1.25m

• Live load = 5.0 kN/m2

• Finishes load = 0.6kN/m2

Assume the stairs to be supported on 230 mm thick masonry walls at the outer edges of the landing, parallel to the risers [Fig. 12.13(a)]. Use M 20 concrete and Fe 415 steel. Assume *mild* exposure conditions.

Solution

Given: *R* = 160 mm, *T* = 270 mm ⇒+*RT*22

= 314 mm Effective span = c/c distance between supports = 5.16 m [Fig below].

• Assume a waist slab thickness ≈*l*20 = 5160/20 = 258 →260 mm.

Assuming 20 mm clear cover (*mild* exposure) and 12 θ main bars,

effective depth *d* = 260 – 20 – 12/2 = 234mm.

The slab thickness in the landing regions may be taken as 200 mm, as the bending moments are relatively low here.

Loads on going [fig. below] on projected plan area:

(1) self-weight of waist slab @ 25 × 0.26 × 314/270 = 7.56kN/m2

(2) self-weight of steps @ 25 × (0.5x0.16) = 2.00kN/m2

(3) finishes (given) = 0.60kN/m2

(4) live load (given) = 5.00 kN/m2

Total =15.16kN/m2

⇒Factored load = 15.16 × 1.5 = 22.74 kN/m2

• *Loads onlanding*

(1) self-weight of slab @ 25 × 0.20 = 5.00kN/m2

(2) finishes @ 0.6kN/m2

(3) live loads @ 5.0 kN/m2

Total =10.60kN/*m*2

⇒Factored load = 10.60 × 1.5 = 15.90 kN/m2

• *Design Moment* [Fig. below]

Reaction *R*= (15.90x1.365)+(22.74x2.43)/2 = 49.33 kN/m

Maximum moment at midspan:

Mu = (49.33 × 2.58) – (15.90 × 1.365) × (2.58 – 1.365/2)

– (22.74) × (2.58 – 1.365)2/2

= 69.30 kNm/m

• *Maiinforcement*

= 1.265 MPa R bd □

Assuming *fck*= 20 MPa, *fy*= 415 MPa,

2 0.381 10 100 100 t st pAx □ □ □

⇒ 2 3 2 ()(0.38110)10 234 892/ st req Axxxmm m □ □ □

Required spacing of 12 θ bars = 127 mm

Required spacing of 16 θ bars = 225 mm

Provide 16 θ @ 220c/c

• *Distributors*

2 ()0.0012312/ st req Abt mm m

spacing 10 θ bars = 251 mm

Provide 10 θ @ 250c/c as distributors.

The superstructure is placed on the top of the foundation structure, designated as substructure as they are placed below the ground level. The elements of the superstructure transfer the loads and moments to its adjacent element below it and finally all loads and moments come to the foundation structure, which in turn, transfers them to the underlying soil or rock. Thus, the foundation structure effectively supports the superstructure. However, all types of soil get compressed significantly and cause the structure to settle. Accordingly, the major requirements of the design of foundation structures are the two as given below (see cl.34.1 of IS456)

:

1. Foundation structures should be able to sustain the applied loads, moments, forces and induced reactions without exceeding the safe bearing capacity of thesoil.

2. The settlement of the structure should be as uniform as possible and it should be within the tolerable limits. It is well known from the structural analysis that differential settlement of supports causes additional moments in statically indeterminate structures. Therefore, avoiding the differential settlement is considered as more important than maintaining uniform overall settlement of thestructure.

Figure 3.3 presents the three modes of failure of columns with different slenderness ratios when loaded axially. In the mode 1, column does not undergo any lateral deformation and collapses due to material failure. This is known as compression failure. Due to the combined effects of axial load and moment a short column may have material failure of mode 2. On the other hand, a slender column subjected to axial load only undergoes deflection due to beam-column effect and may have material failure under the combined action of direct load and bending moment. Such failure is called combined compression and bending failure of mode 2. Mode 3 failure is by elastic instability of very long column even under small load much before the material reaches the yield stresses. This type of failure is known as elasticbuckling.

The slenderness ratio of steel column is the ratio of its effective length *le* to its least radius of gyration *r*. In case of reinforced concrete column, however, IS 456 stipulates the slenderness ratio as the ratio of its effective length *le* to its least lateral dimension. As mentioned earlier in sec. 3.1(a), the effective length *le* is different from the unsupported length, the rectangular reinforced concrete column of cross-sectional dimensions *b* and *D* shall have two effective lengths in the two directions of *b* and *D*. Accordingly, the column may have the possibility of buckling depending on the two values of slenderness ratios as given below:

Slenderness ratio about the major axis $= lex/D$

Slenderness ratio about the minor axis $= ley/b$

Factored concentric load applied on short tied columns is resisted by concrete of area *Ac* and longitudinal steel of areas *Asc*effectively held by lateral ties at intervals. Assuming the design strengths of concrete and steel are 0.4*fck* and 0.67*fy*, respectively, we can write

$Pu = 0.4fck\ Ac + 0.67fy\ Asc$ (1)

Where *Pu*= factored axial load on the member,

fck= characteristic compressive strength of the concrete,

Ac = area of concrete,

fy= characteristic strength of the compression reinforcement, and

Asc= area of longitudinal reinforcement for columns.

The above equation, given in cl. 39.3 of IS 456, has two unknowns *Ac* and *Asc*to be determined from one equation. The equation is recast in terms of *Ag*, the gross area of concrete and *p*, the percentage of compression reinforcement employing

$Asc = pAg/100$ (2)

$Ac = Ag(1 – p/100)$ (3)

Accordingly, we can write

$Pu/Ag = 0.4fck + (p/100)\ (0.67fy – 0.4fck)$ (4)

Equation 4 can be used for direct computation of *Ag* when *Pu*, *fck*and *fy*are known by assuming *p* ranging from 0.8 to 4 as the minimum and maximum percentages of longitudinal reinforcement. Equation 10.4 also can be employed to determine *Ag* and *p* in a similar manner by assuming *p*.

The superstructure is placed on the top of the foundation structure, designated as substructure as they are placed below the ground level. The elements of the superstructure transfer the loads and moments to its adjacent element below it and finally all loads and moments come to the foundation structure, which in turn, transfers them to the underlying soil or rock. Thus, the foundation structure effectively supports the superstructure. However, all types of soil get compressed significantly and cause the structure to settle. Accordingly, the major requirements of the design of foundation structures are the two as given below (see cl.34.1 of IS456)

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UNIT – II

STATIC ANALYSIS OF PLATES

INTRODUCTION

We begin the application of the developed plate bending theory with thin rectangular plates. These plates represent an excellent model for development and as a check of various methods for solving the governing differential equation

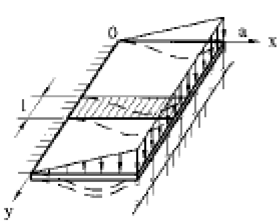
$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D}.$ In this chapter we consider some mathematically ‘‘exact’’ solutions in the form of double and single trigonometric series applied to rectangular plates with various types of supports and transverse loads, plates on an elastic foundation, continuous plates, etc.

THE ELEMENTARY CASES OF PLATE BENDING

Let us consider some elementary examples of plate bending of great importance for understanding how a plate resists the applied loads in bending. In addition, these elementary examples enable one to obtain closed-form solutions of the governing differential equation $\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D}.$

Cylindrical bending of a plate

Consider an infinitely long plate in the y axis direction. Assume that the plate is subjected to a transverse load which is a function of the variable x only, i.e., p ¼ pðxÞ



In this case all the strips of a unit width parallel to the x axis and isolated from the plate will bend identically. The plate as a whole is found to be bent over the cylindrical surface w ¼ wðxÞ. Setting all the derivatives with respect to y equal zeroin Eq $\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{p}{D}.$ we obtain the following equation for thedeflection:

An integration of Eq. (3.1) should present no problems. Let, for example, p ¼ p0xa, then the general solution of Eq. (3.1) is of the following form:

$$w = w_h + w_p = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + \frac{p_0 x^5}{120aD},$$

Pure bending of plates

Consider a rectangular plate with a free boundary and assume that this plate is subjected to istributed bending moments over its edgesMx ¼ m1 ¼ constandMy ¼ m2 ¼ const(Fig. 3.2). In this particular case, the governing differential equation

$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{p}{D}.$ becomes

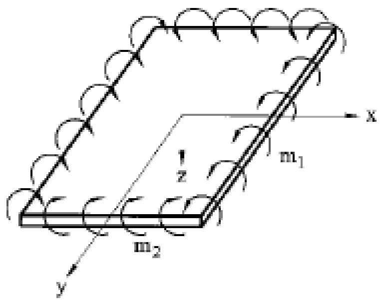
$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}=0.$$

This equation will be satisfied if we make

$$w = 0.5\left(C_1 x^2 + C_2 y^2\right).$$

The constants of integration C1 and C2 may be evaluated from the following boundary conditions:

$$M_x = m_1 \quad \text{and} \quad M_y = m_2.$$



Using Eqs (2.13), (3.8), and (3.9), we obtain

$$C_1 = \frac{\nu m_2 - m_1}{D(1-\nu^2)}, \quad C_2 = \frac{\nu m_1 - m_2}{D(1-\nu^2)}.$$

Substituting the above into Eq. (3.8) yields the deflection surface, as shown below:

$$w = \frac{1}{2D(1-\nu^2)}\left[(\nu m_2 - m_1)x^2 + (\nu m_1 - m_2)y^2\right]$$

Hence, in all sections of the plate parallel to the x and y axes, only the constant bending moments Mx ¼ m1 and My ¼ m2 will act. Other stress resultants and stress couplesare zero,i.e.,

$$M_{xy} = Q_x = Q_y = 0.$$

This case of bending of plates may be referred to as a pure bending. Let

us consider some particular cases of pure bending of plates

(a) Let $m_1 = m_2 = m$.

Then,

$$w = -\frac{m}{2D(1+\nu)}(x^2 + y^2).$$

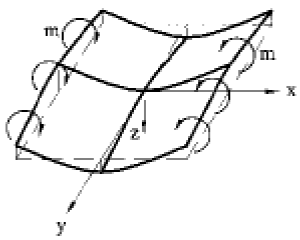
This is an equation of the elliptic paraboloid of revolution. The curved plate in this case represents a part of a sphere because the radii of curvature are the same at all the planes and all the points of the plate.

(b) Let $m_1 = m, m_2 = 0$ (Fig. 3.3).

Then,

$$w = \frac{m}{2D(1-\nu^2)}(-x^2 + \nu y^2).$$

A surface described by this equation has a saddle shape and is called the hyperbolic paraboloid of revolution (Fig. 3.3). Horizontals of this surface are hyperbolas, asymptotes of which are given by the straight linesx $\frac{x}{y} = \pm\sqrt{\nu}$.As is seen, due to the Poisson effect the plate bends not only in the plane of the applied bending moment Mx =m1 =m but it also has an opposite bending in the perpendicularplane



(c) Let $m_1 = m, m_2 = -m$ (Fig. 3.4a).

Then

$$w = \frac{m}{2D(1-\nu)}(-x^2 + y^2).$$

Thus, a part of the plate isolated from the whole plate and equally inclined to the x and y axes will be loaded along its boundary by uniform twisting moments of intensity m. Hence, this part of the plate is subjected to pure twisting (Fig. 3.4b). Let us replace the twisting moments by the effective shear forces V_α rotating these moments through 90 (see Sec. 2.4). Along the whole sides of the isolated part we obtain V_α=0, but at the corner points the concentrated forces S ¼ 2m are applied. Thus, for the model of Kirchhoff's plate, an application of self-balanced concentrated forces at corners of a rectangularplateproducesadeformationofpuretorsionbecauseoverthewholesurface of the plate $\dot{M}_{nt} = m = \text{const.}$

NAVIER'S METHOD (DOUBLE SERIES SOLUTION)

In 1820, Navier presented a paper to the French Academy of Sciences on the solution of bending of simply supported plates by double trigonometric series [1].

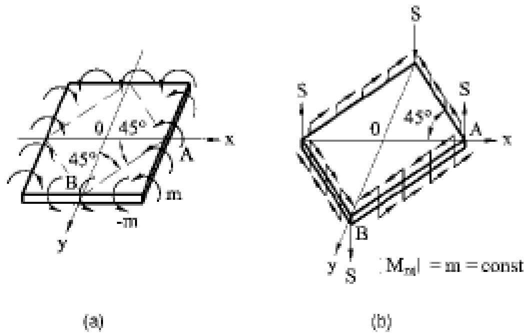


Fig. 3.4

Consider a rectangular plate of sides a and b, simply supported on all edges and subjected to a uniform load pðx; yÞ: The origin of the coordinates is placed at the upper left corner as shown in Fig. 3.5. The boundary conditions for a simply supported plate are the following (see Eqs (2.35):

$$w = 0|_{x=0,a};\ \frac{\partial^2 w}{\partial x^2} = 0\Big|_{x=0,a} \quad \text{and} \quad w = 0|_{y=0,b};\ \frac{\partial^2 w}{\partial y^2} = 0\Big|_{y=0,b}. \tag{3.14}$$

In this case, the solution of the governing differential equation (2.24), i.e., the expressions of the deflection surface, wðx; yÞ, and the distributed surface load, pðx; yÞ, have to be sought in the form of an infinite Fourier series (see Appendix B), as follows:

$$w(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}, \tag{3.15a}$$

$$p(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} p_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}, \tag{3.15b}$$

wherewmn and pmn represent coefficients to be determined. It can be easily verified that the expression for deflections (3.15a) automatically satisfies the prescribed boundary conditions (3.14).

Let us consider a general load configuration. To determine the Fourier coefficients pmn, each side of Eq. (3.15b) is multiplied by sin l_x a sin k_y b and integrated twice between the limits 0;a and 0;b, as follows (see Appendix B):

$$\int_0^a\int_0^b p(x, y) \sin\frac{l\pi x}{a} \sin\frac{k\pi y}{b}\, dx dy = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} p_{mn} \int_0^a\int_0^b \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\frac{l\pi x}{a} \sin\frac{k\pi y}{b}\, dx dy. \tag{a}$$

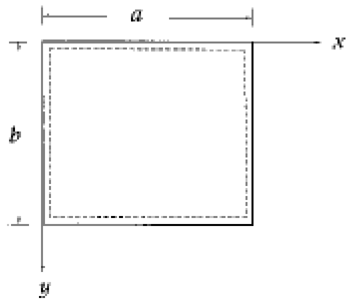


Fig. 3.5

It can be shown by direct integration that

$$\int_0^a \sin\frac{m\pi x}{a}\sin\frac{l\pi x}{a}\mathrm{d}x = \begin{cases} 0 \text{ if } m \neq l \\ a/2 \text{ if } m = l \end{cases}$$

$$\text{and } \int_0^b \sin\frac{n\pi y}{b}\sin\frac{k\pi y}{b}\mathrm{d}y = \begin{cases} 0 \text{ if } n \neq k \\ b/2 \text{ if } n = k. \end{cases} \quad (3.16)$$

The coefficients of the double Fourier expansion are therefore the following

$$p_{mn} = \frac{4}{ab}\int_0^a\int_0^b p(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\mathrm{d}x\mathrm{d}y. \quad (3.17)$$

Since the representation of the deflection (3.15a) satisfies the boundary conditions (3.14), then the coefficients wmn must satisfy Eq. (2.24). Substitution of Eqs (3.15) into Eq. (2.24) results in the following equation

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{w_{mn}\left[\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right] - \frac{p_{mn}}{D}\right\}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = 0.$$

This equation must apply for all values of x and y. We conclude therefore that

$$w_{mn}\pi^4\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 - \frac{p_{mn}}{D} = 0.$$

from which

$$w_{mn} = \frac{1}{\pi^4 D}\frac{p_{mn}}{\left[(m/a)^2 + (n/b)^2\right]^2}.$$

Substituting the above into Eq. (3.15a), one obtains the equation of the deflected surface, as follows:

$$w(x, y) = \frac{1}{\pi^4 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_{mn}}{\left[(m/a)^2 + (n/b)^2\right]^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}, \tag{3.19}$$

wherepmn is given by Eq. (3.17). It can be shown, bynotingthat $|\sin m\pi x/a| \leq 1$ and $|\sin n\pi y/b| \leq 1$ for every x and y and for every m and n, that the series (3.19) isconvergent.

Substituting wðx; yÞ into the Eqs (2.13) and (2.27), we can find the bending moments and the shear forces in the plate, and then using the expressions (2.15), determine the stress components. For the moments in the plate, for instance, we obtain the following

$$\begin{aligned} M_x &= \frac{1}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{mn} \frac{[(m/a)^2 + \nu(n/b)^2]}{[(m/a)^2 + (n/b)^2]^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}, \\ M_y &= \frac{1}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{mn} \frac{[(n/b)^2 + \nu(m/a)^2]}{[(m/a)^2 + (n/b)^2]^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}, \\ M_{xy} &= -\frac{1_{\nu}}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{mn} \frac{mn}{ab[(m/a)^2 + (n/b)^2]^2} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b}. \end{aligned} \tag{3.20}$$

The infinite series solution for the deflection (3.19) generally converges quickly; thus, satisfactory accuracy can be obtained by considering only a few terms. Since the stress resultants and couples are obtained from the second and third derivatives of the deflection wðx; yÞ, the convergence of the infinite series expressions of the internal forces and moments is less rapid, especially in the vicinity of the plate edges. This slow convergence is also accompanied by some loss of accuracy in the process of calculation. The accuracy of solutions and the convergence of series expressions of stress resultants and couples can be improved by considering more terms in the expansions and by using a special technique for an improvement of the convergence of Fourier's series (see Appendix B and Ref. [2]).

RECTANGULAR PLATES SUBJECTED TO A CONCENTRATED LATERAL FORCE P

Let us consider a rectangular plate simply supported on all edges of sides a and b and subjected to concentrated lateral force Pappliedat $x = \xi$ and $y = \eta$, as shown in Fig. 3.7.

Assume first that this force is uniformly distributed over the contact area of sidesu and v (Fig. 3.6) i.e., its load intensity is definedas

$$p_0 = \frac{P}{uv}.$$

Substituting the above into Eq. (3.22), oneobtains

$$p_{mn} = \frac{16P}{\pi^2 mnuv} \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b} \sin\frac{m\pi u}{2a} \sin\frac{n\pi v}{2b}. \qquad (3.24)$$

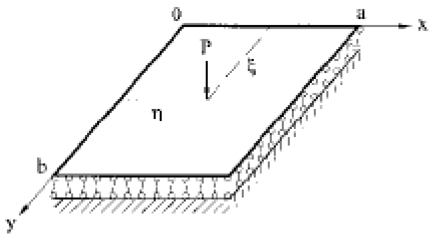


Fig. 3.7

Now we must let the contact area approach zero by permitting u ! 0 and v ! 0. In order to be able to use the limit approach first, Eq. (3.24) must be put in a more suitable form. For this purpose, the right-hand side is multiplied and divided by ab, giving the following:

$$p_{mn} = \lim_{u\to 0, v\to 0} \left[\frac{4P}{ab} \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b} \frac{\sin\frac{m\pi u}{2a} \sin\frac{n\pi v}{2b}}{(m\pi u/2a)(n\pi v/2b)} \right]. \qquad (3.25)$$

Knowing that $\lim_{\alpha\to 0} \frac{\sin\alpha}{\alpha} = 1$, Eq. (3.25) becomes

$$p_{mn} = \frac{4P}{ab} \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b}, \qquad (3.26)$$

The deflected middle surface equation (3.27) in this case becomes

$$w(x,y) = \frac{4P}{\pi^4 Dab} \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{\left[(m/a)^2 + (n/b)^2\right]} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}. \qquad (3.28)$$

Furthermore, if the plate is square (a ¼ b), the maximum deflection, which occurs at the center, is obtained from Eq. (3.28), as follows

$$w_{max} = \frac{4Pa^2}{\pi^4 D} \sum_{m=1,3,...}^{\infty} \sum_{n=1,3,...} \frac{1}{(m^2+n^2)^2}.$$

Retaining the first nine terms of this series $(m = 1,\ n = 1, 3, 5;\ m = 3,\ n = 1, 3, 5;$ $m = 5,\ n = 1, 3, 5)$ we obtain

$$w_{max} = \frac{4Pa^2}{\pi^4 D}\left[\frac{1}{4} + \frac{2}{100} + \frac{1}{324} + \frac{2}{625} + \frac{2}{1156} + \frac{1}{2500}\right] = 0.01142\frac{Pa^2}{D}.$$

The ‘‘exact’’valueis $w_{max} = 0.01159\frac{Pa^2}{D}$ and the error is thus 1.5%[3].

This very simple Navier’s solution, Eq. (3.28), converges sufficiently rapidly for calculating the deflections. However, it is unsuitable for calculating the bending moments and stresses because the series for the secondderivatives $\partial^2 w/\partial x^2$ and $\partial^2 w/\partial y^2$ obtained by differentiating the series (3.28) converge extremely slowly. These series for bending moment, and consequently for stresses as well as for the shear forces, diverge directly at a point of application of a concentrated force, called a singular point. Thus, to calculate stress components in the vicinity of a concentrated force, it is necessary to use a more efficient technique, especially since the maximum stresses occur in the immediate vicinity of the singular point. Therefore, the problem of determining a correctstress
distribution near such types of singular points is of practical interest

We now consider one approach for determining the bending moment distribution near the above-mentioned singular point. Let us write the solution (3.27) in the form

$$w = \frac{4Pb^3}{\pi^4 a} \sum_{m=1}^{\infty} S_m \sin\frac{m\pi\xi}{a} \sin\frac{m\pi x}{a}, \qquad (3.29a)$$

where

$$\beta_m = \frac{1}{2}\sum_{n=1}^{\infty} \frac{\cos\frac{n\pi}{b}(y-\eta) - \cos\frac{n\pi}{b}(y+\eta)}{\left(m^2b^2/a^2+n^2\right)^2}. \tag{3.29b}$$

The series (3.29b) can be summed using the formula [4]:

$$\sum_{n=1}^{\infty} \frac{\cos nz}{(\alpha^2+n^2)^2} = -\frac{1}{2\alpha^4} + \frac{\pi}{4\alpha^3}\frac{\cosh\alpha(\pi-z)}{\sin\pi\alpha} - \frac{\pi(\pi-z)}{4\alpha^2} - \frac{\sinh\alpha(\pi-z)}{\sinh\pi\alpha} + \frac{\pi^2}{4\alpha^2}\frac{\cosh\alpha(\pi-z)\cosh\pi\alpha}{\sin h^2\pi\alpha}. \tag{a}$$

Using the above formula, we can represent the deflection surface (3.28) as

$$w(x,y) = \frac{Pa^2}{\pi^3 D}\sum_{m=1}^{\infty}\left(1+\beta_m\coth\beta_m - \frac{\beta_m y_1}{b}\coth\frac{\beta_m y_1}{b} - \frac{\beta_m\eta}{b}\coth\frac{\beta_m\eta}{b}\right) \times \frac{\sinh\frac{\beta_m\eta}{b}\sin h\frac{\beta_m y_1}{b}\sin\frac{m\pi\xi}{a}\sin\frac{m\pi x}{a}}{m^3\sinh\beta_m}, \tag{3.30}$$

where $\beta_m = \frac{m\pi b}{a,}$ $y_1 = b - y$, and $y \geq \eta$. If $y < \eta$, then the values y_1 and η must be replaced by y and $\eta_1 = b - \eta$, respectively.

Let us use an infinitely long (in the y-direction) plate loaded by a concentrated force P applied at $x = \xi$ and $y = 0$, as shown in Fig. 3.8, to illustrate the above-

LEVY'S SOLUTION (SINGLE SERIES SOLUTION)

In the preceding sections it was shown that the calculation of bending moments and shear forces using Navier's solution is not very satisfactory because of slow convergence of the series.

In 1900 Levy developed a method for solving rectangular plate bending problems with simply supported two opposite edges and with arbitrary conditions of supports on the two remaining opposite edges using single Fourier series [8]. This method is more practical because it is easier to perform numerical calculations for single series that for double series and it is also applicable to plates with various
boundary conditions.

Levy suggested the solution of Eq. (2.24) be expressed in terms of complementary,wh; and particular, wp, parts, each of which consists of a single Fourier series where unknown functions are determined from the prescribed boundary conditions. Thus, the solution is expressed as follows:

$$w = w_h + w_p. \tag{3.40}$$

Consider a plate with opposite edges, x ¼ 0 and x ¼ a, simply supported, and two remaining opposite edges, y ¼ 0 and y ¼ b, which may have arbitrary supports.

The boundary conditions on the simply supported edges are

$$w = 0|_{x=0,x=a} \text{ and } M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) = 0\Big|_{x=0,x=a}. \tag{3.41a}$$

As mentioned earlier, the second boundary condition can be reduced to the following form:

$$\frac{\partial^2 w}{\partial x^2} = 0\Big|_{x=0,x=a} \tag{3.41b}$$

The complementary solution is taken to be

$$w_h = \sum_{m=1}^{\infty} f_m(y) \sin \frac{m\pi x}{a}, \tag{3.42}$$

wherefmðyÞ is a function of y only; wh also satisfies the simply supported boundary conditions (3.41). Substituting (3.42) into the following homogeneous differential equation

$$\nabla^2 \nabla^2 w = 0$$

Gives

$$\left[\left(\frac{m\pi}{a}\right)^4 f_m(y) - 2\left(\frac{m\pi}{a}\right)^2 \frac{d^2 f_m(y)}{dy^2} + \frac{d^4 f_m(y)}{dy^4}\right] \sin \frac{m\pi x}{a} = 0,$$

which is satisfied when the bracketed term is equal to zero. Thus,

$$\frac{d^4 f_m(y)}{dy^4} - 2\left(\frac{m\pi}{a}\right)^2 \frac{d^2 f_m(y)}{dy^2} + \left(\frac{m\pi}{a}\right)^4 f_m(y) = 0 \tag{3.44}$$

The solution of this ordinary differential equation can be expressed as

$$f_m(y) = e^{\lambda y}.$$

Substituting the above into Eq. (3.44), gives the following characteristic equation

$$\lambda^4 - 2\frac{m^2\pi^2}{a^2}\lambda^2 + \frac{m^4\pi^4}{a^4} = 0, \tag{3.46}$$

According to the obtained values of the characteristic exponents, the solution of the homogeneous equation can be expressed in terms of either exponential functions

$$f_m(y) = A'_m e^{m\pi y/a} + B'_m e^{-m\pi y/a} + \frac{m\pi y}{a}\left(C'_m e^{m\pi y/a} + D'_m e^{-m\pi y/a}\right) \tag{3.48}$$

or hyperbolic functions

$$f_m(y) = A_m \sinh\frac{m\pi y}{a} + B_m \cosh\frac{m\pi y}{a} + \frac{m\pi y}{a}\left(C_m \sinh\frac{m\pi y}{a} + D_m \cosh\frac{m\pi y}{a}\right). \tag{3.49}$$

The second form, Eq. (3.49), is more convenient for calculations.

The complementary solution given by Eq. (3.42) becomes

$$w_h = \sum_{m=1}^{\infty}\left[A_m \sinh\frac{m\pi y}{a} + B_m \cosh\frac{m\pi y}{a} + \frac{m\pi y}{a}\left(C_m \sinh\frac{m\pi y}{a} + D_m \cosh\frac{m\pi y}{a}\right)\right]\sin\frac{m\pi x}{a}, \tag{3.50}$$

where the constants Am; Bm;Cm; and Dm are obtained from the boundary conditions on the edges y =0 and y =b:

The particular solution, wp, in Eq. (3.40), can also be expressed in a single Fourier series as

$$w_p(x, y) = \sum_{m=1}^{\infty} g_m(y)\sin\frac{m\pi x}{a}. \tag{3.51}$$

The lateral distributed load pðx; yÞ is taken to be the following (see Appendix B):

$$p(x, y) = \sum_{m=1}^{\infty} p_m(y)\sin\frac{m\pi x}{a}, \tag{3.52}$$

where

$$p_m(y) = \frac{2}{a}\int_0^a p(x, y)\sin\frac{m\pi x}{a}\mathrm{d}x. \quad (3.53)$$

Substituting Eqs (3.51) and (3.52) into Eq. (3.44), gives

$$\frac{\mathrm{d}^4 g_m(y)}{\mathrm{d}y^4} - 2\left(\frac{m\pi}{a}\right)^2\frac{\mathrm{d}^2 g_m(y)}{\mathrm{d}y^2} + \left(\frac{m\pi}{a}\right)^4 g_m(y) = \frac{p_m(y)}{D}. \quad (3.54)$$

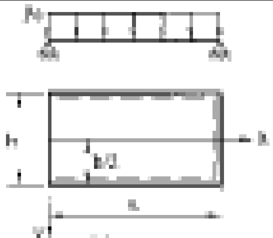
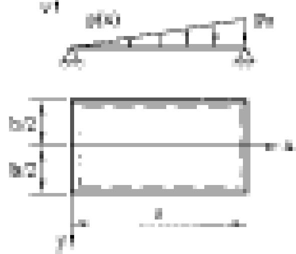
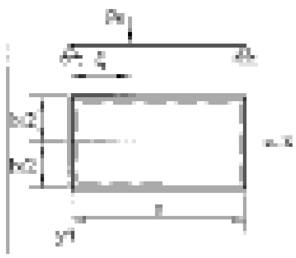
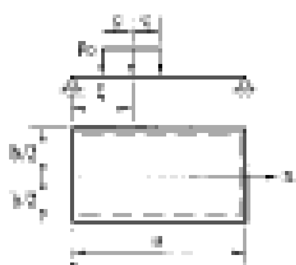
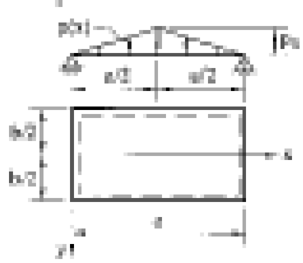
Solving this equation, we can determine gmðyÞ and, finally, find the particular solution, wpðx; yÞ. The complementary components of the stress resultants and stress couples, Mxh;Myh;Mxyh; and Vxh;Vyh; can be expressed in terms of fmðyÞ by substituting Eq. (3.50) into Eqs (2.13) and (2.39), as follows

CONTINUOUS PLATES

When a uniform plate extends over a support and has more than one span along its length or width, it is termed continuous. Such plates are of considerable practical interest. Continuous plates are externally statically indeterminate members (note that a plate itself is also internally statically indeterminate). So, the well-known methods developed in structural mechanics can be used for the analysis of continuous plates. In this section, we consider the force method which is commonly used for the analysis of statically indeterminate systems. According to this method, the continuous plate is subdivided into individual, simple-span panels between intermediate supports by removing all redundant restraints. It can be established, for example, by introducing some fictitious hinges above the intermediate supports. In this way, the redundant moments acting along the intermediate supports are eliminated. Similar fictitious hinges can be used at the ends if those are fixed. The simple-span panel obtained in such a way is referred to as a primary plate. In order to restore the rejected restraints, the unknown redundant moments are applied to the primary plate. These moments can be determined from the solution of simultaneous algebraic equations expressing the compatibility of the slopes between the adjoining panels produced by both external loads and unknown redundant moments. In our further discussion we assume that the supports are unyielding. Forthesakeofsimplicity,weconfineourselvestoarectangularplatecontinuousinone direction only and having the same flexural rigidity. Obviously, the general procedure of the force method discussed below can be applied to plates continuous in bothdirections.

subdivided into individual, simple-span panels between intermediate supports by removing all redundant restraints. It can be established, for example, by introducing some fictitious hinges above the intermediate supports. In this way, the redundant moments acting along the intermediate supports are eliminated. Similar fictitious hinges can be used at the ends if those are fixed. The simple-span panel obtained in such a way is referred to as a primary plate. In order to restore the rejected restraints, the unknown redundant moments are applied to the primary plate. These moments can be determined from the solution of simultaneous algebraic equations expressing the compatibility of the slopes between the adjoining panels produced by both external loads and unknown redundant moments. In our further discussion we assume that the supports are unyielding. For the sake of simplicity, we confine ourselves to a rectangular plate continuous in one direction only and having the same flexural rigidity. Obviously, the general procedure of the force method discussed below can be applied to plates continuous in bothdirections.

Table 3.2

No.	Load Geometry $p(x,y)=\sum_m p_m \sin\frac{m\pi x}{a}$	Expansion coefficient p_m [determined from Eq. (3.53)]
1		Uniform loading, $p_0 = \text{const}$ $p_m = \frac{4p_0}{m\pi}$ $(m = 1, 3, 5, \ldots)$
2		Hydrostatic loading, $p(x) = p_0\frac{x}{a}$ $p_m = \frac{2p_0}{m\pi}(-1)^{m+1}$ $(m = 1, 2, \ldots)$
3		Line load p_0 at $x = \xi$ $p_m = \frac{2p_0}{a}\sin\frac{m\pi\xi}{a}$ $(m = 1, 2, 3, \ldots)$
4		Uniform load from $(\xi - \varepsilon)$ to $(\xi + \varepsilon)$ $p_m = \frac{4p_0}{m\pi}\sin\frac{m\pi\xi}{a}\sin\frac{m\pi\varepsilon}{a}$ $(m = 1, 2, \ldots)$
5		Triangular load $p(x) = 2p_0\frac{x}{a}$ if $x \le a/2$ $p(x) = 2p_0\frac{a-x}{a}$ if $x \ge a/2$ $p_m = \frac{8p_0}{m^2\pi^2}(-1)^{\frac{m-1}{2}}$ $(m = 1, 3, \ldots)$

Consider the three-span simply supported continuous plate. The plate is subjected to a uniform load of intensities p1; p2, and p3 as shown in Fig. 3.14a. Figure 3.14b illustrates the primary plate obtained by introducing the fictitious hinges above the intermediate supports; _1ðyÞ and _2ðyÞ are the redundant, unknown distributed moments replacing the removed restraints at the intermediate supports. These unknown bending moments may be represented by the following Fourier series:

$$\mu_1(y) = \sum_{m=1,3,\ldots}^{\infty} \mu_{1m} \sin \lambda_m y, \quad \mu_2(y) = \sum_{m=1,3,\ldots}^{\infty} \mu_{2m} \sin \lambda_m y, \quad \lambda_m = \frac{m\pi}{b}. \quad (3.63)$$

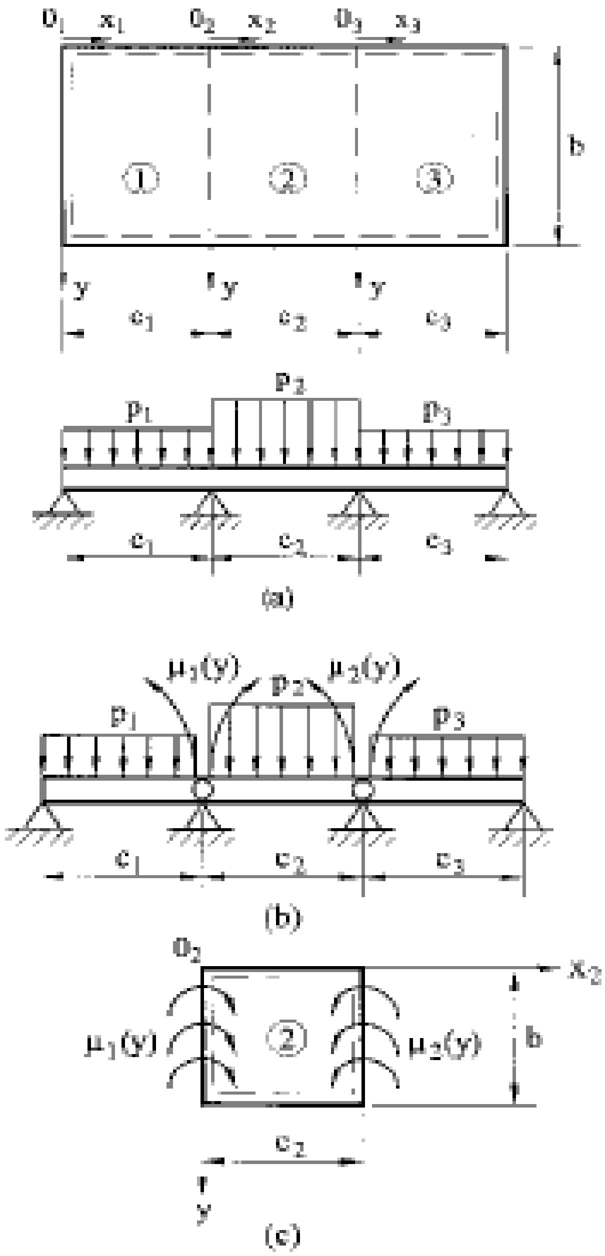


Fig. 3.14

primary plate consists of individual simple-span panels subjected to a given loading and the distributed edge moments _1ðyÞ and _2ðyÞ, as shown in Fig. 3.14c. The deflected surface of each simply supported individual panel may be obtained by application of Levy's method. The general solution for individual panels simply supported on all edges is given by Eqs (3.50) and (3.56a) if y is replaced by xi, x by y, and a by b. We have the following:

$$w_i = \sum_{m=1,3,...}^{\infty} \left(A_m^{(i)} \sinh \lambda_m x_i + B_m^{(i)} \cosh \lambda_m x_i + C_m^{(i)} \lambda_m x_i \sinh \lambda_m x_i + D_m^{(i)} \lambda_m x_i \cosh \lambda_m x_i + \frac{4 p_i b^4}{m^5 \pi^5 D} \right) \sin \lambda_m y, \quad i = 1, 2, 3. \tag{3.64}$$

The boundary conditions of each panel as well as the compatibility of deformations between any two panels across a common boundary (slopes continuity at the intermediate supports) are used to determine the constants in the solution (3.64) and the coefficients in the Fourier series (3.63). The boundary conditions for the plates 1, 2, and 3 are represented as follows:

$$w_1 = 0|_{x_1=0,c_1}, \ \frac{\partial^2 w_1}{\partial x_1^2} = 0\bigg|_{x_1=0}, \quad -D\frac{\partial^2 w_1}{\partial x_1^2} = \sum_{m=1,3,\ldots}^{\infty} \mu_{1m} \sin \lambda_m y\bigg|_{x_1=c_1}, \tag{a}$$

$$w_2 = 0|_{x_2=0,c_2}, -D\frac{\partial^2 w_2}{\partial x_2^2} = \sum_{m=1,3,\ldots}^{\infty} \mu_{1m} \sin \lambda_m y\bigg|_{x_2=0},$$

$$-D\frac{\partial^2 w_2}{\partial x_2^2} = \sum_{m=1,3,\ldots}^{\infty} \mu_{2m} \sin \lambda_m y\bigg|_{x_2=c_2} \tag{b}$$

$$w_3 = 0|_{x_3=0,c_3}, \frac{\partial^2 w_3}{\partial x_3^2} = 0\bigg|_{x_3=c_3}, \quad -D\frac{\partial^2 w_3}{\partial x_3^2} = \sum_{m=1,3,\ldots}^{\infty} \mu_{2m} \sin \lambda_m y\bigg|_{x_3=0}. \tag{c}$$

The compatibility conditions are expressed as follows

$$\left(\frac{\partial w_1}{\partial x_1}\right)_{x_1=c_1} = \left(\frac{\partial w_2}{\partial x_2}\right)_{x_2=0}, \quad \left(\frac{\partial w_2}{\partial x_2}\right)_{x_2=c_2} = \left(\frac{\partial w_3}{\partial x_3}\right)_{x_3=0}. \tag{d}$$

Let us consider the middle panel 2 (Fig. 3.14c). Application of the edge conditions (b) to Eq. (3.64) for i =2 leads to the following values of the constants

$$\begin{aligned} A_m^{(2)} &= \frac{1}{D \sinh \lambda_m c_2}\left[\frac{4p_2 b^4}{m^5\pi^5}(\cosh \lambda_m c_2 - 1)(1 + 0.5\lambda_m \operatorname{csch} \lambda_m c_2) \right. \\ &\left. + \frac{c_2}{2\lambda_m}(\mu_{1m} \operatorname{csch} \lambda_m c_2 - \mu_{2m} \coth \lambda_m c_2)\right], \\ B_m^{(2)} &= -\frac{4p_2 b^4}{m^5\pi^5 D}, \ C_m^{(2)} = \frac{2p_2 b^4}{m^5\pi^5 D} - \frac{\mu_{1m}}{2D\lambda_m^2}, \\ D_m^{(2)} &= \frac{1}{2D \sinh \lambda_m c_2}\left[\frac{4p_2 b^4}{m^5\pi^5}(1 - \cosh \lambda_m c_2) - \frac{1}{\lambda_m^2}(\mu_{2m} - \mu_{1m} \cosh \lambda_m c_2)\right]. \end{aligned} \tag{e}$$

The coefficients $A_m^{(1)}, B_m^{(1)}, C_m^{(1)}$, and $D_m^{(1)}$ and $D_m^{(i)}$ $(i = 1, 2, 3)$are also given by Eqs (e) if p2 is replaced by p_1, c_2 by c_1, μ_{2m} by $\mu_{1m}$by _1m and letting _1m ¼ 0. Similarly, the coefficients Að3Þ m ; Bð3Þ m ;Cð3Þ m ; and Dð3Þ m can be obtained by replacing c2 by c3, p2 by p3, _1m by _2m and letting _2m ¼ 0. Introducing Eqs (3.64) for i ¼ 1 and i ¼ 2 into the compatibility conditions (d), we obtain the following two additionalequations

$$
\begin{aligned}
&A_m^{(1)} \cosh \lambda_m c_1 + B_m^{(1)} \sinh \lambda_m c_1 + C_m^{(1)} (\sinh \lambda_m c_1 + \lambda_m c_1 \cosh \lambda_m c_1) \\
&\qquad + D_m^{(1)} (\cosh \lambda_m c_1 + \lambda_m c_1 \sinh \lambda_m c_1) = A_m^{(2)} + D_m^{(2)}, \\
&A_m^{(2)} \cosh \lambda_m c_2 + B_m^{(2)} \sinh \lambda_m c_2 + C_m^{(2)} (\sinh \lambda_m c_2 + \lambda_m c_2 \cosh \lambda_m c_2) \\
&\qquad + D_m^{(2)} (\cosh \lambda_m c_2 + \lambda_m c_2 \sinh \lambda_m c_2) = A_m^{(3)} + D_m^{(3)}.
\end{aligned}
\tag{f}
$$

Having coefficients $A_m^{(i)}, B_m^{(i)}, C_m^{(i)}$, and $D_m^{(i)}$ $(i = 1, 2, 3)$available, from Eqs (f) we obtain the moment coefficients _1m and _2m. Equations (3.64) then give the deflection ofthe continuous plate from which the moments and stresses can also becomputed.

Application of Navier's solution

Let us consider a simply supported rectangular plate of sides a x b on an elastic foundation (Fig. 3.17). The deflection wðx; yÞ and a given loading pðx; yÞ can be expressed in the form of a double trigonometric series in the form of the expressions (3.15), as follows. Substituting the above expressions into the governing differential equation of the problem (3.67), we obtain the unknown amplitudes of the deflections, wmn, for a specific set of m and n values, as follows

$$w_{mn} = \frac{p_{mn}}{D\pi^4\left(m^2/a^2 + n^2/b^2\right)^2 + k}, \tag{3.68}$$

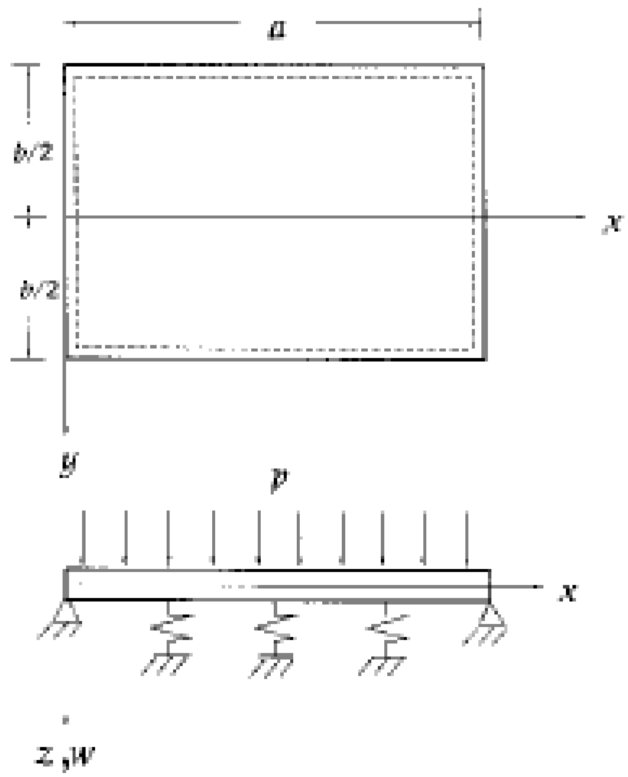


Fig. 3.17

which substituted into Eq. (3.15a) yields

$$w(x, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{p_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}{D\pi^4\left(m^2/a^2 + n^2/b^2\right)^2 + k}. \tag{3.69}$$

If a plate is subjected to a uniform load of intensity p0, then Eq. (3.69) becomes

$$w = \frac{16p_0}{\pi^2} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}{mn\left[\pi^4 D(m^2/a^2 + n^2/b^2)^2 + k\right]}. \tag{3.70}$$

If a plate is loaded by a concentrated force P applied atapoint (ξ, η), then the deflected surface equation of the plate can be represented by the followingform:

$$w = \frac{4P}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}}{\pi^4 D(m^2/a^2 + n^2/b^2)^2 + k}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \qquad (3.71)$$

Having the deflection of the plate produced by a concentrated force, the deflection produced by any kind of lateral loading is obtained by the method of superposition.

Application of Levy's method

If any two opposite edges of the plate on an elastic foundation are simply supported, say, the edges x = 0 and x = a, then Levy's solution presented earlier in Sec. 3.5 can be applied advantageously. The general solution of the governing differential equation of the plate on an elastic foundation, Eq. (3.67), may be represented again in the form of the sum of particular and complementary solutions,i.e.,

$$w = w_p + w_h. \qquad (3.72)$$

Expressing the complementary solution by the Fourier series, yields

$$w_h = \sum_{m=1}^{\infty} f_m(y)\sin\frac{m\pi x}{a}. \qquad (3.73)$$

Substituting the above into the homogeneous part of Eq. (3.66) and solvingitfor $f_m(y)$, weobtain

$$f_m(y) = A_m \sinh\alpha_m y \sin\beta_m y + B_m \sinh\alpha_m y \cos\beta_m y + C_m \cosh\alpha_m y \sin\beta_m y + D_m \cosh\alpha_m y \cos\beta_m y, \qquad (3.74)$$

Where

$$\alpha_m = \sqrt{\frac{1}{2}\left(\frac{m^2\pi^2}{a^2} + \sqrt{\frac{m^4\pi^4}{a^4} + \frac{k}{D}}\right)}; \quad \beta_m = \sqrt{\frac{1}{2}\left(-\frac{m^2\pi^2}{a^2} + \sqrt{\frac{m^4\pi^4}{a^4} + \frac{k}{D}}\right)}. \tag{3.75}$$

Substituting (3.74) into (3.73), we can obtain the expression for complementary solution wh. Similarly, if we express the particular solution as

$$w_p = \sum_{m=1}^{\infty} g_m(y) \sin\frac{m\pi x}{a}, \tag{3.76}$$

and the applied load as

$$p = \sum_{m=1}^{\infty} p_m(y) \sin\frac{m\pi x}{a}, \tag{3.77}$$

and then we substitute these equations into Eq. (3.66), we obtain

$$\frac{d^4 g_m}{dy^4} - 2\left(\frac{m\pi}{a}\right)^2 \frac{d^2 g_m}{dy^2} + \left(\frac{m^4\pi^4}{a^4} + \frac{k}{D}\right) g_m = \frac{p_m(y)}{D}. \tag{3.78}$$

Solving Eq. (3.78) for gm, we can find the particular solution wp. Therefore, Eq. (3.76) together with Eqs (3.73) and (3.74) describes the deflected surface of the plate.

BENDING OF PLATES WITH SMALL INITIAL CURVATURE

Let us assume that a plate has an initial curvature of its middle surface, i.e., there is an initial deflection w0 at any point of the above surface. It is assumed that w0 is small compared with the plate thickness. If the plate is subject to lateral load and then an additional deflection w1 occurs, the total deflection is thus

$$w = w_0 + w_1 . \tag{3.93}$$

Here w1 is the solution of Eq. (2.24) set up for the flat plate, i.e., without the abovementioned initial deflection. It will be valid if the small initial deflection, w0, is considered as a result of the action of some fictitious lateral load. Then, applying the superposition principle (recall that the above principle cannot be applied for large deflections), it is possible to determine the total deflection.

If besides the lateral load, the direct forces are also applied to an initially curved plate, then these forces produce bending also, which depends not only on w1 but also on w0. In order to determine the total deflection, w, we introduce w ¼ w0 þ w1 into the right hand side of Eq. (3.92). The left-hand side of this equation takes into account a change in curvature from the initial curved state due to the given lateral load. Therefore, w1 has to be substituted for w on the left-hand side of Eq. (3.92). Thus, Eq. (3.92) for the initially curved plate is of the following

$$\frac{\partial^4 w_1}{\partial x^4} + 2\frac{\partial^4 w_1}{\partial x^2 \partial y^2} + \frac{\partial^4 w_1}{\partial y^4} = \frac{1}{D}\left[p + N_x \frac{\partial^2 (w_0 + w_1)}{\partial x^2} + N_y \frac{\partial^2 (w_0 + w_1)}{\partial y^2} + 2N_{xy} \frac{\partial^2 (w_0 + w_1)}{\partial x \partial y}\right]. \quad (3.94)$$

As mentioned previously, the influence of the initial curvature on the total deflection of the plate is equivalent to the influence of some fictitious lateral load of intensity pf equal to

$$p_f = N_x \frac{\partial^2 w_0}{\partial x^2} + N_y \frac{\partial^2 w_0}{\partial y^2} + 2N_{xy} \frac{\partial^2 w_0}{\partial x \partial y}. \quad (3.95)$$

Hence, an initially curved plate will experience a bending under action of the direct forces, lying in the plate middle surface only.

The critical section for checking the development length in a footing slab shall be the same planes as those of bending moments in part (c) of this section. Moreover, development length shall be checked at all other sections where they change abruptly. The critical sections for checking the development length are given in cl.34.2.4.3 of IS 456, which further recommends to check the anchorage requirements if the reinforcement is curtailed, which shall be done in accordance with cl.26.2.3 of IS456

.

The distribution of the total tensile reinforcement, calculated in accordance with the moment at critical sections, as specified in part (c) of this section, shall be done as given below for one-way and two-way footing slabs separately.

(i) In one-way reinforced footing slabs like wall footings, the reinforcement shall be distributed uniformly across the full width of the footing i.e., perpendicular to the direction of wall. Nominal distribution reinforcement shall be provided as per cl. 34.5 of IS 456 along the length of the wall to take care of the secondary moment, differential settlement, shrinkage and temperatureeffects.

(ii) In two-way reinforced square footing slabs, the reinforcement extending in each direction shall be distributed uniformly across the full width/length of thefooting.

iii) In two-way reinforced rectangular footing slabs, the reinforcement in the long direction shall be distributed uniformly across the full width of the footing slab. In the short direction, a central band equal to the width of the footing shall be marked along the length of the footing, where the portion of the reinforcement shall be determined as given in the equation below. This portion of the reinforcement shall be distributed across the central band

Structural Classification

Structurally, staircases may be classified largely into two categories, depending on the predominant direction in which the slab component of the stair undergoesflexure:

1. Stair slab spanning transversely (stairwidthwise);
2. Stair slab spanning longitudinally (along theincline).

The slab component of the stair (whether comprising an isolated tread slab, a tread-riser unit or a waist slab) is supported on its side(s) or cantilevers laterally from a central support. The slab supports gravity loads by bending essentially in a *transverse vertical plane*, with the span along the *width* of the stair.

In the case of the cantilevered slabs, it is economical to provide isolated treads (without risers). However, the tread-riser type of arrangement and the waist slab type are also sometimes employed in practice, as cantilevers. The spandrel beam is subjected to torsion (_equilibrium torsion‘), in addition to flexure and shear.

When the slab is supported at the two sides by means of _stringer beams‘ or masonry walls, it may be designed as simply supported, but reinforcement at the top should be provided near the supports to resist the _negative‘ moments that may arise on account of possible partial fixity.

In this case, the supports to the stair slab are provided parallel to the riser at two or more locations, causing the slab to bend longitudinally between the supports. It may be noted that longitudinal bending can occur in configurations other than the straight stair configuration, such as quarter-turn stairs, dog-legged stairs, open well stairs and helicoidalstairs.
The slab arrangement may either be the conventional _waist slab‘ type or the _tread- riser‘ type. The slab thickness depends on the _effective span‘, which should be taken as the centre-to-centre distance between the beam/wall supports, according to the Code (Cl. 33.1a, c).In certain situations, beam or wall supports may not be available parallel to the riser at the landing. Instead, the flight is supported between the landings, which span transversely, parallel to the risers. In such cases, the Code(Cl. 33.1b) specifies that the effective span for the flight (spanning longitudinally) should be taken as *the going of the stairs plus at each end either half the width of the landing or one metre, whichever is smaller*.

Types of Structures

Shallow foundations are used when the soil has sufficient strength within a short depth below the ground level. They need sufficient plan area to transfer the heavy loads to the base soil. These heavy loads are sustained by the reinforced concrete columns or walls(eitherofbricksorreinforcedconcrete)ofmuchlessareasofcross-sectiondueto high strength of bricks or reinforced concrete when compared to that of soil. The strength of the soil, expressed as the safe bearing capacity of the soil is normally supplied by the geotechnical experts to the structural engineer. Shallow foundations are also designated as footings. The different types of shallow foundations or footings are discussedbelow.

As mentioned earlier, the shallow foundations need more plan areas due to the low strength of soil compared to that of masonry or reinforced concrete. However, shallow foundations are selected when the soil has moderately good strength, except the raft foundation which is good in poor condition of soil also. Raft foundations are under the category of shallow foundation as they have comparatively shallow depth than that of deep foundation. It is worth mentioning that the depth of raft foundation is much larger than those of other types of shallowfoundations.

However, for poor condition of soil near to the surface, the bearing capacity is very less and foundation needed in such situation is the pile foundation. Piles are, in fact, small diameter columns which are driven or cast into the ground by suitable means. Precast piles are driven and cast-in-situ are cast. These piles support the structure by the skin friction between the pile surface and the surrounding soil and end bearing force, if such resistance is available to provide the bearing force. Accordingly, they are designated as frictional and end bearing piles. They are normally provided in a group with a pile cap at the top through which the loads of the superstructure are transferred to the piles.

Piles are very useful in marshy land where other types of foundation are impossible to construct. The length of the pile which is driven into the ground depends on the availability of hard soil/rock or the actual load test. Another advantage of the pile foundations is that they can resist uplift also in the same manner as they take the compression forces just by the skin friction in the opposite direction.

However, driving of pile is not an easy job and needs equipment and specially trained persons or agencies. Moreover, one has to select pile foundation in such a situation where the adjacent buildings are not likely to be damaged due to the driving of piles. The choice of driven or bored piles, in this regard, is critical.

Exhaustive designs of all types of foundations mentioned above are beyond the scope of this course. Accordingly, this module is restricted to the design of some of the shallow footings, frequently used for normal low rise buildings only.

subdivided into individual, simple-span panels between intermediate supports by removing all redundant restraints. It can be established, for example, by introducing some fictitious hinges above the intermediate supports. In this way, the redundant moments acting along the intermediate supports are eliminated. Similar fictitious hinges can be used at the ends if those are fixed. The simple-span panel obtained in such a way is referred to as a primary plate. In order to restore the rejected restraints, the unknown redundant moments are applied to the primary plate. These moments can be determined from the solution of simultaneous algebraic equations expressing the compatibility of the slopes between the adjoining panels produced by both external loads and unknown redundant moments. In our further discussion we assume that the supports are unyielding. For the sake of simplicity, we confine ourselves to a rectangular plate continuous in one direction only and having the same flexural rigidity. Obviously, the general procedure of the force method discussed below can be applied to plates continuous in bothdirections.

UNIT – III CIRCULAR PLATES

INTRODUCTION

Circular plates are common in many structures such as nozzle covers, end closures in pressure essels, pump diaphragms, turbine disks, and bulkheads in submarines and airplanes, etc. When circular plates are analyzed, it is convenient to express thegoverning differential equation (2.24) in polar oordinates. This can be readily accomplished by a coordinate transformation. An alternative approach based on the procedure presented in Chapter 3 for rectangular plates to derive the basic relationships for the lateral deflections of circular plates may be usedalso.

BASIC RELATIONS IN POLAR COORDINATES

As mentioned earlier, we use the polar coordinates r and ' in solving the bending problems for circular plates. If the coordinate transformation technique is used, the following geometrical relations between the Cartesian and polar coordinates are applicable (see Fig. 4.1a):

$$x = r\cos\varphi, \; y = r\sin\varphi \text{ and } r^2 = x^2 + y^2, \varphi = \tan^{-1}\frac{y}{x}. \tag{4.1}$$

Referring to the above

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{r} = \cos\varphi, \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\varphi, \\ \frac{\partial \varphi}{\partial x} &= -\frac{y}{r^2} = -\frac{\sin\varphi}{r}, \quad \frac{\partial \varphi}{\partial y} = \frac{x}{r^2} = \frac{\cos\varphi}{r}. \end{aligned} \tag{4.2}$$

Inasmuch as the deflection is a function of r and ', the chain rule together with the relations (4.2) lead to the following

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \varphi}\frac{\partial \varphi}{\partial x} = \frac{\partial w}{\partial r}\cos\varphi - \frac{1}{r}\frac{\partial w}{\partial \varphi}\sin\varphi. \tag{4.3}$$

To evaluatethe expression $\partial^2 w/\partial x^2$, we can repeat the operation (4.3) twice. As a result, we obtain

$$\frac{\partial^2 w}{\partial x^2} = \cos\varphi \frac{\partial}{\partial r}\left(\frac{\partial w}{\partial x}\right) - \frac{1}{r}\sin\varphi \frac{\partial}{\partial \varphi}\left(\frac{\partial w}{\partial x}\right)$$
$$= \frac{\partial^2 w}{\partial r^2}\cos^2\varphi - \frac{\partial^2 w}{\partial\varphi\partial r}\frac{\sin 2\varphi}{r} + \frac{\partial w}{\partial r}\frac{\sin^2\varphi}{r} + \frac{\partial w}{\partial \varphi}\frac{\sin 2\varphi}{r^2} + \frac{\partial^2 w}{\partial \varphi^2}\frac{\sin^2\varphi}{r^2}. \tag{4.4a}$$

Similarly,

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2}\sin^2\varphi + \frac{\partial^2 w}{\partial r\partial\varphi}\frac{\sin 2\varphi}{r} + \frac{\partial w}{\partial r}\frac{\cos^2\varphi}{r} - \frac{\partial w}{\partial \varphi}\frac{\sin 2\varphi}{r^2} + \frac{\partial^2 w}{\partial \varphi^2}\frac{\cos^2\varphi}{r^2}, \tag{4.4b}$$

$$\frac{\partial^2 w}{\partial x\partial y} = \frac{\partial^2 w}{\partial r^2}\frac{\sin 2\varphi}{2} + \frac{\partial^2 w}{\partial r\partial\varphi}\frac{\cos 2\varphi}{r} - \frac{\partial w}{\partial \varphi}\frac{\cos 2\varphi}{r^2} - \frac{\partial w}{\partial r}\frac{\sin 2\varphi}{2r} - \frac{\partial^2 w}{\partial \varphi^2}\frac{\sin 2\varphi}{2r^2}. \tag{4.4c}$$

Adding term by term the relations (4.4a) and (4.4b), yields

$$\nabla_r^2 w \equiv \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}. \tag{4.5}$$

After repeating twicethe operation $\nabla_r^2,$ the governing differential equation for the plate deflection (2.26) in polar coordinates becomes

$$\nabla_r^4 w \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}\right) = \frac{p}{D}, \tag{4.6a}$$

or in the expended form

$$\frac{\partial^4 w}{\partial r^4} + \frac{2}{r}\frac{\partial^3 w}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^3}\frac{\partial w}{\partial r} + \frac{2}{r^2}\frac{\partial^4 w}{\partial r^2\partial\varphi^2} - \frac{2}{r^3}\frac{\partial^3 w}{\partial\varphi^2\partial r} + \frac{4}{r^4}\frac{\partial^2 w}{\partial\varphi^2} + \frac{1}{r^4}\frac{\partial^4 w}{\partial\varphi^4} = \frac{p}{D}. \tag{4.6b}$$

Let us set up the relationships between moments and curvatures. Consider now the state of moment and shear force on an infinitesimal element of thickness h, described in polar coordinates, as shown in Fig. 4.1b. Note that, to simplify the derivations, the x axis is taken in the direction of the radius r, at ’ ¼ 0 (Fig. 4.1b). Then, the radialMr, tangential Mt, twisting Mrt moments, and the vertical shear forces Qr;Qt will have the same values as the moments Mx;My; and Mxy, and shears Qx;Qy at the same point in the plate. Thus, transforming the expressions for moments (2.13) and shear forces (2.27) into polar coordinates, we can write thefollowing:

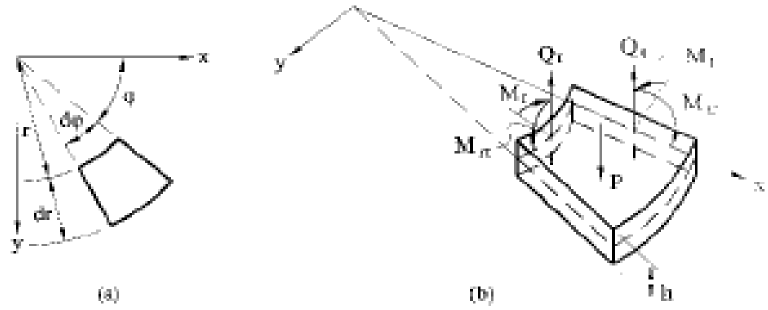


Fig. 4.1

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}\right)\right]; \quad M_t = -D\left[\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2} + \nu\frac{\partial^2 w}{\partial r^2}\right];$$

$$M_{rt} = M_{tr} = -D(1-\nu)\left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \varphi} - \frac{1}{r^2}\frac{\partial w}{\partial \varphi}\right); \qquad (4.7a)$$

$$Q_r = -D\frac{\partial}{\partial r}(\nabla_r^2 w); \; Q_t = -D\frac{1}{r}\frac{\partial}{\partial \varphi}(\nabla_r^2 w). \qquad (4.7b)$$

Similarly, the formulas for the plane stress components, from Eqs (2.15), are written in the following form:

$$\sigma_r = \frac{12M_r}{h^3}z, \quad \sigma_t = \frac{12M_t}{h^3}z, \quad \tau_{rt} = \tau_{tr} = \frac{12M_{tr}}{h^3}z, \qquad (4.8)$$

whereMr;Mt and Mtr are determined by Eqs (4.7). Clearly the maximum stresses take place on the surfaces z ¼ _h=2 of the plate.

Similarly, transforming Eqs (2.38) and (2.39) into polar coordinates gives the effective transverse shear forces. They may be written for an edge with outwardnormal in the r and ’ directions, asfollows:

$$V_r = Q_r + \frac{1}{r}\frac{\partial M_{rt}}{\partial \varphi} = -D\left[\frac{\partial}{\partial r}(\nabla_r^2 w) + \frac{1-\nu}{r}\frac{\partial}{\partial \varphi}\left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \varphi} - \frac{1}{r^2}\frac{\partial w}{\partial \varphi}\right)\right],$$

$$V_t = Q_t + \frac{\partial M_{rt}}{\partial r} = -D\left[\frac{1}{r}\frac{\partial}{\partial \varphi}(\nabla_r^2 w) + (1-\nu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \varphi} - \frac{1}{r^2}\frac{\partial w}{\partial \varphi}\right)\right]. \qquad (4.9)$$

The boundary conditions at the edges of a circular plate of radius a may readily bewritten by referring to Eqs. (2.48),namely:

(a) Clamped edge r =a

$$w = 0|_{r=a}, \frac{\partial w}{\partial r} = 0\Big|_{r=a}. \qquad (4.10)$$

(b) Simply supported edge r =a

$$w = 0|_{r=a}, M_r = 0|_{r=a}. \qquad (4.11)$$

$$w = 0|_{r=a}, M_r = 0|_{r=a}. \qquad (4.11)$$

(c) *Free edge r = a*

$$M_r = 0|_{r=a}, V_r = 0|_{r=a}. \qquad (4.12)$$

If we use the transformations given by Eqs (4.1)–(4.4), the strain energy for a circular plate is

$$U = \frac{1}{2}\iint_S D\left[\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}\right)^2 - 2(1-\nu)\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}\right) + 2(1-\nu)\left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \varphi} - \frac{1}{r^2}\frac{\partial w}{\partial \varphi}\right)^2\right] r\,\mathrm{d}r\,\mathrm{d}\varphi. \qquad (4.13)$$

This has two situations: (a) when D_f/x_udoes not exceed 0.43, the full depth of flange is having the constant stress (Fig. 5.10.9), and (b) when D_f/x_u> 0.43, the constant stress is for a part of the depth offlange

(Fig. 5.10.10).

(i) Neutral axis is in the web and the section is over-reinforced ($x_u > x_{u,max} > D_f$), (Figs. 5.10.7 and 8 a toe)

As mentioned earlier, the value of $x_u$is then taken as $x_{u,max}$when $x_u > x_{u,max}$. Therefore, this case also will have two situations depending on D_f/d not exceeding 0.2 or > 0.2 as in (ii) above. The governing equations of the four different cases are now takenup.

Governing Equations

The following equations are only for the singly reinforced *T*-beams.

Additional terms involving $M_{u,lim}$, M_{u2}, A_{sc}, A_{st1} and A_{st2} are to be included from Eqs. 4.1 to 4.8 of sec. 4.8.3 of Lesson 8 depending on the particular case.

Applications of these terms are explained through the solutions of numerical problems of doubly reinforced *T*-beams in Lessons 11 and 12.

Case (i): When the neutral axis is in the flange ($x_u < D_f$), (Figs. 5.10.6 a to c)

Concrete below the neutral axis is in tension and is ignored. The steel reinforcement takes the tensile force (Fig. 5.10.6). Therefore, *T* and *L*-beams are considered as rectangular beams of width b_f and effective depth *d*. All the equations of singly and doubly reinforced rectangular beams derived in Lessons 4 to 5 and 8 respectively, are also applicable here.

Case (ii): When the neutral axis is in the web and the section is balanced ($x_{u,max} > D_f$), (Figs. 5.10.7 and 8 a to e)

(a) When D_f / d does not exceed 0.2, (Figs. 5.10.7 a toe)

As explained in sec. 5.10.3, the depth of the rectangular portion of the stress block (of constant stress = 0.446 f_{ck}) in this case is greater than D_f (Figs. 5.10.7 a, b and c). The section is split into two parts: (i) rectangular web of width b_w and effective depth *d*, and (ii) flange of width (b_f - b_w) and depth D_f (Figs. 5.10.7 d ande).

Total compressive force = Compressive force of rectangular beam of width b_w and depth *d* + Compressive force of rectangular flange of width (b_f - b_w) and depth D_f.

Thus, total compressive force

The lever arm of the rectangular beam (web part) is (d - 0.42 $x_{u,max}$) and the same for the flanged part is (d - 0.5 D_f).

So, the total moment = Moment due to rectangular web part + Moment due to rectangular flange part

Equation 5.7 is given in G-2.2 of IS 456.

(b) When D_f /d >0.2, (Figs. 5.10.8 a toe)

In this case, the depth of rectangular portion of stress block is within the flange (Figs. 5.10.8 a, b and c). It is assumed that this depth of constant stress (0.45 f_{ck}) is y_f, where

y_f= 0.15 $x_{u,max}$+0.65 D_f, but notgreaterthan D_f

(5.8)

The above expressionof y_f is derived in sec. 5.10.4.5.

Asinthepreviouscase(iia),whenD_f/ddoesnotexceed0.2,equationsof$C$,T and$M_u$areobtainedfromEqs.5.5,6and7bychanging$D_f$toy_f.Thus,wehave(Figs. 5.10.8 d ande)

C =0.36 f_{ck} $b_w x_{u,max}$+ 0.45 $f_{ck}(b_f - b_w)y_f$

T = 0.87 $f_y A_{st}$(5.10)

The lever arm of the rectangular beam (web part) is (d - 0.42 $x_{u,max}$same for the flange part is (d - 0.5 y_f). Accordingly, the expression of follows:

M_u= 0.36($x_{u,max} /d$){1 - 0.42($x_{u,max}/d$)} $f_{ck} b_w d^2$+0.45 $f_{ck}(b_f - b_w) y_f(d - y_f$

/2)

Since D_f does not exceed 0.43 x_u and h (depth of fibre where the strain is 0.002) is at a depth of 0.43 x_u, the entire flange will be under a constant stress of 0.45 f_{ck} (Figs. 5.10.9 a, b and c). The equations of C, T and M_u can be written in the same manner as in sec. 5.10.4.2, case (ii a). The final forms of the equations are obtained from Eqs. 5.5, 6 and 7 by replacing $x_{u,max}$ by x_u. Thus, we have (Figs. 5.10.9 d ande)

$C = 0.36 f_{ck} b_w x_u + 0.45 f_{ck}(b_f - b_w) D_f T =$

$0.87 f_y A_{st}$

(5.13)

$M_u = 0.36(x_u /d)\{1 - 0.42(x_u /d)\} f_{ck} b_w d^2 + 0.45 f_{ck}(b_f - b_w) D_f(d - D_f/2)$

(5.14)

(b) When $D_f/x_u > 0.43$, (Figs. 5.10.10 a toe)

Since $D_f > 0.43 x_u$ and h (depth of fibre where the strain is 0.002) is at a depth of 0.43 x_u, the part of the flange having the constant stress of 0.45 f_{ck} is assumed as y_f (Fig. 5.10.10 a, b and c). The expressions of y_f, C, T and M_u can be written from Eqs. 5.8, 9, 10 and 11 of sec. 5.10.4.2, case (iib), by replacing $x_{u,max}$ by x_u. Thus, we have (Fig. 5.10.10 d ande)

$y_f = 0.15 x_u + 0.65 D_f$, but not greater than D_f

(5.15)

$C = 0.36 f_{ck} b_w x_u + 0.45 f_{ck}(b_f - b_w) y_f$

(5.16)

$T = 0.87 f_y A_{st}$

(5.17)

$$M_u = 0.36(x_u/d)\{1 - 0.42(x_u/d)\} f_{ck} b_w d^2 + 0.45 \quad f_{ck}(b_f - b_w) y_f(d - y_f/2)$$

(5.18)

to 5.7 and 5.9 to 5.11, respectively of sec. 5.10.4.2 (Figs. 5.10.7 and 8). The expression of y_f for (b) is the same as that of Eq. 5.8.

It is clear from the above that the over-reinforced beam will not have additional moment of resistance beyond that of the balanced one. Moreover, it will prevent steel failure. It is, therefore, recommended either to re-design or to go for doubly reinforced flanged beam than designing over-reinforced flanged beam.

Whitney's stress block has been considered to derive Eq. 5.8.Figure

5.10.11 shows the two stress blocks of IS code and ofWhitney.

y_f= Depth of constant portion of the stress block when $D_f/d > 0.2$. As y_f is a function of x_u and D_f and let us assume

$$y_f = A\, x_u + B\, D_f$$

(5.19)

where A and B are to be determined from the following two conditions:

Using the conditions of Eqs. 5.20 and 21 in Eq. 5.19, we get A = 0.15 and B = 0.65. Thus, we have

$$y_f = 0.15\, x_u + 0.65 D_f$$

- When beam depth is restricted and the moment the beam has to carry is greater than the moment capacity of the beam in concretefailure.

- When B.M at the section can change sign.
- When compression steel can substantially improve the ductility of beams and its use is therefore advisable in members when larger amount of tension steel becomes necessary for itsstrength.
- Compression steel is always used in structures in earthquake regions to increase theirductility.
- Compression reinforcement will also aid significantly in reducing the long-term deflections ofbeams.

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Concrete below the neutral axis is in tension and is ignored. The steel reinforcement takes the tensile force (Fig. 5.10.6). Therefore, *T* and *L*-beams are considered as rectangular beams of width $b_f$and effective depth *d*. All the equations of singly and doubly reinforced rectangular beams derived in Lessons 4 to 5 and 8 respectively, are also applicable here.

Case (ii): When the neutral axis is in the web and the section is balanced ($x_{u,max}$>D_f), (Figs. 5.10.7 and 8 a to e)

(a) When D_f /*d* does not exceed 0.2, (Figs. 5.10.7 a toe)

As explained in sec. 5.10.3, the depth of the rectangular portion of the stress block (of constant stress = 0.446 f_{ck}) in this case is greater than D_f(Figs. 5.10.7 a, b and c). The section is split into two parts: (i) rectangular web of width $b_w$and effective depth *d*, and
(ii) flange of width (b_f- b_w) and depth D_f(Figs.
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Total compressive force = Compressive force of rectangular beam of width b_w and depth d + Compressive force of rectangular flange of width (b_f - b_w) and depth D_f.

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So, the total moment = Moment due to rectangular web part + Moment due to rectangular flange part

Equation 5.7 is given in G-2.2 of IS 456.

(b) When D_f /d >0.2, (Figs. 5.10.8 a toe)

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The above expressionof y_f is derived in sec. 5.10.4.5.

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C =0.36 f_{ck} $b_w x_{u,max}$ + 0.45 f_{ck}(b_f - b_w)y_f

T = 0.87 $f_y A_{st}$ (5.10)

The lever arm of the rectangular beam (web part) is (d - 0.42 $x_{u,max}$same for the flange part is (d - 0.5 y_f). Accordingly, the expression of follows:

$M_u = 0.36(x_{u,max}/d)\{1 - 0.42(x_{u,max}/d)\} f_{ck} b_w d^2 + 0.45 \quad f_{ck}(b_f - b_w) y_f(d - y_f$

$/2)$

Since D_f does not exceed 0.43 x_u and h (depth of fibre where the strain is 0.002) is at a depth of $0.43x_u$, the entire flange will be under a constant stress of $0.45f_{ck}$ (Figs. 5.10.9 a, b and c). The equations of C, T and M_u can be written in the same manner as in sec. 5.10.4.2, case (ii a). The final forms of the equations are obtained from Eqs. 5.5, 6 and 7 by replacing $x_{u,max}$ by x_u. Thus, we have (Figs. 5.10.9 d and e)

$C = 0.36 f_{ck} \quad b_w x_u + 0.45 f_{ck}(b_f - b_w) D_f T =$

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(5.13)

$M_u = 0.36(x_u/d)\{1 - 0.42(x_u/d)\} f_{ck} b_w d^2 + 0.45 \quad f_{ck}(b_f - b_w) D_f(d - D_f/2)$

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(b) When $D_f/x_u > \quad 0.43$, (Figs. 5.10.10 a to e)

Since $D_f > 0.43x_u$ and h (depth of fibre where the strain is 0.002) is at a depth of $0.43x_u$, the part of the flange having the constant stress of $0.45f_{ck}$ is assumed as y_f (Fig. 5.10.10 a, b and c). The expressions of y_f, C, T and M_u can be written from Eqs. 5.8, 9, 10 and 11 of sec. 5.10.4.2, case (iib), by replacing $x_{u,max}$ by x_u. Thus, we have (Fig. 5.10.10 d and e)

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(5.15)

$C = 0.36 f_{ck} \quad b_w x_u + 0.45 f_{ck}(b_f - b_w) y_f$

(5.16)

$T = 0.87\, f_y A_{st}$

(5.17)

$M_u = 0.36(x_u /d)\{1 - 0.42(x_u /d)\} f_{ck} b_w d^2 + 0.45 f_{ck}(b_f - b_w) y_f(d - y_f/2)$

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where A and B are to be determined from the following two conditions:

Using the conditions of Eqs. 5.20 and 21 in Eq. 5.19, we get A = 0.15 and B = 0.65. Thus, we have

$y_f = 0.15\, x_u + 0.65 D_f$

- When beam depth is restricted and the moment the beam has to carry is greater than the moment capacity of the beam in concretefailure.

- When B.M at the section can change sign

UNIT – IV

STATIC ANALYSIS OF SHELLS: MEMBRANE THEORY OF SHELLS

CLASSIFICATION OF SHELL SURFACES

We define here some of the surfaces that are commonly used for shell structures in engineering practice. There are several possible classifications of these surfaces. One such classification, associated with the Gaussian curvature, was discussed in Sec. 11.6. Following Ref. [4], we now discuss other categories of shell surfaces associated with their shape and geometric developability.

Classification based on geometric form

(a) Surfaces of revolution (Fig. 11.9)

As mentioned previously, surfaces of revolution are generated by rotating a plane curve, called the meridian, about an axis that is not necessarily intersecting the meridian. Circular cylinders, cones, spherical or elliptical domes, hyperboloids of revolution, and toroids (see Fig. 11.9) are some examples of surfaces of revolution. It can be seen that for the circular cylinder and cone (Fig. 11.9a and b), the meridian is a straight line, and hence, k1 ¼ 0, which gives _¼ 0. These are shells ofzero
Gaussian curvature.For ellipsoids and paraboloids of revolution and spheres (Fig. 11.9c, d, and e), both the principal curvatures are in the same direction and, thus, these surfaces have a positive Gaussian curvature (_ > 0). They are synclastic surfaces. For the hyperboloid of revolution (Fig. 11.9f), the curvatures of the meridian and the second line of curvature are in opposite directions (i.e., the principal radii R1 and R2 lie on opposite sides of the surface for all points on the surface). For
the toroid (Fig. 11.9g), the Gaussian curvature changes from positive to negative as we move along the surface.

(b) Surfaces of translation (Fig.11.10)

A surface of translation is defined as the surface generated by keeping a plane curve parallel to its initial plane as we move it along another plane curve. The two planes containing the two curves are at right angles to each other. An elliptic paraboloid is

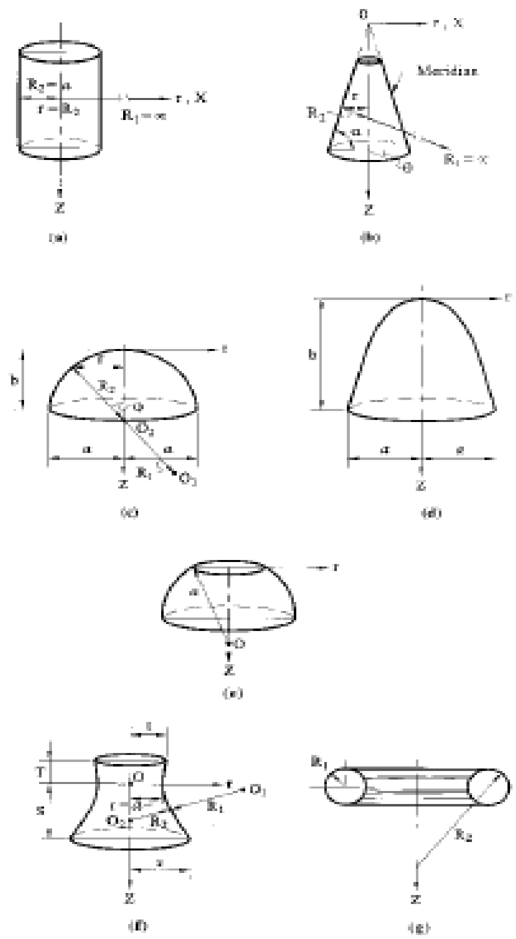

Fig. 11.9

Fig. 11.10

shown in Fig. 11.10 as an example of such a type of surfaces. It is obtained by translation of a parabola on another parabola; both parabolas have their curvatures in the same direction. Therefore, this shell has a positive Gaussian curvature. For this surface sections x ¼ const and y ¼ const are parabolas, whereas a section z ¼ const represents an ellipse: hence its name, ‘‘elliptic paraboloid.’’

(c) Ruled surfaces (Fig.11.11)

Ruled surfaces are obtained by the translation of straight lines over two end curves (Fig. 11.11). The straight lines are not necessarily at right angles to the planes containing the end curves. The frustum of a cone can thus be considered as a ruled surface, since it can be generated by translation of a straight line (the generator) over two curves at its ends. It is also, of course, a shell of revolution. The hyperboloid of revolution of one sheet, shown in Fig. 11.11a, represents another example of ruled surfaces. It can be generated also by the translation of a straight line over two circles at its ends. Figure 11.11b shows a surface generated by a translation of a straight line on a circular curve at one end andon a straight line at the other end. Such surfaces are referred to as conoids. Both surfaces shown in Fig. 11.11 have negative Gaussian curvatures. 11.7.2 Classification based on shellcurvature

Classification based on shell curvature

(a) Singly curvedshells

These shells have a zero Gaussian curvature. Some shells of revolution (circular cylinders, cones), shells of translation, or ruled surfaces (circular or noncircular cylinders and cones) are examples of singly curved shells.

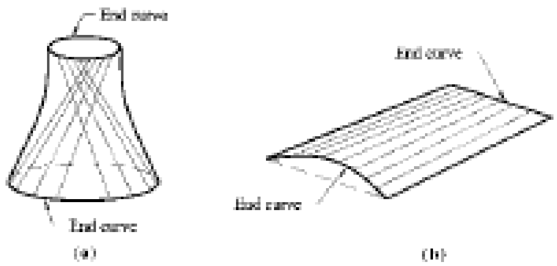

Fig. II.11

(b) Doubly curved shells of positive Gaussiancurvature

Some shells of revolution (circular domes, ellipsoids and paraboloids of revolution) and shells of translation and ruled surfaces (elliptic paraboloids, paraboloids of revolution) can be assigned to this category ofsurfaces.

(c) Doubly curved shells of negative Gaussiancurvature

This category of surfaces consists of some shells of revolution (hyperboloids of revolution of one sheet) and shells of translation or ruled surfaces (paraboloids,conoids, hyperboloids of revolution of one sheet).It is seen from this classification that the same type of shell may appear in more than onecategory.

Classification based on geometrical develop ability

(a) Developablesurfaces

Developable surfaces are defined as surfaces that can be ‘‘developed’’ into a plane form without cutting and/or stretching their middle surface. All singly curved surfaces are examples of developable surfaces.

(b) Non-developablesurfaces

A non-developable surface is a surface that has to be cut and/or stretched in order to be developed into a planar form. Surfaces with double curvature are usually nondevelopable. The classification of shell surfaces into developable and non-developable has a certain mechanical meaning. From a physical point of view, shells with non-developable surfaces require more external energy to be deformed than do developable shells, i.e., to collapse into a plane form. Hence, one may conclude that non-developable shells are, in general, stronger and more stable than the correspondingdevelopable

shells having the same overall dimensions. On the other hand, developable shells have some advantages associated with their technological effectiveness.

SPECIALIZATION OF SHELL GEOMETRY

It is shown in the next chapter that the governing equations and relations of the general theory of thin shells are formulated in terms of the Lame´ parameters A and B as well as of the principal curvatures _1 ¼ 1=R1 and _2 ¼ 1=R2. In the general case of shells having an arbitrary geometry of the middle surface, the coefficients of the first and second quadratic forms and the principal curvatures are some functions of the curvilinear coordinates. We determine the Lame´ parameters for some shell geometries that are commonly encountered in engineering practice

Shells of revolution

The shells of revolution were discussed in Secs 11.2 and 11.7. As for the curvilinear coordinate lines and , the meridians and parallels may be chosen: they are the lines of principal curvatures and form an orthogonal mesh on the shell middle surface. Figure 11.12a shows a surface of revolution where R1 is the principal radius of the meridian, R2 is the principal radius of the parallel circle (as shown in Sec.11.2, R2 is the distance along a normal to the meridional curve drawn from a point of interest to the axis of revolution of the surface), and r is the radius of the parallelcircle.

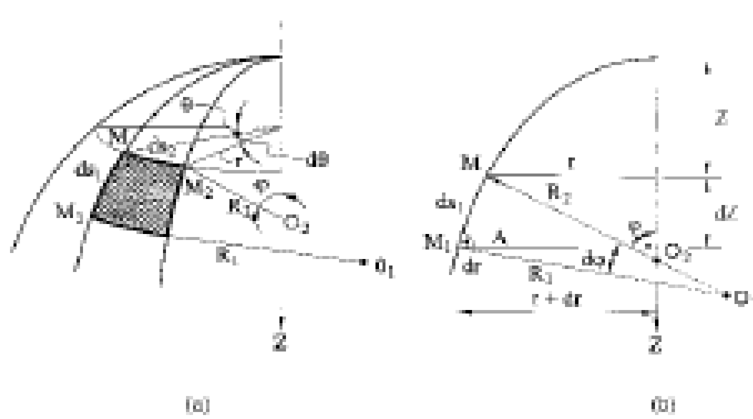

Fig. 11.12

There are several possibilities for a choice of thecurvilinear coordinates and .b

The overall goal is to be able to design reinforced concrete structures thatare:

- Safe
- Economical
- Efficient

Reinforced concrete is one of the principal building materials used in engineered structures because:

- Low cost
- Weathering and fire resistance
- Good compressivestrength
- Formability

- identify the regions where the beam shall be designed as a flanged and where it will be rectangular in normal slab beamconstruction,
- define the effective and actual widths of flangedbeams,
- state the requirements so that the slab part is effectively coupled with the flanged beam,
- write the expressions of effective widths of *T* and *L*-beams both for continuous and isolatedcases,
- derive the expressions of *C, T* and $M_u$for four different cases depending on the location of the neutral axis and depth of theflange.

Reinforced concrete slabs used in floors, roofs and decks are mostly cast monolithic from the bottom of the beam to the top of the slab. Such rectangular beams having slab on top are different from others having either no slab (bracings of elevated tanks, lintels etc.) or having disconnected slabs as in some pre-cast systems (Figs. 5.10.1 a, b and c). Due to monolithic casting, beams and a part of the slab act together. Under the action of positive bending moment, i.e., between the supports of a continuous beam, the slab, up to a certain width greater than the width of the beam, forms the top part of the beam. Such beams having slab on top of the rectangular rib are designated as the flanged beams - either *T* or *L* type depending on whether the slab is on both sides or on one side of the beam (Figs. 5.10.2 a to e) . Over the supports of a continuous beam, the bending moment is negative and the slab, therefore, is in tension while a part of the rectangular beam (rib) is in compression. The continuous beam at support is thus equivalent to a rectangular beam (Figs. 5.10.2 a, c, f and g).

Loads

Loads that act on structures can be divided into three general categories:

Dead Loads

(i) Sufficient development length of the reinforcement shall be provided to transferthe compression or tension to the supporting member in accordance with cl.26.2 of IS 456, when transfer of force is accomplished by reinforcement of column (cl.34.4.2 of IS456).

(ii) Minimum area of extended longitudinal bars or dowels shall be 0.5 per cent ofthe cross-sectional area of the supported column or pedestal (cl.34.4.3 of IS456).

(iii) A minimum of four bars shall be provided (cl.34.4.3 of IS 456).

(iv) The diameter of dowels shall not exceed the diameter of column bars by more than 3 mm.

(v) Column bars of diameter larger than 36 mm, in compression only can be doweled at thefootingswithbarsofsmallersizeofthenecessaryarea.Thedowelshallextendinto

the column, a distance equal to the development length of the column bar and into the footing, a distance equal to the development length of the dowel, as stipulated in cl.34.4.4 of IS 456.

Clause 34.5.1 of IS 456 stipulates the minimum reinforcement and spacing of the bars in footing slabs as per the requirements of solid slab (cls.26.5.2.1 and 26.3.3b(2) of IS 456, respectively).

The staircase is an important component of a building, and often the only means of access between the various floors in the building. It consists of a *flight* of steps, usually with one or more intermediate *landings* (horizontal slab platforms) provided between the floor levels. The horizontal top portion of a step (where the foot rests) is termed *tread* and the vertical projection of the step (i.e., the vertical distance between two neighbouring steps) is called *riser* [Fig. 2.10]. Values of 300 mm and 150 mm are ideally assigned to the tread and riser respectively — particularly in public buildings.

However, lower values of tread (up to 250 mm) combined with higher values of riser (up to 190 mm) are resorted to in residential and factory buildings. The *width* of the stair is generally around 1.1 – 1.6*m*, and in any case, should normally not be less than 850 mm; large stair widths are encountered in entrances to public buildings. The horizontal projection (plan) of an inclined flight of steps, between the first and last risers, is termed *going*. A typical flight of steps consists of two landings and one going, as depicted in Fig. 2.10(a). Generally, risers in a flight should not exceed about 12 in number. The steps in the flight can be designed in a number of ways: with *waist slab*, with *tread-riser* arrangement (without waist slab) or with *isolated tread slabs* — as shown in Fig. 2.10(b), (c), (d)respectively.

Geometrical Configurations

A wide variety of staircases are met with in practice. Some of the more common geometrical configurations are depicted in Fig. 2.11. These include:

• straight stairs (with or without intermediate landing) [Fig. 2.11(a)]

• quarter-turn stairs [Fig. 2.11(b)]

• dog-legged stairs [Fig. 2.11 (c)]

open well stairs [Fig. 2.11(d)]

• spiral stairs [Fig. 2.11(e)]

• helicoidal stairs [Fig. 2.11(f)]

Design Considerations

(a) Minimum nominalcover

The minimum nominal cover for the footings should be more than that of other structural elements of the superstructure as the footings are in direct contact with the soil. Clause 26.4.2.2 of IS 456 prescribes a minimum cover of 50 mm for footings. However, the actual cover may be even more depending on the presence of harmful chemicals or minerals, water table etc.

(b) Thickness at theedge

The minimum thickness at the edge of reinforced and plain concrete footings shall be at least 150 mm for footings on soils and at least 300 mm above the top of piles for footings on piles, as per the stipulation in cl.34.1.2 of IS 456.

footings with bars of smaller size of the necessary area.

The dowel shall extend into the column, a distance equal to the development length of the column bar and into the footing, a distance equal to the development length of the dowel, as stipulated in cl.34.4.4 of IS456.

Clause 34.5.1 of IS 456 stipulates the minimum reinforcement and spacing of the bars in footing slabs as per the requirements of solid slab (cls.26.5.2.1 and 26.3.3b(2) of IS 456, respectively).

The staircase is an important component of a building, and often the only means of access between the various floors in the building. It consists of a *flight* of steps, usually with one or more intermediate *landings* (horizontal slab platforms) provided between the floor levels. The horizontal top portion of a step (where the foot rests) is termed *tread* and the vertical projection of the step (i.e., the vertical distance between two neighbouring steps) is called *riser* [Fig. 2.10]. Values of 300 mm and 150 mm are ideally assigned to the tread and riser respectively — particularly in public buildings. However, lower values of tread (up to 250 mm) combined with higher values of riser (up to 190 mm) are resorted to in residential and factory buildings. The *width* of the stair is generally around 1.1 – 1.6*m*, and in any case, should normally not be less than 850 mm; large stair widths are encountered in entrances to public buildings. The horizontal projection (plan) ofan

inclined flight of steps, between the first and last risers, is termed *going*. A typical flight of steps consists of two landings and one going, as depicted in Fig. 2.10(a). Generally, risers in a flight should not exceed about 12 in number. The steps in the flight can be designed in a number of ways: with *waist slab*, with *tread-riser* arrangement (without waist slab) or with *isolated tread slabs* — as shown in Fig. 2.10(b), (c), (d) respectively.

Geometrical Configurations

A wide variety of staircases are met with in practice. Some of the more common geometrical configurations are depicted in Fig. 2.11. These include:

• straight stairs (with or without intermediate landing) [Fig. 2.11(a)]

• quarter-turn stairs [Fig. 2.11(b)]

• dog-legged stairs [Fig. 2.11 (c)]

open well stairs [Fig. 2.11(d)]

• spiral stairs [Fig. 2.11(e)]

•

• helicoidal stairs [Fig. 2.11(f)]

Design Considerations

(a) Minimum nominalcover

The minimum nominal cover for the footings should be more than that of other structural elements of the superstructure as the footings are in direct contact with the soil. Clause 26.4.2.2 of IS 456 prescribes a minimum cover of 50 mm for footings. However, the actual cover may be even more depending on the presence of harmful chemicals or minerals, water table etc.

(b) Thickness at theedge

The minimum thickness at the edge of reinforced and plain concrete footings shall be at least 150 mm for footings on soils and at least 300 mm above the top of piles for footings on piles, as per the stipulation in cl.34.1.2 of IS 456.

For plain concrete pedestals, the angle α (see Fig.11.28.1) between the plane passing through the bottomedge of thepedestalandthe corresponding junctionedge of the column with pedestal and the horizontal plane shall be determined from the following expression (cl.34.1.3 of IS 456)

0.5 tan0.9{(100/) 1} a ck qf□ □ □

whereq_a= calculated maximum bearing pressure at the base of pedestal in N/mm2, and

f_{ck}= characteristic strength of concrete at 28 days in N/mm2.

(c) Bendingmoments

1. It may be necessary to compute the bending moment at several sections of the footing depending on the type of footing, nature of loads and the distribution of pressure at the base of the footing. However, bending moment at any section shall be determined taking all forces acting over the entire area on one side of the section of the footing, which is obtained by passing a vertical plane at that section extending across the footing (cl.34.2.3.1 of IS 456).

2. The critical section of maximum bending moment for the purpose of designing an isolated concrete footing which supports a column, pedestal or wall shallbe:

(i) at the face of the column, pedestal or wall for footing supporting a concrete column, pedestal or reinforced concrete wall,and

(ii) halfway between the centre-line and the edge of the wall, for footing under masonry wall. This is stipulated in cl.34.2.3.2 of IS 456.

The maximum moment at the critical section shall be determined as mentioned in 1 above.

For round or octagonal concrete column or pedestal, the face of the column or pedestal shall be taken as the side of a square inscribed within the perimeter of the round or octagonal column or pedestal (see cl.34.2.2 of IS 456 and Figs.11.28.13a and b).

(d) Shearforce

Footing slabs shall be checked in one-way or two-way shears depending on the nature of bending. If the slab bends primarily in one-way, the footing slab shall be checked in one-way vertical shear. On the other hand, when the bending is primarily two- way, the footing slab shall be checked in two-way shear or punching shear. The respective critical sections and design shear strengths are givenbelow:

One-way shear has to be checked across the full width of the base slab on a vertical section located from the face of the column, pedestal or wall at a distance equal to

(i) effective depth of the footing slab in case of footing slab on soil,and

(ii) half the effective depth of the footing slab if the footing slab is onpiles.

The design shear strength of concrete without shear reinforcement is given in Table 19 of cl.40.2 of IS 456.

punching shear shall be checked around the column on a perimeter half the effective depth of the footing slab away from the face of the column or pedestal.

The permissible shear stress, when shear reinforcement is not provided, shall not exceed , where *ks*= (0.5 + *c*β), but not greater than one, *c*β being the ratio of short side to long sideofthecolumn,and=0.25(*fck*)1/2inlimitstatemethodofdesign,asstipulatedin

cl.31.6.3 of IS 456. sc k□ c □

Normally, the thickness of the base slab is governed by shear. Hence, the necessary thickness of the slab has to be provided to avoid shear reinforcement.

Reinforcement in the central band = {2/(β+1)} (Total reinforcement in the short direction)
Where β is the ratio of longer dimension to shorter dimension of the footing slab (Fig.3.10).
Each of the two end bands shall be provided with half of the remainingreinforcement, distributed
uniformly across the respective end band.

All forces and moments acting at the base of the column must be transferred to thepedestal,
if any, and then from the base of the pedestal to the footing, (or directly from the base of the column to the footing if there is no pedestal) by compression in concrete and steel and tension
in steel. Compression forces are transferred through direct bearing while tension forces are transferred through developed reinforcement. The permissible bearing stresses on full areaof
concrete shall be taken as given below from cl.34.4
□ br= 0.45fck , in limit state method The
stress of concrete is taken as 0.45fck
while designing the column. Since the area of
footing is much larger, this bearing stress of concrete in column may be increased considering the dispersion of the concentrated load of column to footing. Accordingly, the permissible bearing stress of concrete in footing is given by (cl.34.4 of IS 456):
□ br= 0.45fck($A$1/$A$2)1/2
with a condition that
($A$1/$A$2)1/22.0 (11.8) ≤ 2
where$A$1= maximum supporting area of footing for bearing which is geometrically imilar to and concentric with the loaded area $A$2
.$A$2= loaded area at the base of the column.

The above clause further stipulates that in sloped or stepped footings, $A$1 may be taken as the area of the lower base of the largest frustum of a pyramid or cone contained wholly within the footing and having for its upper base, the area actually loaded and having side slope of one vertical to two horizontal.

If the permissible bearing stress on concrete in column or in footing is exceeded, reinforcement shall be provided for developing the excess force (cl.34.4.1 of IS 456), either by extending the longitudinal bars of columns into the footing (cl.34.4.2 of IS 456) or by providing dowels as stipulated in cl.34.4.3 of IS 456 and given below:

(i) A doubly reinforced concrete beam is reinforced in both compression and tension faces.

4.8.2 When depth of beam is restricted, strength available from a singly reinforced beam isinadequate.

4.8.3 At a support of a continuous beam, the bending moment changes sign, such a situation may also arise in design of a ringbeam.

1 Analysis of a doubly reinforced section involves determination of moment of resistance with given beam width, depth, area of tension and compression steels and theircovers.

2 In doubly reinforced concrete beams the compressive force consists of two parts; both in concrete and steel incompression.

3 Stress in steel at the limit state of collapse may be equal to yield stress or less depending on position of the neutralaxis.

Design Steps

Determine the limiting moment of resistance M_{um} for the given cross-section using the equation for a singly reinforced beam

$$M_{lim} = 0.87f_y.A_{st,1} [d - 0.42x_m] = 0.36 f_{ck}.b.x_m [d - 0.42x_m]$$

(i) If the factored moment M_u exceeds M_{lim}, a doubly reinforced section is required(M_u - M_{lim}) = M_{u2}

Additional area of tension steel A_{st2} is obtained by considering the equilibrium of force of compression in comp. steel and force of tension T_2 in the additional tension steel

A_{sc} = compression steel.

σ_{cc} = Comp. stress in conc at the level of comp. steel = 0.446fck.

Reasons

(ii) When beam section is shallow in depth, and the flexural strength obtained using balanced steel is insufficient i.e. the factored moment is more than thelimiting ultimate moment of resistance of the beam section. Additional steel enhances the moment capacity.

(iii) Steel bars in compression enhances ductility of beam at ultimate strength. Compression steel reinforcement reduces deflection as moment of inertia of the beam section alsoincreases.

(iv) Long-term deflections of beam are reduced by compressionsteel.

(v) Curvature due to shrinkage of concrete are alsoreduced.

(vi) Doubly reinforced beams are also used in reversal of externalload

Examples

(i) A single reinforced rectangular beam is 400mm wide. The effective depth of the beam section is 560mm and its effective cover is 40mm. The steel reinforcement consists of 4 MS 18mm diameter bars in the beam section. The grade of concrete is M20. Locate the neutral axis of the beamsection.

(ii) In example 1, the bending moment at a transverse section of beam is 105 kN-m. Determine the strains at the extreme fibre of concrete in compression and steel bars

provided as reinforcement in tension. Also determine the stress in steelbars.

(iii) *In example 2, the strain in concrete at the extreme fibre in compression ε_{cu} is 0.00069 and the tensile stress in bending in steel is 199.55 N/mm^2. Determine the depth of neutral axis and the moment of resistance of the beamsection.*

(iv) *Determine the moment of resistance of a section 300mm wide and 450mm deep up to the centre of reinforcement. If it is reinforced with (i) 4-12mm fe415 grade bars, (ii) 6-18mm fe415 gradebars.*

(i) A rectangular beam section is 200mm wide and 400mm deep up to the centre of reinforcement. Determine the reinforcement required at the bottom if it has to resist a factored moment of 40kN-m. Use M20 grade concrete and fe415 gradesteel.

(ii) A rectangular beam section is 250mm wide and 500mm deep up to the centre of tension steel which consists of 4-22mm dia. bars. Find the position of the neutral axis, lever arm, forces of compression and tension and safe moment of resistance if concrete is M20 grade and steel is Fe500 grade.

(iii) A rectangular beam is 200mm wide and 450 mm overall depth with an effective cover of 40mm. Find the reinforcement required if it has to resist a moment of 35 kN.m. Assume M20 concrete and Fe250 gradesteel.

- explain the need to check for the limit state of serviceability after designing the structures by limit state ofcollapse,
- differentiate between short- and long-termdeflections,
- state the influencing factors to both short- and long-termdeflections,
- select the preliminary dimensions of structures to satisfy the requirements as per IS456,
- calculate the short- and long-term deflections of designedbeams.

Introduction

Structures designed by limit state of collapse are of comparatively smaller sections than those designed employing working stress method. They, therefore, must be checked for deflection and width of cracks. Excessive deflection of a structure or part thereof adversely affects the appearance and efficiency of the structure, finishes or partitions. Excessive cracking of concrete also seriously affects the appearance and durabilityofthestructure.Accordingly,cl.35.1.1ofIS456stipulatesthatthedesigner

should consider all relevant limit states to ensure an adequate degree of safety and serviceability. Clause 35.3 of IS 456 refers to the limit state of serviceability comprising deflection in cl. 35.3.1 and cracking in cl. 35.3.2. Concrete is said to be durable when it performs satisfactorily in the working environment during its anticipated exposure conditions during service. Clause 8 of IS 456 refers to the durability aspects of concrete. Stability of the structure against overturning and sliding (cl. 20 of IS

456), and fire resistance (cl. 21 of IS 456) are some of the other importance issues to be kept in mind while designing reinforced concrete structures.

This lesson discusses about the different aspects of deflection of beams and the requirements as per IS 456. In addition, lateral stability of beams is also taken up while selecting the preliminary dimensions of beams. Other requirements, however, are beyond the scope of this lesson.

Short- and Long-term Deflections

As evident from the names, short-term deflection refers to the immediate deflection after casting and application of partial or full service loads, while the long-term deflection occurs over a long period of time largely due to shrinkage and creep of the

materials. The following factors influence the short-term deflection of structures:

(ii) magnitude and distribution of liveloads,

(iii) span and type of endsupports,

(iv) cross-sectional area of themembers,

(v) amount of steel reinforcement and the stress developed in the reinforcement,

(vi) characteristic strengths of concrete and steel,and

(vii) amount and extent ofcracking.

The long-term deflection is almost two to three times of the short-term deflection. The following are the major factors influencing the long-term deflection of the structures.

4.8.4 humidity and temperature ranges duringcuring,

4.8.5 age of concrete at the time of loading,and

(c) type and size of aggregates, water-cement ratio, amount of compression reinforcement, size of members etc., which influence the creep and shrinkage of concrete.

Control of Deflection

Clause 23.2 of IS 456 stipulates the limiting deflections under two heads as given below:

The maximum final deflection should not normally exceed span/250 due to all loads including the effects of temperatures, creep and shrinkage and measured from the as-cast level of the supports of floors, roof and all other horizontal members.The maximum deflection should not normally exceed the lesser of span/350 or 20 mm including the effects of temperature, creep and shrinkage occurring after erection of partitions and the application of finish. It is essential that both the requirements are to be fulfilled for every structure.

Selection of Preliminary Dimension.

The two requirements of the deflection are checked after designing the members. However, the structural design has to be revised if it fails to satisfy any one of the two or both the requirements. In order to avoid this, IS 456 recommends the guidelines to assume the initial dimensions of the members which will generally satisfy the deflection limits.

Clause 23.2.1 stipulates different span to effective depth ratios and cl. 23.3 recommends limiting slenderness of

beams, a relation of b and d of the members, to ensure lateral stability. They are given below:

For the deflection requirements

Different basic values of span to effective depth ratios for three different support conditions are prescribed for spans up to 10 m, which should be modified under any or all of the four different situations: (i) for spans above 10 m, (ii) depending on the amount and the stress of tension steel reinforcement, (iii) depending on the amount of compression reinforcement, and (iv) for flanged beams. These are furnished in Table7.1.

(B) For lateralstability

The lateral stability of beams depends upon the slenderness ratio and the support conditions. Accordingly cl. 23.3 of IS code stipulates the following:

For simply supported and continuous beams, the clear distance between the lateral restraints shall not exceed the lesser of 60b or 250b2/d, where d is the effective depth and b is the breadth of the compression face midway between the lateral restraints.

For cantilever beams, the clear distance from the free end of the cantilever to the lateral restraint shall not exceed the lesser of 25b or 100b2/d.

Calculation of Short-Term Deflection

Clause C-2 of Annex C of IS 456 prescribes the steps of calculating the short-term deflection. The code recommends the usual methods for elastic deflections using the short-term modulus of elasticity of concrete Ec and effective moment of inertia Ieff given by the following equation:
Mr = cracking moment equal to (fcrIgr)/yt , where fcr is the modulus of rupture of concrete, Igr is the moment of inertia of the gross section about the centroidal axis neglecting the reinforcement, and yt is the distance from centroidal axis of gross section, neglecting the reinforcement, to extreme fibre in tension,

M = maximum moment under service loads, z
= leverarm,
x = depth of neutralaxis,

d = effectivedepth,

bw= breadth of web, and

b = breadth of compressionface.

For continuous beams, however, the values of Ir, Igr and Mr are to be modified by the following equation:

Figure 3.3 presents the three modes of failure of columns with different slenderness ratios when loaded axially. In the mode 1, column does not undergo any lateral deformationand collapses due to material failure. This is known as compression failure. Due to the combined effects of axial load and moment a short column may have material failure of mode 2. On the other hand, a slender column subjected to axial load only undergoes deflection due to beam-column effect and may have material failure under the combined action of direct load and bending moment. Such failure is called combined compression and bending failure of mode 2. Mode 3 failure is by elastic instability of very long column even under small load much before the material reaches the yield stresses. This type of failure is known as elasticbuckling.

The slenderness ratio of steel column is the ratio of its effective length le to its least radius of gyration r. In case of reinforced concrete column, however, IS 456 stipulates the slenderness ratio as the ratio of its effective length le to its least lateral dimension. As mentioned earlier in sec. 3.1(a), the effective length le is different from the unsupported length, the rectangular reinforced concrete column of cross-sectional dimensions b and D shall have two effective lengths in the two directions of b and D. Accordingly, the column may have the possibility of buckling depending on the two values of slenderness ratios as given below:

Slenderness ratio about the major axis = lex/D

Slenderness ratio about the minor axis = ley/b

Factored concentric load applied on short tied columns is resisted by concrete of area Ac and longitudinal steel of areas Asc effectively held by lateral ties at intervals. Assuming the design strengths of concrete and steel are 0.4fck and 0.67fy, respectively, we can write

Pu = 0.4fck Ac + 0.67fy Asc (1)

Where Pu = factored axial load on the member,

fck = characteristic compressive strength of the concrete, Ac

= area of concrete,

fy = characteristic strength of the compression reinforcement, and

Asc = area of longitudinal reinforcement for columns.

The above equation, given in cl. 39.3 of IS 456, has two unknowns Ac and Asc to be determined from one equation. The equation is recast in terms of Ag, the gross area of concrete and p, the percentage of compression reinforcement employing

$Asc = pAg/100$ (2)

$Ac = Ag(1 – p/100)$ (3)

Accordingly, we can write

$Pu/Ag = 0.4fck + (p/100) (0.67fy – 0.4fck)$ (4)

Equation 4 can be used for direct computation of Ag when Pu, fck and fy are known by assuming p ranging from 0.8 to 4 as the minimum and maximum percentages of longitudinal reinforcement. Equation 10.4 also can be employed to determine Ag and p in a similar manner by assuming p.

The superstructure is placed on the top of the foundation structure, designated as substructure as they are placed below the ground level. The elements of the superstructure transfer the loads and moments to its adjacent element below it and finally all loads and moments come to the foundation structure, which in turn, transfers them to the underlying soil or rock. Thus, the foundation structure effectively supports the superstructure. However, all types of soil get compressed significantly and cause the structure to settle. Accordingly, the major requirements of the design of foundation structures are the two as given below (see cl.34.1 of IS 456)

1. Foundation structures should be able to sustain the applied loads, moments, forcesand induced reactions without exceeding the safe bearing capacity of thesoil.

2. The settlement of the structure should be as uniform as possible and it should bewithin the tolerable limits. It is well known from the structural analysis that differential settlement of supports causes additional moments in statically indeterminate structures. Therefore, avoiding the differential settlement is considered as more important than maintaining uniform overall settlement of thestructure.

UNIT – V
SHELLS OF REVOLUTION: WITH BENDING RESISTANCE

INTRODUCTION

As mentioned, shells of revolution belong to a highly general class of shells frequently used in ngineering. One representative of this class, cylindrical shells, was considered in Chapter 15, and we will not dwell on these shells. The shell types analyzed in this chapter are subclasses of shells of revolution having non-zero Gaussian curvature. As mentioned in Sec. 11.7, such shells have non-developable surfaces. Hence, they are stronger, stiffer, and more stable than shells with zero Gaussian curvature. These shells are frequently used to cover the roofs of sport halls and large liquid storage tanks. The containment shield structures of nuclear power plants also have dome-like roofs. Various pressure vessels are either completely composed of a single rotational shell or have shells of revolution at their end caps. Conical shells with zero Gaussian curvature are also representative of this class of shells: they are used to cover liquid storage tanks and the nose cones of missiles androckets.

In the membrane analysis of shells of revolution considered in Chapters 13 and 14, we saw that the membrane theory alone cannot accommodate all the loads, support conditions, and geometries in actual shells. Thus, in a general case, shells of revolution experience both stretching and bending to resist an applied loading, which distinguishes significantly the bending of shells from the elementary behavior of plates (see also Sec. 10.4).

However, the character of bending deformation may be different. If a shell of revolution is subjected to a concentrated force (Fig. 16.1a), bending exerts a crucial effect on its strength, because, in this case, the bending deformation increases with a growth of the forces until the load-carrying capacity of the shell structure is exhausted. In places of junction of a shell with its supports (Fig. 16.1b) or other structural members (shell of another geometry, ring beam, etc.), or in places of jump change in the radii of curvature (Fig. 16.1c), the bending has another character; here,bending

propagates only if it is needed to eliminate the discrepancies between the membrane displacements or to satisfy the conditions of statics. If a shell material is ductile, the bending deformations of the latter type are usually decreased and do not practically influence the load-carrying capacity of shell structures. If the material of the shell is brittle, the bending deformations remain proportional to the applied loads until failure and can result in a significant decrease in the strength of the shell structure

In this chapter we consider the bending theory of shells of revolution. It should be noted that the solutions of the governing differential equations involve many difficulties for a general shell of revolution, and therefore, we solve these equations for some particular shell geometries and load configurations that are frequently used in engineeringpractice.

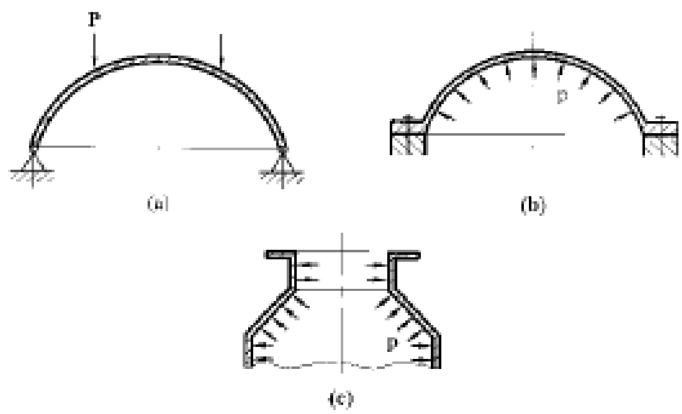

Fig. 16.1

Dead loads are those that are constant in magnitude and fixed in location throughout the lifetime of the structure such as: floor fill, finish floor, and plastered ceiling for buildings and wearing surface, sidewalks, and curbing for bridges

Live Loads

Live loads are those that are either fully or partially in place or not present at all, may also change in location; the minimum live loads for which the floors and roof of a building should be designed are usually specified in building code that governs at the site of construction (see Table1 - "Minimum Design Loads for Buildings and Other Structure.")

Environmental Loads

Environmental Loads consist of wind, earthquake, and snow loads. such as wind, earthquake, and snow loads.

Serviceability

Serviceability requires that

• Deflections be adequatelysmall;

• Cracks if any be kept to a tolerablelimits;

• Vibrations beminimized

Safety

A structure must be safe against collapse; strength of the structure must be dequate for all loads that might act on it. If we could build buildings as designed, and if the loads and their internal effects can be predicted accurately, we do not have to worry about safety. But there are uncertainties in:

- Actual loads;
- Forces/loads might be distributed in a manner different from what weassumed;
- The assumptions in analysis might not be exactlycorrect;
- Actual behavior might be different from thatassumed;
- etc.

Finally, we would like to have the structure safe against

Concrete

Concrete is a product obtained artificially by hardening of the mixture of cement, sand, gravel and water in predetermined proportions.

Depending on the quality and proportions of the ingredients used in the mix the properties of concrete vary almost as widely as different kinds of stones.

Concrete has enough strength in compression, but has little strength in tension. Due to this, concrete is weak in bending, shear and torsion. Hence the use of plain concrete is limited applications where great compressive strength and weight are the principal requirements and where tensile stresses are either totally absent or are extremely low.

Properties of Concrete

The important properties of concrete, which govern the design of concrete mix are as follows

(i) Weight

The unit weights of plain concrete and reinforced concrete made with sand, gravel of crushed natural stone aggregate may be taken as 24 KN/m3 and 25 KN/m3 respectively.

(ii) CompressiveStrength

With given properties of aggregate the compressive strength of concrete depends primarily on age, cement content and the water cement ratio are given Table 2 of IS 456:2000. Characteristic strength are based on the strength at 28 days. The strength at 7 days is about two-thirds of that at 28 days with ordinary portland cement and generally good indicator of strength likely to be obtained.

(iii) Increase in strength withage

There is normally gain of strength beyond 28 days. The quantum of increase depends upon the grade and type of cement curing and environmental conditionsetc.

(iv) Tensile strength ofconcrete

The flexure and split tensile strengths of various concrete are given in IS 516:1959 and IS 5816:1970 respectively when the designer wishes to use an estimate of the tensile strength from compressive strength, the following formula can be used

Flexural strength, $f_{cr}=0.7\sqrt{f_{ck}}$ N/mm2

(v) ElasticDeformation

The modulus of elasticity is primarily influenced by the elastic properties of the aggregate and to lesser extent on the conditions of curing and age of the concrete, the mix proportions and the type of cement. The modulus of elasticity is normally related to the compressive characteristic strength of concrete

$E_c=5000\sqrt{f_{ck}}$ N/mm2

Where E_c= the short-term static modulus of elasticity inN/mm2

f_{ck}=characteristic cube strength of concrete inN/mm2

(vi) Shrinkage of concrete

Shrinkage is the time dependent deformation, generally compressive in nature. The constituents of concrete, size of the member and environmental conditions are the factors on which the total shrinkage of concrete depends. However, the total shrinkage of concrete is most influenced by the total amount of water present in the concrete at the time of mixing for a given humidity and temperature. The cement content, however, influences the total shrinkage of concrete to a lesser extent. The approximate value of the total shrinkage strain for design is taken as 0.0003 in the absence of test data (cl.6.2.4.1).

(vii) Creep ofconcrete

The effective modulus of *Ece*of concrete is used only in the calculation of creep deflection. It is seen that the value of creep coefficient θ is reducing with the age of concrete at loading. It may also be noted that the ultimate creep strain does not include short term strain.

• Properties ofconcrete

• Water/cement ratio

• Humidity and temperature ofcuring

• Humidity during the period ofuse

• Age of concrete at firstloading

• Magnitude of stress and its duration

• Surface-volume ratio of themember

Concrete has very good compressive strength and almost negligible tensile strength. Hence, steel reinforcement is used on the tensile side of concrete. Thus, singly reinforced beams reinforced on the tensile face are good both in compression and tension. However, these beams have their respective limiting moments of resistance with specified width, depth and grades of concrete and steel. The amount of steel reinforcement needed is known as $A_{st,lim}$. Problem will arise, therefore, if such a section is subjected to bending moment greater than its limiting moment of resistance as a singly reinforcedsection.

There are two ways to solve the problem. First, we may increase the depth of the beam, which may not be feasible in many situations. In those cases, it is possible to increase both the compressive and tensile forces of the beam by providing steel reinforcement in compression face and additional reinforcement in tension face of the beam without increasing the depth (Fig. 4.8.1). The total compressive force of such beams comprises (i) force due to concrete in compression and (ii) force due to steel in compression. The tensile force also has two components: (i) the first provided by $A_{st,lim}$which is equal to the compressive force of concrete in compression. The second part is due to the additional steel in tension - its force will be equal to the compressive force of steel incompression.

Such reinforced concrete beams having steel reinforcement both on tensile and compressive faces are known as doubly reinforced beams.

Doubly reinforced beams, therefore, have moment of resistance more than the singly reinforced beams of the same depth for particular grades of steel and concrete. In many practical situations, architectural or functional requirements may restrict the overall depth of the beams. However, other than in doubly reinforced beams compression steel reinforcement is providedwhen:

- some sections of a continuous beam with moving loads undergo change of sign of the bending moment which makes compression zone as tension zone or viceversa.

- the ductility requirement has to befollowed.

- the reduction of long term deflection isneeded.

It may be noted that even in so called singly reinforced beams there would be longitudinal hanger bars in compression zone for locating and fixing stirrups.

GOVERNING EQUATIONS

We present below the governing differential equations of the moment theory of shells of revolution of an arbitrary shape. As curvilinear coordinates and of a point on the shell middle surface, it is convenient to take the spherical coordinates, introduced in Sec. 11.8, and used in the membrane theory of shells of revolution in Chapters 13 and 14. Thus, wetake

¼ ' and ¼ _. As before, the angle ' defines the location of a point along the meridian, whereas _ characterizes the location of a point along the parallel circle (see Fig. 11.12). Let R1 and R2 be the principal radii of curvature of the meridian and parallel circle, respectively. Obviously, R1 and R2 will be functions of ' only, i.e., R1 ¼ R1ð'Þ and R2 ¼ R2ð'Þ. The Lame´ parameters in this case are determined by the following formulas (see Sec.11.8):

$$A = R_1(\varphi), \quad B = R_2(\varphi)\sin\varphi, \quad r = R_2 \sin\varphi. \qquad (16.1)$$

The Codazzi and Gauss conditions are given by Eqs (11.41). Let us consider the kinematic elations of the moment theory of shells of revolution. Displacement components of the middle surface along the given coordinate axes are u (in the meridional direction), v (in the circumferential direction), and w (in the normal direction to the middle surface). The strain–displacement relations (12.23) and (12.24) of the general shell theory – taking into account Eqs (16.1) and (11.41) – take the following form for shells of revolution

$$\varepsilon_1 = \frac{1}{R_1}\left(\frac{\partial u}{\partial \varphi} - w\right),$$
$$\varepsilon_2 = \frac{1}{R_2 \sin\varphi}\left(\frac{\partial v}{\partial \theta} + u\cos\varphi - w\sin\varphi\right),$$
$$\gamma_{12} = \frac{1}{R_1}\frac{\partial v}{\partial \varphi} - \frac{\cos\varphi}{R_2\sin\varphi}v + \frac{1}{R_2\sin\varphi}\frac{\partial u}{\partial \theta}, \qquad (16.2)$$
$$\chi_1 = -\frac{1}{R_1}\frac{\partial}{\partial \varphi}\left[\frac{1}{R_1}\left(u + \frac{\partial w}{\partial \varphi}\right)\right],$$
$$\chi_2 = -\frac{1}{(R_2\sin\varphi)^2}\left(\frac{\partial v}{\partial \theta}\sin\varphi + \frac{\partial^2 w}{\partial \theta^2}\right) - \frac{\cos\varphi}{R_1 R_2 \sin\varphi}\left(u + \frac{\partial w}{\partial \varphi}\right),$$
$$\chi_{12} = \frac{1}{R_2\sin\varphi}\left[\frac{\cos\varphi}{R_2\sin\varphi}\frac{\partial w}{\partial \theta} - \frac{1}{R_1}\frac{\partial^2 w}{\partial \theta \partial \varphi} - \frac{1}{R_1}\frac{\partial u}{\partial \theta} - \frac{\sin\varphi}{R_1}\frac{\partial v}{\partial \varphi} + \frac{\cos\varphi}{R_2}v\right], \qquad (16.3)$$

where "1 and "2 are in-plane meridional and circumferential strain components in the middle surface, _12 characterizes a shear of the middle surface; _1 and _2 represent the changes in curvature of the coordinate lines in the middle surface due to its bending, and _12 characterizes a twist of the middle surface. The rotations of the shell edges that coincide with the coordinate lines _ and ’, respectively, can be obtained from Eqs (12.2) using the relations (16.1) and (11.41). We obtain

$$\vartheta_1 = \frac{u}{R_1} + \frac{1}{R_1}\frac{\partial w}{\partial \varphi}, \quad \vartheta_2 = \frac{1}{R_2\sin\varphi}\left(\frac{\partial w}{\partial \theta} + v\sin\varphi\right). \qquad (16.4)$$

Equations of static equilibrium (12.41) and (12.42) with regard to relations (12.43), (11.41), and (16.2) take the form

$$\frac{\partial}{\partial \varphi}(N_1 R_2 \sin\varphi) + R_1\frac{\partial}{\partial \theta}\left(S - \frac{H}{R_1}\right) - N_2 R_1 \cos\varphi - Q_1 R_2 \sin\varphi$$
$$+ p_1 R_1 R_2 \sin\varphi = 0,$$
$$R_1\frac{\partial N_2}{\partial \theta} + \frac{\partial}{\partial \varphi}\left[R_2\sin\varphi\left(S - \frac{H}{R_2}\right)\right] + \left(S - \frac{H}{R_1}\right) R_1 \cos\varphi$$
$$- Q_2 R_1 \sin\varphi + p_2 R_1 R_2 \sin\varphi = 0,$$
$$\frac{\partial}{\partial \varphi}(Q_1 R_2 \sin\varphi) + \frac{\partial}{\partial \theta}(Q_2 R_1) + N_1 R_2 \sin\varphi + N_2 R_1 \sin\varphi + p_3 R_1 R_2 \sin\varphi = 0,$$
$$\frac{\partial}{\partial \varphi}(H R_2 \sin\varphi) + R_1\frac{\partial M_2}{\partial \theta} + H R_1 \cos\varphi - Q_2 R_1 R_2 \sin\varphi = 0,$$
$$R_1\frac{\partial H}{\partial \theta} + \frac{\partial}{\partial \varphi}(M_1 R_2 \sin\varphi) - M_2 R_1 \cos\varphi - Q_1 R_1 R_2 \sin\varphi = 0.$$
$$(16.5)$$

Taking into account the relations for the effective shear forces (in-plane and out-of-plane) introduced in Sec. 12.5, the following static quantities can be assigned on a boundary coinciding with the edge parallel circle $\varphi = \varphi_e$:

$$N_1, T_1 = S - \frac{2H}{R_2}, \quad M_1, \quad \text{and} \quad V_1$$

Where

$$V_1 = Q_1 + \frac{1}{R_2 \sin\varphi}\frac{\partial H}{\partial\theta} = \frac{1}{R_2 \sin\varphi}\left[2\frac{\partial H}{\partial\theta} + \frac{\partial(M_1 R_2 \sin\varphi)}{R_1 \partial\varphi} - M_2 \cos\varphi\right]. \quad (16.6)$$

The constitutive relations are found to be in the form of Eqs (12.45) and (12.46). The governing equations and relations introduced above allow one to determine the stress and displacement components that occur in a shell of revolution supported along its edges and subjected to given external loads. The unknown functions characterizing the state of stress and strain (deformations and stress resultants and couples) depend upon variables _ and ’ in Eqs (16.3), (16.5), and (12.45), (12.46). These unknowns are periodic functions of variable _ for shells of revolution. Thus, one can apply the separation of variables method for solving the governing differential equations of shells of revolution. This method implies that all loads, displacements, and stress resultants and couples may be represented in theform

$$f_i(\theta,\varphi) = \sum_{k=0}^{\infty} f_{ik}^{(s)} \cos k\theta + \sum_{k=1}^{\infty} f_{ik}^{(a)} \sin k\theta,$$
$$\zeta_i(\theta,\varphi) = \sum_{k=1}^{\infty} \zeta_{ik}^{(s)} \sin k\theta - \sum_{k=0}^{\infty} \zeta_{ik}^{(a)} \cos k\theta, \quad (16.7)$$

here by functions fi are meant the functions p1; p3; u; w; "1; "2; #1; _1; _2;N1;N2; M1; M2; and Q1, and by functions _i are meant the functions p2; v; _12; #2; S;H; and Q2. It can be seen that the functions having the superscripts s and a correspond to symmetric and skew- symmetric of the above-mentioned functions about a zero meridian, respectively. It is easily verified that the symmetric and skewsymmetric components of the displacements, stress resultants, etc., determined by the same system of equations. Therefore, we present the correspondingrelations and equations for functions having the superscript s only, without using this index. So, substituting the expressions (16.7) into the kinematic relations (16.2) and (16.3), equilibrium equations (16.5), and eliminating the shear forces, Q1 and Q2, we obtain some systems of ordinary differential equations for unknown functions of deformations, and stress resultants and couples. Note that the constitutive equations (12.45) and (12.46) remain unchanged. It is required only to provide all the quantities involving in these equations with the index k. Let us present theseequations:

Kinematic equations (strain–displacement relations)

$$
\begin{aligned}
\varepsilon_{1k} &= \frac{1}{R_1}\left(\frac{\mathrm{d}u_k}{\mathrm{d}\varphi} - w_k\right),\\
\varepsilon_{2k} &= \frac{1}{R_2 \sin\varphi}(kv_k + u_k\cos\varphi - w_k \sin\varphi),\\
\gamma_{12k} &= \frac{1}{R_1}\frac{\mathrm{d}v_k}{\mathrm{d}\varphi} - \frac{\cos\varphi}{R_2\sin\varphi}v_k - \frac{ku_k}{R_2\sin\varphi},\\
\chi_{1k} &= -\frac{1}{R_1}\frac{\mathrm{d}}{\mathrm{d}\varphi}\left[\frac{u_k}{R_1} + \frac{1}{R_1}\frac{\mathrm{d}w_k}{\mathrm{d}\varphi}\right],\\
\chi_{2k} &= -\frac{1}{(R_2\sin\varphi)^2}(kv_k\sin\varphi - k^2 w_k) - \frac{\cos\varphi}{R_1R_2\sin\varphi}\left(u_k + \frac{\mathrm{d}w_k}{\mathrm{d}\varphi}\right),\\
\chi_{12k} &= \frac{1}{R_2\sin\varphi}\left[-\frac{\cos\varphi}{R_2\sin\varphi}kw_k + \frac{k}{R_1}\frac{\mathrm{d}w_k}{\mathrm{d}\varphi} + \frac{k}{R_1}u_k - \frac{\sin\varphi}{R_1}\frac{\mathrm{d}v_k}{\mathrm{d}\varphi} + \frac{\cos\varphi}{R_2}v_k\right],\\
\vartheta_{1k} &= \frac{1}{R_1}\left(u_k + \frac{\mathrm{d}w_k}{\mathrm{d}\varphi}\right),\ \vartheta_{2k} = \frac{1}{R_2\sin\varphi}(-kw_k + v_k\sin\varphi)
\end{aligned}
\tag{16.8}
$$

Equations of static equilibrium

$$
\begin{aligned}
&\frac{\mathrm{d}}{d\varphi}(N_{1k}R_2\sin\varphi) + R_1kS_k - \frac{1}{R_1}\frac{\mathrm{d}}{\mathrm{d}\varphi}(M_{1k}R_2\sin\varphi) + M_{2k}\cos\varphi\\
&\quad - 2kH_k - N_{2k}R_1\cos\varphi + p_{1k}R_1R_2\sin\varphi = 0\\
&\quad - R_1kN_{2k} + \frac{1}{R_2\sin\varphi}\frac{\mathrm{d}}{\mathrm{d}\varphi}\left[S_k(R_2\sin\varphi)^2\right]\\
&\quad - \frac{1}{R_2}\left[R_1(-k)M_{2k} + 2R_2\sin\varphi\frac{\mathrm{d}H_k}{\mathrm{d}\varphi} + 2\cos\varphi H_k(R_1+R_2)\right]\\
&\quad + p_{2k}R_1R_2\sin\varphi = 0.\\
&\frac{N_{1k}}{R_1} + \frac{N_{2k}}{R_2} + \frac{1}{R_1R_2\sin\varphi}\left\{\frac{\mathrm{d}}{\mathrm{d}\varphi}\left[H_kk + \frac{1}{R_1}\frac{\mathrm{d}}{\mathrm{d}\varphi}(M_{1k}R_2\sin\varphi) - M_{2k}\cos\varphi\right]\right.\\
&\quad \left. + \frac{1}{R_2\sin\varphi}\left[\frac{k}{R_2\sin\varphi}\frac{\mathrm{d}}{\mathrm{d}\varphi}\left((R_2\sin\varphi)^2H_k\right) - k^2R_1M_{2k}\right]\right\} + p_{3k} = 0.
\end{aligned}
\tag{16.9}
$$

Equations (16.8) and (16.9), together with the constitutive equations (12.45) and (12.46) and proper boundary conditions, form the closed eight-order system of the ordinary differential equations for each kth harmonic of the expansion (16.7). This system of the governing equations describes the state of stress and strain for the general moment theory of shells of revolution having a meridian of an arbitrary shape. This system of ordinary equations may be solved by applying the standard numerical methods intended for a solution of ordinary differential equations and introduced, for example, in Refs. [1,2]. The finite element method can also be applied to the analysis of the state of stress and strain for shells of revolution of a general shape[3,4].

It should be noted that numerical difficulties associated with a solution of the differential equations (16.8) and (16.9) may be partially eliminated for some specific shapes of shells of revolution and loading.

As mentioned in sec. 4.8.1, the moment of resistance $M_u$of the doubly reinforced beam consists of (i) $M_{u,lim}$of singly reinforced beam and (ii) M_{u2}because of equal and opposite compression and tension forces ($C_2$and T_2) due to additional steel reinforcement on compression and tension faces of the beam

(Figs. 4.8.1 and 2). Thus, the moment of resistance $M_u$of a doubly reinforced beam is

The additional moment M_{u2}can be expressed in two ways (Fig. 4.8.2): considering (i) the compressive force C_2due to compression steel and (ii) the tensile force T_2due to additional steel on tension face. In both the equations, the lever arm is (d - d'). Thus, we have

Since the additional compressive force $C_2$is equal to the additional tensile force T_2, wehave

$A_{sc}(f_{sc}$ - $f_{cc}) = A_{st2}(0.87\, f_y)$

(4.6)

Any two of the three equations (Eqs. 4.4 - 4.6) can be employed to determine $A_{sc}$and A_{st2}.

It is seen that the values of $f_{sc}$and f_{cc}should be known before calculating A_{sc}. The following procedure may be followed to determine the value of $f_{sc}$and $f_{cc}$for the design type ofproblems (and not for analysing a given section).For

the design problem the depth of the neutral axis may be taken as $x_{u,max}$as shown in Fig. 4.8.2. From Fig. 4.8.2, the strain at the level of compression steel reinforcement ε_{sc}may be writtenas

The stress in compression steel $f_{sc}$is corresponding to the strain $\varepsilon_{sc}$of Eq. 4.9 and is determined for (a) mild steel and (b) cold worked bars Fe 415 and 500 as given below:

The strain at the design yield stress of 217.39N/mm^2(f_d = 0.87 f_y) is

0.0010869 (= 217.39/E_s). The $f_{sc}$is determined from the idealized stress-strain diagram of mild steel (Fig. 1.2.3 of Lesson 2 or Fig. 23B of IS 456) after computing the value of $\varepsilon_{sc}$from Eq. 4.9 asfollows:

(i) If the computedvalueof $\varepsilon_{sc} \leq 0.0010869$, $f_{sc} = \varepsilon_{sc} E_s = 2\ (10^5)\varepsilon_{sc}$

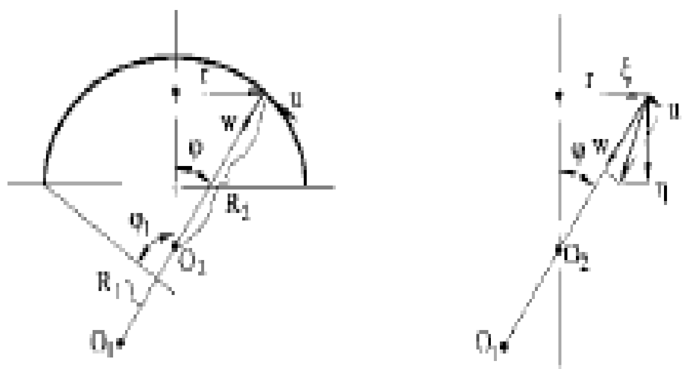

Fig. 16.2

The relative stretching in the circumferential direction is

$$\varepsilon_2 = \frac{\xi}{r} = \frac{\xi}{R_2 \sin\varphi}, \quad \text{from which } \xi = \varepsilon_2 R_2 \sin\varphi. \qquad \text{(a)}$$

Using the above relation (a) and Eqs (16.13), the second boundary condition (Eq. (16.38a)) can be rewritten as follows

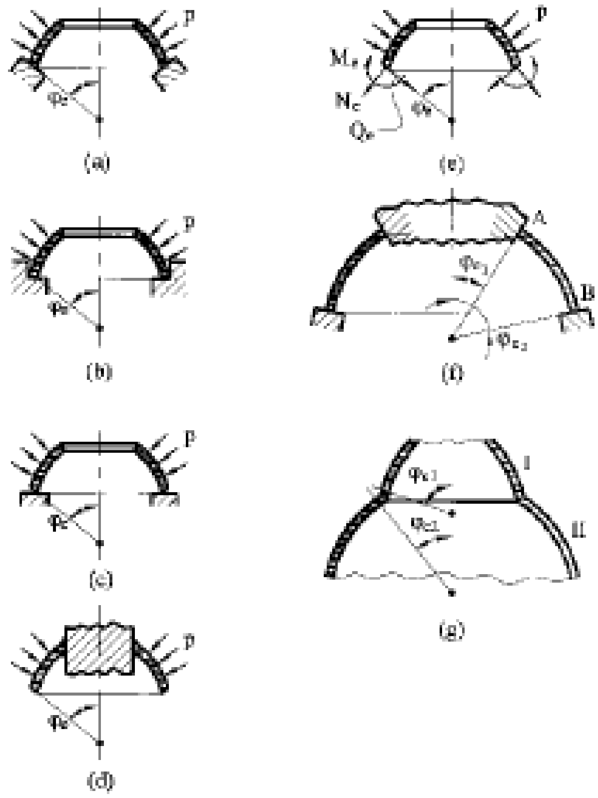

Fig. 16.3

$$\varepsilon_2 = 0|_{\varphi=\varphi_e} \quad \text{or} \quad N_2 - \nu N_1 = 0|_{\varphi=\varphi_e} \qquad (16.38b)$$

(ii) If the computedvalueof ε_{sc}> 0.0010869, f_{sc} = 217.39N/mm^2.

The stress-strain diagram of these bars is given in Fig. 1.2.4 of Lesson 2 and in Fig. 23A of IS 456. It shows that stress is proportional to strain up to a stress of 0.8 f_y. The stress-strain curve for the design purpose is obtained by

substituting$f_{yd}$for $f_y$in the figure up to 0.8 f_{yd}. Thereafter, from 0.8 f_{yd}~~to f_{yd}, Table A of SP~~-16 gives the values of total strains and design stresses for Fe 415

and Fe 500. Table 4.1 presents these values as a ready reference here.

The above procedure has been much simplified for the cold worked bars by presenting the values of $f_{sc}$of compression steel in doubly reinforced beams for different values of d'/d only taking the practical aspects into consideration. In most of the doubly reinforced beams, *d'/d* has been found to be between 0.05 and 0.2. Accordingly, values of f_{sc}can be computed from Table 4.1 after determining the value of $\varepsilon_{sc}$from Eq. 4.9 for known values of *d'/d* as 0.05, 0.10, 0.15 and 0.2. Table F of SP-16 presents these values of $f_{sc}$for four values of *d'/d* (0.05, 0.10, 0.15 and 0.2) of Fe 415 and Fe 500. Table 4.2 below, however, includes Fe 250 also whose f_{sc}values are computed as laid down insec.

4.8.4(a) (i) and (ii) along with those of Fe 415 and Fe 500. This table is very

Here, there is only elastic component of the strain without any inelastic strain.

Minimum and maximum steel in compression

There is no stipulation in IS 456 regarding the minimum compression steel in doubly reinforced beams. However, hangers and other bars provided up to 0.2% of the whole area of cross section may be necessary for creep and shrinkage of concrete. Accordingly, these bars are not considered as compression reinforcement. From the practical aspects of consideration, therefore, the minimum steel as compression reinforcement should be at least 0.4% of the area of concrete in compression or 0.2% of the whole cross -sectional area of the beam so that the doubly reinforced beam can take care of the extra loads in addition to resisting the effects of creep and shrinkage ofconcrete.

The maximum compression steel shall not exceed 4 per cent of the whole area of cross- section of the beam as given in cl. 26.5.1.2 of IS 456.

Minimum and maximum steel in tension

As stipulated in cl. 26.5.1.1(a) and (b) of IS 456, the minimum amount of tensile reinforcement shall be at least (0.85 bd/f_y) and the maximum area of tension reinforcement shall not exceed (0.04 *bD*).

It has been discussed in sec. 3.6.2.3 of Lesson 6 that the singly reinforced

beams shall have A_{st}normally not exceeding 75 to 80% of $A_{st,lim}$so that $x_u$remains less than $x_{u,max}$with a view to ensuring ductile failure. However, in the case of doubly reinforced beams, the ductile failure is ensured with the presence of compression steel. Thus, the depth of the neutral axis may be taken as $x_{u,max}$if the beam is over-reinforced. Accordingly, the A_{st1}part of tension steel can go

up to $A_{st,lim}$and the additional tension steel $A_{st2}$is provided for the additional moment M_u - $M_{u,lim}$. The quantities of $A_{st1}$and A_{st2}together form the total A_{st}, which shall not exceed 0.04 *bD*.

ypes of problems and steps of solution

Similar to the singly reinforced beams, the doubly reinforced beams have two types of problems: (i) design type and (ii) analysis type. The different steps of solutions of these problems are taken up separately.

Design type of problems

In the design type of problems, the given data are *b*, *d*, *D*, grades of concrete and steel. The designer has to determine $A_{sc}$and $A_{st}$of the beam from the given factored moment. These problems can be solved by two ways: (i) use of the equations developed for the doubly reinforced beams, named here as direct computation method, (ii) use of charts and tables of SP-16.

(a) Direct computation method

Step 1: To determine $M_{u,lim}$and $A_{st,lim}$from Eqs. 4.2 and 4.8, respectively.

Step2: To determine M_{u2},A_{sc},A_{st2} andA_{st} from Eqs. 4.1, 4.4, 4.6 and

4.7, respectively.

Step 3: To check for minimum and maximum reinforcement in compression and tension as explained in sec. 4.8.5.

Step4: To select the number and diameter of bars from known valuesof $A_{sc}$and A_{st}.

(b) Use of SPtable

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(a) Direct computation method

Step 1: To determine $M_{u,lim}$ and $A_{st,lim}$ from Eqs. 4.2 and 4.8, respectively.

Step2: To determine M_{u2}, A_{sc}, A_{st2} and A_{st} from Eqs. 4.1, 4.4, 4.6 and 4.7, respectively.

Step 3: To check for minimum and maximum reinforcement in compression and tension as explained in sec. 4.8.5.

Step4: To select the number and diameter of bars from known values of A_{sc} and A_{st}.

(b) Use of SPtable

Tables 45 to 56 present the p_t and p_c of doubly reinforced sections for d'/d = 0.05, 0.10, 0.15 and 0.2 for different f_{ck} and f_y values against M_u/bd^2. The values of p_t and p_c are obtained directly selecting the proper table with known values of M_u/bd^2 and d'/d.

Analysis type of problems

In the analysis type of problems, the data given are b, d, d', D, f_{ck}, f_y, A_{sc} and A_{st}. It is required to determine the moment of resistance M_u of such beams.

These problems can be solved: (i) by direct computation method and (ii) by using tables of SP-16.

(a) Direct computationmethod

Step1: To check if the beam is under-reinforced orover-reinforced.

First, $x_{u,max}$isdetermined assuming ithas reached limiting stageusing

The beam is under-reinforced or over-reinforced if $\varepsilon_{st}$is less than or more than the yield strain.

Step 2: To determine $M_{u,lim}$from Eq. 4.2 and $A_{st,lim}$from the $p_{t,lim}$given in Table 3.1 of Lesson 5.

Step 3: To determine A_{st2} and A_{sc} fromEqs. 4.7 and 4.6, respectively.

Step 4: To determine M_{u2} and M_u fromEqs. 4.4 and 4.1, respectively.

(b) Use of tables

As mentioned earlier Tables 45 to 56 are needed for the doubly reinforced beams. First, the needed parameters d'/d, $p_t$and $p_c$are calculated. Thereafter, $M_u/bd^2$is computed in two stages: first, using d'/d and $p_t$and then using d'/d and p_c. The lower value of $M_u$is the moment of resistance of the beam.

The actual width of the flange is the spacing of the beam, which is the same as the distance between the middle points of the adjacent spans of the slab, as shown in Fig. 5.10.2 b. However, in a flanged beam, a part of the width less than the actual width, is effective to be considered as a part of the beam. This width of the slab is designated as the effective width of the flange.

The following requirements (cl. 23.1.1 of IS 456) are to be satisfied to ensure the combined action of the part of the slab and the rib (rectangular part of the beam).

4.8.6 The slab and the rectangular beam shall be cast integrally or they shall be effectively bonded in any othermanner.

4.8.7 Slabs must be provided with the transverse reinforcement of at least 60 per cent of the main reinforcement at the mid span of the slab if the main reinforcement of the slab is parallel to the transverse beam (Figs. 5.10.3 a andb).

The variation of compressive stress (Fig. 5.10.4) along the actual width of the flange shows that the compressive stress is more in the flange just above the rib than the same at some distance away from it. The nature of variation is complex and, therefore, the concept of effective width has been introduced. The effective width is a convenient hypothetical width of the flange over which the compressive stress is assumed to be uniform to give the same compressive

orce as it would have been in case of the actual width with the true variation of compressive stress.

Clause 23.1.2 of IS 456 specifies the following effective widths of *T* and *L*-beams:

(a) For *T*-beams, the lesser of

(i) $b_f = l_o/6 + b_w + 6D_f$

(iii) b_f=Actual width of the flange For isolated

T-beams, the lesserof

b_f=Actual width of the flange

For*L*-beams, the lesser of

(i) $b_f = l_o/12 + b_w + 3D_f$

b_f=Actual width of theflange

For isolated beams, the lesser of

(ii) b_f = Actual width of the flange

whereb_f = effective width of theflange,

l_o= distance between points of zero moments in the beam, which is the effective span for simply supported beams and 0.7 times the effective span for continuous beams and frames,

b_w= beadth of theweb,

D_f= thickness of the flange,

and b = actual width of theflange.

Four Different Cases

The neutral axis of a flanged beam may be either in the flange or in the web depending on the physical dimensions of the effective width of flange b_f, effective width of web b_w, thickness of flange $D_f$and effective depth of flanged beam d (Fig. 5.10.4). The flanged beam may be considered as a rectangular beam of width $b_f$and effective depth d if the neutral axis is in the flange as the concrete in tension is ignored. However, if the neutral axis is in the web, the compression is taken by the flange and a part of the web.

All the assumptions made in sec. 3.4.2 of Lesson 4 are also applicable for the flanged beams. As explained in Lesson 4, the compressive stress remains constant between the strains of 0.002 and 0.0035. It is important to find the depth h of the beam where the strain is 0.002 (Fig. 5.10.5 b). If it is located in the web, the whole of flange will be under the constant stress level of 0.446 f_{ck}.The

ollowing gives the relation of $D_f$and d to facilitate the determination of the depth h where the strain will be 0.002.

The same relation is obtained below from the values of strains of concrete and steel of Fig. 5.10.5 b.

following gives the relation of $D_f$and d to facilitate the determination of the depth h where the strain will be 0.002.

The same relation is obtained below from the values of strains of concrete and steel of Fig. 5.10.5 b.

It is now clear that the three values of h are around 0.2 d for the three grades of steel. The maximum value of h may be D_f, at the bottom of the flange where the strain will be 0.002, if D_f/d = 0.2. This reveals that the thickness of the flange may be considered small if D_f/d does not exceed 0.2 and in that case, the position of the fibre of 0.002 strain will be in the web and the entire flange will be under a constant compressive stress of 0.446 f_{ck}.

On the other hand, if $D_f$is > 0.2 d, the position of the fibre of 0.002 strain will be in the flange. In that case, a part of the slab will have the constant stress of 0.446 $f_{ck}$where the strain will be more than0.002.

Thus, in the balanced and over-reinforced flanged beams (when $x_u = x_{u,\max}$), the ratio of Df/d is important to determine if the rectangular stress block is for the full depth of the flange (when D_f/d does not exceed 0.2) of for a part of the flange (when D_f/d > 0.2). Similarly, for the under-reinforced flanged beams, the ratio of $D_f/x_u$is considered in place of D_f/d. If D_f/x_udoes not exceed

0.43 (see Eq. 5.1), the constant stress block is for the full depth of the flange. If D_f/x_u> 0.43, the constant stress block is for a part of the depth of the flange.

Based on the above discussion, the four cases of flanged beams are as follows:

The superstructure is placed on the top of the foundation structure, designated as substructure as they are placed below the ground level. The elements of the superstructure transfer the loads and moments to its adjacent element below it and finally all loads and moments cometo

the foundation structure, which in turn, transfers them to the underlying soil or rock. Thus, the foundation structure effectively supports the superstructure. However, all types of soil get compressed significantly and cause the structure to settle. Accordingly, the major requirements of the design of foundation structures are the two as given below (see cl.34.1 of IS 456)
:

1. Foundation structures should be able to sustain the applied loads, moments, forces and induced reactions without exceeding the safe bearing capacity of thesoil.

2. The settlement of the structure should be as uniform as possible and it should be within the tolerable limits. It is well known from the structural analysis that differential settlement of supports causes additional moments in statically indeterminate structures. Therefore, avoiding the differential settlement is considered as more important than maintaining uniform overall settlement of thestructure.

Factored concentric load applied on short tied columns is resisted by concrete of area A_c and longitudinal steel of areas A_{sc}effectively held by lateral ties at intervals. Assuming the design strengths of concrete and steel are $0.4f_{ck}$ and $0.67f_y$, respectively, we can write

$P_u = 0.4f_{ck} A_c + 0.67f_y A_{sc}$(1)

Where P_u= factored axial load on the member,

f_{ck}= characteristic compressive strength of the concrete,

A_c = area of concrete,

f_y= characteristic strength of the compression reinforcement, and

A_{sc}= area of longitudinal reinforcement for columns.

The above equation, given in cl. 39.3 of IS 456, has two unknowns A_c and $A_{sc}$to be determined from one equation. The equation is recast in terms of A_g, the gross area of concrete and p, the percentage of compression reinforcementemploying

$A_{sc} = pA_g/100$ (2)

$A_c = A_g(1 - p/100)$ (3)

Accordingly, we can write

$P_u/A_g = 0.4f_{ck} + (p/100)(0.67f_y - 0.4f_{ck})$ (4)

Equation 4 can be used for direct computation of A_g when P_u, $f_{ck}$and $f_y$are known by assuming p ranging from 0.8 to 4 as the minimum and maximum percentages of longitudinal reinforcement. Equation 10.4 also can be employed to determine *Ag* and *p* in a similar manner by assuming *p*.

The superstructure is placed on the top of the foundation structure, designated as substructure as they are placed below the ground level. The elements of the superstructure transfer the loads and moments to its adjacent element below it and finally all loads and moments come to the foundation structure, which in turn, transfers them to the underlying soil or rock. Thus, the foundation structure effectively supports the superstructure. However, all types of soil get compressed significantly and cause the structure to settle. Accordingly, the major requirements of the design of foundation structures are the two as given below (see cl.34.1 of IS 456):

1. Foundation structures should be able to sustain the applied loads, moments, forces and induced reactions without exceeding the safe bearing capacity of thesoil.

2. The settlement of the structure should be as uniform as possible and it should be within the tolerable limits. It is well known from the structural analysis that differential settlement of supports causes additional moments in statically indeterminate structures. Therefore, avoiding the differential settlement is considered as more important than maintaining uniform overall settlement of thestructure.

(ii)If the computed value of ε_{sc}> 0.0010869, f_{sc} = 217.39 N/mm^2.

The stress-strain diagram of these bars is given in Fig. 1.2.4 of Lesson 2 and in Fig. 23A of IS 456. It shows that stress is proportional to strain up to a stress of 0.8 f_y. The stress-strain curve for the design purpose is obtained by

substituting$f_{yd}$for $f_y$in the figure up to 0.8 f_{yd}. Thereafter, from 0.8 $f_{yd}$to $f_{yd,}$Table A of SP-16 gives the values of total strains and design stresses for Fe 415

and Fe 500. Table 4.1 presents these values as a ready reference here.

The above procedure has been much simplified for the cold worked bars by presenting the values of $f_{sc}$of compression steel in doubly reinforced beams for different values of *d'/d* only taking the practical aspects into consideration. In most of the doubly reinforced beams, *d'/d* has been found to be between 0.05 and 0.2. Accordingly, values of f_{sc}can be computed from Table 4.1 after determining the value of $\varepsilon_{sc}$from Eq. 4.9 for known values of *d'/d* as 0.05, 0.10, 0.15 and 0.2. Table F of SP-16 presents these values of $f_{sc}$for four values of *d'/d* (0.05, 0.10, 0.15 and 0.2) of Fe 415 and Fe 500. Table 4.2 below, however, includes Fe 250 also whose f_{sc}values are computed as laid down insec.

4.8.4(a) (i) and (ii) along with those of Fe 415 and Fe 500. This table is very

Here, there is only elastic component of the strain without any inelastic strain.

Minimum and maximum steel in compression

There is no stipulation in IS 456 regarding the minimum compression steel in doubly reinforced beams. However, hangers and other bars provided up to 0.2% of the whole area of cross section may be necessary for creep and shrinkage of concrete. Accordingly, these bars are not considered as compression reinforcement. From the practical aspects of consideration, therefore, the minimum steel as compression reinforcement should be at least 0.4% of the area of concrete in compression or 0.2% of the whole cross -sectional area of the beam so that the doubly reinforced beam can take care of the extra loads in addition to resisting the effects of creep and shrinkage ofconcrete.

The maximum compression steel shall not exceed 4 per cent of the whole area of cross- section of the beam as given in cl. 26.5.1.2 of IS 456.

Minimum and maximum steel in tension

The superstructure is placed on the top of the foundation structure, designated as substructure as they are placed below the ground level. The elements of the superstructure transfer the loads and moments to its adjacent element below it and finally all loads and moments come to the foundation structure, which in turn, transfers them to the underlying soil or rock. Thus, the foundation structure effectively supports the superstructure. However, all types of soil get compressed significantly and cause the structure to settle. Accordingly, the major requirements of the design of foundation structures are the two as given below (see cl.34.1 of IS456)
:

1. Foundation structures should be able to sustain the applied loads, moments, forces and induced reactions without exceeding the safe bearing capacity of thesoil.

2. The settlement of the structure should be as uniform as possible and it should be within the tolerable limits. It is well known from the structural analysis that differential settlement of supports causes additional moments in statically indeterminate structures. Therefore, avoiding the differential settlement is considered as more important than maintaining uniform overall settlement of thestructure.

Columns are classified into the three following types based on the loadings:

(i) Columns subjected to axial loads only (concentric), as shown in Fig.3.2a.

(ii) Columns subjected to combined axial load and uniaxial bending, as shown in Fig.3.2b.

(iii) Columns subjected to combined axial load and bi-axial bending, as shown in Fig.3.2c.

Classification Based on Slenderness Ratios

Columns are classified into the following two types based on the slenderness ratios:

(i) Short columns

(ii) Slender or longcolumns

Figure 3.3 presents the three modes of failure of columns with different slenderness ratios when loaded axially. In the mode 1, column does not undergo any lateral deformation and collapses due to material failure. This is known as compression failure. Due to the combined effects of axial load and moment a short column may have material failure of mode 2. On the other hand, a slender column subjected to axial load only undergoes deflection due to beam- column effect and may have material failure under the combined action of direct load and bending moment.Such failure is called finally all loads and moments come to the foundation structure, which in turn, transfers them to the underlying soil or rock. Thus, the foundation structure effectively supports the superstructure. However, all types of soil get compressed significantly and cause the structure to settle. Accordingly, the major requirements of the design of foundation structures are the two as given below (see cl.34.1 of IS 456) :

1. Foundation structures should be able to sustain the applied loads, moments, forces and induced reactions without exceeding the safe bearing capacity of thesoil.

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Columns are classified into the three following types based on the loadings:

(i) Columns subjected to axial loads only (concentric), as shown in Fig.3.2a.

(ii) Columns subjected to combined axial load and uniaxial bending, as shown in Fig.3.2b.

(iii) Columnssubjectedtocombinedaxialloadandbi-axialbending,asshowninFig.3.2c.

Classification of Columns Based on Slenderness Ratios

Columns are classified into the following two types based on the slenderness ratios:

(i) Short columns

(ii) Slender or longcolumns

Figure 3.3 presents the three modes of failure of columns with different slenderness ratios when loaded axially. In the mode 1, column does not undergo any lateral deformation and collapses due to material failure. This is known as compression failure. Due to the combined effects of axial load and moment a short column may have material failure of mode 2. On the other hand, a slender column subjected to axial load only undergoes deflection due to beam- column effect and may have material failure under the combined action of direct load and bending moment. Such failure is called combined compression and bending failure of mode 2. Mode 3 failure is by elastic instability of very long column even under small load much before the material reaches the yield stresses. This type of failure is known as elastic buckling. The slenderness ratio of steel column is the ratio of its effective length *le* to its least radius of gyration *r*. In case of reinforced concrete column, however, IS 456 stipulates reinforcement needed is known as $A_{st,lim}$. Problem will arise, therefore, if such a section is subjected to bending moment greater than its limiting moment of resistance as a singly reinforcedsection.