INSTITUTE OF AERONAUTICAL ENGINEERING

(Autonomous)

Dundigal, Hyderabad -500 043

COMPUTER SCIENCE AND ENGINEERING

V Semester

OPTIMIZATION TECHNIQUE

Prepared by:

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COURSE OBJECTIVES (COs):

I: Learn fundamentals of linear programming through optimization.

II: Apply the mathematical results and numerical techniques of optimization theory to concrete Engineering Problems

III: Understand and apply optimization techniques to industrial applications.

IV: Apply the dynamic programming and quadratic approximation to electrical and electronic problems and applications

COURSE LEARNING OUTCOMES

CLO 1: Explain the various characteristics and phases of linear programming.

CLO 2: Formulate the various linear programming problems by using graphical and simplex methods.

CLO 3: Understand the artificial variable techniques like two phase and Big-M methods

CLO 4: Explain Transportation problem and the formulation of the problem by using optimal solution.

CLO 5: Solve the assignment problems by using optimal solutions and the variance of assignment problems.

CLO 6: Describe the travelling sales man problem by using assignment method.

CLO 7: Explain the sequencing and the types of sequencing methods.

CLO 8: Use n jobs through two machines and n jobs through three machines to solve an appropriate problem.

CLO 9: Use two jobs through m machines to solve an appropriate problem.

CLO 10: Understand theory of games and the terminologies used in theory of games concept.

CLO 11: Determine appropriate technique to solve to a given problem.

CLO 12: Solve the problems by using dominance principle and Graphical method.

CLO 13: Understand the Bellman's principle of optimality.

CLO 14: Describe heuristic problem-solving methods with stages.

CLO 15: Understand the mapping of real-world problems to algorithmic solutions.

CLO16:List out the various applications of dynamic programming.

CLO17:Define the shortest path problem with approximate solutions.

CLO 18: Explain the linear programming problem with approximate solutions

CLO 19: Define the various quadratic approximation methods for solving constraint problems.

CLO 20: Explain the direct quadratic approximation for solving the constraint problems.

CLO 21: Explain the quadratic approximation method by using lagrangian function.

CLO 22: Describe the variable metric methods for constrained optimization.

UNIT -I

Running outcomes

CLO 1: Explain the various characteristics and phases of linear programming.

CLO 2: Formulate the various linear programming problems by using graphical and simplex methods.

CLO 3: Understand the artificial variable techniques like two phase and Big-M methods

Operations Research

- **Operations Research** is an Art and Science
- It had its early roots in World War II and is flourishing in business and industry with the aid of computer
- Primary applications areas of Operations Research include forecasting, production scheduling, inventory control, capital budgeting, and transportation.

What is Operations Research?

Operations

- The activities carried out in an organization.

Research

- The process of observation and testing characterized by the scientific method. Situation, problem statement, model construction, validation, experimentation, candidate solutions.
- **Operations Research** is a quantitative approach to decision making based on the scientific method of problem solving.

- **Operations Research** is the scientific approach to execute decision making, which consists of:
  - The art of **mathematical modeling** of complex situations
  - The science of the development of **solution techniques** used to solve these models
  - The ability to effectively **communicate** the results to the decision maker

What do We do

1. OR professionals aim to provide rational bases for decision making by seeking to understand and structure complex situations and to use this understanding to predict system behavior and improve system performance.

2. Much of this work is done using analytical and numerical techniques to develop and manipulate mathematical and computer models of organizational systems composed of people, machines, and procedures.

Terminology

- The British/Europeans refer to **—Operational Research**", the Americans to **—Operations Research**" - but both are often shortened to just "OR".

- Another term used for this field is **—Management Science**" ("MS"). In U.S. OR and MS are combined together to form "OR/MS" or "ORMS".

- Yet other terms sometimes used are **—Industrial Engineering**" ("IE") and **—Decision Science**" ("DS").

History of OR

- OR is a relatively new discipline.
- 70 years ago it would have been possible to study mathematics, physics or engineering at university it would not have been possible to study OR.
- It was really only in the late 1930's that operationas research began in a systematic way.

Features/Characteristics of OR

- Decision-Making
- Scientific Approach
- Inter-Disciplinary Team Approach
- System Approach
- Use of Computers
- Objectives
- Human Factors

Scope of OR

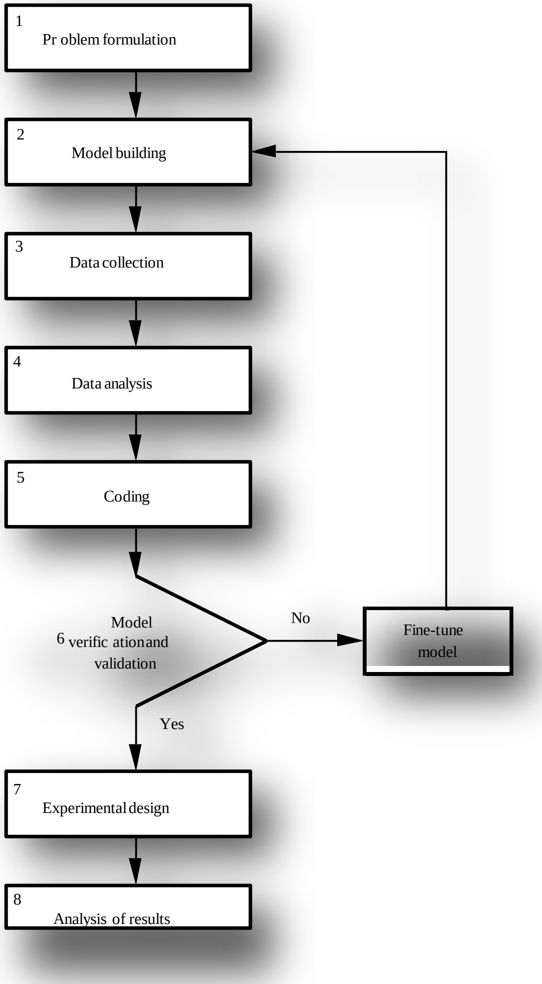
The scope of OR is not only confined to any specific agency like defense services but today it is widely used in all industrial organizations.

It can be used to find the best solution to any problem be it simple or complex. It is useful in every field of human activities. Thus, it attempts to resolve the conflicts of interest among the components of organization in a way that is best for the organization as a whole.

Limitations Of OR

- Magnitude of Computation
- Non-Quantifiable Factors
- Distance between User and Analyst
- Time and Money Costs
- Implementation

Steps in OR Study



What you Should Know about OR

- How decision-making problems are characterized
- OR terminology
- What a model is and how to assess its value
- How to go from a conceptual problem to a quantitative solution

Simplex Method

- Simplex: a linear-programming algorithm that can solve problems having more than two decision variables.
- The simplex technique involves generating a series of solutions in tabular form, called tableaus. By inspecting the bottom row of each tableau, one can immediately tell if it represents the optimal solution. Each tableau corresponds to a corner point of the feasible solution space. The first tableau corresponds to the origin. Subsequent tableaus are developed by shifting to an adjacent corner point in the direction that yields the highest (smallest) rate of profit (cost). This process continues as long as a positive (negative) rate of profit (cost) exists.

Simplex Algorithm

The key solution concepts

- Solution Concept 1: the simplex method focuses on CPF solutions.
- Solution concept 2: the simplex method is an iterative algorithm (a systematic solution procedure that keeps repeating a fixed series of steps, called, an iteration, until a desired result has been obtained) with the following structure:

Simplex algorithm

- Solution concept 3: whenever possible, the initialization of the simplex method chooses the origin point (all decision variables equal zero) to be the initial CPF solution.

- Solution concept 4: given a CPF solution, it is much quicker computationally to gather information about its adjacent CPF solutions than about other CPF solutions. Therefore, each time the simplex method performs an iteration to move from the current CPF solution to a better one, it always chooses a CPF solution that is adjacent to the current one.

- Solution concept 5: After the current CPF solution is identified, the simplex method examines each of the edges of the feasible region that emanate from this CPF solution. Each of these edges leads to an adjacent CPF solution at the other end, but the simplex method doesn‘t even take the time to solve for the adjacent CPF solution. Instead it simply identifies the rate of improvement in Z that would be obtained by moving along the edge. And then chooses to move along the one with largest positive rate of improvement.

- Solution concept 6: A positive rate of improvement in Z implies that the adjacent CPF solution is better than the current one, whereas a negative rate of improvement in Z implies that the adjacent CPF solution is worse. Therefore, the optimality test consists simply of checking whether any of the edges give a positive rate of improvement in Z. if none do, then the current CPF solution is optimal.

Simplex method in tabular form

2. Test for optimality:

Case 1: Maximization problem

the current BF solution is optimal if every coefficient in the objective function row is nonnegative

Case 2: Minimization problem

the current BF solution is optimal if every coefficient in the objective function row is no positive.

3. Iteration

Step 1: determine the entering basic variable by selecting the variable (automatically a non basic variable) with the most negative value (in case of maximization) or with the most positive (in case of minimization) in the last row (Z-row). Put a box around the column below this variable, and call it the ―pivot column‖

References

A Ravindran, “Engineering Optimization”, JohnWiley&Sons Publications, 4th Edition, 2009.

UNIT-II

Running outcomes

CLO 4: Explain Transportation problem and the formulation of the problem by using optimal solution.

CLO 5: Solve the assignment problems by using optimal solutions and the variance of assignment problems.

CLO 6: Describe the travelling sales man problem by using assignment method.

Introduction

- A sequence is the order in which the jobs are processed. Sequence problems arise when we are concerned with situations where there is a choice in which a number of tasks can be performed. A sequencing problem could involve:

- Jobs in a manufacturing plant.
- Aircraft waiting for landing and clearance.
- Maintenance scheduling in a factory.
- Programmes to be run on a computer.
- Customers in a bank & so-on

Terms used:

- **Job :** The jobs or items or customers or orders are the primary stimulus for sequencing. There should be a certain number of jobs say ‗n' to be processed or sequenced.
- **Number of Machines :** A machine is characterized by a certain processing capability or facilities through which a job must pass before it is completed in the shop. It may not be necessarily a mechanical device. Even human being assigned jobs may be taken as machines. There must be certain number of machines say ‗k' to be used for processing the jobs.
- **Processing Time :** Every operation requires certain time at each of machine. If the time is certain then the determination of schedule is easy. When the processing times are uncertain then the schedule is complex.

Terms used:

- **Total Elapsed Time :** It is the time between starting the first job and completing the last one.
- **Idle time :** It is the time the machine remains idle during the total elapsed time.
- **Technological order :** Different jobs may have different technological order. It refers to the order in which various machines are required for completing the jobs.

Types of sequencing problems:

- There can be many types of sequencing problems which are as follows:
- Problem with ‗n' jobs through one machine.
- Problem with ‗n' jobs through two machines.
- Problem with ‗n' jobs through three machines.
- Here the objective is to find out the optimum sequence of the jobs to be processed and starting and finishing time of various jobs through all the machines.
- No passing rule: it implies that passing is not allowed i.e. the same order of jobs is maintained over each machine
- Static arrival pattern. If all the jobs arrive simultaneously.
- Dynamic arrival pattern. Where the jobs arrive continuously.

Basic Assumptions:

Following are the basic assumptions underlying a sequencing problem:

- No machine can process more than one job at a time.
- The processing times on different machines are independent of the order in which they are processed.
- The time involved in moving a job from one machine to another is negligibly small.
- Each job once started on a machine is to be performed up to completion on that machine.
- All machines are of different types.
- All jobs are completely known and are ready for processing.
- A job is processed as soon as possible but only in the order specified.

n jobs through two machines

- Let there be _n‘ jobs each of which is to be processed through two machines say A & B, in the order AB. That is each job will go to machine A first and then to B in other words passing off is not allowed.
- All _n‘ jobs are to be processed on A without any idle time. On the other hand the machine B is subject to its remaining idle at various stages.
- Let A1 A2………….An & B1 B2……..Bn be the expected processing time of n jobs on these two machines.

Steps for n jobs through two machines:

- **Step 1:** Select the smallest processing time occurring in list Ai or Bi, if there is a tie select either of the smallest processing time.
- **Step 2:** If the smallest time is on machine A, then place it at first place if it is for the B machine place the corresponding job at last. Cross off that job.
- **Step 3:** If there is a tie for minimum time on both the machines then select machine A first & machine B last and if there is tie for minimum on machine A (same machine) then select any one of these jobs first and if there is tie for minimum on machine B among and select any of these job in the last.

- **Step 4:** Repeat step 2 & 3 to the reduced set of processing times obtained by deleting the processing time for both the machines corresponding to the jobs already assigned
- **Step 5:** Continue the process placing the job next to the last and so on till all jobs have been placed and it is called optimum sequence.
- **Step 6:** after finding the optimum sequence we can find the followings
- Total elapsed time = Total time between starting the first job of the optimum sequence on machine A and completing the last job on machine B.
- Idle time in machine A = Time when the last job in the optimum sequence is completed on Machine B – Time when the last job in the optimum sequence is completed on Machine A.

Problem:

- In a factory, there are six jobs to process, each of which should go to machines A & B in the order AB. The processing timings in minutes are given, determine the optimal sequencing & total elapsed time.

Jobs	1	2	3	4	5	6
Machine A	7	4	2	5	9	8
Machine B	3	8	6	6	4	1

Solution:

- Step 1: the least of all the times given in for job 6 in machine B. so perform job 6 in the end. It is last in the sequences. Now delete this job from the given data.
- Step 2: Of the remaining timings now the minimum is for job 3 on machine A. so do the job . Now delete this job 3 also.
- Step 3: Now the smallest time is 3 minutes for job first on machine B. thus perform job 1 at the second last before job 6.

n jobs through two machines

- Example 2:Suppose we have five jobs, each of which has to be processed on two machines A & B in the order AB. Processing times are given in the following table:

Job	Machine A	Machine B
1	6	3
2	2	7
3	10	8
4	4	9
5	11	5

Determine an order in which these jobs should be processed so as to minimize the total processing time.

Solution:

The minimum time in the above table is 2, which corresponds to job 2 on machine A.

2				

Now we eliminate job 2 from further consideration. The reduced set of processing times are as follows:

Job	Machine A	Machine B
1	6	3
3	10	8
4	4	9
5	11	5

The minimum time is 3 for job 1 on machine B. Therefore, this job would be done in last. The allocation of jobs till this stage would be

2				1

After deletion of job 1, the reduced set of processing times are as follows:

Job	Machine A	Machine B
3	10	8
4	4	9
5	11	5

Similarly, by repeating the above steps, the optimal sequence is as follows:

2	4	3	5	1

Once the optimal sequence is obtained, the minimum elapsed time may be calculated as follows:

Job	Machine A		Machine B	
	Time in	**Time out**	**Time in**	**Time out**
2	0	2	2	9
4	2	6	9	18
3	6	16	18	26
5	16	27	27	32
1	27	33	33	36

- Idle time for machine A

= total elapsed time - time when the last job is out of machine A = 36-33=3hours.

- Idle time for machine B = 2 + (9 - 9) + (18 - 18) + (27 - 26) + (33 - 32) = 4 hours.

Example3 :Strong Book Binder has one printing machine, one binding machine, and the manuscripts of a number of different books. Processing times are given in the following table:

Book	Time In Hours	
	Printing	Binding
A	5	2
B	1	6
C	9	7
D	3	8
E	10	4

We wish to determine the order in which books should be processed on the machines, in order to minimize the total time required.

Solution.

The minimum time in the above table is 1, which corresponds to the book B on printing machine.

B				

Now book B is eliminated. The reduced set of processing times is as follows:

Book	Time In Hours	
	Printing	Binding
A	5	2
C	9	7
D	3	8
E	10	4

The minimum time is 2 for book A on binding machine. Therefore, this job should be done in last. The allocation of jobs till this stage is:

B				A

The reduced set of processing times is as follows:

Book	**Time In Hours**	
	Printing	**Binding**
C	9	7
D	3	8
E	10	4

- Similarly, by repeating the above steps, the optimal sequence is as follows

B	D	C	E	A

Once the optimal sequence is obtained, the minimum elapsed time may be calculated as follows:

Book	Printing		Binding	
	Time in	Time out	Time in	Time out
B	0	1	1	7
D	1	4	7	15
C	4	13	15	22
E	13	23	23	27
A	23	28	28	30

- Idle time for printing process = total elapsed time - time when the last job is out of machine A 30- 28=2hours.

- Idle time for binding process = 1 + (7 - 7) + (15 - 15) + (23 - 22) + (28 - 27) = 3 hours

References

A Ravindran, “Engineering Optimization”, JohnWiley&Sons Publications, 4th Edition, 2009.

UNIT-III

Running outcomes

CLO 7: Explain the sequencing and the types of sequencing methods.

CLO 8: Use n jobs through two machines and n jobs through three machines to solve an appropriate problem.

CLO 9: Use two jobs through m machines to solve an appropriate problem.

CLO 10: Understand theory of games and the terminologies used in theory of games concept.

CLO 11: Determine appropriate technique to solve to a given problem.

CLO 12: Solve the problems by using dominance principle and Graphical method.

n jobs through three machines

Processing n jobs on 3 Machines:

- There is no solution available for the general sequencing problems of n jobs through 3 machines. However we do have a method under the circumstance that no passing of jobs is permissible and if either or both the following conditions are satisfied.
- 1)The minimum time on machine A is greater than or equal to the maximum time on machine B.
- 2)The minimum time on machine C is greater than or equal to the maximum time on machine B
- Or both are satisfied that the following method can be applied

Method of Procedure:

- Step1: First of all, the given problem is replaced with an equivalent problem involving n jobs and 2 fictitious machines G and H .define the corresponding processing times Gi and Hi by
- Gi=AI+BI
- Hi=Bi+Ci
- Step2: to the problem obtained step1 above ,the method for processing n jobs through 2 machines is applied .The optimal sequence resulting this shall also be optimal for the given problem.

Example 1: There are five jobs which must go through these machines A,B and C the order ABC .Processing times of the jobs on different machines given below.

Jobs	A	B	C
1	7	5	6
2	8	5	8
3	6	4	7
4	5	2	4
5	6	1	3

- Determine a sequence for 5 jobs which will minimize elapsed time(T) .

Solution: according to given information

Min.Ai=5

Max.Bi=5

Min.Ci=3

Here since Min.Ai=Max.Ci, the first of the conditions is satisfied.

We shall now determines Gi and Hi and from them find the optimal sequence.

In accordance with the rules for determining optimal sequence in respect of n jobs processi ng on 2 machines , the sequence for above shall be:

3 2 1 4 5

Table : Calculation of Total Elapsed Time(T) .

Jobs	Machine A In	Machine A out	Machine B In	Machine B out	Machine C In	Machine C out
3	0	6	6	10	10	17
2	6	14	14	19	19	27
1	14	21	21	26	27	33
4	21	26	21	28	33	37
5	26	32	32	33	37	40

Total elapsed time (T) =40 hours.

n jobs through three machines

Example 2: The MDH Masala company has to process five items on three machines:- A, B & C. Processing times are given in the following table:

Item	A_i	B_i	C_i
1	4	4	6
2	9	5	9
3	8	3	11
4	6	2	8
5	3	6	7

Find the sequence that minimizes the total elapsed time.

Solution:

Here, Min. $(A_i) = 3$, Max. $(B_i) = 6$ and Min. $(C_i) = 6$. Since the condition of Max. $(B_i) \leq$ Min. (C_i) is satisfied, the problem can be solved by the above procedure. The processing times for the new problem are given below.

Item	$G_i = A_i + B_i$	$H_i = B_i + C_i$
1	8	10
2	14	14
3	11	14
4	8	10
5	9	13

The optimal sequence is

1	4	5	3	2

Item	Machine A		Machine B		Machine C	
	Time in	Time out	Time in	Time out	Time in	Time out
1	0	4	4	8	8	14
4	4	10	10	12	14	22
5	10	13	13	19	22	29
3	13	21	21	24	29	40
2	21	30	30	35	40	49

Total elapsed time = 49.

Idle time for machine A = 49 – 30 = 19 hours.

Idle time for machine B = 4 + (10 - 8) + (13 - 12) + (21 - 19)+ (30 - 24) + (49 - 35) = 29 hours.

Idle time for machine C = 8 + (14 - 14) + (22 - 22) + (29 - 29)+ (40 - 40) = 8 hours.

n jobs through three machines

Example 3: Shahi Export House has to process five items through three stages of production, via, cutting, sewing & pressing. Processing times are given in the following table:

Item	Cutting (A_i)	Sewing (B_i)	Pressing (C_i)
1	3	3	5
2	8	4	8
3	7	2	10
4	5	1	7
5	2	5	6

Determing an order in which these items should be processed so as to minimize the total processing time.

Solution:

The processing times for the new problem are given below.

Item	$G_i = A_i + B_i$	$H_i = B_i + C_i$
1	6	8
2	12	12
3	9	12
4	6	8
5	7	11

- Thus, the optimal sequence may be formed in any of the two ways.

1	4	5	3	2

4	1	5	3	2

Item	Cutting		Sewing		Pressing	
	Time in	Time out	Time in	Time out	Time in	Time out
1	0	3	3	6	6	11
4	3	8	8	9	11	18
5	8	10	10	15	18	24
3	10	17	17	19	24	34
2	17	25	25	29	34	42

- Total elapsed time = 42
- Idle time for cutting process = 42 – 25 = 17 hours. Idle time for sewing process = 3 + (8 - 6) + (10 - 9) + (17 - 15)+ (25 - 19) + (42 - 29) = 27 hours. Idle time for pressing process = 6 + (11 - 11) + (18 - 18) + (24 - 24)+ (34 - 34) = 6 hours.

Processing n jobs through m machines

- This section focuses on the sequencing problem of processing two jobs through m machines. Problems under this category can be solved with the help of graphical method. The graphical method below is explained with the help of the following example.
- Two jobs are to be performed on five machines A, B, C, D, and E. Processing times are given in the following table.

			Machine				
Job 1	**Sequence**	:	A	B	C	D	E
	Time	:	3	4	2	6	2
Job 2	**Sequence**	:	B	C	A	D	E
	Time	:	5	4	3	2	6

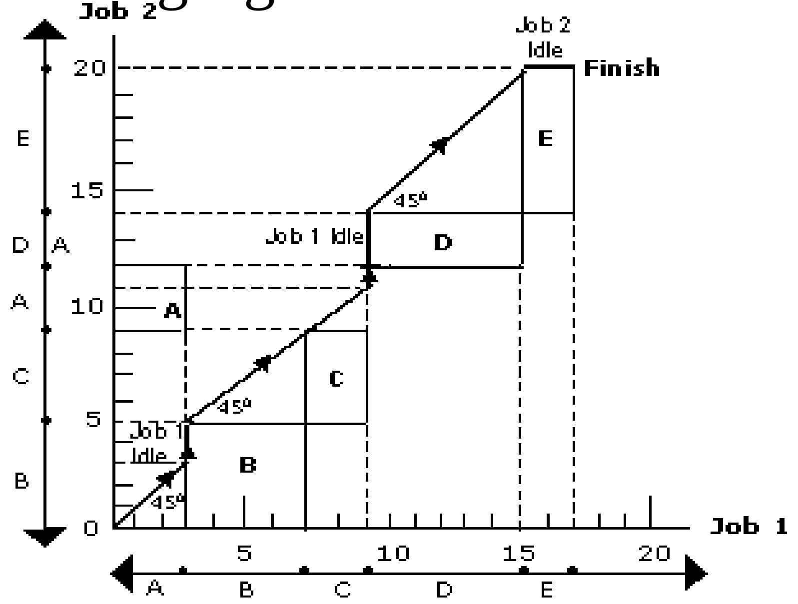
Use graphical method to obtain the total minimum elapsed time.

Solution:

Steps

Mark the processing times of job 1 & job 2 on X-axis & Y-axis respectively.

Draw the rectangular blocks by pairing the same machines as shown in the following figure.



- Starting from origin O, move through the 45^0 line until a point marked finish is obtained.
- The elapsed time can be calculated by adding the idle time for either job to the processing time for that job.
- **Idle time** for **job 1** is 5 hours.
- **Elapsed time** = processing time for job 1+Idle tome of job 1

  = (3+4+2+6+2)+5=22 hours.
- Likewise **idle time** of **job 2** is 2 hours.
- **Elapsed time** =processing time of job 2+Idle time of job 2

  = (5+4+3+2+6)+2=22 hours.

- Example 2:There are 4 job ABCD are required to be processed on four machine M1, M2,M3, M4 in that order. Determine optimal sequence and total elapsed time.

Job	M1	M2	M3	M4
A	13	8	7	14
B	12	6	8	19
C	9	7	5	15
D	8	5	6	15

Given

Job	M1	M2	M3	M4
A	13	8	7	14
B	12	6	8	19
C	9	7	5	15
D	8	5	6	15

- Step 1- 1st we have to convert this problem into two machine problem. For that we have to check following condition:
- Min M1 or Min M4 >= Max M2 or Max M3
- here Min M1=8, Min M4=14, Max M2=7, Max M3=8.
- therefore 8=8
- Min M1=Max M3
- Consolidation or Conversion Table:

JOB	MACHINES 5		MACHINES 6	
	P(M1+M2+M3)	NEW TIME	P(M2+M3+M4)	NEW TIME
A	13+8+7	28	8+7+14	29
B	12+6+8	26	6+8+19	33
C	9+7+5	21	7+5+15	27
D	8+5+6	19	5+6+15	26

- New job timing According to Consolidation Table:

Job	A	B	C	D
New M/c 5	28	26	21	19
New M/c 6	29	33	27	26

- **Sequencing According to consolidation Table:**
- Consolidated table:

Job	A	B	C	D
New M/c 5	28	26	21	19
New M/c 6	29	33	27	26

- **Job sequence:**

JOB	D	B	A	C

	Machine 1			Machine 2			Machine 3			Machine 4		
S.Q	ST	TT	ET	ST	TT	ET	ST	TT	ET	ST	TT	ET
D	0	8	8	8	5	13	13	6	19	19	15	34
B	8	12	20	20	6	26	26	8	34	34	19	53
A	20	13	33	33	8	41	41	7	48	53	14	67
C	33	9	42	42	7	49	49	5	54	67	15	82
T.T		49			26			26			63	

- Total Elapsed time= 82 hrs.
- **Idle Time for M/c 1=**Total Elapsed Time- Total time of M/c 1 =82-42= 40hrs.
- **Idle Time for M/c 2=**Total Elapsed Time- Total time of M/c 2 =82-26= 56hrs.
- **Idle Time for M/c 3=**Total Elapsed Time- Total time of M/c 3 = 82-26= 56hrs.
- **Idle Time for M/c 4=**Total Elapsed Time- Total time of M/c 4 =82-63=19hrs.

Characteristics of Games

Introduction to Game Theory:

Game theory is a kind of decision theory in which one's alternative action is determined after taking into consideration all possible alternatives available to an opponent playing the similar game, rather than just by the possibilities of various outcome results. Game theory does not insist on how a game must be played but tells the process and principles by which a particular action should be chosen. Therefore it is a decision theory helpful in competitive conditions.

- **<u>Properties of a Game</u>**
- There are finite number of competitors known as 'players'
- All the strategies and their impacts are specified to the players but player does not know which strategy is to be selected.

- A game is played when every player selects one of his strategies. The strategies are supposed to be prepared simultaneously with an outcome such that no player recognizes his opponent's strategy until he chooses his own strategy.
- The figures present as the outcomes of strategies in a matrix form are known as 'pay-off matrix'.
- The game is a blend of the strategies and in certain units which finds out the gain or loss.
- The player playing the game always attempts to select the best course of action which results in optimal pay off known as 'optimal strategy'.

Characteristics of Game Theory:

1. Competitive game:

- A competitive situation is known as **competitive game** if it has the four properties
- There are limited number of competitors such that $n \geq 2$. In the case of $n = 2$, it is known as **two-person game** and in case of $n > 2$, it is known as **n-person game**.
- Each player has a record of finite number of possible actions.
- A play is said to takes place when each player selects one of his activities. The choices are supposed to be made simultaneously i.e. no player knows the selection of the other until he has chosen on his own.
- Every combination of activities finds out an outcome which results in a gain of payments to every player, provided each player is playing openly to get as much as possible.

- **2. Strategy**
- The strategy of a player is the determined rule by which player chooses his strategy from his own list during the game. The two types of strategy are
- Pure strategy
- Mixed strategy
- **Pure Strategy**
- If a player knows precisely what another player is going to do, a deterministic condition is achieved and objective function is to maximize the profit. Thus, the pure strategy is a decision rule always to choose a particular startegy.
- **Mixed Strategy**
- If a player is guessing as to which action is to be chosen by the other on any particular instance, a probabilistic condition is achieved and objective function is to maximize the expected profit. Hence the mixed strategy is a choice among pure strategies with fixed probabilities.

Repeated Game Strategies

- In repeated games, the chronological nature of the relationship permits for the acceptance of strategies that are dependent on the actions chosen in previous plays of the game.
- Most contingent strategies are of the kind called as "trigger" strategies.
- For Example trigger strategies
- In prisoners' dilemma: At start, play doesn't confess. If your opponent plays Confess, then you need to play Confess in the next round. If your opponent plays don't confess, then go for doesn't confess in the subsequent round. This is called as the "tit for tat" strategy.
- In the investment game, if you are sender: At start play Send. Play Send providing the receiver plays Return.

3. Number of persons

When the number of persons playing is 'n' then the game is known as 'n' person game. The person here means an individual or a group aims at a particular objective.

Two-person, zero-sum game

- A game with just two players (player A and player B) is known as 'two-person, zero-sum game', if the losses of one player are equal to the gains of the other one so that the sum total of their net gains or profits is zero.
- Two-person, zero-sum games are also known as rectangular games as these are generally presented through a payoff matrix in a rectangular form.

4. Number of activities

The activities can be finite or infinite.

5. Payoff

Payoff is referred to as the quantitative measure of satisfaction a person obtains at the end of each play

6. Payoff matrix

- Assume the player A has 'm' activities and the player B has 'n' activities. Then a payoff matrix can be made by accepting the following rules
- Row designations for every matrix are the activities or actions available to player A
- Column designations for every matrix are the activities or actions available to player B
- Cell entry V_{ij} is the payment to player A in A's payoff matrix when A selects the activity i and B selects the activity j.

7. Value of the game

Value of the game is the maximum guaranteed game to player A (maximizing player) when both the players utilizes their best strategies. It is usually signifies with 'V' and it is unique.

Game Models, Terminology

Classification of Games:

Simultaneous vs. Sequential Move Games

- Games where players select activities simultaneously are simultaneous move games.
- Examples: Sealed-Bid Auctions, Prisoners' Dilemma.
- Must forecast what your opponent will do at this point, finding that your opponent is also doing the same.
- Games where players select activities in a particular series or sequence are sequential move games.
- Examples: Bargaining/Negotiations, Chess.
- Must look forward so as to know what action to select now.
- Many sequential move games have deadlines on moves.

One-Shot versus Repeated Games:

One-shot: play of the game takes place once.

- Players likely not know much about each another.
- Example - tipping on vacation

- Repeated: play of the game is recurring with the same players.
  - Finitely versus Indefinitely repeated games
  - Reputational concerns do matter; opportunities for cooperative behavior may emerge.

Advise: If you plan to follow an *aggressive* strategy, ask yourself whether you are in a one-shot game or in repeated game. If a repeated game then *think again*.

- **Usually games are divided into:**
- Pure strategy games
- Mixed strategy games

- The technique for solving these two types changes. By solving a game, we require to determine best strategies for both the players and also to get the value of the game. Saddle **point method** can be used to solve pure strategy games.
- The diverse methods for solving a mixed strategy game are
- Dominance rule
- Analytical method
- Graphical method
- Simplex method

- **Basic Game Theory Terms:**

- Game : Description of the situation includes the rules of the game.
- Players : Decision makers in the game.
- Payoffs : Expected rewards enjoyed at the end of the game.
- Actions : Possible choices made by the player.
- Strategies : Specified plan of action for every contingency played by other players.

Rule for Game theory(with saddle point and without saddle point)

- Rules for Game theory (with saddle point and without saddle point):
- **Rule 1:** Look for pure strategy (saddle point)
- **Rule 2**:Reduce game by Dominance.
- If no pure strategies exist,the next step is to eliminate certain strategies (row/column) by law of Dominance.
- **Rule 3:**Solve for mixed strategy.

A mixed strategy can be solved by different solution method, such as

1. Arithmetic method
2. Algebraic method
3. Graphical method
4. Matrix method
5. Short cut method

References

A Ravindran, “Engineering Optimization”, JohnWiley&Sons Publications, 4th Edition, 2009.

UNIT-IV

Running outcomes

CLO 13: Understand the Bellman’s principle of optimality.

CLO 14: Describe heuristic problem-solving methods with stages.

CLO 15: Understand the mapping of real-world problems to algorithmic solutions.

CLO16:List out the various applications of dynamic programming.

CLO17:Define the shortest path problem with approximate solutions.

CLO 18: Explain the linear programming problem with approximate solutions

2X2 Games Problems

Pure Strategies (with saddle points):

- In a zero-sum game, the pure strategies of two players constitute a saddle point if the corresponding entry of the payoff matrix is simultaneously a maximum of row minima and a minimum of column maxima. This decision-making is referred to as the **minimax-maximin principle** to obtain the best possible selection of a strategy for the players.
- In a pay-off matrix, the minimum value in each row represents the minimum gain for player A. Player A will select the strategy that gives him the maximum gain among the row minimum values. The selection of strategy by player A is based on maximin principle. Similarly, the same pay-off is a loss for player B. The maximum value in each column represents the maximum loss for Player B. Player B will select the strategy that gives him the minimum loss among the column maximum values.
- The selection of strategy by player B is based on minimax principle. If the maximin value is equal to minimax value, the game has a saddle point (i.e., equilibrium point). Thus the strategy selected by player A and player B are optimal

Example 1: Consider the example to solve the game whose pay-off matrix is given in the following table as follows:

Player B

$$
\begin{array}{ccc}
 & 1 & 2 \\
1 & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{pmatrix} 1 & 3 \\ -1 & 6 \end{pmatrix}$}} \\
2 & &
\end{array}
$$

$$
\textbf{Player A} \quad
\begin{array}{c c}
 & \begin{array}{cc} 1 & 2 \end{array} \\
\begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 3 \\ -1 & 6 \end{pmatrix}
\end{array}
$$

- The game is worked out using minimax procedure. Find the smallest value in each row and select the largest value of these values. Next, find the largest value in each column and select the smallest of these numbers. The procedure is shown in the following table.

Minimax Procedure

		Player B 1	Player B 2	Row Min
Player A →	1	1	3	1
	2	–1	6	–1
Col Max →		1	6	

- If Maximum value in row is equal to the minimum value in column, then saddle point exists.
  Max Min = Min Max
  1 = 1
- Therefore, there is a saddle point.
  The strategies are,
  Player A plays Strategy A1, (AA1).
  Player B plays Strategy B1, (B B1).
- Value of game = 1.

- **Example 2:** Check whether the following game is given in Table, determinable and fair.

$$
\begin{array}{ccc}
 & & \textbf{Player B} \\
 & & \begin{array}{cc} \mathbf{1} & \mathbf{2} \end{array} \\
\textbf{Player A} & \begin{array}{c} \mathbf{1} \\ \mathbf{2} \end{array} & \begin{pmatrix} 7 & 0 \\ 0 & 8 \end{pmatrix}
\end{array}
$$

Solution: The game is solved using maximin criteria as shown in Table.

Maximin Procedure

$$
\begin{array}{lc|cc|c}
 & & \multicolumn{2}{c|}{\textbf{Player B}} & \\
 & & \textbf{1} & \textbf{2} & \textbf{Row Min} \\
\textbf{Player A} & \textbf{1} & 7 & 0 & 0 \\
 & \textbf{2} & 0 & 8 & 0 \\
\textbf{Column Max} & & 7 & 8 &
\end{array}
$$

The game is strictly neither determinable nor fair.

- **Example 3:** Identify the optimal strategies for player A and player B for the game, given below in Table. Also find if the game is strictly determinable and fair.

		Player B		
		1	2	Row Min
Player A	1	4	0	0
	2	1	–3	–3
Col Max		4	0	

- The game is strictly determinable and fair. The saddle point exists and the game has a pure strategy. The optimal strategies are given in the following table.
- **Optimal Strategies**

$$(a)\quad S_A \begin{array}{c} \begin{matrix} 1 & \quad 2 \end{matrix} \\ \begin{pmatrix} p_1 & p_2 \\ 1 & 0 \end{pmatrix} \end{array} \qquad \text{and} \qquad (b)\quad S_B \begin{array}{c} \begin{matrix} 1 & \quad 2 \end{matrix} \\ \begin{pmatrix} q_1 & q_2 \\ 0 & 1 \end{pmatrix} \end{array}$$

2X2 Games Problems

- **Analytical Method:[No saddle point exists so using analytical method]**
- A 2 x2 payoff matrix where there is no saddle point can be solved by analytical method.
- Given the matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Value of the game is

$$V = \frac{a_{11}a_{22} - a_{21}a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

- With the coordinates

$$x_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}, \quad x_2 = \frac{a_{11} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$y_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}, \quad y_2 = \frac{a_{11} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

Alternative procedure to solve the strategy

▪Find the difference of two numbers in column 1 and enter the resultant under column 2. Neglect the negative sign if it occurs.

•Find the difference of two numbers in column 2 and enter the resultant under column 1. Neglect the negative sign if it occurs.

•Repeat the same procedure for the two rows

- **Example 1:**

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

Solution:

It is a 2 x 2 matrix and no saddle point exists. We can solve by analytical method

$$\begin{array}{ccc} & B & \\ A & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} & \begin{array}{c} 1 \\ 4 \end{array} \\ & \begin{array}{cc} 3 & \quad 2 \end{array} & \end{array}$$

$$V = \frac{a_{11}\, a_{22} - a_{21}\, a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{20 - 3}{9 - 4}$$

$V = 17 / 5$

$S_A = (x_1, x_2) = (1/5, 4/5)$

$S_B = (y_1, y_2) = (3/5, 2/5)$

- **Example 2:**

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \end{array}$$

Solution:

$$\begin{array}{cccc} & \multicolumn{2}{c}{B} & \\ A & \left[\begin{array}{cc} 2 & -1 \\ -1 & 0 \end{array}\right] & & \begin{array}{c} 1 \\ 3 \end{array} \\ & \begin{array}{cc} 1 & 3 \end{array} & & \end{array}$$

$$V = \frac{a_{11}\, a_{22} - a_{21}\, a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{0 - 1}{2 + 2}$$

$V = -1 / 4$

$S_A = (x_1, x_2) = (1/4, 3/4)$

$S_B = (y_1, y_2) = (1/4, 3/4)$

Benefits of flow shop sequencing

- Improved process efficiency.
- Improved machine utilization.
- Increased production rate.
- Reduced total processing time.
- Minimum or Zero Ideal Time.
- Potential increase in profits and decrease in costs.

- Idle time for machine A = total elapsed time - time when the last job is out of machine A
  36-33=3hours.
- Idle time for machine B = 2 + (9 - 9) + (18 - 18) + (27 - 26) + (33 - 32) = 4 hours.

Example3 :Strong Book Binder has one printing machine, one binding machine, and the manuscripts of a number of different books. Processing times are given in the following table:

Book	Time In Hours	
	Printing	Binding
A	5	2
B	1	6
C	9	7
D	3	8
E	10	4

3X3 Games Problems

- **Example 1:** Solve the game with the pay-off matrix for player A as given in table.

$$
\begin{array}{ccc}
 & & \textbf{Player B} \\
 & & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\
\textbf{Player A} & \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{pmatrix} -4 & 0 & 4 \\ 1 & 4 & 2 \\ -1 & 5 & -3 \end{pmatrix}
\end{array}
$$

Solution: Find the smallest element in rows and largest elements in columns as shown in table.

Minimax Procedure

Player B

		B_1	B_2	B_3	Row min
Player A	A_1	–4	0	4	–4
	A_2	1	4	2	1
	A_3	–1	5	–3	–3
	Column Max	1	5	4	

Select the largest element in row and smallest element in column.

Check for the minimax criterion,

Max Min = Min Max

1 = 1

Therefore, there is a saddle point and it is a pure strategy.

Optimum Strategy:

Player A A_2 Strategy

Player B B_1 Strategy

The value of the game is 1.

Example 2: Solve the game with the payoff matrix given in table and determine the best strategies for the companies A and B and find the value of the game for them.

$$\begin{array}{cc} & \text{Company B} \\ \text{Company A} & \begin{pmatrix} 2 & 4 & 2 \\ 1 & -5 & -4 \\ 2 & 6 & -2 \end{pmatrix} \end{array}$$

Solution: The matrix is solved using maximin criteria, as shown in table below.

- Maximin Procedure

		Company B			
		1	**2**	**3**	**Row Min**
Company A	1	2	4	2	2
	2	1	–5	–4	–5
	3	2	6	–2	–2
Column Max		2	6	2	

Max Min = Min Max

2 = 2

- Therefore, there is a saddle point.
  Optimum strategy for company A is A_1 and
  Optimum strategy for company B is B_1 or B_3.

3X3 Games Problems

Example 1: *A* and *B* play a game in which each has three coins, *a* 5 paisa, 10 paisa and 20 paisa coins. Each player selects a coin without the knowledge of the other's choice. If the sum of the coins is an odd amount, *A* wins *B*'s coins. If the sum is even, *B* wins *A*'s coins. Find the optimal strategies for the players and the value of the game.

Solution:

The pay of matrix for the given game is: Assume 5 paisa as the I strategy, 10 paisa as the II strategy and the 20 paisa as the III strategy.

$$
\begin{array}{cccc|ccc}
 & & & & 5 & 10 & 20 \\
 & & & & \mathrm{I} & \mathrm{II} & \mathrm{III} \\
\hline
 & 5 & \mathrm{I} & & -10 & 15 & 25 \\
A & 10 & \mathrm{II} & & 15 & -20 & -30 \\
 & 20 & \mathrm{III} & & 25 & -30 & -40
\end{array}
$$

In the problem it is given when the sum is odd, *A* wins *B*'s coins and when the sum is even, *B* will win *A*'s coins. Hence the actual pay of matrix is:

			5 I	10 II	20 III	Row minimum
	5	I	–5	10	20	–5
A	10	II	5	–10	–10	–10
	20	III	5	–20	–20	–20
Column maximum.			5	10	20	

The problem has no saddle point. Column I and II are dominating the column III. Hence it is removed from the game. The reduced matrix is:

			5	10	
			I	II	Row minimum
	5	I	–5	10	–5
A	10	II	5	–10	–10
	20	III	5	–20	–20
Column maximum.			5	10	

The problem has no saddle point. Considering *A*, row III is dominated by row II, hence row III is eliminated from the matrix. The reduced matrix is:

		I	II	Row minimum
A	I	–5	10	–5
	II	5	–10	–10
Column maximum.		5	10	

No saddle point. By application of formulae:

x1 = (a22 – a21) / (a11 + a22) – (a12 + a21) or = 1 – x2 = (–10 – 5) / [–5 + (–10)] – (10 – 5)

= –15 / (–15 – 5) = (–15 / –20) = (15 / 20) = 3 / 4, hence x2 = 1 – (3 / 4) = 1 / 4

y1 = (a22 – a12) / (a11 + a22) – (a12 + a21) or = 1 – y2 = (–10 – 10) / –20 = 20 / 20 = **1** and

y2 = 0

Value of the game = **v = (a11 a22 – a12 a21) / (a11 + a22) – (a12 + a21)** = (50 – 50) / – 20 = **0 Answer is A (3/ 4, 1 / 4, 0), B (1, 0, 0) , v = 0.**

- Idle time for machine A = total elapsed time - time when the last job is out of machine A
36-33=3hours.
- Idle time for machine B = 2 + (9 - 9) + (18 - 18) + (27 - 26) + (33 - 32) = 4 hours.

Example3 :Strong Book Binder has one printing machine, one binding machine, and the manuscripts of a number of different books. Processing times are given in the following table:

Book	**Time In Hours**	
	Printing	**Binding**
A	5	2
B	1	6
C	9	7
D	3	8
E	10	4

2Xn Games or mX2 Games Problems

2*n game problem:

When we can reduce the given payoff matrix to 2×3 or 3×2 we can get the solution by method of **sub games**. If we can reduce the given matrix to $2 \times n$ or $m \times 2$ sizes, then we can get the solution by **graphical method**. A game in which one of the players has two strategies and other player has number of strategies is known as $2 \times n$ or $m \times 2$ games. If the game has saddle point it is solved. If no saddle point, if it can be reduced to 2×2 by method of dominance, it can be solved. When no more reduction by dominance is possible, we can go for Method of Sub games or Graphical method. We have to identify 2×2 sub games within $2 \times n$ or $m \times 2$ games and solve the game.

Problem 1: Solve the game whose payoff matrix is:

$$\begin{array}{cc|ccc} & & & B & \\ & & I & II & III \\ \hline A & I & -4 & 3 & -1 \\ & II & 6 & -4 & -2 \end{array}$$

No saddle point.

The sub games are:

Sub game I:

		B		
		I	II	Row minimum
A	I	–4	3	–4
	II	6	–4	–4
Column Maximum.		6	3	

No saddle point. First let us find the value of the sub games by applying the formula. Then compare the values of the sub games; which ever is favorable for the candidate, that sub game is to be selected. Now here as *A* has only two strategies and *B* has three strategies, the game, which is favorable to *B*, is to be selected.

Value of the game = $\boldsymbol{v1 = (a11\ a22 - a12\ a21) / (a11 + a22) - (a12 + a21)}$
$= (-4 \times -4) - (3 \times 6) / [(-4 + -4] - (3 + 6) = 2 / 17$

Sub game II:

		B		
		I	III	Row minimum
A	I	−4	−1	−4
	II	6	−2	−2
Column maximum.		6	1	

No saddle point, hence value of the game = $v_2 = (a_{11}\, a_{22} - a_{12}\, a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$

$= [(-4) \times (-2)] - [(-1) \times 6] / [(-4) + (-2) - (6 - 1) = -(14/11)$

2Xn Games or mX2 Games Problems

m*2 game problem:

When we can reduce the given payoff matrix to 2×3 or 3×2 we can get the solution by method of **sub games**. If we can reduce the given matrix to $2 \times n$ or $m \times 2$ sizes, then we can get the solution by **graphical method**. A game in which one of the players has two strategies and other player has number of strategies is known as $2 \times n$ or $m \times 2$ games. If the game has saddle point it is solved. If no saddle point, if it can be reduced to 2×2 by method of dominance, it can be solved. When no more reduction by dominance is possible, we can go for Method of Sub games or Graphical method. We have to identify 2×2 sub games within $2 \times n$ or $m \times 2$ games and solve the game.

Problem 1: Solve the following $2 \times n$ sub game:

		B	
		I	II
A	I	1	8
	II	3	5
	III	11	2

Solution: The given game is $m \times 2$ game.

		B: I	B: II	Row minimum
A	I	1	8	1
	II	3	5	3
	III			2
Column maximum.		11	8	

No saddle point. Hence *A*'s Sub games are:

A's sub game No.1.

		B		
		I	II	Row minimum
A	I	1	8	1
	II	3	5	3
Column Maximum.		3	8	

The game has saddle point and hence value of the game is $v1 = 3$ A's sub game No.2.

Graphical method for 2Xn Games or mX2 Games

Graphical method: The graphical method is used to solve the games whose payoff matrix has

- Two rows and n columns (2 x n)
- m rows and two columns (m x2)

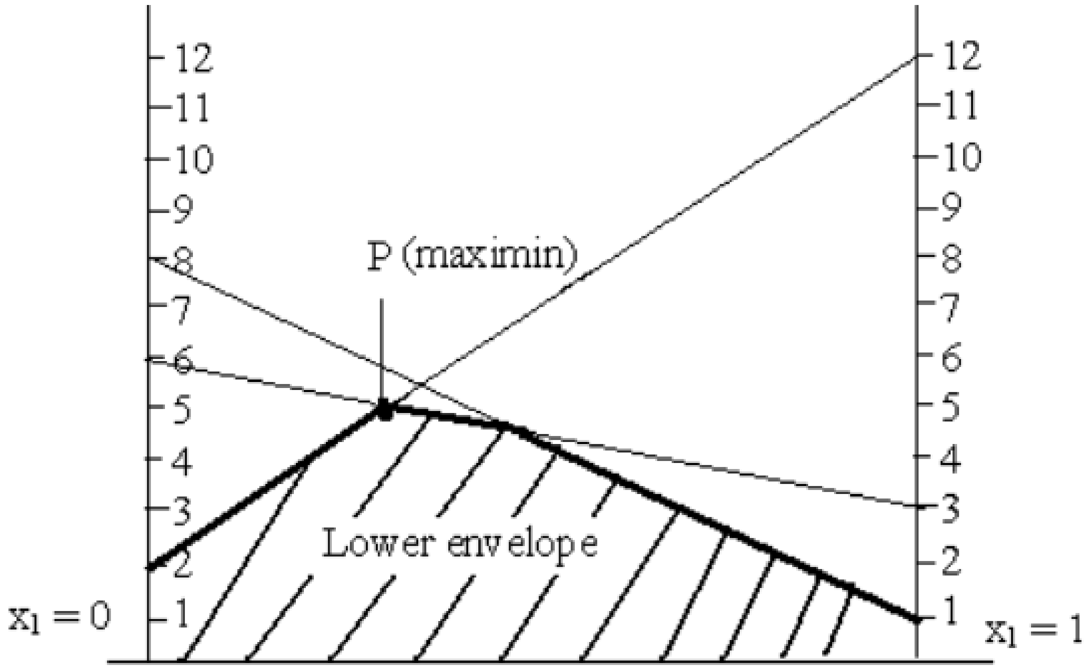
Algorithm for solving 2 x n matrix games:

- Draw two vertical axes 1 unit apart. The two lines are x1= 0, x1= 1
- Take the points of the first row in the payoff matrix on the vertical line x1= 1 and the points of the second row in the payoff matrix on the vertical line x1= 0.
- The point a1jon axis x1= 1 is then joined to the point a2jon the axis x1= 0 to give a straight line. Draw 'n' straight lines for j=1, 2... n and determine the highest point of the lower envelope obtained. This will be the maximin point.
- The two or more lines passing through the maximin point determines the required 2 x 2 payoff matrix. This in turn gives the optimum solution by making use of analytical method.

Example 1: Solve by graphical method

$$\begin{array}{c c} & \begin{array}{ccc} B1 & B2 & B3 \end{array} \\ \begin{array}{c} A1 \\ A2 \end{array} & \begin{bmatrix} 1 & 3 & 12 \\ 8 & 6 & 2 \end{bmatrix} \end{array}$$

Solution:



	B2	B3	
A1	3	12	4
A2	6	2	9
	10	3	

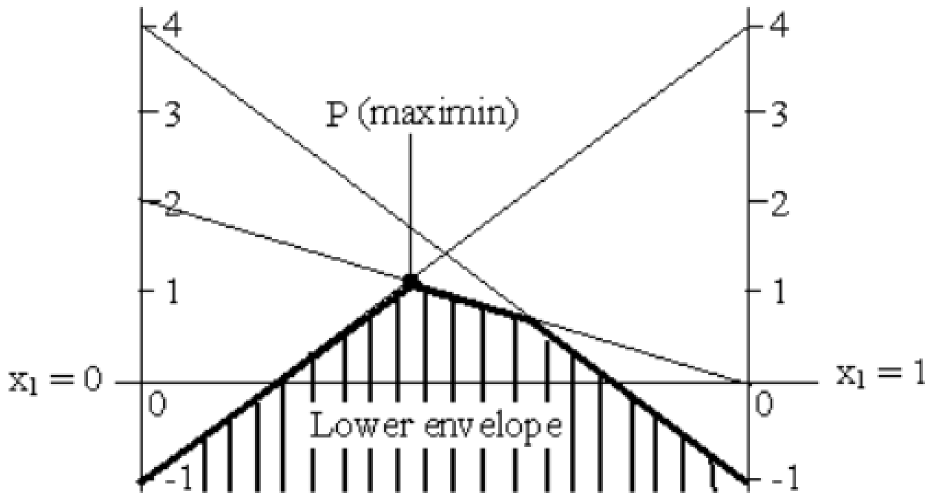
$$V = \frac{a_{11}\, a_{22} - a_{21}\, a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{6 - 72}{5 - 18}$$

$V = 66/13$

$S_A = (4/13, 9/13)$

$S_B = (0, 10/13, 3/13)$

Solution:



$$\begin{array}{cccc} & B1 & B3 & \\ A1 & \left[\begin{array}{c} 4 \\ -1 \end{array}\right. & \left.\begin{array}{c} 0 \\ 2 \end{array}\right] & \begin{array}{c} 3 \\ 4 \end{array} \\ & 2 & 5 & \end{array}$$

A2

$$V = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{8-0}{6+1}$$

$V = 8/7$

$S_A = (3/7,\ 4/7)$

$S_B = (2/7,\ 0,\ 5/7)$

Graphical method for 2Xn Games or mX2 Games

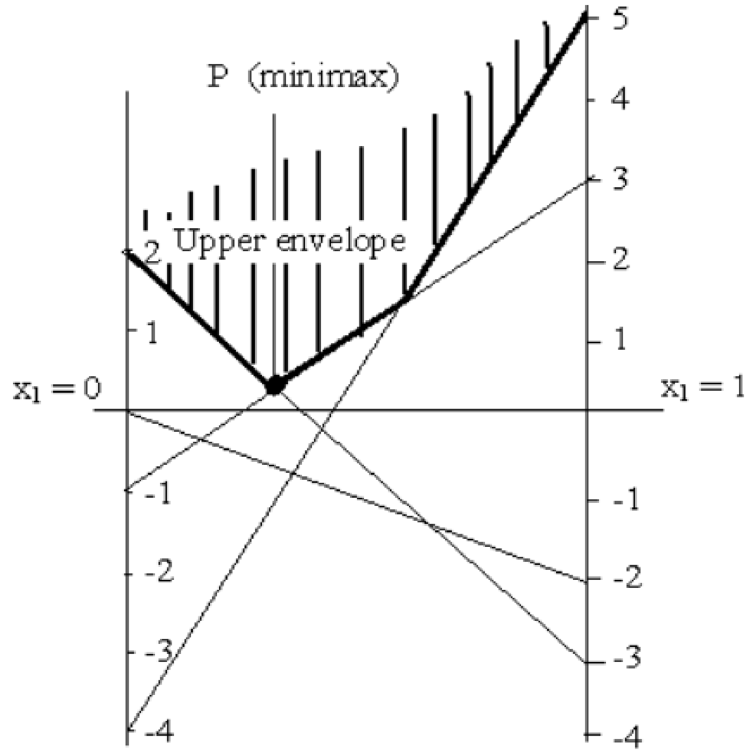
Algorithm for solving m x 2 matrix games:

- Draw two vertical axes 1 unit apart. The two lines are x1=0, x1= 14
- Take the points of the first row in the payoff matrix on the vertical line x1= 1 and the points of the second row in the payoff matrix on the vertical line x1= 0.
- The point a1jon axis x1= 1 is then joined to the point a2jon the axis x1= 0 to give a Straight line. Draw ‘n’ straight lines for j=1, 2... n and determine the lowest point of the upper envelope obtained. This will be the minimax point.
- The two or more lines passing through the minimax point determines the required 2 x 2 payoff matrix. This in turn gives the optimum solution by making use of analytical method.

Example 1: Solve by graphical method

$$\begin{array}{c|cc} & B1 & B2 \\ \hline A1 & -2 & 0 \\ A2 & 3 & -1 \\ A3 & -3 & 2 \\ A4 & 5 & -4 \end{array}$$

Solution:



$$\begin{array}{c|cc|c} & B1 & B2 & \\ \hline A2 & 3 & -1 & 5 \\ A3 & -3 & 2 & 4 \\ \hline & 3 & 6 & \end{array}$$

$$V = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{6-3}{5+4}$$

V = 3/9 = 1/3

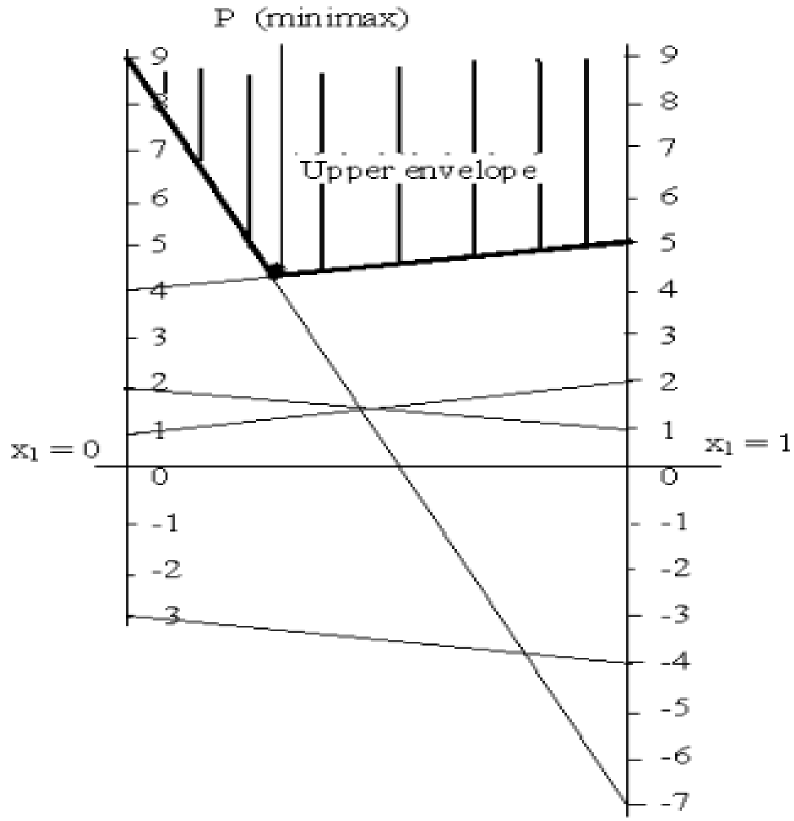
SA=(0, 5/9, 4/9, 0)

SB=(3/9, 6/9)

Example 2: Solve by graphical method

$$\begin{array}{cc} & \begin{array}{cc} B1 & B2 \end{array} \\ \begin{array}{c} A1 \\ A2 \\ A3 \\ A4 \\ A5 \end{array} & \begin{bmatrix} 1 & 2 \\ 5 & 4 \\ -7 & 9 \\ -4 & -3 \\ 2 & 1 \end{bmatrix} \end{array}$$

Solution:



$$\begin{array}{c|cc|c} & B1 & B2 & \\ A2 & 5 & 4 & 16 \\ A3 & -7 & 9 & 1 \\ & 5 & 12 & \end{array}$$

$$V = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{45 + 28}{14 + 3}$$

V = 73/17

S_A = (0, 16/17, 1/17, 0, 0)

S_B = (5/17, 12 /17)

References

A Ravindran, "Engineering Optimization", JohnWiley&Sons Publications, 4thEdition, 2009.

UNIT-V

Running outcomes

CLO 19: Define the various quadratic approximation methods for solving constraint problems.

CLO 20: Explain the direct quadratic approximation for solving the constraint problems.

CLO 21: Explain the quadratic approximation method by using lagrangian function.

CLO 22: Describe the variable metric methods for constrained optimization.

QUADRATIC APPROXIMATION

- Quadratic approximation is an extension of linear approximation, where we are adding one more term, which is related to the second derivative. The formula fo r the quadratic approximation of a function f(x) for values of x near x"0" is :

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 \quad (x \approx x_0)$$

- Compare this to the old formula for the linear approximation of f:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \quad (x \approx x_0).$$

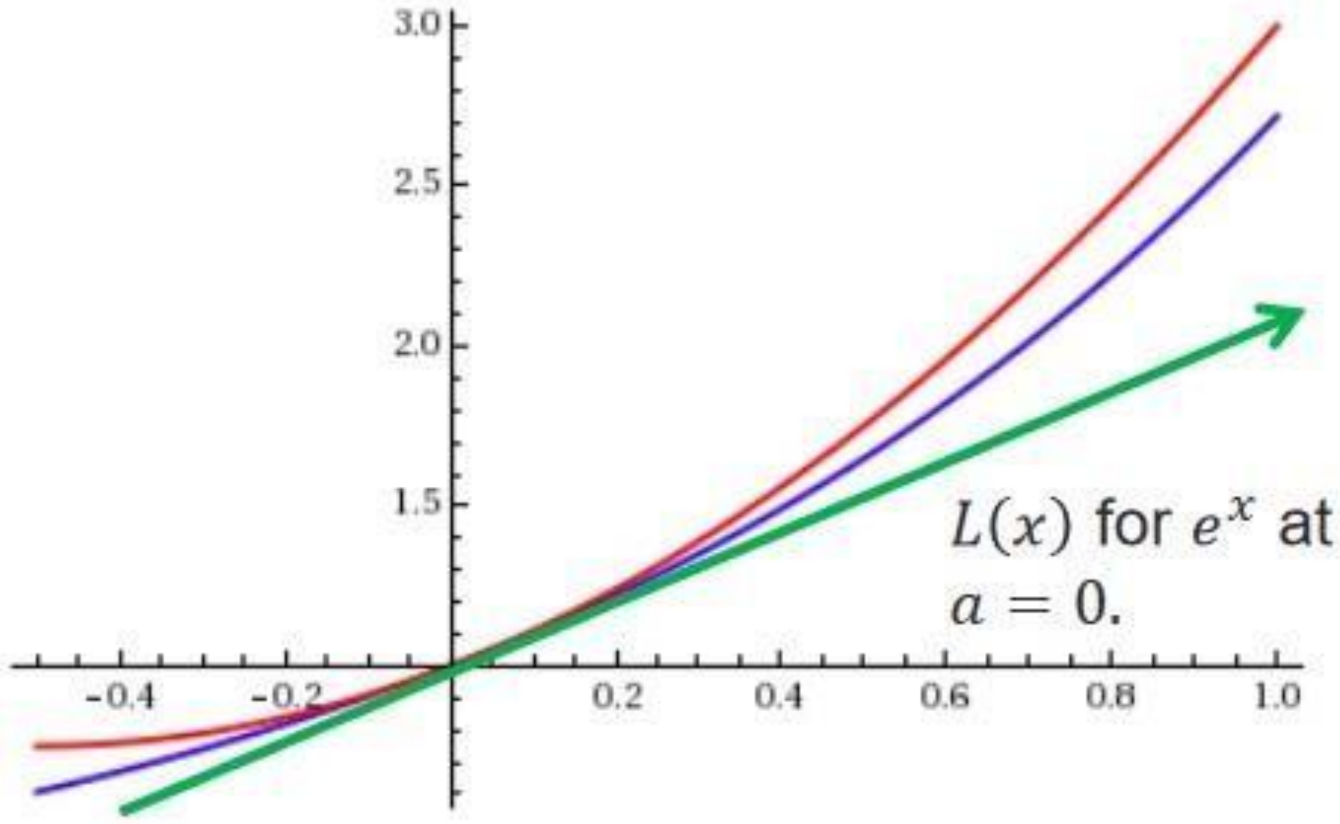
- We got from the linear approximation to the quadratic one by adding one more term that is related to the second derivatives:

$$f(x) \approx \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{Linear Part}} + \underbrace{\frac{f''(x_0)}{2}(x - x_0)^2}_{\text{Quadratic Part}} \quad (x \approx x_0)$$

QUADRATIC APPROXIMATION

- These are more complicated and so are only used when higher accuracy is needed.
- The quadratic approximation also uses the point x=a to approximate nearby values, but uses a parabola instead of just a tangent line to do so.
- This gives a closer approximation because the parabola stays closer to the actual function.

QUADRATIC APPROXIMATION



Algorithm to Convert a CFG into GNF

Algorithm to Convert a CFG into Greibach Normal Form

Step 1 − If the start symbol **S** occurs on some right side, create a new start symbol **S’** and a new production **S’** → **S**.

Step 2 − Remove Null productions. (Using the Null production removal algorithm discussed earlier)

Step 3 − Remove unit productions. (Using the Unit production removal algorithm discussed earlier)

Step 4 − Remove all direct and indirect left-recursion.

Step 5 − Do proper substitutions of productions to convert it into the proper form of GNF.

EXAMPLE1

- Example: Convert the following grammar G into Greibach Normal Form (GNF).

  $S \rightarrow X\ A | BB$

  $B \rightarrow b | SB$

  $X \rightarrow b$

  $A \rightarrow a$

- To write the above grammar G into GNF, we shall follow the following steps:
- Rewrite G in Chomsky Normal Form (CNF) It is already in CNF.
- Re-label the variables

  S with A_1

  X with A_2

  A with A_3

  B with A_4

EXAMPLE1

- After re-labeling the grammar looks like:

  $A1 \rightarrow A2\,A3 | A4A4$

  $A4 \rightarrow b | A1\,A4$

  $A2 \rightarrow b$

  $A3 \rightarrow a$

- Identify all productions which do not conform to any of the types listed below:

  $A_i \rightarrow A_j\, x_k$ such that $j > I$

  $Z_i \rightarrow A_j\, x_k$ such that $j \leq n$

  $A_i \rightarrow a x_k$ such that $x_k \in V*$ and $a \in T$

EXAMPLE1

$A_4 \rightarrow A_1 A_4$............................identified

$A_4 \rightarrow A_1 A_4 | b.$

- To eliminate A_1 we will use the substitution rule $A_1 \rightarrow A_2 A_3 | A_4 A_4$.
- Therefore, we have $A_4 \rightarrow A_2 A_3 A_4 | A_4 A_4 A_4 | b$
- The above two productions still do not conform to any of the types in step 3.

Substituting for $A_2 \rightarrow b$

$A_4 \rightarrow bA_3 A_4 | A_4 A_4 A_4 | b$

- Now we have to remove left recursive production

$A_4 \rightarrow A_4 A_4 A_4$

$A_4 \rightarrow bA_3 A_4 | b | bA_3 A_4 Z | bZ$

$Z \rightarrow A_4 A_4 | A_4 A_4 Z$

EXAMPLE1

At this stage our grammar now looks like

$A_1 \rightarrow A_2 A_3 | A_4 A_4$

$A_4 \rightarrow bA_3 A_4 | b | bA_3 A_4 Z | bZ$

$Z \rightarrow A_4 A_4 | A_4 A_4 Z$

$A_2 \rightarrow b$

$A_3 \rightarrow a$

All rules now conform to one of the types in step 3.

But the grammar is still not in Greibach Normal Form!

EXAMPLE1

All productions for A_2 , A_3 and A_4 are in GNF

for $A_1 \rightarrow A_2 A_3|A_4A_4$

Substitute for A_2 and A_4 to convert it to GNF

$A_1 \rightarrow bA_3 |bA_3 A_4A_4 |bA_4 |bA_3A_4 Z A_4|bZ A_4$

for $Z \rightarrow A_4 A_4 |A_4A_4Z$

Substitute for A_4 to convert it to GNF

$Z \rightarrow bA_3 A_4 A_4 |bA_4|bA_3 A_4 Z A_4|bZ A_4 |bA_3A_4 A_4 Z|bA_4Z|bA_3 A_4Z A_4 Z|bZ A_4Z$ 8. Finally the grammar in GNF is

$A_1 \rightarrow bA_3 |bA_3 A_4A_4 |bA_4 |bA_3A_4 Z A_4|bZ A_4$

$A_4 \rightarrow bA_3 A_4 |b|bA_3 A_4Z|bZ$

$Z \rightarrow bA_3 A_4 A_4 |bA_4|bA_3 A_4 Z A_4|bZ A_4 |bA_3A_4 A_4 Z|bA_4Z|bA_3 A_4Z A_4 Z|bZ A_4Z$

$A_2 \rightarrow b$

$A_3 \rightarrow a$

EXAMPLE2

- Convert the following CFG into CNF
- $S \rightarrow XY \mid Xn \mid p$
- $X \rightarrow mX \mid m$
- $Y \rightarrow Xn \mid o$
- Solution
- Here, **S** does not appear on the right side of any production and there are no unit or null productions in the production rule set. So, we can skip Step 1 to Step 3.
- Step 4
- Now after replacing
- X in $S \rightarrow XY \mid Xo \mid p$
- with
- $mX \mid m$

EXAMPLE2

- we obtain
- $S \rightarrow mXY \mid mY \mid mXo \mid mo \mid p.$
- And after replacing
- X in $Y \rightarrow X_n \mid o$ with the right side of
- $X \rightarrow mX \mid m$
- we obtain
- $Y \rightarrow mXn \mid mn \mid o.$
- Two new productions $O \rightarrow o$ and $P \rightarrow p$ are added to the production set and then we came to the final GNF as the following −
- $S \rightarrow mXY \mid mY \mid mXC \mid mC \mid p$
- $X \rightarrow mX \mid m$
- $Y \rightarrow mXD \mid mD \mid o$
- $O \rightarrow o$
- $P \rightarrow p$

Types of Non-Linear Programming Problems:-

- In the preceding two chapters we considered a number of alternative strategies for exploiting linear approximations to nonlinear problem functions.
- In general we found that, depending upon the strategy employed, linearizations would either lead to vertex points of the linearized constraint sets or generate descent directions for search.
- In either case, some type of line search was required in order to approach the solution of non-corner-point constrained problems.
- Based upon our experience with unconstrained problems, it is reasonable to consider the use of higher order approximating functions since these could lead directly to better approximations of non-corner-point solutions.

TYPES OF NON-LINEAR PROGRAMMING PROBLEMS

- For instance, in the single-variable case we found that a quadratic approximating function could be used to predict the location of optima lying in the interior of the search interval.
- In the multivariable case, the use of a quadratic approximation (e.g., in Newton‘s method) would yield good estimates of unconstrained minimum points.
- Furthermore, the family of quasiNewton methods allowed us to reap some of the benefits of a quadratic approximation without explicitly developing a full second-order approximating function at each iteration.
- In fact, in the previous chapter we did to some extent exploit the acceleration capabilities of quasi-Newton methods by introducing their use within the direction generation mechanics of the reduced gradient and gradient projection methods.

TYPES OF NON-LINEAR PROGRAMMING PROBLEMS

- Thus, much of the discussion of the previous chapters does point to the considerable algorithmic potential of using higher order, specifically quadratic, approximating functions for solving constrained problems.
- In this chapter we examine in some detail various strategies for using quadratic approximations. We begin by briefly considering the consequence of direct quadratic approximation, the analog of the successive LP strategy.
- Then we investigate the use of the second derivatives and Lagrangian constructions to formulate quadratic programming (QP) subproblems, the analog to Newton's method.

TYPES OF NON-LINEAR PROGRAMMING PROBLEMS

- Finally, we discuss the application of quasi-Newton formulas to generate updates of quadratic terms.
- We will find that the general NLP problem can be solved very efficiently via a series of sub problems consisting of a quadratic objective function and linear constraints, provided a suitable line search is carried out from the solution of each such sub problem.
- The resulting class of exterior point algorithms can be viewed as a natural extension of quasi-Newton methods to constrained problems.

DIRECT QUADRATIC APPROXIMATION

- The solution of the general NLP problem is by simply replacing each nonlinear function by its local quadratic approximation at the solution estimate x^0 and solving the resulting series of approximating sub problems.

$$q(x; x^0) = f(x^0) + \nabla f(x^0)^T (x - x^0) + \tfrac{1}{2}(x - x^0)^T \nabla^2 f(x^0)(x - x^0)$$

- If each function $f(x)$ is replaced by its quadratic approximation then the sub problem becomes one of minimizing a quadratic function subject to quadratic equality and inequality constraints.
- While it seems that this sub problem structure ought to be amendable to efficient solution, in fact, it is not.
- To be sure, the previously discussed strategies for constrained problems can solve this sub problem but at no real gain over direct solution of the original problem.

QUADRATIC APPROXIMATION

- The For a sequential strategy using sub problem solutions to be effective, the sub problem solutions must be substantially easier to obtain than the solution of the original problem.
- Recall the problems with a quadratic positive-definite objective function and linear constraints can be solved in a finite number of reduced gradient iterations provided that quasi-Newton or conjugate gradient enhancement of the reduced gradient direction vector is used.
- Of course, while the number of iterations is finite, each iteration requires a line search, which strictly speaking is itself not a finite procedure.

QUADRATIC APPROXIMATION

- However, as there are specialized methods for these so-called QP problems that will obtain a solution in a finite number of iterations without line searching, using instead simplex like pivot operations.
- Given that QP problems can be solved efficiently with truly finite procedures, it appears to be desirable to formulate our approximating sub problems as quadratic programs.
- Thus, assuming that the objective function is twice continuously differentiable, a plausible solution strategy would consist of the following steps:

QUADRATIC APPROXIMATION

DIRECT SUCCESSIVE QUADRATIC PROGRAMMING SOLUTION

Given x^0, an initial solution estimate, and a suitable method for solving QP subproblems.

Step 1: Formulate the QP problem

Step 2: Solve the QP problem and

Step 3: Check for convergence. If not converged, repeat step 1.

QUADRATIC APPROXIMATION

Example

Solve the problem

$$\begin{aligned} \text{Minimize} \quad & f(x) = 6x_1x_2^{-1} + x_2x_1^{-2} \\ \text{Subject to} \quad & h(x) = x_1x_2 - 2 = 0 \\ & g(x) = x_1 + x_2 - 1 \geq 0 \end{aligned}$$

- from the initial feasible estimate $x^0 = (2, 1)$ using the direct successive QP strategy.
- At x^0, $f(x^0) = 12.25$, $h(x^0) = 0$, and $g(x^0) = 2 > 0$. The derivatives required to construct the QP sub problem are

QUADRATIC APPROXIMATION

$$\nabla f(x) = ((6x_2^{-1} - 2x_2x_1^{-3}), (-6x_1x_2^{-2} + x_1^{-2}))^T$$

$$\nabla^2 f = \begin{pmatrix} (6x_2x_1^{-4}) & (-6x_2^{-2} - 2x_1^{-3}) \\ (-6x_2^{-2} - 2x_1^{-3}) & (12x_1x_2^{-3}) \end{pmatrix}$$

$$\nabla h(x) = (x_2, x_1)^T$$

- Thus, the first QP subproblem will be

$$\begin{aligned} &\text{Minimize} && \left(\tfrac{23}{4}, -\tfrac{47}{4}\right)d + \tfrac{1}{2}d^T\begin{pmatrix} \frac{3}{8} & -\frac{25}{4} \\ -\frac{25}{4} & 24 \end{pmatrix}d \\ &\text{Subject to} && (1, 2)d = 0 \\ & && (1, 1)d + 2 \geq 0 \end{aligned}$$

QUADRATIC APPROXIMATION

- Since the first constraint can be used to eliminate one of the variables, that is,

$$d_1 = -2d_2$$

- the resulting single-variable problem can be solved easily analytically to give

$$d^0 = (-0.92079, 0.4604)$$

- Thus, the new point becomes

$$x^{(1)} = x^0 + d^0 = (1.07921, 1.4604)$$

at which point

QUADRATIC APPROXIMATION

$$f(x^{(1)}) = 5.68779$$

$$h(x^{(1)}) = -0.42393$$

$$g(x^{(1)}) > 0$$

- Note that the objective function value has improved substantially but that the equality constraint is violated. Suppose we continue with the solution procedure. The next sub problem is

QUADRATIC APPROXIMATION

Minimize $(1.78475, -2.17750)d + \frac{1}{2}d^T\begin{pmatrix} 6.4595 & -4.4044 \\ -4.4044 & 4.1579 \end{pmatrix}d$

Subject to $(1.4604, 1.07921)d - 0.42393 = 0$

$$(1, 1)d + 1.5396 \geq 0$$

The solution is $d^{(1)} = (-0.03043, 0.43400)$, resulting in the new point, with

$$x^{(2)} = (1.04878, 1.89440)$$

$$f(x^{(2)}) = 5.04401$$

$$h(x^{(2)}) = -0.013208$$

$$g(x^{(2)}) > 0$$

QUADRATIC APPROXIMATION

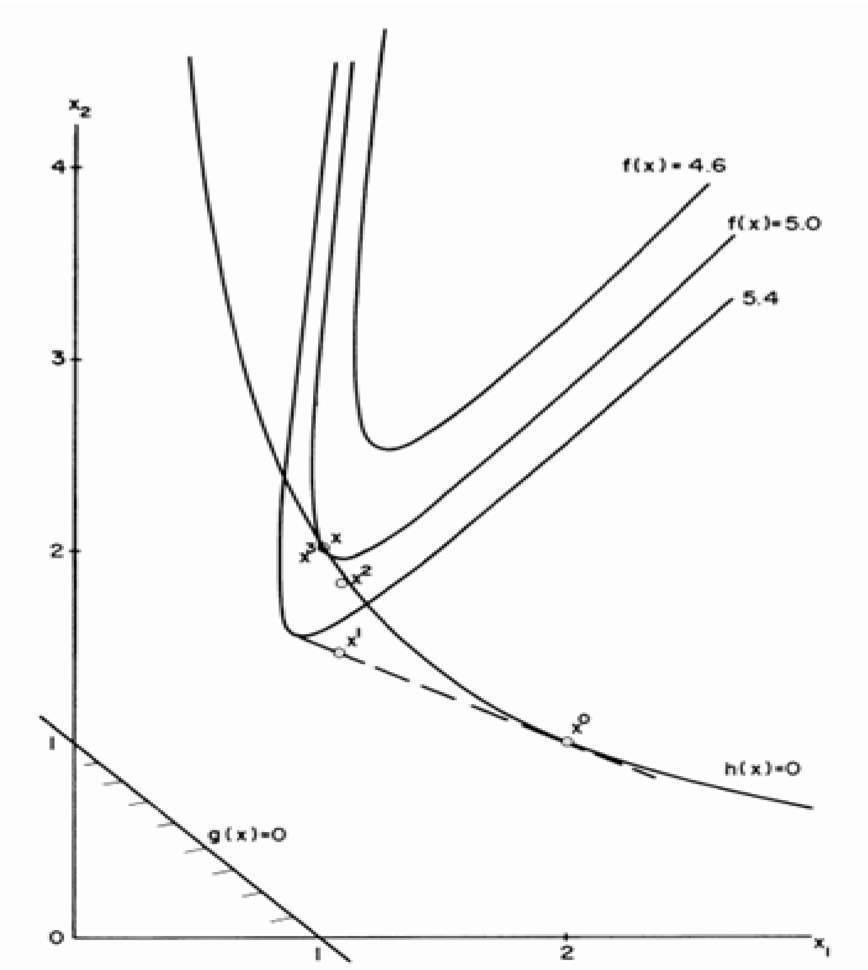
- Note that both the objective function value and the equality constraint violation are reduced. The next two iterations produce the results.

x(3) = (1.00108, 1.99313) f(x(3)) = 5.00457 h(x(3)) = -4.7 * 10-3

x(4) = (1.00014, 1.99971) f(x(4)) = 5.00003 h(x(4)) = -6.2 * 10-6

- The exact optimum is x* (1, 2) with f(x*) 5.0; a very accurate solution has been obtained in four iterations. As is evident from Figure 10.1, the constraint linearizations help to define the search directions, while the quadratic objective function approximation effectively fixes the step length along that direction.

QUADRATIC APPROXIMATION



Example 2:-

- Suppose the objective function and equality constraint of Example 10.1 are interchanged. Thus, consider the problem

$$\begin{aligned} \text{Minimize} \quad & f(x) = x_1 x_2 \\ \text{Subject to} \quad & h(x) = 6x_1 x_2^{-1} + x_2 x_1^{-2} - 5 = 0 \\ & g(x) = x_1 + x_2 - 1 \geq 0 \end{aligned}$$

- The optimal solution to this problem is identical to that of Example 10.1. With the starting point $x^0 = (2, 1)$ the first sub problem becomes

QUADRATIC APPROXIMATION

$$\text{Minimize} \quad (1, 2)d + \tfrac{1}{2}d^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}d$$

$$\text{Subject to} \quad \left(\tfrac{23}{4}, -\tfrac{47}{4}\right)d + \tfrac{29}{4} = 0$$

$$(1, 1)d + 2 \geq 0$$

- The solution to this sub problem is $d^0 = (1.7571, 0.24286)$, at which both constraints are tight. Thus, a sub problem corner point is reached.
- The resulting intermediate solution is

$$x^1 = x^0 + d^0 = (0.24286, 0.75714)$$

with

$$f(x^{(1)}) = 0.18388 \qquad h(x^{(1)}) = 9.7619 \qquad g(x(1)) = 0$$

QUADRATIC APPROXIMATION

- Although the objective function value decreases, the equality constraint vio- lation is worse. The next subproblem becomes

$$\text{Minimize} \quad (0.75714, 0.24286)d + \tfrac{1}{2}d^{T}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}d$$

$$\text{Subject to} \quad -97.795d_1 + 14.413d_2 + 9.7619 = 0$$

$$d_1 + d_2 \text{``} \; 0$$

- The resulting new point is

$$x(2) = (0.32986, 0.67015)$$

$$f(x^{(2)}) = 0.2211 \qquad h(x^{(2)}) = 4.1125 \qquad g(x^{(2)}) = 0$$

QUADRATIC APPROXIMATION

- Again, g(x) is tight. The next few points obtained in this fashion are

$$x(3) = (0.45383, 0.54618)$$
$$x(4) = (—0.28459, 1.28459)$$
$$x(5) = (—0.19183, 1.19183)$$

- These and all subsequent iterates all lie on the constraint g(x) = 0. Since the objective function decreases toward the origin, all iterates will be given by the intersection of the local linearization with g(x) = 0.
- Since the slope of the linearization becomes larger or smaller than -1 depending upon which side of the constraint ‗ elbow“ the linearization point lies, the successive iterates simply follow an oscillatory path up and down the surface g(x)=0.

QUADRATIC APPROXIMATION

- Evidently the problem arises because the linearization cannot take into account the sharp curvature of the constraint $h(x) = 0$ in the vicinity of the optimum.
- Since both the constraint and the objective function shapes serve to define the location of the optimum, both really ought to be taking into ac-count.

QUADRATIC APPROXIMATION OF THE LAGRANGIAN FUNCTION

- The examples of the previous section suggest that it is desirable to incorporate into the sub problem definition not only the curvature of the objective function but also that of the constraints.

- However, based on computational considerations, we also noted that it is preferable to deal with linearly constrained rather than quadratically constrained sub problems.

- Fortunately, this can be accomplished by making use of the Lagrangian function, as will be shown below.

LAGRANGIAN FUNCTION

- For purposes of this discussion, we first consider only the equality- constrained problem.
- The extension to inequality constraints will follow in a straightforward fashion.

Consider the problem

Minimize $f(x)$

Subject to $h(x) = 0$

- Recall that the necessary conditions for a point x^* to be a local minimum are that there exist a multiplier v^* such that

$Qx\ L(x^*, v^*) = Qf^* — v^{*T}\ Qh^* = 0$ and $h(x^*) = 0$

(Ex-1)

LAGRANGIAN FUNCTION

- Sufficient conditions for x* to be a local minimum are that conditions (Ex-1) hold and that the Hessian of the Lagrangian function,

$$Q_x^2 L(x^*, v^*) = Q^2 f^* - (v)^{*T} Q^2 h^*$$

satisfies

$$d^{T} Q_x^2 L d > 0$$

for all d such that (Qh)*Td = 0 (Ex-2)

- Given some point $(\bar{x}, \bar{v})$, we construct the following sub problem expressed in terms of the variables d:

LAGRANGIAN FUNCTION

$$\text{Minimize} \quad Qf(\bar{x})^T d + \tfrac{1}{2} d^T Q^2_x L(\bar{x}, \bar{v}) d \qquad (10.3)$$

$$\text{Subject to} \quad h(\bar{x}) + Qh(\bar{x})^T d = 0 \qquad (10.4)$$

- We now observe that if d* = 0 is the solution to the problem consisting of (10.3) and (10.4), then x must satisfy the necessary conditions for a local minimum of the original problem.
- First note that if d* = 0 solves the sub- problem, then form (10.4) it follows that h(x) = 0; in other words, x is a feasible point.
- Next, there must exist some v* such that the sub problem func- tions satisfy the Lagrangian necessary conditions at d* = 0.

LAGRANGIAN FUNCTION

- Thus, since the gradient of the objective function (10.3) with respect to d at d* = 0 is Qf(x) and that of (10.4) is Qh(x), it follows that

$$d\{\nabla_x^2 L(\bar{x}, v^*) - v^*(0)\}d > 0 \qquad \text{for all } d \text{ such that } \nabla h(x)^T d = 0$$

- Note that the second derivative with respect to d of (10.4) is zero, since it is a linear function in d.
- Consequently, the above inequality implies that d T Qx2L(, v*)d is positive also.
- Therefore, the pair (, v*) satisfies the sufficient conditions for a local minimum of the original problem.

LAGRANGIAN FUNCTION

This demonstration indicates that the sub problem consisting of (10.3) and (10.4) has the following very interesting features:

1. If no further corrections can be found, that is, d = 0, then the local minimum of the original problem will have been obtained.
2. The Lagrange multipliers of the sub problem can be used conveniently as estimates of the multipliers used to formulate the next sub problem.
3. For points sufficiently close to the solution of the original problem the quadratic objective function is likely to be positive definite, and thus the solution of the QP sub problem will be well behaved.

LAGRANGIAN FUNCTION

By making use of the sufficient conditions stated for both equality and in- equality constraints, it is easy to arrive at a QP subproblem formulation for the general case involving K equality and J inequality constraints. If we let

$$L(x, u, v) = f(x) - 3 v_k h_k(x) - 3 u_j g_j(x) \qquad (10.6)$$

then at some point $(\bar{x}, \bar{u}, \bar{v})$ the subproblem becomes

Minimize $\quad q(d; \bar{x}) \div \mathrm{Q} f(\bar{x})^T d + \frac{1}{2} d^T \mathrm{Q}_x L(\bar{x}, \bar{u}, \bar{v}) d \qquad (10.7)$

Subject to $\quad \tilde{h}_k(d; \bar{x}) \div h_k(\bar{x}) + \mathrm{Q} h_k(\bar{x})^T d = 0 \qquad k = 1, \ldots, K \quad (10.8a)$

$\quad \tilde{g}_j(d; \bar{x}) \div g_j(\bar{x}) + \mathrm{Q} g_j(\bar{x})^T d \text{ `` } 0 \qquad j = 1, \ldots, J \quad (10.8b)$

LAGRANGIAN FUNCTION

- The algorithm retains the basic steps outlined for the direct QP case.
- Namely, given an initial estimate x 0 as well as u0 and v0 (the latter could be set equal to zero, we formulate the sub problem [Eqs. (10.7), (10.8a), and (10.8b)];
- solve it; set x(t+1) = x(t) + d; check for convergence; and repeat, using as next estimates of u and v the corresponding multipliers obtained at the solution of the sub problem.

Example 3

Repeat the solution of the problem of Example 10.1 using the Lagrangain QP sub problem with initial estimates $x^0 = (2, 1)T$, $u^0 = 0$, and $v^0 = 0$. The first sub problem becomes

$$\begin{aligned} \text{Minimize} \quad & \left(\tfrac{23}{4}, -\tfrac{47}{4}\right)d + \tfrac{1}{2}d^T \begin{pmatrix} \frac{3}{8} & -\frac{25}{4} \\ -\frac{25}{4} & 24 \end{pmatrix} d \\ \text{Subject to} \quad & (1, 2)d = 0 \\ & (1, 1)d + 2 \geq 0 \end{aligned}$$

This is exactly the same as the fist sub problem of Example 1, because with the initial zero estimates of the multipliers of the constraint terms of the Lagrangian will vanish.

LAGRANGIAN FUNCTION

- The sub problem solution is thus, as before,

$$d^0 = (—0.92079, 0.4604)^T$$

- Since the inequality constraint is loose at this solution, u(1) must equal zero.
-
- The equality constraint multiplier can be found from the solution of the Lagrangian necessity conditions for the sub problem.

- Namely,

$$\nabla q(d^0; x^0) = v\ \nabla\tilde{h}(d^0; x^0)$$

or

$$\left\{\left(\tfrac{23}{4}, -\tfrac{47}{4}\right) + d^{\mathrm{T}}\begin{pmatrix} -\frac{3}{8} & -\frac{25}{4} \\ -\frac{25}{4} & 24 \end{pmatrix}\right\} = v(1, 2)^{\mathrm{T}}$$

- Thus, $v^{(1)} = 2.52723$. Finally, the new estimate of the problem solution will be $x^{(1)} = x^0 + d^0$, or

$x^{(1)} = (1.07921, 1.4604)^T$ $f(x^{(1)}) = 5.68779$ $h(x^{(1)}) = -0.42393$

as was the case before.

- The second subproblem requires the gradients

$$\nabla f(x^{(1)}) = (1.78475,\ -2.17750)^{T} \qquad \nabla h(x^{(1)}) = (1.4604,\ 1.07921)^{T}$$

$$\nabla^2 f(x^{(1)}) = \begin{pmatrix} 6.45954 & -4.40442 \\ -4.40442 & 4.15790 \end{pmatrix}$$

$$\nabla^2 h(x^{(1)}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

LAGRANGIAN FUNCTION

- The quadratic term is therefore equal to

$$\nabla^2 L = \nabla^2 f - v\, \nabla^2 h$$
$$= \nabla^2 f - 2.52723\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 6.45924 & -6.93165 \\ -6.93165 & 4.15790 \end{pmatrix}$$

- The complete problem becomes

$$\text{Minimize} \quad (1.78475, -2.17750)d + \tfrac{1}{2}d^{T}\begin{pmatrix} 6.45924 & -6.93165 \\ -6.93165 & 4.15790 \end{pmatrix}d$$
$$\text{Subject to} \quad 1.4604d_1 + 1.07921d_2 = 0.42393$$
$$d_1 + d_2 + 1.539604 \geqslant 0$$

LAGRANGIAN FUNCTION

- The solution is $d^{(1)} = (0.00614, 0.38450)$. Again, since $\tilde{g}(d^{(1)}; x^{(1)}) > 0$, $u^{(2)} = 0$, and the remaining equality constraint multiplier can be obtained from

$$\nabla q(d^{(1)}; x^{(1)}) = v^T \nabla h(d^{(1)}; x^{(1)})$$

or

$$\begin{pmatrix} -0.84081 \\ -0.62135 \end{pmatrix} = v \begin{pmatrix} 1.46040 \\ 1.07921 \end{pmatrix}$$

Thus,

$$v^{(2)} = -0.57574 \quad \text{and} \quad x^{(2)} = (1.08535, 1.84490)^T$$

with

$$f(x^{(2)}) = 5.09594 \quad \text{and} \quad h(x^{(2)}) = 2.36 \times 10^{-3}$$

- Continuing the calculations for a few more iterations, the results obtained are

LAGRANGIAN FUNCTION

$$x^{(3)} = (0.99266, 2.00463)^T \qquad v^{(3)} = -0.44046$$

$$f^{(3)} = 4.99056 \qquad h^{(3)} = -1.008 \times 10^{-2}$$

and

$$x^{(4)} = (0.99990, 2.00017)^T \qquad v^{(4)} = -0.49997$$

$$f^{(4)} = 5.00002 \qquad h^{(4)} = -3.23 \times 10^{-5}$$

- It is interesting to note that these results are essentially comparable to those obtained in Example 1 without the inclusion of the constraint second derivative terms.
- This might well be expected, because at the optimal solution (1, 2) the constraint contribution to the second derivative of the Lagrangian is small:

LAGRANGIAN FUNCTION

$$\nabla^2 f^* - v^* \nabla^2 h^* = \begin{pmatrix} 12 & -3.5 \\ -3.5 & 1.5 \end{pmatrix} - \left(-\frac{1}{2}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 12 & -3.0 \\ -3.0 & 1.5 \end{pmatrix}$$

- The basic algorithm illustrated in the preceding example can be viewed as an extension of Newton‘s method to accommodate constraints.
- Specifically, if no constraints are present, the subproblem reduces to

$$\text{Minimize} \quad \nabla f(\bar{x})^T d + \tfrac{1}{2} d^T \nabla^2 f(\bar{x}) d$$

Example 10.4

- Consider the problem of Example 10.2 with the initial estimate $x^0 = (2, 2.789)$ and u = v = 0. The first sub problem will be given by

$$\text{Minimize} \quad (2.789, 2)d + \tfrac{1}{2}d^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} d$$

$$\text{Subject to} \quad 1.4540d_1 - 1.2927d_2 = 1.3 \times 10^{-4}$$

$$d_1 + d_2 + 3.789 \geq 0$$

- solution is $d^0 = (-1.78316, -2.00583)$. The inequality constraint is tight, so both constraint multipliers must be computed. The result of solving the system is $v^{(1)} = -0.00343$ and $u^{(1)} = 0.28251$.

$$\nabla \tilde{q}(d^0; x^0) = v\,\nabla \tilde{h}(d^0; x^0) + u\,\nabla \tilde{g}(d^0; x^0)$$

$$\begin{pmatrix} 0.78317 \\ 0.21683 \end{pmatrix} = v\begin{pmatrix} -145.98 \\ 19.148 \end{pmatrix} + u\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

At the corresponding intermediate point,

$$x^{(1)} = (0.21683, 0.78317)^T$$

We have

$$f(x^{(1)}) = 0.1698 \qquad h(x^{(1)}) = 13.318 \qquad g(x^{(1)}) = 0$$

- Note that the objective function decreases substantially, but the equality constraint violation becomes very large. The next sub problem constructed at $x^{(1)}$ with multiplier estimates $u^{(1)}$ and $v^{(1)}$ is

$$\text{Minimize} \quad (0.78317, 0.21683)d + \tfrac{1}{2}d^T\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - v^{(1)}\begin{pmatrix} 2125.68 & -205.95 \\ -205.95 & 5.4168 \end{pmatrix} - u^{(1)}(0)\right\}d$$

$$\text{Subject to} \quad -145.98d_1 + 19.148d_2 = -13.318$$

$$d_1 + d_2 \geq 0$$

- The sub problem solution is $d^{(1)} = (+0.10434,-0.10434)^T$, and the multipliers are $v^{(2)} = -0.02497$, $u^{(2)} = 0.38822$.
- At the new point $x^{(2)} = (0.40183,0.59817)^T$, the objective function value is 0.24036 and the constraint value is 2.7352.
- The results of the next iteration,

$x^{(3)} = (0.74969, 0.25031)^T \qquad f(x^{(3)}) = 0.18766 \qquad h(x^{(3)}) = 13.416$

- indicate that the constraint violation has increased considerably while the objective function value has decreased somewhat.
- Comparing the status at $x^{(1)}$ and $x^{(3)}$, it is evident that the iterations show no real improvement.
- In fact, both the objective function value and the equality constraint violation have increased in proceeding from $x^{(1)}$ to $x^{(3)}$.
- The solution to the problem of unsatisfactory convergence is, as in the unconstrained case, to perform a line search from the previous solution estimate in the direction obtained from the current QP sub problem solution.

QUADRATIC APPROXIMATION OF THE LAGRANGIAN FUNCTION

- However, since in the constrained case both objective function improvement and reduction of the constraint infeasibilities need to be taken into account, the line search must be carried using some type of penalty function.
- For instance, as in the case of the SLP strategy advanced by Palacios-Gomez, an exterior penalty function of the form

$$P(x, R) = f(x) + R\left\{\sum_{k=1}^{K} [h_k(x)]^2 + \sum_{j=1}^{J} [\min(0, g_j(x))]^2\right\}$$

could be used along with some strategy for adjusting the penalty parameter R. This approach is illustrated in the next example.

EXAMPLE

Consider the application of the penalty function line search to the problem of Example 10.4 beginning with the point $x^{(2)}$ and the direction vector $d^{(2)} = (0.34786, -0.34786)^T$ which was previously used to compute the point x(3) directly.

Suppose we use the penalty function

$$P(x, R) = f(x) + 10\{h(x)^2 + [\min(0, g(x)]^2\}$$

and minimize it along the line

$$x = x^{(2)} + \alpha d^{(2)} = \begin{pmatrix} 0.40183 \\ 0.59817 \end{pmatrix} + \begin{pmatrix} 0.34786 \\ -0.34786 \end{pmatrix} \alpha$$

QUADRATIC APPROXIMATION OF THE LAGRANGIAN FUNCTION

- Note that at P 75.05, while at 1, P 1800.0. Therefore, a minimum ought to be found in the range 0 1.
- Using any convenient line search method, the approximate minimum value P 68.11 can be found with 0.1. The resulting point will be

$$x^{(3)} = (0.43662, 0.56338)^T$$

with $f(x^{(3)}) = 0.24682$ and $h(x^{(3)}) = 2.6053$.

- To continue the iterations, updated estimates of the multipliers are required. Since is no longer the optimum solution of the previous sub problem, this value of d cannot be used to estimate the multipliers.
    - The only available updated multiplier values are those associated with $d^{(2)}$, namely $v^{(3)} = 0.005382$ and $u^{(3)} = 0.37291$.

QUADRATIC APPROXIMATION OF THE LAGRANGIAN FUNCTION

- The results of the next four iterations obtained using line searches of the penalty function after each sub problem solution are shown in Table 10.1.
- As is evident from the table, the use of the line search is successful in forcing convergence to the optimum from poor initial estimates.
- The useof the quadratic approximation to the Lagrangian was proposed by Wilson.
- Although the idea was pursued by Beale and by Bard and Greeted, it has not been widely adopted in its direct form.
- As with Newton's method, the barriers to the adoption of this approach in engineering applications have been two fold: first the need to provide second derivative values for all model functions and, second, the sensitivity solution estimates.

Table 10.1 Results for Example 10.5

Iteration	x_1	x_2	f	h	v
3	0.43662	0.56338	0.24682	2.6055	−0.005382
4	0.48569	0.56825	0.27599	2.5372	−0.3584
5	1.07687	1.8666	2.0101	0.07108	−0.8044
6	0.96637	1.8652	1.8025	0.10589	−1.6435
7	0.99752	1.99503	1.9901	0.00498	−1.9755
∞	1.0	2.0	2.0	0.0	−2.0

relevance to defining a good search direction. (For instance, Table 10.1, $v^{(3)} = -5.38 \times 10^{-3}$, while $v^* = -2$.)

Thus, during the initial block of iterations, the considerable computational burden of evaluating all second derivatives may be entirely wasted.

A further untidy feature of the above algorithm involves the strategies required to adjust the penalty parameter of the line search penalty function.

First, a good initial estimate of the parameter must somehow be supplied; second, to guarantee convergence, the penalty parameter must in principle be increased to large values.

Variable metric methods for Constrained optimization:-

- The desirable improved convergence rate of Newton's method could be approached by using suitable update formulas to approximate the matrix of second derivatives.
- Thus, with the wisdom of hindsight, it is not surprising that, as first shown by Garcia Palomares and Mangasarian, similar constructions can be applied to approximate the quadratic portion of our Lagrangian sub problems.
- The idea of approximating using quasi-Newton update formulas that only require differences of gradients of the Lagrangian function was further developed by Han and Powell.

CONSTRAINED VARIABLE METRIC METHOD

- The basic variable metric strategy proceeds as follows.

Constrained Variable Metric Method:-

- Given initial estimates x^0, u^0, v^0 and a symmetric positive-definite matrix H^0.

Step 1: Solve the problem

$$
\begin{aligned}
&\text{Minimize} && \nabla f(x^{(t)})^T d + \tfrac{1}{2} d^T \mathbf{H}^{(t)} d \\
&\text{Subject to} && h_k(x^{(t)}) + \nabla h_k(x^{(t)})^T d = 0 && k = 1, \ldots, K \\
& && g_j(x^{(t)}) + \nabla g_j(x^{(t)})^T d \geq 0 && j = 1, \ldots, J
\end{aligned}
$$

CONSTRAINED VARIABLE METRIC METHOD

- Step 2: Select the step size along $d^{(t)}$ and set $x^{(t+1)} = x^{(t)} + d^{(t)}$.
- Step 3: Check for convergence.
- Step 4: Update $H^{(t)}$ using the gradient difference in such a way that $H^{(t+1)}$ remains positive definite.

$$\nabla_x L(x^{(t+1)}, u^{(t+1)}, v^{(t+1)}) - \nabla_x L(x^{(t)}, u^{(t+1)}, v^{(t+1)})$$

- The key choices in the above procedure involve the update formula for H(t) and the manner of selecting. Han considered the use of several well known update formulas, particularly DFP.

CONSTRAINED VARIABLE METRIC METHOD

- Here it is also showed that if the initial point is sufficiently close, then convergence will be achieved at a superlinear rate without a step-size procedure or line search by setting = 1.
- However, to assure convergence from arbitrary points, a line search is required.
- Specifically, Han recommends the use of the penalty function

$$P(x, R) = f(x) + R\left\{\sum_{k=1}^{K} |h_k(x)| - \sum_{j=1}^{J} \min[0, g_j(x)]\right\}$$

to select α^* so that

$$P(x(\alpha^*)) = \min_{0 \le \alpha \le \delta} P(x^{(t)} + \alpha\, d^{(t)}, R)$$

CONSTRAINED VARIABLE METRIC METHOD

- where R and are suitably selected positive numbers.
- Powell, on the other hand, suggests the use of the BFGS formula together with a conservative check that ensures that $H^{(t)}$ remains positive definite. Thus, if

$$z = x^{(t+1)} - x^{(t)}$$

and

$$y = \nabla_x L(x^{(t+1)}, u^{(t+1)}, v^{(t+1)}) - \nabla_x L(x^{(t)}, u^{(t+1)} v^{(t+1)})$$

- Then define

$$\theta = \begin{cases} 1 & \text{if } z^T y \geq 0.2 z^T \mathbf{H}^{(t)} z \\ \dfrac{0.8 z^T \mathbf{H}^{(t)} z}{z^T \mathbf{H}^{(t)} z - z^T y} & \text{otherwise} \end{cases} \qquad (10.9)$$

- and calculate

$$w = \theta y + (1 - \theta)\mathbf{H}^{(t)} z \qquad (10.10)$$

- Finally, this value of w is used in the BFGS updating formula,

$$\mathbf{H}^{(t+1)} = \mathbf{H}^{(t)} - \frac{\mathbf{H}^{(t)} z z^{\mathrm{T}} \mathbf{H}^{(t)}}{z^{\mathrm{T}} \mathbf{H}^{(t)} z^{\mathrm{T}}} + \frac{w w^{\mathrm{T}}}{z^{\mathrm{T}} w} \qquad (10.11)$$

- Note that the numerical value 0.2 is selected empirically and that the normal BFGS update is usually stated in terms of y rather than w.
- On the basis of empirical testing, Powell proposed that the step-size procedure be carried out using the penalty function

$$P(x, \mu, \sigma) = f(x) + \sum_{k=1}^{K} \mu_k |h_k(x)| - \sum_{j=1}^{J} \sigma_j \min(0, g_j(x)) \qquad (10.12)$$

CONSTRAINED VARIABLE METRIC METHOD

- where for the first iteration $\mu_k = |v_k| \qquad \sigma_j = |u_j|$
- and for all subsequent iterations t

$$\mu_k^{(t)} = \max\{|v_k^{(t)}|, \tfrac{1}{2}(\mu_k^{(t-1)} + |v_k^{(t)}|)\} \qquad (10.13)$$

$$\sigma_j^{(t)} = \max\{|u_j^{(t)}|, \tfrac{1}{2}(\sigma_j^{(t+1)} + |u_j^{(t)}|)\} \qquad (10.14)$$

- The line search could be carried out by selecting the largest value of , 0 <= <=1, such that

$$P(x(\alpha)) < P(x(0)) \qquad (10.15)$$

- However, Powell prefers the use of quadratic interpolation to generate a sequence of values of k until the more conservative condition is met.

CONSTRAINED VARIABLE METRIC METHOD

$$P(x(\alpha_k)) \leq P(x(0)) + 0.1\alpha_k \frac{dP}{d\alpha}(x(0)) \qquad (10.16)$$

- It is interesting to note, however, that examples have been found for which the use of Powell‘s heuristics can lead to failure to converge.
- Further refinements of the step-size procedure have been reported, but these details are beyond the scope of the present treatment.
- We illustrate the use of a variant of the constrained variable metric (CVM) method using update (10.11), penalty function (10.12), and a simple quadratic interpolation-based step-size procedure.

Variable metric methods for Constrained optimization:-

Example

Solve the problem

$$\text{Minimize} \quad f(x) = 6x_1x_2^{-1} + x_2x_1^{-2}$$

$$\text{Subject to} \quad h(x) = x_1x_2 - 2 = 0$$

$$g(x) = x_1 + x_2 - 1 \geq 0$$

using the CVM method with initial metric $H^0 = I$.

At the initial point (2, 1), the function gradients are

$$\nabla f = (\tfrac{23}{4}, -\tfrac{47}{4})^T \qquad \nabla h = (1, 2)^T \qquad \nabla g = (1, 1)^T$$

- Therefore, the first sub problem will take the form

$$\begin{aligned} &\text{Minimize} && \left(\tfrac{23}{4}, -\tfrac{47}{4}\right)d + \tfrac{1}{2}d^T\mathbf{I}d \\ &\text{Subject to} && (1, 2)d \qquad = 0 \\ &&& (1, 1)d + 2 \geq 0 \end{aligned}$$

- It is easy to show that the problem solution lies at the intersection of the two constraints. Thus, $d^0 = (-4, 2)^T$, and the multipliers at this point are solutions of the system

$$\left\{\begin{pmatrix} \frac{23}{4} \\ -\frac{47}{4} \end{pmatrix} + \begin{pmatrix} -4 \\ 2 \end{pmatrix}\right\} = v\begin{pmatrix} 1 \\ 2 \end{pmatrix} + u\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

VARIABLE METRIC METHODS FOR CONSTRAINED

or $$v^{(1)} = -\frac{46}{4},\ u^{(1)} = \frac{53}{4}.$$

- For the first iteration, we use the penalty parameters

$$\mu^{(1)} = \left|-\frac{46}{4}\right| \quad \text{and} \quad \sigma^{(1)} = \left|\frac{53}{4}\right|$$

- The penalty function (10.12) thus take the form

$$P = 6x_1x_2^{-1} + x_2x_1^{-2} + \frac{46}{4}|x_1x_2 - 2| - \frac{53}{4}\min(0, x_1 + x_2 - 1)$$

- We now conduct a one-parameter search of *P* on the line

$$x = (2, 1)^T + \alpha(-4, 2)^T. \text{ At } a = 0, P(0) = 12.25.$$

- Suppose we conduct a bracketing search with $\Delta = 0.1$

  Then P(0 + 0.1) = 9.38875 and p(0.1 + 2(0.1)) = 13.78
- Clearly, the minimum on the line has been bounded.
- Using quadratic interpolation on the three trial points of α = 0, 0.1, 0.3, we obtain α dash = 0.1348 with P(α) = 9.1702. Since this is a reasonable improvement over P(0), the search is terminated with this value of α The new point is

CONSTRAINED VARIABLE METRIC METHOD

- The new point is

$$X^{(1)} = (2, 1)^T + (0.1348)(-4, 2) = (1.46051,\ 1.26974)$$

- We now must proceed to update the matrix H. Following Powell, we calculate

$$z = x^{(1)} - x^{0}\ (-0.53949,\ 0.26974)^T$$

- Then

$$\nabla_x L(x^0, u^{(1)}, v^{(1)}) = \begin{pmatrix} \frac{23}{4} \\ -\frac{47}{4} \end{pmatrix} - \left(-\frac{46}{4}\right)\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \left(\frac{53}{4}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$$\nabla_x L(x^{(1)}, u^{(1)}, v^{(1)}) = \begin{pmatrix} 3.91022 \\ -4.96650 \end{pmatrix} - \left(-\frac{46}{4}\right)\begin{pmatrix} 1.26974 \\ 1.46051 \end{pmatrix} - \frac{53}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5.26228 \\ 4.81563 \end{pmatrix}$$

CONSTRAINED VARIABLE METRIC METHOD

- Note that both gradients are calculated using the same multiplier values u(1) ,v(1). By definition,,

$$y = \nabla_x L(x^{(1)}) - \nabla_x L(x^0) = (1.26228, 6.81563)^T$$

Therefore, $\theta = 1$ and $w = y$. Using (10.11), the update $\mathbf{H}^{(1)}$ is

$$\mathbf{H}^{(1)} = \mathbf{I} - \frac{zz^T}{\|z\|^2} + \frac{yy^T}{z^T y}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - (0.3638)^{-1} \begin{pmatrix} -0.53949 \\ +0.26974 \end{pmatrix} (-0.53949, 0.26974)$$

$$+ (1.15749)^{-1} \begin{pmatrix} 1.26228 \\ 1.26228 \end{pmatrix} (1.26228, 1.26228)$$

$$= \begin{pmatrix} 1.57656 & 7.83267 \\ 7.83267 & 40.9324 \end{pmatrix}$$

- Note that $H^{(1)}$ is positive definite.
- This completes on iteration. We will carry out the second in abbreviated form only.

The sub problem at x(1) is

$$\begin{aligned} &\text{Minimize} && (3.91022, -4.96650)d + \tfrac{1}{2}d^{T}\begin{pmatrix} 1.57656 & 7.83267 \\ 7.83267 & 40.9324 \end{pmatrix}d \\ &\text{Subject to} && (1.26974, 1.46051)d - 0.14552 = 0 \\ &&& d_1 + d_2 + 1.73026 \geq 0 \end{aligned}$$

The solution of the quadratic program is

$$d^{(1)} = (\quad -0.28911, 0.35098)$$

CONSTRAINED VARIABLE METRIC METHOD

- At this solution, the inequality is loose, and hence $u^{(2)} = 0$. The other multiplier value is $v^{(2)} = 4.8857$.
- The penalty function multipliers are updated using (10.13) and (10.14):

$$\mu^{(2)} = \max\left(|4.8857|, \frac{46}{4} + \frac{4.8857}{2}\right) = 8.19284$$

$$\sigma^{(2)} = \max\left(|0|, \frac{\frac{53}{4} + 0}{2}\right) = 6.625$$

- The penalty function now becomes

$$P(x(\alpha)) = f(x) + 8.19284|x_1x_2 - 2| - 6.625 \min(0, x_1 + x_2 - 1)$$

CONSTRAINED VARIABLE METRIC METHOD

- where

$$x(\alpha) = x^{(1)} + \alpha d^{(1)} \qquad 0 \leqslant \alpha \leqslant 1$$

At $\alpha = 0$, $P(0) = 8.68896$, and the minimum occurs at $\alpha = 1$, $P(1) = 6.34906$. The new point is

$$x^{(2)} = (1.17141, 1.62073)$$

with

$$f(x^{(2)}) = 5.5177 \qquad \text{and} \qquad h(x^{(2)}) = -0.10147$$

- The iterations continue with an update of $\mathbf{H}^{(1)}$ .The details will not be elaborated since they are repetitious. The results of the next four iterations are summarized below.

Iteration	x_1	x_2	f	h	v
3	1.14602	1.74619	5.2674	0.001271	−0.13036
4	1.04158	1.90479	5.03668	−0.01603	−0.17090
5	0.99886	1.99828	5.00200	−0.003994	−0.45151
6	1.00007	1.99986	5.00000	-1.9×10^{-6}	−0.50128

- Recall that in Example 10.3, in which analytical second derivatives were used to formulate the QP sub problem, comparable solution accuracy was attained in four iterations.

CONSTRAINED VARIABLE METRIC METHOD

- Thus, the quasi-Newton result obtained using only first derivatives is quite satisfactory, especially in view of the fact that the line searches were all carried out only approximately.
- It should be reemphasized that the available convergence results (super linear rate) [6, 11] assume that the penalty function parameters remain unchanged and that exact line searches are used.
- Powell's modifications (10.13) and (10.14) and the use of approximate searches thus amount to useful heuristics justified solely by numerical experimentation.
- Finally, it is noteworthy that an alternative formulation of the QP sub problem has been reported by Biggs as early as 1972.

CONSTRAINED VARIABLE METRIC METHOD

- The primary differences of that approach lie in the use of an active constraint strategy to select the inequality constraints that are linearized and the fact that the quadratic approximation appearing in the sub problem is that of a penalty function.
- In view of the overall similarity of that approach to the Lagrangian-based construction, we offer no elaboration here, but instead invite the interested reader to study the recent exposition of this approach offered in reference 13 and the references cited therein.

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References

A Ravindran, “Engineering Optimization”, JohnWiley&Sons Publications, 4th Edition, 2009.