AERODYNAMICS - II

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Compressible flow

In a nutshell, the term *compressible flow* refers to the fluids of which there can be found significant variation of density in the flow under consideration.

Compressibility is strongly related to the speed of the flow itself and the thermodynamics of the fluid. A good grasp of thermodynamics is imperative for the study of compressible flow.

For low-speed flow, the kinetic energy is often much smaller than the heat content of the fluid, such that temperature remains more or less constant.

On the other hand, the magnitude of the kinetic energy in a high-speed flow can be very large, able to cause a large variation in the temperature.

Some important phenomena strongly associated with compressibility are the flow discontinuity and *choking* of the flow.

Compressible flow

To illustrate, consider a car at sea-level, $1\,\text{atm}$ and $15\,^\circ\text{C}$, going at a speed of $90\,\text{km/h}$. The density is $1.225\,\text{kg/m}^3$. At a stagnation point, the density there is found to be $1.228\,\text{kg/m}^3$, a mere $0.27\,\%$ difference. The temperature rises by $0.311\,^\circ\text{C}$ and the pressure changes by $0.38\,\%$. Here, the incompressible assumption can be applied.

Now, consider a typical air flow around a cruising jetliner at $10\,\text{km}$ altitude. The speed is now $810\,\text{km/h}$, while the ambient conditions are $0.413\,\text{kg/m}^3$, $0.261\,\text{atm}$ and $-50\,^\circ\text{C}$. At the stagnation point the temperature rises by over $25\,^\circ\text{C}$, while density and pressure changes by more than $30\,\%$ and $45\,\%$, respectively. It is clear that compressibility must now be taken into account.

Compressible flow

Figure 1: *Breaking the sound barrier. . . ?*

An extreme example of compressible flow in action is the re-entry flow. Another is shown here on the left as a jet fighter seemingly punches through the *"sound barrier"*. However, more daily mundane applications can also be found in flows through jet engines, or around a transport aircraft.

Zeroth law of thermodynamics

The concept of *equilibrium* is fundamental to the study of thermodynamics, yet it was not wholly appreciated until later. Thus, it is deemed necessary to define the *zeroth law of thermodynamics*, as a way to say that it precedes even the first law, as follows:

> *When two objects are separately in thermodynamic equilibrium with a third object, they are in equilibrium with each other.*

A system is said to be in equilibrium *if it is free of currents*. The term *"currents"* here refers to the flux of quantities such as mass, momentum, or energy, which is caused by gradients in the system.

First law of thermodynamics

The first law is a statement of energy conservation. It states that the increase in the internal energy level of a system E is equal to the amount of heat Q flowing in from the surroundings and work W done on the system by the surroundings. Mathematically, it can be written as:

$$\Delta E = Q + W \tag{1}$$

Note that E is a variable of state, while Q and W depend on the process involved in the state change. And, for a small change of state the law can be written in a differential form as:

$$\mathrm{d}E = \delta Q + \delta W \tag{2}$$

First law of thermodynamics

Consider this simple idolized system of cylinder-piston arrangement, assuming rigid walls. Note that the only avenue for work done is through the displacement of the piston.
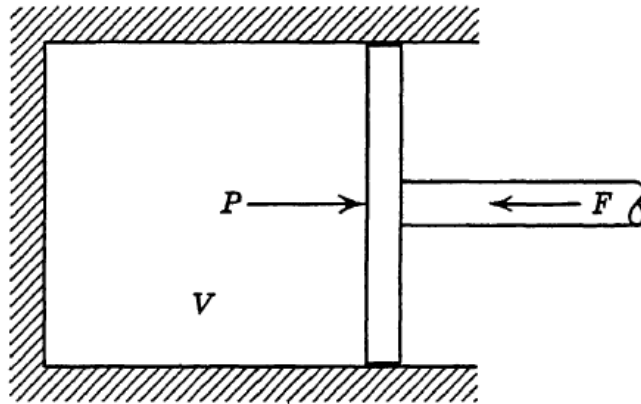
Figure 2: Simple cylinder-piston arrangement

First law of thermodynamics

Thus, for the system considered here, the work term can be written in terms of force vector $\vec{F}$ and displacement $\vec{r}$ as:

$$W = \int \vec{F} \cdot \mathrm{d}\vec{r} = -\int PA\,\mathrm{d}r = -\int P\,\mathrm{d}V \qquad (3)$$

However, it can also be expressed in terms of pressure P and the volume V, since the force is acting parallel to the displacement and is equal to the pressure P multiplied by the piston surface area A. Note that the negative sign indicates that the work is done *by* the system if the system volume *increases*.

First law of thermodynamics

One can also distinguish between *extensive* and *intensive* variables of state. An variable is *extensive* if its value depends on the mass of the system. Such are the system's mass m, E and V.

On the other hand, an *intensive* variable is independent on the mass of the system, temperature ϑ and P being typical examples.

Also, for every extensive quantity, such as E, one can introduce its intensive counterpart e, the energy per unit mass, or *specific* energy. Similarly, *specific* volume $v = V/m$ can be defined.

First law of thermodynamics

Using specific variables, the first law can now be written in its differential form as:

$$\mathrm{d}e = \tfrac{1}{m}\delta Q - P\,\mathrm{d}v = \delta q - P\,\mathrm{d}v \tag{4}$$

Besides the energy E, one can also introduce the concept of *enthalpy H*, which is defined as:

$$H = E + PV \tag{5}$$

Thus, the first law can be expressed using enthalpy as:

$$\mathrm{d}h = \mathrm{d}e + P\,\mathrm{d}v + v\,\mathrm{d}P = \delta q + v\,\mathrm{d}P = \delta q + \frac{\mathrm{d}P}{\rho} \tag{6}$$

where $\rho = 1/v$ is the mass density of the system.

Equation of state — ideal gas

Measurements of thermal properties of gases have shown that for low densities, the relationship among the variables approaches the same form for all gases, which is referred to as the *ideal gas equation*:

$$Pv = \mathcal{R}\left(\vartheta + \vartheta_{\circ}\right) \qquad (7)$$

The characteristic temperature $\vartheta_{\circ}$ turns out to be the *same for all gases*, $\vartheta_{\circ} = 273.15\,°\mathrm{C}$. Thus, it is much more convenient to adopt Kelvin scale in favour of the Celcius scale, such that $T = \vartheta + \vartheta_{\circ}$. Furthermore, the gas constant per unit mass $\mathcal{R}$ is a characteristic for different gases. For air, this value is approximately $287.053\,\mathrm{J/kg \cdot K}$.

Specific heats

Heat capacity of a substance C, an extensive property, is defined as the heat amount required to change its temperature, and can be expressed as:

$$C = \frac{Q}{\Delta T} \qquad (8)$$

For gases, the heat transfer Q can occur by different processes. Hence, the value of C is also dependent on the process path. It is also more convenient to work with the notion of *specific heat*, which is simply the heat capacity per unit mass.

Specific heats

In thermodynamics, it is empirically shown that, for a simple system, any state variable can be expressed as a function of two other variables. Hence, one only needs two values of specific heats to adequately describe the values for all processes, typically the values chosen are those for constant volume *(isochoric)* c_v and constant pressure *(isobaric)* c_p processes:

$$c_v = \left. \frac{\delta q}{\mathrm{d}T} \right|_v \quad \text{and} \quad c_p = \left. \frac{\delta q}{\mathrm{d}T} \right|_P \tag{9}$$

Specific heats

Assume that specific energy e is a funtion of v and T. Then:

$$\mathrm{d}e = \frac{\partial e}{\partial v} \, \mathrm{d}v + \frac{\partial e}{\partial T} \, \mathrm{d}T = \delta q - P \, \mathrm{d}v \tag{10}$$

or:

$$\delta q = \frac{\partial e}{\partial T} \, \mathrm{d}T + \left(\frac{\partial e}{\partial v} + P \right) \mathrm{d}v \tag{11}$$

Hence, the specific heats can be expressed as:

$$c_v = \left. \frac{\delta q}{\mathrm{d}T} \right|_v = \frac{\partial e}{\partial T} \tag{12}$$

$$c_p = \left. \frac{\delta q}{\mathrm{d}T} \right|_P = \frac{\partial e}{\partial T} + \left(\frac{\partial e}{\partial v} + P \right) \left. \frac{\partial v}{\partial T} \right|_P \tag{13}$$

Specific heats

Likewise, using specific enthalpy h as a funtion of P and T:

$$\mathrm{d}h = \frac{\partial h}{\partial P}\,\mathrm{d}P + \frac{\partial h}{\partial T}\,\mathrm{d}T = \delta q + v\,\mathrm{d}P \qquad (14)$$

or:

$$\delta q = \frac{\partial h}{\partial T}\,\mathrm{d}T + \left(\frac{\partial h}{\partial P} - v\right)\mathrm{d}P \qquad (15)$$

And now, the specific heats can be expressed as:

$$c_v = \left.\frac{\delta q}{\mathrm{d}T}\right|_v = \frac{\partial h}{\partial T} + \left(\frac{\partial h}{\partial P} - v\right)\left.\frac{\partial P}{\partial T}\right|_v \qquad (16)$$

$$c_p = \left.\frac{\delta q}{\mathrm{d}T}\right|_P = \frac{\partial h}{\partial T} \qquad (17)$$

The "perfect" gas

No gas is truly *"perfect"*. However, it is the simplest working model. Further, experiences show that for a large range of applications, the gases are under *"nearly perfect"* conditions to justify the use of the model.

A *thermally perfect* gas is characterised by the fact that energy e is solely a function of temperature:

$$e = e(T) \qquad (18)$$

From the definition of enthalpy:

$$h = e + Pv = e(T) + \mathcal{R}T = h(T) \qquad (19)$$

The "perfect" gas

Furthermore, it follows then that the specific heats are also functions of only temperature:

$$c_v = \frac{\mathrm{d}e}{\mathrm{d}T} \tag{20}$$

$$c_p = \frac{\mathrm{d}e}{\mathrm{d}T} + P \left.\frac{\partial v}{\partial T}\right|_P = c_v + \mathcal{R} \tag{21}$$

Similarly:

$$c_v = \frac{\mathrm{d}h}{\mathrm{d}T} - v \left.\frac{\partial P}{\partial T}\right|_v = c_p - \mathcal{R} \tag{22}$$

$$c_p = \frac{\mathrm{d}h}{\mathrm{d}T} \tag{23}$$

The "perfect" gas

The followings then apply for perfect gases:

$$\mathcal{R} = c_p - c_v \tag{24}$$

$$e(T) = \int c_v \, \mathrm{d}T \tag{25}$$

$$h(T) = \int c_p \, \mathrm{d}T \tag{26}$$

$$\gamma = \frac{c_p}{c_v} \tag{27}$$

where γ is the ratio of specific heats. For diatomic gases, *e.g.*, nitrogen and oxygen, which constitute the bulk of the Earth's atmosphere at standard conditions, this value is found to be roughly 1.4.

The "perfect" gas

An important special case is the *calorically perfect* gas, where the specific heats are constants, independent of temperature T. For this special class, integrations can be carried out, resulting in:

$$e(T) = \int c_v \, dT = c_v T + \text{constant} \qquad (28)$$

$$h(T) = \int c_p \, dT = c_p T + \text{constant} \qquad (29)$$

Second law of thermodynamics

The first law of thermodynamics dictates that energy must be conserved in a change of state for a system. It doesn't say in which direction such change of state is allowed. Consider the following.
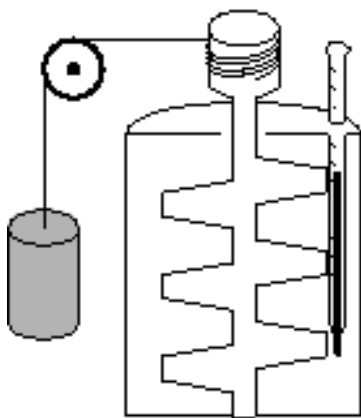
In this experiment, the weight is allowed to drop, turning the paddles, which in turn raises the temperature of the gas in the adiabatic container. It is clear that one cannot induce the wheel by itself to extract the energy from the gas and lift the weight, thus reversing the process.

Figure 3: Joule's paddle-wheel experiment

Second law of thermodynamics

For a *reversible change of state* of an *adiabatically enclosed system*, the following can be shown:

$$P = \frac{\partial E}{\partial V} \tag{30}$$

Now assume that similar relation can be found for T, using a variable S, the entropy. Express E as function of S and V, such that:

$$dE = \frac{\partial E}{\partial S} dS + \frac{\partial E}{\partial v} dV = T dS - P dV \tag{31}$$

Hence, using the first law of thermodynamics, the following is found for a *reversible* process:

$$\delta Q = T dS \tag{32}$$

Second law of thermodynamics

The concept of entropy can be generalized for any arbitrary process between two states 1 and 2:

$$S_2 - S_1 \geq \int \frac{\delta Q}{T} \tag{33}$$

For thermally perfect gas, S, or the specific entropy s, can be given explicitly, relative to some reference state:

$$s = \int \frac{de + P dv}{T} = \int \frac{c_v}{T} dT + \int \frac{\mathcal{R}}{v} dv \tag{34}$$

$$= \int \frac{dh - v dP}{T} = \int \frac{c_p}{T} dT - \int \frac{\mathcal{R}}{P} dP \tag{35}$$

And, for calorically perfect gas:

$$s - s_o = c_v \ln \frac{T}{T_o} + \mathcal{R} \ln \frac{v}{v_o} = c_p \ln \frac{T}{T_o} - \mathcal{R} \ln \frac{P}{P_o} \tag{36}$$

Adiabatic, reversible process

The term *adiabatic* refers to the idea that *no heat passes through the boundary of the system*, $\delta Q = 0$.

The term *reversible* indicates that *the system can be restored to its initial state with no changes in both the system itself and its surroundings*, such that no energy is being dissipated. This is signified as $\mathrm{d}S = \delta Q/T$.

In general, $\mathrm{d}S \geq \delta Q/T$ for natural processes. This means that entropy increase can be attributed to either *heat transfer*, or to *irreversibilities*.

A process that is both *adiabatic* and *reversible* is marked by the fact that the entropy is kept constant, $\mathrm{d}S = 0$.

Though it may sound very restrictive, plenty of important applications in fluid flow can qualify. Consequently, *isentropic* analysis is used rather extensively.

Adiabatic, reversible process

From the first law, the following can be written for isentropic processes:

$$\frac{\partial e}{\partial v}\,\mathrm{d}v + \frac{\partial e}{\partial T}\,\mathrm{d}T = -P\,\mathrm{d}v \qquad (37)$$

$$\frac{\partial h}{\partial P}\,\mathrm{d}P + \frac{\partial h}{\partial T}\,\mathrm{d}T = v\,\mathrm{d}P \qquad (38)$$

Which result in:

$$\frac{\mathrm{d}T}{\mathrm{d}v} = -\frac{1}{c_v}\left(\frac{\partial e}{\partial v} + P\right) \qquad (39)$$

$$\frac{\mathrm{d}T}{\mathrm{d}P} = -\frac{1}{c_p}\left(\frac{\partial h}{\partial P} - v\right) \qquad (40)$$

Adiabatic, reversible process

For perfect gases, where $\mathcal{R} = c_p - c_v$ and $\gamma = c_p/c_v$, these simplify to:

$$\frac{v}{T}\frac{\mathrm{d}T}{\mathrm{d}v} = -\frac{\mathcal{R}}{c_v} = -(\gamma - 1) \tag{41}$$

$$\frac{P}{T}\frac{\mathrm{d}T}{\mathrm{d}P} = \frac{\mathcal{R}}{c_p} = \frac{\gamma - 1}{\gamma} \tag{42}$$

where γ generally depends on temperature T, except when calorically perfect gas is assumed. In any case, these relationships can be easily integrated.

Adiabatic, reversible process

For constant γ:

$$\ln\frac{v}{v_o} = -\int \frac{\mathrm{d}T}{(\gamma - 1)\,T} = -\frac{1}{\gamma - 1}\ln\frac{T}{T_o} \tag{43}$$

$$\ln\frac{P}{P_o} = \int \frac{\gamma\,\mathrm{d}T}{(\gamma - 1)\,T} = \frac{\gamma}{\gamma - 1}\ln\frac{T}{T_o} \tag{44}$$

And since $\rho = 1/v$:

$$\ln\frac{\rho}{\rho_o} = \frac{1}{\gamma - 1}\ln\frac{T}{T_o} \tag{45}$$

The free energy and free enthalpy

In addition to energy E and enthalpy H, one can also define the following two state variables: Helmholtz function (free energy) F and Gibbs function (free enthalpy) G. These are related to E and H as follows:

$$F = E - TS \tag{46}$$
$$G = H - TS \tag{47}$$

In differential forms, one then ends up with the followings:

$$dE = T\,dS - P\,dV \tag{48}$$
$$dF = -S\,dT - P\,dV \tag{49}$$
$$dG = -S\,dT + V\,dP \tag{50}$$
$$dH = T\,dS + V\,dP \tag{51}$$

Entropy and real gas flows

In reality, irreversible state changes increase the entropy. It can be argued that during such process entropy are produced and left in the system. In the case of simple one-dimensional flow the following can be obtained. Let σ be defined as the rate of entropy change per unit time and volume:

$$\sigma_T = \frac{\kappa}{T^2}\left(\frac{dT}{dx}\right)^2 \tag{52}$$

$$\sigma_u = \frac{\tilde{\mu}}{T}\left(\frac{du}{dx}\right)^2 \tag{53}$$

where κ is the coefficient of heat conduction, while $\tilde{\mu}$ denotes a coefficient of viscosity.

Entropy and real gas flows

Experiments have shown that for large regions of the flow, these gradients are small. Consequently, entropy production is very small and may be neglected in these regions. This, however, do not apply to regions where large gradients exist, such as across shock waves, and flows inside the boundary layers, vortex cores and wakes. In this regions, non-isentropic analysis must be employed.

One-dimensional gas dynamics

To begin the study of the motion of compressible fluids, it is advisable to consider the case of *one-dimensional flow*, shown here as a *streamtube* in the figure.
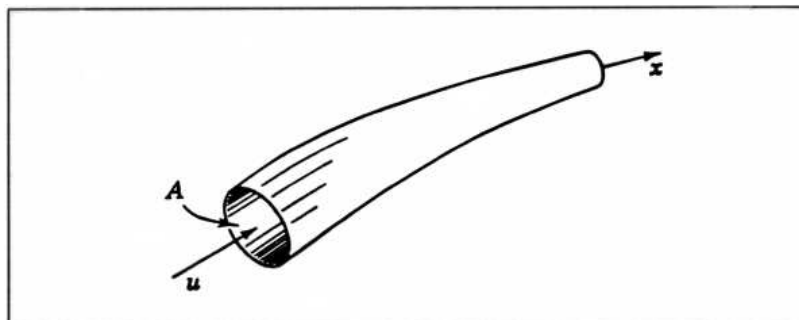

Figure 4: A streamtube

One-dimensional gas dynamics

Note that this definition still holds in the case of *slowly* varying cross-sectional area along the axis x, provided that the variation is a function of the axis $A = A(x)$ only, unchanging in time; and that flow properties, *e.g.*, P and ρ, are uniform across the cross-section, including the velocity $u = u(x)$, which is normal to the cross-sectional area. These flow quantities can also be allowed to vary with time t in case of *non-stationary* or *unsteady* flows.
In a one-dimensional incompressible flow, practically all information is contained in the kinematic relation $uA = $ constant, while pressure is obtained independently using the Bernoulli equation.
On the other hand, compressible flows impose inter-dependency between the conservation of mass and momentum, and the "area-velocity" rule is no longer straightforward.

Conservation of mass — continuity equation
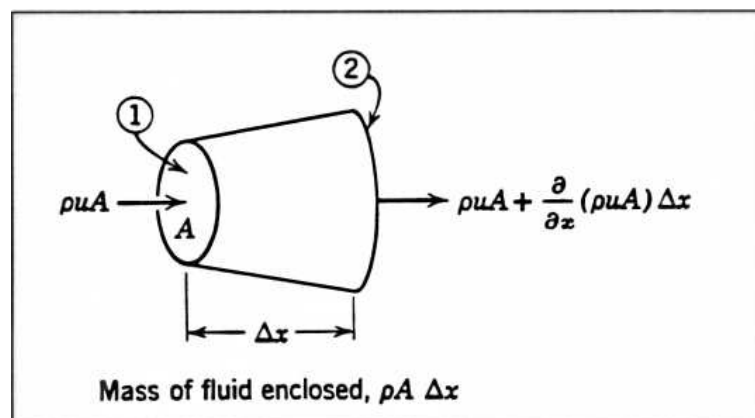
Refer to the figure below.

Figure 5: Flow through a tube segment

Note that the mass flow rate across any cross-section is $\dot{m} = \rho u A$. Also, the mass enclosed within the segment for small enough Δx can be set equal to $\rho A \, \Delta x$.

Conservation of mass — continuity equation

The amount of fluid mass enclosed within the segment increases at the rate of $\partial_t (\rho A \, \Delta x)$. Since mass is conserved, any *sources* or *sinks* can be excluded, and any increases must be balanced by the inflow through 1, minus the outflow through 2, such as:

$$\frac{\partial}{\partial t} (\rho A \, \Delta x) = -\frac{\partial}{\partial x} (\rho u A) \, \Delta x \qquad (54)$$

Since Δx does not depend on time t, and may be divided through, this results in the *continuity* equation:

$$\frac{\partial}{\partial t} (\rho A) + \frac{\partial}{\partial x} (\rho u A) = 0 \qquad (55)$$

And in the case of *steady* flow situations, the time dependency can be dropped, and one is left with:

$$\frac{\mathrm{d}}{\mathrm{d}x} (\rho u A) = 0 \quad \Rightarrow \quad \dot{m} = \rho u A = \text{constant} \qquad (56)$$

Conservation of energy — energy equation

Refer to the figure below.

Figure 6: System for energy calculation
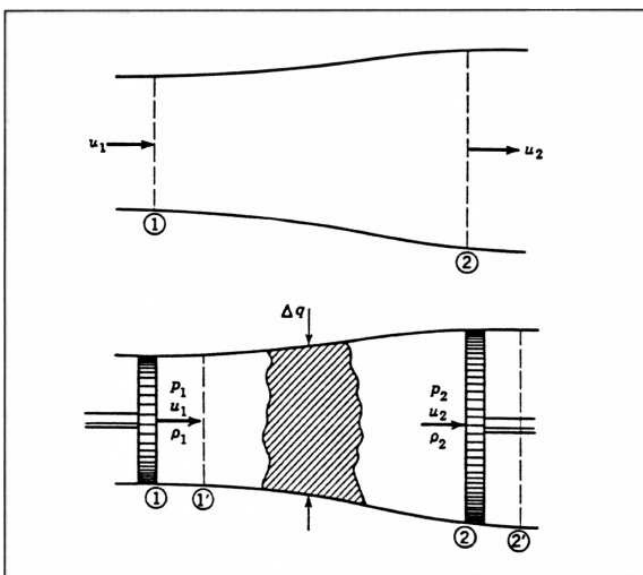
- Assume steady state condition
- Include the kinetic energy contribution $\frac{1}{2} u^2$
- Assume small enough displacements, such that the pressure variations at the pistons are negligible

Conservation of energy — energy equation

During some time increment Δt, the pistons moved from 1–2 positions to $1'$–$2'$ positions. Also, some heat Δq is added to the system. Furthermore, the work done by the pistons is:

$$\Delta w = P_1 v_1 - P_2 v_2 \tag{57}$$

According to the first law, the energy balance, which now includes the kinetic energy contributions, is expressible as:

$$\left(e_2 + \tfrac{1}{2} u_2^2\right) - \left(e_1 + \tfrac{1}{2} u_1^2\right) = \Delta q + \Delta w = \Delta q + \left(P_1 v_1 - P_2 v_2\right) \tag{58}$$

In terms of the enthalpy $h = e + Pv = e + P/\rho$, this balance equation becomes:

$$\left(h_2 + \tfrac{1}{2} u_2^2\right) - \left(h_1 + \tfrac{1}{2} u_1^2\right) = \Delta q \tag{59}$$

Conservation of energy — energy equation

Note that Δq as shown in the figure refers to heat from outside the walls. Heat generated internally is included in h. Also, the energy equation derived here relate the *equilibrium states at sections* 1 *and* 2 in the figure. It is valid even if there are regions on non-equilibrium in-between, *as long as the flow is at equilibrium at the sections themselves*.

In *adiabatic* flows, there is no heat transfer allowed, $\Delta q = 0$, and the equation is refered as the *adiabatic energy equation*:

$$h_2 + \tfrac{1}{2} u_2^2 = h_1 + \tfrac{1}{2} u_1^2 \tag{60}$$

If in addition, equilibrium exists all along the flow, then the equilibrium equation is valid continously, and one ends up with:

$$h + \tfrac{1}{2} u^2 = \text{constant} \tag{61}$$

Conservation of energy — energy equation

In the case of equilibrium flows, one may obtain the differential form:

$$dh + u\,du = 0 \qquad (62)$$

For a thermally perfect gas, this can be rewritten as:

$$c_p\,dT + u\,du = 0 \qquad (63)$$

And, if the gas is calorically perfect, this can be integrated as:

$$c_p T + \tfrac{1}{2}u^2 = \text{constant} \qquad (64)$$

Reservoir conditions

It is helpful to define the *stagnation* or *reservoir* conditions for fluids in equilibrium, where the velocity is negligible and can be set to zero. This allows one to obtain the stagnation / reservoir enthalpy h_o:

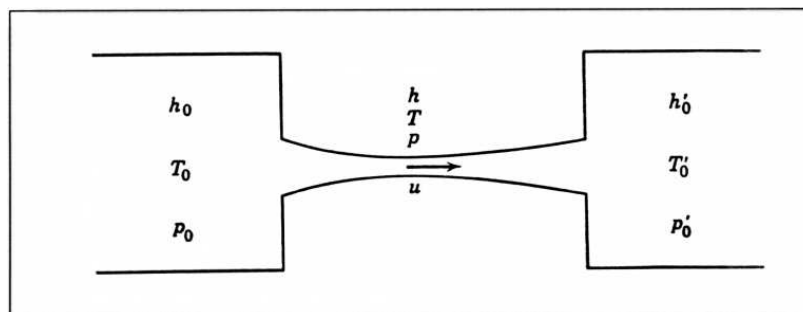$$h + \tfrac{1}{2}u^2 = h_o \qquad (65)$$

Figure 7: Flow between two reservoirs

Reservoir conditions

In the case of adiabatic flow, the reservoir enthalpy in the second reservoir is also equal to that of the first reservoir, as well as along the connecting channel:

$$h'_o = h_o = h + \tfrac{1}{2} u^2 \qquad (66)$$

By assuming (calorically) perfect gas, $h = c_p T$, one is left with:

$$c_p T'_o = c_p T_o \quad \Rightarrow \quad T'_o = T_o \qquad (67)$$

Be aware that this is *only* for the *perfect* gas, otherwise only the constancy of the stagnation enthalpy holds in the adiabatic flows. Also note that often these *stagnation* or *reservoir conditions* are referred to as the *total conditions*.

On isentropic flows

Refering again to the figure of the flow between two reservoirs, note that the first law doesn't forbid the flow to be reversed, *i.e.*, from right to left. This illustrates the application of the second law:

$$s'_o - s_o \geq 0 \qquad (68)$$

For the calorically perfect gas, one has:

$$s'_o - s_o = c_p \ln \frac{T'_o}{T_o} - \mathcal{R} \ln \frac{P'_o}{P_o} = -\mathcal{R} \ln \frac{P'_o}{P_o} \geq 0 \qquad (69)$$

This simply implies what one intuitively would have conclude, which is that the downstream reservoir pressure cannot be greater than upstream one, $P'_o \leq P_o$, in order for a *spontaneous* flow to take place.

On isentropic flows

Looking back at the definition of entropy change:

$$\mathrm{d}s = \frac{1}{T}\left(\mathrm{d}e + P\,\mathrm{d}v\right) = \frac{1}{T}\left(\mathrm{d}h - \frac{1}{\rho}\,\mathrm{d}P\right) \tag{70}$$

The following can be obtained:

$$\left.\frac{\partial s}{\partial P}\right|_{h} = -\frac{1}{\rho T} < 0 \tag{71}$$

This shows that an increase in entropy, at constant stagnation enthalpy, must result in a decrease in stagnation pressure.
The irreversible increase in entropy, with the decrease in total pressure, is due to the *entropy production* between the reservoirs.

On isentropic flows

This *entropy production* will only be absent if there are no dissipative processes, *i.e.*, the flow is in equilibrium throughout.
Only for such *"isentropic flow"* do $s_o' = s_o$ and $P_o' = P_o$.
And now, the term *total condition* can now be broadened to include conditions at any point of the flow, not just at the reservoirs as follows:

> *The local total conditions at any point in the flow are the conditions that would be attained if the flow there were brought to rest isentropically.*

In the light of this definition, the local stagnation, or total entropy is equal to the local static entropy, $s_o' = s'$.

On isentropic flows

For adiabatic flows of perfect gases, this allows for measurement of flow entropy from the measurement of the local total pressure, which under suitable conditions can be made using a simple pitot probe, since the total temperature T_o is constant throughout:

$$s' - s = -\mathcal{R} \ln \frac{P_o'}{P_o} \tag{72}$$

Be fully aware that for stagnation conditions to properly exist, it is not sufficient that the velocity is zero. It is also necessary that equilibrium conditions exist, and there are non-negligible gradients to cause currents.

Euler's equation

Newton's second law states that the rate of change of momentum for a body is proportional to the force acting on it:

$$\vec{F} = \frac{\mathrm{D}}{\mathrm{D}t}(m\vec{u}) = m\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = m\vec{a} \tag{73}$$

where $\vec{a}$ is the acceleration suffered by the body.
It is customary to adopt Eulerian viewpoint when it comes to taking derivatives of some field quantity ϕ associated with the fluid particle, which is assumed to be a function of both time *and its position*. That is to say that $\phi = \phi(t, \vec{x})$. Thus, in one-dimensional flows, the following definition then applies:

$$\mathrm{d}\phi = \frac{\partial\phi}{\partial t}\,\mathrm{d}t + \frac{\partial\phi}{\partial x}\,\mathrm{d}x \tag{74}$$

Euler's equation

On the other hand, a particle's position is also a time-dependent variable. Hence, the rate of change of ϕ is expressible as:

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \left(\frac{dx}{dt}\right)\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} \qquad (75)$$

Be aware of the two parts involved: the *local time derivative* and the *convective derivative* associated with the spatial changes to the quantity ϕ as the particle travels with the fluid velocity u. Together, they are called the *total* or *material* derivative.

Accordingly, the acceleration suffered by the fluid particle in a one-dimensional flow can be written as:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} \qquad (76)$$

Note that the velocity term occurs twice in the expression.
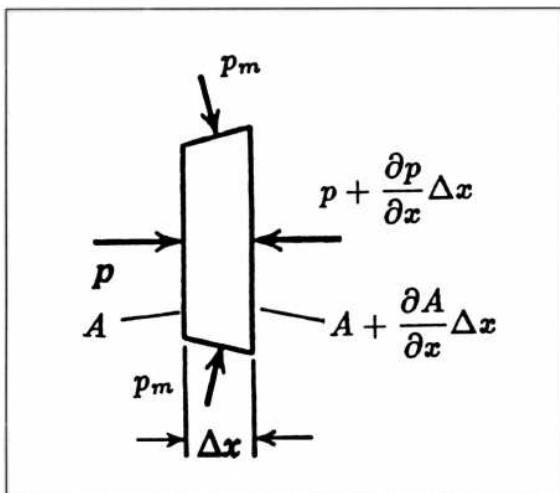
Euler's equation

Figure 8: Pressure on a fluid particle in diverging stream tube

At present, it is assumed that the only forces acting on the fluid are inviscid in nature, no viscous elements are included. Furthermore, gravitational effects are also neglected. Thus, one ends up with only pressure forces.

Euler's equation

From the preceding figure, the pressure forces in the x-direction can be summed up as:

$$\sum F_x = PA + P_m \frac{\partial A}{\partial x} \Delta x - \left(P + \frac{\partial P}{\partial x} \Delta x \right) \left(A + \frac{\partial A}{\partial x} \Delta x \right) \quad (77)$$

where $P_m = P + \frac{1}{2} \partial_x P \Delta x$ is, for lack of better words, *the mean pressure* between the two stations.

By expanding and neglecting the second order terms, one then ends up with:

$$\sum F_x = -\frac{\partial P}{\partial x} A \, \Delta x \qquad (78)$$

Euler's equation

Recall that the mass inside the volume is $m = \rho A \, \Delta x$. Putting terms into Newton's second law, one ends up with:

$$\rho A \, \Delta x \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) = -\frac{\partial P}{\partial x} A \, \Delta x \qquad (79)$$

And by dividing out the term $A \, \Delta x$:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial P}{\partial x} \qquad (80)$$

one ends up with the conservation of momentum for inviscid fluids, often also called as the Euler's equation.

Euler's equation

For steady flows, $\partial_t u = 0$, $\partial_t u = 0$, and the Euler equation becomes:

$$u \, du + \frac{dP}{\rho} = 0 \tag{81}$$

When P is a known function of ρ, one can integrate and obtain the celebrated Bernoulli equation:

$$\tfrac{1}{2} u^2 + \int \frac{dP}{\rho} = \text{constant} \tag{82}$$

For example, incompressible assumption $\rho = \rho_o$ gives:

$$\tfrac{1}{2} \rho_o u^2 + P = \text{constant} \tag{83}$$

An isentropic process for perfect gas gives $P/P_o = (\rho/\rho_o)^\gamma$, resulting in:

$$\tfrac{1}{2} u^2 + \tfrac{\gamma}{\gamma-1} (P/\rho) = \text{constant} \tag{84}$$

Momentum equation

The continuity equation and the Euler's equation can be combined to yield an integral form of momentum balance equation.
By multiplying the continuity equation by u and the Euler's equation by ρA, one obtains:

$$u \frac{\partial}{\partial t} (\rho A) + u \frac{\partial}{\partial x} (\rho u A) = 0 \tag{85}$$

$$\rho A \frac{\partial}{\partial t} (u) + \rho u A \frac{\partial}{\partial x} (u) = -A \frac{\partial}{\partial x} (P) \tag{86}$$

One can then take the sum of the results and apply integration by parts, which results in the following:

$$\frac{\partial}{\partial t} (\rho u A) + \frac{\partial}{\partial x} (\rho u^2 A) = -\frac{\partial}{\partial x} (PA) + P \frac{dA}{dx} \tag{87}$$
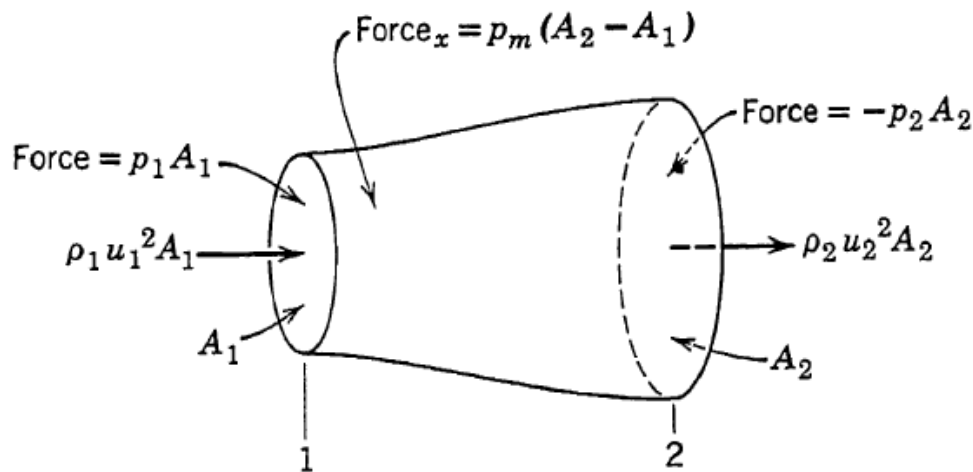
Momentum equation

Figure 9: Control volume for the momentum equation

Refering to the figure above, the previous equation can be integrated with respect to x from 1 to 2.

Momentum equation

Noting that the time differentiation can be taken out from inside the integral sign, one now has:

$$\frac{\partial}{\partial t} \int_1^2 (\rho u A) \, dx + \left(\rho_2 u_2^2 A_2 - \rho_1 u_1^2 A_1 \right)$$
$$= - \left(P_2 A_2 - P_1 A_1 \right) + \int_1^2 P \, dA \tag{88}$$

The first integral is the momentum of the fluid contained in the control volume, while the last integral can be evaluated by defining a *mean pressure* P_m, which can then be taken out of the integral sign, and may include even contributions from *viscous effects*:

$$\int_1^2 P \, dA = P_m \left(A_2 - A_1 \right) \tag{89}$$

Momentum equation

Thus, the left-hand-side is the rate of change of the fluid's momentum, consisting of the local contribution inside the control volume, and the contribution due to the transport (flux) of the momentum through the end sections.

The right-hand-side is the force in the direction of the flow, which here is the x-direction, both from the end sections *and* also from the walls.

This equation turns out to be more general than what the eye catches. It is still valid even if there are regions of dissipation (viscous effects) *within* the control volume, provided that they are absent at the reference points 1 and 2.

Momentum equation

As shown here, the forces and fluxes on adjacent *internal* faces cancel each other. Thus, any *inequlibrium* regions *inside* this volume do not affect the end result.
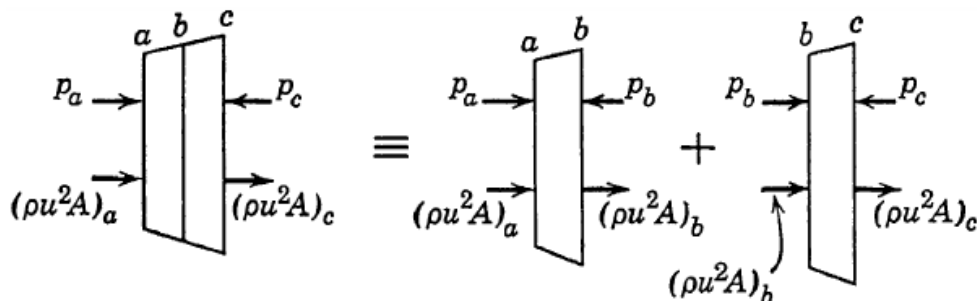

Figure 10: Summing the forces and fluxes in the control volume

Finally, for *steady* flow in a duct of *constant area*:

$$\rho_2 u_2^2 + P_2 = \rho_1 u_1^2 + P_1 \tag{90}$$

A review of equations of conservation

So far, the followings have been derived for a one-dimensional flow: the conservation of mass, of linear momentum, and of total energy ($e_o = e + \frac{1}{2}u^2$):

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho u A) = 0$$

$$\frac{\partial}{\partial t}(\rho u A) + \frac{\partial}{\partial x}(\rho u^2 A) = -\frac{\partial}{\partial x}(P A) + P\frac{\mathrm{d}A}{\mathrm{d}x} \qquad (91)$$

$$\frac{\partial}{\partial t}(\rho e_o A) + \frac{\partial}{\partial x}(\rho u e_o A) = -\frac{\partial}{\partial x}(P u A)$$

Implicit in the expressions are the assumptions of negligible transport coefficients, *e.g.*, viscosity and heat conduction; adiabatic condition at the walls; and that only pressure forces are present.

Isentropic condition

It is stated before that a flow which is both adiabatic and in equilibrium is isentropic. This can be verified from the energy and momentum equations recently derived.

For adiabatic and non-viscous flows, the followings are applicable:

$$\mathrm{d}h + u\,\mathrm{d}u = 0 \qquad \text{and} \qquad u\,\mathrm{d}u + \frac{\mathrm{d}P}{\rho} = 0 \qquad (92)$$

Eliminating $u\,\mathrm{d}u$ between the two:

$$\mathrm{d}h - \frac{\mathrm{d}P}{\rho} = 0 \qquad (93)$$

From the mathematical definition of entropy $T\,\mathrm{d}s = \mathrm{d}h - \mathrm{d}P/\rho$, this is simply saying that entropy is constant.

Isentropic condition

It is obvious that equilibrium cannot be strictly attained in real fluid, since the fluid particle must continuously adjust itself upon encountering new conditions. The rate of adjustments depends on the existing gradients, and is a measure of the degree of non-equilibrium in the flow. It has been mentioned before that entropy production terms in one-dimensional flow are:

$$\frac{\tilde{\mu}}{T}\left(\frac{\partial u}{\partial x}\right)^2 \qquad \text{and} \qquad \frac{\kappa}{T^2}\left(\frac{\partial T}{\partial x}\right)^2 \qquad (94)$$

Isentropic condition

Thus, entropy is always produced, since gradients are always present, and coefficients of viscosity $\tilde{\mu}$ and conductivity κ are finite. However, a large and useful part of fluid mechanics, can be studied using idealized flows, in which the fluid is assumed to be *inviscid* and *non-conducting* ($\tilde{\mu} = 0$ and $\kappa = 0$).

Yet, be fully aware that even if the actual fluid possesses extremely low values of $\tilde{\mu}$ and κ, serious care must be taken in regions where large gradients are present. Such exist as boundary layers, wakes, vortex cores, and in supersonic flows, as shock waves.

Speed of sound — Mach number

In principle, *sound* can be said to be waves of density *and* pressure variations in space and in time.

Note that this is another hallmark of compressible flows, in that small pressure changes can affect appreciable density changes, and *vice versa*.

Furthermore, sound propagates at some finite speed through the medium, while in incompressible flows, it is generally assumed to be infinite.

And one can define the *Mach number* Ma as the ratio of flow velocity *u* to that of the speed of sound *a* associated with the flow.

Speed of sound — Mach number

To learn more about the speed of sound in the fluid, it is instructive to assume several things:

- The disturbance is *weak* enough that only small changes are produced, and the flow properties can be considered more-or-less continuous

- The disturbance is *fast* enough that the process is *adiabatic*, no heat transfer is allowed to take place

- The disturbance is *slow* enough that no significant deviation from equilibrium is taking place during the process, a *quasi-static* condition

Since the process is considered to be both adiabatic and in equilibrium, it is then also *isentropic*.

Speed of sound — Mach number

Consider this thought experiment of a sound pulse caused by a piston moving *slowly* into a still air in a duct of constant area.

(a) Laboratory frame	(b) Moving frame
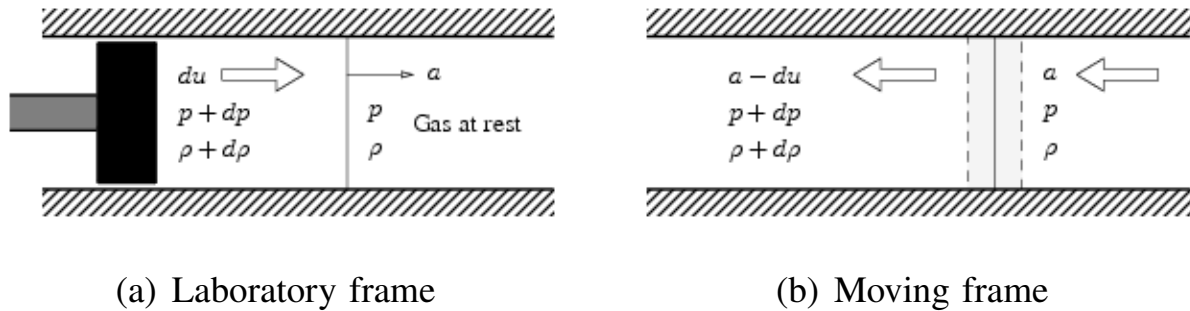
By considering the situation on the right figure as opposed to the left, the problem becomes that of a steady state flow, since now the wave front does not move in the frame of reference.

Speed of sound — Mach number

The continuity equation gives:

$$(\rho + d\rho)(a - du) = \rho a \qquad \Rightarrow \qquad a\,d\rho - \rho\,du = 0 \qquad (95)$$

And the momentum equation gives:

$$[(\rho + d\rho)(a - du)](a - du) + (P + dP) = \rho a^2 + P$$
$$\Rightarrow \qquad dP/a - \rho\,du = 0 \qquad (96)$$

By eliminating $\rho\,du$ from the two expressions, the following expression for a is obtained:

$$a^2 = dP/d\rho \qquad (97)$$

But that still leaves out the question of how to relate the pressure changes to the density changes.

Speed of sound — Mach number

At this point, it is illustrative to show how choosing correct assumptions matter.

Isaac Newton had based his works on Boyle's law, which stated that *for an isothermal process* pressure is linearly dependent on density. When combined with the ideal gas equation, it gives:

$$\frac{\mathrm{d}P}{\mathrm{d}\rho} = \left.\frac{\partial P}{\partial \rho}\right|_T = \mathcal{R}T \tag{98}$$

Applying this to calculate the speed of sound:

$$a = \sqrt{\mathcal{R}T} \tag{99}$$

Imagine his sheer disappointment when experiments showed that he was short by more than 15%.

Speed of sound — Mach number

It shows that the *"fast"* adiabatic process of sound wave propagation does not rule out temperature changes within the localized pressure "pulse". It is not a *"slow"* isothermal process, where the temperature has had the chance to spread out evenly.

Now assume that pressure is a function of two state variables: density and entropy. Earlier, it is stipulated that the sound motion is an isentropic process. Hence, the following:

$$\mathrm{d}P = \left.\frac{\partial P}{\partial \rho}\right|_s \mathrm{d}\rho + \left.\frac{\partial P}{\partial S}\right|_\rho \mathrm{d}s \qquad \Rightarrow \qquad a^2 = \left.\frac{\partial P}{\partial \rho}\right|_s \tag{100}$$

For isentropic processes, the following relationships apply:

$$c_p \ln\left(\frac{T}{T_o}\right) - \mathcal{R} \ln\left(\frac{P}{P_o}\right) = c_v \ln\left(\frac{T}{T_o}\right) - \mathcal{R} \ln\left(\frac{\rho}{\rho_o}\right) = 0 \tag{101}$$

Eliminating temperature from the equations gives the following pressure-density relationship for isentropic process:

$$\frac{P}{P_o} = \left(\frac{\rho}{\rho_o}\right)^{\gamma} \qquad \Rightarrow \qquad \frac{\mathrm{d}P}{P} = \frac{\gamma\,\mathrm{d}\rho}{\rho} \tag{102}$$

When combined with the ideal gas equation, the following *correct* expression is obtained for the speed of sound in a calorically perfect ideal gas:

$$a = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\gamma \mathcal{R} T} \tag{103}$$

Speed of sound — Mach number

As mentioned earlier, one can now define Mach number as the ratio of the flow velocity and the speed of sound $\mathrm{Ma} = u/a$. It is clear that its value depends on the local flow velocity *and* the flow conditions. Generally, Ma is assumed to be a positive value. Based on the value of Mach number, fluid flows can often be divided into several (sometimes overlapping) regimes:

sub / super-sonic Mach number is less / greater than 1

transonic Roughly set around $0.8 \leq \mathrm{Ma} \leq 1.25$

hypersonic Usually said to be $\mathrm{Ma} > 5$, but mostly defined by the high degree of *real gas effects*, such as dissociation of gas molecules

Results from the energy equation

Rewrite c_p in terms of $\mathcal{R}$ and γ:

$$c_p = \frac{c_p}{c_p - c_v}\left(c_p - c_v\right) = \frac{c_p/c_v\left(c_p - c_v\right)}{c_p/c_v - 1} = \frac{\gamma\mathcal{R}}{\gamma - 1} \qquad (104)$$

Use this result to rewrite the adiabatic energy equation for calorically perfect gas:

$$\tfrac{1}{2}u^2 + c_p T = c_p T_o \qquad \Leftrightarrow \qquad \tfrac{1}{2}u^2 + \tfrac{\gamma}{\gamma-1}\mathcal{R}T = \tfrac{\gamma}{\gamma-1}\mathcal{R}T_o \quad (105)$$

By further employing the definition of speed of sound ($a^2 = \gamma\mathcal{R}T$), and Mach number ($\mathrm{Ma} = u/a$), one ends up with:

$$T_o/T = (a_o/a)^2 = 1 + \tfrac{\gamma-1}{2}\mathrm{Ma}^2 \qquad (106)$$

Results from the energy equation

And the *isentropic* relations can also be recast in terms of the Mach number:

$$\rho_o/\rho = \left(1 + \tfrac{\gamma-1}{2}\mathrm{Ma}^2\right)^{1/(\gamma-1)} \qquad (107)$$

$$P_o/P = \left(1 + \tfrac{\gamma-1}{2}\mathrm{Ma}^2\right)^{\gamma/(\gamma-1)} \qquad (108)$$

Be fully aware that in the case of density ρ and pressure P, these relationships are only for *local* reservoir values. They are not necessarily the same throughout the flow field, unlike that of the reservoir temperature T_o, and accordingly, the speed of sound a_o, which are constant throughout.

Results from the energy equation

These equations can have serious implications. For example, getting a Mach 2 air flow in a blow-down wind tunnel with sea-level conditions at the test section would require a reservoir condition of 7.8 atm and 540 K.

These requirements on the reservoir conditions only grow stronger at higher Mach numbers. For air at Mach 7, $T_o/T = 10.8$ and $P_o/P \approx 4140$. Getting sea-level conditions at the test section requires the tank temperature to be hot enough to almost *boil* iron, and the pressure to be almost $4 \times$ the pressure at the bottom of the Mariana Trench.

One can also allow the test section to be very cold, without liquefying the working fluid, and to operate at near-vacuum. Furthermore, specialized gases may be used, such that $\gamma \to 1$, *e.g.*, CF_4 ($\gamma = 1.156$) and C_2F_6 ($\gamma = 1.085$).

Results from the energy equation

It is also convenient to define yet another reference condition, not necessarily at the reservoir, which gives the *stagnation* condition. The *sonic* condition provides an excellent choice, here marked by an asterisk. And since Ma $= 1$, it follows that $u_* = a_*$. The point is that the flow is to be brought to sonic point *isentropically*.
From the energy equation:

$$u^2 + \frac{2}{\gamma - 1} a^2 = u_*^2 + \frac{2}{\gamma - 1} a_*^2 = \frac{\gamma + 1}{\gamma - 1} a_*^2 \qquad (109)$$

And since $T_*/T_o = 2/(\gamma + 1)$, the followings are computed for a $\gamma = 1.4$ gas, which is an excellent approximation for air:

$$\frac{T_*}{T_o} = 0.83333 \quad ; \quad \frac{a_*}{a_o} = 0.91287$$

$$\frac{\rho_*}{\rho_o} = 0.63394 \quad ; \quad \frac{P_*}{P_o} = 0.52828$$

(110)

Results from the energy equation

Another useful measure can be also introduced, the ratio of the actual flow speed to its would-be value *if it were brought to sonic point isentropically*:

$$\mathcal{M}_* = u/u_* = u/a_* \tag{111}$$

Its relationship to the actual Mach number Ma is:

$$\mathcal{M}_*^2 = \frac{u^2}{a^2} \frac{a^2}{a_o^2} \frac{a_o^2}{a_*^2} = \frac{\frac{\gamma+1}{2} \, \text{Ma}^2}{1 + \frac{\gamma-1}{2} \, \text{Ma}^2} = \frac{(\gamma+1)}{2/\text{Ma}^2 + (\gamma-1)} \tag{112}$$

And the inverse:

$$\text{Ma}^2 = \frac{\mathcal{M}_*^2}{\frac{\gamma+1}{2} - \frac{\gamma-1}{2} \mathcal{M}_*^2} = \frac{2}{(\gamma+1)/\mathcal{M}_*^2 - (\gamma-1)} \tag{113}$$

Note that $\mathcal{M}_*$ behaves like Ma, in the sense that $\mathcal{M}_* \lessgtr 1$ when Ma $\lessgtr 1$. And that $\mathcal{M}_*$ is bounded, $\lim_{\text{Ma}\to\infty} \mathcal{M}_* = \frac{\gamma+1}{\gamma-1}$.

The area-velocity relationship

Compressibility introduces new twist to the familiar area-velocity rule, familiar in incompressible flows.
Begin by rearranging the steady one-dimensional continuity equation $\rho u A = \text{constant}$:

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0 \tag{114}$$

Also assuming an adiabatic inviscid flow, Euler's equation gives:

$$u\,du = -\frac{dP}{\rho} = -\left(\frac{dP}{d\rho}\right)\frac{d\rho}{\rho} = -a^2 \frac{d\rho}{\rho} \tag{115}$$

$$\Leftrightarrow \quad \frac{d\rho}{\rho} = -\text{Ma}^2 \frac{du}{u} \tag{116}$$

What this says is that at sub / super-sonic speed, the decrease in density is slower / faster than the increase in velocity.

The area-velocity relationship

Substituting the result for $d\rho/\rho$ into the continuity equation gives the following area-velocity relationship:

$$\left(1 - \mathrm{Ma}^2\right)^{-1} \frac{dA}{A} = -\frac{du}{u} = \frac{dP}{\rho u^2} \qquad (117)$$

$\mathrm{Ma} \ll 1$ This is the incompressible limit: $uA = \text{constant}$

$0 < \mathrm{Ma} < 1$ In the subsonic regime, the flow behaves similarly as incompressible, in which the flow accelerates when cross-sectional area decreases, and *vice versa*

$\mathrm{Ma} > 1$ In the supersonic regime, the opposite behaviour takes place, the flow *decelerates* with the decrease in cross-sectional area, and *accelerates* when A increases

The area-velocity relationship

At $\mathrm{Ma} = 1$, the flow reaches *the sonic point*. The preceding equation demands, that in order for du to have a finite value, the condition $dA = 0$ *must* be fulfilled.

At the sonic point, the flow can cross over, from being subsonic (supersonic) to being supersonic (subsonic).
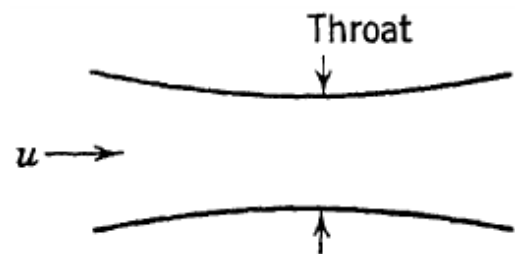
Figure 11: Throat area of a duct

From previous discussion, the sonic point must occurs at the *throat* area. This follows from the fact that the area must decreases to accelerate a subsonic flow, and increases to accelerate it further in the supersonic regime, and *vice versa*.

The area-velocity relationship

However, it does not mean that at the throat area the flow is sonic. It only says that if sonic condition is reached, it is necessarily be at the throat area.

If sonic condition is not attained, the throat area will simply be the location for the maximum velocity for a subsonic flow, or the minimum velocity for a supersonic flow, since $du = 0$ for $\mathrm{Ma} \neq 1$. Also, be aware that in the vicinity of the sonic point, $\mathrm{Ma} \to 1$, the flow is very sensitive to minute area changes, since the denominator $(1 - \mathrm{Ma}^2)$ becomes very small. This is a major concern in designing supersonic nozzles and wind tunnels.

On the equations of state

For a given cross-sectional area $A = A(x)$, and defining $e_o = e + \frac{1}{2}u^2$, the followings *quasi 1-D* flow equations are valid:

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho u A) = 0$$
$$\frac{\partial}{\partial t}(\rho u A) + \frac{\partial}{\partial x}(\rho u^2 A) = -\frac{\partial}{\partial x}(PA) + P\frac{dA}{dx} \qquad (118)$$
$$\frac{\partial}{\partial t}(\rho e_o A) + \frac{\partial}{\partial x}(\rho u e_o A) = -\frac{\partial}{\partial x}(PuA)$$

An immediate problem can be seen in that there are 4 unknowns (ρ, u, P, e) and only 3 equations. To close the system, one needs a state equation.

On the equations of state

In practice, there are actually *TWO* equations of state used:

Thermal equation of state which expresses $P = P(T, v)$

Caloric equation of state which expresses $e = e(T, v)$

The first of these is perhaps the more familiar one. Examples are the ideal gas law and the *van der Waals* gas equation:

$$P = \rho \mathcal{R} T \qquad \text{and} \qquad P = \frac{\rho \mathcal{R} T}{1 - b\rho} - a\rho^2 \qquad (119)$$

From the first law:

$$de = T \, ds - P \, dv$$

$$\left. \frac{\partial e}{\partial v} \right|_T = T \left. \frac{\partial S}{\partial v} \right|_T - P = T \left. \frac{\partial P}{\partial T} \right|_v - P \qquad (120)$$

On the equations of state

However, one also has the following:

$$de = \left. \frac{\partial e}{\partial T} \right|_v dT + \left. \frac{\partial e}{\partial v} \right|_T dv \qquad (121)$$

Thus, one can rewrite it and then integrate to obtain the caloric equation of state $e = e(T, v)$:

$$de = c_v \, dT + \left(T \left. \frac{\partial P}{\partial T} \right|_v - P \right) dv$$

$$\int de = \int c_v \, dT + \int \left(T \left. \frac{\partial P}{\partial T} \right|_v - P \right) dv \qquad (122)$$

For example, when applying ideal gas law $P = \rho \mathcal{R} T$, the second term on the right-hand-side vanishes, leaving $e = e_o + \int_{T_o}^{T} c_v \, dT'$.

Bernoulli equation — dynamic pressure

In compressible flow, the *dynamic pressure* $\mathcal{Q} = \rho u^2/2$ is no longer simply the difference between the stagnation and static pressures. It also depends on the Mach number. For a perfect gas:

$$\tfrac{1}{2}\rho u^2 = \tfrac{1}{2}\rho a^2 \mathrm{Ma}^2 = \tfrac{\gamma}{2}P\mathrm{Ma}^2 \tag{123}$$

For adiabatic flows, applying the ideal gas law to the energy equation gives the compressible Bernoulli equation:

$$\tfrac{1}{2}u^2 + \tfrac{\gamma}{\gamma-1}\left(P/\rho\right) = \tfrac{\gamma}{\gamma-1}\left(P_o/\rho_o\right) \tag{124}$$

In fact, as shown before, this applies also for isentropic flows, which would further allow ρ to be eliminated:

$$\tfrac{1}{2}u^2 = \tfrac{\gamma}{\gamma-1}\left(P_o/\rho_o\right)\left[1 - \left(P/P_o\right)^{(\gamma-1)/\gamma}\right] \tag{125}$$

Bernoulli equation — dynamic pressure

Compressibility also introduces complexities in calculating the pressure coefficient C_p with respect to some reference section 1:

$$C_p = \frac{P - P_\infty}{\tfrac{1}{2}\rho_\infty u_\infty^2} = \frac{2}{\gamma\mathrm{Ma}_\infty^2}\left(\frac{P}{P_\infty} - 1\right) \tag{126}$$

For isentropic flows, this can be further recast as:

$$C_p = \frac{2}{\gamma\mathrm{Ma}_\infty^2}\left[\left(\frac{1 + \tfrac{\gamma-1}{2}\mathrm{Ma}_\infty^2}{1 + \tfrac{\gamma-1}{2}\mathrm{Ma}^2}\right)^{\tfrac{\gamma}{\gamma-1}} - 1\right] \tag{127}$$

Note well that for compressible flows, a value of $C_p > 1$ is possible. Surprisingly, there is a minimum value for C_p as well, found by letting $\mathrm{Ma} \to \infty$, which simply results in:

$$C_{p_{\min}} = -\frac{2}{\gamma\mathrm{Ma}_\infty^2} \tag{128}$$

Bernoulli equation — dynamic pressure

There are two C_p values that are of special interest: the *stagnation* value, and the *critical* value, defined at the location where the local flow turns sonic.

The *critical* C_p is obtainable by setting $\mathrm{Ma} = 1$ as:

$$C_{p_{\text{crit}}} = \frac{P_* - P_\infty}{\frac{1}{2}\rho_\infty u_\infty^2} = \frac{2}{\gamma \mathrm{Ma}_\infty^2}\left[\left(\frac{2}{\gamma+1} + \frac{\gamma-1}{\gamma+1}\mathrm{Ma}_\infty^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right]$$

$$(129)$$

Similarly, setting $\mathrm{Ma} = 0$ gives the *stagnation* C_p:

$$C_{p_{\text{stag}}} = \frac{P_o - P_\infty}{\frac{1}{2}\rho_\infty u_\infty^2} = \frac{2}{\gamma \mathrm{Ma}_\infty^2}\left[\left(1 + \frac{\gamma-1}{2}\mathrm{Ma}_\infty^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right] \quad (130)$$

Bernoulli equation — dynamic pressure

Recall that $C_p = 1 - (u/u_\infty)^2$ for incompressible flows. It remains to be seen how to incorporate this into the compressible C_p.

The energy equation dictates that:

$$\tfrac{1}{2}u^2 + \tfrac{1}{\gamma-1}a^2 = \tfrac{1}{2}u_\infty^2 + \tfrac{1}{\gamma-1}a_\infty^2 \qquad (131)$$

By rearranging terms, one can easily obtain the following relationship:

$$\frac{a^2}{a_\infty^2} = 1 + \frac{\gamma-1}{2}\mathrm{Ma}_\infty^2\left(1 - \frac{u^2}{u_\infty^2}\right) = \frac{1 + \frac{\gamma-1}{2}\mathrm{Ma}_\infty^2}{1 + \frac{\gamma-1}{2}\mathrm{Ma}^2} \qquad (132)$$
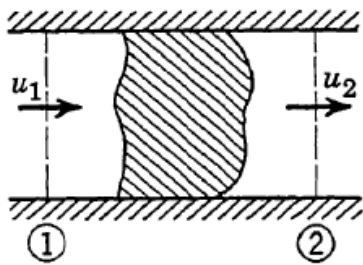
Substituting that into the C_p expression, one finally ends up with:

$$C_p = \frac{2}{\gamma \mathrm{Ma}_\infty^2}\left[\left[1 + \frac{\gamma-1}{2}\mathrm{Ma}_\infty^2\left(1 - \frac{u^2}{u_\infty^2}\right)\right]^{\frac{\gamma}{\gamma-1}} - 1\right] \qquad (133)$$
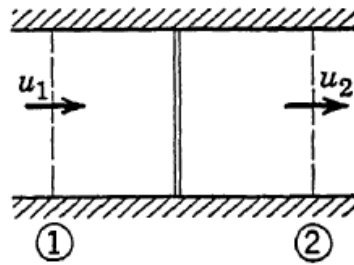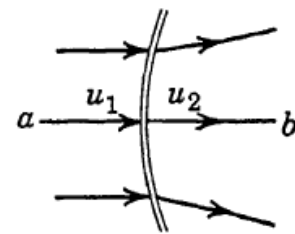
Constant area flows

Consider the control volumes of these steady flows, all enclosing regions of non-equilibrium, in the following three figures.

(a) Finite region   (b) Sharp "jump"   (c) Localized volume

Note that in the third figure, the control volume has been shrank to just enclose the portion of the shock wave normal to the flow, the streamline a–b, where the one-dimensional flow assumption is *locally* valid.

Constant area flows

The figures show that a region of non-equilibrium can be idealized to be a sharp "jump" discontinuity within the flow, a *shock "wave"*. Be aware that now the gradients inside the vanishingly small control volume are greatly increased, thus there will be an increase in the entropy of the flow as it goes through.

Since the reference sections are outside the region of non-equilibrium, the balance equations (mass, momentum and energy) can be expressed as:

$$\rho_1 u_1 = \rho_2 u_2 \tag{134}$$

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2 \tag{135}$$

$$h_1 + \tfrac{1}{2} u_1^2 = h_2 + \tfrac{1}{2} u_2^2 \tag{136}$$

These will form the basis for the derivation of the shock relations for perfect gas in the following section.

Shock relations for perfect gas — Part I

The following figure depicts similar situation encountered in deriving the speed of sound wave. However, the assumption of a weak disturbance has been discarded.

| (a) Moving frame | (b) Laboratory frame |
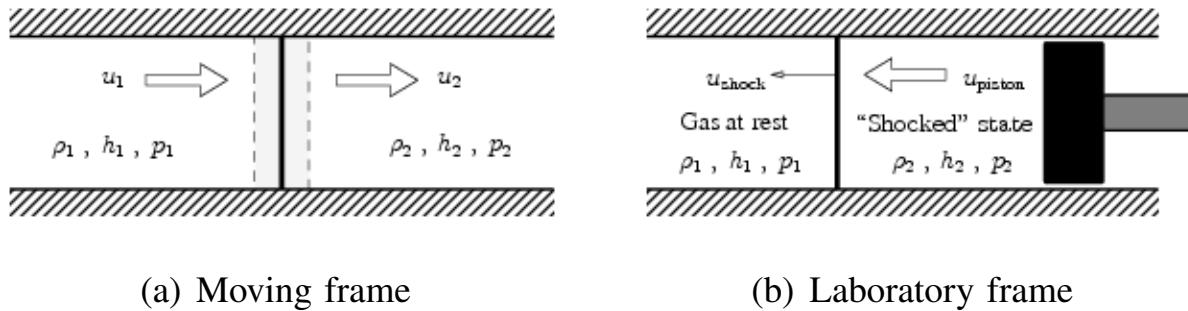
The figure on the right describe the case of moving shock. It is related to the stationary shock through the Galilean transformation. Notably, $u_{\text{shock}} = -u_1$ and $u_{\text{piston}} = u_2 - u_1$.

Shock relations for perfect gas — Part I

From the continuity ($\rho_1 u_1 = \rho_2 u_2$) and momentum equations:

$$P_2 - P_1 = \frac{\rho_1^2 u_1^2}{\rho_1} - \frac{\rho_2^2 u_2^2}{\rho_2} = \rho_1^2 u_1^2 \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \qquad (137)$$

This is the *Rayleigh line* equation in the $P - v$ diagram for a given pre-shock condition.
Further solving for $\rho_1 u_1$ gives:

$$\rho_1^2 u_1^2 = (P_2 - P_1) \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right)^{-1} = \frac{\rho_1 \rho_2}{\rho_2 - \rho_1} (P_2 - P_1) \qquad (138)$$

Eliminate u_2 term from the energy equation:

$$h_2 - h_1 = \frac{1}{2} u_1^2 - \frac{1}{2} \left(\frac{\rho_1 u_1}{\rho_2} \right)^2 = \frac{1}{2} \rho_1^2 u_1^2 \left(\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2} \right) \qquad (139)$$

Shock relations for perfect gas — Part I

Substituting for $\rho_1^2 u_1^2$ and rearranging terms yields the *Hugoniot* equation:

$$h_2 - h_1 = \tfrac{1}{2}(P_2 - P_1)(1/\rho_2 + 1/\rho_1) \tag{140}$$

At this point, it is worth mentioning that both the Rayleigh line and the Hugoniot curve have been derived without any mention of any equation of state. Thus, it can be applied to the general case. The role of the equation of state is then to define the relationship of the enthalpy to the pressure and density.

For example, take the special case of a calorically perfect ideal gas:

$$P = \rho \mathcal{R} T \quad \text{and} \quad h = c_p T$$

$$\Rightarrow \quad h = \frac{c_p P}{\rho \mathcal{R}} = \frac{\gamma}{\gamma - 1}\left(\frac{P}{\rho}\right) \tag{141}$$

Shock relations for perfect gas — Part I

Using it in the Hugoniot curve results in the following equation:

$$\frac{\gamma}{\gamma - 1}\left(\frac{P_2}{\rho_2} - \frac{P_1}{\rho_1}\right) = \frac{1}{2}(P_2 - P_1)\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right)$$

$$\Leftrightarrow \quad \frac{P_2}{\gamma - 1}\left(\frac{\gamma + 1}{\rho_2} - \frac{\gamma - 1}{\rho_1}\right) = \frac{P_1}{\gamma - 1}\left(\frac{\gamma + 1}{\rho_1} - \frac{\gamma - 1}{\rho_2}\right) \tag{142}$$

This is can be put into the Rayleigh line to yield:

$$2\gamma P_1\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) = \rho_1^2 u_1^2\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\left(\frac{\gamma + 1}{\rho_2} - \frac{\gamma - 1}{\rho_1}\right) \tag{143}$$

By discarding the trivial solution $\rho_2 = \rho_1$, one finally obtains the following density and velocity ratios across a shock wave:

$$\frac{\rho_2}{\rho_1} = \left(\frac{u_2}{u_1}\right)^{-1} = \frac{\gamma + 1}{\gamma - 1}\left[1 + \frac{2}{\gamma - 1}\left(\frac{\gamma P_1}{\rho_1 u_1^2}\right)\right]^{-1} \tag{144}$$

Shock relations for perfect gas — Part I

The *shock strength* is obtained from the momentum equation:

$$\frac{P_2}{P_1} - 1 = \frac{\rho_1 u_1^2}{P_1}\left(1 - \frac{u_2}{u_1}\right) = \frac{2\gamma}{\gamma+1}\left(\frac{\rho_1 u_1^2}{\gamma P_1} - 1\right) \qquad (145)$$

Finally, temperature ratio can be obtained from ideal gas equation:

$$\frac{T_2}{T_1} = \frac{P_2}{P_1}\left(\frac{\rho_2}{\rho_1}\right)^{-1} = \frac{P_2}{P_1}\left[\frac{\frac{\gamma+1}{\gamma-1} + (P_2/P_1)}{1 + \frac{\gamma+1}{\gamma-1}(P_2/P_1)}\right] \qquad (146)$$

Recall that the stagnation enthalpy is constant across the shock wave. For perfect gas, that implies that the stagnation temperature is also constant:

$$T_{o_2} = T_{o_1} = T_o \qquad (147)$$

Shock relations for perfect gas — Part I

The second law then states that the entropy change across the shock wave can be found from the ratio of the stagnation pressures:

$$\frac{s_2 - s_1}{\mathcal{R}} = \frac{\gamma}{\gamma-1}\ln\left(\frac{T_2}{T_1}\right) - \ln\left(\frac{P_2}{P_1}\right) = -\ln\left(\frac{P_{o_2}}{P_{o_1}}\right)$$

$$\Rightarrow \quad \frac{P_{o_2}}{P_{o_1}} = \frac{P_2}{P_1}\left(\frac{T_2}{T_1}\right)^{-\frac{\gamma}{\gamma-1}} \qquad (148)$$

$$= \left(\frac{P_2}{P_1}\right)^{-\frac{1}{\gamma-1}}\left[\frac{1 + \frac{\gamma+1}{\gamma-1}(P_2/P_1)}{\frac{\gamma+1}{\gamma-1} + (P_2/P_1)}\right]^{\frac{\gamma}{\gamma-1}}$$

Note that for adiabatic flows, the stagnation pressure also allows one to measure the *available energy* in the flow.

Shock relations for perfect gas — Part II

The shock relationships derived previously can also be expressed in terms of the upstream Mach number Ma_1. By dividing the momentum equation by the continuity, and applying the ideal gas equation, one ends up with:

$$u_2 - u_1 = \frac{P_1}{\rho_1 u_1} - \frac{P_2}{\rho_2 u_2} = \frac{a_1^2}{\gamma u_1} - \frac{a_2^2}{\gamma u_2} \qquad (149)$$

Invoking the sonic condition, energy equation yields:

$$u_2^2 + \frac{2}{\gamma - 1} a_2^2 = u_1^2 + \frac{2}{\gamma - 1} a_1^2 = \frac{\gamma + 1}{\gamma - 1} a_*^2 \qquad (150)$$

Combining these equations gives:

$$u_2 - u_1 = \frac{\gamma + 1}{2\gamma} a_*^2 \left(\frac{1}{u_1} - \frac{1}{u_2} \right) - \frac{\gamma - 1}{2\gamma} (u_1 - u_2) \qquad (151)$$

Shock relations for perfect gas — Part II

Algebraic simplification results in the *Prandtl* or *Meyer* relation:

$$u_1 u_2 = a_*^2 \qquad \Leftrightarrow \qquad \mathcal{M}_{*2} = 1/\mathcal{M}_{*1} \qquad (152)$$

This can be used to express the downstream Mach number Ma_2 as a function of the upstream Mach number Ma_1, and *vice versa*:

$$\frac{\frac{\gamma+1}{2}\mathrm{Ma}_2^2}{1 + \frac{\gamma-1}{2}\mathrm{Ma}_2^2} = \frac{1 + \frac{\gamma-1}{2}\mathrm{Ma}_1^2}{\frac{\gamma+1}{2}\mathrm{Ma}_1^2}$$

$$\Rightarrow \quad \mathrm{Ma}_2^2 = \frac{\frac{2}{\gamma-1} + \mathrm{Ma}_1^2}{\frac{2\gamma}{\gamma-1}\mathrm{Ma}_1^2 - 1} \qquad (153)$$

It is clear that there is a lower limit imposed on Ma_2, such that:

$$\lim_{\mathrm{Ma}_1 \to \infty} \mathrm{Ma}_2 = \frac{\gamma - 1}{2\gamma} \qquad (154)$$

Shock relations for perfect gas — Part II

Now, for the other relationships:

$$\frac{\rho_2}{\rho_1} = \left(\frac{u_2}{u_1}\right)^{-1} = \frac{\frac{\gamma+1}{2}\mathrm{Ma}_1^2}{1 + \frac{\gamma-1}{2}\mathrm{Ma}_1^2}$$

$$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1}\left(\mathrm{Ma}_1^2 - 1\right) = \frac{2\gamma}{\gamma+1}\left(\mathrm{Ma}_1^2 - \frac{\gamma-1}{2\gamma}\right) \qquad (155)$$

$$\frac{T_2}{T_1} = \left[\frac{2}{\gamma+1}\left(\gamma\mathrm{Ma}_1^2\right) - \frac{\gamma-1}{\gamma+1}\right]\left[\frac{2}{\gamma+1}\left(1/\mathrm{Ma}_1^2\right) + \frac{\gamma-1}{\gamma+1}\right]$$

Note that these are valid *only* for *calorically perfect* gases.
These shock relations, however, have not restricted the direction of
the shock wave process. They are applicable when Ma_1 is either
subsonic (an expansion process), or supersonic (a compression
process). One then must refer to the second law for the answer.

Shock relations for perfect gas — Part II

This figure shows the entropy change $\Delta s/\mathcal{R}$ across the shock as a
function of the upstream Mach number Ma_1.
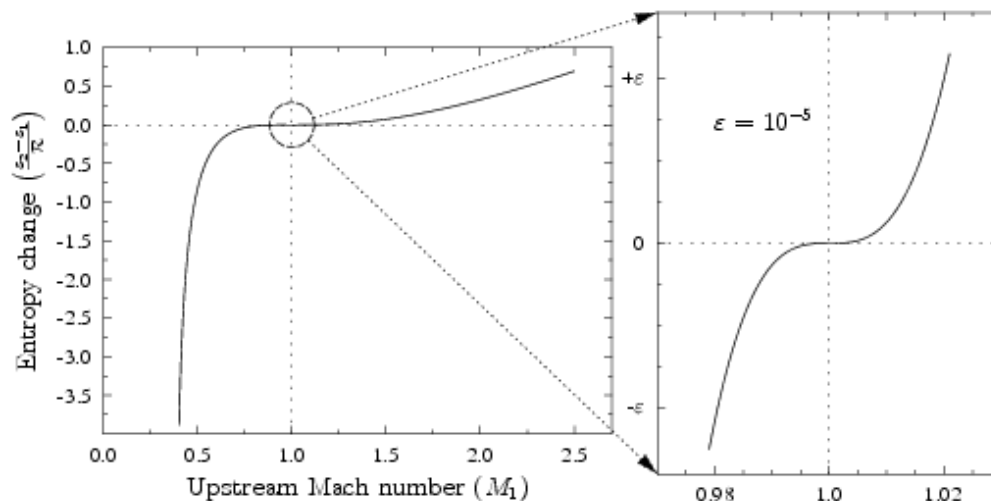
This figure clearly states that *only compression shock wave* is
physically allowed, such that $\mathrm{Ma}_1 \geq 1$ and $\mathrm{Ma}_2 \leq 1$.

Thus, as it was previously mentioned, downstream stagnation pressure must be less than the upstream value, $P_{o_2} \leq P_{o_1}$. Additionally, static pressure P and temperature T, as well as density ρ, all increase across the shock wave, shown in this figure.
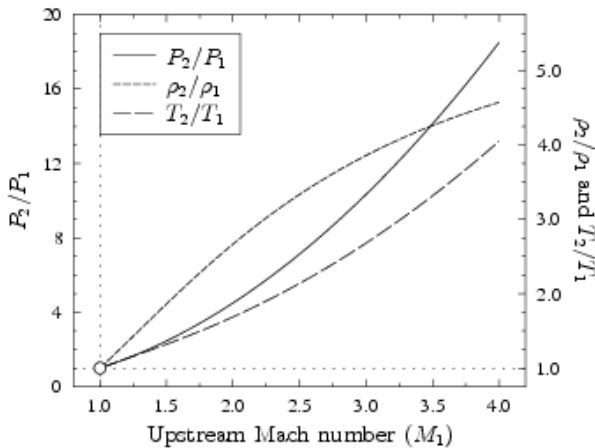
Furthermore, based on these shock relationships, an upper bound exists for the density ratio:

$$\lim_{\text{Ma}_1 \to \infty} \frac{\rho_2}{\rho_1} = \frac{\gamma + 1}{\gamma - 1} \tag{156}$$

Rewrite the pressure ratio equation using $\epsilon = \text{Ma}_1^2 - 1$:

$$P_2/P_1 = 1 + \tfrac{2\gamma}{\gamma+1} \left(\text{Ma}_1^2 - 1 \right) = 1 + \tfrac{2\gamma}{\gamma+1} \epsilon \tag{157}$$

Put this in the equation of entropy change across the shock wave:

$$\frac{s_2 - s_1}{\mathcal{R}} = -\ln \left[\left(\frac{P_2}{P_1} \right)^{-\frac{1}{\gamma-1}} \left[\frac{1 + \frac{\gamma+1}{\gamma-1} (P_2/P_1)}{\frac{\gamma+1}{\gamma-1} + (P_2/P_1)} \right]^{\frac{\gamma}{\gamma-1}} \right]$$

$$= \ln \left[\left(1 + \tfrac{2\gamma}{\gamma+1} \epsilon \right)^{\frac{1}{\gamma}} \left(1 + \tfrac{\gamma-1}{\gamma+1} \epsilon \right) (1 + \epsilon)^{-1} \right]^{\frac{\gamma}{\gamma-1}} \tag{158}$$

For small $\epsilon \ll 1$, the Taylor's expansion gives:

$$\ln (1 + \epsilon) = \epsilon - \tfrac{1}{2}\epsilon^2 + \tfrac{1}{3}\epsilon^3 - \tfrac{1}{4}\epsilon^4 + \cdots \tag{159}$$

# Shock relations for perfect gas — Part II

Applying the expansion:

$$
\begin{aligned}
\frac{s_2 - s_1}{\mathcal{R}} &= \frac{\gamma}{\gamma-1}\left[\frac{1}{\gamma}\left(\frac{2\gamma}{\gamma+1}\right) + \left(\frac{\gamma-1}{\gamma+1}\right) - 1\right]\epsilon \\
&\quad - \frac{\gamma/2}{\gamma-1}\left[\frac{1}{\gamma}\left(\frac{2\gamma}{\gamma+1}\right)^2 + \left(\frac{\gamma-1}{\gamma+1}\right)^2 - 1\right]\epsilon^2 \\
&\quad + \frac{\gamma/3}{\gamma-1}\left[\frac{1}{\gamma}\left(\frac{2\gamma}{\gamma+1}\right)^3 + \left(\frac{\gamma-1}{\gamma+1}\right)^3 - 1\right]\epsilon^3 \\
&\quad - \frac{\gamma/4}{\gamma-1}\left[\frac{1}{\gamma}\left(\frac{2\gamma}{\gamma+1}\right)^4 + \left(\frac{\gamma-1}{\gamma+1}\right)^4 - 1\right]\epsilon^4 + \cdots \\
&= \frac{2\gamma}{3(\gamma+1)^2}\epsilon^3 - \frac{2\gamma^2}{(\gamma+1)^3}\epsilon^4 + \cdots \\
&\approx \frac{2\gamma}{3(\gamma+1)^2}\left(\mathrm{Ma}_1^2 - 1\right)^3 = \frac{\gamma+1}{12\gamma^2}\left(P_2/P_1 - 1\right)^3
\end{aligned}
\tag{160}
$$

Again, it shows that upstream flow *must* be supersonic. However, a weak shock produces a nearly isentropic change of state.
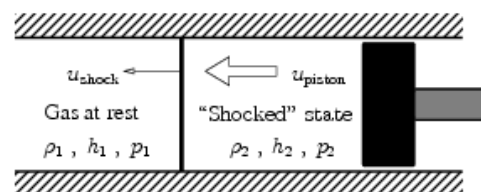
Shock relations for perfect gas — Part III

The previous relationships across a shock wave shown here have been derived under the assumption of stationary shock wave. These can also be applied to the case of moving shock waves through Galilean transformation. The static property ratios stays the same, but not stagnation property ratios.
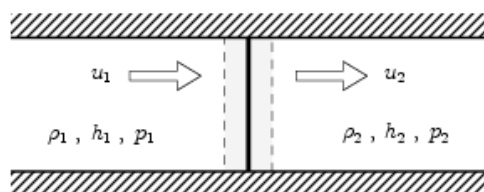
(a) Moving frame   (b) Laboratory frame

For example, note that the gas at rest, the left half of the right figure, has its stagnation properties equal to its static properties, as opposed to its counterpart on the left figure.

The area-velocity relationship — revisited

For a steady one-dimensional flow, the mass flow rate $\dot{m} = \rho u A$ is constant. Suppose that this flow may be adjusted *isentropically* to reach sonic condition (Ma $= 1$, $A = A_*$), one can then write:

$$\rho u A = \rho_* u_* A_*$$

$$\Rightarrow \quad \left(\frac{A_*}{A}\right)^2 = \left(\frac{\rho/\rho_o}{\rho_*/\rho_o}\right)^2 \mathcal{M}_*^2 = \mathrm{Ma}^2 \left(\frac{\frac{\gamma+1}{2}}{1 + \frac{\gamma-1}{2}\mathrm{Ma}^2}\right)^{\frac{\gamma+1}{\gamma-1}}$$

$$(161)$$

For flows that are entirely subsonic, the term A_* refers to the *fictitious* area where the flow would reach Ma $= 1$, it does not necessarily exist. However, if sonic or supersonic conditions are attained, then it would correspond to the actual *throat* area. Be aware that this relationship is valid only for *isentropic flows* due to the use of ρ/ρ_o relation.

The area-velocity relationship — revisited

In this figure, the area ratio A_*/A is plotted as function of the Mach number Ma.
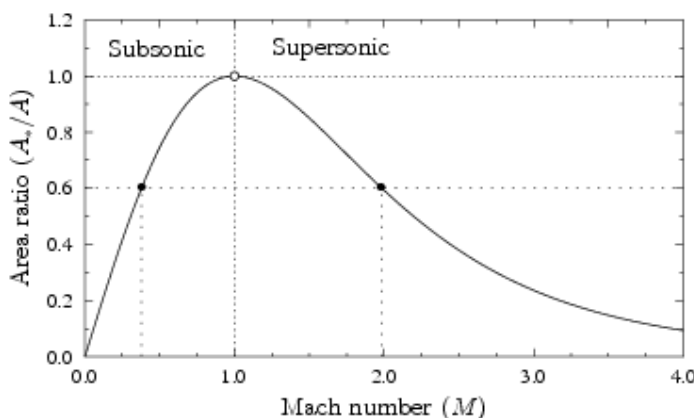
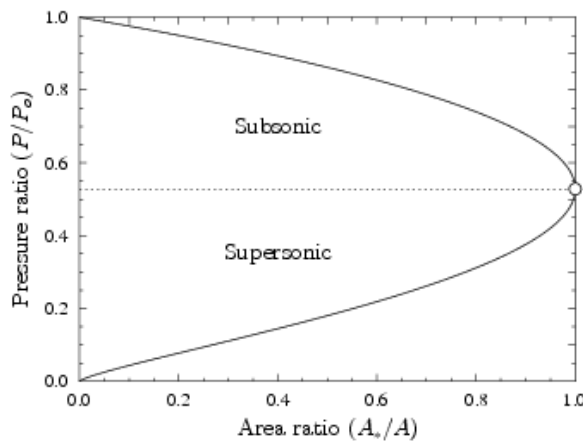It is clear that for a given area ratio, there are two corresponding values of Ma: subsonic, and supersonic.

For example, a value of $A_*/A = 0.6$ would allow a flow with either a Mach number of Ma $= 0.3778$, or Ma $= 1.985$.

The area-velocity relationship — revisited

Additionally, this area relation A_*/A can be expressed in terms of the ratio of local static pressure and stagnation pressure P/P_o:

$$\left(\frac{A_*}{A}\right)^2 = \frac{2}{\gamma-1}\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{\gamma-1}}\left(\frac{P}{P_o}\right)^{\frac{2}{\gamma}}\left[1-\left(\frac{P}{P_o}\right)^{\frac{\gamma-1}{\gamma}}\right] \quad (162)$$

This relationship would prove to become quite useful for the discussion on the nozzle flows in the next sections.

Nozzle flow — converging nozzle

As discussed previously, a subsonic flow cannot be accelerated beyond sonic velocity in a *converging* nozzle. This is true irrespective of pressure difference across the nozzle.

When the ambient pressure at the exit is equal to the reservoir value, no flow will exist, $\dot{m} = 0$.

As this back pressure at the exit is lowered, $\dot{m}$ steadily increases. However, when the flow at the exit reaches $\mathrm{Ma}_e = 1$, any further decrease in the back pressure cannot be communicated to the reservoir, since the signal can only travels at the local speed of sound relative to the moving fluid particle. Thus, $\dot{m}$ stays constant, and the flow is said to be *"choked"*.

The pressure ratio, beyond which no changes in $\dot{m}$ is obtained without changing reservoir condition, or the nozzle geometry, is called the *critical pressure ratio*.

Nozzle flow — converging nozzle

For subsonic flows of a calorically perfect gas in a *converging* nozzle, the mass flow rate $\dot{m} = \rho u A$ can be expressed at the exit in terms of pressure and temperature as:

$$\dot{m} = \frac{P_e A_e \mathrm{Ma}_e \sqrt{\gamma}}{\sqrt{\mathcal{R} T_e}} = \frac{P_o A_e}{\sqrt{\mathcal{R} T_o}} \sqrt{\gamma \mathrm{Ma}_e^2 \left(1 + \frac{\gamma - 1}{2} \mathrm{Ma}_e^2\right)^{\frac{1+\gamma}{1-\gamma}}}$$
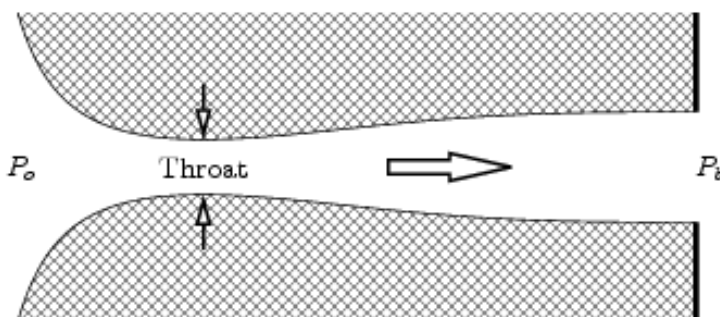
(163)

This value increases as Ma_e increases, and it reaches its maximum when $\mathrm{Ma}_e = 1$:

$$\dot{m}_{\mathrm{max}} = \frac{P_o A_e}{\sqrt{\mathcal{R} T_o}} \sqrt{\gamma \left(\frac{\gamma + 1}{2}\right)^{\frac{1+\gamma}{1-\gamma}}}$$

(164)

Nozzle flow — converging-diverging nozzle

Now consider the following converging-diverging nozzle connected to a reservoir. Define the back pressure P_b as the ambient (static) pressure just outside the exit.

P_o Throat P_b

The diverging part will allow the flow to either decelerate back to subsonic value, or if sonic condition is reached at the throat, to accelerate to supersonic flow regime.

Nozzle flow — converging-diverging nozzle
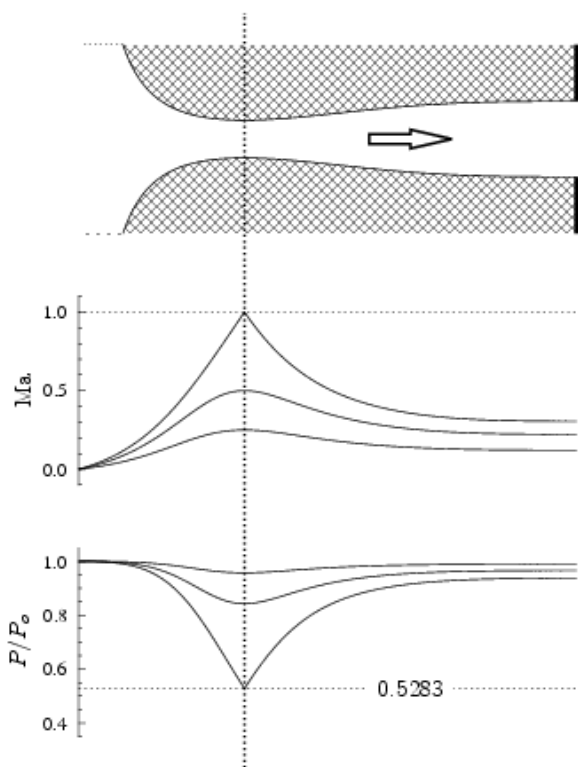
The figures show three different nozzle flows for three different exit pressures, deliberately chosen to correspond to throat Mach numbers of 0.25, 0.5 and 1. All three flows are entirely subsonic in the diverging part. The ratios of the mass flow rate to its maximum value ($\dot{m}/\dot{m}_{\mathrm{max}}$) then are 0.4162, 0.7464 and 1.000, respectively.
Note that isentropic conditions hold throughout the nozzle.

Nozzle flow — converging-diverging nozzle

When the flow at the throat reaches sonic speed, the mass flow can no longer be increased, and the flow is *choked*.
As the back pressure is lowered even further, a region of supersonic flow begins to appear, starting from the throat down the diverging part of the nozzle. This will terminate in a normal shock within the nozzle, which is located farther downstream as the back pressure further decreases, until it reaches the exit plane of the nozzle. *Ergo*, no isentropic solution is possible for this range of back pressures. This is due to the fact that the back pressure P_b is set higher than what the exit pressure P_e would be if the entire nozzle beyond the throat goes supersonic. Thus, a discontinous pressure increase, a shock wave, is required to match the exit condition.

Nozzle flow — converging-diverging nozzle
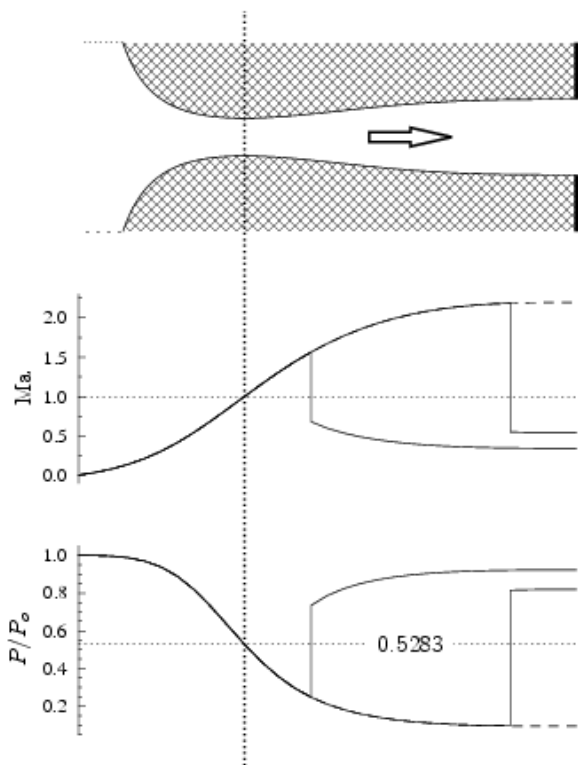
The figures show what can happen if the exit pressure is low enough to allow a supersonic flow region past the nozzle's throat. Here, two of them terminate at shock waves inside the nozzle.

The dashed lines, on the other hand, indicate the flow which is supersonic throughout the nozzle. Several different scenarios, however, may happen at the exit, depending on the back pressure P_b.

Nozzle flow — converging-diverging nozzle

- A normal shock may exist just outside the exit plane.
- At lower P_b, the flow may go through *oblique* shock waves at the exit, which results in smaller pressure increase overall, as opposed of having a single normal shock. The flow is said to be *"over-expanded"*. Note however, that the flow may then go through alternating series of expansion waves and more *oblique* shock waves. This can be seen as a diamond pattern in a rocket's exhaust.
- When $P_b = P_e$, isentropic condition prevails. This is the *design condition* of the nozzle.
- Finally, when $P_b < P_e$, the flow is said to be *"under-expanded"*. Expansion waves will form at the exit. Similar to the *over-expanded* case, alternating series of expansion waves and *oblique* shock waves can then ensue.

Normal shock recovery — diffuser

When the supersonic flow from the nozzle discharges directly into the receiver, such that a normal shock wave stands at the exit, the subsequent subsonic flow can be decelerated isentropically by the use of a *diffuser*. Essentially, it's a diverging extension of the nozzle. The aim is to recover as much stagnation pressure as possible in the receiving chamber, instead of simply obtaining the P_b at the exit. Another configuration utilizes a long duct ahead of the exit. The *recompression* then relies on the interaction of shocks and boundary layers. For supersonic flows, this surprisingly results in a fairly efficient and practical method.

Normal shock recovery — diffuser

A *second throat* at the exit can also be utilized. Ideally, it would decelerate the supersonic flow to sonic condition before subsequent deceleration to subsonic regime. Since there is no losses, no pressure difference is required, hence no power is needed. Of course, real world would never allow this to happen.

However, a pressure difference still is required to start the flow. In principle, this may also require the ability to rapidly adjust the area of the second throat.

As the flow turns supersonic in the test section, the shock would then travel downstream, until it is *"swallowed"* by the second throat, to settle at the diverging part. The throat can then be adjusted to bring the shock closer to the throat, where it would then be of vanishing strength. Then the flow would be close to the ideal condition.

Flow with wall roughness — Fanno flow

Fanno flow refers to the flow through a constant area duct, where the effect of wall friction τ_w is considered.

However, it is further assumed that the wall friction does no work, and does not change the total enthalpy level. Thus no heat transfer from the friction effects.

This model is applicable to flow processes which are very fast compared to heat transfer mechanisms. Thus, heat transfer effects are negligible. This assumption can be used to model flows in relatively short or insulated tubes.
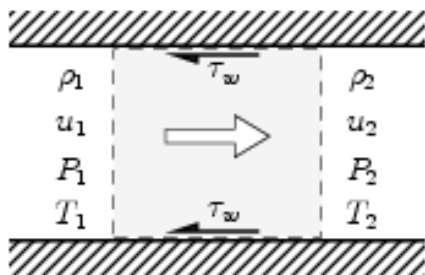
Wall friction will cause both supersonic and subsonic Mach numbers to approach Mach 1.

It can be shown that for calorically perfect flows the maximum entropy occurs at Ma $= 1$.

Flow with wall roughness — Fanno flow

Refer to the figure.

Assume:

- perfect gas
- steady flow
- constant area
- no heat transfer

Continuity equation reads:

$$\rho_1 u_1 = \rho_2 u_2 = \frac{\dot{m}}{A} \qquad (165)$$

Momentum equation reads:

$$\sum F_x = A\left(\rho_2 u_2^2 - \rho_1 u_1^2\right)$$
$$= -A\left(P_2 - P_1\right) - A_w \tau_w \qquad (166)$$

Energy balance dictates:

$$h_2 + \tfrac{1}{2} u_2^2 = h_1 + \tfrac{1}{2} u_1^2 = h_o \qquad (167)$$

Flow with wall roughness — Fanno flow

Define the following *hydraulic diameter*:

$$D_h = \frac{4A}{\text{perimeter}} \tag{168}$$

Additionally, define the (Darcy) friction factor $\mathfrak{f}$, that is dependent on the flow Reynolds number Re and wall roughness:

$$\mathfrak{f} = \frac{4\tau_w}{\frac{1}{2}\rho u^2} \tag{169}$$

By letting the state 2 be a differential change from state 1, the momentum equation can be written as:

$$u\left(\rho\, \mathrm{d}u + u\, \mathrm{d}\rho\right) + \rho u\, \mathrm{d}u = -\mathrm{d}P - \frac{\frac{1}{2}\rho u^2 \mathfrak{f}}{D_h}\, \mathrm{d}x \tag{170}$$

Be aware that some places may use Fanning friction factor, which is $\frac{1}{4}$ the value of Darcy friction factor used here.

Flow with wall roughness — Fanno flow

Some manipulations of the flow equations give:

$$\rho u = \frac{\dot{m}}{A} \quad \Rightarrow \quad \frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}u}{u} = 0 \tag{171}$$

$$P = \rho \mathcal{R} T \quad \Rightarrow \quad \frac{\mathrm{d}P}{P} = \frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}T}{T} \tag{172}$$

$$\mathrm{Ma}^2 = \frac{u^2}{\gamma \mathcal{R} T} \quad \Rightarrow \quad \frac{\mathrm{d}\left(\mathrm{Ma}^2\right)}{\mathrm{Ma}^2} = \frac{2\,\mathrm{d}u}{u} - \frac{\mathrm{d}T}{T} \tag{173}$$

Applying these to the momentum equation:

$$\rho u\, \mathrm{d}u = -\mathrm{d}P - \frac{\frac{1}{2}\rho u^2 \mathfrak{f}}{D_h}\, \mathrm{d}x$$

$$\Rightarrow \quad \frac{\mathfrak{f}\, \mathrm{d}x}{D_h} = \frac{1 - \gamma \mathrm{Ma}^2}{\gamma \mathrm{Ma}^2}\left[\frac{\mathrm{d}\left(\mathrm{Ma}^2\right)}{\mathrm{Ma}^2} + \frac{\mathrm{d}T}{T}\right] - \frac{2}{\gamma \mathrm{Ma}^2}\frac{\mathrm{d}T}{T} \tag{174}$$

Flow with wall roughness — Fanno flow

Due to adiabatic assumptions, the total temperature
$T_o = T\left(1 + \frac{\gamma-1}{2}\mathrm{Ma}^2\right)$ is constant. This results in the following:

$$\frac{\mathrm{d}T}{T} + \left[\frac{\frac{\gamma-1}{2}\mathrm{Ma}^2}{1 + \frac{\gamma-1}{2}\mathrm{Ma}^2}\right]\frac{\mathrm{d}\left(\mathrm{Ma}^2\right)}{\mathrm{Ma}^2} = 0 \qquad (175)$$

Plugging this in, and simplifying gives:

$$\frac{\mathfrak{f}\,\mathrm{d}x}{D_h} = \frac{1}{\gamma\mathrm{Ma}^2}\left(\frac{1 - \mathrm{Ma}^2}{1 + \frac{\gamma-1}{2}\mathrm{Ma}^2}\right)\frac{\mathrm{d}\left(\mathrm{Ma}^2\right)}{\mathrm{Ma}^2} \qquad (176)$$

As will be shown later, the wall friction causes the flow to either speed up or slow down toward Mach 1. Thus, sonic condition is chosen as the reference point, corresponding to the maximum length of the conduit.

Flow with wall roughness — Fanno flow

Define a mean friction factor $\bar{\mathfrak{f}}$, assumed to be constant:

$$\bar{\mathfrak{f}} = \frac{1}{L_{\max}}\int_0^{L_{\max}} \mathfrak{f}\,\mathrm{d}x \qquad (177)$$

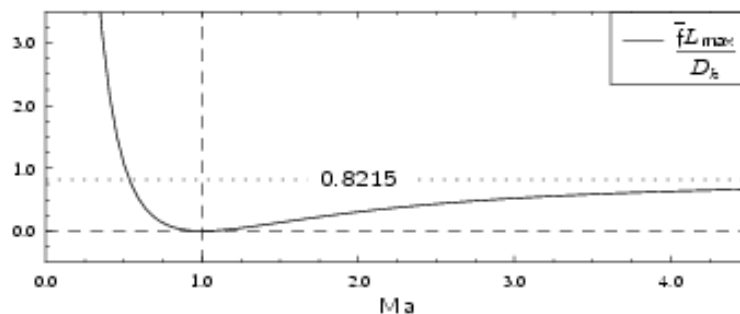Setting $\psi = \mathrm{Ma}^2$ and integrating the previous equation:

$$\begin{aligned}
\frac{\bar{\mathfrak{f}}L_{\max}}{D_h} &= \int_{\mathrm{Ma}^2}^1 \frac{1}{\gamma\psi^2}\left(\frac{1 - \psi}{1 + \frac{\gamma-1}{2}\psi}\right)\mathrm{d}\psi \\
&= \int_{\mathrm{Ma}^2}^1 \frac{1}{\gamma}\left(\frac{1}{\psi^2} + \frac{\frac{\gamma^2-1}{4}}{1 + \frac{\gamma-1}{2}\psi} - \frac{\frac{\gamma+1}{2}}{\psi}\right)\mathrm{d}\psi \qquad (178) \\
&= \frac{1 - \mathrm{Ma}^2}{\gamma\mathrm{Ma}^2} + \frac{\gamma+1}{2\gamma}\ln\left(\frac{\frac{\gamma+1}{2}\mathrm{Ma}^2}{1 + \frac{\gamma-1}{2}\mathrm{Ma}^2}\right)
\end{aligned}$$

Flow with wall roughness — Fanno flow

Surprisingly, for the supersonic flows Ma > 1, there is an asymptotic upper limit. Letting Ma goes to infinity gives:

$$\lim_{\mathrm{Ma}\to\infty} \frac{\bar{f}L_{\mathrm{max}}}{D_h} = \frac{\gamma+1}{2\gamma} \ln\left(\frac{\gamma+1}{\gamma-1}\right) - \frac{1}{\gamma} \tag{179}$$

Flow with wall roughness — Fanno flow

Starting from the equation of the first law for perfect gas:

$$\mathrm{d}s = c_v \frac{\mathrm{d}T}{T} - \mathcal{R}\frac{\mathrm{d}\rho}{\rho} = c_p \frac{\mathrm{d}T}{T} - \mathcal{R}\frac{\mathrm{d}P}{P} \tag{180}$$

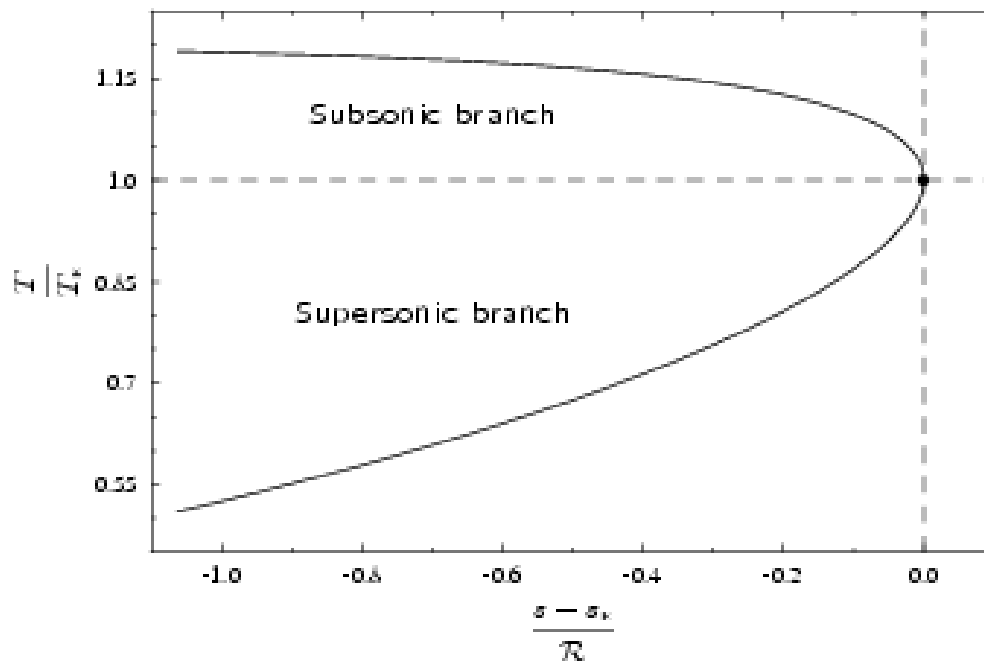Assuming sonic condition as a reference point, and applying the continuity and energy equations:

$$
\begin{aligned}
\frac{s-s_*}{\mathcal{R}} &= \frac{1}{\gamma-1}\ln\frac{T}{T_*} + \ln\frac{u}{u_*} \\
&= \frac{1}{\gamma-1}\ln\frac{T}{T_*} + \ln\frac{\sqrt{T_o-T}}{\sqrt{T_o-T_*}} \\
&= \frac{1}{\gamma-1}\ln\frac{T}{T_*} + \frac{1}{2}\ln\left(\frac{\gamma+1}{2} - \frac{T}{T_*}\right) - \frac{1}{2}\ln\frac{\gamma-1}{2}
\end{aligned}
\tag{181}
$$

This is the *Fanno line* curve in the $T - s$ diagram.

Flow with wall roughness — Fanno flow

The following figure shows an example of a *Fanno line* curve. Note that the second law dictates that the flow is driven toward Mach 1.

Flow with wall roughness — Fanno flow

Other relationships can be derived for the Fanno flow as well. From the momentum equation:

$$
\begin{aligned}
\frac{\mathrm{d}P}{P} &= -\left(\frac{\frac{1}{2} + \frac{\gamma-1}{2}\mathrm{Ma}^2}{1 + \frac{\gamma-1}{2}\mathrm{Ma}^2} \right) \frac{\mathrm{d}\left(\mathrm{Ma}^2\right)}{\mathrm{Ma}^2} \\
&= -\frac{1}{2} \left(\frac{1}{\mathrm{Ma}^2} + \frac{\frac{\gamma-1}{2}}{1 + \frac{\gamma-1}{2}\mathrm{Ma}^2} \right) \mathrm{d}\left(\mathrm{Ma}^2\right)
\end{aligned}
\tag{182}
$$

Again, assuming sonic condition as a reference point:

$$
\ln \frac{P}{P_*} = -\frac{1}{2} \ln \mathrm{Ma}^2 - \frac{1}{2} \ln \frac{1 + \frac{\gamma-1}{2}\mathrm{Ma}^2}{\frac{\gamma+1}{2}}
$$

$$
\Rightarrow \quad \frac{P}{P_*} = \frac{1}{\mathrm{Ma}} \sqrt{\frac{\frac{\gamma+1}{2}}{1 + \frac{\gamma-1}{2}\mathrm{Ma}^2}}
\tag{183}
$$

Flow with heat addition — Rayleigh flow

Rayleigh flow refers to *diabatic* flow through a constant area duct, where the effect of heat addition or rejection is considered.

The heat addition causes a decrease in stagnation pressure, which is known as the Rayleigh effect and is critical in the design of combustion systems.

Heat addition will cause both supersonic and subsonic Mach numbers to approach Mach 1.
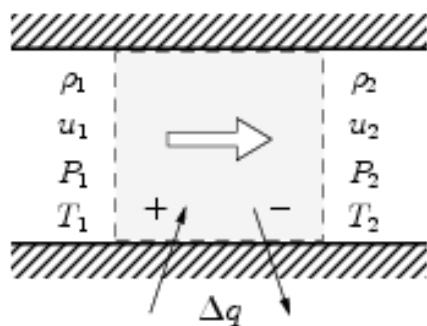
Conversely, heat rejection decreases a subsonic Mach number and increases a supersonic Mach number along the duct.

It can be shown that for calorically perfect flows the maximum entropy occurs at Ma = 1.

Flow with heat addition — Rayleigh flow

Refer to the figure.

Assume:

- perfect gas
- steady flow
- constant area
- no wall friction

Continuity equation reads:

$$\rho_1 u_1 = \rho_2 u_2 = \frac{\dot{m}}{A} \qquad (184)$$

Momentum equation reads:

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2$$

$$\Leftrightarrow \quad P_2 - P_1 = -\frac{\dot{m}}{A}(u_2 - u_1) \qquad (185)$$

Energy balance dictates:

$$h_2 + \tfrac{1}{2}u_2^2 = h_1 + \tfrac{1}{2}u_1^2 + \Delta q$$

$$\Leftrightarrow \quad \Delta q = h_{o_2} - h_{o_1} \qquad (186)$$

$$= c_p(T_{o_2} - T_{o_1})$$

Flow with heat addition — Rayleigh flow

To simplify expressions, it is customary to use sonic condition for reference state 2, and letting the state 1 varies.

For perfect gas, the momentum equation can be rewritten as:

$$P\left(1 + \gamma \mathrm{Ma}^2\right) = P_*\left(1 + \gamma\right) \quad \Rightarrow \quad \frac{P}{P_*} = \frac{1 + \gamma}{1 + \gamma \mathrm{Ma}^2} \qquad (187)$$

Plugging in the continuity equation in the perfect gas law gives:

$$T = \frac{P}{\rho \mathcal{R}} = \frac{P \mathrm{Ma}\sqrt{\gamma T}}{(\dot{m}/A)\sqrt{\mathcal{R}}} = \frac{\gamma P^2 \mathrm{Ma}^2}{\mathcal{R}(\dot{m}/A)^2} \qquad (188)$$

Combining:

$$\frac{T}{T_*} = \frac{P^2 \mathrm{Ma}^2}{P_*^2} = \frac{(1 + \gamma)^2 \, \mathrm{Ma}^2}{\left(1 + \gamma \mathrm{Ma}^2\right)^2} \qquad (189)$$

Flow with heat addition — Rayleigh flow

In addition:

$$\frac{u}{u_*} = \frac{\rho_*}{\rho} = \frac{P_*/T_*}{P/T} = \frac{(1 + \gamma)\,\mathrm{Ma}^2}{1 + \gamma \mathrm{Ma}^2} \qquad (190)$$

$$\frac{T_o}{T_{o*}} = \left[\frac{(1 + \gamma)^2 \, \mathrm{Ma}^2}{\left(1 + \gamma \mathrm{Ma}^2\right)^2}\right] \left[\frac{1 + \frac{\gamma - 1}{2}\mathrm{Ma}^2}{\frac{\gamma + 1}{2}}\right] \qquad (191)$$

$$\frac{P_o}{P_{o*}} = \left[\frac{1 + \gamma}{1 + \gamma \mathrm{Ma}^2}\right] \left[\frac{1 + \frac{\gamma - 1}{2}\mathrm{Ma}^2}{\frac{\gamma + 1}{2}}\right]^{\frac{\gamma}{\gamma - 1}} \qquad (192)$$

It would be instructive to plot the flow on the $T - s$ diagram. To do so, one would need to express P/P_* in terms of T/T_*, since:

$$\frac{s - s_*}{\mathcal{R}} = \frac{\gamma}{\gamma - 1} \ln\left(\frac{T}{T_*}\right) - \ln\left(\frac{P}{P_*}\right) \qquad (193)$$

Flow with heat addition — Rayleigh flow

From the temperature and pressure ratios, one obtains:

$$\frac{T}{T_*} = \left(\frac{P}{P_*}\right)^2 \left[\frac{1+\gamma}{\gamma}\left(\frac{P}{P_*}\right)^{-1} - \frac{1}{\gamma}\right]$$

$$\Leftrightarrow \quad \frac{P}{P_*} = \frac{\gamma+1}{2}\left[1 \pm \sqrt{1 - \frac{4\gamma}{(\gamma+1)^2}\left(\frac{T}{T_*}\right)}\right] \tag{194}$$

Note that there are two entropy values for each possible temperature values, which can be shown to correspond to the subsonic and supersonic branches.
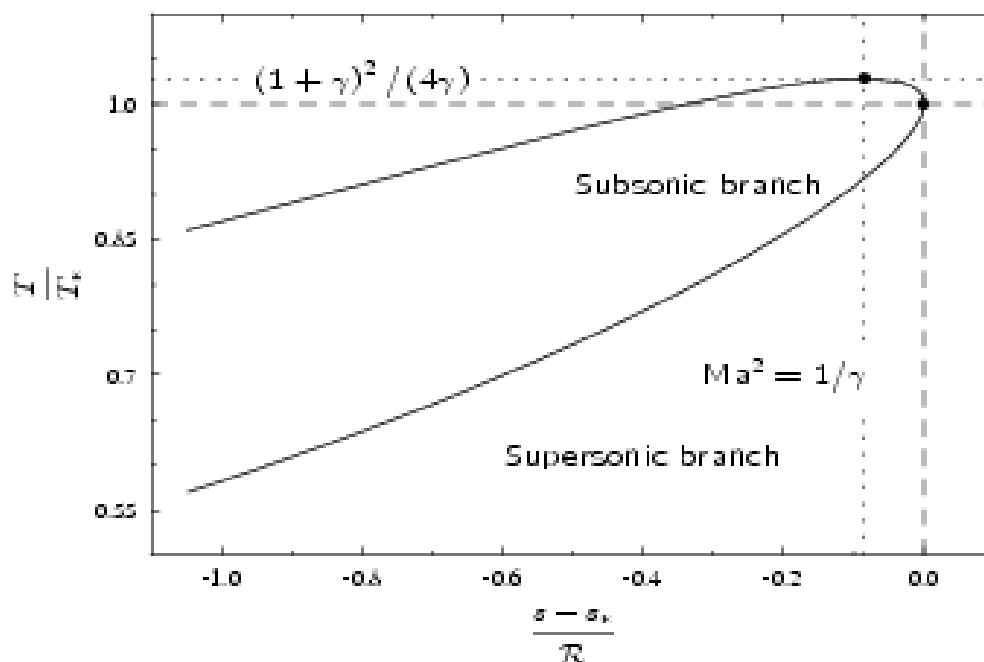
As mentioned before during discussion on shock relations, this *Rayleigh curve* is a line on the $P - v$ diagram:

$$\frac{v}{v_*} = \frac{\gamma+1}{\gamma} - \frac{1}{\gamma}\left(\frac{P}{P_*}\right) \tag{195}$$

Flow with heat addition — Rayleigh flow

The following figure shows a *Rayleigh line* curve. Note that the second law dictates that the flow is driven toward Mach 1.

Flow with heat addition — Rayleigh flow

The maximum temperature point can be found from:

$$\frac{T}{T_*} = \frac{(1+\gamma)^2 \, \mathrm{Ma}^2}{\left(1 + \gamma \mathrm{Ma}^2\right)^2}$$

$$\Rightarrow \quad \frac{\mathrm{d}\left(T/T_*\right)}{\mathrm{d}\left(\mathrm{Ma}^2\right)} = \frac{(1+\gamma)^2 \left(1 - \gamma \mathrm{Ma}^2\right)}{\left(1 + \gamma \mathrm{Ma}^2\right)^3} = 0 \tag{196}$$

Thus, the point is located at $\mathrm{Ma}^2 = 1/\gamma$, which gives a maximum temperature ratio of $T_{\mathrm{max}}/T_* = (1+\gamma)^2/(4\gamma)$. For example, a $\gamma = 1.4$ gas results in $T_{\mathrm{max}}/T_* \approx 1.0286$ at $\mathrm{Ma} \approx 0.845$.

Normal shock, Fanno flow and Rayleigh flow

Capping the discussions on flows in constant area conduits, recall the following jump relations from the stationary normal shock:

$$
\begin{aligned}
\text{mass:} \quad & \left(\rho_2 u_2\right) - \left(\rho_1 u_1\right) = 0 \\
\text{momentum:} \quad & \left(\rho_2 u_2^2 + P_2\right) - \left(\rho_1 u_1^2 + P_1\right) = 0 \\
\text{total enthalpy:} \quad & \left(h_2 + \tfrac{1}{2} u_2^2\right) - \left(h_1 + \tfrac{1}{2} u_1^2\right) = 0
\end{aligned}
\tag{197}
$$

Finally, to complete the system, the ideal gas law $P = \rho \mathcal{R} T$ has been used to provide the link between pressure and temperature, as well as the assumptions of calorically perfect gas, *e.g.*, $h = c_p T = \frac{\gamma}{\gamma - 1} \mathcal{R} T$, and $\mathrm{Ma} = u/\sqrt{\gamma \mathcal{R} T}$.

Normal shock, Fanno flow and Rayleigh flow

Other than the momentum balance equation, Fanno flow uses exactly the same equation set.

Similarly, Rayleigh flow employs the same set, except for the total enthalpy balance equation.

What this entails is that for a given mass flow, the points, corresponding to the states before and after the stationary normal shock wave, must necessarily belong to the Fanno line curve *AND* the Rayleigh line curve. In fact, the two points must be located at the intersections of the two curves.

This will be shown on the next T–s diagram, starting by putting the continuity equation in the entropy change, resulting in:

$$\frac{s_2 - s_1}{\mathcal{R}} = \frac{1}{\gamma - 1} \ln \frac{T_2}{T_1} - \ln \frac{\rho_2}{\rho_1} = \frac{1}{\gamma - 1} \ln \frac{T_2}{T_1} + \ln \frac{u_2}{u_1} \qquad (198)$$

Normal shock, Fanno flow and Rayleigh flow

For Fanno line, the total enthalpy equation gives:

$$
\begin{aligned}
u_2/u_1 &= \sqrt{2c_p\left(T_o - T_2\right)} / \sqrt{2c_p\left(T_o - T_1\right)} \\
&= \left(\tfrac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{-\frac{1}{2}} \sqrt{\left(1 + \tfrac{\gamma-1}{2}\mathrm{Ma}_1^2\right) - (T_2/T_1)}
\end{aligned} \qquad (199)
$$

For Rayleigh line, the momentum equation, along with ideal gas law, gives:

$$
\begin{aligned}
0 &= \left(\rho_2 u_2^2 + \rho_2 \mathcal{R} T_2\right) - \left(\rho_1 u_1^2 + \rho_1 \mathcal{R} T_1\right) \\
0 &= (u_2/u_1)^2 - \tfrac{1+\gamma\mathrm{Ma}_1^2}{\gamma\mathrm{Ma}_1^2}(u_2/u_1) + \tfrac{1}{\gamma\mathrm{Ma}_1^2}(T_2/T_1) \\
u_2/u_1 &= \tfrac{1+\gamma\mathrm{Ma}_1^2}{2\gamma\mathrm{Ma}_1^2}\left[1 \pm \sqrt{1 - \tfrac{4\gamma\mathrm{Ma}_1^2}{\left(1+\gamma\mathrm{Ma}_1^2\right)^2}(T_2/T_1)}\right]
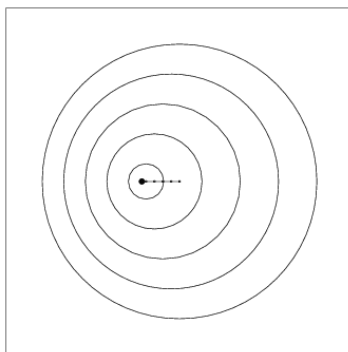\end{aligned} \qquad (200)
$$

Normal shock, Fanno flow and Rayleigh flow

Here is a T–s diagram for a flow at $Ma_1 = 2$. Note that "State 1" refers to supersonic pre-shock condition, while "State 2" refers to subsonic post-shock condition.
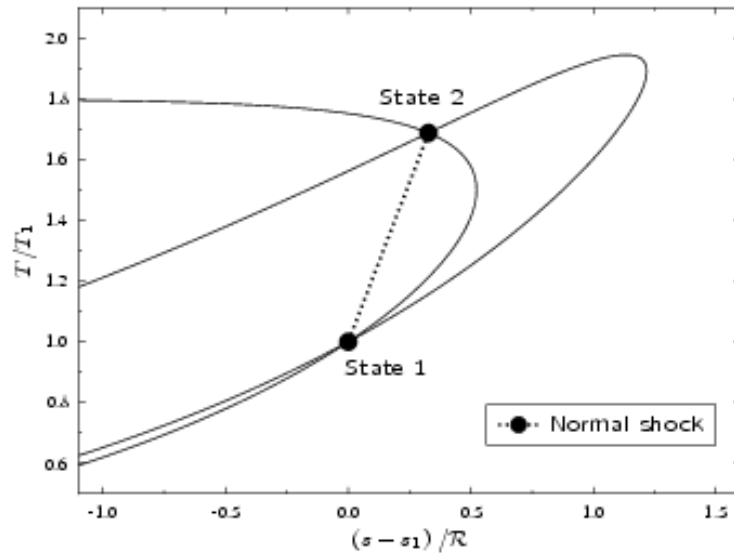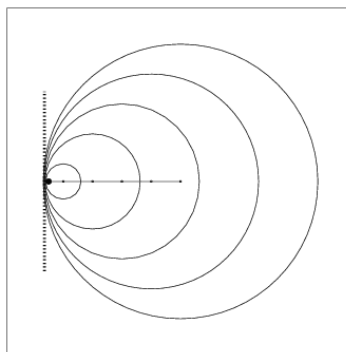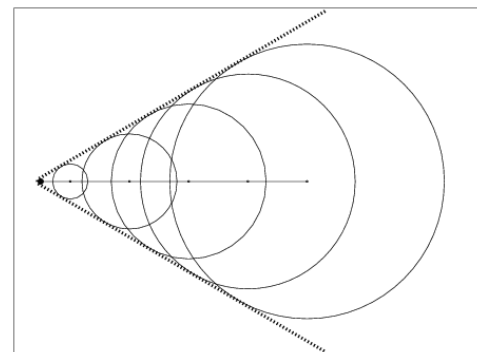
Waves in supersonic flow

The following set of figures is commonly used to illustrate some interesting aspects of wave propagation in supersonic flows:

| (a) Ma < 1 | (b) Ma ≈ 1 | (c) Ma > 1 |

Waves in supersonic flow

Ma < 1   The upstream air is smoothly adjusted for the passing of the object, since the acoustic signals (pressure waves) from it travels faster than the object itself.

Ma ≈ 1   The signals from the object is travelling at about the same speed with the object itself, so the upstream air receives no information about the event. There exists a *"zone of silence"* just ahead of the object. The tight clustering of the pressure waves also gives rise to a thin layer of compression region, which is often visible as a shock. In the past, this sudden pressure increase has given birth to the idea of *"sound barrier"*.

Waves in supersonic flow

Ma > 1   Even larger region in the flow is unaware of the object, and some parts of it may even located physically behind it, as the information carried by the pressure front (*the Mach wave*) does not have enough time to propagate. The angle of this Mach wave is related to the speed as the object as:

$$\beta = \arcsin\left(\frac{1}{\mathrm{Ma}}\right) \tag{201}$$

And in the case of the object being a sphere, this pressure front forms what is often called *"Mach cone"*.

Multi-dimensional equations of the flow

For the following topics, the following simplifying assumptions about the flow is made:

- Steady flow, $\partial/\partial t = 0$.

- Two-dimensional flow, $u_z = 0$ and $\partial/\partial z = 0$. Often, a 3-D flow can be expressed as 2-D by using an appropriate coordinate system.

- No viscous stresses or heat conduction, the flow is isentropic except across shocks.

- Calorically perfect ideal gas, $P = \rho \mathcal{R} T$ and $h = \frac{\gamma}{\gamma-1} \mathcal{R} T$.

# Multi-dimensional equations of the flow

Setting $h_o = h + \frac{1}{2}\left(u_x^2 + u_y^2\right)$, the governing equations for the steady *inviscid* 2-D flows considered here are:

$$
\frac{\partial}{\partial x}\left(\rho u_x\right) + \frac{\partial}{\partial y}\left(\rho u_y\right) = 0
$$
$$
\frac{\partial}{\partial x}\left(\rho u_x^2 + P\right) + \frac{\partial}{\partial y}\left(\rho u_y u_x\right) = 0
$$
$$
\frac{\partial}{\partial x}\left(\rho u_x u_y\right) + \frac{\partial}{\partial y}\left(\rho u_y^2 + P\right) = 0
$$
$$
\frac{\partial}{\partial x}\left(\rho u_x h_o\right) + \frac{\partial}{\partial y}\left(\rho u_y h_o\right) = 0
$$

(202)

Oblique shocks

Consider a wedge in a supersonic flow, making an angle θ with the flow. As a result, the flow is deflected by that much. Since this is a supersonic flow, the deflection is accomplished through the means of a shock wave.
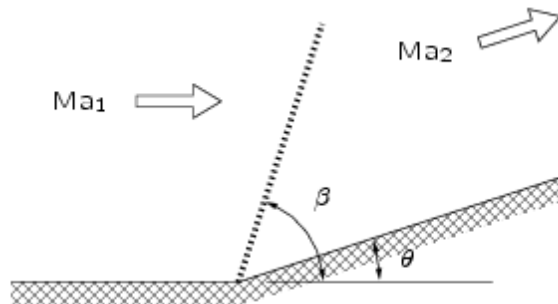

For a small enough deflection, the shock wave is a straight line, making an angle β with respect to the incoming flow. The task now is to relate the two angles for a given upstream Mach number.

Oblique shocks

Refer to the following schematic figure of a flow across a 2-D oblique shock. Note how a coordinate system can be set up such that the flow velocity can be resolved into its components, one that is normal to the shock, and another that is tangential to it. The equations governing the flow will be applied to this rotated system.
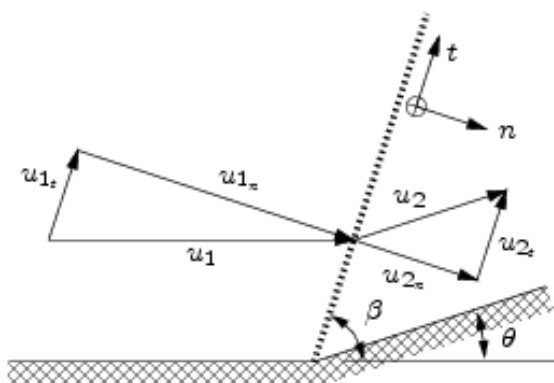
From geometry:

$$
\begin{aligned}
u_{1_n} &= u_1 \sin \beta \\
&= u_{1_t} \tan \beta \\
u_{2_n} &= u_2 \sin (\beta - \theta) \\
&= u_{2_t} \tan (\beta - \theta)
\end{aligned}
\qquad (203)
$$

Oblique shocks

In the following analysis, another assumption is made, that is $\partial/\partial t = 0$. This is to say that the shock will *NOT* influence the tangential components of the flow, further reducing it to a 1-D system:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\rho u_n\right) = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\rho u_n^2 + P\right) = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\rho u_n u_t\right) = 0 \tag{204}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\rho u_n h_o\right) = 0$$

Oblique shocks

Upon integration, and application of boundary conditions, those equations result in the following:

$$\rho_1 u_{1_n} = \rho_2 u_{2_n}$$
$$\rho_1 u_{1_n}^2 + P_1 = \rho_2 u_{2_n}^2 + P_2$$
$$u_{1_t} = u_{2_t} \tag{205}$$
$$\tfrac{1}{2} u_{1_n}^2 + \tfrac{\gamma}{\gamma - 1}\left(P_1/\rho_1\right) = \tfrac{1}{2} u_{2_n}^2 + \tfrac{\gamma}{\gamma - 1}\left(P_2/\rho_2\right)$$

However, these are similar to the relations across a normal shock. Thus, all the properties can be obtained in a similar fashion.

Oblique shocks

- The tangential velocity stays constant across the oblique shock:

$$u_{1_t} = u_{1_t} = u_t \tag{206}$$

- Jumps in the other flow properties are now dependent on the normal component of the upstream Mach number:

$$\mathrm{Ma}_{1_n} = u_{1_n}/a_1 = u_{1_n}/\sqrt{\gamma P_1/\rho_1} = \mathrm{Ma}_1 \sin\beta \tag{207}$$

- From geometry, the wave angle β is found to be:

$$\tan\beta = u_{1_n}/u_t \tag{208}$$

- Similarly, the wedge angle θ can be calculated as:

$$\tan(\beta - \theta) = u_{2_n}/u_t \tag{209}$$

Oblique shocks

Thus, one has the following relationships for the oblique shocks:

$$
\begin{aligned}
\frac{\rho_2}{\rho_1} &= \left(\frac{u_{2_n}}{u_{1_n}}\right)^{-1} = \frac{\frac{\gamma+1}{2}\mathrm{Ma}_{1_n}^2}{1 + \frac{\gamma-1}{2}\mathrm{Ma}_{1_n}^2} = 1 + \frac{\mathrm{Ma}_{1_n}^2 - 1}{1 + \frac{\gamma-1}{2}\mathrm{Ma}_{1_n}^2} \\
\frac{P_2}{P_1} &= \frac{2\gamma}{\gamma+1}\left(\mathrm{Ma}_{1_n}^2 - \frac{\gamma-1}{2\gamma}\right) = 1 + \frac{2\gamma}{\gamma+1}\left(\mathrm{Ma}_{1_n}^2 - 1\right) \\
\frac{T_2}{T_1} &= \left[\frac{2}{\gamma+1}\left(\gamma\mathrm{Ma}_{1_n}^2\right) - \frac{\gamma-1}{\gamma+1}\right]\left[\frac{2}{\gamma+1}\left(1/\mathrm{Ma}_{1_n}^2\right) + \frac{\gamma-1}{\gamma+1}\right] \\
\frac{\mathrm{Ma}_{t_2}}{\mathrm{Ma}_{t_1}} &= \frac{u_t/a_2}{u_t/a_1} = \left(\frac{a_2}{a_1}\right)^{-1} = \left(\frac{T_2}{T_1}\right)^{-\frac{1}{2}}
\end{aligned}
\tag{210}
$$

Oblique shocks

Since it is required that $\mathrm{Ma}_{1_n} \geq 1$, there is a minimum value imposed for the wave angle β, such that $\beta_{\min} = \arcsin\left(1/\mathrm{Ma}_{1_n}\right)$. Recall that this expression is encountered earlier in the discussion on Mach wave.

Naturally, the maximum value for β corresponds to the case of normal shock, $\beta_{\max} = \pi/2$.

In addition, the following relationships can also be established:

$$\mathrm{Ma}_{2_n}^2 = \mathrm{Ma}_2^2 \sin^2\left(\beta - \theta\right) = \frac{\frac{2}{\gamma - 1} + \mathrm{Ma}_1^2 \sin^2 \beta}{\frac{2\gamma}{\gamma - 1}\mathrm{Ma}_1^2 \sin^2 \beta - 1}$$

$$\frac{s_2 - s_1}{\mathcal{R}} = \frac{1}{\gamma - 1}\ln\left(P_2/P_1\right) - \frac{\gamma}{\gamma - 1}\ln\left(\rho_2/\rho_1\right) = -\ln\left(P_{o_2}/P_{o_1}\right)$$

$$\tag{211}$$

Relationship between wedge angle and wave angle

From the shock relationship and geometry, one obtains the following:

$$\frac{u_{2_n}}{u_{1_n}} = \frac{u_{2_n}/u_t}{u_{1_n}/u_t} = \frac{\tan\left(\beta - \theta\right)}{\tan \beta} = \frac{2 + \left(\gamma - 1\right)\mathrm{Ma}_1^2 \sin^2 \beta}{\left(\gamma + 1\right)\mathrm{Ma}_1^2 \sin^2 \beta} = \chi \quad \text{(212)}$$

Expanding:

$$\chi = \frac{\tan\left(\beta - \theta\right)}{\tan \beta} = \frac{\tan \beta - \tan \theta}{\tan \beta\left(1 + \tan \beta \tan \theta\right)}$$

$$\tan \theta = \frac{\left(1 - \chi\right)\sin \beta \cos \beta}{\cos^2 \beta + \chi \sin^2 \beta} = \frac{2\cos \beta\left(\mathrm{Ma}_1^2 \sin^2 \beta - 1\right)}{\sin \beta\left[\mathrm{Ma}_1^2\left(\gamma + \cos 2\beta\right) + 2\right]}$$

$$\tag{213}$$

This equation can also be inverted to give a cubic equation for $\sin^2 \beta$, allowing an explicit solution for β, for a given Ma_1 and θ.

Relationship between wedge angle and wave angle

The following is the chart showing the relationship. Note that there are a maximum wedge angle $\theta_{\max}$ allowed for attached oblique shock. A greater value would result in detached shock. This maximum wedge angle divides the solution for β into two branches.
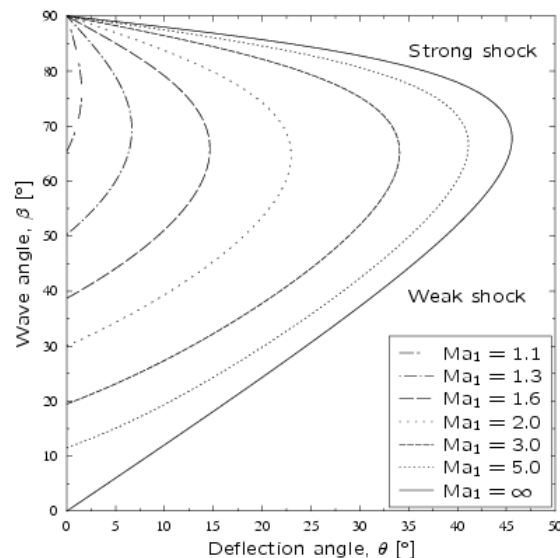
Relationship between wedge angle and wave angle

- There is a maximum deflection angle $\theta_{\max}$ associated with each Mach number, found by setting $d\theta/d\beta = 0$.
- Smaller wave angles correspond to the *"weak shock"*.
  - $\lim_{\theta \to 0} \beta = \arcsin(1/\mathrm{Ma}_1)$, a Mach wave.
  - Applicable to most external flows, which must matches to acoustic wave in far-field, but may exist in some internal flows.
  - Except for some small region close to $\theta_{\max}$, the downstream Mach number is supersonic, $\mathrm{Ma}_2 \geq 1$.
- Larger wave angles correspond to the *"strong shock"*.
  - $\lim_{\theta \to 0} \beta = \pi/2$, a normal shock wave.
  - Applicable mostly to internal flows, depending on the level of the back pressure.
  - The downstream Mach number is subsonic, $\mathrm{Ma}_2 \leq 1$.
- An important point is that if oblique shock waves are attached to the leading edge of an object, *the upper and lower surfaces are independent of each other*.

Small angle approximation

Rearranging the previous equation for $\chi = u_{2_n}/u_{1_n}$:

$$\left(\mathrm{Ma}_1^2 \sin^2 \beta\right)^{-1} = \tfrac{\gamma+1}{2} \cot \beta \tan\left(\beta - \theta\right) - \tfrac{\gamma-1}{2}$$
$$\Leftrightarrow \quad \mathrm{Ma}_1^2 \sin^2 \beta - 1 = \tfrac{\gamma+1}{2}\mathrm{Ma}_1^2 \sin \beta \sec\left(\beta - \theta\right) \sin \theta \tag{214}$$

For small deflection angle θ, such that $\beta - \theta \approx \beta$, this may be approximated as:

$$\mathrm{Ma}_1^2 \sin^2 \beta - 1 \approx \left(\tfrac{\gamma+1}{2}\mathrm{Ma}_1^2 \tan \beta\right) \theta \tag{215}$$

Furthermore, for large upstream Mach number, this can be further simplified into a *"hypersonic approximation"*, noting that $\beta \ll 1$ while $\mathrm{Ma}_1 \beta \gg 1$:

$$\beta \approx \tfrac{\gamma+1}{2}\theta \tag{216}$$

Mach lines

From the earlier discussions, it has been mentioned that for the weak oblique shock, the wave angle corresponding to an ever-smaller deflection angle approaches a non-zero limit μ, such that $\lim_{\theta \to 0} \beta = \mu = \arcsin\left(1/\mathrm{Ma}_1\right)$.

At this point, the flow essentially experiences no changes whatsoever. There is nothing unique about the point, it could be anywhere in the flow. In fact, the angle μ is simply a *characteristic angle* associated with the Mach number Ma_1. Thus, the name *Mach angle*, which is the angle of the characteristic line emanating from some point in the flow, relative to the direction of the flow.

Mach lines

In 2-D supersonic flows, there exist 2 families of characteristic lines, the *left-running*, given the negative sign, and the positive *right-running*. These are named relative to the streamline. And if the flow is non-uniform, these lines are curved, since μ would vary accordingly.

The name *"characteristics"* come from the mathematical theory of partial differential equation. Supersonic flows are considered hyperbolic, thus they have real characteristics. Subsonic flows are considered elliptic, and the characteristics are imaginary.

This attribute manifests itself in the fact that Mach lines have a sense of direction, which is toward *"increasing time"*. Hence, the fact that there are no information going *"upstream"*.
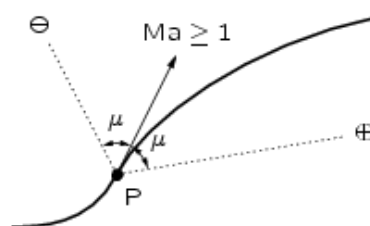
Mach lines

This concept of Mach lines is depicted in the following figures.
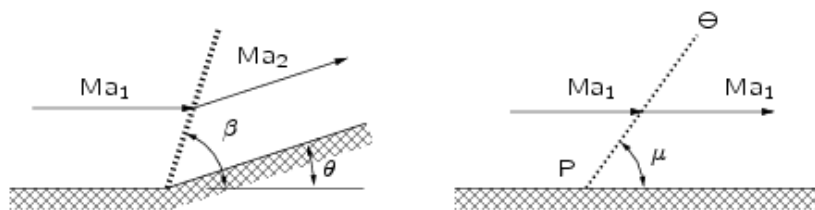
(a)   (b)

(c)

Weak oblique shocks

For small deflection angles, the following relationship has been derived:

$$\text{Ma}_1^2 \sin^2 \beta - 1 = \left(\tfrac{\gamma+1}{2} \text{Ma}_1^2 \tan \beta \right) \theta \qquad (217)$$

And since $\theta \to 0$, the right-hand-side can be approximated in terms of Mach angle μ as:

$$\text{Ma}_1^2 \sin^2 \beta - 1 \approx \left(\frac{\gamma+1}{2} \text{Ma}_1^2 \tan \mu \right) \theta = \left[\frac{(\gamma+1)\,\text{Ma}_1^2}{2\sqrt{\text{Ma}_1^2 - 1}} \right] \theta \quad (218)$$

This equation shall be used as the starting point for the relationship for weak oblique shock waves.

Weak oblique shocks

From the oblique shock relations, the shock strength is found to be in the order of $\mathcal{O}\left(\theta\right)$:

$$\frac{\Delta P}{P_1} = \frac{2\gamma}{\gamma+1} \left(\text{Ma}_1^2 \sin^2 \beta - 1 \right) \approx \left(\frac{\gamma \text{Ma}_1^2}{\sqrt{\text{Ma}_1^2 - 1}} \right) \theta \qquad (219)$$

In addition, the change in entropy can be shown to be proportional to the third order of the shock strength for weak oblique shock, $\Delta s = \mathcal{O}\left(\theta^3\right)$:

$$\frac{s_2 - s_1}{\mathcal{R}} \approx \frac{\gamma+1}{12\gamma^2} \left(\Delta P/P_1 \right)^3 \approx \frac{\gamma+1}{12\gamma^2} \left(\frac{\gamma \text{Ma}_1^2}{\sqrt{\text{Ma}_1^2 - 1}} \right)^3 \theta^3 \qquad (220)$$

Weak oblique shocks

The other properties can also be shown to be proportional to the deflection angle θ as well. For example:

$$\frac{T_2}{T_1} = [1 + (\Delta P / P_1)] \frac{\left[1 + \frac{\gamma-1}{2\gamma} (\Delta P / P_1)\right]}{\left[1 + \frac{\gamma+1}{2\gamma} (\Delta P / P_1)\right]}$$

$$\approx \left[1 + \left(\frac{\gamma \mathrm{Ma}_1^2}{\sqrt{\mathrm{Ma}_1^2 - 1}}\right) \theta\right] \frac{\left[1 + \frac{\gamma-1}{2\gamma} \left(\frac{\gamma \mathrm{Ma}_1^2}{\sqrt{\mathrm{Ma}_1^2 - 1}}\right) \theta\right]}{\left[1 + \frac{\gamma+1}{2\gamma} \left(\frac{\gamma \mathrm{Ma}_1^2}{\sqrt{\mathrm{Ma}_1^2 - 1}}\right) \theta\right]} \tag{221}$$

Weak oblique shocks

For a closer look, one can define the wave angle β as a slight increase from the Mach angle μ, such that $\beta = \mu + \epsilon$ for $\epsilon \ll 1$. With this approximation, one can obtain the following expressions:

$$\sin\beta \approx \sin\mu + \epsilon \cos\mu = \frac{1 + \epsilon \sqrt{\mathrm{Ma}_1^2 - 1}}{\mathrm{Ma}_1}$$

$$\mathrm{Ma}_1^2 \sin^2\beta \approx 1 + 2\epsilon \sqrt{\mathrm{Ma}_1^2 - 1} \tag{222}$$

$$\epsilon = \left[\frac{(\gamma + 1) \mathrm{Ma}_1^2}{4 (\mathrm{Ma}_1^2 - 1)}\right] \theta$$

Hence, *for a small finite deflection angle θ, the direction of the shock wave β differs from the Mach wave μ by an amount $\epsilon = \mathcal{O}(\theta)$.*

Weak oblique shocks

The expression relating the change of flow speed Δu across the wave can also be developed as follows:

$$\frac{u_2}{u_1} = \frac{\sqrt{(u_{2_n}/u_t)^2 + 1}}{\sqrt{(u_{1_n}/u_t)^2 + 1}} = \frac{\sqrt{\tan^2(\beta - \theta) + 1}}{\sqrt{\tan^2\beta + 1}} = \frac{\cos\beta}{\cos(\beta - \theta)} \quad (223)$$

Expanding the cosine terms, assuming small angles for ϵ and θ:

$$\cos(\beta - \theta) = \cos(\mu + \epsilon - \theta) = \cos\mu - (\epsilon - \theta)\sin\mu$$
$$= \frac{\sqrt{\text{Ma}_1^2 - 1} - (\epsilon - \theta)}{\text{Ma}_1} = \frac{\sqrt{\text{Ma}_1^2 - 1}}{\text{Ma}_1}\left(1 - \frac{\epsilon - \theta}{\sqrt{\text{Ma}_1^2 - 1}}\right) \quad (224)$$

Weak oblique shocks

Thus, the expression for the speed ratio becomes:

$$\frac{u_2}{u_1} = \frac{1 - \frac{\epsilon}{\sqrt{\text{Ma}_1^2 - 1}}}{1 - \frac{\epsilon - \theta}{\sqrt{\text{Ma}_1^2 - 1}}}$$
$$= \left[1 - \frac{\epsilon}{\sqrt{\text{Ma}_1^2 - 1}}\right]\left[1 + \frac{\epsilon - \theta}{\sqrt{\text{Ma}_1^2 - 1}} + \cdots\right] \quad (225)$$
$$= 1 - \frac{\epsilon}{\sqrt{\text{Ma}_1^2 - 1}} + \frac{\epsilon - \theta}{\sqrt{\text{Ma}_1^2 - 1}} + \cdots$$
$$\approx 1 - \frac{\theta}{\sqrt{\text{Ma}_1^2 - 1}}$$

In terms of speed change, this becomes:

$$\frac{\Delta u}{u_1} \approx -\frac{\theta}{\sqrt{\text{Ma}_1^2 - 1}} \quad (226)$$

Supersonic compression by turning

Supersonic flow may be compressed by turning it by an angle θ, and forcing it to go through an oblique shock. However, this process of turning may be done by a succession of small turns, each with a magnitude of $\Delta\theta$.

The series of these oblique shocks will eventually intersect one another and coalesce into a single wave. However, near the wall, each can be treated as independent from of the following one, and the flow may be constructed step-by-step. This is as long as the deflection does not become too great that the flow becomes subsonic.

And as these turns become ever smaller, $\Delta\theta \to 0$, one then obtains a *smooth turn*. Along with this, the compression approaches isentropic limit.

Supersonic compression by turning

The situation can be depicted as follows.
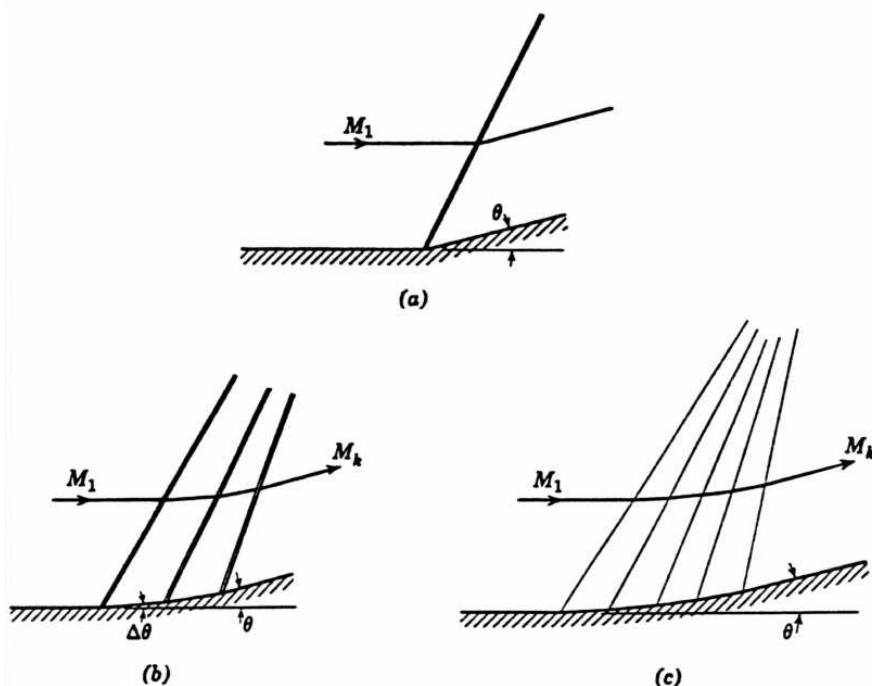
Supersonic compression by turning

For each wave associated with each $\Delta\theta$, one has the followings:
$\Delta P = \mathcal{O}\left(\Delta\theta\right)$ and $\Delta s = \mathcal{O}\left(\Delta\theta^3\right)$.
For the complete turn, $\theta = n\Delta\theta$, the overall pressure and entropy changes are:

$$P_k - P_1 \sim n\Delta\theta \sim \theta$$

$$s_k - s_1 \sim n\Delta\theta^3 \sim \frac{1}{n^2}\theta^3 \tag{227}$$

Hence, as the deflections become a smooth turn, the entropy rise decreases by a factor of $1/n^2$, to finally become vanishingly small, approaching the limit of *isentropic compression*.

Supersonic compression by turning

When the compression becomes isentropic, the change of speed across a weak oblique shock becomes a *differential* expression:

$$\frac{\mathrm{d}u}{u} = -\frac{\mathrm{d}\theta}{\sqrt{\mathrm{Ma}^2 - 1}} \tag{228}$$

This equation applies continously through the isentropic turn, and when integrated, will give a relation between Ma and θ.
Moving further away from the wall, the Mach lines will ultimately converge to form a shock wave. This is due to the fact that an intersection of two Mach lines would imply an infinitely high gradient, since the Mach number is double-valued at this point. Physically, however, as the lines approach each other, the gradients would become large enough to invalidate isentropic assumption.

This notion of smoothness ultimately relies also on the scale of the geometry. From far enough away, any smooth bend would still appear as a sharp corner.

In the case of a channel flow, however, if the upper wall is located close enough, then the gradients encountered during the turning may still be small enough to warrant the use of isentropic compression. And since the flow is isentropic, it may be reversed without violating the second law of thermodynamics. In this case, the reversing flow is an expansion flow.

Supersonic compression by turning

The converging of the Mach waves can be seen in these figures.
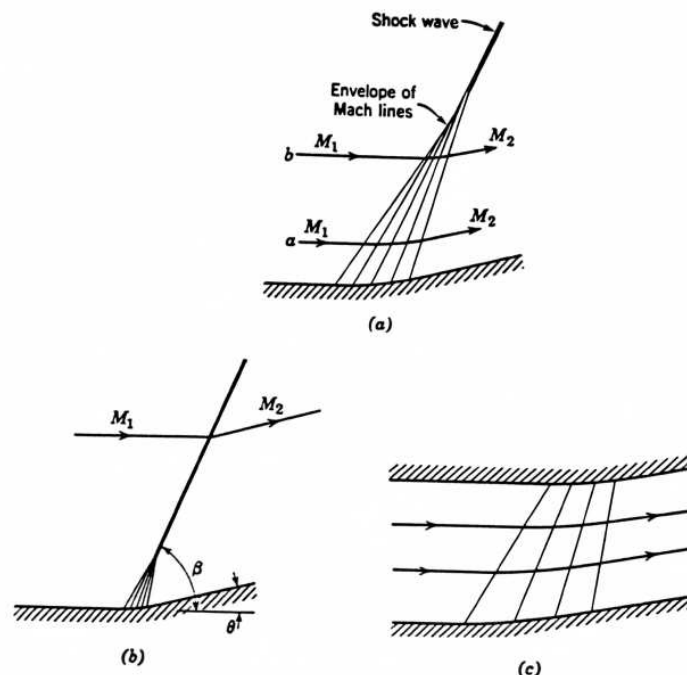
Supersonic expansion by turning

Since a supersonic flow can be compressed by turning it, then it also can be expanded by turning it the other way.
Unlike compression turning, there is no such thing as an expansion oblique shock wave, since it requires a decrease in entropy. Instead, the Mach lines are being directed *away* from each other, resulting in a divergent pattern. This, in effect, would *decrease* the existing gradients. Hence, *an expansion flow is isentropic* throughout.
For a sharp corner, this series would be centered at the corner, while for a smooth turn, the waves would be spread out throughout the turn.

Supersonic expansion by turning

The following figures illustrate this concept of supersonic expansion.
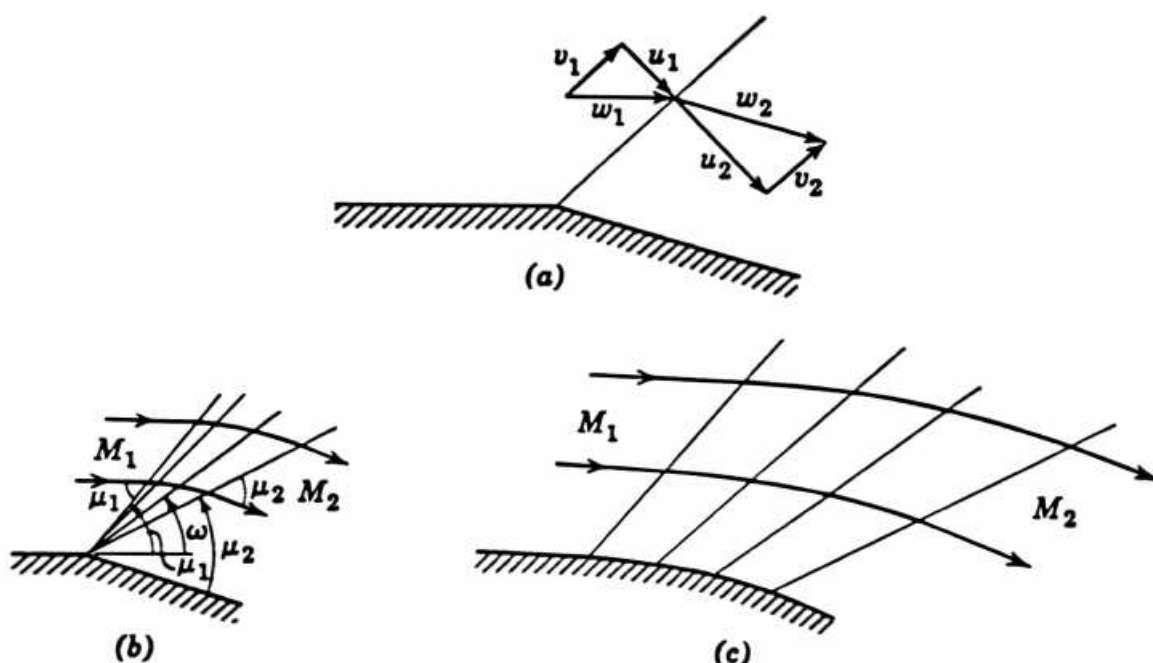
The Prandtl-Meyer function

The relationship between θ and Ma can be integrated as:

$$-\int \mathrm{d}\theta = \int \tfrac{1}{2}\sqrt{\mathrm{Ma}^2 - 1}\,\frac{\mathrm{d}\left(u^2\right)}{u^2} \qquad (229)$$

Here, $\theta > 0$ indicates compression turn, and $\theta < 0$ indicates expansion turn. One can also define the *Prandtl-Meyer function* $\nu\left(\mathrm{Ma}\right) = \theta_o - \theta$, such that $\nu = 0$ correspond to $\mathrm{Ma} = 1$. The term $\mathrm{d}\left(u^2\right)/u^2$ can be rewritten as a function of Ma using the following relations:

$$a^2 = \frac{a_o^2}{1 + \frac{\gamma - 1}{2}\mathrm{Ma}^2} \quad \Rightarrow \quad \frac{\mathrm{d}a}{a} = -\frac{\frac{\gamma - 1}{2}\mathrm{Ma}^2}{1 + \frac{\gamma - 1}{2}\mathrm{Ma}^2}\frac{\mathrm{d}\mathrm{Ma}}{\mathrm{Ma}}$$

$$u = a\mathrm{Ma} \quad \Rightarrow \quad \frac{\mathrm{d}u}{u} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}^2\right)^{-1}\frac{\mathrm{d}\mathrm{Ma}}{\mathrm{Ma}} \qquad (230)$$

The Prandtl-Meyer function

Now integrate the relationship to obtain explicit form of $\nu\left(\mathrm{Ma}\right)$:

$$\nu\left(\mathrm{Ma}\right) = \int_1^{\mathrm{Ma}} \left[\frac{\sqrt{\mathrm{Ma}'^2 - 1}}{1 + \frac{\gamma - 1}{2}\mathrm{Ma}'^2}\right]\frac{\mathrm{d}\mathrm{Ma}'}{\mathrm{Ma}'} \qquad (231)$$

Substituting $\phi^2 = \mathrm{Ma}^2 - 1$:

$$\nu\left(\mathrm{Ma}\right) = \int_0^{\sqrt{\mathrm{Ma}^2 - 1}} \left[\frac{\frac{2}{\gamma + 1}\phi^2}{1 + \frac{\gamma - 1}{\gamma + 1}\phi^2}\right]\frac{\mathrm{d}\phi}{1 + \phi^2}$$

$$= \int_0^{\sqrt{\mathrm{Ma}^2 - 1}} \left[\frac{1}{1 + \frac{\gamma - 1}{\gamma + 1}\phi^2} - \frac{1}{1 + \phi^2}\right]\mathrm{d}\phi \qquad (232)$$

$$= \sqrt{\frac{\gamma + 1}{\gamma - 1}}\arctan\sqrt{\frac{\gamma - 1}{\gamma + 1}\left(\mathrm{Ma}^2 - 1\right)}$$

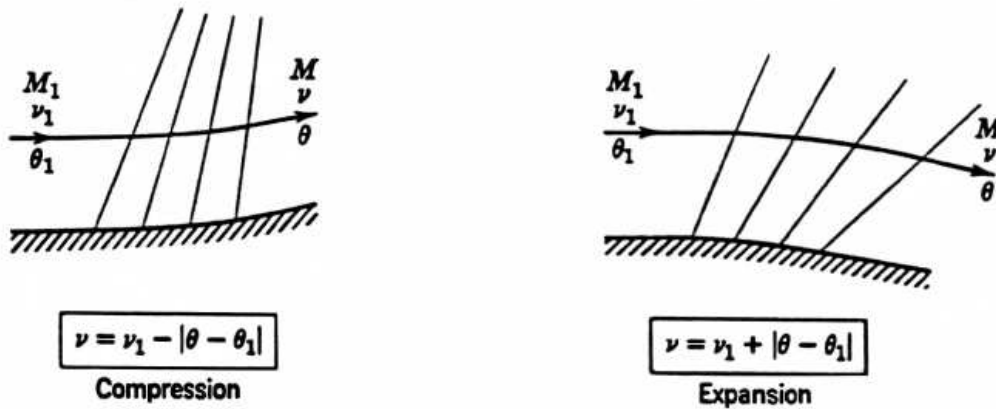$$- \arctan\sqrt{\mathrm{Ma}^2 - 1}$$

The Prandtl-Meyer function

As the Mach number varies from 1 to ∞, ν increases
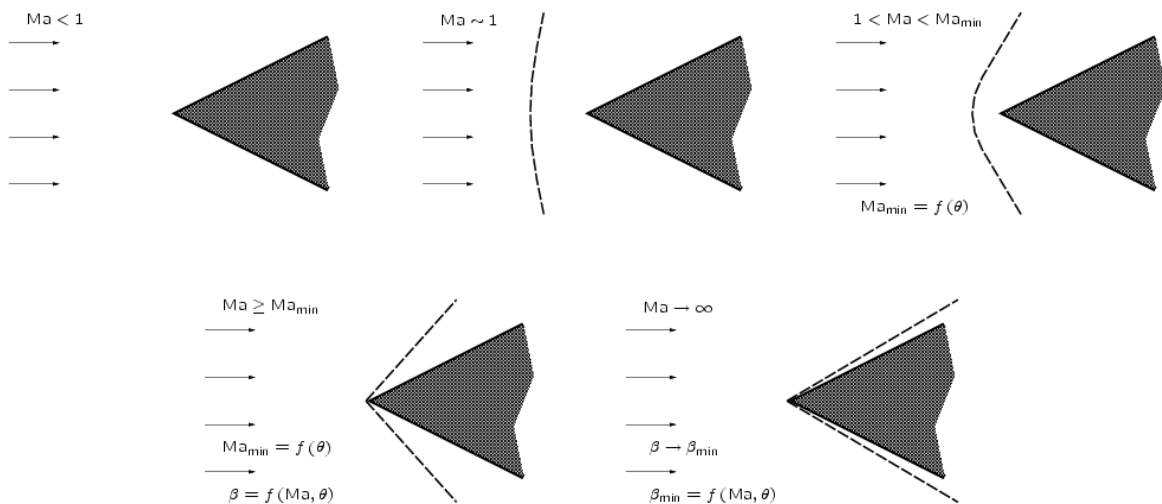monotonically, until it reaches a maximum value:

$$\nu_{\max} = \tfrac{1}{2}\pi\left(\sqrt{\tfrac{\gamma+1}{\gamma-1}} - 1\right) \tag{233}$$

This figure illustrate the relation between ν and θ in simple
isentropic turns.

$$\boxed{\nu = \nu_1 - |\theta - \theta_1|}$$
Compression

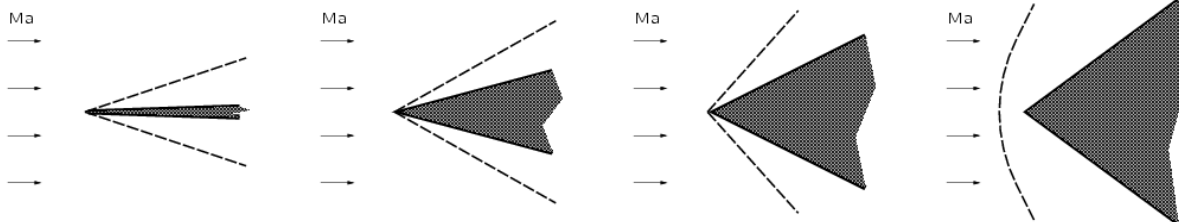$$\boxed{\nu = \nu_1 + |\theta - \theta_1|}$$
Expansion

Detached shocks

Given a wedge at zero angle of attack, with a half-angle $\delta < \delta_{\max}$,
the following phenomena can be described as the free stream flow
changes from subsonic to supersonic.

Detached shocks

Alternatively, one can increase the half-angle of the wedge to detach the shock from it as follows.

The common theme between these two series of figures is that the deflection angle θ for the detached shock is greater than the value allowed in the β-θ-Ma equation.

Detached shocks

This angle θ_{max} can be found by setting $\mathrm{d}\theta/\mathrm{d}\beta = 0$ for a given Mach number. Allowing $\phi = \cos 2\beta$, the β-θ-Ma equation becomes:

$$\tan\theta = \sqrt{\frac{1+\phi}{1-\phi}\left[\frac{(1-2/\mathrm{Ma}^2)-\phi}{(\gamma+2/\mathrm{Ma}^2)+\phi}\right]} = f(\phi)$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\phi} = \frac{f'}{1+f^2} = \left[\frac{f}{1+f^2}\right]g(\phi) \tag{234}$$

$$g(\phi) = \frac{1}{1-\phi^2} - \frac{\gamma+1}{(1-2/\mathrm{Ma}^2-\phi)(\gamma+2/\mathrm{Ma}^2+\phi)}$$

Setting $g(\phi) = 0$ will yield a quadratic equation in ϕ. The correct solution is chosen by noting that $|\phi| \leq 1$. Plugging it back into the β-θ-Ma equation will yield the value for θ_{max}.

Detached shocks

In practice, there is no analytical treatment for the case of detached shocks. Studies are conducted empirically, mainly through experimental and numerical methods.

A detached shock exhibits the whole spectrum of oblique shock behaviour, from normal shock, to strong and weak oblique shock, all the way to Mach angle in the far-field. The shape and its *detachment / stand-off distance* depend on the Mach number and *the geometry of the body*.

During the early years of hypersonic flight research, it was found that blunt nose is preferable to sharp nose, for survivability of a re-entry vehicle, a finding deemed important enough to stay classified for quite a long time. Thus, one sees that the Space Shuttles have blunt leading edges, while supersonic fighters have razor-sharp leading edges for lower drag.
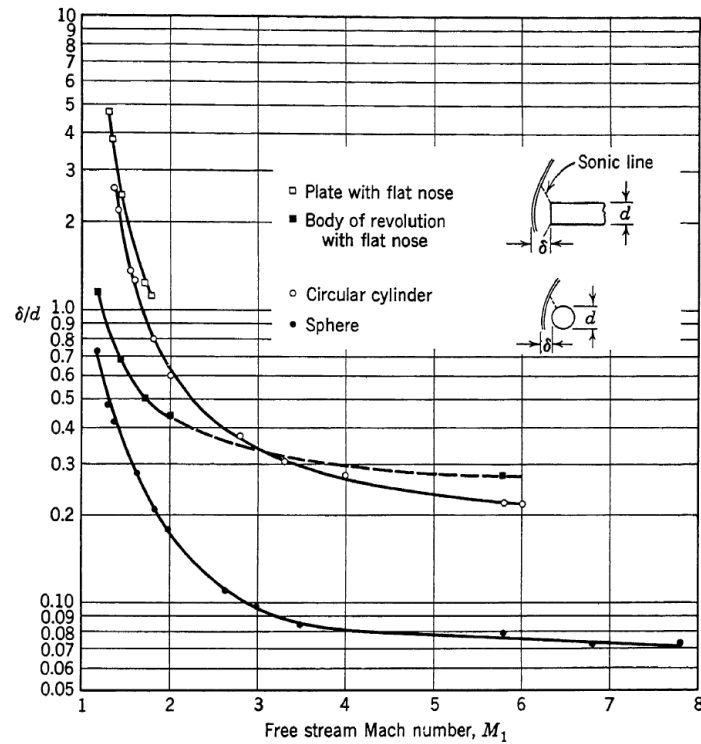
Detached shocks

The complexities associated with detached shocks are mainly caused by the presence of subsonic flow regimes. Hence, the shock now also depends on the interaction with the downstream condition. The following figures show sketches of what is usually observed.
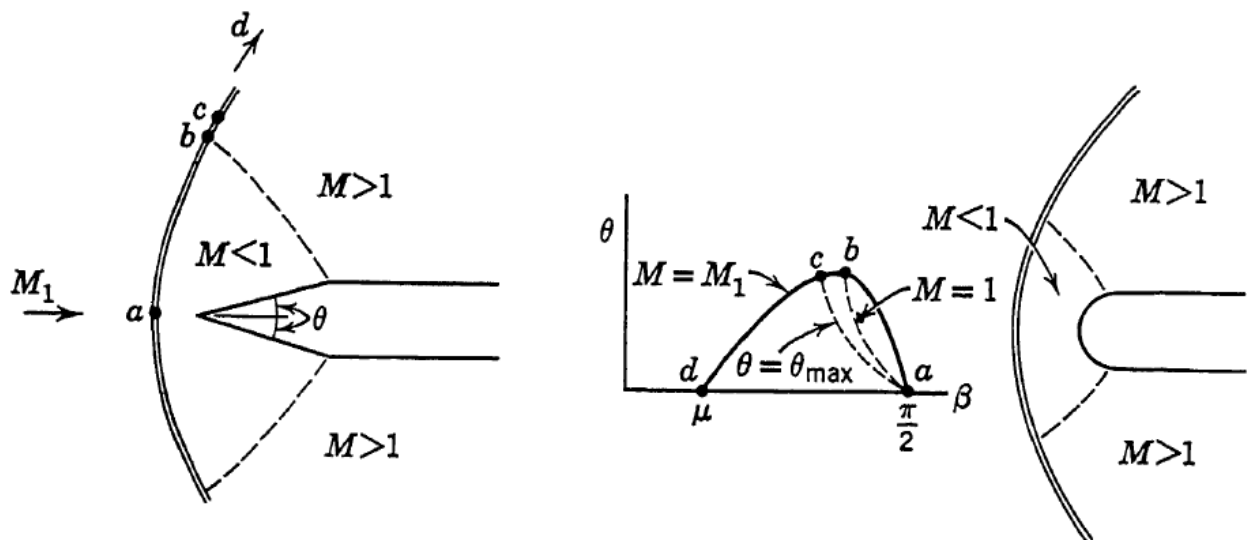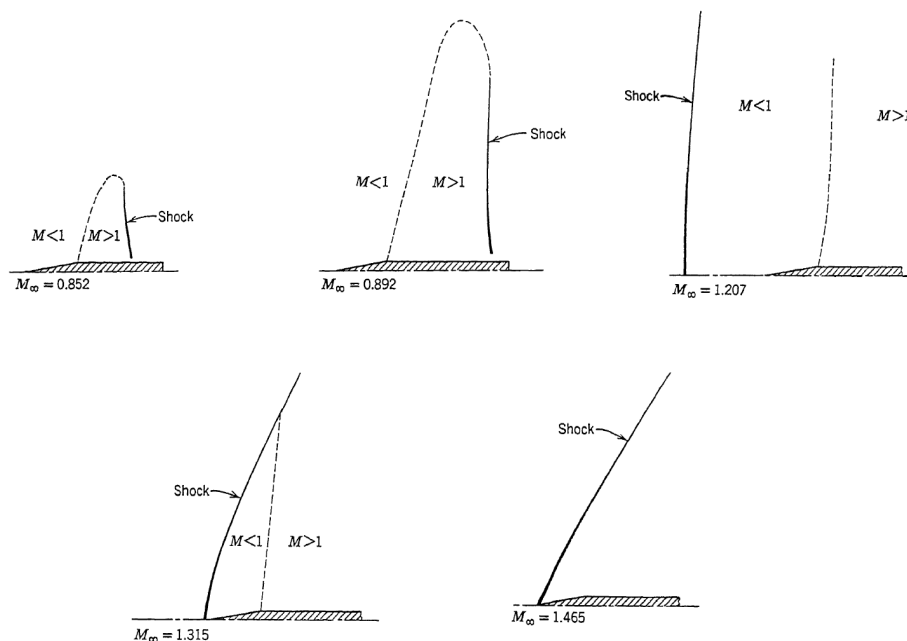
Detached shocks

Some experimental results can be seen in this figure.

Detached shocks

As illustrated earlier, detached shocks are often encountered in transonic flow regime, shown in the following series of figures.

Shock-expansion theory

The oblique shock wave and the simple isentropic wave provide the basic tools for analysing many cases in supersonic flows, by patching together the solutions, of which some are shown in this figure.
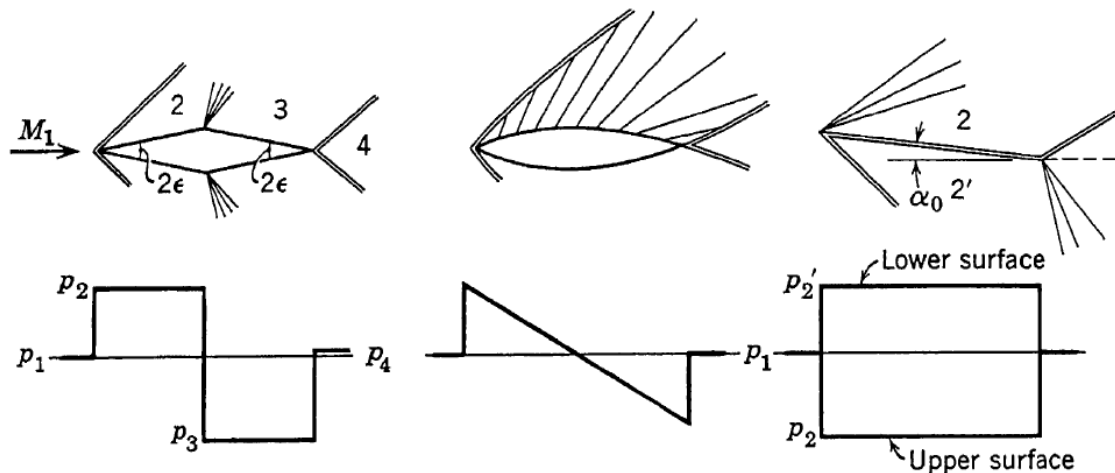
For example, the drag for the diamond airfoil is found as $\mathcal{D} = (P_2 - P_3)\, t$, where t is the thickness at the shoulder.

Shock-expansion theory

Recall the *d'Alembert* paradox, which states that a symmetrical body immersed in an inviscid fluid experiences no drag. Yet, here the airfoil clearly suffers from drag forces, while being put in an inviscid fluid.

This is an example of *supersonic wave drag*. In supersonic flows, drag force can exist, even in the idealized inviscid flows. It is fundamentally different from the frictional drag and separation drag, commonly associated with boundary layers in viscous flows.

It is worth mentioning that in real fluids, the shock wave and the boundary layers do interact. In fact, the wave drag is ultimately *"dissipated"* by viscosity within the shock wave, but it does not depend on the value of the viscosity coefficient. Thus, the entropy change across the shock wave is largely independent from the detailed non-equilibrium process inside the shock wave itself.

Shock-expansion theory

Another example is the curved airfoil, which has *continuous* expansion along its upper and lower surfaces. For the shock waves to be attached, the leading edge and the trailing edge should be wedge-shaped, with half-angle less than $\theta_{\max}$.

For a flat plate at an angle of attack of α_o, the pressure on the upper surface is independent from the lower surface. In fact, the entropy rise experienced by flows along the two surfaces are not equal. Hence, a *slipstream* emanates from the trailing edge, inclined at some small angle relative to the free stream, across which the static pressure is equal.

For this flat plate having a chord length of c, the lift and the drag forces can be found as:

$$\mathcal{L} = \left(P_2' - P_2\right) c \cos \alpha_o \approx \left(P_2' - P_2\right) c$$
$$\mathcal{D} = \left(P_2' - P_2\right) c \sin \alpha_o \approx \left(P_2' - P_2\right) c \alpha_o$$

(235)

Shock-expansion theory

The shock waves and the expansion waves also interact with each other. Here in the figure, it is shown that expansion waves intersect the shock wave, curving and weakening it, and ultimately reducing it to mere Mach wave in the far-field.
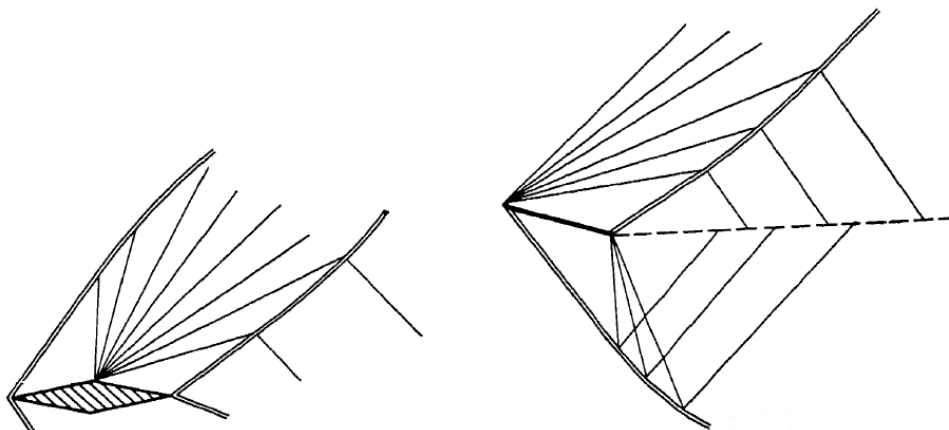

For most part, the reflected waves are very weak, and thus do not affect the shock-expansion result for the pressure distribution.

Reflection and intersection of oblique shocks

An oblique shock wave can also be reflected by a wall. And since the wall can alternatively be represented by a streamline, the reflection can also be seen as an intersection of identical oblique shocks of the opposing family. These are shown in the following figures.
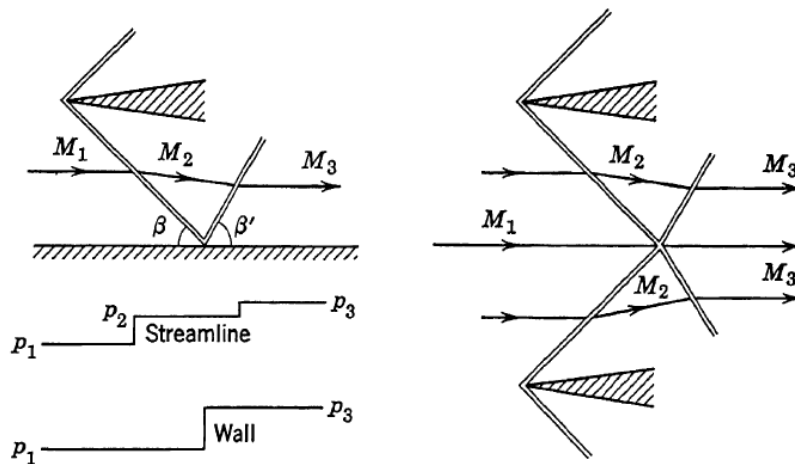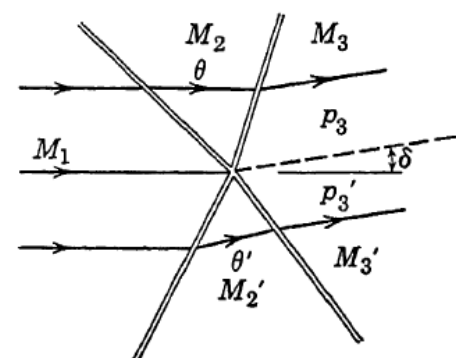
Reflection and intersection of oblique shocks

If the intersecting shocks are of *unequal* strength, a slipstream is seen to divide the flow into two portions, across which the *pressure* and the *flow direction* are identical.
However, the *velocity magnitude* can be different across, thus it is also a *shear layer*.
Additionally, density and temperature may also differ, making the slipstream also a *contact surface*.
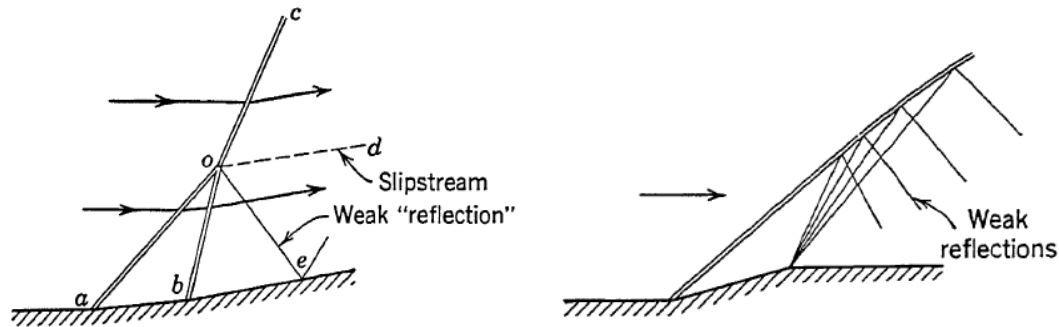

This is due to different entropy rise experienced by the two portions of the flow.

Reflection and intersection of oblique shocks

When the two intersecting shocks are from the same family, it results in the merging of the two, shown in this figure. Note that a wave of the opposing family is needed to equalize the pressure across the slipstream.


The other figure shows what occurs when the expansion waves intersect an oblique shock wave. Along with the shock weakening, there are also weak reflected waves.

Reflection and intersection of oblique shocks

A more complex phenomenon in the shock reflection occurs when the incident shock is strong enough such that the Mach number is lower than the detachment value for the given deflection angle. A nearly normal shock wave then appears near the wall, giving rise a subsonic region. This is called the *Mach reflection*, and the normal leg is often referred to as the *Mach stem*.
Note that since the flow now also depends on the downstream conditions, no simple solutions can be obtained.

A slipstream dividing the subsonic and the supersonic regions emanates from the triple point of intersection, shown in this figure.
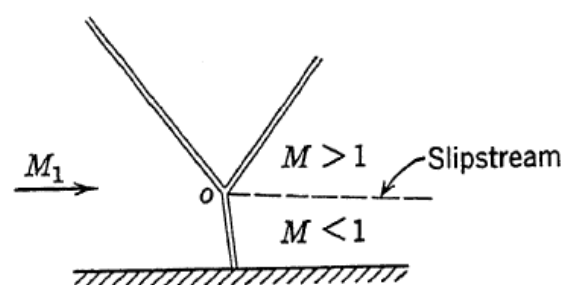
Cones in supersonic flow

The flow field for a cone in supersonic flow at zero angle of attack is not as simple as the one for wedge, where a *uniform* flow region is found between the shock and the wedge. In this 3-D case, uniform flow behind the shock would violate the continuity equation. However, the flow possesses a property that can be exploited for the analysis of the flow field, that is the cone can be considered to be *semi-infinite*.

With this assumption, it can be argued that the flow conditions are *constant along each ray emanating from the vertex of the cone*.
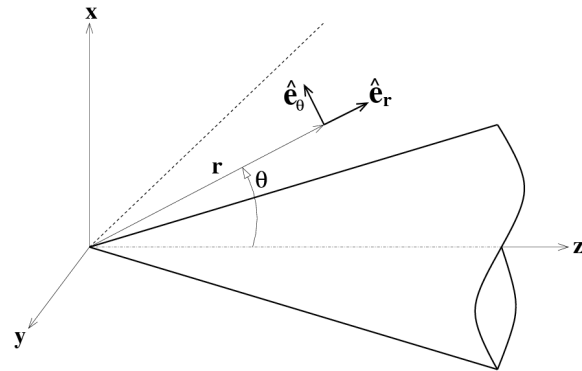
Cones in supersonic flow

An example of the this type of flow is shown in this figure. Note that the flow possesses also azimuthal symmetry, thus one only need to consider the flow along a meridian plane.
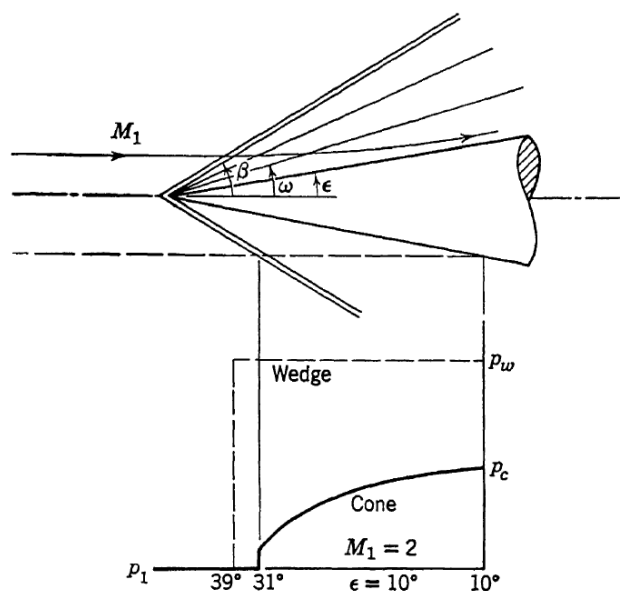
Cones in supersonic flow

The solution has been given, first by Busemann, and later by Taylor and MacColl. The procedure consists of fitting an isentropic conical flow to the conical shock. The 3-D flow equations are rewritten in terms of a single conical variable ω, which is then solved numerically.

At the shock, the simple oblique shock *"jump"* relation can be used, since *locally it always applies to any shock surface*. And the flow behind the shock must be matched with the isentropic conical flow, which then determines the solution.

Note that behind the shock, the streamlines are curved. And compared to the flow over a wedge, the compression is lower due to the 3-D relieving effect. Also, the detachment occurs at a much lower Mach number.

Cones in supersonic flow

The following figure shows the variation of shock wave angle β as a function of Mach number Ma and the cone angle θ.
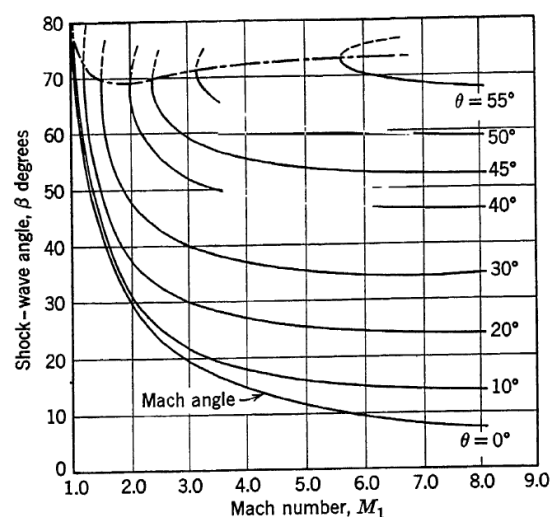
Cones in supersonic flow

Additionally, this figure shows the variation of pressure coefficient C_p as a function of Mach number Ma and the cone angle θ.
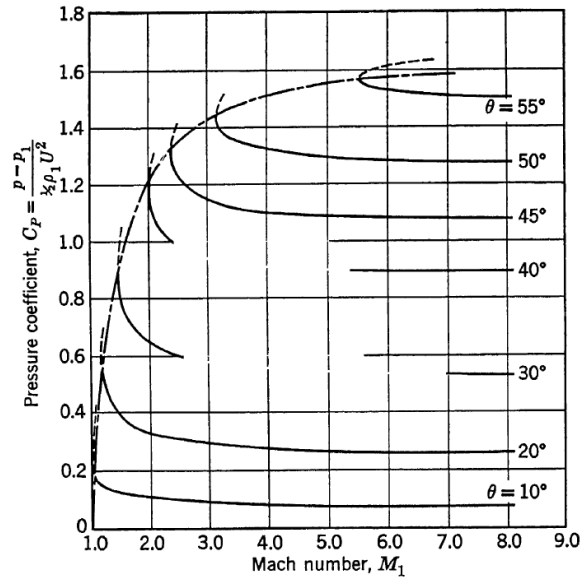
Derivation of perturbation equation

In many problems in aerodynamics, the interest lies in knowing the effect of small perturbation applied to a known fluid motion. This may result in great simplification of the problems at hand.
The most common and obvious example is how a body placed in a unform flow would change the flow field, as shown in this figure.
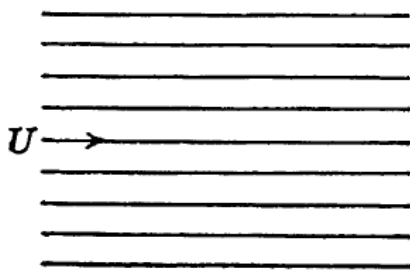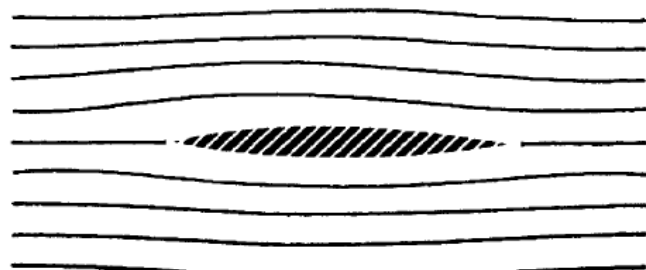
(a) Uniform flow   (b) Perturbed flow

Derivation of perturbation equation

In this example, the coordinate system can be chosen to align with the free stream flow velocity U_∞. Furthermore, it is assumed for the undisturbed flow, the properties are constant, *e.g.*, ρ_∞, p_∞ and T_∞. This results in $\mathrm{Ma}_\infty = U_\infty/a_\infty$. Correspondingly, the velocity field can be expressed as $\vec{u} = \langle U_\infty, 0, 0 \rangle$.

Now, assume that the flow field is perturbed by some small amount, caused by the presence of a solid body in the flow, such that the pertubed velocity field becomes:

$$\vec{u} = \langle U_\infty + u_x, u_y, u_z \rangle \tag{236}$$

Note that these are small compared to free stream velocity:

$$\frac{u_x}{U_\infty}, \frac{u_y}{U_\infty}, \frac{u_z}{U_\infty} \ll 1 \tag{237}$$

Irrotational flow

Refer to the following figure, showing the rotations of two elements $\overline{PR}$ and $\overline{PS}$ on the xy-plane, about the z-axis:
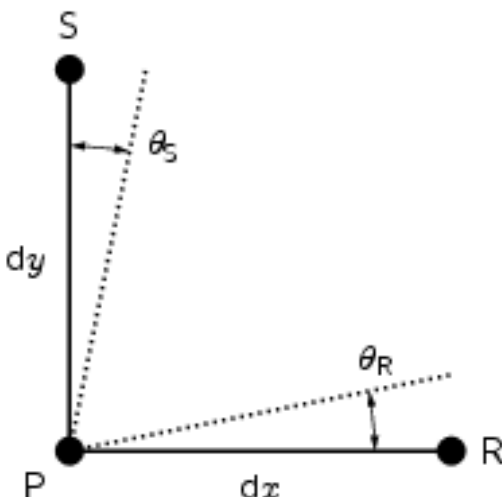
During a small time increment dt, point P would have moved $u_x \, dt$ in the x-direction and $u_y \, dt$ in the y-direction.

Point R would have moved $(\partial u_x/\partial x) \, dx \, dt$ in the x-direction and $(\partial u_y/\partial x) \, dx \, dt$ in the y-direction, relative to P.

Similarly, point S would have moved $(\partial u_x/\partial y) \, dy \, dt$ in the x-direction and $(\partial u_y/\partial y) \, dy \, dt$ in the y-direction, relative to P.

The angles θ_R and θ_S can then be described in terms of the velocity increments, assuming right-hand-rule.

Irrotational flow

Define the angular velocity vector $\vec{\omega}$ as the average of the rotational deformation rate of two mutually perpendicular elements, *e.g.*, in z-direction:

$$\omega_z = \frac{1}{2\,dt}\left(\theta_R - \theta_S\right) = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \tag{238}$$

Similar expressions can also be obtained for x-direction and y-direction, which yields the following, recognizing that ε_{ijk} is the permutation symbol:

$$\vec{\omega} = \frac{1}{2}\nabla \times \vec{u} \quad \Leftrightarrow \quad \omega_k = \frac{1}{2}\varepsilon_{ijk}\left(\partial u_j/\partial x_i\right) \tag{239}$$

The vorticity vector is defined as twice the angular velocity $\vec{\zeta} = 2\vec{\omega}$. For an *irrotational flow*, the vorticity is zero, which yields:

$$\partial u_i/\partial x_j = \partial u_j/\partial x_i \tag{240}$$

Irrotational flow

The 3-D momentum equation for steady, inviscid flow is:

$$\left(\vec{u} \cdot \nabla\right)\vec{u} + \frac{1}{\rho}\nabla P = \mathbf{0} \tag{241}$$

Furthermore, the following can be obtained from a vector identity:

$$\frac{1}{2}\nabla\left(\vec{u} \cdot \vec{u}\right) = \left(\vec{u} \cdot \nabla\right)\vec{u} + \vec{u} \times \left(\nabla \times \vec{u}\right) = \left(\vec{u} \cdot \nabla\right)\vec{u} + 2\left(\vec{u} \times \vec{\omega}\right) \tag{242}$$

The second law of thermodynamics for an adiabatic flow, where h_o is constant, can be expressed as:

$$\begin{aligned} T\,\nabla s = \nabla h - \frac{1}{\rho}\nabla P &= -\frac{1}{2}\nabla\left(\vec{u} \cdot \vec{u}\right) - \frac{1}{\rho}\nabla P \\ &= -\left[\left(\vec{u} \cdot \nabla\right)\vec{u} + 2\left(\vec{u} \times \vec{\omega}\right)\right] - \left[-\left(\vec{u} \cdot \nabla\right)\vec{u}\right] = \vec{\zeta} \times \vec{u} \end{aligned} \tag{243}$$

Hence, an irrotational, adiabatic flow implies isentropicity.

Starting with the continuity equation and the momentum equation for 3-D flows, here written in Cartesian tensor notation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0$$
$$\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} + \frac{1}{\rho}\frac{\partial P}{\partial x_j} = 0 \tag{244}$$

For this topic, only steady, adiabatic, irrotational ($\nabla \times \vec{u} = 0$) flows are considered. Hence, according to *Crocco's theorem*, the flows are isentropic:

$$\frac{\partial P}{\partial x_j} = \left(\left.\frac{\partial P}{\partial \rho}\right|_s\right)\frac{\partial \rho}{\partial x_j} = a^2 \frac{\partial \rho}{\partial x_j} \tag{245}$$

Plugging this back in the momentum equation, and taking scalar product with the velocity vector, gives:

$$u_i u_j \frac{\partial u_j}{\partial x_i} + \frac{a^2}{\rho} u_j \frac{\partial \rho}{\partial x_j} = 0 \tag{246}$$

Combining this with the continuity equation results in the following equation for steady inviscid flow:

$$u_i u_j \frac{\partial u_j}{\partial x_i} = a^2 \frac{\partial u_j}{\partial x_j} \tag{247}$$

And the speed of sound a can be obtained from the energy equation for perfect gas as follows:

$$\tfrac{1}{2}U_\infty^2 + \tfrac{1}{\gamma-1}a_\infty^2 = \tfrac{1}{2}\left[(U_\infty + u_x)^2 + u_y^2 + u_z^2\right] + \tfrac{1}{\gamma-1}a^2$$
$$\Rightarrow \quad a^2 = a_\infty^2 - \tfrac{\gamma-1}{2}\left(2U_\infty u_x + u_x^2 + u_y^2 + u_z^2\right) \tag{248}$$

Expanding the equation for steady inviscid flow, and dividing by a_∞^2:

$$\left(1 - \mathrm{Ma}_\infty^2\right) \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

$$= \mathrm{Ma}_\infty^2 \left[\frac{u_x^2}{U_\infty^2} + \frac{\gamma-1}{2} \left(\frac{2u_x}{U_\infty} + \frac{u_x^2}{U_\infty^2} + \frac{u_y^2}{U_\infty^2} + \frac{u_z^2}{U_\infty^2} \right) \right] \frac{\partial u_x}{\partial x}$$

$$+ \mathrm{Ma}_\infty^2 \left[\frac{u_y^2}{U_\infty^2} + \frac{\gamma-1}{2} \left(\frac{2u_x}{U_\infty} + \frac{u_x^2}{U_\infty^2} + \frac{u_y^2}{U_\infty^2} + \frac{u_z^2}{U_\infty^2} \right) \right] \frac{\partial u_y}{\partial y}$$

$$+ \mathrm{Ma}_\infty^2 \left[\frac{u_z^2}{U_\infty^2} + \frac{\gamma-1}{2} \left(\frac{2u_x}{U_\infty} + \frac{u_x^2}{U_\infty^2} + \frac{u_y^2}{U_\infty^2} + \frac{u_z^2}{U_\infty^2} \right) \right] \frac{\partial u_z}{\partial z}$$

$$+ \mathrm{Ma}_\infty^2 \left[\frac{u_x}{U_\infty} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_x}{\partial x} \right) \right] + \mathrm{Ma}_\infty^2 \left[\frac{u_y u_z}{U_\infty^2} \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) \right]$$

$$+ \mathrm{Ma}_\infty^2 \left[\frac{u_y}{U_\infty} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \right] + \mathrm{Ma}_\infty^2 \left[\frac{u_z u_x}{U_\infty^2} \left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} \right) \right]$$

$$+ \mathrm{Ma}_\infty^2 \left[\frac{u_z}{U_\infty} \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \right] + \mathrm{Ma}_\infty^2 \left[\frac{u_x u_y}{U_\infty^2} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \right]$$

$$\tag{249}$$

This is the full, *exact* equation in terms of the perturbation velocities.

Linear terms are collected in the left-hand-side, while the non-linear terms are all on the right-hand-side.

For small perturbations, many terms can be neglected, *e.g.*, the second-order terms in perturbation velocities u_x, u_y and u_z can be dropped out, in comparison with the first order terms.

Depending on the flow regime of interest, whether it be *subsonic*, *transonic*, *supersonic* or *hypersonic*, even further simplification is possible.

For example, the completely linear equation, valid for subsonic and supersonic flows, but *not for transonic flows*, can be obtained by dropping the entire right-hand-side:

$$\left(1 - \mathrm{Ma}_\infty^2\right) \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0 \tag{250}$$

Governing equations for small perturbation flows — Part II

Earlier, the full, *exact* equation has been obtained:

$$
\left(1 - \mathrm{Ma}_\infty^2\right) \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}
$$

$$
= \mathrm{Ma}_\infty^2 \left[\frac{u_x^2}{U_\infty^2} + \frac{\gamma-1}{2} \left(\frac{2u_x}{U_\infty} + \frac{u_x^2}{U_\infty^2} + \frac{u_y^2}{U_\infty^2} + \frac{u_z^2}{U_\infty^2} \right) \right] \frac{\partial u_x}{\partial x}
$$

$$
+ \mathrm{Ma}_\infty^2 \left[\frac{u_y^2}{U_\infty^2} + \frac{\gamma-1}{2} \left(\frac{2u_x}{U_\infty} + \frac{u_x^2}{U_\infty^2} + \frac{u_y^2}{U_\infty^2} + \frac{u_z^2}{U_\infty^2} \right) \right] \frac{\partial u_y}{\partial y}
$$

$$
+ \mathrm{Ma}_\infty^2 \left[\frac{u_z^2}{U_\infty^2} + \frac{\gamma-1}{2} \left(\frac{2u_x}{U_\infty} + \frac{u_x^2}{U_\infty^2} + \frac{u_y^2}{U_\infty^2} + \frac{u_z^2}{U_\infty^2} \right) \right] \frac{\partial u_z}{\partial z}
$$

$$
+ \mathrm{Ma}_\infty^2 \left[\frac{u_x}{U_\infty} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_x}{\partial x} \right) \right] + \mathrm{Ma}_\infty^2 \left[\frac{u_y u_z}{U_\infty^2} \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) \right]
$$

$$
+ \mathrm{Ma}_\infty^2 \left[\frac{u_y}{U_\infty} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \right] + \mathrm{Ma}_\infty^2 \left[\frac{u_z u_x}{U_\infty^2} \left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} \right) \right]
$$

$$
+ \mathrm{Ma}_\infty^2 \left[\frac{u_z}{U_\infty} \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \right] + \mathrm{Ma}_\infty^2 \left[\frac{u_x u_y}{U_\infty^2} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \right] \tag{251}
$$

Governing equations for small perturbation flows — Part II

The first simplification that can be done involves neglecting second-order perturbation terms, to obtain:

$$
\left(1 - \mathrm{Ma}_\infty^2\right) \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}
$$

$$
= \mathrm{Ma}_\infty^2 \left[\frac{(\gamma-1)u_x}{U_\infty} \left(\frac{\partial u_x}{\partial x} \right) + \frac{u_x}{U_\infty} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_x}{\partial x} \right) \right]
$$

$$
+ \mathrm{Ma}_\infty^2 \left[\frac{(\gamma-1)u_x}{U_\infty} \left(\frac{\partial u_y}{\partial y} \right) + \frac{u_y}{U_\infty} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \right] \tag{252}
$$

$$
+ \mathrm{Ma}_\infty^2 \left[\frac{(\gamma-1)u_x}{U_\infty} \left(\frac{\partial u_z}{\partial z} \right) + \frac{u_z}{U_\infty} \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \right]
$$

Furthermore, the products of the the perturbation velocities and their derivatives can also be dropped. However, one needs to first look at the situation as $\mathrm{Ma}_\infty \to 1$.

As $\text{Ma}_\infty \to 1$, the $\partial u_x / \partial x$ term on the left-hand-side becomes ever smaller, such that it may not be justifiable to neglect the $\partial u_x / \partial x$ term on the right-hand-side.

On the other hand, this does not affect the terms involving the $\partial u_y / \partial y$ and $\partial u_z / \partial z$, as well as the cross-differentiation terms, and these can be dropped from the right-hand-side of the equation to give this non-linear equation:

$$\left(1 - \text{Ma}_\infty^2\right) \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \text{Ma}_\infty^2 \frac{(\gamma+1)u_x}{U_\infty} \frac{\partial u_x}{\partial x} \tag{253}$$

Note that this equation is valid for both *subsonic* and *supersonic*, as well as *transonic* flow regimes, but not the *hypersonic* flow regime, which requires different approximation to be used due to the large value of Ma_∞.

And if the transonic regime is excluded, a fully-linear equation then can be used, obtained by neglecting the entire right-hand-side.

It has been assumed that the flow is *inviscid*. An additional condition, that the flow is *irrotational*, can be assumed. This imposes the following condition on the velocity field:

$$\nabla \times \vec{u} = \mathbf{0} \tag{254}$$

This condition allows the existence of a perturbation velocity potential ϕ, such that:

$$u_x = \frac{\partial \phi}{\partial x}, \quad u_y = \frac{\partial \phi}{\partial y}, \quad u_z = \frac{\partial \phi}{\partial z} \tag{255}$$

Hence, the small-perturbation equation, which includes the transonic flow regime, can be rewritten as:

$$\left(1 - \text{Ma}_\infty^2\right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \text{Ma}_\infty^2 \frac{\gamma+1}{U_\infty} \frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial x^2} \tag{256}$$

Pressure coefficient

The definition of the pressure coefficient is:

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = \frac{2}{\gamma \mathrm{Ma}_\infty^2}\left(\frac{p - p_\infty}{p_\infty}\right) \tag{257}$$

In terms of *local velocity* $\vec{u} = \langle U_\infty + u_x, u_y, u_z\rangle$:

$$
\begin{aligned}
C_p &= \frac{2}{\gamma \mathrm{Ma}_\infty^2}\left[\left[1 + \frac{\gamma-1}{2}\mathrm{Ma}_\infty^2\left(1 - \frac{\vec{u}\cdot\vec{u}}{U_\infty^2}\right)\right]^{\gamma/(\gamma-1)} - 1\right] \\
&= \frac{2}{\gamma \mathrm{Ma}_\infty^2}\left[\left[1 - \frac{\gamma-1}{2}\mathrm{Ma}_\infty^2\left(\frac{2u_x}{U_\infty} + \frac{u_x^2+u_y^2+u_z^2}{U_\infty^2}\right)\right]^{\gamma/(\gamma-1)} - 1\right]
\end{aligned}
\tag{258}
$$

Using binomial theorem, the following is obtained:

$$C_p = -\left[\left(\frac{2u_x}{U_\infty} + \frac{u_x^2+u_y^2+u_z^2}{U_\infty^2}\right) - \frac{\mathrm{Ma}_\infty^2}{4}\left(\frac{2u_x}{U_\infty} + \frac{u_x^2+u_y^2+u_z^2}{U_\infty^2}\right)^2 + \cdots\right] \tag{259}$$

Pressure coefficient

Neglecting the cubic and higher-order terms, the previous binomial expansion can be simplified to:

$$C_p = -\left[\frac{2u_x}{U_\infty} + \left(1 - \mathrm{Ma}_\infty^2\right)\frac{u_x^2}{U_\infty^2} + \frac{u_y^2}{U_\infty^2} + \frac{u_z^2}{U_\infty^2}\right] \tag{260}$$

For 2-D and planar flows, consistency with the first-order (linear) perturbation equation can be maintained by retaining only the first term:

$$C_p = -\frac{2u_x}{U_\infty} \tag{261}$$

On the other hand, for axi-symmetric flows, or flows over elongated bodies, it is necessary to include also the *transverse* velocity terms u_y and u_z:

$$C_p = -\left[\frac{2u_x}{U_\infty} + \frac{u_y^2}{U_\infty^2} + \frac{u_z^2}{U_\infty^2}\right] \tag{262}$$

Boundary conditions

The inviscid condition dictates that the flow velocity must be *tangential* to the surface of a solid body. This is to say that the velocity vector must be *at the right-angle* $(90\,°)$ with respect to the *normal vector* of the solid surface.

If the solid surface is expressed functionally as $f(x, y, z) = 0$, this *flow tangency* condition can be expressed mathematically in Cartesian tensor notation as:

$$\vec{u} \cdot \nabla f = u_i \frac{\partial f}{\partial x_i} = (U_\infty + u_x) \frac{\partial f}{\partial x} + u_y \frac{\partial f}{\partial y} + u_z \frac{\partial f}{\partial z} = 0 \quad (263)$$

Since u_x can be neglected relative to U_∞, this condition can be further simplified as:

$$U_\infty \frac{\partial f}{\partial x} + u_y \frac{\partial f}{\partial y} + u_z \frac{\partial f}{\partial z} = 0 \quad (264)$$

Boundary conditions

As an illustration, a two-dimensional case shall be considered, for which $u_z = 0$, and $\dfrac{\partial f}{\partial z} = 0$. This gives:

$$\frac{u_y}{U_\infty} = -\frac{\partial f / \partial x}{\partial f / \partial y} = \frac{\mathrm{d}y}{\mathrm{d}x} \quad (265)$$

Since $\mathrm{d}y/\mathrm{d}x \perp \nabla f$ is the slope of the body, which is perpendicular to normal vector of the body surface, and u_y/U_∞ approximates the slope of the streamline, the boundary condition requires that the velocity component u_y at the body surface should be equal to U_∞ multiplied by the slope of the body at that point.

To justify the use of small-perturbation assumption, the body needs to be thin, $|y| \ll 1$. Applying Taylor series expansion:

$$u_y(x, y) = u_y(x, 0) + y \left.\frac{\partial u_y}{\partial y}\right|_{(x,0)} + \frac{y^2}{2} \left.\frac{\partial^2 u_y}{\partial y^2}\right|_{(x,0)} + \cdots \quad (266)$$

Boundary conditions

For the 2-D case, it is only necessary to keep the first term, leading to the following boundary condition:

$$u_y(x,0) = U_\infty \left. (dy/dx) \right|_{\text{body}} \tag{267}$$

The previous expression is still valid for a *planar* 3-D case, since it requires $\partial f/\partial z \approx 0$. In this case, the boundary condition becomes:

$$u_y(x,0,z) = U_\infty \left. (\partial y/\partial x) \right|_{\text{body}} \tag{268}$$

Note that this does not apply to axisymmetric cases, since the transverse perturbation velocity cannot be expanded in power series in the neighbourhood of the axis.
Finally, the far-field boundary condition can be set by requiring the perturbation velocities to die out, or approaching some finite value, depending on the nature of the problem.

Flow past a wave-shaped wall — an example

A simple application of the small-perturbation theory in 2-D can be served by the following example of a flow over a wave-shaped wall. The surface of wall can be described as a sinusoidal shape of a magnitude ϵ and a wavelength $l = 2\pi/\alpha$:

$$y - \epsilon \sin(\alpha x) = 0 \tag{269}$$

Flow past a wave-shaped wall — an example

Since this example only deals with subsonic and moderate supersonic flows, the linear small-perturbation can be used, subject to finite values of u_x and u_y at far-field:

$$\left(1 - \text{Ma}_\infty^2\right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (270)$$

At the wall, flow tangency is enforced:

$$u_y\left(x, 0\right) = U_\infty \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\text{body}} = U_\infty \epsilon \alpha \cos\left(\alpha x\right) \qquad (271)$$

Due to vastly different behaviours of the two flow regimes, the subsonic case and the supersonic case shall be investigated separately.

Flow past a wave-shaped wall — subsonic case

Let $m^2 = 1 - \text{Ma}^2$, and rewrite:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{1}{m^2} \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (272)$$

Using the separation of variable technique, $\phi\left(x, y\right) = \mathbb{X}\left(x\right) \mathbb{Y}\left(y\right)$, the following is obtained:

$$\frac{\mathbb{X}''}{\mathbb{X}} = -\frac{1}{m^2} \frac{\mathbb{Y}''}{\mathbb{Y}} = -\mathsf{k}^2 \qquad (273)$$

This gives two ordinary differential equations, whose solutions are:

$$\mathbb{X} = \mathbb{X}_1 \cos\left(\mathsf{k}x\right) + \mathbb{X}_2 \sin\left(\mathsf{k}x\right) \qquad (274)$$
$$\mathbb{Y} = \mathbb{Y}_1 \cosh\left(m\mathsf{k}y\right) + \mathbb{Y}_2 \sinh\left(m\mathsf{k}y\right) \qquad (275)$$

Flow past a wave-shaped wall — supersonic case

In the supersonic case, define $\lambda^2 = \text{Ma}^2 - 1$, and rewrite:

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{\lambda^2} \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (276)$$

This equation is in the form of what is commonly referred to as the *second order wave equation*. The solutions are simply the sum of two arbitrary functions f and g:

$$\phi(x, y) = f(x - \lambda y) + g(x + \lambda y) \qquad (277)$$

The lines $x \mp \lambda y = \text{constant}$ are recognizable as the characteristics of the flow, carrying information toward downstream and upstream, respectively.

But since this is a supersonic flow with limited upstream influence, only one solution is valid, which is f.