MATHEMATICAL METHODS IN ENGINEERING

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MODULE– I

INTRODUCTION TO PROBABILITY

Probability (Mathematical Definition)

Definition: If a trial results in n-exhaustive mutually exclusive, and equally likely cases and m of them are favorable to the happening of an event E then the probability of an event E is denoted by P(E) and is defined as

$$P(E) = \frac{no\,of\;\; favourable cases\, to\, event}{Total\, no\, of\;\; exaustive cases} = \frac{m}{n}$$

Statistical or Empirical Probability:

If a trial is repeated a no. of times under essential homogenous and identical conditions, then the limiting value of the ratio of the no. of times the event happens to the total no. of trials, as the number of trials become indefinitely large, is called the probability of happening of the event.(It is assumed the limit is finite and unique)

Random Variables

- A random variable X on a sample space S is a function $X : S \to R$ from S onto the set of real numbers R, which assigns a real number X (s) to each sample point 's' of S.
- Random variables (r.v.) bare denoted by the capital letters X,Y,Z,etc..
- Random variable is a single valued function.
- Sum, difference, product of two random variables is also a random variable .Finite linear combination of r.v is also a r.v .Scalar multiple of a random variable is also random variable.

- A random variable, which takes at most a countable number of values, it is called a discrete r.v. In other words, a real valued function defined on a discrete sample space is called discrete r.v.
- A random variable X is said to be continuous if it can take all possible values between certain limits .In other words, a r.v is said to be continuous when it's different values cannot be put in 1-1 correspondence with a set of positive integers.

- A continuous r.v is a r.v that can be measured to any desired degree of accuracy. Ex : age , height, weight etc..
- Discrete Probability distribution: Each event in a sample has a certain probability of occurrence . A formula representing all these probabilities which a discrete r.v. assumes is known as the discrete probability distribution.

- The probability function or probability mass function (p.m.f) of a discrete random variable X is the function f(x) satisfying the following conditions.
  - i) $f(x) \geq 0$
  - ii) $\sum_{x} f(x) = 1$
  - iii) $P(X = x) = f(x)$
- Cumulative distribution or simply distribution of a discrete r.v. X is F(x) defined by $F(x) = P(X \leq x) = \sum_{t \leq x} f(t)$ for $-\infty < x < \infty$

- For a continuous r.v. X, the function f(x) satisfying the following is known as the probability density function(p.d.f.) or simply density function:

   i) $f(x) \geq 0$,$-\infty < x < \infty$

   ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

   iii) $P(a<X<b) = \int_{a}^{b} f(x)dx$ = Area under f(x) between ordinates x=a and x=b

- Cumulative distribution for a continuous r.v. X with p.d.f. f(x), the cumulative distribution F(x) is defined as

$$F(x) = P(X \leq x) = \int_{-\infty}^{\infty} f(t)\,dt \quad -\infty < x < \infty$$

It follows that $F(-\infty) = 0$, $F(\infty)=1$, $0 \leq F(x) \leq 1$

for $-\infty < x < \infty$

$f(x) = d/dx(F(x)) = F^1(x) \geq 0$ and $P(a < x < b) = F(b) - F(a)$

- In case of discrete r.v. the probability at a point i.e., P(x=c) is not zero for some fixed c however in case of continuous random variables the probability at appoint is always zero. I.e., P(x=c) = 0 for all possible values of c.

P(E) = 0 does not imply that the event E is null or impossible event.

- If X and Y are two discrete random variables the joint probability function of X and Y is given by $P(X=x, Y=y) = f(x,y)$ and satisfies

(i) $f(x,y) \geq 0$ (ii) $\sum_{x}\sum_{y} f(x,y) = 1$

The joint probability function for X and Y can be reperesented by a joint probability table.

Table

X \ Y	y_1	y_2	……	y_n	Totals
x_1	$f(x_1,y_1)$	$f(x_1,y_2)$	……..	$f(x_1,y_n)$	$f_1(x_1) = P(X=x_1)$

x_2	$F(x_2,y_1)$	$f(x_2,y_2)$	……..	$f(x_2,y_n)$	$f_1(x_2)$ $=P(X=x_2)$
……..	…….	………	………	………	……..
x_m	$f(x_m,y_1)$	$f(x_m,y_2)$	…….	$f(x_m,y_n)$	$f_1(x_m)$ $=P(X=x_m)$
Totals	$f_2(y_1)$ $=P(Y=y_1)$	$f_2(y_2)$ $=P(Y=y_2)$	……..	$f_2(y_n)$ $=P(Y=y_n)$	1

The probability of$X = x_j$ is obtained by adding all entries in arrow corresponding to $X = x_j$

Similarly the probability of $Y = y_k$ is obtained by all entries in the column corresponding to $Y = y_k$

$f_1(x)$ and $f_2(y)$ are called marginal probability functions of X and Y respectively.

The joint distribution function of X and Y is defined by $F(x,y) = P(X \le x, Y \le y) = \sum_{u \le x} \sum_{v \le y} f(u,v)$

The probability of$X = x_j$ is obtained by adding all entries in arrow corresponding to $X = x_j$

Similarly the probability of $Y = y_k$ is obtained by all entries in the column corresponding to $Y = y_k$

$f_1(x)$ and $f_2(y)$ are called marginal probability functions of X and Y respectively.

- Two discrete random variables X and Y are independent iff

$$P(X = x, Y = y) = P(X = x)P(Y = y) \ \forall\ x,y \quad \text{(or)}$$

$$f(x,y) = f_1(x)f_2(y) \quad \forall\ x, y$$

- Two continuous random variables X and Y are independent iff

$$P(X \le x, Y \le y) = P(X \le x)P(Y \le y) \ \forall\ x,y$$

(or)

$$f(x,y) = f_1(x)f_2(y) \quad \forall\ x, y$$

If X and Y are two discrete r.v. with joint probability function f(x,y) then

$$P(Y = y|X=x) = \frac{f(x,y)}{f_1(x)} = f(y|x)$$

Similarly, $$P(X = x|Y=y) = \frac{f(x,y)}{f_2(y)} = f(x|y)$$

- Median is the point, which divides the entire distribution into two equal parts. In case of continuous distribution median is the point, which divides the total area into two equal parts. Thus, if M is the median then $\int_{-\infty}^{M} f(x)dx$ = $\int_{M}^{\infty} f(x)dx$=1/2. Thus, solving any one of the equations for M we get the value of median. Median is unique

Mode: Mode is the value for $f(x)$ or $P(x_i)$ at attains its maximum

For continuous r.v. X mode is the solution of $f^1(x) = 0$ and $f^{11}(x) < 0$ provided it lies in the given interval. Mode may or may not be unique.

- Variance: Variance characterizes the variability in the distributions with same mean can still have different dispersion of data about their means

Variance of r.v. X denoted by Var(X) and is defined as

$$\mathrm{Var}(X) = E\left[(X - \mu)^2 \right] = \begin{cases} \sum_x (x-\mu)^2 f(x) & \text{for discrete} \\ \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx & \text{for continuous} \end{cases}$$

where $\mu = E(X)$

- If c is any constant then $E(cX) = c\,E(X)$
- If X and Y are two r.v.'s then $E(X+Y) = E(X)+E(Y)$
- IF X,Y are two independent r.v.'s then $E(XY) = E(X)E(Y)$

- If $X_1, X_2, \text{-------}, X_n$ are random variables then $E(c_1X_1 + c_2X_2 + \text{------} + c_nX_n) = c_1E(X_1) + c_2E(X_2) + \text{-----} + c_nE(X_n)$ for any scalars $c_1, c_2, \text{------}, c_n$ If all expectations exists
- If $X_1, X_2, \text{-------}, X_n$ are independent r.v's then $E\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} E(X_i)$ if all expectations exists.
- $Var(X) = E(X^2) - [E(X)]^2$
- If 'c' is any constant then $var(cX) = c^2 var(X)$
- The quantity $E[(X-a)^2]$ is minimum when $a = \mu = E(X)$
- If X and Y are independent r.v.'s then $Var(X \pm Y) = Var(X) \pm Var(Y)$

1:A random variable x has the following probability function:

x	0	1	3	4	5	6	7
P(x)	0	k	2k	2k	3k	k^2	$7k^2$+k

Find (i) k (ii) P(x<6) (iii) P(x>6)

Solution:since the total probability is unity, we have $\sum_{x=0}^{n} p(x)=1$

i.e., 0 + k +2k+ 2k+ 3k+ $k^2 +$ $7k^2$+k=1

i.e., $8k^2 +$ 9k $-1=0$

$k=1,-1/8$

$P(x<6) = 0 + k + 2k + 2k + 3k$

$= 1+2+2+3 = 8$

iii) $P(x>6) = k^2 + 7k^2 + k$

$= 9$

2. Let X denotes the minimum of the two numbers that appear when a pair of fair dice is thrown once. Determine (i) Discrete probability distribution (ii) Expectation (iii) Variance

Solution:

When two dice are thrown, total number of outcomes is 6x6=36

In this case, sample space S=
$$\begin{Bmatrix} (1,1)(1,2)(1,3)(1,4)(1,5)(1,6) \\ (2,1)(2,2)(2,3)(2,4)(2,5)(2,6) \\ (3,1)(3,2)(3,3)(3,4)(3,5)(3,6) \\ (4,1)(4,2)(4,3)(4,4)(4,5)(4,6) \\ (5,1)(5,2)(5,3)(5,4)(5,5)(5,6) \\ (6,1)(6,2)(6,3)(6,4)(6,5)(6,6) \end{Bmatrix}$$

If the random variable X assigns the minimum of its number in S, then the sample space S=

$$\left\{\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}\right\}$$

The minimum number could be 1,2,3,4,5,6

For minimum 1, the favorable cases are 11

Therefore, $P(x=1)=11/36$

$P(x=2)=9/36$, $P(x=3)=7/36$, $P(x=4)=5/36$, $P(x=5)=3/36$, $P(x=6)=1/36$

The probability distribution is

X	1	2	3	4	5	6
P(x)	11/36	9/36	7/36	5/36	3/36	1/36

(i)Expectation mean =$\sum p_i x_i$

$$E(x)=1\frac{11}{36}+2\frac{9}{36}+3\frac{7}{36}+4\frac{5}{36}+5\frac{3}{36}+6\frac{1}{36}$$

Or $\mu=\frac{1}{36}\left[11+8+21+20+15+6\right]=\frac{9}{36}=2.5278$

ii) variance = $\sum p_i x^2 i - \mu^2$

$$E(x) = \frac{11}{36}1 + \frac{9}{36}4 + \frac{7}{36}9 + \frac{5}{36}16 + \frac{3}{36}25 + \frac{1}{36}36 - (2.52$$

=1.9713

A continuous random variable has the probability density function

$$f(x)=\begin{cases} kxe^{-\lambda x},\ for\ x\ge 0, \lambda>0 \\ 0, otherwise \end{cases}$$

Determine (i) k (ii) Mean (iii) Variance

Solution:

(i) since the total probability is unity, we have $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{0} 0dx + \int_{0}^{\infty} kxe^{-\lambda x}dx = 1$$

i.e., $\int_{0}^{\infty} kxe^{-\lambda x}dx = 1$

$$k\left[x\left(\frac{e^{-\lambda x}}{-\lambda}\right) - 1\left(\frac{e^{-\lambda x}}{\lambda^2}\right)\right]_0^{\infty} or\, k=\lambda^2$$

(i) mean of the distribution $\mu = \int_{-\infty}^{\infty} x f(x) dx$

$$\int_{-\infty}^{0} 0 dx + \int_{0}^{\infty} k x^2 e^{-\lambda x} dx$$

$$\lambda^2 \left[x^2 \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 2x \left(\frac{e^{-\lambda x}}{\lambda^2} \right) + 2 \left(\frac{e^{-\lambda x}}{\lambda^3} \right) \right]_0^{\infty}$$

$$= \frac{2}{\lambda}$$

Variance of the distribution

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$$

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x) dx - \frac{4}{\lambda^2}$$

$$= \lambda^2 \left[x^3 \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 3x^2 \left(\frac{e^{-\lambda x}}{\lambda^2} \right) + 6x \left(\frac{e^{-\lambda x}}{\lambda^3} \right) - 6 \left(\frac{e^{-\lambda x}}{\lambda^4} \right) \right]_0^{\propto}$$

$$= \frac{2}{\lambda^2}$$

Out of 800 families with 5 children each, how many would you expect to have (i)3 boys (ii)5girls (iii)either 2 or 3 boys ? Assume equal probabilities for boys and girls

Solution(i)

P(3boys)=P(r=3)=P(3)=$\frac{1}{2^5}{}^5C_3=\frac{5}{16}$per family

Thus for 800 families the probability of number of families having 3 boys=$\frac{5}{16}(800)=250$families

P(5 girls)=P(no boys)=P(r=0)= $\frac{1}{2^5}{}^5C_0=\frac{1}{32}$ per

Family Thus for 800 families the probability of

number

of families having 5girls= $\frac{1}{32}(800)=25$ families

P(either 2 or 3 boys =P(r=2)+P(r=3)=P(2)+P(3)

$\frac{1}{2^5}{}^5C_2+\frac{1}{2^5}{}^5C_3$ =5/8 per family

Expected number of families with 2 or 3 boys = $\frac{5}{8}(800)=500$ families.

4. In 1000 sets of trials per an event of small probability the frequencies f of the number of x of successes are

X	0	1	2	3	4	5	6	7	Total
P(X)	305	365	210	80	28	9	2	1	1000

of 10 randomly chosen tape recorders. Find (i) $P(X=0)$ (ii) $P(X=1)$ (iii) $P(X=2)$ (iv) $P\ (1<X<4)$.

Bionomial Distribution

- A random variable X is said to follow binomial distribution if it assumes only non-negative values and its probability mass function is given by

$$P(X = x) = \begin{cases} P(x) = \binom{n}{x} p^x q^{n-x} & \text{where } x = 0,1,2,3,\ldots.n \quad q = 1\text{-}p \\ 0 & \text{other wise} \end{cases}$$

where n, p are known as parameters, n- number of independent trials p- probability of success in each trial, q- probability of failure.

Mode of the Binomial distribution: Mode of B.D. Depending upon the values of (n+1)p

(i) If (n+1)p is not an integer then there exists a unique modal value for binomial distribution and it is 'm'= integral part of (n+1)p

(ii) If (n+1)p is an integer say m then the distribution is Bi-Modal and the two modal values are m and m-1

Poisson distribution

Poisson distribution:

- Poisson Distribution is a limiting case of the Binomial distribution under the following conditions:

(i) n, the number of trials is infinitely large.

(ii) P, the constant probability of success for each trial is indefinitely small.

(iii) np= λ, is finite where λ is a positive real number.

- A random variable X is said to follow a Poisson distribution if it assumes
- only non-negative values and its p.m.f. is given by

$$P(x,\lambda) = \begin{cases} P(X = x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!} & : \quad x = 0,1,2,3,\ldots\ldots \ \lambda > 0 \\ 0 & \text{Other wise} \end{cases}$$

Here λ is known as the parameter of the distribution.

- We shall use the notation X~ P(λ) to denote that X is a Poisson variate with parameter λ
- Mean and variance of Poisson distribution are equal to λ.
- The coefficient of skewness and kurtosis of the poisson distribution are $\gamma_1 = \sqrt{\beta_1} = 1/\sqrt{\lambda}$ and $\gamma_2 = \beta_2 - 3 = 1/\lambda$. Hence the poisson distribution is always a skewed distribution. Proceeding to limit as λ tends to infinity we get $\beta_1 = 0$ and $\beta_2 = 3$

Mode of Poisson Distribution: Mode of P.D. Depending upon the value of λ

(i) when λ is not an integer the distribution is uni- modal and integral part of λ is the unique modal value.

(ii) When $\lambda = k$ is an integer the distribution is bi-modal and the two modals are k-1 and k.

(iii) Sum of independent poisson variates is also poisson variate.

(iii) The difference of two independent poisson variates is not a poisson
(iv) variate.

- **Moment generating function of the P.D.**

  If X~ P(λ) then $M_X(t)$ = $e^{\lambda(e^t-1)}$

- Recurrence formula for the probabilities of P.D. (Fitting of P.D.)

  P(x+1) = $\frac{\lambda}{x+1}p(x)$

- Recurrence relation for the probabilities of B.D. (Fitting of B.D.)

  P(x+1) = $\left\{\frac{n-x}{x+1}.\frac{p}{q}\right\}p(x)$

1. Average number of accidents on any day on a national highway is 1.8. Determine th probability that the number of accidents is (i) at least one (ii) at most one

Solution:

Mean= $\lambda = 1.8$

We have P(X=x)=p(x)$\frac{e^{-\lambda}\lambda^{x}}{x!}$ =$\frac{e^{-1.8}1.8^{x}}{x!}$

i) P (at least one) $=P(x \geq 1)=1-P(x=0)$

$=1-0.1653$

$=0.8347$

ii) P (at most one) $=P(x \leq 1)$

$=P(x=0)+P(x=1)$

$= 0.4628$

Exercise problems:

2.If a Poisson distribution is such that $P(X=1)=\frac{3}{2}P(X=3)$ then find

(i) $P(X \geq 1)$

(ii) $P(X \leq 3)$ (iii) $P(2 \leq X \leq 5)$.

Normal Distribution

- A random variable X is said to have a normal distribution with parameters μ called mean and σ^2 called variance if its density function is given by the probability law

$$f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-1}{2}\left\{\frac{x-\mu}{\sigma}\right\}^2\right], \quad -\infty < x < \infty,\ -\infty < \mu < \infty,\ \sigma > 0$$

- A r.v. X with mean μ and variance σ^2 follows the normal distribution is denoted by

  X~ N(μ, σ^2)

- If X~ N(μ, σ^2) then Z = $\frac{X-\mu}{\sigma}$ is a standard normal variate with E(Z) = 0 and var(Z)=0 and we write Z~ N(0,1)

- The p.d.f. of standard normal variate Z is given by f(Z) = $\dfrac{1}{\sqrt{2\pi}} e^{-z^2/2}$,
- -∞ < Z< ∞
- The distribution function F(Z) = P(Z ≤ z) = $\dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$
- F(-z) = 1 – F(z)
- P(a < z ≤ b) = P(a ≤ z < b)= P(a <z < b)= P(a ≤ z ≤ b)= F(b) – F(a)

- If X~ N(μ, σ^2) then Z = $\frac{X-\mu}{\sigma}$ then P(a $\leq$ X $\leq$ b) = $F\left(\frac{b-\mu}{\sigma}\right) - F\left(\frac{a-\mu}{\sigma}\right)$
- N.D. is another limiting form of the B.D. under the following conditions:

  i) n , the number of trials is infinitely large.

  ii) Neither p nor q is very small

i) The maximum probability occurring at the point x= μ and is given by

$[P(x)]_{max} = 1/\sigma\sqrt{2\Pi}$

ii) $\beta_1 = 0$ and $\beta_2 = 3$

$\mu_{2r+1} = 0$ (r = 0,1,2......) and $\mu_{2r} = 1.3.5....(2r-1)\sigma^{2r}$

iii) Since f(x) being the probability can never be negative no portion of the curve lies below x- axis.

i) Linear combination of independent normal variate is also a normal variate.

ii) X- axis is an asymptote to the curve.

iii) The points of inflexion of the curve are given by x = $\mu \pm \sigma$, f(x) = $\frac{1}{\sigma\sqrt{2\pi}} e^{-1/2}$

iv) Q.D. : M.D.: S.D. :: $\frac{2}{3}\sigma : \frac{4}{5}\sigma : \sigma$ ::: $\frac{2}{3}\ \frac{4}{5} : 1$ Or Q.D. : M.D.: S.D. **::10:12:15**

Problems:

1. The mean weight of 800 male students at a certain college is 140kg and the standard deviation is 10kg assuming that the weights are normally distributed find how many students weigh I) Between 130 and 148kg ii) more than 152kg

Solution:

Let μ be the mean and σ be the standard deviation. Then μ =140kg and σ =10pounds

(i) When x= 138, $z = \frac{x - \mu}{\sigma} = \frac{138 - 140}{10} = -0.2 = z_1$

When x= 138, $z = \frac{x - \mu}{\sigma} = \frac{148 - 140}{10} = 0.8 = z_2$

ii) When x=152, $\frac{x-\mu}{\sigma}=\frac{152-140}{10}=1.2=z_1$

Therefore $P(x>152)=P(z>z_1)=0.5-A(z_1)$

=0.5-0.3849=0.1151

Therefore number of students whose weights are more than 152kg

=800x0.1151=92.

Exercise Problems:

1. Two coins are tossed simultaneously. Let X denotes the number of heads then find i) E(X) ii) $E(X^2)$ iii)$E(X^3)$ iv) V(X)
2. If f(x)=k$e^{-|x|}$ is probability density function in the interval, $-\infty < x < \infty$, then find i) k ii) Mean iii) Variance iv) P(0<x<4)

3. If X is a normal variate with mean 30 and standard deviation 5. Find the probabilities that i) $P(26 \leq X \leq 40)$ ii) $P(X \geq 45)$

4. The marks obtained in Statistics in a certain examination found to be normally distributed. If 15% of the students greater than or equal to 60 marks, 40% less than 30 marks. Find the mean and standard deviation.

t-distribution

- If $\overline{X}$ is the mean of a random sample of size n taken from a normal population having the mean μ and the variance σ^2 ,

  and $s^2 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n-1}$ then t= $\frac{\overline{X} - \mu}{\frac{s}{\sqrt{n}}}$ is a r.v. having the

  t- distribution with the parameter υ = (n-1)dof

- The overall shape of a t-distribution is similar to that of a normal distribution both are bell shaped and symmetrical about the mean. Like the standard normal distribution t-distribution has the mean 0, but its variance depends on the parameter υ (nu), called the number of degrees of freedom. The variance of t- distribution exceeds1, but it approaches 1 as $n \to \infty$. The t-distribution with υ-degree of freedom approaches the standard normal distribution as $\upsilon \to \infty$.

  The standard normal distribution provides a good approximation to the t-

  distribution for samples of size 30 or more

Producer of ‘gutkha’ claims that the nicotine content in his ‘gutkha’ on the average is 83 mg. can this claim be accepted if a random sample of 8 ‘gutkhas’ of this type have the nicotine contents of 2.0,1.7,2.1,1.9,2.2,2.1,2.0,1.6 mg.

Solution: Given n=8 and μ=1.83 mg

1. Null hypothesis(H_0): μ=1.83
2. Alternative hypothesis(H_1): $\mu \neq$ 1.83
3. Level of significance: α=0.05

t_α for n-1 degrees of freedom $t_{0.05}$ for 8-1 degrees of freedom is 1.895

Test statistic:

$$t = \frac{\overline{x} - \mu}{\frac{S}{\sqrt{n}}}$$

x	$(x - \bar{x})$	$(x - \bar{x})^2$
2.0	0.05	0.0025
1.7	-0.25	0.0625
2.1	0.15	0.0225
1.9	-0.05	0.0025
2.2	0.25	0.0625
2.1	0.15	0.0225
2.0	0.05	0.0025
1.6	-0.35	0.1225
Total=15.6		

$\bar{x} = \frac{15.6}{8} = 1.95$ and $S^2 = \sum \frac{(x-\bar{x})^2}{n-1} = \frac{0.3}{7}$ S=0.21

$t = \frac{\bar{x}-\mu}{\frac{S}{\sqrt{n}}} = \frac{1.95-1.83}{\frac{0.21}{\sqrt{8}}}$ =1.62 $|t| =$ 1.62

Conclusion:

$\therefore$ $|t| < t_\alpha$ $\therefore$ We accept the Null hypothesis.

The means of two random samples of sizes 9,7 are 196.42 and 198.82.the sum of squares of deviations from their respective means are 26.94,18.73.can the samples be considered to have been the same population?

Solution: Given n_1=9, n_2=7, $\bar{x}_1$=196.42, $\bar{x}_2$=198.82 and

$\sum(x_i-\bar{x}_1)^2$=26.94,

$\sum(x_i-\bar{x}_2)^2$=18.73

$$\therefore S^2=\frac{\sum(x_i-x_1)^2+\sum(x_i-x_2)^2}{n_1+n_2-2}=3.26$$

$\Rightarrow$S=1.81

Null hypothesis(H_0): $\bar{x}_1 = \bar{x}_2$

Alternative hypothesis(H_1): $\bar{x}_1 \neq \bar{x}_2$

Level of significance: α=0.05

t_α for n_1+n_2-2 degrees of freedom

$t_{0.05}$ for 9+7-2=14 degrees of freedom is 2.15

Test statistic: $t = \dfrac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\dfrac{1}{n_1}+\dfrac{1}{n_2}}}$ = $\dfrac{196.42 - 198.82}{(1.81)\sqrt{\dfrac{1}{9}+\dfrac{1}{7}}}$ **=**-2.63 $|t| =$ 2.63

Conclusion:

$\therefore$ $|t| > t_\alpha$ $\therefore$ We reject the Null hypothesis.

F-distribution

- If S_1^2 and S_2^2 are the variances of independent random samples of size n_1 and n_2 respectively, taken from two normal populations having the same variance, then $F = \frac{S_1^2}{S_2^2}$ is a r.v.

  having the F- distribution with the parameter's $\upsilon_1=n_1-1$ and $\upsilon_2=n_2-1$ are called the numerator and denominator degrees of freedom respectively.
- $F_{1-\alpha}(\upsilon_1,\upsilon_2)=\frac{1}{F_\alpha(\upsilon_2,\upsilon_1)}$

In one sample of 8 observations the sum of squares of deviations of the sample values from the sample mean was 84.4 and another sample of 10 observations it was 102.6 .test whether there is any significant difference between two sample variances at at 5% level of significance.

Solution: Given n_1=8, n_2=10, $\sum(x_i-\bar{x}_1)^2$=84.4 and $\sum(x_i-\bar{x}_2)^2$=102.6

$$S_1^{\,2}=\frac{\sum(x_i-x_1)^2}{n_1-1}=\frac{84.4}{7}=12.057$$

$$S_2^{\,2}=\frac{\sum(x_i-x_1)^2}{n_2-1}=\frac{102.6}{9}=11.4$$

Null hypothesis(H_0): $S_1^2 = S_2^2$

Alternative hypothesis(H_1): $S_1^2 \neq S_2^2$

Level of significance: α=0.05

F_α For $(n_1 - 1, n_2 - 1)$ degrees of freedom

$F_{0.05}$ For (7,9) degrees of freedom is 3.29

Test statistic: $F = \frac{S_1^2}{S_2^2}$ $= \frac{12.057}{11.4}$ =1.057

$|F| = 1.057$

Conclusion: $\therefore$ $|F| < F_\alpha$ $\therefore$ We accept the Null hypothesis.

Chi-square test

- If S^2 is the variance of a random sample of size n taken from a normal population having the variance σ^2 , then

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{\sigma^2}$$

is a r.v. having the chi-square distribution with the parameter υ = n-1

- The chi-square distribution is not symmetrical

The following table gives the classification of 100 workers according to gender and nature of work. Test whether the nature of work is independent of the gender of the worker.

	Stable	Unstable	Total
Male	40	20	60
Female	10	30	40
Total	50	50	100

Solution: Given that

$$\text{Expected frequencies} = \frac{\text{row total} \times \text{column total}}{\text{grand total}}$$

$\frac{90\times100}{200}=45$	$\frac{90\times100}{200}=45$	90
$\frac{90\times100}{200}=55$	$\frac{90\times100}{200}=55$	110
100	100	200

Calculation of χ^2 :

Observed Frequency(O_i)	Expected Frequency(E_i)	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
60	45	225	5
30	45	225	5
40	55	225	4.09
70	55	225	4.09
			18.18

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = 18.18$$

1. Null hypothesis(H_0): $O_i = E_i$
2. Alternative hypothesis(H_1): $O_i \neq E_i$
3. Level of significance: $\alpha = 0.05$

χ_{α}^2 For (r-1)(c-1) degrees of freedom

$\chi_{0.05}^2$ For (2-1)(2-1)=1 degrees of freedom is 3.84

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = 18.18$$

1. Null hypothesis(H_0): $O_i = E_i$
2. Alternative hypothesis(H_1): $O_i \neq E_i$
3. Level of significance: $\alpha = 0.05$

   ${\chi_\alpha}^2$ For (r-1)(c-1) degrees of freedom

   ${\chi_{0.05}}^2$ For (2-1)(2-1)=1 degrees of freedom is 3.84
4. Test statistic: $\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = 18.18,$

$$|\chi^2| = 1.057$$

Conclusion: $\therefore \ |\chi^2| > {\chi_\alpha}^2$

$\therefore$ We reject the Null hypothesis.

MODULE– II

TESTING OF STATISTICAL HYPOTHESIS

- **Null Hypothesis (N.H)** denoted by H_0 is statistical hypothesis, which is to be actually tested for acceptance or rejection. NH is the hypothesis, which is tested for possible rejection under the assumption that it is true.
- Any Hypothesis which is complimentary to the N.H is called an **Alternative Hypothesis** denoted by H_1
- Simple Hypothesis is a statistical Hypothesis which completely specifies an exact parameter. N.H is always simple hypothesis stated as a equality specifying an exact value of the parameter. E.g. N.H = $H_0 : \mu = \mu_0$ N.H. = $H_0 : \mu_1 - \mu_2 = \delta$
- Composite Hypothesis is stated in terms of several possible values.
- Alternative Hypothesis(A.H) is a composite hypothesis involving statements expressed as inequalities such as < , > or $\neq$

  i) A.H : $H_1: \mu > \mu_0$ (Right tailed) ii) A.H : $H_1: \mu < \mu_0$ (Left tailed)

  iii) A.H : $H_1: \mu \neq \mu_0$ (Two tailed alternative)

ERRORS IN SAMPLING

- **Errors in sampling:**

  Type I error: Reject H_0 when it is true

  Type II error: Accept $\mathbf{H_0}$ when it is wrong (i.e) accept if when H_1 is true.

	Accept H_0	Reject H_0
H_0 is True	Correct Decision	Type 1 error
H_0 is False	Type 2 error	Correct Decision

- If P{ Reject H_0 when it is true}= P{ Reject H_0 | H_0}=α and P{ Accept H_0 when it is false}= P{ Accept H_0 | H_1} = β then α,β are called the sizes of Type I error and Type II error respectively. In practice, type I error amounts to rejecting a lot when it is good and type II error may be regarded as accepting the lot when it is bad.

- α and β are referred to as producers risk and consumers risk respectively.
- A region (corresponding to a statistic t) in the sample space S that amounts to rejection of H_0 is called critical region of rejection.
- Level of significance is the size of the type I error (or maximum producer’s risk)
- The levels of significance usually employed in testing of hypothesis are 5% and 1% and is always fixed in advance before collecting the test information.
- A test of any statistical hypothesis where AH is one tailed(right tailed or left tailed) is called a **one–tailed test.** If AH is two-tailed such as: H_0: $\mu = \mu_0$, against the AH. H_1 **:** $\mu \neq \mu_0$ ($\mu > \mu_0$ and $\mu < \mu_0$) is called **Two-Tailed Test.**

- The value of test statistics which separates the critical (or rejection) region and the acceptance region is called **Critical value or Significant value**. It depends upon (i) The level of significance used and (ii) The Alternative Hypothesis, whether it is two-tailed or single tailed

Critical Value (Z_α)	Level of significance (α) 1%	5%	10%
Two-Tailed test	$-Z_{\alpha/2} = -2.58$ $Z_{\alpha/2} = 2.58$	$-Z_{\alpha/2} = -1.96$ $Z_{\alpha/2} = 1.96$	$-Z_{\alpha/2} = -1.645$ $Z_{\alpha/2} = 1.645$
Right-Tailed test	$Z_\alpha = 2.33$	$Z_\alpha = 1.645$	$Z_\alpha = 1.28$
Left-Tailed Test	$-Z_\alpha = -2.33$	$-Z_\alpha = -1.645$	$-Z_\alpha = -1.28$

- When the size of the sample is increased, the probability of committing both types of error I and II (i.e) α and β are small, the test procedure is good one giving good chance of making the correct decision.
- P-value is the lowest level (of significance) at which observed value of the test statistic is significant.
- A test of Hypothesis (T. O.H) consists of

1. Null Hypothesis (NH) : H_0
2. Alternative Hypothesis (AH) : H_1
3. Level of significance: α
4. Critical Region pre determined by α
5. Calculation of test statistic based on the sample data.
6. Decision to reject NH or to accept it.

CONFIDENCE INTERVALS

Maximum error E of a population mean μ by using large sample mean is
E = $Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$

The most widely used values for 1-α are 0.95 and 0.99 and the corresponding values of $Z_{\alpha/2}$ are $Z_{0.025}$ = 1.96 and $Z_{0.005}$ = 2.575
Sample size n = $\left[Z_{\alpha/2} \frac{\sigma}{E} \right]^2$

- Confidence interval for μ (for large samples n ≥ 30) σ known

$$\bar{x} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

- If the sampling is without replacement from a population of finite size N then the confidence interval for μ with known is

$$\bar{x} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}} < \mu < \bar{x} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}$$

1. Large sample confidence interval for μ - σ unknown is

$$\bar{x} - Z_{\alpha/2}\frac{S}{\sqrt{n}} < \mu < \bar{x} + Z_{\alpha/2}\frac{S}{\sqrt{n}}$$

Large sample confidence interval for μ_1 - μ_2 (where σ_1 and σ_2 are unknowns)

$$(\bar{x}_1 - \bar{x}_2) \pm Z_{\alpha/2}\sqrt{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)}$$

The end points of the confidence interval are called **Confidence Limits.**

Test statistic for T.O.H. in several cases are

Statistic for test concerning mean σ known

Z = $\frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}}$

Statistic for large sample test concerning mean with σ unknown

Z = $\frac{\overline{X} - \mu_0}{S / \sqrt{n}}$

Statistic for test concerning difference between the means

Z = $\frac{(\overline{X}_1 - \overline{X}_2) - \delta}{\sqrt{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)}}$ under NH $H_0 : \mu_1 - \mu_2 = \delta$ against the AH, $H_1 : \mu_1 - \mu_2 > \delta$ or $H_1 : \mu_1 - \mu_2 < \delta$ or $H_1 : \mu_1 - \mu_2 \neq \delta$

Statistic for large samples concerning the difference between two means (σ_1 and σ_2 are unknown)

Z = $\frac{(\overline{X}_1 - \overline{X}_2) - \delta}{\sqrt{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)}}$

Statistics for large sample test concerning one proportion

Z = $\frac{X - np_o}{\sqrt{np_0(1-p_0)}}$ under the N.H: H_0**:** $p = p_o$ against H_1: $p \neq p_0$ or $p > p_0$ or

$p < P_0$

Statistic for test concerning the difference between two proportions

Z= $\frac{\frac{X_1}{n_1} - \frac{X_2}{n_2}}{\sqrt{\hat{p}(1-\hat{p})(\frac{1}{n_1}+\frac{1}{n_2})}}$ with $\hat{p} = \frac{X_1 + X_2}{n_1 + n_2}$ under the NH : H_0: $p_1 = p_2$ against the AH

H_1: $p_1 < p_2$ or $p_1 > p_2$ or $p_1 \neq p_2$

- Large sample confidence interval for difference of two proportions $(p_1 - p_2)$ is

$$\left(\frac{x_1}{n_1}-\frac{x_2}{n_2}\right)\pm Z_{\alpha/2}\sqrt{\frac{\frac{x_1}{n_1}\left(1-\frac{x_1}{n_1}\right)}{n_1}+\frac{\frac{x_2}{n_2}\left(1-\frac{x_2}{n_2}\right)}{n_2}}$$

- Maximum error of estimate $E = Z_{\alpha/2}\sqrt{\frac{p(1-p)}{n}}$ with observed value x/n substituted for p we obtain an estimate of E
- Sample size $n = p(1-p)\left(\frac{Z_{\alpha/2}}{E}\right)^2$ when p is known

  $n = \frac{1}{4}\left(\frac{Z_{\alpha/2}}{E}\right)^2$ when p is unknown

- One sided confidence interval is of the form $p < (1/2n)\chi_\alpha^2$ with (2n+1) degrees of freedom.

1. A sample of 400 items is taken from a population whose standard deviation is 10.The mean of sample is 40.Test whether the sample has come from a population with mean 38 also calculate 95% confidence interval for the population.

Solution: Given n=400, $\bar{x} = 40$ and μ=38 and σ=10

1. **Null hypothesis(H_0):** μ=38
2. **Alternative hypothesis(H_1):** $\mu \neq 38$
3. **Level of significance:** α=0.05 and Z_α=1.96
4. **Test statistic:** $Z = \dfrac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}$

$$Z = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} = \frac{40-38}{\frac{10}{\sqrt{400}}} = 4$$

$$|Z| = 4$$

Confidence interval = $\left(\bar{x} - Z_\alpha \frac{\sigma}{\sqrt{n}}, \bar{x} + Z_\alpha \frac{\sigma}{\sqrt{n}}\right)$

$$= \left(40 - 1.96\frac{10}{\sqrt{400}}, 40 + 1.96\frac{10}{\sqrt{400}}\right)$$

Samples of students were drawn from two universities and from their weights in kilograms mean and S.D are calculated and shown below make a large sample test to the significance of difference between means.

	MEAN	S.D	SAMPLE SIZE
University-A	55	10	400
University-B	57	15	100

Solution: *Given n_1=400, n_2=100, $\bar{x}_1$=55, $\bar{x}_2$=57*
S_1=10 and S_2=15

1. **Null hypothesis(H_0):** $\bar{x}_1 = \bar{x}_2$
2. **Alternative hypothesis(H_1):** $\bar{x}_1 \neq \bar{x}_2$
3. **Level of significance:** α=0.05 and Z_α=1.96
4. **Test statistic:** $Z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}} = \dfrac{55-57}{\sqrt{\dfrac{100}{400} + \dfrac{225}{100}}}$ =-1.26

   $|Z| = 1.26$

5. **Conclusion:**

$\therefore$ $|Z| < Z_\alpha$

$\therefore$We accept the Null hypothesis.

1. A sample of 400 items is taken from a population whose standard deviation is 10.The mean of sample is 40.Test whether the sample has come from a population with mean 38 also calculate 95% confidence interval for the population.

Solution: Given n=400, $\bar{x}=40$ and μ=38 and σ=10

1. **Null hypothesis(H_0):** μ=38
2. **Alternative hypothesis(H_1):** $\mu \neq 38$
3. **Level of significance:** α=0.05 and z_α=1.96
4. **Test statistic:** $Z=\frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}$

$Z=\frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}=\frac{40-38}{\frac{10}{\sqrt{400}}}=4 \quad |Z|=4$

5. **Conclusion:**

$\therefore |Z|>z_\alpha$

$\therefore$ We reject the Null hypothesis.

Confidence interval = $\left(\bar{x} - Z_{\alpha}\frac{\sigma}{\sqrt{n}}, \bar{x} + Z_{\alpha}\frac{\sigma}{\sqrt{n}}\right)$

$$= \left(40 - 1.96\frac{10}{\sqrt{400}}, 40 + 1.96\frac{10}{\sqrt{400}}\right)$$

$$= (39.02, 40.98)$$

1. Samples of students were drawn from two universities and from their weights in kilograms mean and S.D are calculated and shown below make a large sample test to the significance of difference between means.

	MEAN	S.D	SAMPLE SIZE
University-A	55	10	400
University-B	57	15	100

Solution: Given n_1=400, n_2=100, $\bar{x}_1$=55, $\bar{x}_2$=57

S_1=10 and S_2=15

1. **Null hypothesis(H_0):** $\bar{x}_1 = \bar{x}_2$
2. **Alternative hypothesis(H_1):** $\bar{x}_1 \neq \bar{x}_2$
3. **Level of significance:** α=0.05 and z_α=1.96
4. **Test statistic:** $Z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}}$ = $\dfrac{55-57}{\sqrt{\dfrac{100}{400} + \dfrac{225}{100}}}$ =-1.26

$|Z| =$ 1.26

5. **Conclusion:**

$\therefore\ |Z| < z_\alpha$

$\therefore$ We accept the Null hypothesis.

1. In a sample of 1000 people in Karnataka 540 are rice eaters and the rest are wheat eaters. Can we assume that both rice and wheat are equally popular in this state at 1% level of significance?

Solution: Given n=400, x=540

$p = \frac{x}{n} = \frac{540}{1000} = 0.54$

$P = \frac{1}{2} = 0.5$, $Q = 0.5$

1. **Null hypothesis(H_0):** P=0.5
2. **Alternative hypothesis(H_1):** $P \neq 0.5$
3. **Level of significance:** α=1% and z_α=2.58
4. **Test statistic:** $z = \frac{P-p}{\sqrt{\frac{PQ}{n}}}$

$z = \frac{P-p}{\sqrt{\frac{PQ}{n}}} = \frac{0.54-0.5}{\sqrt{\frac{0.5\times 0.5}{1000}}} = 2.532$ $|z| = 2.532$ $\therefore$ $|z| < z_\alpha$ $\therefore$ We accept the Null hypothesis.

4.Random sample of 400 men and 600 women were asked whether they would like to have flyover near their residence .200 men and 325 women were in favour of proposal. Test the hypothesis that the proportion of men and women in favour of proposal are same at 5% level.

Solution: Given $n_1=400$, $n_2=600$, $x_1 = 200$ and $x_2 = 325$

$$p_1 = \frac{200}{400} = 0.5$$

$$p_2 = \frac{325}{600} = 0.541$$

$$p = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{400 \times \frac{200}{400} + 600 \times \frac{325}{600}}{400 + 600} = 0.525$$

1. **Null hypothesis(H_0):** $p_1 = p_2$
2. **Alternative hypothesis(H_1):** $p_1 \neq p_2$
3. **Level of significance:** α=0.05 and z_α=1.96
4. **Test statistic:** $Z = \dfrac{p_1 - p_2}{\sqrt{pq\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$ = $\dfrac{0.5 - 0.541}{\sqrt{0.525 \times 0.425\left(\dfrac{1}{400} + \dfrac{1}{600}\right)}} = -1.28$

$|Z| = 1.28$

5. **Conclusion:**

$\therefore$ $|Z| < z_\alpha$

$\therefore$ We accept the Null hypothesis.

ANALYSIS OF VARIANCE

ANOVA:

It is abbreviated form for ANALYSIS OF VARIANCE which is a method for comparing several population means at the same time. It is performed using F-distribution

Assumptions of ANALYSIS OF VARIANCE:

1. The data must be normally distributed.
2. The samples must draw from the population randomly and independently.
3. The variances of population from which samples have been drawn are equal.

Types of Classification:

There are two types of model for analysis of variance

1. One-Way Classification
2. Two-Way Classification.

ONE –WAY CLASSIFICATION:

PROCEDURE FOR ANOVA

Step 1 : State the null and alternative hypothesis.

H_0: $\mu_1 = \mu_2 = \mu_3$ (The means for three groups are equal).

H_1 : At least one pair is unequal.

Step 2: Select the test criterion to be used.

We have to decide which test criterion or distribution should be used. As our assumption involves means for three normally distributed populations. We should use the F-distribution to test the hypothesis.

Step 3. Determine the rejection and non-rejection regions

We decide to use 0.05 level of significance. As on one-way ANOVA test is always right-tail, the area in the right tail of the F-distribution curve is 0.05, which is the rejection region. Now, we need to know the degrees of freedom for the numerator and the denominator. Degrees of freedom for the numerator=k-1, where k is the number of groups. Degree of freedom for denominator =n-k where n is total number of observations

Step 4. Calculate the value of the test statistics by applying ANOVA. i.e., F Calculated

Step 5: conclusion

I) If $F_{\text{Calculated}} < F_{\text{Critical}}$, then H_0 is accepted

ii) if $F_{\text{calculated}} < F_{\text{critical}}$, then H_0 is rejected

TWO –WAY CLASSIFCATION:

The analysis of variance table for two-way classification is taken as follows;

Source of variation	Sum of squares SS	Degree of freedom df	Mean squares Ms
Between columns	SSC	(c-1)	MSC=SSC/(c-1)
Within rows	SSR	(r-10	MSR=SSR/(r-1)
Residual(ERROR)	SSE	(c-1)(r-1)	MSE=SSE/(c-1)(r-1)
total	SST	Cr-1	

The abbreviations used in the table are:

SSC= sum of squares between column s.

SSR= sum of square between rows.

SST=total sum of squares;

SSE= sum of squares of error, it is obtained by subtracting SSR and SSC from SST.

(c-1)=number of degrees of freedom between columns.

(r-1)=number of degrees of freedom between rows.

(c-1)(r-1)=number of degree of freedom for residual.

MSC=mean of sum of squares between columns

MSR= mean of sum of squares between rows.

MSE= mean of sum of squares between residuals.

It may be noted that total number of degrees of freedom are =(c-1)+(r-1)+(c-1)(r-1)=cr-1=N-1

PROBLEMS:

1. **There are three different methods of teaching English that are used on three groups of students. Test by using analysis of variance whether this method s of teaching had an effect on the performance of students. Random sample of size 4 are taken from each group and the marks obtained by the sample students in each group are given below**

Marks obtained the students

Group A	Group B	Group C
16	15	15
17	15	14
13	13	13
18	17	14
Total 64	**Total 60**	**Total 56**

Solution:

It is assumed that the marks obtained by the students are distributed normally with means μ_1, μ_2, μ_3 for the three groups A, B and C. respectively. Further, is is assumed that the standard deviation of the distribution of marks for groups A,B and C are equal and constant. This assumption implies that the mean marks of the groups may differ on account of using different methods of teaching, but they do not affect the dispersion of marks.

PROCEDURE FOR ANOVA

Step 1 : State the null and alternative hypothesis.

H_0: $\mu_1 = \mu_2 = \mu_3$ (The means for three groups are equal).

H_1 : At least one pair is unequal.

Step 2: Select the test criterion to be used.

We have to decide which test criterion or distribution should be used. As our assumption involves means for three normally distributed populations. We should use the F-distribution to test the hypothesis.

Step 3. Determine the rejection and non-rejection regions

We decide to use 0.05 level of significance. As on one-way ANOVA test is always right-tail, the area in the right tail of the F-distribution curve is 0.05, which is the rejection region. Now, we need to know the degrees of freedom for the numerator and the denominator. Degrees of freedom for the numerator=k-1=3-1=2, where k is the number of groups. Degree of freedom for denominator =n-k=12-3=9, where n is total number of observations.

Step 4. Calculate the value of the test statistics by applying ANOVA. i.e., F Calculated

Worksheet for calculating Variances

Group A Group B Group C

X_{1j}	$(x_{1j}-x_i)$	$(x_{1j}-x_i)^2$	X_{2j}	$(x_{2j}-x_i)$	$(x_{2j}-x_i)^2$	X_{3j}	$(x_{3j}-x_i)$	$(x_{3j}-x_i)^2$
16	0	0	15	0	0	15	1	1
17	1	1	15	0	0	14	0	0
13	-3	9	13	-2	4	13	-1	1
18	2	4	17	2	4	14	0	0
Total 64			Total 60			Total 56		
Mean 16			Mean 15			Mean 14		

The sample variances for the groups are

$$S_1^2 = \frac{1}{n_1}\sum_{j=1}^{n_1}\left(x_{1j}-\overline{x_1}\right)^2 = \frac{1}{4}(14) = 3.5$$

$$S_2^2 = \frac{1}{n_2}\sum_{j=1}^{n_2}\left(x_{2j}-\overline{x_2}\right)^2 = \frac{1}{4}(8) = 2$$

$$S_3^2 = \frac{1}{n_3}\sum_{j=1}^{n_{13}}\left(x_{3j}-\overline{x_3}\right)^2 = \frac{1}{4}(14) = 0.5$$

We can now estimate the variance by the pooled variance method as follows;

$$\sigma^2 = \frac{\sum\sum(x_{ij} - x_i)^2}{n-3}$$

The denominator is $n_1+n_2+n_3=3$

Applying the value in the formulas,

$$\sigma^2 = \frac{\sum\sum(x_{ij} - x_i)^2}{n-3} = \frac{4[(16-15)^2 + (15-15)^2 + (14-15)^2]}{3-1}$$

=4(This is the variance between the samples)

Now, F is to be calculated . F=ratio of two variances

$$= \frac{estimateof\ \ \sigma^2 between\ \ samples}{estimateof\ \ \sigma^2 within\ \ samples} = \frac{4}{2.67} = 1.498$$

The foregoing calculations can be summarized in the form of an ANOVA TABLE.

Source of variation	Sum of squares SS	Degrees of freedom df	Mean of equares	Variance ratio F
Between sampling	SSB	k-1	MSB=SSB/(k-1)	
Within sampling	SSW	n-k	MSW=SSW/(n-k)	F=MSB/MSW
total	SST	n-1		

Source of variation	Sum of squares SS	Degrees of freedom df	Mean of equares	Variance ratio F
Between sampling	6	3-1	8/2=4	
Within sampling	24	12-3	24/8=2.67	4/2.67=1.498

Step: conclusion: The critical value of F for 2 and 9 degrees of freedom at 5 percent level of significance is 4.26. As the calculated value of F=1.0498 is less than critical values of F.

i.e.,$F_{calculated} < F_{critical.}$ The null hypothesis is accepted.

7. A company has appointed four salesman, A,B,C and D. observed their sales in three seasons-summer, winter, monsoon. The figures (in Rs lakh) are given in the following table.

SALESMEN

seasons	A	B	C	D	Seasons totals
summer	36	36	21	35	128
winter	28	29	31	32	120
monsoon	26	28	29	29	112
Sales man totals	90	93	81	96	360

Using 5 percent level of significance, perform an analysis of variance on the above data and interpret the result.

Solution:

Step 1 : State the null and alternative hypothesis.

H_0: there is no difference in the mean sales performance of A, B, C and D in the three seasons.

H_1 : there is difference in the mean sales performance of A ,B, C and D in the three season.

Step 2: Select the test criterion to be used.

We have to decide which test criterion or distribution should be used. As our assumption involves means for three normally distributed populations. We should use the F-distribution to test the hypothesis.

Step 3. Determine the rejection and non-rejection regions

We decide to use 0.05 level of significance. The degrees of freedom for rows are (r-1) =2 and for columns are (c-1)=3 and for residual (r-1)(c-1)=2x3=6. Thus, we have to compare the calculated value of F with the critical value of F for a) 2 and 6 df at 5% l. o. s b)3 and 6 df at 5% .l. o. s.

Step 4;

Coded Data for ANOVA

SALESMEN

seasons	A	B	C	D	Seasons totals
summer	6	6	-9	5	8
winter	-2	-1	1	2	0
monsoon	-4	-2	-1	-1	-8
Sales man totals	0	3	-9	6	0

Correction factor $C=T^2/N=(0)^2/12=0$

Sum os squares between salesmen

$=0^2/3+3^2/3+(-9^2/3)+6^2/3=0+3+27+12=42$

Sum of squares between seasons=$8^2/4+0^2/4+(-8^2/4)=16+0+16=32$

Total sum of squares

$=(6)^2+(-2)^2+(-4)^2+(6)^2+(-1)^2+(-2)^2+(-9)^2+(1)^2+(-1)^2+(5)^2+(2)^2+(-1)^2$

$=210$

Analysis of variance table

Source of variation	Sum of squares SS	Degree of freedom df	Mean squares Ms
Between columns	42	4-1=3	14.00
Within rows	32	3-1=2	16.00
Residual(ERROR)	136	3x2=6	22.67
total	210	12-1=11	

We now test the hypothesis (i) that there is no difference in the sales performance among the four salesmen and (ii) there is no difference in the mean sales in the three seasons. For this, we have to first compare the salesman variance estimate with the residual estimate. This is shown below:

F_A =14/22.67=0.62

In the same manner, we have to compare the season variance estimate with the residual variances estimate. This is shown below;

F_B=16/22.67=0.71

Step 5:

It may noted that the critical value of F for 3 and 6 degree of freedom at 5 percent level of significance is 4.76. Since the calculated value of F_A is 0.62 is less than critical value of F. Therefore there is no significance difference among salesmen.

Also the critical value of F for 2 and 6 degree of freedom at 5 percent level of significance is 4.76. Since the calculated values of F_B=16/22.67=0.71 is less than critical value of F. Therefore there is no significance difference among seasons

The overall conclusion is that the salesmen and seasons are alike in respect of sales.

Exercise problems:

1. A company has derived three training methods to train its workers. It is keen to know which of these three training methods would lead to greatest productivity after training. Given below are productivity measures for individual workers trained by each method.

Method 1	30	40	45	38	48	55	52
Method 2	55	46	37	43	52	42	40
Method 3	42	38	49	40	55	36	41

Find out whether the three training methods lead to different levels of productivity at the 0.05 level of significance.

1. Consider the following ANOVA TABLE, based on information obtained for three randomly selected samples from three independent population, which are normally distributed with equal variances.

Source of variance	Sum of squares SS	Degree of freedom df	Mean squares MS	Value of test statistics
Between samples	60	?	20	F=
Within samples	?	14	?	

(A) Complete the ANOVA table by filling in missing values.

(B) test the null hypothesis that the means of the three population are all equal, using 0.01 level of significance.

3. The following represent the number of units of production per day turned out by four different workers using five different types of machines

Machine type						
Worker	A	B	C	D	E	TOTAL
1	4	5	3	7	6	25
2	5	7	7	4	5	28
3	7	6	7	8	8	36
4	3	5	4	8	2	22
TOTAL	19	23	21	27	21	111

On the basis of this information, can it be concluded that (i) The mean productivity is the same for different machines. (ii) The workers don’t differ with regard to productivity.

MODULE– III

ORDINARY DIFFERENTIAL EQUATIONS

1. The important methods of solving ordinary differential equations of first order numerically are as follows

   1) Taylors series method
   2) Euler's method
   3) Modified Euler's method of successive approximations
   4) Runge- kutta method

To describe various numerical methods for the solution of ordinary differential eqn's,we consider the general 1st order differential eqn

$dy/dx=f(x,y)$-------(1)

with the initial condition $y(x_0)=y_0$

The methods will yield the solution in one of the two forms:

i) A series for y in terms of powers of x,from which the value of y can be obtained by direct substitution.

ii) A set of tabulated values of y corresponding to different values of x

The methods of Taylor and picard belong to class(i)

The methods of Euler, Runge - kutta method, Adams, Milne etc, belong to class (ii)

TAYLOR'S SERIES METHOD

To find the numerical solution of the differential equation

$$\frac{dy}{dx} = f(x, y) \rightarrow (1)$$

With the initial condition $y(x_0) = y_0 \rightarrow (2)$

$y(x)$ can be expanded about the point x_0 in a Taylor's series in powers of $(x - x_0)$ as

$$y(x) = y(x_0) + \frac{(x-x_0)}{1} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + ... + \frac{(x-x_0)^n}{n!} y^n(x_0) \rightarrow (3)$$

In equ3, $y(x_0)$ is known from I.C equ2. The remaining coefficients $y'(x_0), y''(x_0), ...y^n(x_0)$ etc are obtained by successively differentiating equ1 and evaluating at x_0. Substituting these values in equ3, $y(x)$ at any point can be calculated from equ3. Provided $h = x - x_0$ is small.

When $x_0 = 0$, then Taylor's series equ3 can be written as

$$y(x) = y(0) + x.y'(0) + \frac{x^2}{2!} y''(0) + ... + \frac{x^n}{n!} y^n(0) + ... \rightarrow (4)$$

1. Using Taylor's expansion evaluate the integral of $y' - 2y = 3e^x, y(0) = 0$**, at a)** $x = 0.2$

b) compare the numerical solution obtained with exact solution .

Sol:Given equation can be written as $2y + 3e^x = y', y(0) = 0$

Differentiating repeatedly w.r.t to 'x' and evaluating at $x = 0$

$$y'(x) = 2y + 3e^x, y'(0) = 2y(0) + 3e^0 = 2(0) + 3(1) = 3$$
$$y''(x) = 2y' + 3e^x, y''(0) = 2y'(0) + 3e^0 = 2(3) + 3 = 9$$
$$y'''(x) = 2.y''(x) + 3e^x, y'''(0) = 2y''(0) + 3e^0 = 2(9) + 3 = 21$$
$$y^{iv}(x) = 2.y'''(x) + 3e^x, y^{iv}(0) = 2(21) + 3e^0 = 45$$
$$y^{v}(x) = 2.y^{iv} + 3e^x, y^{v}(0) = 2(45) + 3e^0 = 90 + 3 = 93$$

In general, $y^{(n+1)}(x) = 2.y^{(n)}(x) + 3e^{x}$ or $y^{(n+1)}(0) = 2.y^{(n)}(0) + 3e^{0}$

The Taylor's series expansion of $y(x)$ about $x_0 = 0$ is

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y''''(0) + \frac{x^5}{5!}y'''''(0) + ...$$

Substituting the values of $y(0), y'(0), y''(0), y'''(0),..........$

$$y(x) = 0 + 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{45}{24}x^4 + \frac{93}{120}x^5 + ...$$

$$y(x) = 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \frac{15}{8}x^4 + \frac{31}{40}x^5 + ... \rightarrow \text{ equ1}$$

Now put $x = 0.1$ in equ1

$$y(0.1)=3(0.1)+\frac{9}{2}(0.1)^2+\frac{7}{2}(0.1)^3+\frac{15}{8}(0.1)^4+\frac{31}{40}(0.1)^5=0.34869$$

Now put $x=0.2$ in equ1

$$y(0.2)=3(0.2)+\frac{9}{2}(0.2)^2+\frac{7}{2}(0.2)^3+\frac{15}{8}(0.2)^4+\frac{31}{40}(0.2)^5=0.811244$$

$$y(0.3)=3(0.3)+\frac{9}{2}(0.3)^2+\frac{7}{2}(0.3)^3+\frac{15}{8}(0.3)^4+\frac{31}{40}(0.3)^5=1.41657075$$

Analytical Solution:

The exact solution of the equ $\frac{dy}{dx}=2y+3e^{x}$ with $y(0)=0$ can be found as follows

$\frac{dy}{dx}-2y=3e^{x}$ Which is a linear in y.

Here $P=-2, Q=3e^{x}$

I.F = $\int_{e}^{pdx}=\int_{e}^{-2dx}=e^{-24}$

General solution is $y.e^{-2x}=\int 3e^{x}.e^{-2x}dx+c=-3e^{-x}+c$

$\therefore y=-3e^{x}+ce^{2x}$ where $x=0, y=0$ $0=-3+c \Rightarrow c=3$

The particular solution is $y=3e^{2x}-3e^{x}$ or $y(x)=3e^{2x}-3e^{x}$

Put $x = 0.1$ in the above particular solution,

$$y = 3.e^{0.2} - 3e^{0.1} = 0.34869$$

Similarly put $x = 0.2$

$$y = 3e^{0.4} - 3e^{0.2} = 0.811265$$

put $x = 0.3$

$$y = 3e^{0.6} - 3e^{0.3} = 1.416577$$

EULER'S METHOD

It is the simplest one-step method and it is less accurate. Hence it has a limited application.

Consider the differential equation $\frac{dy}{dx}$ = f(x,y) →(1)

With $y(x_0) = y_0$ →(2)

Consider the first two terms of the Taylor's expansion of y(x) at $x = x_0$

$$y(x) = y(x_0) + (x - x_0)\, y^1(x_0) \qquad \rightarrow(3)$$

from equation (1) $y^1(x_0) = f(x_0,y(x_0)) = f(x_0,y_0)$

Substituting in equation (3)

$\therefore y(x) = y(x_0) + (x - x_0)\, f(x_0,y_0)$

At $x = x_1$, $y(x_1) = y(x_0) + (x_1 - x_0)\, f(x_0,y_0)$

$\therefore y_1 = y_0 + h\, f(x_0,y_0)$ where $h = x_1 - x_0$

Similarly at $x = x_2$, $y_2 = y_1 + h\, f(x_1,y_1)$,

Proceeding as above, $y_{n+1} = y_n + h\, f(x_n,y_n)$

This is known as Euler's Method

Modified Euler's method

It is given by $y^{(i)}{}_{k+1} = y_k + h/2\, f\left[(x_k, y_k) + f\,(x_{k+1}, 1)_{k+1}{}^{(i-1)}\right], i = 1, 2....., ki = 0, 1.....$

Working rule :

i)Modified Euler's method

$$y^{(i)}{}_{k+1} = y_k + h/2\, f\left[(x_k, y_k) + f\,(x_{k+1}, 1)_{k+1}{}^{(i-1)}\right], i = 1, 2....., ki = 0, 1.....$$

ii) When $i = 1$ $y^{0}{}_{k+1}$ can be calculated from Euler's method

iii) K=0, 1……… gives number of iteration. $i=1,2...$

gives number of times, a particular iteration k is repeated

Suppose consider dy/dx=f(x, y) -------- (1) with $y(x_0)=y_0$----------- (2)

To find $y(x_1)=y_1$ at $x=x_1=x_0+h$

Now take k=0 in modified Euler's method

We get $y_1^{(1)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(i-1)}\right)\right]$……………………….. (3)

Taking i=1, 2, 3...k+1 in eqn (3), we get

$y_1^{(0)} = y_0 + h/2\left[f(x_0, y_0)\right]$ (By Euler's method)

$y_1^{(1)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$

$y_1^{(2)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$

$y_1^{(k+1)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(k)}\right)\right]$

If two successive values of $y_1^{(k)}, y_1^{(k+1)}$ are sufficiently close to one another, we will take the common value as $y_2 = y(x_2) = y(x_1 + h)$

We use the above procedure again

1) using modified Euler's method find the approximate value of x when $x = 0.3$ given that $dy/dx = x + y \ and \ y(0) = 1$

sol: Given $dy/dx = x + y \ and \ y(0) = 1$

Here $f(x, y) = x + y, x_0 = 0, \ and \ y_0 = 1$

Take h = 0.1 which is sufficiently small

Here $x_0 = 0, x_1 = x_0 + h = 0.1, x_2 = x_1 + h = 0.2, x_3 = x_2 + h = 0.3$

The formula for modified Euler's method is given by

$$y_{k+1}{}^{(i)} = y_k + h/2\left[f(x_k + y_k) + f\left(x_{k+1}, y_{k+1}{}^{(i-1)}\right)\right] \rightarrow (1)$$

Step1: To find $y_1 = y(x_1) = y\ (0.1)$

Taking k = 0 in eqn(1)

$$y_{k+1}^{(i)} = y_0 + h/2\left[f\left(x_0 + y_0\right) + f\left(x_1, y_1^{(i-1)}\right)\right] \rightarrow (2)$$

when i = 1 in eqn (2)

$$y_1^{(i)} = y_0 + h/2\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(0)}\right)\right]$$

First apply Euler's method to calculate $y_1^{(0)}$ = y_1

$$\therefore\ y_1^{(0)} = y_0 + h\, f\left(x_0, y_0\right)$$

$= 1+(0.1)f(0.1)$

$= 1+(0.1)$

$= 1.10$

$now\left[x_0 = 0, y_0 = 1, x_1 = 0.1, y_1(0) = 1.10\right]$

$\therefore\ y_1^{(1)} = y_0 + 0.1/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$

$= 1+0.1/2[f(0,1) + f(0.1,1.10)$

$= 1+0.1/2[(0+1)+(0.1+1.10)]$

$= 1.11$

When i=2 in eqn (2)

$$y_1^{(2)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

= 1+0.1/2[f(0.1)+f(0.1,1.11)]

= 1 + 0.1/2[(0+1)+(0.1+1.11)]

= 1.1105

$$y_1^{(3)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right]$$

= 1+0.1/2[f(0,1)+f(0.1 , 1.1105)]

= 1+0.1/2[(0+1)+(0.1+1.1105)]

= 1.1105

Since $y_1^{(2)} = y_1^{(3)}$

$\therefore$ $y_1 = 1.1105$

Step:2 To find $y_2 = y(x_2) = y(0.2)$

Taking k = 1 in eqn (1) , we get

$$y_2^{(i)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2, y_2^{(i-1)}\right)\right] \to (3)$$

i = 1,2,3,4,.....

For i = 1

$$y_2^{(1)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2, y_2^{(0)}\right)\right]$$

$y_2^{(0)}$ is to be calculate from Euler's method

$$y_2^{(0)} = y_1 + h\ f(x_1, y_1)$$

= 1.1105 + (0.1) f(0.1 , 1.1105)

= 1.1105+(0.1)[0.1+1.1105]

= 1.2316

$$\therefore\ y_2^{(1)} = 1.1105 + 0.1/2\left[f(0.1, 1.1105) + f(0.2, 1.2316)\right]$$

= 1.1105 +0.1/2[0.1+1.1105+0.2+1.2316]

= 1.2426

$$y_2^{(2)} = y_1 + h/2\left[f\left(x_1, y_1\right) + f\left(x_2 y_2^{(1)}\right)\right]$$

= 1.1105 + 0.1/2[f(0.1 , 1.1105) , f(0.2 . 1.2426)]

= 1.1105 + 0.1/2[1.2105 + 1.4426]

= 1.1105 + 0.1(1.3266)

= 1.2432

$$y_2^{(3)} = y_1 + h/2\left[f\left(x_1, y_1\right) + f\left(x_2 y_2^{(2)}\right)\right]$$

= 1.1105+0.1/2[f(0.1,1.1105)+f(0.2 , 1.2432)]

= 1.1105+0.1/2[1.2105+1.4432)]

= 1.1105 + 0.1(1.3268)

= 1.2432

Since $y_2^{(3)} = y_2^{(3)}$

Hence y_2 = 1.2432

Step:3

To find $y_3 = y(x_3) = y\ y(0.3)$

Taking k =2 in eqn (1) we get

$$y_3^{(1)} = y_2 + h/2\left[f(x_2, y_2) + f\left(x_3, y_3^{(i-1)}\right)\right] \rightarrow (4)$$

For i = 1 ,

$$y_3^{(1)} = y_2 + h/2\left[f(x_2, y_2) + f\left(x_3, y_3^{(0)}\right)\right]$$

$y_3^{(0)}$ is to be evaluated from Euler’s method .

$y_3^{(0)} = y_2 + h\, f\,(x_2, y_2)$

= 1.2432 +(0.1) f(0.2 , 1.2432)

= 1.2432+(0.1)(1.4432)

= 1.3875

$\therefore\ y_3^{(1)}$ = 1.2432+0.1/2[f(0.2 , 1.2432)+f(0.3, 1.3875)]

= 1.2432 + 0.1/2[1.4432+1.6875]

= 1.2432+0.1(1.5654)

= 1.3997

$$y_3^{(2)} = y_2 + h/2\left[\ f\left(x_2, y_2\right) + f\left(x_3, y_3^{(1)}\right)\right]$$

= 1.2432+0.1/2[1.4432+(0.3+1.3997)]

= 1.2432+ (0.1) (1.575)

= 1.4003

$$y_3^{(3)} = y_2 + h/2\left[\ f\left(x_2, y_2\right) + f\left(x_3, y_3^{(2)}\right)\right]$$

= 1.2432+0.1/2[f(0.2 , 1.2432)+f(0.3 , 1.4003)]

= 1.2432 + 0.1(1.5718)

= 1.4004

$$y_3^{(4)} = y_2 + h/2\left[\ f\left(x_2, y_2\right) + f\left(x_3, y_3^{(3)}\right)\right]$$

= 1.2432 + 0.1/2[1.4432+1.7004]

= 1.2432+(0.1)(1.5718)

= 1.4004

Since $y_3^{(3)} = y_3^{(4)}$

Hence $y_3 = 1.4004$ $\therefore$ The value of y at x = 0.3 is 1.4004

Runge – Kutta Methods

I. Second order R-K Formula

$y_{i+1} = y_i + 1/2\ (K_1 + K_2)$,

Where $K_1 = h\ (x_i, y_i)$

$K_2 = h\ (x_i + h, y_i + k_1)$

For i= 0,1,2-------

II. Third order R-K Formula

$y_{i+1} = y_i + 1/6\ (K_1 + 4K_2 + K_3)$,

Where $K_1 = h\ (x_i, y_i)$

$K_2 = h\ (x_i + h/2,\ y_0 + k_1/2)$

$K_3 = h\ (x_i + h,\ y_i + 2k_2 - k_1)$

For i= 0,1,2-------

III. Fourth order R-K Formula

$y_{i+1} = y_i + 1/6\,(K_1 + 2K_2 + 2K_3 + K_4)$,

Where $K_1 = h\,(x_i, y_i)$

$K_2 = h\,(x_i + h/2,\ y_i + k_1/2)$

$K_3 = h\,(x_i + h/2,\ y_i + k_2/2)$

$K_4 = h\,(x_i + h,\ y_i + k_3)$

For i= 0,1,2-------

1. Using Runge-Kutta method of second order, find $y(2.5)$ from $\frac{dy}{dx} = \frac{x+y}{x}$, y(2)=2 , h = 0.25 .

Sol: Given $\frac{dy}{dx} = \frac{x+y}{x}$, y(2) = 2 .

Here f(x, y) = $\frac{x+y}{x}$, $x_0 = 0$, $y_0=2$ and h = 0.25

$\therefore$ $x_1 = x_0+h = 2+0.25 = 2.25$, $x_2 = x_1+h = 2.25+0.25 = 2.5$

By R-K method of second order,

$$y_{i+1} = y_i + 1/2(k_1 + k_2), k_1 - hf\,(x_i + h, y_i + k_1), i = 0,1.... \rightarrow (1)$$

Step -1:-

To find $y(x_1)$ i.e $y(2.25)$ by second order R - K method taking i=0 in eqn(i)

We have $y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

Where $k_1 = hf(x_0, y_0)$, $k_2 = hf(x_0+h, y_0+k_1)$

$f(x_0, y_0) = f(2,2) = 2+2/2 = 2$

$k_1 = hf(x_0, y_0) = 0.25(2) = 0.5$

$k_2 = hf(x_0+h, y_0+k_1) = (0.25)f(2.25, 2.5)$

$= (0.25)(2.25+2.5/2.25) = 0.528$

$\therefore y_1 = y(2.25) = 2+1/2(0.5+0.528)$

$= 2.514$

Step2:

To find $y(x_2)$ i.e., y(2.5)

i=1 in (1)

$x_1=2.25, y_1=2.514$, and h=0.25

$y_2=y_1+1/2(k_1+k_2)$

where $k_1=h\, f((x_1,y_1))=(0.25)f(2.25,2.514)$

=(0.25)[2.25+2.514/2.25]=0.5293

$k_2 = h\, f\,(x_0+h, y_0+k_1) = (0.1)\, f\,(0.1, 1-0.1) = (0.1)(-0.9) = -0.09$

=(0.25)[2.5+2.514+0.5293/2.5]

=0.55433

$y_2 = y$ (2.5)=2.514+1/2(0.5293+0.55433)

=3.0558

$\therefore$ y =3.0558 when x = 2.5

9.using Runge-kutta method of order 4,compute y(1.1) for the eqn $y^1=3x+y^2$,y(1)=1.2 h = 0.05

Ans:1.7278

10. using Runge-kutta method of order 4,compute y(2.5) for the eqn dy/dx = x+y/x, y(2)=2 [hint h = 0.25(2 steps)]

Ans:3.058

MODULE– IV

PARTIAL DIFFERENTIAL EQUATIONS AND CONCEPTS IN SOLUTION TO BOUNDARY VALUE PROBLEMS

Introduction

The concept of a differential equation to include equations that involve partial derivatives, not just ordinary ones. Solutions to such equations will involve functions not just of one variable, but of several variables. Such equations arise naturally, for example, when one is working with situations that involve positions in space that vary over time. To model such a situation, one needs to use functions that have several variables to keep track of the spatial dimensions and an additional variable for time.

Examples of some important PDEs:

(1) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ *One-dimensional wave equation*

(2) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ *One-dimensional heat equation*

(3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ *Two-dimensional Laplace equation*

(4) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ *Two-dimensional Poisson equation*

Partial differential equations: An equation involving partial derivatives of one dependent variable with respective more than one independent variables.

Notations which we use in this unit:

$$p = \frac{\partial z}{\partial x} \;, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, = \frac{\partial^2 z}{\partial x\ \partial y}, \text{ t}= \frac{\partial^2 z}{\partial y^2},$$

Formation of partial differential equation:

A partial differential equation of given curve can be formed in two ways

1. By eliminating arbitrary constants
2. By eliminating arbitrary functions

Problems

Form a partial differential equation by eliminating a,b,c from

$$\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$$

Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Differentiating partially w.r.to x and y, we have

$\frac{1}{a^2}(2x) + \frac{1}{c^2}(2z)\frac{\partial z}{\partial x}$=o

$\frac{1}{a^2}(x) + \frac{1}{c^2}(z)\,p$ =o ________(1)

And $\frac{1}{b^2}(2x) + \frac{1}{c^2}(2z)\frac{\partial z}{\partial x}$=o

$\frac{1}{b^2}(y) + \frac{1}{c^2}(z)$ q = o ________(2)

Diff (1) partially w.r.to x, we have

$\frac{1}{a^2} + \frac{p}{c^2}\frac{\partial z}{\partial x} + \frac{z}{c^2}\frac{\partial p}{\partial x}$=o _______(3)

$\frac{1}{a^2} + \frac{p^2}{c^2} + \frac{z}{c^2}r$ =O

Multiply this equation by x and then subtracting (1) from it

$$\frac{1}{c^2}(xzr + xp^2 - pz) = 0$$

Form a partial differential equation by eliminating the constants from $(x-a)^2+(y-b)^2 = z^2cot^2$α, where α is a parameter

Given $(x-a)^2+(y-b)^2 = z^2cot^2$α ____________(1)

Differentiating partially w.r.to x and y, we have

2 $(x-a)$+0 = 2 z p$cot^2\alpha$

$(x-a)$ = Zp$cot^2\alpha$

And 0+2(y-b) = 2zq$cot^2\alpha$

(Y-b) = zq$cot^2\alpha$

Substituting the values of (x-a) and (y-b) in (1),we get

$$(zpcot^2\alpha)^2+(zqcot^2\alpha)^2 = z^2cot^2\alpha$$

$$(p^2+q^2)(cot^2\alpha)^2 = cot^2\alpha$$

$$p^2+q^2 = tan^2\alpha$$

Linear partial differential equations of first order :

Lagrange's linear equation: An equation of the form Pp + Qq = R is called Lagrange's linear equation.

To solve Lagrange's linear equation consider auxiliary equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Non-linear partial differential equations of first order :

Complete Integral : A solution in which the number of arbitrary constants is equal to the number of independent variables is called complete integral or complete solution of the given equation.

Particular Integral: A solution obtained by giving particular values to the arbitrary constants in the complete integral is called a particular integral.

Singular Integral: let f(x,y,z,p,q) = 0 be a partial differential equation whose complete integral is
To solve non-linear pde we use Charpit's Method :

There are six types of non-linear partial differential equations of first order as given below.
1. f (p,q) = 0
 2. f (z,p,q) = 0
 3. f1 (x,p) = f_2 (y,q)

4. $z = px + qy + f(p,q)$

5. $f(x^m p, y^n q) = 0$ and $f(m^y p, y^n q, z) = 0$

6. $f(pz^m, qz^m) = 0$ and $f_1(x, pz^m) = f_2(y, qz^m)$

Charpit's Method:

We present here a general method for solving non-linear partial differential equations. This is known as Charpit's method.

Let $F(x,y,u, p.q)=0$ be a general nonlinear partial differential equation of first-order. Since u depends on x and y, we have

$du=u_x dx+u_y dy = pdx+qdy$ where $p=u_x=\frac{\partial u}{\partial x}$, $q = u_y=\frac{\partial u}{\partial y}$

If we can find another relation between x,y,u,p,q such that $f(x,y,u,p,q)=0$ then we can solve for p and q and substitute them in equation This will give the solution provided is integrable.

To determine f, differentiate w.r.t. x and y so that

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial u}p+\frac{\partial F}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial x}=0$$

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}p+\frac{\partial f}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial x}=0$$

$$\frac{\partial F}{\partial y}+\frac{\partial F}{\partial u}q+\frac{\partial F}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial y}=0$$

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial u}q+\frac{\partial f}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial y}=0$$

Eliminating $\frac{\partial p}{\partial x}$ from, equations and $\frac{\partial q}{\partial y}$ from equations we obtain

$$\left(\frac{\partial F}{\partial x}\frac{\partial f}{\partial p}-\frac{\partial f}{\partial x}\frac{\partial F}{\partial p}\right)+\left(\frac{\partial F}{\partial u}\frac{\partial f}{\partial p}-\frac{\partial f}{\partial u}\frac{\partial F}{\partial p}\right)p+\left(\frac{\partial F}{\partial q}\frac{\partial f}{\partial p}-\frac{\partial f}{\partial q}\frac{\partial F}{\partial p}\right)\frac{\partial q}{dx}=0$$

$$\left(\frac{\partial F}{\partial y}\frac{\partial f}{\partial q}-\frac{\partial f}{\partial y}\frac{\partial F}{\partial q}\right)+\left(\frac{\partial F}{\partial u}\frac{\partial f}{\partial q}-\frac{\partial f}{\partial u}\frac{\partial F}{\partial q}\right)q+\left(\frac{\partial F}{\partial p}\frac{\partial f}{\partial q}-\frac{\partial f}{\partial p}\frac{\partial F}{\partial q}\right)\frac{\partial p}{dy}=0$$

Adding these two equations and using

$$\frac{\partial q}{\partial x}=\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial p}{\partial y}$$

and rearranging the terms, we get

$$\left(-\frac{\partial F}{\partial p}\right)\frac{\partial f}{\partial x}+\left(-\frac{\partial F}{\partial q}\right)\frac{\partial f}{\partial y}+\left(-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}\right)\frac{\partial f}{\partial u}+\left(\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial u}\right)\frac{\partial f}{\partial p}$$

$$+\left(\frac{\partial F}{\partial y}+q\frac{\partial f}{\partial u}\right)\frac{\partial f}{\partial q}=0$$

We get the auxiliary system of equations

$$\frac{dx}{\frac{-\partial F}{\partial p}}=\frac{dy}{\frac{-\partial F}{\partial q}}=\frac{du}{-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}}=\frac{dp}{\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial u}}=\frac{dq}{\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial u}}=\frac{df}{0}$$

An Integral of these equations, involving p or q or both, can be taken as the required equation.

Problems

solve $(x^2 - y^2 - yz)p + (x^2 - y^2 - zx)q = z(x - y)$

Here

P=$(x^2 - y^2 - yz), Q = (x^2 - y^2 - zx), R = z(x - y)$

The subsidiary equations are

$$\frac{dx}{(x^2-y^2-yz)} = \frac{dy}{(x^2-y^2-zx)} = \frac{dz}{z(x-y)}$$

Using 1,-1,0 and x,-y,0 as multipliers , we have

$$\frac{dz}{z(x-y)} = \frac{dx-dy}{z(x-y)} = \frac{x\,dx-ydy}{(x^2-y^2)(x-y)}$$

From the first two rations Of ,we have

dz= dx-dy

integrating , z=x-y-c_1 or x-y-z = c_1

now taking first and last ratios in (2) ,we get

Heat Equation

For this next PDE, we create a mathematical model of how heat spreads, or diffuses through an object, such as a metal rod, or a body of water. To do this we take advantage of our knowledge of vector calculus and the divergence theorem to set up a PDE that models such a situation. Knowledge of this particular PDE can be used to model situations involving many sorts of diffusion processes, not just heat. For instance the PDE that we will derive can be used to model the spread of a drug in an organism, of the diffusion of pollutants in a water supply.

Solving the Heat Equation in the one-dimensional case

We simplify our heat diffusion modeling by considering the specific case of heat flowing in a long thin bar or wire, where the cross-section is very small, and constant, and insulated in such a way that the heat flow is just along the length of the bar or wire. In this slightly contrived situation, we can model the heat flow by keeping track of the temperature at any point along the bar using just one spatial dimension, measuring the position along the bar.

This means that the function, *u*, that keeps track of the temperature, just depends on *x*, the position along the bar, and *t*, time, and so the heat equation from the previous section becomes the so-called ***one-dimensional heat equation***:

(1) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

MODULE– IV

MAJOR EQUATION TYPES ENCOUNTERED IN ENGINEERING AND PHYSICAL SCIENCES

Wave Equation

For the rest of this introduction to PDEs we will explore PDEs representing some of the basic types of linear second order PDEs: heat conduction and wave propagation. These represent two entirely different physical processes: the process of diffusion, and the process of oscillation, respectively. The field of PDEs is extremely large, and there is still a considerable amount of undiscovered territory in it, but these two basic types of PDEs represent the ones that are in some sense, the best understood and most developed of all of the PDEs. Although there is no one way to solve all PDEs explicitly, the main technique that we will use to solve these various PDEs represents one of the most important techniques used in the field of PDEs, namely separation of variables (which we saw in a different form while studying ODEs). The essential manner of using separation of variables is to try to break up a differential equation involving several partial derivatives into a series of simpler, ordinary differential equations.

Let’s go back to the original idea – start by breaking up the vibrating string into little segments, examine each such segment using Newton’s $F = ma$ equation, and finally figure out what happens as we let the length of the little string segment dwindle to zero, i.e. examine the result as Δx goes to 0. Do you see any limit definitions of derivatives kicking around in equation (7)? As Δx goes to 0, the left-hand side of the equation is in fact just equal to $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$, so the whole thing boils down to:

(8) $\frac{\partial^2 u}{\partial x^2} = \frac{\delta}{T}\frac{\partial^2 u}{\partial t^2}$

which is often written as

(9) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

by bringing in a new constant $c^2 = \frac{T}{\delta}$ (typically written with c^2, to show that it’s a positive constant).

This equation, which governs the motion of the vibrating string over time, is called the ***one-dimensional wave equation***. It is clearly a second order PDE, and it’s linear and homogeneous.

Solution of the Wave Equation by Separation of Variables

There are several approaches to solving the wave equation. The first one we will work with, using a technique called separation of variables, again, demonstrates one of the most widely used solution techniques for PDEs. The idea behind it is to split up the original PDE into a series of simpler ODEs, each of which we should be able to solve readily using tricks already learned. The second technique, which we will see in the next section, uses a transformation trick that also reduces the complexity of the original PDE, but in a very different manner. This second solution is due to Jean Le Rond D’Alembert (an 18th century French mathematician), and is called D’Alembert’s solution, as a result.

First, note that for a specific wave equation situation, in addition to the actual PDE, we will also have boundary conditions arising from the fact that the endpoints of the string are attached solidly, at the left end of the string, when $x = 0$ and at the other end of the string, which we suppose has overall length l. Let's start the process of solving the PDE by first figuring out what these boundary conditions imply for the solution function, $u(x,t)$.

Answer: for all values of t, the time variable, it must be the case that the vertical displacement at the endpoints is 0, since they don't move up and down at all, so that

(1) $u(0,t)=0$ and $u(l,t)=0$ for all values of t

are the ***boundary conditions*** for our wave equation. These will be key when we later on need to sort through possible solution functions for functions that satisfy our particular vibrating string set-up.

You might also note that we probably need to specify what the shape of the string is right when time *t* = 0, and you're right - to come up with a particular solution function, we would need to know $u(x,0)$. In fact we would also need to know the initial velocity of the string, which is just $u_t(x,0)$. These two requirements are called the ***initial conditions*** for the wave equation, and are also necessary to specify a particular vibrating string solution. For instance, as the simplest example of initial conditions, if no one is plucking the string, and it's perfectly flat to start with, then the initial conditions would just be $u(x,0)=0$ (a perfectly flat string) with initial velocity, $u_t(x,0)=0$. Here, then, the solution function is pretty unenlightening – it's just $u(x,t)=0$, i.e. no movement of the string through time.

To start the separation of variables technique we make the key assumption that whatever the solution function is, that it can be written as the product of two independent functions, each one of which depends on just one of the two variables, *x* or *t*. Thus, imagine that the solution function, $u(x,t)$ can be written as

(2) $u(x,t) = F(x)G(t)$

where*F*, and *G*, are single variable functions of *x* and *t* respectively. Differentiating this equation for $u(x,t)$ twice with respect to each variable yields

(3) $\frac{\partial^2 u}{\partial x^2} = F''(x)G(t)$ and $\frac{\partial^2 u}{\partial t^2} = F(x)G''(t)$

Thus when we substitute these two equations back into the original wave equation, which is

(4) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

then we get

(5) $\frac{\partial^2 u}{\partial t^2} = F(x)G''(t) = c^2 \frac{\partial^2 u}{\partial x^2} = c^2 F''(x)G(t)$

Here's where our separation of variables assumption pays off, because now if we separate the equation above so that the terms involving *F* and its second derivative are on one side, and likewise the terms involving *G* and its derivative are on the other, then we get

Here’s where our separation of variables assumption pays off, because now if we separate the equation above so that the terms involving *F* and its second derivative are on one side, and likewise the terms involving *G* and its derivative are on the other, then we get

(6) $\frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)}$

Now we have an equality where the left-hand side just depends on the variable *t*, and the right-hand side just depends on *x*. Here comes the critical observation - how can two functions, one just depending on *t*, and one just on *x*, be equal for all possible values of *t* and *x*? The answer is that they must each be constant, for otherwise the equality could not possibly hold for all possible combinations of *t* and *x*. Aha! Thus we have

(7) $\frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k$

where*k* is a constant. First let’s examine the possible cases for *k*.

Case One: k = 0

Suppose k equals 0. Then the equations in (7) can be rewritten as

(8) $G''(t) = 0 \cdot c^2 G(t) = 0$ and $F''(x) = 0 \cdot F(x) = 0$

yielding with very little effort two solution functions for F and G:

(9) $G(t) = at + b$ and $F(x) = px + r$

where a, b, p and r, are constants (note how easy it is to solve such simple ODEs versus trying to deal with two variables at once, hence the power of the separation of variables approach).

Putting these back together to form $u(x,t) = F(x)G(t)$, then the next thing we need to do is to note what the boundary conditions from equation (1) force upon us, namely that

(10) $u(0,t) = F(0)G(t) = 0$ and $u(l,t) = F(l)G(t) = 0$ for all values of t

Unless $G(t)=0$ (which would then mean that $u(x,t)=0$, giving us the very dull solution equivalent to a flat, unplucked string) then this implies that

(11) $$F(0)=F(l)=0.$$

But how can a linear function have two roots? Only by being identically equal to 0, thus it must be the case that $F(x)=0$. Sigh, then we still get that $u(x,t)=0$, and we end up with the dull solution again, the only possible solution if we start with $k = 0$.

So, let's see what happens if…

Case Two: k > 0

So now if k is positive, then from equation (7) we again start with

(12) $G''(t) = kc^2 G(t)$

and

(13) $F''(x) = kF(x)$

Try to solve these two ordinary differential equations. You are looking for functions whose second derivatives give back the original function, multiplied by a positive constant. Possible candidate solutions to consider include the exponential and sine and cosine functions. Of course, the sine and cosine functions don't work here, as their second derivatives are *negative* the original function, so we are left with the exponential functions.

Let's take a look at (13) more closely first, as we already know that the boundary conditions imply conditions specifically for $F(x)$, i.e. the conditions in (11). Solutions for $F(x)$ include anything of the form

(14) $F(x) = Ae^{\alpha x}$

where $\omega^2 = k$ and *A* is a constant. Since ω could be positive or negative, and since solutions to (13) can be added together to form more solutions (note (13) is an example of a second order linear homogeneous ODE, so that the superposition principle holds), then the general solution for (13) is

(14) $$F(x) = Ae^{\omega x} + Be^{-\omega x}$$

where now *A* and *B* are constants and $\omega = \sqrt{k}$. Knowing that $F(0) = F(l) = 0$, then unfortunately the only possible values of *A* and *B* that work are $A = B = 0$, i.e. that $F(x) = 0$. Thus, once again we end up with $u(x,t) = F(x)G(t) = 0 \cdot G(t) = 0$, i.e. the dull solution once more. Now we place all of our hope on the third and final possibility for *k*, namely...

Case Three: k < 0

So now we go back to equations (12) and (13) again, but now working with *k* as a negative constant. So, again we have

(12) $G''(t) = kc^2 G(t)$

and

(13) $F''(x) = kF(x)$

Exponential functions won't satisfy these two ODEs, but now the sine and cosine functions will. The general solution function for (13) is now

(15) $F(x) = A\cos(\omega x) + B\sin(\omega x)$

where again *A* and *B* are constants and now we have $\omega^2 = -k$. Again, we consider the boundary conditions that specified that $F(0) = F(l) = 0$. Substituting in 0 for *x* in (15) leads to

(16) $F(0) = A\cos(0) + B\sin(0) = A = 0$

so that $F(x) = B\sin(\omega x)$. Next, consider $F(l) = B\sin(\omega l) = 0$. We can assume that *B* isn't equal to 0, otherwise $F(x) = 0$ which would mean that $u(x,t) = F(x)G(t) = 0 \cdot G(t) = 0$, again, the trivial unplucked string solution. With $B \neq 0$, then it must be the case that $\sin(\omega l) = 0$ in order to have $B\sin(\omega l) = 0$. The only way that this can happen is for ωl to be a multiple of π. This means that

(17) $\omega l = n\pi$ or $\omega = \frac{n\pi}{l}$ (where *n* is an integer)

This means that there is an infinite set of solutions to consider (letting the constant *B* be equal to 1 for now), one for each possible integer *n*.

(18) $F(x) = \sin\left(\frac{n\pi}{l}x\right)$

Well, we would be done at this point, except that the solution function $u(x,t) = F(x)G(t)$ and we've neglected to figure out what the other function, $G(t)$, equals. So, we return to the ODE in (12):

(12) $G''(t) = kc^2G(t)$

where, again, we are working with *k,* a negative number. From the solution for $F(x)$ we have determined that the only possible values that end up leading to non-trivial solutions are with $k = -\omega^2 = -\left(\frac{n\pi}{l}\right)^2$ for*n* some integer. Again, we get an infinite set of solutions for (12) that can be written in the form

(19) $$G(t) = C\cos(\lambda_n t) + D\sin(\lambda_n t)$$

where*C* and *D* are constants and $\lambda_n = c\sqrt{-k} = c\omega = \frac{cn\pi}{l}$, where *n* is the same integer that showed up in the solution for $F(x)$ in (18) (we're labeling λ with a subscript "*n*" to identify which value of *n* is used).

Now we really are done, for all we have to do is to drop our solutions for $F(x)$ and $G(t)$ into $u(x,t)=F(x)G(t)$, and the result is

(20) $$u_n(x,t)=F(x)G(t)=\left(C\cos(\lambda_n t)+D\sin(\lambda_n t)\right)\sin\left(\frac{n\pi}{l}x\right)$$

where the integer *n* that was used is identified by the subscript in $u_n(x,t)$ and λ_n, and *C* and *D* are arbitrary constants.

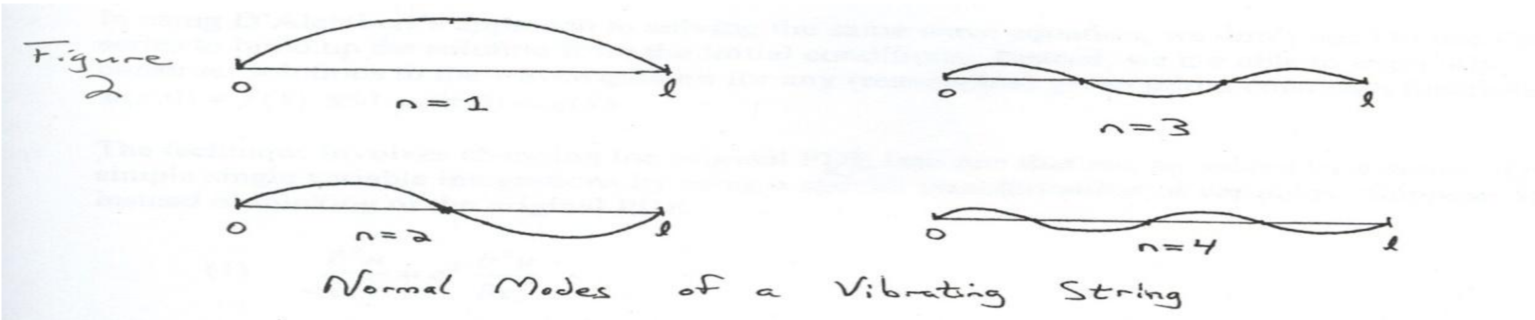
At this point you should be in the habit of immediately checking solutions to differential equations. Is (20) really a solution for the original wave equation

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$$

and does it actually satisfy the boundary conditions $u(0,t)=0$ and $u(l,t)=0$ for all values of *t*

The solution given in the last section really does satisfy the one-dimensional wave equation. To think about what the solutions look like, you could graph a particular solution function for varying values of time, *t*, and then examine how the string vibrates over time for solution functions with different values of *n* and constants *C* and *D*. However, as the functions involved are fairly simple, it's possible to make sense of the solution $u_n(x,t)$ functions with just a little more effort.

For instance, over time, we can see that the $G(t) = (C\cos(\lambda_n t) + D\sin(\lambda_n t))$ part of the function is periodic with period equal to $\frac{2\pi}{\lambda_n}$. This means that it has a frequency equal to $\frac{\lambda_n}{2\pi}$ cycles per unit time. In music one cycle per second is referred to as one *hertz*. Middle C on a piano is typically 263 hertz (i.e. when someone presses the middle C key, a piano string is struck that vibrates predominantly at 263 cycles per second), and the A above middle C is 440 hertz. The solution function when *n* is chosen to equal 1 is called the ***fundamental mode*** (for a particular length string under a specific tension). The other ***normal modes*** are represented by different values of *n*. For instance one gets the 2nd and 3rd normal modes when *n* is selected to equal 2 and 3, respectively. The fundamental mode, when *n* equals 1 represents the simplest possible oscillation pattern of the string, when the whole string swings back and forth in one wide swing. In this fundamental mode the widest vibration displacement occurs in the center of the string (see the figures below).



Thus suppose a string of length *l*, and string mass per unit length δ, is tightened so that the values of *T*, the string tension, along the other constants make the value of $\lambda_1 = \frac{\sqrt{T}}{2l\sqrt{\delta}}$ equal to 440. Then if the string is made to vibrate by striking or plucking it, then its fundamental (lowest) tone would be the A above middle C.

Now think about how different values of *n* affect the other part of $u_n(x,t) = F(x)G(t)$, namely $F(x) = \sin\left(\frac{n\pi}{l}x\right)$. Since $\sin\left(\frac{n\pi}{l}x\right)$function vanishes whenever *x* equals a multiple of $\frac{l}{n}$, then selecting different values of *n* higher than 1 has the effect of identifying which parts of the vibrating string do not move. This has the affect musically of producing *overtones*, which are musically pleasing higher tones relative to the fundamental mode tone. For instance picking *n* = 2 produces a vibrating string that appears to have two separate vibrating sections, with the middle of the string standing still. This mode produces a tone exactly an octave above the fundamental mode. Choosing *n* = 3 produces the 3rd normal mode that sounds like an octave and a fifth above the original fundamental mode tone, then 4th normal mode sounds an octave plus a fifth plus a major third, above the fundamental tone, and so on.

It is this series of fundamental mode tones that gives the basis for much of the tonal scale used in Western music, which is based on the premise that the lower the fundamental mode differences, down to octaves and fifths, the more pleasing the relative sounds. Think about that the next time you listen to some Dave Matthews!

Finally note that in real life, any time a guitar or violin string is caused to vibrate, the result is typically a combination of normal modes, so that the vibrating string produces sounds from many different overtones. The particular combination resulting from a particular set-up, the type of string used, the way the string is plucked or bowed, produces the characteristic tonal quality associated with that instrument. The way in which these different modes are combined makes it possible to produce solutions to the wave equation with different initial shapes and initial velocities of the string. This process of combination involves ***Fourier Series*** which will be covered at the end of Math 21b (come back to see it in action!)

Finally, finally, note that the solutions to the wave equations also show up when one considers acoustic waves associated with columns of air vibrating inside pipes, such as in organ pipes, trombones, saxophones or any other wind instruments (including, although you might not have thought of it in this way, your own voice, which basically consists of a vibrating wind-pipe, i.e. your throat!). Thus the same considerations in terms of fundamental tones, overtones and the characteristic tonal quality of an instrument resulting from solutions to the wave equation also occur for any of these instruments as well. So, the wave equation gets around quite a bit musically!

D'Alembert's Solution of the Wave Equation

As was mentioned previously, there is another way to solve the wave equation, found by Jean Le Rond D'Alembert in the 18th century. In the last section on the solution to the wave equation using the separation of variables technique, you probably noticed that although we made use of the boundary conditions in finding the solutions to the PDE, we glossed over the issue of the initial conditions, until the very end when we claimed that one could make use of something called Fourier Series to build up combinations of solutions. If you recall, being given specific initial conditions meant being given both the shape of the string at time t = 0, i.e. the function $u(x,0) = f(x)$, as well as the initial velocity, $u_t(x,0) = g(x)$ (note that these two initial condition functions are functions of x alone, as t is set equal to 0). In the separation of variables solution, we ended up with an infinite set, or family, of solutions, $u_n(x,t)$ that we said could be combined in such a way as to satisfy any reasonable initial conditions.

In using D'Alembert's approach to solving the same wave equation, we don't need to use Fourier series to build up the solution from the initial conditions. Instead, we are able to explicitly construct solutions to the wave equation for any (reasonable) given initial condition functions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$.

The technique involves changing the original PDE into one that can be solved by a series of two simple single variable integrations by using a special transformation of variables. Suppose that instead of thinking of the original PDE

(1) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

in terms of the variables *x,* and *t,* we rewrite it to reflect two new variables

(2) $v = x + ct$ and $z = x - ct$

This then means that *u*, originally a function of *x,* and *t,* now becomes a function of *v* and *z*, instead. How does this work? Note that we can solve for *x* and *t* in (2), so that

(3) $x = \frac{1}{2}(v+z)$ and $t = \frac{1}{2c}(v-z)$

Now using the chain rule for multivariable functions, you know that

(4) $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} = c\frac{\partial u}{\partial v} - c\frac{\partial u}{\partial z}$

since $\frac{\partial v}{\partial t} = c$ and $\frac{\partial z}{\partial t} = -c$, and that similarly

(5) $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}=\frac{\partial u}{\partial v}+\frac{\partial u}{\partial z}$$

since $\frac{\partial v}{\partial x}=1$ and $\frac{\partial z}{\partial x}=1$. Working up to second derivatives, another, more involved application of the chain rule yields that

(6) $$\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial t}\left(c\frac{\partial u}{\partial v}-c\frac{\partial u}{\partial z}\right)=c\left(\frac{\partial^2 u}{\partial v^2}\frac{\partial v}{\partial t}+\frac{\partial^2 u}{\partial z\partial v}\frac{\partial z}{\partial t}\right)-c\left(\frac{\partial^2 u}{\partial z^2}\frac{\partial z}{\partial t}+\frac{\partial^2 u}{\partial v\partial z}\frac{\partial v}{\partial t}\right)$$

$$=c^2\left(\frac{\partial^2 u}{\partial v^2}-\frac{\partial^2 u}{\partial z\partial v}\right)+c^2\left(\frac{\partial^2 u}{\partial z^2}-\frac{\partial^2 u}{\partial v\partial z}\right)=c^2\left(\frac{\partial^2 u}{\partial v^2}-2\frac{\partial^2 u}{\partial z\partial v}+\frac{\partial^2 u}{\partial z^2}\right)$$

Another almost identical computation using the chain rule results in the fact that

(7) $$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial v}+\frac{\partial u}{\partial z}\right)=\left(\frac{\partial^2 u}{\partial v^2}\frac{\partial v}{\partial x}+\frac{\partial^2 u}{\partial z\partial v}\frac{\partial z}{\partial x}\right)+\left(\frac{\partial^2 u}{\partial z^2}\frac{\partial z}{\partial x}+\frac{\partial^2 u}{\partial v\partial z}\frac{\partial v}{\partial x}\right)$$

$$=\frac{\partial^2 u}{\partial v^2}+2\frac{\partial^2 u}{\partial z\partial v}+\frac{\partial^2 u}{\partial z^2}$$

Now we revisit the original wave equation

(8) $$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$$

and substitute in what we have calculated for $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^2 u}{\partial x^2}$ in terms of $\frac{\partial^2 u}{\partial v^2}$, $\frac{\partial^2 u}{\partial z^2}$ and $\frac{\partial^2 u}{\partial z\partial v}$. Doing this gives the following equation, ripe with cancellations:

(9) $$\frac{\partial^2 u}{\partial t^2}=c^2\left(\frac{\partial^2 u}{\partial v^2}-2\frac{\partial^2 u}{\partial z\partial v}+\frac{\partial^2 u}{\partial z^2}\right)=c^2\frac{\partial^2 u}{\partial x^2}=c^2\left(\frac{\partial^2 u}{\partial v^2}+2\frac{\partial^2 u}{\partial z\partial v}+\frac{\partial^2 u}{\partial z^2}\right)$$

Dividing by c^2 and canceling the terms involving $\frac{\partial^2 u}{\partial v^2}$ and $\frac{\partial^2 u}{\partial z^2}$ reduces this series of equations to

(10) $$-2\frac{\partial^2 u}{\partial z \partial v} = +2\frac{\partial^2 u}{\partial z \partial v}$$

which means that

(11) $$\frac{\partial^2 u}{\partial z \partial v} = 0$$

So what, you might well ask, after all, we still have a second order PDE, and there are still several variables involved. But wait, think about what (11) implies. Picture (11) as it gives you information about the partial derivative of a partial derivative:

(12) $$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial v}\right) = 0$$

In this form, this implies that $\frac{\partial u}{\partial v}$ considered as a function of *z* and *v* is a constant in terms of the variable *z*, so that $\frac{\partial u}{\partial v}$ can only depend on *v*, i.e.

(13) $$\frac{\partial u}{\partial v} = M(v)$$

Now, integrating this equation with respect to *v* yields that

(14) $$u(v,z) = \int M(v)dv$$

This, as an indefinite integral, results in a constant of integration, which in this case is just constant from the standpoint of the variable *v*. Thus, it can be any arbitrary function of *z* alone, so that actually

(15) $$u(v,z) = \int M(v)dv + N(z) = P(v) + N(z)$$

where $P(v)$ is a function of *v* alone, and $N(z)$ is a function of *z* alone, as the notation indicates.

Substituting back the original change of variable equations for *v* and *z* in (2) yields that

(16) $u(x,t) = P(x+ct) + N(x-ct)$

where*P* and *N* are arbitrary single variable functions. This is called D'Alembert's solution to the wave equation. Except for the somewhat annoying but easy enough chain rule computations, this was a pretty straightforward solution technique. The reason it worked so well in this case was the fact that the change of variables used in (2) were carefully selected so as to turn the original PDE into one in which the variables basically had no interaction, so that the original second order PDE could be solved by a series of two single variable integrations, which was easy to do.

Check out that D'Alembert's solution really works. According to this solution, you can pick any functions for *P* and *N* such as $P(v)=v^2$ and $N(v)=v+2$. Then

(17) $$u(x,t)=(x+ct)^2+(x-ct)+2=x^2+x+ct+c^2t^2+2$$

Now check that

(18) $$\frac{\partial^2 u}{\partial t^2}=2c^2$$

and that

(19) $$\frac{\partial^2 u}{\partial x^2}=2$$

so that indeed

(20) $$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$$

and so this is in fact a solution of the original wave equation.

This same transformation trick can be used to solve a fairly wide range of PDEs. For instance one can solve the equation

(21) $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y^2}$$

by using the transformation of variables

(22) $v = x$ and $z = x + y$

(Try it out! You should get that $u(x, y) = P(x) + N(x + y)$ with arbitrary functions *P* and *N*)

Note that in our solution (16) to the wave equation, nothing has been specified about the initial and boundary conditions yet, and we said we would take care of this time around. So now we take a look at what these conditions imply for our choices for the two functions *P* and *N.*

If we were given an initial function $u(x,0)=f(x)$ along with initial velocity function $u_t(x,0)=g(x)$ then we can match up these conditions with our solution by simply substituting in $t=0$ into (16) and follow along. We start first with a simplified set-up, where we assume that we are given the initial displacement function $u(x,0)=f(x)$, and that the initial velocity function $g(x)$ is equal to 0 (i.e. as if someone stretched the string and simply released it without imparting any extra velocity over the string tension alone).

Now the first initial condition implies that

(23) $$u(x,0)=P(x+c\cdot 0)+N(x-c\cdot 0)=P(x)+N(x)=f(x)$$

We next figure out what choosing the second initial condition implies. By working with an initial condition that $u_t(x,0)=g(x)=0$, we see that by using the chain rule again on the functions *P* and *N*

(24) $$u_t(x,0)=\frac{\partial}{\partial t}\left(P(x+ct)+N(x-ct)\right)=cP'(x+ct)-cN'(x-ct)$$

(remember that *P* and *N* are just single variable functions, so the derivative indicated is just a simple single variable derivative with respect to their input). Thus in the case where $u_t(x,0) = g(x) = 0$, then

(25) $$cP'(x+ct) - cN'(x+ct) = 0$$

Dividing out the constant factor *c* and substituting in $t = 0$

(26) $$P'(x) = N'(x)$$

and so $P(x) + k = N(x)$ for some constant *k*. Combining this with the fact that $P(x) + N(x) = f(x)$, means that $2P(x) + k = f(x)$, so that $P(x) = (f(x) - k)/2$ and likewise $N(x) = (f(x) + k)/2$. Combining these leads to the solution

(27) $$u(x,t) = P(x+ct) + N(x-ct) = \frac{1}{2}(f(x+ct) + f(x-ct))$$

To make sure that the boundary conditions are met, we need

(28) $u(0,t)=0$ and $u(l,t)=0$ for all values of t

The first boundary condition implies that

(29) $u(0,t)=\frac{1}{2}\left(f(ct)+f(-ct)\right)=0$

or

(30) $f(-ct)=-f(ct)$

so that to meet this condition, then the initial condition function f must be selected to be an odd function. The second boundary condition that $u(l,t)=0$ implies

(31) $$u(l,t)=\frac{1}{2}\left(f(l+ct)+f(l-ct)\right)=0$$

so that $f(l+ct)=-f(l-ct)$. Next, since we've seen that f has to be an odd function, then $-f(l-ct)=f(-l+ct)$. Putting this all together this means that

(32) $f(l+ct)=f(-l+ct)$ for all values of t

which means that f must have period $2l$, since the inputs vary by that amount. Remember that this just means the function repeats itself every time $2l$ is added to the input, the same way that the sine and cosine functions have period 2π.

What happens if the initial velocity isn't equal to 0? Thus suppose $u_t(x,0)=g(x)\neq 0$. Tracing through the same types of arguments as the above leads to the solution function

(33) $$u(x,t)=\frac{1}{2}\left(f(x+ct)+f(x-ct)\right)+\frac{1}{2c}\int_{x-ct}^{x+ct}g(s)\,ds$$


Thank you