Course Code: AHSB02

LINEAR ALGEBRA AND CALCULUS

Regulation: IARE-R18

B . TECH: I SEM

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THEORY OF MATRICES
AND HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

Matrix: A system of mn numbers (real or complex) arranged in a rectangular array of m horizontal lines (Called rows) and n vertical lines (called columns) is known as matrix of order mxn [read as "m by n matrix"]. These numbers are called elements being enclosed in brackets [ ] or ( ) .

1.Real Matrix: A matrix whose elements are real numbers is called a real matrix.

Example: $\begin{bmatrix} 6 & 0 & -1 \\ 4 & \sqrt{3} & 2 \end{bmatrix}$ is a real matrix**.**

2.Symmetric Matrix: A square matrix
A = $\left[a_{ij}\right]$
is called symmetric , if A = A^T
Thus, for a symmetric matrix A, we have

$a_{ij} = a_{ji}$ for all i and j.

3.Skew-Symmetric Matrix: A square matrix

A = $\left[a_{ij}\right]$ is called skew-symmetric, if

$$A^T = -A$$

Thus for a skew-symmetric matrix A

$a_{ij} = -a_{ji}$ for all i and j.

Example: $\begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$

Note: If A is a skew-symmetric matrix then :

$$a_{ij} = -a_{ji}$$

$$a_{ii} = -a_{ii} \forall i$$

$$2a_{ii} = 0$$

Thus, the diagonal elements of a skew-symmetric matrix are all zero.

4. Orthogonal Matrix: A square matrix with real elements is said to be orthogonal if

$$A^T A = I$$

$$\left[A^T = A^{-1} \right]$$

Example: Show That
$$\begin{bmatrix} \text{Cos} & 0 & \text{Sin}\,\phi \\ \text{Sin}\,\theta\,\text{Sin}\,\phi & \text{Cos}\,\theta & -\text{Sin}\,\theta\,\text{Cos}\,\phi \\ -\text{Cos}\,\theta\,\text{Sin}\,\phi & \text{Sin}\,\theta & \text{Cos}\,\theta\,\text{Cos}\,\phi \end{bmatrix}$$

Is an orthogonal matrix

Solution: Let A =
$$\begin{bmatrix} \text{Cos} & 0 & \text{Sin}\,\phi \\ \text{Sin}\,\theta\,\text{Sin}\,\phi & \text{Cos}\,\theta & -\text{Sin}\,\theta\,\text{Cos}\,\phi \\ -\text{Cos}\,\theta\,\text{Sin}\,\phi & \text{Sin}\,\theta & \text{Cos}\,\theta\,\text{Cos}\,\phi \end{bmatrix}$$

$$A^T = \begin{bmatrix} \text{Cos}\,\phi & \text{Sin}\,\theta\,\text{Sin}\,\phi & -\text{Cos}\,\theta\,\text{Sin}\,\phi \\ 0 & \text{Cos}\,\theta & \text{Sin}\,\theta \\ \text{Sin}\,\phi & \text{Sin}\,\theta & \text{Cos}\,\theta\,\text{Cos}\,\phi \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \text{Cos} & 0 & \text{Sin}\,\phi \\ \text{Sin}\,\theta\,\text{Sin}\,\phi & \text{Cos}\,\theta & -\text{Sin}\,\theta\,\text{Cos}\,\phi \\ -\text{Cos}\,\theta\,\text{Sin}\,\phi & \text{Sin}\,\theta & \text{Cos}\,\theta\,\text{Cos}\,\phi \end{bmatrix} \begin{bmatrix} \text{Cos}\,\phi & \text{Sin}\,\theta\,\text{Sin}\,\phi & -\text{Cos}\,\theta\,\text{Sin}\,\phi \\ 0 & \text{Cos}\,\theta & \text{Sin}\,\theta \\ \text{Sin}\,\phi & \text{Sin}\,\theta & \text{Cos}\,\theta\,\text{Cos}\,\phi \end{bmatrix}$$

$$= \begin{bmatrix} \text{Cos}^2\phi + \text{Sin}^2\phi & \begin{matrix}\text{Cos}\,\phi\,\text{Sin}\,\theta\,\text{Sin}\,\phi \\ -\text{Sin}\,\phi\,\text{Sin}\,\theta\,\text{Cos}\,\phi\end{matrix} & \begin{matrix}-\text{Cos}\,\theta\,\text{Cos}\,\phi\,\text{Sin}\,\phi + \\ \text{Sin}\,\phi\,\text{Cos}\,\theta\,\text{Cos}\,\phi\end{matrix} \\ \begin{matrix}\text{Sin}\,\theta\,\text{Sin}\,\phi\,\text{Cos}\,\phi \\ -\text{Sin}\,\theta\,\text{Cos}\,\phi\,\text{Sin}\,\phi\end{matrix} & \begin{matrix}\text{Sin}^2\theta\,\text{Sin}^2\phi + \text{Cos}^2\phi \\ +\text{Sin}^2\theta\,\text{Cos}^2\phi\end{matrix} & \begin{matrix}-\text{Sin}\,\theta\,\text{Cos}\,\theta\,\text{Sin}^2\phi + \text{Cos}\,\theta\,\text{Sin}\,\theta \\ -\text{Sin}\,\theta\,\text{Cos}\,\phi\,\text{Cos}^2\theta\end{matrix} \\ \begin{matrix}-\text{Cos}\,\theta\,\text{Sin}\,\phi\,\text{Cos}\,\phi \\ \text{Cos}\,\theta\,\text{Cos}\,\phi\,\text{Sin}\,\phi\end{matrix} & \begin{matrix}-\text{Cos}\,\theta\,\text{Sin}\,\theta\,\text{Sin}^2\phi + \text{Sin}\,\theta\,\text{Cos}\,\theta \\ -\text{Cos}\,\theta\,\text{Sin}\,\theta\,\text{Cos}^2\phi\end{matrix} & \begin{matrix}\text{Cos}^2\theta\,\text{Sin}^2\phi + \text{Sin}^2\theta \\ +\text{Cos}^2\theta\,\text{Cos}^2\phi\end{matrix} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \text{Sin}^2(\text{Sin}^2\theta+\text{Cos}^2\theta)+\text{Cos}^2\theta & -\text{Sin}\,\theta\text{Cos}\,\theta(\text{Sin}^2\theta+\text{Cos}^2\theta)+\text{Cos}\,\theta\text{Sin}\,\theta(\text{Sin}^2\theta+\text{Cos}^2\theta)+\text{Cos}^2\theta \\ 0 & -\text{Cos}\,\theta\text{Sin}\,\theta(\text{Sin}^2\theta+\text{Cos}^2\theta)+\text{Sin}\,\theta\text{Cos}\,\theta & \text{Cos}^2\theta(\text{Sin}^2\theta+\text{Cos}^2\theta)+\text{Sin}^2\theta \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since, $AA^T = I$

A is an Orthogonal Matrix.

Exercise

Q.1 Express the following matrices

$$\begin{bmatrix} 2 & 4 & 8 \\ 6 & 2 & 8 \\ 2 & 2 & 2 \end{bmatrix} \qquad \begin{bmatrix} 3 & -4 & -1 \\ 6 & 0 & -1 \\ -3 & 13 & -4 \end{bmatrix}$$

as the sum of a symmetric matrix and a skew-symmetric matrix

Q.3 Verify the matrix $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$ is orthogonal or not.

Q.4 Show that the matrix

$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$$

is orthogonal.

Q.5 Show that the matrix

$$A = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

is orthogonal

COMPLEX MATRICES: So far we discussed about real numbers whose elements were real. In this topic we will be considering the matrices whose elements are complex numbers. Complex matrices have a very wide applications in many areas of Engineering Such as quantum mechanics etc.

Complex Matrix: A matrix in which at least one element is imaginary is called a Complex Matrix

Example:

$$\begin{bmatrix} 4 & 0 & i \\ -5i & 0 & 2 \end{bmatrix}$$

6.Conjugate of a Matrix: The matrix obtained from any given
matrix A on replacing its elements by the corresponding conjugate complex numbers is called the conjugate of

A denoted by $\overline{A}$

Thus, if $A=\left[a_{ij}\right]_{m\times n}$ then $\overline{A}=\left[\overline{a}_{ij}\right]_{m\times n}$ Where, $\overline{a}_{ij}$

denotes the conjugate complex of a_{ij}

Example: If $A=\begin{bmatrix} 2+3i & 5 \\ 6-2i & 5+i \end{bmatrix}$ then $\overline{A}=\begin{bmatrix} 2-3i & 5 \\ 6+2i & 5-i \end{bmatrix}$

7.Transposed Conjugate of a Matrix: The transpose of the conjugate of a matrix A is called transposed conjugate of A and is denoted by A^{θ}

$$A^{\theta} = \left[\bar{A}\right]^{T} = \overline{\left[A^{T}\right]}$$

i.e., The transpose of the conjugate of a square matrix is same as the conjugate of its transpose

Example: Let

$$A=\begin{bmatrix}1+2i & 2-3i & 5\\ 5+2i & 5-2i & 8+5i\\ 2 & 6 & 9-i\end{bmatrix}$$

then.

$$\bar{A}=\begin{bmatrix}1-2i & 2+3i & 5\\ 5-2i & 5+2i & 8-5i\\ 2 & 6 & 9+i\end{bmatrix}$$

$$A^{\theta}=\left(\bar{A}\right)^{T}=\begin{bmatrix}1-2i & 5-2i & 2\\ 2+3i & 5+2i & 6\\ 5 & 8-5i & 9+i\end{bmatrix}$$

Example: Let
$$A=\begin{bmatrix}1+2i & 2-3i & 5\\ 5+2i & 5-2i & 8+5i\\ 2 & 6 & 9-i\end{bmatrix}$$

then.
$$\bar{A}=\begin{bmatrix}1-2i & 2+3i & 5\\ 5-2i & 5+2i & 8-5i\\ 2 & 6 & 9+i\end{bmatrix}$$

$$A^{\theta}=\left(\bar{A}\right)^{T}=\begin{bmatrix}1-2i & 5-2i & 2\\ 2+3i & 5+2i & 6\\ 5 & 8-5i & 9+i\end{bmatrix}$$

Hermitian Matrix: If the transpose of the conjugate matrix is equal to the matrix itself i.e.,

$$A^{\theta} = A$$

then the matrix A is said to be a Hermitian Matrix.

Thus, $A = \left[a_{ij} \right]$ is Hermitian, if $a_{ij} = \overline{a_{ji}}$ $\forall$ i, j.

Thus every diagonal element of a Hermitian matrix is real.

Example: $\begin{bmatrix} 1 & 2+i & 3-2i \\ 2-i & 0 & 2i \\ 3+2i & -2i & 4 \end{bmatrix}$ is a Hermitian Matrix.

Skew-Hermitian matrix: A square matrix $A = [a_{ji}]$

is said to be Skew-Hermitian if $A^{\theta} = -A$ i.e., $a_{ij} = -\bar{a}_{ji}$

If A is a Skew-Hermitian matrix, then $a_{ii} = -\bar{a}_{ii}$

$$\Rightarrow \quad a_{ii} + a_{ii} = 0$$

So, that a_{ii}

is either a purely imaginary number or zero. Thus the diagonal elements of a Skew-Hermitian matrix must be a purely imaginary number or zero.

Example: $\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 1-i & 2 \\ -1-i & 3i & i \\ -2 & i & 0 \end{bmatrix}$

are Skew-Hermitian matrices.

Unitary matrix: A square matrix A with complex elements is said to be unitary if

$$A^{\theta} A = I$$

$$\begin{bmatrix} \frac{i}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{i}{2} \end{bmatrix}$$

the matrix Is an example for a unitary matrix.

Theorem 8: If A is any square matrix, then prove that :

(a) $A+A^{\theta}$ is Hermitian.

(b) $AA^{\theta}, A^{\theta}A$ are Hermitian.

(c) $A-A^{\theta}$ is Skew-Hermitian.

Proof

(a) $$\left[A+A^{\theta}\right]^{\theta}=A^{\theta}+\left[A^{\theta}\right]^{\theta}$$

$$=A^{\theta}+A$$

$$=A+A^{\theta}$$

$\therefore$ $A+A^{\theta}$ is Hermitian.

$$\left[AA^{\theta}\right] = \left[A^{\theta}\right]^{\theta} A^{\theta} = AA^{\theta}$$

$$\left[A - A^{\theta}\right]^{\theta} = A^{\theta} - \left[A^{\theta}\right]^{\theta}$$

$$= A^{\theta} - A$$

$$= -\left[A - A^{\theta}\right]$$

$\therefore$ $A - A^{\theta}$ is Skew-Hermitian.

Exercise Q.1 If A is Hermitian Matrix, then show that iA is a Skew-Hermitian Matrix.

Q.2 Show that the matrix $\begin{bmatrix} 15 & 8i & 6-2i \\ -8i & 0 & -4+i \\ 6+2i & -4-i & -3 \end{bmatrix}$ is Hermitian.

Q.3 Show the matrix $\begin{bmatrix} 0 & 8i & 2i \\ 8i & 0 & 4i \\ 2i & 4i & 0 \end{bmatrix}$ is Skew-Hermitian.

Q.4 Express the matrix $\begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$ as the sum of a Hermitian and Skew-Hermitian Matrix.

Q.5 If $A = \begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix}$ Show that A is Hermitian and iA is a Skew-Hermitian Matrix.

ELEMENTARY ROW AND COLUMN TRANSFORMATIONS

Let, $R_1, R_2 \ldots. R_n$ be the row vectors of matrix A of order $m \times n$ and $C_1, C_2 \ldots. C_n$ be the column vectors of A

An **elementary row operation** of A is of any one of the following three operations of transformation

ROW OPERATIONS

*The interchange of any two rows.
*Multiplication of a row by a non-zero scalar K.
*Replace a row by adding to itself any non-zero scalar multiple of any other row

The notations we shall follow for these three elementary row operations is as follows :

1. Interchange of i^{th} and j^{th} row is denoted by $R_i \leftrightarrow R_j$.
2. Multiplication of i^{th} row by a non-zero scalar K is denoted by $R_i \to KR_i$
3. Addition of K times the j^{th} row to the i^{th} row is denoted by $R_i \to R_i + KR_j$.

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COLUMN OPERATIONS

Similarly we can define an **elementary column operation** of A as one of the following three operations.

*The interchange of any two columns.

*Multiplication of a column by a non-zero scalar K.

*Replace a column by adding to itself any non-zero scalar multiple of any other column.

*The notations we shall follow for these three elementary column operations is as follows

COLUMN OPERATIONS

1. Interchange of i^{th} and j^{th} column is denoted by $C_i \leftrightarrow C_j$

2. Multiplication of i^{th} column by a non-zero scalar K will be denoted by $C_i \to K$

3. Addition of K times the j^{th} column to the i^{th} column will be denoted by $C_i \to C_i + KC_j$

Rank of a Matrix:

Let A be mxn matrix. If A is a null matrix, we define its rank to be ‘0’. If A is a non-zero matrix, we say that r is the rank of A if

(i) Every $(r+1)^{th}$ order minor of A is ‘0’ (zero) &

(ii) At least one r^{th} order minor of A which is not zero.

Note: 1. It is denoted by $\rho(A)$

RANK OF A MATRIX

Note: 1. It is denoted by $\rho(A)$

2. Rank of a matrix is unique.

3. Every matrix will have a rank.

4. If A is a matrix of order mxn,

   Rank of A $\leq$ min(m,n)

5. If $\rho(A) = r$ then every minor of A of order r+1, or more is zero.

6. Rank of the Identity matrix I_n is n.

7. If A is a matrix of order n and A is non-singular then $\rho(A) = n$

1. Find the rank of the given matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$$

Given matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

det A = 1(48-40)-2(36-28)+3(30-28)

= 8-16+6 = -2 $\neq$ 0

We have minor of order 3

$\rho(A) = 3$

2. Find the rank of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$$

Sol: Given the matrix is of order 3x4

Its Rank ≤ min(3,4) = 3

Highest order of the minor will be 3.

Let us consider the minor $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{bmatrix}$

RANK OF A MATRIX

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{bmatrix}$$

Determinant of minor is 1(-49)-2(-56)+3(35-48)

$= -49+112-39 = 24 \neq 0.$

Hence rank of the given matrix is '3'.

Echelon form of a matrix:

A matrix is said to be in Echelon form, if

(i). Zero rows, if any exists, they should be below the non-zero row.

(ii). The first non-zero entry in each non-zero row is equal to ‘1’.

(iii). The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

Note: 1. The number of non-zero rows in echelon form of A is the rank of ‘A’.

1. The rank of the transpose of a matrix is the same as that of original matrix.
2. The condition (ii) is optional.

1. Find the rank of the matrix A = $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$ **by reducing it to Echelon form.**

sol: Given A = $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

Applying row transformations on A.

$$A \sim \begin{bmatrix} 1 & -3 & -1 \\ 3 & -2 & 4 \\ 2 & 3 & 7 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

ECHELON FORM

$$\sim\begin{bmatrix} 1 & -3 & -1 \\ 0 & 7 & 7 \\ 0 & 9 & 9 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$\sim R_2 \to R_2/7,$

$R_3 \to R_3/9$

This is the Echelon form of matrix A.

The rank of a matrix A.

= Number of non – zero rows =2

$$\sim\begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

1. For what values of k the matrix

$$\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$$ **has rank '3'.**

Sol: The given matrix is of the order 4x4

If its rank is 3 ⇨ det A =0

$$A = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$$

Applying $R_2 \rightarrow 4R_2 - R_1$, $R_3 \rightarrow 4R_3 - kR_1$, $R_4 \rightarrow 4R_4 - 9R_1$

We get $A \sim \begin{bmatrix} 4 & 4 & -3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 8-4k & 8+3k & 8-k \\ 0 & 0 & 4k+27 & 3 \end{bmatrix}$

Since Rank A = 3 ⇨ det A =0

$$\Rightarrow 4 \begin{vmatrix} 0 & -1 & -1 \\ 8-4k & 8+3k & 8-k \\ 0 & 4k+27 & 3 \end{vmatrix} = 0$$

RANK OF A MATRIX

- ⇨ $1[(8-4k)3]-1(8-4k)(4k+27)] = 0$
- ⇨ $(8-4k)\ (3-4k-27) = 0$
- ⇨ $(8-4k)(-24-4k) = 0$
- ⇨ $(2-k)(6+k)=0$
- ⇨ $k = 2$ or $k = -6$

Normal Form:

Every mxn matrix of rank r can be reduced to the form $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$

(or) (I_r) (or) $\begin{pmatrix} I_r \\ 0 \end{pmatrix}$ (or) $\begin{pmatrix} I_r & 0 \end{pmatrix}$

by a finite number of elementary transformations, where I_r is the r – rowed unit matrix.

Normal form or canonical form

e.g: By reducing the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$$

into normal form, find its rank.

NORMAL FORM

Sol: Given $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

$R_2 \rightarrow R_2 - 2R_1$

$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & 5 \\ 0 & -6 & -4 & -22 \end{bmatrix}$ $R_3 \rightarrow R_3 - 3R_1$

$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 3 & 2 & 11 \end{bmatrix}$ $R_3 \rightarrow R_3/-2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$

$$A \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$c_2 \rightarrow c_2 - 2c_1$, $c_3 \rightarrow c_3 - 3c_1$, $c_4 \rightarrow c_4 - 4c_1$

NORMAL FORM

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 18 \end{bmatrix}$$

$c_3 \rightarrow 3\ c_3\ -2c_2,\ c_4 \rightarrow 3c_4-5c_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$c_2 \rightarrow c_2/-3,\ c_4 \rightarrow c_4/18$

GAUSS JORDAN METHOD

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$c_4 \leftrightarrow c_3$

This is in normal form $[I_3 \ 0]$

Hence Rank of A is ‘3’.

Gauss – Jordan method

- The inverse of a matrix by elementary Transformations: **(Gauss – Jordan method)**

1. suppose A is a non-singular matrix of order 'n' then we write $A = I_n A$
2. Now we apply elementary row-operations only to the matrix A and the pre-factor I_n of the R.H.S
3. We will do this till we get $I_n = BA$ then obviously B is the inverse of A.

*Find the inverse of the matrix A using elementary operations where

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

Sol:

$$\text{Given } A = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

We can write A = I_3 A

$$\begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \rightarrow 2R_3 - R_2$, we get

$$\begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} A$$

Applying $R_1 \rightarrow R_1 - 3R_2$, we get

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} A$$

Applying $R_1 \rightarrow R_1 + 5R_3$, $R_2 \rightarrow R_2 - 3R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -8 & 10 \\ 0 & 4 & -6 \\ 0 & -1 & 2 \end{bmatrix} A$$

Applying $R_2 \to R_2/2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -8 & 10 \\ 0 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix} A$$

⇨ $I_3 = BA$

B is the inverse of A.

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Def:

An equation of the form $\frac{d^n y}{dx^n} + P_1 \cdot \frac{d^{n-1} y}{dx^{n-1}} + P_2 \cdot \frac{d^{n-2} y}{dx^{n-2}} + \text{---------} + P_n .y = Q(x)$ where P_1, P_2, P_3,.....P_n, are real constants and Q(x) is a continuous function of x is called an linear differential equation of order ' n' with constant coeffin

<u>To find the general solution of f(D).y = 0</u> :

Where $f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2}$ +-----------
$+P_n$ is a polynomial in D.

Now consider the auxiliary equation : f(m) = 0

i.e $f(m) = m^n + P_1 m^{n-1} + P_2 m^{n-2}$ +-----------$+P_n = 0$

where p_1, p_2, p_3 ....p_n are real constants.

Let the roots of f(m) =0 be m_1, m_2, m_3,.....m_n.

Depending on the nature of the roots we write the complementary function as follows:

S.No	Roots of A.E f(m) =0	Complementary function(C.F)
1.	$m_1, m_2, ..m_n$ are real and distinct.	$y_c = c_1e^{m_1x} + c_2e^{m_2x} + ... + c_ne^{m_nx}$
2.	$m_1, m_2, ..m_n$ are and two roots are equal i.e., m_1, m_2 are equal and real(i.e repeated twice) & the rest are real and different.	$y_c = (c_1+c_2x)e^{m_1x} + c_3e^{m_3x} + ... + c_ne^{m_nx}$
3.	$m_1, m_2, ..m_n$ are real and three roots are equal i.e., m_1, m_2, m_3 are equal and real(i.e repeated thrice) & the rest are real and different.	$y_c = (c_1+c_2x+c_3x^2)e^{m_1x} + c_4e^{m_4x} + ... + c_ne^{m_nx}$
4.	Two roots of A.E are complex say $\alpha+i\beta$ $\alpha-i\beta$ and rest are real and distinct.	$y_c = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x) + c_3e^{m_3x} + ... + c_ne^{m_nx}$
5.	If $\alpha\pm i\beta$ are repeated twice & rest are real and distinct	$y_c = e^{\alpha x}[(c_1+c_2x)\cos\beta x + (c_3+c_4x)\sin\beta x)] + c_5e^{m_5x} + ... + c_ne^{m_nx}$
6.	If $\alpha\pm i\beta$ are repeated thrice & rest are real and distinct	$y_c = e^{\alpha x}[(c_1+c_2x+c_3x^2)\cos\beta x + (c_4+c_5x+c_6x^2)\sin\beta x)] + c_7e^{m_7x} + ... + c_ne^{m_nx}$
7.	If roots of A.E. irrational say $\alpha\pm\sqrt{\beta}$ and rest are real and distinct	$y_c = e^{\alpha x}[c_1\cosh\sqrt{\beta}x + c_2\sinh\sqrt{\beta}x] + c_3e^{m_3x} + ... + c_ne^{m_nx}$

Solve $\frac{d^3y}{dx^3}$ - 3$\frac{dy}{dx}$ + 2y = 0

Given equation is of the form f(D)y = 0

Where f(D) = $(D^3 - 3D + 2)\, y = 0$

consider the auxiliary equation f(m) = 0

$f(m) = m^3 - 3m + 2 = 0 \Rightarrow (m-1)(m-1)(m+2) = 0$

$\Rightarrow m = 1, 1, -2$

Since m_1 and m_2 are equal and m_3 is -2

We have $y_c = (c_1 + c_2 x)e^x + c_3 e^{-2x}$

3. Solve $(D^4 + 8D^2 + 16)\, y = 0$

Sol: Given $f(D) = (D^4 + 8D^2 + 16)\, y = 0$

Auxiliary equation $f(m) = (m^4 + 8m^2 + 16) = 0$

$\Rightarrow (m^2 + 4)^2 = 0$

$\Rightarrow (m+2i)^2\,(m+2i)^2 = 0$

$\Rightarrow m = 2i, 2i, -2i, -2i$

$Y_c = e^{0x}\,[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x)]$

4. Solve $y^{11}+6y^{1}+9y = 0$; $y(0) = -4$, $y^{1}(0) = 14$

Sol: Given equation is $y^{11}+6y^{1}+9y = 0$

Auxiliary equation f(D) y = 0 $\Rightarrow$ $(D^2 +6D +9)\ y = 0$

A.equation f(m) = 0 $\Rightarrow$ $(m^2 +6m +9) = 0$

$\Rightarrow$ m = -3 ,-3

$y_c = (c_1+c_2x)e^{-3x}$ -------------------> (1)

Differentiate of (1) w.r.to x $\Rightarrow$ $y^1 =(c_1+c_2x)(-3e^{-3x}) + c_2(e^{-3x})$

Given y_1 (0) =14 $\Rightarrow$ $c_1 = -4$ & $c_2 =2$

Hence we get $y=(-4 + 2x)\ (e^{-3x})$

5. Solve $4y^{111} + 4y^{11} +y^{1} = 0$

Sol: Given equation is $4y^{111} + 4y^{11} +y^{1} = 0$

That is $(4D^3+4D^2+D)y=0$

Auxiliary equation f(m) = 0

$4m^3 +4m^2 + m = 0$

$m(4m^2 +4m + 1) = 0$

$m(2m + 1)^2 = 0$

m = 0 , -1/2 ,-1/2

$y =c_1+ (c_2+ c_3x)\ e^{-x/2}$

Is given by $y = y_c + y_p$

i.e. y = C.F+P.I

Where the P.I consists of no arbitrary constants and P.I of $f(D)\, y = Q(x)$

Is evaluated as $P.I = \frac{1}{f(D)} . Q(x)$

Depending on the type of function of Q(x).

P.I is evaluated as follows:

1. P.I of f (D) y = Q(x) where Q(x) =e^{ax} for (a) ≠ 0

Case1: P.I = $\frac{1}{f(D)}$. Q(x) = $\frac{1}{f(D)}e^{ax}$ = $\frac{1}{f(a)}e^{ax}$

Provided f(a) ≠ 0

Case 2: If f(a) = 0 then the above method fails. Then

if $f(D) = (D-a)^k \, \emptyset(D)$

(i.e ' a' is a repeated root k times).

Then P.I = $\frac{1}{\emptyset(a)}e^{ax} \cdot \frac{1}{k!}x^k$ provided $\emptyset$ (a) ≠ 0

Express $\frac{1}{f(D)} = \frac{1}{1 \pm \emptyset(D)} = [1 \pm \emptyset(D)]^{-1}$

Hence P.I = $\frac{1}{1 \pm \emptyset(D)}$ Q(x).

$= [1 \pm \emptyset(D)]^{-1} \cdot x^k$

Solve the Differential equation $(D^2+5D+6)y=e^x$

Sol : Given equation is $(D^2+5D+6)y=e^x$

Here $Q(x) = e^x$

Auxiliary equation is $f(m) = m^2+5m+6=0$

$m^2+3m+2m+6=0$

$m(m+3)+2(m+3)=0$

$m=-2$ or $m=-3$

The roots are real and distinct

$C.F = y_c = c_1 e^{-2x} + c_2 e^{-3x}$

Particular Integral = $y_p = \frac{1}{f(D)} \cdot Q(x)$

$= \frac{1}{D2+5D+6} e^x \quad = \frac{1}{(D+2)(D+3)} e^x$

Put D = 1 in f(D)

P.I. = $\frac{1}{(3)(4)} e^x$

Particular Integral = $y_p = \frac{1}{12} \cdot e^x$

General solution is $y = y_c + y_p$

$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{e^x}{12}$

Solve $y^{11}-4y^{1}+3y=4e^{3x}$, $y(0) = -1$, $y^{1}(0) = 3$

Sol : Given equation is $y^{11}-4y^{1}+3y=4e^{3x}$

i.e. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 4e^{3x}$

it can be expressed as

$D^2y-4Dy+3y=4e^{3x}$

$(D^2-4D+3)y=4e^{3x}$

Here $Q(x)=4e^{3x}$; $f(D)= D^2-4D+3$

Auxiliary equation is $f(m)=m^2-4m+3 = 0$

$m^2-3m-m+3 = 0$

$m(m-3) -1(m-3)=0 \Rightarrow m=3$ or 1

The roots are real and distinct.

C.F= $y_c=c_1e^{3x}+c_2e^{x}$ ----→ (2)

$$P.I. = y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= y_p = \frac{1}{D^2 - 4D + 3} \cdot 4e^{3x}$$

$$= y_p = \frac{1}{(D-1)(D-3)} \cdot 4e^{3x}$$

Put D=3

$$y_p = \frac{4e^{3x}}{(3-1)(D-3)} = \frac{4}{2}\frac{e^{3x}}{(D-3)} = 2\frac{x^1}{1!}e^{3x} = 2xe^{3x}$$

General solution is $y = y_c + y_p$

$y = c_1 e^{3x} + c_2 e^{x} + 2xe^{3x}$ ------------------→ (3)

Equation (3) differentiating with respect to 'x'

$y^1 = 3c_1 e^{3x} + c_2 e^{x} + 2e^{3x} + 6x e^{3x}$ ----------→ (4)

By data, $y(0) = -1$, $y^1(0) = 3$

From (3), $-1 = c_1 + c_2$ ------------------→ (5)

From (4), $3 = 3c_1 + c_2 + 2$

$3c_1 + c_2 = 1$ ------------------→ (6)

Solving (5) and (6) we get $c_1 = 1$ and $c_2 = -2$

$y = -2e^{x} + (1+2x)e^{3x}$

The general solution is

y= C.F + P.I

$$y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{-2x} + [2x^3 - 3x^2 + 15x - 8]$$

P.I of f(D) y = Q(x) when Q(x) = e^{ax} V

P.I of f(D) y = Q(x) when Q(x) = e^{ax} V where 'a' is a constant and V is function of x. where

V = sin ax or cos ax or x^k

$$\text{Then P.I} = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{f(D)} e^{ax} V$$

$$= e^{ax} \left[\frac{1}{f(D+a)}(V)\right]$$

& $\frac{1}{f(D+a)}$ V is evaluated depending on V.

Solve $(D^3 - 7D^2 + 14D - 8)y = e^x \cos 2x$

Given equation is

$(D^3 - 7D^2 + 14D - 8)y = e^x \cos 2x$

A.E is $(m^3 - 7m^2 + 14m - 8) = 0$

$(m-1)(m-2)(m-4) = 0$

Then $m = 1, 2, 4$

C.F $= c_1 e^x + c_2 e^{2x} + c_3 e^{4x}$

$$P.I = \frac{e^{x}\cos 2x}{(D^3 - 7D^2 + 14D - 8)}$$

$$= e^{x} . \frac{1}{(D+1)^3 - 7(D+1)^2 + 14(D+1) - 8} . \text{Cos}2x$$

$$\left[\because P.I = \frac{1}{f(D)} e^{ax} v = e^{ax} \frac{1}{f(D+a)} v\right]$$

$$= e^{x} . \frac{1}{(D^3 - 4D^2 + 3D)} . \cos 2x$$

$$= e^{x} . \frac{1}{(-4D + 3D + 16)} . \cos 2x \text{ (Replacing } D^2 \text{ with } -2^2)$$

$= e^{x}.\frac{1}{(16-D)}.\cos 2x$

$= e^{x}.\frac{16+D}{(16-D)(16+D)}.\cos 2x$

$= e^{x}.\frac{16+D}{256-D^2}.\cos 2x$

$= e^{x}.\frac{16+D}{256-(-4)}.\cos 2x$

$= \frac{e^{x}}{260}(16\cos 2x - 2\sin 2x)$

$= \frac{2e^{x}}{260}(8\cos 2x - \sin 2x)$

$= \frac{e^{x}}{130}(8\cos 2x - \sin 2x)$

General solution is $y = y_c + y_p$

$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{4x} + \frac{e^{x}}{130}(8\cos 2x - \sin 2x)$

Solve $(D^2 - 4D+4)y = x^2 sinx + e^{2x}+3$

Sol:Given $(D^2 - 4D+4)y = x^2\ sinx + e^{2x}+3$

A.E is $(m^2 - 4m + 4) = 0$

$(m - 2)^2 = 0$ then m=2,2

C.F. = $(c_1 + c_2x)e^{2x}$

P.I = $\frac{x^2 sinx+ e^{2x} +3}{(D-2)^2} = \frac{1}{(D-2)^2}(x^2 sinx) + \frac{1}{(D-2)^2}e^{2x} + \frac{1}{(D-2)^2}(3)$

Now $\frac{1}{(D-2)^2}(x^2 sinx) = \frac{1}{(D-2)^2}(x^2)$ (I.P of e^{ix})

$= \text{I.P of } \frac{1}{(D-2)^2}(x^2)(e^{ix})$

$= \text{I.P of } (e^{ix}) . \frac{1}{(D+i-2)^2} (x^2)$

On simplification, we get

$\frac{1}{(D+i-2)^2} (x^2 \; sinx) = \frac{1}{625} [(220x+244)cosx+(40x+33)sinx]$

and $\frac{1}{(D-2)^2} (e^{2x}) = \frac{x^2}{2} (e^{2x})$,

$\frac{1}{(D-2)^2} (3) = \frac{3}{4}$

$P.I = \frac{1}{625} [(220x+244)cosx+(40x+33)sinx] + \frac{x^2}{2} (e^{2x}) + \frac{3}{4}$

$y = y_c + y_p$

$y = (c_1 + c_2x)e^{2x} + \frac{1}{625} [(220x+244)cosx+(40x+33)sinx] + \frac{x^2}{2} (e^{2x}) + \frac{3}{4}$

P.I. of f(D)y=Q(x) where $Q(x)=x^m v$ where v is a function of x.

$$\text{Then P.I.} = \frac{1}{f(D)} \times Q(x) = \frac{1}{f(D)} x^m v = I.P. of \frac{1}{f(D)} x^m (\cos ax + i \sin ax)$$

$$= I.P. of \frac{1}{f(D)} x^m e^{iax}$$

ii. $\text{P.I.} = \frac{1}{f(D)} x^m \cos ax = R.P. of \frac{1}{f(D)} x^m e^{iax}$

Solve $(D^2 - 4D + 4)y = x^2 sinx + e^{2x} + 3$

Sol:Given $(D^2 - 4D + 4)y = x^2 sinx + e^{2x} + 3$

A.E is $(m^2 - 4m + 4) = 0$

$(m - 2)^2 = 0$ then m=2,2

C.F. = $(c_1 + c_2 x)e^{2x}$

P.I = $\frac{x^2 sinx + e^{2x} + 3}{(D-2)^2} = \frac{1}{(D-2)^2}(x^2 sinx) + \frac{1}{(D-2)^2}e^{2x} + \frac{1}{(D-2)^2}(3)$

$$P.I = \frac{1}{625}[(220x+244)\cos x+(40x+33)\sin x] + \frac{x^2}{2}(e^{2x}) + \frac{3}{4}$$

$$y = y_c + y_p$$

$$y = (c_1 + c_2x)e^{2x} + \frac{1}{625}[(220x+244)\cos x+(40x+33)\sin x] + \frac{x^2}{2}(e^{2x}) + \frac{3}{4}$$

Working Rule :

1. Reduce the given equation of the form $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R$
2. Find C.F.
3. Take P.I. y_p=Au+Bv where A= $-\int\frac{vRdx}{uv^1-vu^1}$ and $B=\int\frac{uRdx}{uv^1-vu^1}$
4. Write the G.S. of the given equation $y=y_c+y_p$

Problems:

1. Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2}$ + y = cosecx

Sol: Given equation in the operator form is $(D^2+1)y = cosecx$--------------(1)

A.E is (m^2+1) = 0

$\therefore m = \pm i$

The roots are complex conjugate numbers.

$\therefore$ C.F. is $y_c = c_1\cos x + c_2\sin x$

Let y_p = Acosx + Bsinx be P.I. of (1)

$u\frac{dv}{dx} - v\frac{du}{dx} = \cos^2 x + \sin^2 x = 1$

A and B are given by

$$A = -\int \frac{vRdx}{uv^1 - vu^1} = -\int \frac{\sin x \ \operatorname{cosec} x}{1} dx = -\int dx = -x$$

$$B = \int \frac{uRdx}{uv^1 - vu^1} = \int \cos x . \operatorname{cosec} x \, dx = \int \cot x \, dx = \log(\sin x)$$

$\therefore y_p = -x\cos x + \sin x . \log(\sin x)$

$\therefore$ General solution is $y = y_c + y_p$.

$y = c_1\cos x + c_2\sin x - x\cos x + \sin x . \log(\sin x)$

2. Solve $(4D^2 - 4D + 1)y = 100$

Sol: A.E is $(4m^2 - 4m + 1) = 0$

$(2m - 1)^2 = 0$ then $m = \frac{1}{2}, \frac{1}{2}$

C.F = $(c_1 + c_2 x)\, e^{\frac{x}{2}}$

P.I = $\frac{100}{(4D^2 - 4D + 1)} = \frac{100\, e^{0.x}}{(2D-1)^2} = \frac{100}{(0-1)^2} = 100$

Hence the general solution is y = C.F + P.I

$y = (c_1 + c_2 x)\, e^{\frac{x}{2}} + 100$

MODULE-II

MATRIX LINEAR TRANSFORMATION
AND
DOUBLE INTEGRALS

Eigen Values & Eigen Vectors

Def: Characteristic vector of a matrix:

Let A= [a_{ij}] be an n x n matrix. A non-zero vector X is said to be a Characteristic Vector of A if there exists a scalar such that $AX=\lambda X$.

Note: If AX=λX (X≠0), then we say 'λ' is the eigen value (or) characteristic root of 'A'.Eg: Let A= $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ X = $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

AX = $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ = $\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1.\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$= 1.X$

.

Here Characteristic vector of A is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and Characteristic root of A is "1".

<u>Note:</u> We notice that an eigen value of a square matrix A can be 0. But a zero vector cannot be an eigen vector of A.

Method of finding the Eigen vectors of a matrix.

Let $A = [a_{ij}]$ be a nxn matrix. Let X be an eigen vector of A corresponding to the eigen value λ.

Then by definition $AX = \lambda X$.

$\Rightarrow \quad AX = \lambda IX$

$\Rightarrow \quad AX - \lambda IX = 0$

$\Rightarrow \quad (A-\lambda I)X = 0$ -------- (1)

This is a homogeneous system of n equations in n unknowns.

Will have a non-zero solution X if and only $|A-\lambda I| = 0$

$A-\lambda I$ is called characteristic matrix of A

$|A-\lambda I|$ is a polynomial in λ of degree n and is called the characteristic polynomial of A

$|A-\lambda I|=0$ is called the characteristic equation

Solving characteristic equation of A, we get the roots , $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots \lambda_n$, These are called the characteristic roots or eigen values of the matrix.

- Corresponding to each one of these n eigen values, we can find the characteristic vectors.

  - **Procedure to find eigen values and eigen vectors**

  - Let A = $\begin{bmatrix} a_{11} & a_{12} \dots \dots & a_{1n} \\ a_{21} & a_{22} \dots \dots & a_{2n} \\ \dots \dots & \dots \dots & \dots \dots \\ a_{n1} & a_{n2} \dots \dots & a_{nn} \end{bmatrix}$ *be a given matrix*

Characteristic matrix of A is $A - \lambda I$

$$i.e., A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$$

Then the characterstic polynomial is $|A-\lambda I|$

$$say\phi(\lambda)=|A-\lambda I|=\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix}$$

The characteristic equation is $|A-\lambda I| = 0$ we solve the $\phi(\lambda) = |A - \lambda I| = 0$, we get n roots, these are called eigen values or latent values or proper values.

Let each one of these eigen values say λ their eigen vector X corresponding the given value λ is obtained by solving Homogeneous system

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ 0 \end{bmatrix}$$

And determining the non-trivial solution.

1. **Find the eigen values and the corresponding eigen vectors of matrix** $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

The characteristic equation is $|A-\lambda I|=0$

i.e. $|A-\lambda I| = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$

$$\Rightarrow (2-\lambda)(2-\lambda)^2 - 0 + [-(2-\lambda)] = 0$$

$$\Rightarrow (2-\lambda)^3 — (\lambda - 2) = 0$$

$$\Rightarrow \lambda - 2\,[-(\lambda-2)^2 - 1] = 0$$

$$\Rightarrow \lambda - 2\,[-\lambda^2 + 4\lambda - 3] = 0$$

$$\Rightarrow (\lambda-2)(\lambda-3)(\lambda-1) = 0$$

⇨ $\lambda = 1,2,3$

The eigen values of A is 1,2,3.

EIGEN VALUES AND EIGEN VECTORS

For finding eigen vector the system is $(A-\lambda I)X=0$

$$\Rightarrow \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigen vector corresponding to $\lambda = 1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_3 = 0$$

$$x_2 = 0$$

$$x_1 + x_3 = 0$$

$$x_1 = -x_3, x_2 = 0$$

$$say\ x_3 = \alpha$$

$$x_1 = -\alpha \quad x_2 = 0, \quad x_3 = \alpha$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\alpha \\ 0 \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \textit{ is Eigen vector}$$

Eigen vector corresponding to $\lambda = 2$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here $x_1 = 0$ *and* $x_3 = 0$ *and we can take any arbitary value* x_2 *i.e* $x_2 = \alpha$ *(say)*

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Eigen vector is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Eigen vector corresponding to $\lambda = 3$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_3 = 0$$

$$-x_2 = 0$$

$$x_1 - x_3 = 0$$

here by solving we get $x_1 = x_3, x_2 = 0$ *say* $x_3 = \propto$

$x_1 = \propto$, $x_2 = 0$, $x_3 = \propto$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Eigen vector is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Properties of Eigen Values:

Theorem 1: The sum of the eigen values of a square matrix is equal to its trace and product of the eigen values is equal to its determinant.

Proof: Characteristic equation of A is

$$|A-\lambda I|=0$$

, $$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$$ *expanding this we get*

$$(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)-a_{12}$$

$(a \text{ polynomial of degree } n-2)$

$+ a_{13}$ $(a \text{ polynomial of degree } n -2) + \ldots = 0$

$$\Rightarrow (-1)^n\left[\lambda^n - (a_{11}+a_{22}+....+a_{nn})\lambda^{n-1} + a\ polynomial\ \ of\ \deg ree\,(n-2)\right]=0$$

$$(-1)^n\lambda^n + (-1)^{n+1}(Trace\,A)\lambda^{n-1} + a\ polynomial\ of\ \deg ree\,(n-2)\, in\,\lambda = 0$$

$If\ \lambda_1, \lambda_2 \ ..... \lambda_n\ are\ the\ roots\ of\ this\ equation$

$sum\ of\ the\ roots = \frac{(-1)^{n+1}Tr(A)}{(-1)^n} = Tr(A)$s

$Further\ |A - \lambda I| = (-1)^n\lambda^n + \cdots . + a_0$

$put\ \lambda = 0\ then\ |A| = a_0$

$(-1)^n\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \ .... + a_0 = 0$

$Product\ of\ the\ roots = \frac{(-1)^n\, a_0}{(-1)^n} = a_0$

$but\ a_0 = \ |A| = \det A$

$Hence\ the\ result$

Theorem 2: If λ is an eigen value of A corresponding to the eigen vector X, then λ^n is eigen value A^n corresponding to the eigen vector X.

Proof: Since λ is an eigen value of A corresponding to the eigen value X, we have

PROPERTIES OF EIGEN VALUES

$AX = \lambda X$ -----------(1) Pre multiply (1) by A,

$A(AX) = A(\lambda X)$

$(AA)X = \lambda(AX)$

$A^2X = \lambda(\lambda X)$

$A^2X = \lambda^2 X$

λ^2 is eigen value of A^2 with X itself as the corresponding eigen vector.

λ^n is an eigen value of A^n

PROPERTIES OF EIGEN VALUES

Theorem 3: A Square matrix A and its transpose A^T have the same eigen values.

Theorem 4: If A and B are n-rowed square matrices and If A is invertible show that $A^{-1}B$ and BA^{-1} have same eigen values.

Theorem 5: If $\lambda_1, \lambda_2, \ldots\ldots\ldots\lambda_n$ are the eigen values of a matrix A then $k\lambda_1$, $k\lambda_2$, ... $k\lambda_n$ are the eigen value of the matrix KA, where K is a non-zero scalar.

Theorem 6: If λ is an eigen values of the matrix A then $\lambda+K$ is an eigen value of the matrix A+KI

Theorem 7: If $\lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of A, then

$\lambda_1 - K, \quad \lambda_2 - K, \ldots \quad \lambda_n - K,$

are the eigen values of the matrix $(A - KI)$, where K is a non-zero scalar

Theorem 8: If $\lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of A, find the eigen values of the matrix $(A - \lambda I)^2$

<u>Theorem 9:</u> If λ is an eigen value of a non-singular matrix A corresponding to the eigen vector X, then λ^{-1} *is an eigen value of* A^{-1} *and corresponding eigen vector X itself.*

PROPERTIES OF EIGEN VALUES

Theorem 15: The eigen values of a triangular matrix are just the diagonal elements of the matrix.

Theorem 16: The eigen values of a real symmetric matrix are always real.

Theorem 17: For a real symmetric matrix, the eigen vectors corresponding to two distinct eigen values are orthogonal.

1. Find the Eigen values of $3A^3 + 5A^2 - 6A + 2I$ where $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

Sol: The characteristic equation of A is $|A - \lambda I| = 0$

i.e, $\begin{vmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$

$\Longrightarrow [(1-\lambda)(3-\lambda)(-2-\lambda) - 0] = 0$

$\Longrightarrow (1-\lambda)(3-\lambda)(2+\lambda) = 0$ $\quad\quad \lambda = 1, 3, -2$

Eigen values of A are $1, 3, -2$

We know that if λ is an eigen value of A and $f(A)$ is a polynomial in A.

then the eigen value of $f(A)$ is $f(\lambda)$

Let f(A) = $3A^3 + 5A^2 - 6A + 2I$

Then eigen values of f(A) are f(1), f(3) and f(-2)

$f(1) = 3(1)^3+5(1)^2-6(1)+2(1) = 4$

$f(3) = 3(3)^3+5(3)^2-6(3)+2(1) = 110$

$f(-2) = 3(-2)^3+5(-2)^2-6(-2)+2(1) = 10$

Eigen values of $3A^3 + 5A^2 - 6A + 2I$ are 4,110,10

Cayley - Hamilton Theorem: Every square matrix satisfies its own characterstic equation.

Q)Show that the matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ satisfies its characteristic equation Hence find A^{-1}

Sol: Characteristic equation of A is det

(A-λI) = 0

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 & 2 \\ 1 & -2-\lambda & 3 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$$

C2 → C2+C3

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1-\lambda & 3 \\ 0 & 1-\lambda & 2-\lambda \end{vmatrix} = 0$$

PROBLEM

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$

By Cayley – Hamilton theorem, we have

A^3-A^2+A-I=0

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} \quad A^2 = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \quad A^3 = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$A^3 - A^2 + A - I = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Multiplying with A^{-1} we get $A^2 - A + I = A^{-1}$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

PROBLEM

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Multiplying with A^{-1} we get $A^2 - A + I = A^{-1}$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

1.Using Cayley - Hamilton Theorem find the inverse and A^4 of the matrix $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

2.Verify Cayley – Hamilton Theorem for $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$.

Hence find A^{-1}.

Diagonalization of a matrix:

Theorem: If a square matrix A of order n has n linearly independent eigen vectors $(X_1, X_2 \ldots X_n)$ corresponding to the n eigen values $\lambda_1, \lambda_2 \ldots \lambda_n$ respectively then a matrix P can be found such that

$P^{-1}AP$ is a diagonal matrix.

Proof: Given that $(X_1, X_2 \ldots X_n)$ be eigen vectors of A corresponding to the eigen values $\lambda_1, \lambda_2 \ldots \lambda_n$ respectively and these eigen vectors are linearly independent

Define $P = (X_1, X_2 \ldots X_n)$

Since the n columns of P are linearly independent $|P| \neq 0$

Hence P^{-1} exists

Consider $AP = A[X_1, X_2 \ldots X_n]$

$= [AX_1, AX_2 \ldots\ldots AX_n]$

$= [\lambda X_1, \lambda_2 X_2 \ldots\ldots \lambda_n X_n]$

$$[X_1, X_2 \ldots X_n] \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

$$= PD$$

Where D = diag $(\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\ldots\ldots \lambda_n)$

AP=PD

$$P^{-1}(AP) = P^{-1}\ (PD) \Rightarrow P^{-1}AP = (P^{-1}P)D$$

$$\Rightarrow P^{-1}AP = (I)D$$

$$= D$$

$= \text{diag}\ (\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\ldots\ldots \lambda_n)$

Hence the theorem is proved.

Modal and Spectral matrices:

The matrix P in the above result which diagonalize the square matrix A is called modal matrix of A and the resulting diagonal matrix D is known as spectral matrix.

Note 1: If $X_1, X_2 \ldots X_n$ are not linearly independent this result is not true.

2: Suppose A is a real symmetric matrix with n pair wise distinct eigen values $\lambda_1, \lambda_2 \cdots \lambda_n$ then the corresponding i.e, $P^T P = P P^T = I$

Hence $P^{-1} = P^T$

$P^{-1}AP = D \Rightarrow P^T AP = D$

Calculation of powers of a matrix:

We can obtain the power of a matrx by using diagonalization

Let A be the square matrix then a non-singular matrix P can be found such that $D = P^{-1}AP$

$D^2 = (P^{-1}AP)\,(P^{-1}AP)$

$= P^{-1}A(PP^{-1})AP$

POWERS OF A MATRIX

$= P^{-1}A^2P$ (since $PP^{-1}=I$)

Simlarly $D^3 = P^{-1}A^3P$

In general $D^n = P^{-1}A^nP$........(1)

To obtain A^n, Premultiply (1) by P and post multiply by P^{-1}

Then $PD^nP^{-1} = P(P^{-1}A^nP)P^{-1}$

$= (PP^{-1})A^n(PP^{-1}) = A^n \Rightarrow A^n = PD^nP^{-1}$

$$\text{Hence } A^n = P\begin{bmatrix} \lambda_1^n & 0 & 0\cdots & 0 \\ 0 & \lambda_2^n & 0\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \lambda_n^n \end{bmatrix}P^{-1}$$

PROBLEM

1. Determine the modal matrix P of $= \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. Verify that $P^{-1}AP$ is a diagonal matrix.

Sol: The characteristic equation of A is

$|A-\lambda I| = 0$

i.e, $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

which gives $(\lambda-5)(\lambda+3)^2 = 0$

Thus the eigen values are $\lambda=5$, $\lambda=-3$ and $\lambda=-3$

when λ=5 $\Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By solving above we get $X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Similarly, for the given eigen value λ=-3

we can have two linearly independent

eigen vectors $X_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ $and\ X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

$P = (X_1\ X_2\ \ X_3)$

$P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = modal\ matrix\ of\ A$

$Now \det P = 1(-1) - 2(2) + 3(0-1) = -8$

$$P^{-1} = \frac{adj\ P}{\det P} = -\frac{1}{8}\begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$-\frac{1}{8}\begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$= -\frac{1}{8}\begin{bmatrix} -5 & -10 & 15 \\ 6 & -12 & -18 \\ 3 & 6 & 15 \end{bmatrix}$$

$$P^{-1}AP = -\frac{1}{8}\begin{bmatrix} -40 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = diag\,(5, -3, -3)$$

Hence $P^{-1}AP$ is a diagonal matrix.

Problems

1. Diagonalize the matrix

(i) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

MULTIPLE INTEGRALS

- Double integrals
- Triple integrals
- Change of order of integration
- Transformation of coordinate system;
- Determination of areas by double integrals

Double integrals

The expression:

$$\int_{y=y_1}^{y_2}\int_{x=x_1}^{x_2} f(x,y)dx.dy$$

is called a *double integral* and indicates that $f(x, y)$ is first integrated with respect to x and the result is then integrated with respect to y

If the four limits on the integral are all constant the order in which the integrations are performed does not matter.

If the limits on one of the integrals involve the other variable then the order in which the integrations are performed is crucial.

Double Integral :

I. When y_1, y_2 are functions of x and x_1 and x_2 are constants. f(x,y)is first integrated w.r.t y keeping 'x' fixed between limits y_1, y_2 and then the resulting expression is integrated w.r.t 'x' with in the limits x_1, x_2 i.e.,

$$\iint_R f(x,y)\,dx\,dy = \int_{x=x_1}^{x=x_2} \int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x,y)\,dy\,dx$$

II. When x_1, x_2 are functions of y and y_1, y_2 are constants, f(x,y)is first integrated w.r.t 'x' keeping 'y' fixed, with in the limits x_1, x_2 and then resulting expression is integrated w.r.t 'y' between the limits y_1, y_2 i.e.,

$$\iint_R f(x,y)\,dx\,dy = \int_{y=y_1}^{y=y_2} \int_{x=\phi_1(y)}^{x=\phi_2(y)} f(x,y)\,dx\,dy$$

III. When x_1, x_2, y_1, y_2 are all constants. Then

$$\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x,y)\,dx\,dy = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y)\,dy\,dx$$

1. Evaluate $\int_1^2 \int_1^3 xy^2 dx\, dy$

Sol. $\int_1^2 \left[\int_1^3 xy^2 dx \right] dy$

$$= \int_1^2 \left[y^2 . \frac{x^2}{2} \right]_1^3 dy = \int_1^2 \frac{y^2}{2} dy [9-1]$$

$$= \frac{8}{2} \int_1^2 y^2 dy = 4. \int_1^2 y^2 dy$$

$$= 4. \left[y^3 / 3 \right]_1^2 = \frac{4}{3} [8-1] = \frac{4.7}{3}$$

$$= \frac{28}{3}$$

Evaluate $\int_0^2 \int_0^x y \, dy \, dx$

Sol. $\int_{x=0}^{2} \int_{y=0}^{x} y \, dy \, dx = \int_{x=0}^{2} \left[\int_{y=0}^{x} y \, dy \right] dx$

$$= \int_{x=0}^{2} \left[\frac{y^2}{2} \right]_0^x dx = \int_{x=0}^{2} \frac{1}{2}\left(x^2 - 0\right) dx = \frac{1}{2} \int_{x=0}^{2} x^2 dx = \frac{1}{2}\left[\frac{x^3}{3} \right]_0^2 = \frac{1}{6}\left(8 - 0\right)$$

Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2}$

Sol: $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2} = \int_{x=0}^{1} \left[\int_{y=0}^{\sqrt{1+x^2}} \frac{1}{\left(1+x^2\right)+y^2} dy \right] dx$

$$= \int_{x=0}^{1} \left[\int_{y=0}^{\sqrt{1+x^2}} \frac{1}{\left(\sqrt{1+x^2}\right)^2 + y^2} dy \right] dx = \int_{x=0}^{1} \frac{1}{\sqrt{1+x^2}} \left[Tan^{-1} \frac{y}{\sqrt{1+x^2}} \right]_{y=0}^{\sqrt{1+x^2}} dx$$

$[\because \int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}(x/a)]$

$$= \int_{x=0}^{1} \frac{1}{\sqrt{1+x^2}} \left[Tan^{-1}1 - Tan^{-1}0 \right] dx$$

$$or \quad \frac{\pi}{4} (\sinh^{-1} x)_0^1 = \frac{\pi}{4} (\sinh^{-1} 1)$$

$$= \frac{\pi}{4} \int_{x=0}^{1} \frac{1}{\sqrt{1+x^2}} dx = \frac{\pi}{4} \left[\log(x + \sqrt{x^2+1}) \right]_{x=0}^{1}$$

$$= \frac{\pi}{4} \log(1+\sqrt{2})$$

Evaluate $\int_0^1 \int_0^{\pi/2} r \sin\theta \, d\theta \, dr$

Sol. $\int_{r=0}^{1} r \left[\int_{\theta=0}^{\pi/2} \sin\theta \, d\theta \right] dr$

$$\int_{r=0}^{1} r \, (-\cos\theta)_{\theta=0}^{\pi/2} \, dr$$

$$\int_{r=0}^{1} -r \left(\cos \pi/2 - \cos 0\right) dr$$

$$\int_{r=0}^{1} -r(0-1)\,dr = \int_0^1 r\,dr = \left(\frac{r^2}{2}\right)_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

Evaluate $\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}$

Sol.

$$\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}} =$$

$$\int_0^{\pi/4}\left\{\int_0^{a\sin\theta} \frac{r}{\sqrt{a^2-r^2}}\,dr\right\}d\theta =$$

$$-1/2\int_0^{\pi/4}\left\{\int_0^{a\sin\theta} \frac{-2r}{\sqrt{a^2-r^2}}\,dr\right\}d\theta$$

$$\frac{-1}{2}\int_0^{\pi/4} 2\left(\sqrt{a^2-r^2}\right)_0^{a\sin\theta} d\theta = (-1)\int_0^{\pi/4} 2\left[\sqrt{a^2-a^2\sin^2\theta} - \sqrt{}\right.$$

$$(-a)\int_0^{\pi/4}(\cos\theta-1)\,d\theta = (-a)(\sin\theta-\theta)_0^{\pi/4}$$

$$(-a)\left[\left[\sin \pi/4 - \pi/4\right] - (0-0)\right]$$

$$(-a)\left[1/\sqrt{2} - \pi/4\right] = 2\left[\pi/4 - 1/\sqrt{2}\right]$$

Problems

1. Evaluate $\int_1^2 \int_1^3 xy^2 dx\, dy$

Sol. $\int_1^2 \left[\int_1^3 xy^2 dx \right] dy$

$$= \int_1^2 \left[y^2 . \frac{x^2}{2} \right]_1^3 dy = \int_1^2 \frac{y^2}{2} dy [9-1]$$

$$= \frac{8}{2} \int_1^2 y^2 dy = 4. \int_1^2 y^2 dy$$

$$= 4. \left[{y^3}/{3} \right]_1^2 = \frac{4}{3}[8-1] = \frac{4.7}{3} = \frac{28}{3}$$

4. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2}$

Sol: $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2} = \int_{x=0}^1 \left[\int_{y=0}^{\sqrt{1+x^2}} \frac{1}{(1+x^2)+y^2} dy \right] dx$

$$= \int_{x=0}^1 \left[\int_{y=0}^{\sqrt{1+x^2}} \frac{1}{\left(\sqrt{1+x^2}\right)^2 + y^2} dy \right] dx = \int_{x=0}^1 \frac{1}{\sqrt{1+x^2}} \left[Tan^{-1} \frac{y}{\sqrt{1+x^2}} \right]_{y=0}^{\sqrt{1+x^2}} dx \; [\because \int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}(x/a)]$$

$$= \int_{x=0}^1 \frac{1}{\sqrt{1+x^2}} \left[Tan^{-1}1 - Tan^{-1}0 \right] dx \quad or \quad \frac{\pi}{4}(\sinh^{-1} x)_0^1 = \frac{\pi}{4}(\sinh^{-1} 1)$$

$$= \frac{\pi}{4} \int_{x=0}^1 \frac{1}{\sqrt{1+x^2}} dx = \frac{\pi}{4} \left[\log(x + \sqrt{x^2+1}) \right]_{x=0}^1$$

$$= \frac{\pi}{4} \log(1+\sqrt{2})$$

10. Evaluate $\iint xy(x+y)dxdy$ over the region R bounded by y=x^2 and y=x

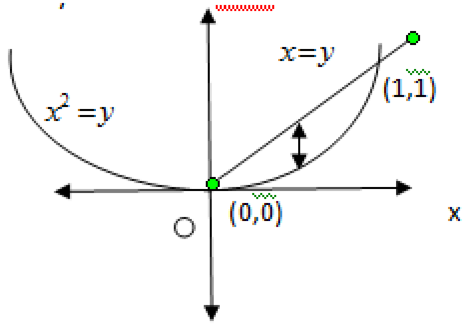
Sol: y=x^2 is a parabola through (0, 0) symmetric about y-axis y=x is a straight line through (0,0) with slope1.

Let us find their points of intersection solving y=x^2, y=x we get x^2=x $\Rightarrow$ x=0,1Hence y=0, 1

$\therefore$ The point of intersection of the curves are

(0,0), (1,1)

Consider $\iint_R xy(x+y)dxdy$



$$= \int_{x=0}^{1} \left(x^2 \frac{y^2}{2} + \frac{xy^3}{3} \right)_{y=x^2}^{x} dx$$

$$= \int_{x=0}^{1} \left(\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3} \right) dx$$

$$= \int_{x=0}^{1} \left(\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3} \right) dx$$

$$= \left(\frac{5}{6} \cdot \frac{x^5}{5} - \frac{x^7}{14} - \frac{x^8}{24} \right)_0^1 = \frac{1}{6} - \frac{1}{14} - \frac{1}{24} = \frac{28-12-7}{168} = \frac{28-19}{168} = \frac{9}{168} = \frac{3}{56}$$

11. Evaluate $\iint_R xy\,dx\,dy$ where R is the region bounded by x-axis and x=2a and the curve $x^2=4ay$.

Sol. The line x=2a and the parabola $x^2=4ay$ intersect at B(2a,a)

∴ The given integral = $\iint_R xy\ dx\ dy$

Let us fix 'y'

For a fixed 'y', x varies from $2\sqrt{ay}$ to 2a.

Then y varies from 0 to a.

Hence the given integral can also be written as

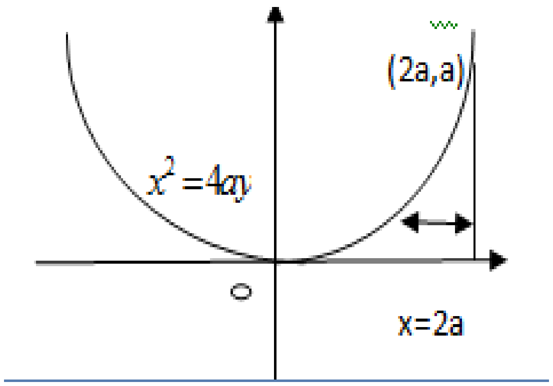
$$\int_{y=0}^{a}\int_{x=2\sqrt{ay}}^{x=2a} xy\,dx\,dy = \int_{y=0}^{a}\left[\int_{x=2\sqrt{ay}}^{x=2a} xdx\right]ydy$$

$$= \int_{y=0}^{a}\left[\frac{x^2}{2}\right]_{x=2\sqrt{ay}}^{2a} ydy$$

$$= \int_{y=0}^{a}\left[2a^2 - 2ay\right]y\,dy$$

$$= \left[\frac{2a^2y^2}{2} - \frac{2ay^3}{3}\right]_0^a$$

$$= a^4 - \frac{2a^4}{3} = \frac{3a^4 - 2a^4}{3} = \frac{a^4}{3}$$



12 Evaluate $\int_0^1 \int_0^{\pi/2} r \sin\theta \, d\theta \, dr$

Sol. $\int_{r=0}^{1} r \left[\int_{\theta=0}^{\pi/2} \sin\theta \, d\theta \right] dr$

$$= \int_{r=0}^{1} r\,(-\cos\theta)_{\theta=0}^{\pi/2}\, dr$$

$$= \int_{r=0}^{1} -r\left(\cos \pi/2 - \cos 0\right) dr$$

$$= \int_{r=0}^{1} -r(0-1)\, dr = \int_0^1 r\,dr = \left(\frac{r^2}{2}\right)_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

MODULE III

FUNCTIONS OF SINGLE VARIABLE AND TRIPLE INTEGRALS

And we also introduce function of several variables which are essential for the discussion of transcendental function and also maxima and minima of function of more than one variable with and without Constraints. In many engineering problems change of variables and transformation of co-ordinates play an important role in solving the problems. For such problems, Jacobian of functions of more than one variable and functional dependence are introduced.

Limits, Continuity and Differentiability:

The reader familiar with the concept of limit, continuity and differentiability for real valued functions. In this section, we give a brief review of these concepts, which form the basis of differential calculus.

Throughout this section we consider $f : A \to R$ where A is an interval in R. It may happen that for a function f,

As x approaches closer to a, the value $f(x)$ approaches closer to a definite real number l

(1) **Note** : The following are some fundamental properties of continuous functions.

(2) **Definition:** A function f is said to approach to a limit l as x tends to a, if given $\in > 0$ there exists $\delta > 0$ such that

$0 < |x - a| < \delta |f(x) - l| < \in$.

We write f(x) = 1

$x \to a$

(1) Definition: A function f is said to be continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$

If f is not continuous at $x = a$. We say that f is discontinuous at $x = a$.

A function f is said to be continuous if it is continuous at every point of its domain.

(a) If f and g are continuous at 'a', then $f+g$, $f-g$, fg, kf and f/g (if $g \neq 0$) are all continuous at 'a'.

(b) Intermediate Value Theorem: Let f be a continuous function defined on a closed interval $[a,b]$ and let $f(a) \neq f(b)$. Let c be any real number lying between $f(a)$ and $f(b)$. Then there exists $\propto \in (a,b)$ such that $f(\propto) = c$.

In other words any continuous function defined on a closed interval [a, b] assumes every value lying between f(a) and f(b) is **bounded.**

(a) **Let f be a continuous function defined on a closed interval [a, b]. Then there exists a real number M such that** $|f(x)| \leq M$ for all $x \in [a,b]$

In other words any continous function defined on a closed interval is bounded.

(3) **Definition:** A function f is said to be **differentiable** at x if $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ exists and is finite. The value of the limit is called the **derivative** or **differential coefficient** of f at x and is denoted by $f'(x)$ or $\frac{df}{dx}$ or $\frac{dy}{dx}$ where $y = f(x)$.

If the derivative of $f'(x)$ is differentiable, then the derivative $f'(x)$ is called the second derivative of $f(x)$ and is denoted by $f''(x)$ or $\frac{d^2 f}{dx^2}$ or $\frac{d^2 y}{dx^2}$ or y_2. Continuing this process, one can define n^{th} derivative of the function $y = f(x)$, which is denoted by $f^n(x)$, or $\frac{d^n f}{dx^n}$ or $\frac{d^n y}{dx^n}$ or y_n.

Note : If a function f is differentiable at x, then f is continuous at x. However the converse is not true.

For example the function $f(x) = |x|$ is continuous but not differentiable at $x = 0$.

Rolle's Theorem

Statement: Let $f(x)$ be a function defined on $[a,b]$ satisfying the following conditions.

(a) f is Continuous on (a,b)

(b) f is differentiable on (a,b)

(c) $f(a)=f(b)$

Then there exists at least one point $C\in(a,b)$ such that $f'(c)=0$

Geometrical Interpretation of Rolle's Theorem:

Interpreted geometrically in the following figure.

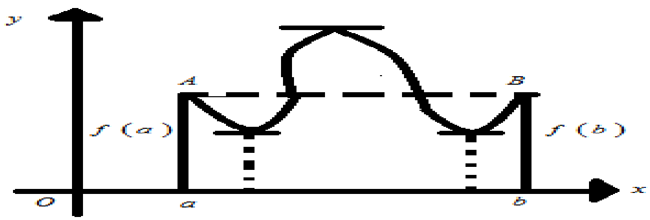


Fig.2.1

Rolle's Theorem says that the curve representing the graph of the function $y = f(x)$ must have a **tangent** parallel to the x-axis at same point between a and b.

Daily life application of rolles theorem

Since Rolle's theorem asserts the existence of a point where the derivative vanishes, I assume your students already know basic notions like continuity and differentiability. One way to illustrate the theorem in terms of a practical example is to look at the calendar listing the precise time for sunset each day. One notices that around the precise date in the summer when sunset is the latest, the precise hour changes very little from day to day in the vicinity of the precise date. This is an illustration of Rolle's theorem because near a point where the derivative vanishes, the function changes very little.

ROLLES THOEREM

Example 1:

Verify Rolle's Theorem for $f(x) = x^2 - 1$

in [-1,1]

Solution:

Given $f(x) = x^2 - 1$, Which is a polynomial in 'x'

(i) $f(x)$ is continuous in [-1,1], since it is polynomial function.

(ii) $f(x)$ is also derivable in (-1,1), since it is polynomial function

iii) $f(-1)=0$, $f(1)=0$

i.e. $f(-1)=f(1)$

Hence all the conditions of Rolle's theorem are satisfied for the function $f(x)=x^2-1$. Therefore there exists a constant, C such that $f'(c)=0$.

$$\therefore \quad f'(x)=2x$$

$$\Rightarrow \quad f(c)=2c=0$$

$$\therefore \quad C=0\in(-1,1)$$

i.e. C lies in the interval (-1,1)

Hence Rolle's theorem is verified.

Verify Rolle's theorem for the function $f(x) = (x-a)^m (x-b)^n$ in [a, b]

Solution:

Given $f(x) = (x-a)^m (x-b)^n$

(i) Since $f(x)$ is the product of two polynomial in x hence $f(x)$ is continuous in [a, b].

(ii) $f'(x) = m(x-a)^{m-1}(x-b)^n + (x-a)^m . \; n(x-b)^{n-1}$

$$= (x-a)^{m-1}(x-b)^{n-1}\left[n(x-a) + m(x-b)\right]$$

$\therefore$ $f'(x)$ exists for all $x \in (a,b)$

$\therefore$ $f(x)$ is differentiable in (a,b)

(iii) Also $f(x) = f(b) = 0$

$\therefore$ $f(x)$ satisfies all the conditions of Roll's Theorem.

Then $\exists$ $C \in (a,b)$ such that

$$f'(c) = 0$$

$\Rightarrow \quad (c-a)^{m-1}(c-b)^{n-1}\{n(c-a)+m(c-b)\}=0$

$\Rightarrow C = a, c = b, n(c-a)+m(c-b)=0$

$\Rightarrow \quad C = \dfrac{na + mb}{m + n}$

$\therefore \quad C = \dfrac{na + mb}{m + n} \in (a,b)$

Hence Rolle's Theorem is verified.

Verify whether Rolle's Theorem can be applied to the following function in the intervals cited :

(i) $f(x) = \tan x$ in $[0, \pi]$

Solution:

$f(x)$ is discontinuous at $x = \pi/2$ as, it is not defined there.

$\therefore$ The condition (1) of Roll's Theorem is not satisfied. Hence we cannot apply Rolle's theorem.

(ii) $f(x) = \frac{1}{x^2}$ in [-1, 1]

It is discontinuous at $x = 0$. Hence we cannot apply.

Verify Rolle's theorem for $f(x) = |x|$ in [-1 ,1]

Solution:

We have $f(x) = |x|$

i.e. $f(x) = x$, for $x \geq 0$

$= -x$, for $x < 0$

(i) $f(x)$ is continuous for all values of x.

$\therefore$ $f(x)$ is continuous in the closed interval

[-1, 1]

(ii) $f(x)$ is not derivable at $x = 0$

We have $f(0) = |0| = 0$

L.H.D. $= f'(0) = \lim_{x \to 0^-} = \dfrac{f(x) - f(0)}{x - 0}$

$$= \lim_{x \to 0^-} = \frac{|x| - 0}{x}$$

$$= \lim_{x \to 0^-} = \frac{-x}{x} = -1$$

R.H.S. $f'(0) = \lim_{x \to 0^+} = \frac{f(x) - f(0)}{x - 0}$

$$= \underset{x \to 0^+}{Lt} = \frac{|x| - 0}{x}$$

$$= \underset{x \to 0^+}{Lt} = \frac{x}{x} = 1$$

$\therefore$ L.H.D. $\neq$ R.H.D. $f(x)$ is not derivable in the open interval (-1, 1)

$\therefore$ Roll's Theorem is not applicable.

EXERCISE

Verify Rolle's Theorem for the following functions in the intervals indicated.

(i) $f(x)=(x-a)^3(x-b)^4$ in $[a,b]$

(ii) $f(x)=e^{-x}Sinx$ in $[0,\pi]$

(iii) $f(x)=x^2-2x$ in $[0,2]$

(iv) $f(x)=x(x+3)e^{-x/2}$ in $[-3,0]$

Daily life application of lagranges method

Well, for Lagrange's theorem (if you mean the mean value theorem) there's always the story about the hiker who goes up a mountain one day and down again the other. The question is, as he's walking down, will he ever be at some point on the path exactly 24 hours after he was there last? This is without assuming he walks at an even pace. He can walk slowly uphill and run downhill if he wants. The only thing he's not allowed to do is deviate from the path, and teleport.

2 Lagrange's Mean Value Theorem

Statement: Let $f(x)$ be a function defined on [a, b] satisfying the following conditions.

(a) f is continuous on (a, b)

(b) f is differentiable on (a, b)

Then, there at least one point $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Geometrical Interpretation of Lagrange's mean value theorem:

Consider the graph of the curve $y = f(x)$, $P[a, f(a)]$ and $Q[b, f(b)]$ are two points on the curve. Hence slope of the chord PA is $\frac{f(b) - f(a)}{b - a}$.

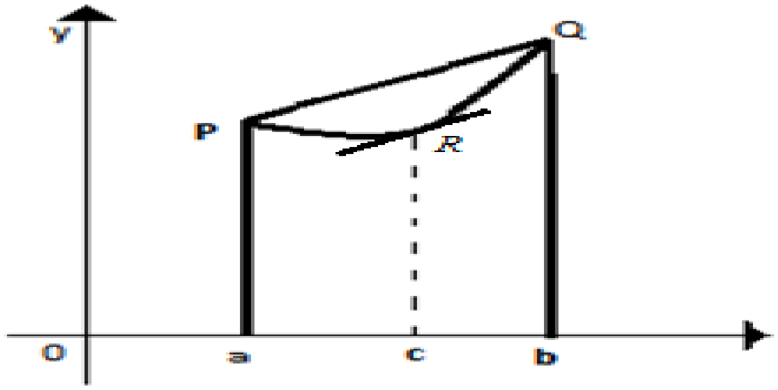


Fig.2.2

Also $f'(c)$ represents the slope of the tangent of the curve $f(x)$ at $R[c, f(c)]$.

The relation $\frac{f(b)-f(a)}{b-a} = f'(c)$ means that the tangent at R is parallel to the chord PQ.

Find C of Lagrange's mean value theorem (L.M.V.T) for the function $f(x)=e^x$ in [0, 1]

Solution:

Here we have

$$f(x)=e^x, a=0, b=1$$

(i) $f(x)$ is continuous in [0, 1]and

(ii) $f(x)$ is derivable in (0, 1) and $f'(x)=e^x$

$x \in (0,1)$

$f(x)$ satisfies both the conditions of L.M.V.T.

Therefore, there must be atleast one value $C \in (0,1)$ such that

$$F'(c) = \frac{f(b) - f(a)}{b - a}$$

i.e. $$e^c = \frac{e^1 - e^0}{1 - 0} = \frac{e - 1}{1}$$

i.e. $$e^c = e - 1$$

i.e. $$c = \log(e - 1) \in (0,1)$$

Hence, Lagrange mean value Theorem is verified

Example: S.T. for $0 < a < b < 1$

$$\frac{1}{1+a^2} > \frac{\tan^{-1} b - \tan^{-1} a}{1+a^2} > \frac{1}{1+b^2}$$

Solution:

Consider $f(x) = \tan^{-1} x$ in [a, b] for $0 < a < b < 1$

Since $f(x)$ is continuous in [a, b] and derivable in (a, b)

We can apply L.M.V.T. here

Hence there exists a pt c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Here $f'(x) = \frac{1}{1+x^2}$

And hence $f'(c) = \frac{1}{1+c^2}$

Thus, there exist a point $c, a < c < b$

Such that $\frac{1}{1+c^2} = \frac{\tan^{-1} b - \tan^{-1} a}{b - a}$

We have $$1+a^2 < 1+c^2 < 1+b^2$$

$\therefore$ $$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$

Using (1) and (2) we have

$$\frac{1}{1+a^2} > \frac{\tan^{-1} b - \tan^{-1} a}{1+a^2} > \frac{1}{1+b^2}$$

Hence the result

Calculate approximately $\sqrt[5]{245}$ by using L.M.V.T.

Solution:

Let $f(x) = \sqrt[5]{x}$ and $a = 243, b = 245$

Then $f'(x) = \frac{1}{5} x^{-4/5}$

And $f'(c) = \frac{1}{5} c^{-4/5}$

$\therefore$ By L.M.V.T. we have $\frac{f(b) - f(a)}{b - a} = f'(c)$

$$\Rightarrow \quad \frac{f(245) - f(243)}{245 - 243} = \frac{1}{5} c^{-4/5}$$

$$\Rightarrow \quad f(245) = f(243) + \frac{2}{5} c^{-4/5}$$

$$\sqrt[5]{245} = (243)^{1/5} + \frac{2}{5} c^{-4/5}$$

$\therefore$ C lies between 243 and 245. [Take c=244]

$$\therefore \quad \sqrt[5]{245} = 3.0049$$

Prove that $\frac{\pi}{3}-\frac{1}{5\sqrt{3}} > \cos^{-1}(3/5) > \frac{\pi}{3}-\frac{1}{8}$ using L.M.V.T.

Solution:

Let $f(x)=\cos^{-1}x$ and an interval [a, b]

Then $f'(x)=\frac{-1}{\sqrt{1-x^2}}$,

By L.M.V.T.

$$\frac{Cos^{-1}b-Cos^{-1}a}{b-a}=\frac{-1}{\sqrt{1-c^2}}$$ where $a<c<b$

LAGRANGES MEAN VALUE THEOREM

$\because \quad C \in (a,b)$

$$\therefore \quad a < c < b \Rightarrow a^2 < c^2 < b^2$$

$$\Rightarrow \quad -a^2 < -c^2 < -b^2$$

$$\Rightarrow \quad 1-a^2 > 1-c^2 > 1-b^2$$

$$\Rightarrow \quad \frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\Rightarrow \quad \frac{-1}{\sqrt{1-a^2}} > \frac{-1}{\sqrt{1-c^2}} > \frac{-1}{\sqrt{1-b^2}}$$

$$\frac{-1}{\sqrt{1-a^2}} > \frac{Cos^{-1}b - Cos^{-1}a}{b-a} > \frac{-1}{\sqrt{1-b^2}}$$

Let a=1/2 and b=3/5. Then

$$\frac{-2}{\sqrt{3}} > \frac{Cos^{-1}(3/5) - Cos^{-1}(1/2)}{\frac{3}{5}-\frac{1}{2}} > -5/4$$

$$\frac{-2}{\sqrt{3}} > \frac{Cos^{-1}(3/5) - Cos^{-1}(1/2)}{1/10} > -5/4$$

$$\frac{-2}{10\sqrt{3}} > \cos^{-1}(3/5) - \pi/3 > \frac{-5}{4} - \frac{1}{10}$$

$\Rightarrow$

$$\frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}\left(3/5\right) > \frac{\pi}{3} - \frac{1}{8}$$

Hence the result

Using Mean Value Theorem prove that $\tan x > x$ in $0 < x < \pi / 2$

Solution:

Consider $f(x) = \tan x$ in $0 < x < \pi / 2$

Take $f(x) = \tan x$ in $[\in, x]$, where

$0 < \in < x < \pi / 2$

Applying Lagranges Mean-Value Theorem to $f(x)$

There exists a point C such that

There exists a point C such that

$0 < \in < c < x < \pi / 2$

Such that $\frac{\tan x - \tan \in}{x - \in} = Sec^2 C$

$\Rightarrow \tan x - \tan \in = (x - \in)\sec^2 c$

Take $\in \to 0$, Then $\tan x = x \sec^2 c$

But $\sec^2 c > 1$. Hence $\tan x > x$

3 Cauchys' Mean Value Theorem (C.M.V.T)

Statement: Let $f(x)$ and $g(x)$ be functions defined on $[a,b]$ satisfying the following conditions.

(a) f and g are continuous on $[a,b]$

(b) f and g are differentiable on $[a,b]$

(c) $g'(x)$ does not vanish at any pt in $[a,b]$

[i.e. $g'(x) \neq 0 \forall x \in (a,b)$]

Then, there exists at least one point $c \in (a,b)$ such that,

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Example 1:

Verify Cauchy's mean value theorem for the function x^2 and x^3 in the interval [1, 2]

Solution:

Let $f(x) = x^2$ and $g(x) = x^3$

(i) $f(x)$ and $g(x)$ are continuous in [1, 2]

(ii) $f(x)$ and $g(x)$ are differentiable in [1, 2]

(iii) $g'(x) = 3x^2 \neq 0 \quad \forall \quad x \in [1, 2]$

$\therefore$ $f(x)$ and $g(x)$ satisfy all the conditions of Cauchy's mean value theorem.
Hence there exist at least one real number c in (1, 2) such that,

$$\frac{f'(c)}{g'(c)} = \frac{f(2)-f(1)}{g(2)-g(1)}$$

$$\Rightarrow \quad \frac{2c}{3c^2} = \frac{2^2-1^2}{2^3-1^3} = \frac{4-1}{8-1}$$

$$\Rightarrow \quad \frac{2}{3c} = \frac{3}{7}$$

$$\Rightarrow \quad C = \frac{14}{9}$$

$\therefore$ The value of $C = \frac{14}{9}$ lies in (1, 2)

$\therefore$ The value of $c = \frac{14}{9}$ lies in (1, 2)

Hence, Cauchy's mean value theorem is verified.

Example 2:

Verify Cauchy's mean value theorem for the functions $\log x$ and $\frac{1}{x}$ in [1, e]

Solution:

Here, we have

$f(x) = \log x, g(x) = \frac{1}{x}$, [a, b] = [1, e]

(i) Both $f(x)$ and $g(x)$ are continuous in [1, e]

(ii) Differentiable in (1, e)

(iii) Also $g'(x) = -\frac{1}{x^2} \neq 0$ in (1, e)

Since $f(x), g(x)$ satisfy all the functions of C.M.V.T. there exist at least one real number c in (1, e) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f(e)-f(1)}{g(e)-g(1)}$$

CAUCHY MEAN VALUE THOERM

e. $$\frac{1/2}{\left(-1/c^2\right)} = \frac{\log e - \log 1}{1/2 - 1}$$

$\Rightarrow$ $$-c = \frac{1-0}{\left(\frac{1-e}{e}\right)} = \frac{e}{1-e}$$

$\therefore$ $$c = \frac{e}{1-e} \in \left(1, e\right)$$

Example:If $f(x)=\log x$ and $g(x)=x^2$ in [a, b] with $b>a>1$, using C.M.V.T. Prove that

$$\frac{\log b-\log a}{b-a}=\frac{a+b}{2c^2}$$

Solution:

We are given $f(x)=\log x$

$\Rightarrow$ $f(a)=\log a,\ f(b)=\log b$

And $g(x)=x^2$

$\Rightarrow$ $g(a)=a^2, g(b)=b^2$

Also $$f'(x) = \frac{1}{x}$$

And $$g'(x) = 2x$$

$\therefore$ By Cauchy's mean value theorem

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \quad \frac{\log b - \log a}{b^2 - a^2} = \frac{1/c}{2c}$$

$$\Rightarrow \quad \frac{\log b - \log a}{(b-a)(b+a)} = \frac{1}{2c^2}$$

$$\frac{\log b - \log a}{b-a} = \frac{a+b}{2c^2}$$

Hence the result.

Triple integrals :

$_1$,x_2 are constants. y_1,y_2 are functions of x and z_1,z_2 are functions of x and y, then

,y,z) is first integrated w.r.t. 'z' between the limits z_1 and z_2 keeping x and y fixed.

e resulting expression is integrated w.r.t 'y' between the limits y_1 and y_2 keeping x

nstant. The resulting expression is integrated w.r.t. 'x' from x_1 to x_2

$$\iiint_V f(x,y,z)\,dx\,dy\,dz =$$

$$\int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x,y,z)\,dz\ \ dy\ \ dx$$

Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\, dx\, dy\, dz$

Sol

$$\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\, dx\, dy\, dz$$

$$= \int_{x=0}^{1} dx \int_{y=0}^{\sqrt{1-x^2}} dy \int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\, dz$$

$$= \int_{x=0}^{1} dx \int_{y=0}^{\sqrt{1-x^2}} xy \left(\frac{z^2}{2} \right)_{z=0}^{\sqrt{1-x^2-y^2}} dy$$

$$= \frac{1}{2} \int_{x=0}^{1} dx \int_{y=0}^{\sqrt{1-x^2}} xy \left(1 - x^2 - y^2\right) dy$$

$$= \frac{1}{2} \int_{x=0}^{1} dx \int_{y=0}^{\sqrt{1-x^2}} x \left[\left(1 - x^2\right) y - y^3 \right] dy$$

$$=\frac{1}{2}\int_{x=0}^{1} x\left[\left(1-x^2\right)\frac{y^2}{2}-\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$=\frac{1}{2}.\int_{x=0}^{1} x\left[\frac{y^2}{2}-\frac{x^2y^2}{2}-\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$=\frac{1}{8}.\int_{x=0}^{1} x\left[2\left(1-x^2\right)-2x^2\left(1-x^2\right)-\left(1-x^2\right)^2\right]dx$$

$$=\frac{1}{8}\int_{x=0}^{1}\left(x-2x^3+x^5\right)dx=\frac{1}{8}\left[\frac{x^2}{2}-\frac{2x^4}{4}+\frac{x^6}{6}\right]_0^1$$

$$=\frac{1}{8}\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right)=\frac{1}{48}$$

$\Rightarrow$

Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)dx\,dy\,dz$

' :

$$\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+y)\,dx\,dydz$$

$$=\int_{-1}^{1}\int_{0}^{z}\left[\left(xy+{y^2}\!\!\diagup_{2}+zy\right)_{x-z}^{x+z}\right]dx\,dz$$

$$=\int_{-1}^{1}\int_{0}^{z}x(x+z)-x(x-z)+\left[\frac{x+z}{2}\right]^2-\left[\frac{x-z}{2}\right]^2+z(x+z)-z(x-z)dx\,dz$$

$$=\int_{-1}^{1}\int_{0}^{z}\left[2z(x+z)+\frac{1}{2}4xz\right]dx\,dz$$

$$=2\int_{-1}^{1}\left[z.\frac{x^2}{2}+z^2x+z.\frac{x^2}{2}\right]_0^z dz$$

$$=2.\int_{-1}^{1}\left[\frac{z^3}{2}+z^3+\frac{z^3}{2}\right]dz=4.\left(\frac{z^4}{4}\right)_{-1}^{1}=0$$

MODULE IV

FUNCTIONS SEVERAL VARIBLES

PARTIAL DIFFERENTIATION

The partial differential coefficients of f_x and f_y are f_{xx}, f_{xy}, f_{yx}, f_{yy} or $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}$, respectively.

It should be specially noted that $\frac{\partial^2 f}{\partial y \partial x}$ means $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ and $\frac{\partial^2 f}{\partial x \partial y}$ means $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$.

The student will be able to convince himself that in all ordinary cases

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

CHAIN RULE OF PARTIAL DIFFERENTIATION

Change of Variables : If u is a function of x, y and x, y are functions of t and r, then u is called a composite function of t and r.

Let u =f(x, y) and x = g(t, r), y = h(t, r) then the continuous first order partial derivatives are

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

PROBLEMS

If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$ show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

Solution : Here given $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$

$= u\,(r, s)$

where $r = \frac{y-x}{xy}$ and $s = \frac{z-x}{zx}$

PROBLEMS

$$\Rightarrow r = \frac{1}{x} - \frac{1}{y} \text{ and } s = \frac{1}{x} - \frac{1}{z} \quad \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

we know that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial x}$$

$$= \frac{\partial u}{\partial r}\left(-\frac{1}{x^2}\right) + \frac{\partial u}{\partial s}\left(-\frac{1}{x^2}\right) \qquad \because r = \frac{1}{x} - \frac{1}{y}$$

$$\Rightarrow \frac{\partial r}{\partial x} = -\frac{1}{x^2}$$

$$= -\frac{1}{x^2}\frac{\partial u}{\partial r} - \frac{1}{x^2}\frac{\partial u}{\partial s} \qquad \because s = \frac{1}{x} - \frac{1}{z}$$

$$\Rightarrow \frac{\partial s}{\partial x} = -\frac{1}{x^2}$$

$$\text{or } x^2\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial r} - \frac{\partial u}{\partial s} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Similarly $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial y}$

$= \frac{\partial u}{\partial y}.\frac{1}{y^2} + \frac{\partial u}{\partial s}.0$ from (i)

or $y^2 \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}$ ....(iii)

and $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial z}$

$= \frac{\partial u}{\partial r}.0 + \frac{\partial u}{\partial s}.\frac{1}{z^2}$

$\Rightarrow z^2 \frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}$ ....(iv)

Adding (i) (ii) and (iii) we get

$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$ Hence Proved.

To find the Maxima & Minima of f(x) we use the following procedure.

(i) Find $f^{1}(x)$ and equate it to zero

(ii) Solve the above equation we get x_0, x_1 as roots.

(iii) Then find $f^{11}(x)$.

If $f^{11}(x)_{(x=x_0)>0,}$ then f(x) is minimum at x_0

If $f^{11}(x)_{(x=x_0)<0,}$ f(x) is maximum at x_0. Similarly we do this for other stationary points.

1. Find the max & min of the function

$f(x) = x^5 - 3x^4 + 5$

Sol: Given $f(x) = x^5 - 3x^4 + 5$

$f^1(x) = 5x^4 – 12x^3$

for maxima or minima $f^1(x) = 0$

$5x^4 – 12x^3 = 0$ $x = 0, x = 12/5$

$f^{11}(x) = 20x^3 – 36x^2$

PROBLEM

At x = 0 => $f^{11}(x) = 0$. So f is neither maximum nor minimum at x = 0 At x = (12/5) =>

$f^{11}(x) = 20\,(12/5)^3 - 36(12/5)$

$= 144(48-36)/25 \quad = 1728/25 > 0$

So f(x) is minimum at x = 12/5

The minimum value is $f\,(12/5) = (12/5)^5 - 3(12/5)^4 + 5$

Working procedure:

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ Equate each to zero. Solve these equations for x & y we get the pair of values (a_1, b_1) (a_2,b_2) (a_3 ,b_3) ....

2. Find $l=\frac{\partial^2 f}{\partial x^2}, m=\frac{\partial^2 f}{\partial x \partial y}$, n = $\frac{\partial^2 f}{\partial y^2}$

3.
   i. If $ln - m^2 > 0$ and $l < 0$ at (a_1,b_1) then f(x ,y) is maximum at (a_1,b_1) and maximum value is $f(a_1,b_1)$
   ii. If $ln - m^2 > 0$ and $l > 0$ at (a_1,b_1) then f(x ,y) is minimum at (a_1,b_1) and minimum value is $f(a_1,b_1)$.

ii. If $ln - m^2 < 0$ and at (a_1, b_1) then $f(x, y)$ is neither maximum nor minimum at (a_1, b_1). In this case (a_1, b_1) is saddle point.

iii. If $ln - m^2 = 0$ and at (a_1, b_1) , no conclusion can be drawn about maximum or minimum and needs further investigation. Similarly we do this for other stationary points.

Locate the stationary points & examine their nature of the following functions.

$u = x^4 + y^4 - 2x^2 + 4xy - 2y^2, \quad (x > 0,\ y > 0)$

Sol: Given $u(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

For maxima & minima $\frac{\partial u}{\partial x} = 0,\ \frac{\partial u}{\partial y} = 0$

$\frac{\partial u}{\partial x} = 4x^3 - 4x + 4y = 0 \Rightarrow x^3 - x + y = 0$ ---------------------> (1)

$\frac{\partial u}{\partial y} = 4y^3 + 4x - 4y = 0 \Rightarrow y^3 + x - y = 0$ ---------------------> (2)

Adding (1) & (2),

$x^3 + y^3 = 0 \Rightarrow x = -y$ ----------------------> (3)

(1) $\Rightarrow x^3 - 2x \Rightarrow x = 0, \sqrt{2}, -\sqrt{2}$

Hence (3) $\Rightarrow y = 0, \sqrt{2}, -\sqrt{2}$

PROBLEM

$l = \frac{\partial^2 u}{\partial x^2}$ = $12x^2 - 4$,

m = $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 4$

n = $\frac{\partial^2 u}{\partial y^2}$ = $12y^2 - 4$

$ln - m^2 = (12x^2 - 4)(12y^2 - 4) - 16$

At $\left(-\sqrt{2}, \sqrt{2}\right)$ $ln - m^2 = (24 - 4)(24 - 4) - 16 = (20)(20) - 16 > 0$

and l=20>0

The function has minimum value at $\left(-\sqrt{2}, \sqrt{2}\right)$

At (0,0) , $ln - m^2 = (0 - 4)(0 - 4) - 16 = 0$

(0,0) is not a extreme value.

Scalar and vector point functions: Consider a region in three dimensional space. To each point $p(x,y,z)$, suppose we associate a unique real number (called scalar) say ϕ. This $\phi(x,y,z)$ is called a scalar point function. Scalar point function defined on the region. Similarly if to each point $p(x,y,z)$we associate a unique vector $\bar{f}(x,y,z)$, $\bar{f}$ is called a ***vector point function*.**

Examples:

For example take a heated solid. At each point p(x,y,z)of the solid, there will be temperature T(*x,y,z*). This T is a scalar point function.

Suppose a particle (or a very small insect) is tracing a path in space. When it occupies a position p(*x,y,z*) in space, it will be having some speed, say, *v*. This **speed***v* is a scalar point function.

Consider a particle moving in space. At each point P on its path, the particle will be having a velocity $\bar{v}$ which is vector point function. Similarly, the acceleration of the particle is also a vector point function.

In a magnetic field, at any point P(x,y,z) there will be a magnetic force $\bar{f}$(*x,y,z*). This is called magnetic force field. This is also an example of a vector point function.

Vector Calculus and Vector Operators

INTRODUCTION

In this chapter, vector differential calculus is considered, which extends the basic concepts of differential calculus, such as, continuity and differentiability to vector functions in a simple and natural way. Also, the new concepts of gradient, divergence and curl are introduced.

DIFFERENTIATION OF A VECTOR FUNCTION

Let S be a set of real numbers. Corresponding to each scalar t ε S, let there be associated a unique vector $\bar{f}$. Then $\bar{f}$ is said to be a vector (vector valued) function. S is called the domain of $\bar{f}$. We write $\bar{f} = \bar{f}(t)$.

Let $\bar{i}, \bar{j}, \bar{k}$ be three mutually perpendicular unit vectors in three dimensional spaces. We can write $\bar{f} = \bar{f}(t) = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$, where $f_1(t)$, $f_2(t)$, $f_3(t)$ are real valued functions (which are called components of $\bar{f}$). (we shall assume that $\bar{i}, \bar{j}, \bar{k}$ are constant vectors).

4. Properties

1) $\frac{\partial}{\partial t}(\phi\overline{a}) = \frac{\partial \phi}{\partial t}\overline{a} + \phi\frac{\partial \overline{a}}{\partial t}$

2). If λ is a constant, then $\frac{\partial}{\partial t}(\lambda\overline{a}) = \lambda\frac{\partial \overline{a}}{\partial t}$

3). If $\overline{c}$ is a constant vector, then $\frac{\partial}{\partial t}(\phi\overline{c}) = \overline{c}\frac{\partial \phi}{\partial t}$

4). $\frac{\partial}{\partial t}(\overline{a} \pm \overline{b}) = \frac{\partial \overline{a}}{\partial t} \pm \frac{\partial \overline{b}}{\partial t}$

5). $\frac{\partial}{\partial t}(\overline{a}.\overline{b}) = \frac{\partial \overline{a}}{\partial t}.\overline{b} + \overline{a}.\frac{\partial \overline{b}}{\partial t}$

6). $\frac{\partial}{\partial t}(\overline{a} \times \overline{b}) = \frac{\partial \overline{a}}{\partial t} \times \overline{b} + \overline{a} \times \frac{\partial \overline{b}}{\partial t}$

7). Let $\overline{f} = f_1\overline{i} + f_2\overline{j} + f_3\overline{k}$, where f_1, f_2, f_3 are differential scalar functions of more than one variable, Then $\frac{\partial \overline{f}}{\partial t} = \overline{i}\frac{\partial f_1}{\partial t} + \overline{j}\frac{\partial f_2}{\partial t} + \overline{k}\frac{\partial f_3}{\partial t}$ (treating $\overline{i}, \overline{j}, \overline{k}$ as fixed directions)

5. Higher order partial derivatives

Let $\bar{f} = \bar{f}(p,q,t)$. Then $\frac{\partial^2 \bar{f}}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \bar{f}}{\partial t}\right), \frac{\partial^2 \bar{f}}{\partial p \partial t} = \frac{\partial}{\partial p}\left(\frac{\partial \bar{f}}{\partial t}\right) etc.$

6.Scalar and vector point functions: Consider a region in three dimensional space. To each point $p(x,y,z)$, suppose we associate a unique real number (called scalar) say ϕ. This $\phi(x,y,z)$ is called a scalar point function. Scalar point function defined on the region. Similarly if to each point $p(x,y,z)$ we associate a unique vector $\bar{f}(x,y,z)$, $\bar{f}$ is called a ***vector point function***.

7. Tangent vector to a curve in space.

Consider an interval [a,b].

Let x = x(t),y=y(t),z=z(t)be continuous and derivable for a$\leq$ t $\leq$b.

Then the set of all points (x(t),y(t),z(t)) is called a curve in a space.

Let A = (x(a),y(a),z(a)) and B = (x(b),y(b),z(b)). These A,B are called the end points of the curve. If A =B, the curve in said to be a closed curve.

Let P and Q be two neighbouring points on the curve.

Let $\overline{OP} = \bar{r}(t), \overline{OQ} = \bar{r}(t+\delta t) = \bar{r} + \delta\bar{r}. Then\ \delta\bar{r} = \overline{OQ} - \overline{OP} = \overline{PQ}$

Then $\frac{\delta\bar{r}}{\delta t}$ is along the vector $\overline{PQ}$. As Q$\rightarrow$P, PQ and hence $\frac{\overline{PQ}}{\delta t}$ tends to be along the tangent to the curve at P.

Hence $\underset{\delta t\to 0}{lt}\frac{\delta\bar{r}}{\delta t} = \frac{d\bar{r}}{dt}$ will be a tangent vector to the curve at P. (This $\frac{d\bar{r}}{dt}$ may not be a unit vector)

CURL OF A VECTOR

Def: Let $\bar{f}$ be any continuously differentiable vector point function. Then the vector function defined by $\bar{i}\times\frac{\partial\bar{f}}{\partial x}+\bar{j}\times\frac{\partial\bar{f}}{\partial y}+\bar{k}\times\frac{\partial\bar{f}}{\partial z}$ is called curl of $\bar{f}$ and is denoted by curl $\bar{f}$ or (∇x $\bar{f}$).

$$\text{Curl } \bar{f} = \bar{i}\times\frac{\partial\bar{f}}{\partial x}+\bar{j}\times\frac{\partial\bar{f}}{\partial y}+\bar{k}\times\frac{\partial\bar{f}}{\partial z}=\sum\left(\bar{i}\times\frac{\partial\bar{f}}{\partial x}\right)$$

Theorem 1: If $\bar{f}$ is differentiable vector point function given by $\bar{f} = f_1\bar{i} + f_2\,\bar{j} + f_3\,\bar{k}$ then curl $\bar{f}$

$$= \left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)\bar{i}+\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)\bar{j}+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)\bar{k}$$

Proof : curl $\bar{f}$ $= \sum\bar{i}\times\frac{\partial}{\partial x}(\bar{f}) = \sum\bar{i}\times\frac{\partial}{\partial x}(f_1\bar{i}+f_2\bar{j}+f_3\bar{k}) = \sum\left(\frac{\partial f_2}{\partial x}\bar{k}-\frac{\partial f_3}{\partial x}\bar{j}\right)$

$$=\left(\frac{\partial f_2}{\partial x}\bar{k}-\frac{\partial f_3}{\partial x}\bar{j}\right)+\left(\frac{\partial f_3}{\partial y}\bar{i}-\frac{\partial f_1}{\partial y}\bar{k}\right)+\left(\frac{\partial f_1}{\partial z}\bar{j}-\frac{\partial f_2}{\partial z}\bar{i}\right)$$

$$=\bar{i}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)+\bar{j}\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)+\bar{k}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)$$

Note : (1) The above expression for curl $\bar{f}$ can be remembered easily through the representation.

$$\textbf{curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \nabla \times \bar{f}$$

Note (2) : If $\bar{f}$ is a constant vector then curl $\bar{f} = \bar{o}$.

Physical Interpretation of curl

If $\overline{w}$ is the angular velocity of a rigid body rotating about a fixed axis and $\bar{v}$ is the velocity of any point P(x,y,z) on the body, then $\overline{w} = \frac{1}{2}$ curl $\bar{v}$. Thus the angular velocity of rotation at any point is equal to half the curl of velocity vector. This justifies the use of the word "curl of a vector".

2. Irrotational Motion, Irrotational Vector

Any motion in which curl of the velocity vector is a null vector i.e curl $\bar{v} = \bar{0}$ is said to be Irrotational.

Def: A vector $\bar{f}$ is said to be Irrotational if curl $\bar{f} = \bar{0}$.

If $\bar{f}$ is Irrotational, there will always exist a scalar function φ(x,y,z) such that $\bar{f}$ =grad ϕ. Thisϕ is called scalar potential of $\bar{f}$.

It is easy to prove that, if $\bar{f}$ = grad ϕ, then curl $\bar{f} = 0$.

Hence $\nabla \times \bar{f} = 0 \Leftrightarrow$ there exists a scalar function ϕ such that $\bar{f} = \nabla\phi$.

This idea is useful when we study the "work done by a force" later.

1: If $\bar{f} = xy^2\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$ **find curl** $\bar{f}$ **at the point (1,-1,1).**

Sol:- Let $\bar{f} = xy^2\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$. Then

$$\text{curl } \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right) + \bar{j}\left(\frac{\partial}{\partial z}(xy^2) - \frac{\partial}{\partial x}(-3yz^2)\right) + \bar{k}\left(\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right)$$

$$= \bar{i}\left(-3z^2 - 2x^2z\right) + \bar{j}(0-0) + \bar{k}(4xyz - 2xy) = -\left(3z^2 + 2x^2y\right)\bar{i} + (4xyz - 2xy)\bar{k}$$

= curl $\bar{f}$ at (1,-1,1) = $-\bar{i} - 2\bar{k}.$

Prove that div $curl\,\bar{f} = 0$

Proof : Let $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\therefore curl\,\bar{f} = \nabla\times\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\bar{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\bar{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\bar{k}$$

$$\therefore \quad div\ curl\ \bar{f} = \nabla.(\nabla\times\bar{f}) = \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \frac{\partial^2 f_3}{\partial x\partial y} - \frac{\partial^2 f_2}{\partial x\partial z} - \frac{\partial^2 f_3}{\partial y\partial x} + \frac{\partial^2 f_1}{\partial y\partial z} + \frac{\partial^2 f_2}{\partial z\partial x} - \frac{\partial^2 f_1}{\partial z\partial y} = 0$$

Note : Since $div(curl\,\bar{f}) = 0$, we have $curl\,\bar{f}$ is always solenoidal.

Thus the operator $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called Laplacian operator.

Note : (i). $\nabla^2\phi = \nabla.(\nabla\phi)$ = div(grad ϕ)

(ii). If $\nabla^2\phi = 0$ then ϕ is said to satisfy Laplacian equation. This ϕ is called a harmonic function

Find div $\bar{F}$, where $\bar{F}$ = grad $(x^3+y^3+z^3-3xyz)$

Sol: Let $\phi = x^3+y^3+z^3-3xyz$. Then

$\bar{F}$ = grad ϕ

$$= \sum i \frac{\partial \phi}{\partial x} = 3(x^2 - yz)\bar{i} + 3(y^2 - zx)\,\bar{j} + 3(x^2 - xy)\,\bar{k} =$$

$$F_1 i + F_2 j + F_3 k \ (say)$$

$$\therefore \text{div } \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 6x+6y+6z = 6(x+y+z)$$

i.e div[grad$(x^3+y^3+z^3-3xyz)$]= $\nabla^2(x^3+y^3+z^3-3xyz) = 6(x+y+z)$.

DIVERGENCE

Prove that div $curl\,\bar{f} = 0$

$Proof : Let\ \bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\therefore\ curl\ \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\bar{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\bar{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\bar{k}$$

$$\therefore\quad div\ \ curl\ \bar{f} = \nabla.(\nabla \times \bar{f}) = \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial x \partial z} - \frac{\partial^2 f_3}{\partial y \partial x} + \frac{\partial^2 f_1}{\partial y \partial z} + \frac{\partial^2 f_2}{\partial z \partial x} - \frac{\partial^2 f_1}{\partial z \partial y} = 0$$

Note : Since $div(curl\,\bar{f}) = 0,$ we have $curl\,\bar{f}$ is always solenoidal.

If $\bar{F} = (x^2-27)\,\bar{i} - 6yz\,\bar{j} + 8xz^2\,\bar{k}$, evaluate $\int_C \bar{F}\cdot d\bar{r}$ from the point (0,0,0) to the point (1,1,1) along the Straight line from (0,0,0) to (1,0,0), (1,0,0) to (1,1,0) and (1,1,0) to (1,1,1).

Solution : Given $\bar{F} = (x^2-27)\,\bar{i} - 6yz\,\bar{j} + 8xz^2\,\bar{k}$

Now $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\therefore \quad \bar{F}\,.\,d\bar{r} = (x^2-27)dx - (6yz)dy + 8xz^2dz$

(i) Along the straight line from O = (0,0,0) to A = (1,0,0)

Here y =0 =z and dy=dz=0. Also x changes from 0 to 1.

$$\therefore \int_{OA} \bar{F}\,.\,d\bar{r} = \int_0^1 (x^2-27)dx = \left[\frac{x^3}{3} - 27x\right]_0^1 = \frac{1}{3} - 27 = \frac{-80}{3}$$

(ii) Along the straight line from A = (1,0,0) to B = (1,1,0)

Here x =1, z=0 $\Rightarrow$ dx=0, dz=0. y changes from 0 to 1.

Along the straight line from B = (1,1,0) to C = (1,1,1)

x =1 =y $\Rightarrow$ dx=dy=0 and z changes from 0 to 1.

$$\therefore \int_{BC} \bar{F} \cdot d\bar{r} = \int_{z=0}^{1} 8xz^2 dz = \int_{z=0}^{1} 8xz^2 dz = \left[\frac{8z^3}{3}\right]_0^1 = \frac{8}{3}$$

$$(i)+(ii)+(iii) \Rightarrow \int_C \bar{F} \cdot d\bar{r} = \frac{88}{3}$$

Find the work done by the force $\bar{F} = z\bar{i} + x\bar{j} + y\bar{k}$, when it moves a particle along the arc of the curve $\bar{r}$ = cost $\bar{i}$ + sint $\bar{j}$-t $\bar{k}$ from t = 0 to t = 2π

Solution : Given force $\bar{F}$ = z$\bar{i}$+ x$\bar{j}$ +y $\bar{k}$ and the arc is $\bar{r}$ = cost $\bar{i}$ + sin t $\bar{j}$-t$\bar{k}$

i.e., x = cost, y= sin t, z = -t

$\therefore d\bar{r} = (-\sin t\,\bar{i} + \cos t\,\bar{j} - \bar{k})dt$

$\therefore \bar{F}.d\bar{r} = (-t\,\bar{i} + \cos t\,\bar{j} + \sin t\,\bar{k}).(-\sin t\,\bar{i} + \cos t\,\bar{j} - \bar{k})dt = (t \sin t + \cos^2 t - \sin t)dt$

Hence work done = $\int_0^{2\pi} \bar{F} . d\bar{r} = \int_0^{2\pi} (t \sin t + \cos^2 t - \sin t)\, dt$

$$= \left[t(-\cos t)\right]_0^{2\pi} - \int_0^{2\pi} (-\sin t)dt + \int_0^{2\pi} \frac{1+\cos 2t}{2} dt - \int_0^{2\pi} \sin t \, dt$$

$$= -2\pi - (\cos t)_0^{2\pi} + \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right)_0^{2\pi} + (\cos t)_0^{2\pi}$$

$$= -2\pi - (1-1) + \frac{1}{2}(2\pi) + (1-1) = -2\pi + \pi = -\pi$$

Surface integral

$\int_S \bar{F}.\bar{n}\,ds$ is called surface integral

Evaluate $\int \overline{F}.ndS$ where $\overline{F} = zi + xj - 3y^2zk$ and S is the surface $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5.

Sol. The surface S is $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5.

Let $\phi = x^2 + y^2 = 16$

Then $$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = 2x\bar{i} + 2y\bar{j}$$

$\therefore$ unit normal $$\bar{n} = \frac{\nabla\phi}{|\nabla\varphi|} = \frac{x\bar{i} + y\bar{j}}{4} \quad (\because x^2 + y^2 = 16)$$

Let R be the projection of S on yz-plane

Then $$\int_S \overline{F}.ndS = \iint_R \overline{F}.\bar{n}\frac{dydz}{|\bar{n}\,.\,\bar{i}|} \quad \ldots\ldots\ldots\ldots\ldots *$$

Given $\bar{F} = zi + xj - 3y^2zk$

$\therefore \quad \bar{F} \cdot \bar{n} = \frac{1}{4}(xz + xy)$

and $\quad \bar{n} \cdot \bar{i} = \frac{x}{4}$

In yz-plane, x = 0, y = 4

In first octant, y varies from 0 to 4 and z varies from 0 to 5.

$$\int_S \bar{F}.ndS = \int_{y=0}^{4} \int_{z=0}^{5} \left(\frac{xz + xy}{4}\right) \frac{dydz}{\left|\frac{x}{4}\right|}$$

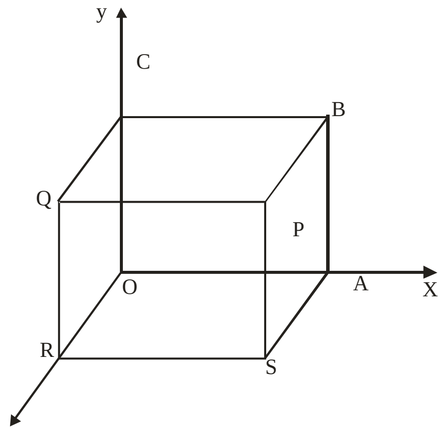
$$= \int_{y=0}^{4} \int_{z=0}^{5} (y + z)dz\ dy$$

$$= 90.$$

SURFACE INTEGRAL

If $\bar{F} = zi + xj - 3y^2zk$, evaluate $\int_S \bar{F}.\bar{n}dS$ where S is the surface of the cube bounded by x = 0, x = a, y = 0, y= a, z = 0, z = a.

Sol. Given that S is the surface of the x = 0, x = a, y = 0, y = a, z = 0, z = a, and $\bar{F} = zi + xj - 3y^2zk$ we need to evaluate $\int_S \bar{F}.\bar{n}dS$.



the surface $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

Sol. The surface S is $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

Let ⍰ $= x^2 + y^2 = 16$

Then $\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = 2x\bar{i} + 2y\bar{j}$

⍰ unit normal $\bar{n} = \frac{\nabla\phi}{|\nabla\varphi|} = \frac{x\bar{i} + y\bar{j}}{4}$ $(\because x^2 + y^2 = 16)$

Let R be the projection of S on yz-plane

Then $\int_S \bar{F}.\bar{n}dS = \iint_R \bar{F}.\bar{n}\frac{dydz}{|\bar{n}\,.\,\bar{i}|}$ .... *

$$\int_S\int \bar{F}.\bar{n}ds = \int_{S_1}\int \bar{F}.\bar{n}ds + \int_{S_2}\int \bar{F}.\bar{n}ds + \cdots + \int_{S_6}\int \bar{F}.\bar{n}ds$$

$On\ S_1, we\ have\ \bar{n} = \bar{i}, x = a$

$$\therefore \iint_{s_1} \overline{F}.\bar{n}ds = \int_{z=0}^{a}\int_{y=0}^{a} \left(a^3\bar{i} + y^3\bar{j} + z^3\bar{k}\right).\bar{i}dydz$$

$$\int_{S_1}\int \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{y=0}^{a} (a^3\bar{i} + y^3\bar{j} + z^3\bar{k}).\bar{i}\,dy\,dz$$

$$= \int_{z=0}^{a}\int_{y=0}^{a} a^3 dy\,dz = a^3 \int_0^a (y)_0^a\,dz$$

$$= a^4(z)_0^a = a^5$$

SURFACE INTEGRAL

On S_2, we have $\bar{n} = -\bar{i}, x = 0$

$$\iint_{S_2} \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{y=0}^{a}\left(y^3\bar{j}+z^3\bar{k}\right).\left(\overline{-i}\right)dydz = 0$$

On S_3, we have $\bar{n} = \bar{j}, y = a$

$$\iint_{S_3} \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{x=0}^{a}\left(x^3\bar{i}+a^3\bar{j}+z^3\bar{k}\right).\bar{j}dxdz = a^3\int_{z=0}^{a}\int_{x=0}^{a}dxdz = a^3\int_{0}^{a}adz = a^4\left(z\right)_0^a$$

$$= a^5$$

On S_4, we have $\bar{n} = -\bar{j}, y = 0$

PROBLEM

$$On\ S_4, we\ have\ \bar{n} = -\bar{j}, y = 0$$

$$\int_{S_4}\int \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{x=0}^{a}\left(x^3\bar{i} + z^3\bar{k}\right).(-\bar{j})dx\,dz = 0$$

$$On\ S_5, we\ have\ \bar{n} = \bar{k}, z = a$$

$$\int_{S_5}\int \bar{F}.\bar{n}ds = \int_{y=0}^{a}\int_{x=0}^{a}\left(x^3\bar{i} + y^3\bar{j} + a^3\bar{k}\right).\bar{k}\,dx\,dy$$

$$= \int_{y=0}^{a}\int_{x=0}^{a} a^3\,dx\,dy = a^3\int_{0}^{a}(x)_0^a dy = a^4(y)_0^a = a^5$$

PROBLEM

$On\ S_6, we\ have\ \bar{n} = -\bar{k}, z = 0$

$$\iint_{S_6} F.\bar{n}ds = \int_{y=0}^{a}\int_{x=0}^{a}(x^3\bar{i} + y^3\bar{j}).(-\bar{k})dx\,dy = 0$$

$$Thus \iint_{S} \bar{F}.\bar{n}ds = a^5 + 0 + a^5 + 0 + a^5 + 0 = 3a^5$$

GAUSS'S DIVERGENCE THEOREM

(Transformation between surface integral and volume integral)

Let S be a closed surface enclosing a volume V. If $\bar{F}$ is a continuously differentiable vector point function, then

$$\int_V \operatorname{div} F dv = \int_S \bar{F} . \bar{n} \, dS$$

When $\bar{n}$ is the outward drawn normal vector at any point of S.

GAUSS DIVERGENCE THEOREM

Verify Gauss Divergence theorem for

$\bar{F} = (x^3 - yz)\bar{i} - 2x^2 y\bar{j} + z\bar{k}$ taken over the surface of the

cube bounded by the planes x = y = z = a and

coordinate planes.

Sol: By Gauss Divergence theorem we have

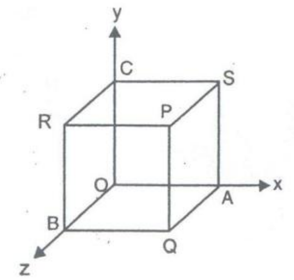
$$\int_S \bar{F}.\bar{n}dS = \int_V div\bar{F}dv$$

$$RHS = \int_0^a\int_0^a\int_0^a (3x^2 - 2x^2 + 1)dx\,dy\,dz = \int_0^a\int_0^a\int_0^a (x^2 + 1)\,dx\,dy\,dz = \int_0^a\int_0^a \left(\frac{x^3}{3} + x\right)_0^a dy\,dz$$

$$\int_0^a\int_0^a \left[\frac{a^3}{3} + a\right]dydz = \int_0^a \left[\frac{a^3}{3} + a\right](y)_0^a dz = \left(\frac{a^3}{3} + a\right)a\int_0^a dz = \left(\frac{a^3}{3} + a\right)(a^2) = \frac{a^5}{3} + a^3 \text{......(1)}$$

(i) For S_1 = PQAS; unit outward drawn normal $\bar{n} = \bar{i}$

x=a; ds=dy dz; 0≤y≤a, 0≤z≤a



$$\therefore \overline{F}.\overline{n} = x^3 - yz = a^3 - yz \sin cex = a$$

$$\therefore \iint_{S_1} \overline{F}.\overline{n}dS = \int_{z=0}^{a}\int_{y=0}^{a} (a^3 - yz)dydz$$

$$= \int_{z=0}^{a} \left[a^3 y - \frac{y^2}{2} z\right]_{y=0}^{a} dz$$

$$= \int_{z=0}^{a} \left(a^4 - \frac{a^2}{2} z\right) dz$$

$$= a^5 - \frac{a^4}{4} \ldots(2)$$

GAUSS DIVERGENCE THEOREM

For S_2 = OCRB; unit outward drawn normal

$$\bar{n} = -\bar{i}$$

x=0; ds=dy dz; 0≤y≤a, y≤z≤a

$$\bar{F}.\bar{n} = -(x^3 - yz) = yz \; since \; x = 0$$

$$\int_{S_2}\int \bar{F}.\bar{n}\,dS = \int_{z=0}^{a}\int_{y=0}^{a} yz\,dy\,dz = \int_{z=0}^{a}\left[\frac{y^2}{2}\right]_{y=0}^{a} z\,dz$$

$$= \frac{a^2}{2}\int_{z=0}^{a} z\,dz = \frac{a^4}{4} \ldots (3)$$

For S_3 = RBQP; Z = a; ds = dxdy; $\bar{n} = \bar{k}$

$0 \le x \le a,\ 0 \le y \le a$

$\bar{F}.\bar{n} = z = a \ \ since\ z = a$

$$\therefore \iint_{S_3} \overline{F}\,\overline{n}\,dS = \int_{y=0}^{a}\int_{x=0}^{a} a\,dx\,dy = a^3 \text{.....(4)}$$

Verify divergence theorem for $\bar{F} = x^2 i + y^2 j + z^2 k$ over the surface S of the solid cut off by the plane x+y+z=a in the first octant.

Sol; By Gauss theorem, $\int_s \bar{F}.\bar{n}dS = \int_v div\bar{F}dv$

Let $\phi = x + y + z - a$ *be the given plane then*

$$\frac{\partial\phi}{\partial x} = 1, \frac{\partial\phi}{\partial y} = 1, \frac{\partial\phi}{\partial z} = 1$$

$$\therefore grad\phi = \sum \bar{i}\frac{\partial\phi}{\partial x} = \bar{i} + \bar{j} + \bar{k}$$

$$Unit\ normal = \frac{grad\ \phi}{|grad\ \phi|} = \frac{\bar{i} + \bar{j} + \bar{k}}{\sqrt{3}}$$

Let R be the projection of S on xy-plane

Then the equation of the given plane will be x+y=a $\Rightarrow$ y=a-x

Also when y=0, x=a

$$\therefore \int_S \overline{F}.\overline{n}dS = \iint_R \frac{\overline{F}.\overline{n}dxdy}{|\overline{n}.\overline{k}|}$$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} \frac{\frac{x^2+y^2+z^2}{\sqrt{3}}}{1/\sqrt{3}} dx\ dy \quad = \int_{0}^{a} \int_{y=0}^{a-x} [x^2+y^2+(a-x-y)^2] dx\ dy\ [\ since\ x+y+z=a]$$

$$= \int_0^a \int_0^{a-x} [2x^2 + 2y^2 - 2ax + 2xy - 2ay + a^2]\,dx\,dy$$

$$= \int_{x=0}^{a} \left[2x^2y + \frac{2y^3}{3} + xy^2 - 2axy - ay^2 + a^2y\right]_0^{a-x} dx$$

$$= \int_{x=0}^{a} [2x^2(a-x) + \frac{2}{3}(a-x)^3 + x(a-x)^2 - 2ax(a-x) - a(a-x)^2 + a^2(a-x)dx$$

$$\therefore \int_s \overline{F}.\overline{n}dS = \int_0^a \left(-\frac{5}{3}x^3 + 3ax^2 - 2a^2x + \frac{2}{3}a^3\right)dx = \frac{a^4}{4},$$ on simplification…(1)

GAUSS DIVERGENCE THEOREM

Given $\overline{F} = x^2\overline{i} + y^2\overline{j} + z^2\overline{k}$

$$\therefore div\ \overline{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) = 2(x+y+z)$$

$$Now \iiint div\overline{F}.dv = 2\int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y}(x+y+z)dxdydz$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{a-x}\left[z(x+y)+\frac{z^2}{2}\right]_0^{a-x-y} dx\ dy$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{a-x}(a-x-y)\left[x+y+\frac{a-x-y}{2}\right]dx\ dy$$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} (a-x-y)[a+x+y]dx\,dy$$

$$= \int_{0}^{a} \int_{0}^{a-x} [a^2-(x+y)^2]\ dy\,dx = \int_{0}^{a} \int_{0}^{a-x} (a^2-x^2-y^2-2xy)dx\,dy$$

$$= \int_{0}^{a} [a^2y - x^2y - \frac{y^3}{3} - xy^2]_{0}^{a-x}\ dx$$

$$= \int_{0}^{a} (a-x)(2a^2-x^2-ax)dx = \frac{a^4}{4} \ldots\ldots (2)$$

Hence from (1) and (2), the Gauss Divergence theorem is verified.

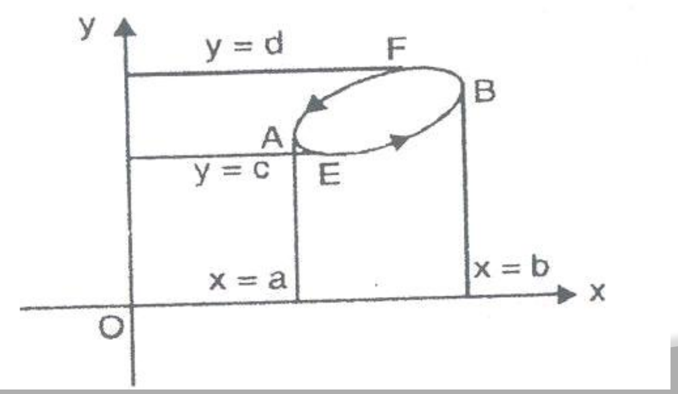
GREENS THEOREM

(Transformation Between Line Integral and Surface Integral)   [JNTU 2001S].

If S is Closed region in xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy.$$

Where C is traversed in the positive(anti clock-wise) direction

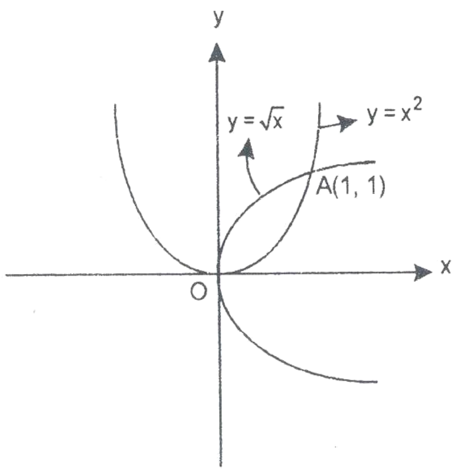


GREENS THEOREM

Verify Green's theorem in plane for $\oint(3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the region bounded by y=$\sqrt{x}$ and y=x^2 .

Solution: Let M=3x^2-8y^2 and N=4y-6xy. Then

$\frac{\partial M}{\partial y} = -16y, \frac{\partial N}{\partial x} = -6y$



GREENS THEOREM

We have by Green's theorem,

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$$

Now $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \iint_R (16y - 6y) dxdy$

$$= 10 \iint_R y dxdy = 10 \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} y dy dx = 10 \int_{x=0}^{1} \left(\frac{y^2}{2} \right)_{x^2}^{\sqrt{x}} dx$$

Verification:

We can write the line integral along c

=[line integral along y=x^2(from O to A) + [line integral along y^2=x(from A to O)]

=I_1+I_2(say)

Now $I_1=\int_{x=0}^{1}\{[3x^2-8(x^2)^2]dx+[4x^2-6x(x^2)]2xdx\}\left[\because y=x^2\Rightarrow\frac{dy}{dx}=2x\right]$

$$=\int_0^1 (3x^3 + 8x^3 - 20x^4)dx = -1$$

And

$$l_2 = \int_1^0 \left[\left(3x^2 - 8x\right)dx + \left(4\sqrt{x} - 6x^{3/2}\right)\frac{1}{2\sqrt{x}}\, dx \right] = \int_1^0 \left(3x^2 - 11x + 2\right)dx = \frac{5}{2}$$

$\therefore\ I_1 + I_{2=-1+5/2=3/2}.$

From(1) and (2), we have $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$

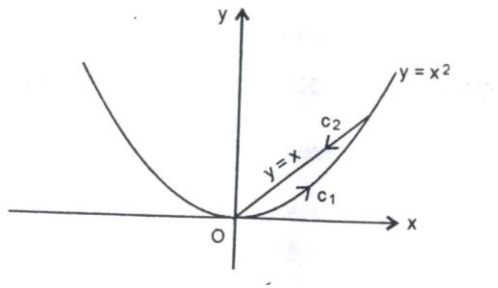
Hence the verification of the Green's theorem.

GREENS THEOREM

Verify Green's theorem for $\int_c [(xy+y^2)dx + x^2 dy]$, where C is bounded by y=x and y=x^2

Solution: By Green's theorem, we have $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

Here M=xy +y^2 and N=x^2



GREENS THEOREM

The line y=x and the parabola y=x^2 intersect at O(0,0) and A(1,1)

Now $\oint_c M\,dx + N\,dy = \int_{c_1} Mdx + Ndy + \int_{c_2} Mdx + Ndy$......(1) .....(1)

Along C_1 $(i.e.\, y = x^2)$, the line integral is

$$\int_{c_1} Mdx + Ndy = \int_{c_1} [x(x^2) + x^4]dx + x^2 d(x^2) \int_c (x^3 + x^4 + 2x^3)dx = \int_0^1 (3x^3 + x^4)dx$$

$$=\left(3.\frac{x^4}{4} + \frac{x^5}{5}\right)_0^1$$

$$=\frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$ ....(2)

GREENS THEOREM

Along C_2 $(i.e.\, y = x)$ from $(1,1)$ to $(0,0)$, the line integral is

$$\int_{c_2} Mdx + Ndy = \int_{c_2} (x.x + x^2)dx + x^2 dx \; [\because dy = dx]$$

$$= \int_{c_2} 3x^2 dx = 3 \int_1^0 x^2 dx = 3\left(\frac{x^3}{3}\right)_1^0 = (x^3)_1^0 = 0-1 = -1 \qquad \ldots.(3)$$

GREENS THEOREM

From (1), (2) and (3), we have

$$\int_c M dx + N dy = \frac{19}{20} - 1 = \frac{-1}{20}$$

…(4)

Now

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \iint_R (2x - x - 2y) dxdy$$

$$= \int_0^1 [(x^2 - x^2) - (x^3 - x^4)] dx = \int_0^1 (x^4 - x^3) dx$$

$$= \left(\frac{x^5}{5} + \frac{x^4}{4} \right)_0^1 = \frac{1}{5} - \frac{1}{4} = \frac{-1}{20}$$

….(5)

From $(4) and (5), We\ have \oint_c M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$ dxdy

Hence the verification of the Green’s theorem.

III. STOKE'S THEOREM

(Transformation between Line Integral and Surface Integral)

Let S be a open surface bounded by a closed, non intersecting curve C.

If $\bar{F}$ is any

differentieable vector point function then $\oint_C \bar{F}.d\,\bar{r}$=

$\int_S curl\ \bar{F}.\bar{n}\ ds$ *where c is traversed in the positive* direction and

$\bar{n}$ *is unit outward drawn normal at any point of the surface.*

Verify Stokes theorem for $\bar{F} = -y^3\bar{\imath} + x^3\bar{\jmath}$, Where S is the circular disc

$x^2 + y^2 \leq 1, z = 0.$

Solution: Given that $\bar{F} = -y^3\bar{\imath} + x^3\bar{\jmath}$. The boundary of C of S is a circle in xy plane.

$x^2 + y^2 \leq 1, z = 0.$ We use the parametric co-ordinates x=cos$\theta, y = sin\theta, z = 0, 0 \leq \theta \leq 2\pi;$

dx=-sin$\theta\ d\theta$ and dy =cos$\theta\ d\theta$

$\therefore \oint_c \bar{F}.dr = \int_c F_1 dx + F_2 dy + F_3 dz = \int_c -y^3 dx + x^3 dy$

$$= \int_0^{2\pi} [-sin^3\theta(-sin\theta) + cos^3\theta cos\theta] d\theta = \int_0^{2\pi} (cos^4\theta + sin^4\theta) d\theta$$

$$= \int_0^{2\pi} (1 - 2sin^2\theta\ cos^2\theta) d\theta = \int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi} (2sin\theta\ cos\theta)^2\ d\theta$$

$$= \int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi} sin^2 2d\theta = (2\pi - 0) - \frac{1}{4}\int_0^{2\pi} (1 - cos4\theta) d\theta$$

$$= 2\pi + \left[-\frac{1}{4}\theta + \frac{1}{16}sin4\theta\right]_0^{2\pi} = 2\pi - \frac{2\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$$

Now $\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = \bar{k}(3x^2 + 3y^2)$

$\therefore \int_s (\nabla \times \bar{F}).\bar{n}ds = 3\int_s (x^2 + y^2)\bar{k}.\bar{n}ds$

We have $(\bar{k}.\bar{n})ds = dxdy$ and R is the region on xy-plane

$\therefore \iint_s (\nabla \times \bar{F}).\bar{n}ds = 3\iint_R (x^2 + y^2)\, dx\, dy$

Put x=r cos$\emptyset$, $y = r\ sin\emptyset \therefore\ dxdy = rdr\ d\emptyset$

r is varying from 0 to 1 and 0$\leq \emptyset \leq 2\pi$.

$\therefore \int (\nabla \times \bar{F}).\ \bar{n}ds = 3\int_{\emptyset=0}^{2\pi}\int_{r=0}^{1} r^2$.rdr d$\emptyset = \frac{3\pi}{2}$

L.H.S=R.H.S.Hence the theorem is verified.

STOKES THEOREM

Verify Stokes theorem for $\bar{F} = (2x - y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$ over the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$bounded by the projection of the xy-plane.

Solution: The boundary C of S is a circle in xy plane i.e $x^2 + y^2$=1, z=0

The parametric equations are x=$cos\theta, y = sin\theta, \theta = 0 \rightarrow 2\pi$

$\therefore \quad dx = -sin\theta \; d\theta, dy = cos\theta \; d\theta$

$$\int_c \bar{F}.d\bar{r} = \int_c \bar{F_1}dx + \bar{F_2}dy + \bar{F_3}dz = \int_c (2x - y)dx - yz^2dy - y^2zdz$$

$$= \int_c (2x - y)dx (since \; z = 0 \; and \; dz = 0)$$

$$= -\int_0^{2\pi} (2\cos\theta - \sin\theta)\sin\theta d\theta = \int_0^{2\pi} \sin^2\theta d\theta - \int_0^{2\pi} \sin 2\theta d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{1 - cos2\theta}{2} d\theta - \int_0^{2\pi} sin2\theta \; d\theta = \left[\frac{1}{2}\theta - \frac{1}{4}sin2\theta + \frac{1}{2}.cos2\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}(2\pi - 0) + 0 + \frac{1}{2}.(cos4\pi - cos0) = \pi$$

Again $\nabla \times \bar{F}=\begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \bar{\imath}(-2yz+2yz) - \bar{\jmath}(0-0) + \bar{k}(0+1) = \bar{k}$

$\therefore \int_S (\nabla \times \bar{F}).\bar{n}ds = \int_S \bar{k}.\bar{n}ds = \int_R \int dxdy$

Where R is the projection of S on xy plane and $\bar{k}.\bar{n}ds = dxdy$

Now $\int \int_R dxdy = 4\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} dydx = 4\int_{x=0}^{1}\sqrt{1-x^2}\,dx = 4\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\right]_0^1$

$=4\left[\frac{1}{2}\sin^{-1}1\right] = 2\frac{\pi}{2} = \pi$

$\therefore The$ Stokes theorem is verified.

III. <u>STOKE'S THEOREM</u>

(Transformation between Line Integral and Surface Integral)

Let S be a open surface bounded by a closed, non intersecting curve C.

If $\bar{F}$ is any

differentieable vector point function then $\oint_C \bar{F}.d\,\bar{r}$=

$\int_S curl\ \ \bar{F}.\bar{n}\ ds\ where\ c\ is\ traversed\ in\ the\ positive$ direction and

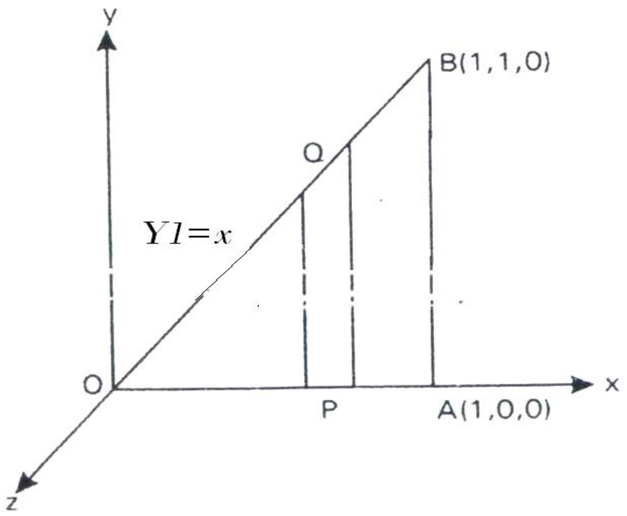
$\bar{n}\ is\ unit\ outward\ drawn\ normal\ at\ any\ point\ of\ the\ surface.$

Evaluate by Stokes theorem $\oint_c (x+y)dx + (2x-z)dy + (y+z)dz$ where C is the boundary of the triangle with vertices (0,0,0), (1,0,0) and (1,1,0).

Solution: Let $\bar{F}.d\bar{r} = \bar{F}.(\bar{\imath}dx + \bar{\jmath}dy + \bar{k}dz) = (x+y)dx + (2x-z)dy + (y+z)dz$

Then $\bar{F} = (x+y)\bar{\imath} + (2x-z)\bar{\jmath} + (y+z)\bar{k}$

By Stokes theorem, $\oint_c \bar{F}.d\bar{r} = \int\int_S curl\ \bar{F}.\bar{n}\, ds$



Where S is the surface of the triangle OAB which lies in the xy plane. Since the z Co-ordinates of O,A and B Are zero. Therefore$\bar{n} = \bar{k}$. Equation of OA is y=0 and that of OB, y=x in the xy plane.

Now $\nabla \times F = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = \bar{k}(3x^2 + 3y^2)$

$\therefore \int_s (\nabla \times F).\bar{n}ds = 3\int_s (x^2 + y^2)\bar{k}.\bar{n}ds$

We have $(\bar{k}.\bar{n})ds = dxdy$ and R is the region on xy-plane

$\therefore \iint_s (\nabla \times F).\bar{n}ds = 3\iint_R (x^2 + y^2)\, dx\, dy$

Put x=r cos$\theta, y = r\sin\theta \therefore dxdy = rdr\, d\theta$

r is varying from 0 to 1 and 0$\leq \theta \leq 2\pi$.

$\therefore \int (\nabla \times F).\, \bar{n}ds = 3\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2$.rdr d$\theta = \frac{3\pi}{2}$

L.H.S=R.H.S.Hence the theorem is verified.

STOKES THEOREM

$$\therefore\ curl\ \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = 2\bar{i} + \bar{k}$$

$$\therefore\ curl\ \bar{F}.\bar{n}\,ds = curl\ \bar{F}.\bar{K}\,dx\,dy = dx\,dy$$

$$\therefore \oint_c \bar{F}.d\bar{r} = \int\int_s dx\,dy = \int\int_s dA = A = area\ of\ \text{the}\ \Delta OAB$$

$$= \frac{1}{2} OA \times AB = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$$


Thank you