INSTITUTE OF AERONAUTICAL ENGINEERING

(Autonomous)

Dundigal, Hyderabad-500043

AEROSPACE ENGINEERING

COURSE LECTURE NOTES

Course Title	**ADVANCED MATHEMATICS IN AEROSPACE ENGINEERING**				
Course Code	BAEB01				
Programme	M.Tech				
Semester	I	AE			
Course Type	Core				
Regulation	**IARE - R18**				
Course Structure	**Theory**			**Practical**	
	Lectures	**Tutorials**	**Credits**	**Laboratory**	**Credits**
	3	-	3	-	-
Chief Coordinator	Ms. P Srilatha, Assistant Professor				
Course Faculty	Ms. P Srilatha, Assistant Professor				

COURSE OBJECTIVES:

The course should enable the students to:	
I	Develop a basic understanding of a range of mathematics tools with emphasis on engineering applications.
II	Solve problems with techniques from advanced linear algebra, ordinary differential equations and multivariable differentiation.
III	Develop skills to think quantitatively and analyze problems critically.

COURSE OUTCOMES (COs):

CO 1	Describe the basic concepts of probability, discrete, continuous random variables and determine probability distribution, sampling distribution of statistics like t, F and chi-square.
CO 2	Understand the foundation for hypothesis testing to predict the significance difference in the sample means and the use of ANOVA technique.
CO 3	Determine Ordinary linear differential equations solvable by nonlinear ODE's.
CO 4	Explore First and second order partial differential equations.
CO 5	Analyze the methods for partial differential equations.

COURSE LEARNING OUTCOMES (CLOs):

BAEB01.01	Describe the basic concepts of probability, discrete and continuous random variables
BAEB01.02	Determine the probability distribution to find mean and variance.
BAEB01.03	Discuss the concept of sampling distribution of statistics like t, F and chi-square.
BAEB01.04	Understand the foundation for hypothesis testing.
BAEB01.05	Apply testing of hypothesis to predict the significance difference in the sample means.
BAEB01.06	Understand the assumptions involved in the use of ANOVA technique.
BAEB01.07	Solve differential equation using single step method.
BAEB01.08	Solve differential equation using multi step methods.
BAEB01.09	Understand the concept of non- linear ordinary differential equations.
BAEB01.10	Understand partial differential equation for solving linear equations.
BAEB01.11	Solving the first order ordinary differential equations subject to boundary conditions.
BAEB01.12	Solving the higher order ordinary differential equations subject to boundary conditions.
BAEB01.13	Understand the concept of methods for elliptic partial differential equations.
BAEB01.14	Understand the concept of Neumann and mixed problems.
BAEB01.15	Analyze the concept of parabolic and hyperbolic partial differential equations.

SYLLABUS

Unit-I	PROBABILITY THEORY AND DISTRIBUTIONS
Theory Probability Theory and Sampling Distributions. Basic probability theory along with examples. Standard discrete and continuous distributions like Binomial, Poisson, Normal, Exponential etc. Central Limit Theorem and its significance. Some sampling distributions like chi-square , t, F distributions.	
Unit-II	**TESTING OF STATISTICAL HYPOTHESIS**
Testing a statistical hypothesis, tests on single sample and two samples concerning means and variances. ANOVA: One – way, Two – way with/without interactions.	
Unit-III	**ORDINARY DIFFERENTIAL EQUATIONS**
Ordinary linear differential equations solvable by direct solution methods. Non-linear differential equations solvable by direct solution methods.	
Unit-IV	**PARTIAL DIFFERENTIAL EQUATIONS AND CONCEPTS IN SOLUTION TO BOUNDARY VALUE PROBLEMS**
First and second order partial differential equations; canonical forms	
Unit-V	**NUMERIC'S FOR ORDINARY DIFFERENTIAL EQUATIONS AND PARTIAL DIFFERENTIAL EQUATIONS**
Methods for first order ordinary differential equations, multistep methods, methods for systems and higher order ordinary differential equations, methods for elliptic partial differential equations, Neumann and mixed problems, irregular boundary, methods for parabolic and hyperbolic partial differential equations.	
Text Books:	
1. J. B. Doshi, "Differential Equations for Scientists and Engineers", Narosa, New Delhi. 2. B. S. Grewal, "Higher Engineering Mathematics", Khanna Publishers, 43rd Edition, Delhi.	
Reference Books:	
1. S. P. Gupta, "Statistical Methods", S. Chand & Sons, 37th revised edition. 2. Erwin Kreyszig, "Advanced Engineering Mathematics", Wiley India, 9th Edition, 2010.	

UNIT-I
PROBABILITY THEORY AND DISTRIBUTIONS

Probability:

Probability is a branch of mathematics that deals with calculating the likelihood of a given event's occurrence, which is expressed as a number between 1 and 0. An event with a probability of 1 can be considered a certainty: for example, the probability of a coin toss resulting in either "heads" or "tails" is 1, because there are no other options, assuming the coin lands flat. An event with a probability of .5 can be considered to have equal odds of occurring or not occurring: for example, the probability of a coin toss resulting in "heads" is .5, because the toss is equally as likely to result in "tails." An event with a probability of 0 can be considered an impossibility: for example, the probability that the coin will land (flat) without either side facing up is 0, because either "heads" or "tails" must be facing up. A little paradoxical, probability theory applies precise calculations to quantify uncertain measures of random events.

In its simplest form, probability can be expressed mathematically as: the number of occurrences of a targeted event divided by the number of occurrences *plus* the number of failures of occurrences (this adds up to the total of possible outcomes):

$$p(a) = p(a)/[p(a) + p(b)]$$

Calculating probabilities in a situation like a coin toss is straightforward, because the outcomes are mutually exclusive: either one event or the other must occur. Each coin toss is an *independent*event; the outcome of one trial has no effect on subsequent ones. No matter how many consecutive times one side lands facing up, the probability that it will do so at the next toss is always .5 (50-50). The mistaken idea that a number of consecutive results (six "heads" for example) makes it more likely that the next toss will result in a "tails" is known as the *gambler's fallacy* , one that has led to the downfall of many a bettor.

Probability theory had its start in the 17th century, when two French mathematicians, Blaise Pascal and Pierre de Fermat carried on a correspondence discussing mathematical problems dealing with games of chance. Contemporary applications of probability theory run the gamut of human inquiry, and include aspects of computer programming, astrophysics, music, weather prediction, and medicine.

Trial and Event: Consider an experiment, which though repeated under essential and identical conditions, does not give a unique result but may result in any one of the several possible outcomes. The experiment is known as **Trial** and the outcome is called **Event**

E.g. (1) Throwing a dice experiment getting the no’s 1,2,3,4,5,6 (event)

(2) Tossing a coin experiment and getting head or tail (event)

Exhaustive Events:

The total no. of possible outcomes in any trial is called exhaustive event.

E.g.: (1) In tossing of a coin experiment there are two exhaustive events.

(2) In throwing an n-dice experiment, there are 6^n exhaustive events.

Favorable event:

The no of cases favorable to an event in a trial is the no of outcomes which entities the happening of the event.

E.g. (1) In tossing a coin, there is one and only one favorable case to get either head or tail.

Mutually exclusive Event: If two or more of them cannot happen simultaneously in the same trial then the event are called mutually exclusive event.

E.g. In throwing a dice experiment, the events 1,2,3,------6 are M.E. events

Equally likely Events: Outcomes of events are said to be equally likely if there is no reason for one to be preferred over other. E.g. tossing a coin. Chance of getting 1,2,3,4,5,6 is equally likely.

Independent Event:

Several events are said to be independent if the happening or the non-happening of the event is not affected by the concerning of the occurrence of any one of the remaining events.

An event that always happen is called **Certain event,** it is denoted by 'S'.

An event that never happens is called **Impossible event**, it is denoted by 'ϕ'.

Eg: In tossing a coin and throwing a die, getting head or tail is independent of getting no's 1 or 2 or 3 or 4 or 5 or 6.

Definition:probability (Mathematical Definition)

If a trial results in n-exhaustive mutually exclusive, and equally likely cases and m of them are favorable to the happening of an event E then the probability of an event E is denoted by P(E) and is defined as

$$P(E) = \frac{no\ of\ \ favourable\ cases\ to\ event}{Total\ no\ of\ \ exaustive\ cases} = \frac{m}{n}$$

Sample Space:

The set of all possible outcomes of a random experiment is called Sample Space .The elements of this set are called sample points. Sample Space is denoted by S.

Eg. (1) In throwing two dies experiment, Sample S contains 36 Sample points.

S = {(1,1) ,(1,2) ,----------(1,6), --------(6,1),(6,2),--------(6,6)}

Eg. (2) In tossing two coins experiment ,   S = {HH ,HT,TH,TT}

A sample space is called **discrete** if it contains only finitely or infinitely many points which can be arranged into a simple sequence $w_1, w_2, \ldots\ldots$ .while a sample space containing non denumerable no. of points is called a continuous sample space.

Statistical or Empirical Probability:

If a trial is repeated a no. of times under essential homogenous and identical conditions, then the limiting value of the ratio of the no. of times the event happens to the total no. of trials, as the number of trials become indefinitely large, is called the probability of happening of the event.(It is assumed the limit is finite and unique)

Symbolically, if in 'n' trials and events E happens 'm' times , then the probability 'p' of the happening of E is given by p = P(E) = $\lim_{n\to\infty} \frac{m}{n}$.

An event E is called **elementary event** if it consists only one element.

An event, which is not elementary, is called **compound event.**

Example 1: What is the probability of getting a 2 <u>or</u> a 5 when a die is rolled?

<u>Solution:</u>

Taking the individual probabilities of each number, getting a 2 is 1/6 and so is getting a 5.

Applying the formula of compound probability,

Probability of getting a 2 **or** a 5,

P(2 or 5) = P(2) + P(5) – P(2 and 5)

==> 1/6 + 1/6 – 0

==> 2/6 = 1/3.

Example 2: Consider the example of finding the probability of selecting a black card or a 6 from a deck of 52 cards.

Solution:

We need to find out P(B or 6)

Probability of selecting a black card = 26/52

Probability of selecting a 6 = 4/52

Probability of selecting both a black card and a 6 = 2/52

P(B or 6) = P(B) + P(6) – P(B and 6)

= 26/52 + 4/52 – 2/52

= 28/52

= 7/13.

Conditional probability:

Conditional probability is calculating the probability of an event given that another event has already occured .

The formula for conditional probability P(A|B), read as P(A given B) is

P(A|B) = P (A and B) / P(B)

Consider the following example:

Example: In a class, 40% of the students study math and science. 60% of the students study math. What is the probability of a student studying science given he/she is already studying math?

Solution

P(M and S) = 0.40

P(M) = 0.60

P(S|M) = P(M and S)/P(S) = 0.40/0.60 = 2/3 = 0.67

Complement of an event

A complement of an event A can be stated as that which does NOT contain the occurrence of A.

A complement of an event is denoted as $P(A^c)$ or P(A’).

$$P(A^c) = 1 – P(A)$$

or it can be stated, $P(A)+P(A^c) = 1$

For example,

if A is the event of getting a head in coin toss, A^c is not getting a head i.e., getting a tail.

if A is the event of getting an even number in a die roll, A^c is the event of NOT getting an even number i.e., getting an odd number.

if A is the event of randomly choosing a number in the range of -3 to 3, A^c is the event of choosing every number that is NOT negative i.e., 0,1,2 & 3 (0 is neither positive or negative).

Consider the following example:

Example: A single coin is tossed 5 times. What is the probability of getting at least one head?

Solution:

Consider solving this using complement.

Probability of getting no head = P(all tails) = 1/32

P(at least one head) = 1 – P(all tails) = 1 – 1/32 = 31/32.

Example 1: A dice is thrown 3 times .what is the probability that atleast one head is obtained?

Sol: Sample space = [HHH, HHT, HTH, THH, TTH, THT, HTT, TTT]

Total number of ways = 2 × 2 × 2 = 8. Fav. Cases = 7

P (A) = 7/8

OR

P (of getting at least one head) = 1 – P (no head)⇒ 1 – (1/8) = 7/8

Example 2: Find the probability of getting a numbered card when a card is drawn from the pack of 52 cards.

Sol: Total Cards = 52. Numbered Cards = (2, 3, 4, 5, 6, 7, 8, 9, 10) 9 from each suit 4 × 9 = 36

P (E) = 36/52 = 9/13

Example 3: There are 5 green 7 red balls. Two balls are selected one by one without replacement. Find the probability that first is green and second is red.

Sol: P (G) × P (R) = (5/12) x (7/11) = 35/132

Example 4: What is the probability of getting a sum of 7 when two dice are thrown?

Sol: Probability math - Total number of ways = 6 × 6 = 36 ways. Favorable cases = (1, 6) (6, 1) (2, 5) (5, 2) (3, 4) (4, 3) --- 6 ways. P (A) = 6/36 = 1/6

Example 5: 1 card is drawn at random from the pack of 52 cards.

(i) Find the Probability that it is an honor card.

(ii) It is a face card.

Sol: (i) honor cards = (A, J, Q, K) 4 cards from each suits = 4 × 4 = 16

P (honor card) = 16/52 = 4/13

(ii) face cards = (J,Q,K) 3 cards from each suit = 3 × 4 = 12 Cards.

P (face Card) = 12/52 = 3/13

Example 6: Two cards are drawn from the pack of 52 cards. Find the probability that both are diamonds or both are kings.

Sol: Total no. of ways = ${}^{52}C_2$

Case I: Both are diamonds = ${}^{13}C_2$

Case II: Both are kings = ${}^{4}C_2$

P (both are diamonds or both are kings) = $({}^{13}C_2 + {}^{4}C_2) / {}^{52}C_2$

Example 7: Three dice are rolled together. What is the probability as getting at least one '4'?

Sol: Total number of ways = 6 × 6 × 6 = 216. Probability of getting number ‘4’ at least one time = 1 – (Probability of getting no number 4) = 1 – (5/6) x (5/6) x (5/6) = 91/216

Example 8: A problem is given to three persons P, Q, R whose respective chances of solving it are 2/7, 4/7, 4/9 respectively. What is the probability that the problem is solved?

Sol: Probability of the problem getting solved = 1 – (Probability of none of them solving the problem)

$$P(P)=\frac{2}{7} \Rightarrow P(\bar{P})=1-\frac{2}{7}=\frac{5}{7},\ P(Q)=\frac{4}{7} \Rightarrow P(\bar{Q})=1-\frac{4}{7}=\frac{3}{7},\ P(R)=\frac{4}{9} \Rightarrow P(\bar{R})=1-\frac{4}{9}=\frac{5}{9}$$

Probability of problem getting solved = 1 – (5/7) x (3/7) x (5/9) = (122/147)

Example 9: Find the probability of getting two heads when five coins are tossed.

Sol: Number of ways of getting two heads = ${}^{5}C_2$ = 10. Total Number of ways = 2^5 = 32

P (two heads) = 10/32 = 5/16

Example 10: What is the probability of getting a sum of 22 or more when four dice are thrown?

Sol: Total number of ways = 6^4 = 1296. Number of ways of getting a sum 22 are 6,6,6,4 = 4! / 3!

= 4

6,6,5,5 = 4! / 2!2! = 6. Number of ways of getting a sum 23 is 6,6,6,5 = 4! / 3! = 4.

Number of ways of getting a sum 24 is 6,6,6,6 = 1.

Fav. Number of cases = 4 + 6 + 4 + 1 = 15 ways. P (getting a sum of 22 or more) = 15/1296 = 5/432

Example 11: Two dice are thrown together. What is the probability that the number obtained on one of the dice is multiple of number obtained on the other dice?

Sol:Total number of cases = 6^2 = 36

Since the number on a die should be multiple of the other, the possibilities are

(1, 1) (2, 2) (3, 3) ------ (6, 6) --- 6 ways

(2, 1) (1, 2) (1, 4) (4, 1) (1, 3) (3, 1) (1, 5) (5, 1) (6, 1) (1, 6) --- 10 ways

(2, 4) (4, 2) (2, 6) (6, 2) (3, 6) (6, 3) -- 6 ways

Favorable cases are = 6 + 10 + 6 = 22. So, P (A) = 22/36 = 11/18

Example 12: From a pack of cards, three cards are drawn at random. Find the probability that each card is from different suit.

Sol: Total number of cases = ${}^{52}C_3$

One card each should be selected from a different suit. The three suits can be chosen in ${}^{4}C_3$ was

The cards can be selected in a total of (${}^{4}C_3$) x (${}^{13}C_1$) x (${}^{13}C_1$) x (${}^{13}C_1$)

Probability = ${}^{4}C_3$ x $({}^{13}C_1)^3$ / ${}^{52}C_3$

= 4 x $(13)^3$ / ${}^{52}C_3$

Example 13: Find the probability that a leap year has 52 Sundays.

Sol: A leap year can have 52 Sundays or 53 Sundays. In a leap year, there are 366 days out of which there are 52 complete weeks & remaining 2 days. Now, these two days can be (Sat, Sun) (Sun, Mon) (Mon, Tue) (Tue, Wed) (Wed, Thur) (Thur, Friday) (Friday, Sat).

So there are total 7 cases out of which (Sat, Sun) (Sun, Mon) are two favorable cases. So, P (53 Sundays) = 2 / 7

Now, P(52 Sundays) + P(53 Sundays) = 1

So, P (52 Sundays) = 1 - P(53 Sundays) = 1 – (2/7) = (5/7)

Example 14: Fifteen people sit around a circular table. What are odds against two particular people sitting together?

Sol: 15 persons can be seated in 14! Ways. No. of ways in which two particular people sit together is 13! × 2!

The probability of two particular persons sitting together 13!2! / 14! = 1/7

Odds against the event = 6 : 1

Example 15: Three bags contain 3 red, 7 black; 8 red, 2 black, and 4 red & 6 black balls respectively. 1 of the bags is selected at random and a ball is drawn from it. If the ball drawn is red, find the probability that it is drawn from the third bag.

Sol: Let E1, E2, E3 and A are the events defined as follows.

E1 = First bag is chosen

E2 = Second bag is chosen

E3 = Third bag is chosen

A = Ball drawn is red

Since there are three bags and one of the bags is chosen at random, so P (E1) = P(E2) = P(E3) = 1 / 3

If E1 has already occurred, then first bag has been chosen which contains 3 red and 7 black balls. The probability of drawing 1 red ball from it is 3/10. So, P (A/E_1) = 3/10, similarly $P(A/E_2)$ = 8/10, and $P(A/E_3)$ = 4/10. We are required to find $P(E_3/A)$ i.e. given that the ball drawn is red, what is the probability that the ball is drawn from the third bag by Baye's rule

$$= \frac{\frac{1}{3}\times\frac{4}{10}}{\frac{1}{3}\times\frac{3}{10}+\frac{1}{3}\times\frac{8}{10}+\frac{1}{3}\times\frac{4}{10}} = \frac{4}{15}.$$

Derivation of Bayes Theorem:

Statement:Let $E_1, E_2, \ldots, E_n\}$ be a set of events associated with a sample space S, where all the events $E_1, E_2, \ldots, E_n$ have nonzero probability of occurrence and they form a partition of S. Let A be any event associated with S, then according to Bayes theorem,

$$P(E_i \mid A) = \frac{P(E_i)P(E_i \mid A)}{\sum_{k=0}^{n} P(E_k)P(A|E_k)}$$

Proof:According to conditional probability formula,

$$P(E_i \mid A) = \frac{P(E_i \cap A)}{P(A)} \quad \cdots\cdots\cdots\cdots(1)$$

Using multiplication rule of probability,

$$P(E_i \cap A) = P(E_i)P(E_i \mid A) \cdots\cdots\cdots\cdots(2)$$

Using total probability theorem,

$$P(A) = \sum_{k=0}^{n} P(E_k)P(A|E_k) \cdots\cdots\cdots\cdots(3)$$

Putting the values from equations (2) and (3) in equation 1, we get

$$P(E_i \mid A) = \frac{P(E_i)P(E_i \mid A)}{\sum_{k=0}^{n} P(E_k)P(A|E_k)}$$

Examples:

Some illustrations will improve the understanding of the concept.

Example 1:Bag I contains 4 white and 6 black balls while another Bag II contains 4 white and 3 black balls. One ball is drawn at random from one of the bags and it is found to be black. Find the probability that it was drawn from Bag I.

Solution:Let E_1 be the event of choosing the bag I, E_2 the event of choosing the bag II and A be the event of drawing a black ball.

Then, $P(E_1) = P(E_2) = \frac{1}{2}$

Also, $P(A|E_1) = P$(drawing a black ball from Bag I) = $\frac{6}{10} = \frac{3}{5}$

$P(A|E_2) = P$(drawing a black ball from Bag II) = $\frac{3}{7}$

By using Bayes' theorem, the probability of drawing a black ball from bag I out of two bags,

$$P(E_1|A) = \frac{P(E_1)P(A|E_1)}{P(E_1)P(A|E_1)+P(E_2)P(A|E_2)}$$

$$= \frac{\frac{1}{2} \times \frac{3}{5}}{\frac{1}{2} \times \frac{3}{7} + \frac{1}{2} \times \frac{3}{5}} = \frac{7}{12}$$

Example 2:A man is known to speak truth 2 out of 3 times. He throws a die andreports that number obtained is a four. Find the probability that the number obtained is actually a four.

Solution:Let A be the event that the man reports that number four is obtained.

Let E_1 be the event that four is obtained and E_2 be its complementary event.

Then, $P(E_1)$ = Probability that four occurs = $\frac{1}{6}$

$P(E_2)$ = Probability that four does not occurs = $1 - P(E_1) = 1 - \frac{1}{6} = \frac{5}{6}$

Also, $P(A|E_1)$ = Probability that man reports four and it is actually a four = $\frac{2}{3}$

$P(A|E_2)$ = Probability that man reports four and it is not a four = $\frac{1}{3}$

By using Bayes' theorem, probability that number obtained is actually a four,

$$P(E_1|A) = \frac{P(E_1)P(A|E_1)}{P(E_1)P(A|E_1) + P(E_2)P(A|E_2)} = \frac{\frac{1}{6} \times \frac{2}{3}}{\frac{1}{6} \times \frac{2}{3} + \frac{5}{6} \times \frac{1}{3}} = \frac{2}{7}$$

Students, are you struggling to find a solution to a specific question from Bayes theorem? We will make it easy for you. For detailed discussion on the concept of Bayes' theorem, download Byju's-the learning app

Random Variables

- A random variable X on a sample space S is a function $X : S \rightarrow R$ from S onto the set of real numbers R, which assigns a real number X (s) to each sample point 's' of S.
- Random variables (r.v.) bare denoted by the capital letters X,Y,Z,etc..
- Random variable is a single valued function.
- Sum, difference, product of two random variables is also a random variable .Finite linear combination of r.v is also a r.v .Scalar multiple of a random variable is also random variable.
- A random variable, which takes at most a countable number of values, it is called a discrete r.v. In other words, a real valued function defined on a discrete sample space is called discrete r.v.
- A random variable X is said to be continuous if it can take all possible values between certain limits .In other words, a r.v is said to be continuous when it's different values cannot be put in 1-1 correspondence with a set of positive integers.
- A continuous r.v is a r.v that can be measured to any desired degree of accuracy. Ex : age , height, weight etc..
- Discrete Probability distribution: Each event in a sample has a certain probability of occurrence . A formula representing all these probabilities which a discrete r.v. assumes is known as the discrete probability distribution.
- The probability function or probability mass function (p.m.f) of a discrete random variable X is the function f(x) satisfying the following conditions.

i) $f(x) \geq 0$

ii) $\sum_{x} f(x) = 1$

iii) $P(X = x) = f(x)$

- Cumulative distribution or simply distribution of a discrete r.v. X is F(x) defined by F(x) = $P(X \leq x) = \sum_{t \leq x} f(t)$ for $-\infty < x < \infty$

$F(-\infty) = 0$, $F(\infty)=1$, $0 \leq F(x) \leq 1$, $F(x) \leq F(y)$ if $x<y$

$P(x_k) = P(X = x_k) = F(x_k) - F(x_{k-1})$

- For a continuous r.v. X, the function f(x) satisfying the following is known as the probability density function(p.d.f.) or simply density function:

  i) $f(x) \geq 0$, $-\infty < x < \infty$

  ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

  iii) $P(a<X<b) = \int_a^b f(x)dx$ = Area under f(x) between ordinates x=a and x=b

$P(a<X<b) = P(a \leq x < b) = P(a < X \leq b) = P(a \leq X \leq b)$

(i.e) In case of continuous it does not matter weather we include the end points of the interval from a to b.This result in general is not true for discrete r.v.

- Probability at a point P(X=a) = $\int_{a-\Delta x}^{a+\Delta x} f(x)dx$

- Cumulative distribution for a continuous r.v. X with p.d.f. f(x), the cumulative distribution F(x) is defined as

$F(x) = P(X \leq x) = \int_{-\infty}^{\infty} f(t)dt \quad -\infty < x < \infty$

It follows that $F(-\infty) = 0$, $F(\infty)=1$, $0 \leq F(x) \leq 1$ for $-\infty < x < \infty$

$f(x) = d/dx(F(x)) = F^1(x) \geq 0$ and $P(a < x < b) = F(b) - F(a)$

- In case of discrete r.v. the probability at a point i.e., P(x=c) is not zero for some fixed c however in case of continuous random variables the probability at appoint is always zero. I.e., P(x=c) = 0 for all possible values of c.
- P(E) = 0 does not imply that the event E is null or impossible event.
- If X and Y are two discrete random variables the joint probability function of X and Y is given by P(X=x,Y=y) = f(x,y) and satisfies

  (i) $f(x,y) \geq 0$ (ii) $\sum_x \sum_y f(x,y) = 1$

- If c is any constant then E(cX) = c E(X)
- If X and Y are two r.v.'s then E(X+Y) = E(X)+E(Y)
- IF X,Y are two independent r.v.'s then E(XY) = E(X)E(Y)
- If $X_1,X_2,\text{-------},X_n$ are random variables then $E(c_1X_1 + c_2X_2 + \text{------} + c_nX_n) = c_1E(X_1)+c_2E(X_2)+\text{-----}+c_nE(X_n)$ for any scalars $c_1,c_2,\text{------},c_n$ If all expectations exists
- If $X_1,X_2,\text{-------},X_n$ are independent r.v's then $E\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} E(X_i)$ if all expectations exists.
- Var (X) = E (X^2) –[E (X)]^2
- If 'c' is any constant then var (cX) = c^2var(X)
- The quantity $E[(X-a)^2]$ is minimum when a =μ= E(X)
- If X and Y are independent r.v.'s then Var(X ± Y) = Var(X) ± Var(Y)

Binomial Distribution

- A random variable X is said to follow binomial distribution if it assumes only non-negative values and its probability mass function is given by

  $$P(X = x) = P(x) = \binom{n}{x} p^x q^{n-x} \quad \text{where } x = 0,1,2,3,\ldots.n \quad q = 1-p$$

  where n, p are known as parameters, n- number of independent trials p- probability of success in each trial, q- probability of failure.

- Binomial distribution is a discrete distribution.
- The notation X ~B(n,p) is the random variable X which follows the binomial distribution with parameters n and p
- If n trials constitute an experiment and the experiment is repeated N times the frequency function of the binomial distribution is given by f(x) = NP(x). The expected frequencies of 0,1,2,….. n successes are the successive terms of the binomial expansion $N(p+q)^n$
- The mean and variance of Binomial distribution are np , npq respectively.
- **Mode of the Binomial distribution**: Mode of B.D. Depending upon the values of (n+1)p

  (i) If (n+1)p is not an integer then there exists a unique modal value for binomial distribution and it is 'm'= integral part of (n+1)p

  (ii) If (n+1)p is an integer say m then the distribution is Bi-Modal and the two modal values are m and m-1

- Moment generating function of Binomial distribution: If X ~B(n,p)then $M_X(t)=(q+pe^t)^n$
- The sum of two independent binomial variates is not a binomial varaite. In other words, Binomial distribution does not posses the additive or reproductive property.
- For B.D. $\gamma_1= \sqrt{\beta_1} = \frac{1-2p}{\sqrt{npq}}$ $\gamma_2= \beta_2 - 3 = \frac{1-6pq}{npq}$
- If X_1~ $B(n_1,p)$ and X_2~ $B(n_2,p)$ then $X_1+X_2 \sim B(n_1+n_2,p)$.Thus the B.D. Possesses the additive or reproductive property if $p_1=p_2$

Poisson Distribution

- Poisson Distribution is a limiting case of the Binomial distribution under the following conditions:

  (i) n, the number of trials is infinitely large.

  (ii) P, the constant probability of success for each trial is indefinitely small.

  (iii) np= λ, is finite where λ is a positive real number.

- A random variable X is said to follow a Poisson distribution if it assumes only non-negative values and

  its p.m.f. is given by

$$P(x,\lambda) = P(X=x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!} & x = 0,1,2,3,\ldots\ldots \ \lambda > 0 \\ 0 & \text{Other wise} \end{cases}$$

Here λ is known as the parameter of the distribution.

- We shall use the notation X~ P(λ) to denote that X is a Poisson variate with parameter λ
- Mean and variance of Poisson distribution are equal to λ.
- The coefficient of skewness and kurtosis of the poisson distribution are $\gamma_1 = \sqrt{\beta_1} = 1/\sqrt{\lambda}$ and $\gamma_2 = \beta_2 - 3 = 1/\lambda$. Hence the poisson distribution is always a skewed distribution. Proceeding to limit as λ tends to infinity we get $\beta_1 = 0$ and $\beta_2 = 3$
- Mode of Poisson Distribution: Mode of P.D. Depending upon the value of λ

(i) when λ is not an integer the distribution is uni- modal and integral part of λ is the unique modal value.

(ii) When λ = k is an integer the distribution is bi-modal and the two modals are k-1 and k.

- Sum of independent poisson variates is also poisson variate.
- The difference of two independent poisson variates is not a poisson variate.
- **Moment generating function of the P.D.**

If X~ P(λ) then $M_X(t) = e^{\lambda(e^t - 1)}$

- Recurrence formula for the probabilities of P.D. (Fitting of P.D.)

$$P(x+1) = \frac{\lambda}{x+1} p(x)$$

- Recurrence relation for the probabilities of B.D. (Fitting of B.D.)

$$P(x+1) = \left\{\frac{n-x}{x+1} \cdot \frac{p}{q}\right\} p(x)$$

Normal Distribution

- A random variable X is said to have a normal distribution with parameters μ called mean and σ^2 called variance if its density function is given by the probability law

$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-1}{2}\left\{\frac{x-\mu}{\sigma}\right\}^2\right], \quad -\infty < x < \infty, -\infty<\mu<\infty, \sigma> 0$$

- A r.v. X with mean μ and variance σ^2 follows the normal distribution is denoted by

  $X \sim N(\mu, \sigma^2)$

- If $X \sim N(\mu, \sigma^2)$ then $Z = \frac{X-\mu}{\sigma}$ is a standard normal variate with E(Z) = 0 and var(Z)=0 and we write $Z \sim N(0,1)$

- The p.d.f. of standard normal variate Z is given by $f(Z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$, $-\infty < Z < \infty$

- The distribution function $F(Z) = P(Z \le z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$

- $F(-z) = 1 - F(z)$

- $P(a < z \le b) = P(a \le z < b)= P(a <z < b)= P(a \le z \le b)= F(b) - F(a)$

- If $X \sim N(\mu, \sigma^2)$ then $Z = \frac{X-\mu}{\sigma}$ then $P(a \le X \le b) = F\left(\frac{b-\mu}{\sigma}\right) - F\left(\frac{a-\mu}{\sigma}\right)$

- N.D. is another limiting form of the B.D. under the following conditions:

  i) n , the number of trials is infinitely large.

  ii) Neither p nor q is very small

- **Chief Characteristics of the normal distribution and normal probability curve:**

  i) The curve is bell shaped and symmetrical about the line x = μ

  ii) Mean median and mode of the distribution coincide.

  iii) As x increases numerically f(x) decreases rapidly.

  iv) The maximum probability occurring at the point x= μ and is given by

  $[P(x)]_{max} = 1/\sigma\sqrt{2\Pi}$

  v) $\beta_1 = 0$ and $\beta_2 = 3$

vi) $\mu_{2r+1} = 0$ (r = 0,1,2……) and $\mu_{2r} = 1.3.5….(2r-1)\sigma^{2r}$

vii) Since f(x) being the probability can never be negative no portion of the curve lies below x- axis.

viii) Linear combination of independent normal variate is also a normal variate.

ix) X- axis is an asymptote to the curve.

x) The points of inflexion of the curve are given by $x = \mu \pm \sigma$, $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2}$

xi) Q.D. **:** M.D.**:** S.D. **::** $\frac{2}{3}\sigma$ **:** $\frac{4}{5}\sigma$ **:** σ**::** $\frac{2}{3}$ **:** $\frac{4}{5}$ **:** 1 Or Q.D. **:** M.D.**:** S.D. **::10:12:15**

xii) Area property: $P(\mu - \sigma < X < \mu + \sigma) = 0.6826 = P(-1 < Z < 1)$

$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9544 = P(-2 < Z < 2)$

$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973 = P(-3 < Z < 3)$

$P(|Z| > 3) = 0.0027$

- m.g.f. of N.D. If $X \sim N(\mu, \sigma^2)$ then $M_X(t) = e^{\mu t} + t^2\sigma^2/2$

  If $Z \sim N(0,1)$ then $M_Z(t) = e^{t^2/2}$

Continuity Correction:

- The N.D. applies to continuous random variables. It is often used to approximate distributions of discrete r.v. Provided that we make the continuity correction.
- If we want to approximate its distribution with a N.D. we must spread its values over a continuous scale. We do this by representing each integer k by the interval from k-1/2 to k+1/2 and at least k is represented by the interval to the right of k-1/2 to at most k is represented by the interval to the left of k+1/2.
- **Normal approximation to the B.D:**

  $X \sim B(n, p)$ and if $Z = \frac{X - np}{\sqrt{np(1-p)}}$ then $Z \sim N(0,1)$ as n tends to infinity and F(Z) =

$$F(Z) = P(Z \le z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt \quad -\infty < Z < \infty$$

- Use the normal approximation to the B.D. only when (i) np and n(1-p) are both greater than 15 (ii) n is small and p is close to ½

- **Poisson process:** Poisson process is a random process in which the number of events (successes) x occurring in atime interval of length T is counted. It is continuous parameter, discrete stable process. By dividing T into n equal parts of length Δt we have T = n **.**ΔT. Assuming that (i) $P \propto \Delta T$ or $P = \alpha\Delta t$ (ii) The occurrence of events are independent (iii) The probability of more than one substance during a small time interval Δt is negligible.

  As $n \to \infty$**,** the probability of x success during a time interval T follows the P.D. with parameter $\lambda = np = \alpha \mathbf{T}$ where α is the average(mean) number of successes for unit time.

  PROBLEMS:

1:A random variable x has the following probability function:

x	0	1	3	4	5	6	7
P(x)	0	k	2k	2k	3k	k^2	$7k^2$+k

Find (i) k (ii) P(x<6) (iii) P(x>6)

Solution:

(i) since the total probability is unity, we have $\sum_{x=0}^{n} p(x) = 1$

i.e., 0 + k +2k+ 2k+ 3k+ k^2+7k^2+k=1

i.e., $8k^2+9k-1=0$

$k=1,-1/8$

(ii) P(x<6)= 0 + k +2k+ 2k + 3k

=1+2+2+3=8

iii) P(x>6)= k^2 +$7k^2$+k

=9

2. Let X denotes the minimum of the two numbers that appear when a pair of fair dice is thrown once. Determine (i) Discrete probability distribution (ii) Expectation (iii) Variance

Solution:

When two dice are thrown, total number of outcomes is 6x6-36

In this case, sample space S= $\begin{Bmatrix} (1,1)(1,2)(1,3)(1,4)(1,5)(1,6) \\ (2,1)(2,2)(2,3)(2,4)(2,5)(2,6) \\ (3,1)(3,2)(3,3)(3,4)(3,5)(3,6) \\ (4,1)(4,2)(4,3)(4,4)(4,5)(4,6) \\ (5,1)(5,2)(5,3)(5,4)(5,5)(5,6) \\ (6,1)(6,2)(6,3)(6,4)(6,5)(6,6) \end{Bmatrix}$

If the random variable X assigns the minimum of its number in S, then the sample space S=

$$\begin{Bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{Bmatrix}$$

The minimum number could be 1,2,3,4,5,6

For minimum 1, the favorable cases are 11

Therefore, P(x=1)=11/36

P(x=2)=9/36, P(x=3)=7/36, P(x=4)=5/36, P(x=5)=3/36, P(x=6)=1/36

The probability distribution is

X	1	2	3	4	5	6
P(x)	11/36	9/36	7/36	5/36	3/36	1/36

(ii)Expectation mean = $\sum p_i x_i$

$$E(x) = 1\frac{11}{36} + 2\frac{9}{36} + 3\frac{7}{36} + 4\frac{5}{36} + 5\frac{3}{36} + 6\frac{1}{36}$$

Or $\mu = \frac{1}{36}[11 + 8 + 21 + 20 + 15 + 6] = \frac{9}{36} = 2.5278$

(ii) variance = $\sum p_i x^2{}_i - \mu^2$

$$E(x)=\frac{11}{36}1+\frac{9}{36}4+\frac{7}{36}9+\frac{5}{36}16+\frac{3}{36}25+\frac{1}{36}36-(2.5278)^2$$

=1.9713

3: A continuous random variable has the probability density function

$$f(x)=\begin{cases} kxe^{-\lambda x},\ for\ x\geq 0, \lambda>0 \\ 0, otherwise \end{cases}$$

Determine (i) k (ii) Mean (iii) Variance

Solution:

(i) since the total probability is unity, we have $\int_{-\infty}^{\infty} f(x)dx=1$

$$\int_{-\infty}^{0} 0dx+\int_{0}^{\infty} kxe^{-\lambda x}dx=1$$

i.e., $\int_{0}^{\infty} kxe^{-\lambda x}dx=1$

$$k\left[x\left(\frac{e^{-\lambda x}}{-\lambda}\right)-1\left(\frac{e^{-\lambda x}}{\lambda^2}\right)\right]_0^{\infty} \ or\ k=\lambda^2$$

(ii) mean of the distribution $\mu=\int_{-\infty}^{\infty} xf(x)dx$

$$\int_{-\infty}^{0} 0dx+\int_{0}^{\infty} kx^2e^{-\lambda x}dx$$

$$\lambda^2\left[x^2\left(\frac{e^{-\lambda x}}{-\lambda}\right)-2x\left(\frac{e^{-\lambda x}}{\lambda^2}\right)+2\left(\frac{e^{-\lambda x}}{\lambda^3}\right)\right]_0^{\infty}$$

$$=\frac{2}{\lambda}$$

Variance of the distribution $\sigma^2=\int_{-\infty}^{\infty} x^2 f(x)dx-\mu^2$

$$\sigma^2=\int_{-\infty}^{\infty} x^2 f(x)dx-\frac{4}{\lambda^2}$$

$$\lambda^2\left[x^3\left(\frac{e^{-\lambda x}}{-\lambda}\right)-3x^2\left(\frac{e^{-\lambda x}}{\lambda^2}\right)+6x\left(\frac{e^{-\lambda x}}{\lambda^3}\right)-6\left(\frac{e^{-\lambda x}}{\lambda^4}\right)\right]_0^{\infty}-\frac{4}{\lambda^2}$$

$$= \frac{2}{\lambda^2}$$

4:

Out of 800 families with 5 children each, how many would you expect to have (i)3 boys (ii)5girls (iii)either 2 or 3 boys ? Assume equal probabilities for boys and girls

Solution

P(3boys)=P(r=3)=P(3)= $\frac{1}{2^5}\,{}^5C_3 = \frac{5}{16}$ per family

Thus for 800 families the probability of number of families having 3 boys= $\frac{5}{16}(800) = 250$ families

P(5 girls)=P(no boys)=P(r=0)= $\frac{1}{2^5}\,{}^5C_0 = \frac{1}{32}$ per family

Thus for 800 families the probability of number of families having 5girls= $\frac{1}{32}(800) = 25$ families

P(either 2 or 3 boys =P(r=2)+P(r=3)=P(2)+P(3)

$\frac{1}{2^5}\,{}^5C_2 + \frac{1}{2^5}\,{}^5C_3$=5/8 per family

Expected number of families with 2 or 3 boys = $\frac{5}{8}(800) = 500$ families.

5: Average number of accidents on any day on a national highway is 1.8. Determine the probability that the number of accidents is (i) at least one (ii) at most one

Solution:

Mean= $\lambda = 1.8$

We have P(X=x)= p(x)$\frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-1.8}1.8^x}{x!}$

P (at least one) =P(x≥1)=1-P(x=0)

=1-0.1653

=0.8347

P (at most one) =P ($x\leq 1$)

=P(x=0)+P(x=1)

= 0.4628

6: The mean weight of 800 male students at a certain college is 140kg and the standard deviation is 10kg assuming that the weights are normally distributed find how many students weigh I) Between 130 and 148kg ii) more than 152kg

Solution:

Let μ be the mean and σ be the standard deviation. Then μ =140kg and σ =10pounds

(i) When x= 138, $z=\dfrac{x-\mu}{\sigma}=\dfrac{138-140}{10}=-0.2=z_1$

When x= 138, $z=\dfrac{x-\mu}{\sigma}=\dfrac{148-140}{10}=0.8=z_2$

$\therefore$ P($138\leq x\leq 148$)=P($-0.2\leq z\leq 0.8$)

=A(z_2)+A(z_1)

=A(0.8)+A(0.2)=0.2881+0.0793=0.3674

Hence the number of students whose weights are between 138kg and 140kg =0.3674x800=294

(ii) When x=152,$\frac{x-\mu}{\sigma}$=$\frac{152-140}{10}=1.2$=z_1

Therefore P(x>152)=P($z>z_1$)=0.5-A(z_1)

=0.5-0.3849=0.1151

Therefore number of students whose weights are more than 152kg =800x0.1151=92.

Exercise Problems:

1. Two coins are tossed simultaneously. Let X denotes the number of heads then find i) E(X) ii) E(X^2) iii)E(X^3) iv) V(X)
2. If f(x)=k$e^{-|x|}$ is probability density function in the interval, $-\infty<x<\infty$, then find i) k ii) Mean iii) Variance iv) P(0<x<4)
3. Out of 20 tape recorders 5 are defective. Find the standard deviation of defective in the sample of 10 randomly chosen tape recorders. Find (i) P(X=0) (ii) P(X=1) (iii) P(X=2) (iv) P (1<X<4).

4. In 1000 sets of trials per an event of small probability the frequencies f of the number of x of successes are

f	0	1	2	3	4	5	6	7	Total
x	305	365	210	80	28	9	2	1	1000

Fit the expected frequencies.

5. If X is a normal variate with mean 30 and standard deviation 5. Find the probabilities that

i) $P(26 \leq X \leq 40)$ ii) $P(X \geq 45)$

6. The marks obtained in Statistics in a certain examination found to be normally distributed. If

15% of the students greater than or equal to 60 marks, 40% less than 30 marks. Find the

mean and standard deviation.

7.If a Poisson distribution is such that $P(X=1)=\frac{3}{2}P(X=3)$ then find (i) $P(X \geq 1)$ (ii) $P(X \leq 3)$ (iii) $P(2 \leq X \leq 5)$.

. A random variable X has the following probability function:

X	-2	-1	0	1	2	3
P(x)	0.1	K	0.2	2K	0.3	K

Then find (i) k (ii) mean (iii) variance (iv) P(0 < x < 3)

- Maximum error E of estimate of a normal population mean μ with σ unknown by using small sample

mean $\overline{X}$ is E = $t_{\alpha/2}\frac{S}{\sqrt{n}}$ sample size n = $\left[t_{\alpha/2}\frac{S}{E}\right]^2$ here the percentage of confidence is (1 - α)100% and the degree of confidence is 1-α

- Small sample confidence interval for μ

$\overline{x} - t_{\alpha/2}\frac{S}{\sqrt{n}} < \mu < \overline{x} + t_{\alpha/2}\frac{S}{\sqrt{n}}$

- If $\overline{X}$ is the mean of a random sample of size n taken from a normal population having the mean μ and the variance σ^2 , and $S^2 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n-1}$ then $t = \frac{\overline{X} - \mu}{\frac{s}{\sqrt{n}}}$ is a r.v. having the

t- distribution with the parameter υ = (n-1)dof

- The overall shape of a t-distribution is similar to that of a normal distribution both are bell shaped and symmetrical about the mean. Like the standard normal distribution t-distribution has the mean 0, but its variance depends on the parameter υ (nu), called the number of degrees of freedom. The variance of t- distribution exceeds1, but it approaches 1 as $n \to \infty$. The t-distribution with υ-degree of freedom approaches the standard normal distribution as $\upsilon \to \infty$.

- The standard normal distribution provides a good approximation to the t- distribution for samples of size 30 or more.

- If S^2 is the variance of a random sample of size n taken from a normal population having the

variance σ^2 , then $\chi^2 = \frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{\sigma^2}$ is a r.v. having the chi-square distribution with the parameter υ = n-1

- The chi-square distribution is not symmetrical
- If S_1^2 and S_2^2 are the variances of independent random samples of size n_1 and n_2 respectively, taken from two normal populations having the same variance, then $F = \frac{S_1^2}{S_2^2}$ is a r.v. having the F- distribution with the parameter's $\upsilon_1 = n_1 - 1$ and $\upsilon_2 = n_2 - 1$ are called the numerator and denominator degrees of freedom respectively.
- $F_{1-\alpha}(\upsilon_1, \upsilon_2) = \frac{1}{F_\alpha(\upsilon_2, \upsilon_1)}$

Problems:

1. **Producer of 'gutkha' claims that the nicotine content in his 'gutkha' on the average is 83 mg. can this claim be accepted if a random sample of 8 'gutkhas' of this type have the nicotine contents of 2.0,1.7,2.1,1.9,2.2,2.1,2.0,1.6 mg.**

Solution: Given n=8 and μ =1.83 mg

1. **Null hypothesis(H_0):** μ =1.83

2. **Alternative hypothesis(H_1):** $\mu \neq 1.83$

3. **Level of significance:** α =0.05

   t_α for n-1 degrees of freedom

   $t_{0.05}$ for 8-1 degrees of freedom is 1.895

4. **Test statistic:** $t = \dfrac{\bar{x}-\mu}{\frac{S}{\sqrt{n}}}$

x	$(x - \bar{x})$	$(x - \bar{x})^2$
2.0	0.05	0.0025
1.7	-0.25	0.0625
2.1	0.15	0.0225
1.9	-0.05	0.0025
2.2	0.25	0.0625
2.1	0.15	0.0225
2.0	0.05	0.0025
1.6	-0.35	0.1225
Total=15.6		

$\bar{x} = \dfrac{15.6}{8} = 1.95$ and $S^2 = \sum \dfrac{(x-\bar{x})^2}{n-1} = \dfrac{0.3}{7}$

S=0.21

$$t = \frac{\bar{x}-\mu}{\frac{S}{\sqrt{n}}} = \frac{1.95-1.83}{\frac{0.21}{\sqrt{8}}} = 1.62$$

$|t| = 1.62$

5. **Conclusion:**

   $\therefore |t| < t_\alpha$

   $\therefore$ We accept the Null hypothesis.

2. **The means of two random samples of sizes 9,7 are 196.42 and 198.82.the sum of squares of deviations from their respective means are 26.94,18.73.can the samples be considered to have been the same population?**

Solution: Given n_1=9, n_2=7, $\bar{x}_1$=196.42, $\bar{x}_2$=198.82 and $\sum(x_i-\bar{x}_1)^2$=26.94, $\sum(x_i-\bar{x}_2)^2$=18.73

$$\therefore S^2 = \frac{\sum(x_i-x_1)^2+\sum(x_i-x_2)^2}{n_1+n_2-2} = 3.26$$

$\Rightarrow$ S=1.81

Null hypothesis(H_0): $\bar{x}_1 = \bar{x}_2$

Alternative hypothesis(H_1): $\bar{x}_1 \neq \bar{x}_2$

Level of significance: α =0.05

t_α for n_1+n_2-2 degrees of freedom

$t_{0.05}$ for 9+7-2=14 degrees of freedom is 2.15

Test statistic: $t = \dfrac{\bar{x}_1-\bar{x}_2}{S\sqrt{\dfrac{1}{n_1}+\dfrac{1}{n_2}}} = \dfrac{196.42-198.82}{(1.81)\sqrt{\dfrac{1}{9}+\dfrac{1}{7}}}$ =-2.63

$|t| = 2.63$

Conclusion:

$\therefore |t| > t_\alpha$

$\therefore$ We reject the Null hypothesis.

3. **In one sample of 8 observations the sum of squares of deviations of the sample values from the sample mean was 84.4 and another sample of 10 observations it was 102.6 .test whether there is any significant difference between two sample variances at at 5% level of significance.**

Solution: Given n_1=8, n_2=10, $\sum(x_i-\bar{x}_1)^2$=84.4 and $\sum(x_i-\bar{x}_2)^2$=102.6

$$S_1^{\,2} = \frac{\sum (x_i - x_1)^2}{n_1 - 1} = \frac{84.4}{7} = 12.057$$

$$S_2^{\,2} = \frac{\sum (x_i - x_1)^2}{n_2 - 1} = \frac{102.6}{9} = 11.4$$

1. **Null hypothesis(H_0):** $S_1^{\,2} = S_2^{\,2}$
2. **Alternative hypothesis(H_1):** $S_1^{\,2} \neq S_2^{\,2}$
3. **Level of significance:** α =0.05

   F_α For $(n_1 - 1, n_2 - 1)$ degrees of freedom

   $F_{0.05}$ For (7,9) degrees of freedom is 3.29
4. **Test statistic:** $F = \frac{S_1^2}{S_2^2} = \frac{12.057}{11.4}$ =1.057

   $|F| = 1.057$
5. **Conclusion:**

   $\therefore |F| < F_\alpha$

   $\therefore$ We accept the Null hypothesis.

4. The following table gives the classification of 100 workers according to gender and nature of work. Test whether the nature of work is independent of the gender of the worker.

	Stable	**Unstable**	**Total**
Male	**40**	**20**	**60**
Female	**10**	**30**	**40**
Total	**50**	**50**	**100**

Solution: Given that

Expected frequencies $= \frac{\text{row total} \times \text{column total}}{\text{grand total}}$

$\frac{90\times100}{200}$ =45	$\frac{90\times100}{200}$ =45	90
$\frac{90\times100}{200}$ =55	$\frac{90\times100}{200}$ =55	110
100	100	200

Calculation of χ^2 :

Observed Frequency(O_i)	Expected Frequency(E_i)	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
60	45	225	5
30	45	225	5
40	55	225	4.09
70	55	225	4.09
			18.18

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = 18.18$$

1. **Null hypothesis(H_0):** $O_i = E_i$
2. **Alternative hypothesis(H_1):** $O_i \neq E_i$
3. **Level of significance:** α **=**0.05

   $\chi_\alpha{}^2$ For (r-1)(c-1) degrees of freedom

   $\chi_{0.05}{}^2$ For (2-1)(2-1)=1 degrees of freedom is 3.84
4. **Test statistic:** $\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = 18.18$

$$|\chi^2| = 1.057$$

5. **Conclusion:**

   $\therefore |\chi^2| > \chi_\alpha{}^2$

   $\therefore$ We reject the Null hypothesis.

Exercise problems:

1. Two random samples gave the following results

Sample	size	Sample mean	Sum of squares of deviations from mean
I	10	15	90
II	12	14	108

Test whether the samples came from the same population or not?

200 digits were chosen at random from set of tables the frequency of the digits are

2. Use chi square test to asset the correctness of the hypothesis that the digits are distributed in equal number in the table

digit	0	1	2	3	4	5	**6**	**7**	**8**	**9**
frequency	18	19	23	21	16	25	22	20	21	15

3. 5 dice were thrown 96 times the number of times showing 4,5 or 6 obtain is given below
   Fit a binomial distribution and test for goodness of fit

x	0	1	2	3	4	5
frequency	1	10	24	35	18	8

UNIT-II
TESTING OF STATISTICAL HYPOTHESIS

Testing of Hypothesis

- Statistical decisions are decisions or conclusions about the population parameters on the basis of a random sample from the population.
- Statistical hypothesis is an assumption or conjecture or guess about the parameters of the population distribution
- **Null Hypothesis (N.H)** denoted by H_0 is statistical hypothesis, which is to be actually tested for acceptance or rejection. NH is the hypothesis, which is tested for possible rejection under the assumption that it is true.
- Any Hypothesis which is complimentary to the N.H is called an **Alternative Hypothesis** denoted by H_1
- Simple Hypothesis is a statistical Hypothesis which completely specifies an exact parameter. N.H is always simple hypothesis stated as a equality specifying an exact value of the parameter. E.g. N.H = $H_0 : \mu = \mu_0$ N.H. = $H_0 : \mu_1 - \mu_2 = \delta$
- Composite Hypothesis is stated in terms of several possible values.
- Alternative Hypothesis(A.H) is a composite hypothesis involving statements expressed as inequalities such as < , > or ≠

  i) A.H : $H_1: \mu > \mu_0$ (Right tailed) ii) A.H : $H_1: \mu < \mu_0$ (Left tailed)

  iii) A.H : $H_1: \mu \neq \mu_0$ (Two tailed alternative)

- **Errors in sampling**

  Type I error: Reject H_0 when it is true

  Type II error: Accept $\mathbf{H_0}$ when it is wrong (i.e) accept if when H_1 is true.

	Accept H_0	Reject H_0
H_0 is True	Correct Decision	Type 1 error
H_0 is False	Type 2 error	Correct Decision

- If P{ Reject H_0 when it is true}= P{ Reject $H_0 \mid H_0$}=α and P{ Accept H_0 when it is false}= P{ Accept $H_0 \mid H_1$} = β then α,β are called the sizes of Type I error and Type II error respectively. In practice, type I error amounts to rejecting a lot when it is good and type II error may be regarded as accepting the lot when it is bad.

- α and β are referred to as producers risk and consumers risk respectively.
- A region (corresponding to a statistic t) in the sample space S that amounts to rejection of H_0 is called critical region of rejection.
- Level of significance is the size of the type I error (or maximum producer's risk)

- The levels of significance usually employed in testing of hypothesis are 5% and 1% and is always fixed in advance before collecting the test information.
- A test of any statistical hypothesis where AH is one tailed(right tailed or left tailed) is called a **one–tailed test.** If AH is two-tailed such as: H_0**:** $\mu = \mu_0$, against the AH. H_1 **:** $\mu \neq \mu_0$ ($\mu > \mu_0$ and $\mu < \mu_0$) is called **Two-Tailed Test.**
- The value of test statistics which separates the critical (or rejection) region and the acceptance region is called **Critical value or Significant value**. It depends upon (i) The level of significance used and (ii) The Alternative Hypothesis, whether it is two-tailed or single tailed
- From the normal probability tables we get

Critical Value (Z_α)	Level of significance (α) 1%	5%	10%
Two-Tailed test	$-Z_{\alpha/2} = -2.58$ $Z_{\alpha/2} = 2.58$	$-Z_{\alpha/2} = -1.96$ $Z_{\alpha/2} = 1.96$	$-Z_{\alpha/2} = -1.645$ $Z_{\alpha/2} = 1.645$
Right-Tailed test	$Z_\alpha = 2.33$	$Z_\alpha = 1.645$	$Z_\alpha = 1.28$
Left-Tailed Test	$-Z_\alpha = -2.33$	$-Z_\alpha = -1.645$	$-Z_\alpha = -1.28$

- When the size of the sample is increased, the probability of committing both types of error I and II (i.e) α and β are small, the test procedure is good one giving good chance of making the correct decision.
- P-value is the lowest level (of significance) at which observed value of the test statistic is significant.
- A test of Hypothesis (T. O.H) consists of
  1. Null Hypothesis (NH) : H_0
  2. Alternative Hypothesis (AH) : H_1
  3. Level of significance: α
  4. Critical Region pre determined by α
  5. Calculation of test statistic based on the sample data.
  6. Decision to reject NH or to accept it.
- Test statistic for T.O.H. in several cases are

1. Statistic for test concerning mean σ known

$$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}}$$

2. Statistic for large sample test concerning mean with σ unknown

$$Z = \frac{\overline{X} - \mu_0}{S / \sqrt{n}}$$

3. Statistic for test concerning difference between the means

$$Z = \frac{\left(\overline{X}_1 - \overline{X}_2\right) - \delta}{\sqrt{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)}}$$ under NH $H_0: \mu_1 - \mu_2 = \delta$ against the AH, $H_1: \mu_1 - \mu_2 > \delta$ or $H_1:$ $\mu_1 - \mu_2 < \delta$ or $H_1: \mu_1 - \mu_2 \neq \delta$

4. Statistic for large samples concerning the difference between two means (σ_1 and σ_2 are unknown)

$$Z = \frac{\left(\overline{X}_1 - \overline{X}_2\right) - \delta}{\sqrt{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)}}$$

Statistics for large sample test concerning one proportion

$$Z = \frac{X - np_o}{\sqrt{np_0(1 - p_0)}}$$ under the N.H: $H_0: p = p_o$ against H_1: $p \neq p_0$ or $p > p_0$ or $p < P_0$

Statistic for test concerning the difference between two proportions

$$Z = \frac{\frac{X_1}{n_1} - \frac{X_2}{n_2}}{\sqrt{\hat{p}(1-\hat{p})(\frac{1}{n_1} + \frac{1}{n_2})}}$$ with $\hat{p} = \frac{X_1 + X_2}{n_1 + n_2}$ under the NH : H_0: $p_1 = p_2$ against the AH $H_1: p_1 < p_2$ or $p_1 > p_2$ or $p_1 \neq p_2$

- To determine if a population follows a specified known theoretical distribution such as ND,BD,PD the χ^2(chi-square) test is used to assertion how closely the actual distribution approximate the assumed theoretical distribution. This test is based on how good a fit is there between the observed frequencies and the expected frequencies is known as "**goodness-of-fit-test**".
- Large sample confidence interval for p

  $$\frac{x}{n} - Z_{\alpha/2}\sqrt{\frac{\frac{x}{n}\left(1 - \frac{x}{n}\right)}{n}} < p < \frac{x}{n} + Z_{\alpha/2}\sqrt{\frac{\frac{x}{n}\left(1 - \frac{x}{n}\right)}{n}}$$ where the degree of confidence is $1 - \alpha$

- Large sample confidence interval for difference of two proportions ($p_1 - p_2$) is

  $$\left(\frac{x_1}{n_1} - \frac{x_2}{n_2}\right) \pm Z_{\alpha/2}\sqrt{\frac{\frac{x_1}{n_1}\left(1 - \frac{x_1}{n_1}\right)}{n_1} + \frac{\frac{x_2}{n_2}\left(1 - \frac{x_2}{n_2}\right)}{n_2}}$$

- Maximum error of estimate E = $Z_{\alpha/2}\sqrt{\frac{p(1-p)}{n}}$ with observed value x/n substituted for p we obtain an estimate of E
- Sample size n = $p(1-p)\left(\frac{Z_{\alpha/2}}{E}\right)^2$ when p is known

  n= $\frac{1}{4}\left(\frac{Z_{\alpha/2}}{E}\right)^2$ when p is unknown

- One sided confidence interval is of the form $p < (1/2n)\chi_\alpha^2$ with (2n+1) degrees of freedom.

Problems:

1. **A sample of 400 items is taken from a population whose standard deviation is 10.The mean of sample is 40.Test whether the sample has come from a population with mean 38 also calculate 95% confidence interval for the population.**

Solution: Given n=400, $\bar{x} = 40$ and μ =38 and σ =10

6. **Null hypothesis(H_0):** μ =38
7. **Alternative hypothesis(H_1):** $\mu \neq 38$
8. **Level of significance:** α =0.05 and Z_α =1.96
9. **Test statistic:** $Z = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}$

$$Z = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} = \frac{40-38}{\frac{10}{\sqrt{400}}} = 4$$

$$|Z| = 4$$

10. **Conclusion:**

$\therefore$ $|Z| > Z_\alpha$

$\therefore$ We reject the Null hypothesis.

Confidence interval = $\left(\bar{x} - Z_\alpha \frac{\sigma}{\sqrt{n}}, \bar{x} + Z_\alpha \frac{\sigma}{\sqrt{n}}\right)$

$$= \left(40 - 1.96\frac{10}{\sqrt{400}}, 40 + 1.96\frac{10}{\sqrt{400}}\right)$$

$$=(39.02, 40.98)$$

2. Samples of students were drawn from two universities and from their weights in kilograms mean and S.D are calculated and shown below make a large sample test to the significance of difference between means.

	MEAN	S.D	SAMPLE SIZE
University-A	55	10	400
University-B	57	15	100

Solution: Given n_1=400, n_2=100, $\bar{x}_1$=55, $\bar{x}_2$=57
S_1=10 and S_2=15

1. **Null hypothesis(H_0):** $\bar{x}_1 = \bar{x}_2$
2. **Alternative hypothesis(H_1):** $\bar{x}_1 \neq \bar{x}_2$
3. **Level of significance:** α =0.05 and Z_α =1.96
4. **Test statistic:** $Z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}} = \dfrac{55-57}{\sqrt{\dfrac{100}{400} + \dfrac{225}{100}}}$ =-1.26

$$|Z| = 1.26$$

5. **Conclusion:**

$\therefore$ $|Z| < Z_\alpha$

$\therefore$ We accept the Null hypothesis.

3. **In a sample of 1000 people in Karnataka 540 are rice eaters and the rest are wheat eaters. Can we assume that both rice and wheat are equally popular in this state at 1% level of significance?**

Solution: Given n =400, x =540

$$p = \frac{x}{n} = \frac{540}{1000} = 0.54$$

$P = \frac{1}{2} = 0.5$, $Q = 0.5$

1. **Null hypothesis(H_0):** $P = 0.5$
2. **Alternative hypothesis(H_1):** $P \neq 0.5$
3. **Level of significance:** $\alpha = 1\%$ and $Z_\alpha = 2.58$
4. **Test statistic:** $Z = \frac{P - p}{\sqrt{\frac{PQ}{n}}}$

$$Z = \frac{P - p}{\sqrt{\frac{PQ}{n}}} = \frac{0.54 - 0.5}{\sqrt{\frac{0.5 \times 0.5}{1000}}} = 2.532$$

$|Z| = 2.532$

5. **Conclusion:**

$\therefore \; |Z| < Z_\alpha$

$\therefore$ We accept the Null hypothesis.

4. **Random sample of 400 men and 600 women were asked whether they would like to have flyover near their residence .200 men and 325 women were in favour of proposal. Test the hypothesis that the proportion of men and women in favour of proposal are same at 5% level.**

Solution: Given n_1=400, n_2=600 , $x_1 = 200$ and $x_2 = 325$

$$p_1 = \frac{200}{400} = 0.5$$

$$p_2 = \frac{325}{600} = 0.541$$

$$p = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{400 \times \frac{200}{400} + 600 \times \frac{325}{600}}{400 + 600} = 0.525$$

$$q = 1 - p = 1 - 0.525 = 0.475$$

1. **Null hypothesis(H_0):** $p_1 = p_2$

2. **Alternative hypothesis(H_1):** $p_1 \neq p_2$

3. **Level of significance:** α =0.05 and Z_α =1.96

4. **Test statistic:** $Z = \dfrac{p_1 - p_2}{\sqrt{pq\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}} = \dfrac{0.5 - 0.541}{\sqrt{0.525 \times 0.425\left(\dfrac{1}{400} + \dfrac{1}{600}\right)}} = -1.28$

$$|Z| = 1.28$$

5. **Conclusion:**

$\therefore$ $|Z| < Z_\alpha$

$\therefore$ We accept the Null hypothesis.

Exercise Problems:

1. An ambulance service claims that it takes on the average 8.9 minutes to reach its destination In emergency calls. To check on this claim the agency which issues license to Ambulance service has then timed on fifty emergency calls getting a mean of 9.2 minutes with 1.6 minutes. What can they conclude at 5% level of significance?

2.According to norms established for a mechanical aptitude test persons who are 18 years have an average weight of 73.2 with S.D 8.6 if 40 randomly selected persons have average 76.7 test the hypothesis H_0 : μ =73.2 againist alternative hypothesis : μ >73.2.

3.A cigarette manufacturing firm claims that brand A line of cigarettes outsells its brand B by 8% .if it is found that 42 out of a sample of 200 smokers prefer brand A and 18 out of another sample of 100 smokers prefer brand B. Test whether 8% difference is a valid claim.

4. The nicotine in milligrams of two samples of tobacco were found to be as follows. Test the hypothesis for the difference between means at 0.05 level

Sample-A	24	27	26	23	25	
Sample-B	29	30	30	31	24	36

5. A machine puts out of 16 imperfect articles in a sample of 500 articles after the machine is overhauled it puts out 3 imperfect articles in a sample of 100 articles. Has the machine improved?

ANALYSIS OF VARIANCE

ANOVA:

It is abbreviated form for ANALYSIS OF VARIANCE which is a method for comparing several population means at the same time. It is performed using F-distribution

Assumptions of ANALYSIS OF VARIANCE:

1. The data must be normally distributed.

2. The samples must draw from the population randomly and independently.

3. The variances of population from which samples have been drawn are equal.

Types of Classification:

There are two types of model for analysis of variance

1. One-Way Classification

2. Two-Way Classification.

ONE –WAY CLASSIFICATION:

PROCEDURE FOR ANOVA

Step 1 : State the null and alternative hypothesis.

H_0: $\mu_1 = \mu_2 = \mu_3$ (The means for three groups are equal).

H_1 : At least one pair is unequal.

Step 2: Select the test criterion to be used.

We have to decide which test criterion or distribution should be used. As our assumption involves means for three normally distributed populations. We should use the F-distribution to test the hypothesis.

Step 3. Determine the rejection and non-rejection regions

We decide to use 0.05 level of significance. As on one-way ANOVA test is always right-tail, the area in the right tail of the F-distribution curve is 0.05, which is the rejection region. Now, we need to know the degrees of freedom for the numerator and the denominator. Degrees of freedom for the numerator=k-1, where k is the number of groups. Degree of freedom for denominator =n-k where n is total number of observations

Step 4. Calculate the value of the test statistics by applying ANOVA. i.e., $F_{Calculated}$

Step 5: conclusion

I) If $F_{Calculated}$<$F_{Critical}$, then H_0 is accepted

ii) if $F_{calculated}$<$F_{critical}$, then H_0 is rejected

TWO –WAY CLASSIFCATION:

The analysis of variance table for two-way classification is taken as follows;

Source of variation	Sum of squares SS	Degree of freedom df	Mean squares Ms
Between columns	SSC	(c-1)	MSC=SSC/(c-1)
Within rows	SSR	(r-10	MSR=SSR/(r-1)
Residual(ERROR)	SSE	(c-1)(r-1)	MSE=SSE/(c-1)(r-1)
total	SST	Cr-1	

The abbreviations used in the table are:

SSC= sum of squares between column s.

SSR= sum of square between rows.

SST=total sum of squares;

SSE= sum of squares of error, it is obtained by subtracting SSR and SSC from SST.

(c-1)=number of degrees of freedom between columns.

(r-1)=number of degrees of freedom between rows.

(c-1)(r-1)=number of degree of freedom for residual.

MSC=mean of sum of squares between columns

MSR= mean of sum of squares between rows.

MSE= mean of sum of squares between residuals.

It may be noted that total number of degrees of freedom are =(c-1)+(r-1)+(c-1)(r-1)=cr-1=N-1

PROBLEMS:

1. **There are three different methods of teaching English that are used on three groups of students. Test by using analysis of variance whether this method s of teaching had an effect on the performance of students. Random sample of size 4 are taken from each group and the marks obtained by the sample students in each group are given below**

Marks obtained the students

Group A	Group B	Group C
16	15	15
17	15	14
13	13	13
18	17	14
Total 64	**Total 60**	**Total 56**

Solution:

It is assumed that the marks obtained by the students are distributed normally with means μ_1, μ_2, μ_3 for the three groups A, B and C. respectively. Further, is is assumed that the standard deviation of the distribution of marks for groups A,B and C are equal and constant. This assumption implies that the mean marks of the groups may differ on account of using different methods of teaching, but they do not affect the dispersion of marks.

PROCEDURE FOR ANOVA

Step 1 : State the null and alternative hypothesis.

H_0: $\mu_1 = \mu_2 = \mu_3$ (The means for three groups are equal).

H_1 : At least one pair is unequal.

Step 2: Select the test criterion to be used.

We have to decide which test criterion or distribution should be used. As our assumption involves means for three normally distributed populations. We should use the F-distribution to test the hypothesis.

Step 3. Determine the rejection and non-rejection regions

We decide to use 0.05 level of significance. As on one-way ANOVA test is always right-tail, the area in the right tail of the F-distribution curve is 0.05, which is the rejection region. Now, we need to know the degrees of freedom for the numerator and the denominator. Degrees of freedom for the numerator=k-1=3-1=2, where k is the number of groups. Degree of freedom for denominator =n-k=12-3=9, where n is total number of observations.

Step 4. Calculate the value of the test statistics by applying ANOVA. i.e., $F_{\text{Calculated}}$

Worksheet for calculating Variances

Group A Group B

Group C

X_{1j}	$(x_{1j}- x_i)$	$(x_{1j}- x_i)^2$	X_{2j}	$(x_{2j}- x_i)$	$(x_{2j}- x_i)^2$	X_{3j}	$(x_{3j}- x_i)$	$(x_{3j}- x_i)^2$
16	0	0	15	0	0	15	1	1
17	1	1	15	0	0	14	0	0
13	-3	9	13	-2	4	13	-1	1
18	2	4	17	2	4	14	0	0
Total 64			Total 60			Total 56		
Mean 16			Mean 15			Mean 14		

The sample variances for the groups are

$$S_1^2 = \frac{1}{n_1}\sum_{j=1}^{n_1}\left(x_{1j} - \overline{x_1}\right)^2 = \frac{1}{4}(14) = 3.5$$

$$S_2^2 = \frac{1}{n_2}\sum_{j=1}^{n_2}\left(x_{2j} - \overline{x_2}\right)^2 = \frac{1}{4}(8) = 2$$

$$S_3^2 = \frac{1}{n_3}\sum_{j=1}^{n_{13}}\left(x_{3j} - \overline{x_3}\right)^2 = \frac{1}{4}(14) = 0.5$$

We can now estimate the variance by the pooled variance method as follows;

$$\sigma^2 = \frac{\sum\sum\left(x_{ij} - x_i\right)^2}{n-3}$$

The denominator is $n_1+n_2+n_3=3$

Applying the value in the formulas,

$$\sigma^2 = \frac{\sum\sum\left(x_{ij} - x_i\right)^2}{n-3} = \frac{4[(16-15)^2 + (15-15)^2 + (14-15)^2]}{3-1}$$

=4(This is the variance between the samples)

Now, F is to be calculated . F=ratio of two variances

$$= \frac{estimate of \ \sigma^2 between \ samples}{estimate of \ \sigma^2 within \ samples} = \frac{4}{2.67} = 1.498$$

The foregoing calculations can be summarized in the form of an ANOVA TABLE.

Source of variation	Sum of squares SS	Degrees of freedom df	Mean of equares	Variance ratio F
Between sampling	SSB	k-1	MSB=SSB/(k-1)	
Within sampling	SSW	n-k	MSW=SSW/(n-k)	F=MSB/MSW
total	SST	n-1		

Source of variation	Sum of squares SS	Degrees of freedom df	Mean of equares	Variance ratio F
Between sampling	6	3-1	8/2=4	
Within sampling	24	12-3	24/8=2.67	4/2.67=1.498
total	32	12-1	32/11=2.9	

Step: conclusion: The critical value of F for 2 and 9 degrees of freedom at 5 percent level of significance is 4.26. As the calculated value of F=1.0498 is less than critical values of F.

i.e.,$F_{calculated} < F_{critical.}$ The null hypothesis is accepted.

7. A company has appointed four salesman, A,B,C and D. observed their sales in three seasons-summer, winter, monsoon. The figures (in Rs lakh) are given in the following table.

SALESMEN

seasons	A	B	C	D	Seasons totals
summer	36	36	21	35	128
winter	28	29	31	32	120
monsoon	26	28	29	29	112
Sales man totals	90	93	81	96	360

Using 5 percent level of significance, perform an analysis of variance on the above data and interpret the result.

Solution:

Step 1 : State the null and alternative hypothesis.

H_0: there is no difference in the mean sales performance of A, B, C and D in the three seasons.

H_1 : there is difference in the mean sales performance of A ,B, C and D in the three season.

Step 2: Select the test criterion to be used.

We have to decide which test criterion or distribution should be used. As our assumption involves means for three normally distributed populations. We should use the F-distribution to test the hypothesis.

Step 3. Determine the rejection and non-rejection regions

We decide to use 0.05 level of significance. The degrees of freedom for rows are (r-1) =2 and for columns are (c-1)=3 and for residual (r-1)(c-1)=2x3=6. Thus, we have to compare the calculated value of F with the critical value of F for a) 2 and 6 df at 5% l. o. s b)3 and 6 df at 5% .l. o. s.

Step 4;

Coded Data for ANOVA

SALESMEN

seasons	A	B	C	D	Seasons totals
summer	6	6	-9	5	8
winter	-2	-1	1	2	0
monsoon	-4	-2	-1	-1	-8
Sales man totals	0	3	-9	6	0

Correction factor $C=T^2/N=(0)^2/12=0$

Sum os squares between salesmen

$=0^2/3+3^2/3+(-9^2/3)+6^2/3=0+3+27+12=42$

Sum of squares between seasons$=8^2/4+0^2/4+(-8^2/4)=16+0+16=32$

Total sum of squares

$=(6)^2+(-2)^2+(-4)^2+(6)^2+(-1)^2+(-2)^2+(-9)^2+(1)^2+(-1)^2+(5)^2+(2)^2+(-1)^2$

$=210$

Analysis of variance table

Source of variation	Sum of squares SS	Degree of freedom df	Mean squares Ms
Between columns	42	4-1=3	14.00
Within rows	32	3-1=2	16.00
Residual(ERROR)	136	3x2=6	22.67
total	210	12-1=11	

We now test the hypothesis (i) that there is no difference in the sales performance among the four salesmen and (ii) there is no difference in the mean sales in the three seasons. For this, we have to first compare the salesman variance estimate with the residual estimate. This is shown below:

F_A =14/22.67=0.62

In the same manner, we have to compare the season variance estimate with the residual variances estimate. This is shown below;

F_B=16/22.67=0.71

Step 5:

It may noted that the critical value of F for 3 and 6 degree of freedom at 5 percent level of significance is 4.76. Since the calculated value of F_A is 0.62 is less than critical value of F. Therefore there is no significance difference among salesmen.

Also the critical value of F for 2 and 6 degree of freedom at 5 percent level of significance is 4.76. Since the calculated values of F_B=16/22.67=0.71 is less than critical value of F. Therefore there is no significance difference among seasons

The overall conclusion is that the salesmen and seasons are alike in respect of sales.

Exercise problems:

1. A company has derived three training methods to train its workers. It is keen to know which of these three training methods would lead to greatest productivity after training. Given below are productivity measures for individual workers trained by each method.

Method 1	30	40	45	38	48	55	52
Method 2	55	46	37	43	52	42	40
Method 3	42	38	49	40	55	36	41

Find out whether the three training methods lead to different levels of productivity at the 0.05 level of significance.

2. Consider the following ANOVA TABLE, based on information obtained for three randomly selected samples from three independent population, which are normally distributed with equal variances.

Source of variance	Sum of squares SS	Degree of freedom df	Mean squares MS	Value of test statistics
Between samples	60	?	20	F=
Within samples	?	14	?	

(A) Complete the ANOVA table by filling in missing values.

(B) test the null hypothesis that the means of the three population are all equal, using 0.01 level of significance.

3. The following represent the number of units of production per day turned out by four different workers using five different types of machines

Machine type						
Worker	A	B	C	D	E	TOTAL
1	4	5	3	7	6	25
2	5	7	7	4	5	28
3	7	6	7	8	8	36
4	3	5	4	8	2	22
TOTAL	19	23	21	27	21	111

On the basis of this information, can it be concluded that (i) The mean productivity is the same for different machines. (ii) The workers don't differ with regard to productivity.

UNIT-III
ORDINARY DIFFERENTIAL EQUATIONS

1. The important methods of solving ordinary differential equations of first order numerically are as follows
   1) Taylors series method
   2) Euler's method
   3) Modified Euler's method of successive approximations
   4) Runge- kutta method

To describe various numerical methods for the solution of ordinary differential eqn's,we consider the general 1^{st} order differential eqn

dy/dx=f(x,y)-------(1)

with the initial condition $y(x_0)=y_0$

The methods will yield the solution in one of the two forms:

i) A series for y in terms of powers of x,from which the value of y can be obtained by direct substitution.

ii) A set of tabulated values of y corresponding to different values of x

The methods of Taylor and picard belong to class(i)

The methods of Euler, Runge - kutta method, Adams, Milne etc, belong to class (ii)

TAYLOR'S SERIES METHOD

To find the numerical solution of the differential equation

$$\frac{dy}{dx} = f(x,y) \rightarrow (1)$$

With the initial condition $y(x_0) = y_0 \rightarrow (2)$

$y(x)$ can be expanded about the point x_0 in a Taylor's series in powers of $(x - x_0)$ as

$$y(x) = y(x_0) + \frac{(x-x_0)}{1} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + ... + \frac{(x-x_0)^n}{n!} y^n(x_0) \rightarrow (3)$$

In equ3, $y(x_0)$ is known from I.C equ2. The remaining coefficients $y'(x_0), y''(x_0), ...y^n(x_0)$ etc are obtained by successively differentiating equ1 and evaluating at x_0. Substituting these values in equ3, $y(x)$ at any point can be calculated from equ3. Provided $h = x - x_0$ is small.

When $x_0 = 0$, then Taylor's series equ3 can be written as

$$y(x) = y(0) + x.y'(0) + \frac{x^2}{2!}y''(0) + ... + \frac{x^n}{n!}y^n(0) + ... \rightarrow (4)$$

1. Using Taylor's expansion evaluate the integral of $y' - 2y = 3e^x, y(0) = 0$**, at a)** $x = 0.2$

b) compare the numerical solution obtained with exact solution .

Sol: Given equation can be written as $2y + 3e^x = y', y(0) = 0$

Differentiating repeatedly w.r.t to 'x' and evaluating at $x = 0$

$$y'(x) = 2y + 3e^x, y'(0) = 2y(0) + 3e^0 = 2(0) + 3(1) = 3$$
$$y''(x) = 2y' + 3e^x, y''(0) = 2y'(0) + 3e^0 = 2(3) + 3 = 9$$
$$y'''(x) = 2.y''(x) + 3e^x, y'''(0) = 2y''(0) + 3e^0 = 2(9) + 3 = 21$$
$$y^{iv}(x) = 2.y'''(x) + 3e^x, y^{iv}(0) = 2(21) + 3e^0 = 45$$
$$y^v(x) = 2.y^{iv} + 3e^x, y^v(0) = 2(45) + 3e^0 = 90 + 3 = 93$$

In general, $y^{(n+1)}(x) = 2.y^{(n)}(x) + 3e^x$ or $y^{(n+1)}(0) = 2.y^{(n)}(0) + 3e^0$

The Taylor's series expansion of $y(x)$ about $x_0 = 0$ is

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y''''(0) + \frac{x^5}{5!}y'''''(0) + ...$$

Substituting the values of $y(0), y'(0), y''(0), y'''(0), ...$

$$y(x) = 0 + 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{45}{24}x^4 + \frac{93}{120}x^5 + ...$$

$$y(x) = 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \frac{15}{8}x^4 + \frac{31}{40}x^5 + ... \rightarrow \text{equ1}$$

Now put $x = 0.1$ in equ1

$$y(0.1) = 3(0.1) + \frac{9}{2}(0.1)^2 + \frac{7}{2}(0.1)^3 + \frac{15}{8}(0.1)^4 + \frac{31}{40}(0.1)^5 = 0.34869$$

Now put $x = 0.2$ in equ1

$$y(0.2) = 3(0.2) + \frac{9}{2}(0.2)^2 + \frac{7}{2}(0.2)^3 + \frac{15}{8}(0.2)^4 + \frac{31}{40}(0.2)^5 = 0.811244$$

$$y(0.3) = 3(0.3) + \frac{9}{2}(0.3)^2 + \frac{7}{2}(0.3)^3 + \frac{15}{8}(0.3)^4 + \frac{31}{40}(0.3)^5 = 1.41657075$$

Analytical Solution:

The exact solution of the equ $\frac{dy}{dx} = 2y + 3e^x$ with $y(0) = 0$ can be found as follows

$\frac{dy}{dx} - 2y = 3e^x$ Which is a linear in y.

Here $P = -2, Q = 3e^x$

I.F = $\int_e^{pdx} = \int_e^{-2dx} = e^{-24}$

General solution is $y.e^{-2x} = \int 3e^x.e^{-2x}dx + c = -3e^{-x} + c$

$\therefore y = -3e^x + ce^{2x}$ where $x = 0, y = 0 \quad 0 = -3 + c \Rightarrow c = 3$

The particular solution is $y = 3e^{2x} - 3e^x$ or $y(x) = 3e^{2x} - 3e^x$

Put $x = 0.1$ in the above particular solution,

$$y = 3.e^{0.2} - 3e^{0.1} = 0.34869$$

Similarly put $x = 0.2$

$$y = 3e^{0.4} - 3e^{0.2} = 0.811265$$

put $x = 0.3$

$$y = 3e^{0.6} - 3e^{0.3} = 1.416577$$

2. Using Taylor's series method, solve the equation $\frac{dy}{dx} = x^2 + y^2$ for $x = 0.4$ given that $y = 0$ when $x = 0$

Sol: Given that $\frac{dy}{dx} = x^2 + y^2$ and $y = 0$ when $x = 0$ i.e. $y(0) = 0$

Here $y_0 = 0$, $x_0 = 0$

Differentiating repeatedly w.r.t 'x' and evaluating at $x = 0$

$$y'(x) = x^2 + y^2, y'(0) = 0 + y^2(0) = 0 + 0 = 0$$
$$y''(x) = 2x + y'.2y, y''(0) = 2(0) + y'(0)2.y = 0$$
$$y'''(x) = 2 + 2yy'' + 2y'.y', y'''(0) = 2 + 2.y(0).y''(0) + 2.y'(0)^2 = 2$$
$$y''''(x) = 2.y.y''' + 2.y''.y' + 4.y''.y', y''''(0) = 0$$

The Taylor's series for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y''''(0) + ...$$

Substituting the values of $y(0), y'(0), y''(0),.....$

$$y(x) = 0 + x(0) + 0 + \frac{2x^3}{3!} + 0 + ... = \frac{x^3}{3} + \text{(Higher order terms are neglected)}$$

$$\therefore y(0.4) = \frac{(0.4)^3}{3} = \frac{0.064}{3} = 0.02133$$

3. Solve $y' = x - y^2, y(0) = 1$ **using Taylor's series method and compute y(0.1),y(0.2)**

Sol: Given that $y' = x - y^2, y(0) = 1$

Here $y_0 = 1$, $x_0 = 0$

Differentiating repeatedly w.r.t 'x' and evaluating at x=0

$$y'(x) = x - y^2, y'(0) = 0 - y(0)^2 = 0 - 1 = -1$$
$$y''(x) = 1 - 2y.y', y''(0) = 1 - 2.y(0)y'(0) = 1 - 2(-1) = 3$$
$$y'''(x) = 1 - 2yy' - 2(y')^2, y'''(0) = -2.y(0).y''(0) - 2.(y'(0))^2 = -6 - 2 = -8$$
$$y''''(x) = -2.y.y''' - 2.y''.y' - 4.y''.y', y''''(0) = -2.y(0).y'''(0) - 6.y''(0).y'(0) = 16 + 18 = 34$$

The Taylor's series for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + \frac{x}{1!}y^1(0) + \frac{x^2}{2!}y^{11}(0) + \frac{x^3}{3!}y^{111}(0) + ...$$

Substituting the value of $y(0)$, $y^1(0)$, $y^{11}(0)$,…..

$$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{8}{6}x^3 + \frac{34}{24}x^4 + ...$$

$$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 + \frac{17}{12}x^4 + ... \quad \rightarrow (1)$$

now put x = 0.1 in (1)

$$y(0.1) = 1 - 0.1 + \frac{3}{2}(0.1)^2 + \frac{4}{3}(0.1)^3 + \frac{17}{12}(0.1)^4 + ...$$

$= 0.91380333 \underset{\sim}{} 0.91381$

Similarly put x = 0.2 in (1)

$$y(0.2) = 1 – 0.2 + \frac{3}{2}(0.2)^2 - \frac{4}{3}(0.2)^3 + \frac{17}{12}(0.2)^4 + \ldots..$$

$= 0.8516.$

4. Solve $y^1 = x^2 – y$, y(0) = 1, using Taylor's series method and compute y(0.1), y(0.2), y(0.3) and y(0.4) (correct to 4 decimal places).

Sol. Given that $y^I = x^2 – y$ and y(0) = 1

Here $x_0 = 0$, $y_0 = 1$ or y =1 when x=0

Differentiating repeatedly w.r.t 'x' and evaluating at x = 0.

$Y^I(x) = x^2 – y$, $y^I(0) = 0 – 1 = -1$

$y^{II}(x) = 2x – y^I$, $\quad y^{II}(0) = 2(0) – y^I(0) = 0 – (-1) = 1$

$y^{III}(x) = 2 – y^{II}$, $\quad y^{III}(0) = 2 – y^{II}(0) = 2 – 1 = 1$,

$y^{IV}(x) = - y^{III}$, $\quad y^{IV}(0) = -y^{III}(0) = -1$.

The Taylor's servies for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + \frac{x}{1!}y^I(0) + \frac{x^2}{2!}y^{II}(0) + \frac{x^3}{3!}y^{III}(0) + \frac{x^4}{4!}y^{IV}(0) + \ldots\ldots$$

substituting the values of $y(0)$, $y^1(0)$, $y^{11}(0)$, $y^{111}(0)$,……

$$y(x) = 1 + x(-1) + \frac{x^2}{2}(1) + \frac{x^3}{6}(1) + \frac{x^4}{24}(-1) + \ldots\ldots$$

$$y(x) = 1 – x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + \ldots\ldots \qquad \rightarrow(1)$$

Now put x = 0.1 in (1),

$$y(0.1) = 1 – 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{6} - \frac{(0.1)^4}{24} + \ldots.$$

$= 1 – 0.1 + 0.005 + 0.01666 – 0.0000416 -0.905125 \sim 0.9051$

(4 decimal places)

Now put x = 0.2 in eq (1),

$$y(0.2) = 1 - 0.2 + \frac{(0.2)^2}{2} + \frac{(0.2)^3}{6} - \frac{(0.2)^4}{64}$$

$= 1 - 0.2 + 0.02 + 0.001333 - 0.000025$

$= 1.021333 - 0.200025$

$= 0.821308 \sim 0.8213$ (4 decimals)

Similarly y(0.3) = 0.7492 and y (0.4) = 0.6897 (4 decimal places).

5. Solve $\frac{dy}{dx}$ -1 = xy and y(0) = 1 using Taylor's series method and compute y(0.1).

Sol. Given that $\frac{dy}{dx} - 1 = xy$ and $y(0) = 1$

Here $\frac{dy}{dx} = 1 + xy$ and $y_0 = 1$, $x_0 = 0$.

Differentiating repeatedly w.r.t 'x' and evaluating at $x_0 = 0$

$y^{I}(x) = 1 + xy,$ $\quad y^{I}(0) = 1+0(1) = 1$.

$y^{II}(x) = x.y' + y,$ $\quad y^{II}(0) = 0+1=1$

$y^{III}(x) = x.y'' + y^{I} + y^{I},$ $\quad y^{III}(0) = 0.(1) + 2 \square(1) = 2$

$y^{IV}(x) = xy^{III} + y^{II} + 2y^{II},$ $\quad y^{IV}(0) = 0+3(1) = 3.$

$y^{V}(x) = xy^{IV} + y^{III} + 2y^{III},$ $\quad y^{V}(0) = 0 + 2 + 2(3) = 8$

The Taylor series for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + x.y^{I}(0) + \frac{x^2}{2!} y^{II}(0) + \frac{x^3}{3!} y^{III}(0) + \frac{x^4}{4!} y^{IV}(0) + \frac{x^5}{5!} y^{V}(0) + \ldots..$$

Substituting the values of $y(0)$, $y^{I}(0)$, $y^{II}(0)$, ….

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}(2) + \frac{x^4}{24}(3) + \frac{x^5}{120}(8) + \ldots.$$

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \ldots. \qquad \rightarrow (1)$$

Now put x = 0.1 in equ (1),

$$y(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{8} + \frac{(0.1)^5}{15} + \ldots..$$

$= 1 + 0.1 + 0.005 + 0.000333 + 0.0000125 + 0.0000006$

$= 1.1053461$

H.W

6. Given the differential equ $y^1 = x^2 + y^2$, $y(0) = 1$.Obtain $y(0.25)$, and $y(0.5)$ by Taylor's Series method.

Ans: 1.3333, 1.81667

7. Solve $y^1 = xy^2 + y$, $y(0) =1$ using Taylor's series method and compute $y(0.1)$ and $y(0.2)$.

Ans: 1.111, 1.248.

Note: We know that the Taylor's expansion of $y(x)$ about the point x_0 in a power of $(x - x_0)$is.

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y^{I}(x_0) + \frac{(x-x_0)^2}{2!} y^{II}(x_0) + \frac{(x-x_0)^3}{3!} y^{III}(x_0) + \ldots \rightarrow (1)$$

Or

$$y(x) = y_0 + \frac{(x-x_0)}{1!} y_0^{I} + \frac{(x-x_0)^2}{2!} y_0^{II} + \frac{(x-x_0)^3}{3!} y_0^{III} + \ldots..$$

If we let $x - x_0 = h$. (i.e. $x = x_0 + h = x_1$) we can write the Taylor's series as

$$y(x) = y(x_1) = y_0 + \frac{h}{1!} y_0^{I} + \frac{h^2}{2!} y_0^{II} + \frac{h^3}{3!} y_0^{III} + \frac{h^4}{4!} y_0^{IV} + \ldots.$$

$$\text{i.e. } y_1 = y_0 + \frac{h}{1!} y_0^{I} + \frac{h^2}{2!} y_0^{II} + \frac{h^3}{3!} y_0^{III} + \frac{h^{IV}}{4!} y_0^{IV} + \ldots.. \rightarrow (2)$$

Similarly expanding $y(x)$ in a Taylor's series about $x = x_1$. We will get.

$$y_2 = y1 + \frac{h}{1!} y_1^{I} + \frac{h^2}{2!} y_1^{II} + \frac{h^3}{3!} y_1^{III} + \frac{h^4}{4!} y_1^{IV} + \ldots\ldots. \rightarrow (3)$$

Similarly expanding $y(x)$ in a Taylor's series about $x = x_2$ We will get.

$$y_3 = y_2 + \frac{h}{1!} y_2^{I} + \frac{h^2}{2!} y_2^{II} + \frac{h^3}{3!} y_2^{III} + \frac{h^4}{4!} y_2^{IV} + \ldots... \rightarrow (4)$$

In general, Taylor's expansion of y(x) at a point $x= x_n$ is

$$y_{n+1} = y_n + \frac{h}{1!} y_n^{I} + \frac{h^2}{2!} y_n^{II} + \frac{h^3}{3!} y_n^{III} + \frac{h^4}{4!} y_n^{IV} + \ldots.. \rightarrow(5)$$

8. Solve $y^1 = x-y^2$, y(0) = 1 using Taylor's series method and evaluate y(0.1), y(0.2).

Sol: Given $y^1 = x - y^2$ $\rightarrow$(1)

and y(0) = 1 $\rightarrow$(2)

Here $x_0 = 0$, $y_0 = 1$.

Differentiating (1) w.r.t 'x', we get.

$y^{II} = 1 - 2yy^{I}$ $\rightarrow$(3)

$y^{III} = -2(y.\, y^{II} + (y^{I})^2)$ $\rightarrow$ (4)

$y^{IV} = -2[y.\, y^{III} + y.\, y^{II} + 2y^{I}.\, y^{II}]$ $\rightarrow$(5)

$= -2(3y^{I}.\, y^{II} + y.\, y^{III})$ …...

Put $x_0 = 0$, $y_0 = 1$ in (1),(3),(4) and (5),

We get

$y_0^{I} = 0-1 = -1,$

$y_0^{II} = 1-2(1)(-1) = 3,$

$y_0^{III} = -2[(-1)^2) + (1)(3)] = -8$

$y_0^{IV} = -2[3(-1)(3) + (1)(-8)] = -2(-9 -8) = 34.$

Take h=0.1

Step1: By Taylor's series, we have

$$y_1 = y_0 + \frac{h}{1!} y_0^{I} + \frac{h^2}{2!} y_0^{II} + \frac{h^3}{3!} y_0^{III} + \frac{h^4}{4!} y_0^{IV} + \ldots.. \rightarrow\mathbf{(6)}$$

on substituting the values of y_0, y_0^{I}, y_0^{II}, etc in equ (6) we get

$$y(0.1) = y_1 = 1 + \frac{0.1}{1}(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-8) + \frac{(0.1)^4}{24}(34) + \ldots.$$

$= 1 - 0.1 + 0.015 - 0.00133 + 0.00014 + \ldots$

$= 0.91381$

Step2: Let us find y(0.2), we start with (x_1,y_1) as the starting value.

Here $x_1 = x_0 + h = 0+0.1 = 0.1$ and $y_1 = 0.91381$

Put these values of x_1 and y_1 in (1),(3),(4) and (5),we get

$$y_1^{I} = x_1 - y_1^2 = 0.1 – (0.91381)^2 = 0.1 – 0.8350487 = -0.735$$

$$y_1^{II} = 1 – 2y_1 \cdot y_1^{I} = 1- 2(0.91381)(-0.735) = 1 + 1.3433 = 2.3433$$

$$y_1^{III} = - 2[(y_1^{I})^2 + y_1 \cdot y_1^{II}] = - 2[(-0.735)^2 + (0.91381)(2.3433)] = -5.363112$$

$$y_1^{IV} = - 2[3.\ y_1^{I}\ y_1^{II} + y_1\ y_1^{III}] = - 2[3.(-0.735)(2.3433) + (0.91381)(-5.363112)]$$

$$= -2[(-5.16697) – 4.9] = 20.133953$$

By Taylor's series expansion,

$$y_2 = y_1 + \frac{h}{1!} y_1^{I} + \frac{h^2}{2!} y_1^{II} + \frac{h^3}{3!} y_1^{III} + \frac{h^4}{4!} y_1^{IV} + \ldots.$$

$$\therefore y(0.2) = y_2 = 0.91381 + (0.1)(-0.735) + \frac{(0.1)^2}{2}(2.3433) +$$

$$\frac{(0.1)^3}{6}(-5.363112) + \frac{(0.1)^4}{24}(20.133953) + \ldots.$$

$$y(0.2) = 0.91381 – 0.0735 + 0.0117 – 0.00089 + 0.00008$$

$$= 0.8512$$

9. Tabulate y(0.1), y(0.2) and y(0.3) using Taylor's series method given that $y^1 = y^2 + x$ and y(0) = 1

Sol: Given $y^1 = y^2 + x$ →(1)

and $y(0) = 1$ →(2)

Here $x_0 = 0$, $y_0 = 1$.

Differentiating (1) w.r.t 'x', we get

$$y^{II} = 2y \cdot y^{I} + 1 \quad \rightarrow(3)$$

$$y^{III} = 2[y \cdot y^{II} + (y^{I})^2] \quad \rightarrow(4)$$

$$y^{IV} = 2[y \cdot y^{III} + y^{I}\ y^{II} + 2\ y^{I}\ y^{II}]$$

$$= 2[y \cdot y^{III} + 3\ y^{I}\ y^{II}] \quad \rightarrow(5)$$

Put $x_0 = 0$, $y_0 = 1$ in (1), (3), (4) and (5), we get

$$y_0^{I} = (1)^2 + 0 = 1$$

$$y_0^{II} = 2(1)(1) + 1 = 3,$$

$$y_0^{III} = 2((1)(3) + (1)^2) = 8$$

y_0^{IV} = 2[(1)(8) + 3(1)(3)]

= 34

Take h = 0.1.

Step1: By Taylor's series expansion, we have

$$y(x_1) = y_1 = y_0 + \frac{h}{1!} y_0^{I} + \frac{h^2}{2!} y_0^{II} + \frac{h^3}{3!} y_0^{III} + \frac{h^4}{4!} y_0^{IV} + \ldots \quad \rightarrow (6)$$

on substituting the values of y_o, y_0^I , y_0^{II} etc in (6),we get

$$y(0.1) = y_1 = 1 + (0.1)(1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(8) + \frac{(0.1)^4}{24}(34) + \ldots$$

= 1 + 0.1 + 0.015 + 0.001333 + 0.000416

y_1 = 1.116749

Step2: Let us find y(0.2),we start with (x_1,y_1) as the starting values

Here $x_1 = x_0 + h = 0 + 0.1 = 0.1$ and y_1 = 1.116749

Putting these values in (1),(3),(4) and (5), we get

$y_1^I = y_1^2 + x_1 = (1.116749)^2 + 0.1 = 1.3471283$

$y_1^{II} = 2y_1\, y_1^I + 1 = 2(10116749)\,(1.3471283) + 1 = 4.0088$

$y_1^{III} = 2(y_1\, y_1^{II} + (y_1^I)^2) = 2((1.116749)\,(4.0088) + (1.3471283)^2] = 12.5831$

$y_1^{IV} = 2y_1\, y_1^{III} + 6\, y_1^I\, y_1^{II} = 2(1.116749)\,(12.5831) + 6(1.3471283)\,(4.0088) =$ 60.50653

By Taylor's expansion

$$y(x_2) = y_2 = y_1 + \frac{h}{1!} y_1^{I} + \frac{h^2}{2!} y_1^{II} + \frac{h^3}{3!} y_1^{III} + \frac{h^4}{4!} y_1^{IV} + \ldots$$

$$\therefore y(0.2) = y_2 = 1.116749 + (0.1)\,(1.3471283) + \frac{(0.1)^2}{2}\,(4.0088) + \frac{(0.1)^3}{6}\,(12.5831)$$

$$+ \frac{(0.1)^4}{24}\,(60.50653)$$

y_2 = 1.116749 + 0.13471283 + 0.020044 + 0.002097 + 0.000252

= 1.27385

y(0.2) = 1.27385

Step3: Let us find y(0.3),we start with (x_2,y_2) as the starting value.

Here $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$ and $y_2 = 1.27385$

Putting these values of x_2 and y_2 in eq (1), (3), (4) and (5), we get

$$y_2^I = y_2^2 + x_2 = (1.27385)^2 + 0.2 = 1.82269$$

$$y_2^{II} = 2y_2\, y_2^1 + 1 = 2(1.27385)\,(1.82269) + 1 = 5.64366$$

$$y_2^{III} = 2[y_2\, y_2^{II} + (y_2^I)^2] = 2[(1.27385)\,(5.64366) + (1.82269)^2]$$

$$= 14.37835 + 6.64439 = 21.02274$$

$$y_2^{IV} = 2y_2 + y_2^{III} + 6\, y_2^I \cdot y_1^{II} = 2(1.27385)\,(21.00274) + 6(1.82269)\,(5.64366)$$

$$= 53.559635 + 61.719856 = 115.27949$$

By Taylor's expansion,

$$y(x_3) = y_3 = y_2 + \frac{h}{1!} y_2^I + \frac{h^2}{2!} y_2^{II} + \frac{h^3}{3!} y_2^{III} + \frac{h^4}{4!} y_2^{IV} + \ldots..$$

$$y(0.3) = y_3 = 1.27385 + (0.1)\,(1.82269) + \frac{(0.1)^2}{2}(5.64366) + \frac{(0.1)^3}{6}(21.02274)$$

$$+ \frac{(0.1)^4}{24}(115.27949)$$

$$= 1.27385 + 0.182269 + 0.02821 + 0.0035037 + 0.00048033$$

$$= 1.48831$$

$$y(0.3) = 1.48831$$

10. Solve $y^1 = x^2 - y$, $y(0) = 1$ using Taylor's series method and evaluate

y(0.1),y(0.2),y(0.3) and y(0.4) (correct to 4 decimal places)

Sol: Given $y^1 = x^2 - y$ →(1)

and $y(0) = 1$ →(2)

Here $x_0 = 0$, $y_0 = 1$

Differentiating (1) w.r.t 'x', we get

$y^{II} = 2x - y^1$ →(3)

$y^{III} = 2 - y^{II}$ →(4)

$y^{IV} = -y^{III}$ →(5)

put $x_0 = 0$, $y_0 = 1$ in (1),(3),(4) and (5), we get

$y_0^I = x_0^2 - y_0 = 0-1 = -1$,

$y_0^{II} = 2x_0 - y_0^I = 2(0) - (-1) = 1$

y_0^{III} = 2- y_0^{II} = 2-1 = 1,

y_0^{IV} = - y_0^{III} = -1 Take h = 0.1

Step1: by Taylor's series expansion

$$y(x_1) = y_1 = y_0 + \frac{h}{1!} y_0^I + \frac{h^2}{2!} y_0^{II} + \frac{h^3}{3!} y_0^{III} + \frac{h^4}{4!} y_0^{IV} + \ldots \quad \rightarrow(6)$$

On substituting the values of y_0, y_0^I, y_0^{II} etc in (6), we get

$$y(0.1) = y_1 = 1 + (0.1)(-1) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(1) + \frac{(0.1)^4}{24}(-1) + \ldots$$

$$= 1 - 0.1 + 0.005 + 0.01666 - 0.0000416$$

$$= 0.905125 \simeq 0.9051 \text{ (4 decimal place).}$$

Step2: Let us find y(0.2) we start with (x_1,y_1) as the starting values

Here x = x_0 + h = 0 + 0.1 = 0.1 and y_1 = 0.905125,

Putting these values of x_1 and y_1 in (1), (3), (4) and (5), we get

$y_1^1 = x_1^2 - y_1 = (0.1)^2 - 0.905125 = -0.895125$

$y_1^{II} = 2x_1 - y_1^1 = 2(0.1) - (-0.895125) = 1.095125$,

$y_1^{III} = 2 - y_1^{II} = 2 - 1.095125 = 0.90475$,

$y_1^{IV} = - y_1^{III} = -0.904875$,

By Taylor's series expansion,

$$y(x_2) = y_2 = y_1 + \frac{h}{1!} y_1^I + \frac{h^2}{2!} y_1^{II} + \frac{h^3}{3!} y_1^{III} + \frac{h^4}{4!} y_1^{IV} + \ldots$$

$$y(0.2) = y_2 = 0.905125 + (0.1)(-0.895125) + + \frac{(0.1)^2}{2}(1.09125)$$

$$+ \frac{(0.1)^3}{6}(1.095125) + \frac{(0.1)^4}{24}(-0.904875) + \ldots$$

$$y(0.2) = y_2 = 0.905125 - 0.0895125 + 0.00547562 + 0.000150812 - 0.0000377$$

$$= 0.8212351 \simeq 0.8212 \text{ (4 decimal places)}$$

Step3: Let us find y(0.3), we start with (x_2,y_2) as the starting value

Here $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$ and $y_2 = 0.8212351$

Putting these values of x_2 and y_2 in (1),(3),(4), and (5) we get

$y_2^1 = x_2^2$ - y_2 = $(0.2)^2$ – 0.8212351= 0.04 – 0.8212351 = - 0.7812351

y_2^{II} = $2x_2$ - y_2^1 = 2(0.2) + (0.7812351) = 1.1812351,

y_2^{III} = 2 - y_2^{II} = 2 – 1. 1812351 = 0.818765,

y_2^{IV} = - y_2^{III} = -0.818765,

By Taylor's series expansion,

$$y(x_3) = y_3 = y_2 + \frac{h}{1!} y_2^I + \frac{h^2}{2!} y_2^{II} + \frac{h^3}{3!} y_2^{III} + \frac{h^4}{4!} y_2^{IV} + \ldots.$$

$$y(0.3) = y_3 = 0.\ 8212351 + (0.1)(-0.7812351) + \frac{(0.1)^2}{2}$$

$$(1.1812351) + \frac{(0.1)^3}{6}(0.818765) + \frac{(0.1)^4}{24}(-0.818765) + \ldots.$$

y(0.3) = y_3 = 0. 8212351– 0.07812351+ 0.005906 + 0.000136 – 0.0000034

= 0.749150 $\simeq$ 0.7492 (4 decimal places)

Step4: Let us find y(0.4), we start with (x_3,y_3) as the starting value

Here $x_3 = x_2 + h$ = 0.2+ 0.1 = 0.3 and y_3 = 0.749150

Putting these values of x_3 and y_3 in (1),(3),(4), and (5) we get

$y_3^1 = x_3^2$ - y_3 = $(0.3)^2$ – 0.749150= -0.65915,

y_3^{II} = $2x_3$ - y_3^1 = 2(0.3) + (0.65915) = 1.25915,

y_3^{III} = 2 - y_3^{II} = 2 – 1. 25915 = 0.74085,

y_3^{IV} = - y_3^{III} = -0.74085,

By Taylor's series expansion,

$$y(x_4) = y_4 = y_3 + \frac{h}{1!} y_3^I + \frac{h^2}{2!} y_3^{II} + \frac{h^3}{3!} y_3^{III} + \frac{h^4}{4!} y_3^{IV} + \ldots.$$

$$y(0.4) = y_4 = 0.\ 749150 + (0.1)(-0.65915) + \frac{(0.1)^2}{2}(1.25915) +$$

$$\frac{(0.1)^3}{6}(0.74085) + \frac{(0.1)^4}{24}(-0.74085) + \ldots.$$

$y(0.4) = y_4 = 0.\ 749150 – 0.065915 + 0.0062926 + 0.000123475 – 0.0000030$

$= 0.6896514 \simeq 0.6896$ (4 decimal places)

11. Solve $y^1 = x^2 – y$, $y(0) = 1$using T.S.M and evaluate y(0.1),y(0.2),y(0.3) and y(0.4) (correct to 4 decimal place) 0.9051, 0.8212, 07492, 0.6896

12. Given the differentiating equation $y^1 = x^1 + y^2$, $y(0) = 1$. Obtain y(0.25) and y(0.5) by T.S.M.

Ans: 1.3333, 1.81667

13. Solve $y^1 = xy^2 + y$, $y(0) = 1$ using Taylor's series method and evaluate y(0.1) and y(0.2)

Ans: 1.111, 1.248.

EULER'S METHOD

It is the simplest one-step method and it is less accurate. Hence it has a limited application.

Consider the differential equation $\frac{dy}{dx} = f(x,y)$ →(1)

With $y(x_0) = y_0$→(2)

Consider the first two terms of the Taylor's expansion of y(x) at $x = x_0$

$y(x) = y(x_0) + (x – x_0)\, y^1(x_0)$ →(3)

from equation (1) $y^1(x_0) = f(x_0,y(x_0)) = f(x_0,y_0)$

Substituting in equation (3)

$\therefore y(x) = y(x_0) + (x – x_0)\, f(x_0,y_0)$

At $x = x_1$, $y(x_1) = y(x_0) + (x_1 – x_0)\, f(x_0,y_0)$

$\therefore y_1 = y_0 + h\, f(x_0,y_0)$ where $h = x_1 – x_0$

Similarly at $x = x_2$, $y_2 = y_1 + h\, f(x_1,y_1)$,

Proceeding as above, $y_{n+1} = y_n + h\, f(x_n,y_n)$

This is known as Euler's Method

1. **Using Euler's method solve for x = 2 from $\frac{dy}{dx} = 3x^2 + 1$,y(1) = 2,taking step size** (I) h = 0.5
and (II) h=0.25

Sol: here $f(x,y) = 3x^2 + 1$, $x_0 = 1$, $y_0 = 2$

Euler's algorithm is $y_{n+1} = y_n + h\ f(x_n,y_n)$, n = 0,1,2,3,….. →(1)

(I) h = 0.5 $\therefore x_1 = x_0 + h = 1+0.5 = 1.5$

Taking n = 0 in (1) , we have $x_2 = x_1 + h = 1.5 + 0.5 = 2$

$y_1 = y_0 + h\ f(x_0,y_0)$

i.e. $y_1 = y(0.5) = 2 + (0.5)\ f(1,2) = 2 + (0.5)\ (3 + 1) = 2 + (0.5)(4)$

Here $x_1 = x_0 + h = 1 + 0.5 = 1.5$

$\therefore y(1.5) = 4 = y_1$

Taking n = 1 in (1),we have

$y_2 = y_1 + h\ f(x_1,y_1)$

i.e. $y(x_2) = y_2 = 4 + (0.5)\ f(1.5,4) = 4 + (0.5)[3(1.5)^2 + 1] = 7.875$

Here $x_2 = x_4 + h = 1.5 + 0.5 = 2$

$\therefore y(2) = 7.875$

(II) h = 0.25 $\therefore x_1 = 1.25, x_2 = 1.50, x_3 = 1.75, x_4 = 2$

Taking n = 0 in (1), we have

$y_1 = y_0 + h\ f(x_0,y_0)$

i.e. $y(x_1) = y_1 = 2 + (0.25)\ f(1,2) = 2 + (0.25)\ (3 + 1) = 3$

$y(x_2) = y_2 = y_1 + h\ f(x_1,y_1)$

i.e. $y(x_2) = y_2 = 3 + (0.25)\ f(1.25,3)$

$= 3 + (0.25)[3(1.25)^2 + 1]$

$= 4.42188$

Here $x_2 = x_1 + h = 1.25 + 0.25 = 1.5$

$\therefore y(1.5) = 5.42188$

Taking n = 2 in (1), we have

i.e. $y(x_3) = y_3 = h\ f(x_2,y_2)$

$= 5.42188 + (0.25)\ f(1.5,2)$

$= 5.42188 + (0.25)\ [3(1.5)^2 + 1]$

$= 6.35938$

Here $x_3 = x_2 + h = 1.5 + 0.25 = 1.75$

$\therefore y(1.75) = 7.35938$

Taking n = 4 in (1),we have

$y(x_4) = y_4 = y_3 + h\, f(x_3,y_3)$

i.e. $y(x_4) = y_4 = 7.35938 + (0.25)\, f(1.75,2)$

$= 7.35938 + (0.25)[3(1.75)^2 + 1]$

$= 8.90626$

Note that the difference in values of y(2) in both cases
(i.e. when h = 0.5 and when h = 0.25).The accuracy is improved significantly when h is reduced to 0.25 (Example significantly of the equ is $y = x^3 + x$ and with this $y(2) = y_2 = 10$

2. **Solve by Euler's method,$y^1 = x + y$, $y(0) = 1$ and find y(0.3) taking step size h = 0.1. compare the result obtained by this method with the result obtained by analytical solution**

Sol: $y_1 = 1.1 = y(0.1)$,

$y_2 = y(0.2) = 1.22$

$y_3 = y(0.3) = 1.362$

Particular solution is $y = 2e^x - (x + 1)$

Hence y(0.1) = 1.11034, y(0.2) = 1.3428, y(0.3) = 1.5997

We shall tabulate the result as follows

X	0	0.1	0.2	0.3
Euler y	1	1.1	1.22	1.362
Euler y	1	1.11034	1.3428	1.3997

The

value of y deviate from the execute value as x increases. This indicate that the method is not accurate

3. **Solve by Euler's method $y^1 + y = 0$ given y(0) = 1 and find y(0.04) taking step size**

h = 0.01 Ans: 0.9606

4. **Using Euler's method, solve y at x = 0.1 from $y^1 = x + y + xy$, y()) = 1 taking step size h = 0.025.**

5. **Given that $\frac{dy}{dx}$ = xy ,y(0) = 1 determine y(0.1),using Euler's method.** h =0.1

Sol: The given differentiating equation is $\frac{dy}{dx} = xy$, y(0) = 1

a = 0

Here f(x,y) = xy , $x_0 = 0$ and $y_0 = 1$

Since h is not given much better accuracy is obtained by breaking up the interval (0,0.1) in to five steps.

i.e. $h = \frac{b-a}{5} = \frac{0.1}{5} = 0.02$

Euler's algorithm is $y_{n+1} = y_n + h\, f(x_n,y_n)$ → (1)

$\therefore$ From (1) form = 0, we have

$$
\begin{aligned}
y_1 &= y_0 + h\, f(x0,y_0) \\
&= 1 + (0.02)\, f(0,1) \\
&= 1 + (0.02)\,(0) \\
&= 1
\end{aligned}
$$

Next we have $x_1 = x_0 + h = 0 + 0.02 = 0.02$

$\therefore$ From (1), form = 1,we have

$$
\begin{aligned}
y_2 &= y_1 + h\, f(x_1,y_1) \\
&= 1 + (0.02)\, f(0.02,1) \\
&= 1 + (0.02)\,(0.02) \\
&= 1.0004
\end{aligned}
$$

Next we have $x_2 = x_1 + h = 0.02 + 0.02 = 0.04$

$\therefore$ From (1), form = 2,we have

$y_3 = y_2 + h\ f(x_2,y_2)$

$= 1.004 + (0.02)\ (0.04)\ (1.0004)$

$= 1.0012$

Next we have $x_3 = x_2 + h = 0.04 + 0.02 = 0.06$

$\therefore$ From (1), form = 3,we have

$y_4 = y_3 + h\ f(x_3,y_3)$

$= 1.0012 + (0.02)\ (0.06)\ (1.00012)$

$= 1.0024.$

Next we have $x_4 = x_3 + h = 0.06 + 0.02 = 0.08$

$\therefore$ From (1), form = 4,we have

$y_5 = y_4 + h\ f(x_4,y_4)$

$= 1.0024 + (0.02)\ (0.08)\ (1.00024)$

$= 1.0040.$

Next we have $x_5 = x_4 + h = 0.08 + 0.02 = 0.1$

When $x = x_5$, $y \simeq y_5$

$\therefore y = 1.0040$ when $x = 0.1$

6. **Solve by Euler's method $y^1 = \frac{2y}{x}$ given y(1) = 2 and find y(2).**

7. **Given that $\frac{dy}{dx} = 3x^2 + y$, y(0) = 4.Find y(0.25) and y(0.5) using Euler's method**

Sol: given $\frac{dy}{dx} = 3x^2 + y$ and $y(1) = 2$.

Here $f(x,y) = 3x^2 + y$, $x_0 = (1)$, $y_0 = 4$

Consider $h = 0.25$

Euler's algorithm is $y_{n+1} = y_n + h\ f(x_n,y_n)$ $\rightarrow$(1)

$\therefore$ From (1), for n = 0, we have

$y_1 = y_0 + h\ f(x_0,y_0)$

$= 2 + (0.25)[0 + 4]$

$= 2 + 1$

$= 3$

Next we have $x_1 = x_0 + h = 0 + 0.25 = 0.25$

When x = x_1, $y_1 \simeq$ y

$\therefore$ y = 3 when x = 0.25

$\therefore$ From (1), for n = 1, we have

$y_2 = y_1 + h\, f(x_1, y_1)$

$= 3 + (0.25)[3.(0.25)^2 + 3]$

$= 3.7968$

Next we have $x_2 = x_1 + h = 0.25 + 0.25 = 0.5$

When x = x_2, y $\simeq$ y_2

$\therefore$ y = 3.7968 when x = 0.5.

8. **Solve first order diff equation $\frac{dy}{dx} = \frac{y-x}{y+x}$, y(0) = 1 and estimate y(0.1) using Euler's method (5 steps)** Ans: 1.0928

9. **Use Euler's method to find approximate value of solution of $\frac{dy}{dx}$ = y-x + 5 at x = 2-1 and 2-2with initial contention y(0.2) = 1**

<u>Modified Euler's method</u>

It is given by $y^{(i)}{}_{k+1} = y_k + h/2\, f\left[\left(x_k, y_k\right) + f\left(x_{k+1}, 1\right)_{k+1}{}^{(i-1)}\right], i = 1, 2....., ki = 0, 1.....$

<u>Working rule :</u>

i)Modified Euler's method

$y^{(i)}{}_{k+1} = y_k + h/2\, f\left[\left(x_k, y_k\right) + f\left(x_{k+1}, 1\right)_{k+1}{}^{(i-1)}\right], i = 1, 2....., ki = 0, 1.....$

ii) When $i = 1$ $y^{0}{}_{k+1}$ can be calculated from Euler's method

iii) K=0, 1……… gives number of iteration. $i = 1, 2...$

gives number of times, a particular iteration k is repeated

Suppose consider dy/dx=f(x, y) -------- (1) with $y(x_0) = y_0$----------- (2)

To find $y(x_1) = y_1$ at $x = x_1 = x_0 + h$

Now take k=0 in modified Euler's method

We get $y_1^{(1)} = y_0 + h/2\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(i-1)}\right)\right]$……………………… (3)

Taking i=1, 2, 3...k+1 in eqn (3), we get

$y_1^{(0)} = y_0 + h/2\left[f\left(x_0, y_0\right)\right]$ (By Euler's method)

$y_1^{(1)} = y_0 + h/2\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(0)}\right)\right]$

$y_1^{(2)} = y_0 + h/2\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(1)}\right)\right]$

$y_1^{(k+1)} = y_0 + h/2\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(k)}\right)\right]$

If two successive values of $y_1^{(k)}, y_1^{(k+1)}$ are sufficiently close to one another, we will take the common value as $y_2 = y\left(x_2\right) = y\left(x_1 + h\right)$

We use the above procedure again

1) using modified Euler's method find the approximate value of x when $x = 0.3$ given that $dy/dx = x + y$ and $y(0) = 1$

sol: Given $dy/dx = x + y$ and $y(0) = 1$

Here $f\left(x, y\right) = x + y, x_0 = 0,$ and $y_0 = 1$

Take h = 0.1 which is sufficiently small

Here $x_0 = 0, x_1 = x_0 + h = 0.1, x_2 = x_1 + h = 0.2, x_3 = x_2 + h = 0.3$

The formula for modified Euler's method is given by

$$y_{k+1}^{(i)} = y_k + h/2\left[f\left(x_k + y_k\right) + f\left(x_{k+1}, y_{k+1}^{(i-1)}\right)\right] \rightarrow (1)$$

<u>Step1:</u> To find $y_1 = y(x_1) = y\ (0.1)$

Taking k = 0 in eqn(1)

$$y_{k+1}^{(i)} = y_0 + h/2\left[f\left(x_0 + y_0\right) + f\left(x_1, y_1^{(i-1)}\right)\right] \rightarrow (2)$$

when i = 1 in eqn (2)

$$y_1^{(i)} = y_0 + h/2\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(0)}\right)\right]$$

First apply Euler's method to calculate $y_1^{(0)}$ = y_1

$$\therefore\ y_1^{(0)} = y_0 + h\, f\left(x_0, y_0\right)$$

= 1+(0.1)f(0.1)

= 1+(0.1)

= 1.10

$now\left[x_0 = 0, y_0 = 1, x_1 = 0.1, y_1(0) = 1.10\right]$

$\therefore\ y_1^{(1)} = y_0 + 0.1/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$

= 1+0.1/2[f(0,1) + f(0.1,1.10)

= 1+0.1/2[(0+1)+(0.1+1.10)]

= 1.11

When i=2 in eqn (2)

$y_1^{(2)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$

= 1+0.1/2[f(0.1)+f(0.1,1.11)]

= 1 + 0.1/2[(0+1)+(0.1+1.11)]

= 1.1105

$y_1^{(3)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right]$

= 1+0.1/2[f(0,1)+f(0.1 , 1.1105)]

= 1+0.1/2[(0+1)+(0.1+1.1105)]

= 1.1105

Since $y_1^{(2)} = y_1^{(3)}$

$\therefore$ y_1 = 1.1105

Step:2 To find $y_2 = y(x_2) = y(0.2)$

Taking k = 1 in eqn (1) , we get

$y_2^{(i)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2, y_2^{(i-1)}\right)\right] \rightarrow (3)$

i = 1,2,3,4,…..

For i = 1

$y_2^{(1)} = y_1 + h/2\left[f\left(x_1, y_1\right) + f\left(x_2, y_2^{(0)}\right)\right]$

$y_2^{(0)}$ is to be calculate from Euler's method

$y_2^{(0)} = y_1 + h\ f\left(x_1, y_1\right)$

= 1.1105 + (0.1) f(0.1 , 1.1105)

= 1.1105+(0.1)[0.1+1.1105]

= 1.2316

$\therefore\ y_2^{(1)}$ = $1.1105 + 0.1/2\left[f\left(0.1, 1.1105\right) + f\left(0.2, 1.2316\right)\right]$

= 1.1105 +0.1/2[0.1+1.1105+0.2+1.2316]

= 1.2426

$y_2^{(2)} = y_1 + h/2\left[f\left(x_1, y_1\right) + f\left(x_2 y_2^{(1)}\right)\right]$

= 1.1105 + 0.1/2[f(0.1 , 1.1105) , f(0.2 . 1.2426)]

= 1.1105 + 0.1/2[1.2105 + 1.4426]

= 1.1105 + 0.1(1.3266)

= 1.2432

$y_2^{(3)} = y_1 + h/2\left[f\left(x_1, y_1\right) + f\left(x_2 y_2^{(2)}\right)\right]$

= 1.1105+0.1/2[f(0.1,1.1105)+f(0.2 , 1.2432)]

= 1.1105+0.1/2[1.2105+1.4432)]

= 1.1105 + 0.1(1.3268)

= 1.2432

Since $y_2^{(3)} = y_2^{(3)}$

Hence y_2 = 1.2432

<u>Step:3</u>

To find $y_3 = y(x_3) = y\ y(0.3)$

Taking k =2 in eqn (1) we get

$$y_3^{(1)} = y_2 + h/2\left[f\left(x_2, y_2\right) + f\left(x_3, y_3^{(i-1)}\right)\right] \rightarrow (4)$$

For i = 1 ,

$$y_3^{(1)} = y_2 + h/2\left[f\left(x_2, y_2\right) + f\left(x_3, y_3^{(0)}\right)\right]$$

$y_3^{(0)}$ is to be evaluated from Euler's method .

$$y_3^{(0)} = y_2 + h\, f\left(x_2, y_2\right)$$

= 1.2432 +(0.1) f(0.2 , 1.2432)

= 1.2432+(0.1)(1.4432)

= 1.3875

$\therefore\ y_3^{(1)}$ = 1.2432+0.1/2[f(0.2 , 1.2432)+f(0.3, 1.3875)]

= 1.2432 + 0.1/2[1.4432+1.6875]

= 1.2432+0.1(1.5654)

= 1.3997

$$y_3^{(2)} = y_2 + h/2\left[f\left(x_2, y_2\right) + f\left(x_3, y_3^{(1)}\right)\right]$$

= 1.2432+0.1/2[1.4432+(0.3+1.3997)]

= 1.2432+ (0.1) (1.575)

= 1.4003

$$y_3^{(3)} = y_2 + h/2\left[f\left(x_2, y_2\right) + f\left(x_3, y_3^{(2)}\right)\right]$$

= 1.2432+0.1/2[f(0.2 , 1.2432)+f(0.3 , 1.4003)]

= 1.2432 + 0.1(1.5718)

= 1.4004

$$y_3^{(4)} = y_2 + h/2\left[\ f\left(x_2, y_2\right) + f\left(x_3, y_3^{(3)}\right)\right]$$

= 1.2432 + 0.1/2[1.4432+1.7004]

= 1.2432+(0.1)(1.5718)

= 1.4004

Since $y_3^{(3)} = y_3^{(4)}$

Hence $y_3 = 1.4004$ $\therefore$ The value of y at x = 0.3 is 1.4004

2 . Find the solution of $\frac{dy}{dx}$ = x-y , y(0)=1 at x =0.1 , 0.2 ,0.3 , 0.4 and 0.5 . Using modified Euler's method

Sol . Given $\frac{dy}{dx}$ = x-y and y(0) = 1

Here f(x,y) = x-y , x_0 = 0 and y_0 = 1

Consider h = 0.1 so that

x = 0.1 , x_2 = 0.2 , x_3 =0.3 , x_4 = 0.4 and x_5 = 0.5

The formula for modified Euler's method is given by

$$y_{k+1}^{(i)} = y_k + h/2\ f\left(x_k, y_k\right) + f x_{k+1}, y_{k+1}^{(i-1)} \rightarrow (1)$$

Where k = 0,1, 2, 3,….. i = 1, 2, 3,…..

x	$f\left(x_k, y_k\right) = x_k - y_k$	$\frac{1}{2}\left[f\left(x_k, y_k\right) + f\left(x_{k+1}, y_{k+1}^{(i-1)}\right)\right]$	$y_{k+1}^{(i)} = y_k + h/2\left[f\left(x_k, y_k\right) + f\left(x_{k+1}, y_{k+1}^{(i-1)}\right)\right]$
K = 0			
0. 0	0-1=-1	-	1+(0.1)(-1)=0.9 = $y_1^{(0)}$

0.1(i=1)	0-1=-1	½(-1-0.8) = -0.9	1+(0.1)(-0.9)=0.91
0.1(i=2)	0-1=-1	½(-1-0.81)= -0.905	1+(0.1)(-0.905)=0.9095
0.1(i=3)	0-1=-1	½(-1-0.80.95)= -0.90475	1+(0.1)(-0.90475)=0.9095
K=1			
0.1	0.1-0.9095= -0.8095	-	0.9095+(0.1)(-0.8095)=0.82855
0.2(i=1)	-0.8095	½(-0.8095-0.62855)	0.9095+(0.1)(-0.719025)=0.8376
0.2(i=2)	-0.8095	½(-0.8095-0.6376)	0.9095+(0.1)(-0.72355)=0.8371
0.2(i=3)	-0.8095	½(-0.8095-0.6371)	0.9095+(0.1)(-0.7233)=0.8372
0.2(i=4)	-0.8095	½(-0.8095-0.6372)	0.9095+(0.1)(-0.72355)=0.8371
K=2			
0.2	0.2-0.8371=-0.6371	-	0.8371+(0.1)(-0.6371)=0.7734
0.3(i=1)	= -0.6371	½(-0.6371-0.4734)	0.8371+(0.1)(-0.555)=0.7816
0.3(i=2)	= -0.6371	½(-0.6371-0.4816)	0.8371-0.056=0.7811
0.3(i=3)	= -0.6371	½(-0.6371-0.4811)	0.8371-0.05591=0.7812
0.3(i=4)	= -0.6371	½(-0.6371-0.4812)	0.8371-0.055915 = 0.7812
K =3			
0.3(i=1)	0.3-0.7812	-	0.7812+(0.1)(-0.4812) = 0.7331

0.4(i=1)	-0.4812	½(-0.4812-0.4311)	0.7812-0.0457 = 0.7355
0.4(i=2)	-0.4812	½(-0.4812-0.4355)	0.7812-0.0458 = 0.7354
0.4(i=3)	-0.4812	½(-0.4812-0.4354)	0.7812-0.0458 = 0.7354
K=4			
0.4	-0.3354	-	0.7354-0.03354 = 0.70186
0.5	-0.3354	½(-0.3354-0.301816)	0.7354-0.03186 = 0.7035
0.5	-0.3354	½(-0.3354-0.30354)	0.7354-0.0319 = 0.7035

3. Find y(0.1) and y(0.2) using modified Euler's formula given that $dy/dx=x^2-y, y(0)=1$

   [consider h=0.1, $y_1=0.90523, y_2=0.8214$]

4. Given $dy/dx = -xy^2, y(0) = 2$ compute y(0.2) in steps of 0.1

   Using modified Euler's method

   [h=0.1, $y_1=1.9804$, $y_2=1.9238$]

5. Given $y^1 = x+\sin y$, y(0)=1 compute y(0.2) and y(0.4) with h=0.2 using modified Euler's method

   [$y_1=1.2046$, $y_2=1.4644$]

Runge – Kutta Methods

I. Second order R-K Formula

$y_{i+1} = y_i+1/2\ (K_1+K_2)$,

Where $K_1 = h\ (x_i, y_i)$

$K_2 = h\,(x_i+h,\ y_i+k_1)$

For i= 0,1,2-------

II. Third order R-K Formula

$y_{i+1} = y_i+1/6\,(K_1+4K_2+K_3)$,

Where $K_1 = h\,(x_i,\ y_i)$

$K_2 = h\,(x_i+h/2,\ y_0+k_1/2)$

$K_3 = h\,(x_i+h,\ y_i+2k_2-k_1)$

For i= 0,1,2-------

III. Fourth order R-K Formula

$y_{i+1} = y_i+1/6\,(K_1+2K_2+2K_3+K_4)$,

Where $K_1 = h\,(x_i,\ y_i)$

$K_2 = h\,(x_i+h/2,\ y_i+k_1/2)$

$K_3 = h\,(x_i+h/2,\ y_i+k_2/2)$

$K_4 = h\,(x_i+h,\ y_i+k_3)$

For i= 0,1,2-------

1. Using Runge-Kutta method of second order, find $y(2.5)$ from $\frac{dy}{dx} = \frac{x+y}{x}$, y(2)=2 , h = 0.25 .

Sol: Given $\frac{dy}{dx} = \frac{x+y}{x}$, y(2) = 2 .

Here f(x, y) = $\frac{x+y}{x}$, $x_0 = 0$, $y_0=2$ and h = 0.25

$\therefore\ x_1 = x_0+h = 2+0.25 = 2.25$, $x_2 = x_1+h = 2.25+0.25 = 2.5$

By R-K method of second order,

$$y_{i+1} = y_i + 1/2\left(k_1+k_2\right), k_1 - hf\left(x_i+h, y_i+k_1\right), i = 0,1.... \rightarrow (1)$$

Step -1:-

To find $y(x_1)$i.e y(2.25) by second order R - K method taking i=0 in eqn(i)

We have $y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

Where k_1= hf (x_0,y_0), k_2= hf (x_0+h,y_0+k_1)

f (x_0,y_0)=f(2,2)=2+2/2=2

k_1=hf (x_0,y_0)=0.25(2)=0.5

k_2= hf (x_0+h,y_0+k_1)=(0.25)f(2.25,2.5)

=(0.25)(2.25+2.5/2.25)=0.528

$\therefore y_1$=y(2.25)=2+1/2(0.5+0.528)

=2.514

Step2:

To find $y(x_2)$ i.e., y(2.5)

i=1 in (1)

x_1=2.25,y_1=2.514,and h=0.25

$y_2=y_1+1/2(k_1+k_2)$

where k_1=h f((x_1,y_1)=(0.25)f(2.25,2.514)

=(0.25)[2.25+2.514/2.25]=0.5293

$$k_2 = h\,f\,(x_0 + h, y_0 + k_1) = (0.1)\,f\,(0.1,1 - 0.1) = (0.1)(-0.9) = -0.09$$

=(0.25)[2.5+2.514+0.5293/2.5]

=0.55433

$y_2 = y$ (2.5)=2.514+1/2(0.5293+0.55433)

=3.0558

$\therefore$ y =3.0558 when x = 2.5

Obtain the values of y at x=0.1,0.2 using R-K method of

(i)second order (ii)third order (iii)fourth order for the diff eqn y^1+y=0,y(0)=1

Sol: Given dy/dx = -y, y(0)=1

f(x,y) = -y, x_0 = 0, y_0 = 1

Here f (x,y) = -y, x_0 = 0, y_0 = 1 take h = 0.1

$\therefore$ $x_1 = x_0+h = 0.1$,

$x_2 = x_1+h = 0.2$

Second order:

step1: To find $y(x_1)$ i.e y(0.1) or y_1

by second-order R-K method,we have

$y_1 = y_0+1/2(k_1+k_2)$

where $k_1=hf(x_0,y_0)=(0.1)\ f(0,1) = (0.1)(-1)= - 0.1$

$k_2= hf\ (x_0+h,\ y_0+k_1)= (0.1)\ f\ (0.1,\ 1-0.1) = (0.1)(-0.9) = -0.09$

$y_1=y(0.1)=1+1/2(-0.1-0.09)=1-0.095=0.905$

$\therefore$ y =0.905 when x=0.1

Step2:

To find y_2 i.e $y(x_2)$ i.e y(0.2)

Here x_1 = 0.1, y_1 = 0.905 and h=0.1

By second-order R-K method, we have

$y_2 = y(x_2)= y_1+1/2(k_1+k_2)$

Where $k_1 = h\ f\left(x_1, y_1\right)$ =(0.1)f(0.1,0.905)=(0.1)(-0.905)=-0.0905

$$k_2 = h\ f\left(x_1 + h, y_1 + k_1\right) = \left(0.1\right) f\left(0.2, 0.905 - 0.0905\right)$$
$$= \left(0.1\right) f\left(0.2, 0.8145\right) = \left(0.1\right)\left(-0.8145\right)$$
$$= -0.08145$$

$y_2 =$ y(0.2)=0.905+1/2(-0.0905-0.08145)

= 0.905- 0.085975 = 0819025

<u>Third order</u>

Step1:

To find y_1 i.e $y(x_1)$= y(0.1)

By Third order Runge kutta method

$$y_1 = y_0 + 1/6\left(k_1 + 4k_2 + k_3\right)$$

where k_1 = h f(x_0, y_0) = (0.1) f (0.1) = (0.1) (-1) = -0.1

$$k_2 = h\ f\left(x_0 + h/2, y_0 + k_1/2\right) = \left(0.1\right) f\left(0.1/2, 1 - 0.1/2\right) = \left(0.1\right) f\left(0.05, 0.95\right)$$
$$= \left(0.1\right)\left(-0.95\right) = -0.095$$

and k_3 = h f((x_0+h,y_0+2k_2-k_1)

(0.1) f (0.1,1+2(-0.095)+0.1)= -0.905

Hence y_1 = 1+1/6(-0.1+4(-0.095)-0.09) = 1+1/6 (-0.57) = 0.905

y_1=0.905 i.e y(0.1)= 0.905

Step2:

To find y_2,i.e $y(x_2)$= y(0.2)

Here x_1=0.1,y_1=0.905 and h = 0.1

Again by 2^{nd} order R-K method

y_2 = y_1+1/6(k_1+4k_2+k_3)

Where k_1=h f(x_1, y_1) = (0.1)f (0.1,0.905)= -0.0905

k_2 = h f (x_1+h/2,y_1+k_1/2)=(0.1)f(0.1+0.2,0.905 - 0.0905)= -(0.1) f (0.15, 0.85975)= (0.1) (-0.85975)

and k_3 = h f((x_1+h,y_1+2k_2-k_1)=(0.1)f(0.2,0.905+2(0.08975)+0.0905= -0.082355

hence y_2 = 0.905+1/6(-0.0905+4(-0.085975)-0.082355)=0.818874

$\therefore$ y = 0.905 when x = 0.1

And y =0.818874 when x =0.2

fourth order:

step1:

x_0=0,y_0=1,h=0.1 To find y_1 i.e y(x_1)=y(0.1)
By 4^{th} order R-K method, we have
y_1 = y_0+1/6(k_1+2k_2+2k_3+k_4)
Where k_1=h f(x_0,y_0)=(0.1)f(0.1)= -0.1
k_2= h f (x_0+h/2, y_0+k1/2) = -0.095
and k_3= h f((x_0+h/2,y_0+k_2/2)=(0.1)f (0.1/2,1-0.095/2)
= (0.1)f(0.05,0.9525)
= -0.09525
and k_4= h f(x_0+h,y_0+k_3)
= (0.1) f(0.1,1-0.09525)=(0.1)f(0.1,0.90475)

=-0.090475

Hence y_1=1+1/6(-0.1)+2(-0.095)+2(0.09525)-0.090475)

=1+1/6(-0.570975)+1-0.951625 = 0.9048375

Step2:

To find $y_2, i.e., y\left(x_2\right) = y\left(0.2\right), y_1 = 0.9048375, i.e., y\left(0.1\right) = 0.9048375$

Here x_1 = 0.1, y_1=0.9048375 and h = 0.1

Again by 4^{th} order R-K method, we have

y_2 = y_1+1/6(k_1+2k_2+2k_3+k_4)

Where k_1=h $f(x_1,y_1)$=(0.1)f(0.1,0.9048375)=-0.09048375

k_2= hf $(x_1+h/2,y_1+k_1/2)$=(0.1)f(0.1+0.1/2,0.9048375 -0.09048375 /2)=-0.08595956

and k_3=hf$(x_1+h/2, y_1+k_2/2)$=(0.1)f(0.15,0. 8618577)= -0.08618577

k_4 =h $f(x_1+h,y_1+k_3)$=(0.1)f(0.2,0.86517)

= -0.08186517

Hence y_2 = 0.09048375+1/6(-0.09048375-2(0.08595956)-2(0.08618577)- 0.08186517

=0.9048375-0.0861065

=0.818731

y = 0.9048375 when x =0.1 and y =0.818731

3. Apply the 4^{th} order R-K method to find an approximate value of y when x=1.2 in steps of 0.1,given that

$y^1 = x^2+y^2$,y (1)=1.5

sol. Given $y^1= x^2+y^2$,and y(1)=1.5

Here $f(x,y)= x^2+y^2$, y_0 =1.5 and x_0=1,h=0.1

So that x_1=1.1 and x_2=1.2

Step1:

To find y_1 i.e., $y(x_1)$

by 4^{th} order R-K method we have

$y_1=y_0+1/6 (k_1+2k_2+2k_3+k_4)$

$k_1=hf(x_0,y_0)$=(0.1)f(1,1.5)=(0.1) $[1^2+(1.5)^2]$=0.325

k_2= hf $(x_0+h/2,y_0+k_1/2)$=(0.1)f(1+0.05,1.5+0.325)=0.3866

and k_3=hf($(x_0+h/2,y_0+k_2/2)$=(0.1)f(1.05,1.5+0. 3866/2)=(0.1)$[(1.05)^2+(1.6933)^2]$

=0.39698

k_4=hf(x_0+h,y_0+k_3)=(0.1)f(1.0,1.89698)

=0.48085

Hence

$$y_1 = 1.5 + \frac{1}{6}\left[0.325 + 2\left(0.3866\right) + 2\left(0.39698\right) + 0.48085\right]$$
$$= 1.8955$$

Step2:

To find y_2, i.e., $y\left(x_2\right) = y\left(1.2\right)$

Here x_1=0.1,y_1=1.8955 and h=0.1

by 4th order R-K method we have

y_2 = y_1+1/6(k_1+2k_2+2k_3+k_4)

k_1=hf(x_1,y_1)=(0.1)f(0.1,1.8955)=(0.1) [1^2+$(1.8955)^2$]=0.48029

k_2= hf (x_1+h/2,y_1+k_1/2)=(0.1)f(1.1+0.1,1.8937+0.4796) =0.58834

and k_3=hf((x_1+h/2,y_1+k_2/2)=(0.1)f(1.5,1.8937+0.58743) =(0.1)[$(1.05)^2$+$(1.6933)^2$]

=0.611715

k_4=hf(x_1+h,y_1+k_3)=(0.1)f(1.2,1.8937+0.610728)

=0.77261

Hence y_2=1.8937+1/6(0.4796+2(0.58834)+2(0.611715)+0.7726) =2.5043

$\therefore$ y =2.5043 where $x = 0.2$

4. using R-K method, find y(0.2) for the eqn dy/dx=y-x,y(0)=1,take h=0.2

Ans:1.15607

5.Given that y^1=y-x,y(0)=2 find y(0.2) using R- K method take h=0.1

Ans: 2.4214

6. Apply the 4th order R-K method to find $y(0.2)$ *and* $y(0.4)$ **for one equation**

$10\frac{dy}{dx} = x^2 + y^2, y(0) = 1$ **take h = 0.1** Ans. 1.0207, 1.038

7. using R-K method, estimate y(0.2) and y(0.4) for the eqn $dy/dx=y^2-x^2/ y^2+x^2$,y(0)=1,h=0.2

Ans:1.19598,1.3751

8. use R-K method, to approximate y when x=0.2 given that $y^1=x+y$,y(0)=1

Sol: Here f(x,y)=x+y,$y_0=1$,$x_0=0$

Since h is not given for better approximation of y

Take h=0.1

$\therefore x_1=0.1,\ x_2=0.2$

Step1

To find y_1 i.e $y(x_1)=y(0.1)$

By R-K method,we have

$y_1=y_0+1/6\ (k_1+2k_2+2k_3+k_4)$

Where $k_1=hf(x_0,y_0)=(0.1)f(0,1)=(0.1)\ (1)=0.1$

$k_2=\ hf\ (x_0+h/2,y_0+k_1/2)=(0.1)f(0.05,1.05)=0.11$

and $k_3=hf((x_0+h/2,y_0+k_2/2)=(0.1)f(0.05,1+0.\ 11/2)=(0.1)[(0.05)\ +(4.0.11/2)]$

=0.1105

$k_4=h\ f\ (x_0+h,y_0+k_3)=(0.1)f(0.1,1.1105)=(0.1)[0.1+1.1105]$

=0.12105

Hence $\therefore\ y_1 = y(0.1) = 1 + \frac{1}{6}(0.1 + 0.22 + 0.240 + 0.12105)$

y = 1.11034

Step2:

To find y_2 i.e $y(x_2) = y(0.2)$

Here x_1=0-1, y_1=1.11034 and h=0.1

Again By R-K method,we have

$y_2=y_1+1/6(k_1+2k_2+2k_3+k_4)$

k_1=h f(x_1,y_1)=(0.1)f(0.1,1.11034)=(0.1) [1.21034]=0.121034

k_2= h f (x_1+h/2, y_1+k_1/2)=(0.1)f(0.1+0.1/2,1.11034+0.121034/2)

=0.1320857

and k_3=h f((x_1+h/2,y_1+k_2/2)=(0.1)f(0.15,1.11034+0.1320857/2)

=0.1326382

k_4=h f(x_1+h,y_1+k_3)=(0.1)f(0.2,1.11034+0.1326382)

(0.1)(0.2+1.2429783)=0.1442978

Hence y_2=1.11034+1/6(0.121034+0.2641714+0.2652764+0.1442978

=1.11034+0.1324631 =1.242803

$\therefore$ y =1.242803 when x=0.2

9.using Runge-kutta method of order 4,compute y(1.1) for the eqn $y^1=3x+y^2$,y(1)=1.2 h = 0.05

Ans:1.7278

10. using Runge-kutta method of order 4,compute y(2.5) for the eqn dy/dx = x+y/x, y(2)=2 [hint h = 0.25(2 steps)]

Ans:3.058

UNIT –IV
PARTIAL DIFFERENTIAL EQUATIONS AND CONCEPTS IN SOLUTION TO BOUNDARY VALUE PROBLEMS

Introduction

The concept of a differential equation to include equations that involve partial derivatives, not just ordinary ones. Solutions to such equations will involve functions not just of one variable, but of several variables. Such equations arise naturally, for example, when one is working with situations that involve positions in space that vary over time. To model such a situation, one needs to use functions that have several variables to keep track of the spatial dimensions and an additional variable for time.

Examples of some important PDEs:

(1) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ *One-dimensional wave equation*

(2) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ *One-dimensional heat equation*

(3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ *Two-dimensional Laplace equation*

(4) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ *Two-dimensional Poisson equation*

Partial differential equations: An equation involving partial derivatives of one dependent variable with respective more than one independent variables.

Notations which we use in this unit:

$p = \frac{\partial z}{\partial x}$,$q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $= \frac{\partial^2 z}{\partial x\ \partial y}$, t= $\frac{\partial^2 z}{\partial y^2}$,

Formation of partial differential equation:

A partial differential equation of given curve can be formed in two ways

1. By eliminating arbitrary constants
2. By eliminating arbitrary functions

Problems

1 Form a partial differential equation by eliminating a,b,c from

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Sol Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Differentiating partially w.r.to x and y, we have

$\frac{1}{a^2}(2x)+\frac{1}{c^2}(2z)\frac{\partial z}{\partial x}$=o

$\frac{1}{a^2}(x)+\frac{1}{c^2}(z)\,p$ =o _______(1)

And $\frac{1}{b^2}(2x)$ + $\frac{1}{c^2}(2z)\frac{\partial z}{\partial x}$=o

$\frac{1}{b^2}(y)+\frac{1}{c^2}(z)$ q = o _______(2)

Diff (1) partially w.r.to x, we have

$\frac{1}{a^2}+\frac{p}{c^2}\frac{\partial z}{\partial x}+\frac{z}{c^2}\frac{\partial p}{\partial x}$=o ______(3)

$\frac{1}{a^2}+\frac{p^2}{c^2}+\frac{z}{c^2}r$ =O

Multiply this equation by x and then subtracting (1) from it

$$\frac{1}{c^2}(xzr+xp^2-pz)=0$$

2 **Form a partial differential equation by eliminating the constants from $(x-a)^2+(y-b)^2=z^2cot^2\alpha$, where α is a parameter**

Sol Given $(x-a)^2+(y-b)^2=z^2cot^2$α ___________(1)

Differentiating partially w.r.to x and y, we have

2 $(x-a)$+0 = 2 z p$cot^2\alpha$

$(x-a)$ = Zp$cot^2\alpha$

And 0+2(y-b) = 2zq$cot^2\alpha$

(Y-b) = zq$cot^2\alpha$

Substituting the values of (x-a) and (y-b) in (1),we get

$$(zpcot^2\alpha)^2+(zqcot^2\alpha)^2=z^2cot^2\alpha$$

$$(p^2+q^2)(cot^2\alpha)^2=cot^2\alpha$$

$$p^2+q^2=tan^2\alpha$$

3 **Form the partial differential equation by eliminating a and b from log (az-1)=x+ay+b**

Sol Given equation is

Log (az-1)=x+ay+b

Differentiating partially w.r.t. x and y ,we get

$\frac{1}{az-1}(ap)=1\Rightarrow ap=az-1$ --------------- (1)

$\frac{1}{az-1}(aq)=a\Rightarrow aq=a(az-1)$ ------------- (2)

(2)/(1) gives

$\frac{q}{p}=a$ $or\ ap=q$---------------- (3)

Substituting (3) in (1), we get

$$q = \frac{q}{p}.(z-1)$$

i.e. pq=qz-p

$$p(q+1) = qz$$

4 **Find the differential equation of all spheres whose centers lie on z-axis with a given radius r.**

Sol The equation of the family of spheres having their centers on z-axis and having radius r is

$$x^2 + y^2 + (z-c)^2 = r^2$$

Where c and r are arbitrary constants

Differentiating this eqn partially w.r.t. x and y ,we get

$2x + 2(z-c)\frac{\partial z}{\partial x} = 0 \Rightarrow x + (z-c)p = 0$ _________(1)

$2y + 2(z-c)\frac{\partial z}{\partial y} = 0 \Longrightarrow y + (z-c)q = 0$_________(2)

From (1),$(z-c) = -\frac{x}{p}$ ___________(3)

From (2),$(z-c) = -\frac{y}{q}$ ___________(4)

From (3) and (4)

We get $-\frac{x}{p} = -\frac{y}{q}$

i.e. $xq - yp = 0$

Linear partial differential equations of first order :

Lagrange's linear equation: An equation of the form Pp + Qq = R is called Lagrange's linear equation.

To solve Lagrange's linear equation consider auxiliary equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Non-linear partial differential equations of first order :

Complete Integral : A solution in which the number of arbitrary constants is equal to the number of independent variables is called complete integral or complete solution of the given equation.

Particular Integral: A solution obtained by giving particular values to the arbitrary constants in the complete integral is called a particular integral.

Singular Integral: let f(x,y,z,p,q) = 0 be a partial differential equation whose complete integral is

To solve non-linear pde we use Charpit's Method :

There are six types of non-linear partial differential equations of first order as given below.

1. f (p,q) = 0
2. f (z,p,q) = 0
3. f1 (x,p) = f_2 (y,q)

4. $z = px + qy + f(p,q)$

5. $f(x^m p, y^n q) = 0$ and $f(m^y p, y^n q, z) = 0$

6. $f(pz^m, qz^m) = 0$ and $f_1(x, pz^m) = f_2(y, qz^m)$

Charpit's Method:

We present here a general method for solving non-linear partial differential equations. This is known as Charpit's method.

Let$F(x,y,u, p.q)=0$be a general nonlinear partial differential equation of first-order. Since u depends on x and y, we have

$du=u_x dx+u_y dy = pdx+qdy$ where $p=u_x=\frac{\partial u}{\partial x}$, $q = u_y=\frac{\partial u}{\partial y}$

If we can find another relation between x,y,u,p,q such that $f(x,y,u,p,q)=0$then we can solve for p and q and substitute them in equation This will give the solution provided is integrable.

To determine f, differentiate w.r.t. x and y so that

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial u}p+\frac{\partial F}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial x}=0$$

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}p+\frac{\partial f}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial x}=0$$

$$\frac{\partial F}{\partial y}+\frac{\partial F}{\partial u}q+\frac{\partial F}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial y}=0$$

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial u}q+\frac{\partial f}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial y}=0$$

Eliminating $\frac{\partial p}{\partial x}$ from, equations and $\frac{\partial q}{\partial y}$ from equations we obtain

$$\left(\frac{\partial F}{\partial x}\frac{\partial f}{\partial p}-\frac{\partial f}{\partial x}\frac{\partial F}{\partial p}\right)+\left(\frac{\partial F}{\partial u}\frac{\partial f}{\partial p}-\frac{\partial f}{\partial u}\frac{\partial F}{\partial p}\right)p+\left(\frac{\partial F}{\partial q}\frac{\partial f}{\partial p}-\frac{\partial f}{\partial q}\frac{\partial F}{\partial p}\right)\frac{\partial q}{dx}=0$$

$$\left(\frac{\partial F}{\partial y}\frac{\partial f}{\partial q}-\frac{\partial f}{\partial y}\frac{\partial F}{\partial q}\right)+\left(\frac{\partial F}{\partial u}\frac{\partial f}{\partial q}-\frac{\partial f}{\partial u}\frac{\partial F}{\partial q}\right)q+\left(\frac{\partial F}{\partial p}\frac{\partial f}{\partial q}-\frac{\partial f}{\partial p}\frac{\partial F}{\partial q}\right)\frac{\partial p}{dy}=0$$

Adding these two equations and using

$$\frac{\partial q}{\partial x}=\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial p}{\partial y}$$

and rearranging the terms, we get

$$\left(-\frac{\partial F}{\partial p}\right)\frac{\partial f}{\partial x}+\left(-\frac{\partial F}{\partial q}\right)\frac{\partial f}{\partial y}+\left(-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}\right)\frac{\partial f}{\partial u}+\left(\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial u}\right)\frac{\partial f}{\partial p}$$

$$+\left(\frac{\partial F}{\partial y}+q\frac{\partial f}{\partial u}\right)\frac{\partial f}{\partial q}=0$$

We get the auxiliary system of equations

$$\frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}} = \frac{du}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial u}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial u}} = \frac{df}{0}$$

An Integral of these equations, involving p or q or both, can be taken as the required equation.

Problems

1 **solve** $(x^2 - y^2 - yz)p + (x^2 - y^2 - zx)q = z(x - y)$

Sol Here

$P=(x^2 - y^2 - yz), Q = (x^2 - y^2 - zx), R = z(x - y)$

The subsidiary equations are $\frac{dx}{(x^2-y^2-yz)} = \frac{dy}{(x^2-y^2-zx)} = \frac{dz}{z(x-y)}$

Using 1,-1,0 and x,-y,0 as multipliers , we have

. $\frac{dz}{z(x-y)} = \frac{dx-dy}{z(x-y)} = \frac{x\,dx-ydy}{(x^2-y^2)(x-y)}$

From the first two rations 0f ,we have

dz= dx-dy

integrating , $z=x-y-c_1$ or $x-y-z = c_1$

now taking first and last ratios in (2) ,we get

$$\frac{dz}{z} = \frac{x\,dx - y\,dy}{x^2 - y^2} \quad \text{or} \quad \frac{2dz}{z} = \frac{2x\,dx - 2y\,dy}{x^2 - y^2}$$

Integrating ,2 log z = $\log(x^2 - y^2) - \log c_2$

$$\Rightarrow \frac{x^2 - y^2}{z^2} = c_2$$

The required general solution is $f\left(x - y - z, \frac{x^2-y^2}{z^2}\right) = 0$

2 **solve** $(\mathbf{mz - ny})\mathbf{p} + (\mathbf{nx - lz})\mathbf{q} = \mathbf{ly - mx}$

Sol The equation is

$$(mz - ny)p + (nx - lz)q = ly - mx$$

Here $P= (mz - ny), Q = (nx - lz)$, $R = ly - mx$

The Auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e. $\frac{dx}{(mz-ny)} = \frac{dy}{(nx-lz)} = \frac{dz}{ly-mx}$

Choosing x,y,z as multipliers ,we get

Each fraction $=\frac{x\,dx+ydy+zdz}{0}$, which gives $x\,dx + ydy + zdz = 0$

Integrating,$x^2 + y^2 + z^2 = a$

Again choosing l, m, n as multipliers ,we get

Each fraction $=\frac{l\,dx+mdy+ndz}{0}$, which gives $l\,dx + mdy + ndz = 0$

Integrating, $lx + my + nz = b$

Hence the solution is

$$f(x^2 + y^2 + z^2, lx + my + nz) = 0$$

3 **Solve**$(\mathbf{z^2 - 2yz - y^2})\mathbf{p} + (\mathbf{xy + zx})\mathbf{q} = \mathbf{xy - zx.}$

Sol The subsidiary equations are $\frac{dx}{z^2-2yz-y^2} = \frac{dy}{xy+zx} = \frac{dz}{xy-zx}$

Each fraction $= \frac{xdx+ydy+zdz}{0}$

$$\therefore\ xdx + ydy + zdz = 0$$

Integrating, $x^2 + y^2 + z^2 = a$

Taking second and third terms ,we get $(y-z)dy = (y+z)dz$

i.e. $ydy - zdy - ydz - zdz = 0$

$$ydy - (ydz + zdy) - zdz = 0$$

$$d\left(\frac{y^2}{2}\right) - d(yz) - d\left(\frac{z^2}{2}\right) = 0$$

integrating, $\frac{y^2}{2} - yz - \frac{z^2}{2} = b or y^2 - 2yz - z^2 = b$

Hence the general solution is $\varphi(x^2 + y^2 + z^2, y^2 - 2yz - z^2) = b$

4 Find the integral surface of $\mathbf{x(y^2 + z)p - y(x^2 + z)q = (x^2 + y^2)z}$ Which contains the straight line x+y=0 ,z=1

Sol The subsidiary equations are $\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z}$

Each fraction $= \frac{xdx+ydy+zdz}{0}$

And also $= \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$

$$xdx + ydy + zdz = 0\ and \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating $x^2 + y^2 - 2z = a and xyz = b$

The straight line is x+y =0 , z=1

$$\therefore\ x^2 + y^2 - 2 = a and xy = b$$

Now a + 2b = $x^2 + y^2 - 2 + 2xy = (x+y)^2 - 2 = -2$ $(since x + y = 0)$

a+2b+2=0

Hence the required surface is $x^2 + y^2 - 2z + 2xyz + 2 = 0$

5 Find the general solution of the first-order linear partial differential equation with the constant coefficients: $4u_x+u_y=x^2y$

Sol The auxiliary system of equations is

$$\frac{dx}{4} = \frac{dy}{1} = \frac{du}{x^2y}$$

From here we get

$\frac{dx}{4} = \frac{dy}{1}$ or dx-4dy=0. Integrating both sides

we get x-4y=c. Also $\frac{dx}{4} = \frac{du}{x^2y}$ or x^2y dx=4du

or $x^2(\frac{x-c}{4})\,dx$=4du or

$\frac{1}{16}(x^3 - cx^2)\,dx = du$

Integrating both sides we get

$$u=c_1+\frac{3x^4-4cx^3}{192}$$

$$= f(c)+\frac{3x^4-4cx^3}{192}$$

After replacing c by x-4y, we get the general solution

$$u=f(x-4y)+\frac{3x^4-4(x-4y)x^3}{192}$$

$$=f(x-4y)-\frac{x^4}{192}+\frac{x^3y}{12}$$

6 Find the general solution of the partial differential equation $y^2up + x^2uq = y^2x$

Sol The auxiliary system of equations is

$$\frac{dx}{y^2u}=\frac{dy}{x^2u}=\frac{du}{xy^2}$$

Taking the first two members we have $x^2dx = y^2dy$ which on integration given $x^3-y^3 = c_1$. Again taking the first and third members,

we have x dx = u du

which on integration given $x^2-u^2 = c_2$

Hence, the general solution is

$F(x^3-y^3,x^2-u^2) = 0$

7 Find the general solution of the partial differential equation.

$$\left(\frac{\partial u}{\partial x}\right)^2 x+\left(\frac{\partial u}{\partial y}\right)^2 y-u=0$$

Sol : Let $p=\frac{\partial u}{\partial x}$, $q=\frac{\partial u}{\partial y}$

The auxiliary system of equations is

$$\frac{dx}{2px}=\frac{dy}{2qy}=\frac{du}{2(p^2x+q^2y)}=\frac{dp}{p-p^2}=\frac{dq}{q-q^2}$$

which we obtain from putting values of

$$\frac{\partial F}{\partial p}=2px\ ,\frac{\partial F}{\partial q}=2qy,\frac{\partial F}{\partial x}=p^2,\frac{\partial F}{\partial u}=-1,\frac{\partial F}{\partial y}=q^2$$

and multiplying by -1 throughout the auxiliary system. From first and 4th expression in (11.38) we get

$dx=\frac{p^2dx+2pxdp}{py}$. From second and 5th expression

$$dy=\frac{q^2dy+2qydq}{qy}$$

Using these values of dx and dy we get

$$\frac{p^2dx+2pxdp}{p^2x} = \frac{q^2dy+2qydq}{q^2y}$$

or $\frac{dx}{x}+\frac{2}{p}dp=\frac{dy}{y}+\frac{2dq}{q}$

Taking integral of all terms we get

ln|x| + 2ln|p| = ln|y|+2ln|q|+lnc

or ln|x| p^2 = ln|y|q^2c

or $p^2x=cq^2y$, where c is an arbitrary constant.

Solving for p and q we get $cq^2y+q^2y-u=0$

$(c+1)q^2y=u$

$$q=\left\{\frac{u}{(c+1)y}\right\}^{1/2}$$

$$p=\left\{\frac{cu}{(c+1)x}\right\}^{1/2}$$

$$du=\left\{\frac{cu}{(c+1)x}\right\}^{1/2}dx+\left\{\frac{u}{(c+1)y}\right\}^{1/2}dy$$

or $\left(\frac{1+c}{u}\right)^{1/2}du=\left(\frac{c}{x}\right)^{1/2}dx+\left(\frac{i}{y}\right)^{1/2}dy$

By integrating this equation we obtain $((1+c)u)^{1/2}=(cx)^{1/2}+(y)^{1/2}+c_1$

This is a complete solution.

8 Solve $p^2+q^2=1$

Sol The auxiliary system of equation is

$$-\frac{dx}{-2p}=\frac{dy}{2q}=\frac{du}{-2p^2-2q^2}=\frac{dp}{0}=\frac{dq}{0}$$

or $\frac{dx}{p}=\frac{dy}{q}=\frac{du}{p^2+q^2}=\frac{dp}{0}=\frac{dq}{0}$

Using dp =0, we get p=c and q=$\sqrt{1-c^2}$, and these two combined with du =pdx+qdy yield

u=cx+y$\sqrt{1-c^2}$ + c_1 which is a complete solution.

Using $\frac{dx}{du}$ = p , we get du = $\frac{dx}{c}$ where p= c

Integrating the equation we get u = $\frac{x}{c}$ + c_1

Also du = $\frac{dy}{q}$, where q = $\sqrt{1-p^2}$ = $\sqrt{1-c^2}$

or du = $\frac{dy}{\sqrt{1-c^2}}$. Integrating this equation we get u = $\frac{1}{\sqrt{1-c^2}}$ y +c_2

This cu = x+cc_1 and $u\sqrt{1-c^2}$ = y + $c_2\sqrt{1-c^2}$

Replacing cc_1 and $c_2\sqrt{1-c^2}$ by - α and -β respectively, and eliminating c, we get

$$u^2 = (x-\alpha)^2 + (y-\beta)^2$$

9 Solve u^2+pq – 4 = 0

Sol The auxiliary system of equations is

$$\frac{dx}{q} = \frac{dy}{p} = \frac{du}{2pq} = \frac{dp}{-2up} = \frac{dq}{-2uq}$$

The last two equations yield p = a^2q.

Substituting in u^2+pq – 4 = 0 gives

$q = \pm\frac{1}{a}\sqrt{4-u^2}$ and $p = \pm a\sqrt{4-u^2}$

Then du = pdx+qdy yields

$$du = \pm\sqrt{4-u^2}\left(adx + \frac{1}{a}dy\right)$$

or $$\frac{du}{\sqrt{4-u^2}} = \pm adx + \frac{1}{a}dy$$

Integrating we get $\sin^{-1}\frac{u}{2} = \pm\left(adx + \frac{1}{a}y + c\right)$

or $u = \pm 2\sin\left(ax + \frac{1}{a}y + c\right)$

10 Solve $p^2(1-x^2)-q^2(4-y^2) = 0$

Sol Let $p^2(1-x^2) = q^2(4-y^2) = a^2$

This gives $p = \frac{a}{\sqrt{1-x^2}}$ and $q = \frac{a}{\sqrt{4-y^2}}$

(neglecting the negative sign).

Substituting in du = pdx + q dy we have

$$du = \frac{a}{\sqrt{1-x^2}}dx + \frac{a}{\sqrt{4-y^2}}dy$$

Integration gives $u = a\left(\sin' x + \sin'\frac{y}{2}\right) + c$.

Heat Equation

For this next PDE, we create a mathematical model of how heat spreads, or diffuses through an object, such as a metal rod, or a body of water. To do this we take advantage of our knowledge of vector calculus and the divergence theorem to set up a PDE that models such a situation. Knowledge of this particular PDE can be used to model situations involving many sorts of diffusion processes, not just heat. For instance the PDE that we will derive can be used to model the spread of a drug in an organism, of the diffusion of pollutants in a water supply.

The key to this approach will be the observation that heat tends to flow in the direction of decreasing temperature. The bigger the difference in temperature, the faster the heat flow, or heat loss (remember Newton's heating and cooling differential equation). Thus if you leave a hot drink outside on a freezing cold day, then after ten minutes the drink will be a lot colder than if you'd kept the drink inside in a warm room - this seems pretty obvious!

If the function $u(x, y, z, t)$ gives the temperature at time t at any point (x, y, z) in an object, then in mathematical terms the direction of fastest decreasing temperature away from a specific point (x, y, z), is just the gradient of u (calculated at the point (x, y, z) and a particular time t). Note that here we are considering the gradient of u as just being with respect to the spatial coordinates x, y and z, so that we write

$$(1) \qquad grad(u) = \nabla u = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}$$

Thus the rate at which heat flows away (or toward) the point is proportional to this gradient, so that if **F** is the vector field that gives the velocity of the heat flow, then

$$(2) \qquad \mathbf{F} = -k(grad(u))$$

(negative as the flow is in the direction of fastest *decreasing* temperature).

The constant, k, is called the *thermal conductivity* of the object, and it determines the rate at which heat is passed through the material that the object is made of. Some metals, for instance, conduct heat quite rapidly, and so have high values for k, while other materials act more like insulators, with a much lower value of k as a result.

Now suppose we know the temperature function, $u(x, y, z, t)$, for an object, but just at an initial time, when $t = 0$, i.e. we just know $u(x, y, z, 0)$. Suppose we also know the thermal conductivity of the material. What we would like to do is to figure out how the temperature of the object, $u(x, y, z, t)$, changes over time. The goal is to use the observation about the rate of heat flow to set up a PDE involving the function $u(x, y, z, t)$ (i.e. the Heat Equation), and then solve the PDE to find $u(x, y, z, t)$.

Deriving the Heat Equation

To get to a PDE, the easiest route to take is to invoke something called the Divergence Theorem. As this is a multivariable calculus topic that we haven't even gotten to at this point in the semester, don't worry! (It will be covered in the vector calculus section at the end of the course in Chapter 13 of Stewart). It's such a neat application of the use of the Divergence Theorem, however, that at this point you should just skip to the end of this short section and take it on faith that we will get a PDE in this situation (i.e. skip to equation (10) below. Then be sure to come back and read through this section once you've learned about the divergence theorem.

First notice if *E* is a region in the body of interest (the metal bar, the pool of water, etc.) then the amount of heat that leaves *E* per unit time is simply a surface integral. More exactly, it is the flux integral over the surface of *E* of the heat flow vector field, **F**. Recall that **F** is the vector field that gives the velocity of the heat flow - it's the one we wrote down as $\mathbf{F} = -k\nabla u$ in the previous section. Thus the amount of heat leaving *E* per unit time is just

(1) $$\iint_{S} \mathbf{F}\cdot d\mathbf{S}$$

where*S* is the surface of *E*. But wait, we have the highly convenient divergence theorem that tells us that

(2) $$\iint_{S} \mathbf{F}\cdot d\mathbf{S} = -k\iiint_{E} div(grad(u))dV$$

Okay, now what is *div(grad(u))*? Given that

(3) $$grad(u) = \nabla u = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}$$

then*div(grad(u))* is just equal to

(4) $$div(grad(u)) = \nabla\cdot\left(\nabla u\right) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Incidentally, this combination of divergence and gradient is used so often that it's given a name, the *Laplacian*. The notation $div(grad(u) = \nabla\cdot\left(\nabla u\right)$ is usually shortened up to simply $\nabla^2 u$. So we could rewrite (2), the heat leaving region *E* per unit time as

(5) $$\iint_{S} \mathbf{F}\cdot d\mathbf{S} = -k\iiint_{E} (\nabla^2 u)dV$$

On the other hand, we can calculate the total amount of heat, *H*, in the region, *E*, at a particular time, *t*, by computing the triple integral over *E*:

(6) $$H = \iiint_{E} (\sigma\delta)u(x,y,z,t)dV$$

where δ is the *density* of the material and the constant σ is the *specific heat* of the material (don't worry about all these extra constants for now - we will lump them all together in one place in the end). How does this relate to the earlier integral? On one hand (5) gives the rate of heat leaving E per unit time. This is just the same as $-\frac{\partial H}{\partial t}$, where H gives the total amount of heat in E. This means we actually have two ways to calculate the same thing, because we can calculate $\frac{\partial H}{\partial t}$ by differentiating equation (6) giving H, i.e.

$$(7) \qquad -\frac{\partial H}{\partial t} = -\iiint_E (\sigma\delta)\frac{\partial u}{\partial t}\,dV$$

Now since both (5) and (7) give the rate of heat leaving E per unit time, then these two equations must equal each other, so…

$$(8) \qquad -\frac{\partial H}{\partial t} = -\iiint_E (\sigma\delta)\frac{\partial u}{\partial t}\,dV = -k\iiint_E (\nabla^2 u)dV$$

For these two integrals to be equal means that their two integrands must equal each other (since this integral holds over any arbitrary region E in the object being studied), so…

$$(9) \qquad (\sigma\delta)\frac{\partial u}{\partial t} = k(\nabla^2 u)$$

or, if we let $c^2 = \frac{k}{\sigma\delta}$, and write out the Laplacian, $\nabla^2 u$, then this works out simply as

$$(10) \qquad \frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

This, then, is the PDE that models the diffusion of heat in an object, i.e. the Heat Equation! This particular version (10) is the ***three-dimensional heat equation.***

Solving the Heat Equation in the one-dimensional case

We simplify our heat diffusion modeling by considering the specific case of heat flowing in a long thin bar or wire, where the cross-section is very small, and constant, and insulated in such a way that the heat flow is just along the length of the bar or wire. In this slightly contrived situation, we can model the heat flow by keeping track of the temperature at any point along the bar using just one spatial dimension, measuring the position along the bar.

This means that the function, u, that keeps track of the temperature, just depends on

x, the position along the bar, and t, time, and so the heat equation from the previous section becomes the so-called ***one-dimensional heat equation***:

(1) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

One of the interesting things to note at this point is how similar this PDE appears to the wave equation PDE. However, the resulting solution functions are remarkably different in nature. Remember that the solutions to the wave equation had to do with oscillations, dealing with vibrating strings and all that. Here the solutions to the heat equation deal with temperature flow, not oscillation, so that means the solution functions will likely look quite different. If you're familiar with the solution to Newton's heating and cooling differential equations, then you might expect to see some type of exponential decay function as part of the solution function.

Before we start to solve this equation, let's mention a few more conditions that we will need to know to nail down a specific solution. If the metal bar that we're studying has a specific length, l, then we need to know the temperatures at the ends of the bars. These temperatures will give us boundary conditions similar to the ones we worked with for the wave equation. To make life a bit simpler for us as we solve the heat equation, let's start with the case when the ends of the bar, at $x = 0$ and $x = l$ both have temperature equal to 0 for all time (you can picture this situation as a metal bar with the ends stuck against blocks of ice, or some other cooling apparatus keeping the ends exactly at 0 degrees). Thus we will be working with the same boundary conditions as before, namely

(2) $u(0,t) = 0$ and $u(l,t) = 0$ for all values of t

Finally, to pick out a particular solution, we also need to know the initial starting temperature of the entire bar, namely we need to know the function $u(x,0)$. Interestingly, that's all we would need for an initial condition this time around (recall that to specify a particular solution in the wave equation we needed to know two initial conditions, $u(x,0)$ and $u_t(x,0)$).

The nice thing now is that since we have already solved a PDE, then we can try following the same basic approach as the one we used to solve the last PDE, namely separation of variables. With any luck, we will end up solving this new PDE. So, remembering back to what we did in that case, let's start by writing

(3) $$u(x,t) = F(x)G(t)$$

where*F*, and *G*, are single variable functions. Differentiating this equation for $u(x,t)$ with respect to each variable yields

(4) $$\frac{\partial^2 u}{\partial x^2} = F''(x)G(t) \text{ and } \frac{\partial u}{\partial t} = F(x)G'(t)$$

When we substitute these two equations back into the original heat equation

(5) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

we get

(6) $$\frac{\partial u}{\partial t} = F(x)G'(t) = c^2 \frac{\partial^2 u}{\partial x^2} = c^2 F''(x)G(t)$$

If we now separate the two functions *F* and *G* by dividing through both sides, then we get

(7) $$\frac{G'(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)}$$

Just as before, the left-hand side only depends on the variable *t*, and the right-hand side just depends on *x*. As a result, to have these two be equal can only mean one thing, that they are both equal to the same constant, *k*:

(8) $$\frac{G'(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k$$

As before, let's first take a look at the implications for $F(x)$ as the boundary conditions will again limit the possible solution functions. From (8) we get that $F(x)$ has to satisfy

(9) $$F''(x) - kF(x) = 0$$

Just as before, one can consider the various cases with *k* being positive, zero, or negative. Just as before, to meet the boundary conditions, it turns out that *k* must in fact be negative (otherwise $F(x)$ ends up being identically equal to 0, and we end up with the trivial solution $u(x,t) = 0$). So skipping ahead a bit, let's assume we have figured out that *k* must be negative (you should check the other two cases just as before to see that what we've just written is true!). To indicate this, we write, as before, that $k = -\omega^2$, so that we now need to look for solutions to

(10) $$F''(x) + \omega^2 F(x) = 0$$

These solutions are just the same as before, namely the general solution is:

(11) $F(x) = A\cos(\omega x) + B\sin(\omega x)$

where again *A* and *B* are constants and now we have $\omega = \sqrt{-k}$. Next, let's consider the boundary conditions $u(0,t) = 0$ and $u(l,t) = 0$. These are equivalent to stating that $F(0) = F(l) = 0$. Substituting in 0 for *x* in (11) leads to

(12) $F(0) = A\cos(0) + B\sin(0) = A = 0$

so that $F(x) = B\sin(\omega x)$. Next, consider $F(l) = B\sin(\omega l) = 0$. As before, we check that *B* can't equal 0, otherwise $F(x) = 0$ which would then mean that $u(x,t) = F(x)G(t) = 0 \cdot G(t) = 0$, the trivial solution, again. With $B \neq 0$, then it must be the case that $\sin(\omega l) = 0$ in order to have $B\sin(\omega l) = 0$. Again, the only way that this can happen is for ωl to be a multiple of π . This means that once again

(13) $\omega l = n\pi$ or $\omega = \dfrac{n\pi}{l}$ (where *n* is an integer)

and so

(14) $F(x) = \sin\left(\dfrac{n\pi}{l}x\right)$

where*n* is an integer. Next we solve for $G(t)$, using equation (8) again. So, rewriting (8), we see that this time

(15) $G'(t) + \lambda_n{}^2 G(t) = 0$

where $\lambda_n = \dfrac{cn\pi}{l}$, since we had originally written $k = -\omega^2$, and we just determined that $\omega = \dfrac{n\pi}{l}$ during the solution for $F(x)$. The general solution to this first order differential equation is just

(16) $G(t) = Ce^{-\lambda_n{}^2 t}$

So, now we can put it all together to find out that

(17) $u(x,t) = F(x)G(t) = C\sin\left(\dfrac{n\pi}{l}x\right)e^{-\lambda_n{}^2 t}$

Where*n* is an integer, *C* is an arbitrary constant, and $\lambda_n = \dfrac{cn\pi}{l}$. As is always the case, given a supposed solution to a differential equation, you should check to see that this indeed is a solution to the original heat equation, and that it satisfies the two boundary conditions we started with.

The next question is how to get from the general solution to the heat equation

$$(1) \qquad u(x,t) = C \sin\left(\frac{n\pi}{l} x\right) e^{-\lambda_n{}^2 t}$$

that we found in the last section, to a specific solution for a particular situation. How can one figure out which values of *n* and *C* are needed for a specific problem? The answer lies not in choosing one such solution function, but more typically it requires setting up an infinite series of such solutions. Such an infinite series, because of the principle of superposition, will still be a solution function to the equation, because the original heat equation PDE was linear and homogeneous. Using the superposition principle, and by summing together various solutions with carefully chosen values of *C*, then it is possible to create a specific solution function that will match any (reasonable) given starting temperature function $u(x,0)$.

Wave Equation
For the rest of this introduction to PDEs we will explore PDEs representing some of the basic types of linear second order PDEs: heat conduction and wave propagation. These represent two entirely different physical processes: the process of diffusion, and the process of oscillation, respectively. The field of PDEs is extremely large, and there is still a considerable amount of undiscovered territory in it, but these two basic types of PDEs represent the ones that are in some sense, the best understood and most developed of all of the PDEs. Although there is no one way to solve all PDEs explicitly, the main technique that we will use to solve these various PDEs represents one of the most important techniques used in the field of PDEs, namely separation of variables (which we saw in a different form while studying ODEs). The essential manner of using separation of variables is to try to break up a differential equation involving several partial derivatives into a series of simpler, ordinary differential equations.

We start with the wave equation. This PDE governs a number of similarly related phenomena, all involving oscillations. Situations described by the wave equation include acoustic waves, such as vibrating guitar or violin strings, the vibrations of drums, waves in fluids, as well as waves generated by electromagnetic fields, or any other physical situations involving oscillations, such as vibrating power lines, or even suspension bridges in certain circumstances. In short, this one type of PDE covers a lot of ground.

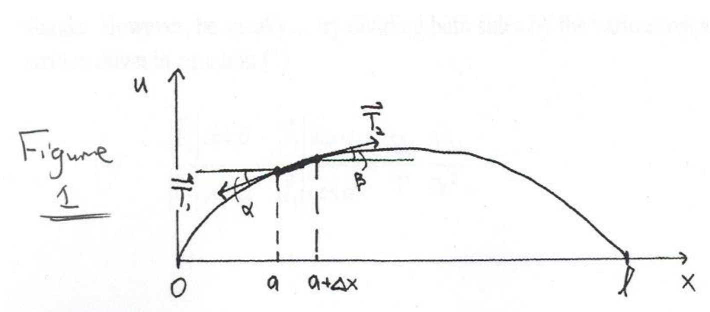
We begin by looking at the simplest example of a wave PDE, the one-dimensional wave equation. To get at this PDE, we show how it arises as we try to model a simple vibrating string, one that is held in place between two secure ends. For instance, consider plucking a guitar string and watching (and listening) as it vibrates. As is typically the case with modeling, reality is quite a bit more complex than we can deal with all at once, and so we need to make some simplifying assumptions in order to get started.

First off, assume that the string is stretched so tightly that the only real force we need to consider is that due to the string's tension. This helps us out as we only have to deal with one force, i.e. we can safely ignore the effects of gravity if the tension force is orders of magnitude greater than that of gravity. Next we assume that the string is as uniform, or homogeneous, as possible, and that it is perfectly elastic. This makes it possible to predict the motion of the string more readily since we don't need to keep track of kinks that might occur if the string wasn't uniform. Finally, we'll assume that the vibrations are pretty minimal in relation to the overall length of the string, i.e. in terms of displacement, the amount that the string bounces up and down is pretty small. The reason this will help us out is that we can concentrate on the simple up and down motion of the string, and not worry about any possible side to side motion that might occur.

Now consider a string of a certain length, l, that's held in place at both ends. First off, what exactly are we trying to do in "modeling the string's vibrations"? What kind of function do we want to solve for to keep track of the motion of string? What will it be a function of? Clearly if the string is vibrating, then its motion changes over time, so *time* is one variable we will want to keep track of. To keep track of the actual motion of the string we will need to have a function that tells us the shape of the string at any particular time. One way we can do this is by looking for a function that tells us the *vertical displacement* (positive up, negative down) that exists at any point along the string – how far away any particular point on the string is from the undisturbed resting position of the string, which is just a straight line. Thus, we would like to find a function $u(x,t)$ of two variables. The variable *x* can measure distance along the string, measured away from one chosen end of the string (i.e. *x* = 0 is one of the tied down endpoints of the string), and *t* stands for time. The function $u(x,t)$ then gives the vertical displacement of the string at any point, *x*, along the string, at any particular time *t*.

As we have seen time and time again in calculus, a good way to start when we would like to study a surface or a curve or arc is to break it up into a series of very small pieces. At the end of our study of one little segment of the vibrating string, we will think about what happens as the length of the little segment goes to zero, similar to the type of limiting process we've seen as we progress from Riemann Sums to integrals.

Suppose we were to examine a very small length of the vibrating string as shown in figure 1:



Now what? How can we figure out what is happening to the vibrating string? Our best hope is to follow the standard path of modeling physical situations by studying all of the forces involved and then turning to Newton's classic equation $F = ma$. It's not a surprise that this will help us, as we have already pointed out that this equation is itself a differential equation (acceleration being the second derivative of position with respect to time). Ultimately, all we will be doing is substituting in the particulars of our situation into this basic differential equation.

Because of our first assumption, there is only one force to keep track of in our situation, that of the string tension. Because of our second assumption, that the string is perfectly elastic with no kinks, we can assume that the force due to the tension of the string is tangential to the ends of the small string segment, and so we need to keep track of the string tension forces T_1 and T_2 at each end of the string segment. Assuming that the string is only vibrating up and down means that the horizontal components of the tension forces on each end of the small segment must perfectly balance each other out. Thus

(1) $$\left|\vec{T}_1\right|\cos\alpha = \left|\vec{T}_2\right|\cos\beta = T$$

where*T* is a string tension constant associated with the particular set-up (depending, for instance, on how tightly strung the guitar string is). Then to keep track of all of the forces involved means just summing up the vertical components of T_1 and T_2. This is equal to

(2) $\left|\vec{T}_2\right| \sin\beta - \left|\vec{T}_1\right| \sin\alpha$

where we keep track of the fact that the forces are in opposite direction in our diagram with the appropriate use of the minus sign. That's it for "Force," now on to "Mass" and "Acceleration." The mass of the string is simple, just $\delta\Delta x$, where δ is the mass per unit length of the string, and Δx is (approximately) the length of the little segment. Acceleration is the second derivative of position with respect to time. Considering that the position of the string segment at a particular time is just $u(x,t)$, the function we're trying to find, then the acceleration for the little segment is $\frac{\partial^2 u}{\partial t^2}$ (computed at some point between *a* and $a + \Delta x$). Putting all of this together, we find that:

$$(3) \qquad \left|\vec{T}_2\right| \sin\beta - \left|\vec{T}_1\right| \sin\alpha = \delta\Delta x \frac{\partial^2 u}{\partial t^2}$$

Now what? It appears that we've got nowhere to go with this – this looks pretty unwieldy as it stands. However, be sneaky… try dividing both sides by the various respective equal parts written down in equation (1):

$$(4) \qquad \frac{\left|\vec{T}_2\right| \sin\beta}{\left|\vec{T}_2\right| \cos\beta} - \frac{\left|\vec{T}_1\right| \sin\alpha}{\left|\vec{T}_1\right| \cos\alpha} = \frac{\delta\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$$

or more simply:

$$(5) \qquad \tan\beta - \tan\alpha = \frac{\delta\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$$

Now, finally, note that $\tan\alpha$ is equal to the slope at the left-hand end of the string segment, which is just $\frac{\partial u}{\partial x}$ evaluated at *a*, i.e. $\frac{\partial u}{\partial x}(a,t)$ and similarly $\tan\beta$ equals $\frac{\partial u}{\partial x}(a+\Delta x,t)$, so (5) becomes…

$$(6) \qquad \frac{\partial u}{\partial x}(a+\Delta x,t) - \frac{\partial u}{\partial x}(a,t) = \frac{\delta\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$$

or better yet, dividing both sides by Δx …

$$(7) \qquad \frac{1}{\Delta x}\left(\frac{\partial u}{\partial x}(a+\Delta x,t) - \frac{\partial u}{\partial x}(a,t)\right) = \frac{\delta}{T}\frac{\partial^2 u}{\partial t^2}$$

Now we're ready for the final push. Let's go back to the original idea – start by breaking up the vibrating string into little segments, examine each such segment using Newton's $F = ma$ equation, and finally figure out what happens as we let the length of the little string segment dwindle to zero, i.e. examine the result as Δx goes to 0. Do you see any limit definitions of derivatives kicking around in equation (7)? As Δx goes to 0, the left-hand side of the equation is in fact just equal to $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$, so the whole thing boils down to:

(8) $$\frac{\partial^2 u}{\partial x^2} = \frac{\delta}{T}\frac{\partial^2 u}{\partial t^2}$$

which is often written as

(9) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

by bringing in a new constant $c^2 = \frac{T}{\delta}$ (typically written with c^2, to show that it's a positive constant).

This equation, which governs the motion of the vibrating string over time, is called the ***one-dimensional wave equation***. It is clearly a second order PDE, and it's linear and homogeneous.

Solution of the Wave Equation by Separation of Variables

There are several approaches to solving the wave equation. The first one we will work with, using a technique called separation of variables, again, demonstrates one of the most widely used solution techniques for PDEs. The idea behind it is to split up the original PDE into a series of simpler ODEs, each of which we should be able to solve readily using tricks already learned. The second technique, which we will see in the next section, uses a transformation trick that also reduces the complexity of the original PDE, but in a very different manner. This second solution is due to Jean Le Rond D'Alembert (an 18^{th} century French mathematician), and is called D'Alembert's solution, as a result.

First, note that for a specific wave equation situation, in addition to the actual PDE, we will also have boundary conditions arising from the fact that the endpoints of the string are attached solidly, at the left end of the string, when $x = 0$ and at the other end of the string, which we suppose has overall length l. Let's start the process of solving the PDE by first figuring out what these boundary conditions imply for the solution function, $u(x,t)$.

Answer: for all values of t, the time variable, it must be the case that the vertical displacement at the endpoints is 0, since they don't move up and down at all, so that

(1) $u(0,t) = 0$ and $u(l,t) = 0$ for all values of t

are the ***boundary conditions*** for our wave equation. These will be key when we later on need to sort through possible solution functions for functions that satisfy our particular vibrating string set-up.

You might also note that we probably need to specify what the shape of the string is right when time $t = 0$, and you're right - to come up with a particular solution function, we would need to know $u(x,0)$. In fact we would also need to know the initial velocity of the string, which is just $u_t(x,0)$. These two requirements are called the ***initial conditions*** for the wave equation, and are also necessary to specify a particular vibrating string solution. For instance, as the simplest example of initial conditions, if no one is plucking the string, and it's perfectly flat to start with, then the initial conditions would just be $u(x,0)=0$ (a perfectly flat string) with initial velocity, $u_t(x,0)=0$. Here, then, the solution function is pretty unenlightening – it's just $u(x,t)=0$, i.e. no movement of the string through time.

To start the separation of variables technique we make the key assumption that whatever the solution function is, that it can be written as the product of two independent functions, each one of which depends on just one of the two variables, *x* or *t*. Thus, imagine that the solution function, $u(x,t)$ can be written as

(2) $$u(x,t) = F(x)G(t)$$

where*F*, and *G*, are single variable functions of *x* and *t* respectively. Differentiating this equation for $u(x,t)$ twice with respect to each variable yields

(3) $$\frac{\partial^2 u}{\partial x^2} = F''(x)G(t) \text{ and } \frac{\partial^2 u}{\partial t^2} = F(x)G''(t)$$

Thus when we substitute these two equations back into the original wave equation, which is

(4) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

then we get

(5) $$\frac{\partial^2 u}{\partial t^2} = F(x)G''(t) = c^2 \frac{\partial^2 u}{\partial x^2} = c^2 F''(x)G(t)$$

Here's where our separation of variables assumption pays off, because now if we separate the equation above so that the terms involving *F* and its second derivative are on one side, and likewise the terms involving *G* and its derivative are on the other, then we get

(6) $$\frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)}$$

Now we have an equality where the left-hand side just depends on the variable *t*, and the right-hand side just depends on *x*. Here comes the critical observation - how can two functions, one just depending on *t*, and one just on *x*, be equal for all possible values of *t* and *x*? The answer is that they must each be constant, for otherwise the equality could not possibly hold for all possible combinations of *t* and *x*. Aha! Thus we have

$$(7) \qquad \frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k$$

where*k* is a constant. First let's examine the possible cases for *k*.

Case One: k = 0

Suppose *k* equals 0. Then the equations in (7) can be rewritten as

$$(8) \qquad G''(t) = 0 \cdot c^2 G(t) = 0 \text{ and } F''(x) = 0 \cdot F(x) = 0$$

yielding with very little effort two solution functions for *F* and *G:*

$$(9) \qquad G(t) = at + b \text{ and } F(x) = px + r$$

where*a*,*b*, *p* and *r*, are constants (note how easy it is to solve such simple ODEs versus trying to deal with two variables at once, hence the power of the separation of variables approach).

Putting these back together to form $u(x,t) = F(x)G(t)$, then the next thing we need to do is to note what the boundary conditions from equation (1) force upon us, namely that

$$(10) \qquad u(0,t) = F(0)G(t) = 0 \text{ and } u(l,t) = F(l)G(t) = 0 \quad \text{for all values of } t$$

Unless $G(t) = 0$ (which would then mean that $u(x,t) = 0$, giving us the very dull solution equivalent to a flat, unplucked string) then this implies that

$$(11) \qquad F(0) = F(l) = 0\,.$$

But how can a linear function have two roots? Only by being identically equal to 0, thus it must be the case that $F(x) = 0$. Sigh, then we still get that $u(x,t) = 0$, and we end up with the dull solution again, the only possible solution if we start with *k* = 0.

So, let's see what happens if…

Case Two: k > 0

So now if *k* is positive, then from equation (7) we again start with

(12) $G''(t) = kc^2 G(t)$

and

(13) $F''(x) = kF(x)$

Try to solve these two ordinary differential equations. You are looking for functions whose second derivatives give back the original function, multiplied by a positive constant. Possible candidate solutions to consider include the exponential and sine and cosine functions. Of course, the sine and cosine functions don't work here, as their second derivatives are *negative* the original function, so we are left with the exponential functions.

Let's take a look at (13) more closely first, as we already know that the boundary conditions imply conditions specifically for $F(x)$, i.e. the conditions in (11). Solutions for $F(x)$ include anything of the form

(14) $F(x) = Ae^{\omega x}$

where $\omega^2 = k$ and *A* is a constant. Since ω could be positive or negative, and since solutions to (13) can be added together to form more solutions (note (13) is an example of a second order linear homogeneous ODE, so that the superposition principle holds), then the general solution for (13) is

(14) $F(x) = Ae^{\omega x} + Be^{-\omega x}$

where now *A* and B are constants and $\omega = \sqrt{k}$. Knowing that $F(0) = F(l) = 0$, then unfortunately the only possible values of A and B that work are $A = B = 0$, i.e. that $F(x) = 0$. Thus, once again we end up with $u(x,t) = F(x)G(t) = 0 \cdot G(t) = 0$, i.e. the dull solution once more. Now we place all of our hope on the third and final possibility for *k*, namely…

Case Three: k < 0

So now we go back to equations (12) and (13) again, but now working with *k* as a negative constant. So, again we have

(12) $G''(t) = kc^2 G(t)$

and

(13) $F''(x) = kF(x)$

Exponential functions won't satisfy these two ODEs, but now the sine and cosine functions will. The general solution function for (13) is now

(15) $F(x) = A\cos(\omega x) + B\sin(\omega x)$

where again *A* and *B* are constants and now we have $\omega^2 = -k$. Again, we consider the boundary conditions that specified that $F(0) = F(l) = 0$. Substituting in 0 for *x* in (15) leads to

$$(16) \qquad F(0) = A\cos(0) + B\sin(0) = A = 0$$

so that $F(x) = B\sin(\omega x)$. Next, consider $F(l) = B\sin(\omega l) = 0$. We can assume that *B* isn't equal to 0, otherwise $F(x) = 0$ which would mean that $u(x,t) = F(x)G(t) = 0 \cdot G(t) = 0$, again, the trivial unplucked string solution. With $B \neq 0$, then it must be the case that $\sin(\omega l) = 0$ in order to have $B\sin(\omega l) = 0$. The only way that this can happen is for ωl to be a multiple of π. This means that

$$(17) \qquad \omega l = n\pi \text{ or } \omega = \frac{n\pi}{l} \quad \text{(where } n \text{ is an integer)}$$

This means that there is an infinite set of solutions to consider (letting the constant *B* be equal to 1 for now), one for each possible integer *n*.

$$(18) \qquad F(x) = \sin\left(\frac{n\pi}{l}x\right)$$

Well, we would be done at this point, except that the solution function $u(x,t) = F(x)G(t)$ and we've neglected to figure out what the other function, $G(t)$, equals. So, we return to the ODE in (12):

$$(12) \qquad G''(t) = kc^2 G(t)$$

where, again, we are working with *k*, a negative number. From the solution for $F(x)$ we have determined that the only possible values that end up leading to non-trivial solutions are with $k = -\omega^2 = -\left(\frac{n\pi}{l}\right)^2$ for*n* some integer. Again, we get an infinite set of solutions for (12) that can be written in the form

$$(19) \qquad G(t) = C\cos(\lambda_n t) + D\sin(\lambda_n t)$$

where*C* and *D* are constants and $\lambda_n = c\sqrt{-k} = c\omega = \frac{cn\pi}{l}$, where *n* is the same integer that showed up in the solution for $F(x)$ in (18) (we're labeling λ with a subscript "*n*" to identify which value of *n* is used).

Now we really are done, for all we have to do is to drop our solutions for $F(x)$ and $G(t)$ into $u(x,t) = F(x)G(t)$, and the result is

(20) $$u_n(x,t) = F(x)G(t) = \left(C\cos(\lambda_n t) + D\sin(\lambda_n t)\right)\sin\left(\frac{n\pi}{l}x\right)$$

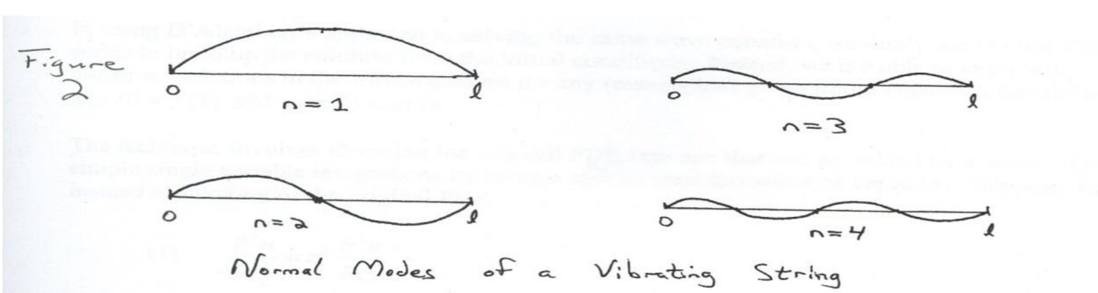
where the integer *n* that was used is identified by the subscript in $u_n(x,t)$ and λ_n, and *C* and *D* are arbitrary constants.

At this point you should be in the habit of immediately checking solutions to differential equations. Is (20) really a solution for the original wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

and does it actually satisfy the boundary conditions $u(0,t) = 0$ and $u(l,t) = 0$ for all values of *t*

The solution given in the last section really does satisfy the one-dimensional wave equation. To think about what the solutions look like, you could graph a particular solution function for varying values of time, *t*, and then examine how the string vibrates over time for solution functions with different values of *n* and constants *C* and *D*. However, as the functions involved are fairly simple, it's possible to make sense of the solution $u_n(x,t)$ functions with just a little more effort.
For instance, over time, we can see that the $G(t) = \left(C\cos(\lambda_n t) + D\sin(\lambda_n t)\right)$ part of the function is periodic with period equal to $\frac{2\pi}{\lambda_n}$. This means that it has a frequency equal to $\frac{\lambda_n}{2\pi}$ cycles per unit time. In music one cycle per second is referred to as one *hertz*. Middle C on a piano is typically 263 hertz (i.e. when someone presses the middle C key, a piano string is struck that vibrates predominantly at 263 cycles per second), and the A above middle C is 440 hertz. The solution function when *n* is chosen to equal 1 is called the ***fundamental mode*** (for a particular length string under a specific tension). The other ***normal modes*** are represented by different values of *n*. For instance one gets the 2nd and 3rd normal modes when *n* is selected to equal 2 and 3, respectively. The fundamental mode, when *n* equals 1 represents the simplest possible oscillation pattern of the string, when the whole string swings back and forth in one wide swing. In this fundamental mode the widest vibration displacement occurs in the center of the string (see the figures below).



Thus suppose a string of length *l*, and string mass per unit length δ, is tightened so that the values of *T*, the string tension, along the other constants make the value of $\lambda_1 = \frac{\sqrt{T}}{2l\sqrt{\delta}}$ equal to 440. Then if the string is made to vibrate by striking or plucking it, then its fundamental (lowest) tone would be the A above middle C.

Now think about how different values of *n* affect the other part of $u_n(x,t) = F(x)G(t)$, namely $F(x) = \sin\left(\frac{n\pi}{l}x\right)$. Since $\sin\left(\frac{n\pi}{l}x\right)$ function vanishes whenever *x* equals a multiple of $\frac{l}{n}$, then selecting different values of *n* higher than 1 has the effect of identifying which parts of the vibrating string do not move. This has the affect musically of producing *overtones*, which are musically pleasing higher tones relative to the fundamental mode tone. For instance picking *n* = 2 produces a vibrating string that appears to have two separate vibrating sections, with the middle of the string standing still. This mode produces a tone exactly an octave above the fundamental mode. Choosing *n* = 3 produces the 3rd normal mode that sounds like an octave and a fifth above the original fundamental mode tone, then 4th normal mode sounds an octave plus a fifth plus a major third, above the fundamental tone, and so on.

It is this series of fundamental mode tones that gives the basis for much of the tonal scale used in Western music, which is based on the premise that the lower the fundamental mode differences, down to octaves and fifths, the more pleasing the relative sounds. Think about that the next time you listen to some Dave Matthews!

Finally note that in real life, any time a guitar or violin string is caused to vibrate, the result is typically a combination of normal modes, so that the vibrating string produces sounds from many different overtones. The particular combination resulting from a particular set-up, the type of string used, the way the string is plucked or bowed, produces the characteristic tonal quality associated with that instrument. The way in which these different modes are combined makes it possible to produce solutions to the wave equation with different initial shapes and initial velocities of the string. This process of combination involves ***Fourier Series*** which will be covered at the end of Math 21b (come back to see it in action!)

Finally, finally, note that the solutions to the wave equations also show up when one considers acoustic waves associated with columns of air vibrating inside pipes, such as in organ pipes, trombones, saxophones or any other wind instruments (including, although you might not have thought of it in this way, your own voice, which basically consists of a vibrating wind-pipe, i.e. your throat!). Thus the same considerations in terms of fundamental tones, overtones and the characteristic tonal quality of an instrument resulting from solutions to the wave equation also occur for any of these instruments as well. So, the wave equation gets around quite a bit musically!

D'Alembert's Solution of the Wave Equation

As was mentioned previously, there is another way to solve the wave equation, found by Jean Le Rond D'Alembert in the 18th century. In the last section on the solution to the wave equation using the separation of variables technique, you probably noticed that although we made use of the boundary conditions in finding the solutions to the PDE, we glossed over the issue of the initial conditions, until the very end when we claimed that one could make use of something called Fourier Series to build up combinations of solutions. If you recall, being given specific initial conditions meant being given both the shape of the string at time t = 0, i.e. the function $u(x,0) = f(x)$, as well as the initial velocity, $u_t(x,0) = g(x)$ (note that these two initial condition functions are functions of x alone, as t is set equal to 0). In the separation of variables solution, we ended up with an infinite set, or family, of solutions, $u_n(x,t)$ that we said could be combined in such a way as to satisfy any reasonable initial conditions.

In using D'Alembert's approach to solving the same wave equation, we don't need to use Fourier series to build up the solution from the initial conditions. Instead, we are able to explicitly construct solutions to the wave equation for any (reasonable) given initial condition functions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$.

The technique involves changing the original PDE into one that can be solved by a series of two simple single variable integrations by using a special transformation of variables. Suppose that instead of thinking of the original PDE

(1) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

in terms of the variables x, and t, we rewrite it to reflect two new variables

(2) $v = x + ct$ and $z = x - ct$

This then means that u, originally a function of x, and t, now becomes a function of v and z, instead. How does this work? Note that we can solve for x and t in (2), so that

(3) $$x = \frac{1}{2}(v+z) \text{ and } t = \frac{1}{2c}(v-z)$$

Now using the chain rule for multivariable functions, you know that

(4) $$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} = c\frac{\partial u}{\partial v} - c\frac{\partial u}{\partial z}$$

since $\frac{\partial v}{\partial t} = c$ and $\frac{\partial z}{\partial t} = -c$, and that similarly

(5) $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial u}{\partial v} + \frac{\partial u}{\partial z}$$

since $\frac{\partial v}{\partial x} = 1$ and $\frac{\partial z}{\partial x} = 1$. Working up to second derivatives, another, more involved application of the chain rule yields that

(6) $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(c\frac{\partial u}{\partial v} - c\frac{\partial u}{\partial z}\right) = c\left(\frac{\partial^2 u}{\partial v^2}\frac{\partial v}{\partial t} + \frac{\partial^2 u}{\partial z \partial v}\frac{\partial z}{\partial t}\right) - c\left(\frac{\partial^2 u}{\partial z^2}\frac{\partial z}{\partial t} + \frac{\partial^2 u}{\partial v \partial z}\frac{\partial v}{\partial t}\right)$$

$$= c^2\left(\frac{\partial^2 u}{\partial v^2} - \frac{\partial^2 u}{\partial z \partial v}\right) + c^2\left(\frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial v \partial z}\right) = c^2\left(\frac{\partial^2 u}{\partial v^2} - 2\frac{\partial^2 u}{\partial z \partial v} + \frac{\partial^2 u}{\partial z^2}\right)$$

Another almost identical computation using the chain rule results in the fact that

(7) $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial z}\right) = \left(\frac{\partial^2 u}{\partial v^2}\frac{\partial v}{\partial x} + \frac{\partial^2 u}{\partial z \partial v}\frac{\partial z}{\partial x}\right) + \left(\frac{\partial^2 u}{\partial z^2}\frac{\partial z}{\partial x} + \frac{\partial^2 u}{\partial v \partial z}\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial^2 u}{\partial v^2} + 2\frac{\partial^2 u}{\partial z \partial v} + \frac{\partial^2 u}{\partial z^2}$$

Now we revisit the original wave equation

(8) $$\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2}$$

and substitute in what we have calculated for $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^2 u}{\partial x^2}$ in terms of $\frac{\partial^2 u}{\partial v^2}$, $\frac{\partial^2 u}{\partial z^2}$ and $\frac{\partial^2 u}{\partial z \partial v}$.
Doing this gives the following equation, ripe with cancellations:

(9) $$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial v^2} - 2\frac{\partial^2 u}{\partial z \partial v} + \frac{\partial^2 u}{\partial z^2}\right) = c^2\,\frac{\partial^2 u}{\partial x^2} = c^2\left(\frac{\partial^2 u}{\partial v^2} + 2\frac{\partial^2 u}{\partial z \partial v} + \frac{\partial^2 u}{\partial z^2}\right)$$

Dividing by c^2 and canceling the terms involving $\frac{\partial^2 u}{\partial v^2}$ and $\frac{\partial^2 u}{\partial z^2}$ reduces this series of equations to

(10) $$-2\frac{\partial^2 u}{\partial z \partial v} = +2\frac{\partial^2 u}{\partial z \partial v}$$

which means that

(11) $$\frac{\partial^2 u}{\partial z \partial v} = 0$$

So what, you might well ask, after all, we still have a second order PDE, and there are still several variables involved. But wait, think about what (11) implies. Picture (11) as it gives you information about the partial derivative of a partial derivative:

(12) $$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial v}\right) = 0$$

In this form, this implies that $\frac{\partial u}{\partial v}$ considered as a function of *z* and *v* is a constant in terms of the variable *z*, so that $\frac{\partial u}{\partial v}$ can only depend on *v*, i.e.

(13) $$\frac{\partial u}{\partial v} = M(v)$$

Now, integrating this equation with respect to *v* yields that

(14) $$u(v,z) = \int M(v)dv$$

This, as an indefinite integral, results in a constant of integration, which in this case is just constant from the standpoint of the variable *v*. Thus, it can be any arbitrary function of *z* alone, so that actually

(15) $$u(v,z) = \int M(v)dv + N(z) = P(v) + N(z)$$

where $P(v)$ is a function of *v* alone, and $N(z)$ is a function of *z* alone, as the notation indicates.

Substituting back the original change of variable equations for *v* and *z* in (2) yields that

(16) $$u(x,t) = P(x+ct) + N(x-ct)$$

where*P* and *N* are arbitrary single variable functions. This is called D'Alembert's solution to the wave equation. Except for the somewhat annoying but easy enough chain rule computations, this was a pretty straightforward solution technique. The reason it worked so well in this case was the fact that the change of variables used in (2) were carefully selected so as to turn the original PDE into one in which the variables basically had no interaction, so that the original second order PDE could be solved by a series of two single variable integrations, which was easy to do.

Check out that D'Alembert's solution really works. According to this solution, you can pick any functions for *P* and *N* such as $P(v) = v^2$ and $N(v) = v + 2$. Then

(17) $$u(x,t) = (x+ct)^2 + (x-ct) + 2 = x^2 + x + ct + c^2t^2 + 2$$

Now check that

(18) $$\frac{\partial^2 u}{\partial t^2} = 2c^2$$

and that

(19) $$\frac{\partial^2 u}{\partial x^2} = 2$$

so that indeed

(20) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

and so this is in fact a solution of the original wave equation.

This same transformation trick can be used to solve a fairly wide range of PDEs. For instance one can solve the equation

(21) $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y^2}$$

by using the transformation of variables

(22) $v = x$ and $z = x + y$

(Try it out! You should get that $u(x, y) = P(x) + N(x + y)$ with arbitrary functions P and N)

Note that in our solution (16) to the wave equation, nothing has been specified about the initial and boundary conditions yet, and we said we would take care of this time around. So now we take a look at what these conditions imply for our choices for the two functions *P* and *N*.

If we were given an initial function $u(x,0) = f(x)$ along with initial velocity function $u_t(x,0) = g(x)$ then we can match up these conditions with our solution by simply substituting in $t = 0$ into (16) and follow along. We start first with a simplified set-up, where we assume that we are given the initial displacement function $u(x,0) = f(x)$, and that the initial velocity function $g(x)$ is equal to 0 (i.e. as if someone stretched the string and simply released it without imparting any extra velocity over the string tension alone).

Now the first initial condition implies that

(23) $$u(x,0) = P(x + c \cdot 0) + N(x - c \cdot 0) = P(x) + N(x) = f(x)$$

We next figure out what choosing the second initial condition implies. By working with an initial condition that $u_t(x,0) = g(x) = 0$, we see that by using the chain rule again on the functions P and N

(24) $u_t(x,0) = \frac{\partial}{\partial t}\left(P(x+ct) + N(x-ct)\right) = cP'(x+ct) - cN'(x-ct)$

(remember that *P* and *N* are just single variable functions, so the derivative indicated is just a simple single variable derivative with respect to their input). Thus in the case where $u_t(x,0) = g(x) = 0$, then

(25) $cP'(x+ct) - cN'(x+ct) = 0$

Dividing out the constant factor *c* and substituting in $t = 0$

(26) $P'(x) = N'(x)$

and so $P(x) + k = N(x)$ for some constant *k*. Combining this with the fact that $P(x) + N(x) = f(x)$, means that $2P(x) + k = f(x)$, so that $P(x) = \left(f(x) - k\right)/2$ and likewise $N(x) = \left(f(x) + k\right)/2$. Combining these leads to the solution

(27) $u(x,t) = P(x+ct) + N(x-ct) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right)$

To make sure that the boundary conditions are met, we need

(28) $u(0,t) = 0$ and $u(l,t) = 0$ for all values of *t*

The first boundary condition implies that

(29) $u(0,t) = \frac{1}{2}\left(f(ct) + f(-ct)\right) = 0$

or

(30) $f(-ct) = -f(ct)$

so that to meet this condition, then the initial condition function *f* must be selected to be an odd function. The second boundary condition that $u(l,t) = 0$ implies

(31) $u(l,t) = \frac{1}{2}\left(f(l+ct) + f(l-ct)\right) = 0$

so that $f(l+ct) = -f(l-ct)$. Next, since we've seen that *f* has to be an odd function, then $-f(l-ct) = f(-l+ct)$. Putting this all together this means that

(32) $f(l+ct) = f(-l+ct)$ for all values of *t*

which means that *f* must have period 2*l*, since the inputs vary by that amount. Remember that this just means the function repeats itself every time 2*l* is added to the input, the same way that the sine and cosine functions have period 2π.

What happens if the initial velocity isn't equal to 0? Thus suppose $u_t(x,0) = g(x) \neq 0$. Tracing through the same types of arguments as the above leads to the solution function

$$(33) \qquad u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

UNIT – V
NUMERIC'S FOR ORDINARY DIFFERENTIAL EQUATIONS AND PARTIAL DIFFERENTIAL EQUATIONS

Introduction:

Second-order partial differential equations (PDEs) may be classified as parabolic, hyperbolic or elliptic.Parabolic and hyperbolic PDEs often model time dependent processes involving initial data.

Partial differential equation:

Partial differential equation, in mathematics, equation relating a function of several variables to its partial derivatives. A partial derivative of a function of several variables expresses how fast the function changes when one of its variables is changed, the others being held constant (compare ordinary differential equation). The partial derivative of a function is again a function, and, if f(x, y) denotes the original function of the variables x and y, the partial derivative with respect to x—i.e., when only x is allowed to vary—is typically written as fx(x, y) or $\partial f/\partial x$. The operation of finding a partial derivative can be applied to a function that is itself a partial derivative of another function to get what is called a second-order partial derivative. For example, taking the partial derivative of fx(x, y) with respect to y produces a new function fxy(x, y), or $\partial^2 f/\partial y\partial x$. The order and degree of partial differential equations are defined the same as for ordinary differential equations.

In general, partial differential equations are difficult to solve, but techniques have been developed for simpler classes of equations called linear, and for classes known loosely as "almost" linear, in which all derivatives of an order higher than one occur to the first power and their coefficients involve only the independent variables.

Many physically important partial differential equations are second-order and linear.

The behaviour of such an equation depends heavily on the coefficients a, b, and c of auxx + buxy + cuyy. They are called elliptic, parabolic, or hyperbolic equations according as $b^2 - 4ac < 0$, $b^2 - 4ac = 0$, or $b^2 - 4ac > 0$, respectively. Thus, the Laplace equation is elliptic, the heat equation is parabolic, and the wave equation is hyperbolic.

Elliptic equation:

Elliptic equation, any of a class of partial differential equations describing phenomena that do not change from moment to moment, as when a flow of heat or fluid takes place within a medium with no accumulations. The Laplace equation, uxx + uyy = 0, is the simplest such equation describing this condition in two dimensions. In addition to satisfying a differential equation within the region, the elliptic equation is also determined by its values (boundary values) along the boundary of the region, which represent the effect from outside the region. These conditions can be either those of a fixed temperature distribution at points of the boundary (Dirichlet problem) or those in which heat is being supplied or removed across the boundary in such a way as to maintain a constant temperature distribution throughout (Neumann problem).

If the highest-order terms of a second-order partial differential equation with constant coefficients are linear and if the coefficients a, b, c of the uxx, uxy, uyy terms satisfy the inequality $b^2 - 4ac < 0$, then, by a change of coordinates, the principal part (highest-order terms) can be written as the Laplacian uxx + uyy. Because the properties of a physical system are independent of the coordinate system used to formulate the problem, it is expected that the properties of the solutions of these elliptic equations should be similar to the properties of the solutions of Laplace's equation (see harmonic function). If the coefficients a, b, and c are not constant but depend on x and y, then the equation is called elliptic in a given region if $b^2 - 4ac < 0$ at all points in the region. The functions $x^2 - y^2$ and excos y satisfy the Laplace equation, but the solutions to this equation are usually more complicated because of the boundary conditions that must be satisfied as well.

Parabolic equation:

Parabolic equation, any of a class of partial differential equations arising in the mathematical analysis of diffusion phenomena, as in the heating of a slab. The simplest such equation in one dimension, uxx = ut, governs the temperature distribution at the various points along a thin rod from moment to moment. The solutions to even this simple problem are complicated, but they are constructed largely from a function called the fundamental solution of the equation, given by an exponential function, exp $[(-x^2/4t)/t^{1/2}]$. To determine the complete solution to this type of problem, the initial temperature distribution along the rod and the manner in which the temperature at the ends of the rod is changing must also be known. These additional conditions are called initial values and boundary values, respectively, and together are sometimes called auxiliary conditions.

In the analogous two- and three-dimensional problems, the initial temperature distribution throughout the region must be known, as well as the temperature distribution along the boundary from moment to moment. The differential equation in two dimensions is, in the simplest case,uxx + uyy = ut,with an additional uzz term added for the three-dimensional case. These equations are appropriate only if the medium is of uniform composition throughout, while, for problems of nonuniform composition or for some other diffusion-type problems, more complicated equations may arise. These equations are also called parabolic in the given region if they can be written in the simpler form described above by using a different coordinate system. An equation in one dimension the higher-order terms of which areauxx + buxt + cuttcan be so transformed if $b^2 - 4ac = 0$. If the coefficients a, b, c depend on the values of x, the equation will be parabolic in a region if $b^2 - 4ac = 0$ at each point of the region.

Boundary value:

Boundary value, condition accompanying a differential equation in the solution of physical problems. In mathematical problems arising from physical situations, there are two considerations involved when finding a solution: (1) the solution and its derivatives must satisfy a differential equation, which describes how the quantity behaves within the region; and (2) the solution and its derivatives must satisfy other auxiliary conditions either describing the influence from outside the region (boundary values) or giving information about the solution at a specified time (initial values), representing a compressed history of the system as it affects its future behaviour. A simple example of a boundary-value problem may be demonstrated by the assumption that a function satisfies the equation $f'(x) = 2x$ for any x between 0 and 1 and that it is known that the function has the boundary value of 2 when x = 1. The function f(x)

= x^2 satisfies the differential equation but not the boundary condition. The function $f(x) = x^2 + 1$, on the other hand, satisfies both the differential equation and the boundary condition. The solutions of differential equations involve unspecified constants, or functions in the case of several variables, which are determined by the auxiliary conditions.

The relationship between physics and mathematics is important here, because it is not always possible for a solution of a differential equation to satisfy arbitrarily chosen conditions; but if the problem represents an actual physical situation, it is usually possible to prove that a solution exists, even if it cannot be explicitly found. For partial differential equations, there are three general classes of auxiliary conditions: (1) initial-value problems, as when the initial position and velocity of a traveling wave are known, (2) boundary-value problems, representing conditions on the boundary that do not change from moment to moment, and (3) initial- and boundary-value problems, in which the initial conditions and the successive values on the boundary of the region must be known to find a solution. See also Sturm-Liouville problem.

Hyperbolic functions:

Hyperbolic functions, also called hyperbolic trigonometric functions, the hyperbolic sine of z (written sinh z); the hyperbolic cosine of z (cosh z); the hyperbolic tangent of z (tanh z); and the hyperbolic cosecant, secant, and cotangent of z. These functions are most conveniently defined in terms of the exponential function, with sinh z = $^1/_2$(ez − e$^-$z) and cosh z = $^1/_2$(ez + e$^-$z) and with the other hyperbolic trigonometric functions defined in a manner analogous to ordinary trigonometry.

Just as the ordinary sine and cosine functions trace (or parameterize) a circle, so the sinh and cosh parameterize a hyperbola—hence the hyperbolic appellation. Hyperbolic functions also satisfy identities analogous to those of the ordinary trigonometric functions and have important physical applications. For example, the hyperbolic cosine function may be used to describe the shape of the curve formed by a high-voltage line suspended between two towers (see catenary). Hyperbolic functions may also be used to define a measure of distance in certain kinds of non-Euclidean geometry.

Classifications of Partial Differential Equations:

The most general form of linear second-order partial differential equations, when restricted to two independent variables and constant coefficients, is auxx + buxy + cuyy + dux + euy + fu = g(x, y), (1.25) where g is a known forcing function; a, b, c, . . . , are given constants, and subscripts denote partial differentiation. In the homogeneous case, i.e., g ≡ 0, this form is reminiscent of the general quadratic form from high school analytic geometry: ax2 + bxy + cy2 + dx + ey + f = 0 . (1.26) Equation (1.26) is said to be an ellipse, a parabola or a hyperbola according as the discriminant b 2 − 4ac is less than, equal to, or greater than zero. This same classification—elliptic, parabolic, or hyperbolic—is employed for the PDE (1.25), independent of the nature of g(x, y). In fact, it is clear that the classification of linear PDEs depends only on the coefficients of the highest-order derivatives. This grouping of terms, auxx + buxy + cuyy , is called the principal part of the differential operator in (1.25), i.e., the collection of highest-order derivative terms with respect to each independent variable; and this notion can be extended in a natural way to more complicated operators. Thus, the type of a linear equation is completely determined by its principal part. 1.2. It should be mentioned that if the coefficients of (1.25) are permitted to vary with x and y, its type may change from point to point within the solution domain. This can pose significant difficulties, both analytical and numerical. We will not specifically deal with these in the current lectures. We next note that corresponding to each of the three types of equations there is a unique canonical form to which (1.25) can always be reduced. We shall not present the details of the transformations needed to achieve these reductions, as they can be found in many standard texts on elementary PDEs (e.g., Berg and MacGregor [5]). On the other hand, it is important to be aware of the possibility of simplifying (1.25), since this may also simplify the numerical analysis required to construct a solution algorithm. Elliptic. It can be shown when $b\ 2 - 4ac < 0$, the elliptic case, that (1.25) collapses to the form $uxx + uyy + Au = g(x, y)$, (1.27) with $A = 0, \pm 1$. When $A = 0$ we obtain Poisson's equation, or Laplace's equation in the case $g \equiv 0$; otherwise, the result is usually termed the Helmholtz equation. Parabolic. For the parabolic case, $b\ 2 - 4ac = 0$, we have $ux - uyy = g(x, y)$, (1.28) which is the heat equation, or the diffusion equation. We remark that $b\ 2 - 4ac = 0$ can also imply a "degenerate" form which is only an ordinary differential equation (ODE). We will not treat this case in the present lectures. Hyperbolic. For the hyperbolic case, $b\ 2 - 4ac > 0$, Eq. (1.25) can always be transformed to $uxx - uyy + Bu = g(x, y)$, (1.29) where $B = 0$ or 1. If $B = 0$, we have the wave equation, and when $B = 1$ we obtain the linear Klein–Gordon equation. Finally, we note that determination of equation type in dimensions greater than two requires a different approach. The details are rather technical but basically involve the fact that elliptic and hyperbolic operators have definitions that are independent of dimension, and usual parabolic operators can then be identified as a combination of an elliptic "spatial" operator and a first-order evolution operator.

Gridding methods:

As implied above, there are two main types of gridding techniques in wide use, corresponding to structured and unstructured gridding—with many different variations available, especially for the former of these. Here, we will briefly outline some of the general features of these approaches, and leave details.

Structured Grids. Use of structure grids involves labeling of grid points in such a way that if the indices of any one point are known, then the indices of all points within the grid can be easily determined. For many years this was the preferred (in fact, essentially only) approach utilized. It leads to very efficient, readily parallelized numerical algorithms and straightforward post processing. But generation of structured grids for complicated problem domains, as arise in many engineering applications, is very time consuming in terms of human time—and thus, very expensive.

Unstructured Grids. Human time required for grid generation has been dramatically reduced with use of unstructured grids, but this represents their only advantage. Such grids produce solutions that arefar less accurate, and the required solution algorithms are less efficient, than is true for a structured grid applied to the same problem. This arises from the fact that the grid points comprising an unstructured grid can be ordered in many different ways, and knowing indexing for any one point provides no information regarding the indexing of any other points—often, even of nearest neighbors. In particular, indexing of points is handled via "pointers" which are vectors of indices. Each point possesses a unique pointer, but there is no implied canonical ordering of these pointers. This leads to numerous difficulties in essentially all aspects of the solution process.

The interested reader is referred to Thompson for comprehensive treatments of both structured and unstructured gridding techniques, generally referred to as “grid generation.”