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COMPUTATIONAL MATHEMATICS
AND
INTEGRAL CALCULUS

Prepared
by
Mr. G. Nagendra Kumar
Assistant Professor
Freshman Engineering

COMPUTATIONAL MATHEMATICS AND INTEGRAL CALCULUS (CMIC)

CONTENTS

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UNIT-I

ROOT FINDING TECHNIQUES AND INTERPOLATION

ROOT FINDING TECHNIQUES

BISECTION METHOD

If a function f(x) is continuous b/w x_0 and x_1 and $f(x_0)$ & $f(x_1)$ are of opposite signs, then there exsist at least one root b/w x_0 and x_1

- Let $f(x_0)$ be –ve and $f(x_1)$ be +ve ,then the root lies b/w xo and x_1 and its approximate value is given by $x_2=(x_0+x_1)/2$
- If $f(x_2)=0$,we conclude that x_2 is a root of the equ $f(x)=0$
- Otherwise the root lies either b/w x_2 and x_1 (or) b/w x_2 and x_0 depending on wheather $f(x_2)$ is +ve or –ve

Then as before, we bisect the interval and repeat the process untill the root is known to the desired accuracy

REGULA-FALSI

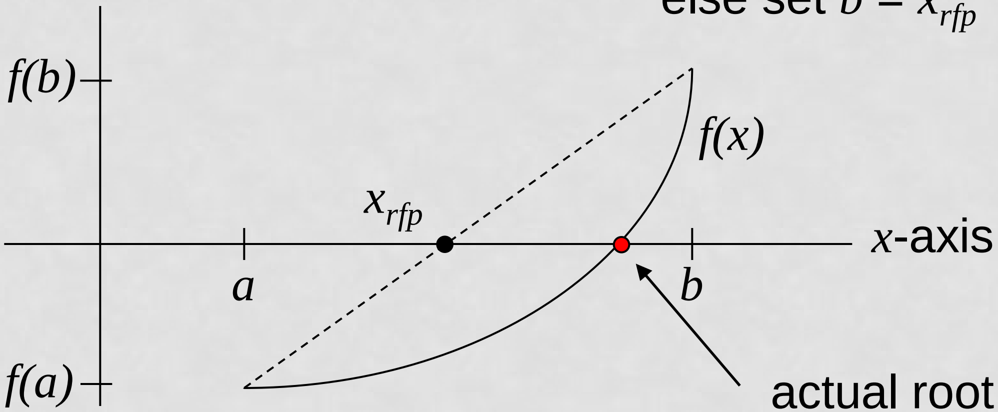
The idea for the Regula-Falsi method is to connect the points $(a,f(a))$ and $(b,f(b))$ with a straight line.Since linear equations are the simplest equations to solve for find the regula-falsi point (x_{rfp}) which is the solution to the linear equation connecting the endpoints.

Look at the sign of $f(x_{rfp})$:

If $sign(f(x_{rfp})) = 0$ then end algorithm

else If $sign(f(x_{rfp})) = sign(f(a))$ then set $a = x_{rfp}$

else set $b = x_{rfp}$



equation of line:

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

solving for x_{rfp}

$$0 - f(a) = \frac{f(b) - f(a)}{b - a}(x_{rfp} - a)$$

$$\frac{-f(a)(b - a)}{f(b) - f(a)} = x_{rfp} - a$$

$$x_{rfp} = a - \frac{f(a)(b - a)}{f(b) - f(a)}$$

NEWTON-RAPHSON METHOD OR NEWTON ITERATION METHOD

Let the given equation be f(x)=0

Find $f^1(x)$ and initial approximation x_0

The first approximation is $x_1 = x_0 - f(x_0)/ f^1(x_0)$

The second approximation is $x_2 = x_1 - f(x_1)/ f^1(x_1)$

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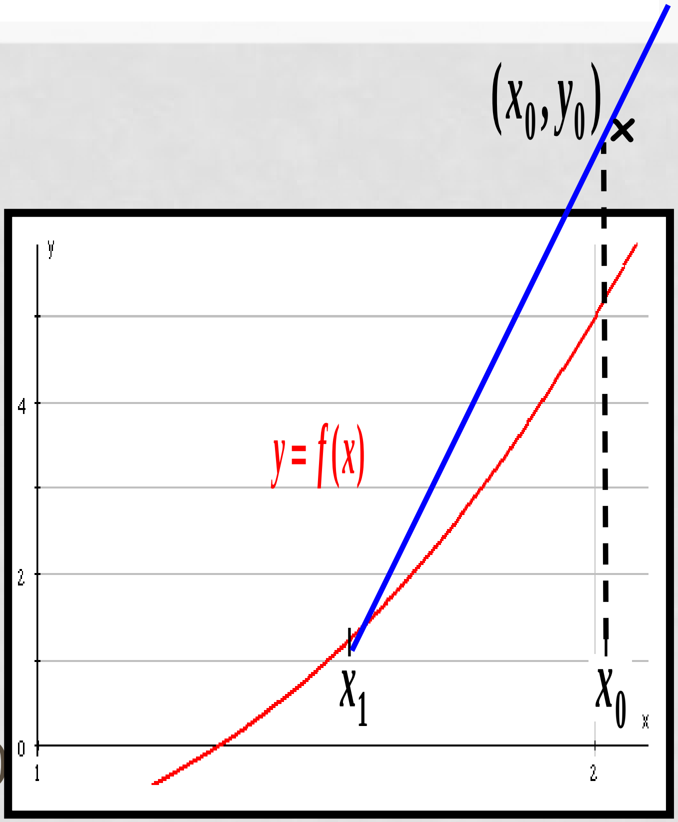
The n^{th} approximation is $x_n = x_{n-1} - f(x_n)/ f^1(x_n)$



INTERPOLATION

INTERPOLATION : The process of finding a missed value in the given table values of X, Y.

FINITE DIFFERENCES : We have three finite differences

1. Forward Difference
2. Backward Difference
3. Central Difference

FINITE DIFFERENCE METHODS

Let (xi,yi),i=0,1,2…………..n be the equally spaced data of the unknown function y=f(x) then much of the f(x) can be extracted by analyzing the differences of f(x).

Let $x_1 = x_0+h$

$x_2 = x_0+2h$

.

.

.

$x_n = x_0+nh$ be equally spaced points where the function value of f(x) be $y_0, y_1, y_2 \ldots\ldots y_n$

SYMBOLIC OPERATORS

Forward shift operator(E) :

It is defined as $Ef(x)=f(x+h)$ (or) $Ey_x = y_{x+h}$

The second and higher order forward shift operators are defined in similar manner as follows

$E^2f(x)= E(Ef(x))= E(f(x+h)= f(x+2h)= y_{x+2h}$

$E^3f(x)= f(x+3h)$

.

.

$E^kf(x)= f(x+kh)$

BACKWARD SHIFT OPERATOR(E^{-1}) :

It is defined as $E^{-1}f(x)=f(x-h)$ (or) $Ey_x = y_{x-h}$

The second and higher order backward shift operators are defined in similar manner as follows

$$E^{-2}f(x)= E^{-1}(E^{-1}f(x))= E^{-1}(f(x-h)= f(x-2h)= y_{x-2h}$$

$$E^{-3}f(x)= f(x-3h)$$

.

.

$$E^{-k}f(x)= f(x-kh)$$

FORWARD DIFFERENCE OPERATOR (Δ) :

The first order forward difference operator of a function f(x) with increment h in x is given by

$\Delta f(x)=f(x+h)-f(x)$ (or) $\Delta f_k = f_{k+1} - f_k$; k=0,1,2………

$\Delta^2 f(x)= \Delta[\Delta f(x)]= \Delta[f(x+h)-f(x)]= \Delta f_{k+1} - \Delta f_k$; k=0,1,2…………

…………………………………

…………………………………

Relation between E and Δ :

$$\Delta f(x)=f(x+h)-f(x)$$

$$=Ef(x)-f(x) \qquad [Ef(x)=f(x+h)]$$

$$=(E-1)f(x)$$

$$\Delta=E-1 \qquad E=1+\Delta$$

BACKWARD DIFFERENCE OPERATOR (NABLA) :

The first order backward difference operator ∇ of a function f(x) with increment h in x is given by

∇ f(x)=f(x)-f(x-h) (or) $\nabla f_k = f_{k+1} - f_k$; k=0,1,2………

∇ f(x)= $\nabla\nabla$[f(x)]= [f(x+h)-f(x)]= $\nabla f_{k+1} - \nabla f_k$; k=0,1,2…………

………………………………

………………………………

<u>Relation between E and nabla</u> :

nabla f(x)=f(x+h)-f(x)

=Ef(x)-f(x) [Ef(x)=f(x+h)]

=(E-1)f(x)

nabla=E-1 E=1+ nabla

CENTRAL DIFFERENCE OPERATOR

The central difference operator is defined as

$\delta f(x) = f(x+h/2) - f(x-h/2)$

$\delta f(x) = E^{1/2} f(x) - E^{-1/2} f(x)$

$= [E^{1/2} - E^{-1/2}] f(x)$

$\delta = E^{1/2} - E^{-1/2}$

RELATIONS BETWEEN THE OPERATORS IDENTITIES

1. $\Delta = E - 1$ or $E = 1 + \Delta$
2. $\nabla = 1 - E^{-1}$
3. $\delta = E^{1/2} - E^{-1/2}$
4. $\mu = \frac{1}{2}(E^{1/2} - E^{-1/2})$
5. $\Delta = E\nabla = \nabla E = \delta E^{1/2}$
6. $(1 + \Delta)(1 - \nabla) = 1$

NEWTONS INTERPOLATION FORMULA

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Newtons Forward interpolation formula

$y=f(x)=f(x_0+ph)= y_0 + p\Delta y_0 + p(p-1)/2!\ \Delta^2 y_0 + p(p-1)(p-2)/3!\ \Delta^3 y_0 + \ldots\ldots\ldots\ldots + p(p-1)(p2)\ \ldots\ldots\ [p-(n-1)]/n!\ \Delta^n y_0$.

Newtons Backward interpolatin formula :

$y=f(x)=f(x_n+ph)= y_n + p\nabla y_n + p(p+1)/2!\ \nabla^2 y_n + p(p+1)(p+2)/3!\ \nabla^3 y_n + \ldots\ldots\ldots\ldots + p(p+1)(p+2)\ \ldots\ldots\ [p+(n-1)]/n!\ \nabla^n y_n$.

GAUSS INTERPOLATION

The Guass forward interpolation is given by $y_{p} = y_{o} + p\,\Delta y_0 + p(p-1)/2!\,\Delta^2 y_{-1} + (p+1)p(p-1)/3!\,\Delta^3 y_{-1} + (p+1)p(p-1)(p-2)/4!\,\Delta^4 y_{-2} + \ldots\ldots$

The Guass backward interpolation is given by

$y_{p} = y_{o} + p\,\Delta y_{-1} + p(p+1)/2!\,\Delta^2 y_{-1} + (p+1)p(p-1)/3!\,\Delta^3 y_{-2} + (p+2)(p+1)p(p-1)/4!\,\Delta^4 y_{-2} + \ldots\ldots$

INTERPOLATION WITH UNEQUAL INTERVALS

The various interpolation formulae Newton's forward formula, Newton's backward formula possess can be applied only to equal spaced values of argument. It is therefore, desirable to develop interpolation formula for unequally spaced values of x. We use Lagrange's interpolation formula.

LAGRANGE'S INTERPOLATION

The Lagrange's interpolation formula is given by

$$Y = \frac{(X-X_1)(X-X_2)\ldots\ldots..(X-X_n)}{(X_0-X_1)(X_0-X_2)\ldots\ldots..(X_0-X_n)} Y_0 +$$

$$\frac{(X-X_0)(X-X_2)\ldots\ldots..(X-X_n)}{(X_1-X_0)(X_1-X_2)\ldots\ldots..(X_1-X_n)} Y_1 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots..$$

$$\frac{(X-X_0)(X-X_1)\ldots\ldots..(X-X_{n-1})}{(X_n-X_0)(X_n-X_1)\ldots\ldots..(X_n-X_{n-1})} Y_n +$$

UNIT-II

CURVE FITTING AND NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

CURVE FITTING

INTRODUCTION :

In interpolation, We have seen that when exact values of the function Y=f(x) is given we fit the function using various interpolation formulae. But sometimes the values of the function may not be given. In such cases, the values of the required function may be taken experimentally. Generally these expt. Values contain some errors. Hence by using these experimental values . We can fit a curve just approximately which is known as approximating curve. Now our aim is to find this approximating curve as much best as through minimizing errors of experimental values this is called best fit otherwise it is a bad fit. In brief by using experimental values the process of establishing a mathematical relationship between two variables is called CURVE FITTING.

METHOD OF LEAST SQUARES

Let y_1, y_2, y_3 ….y_n be the experimental values of $f(x_1)$,$f(x_2)$,…..$f(x_n)$ be the exact values of the function y=f(x). Corresponding to the values of $x=x_o,x_1,x_2….x_n$ .Now error=experimental values –exact value. If we denote the corresponding errors of y_1, y_2,….y_n as $e_1,e_2,e_3,…..e_n$, then $e_1=y_1-f(x_1)$, $e_2=y_2-f(x_2)$

- $e_3=y_3-f(x_3)$…….$e_n=y_n-f(x_n)$.These errors $e_1,e_2,e_3,$……e_n,may be either positive or negative.For our convenient to make all errors into +ve to the square of errors i.e $e_1^2,e_2^2,$……e_n^2.In order to obtain the best fit of curve we have to make the sum of the squares of the errors as much minimum i.e $e_1^2+e_2^2+$……$+e_n^2$ is minimum.

METHOD OF LEAST SQUARES

- Let $S=e_1^2+e_2^2+\ldots\ldots+e_n^2$, S is minimum.When S becomes as much as minimum.Then we obtain a best fitting of a curve to the given data, now to make S minimum we have to determine the coefficients involving in the curve, so that S minimum.It will be possible when differentiating S with respect to the coefficients involving in the curve and equating to zero.

FITTING OF STRAIGHT LINE

Let $y = a + bx$ be a straight line

By using the principle of least squares for solving the straight line equations.

The normal equations are

$$\sum y = na + b \sum x$$

$$\sum xy = a \sum x + b \sum x^2$$

solving these two normal equations we get the values of a & b ,substituting these values in the given straight line equation which gives the best fit.

FITTING OF PARABOLA

Let $y = a + bx + cx^2$ be the parabola or second degree polynomial.

By using the principle of least squares for solving the parabola

The normal equations are

$$\sum y = na + b \sum x + c \sum x^2$$

$$\sum xy = a \sum x + b \sum x^2 + c \sum x^3$$

$$\sum x^2 y = a \sum x^2 + b \sum x^3 + c \sum x^4$$

solving these normal equations we get the values of a,b & c, substituting these values in the given parabola which gives the best fit.

FITTING OF AN EXPONENTIAL CURVE

The exponential curve of the form

$$y = a\,e^{bx}$$

taking log on both sides we get

$\log_e y = \log_e a + \log_e e^{bx}$

$\log_e y = \log_e a + bx \log_e e$

$\log_e y = \log_e a + bx$

- $Y = A + bx$
- Where $Y = \log_e y$, $A = \log_e a$
- This is in the form of straight line equation and this can be solved by using the straight line normal equations we get the values of A & b, for $a = e^{A}$, substituting the values of a & b in the given curve which gives the best fit.

EXPONENTIAL CURVE

The equation of the exponential form is of the form $y = ab^x$

taking log on both sides we get

$\log_e y = \log_e a + \log_e b^x$

$Y = A + Bx$

where $Y = \log_e y$, $A = \log_e a$, $B = \log_e b$

this is in the form of the straight line equation which can be solved by using the normal equations we get the values of A & B for $a = e^A$

$b = e^B$ substituting these values in the equation which gives the best fit.

FITTING OF POWER CURVE

Let the equation of the power curve be

$$y = a\,x^b$$

taking log on both sides we get

$$\log_e y = \log_e a + \log_e x^b$$

$$Y = A + Bx$$

this is in the form of the straight line equation which can be solved by using the normal equations we get the values of A & B, for $a = e^A$ $b = e^B$, substituting these values in the given equation which gives the best fit.

NUMERICAL DIFFERENTIAL OF ORDINARY DIFFERENTIAL EQUATIONS

NUMERICAL DIFFERIATION

- Consider an ordinary differential equation of first order and first degree of the form
- dy/dx = f (x,y) ....(1)
- with the intial condition $y(x_0) = y_0$ which is called initial value problem.
- T0 find the solution of the initial value problem of the form (1) by numerical methods, we divide the interval (a,b) on whcich the solution is derived in finite number of sub- intervals by the points

TAYLOR'S SERIES METHOD

- Consider the first order differential equation
- $dy/dx = f(x, y)$………(1)

  with initial conditions $y(x_o) = y_0$ then expanding $y(x)$ i.e $f(x)$ in a Taylor's series at the point x_o we get

  $y(x_o + h) = y(x_0) + hy^1(x_0) + h^2/2!\, Y^{11}(x_0) + \ldots\ldots.$

Note : Taylor's series method can be applied only when the various derivatives of f(x,y) exist and the value of f (x-x0) in the expansion of $y = f(x)$ near x_0 must be very small so that the series is convergent.

EULER'S METHOD

- Consider the differential equation
- $dy/dx = f(x,y)$...........(1)
- With the initial conditions $y(x_0) = y_0$
- The first approximation of y_1 is given by
- $y_1 = y_0 + h\, f(x_{0,}, y_0)$
- The second approximation of y_2 is given by
- $y_2 = y_1 + h\, f(x_0 + h, y_1)$

EULER'S METHOD

- The third approximation of y_3 is given by
- $y_3 = y_2 + h\, f\,(x_0 + 2h, y_2)$
- ………………………………….
- ………………………………….
- ………………………………….
- $y_n = y_{n-1} + h\, f\,[x_0 + (n-1)h, y_{n-1}]$
- This is Eulers method to find an appproximate solution of the given differential equation.

IMPORTANT NOTE

- Note : In Euler's method, we approximate the curve of solution by the tangent in each interval i.e by a sequence of short lines. Unless h is small there will be large error in y_n . The sequence of lines may also deviate considerably from the curve of solution. The process is very slow and the value of h must be smaller to obtain accuracy reasonably.

MODIFIED EULER'S METHOD

- By using Euler's method, first we have to find the value of $y_1 = y_0 + hf(x_0 , y_0)$
- <u>WORKING RULE</u>
- Modified Euler's formula is given by
- $y^i_{k+1} = y_k + h/2 \, [\, f(x_k , y_k) + f(x_{k+1}, y_{k+1}$
- when $i=1, y(0)_{k+1}$ can be calculated
- from Euler's method.

MODIFIED EULER'S METHOD

- When k=o,1,2,3,……..gives number of iterations
- i = 1,2,3,…….gives number of times a particular iteration k is repeated when
- i=1
- $Y^{1}_{k+1} = y_k + h/2\ [\ f(x_k,y_k) + f(x_{k+1},y_{\ k+1})],,,,,,$

RUNGE-KUTTA METHOD

The basic advantage of using the Taylor series method lies in the calculation of higher order total derivatives of y. Euler's method requires the smallness of h for attaining reasonable accuracy. In order to overcomes these disadvantages, the Runge-Kutta methods are designed to obtain greater accuracy and at the same time to avoid the need for calculating higher order derivatives.The advantage of these methods is that the functional values only required at some selected points on the subinterval.

R-K METHOD

- Consider the differential equation
- $dy/dx = f(x, y)$
- With the initial conditions $y(x_o) = y_0$
- First order R-K method :
- $y_1 = y(x_0 + h)$
- Expanding by Taylor's series
- $y_1 = y_0 + h\, y^1{}_0 + h^2/2\, y^{11}{}_0 + \ldots\ldots.$

- Also by Euler's method
- $y_1 = y_0 + h\, f\,(x_0\,, y_0)$
- $= y_0 + h\, y^1{}_0$
- It follows that the Euler's method agrees with the Taylor's series solution upto the term in h. Hence, Euler's method is the Runge-Kutta method of the first order.

- The second order R-K method is given by
- $y_1 = y_0 + ½ (k_1 + k_2)$
- where $k_1 = h f (x_{0,}, y_0)$
- $k_2 = h f (x_0 + h , y_0 + k_1)$

- Third order R-K method is given by
- $y_1 = y_0 + 1/6 (k_1 + k_2 + k_3)$
- where $k_1 = h f (x_0 , y_0)$
- $k_2 = hf(x_0 + 1/2h , y_0 + ½ k_1)$
- $k_3 = hf(x_0 + ½h , y_0 + ½ k_2)$

The Fourth order R – K method

This method is most commonly used and is often referred to as Runge – Kutta method only. Proceeding as mentioned above with local discretisation error in this method being O (h5), the increment K of y correspoding to an increment h of x by Runge – Kutta method from

$dy/dx = f(x,y), y(x_0)=Y_0$ is given by

- $K_1 = h f (x_0 , y_0)$
- $k_2 = h f (x_0 + ½ h, y_0 + ½ k_1)$
- $k_3 = h f (x_0 + ½ h, y_0 + ½ k_2)$
- $k_4 = h f (x_0 + ½ h, y_0 + ½ k_3)$

and finally computing

- $k = 1/6 (k_1 +2 k_2 +2 k_3 +k_4)$

Which gives the required approximate value as

- $y_1 = y_0 + k$

Note 1 : k is known as weighted mean of

k_1, k_2, k_3 and k_4.

Note 2 : The advantage of these methods is that the operation is identical whether the differential equation is linear or non-linear.

UNIT-III

MULTIPLE INTEGRATION

MULTIPLE INTEGRALS

- Let y=f(x) be a function of one variable defined and bounded on [a,b]. Let [a,b] be divided into n subintervals by points $x_0,\ldots,x_n$ such that $a=x_0,\ldots\ldots\ldots x_n=b$. The generalization of this definition ;to two dimensions is called a double integral and to three dimensions is called a triple integral.

DOUBLE INTEGRALS

- Double integrals over a region R may be evaluated by two successive integrations. Suppose the region R cannot be represented by those inequalities, and the region R can be subdivided into finitely many portions which have that property, we may integrate f(x,y) over each portion separately and add the results. This will give the value of the double integral.

CHANGE OF VARIABLES IN DOUBLE INTEGRAL

- Sometimes the evaluation of a double or triple integral with its present form may not be simple to evaluate. By choice of an appropriate coordinate system, a given integral can be transformed into a simpler integral involving the new variables. In this case we assume that $x=r\cos\theta$, $y=r\sin\theta$ and $dxdy=rdrd\theta$

CHANGE OF ORDER OF INTEGRATION

- Here change of order of integration implies that the change of limits of integration. If the region of integration consists of a vertical strip and slide along x-axis then in the changed order a horizontal strip and slide along y-axis then in the changed order a horizontal strip and slide along y-axis are to be considered and vice-versa. Sometimes we may have to split the region of integration and express the given integral as sum of the integrals over these sub-regions. Sometimes as commented above, the evaluation gets simplified due to the change of order of integration. Always it is better to draw a rough sketch of region of integration.

TRIPLE INTEGRALS

- The triple integral is evaluated as the repeated integral where the limits of z are z_1 , z_2 which are either constants or functions of x and y; the y limits y_1 , y_2 are either constants or functions of x; the x limits x_1, x_2 are constants. First f(x,y,z) is integrated w.r.t. z between z limits keeping x and y are fixed. The resulting expression is integrated w.r.t. y between y limits keeping x constant. The result is finally integrated w.r.t. x from x_1 to x_2.

CHANGE OF VARIABLES IN TRIPLE INTEGRAL

- In problems having symmetry with respect to a point O, it would be convenient to use spherical coordinates with this point chosen as origin. Here we assume that $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$ and $dxdydz=r^2\sin\theta\, drd\theta d\phi$
- *Example*: By the method of change of variables in triple integral the volume of the portion of the sphere $x^2+y^2+z^2=a^2$ lying inside the cylinder $x^2+y^2=ax$ is $2a^3/9(3\pi-4)$

UNIT-IV

VECTOR CALCULUS

INTRODUCTION

- In this chapter, vector differential calculus is considered, which extends the basic concepts of differential calculus, such as, continuity and differentiability to vector functions in a simple and natural way. Also, the new concepts of gradient, divergence and curl are introducted.
- *Example*: i,j,k are unit vectors.

VECTOR DIFFERENTIAL OPERATOR

- The vector differential operator Δ is defined as $\Delta = i\,\partial/\partial x + j\,\partial/\partial y + k\,\partial/\partial z$. This operator possesses properties analogous to those of ordinary vectors as well as differentiation operator. We will define now some quantities known as gradient, divergence and curl involving this operator.

∂/∂

GRADIENT

- Let f(x,y,z) be a scalar point function of position defined in some region of space. Then gradient of f is denoted by grad f or Δf and is defined as

  grad f=i∂f/∂x + j ∂f/∂y + k ∂f/∂z

- *Example*: If f=2x+3y+5z then gradf= 2i+3j+5k

DIRECTIONAL DERIVATIVE

- The directional derivative of a scalar point function f at a point P(x,y,z) in the direction of g at P and is defined as grad g/|grad g|.grad f
- *Example*: The directional derivative of f=xy+yz+zx in the direction of the vector i+2j+2k at the point (1,2,0) is 10/3

DIVERGENCE OF A VECTOR

- Let f be any continuously differentiable vector point function. Then divergence of f and is written as div f and is defined as

  Div f = $\partial f_1/\partial x + j\ \partial f_2/\partial y + k\ \partial f_3/\partial z$

- *Example* 1: The divergence of a vector 2xi+3yj+5zk is 10
- *Example* 2: The divergence of a vector $f=xy^2i+2x^2yzj-3yz^2k$ at (1,-1,1) is 9

SOLENOIDAL VECTOR

- A vector point function f is said to be solenoidal vector if its divergent is equal to zero i.e., div f=0
- *Example* 1: The vector f=(x+3y)i+(y-2z)j+(x-2z)k is solenoidal vector.
- *Example* 2: The vector $f=3y^4z^2i+z^3x^2j-3x^2y^2k$ is solenoidl vector.

CURL OF A VECTOR

- Let f be any continuously differentiable vector point function. Then the vector function curl of f is denoted by curl f and is defined as curl f= $i \times \partial f/\partial x + j \times \partial f/\partial y + k \times \partial f/\partial z$
- *Example* 1: If $f=xy^2 i + 2x^2 yzj - 3yz^2 k$ then curl f at (1,-1,1) is –i-2k
- *Example* 2: If r=xi+yj+zk then curl r is 0

IRROTATIONAL VECTOR

- Any motion in which curl of the velocity vector is a null vector i.e., curl v=0 is said to be irrotational. If f is irrotational, there will always exist a scalar function f(x,y,z) such that f=grad g. This g is called scalar potential of f.
- *Example*: The vector f=(2x+3y+2z)i+(3x+2y+3z)j+(2x+3y+3z)k is irrotational vector.

VECTOR INTEGRATION

- INTRODUCTION: In this chapter we shall define line, surface and volume integrals which occur frequently in connection with physical and engineering problems. The concept of a line integral is a natural generalization of the concept of a definite integral of f(x) exists for all x in the interval [a,b]

WORK DONE BY A FORCE

- If F represents the force vector acting on a particle moving along an arc AB, then the work done during a small displacement F.dr. Hence the total work done by F during displacement from A to B is given by the line integral ∫F.dr
- *Example*: If f=$(3x^2+6y)$i-14yzj+$20xz^2$k along the lines from (0,0,0) to (1,0,0) then to (1,1,0) and then to (1,1,1) is 23/3

SURFACE INTEGRALS

- The surface integral of a vector point function F expresses the normal flux through a surface. If F represents the velocity vector of a fluid then the surface integral $\int F.n\, dS$ over a closed surface S represents the rate of flow of fluid through the surface.
- *Example*: The value of $\int F.n\, dS$ where $F=18zi-12j+3yk$ and S is the part of the surface of the plane $2x+3y+6z=12$ located in the first octant is 24.

VOLUME INTEGRAL

- Let f (r) = $f_1i+f_2j+f_3k$ where f_1,f_2,f_3 are functions of x,y,z. We know that dv=dxdydz. The volume integral is given by $\int f\, dv=\iiint(f_1i+f_2j+f_3k)dxdydz$
- *Example*: If $F=2xzi-xj+y^2k$ then the value of $\int f\, dv$ where v is the region bounded by the surfaces $x=0,x=2,y=0,y=6,z=x^2,z=4$ is128i-24j-384k

VECTOR INTEGRAL THEOREMS

- In this chapter we discuss three important vector integral theorems.
- 1)Gauss divergence theorem
- 2)Green's theorem
- 3)Stokes theorem

GAUSS DIVERGENCE THEOREM

- This theorem is the transformation between surface integral and volume integral. Let S be a closed surface enclosing a volume v. If f is a continuously differentiable vector point function, then
- $\int \text{div } f \, dv = \int f.n \, dS$
- Where n is the outward drawn normal vector at any point of S.

GREEN'S THEOREM

- This theorem is transformation between line integral and double integral. If S is a closed region in xy plane bounded by a simple closed curve C and in M and N are continuous functions of x and y having continuous derivatives in R, then
- $\int M dx + N dy = \iint (\partial N/\partial x - \partial M/\partial y) dx dy$

STOKES THEOREM

- This theorem is the transformation between line integral and surface integral. Let S be a open surface bounded by a closed, non-intersecting curve C. If F is any differentiable vector point function then
- $\int F.dr = \int Curl\ F.n\ ds$

UNIT-V

SPECIAL FUNCTIONS

GAMMA FUNCTION

- Definition

For $\alpha > 0$, the gamma function $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_0^{\alpha} x^{\alpha-1} e^{-x} dx$$

- Properties of the gamma function:

1. For any $\alpha > 1, \Gamma(\alpha) = (\alpha - 1) \cdot \Gamma(\alpha - 1)$
   [via integration by parts]

2. For any positive integer, $n, \Gamma(n) = (n-1)!$

3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

BESSEL'S EQUATION

- Bessel Equation of order ν:

$$x^2 y'' + xy' + \left(x^2 - \nu^2\right)y = 0$$

- Note that x = 0 is a regular singular point.
- Friedrich Wilhelm Bessel (1784 – 1846) studied disturbances in planetary motion, which led him in 1824 to make the first systematic analysis of solutions of this equation. The solutions became known as Bessel functions.
- In this section, we study the following cases:
  - Bessel Equations of order zero: $\nu = 0$
  - Bessel Equations of order one-half: $\nu = ½$
  - Bessel Equations of order one: $\nu = 1$

BESSEL EQUATION OF ORDER ZERO

- The Bessel Equation of order zero is

$$x^2 y'' + xy' + x^2 y = 0$$

- We assume solutions have the form

$$y(x) = \phi(r, x) = \sum_{n=0}^{\infty} a_n x^{r+n}, \text{ for } a_0 \neq 0,\ x > 0$$

- Taking derivatives,

$$y(x) = \sum_{n=0}^{\infty} a_n x^{r+n},\ \ y'(x) = \sum_{n=0}^{\infty} a_n (r+n) x^{r+n-1},$$

$$y''(x) = \sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n-2}$$

- Substituting these into the differential equation, we obtain

$$\sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n} + \sum_{n=0}^{\infty} a_n (r+n) x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0$$

INDICIAL EQUATION

- From the previous slide,

$$\sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n} + \sum_{n=0}^{\infty} a_n (r+n) x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0$$

- Rewriting,

$$a_0 [r(r-1) + r] x^r + a_1 [(r+1)r + (r+1)] x^{r+1}$$
$$+ \sum_{n=2}^{\infty} \{ a_n [(r+n)(r+n-1) + (r+n)] + a_{n-2} \} x^{r+n} = 0$$

- or

$$a_0 r^2 x^r + a_1 (r+1)^2 x^{r+1} + \sum_{n=2}^{\infty} \left\{ a_n (r+n)^2 + a_{n-2} \right\} x^{r+n} = 0$$

- The indicial equation is $r^2 = 0$, and hence $r_1 = r_2 = 0$.

RECURRENCE RELATION

- From the previous slide,

$$a_0 r^2 x^r + a_1(r+1)^2 x^{r+1} + \sum_{n=2}^{\infty}\left\{a_n(r+n)^2 + a_{n-2}\right\}x^{r+n} = 0$$

- Note that $a_1 = 0$; the recurrence relation is

$$a_n = -\frac{a_{n-2}}{(r+n)^2},\ n = 2, 3, \ldots$$

- We conclude $a_1 = a_3 = a_5 = \ldots = 0$, and since $r = 0$,

$$a_{2m} = -\frac{a_{2m-2}}{(2m)^2},\ m = 1, 2, \ldots$$

- Note: Recall dependence of a_n on r, which is indicated by $a_n(r)$. Thus we may write $a_{2m}(0)$ here instead of a_{2m}.

FIRST SOLUTION

- From the previous slide,

$$a_{2m} = -\frac{a_{2m-2}}{(2m)^2},\; m = 1, 2, \ldots$$

- Thus

$$a_2 = -\frac{a_0}{2^2},\; a_4 = -\frac{a_2}{4^2} = \frac{a_0}{4^2 2^2} = \frac{a_0}{2^4 (2\cdot 1)^2},\; a_6 = -\frac{a_0}{2^6 (3\cdot 2\cdot 1)^2}, \ldots$$

and in general,

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} (m!)^2},\; m = 1, 2, \ldots$$

- Thus

$$y_1(x) = a_0 \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} (m!)^2} \right], \quad x > 0$$

BESSEL FUNCTION OF FIRST KIND ORDER ZERO

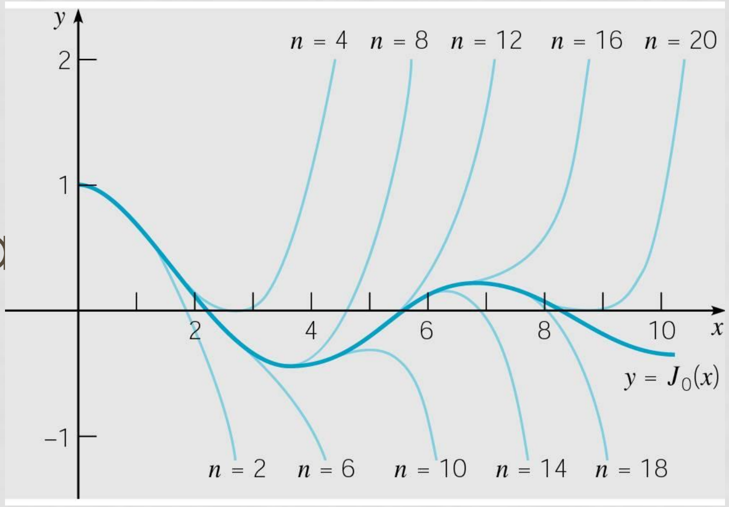
- Our first solution of Bessel's Equation of order zero is

$$y_1(x) = a_0\left[1 + \sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right], \quad x > 0$$

- The series converges for all x, and is called the **Bessel function of the first kind of order zero**, denoted by

$$J_0(x) = \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}, \quad x > 0$$

- The graphs of J_0 and several partial sum approximations are given here.



SECOND SOLUTION: ODD COEFFICIENTS

- Since indicial equation has repeated roots, recall from Section 5.7 that the coefficients in second solution can be found using

$$a_n'(r)\big|_{r=0}$$

- Now

$$a_0(r)r^2x^r + a_1(r)(r+1)^2x^{r+1} + \sum_{n=2}^{\infty}\left\{a_n(r)(r+n)^2 + a_{n-2}(r)\right\}x^{r+n} = 0$$

- Thus

$$a_1(r) = 0 \quad \Rightarrow a_1'(0) = 0$$

- Also,

$$a_n(r) = -\frac{a_{n-2}(r)}{(r+n)^2}, \quad n = 2, 3, \ldots$$

and hence

$$a_{2m+1}'(0) = 0, \quad m = 1, 2, \ldots$$

SECOND SOLUTION: EVEN COEFFICIENTS

- Thus we need only compute derivatives of the even coefficients, given by

$$a_{2m}(r) = -\frac{a_{2m-2}(r)}{(r+2m)^2} \Rightarrow a_{2m}(r) = \frac{(-1)^m a_0}{(r+2)^2 \cdots (r+2m)^2}, \quad m \geq 1$$

- It can be shown that

$$\frac{a'_{2m}(r)}{a_{2m}(r)} = -2\left[\frac{1}{r+2} + \frac{1}{r+4} + \cdots + \frac{1}{r+2m}\right]$$

and hence

$$a'_{2m}(0) = -2\left[\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2m}\right] a_{2m}(0)$$

SECOND SOLUTION: SERIES REPRESENTATION

- Thus

$$a'_{2m}(0) = -H_m \frac{(-1)^m a_0}{2^{2m}(m!)^2}, \quad m = 1, 2, \ldots$$

where

$$H_m = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2m}$$

- Taking a_0 = 1 and using results of Section 5.7,

$$y_2(x) = J_0(x)\ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} x^{2m}, \quad x > 0$$

BESSEL FUNCTION OF SECOND KIND, ORDER ZERO

- Instead of using y_2, the second solution is often taken to be a linear combination Y_0 of J_0 and y_2, known as the **Bessel function of second kind of order zero**. Here, we take

$$Y_0(x) = \frac{2}{\pi}\left[y_2(x) + (\gamma - \ln 2)J_0(x)\right]$$

- The constant γ is the Euler-Mascheroni constant, defined by

$$\gamma = \lim_{n\to\infty}(H_n - \ln n) \cong 0.5772$$

- Substituting the expression for y_2 from previous slide into equation for Y_0 above, we obtain

$$Y_0(x) = \frac{2}{\pi}\left[\left(\gamma + \ln\frac{x}{2}\right)J_0(x) + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}H_m}{2^{2m}(m!)^2}x^{2m}\right], \quad x > 0$$

GENERAL SOLUTION OF BESSEL'S EQUATION, ORDER ZERO

- The general solution of Bessel's equation of order zero, $x > 0$, is given by

$$y(x) = c_1 J_0(x) + c_2 Y_0(x)$$

where

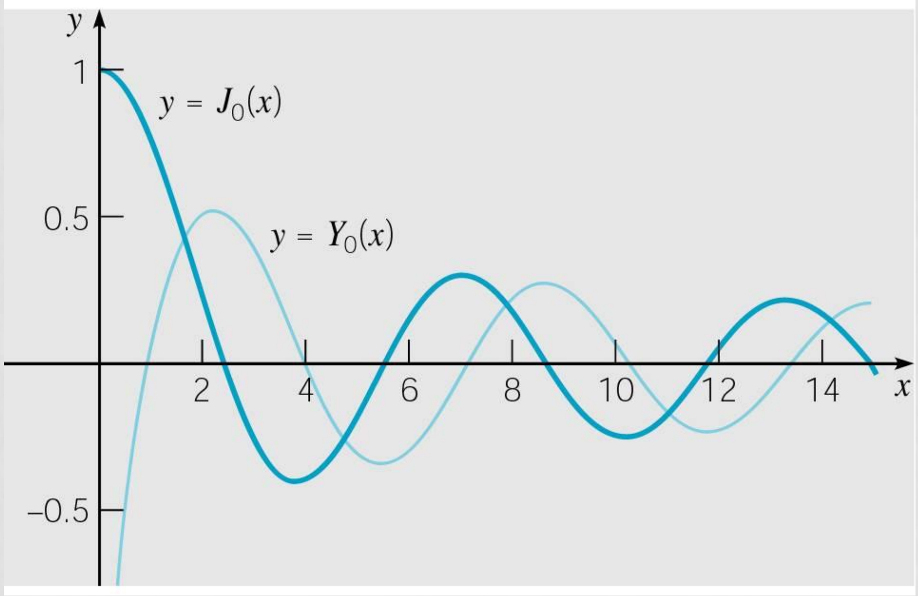
$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2},$$

$$Y_0(x) = \frac{2}{\pi}\left[\left(\gamma + \ln\frac{x}{2}\right)J_0(x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} x^{2m}\right]$$

- Note that $J_0 \to 0$ as $x \to 0$ while Y_0 has a logarithmic singularity at $x = 0$. If a solution which is bounded at the origin is desired, then Y_0 must be discarded.

GRAPHS OF BESSEL FUNCTIONS, ORDER ZERO

- The graphs of J_0 and Y_0 are given below.
- Note that the behavior of J_0 and Y_0 appear to be similar to $\sin x$ and $\cos x$ for large x, except that oscillations of J_0 and Y_0 decay to zero.



APPROXIMATION OF BESSEL FUNCTIONS, ORDER ZERO

- The fact that J_0 and Y_0 appear similar to sin x and cos x for large x may not be surprising, since ODE can be rewritten as

$$x^2 y'' + xy' + \left(x^2 - v^2\right) y = 0 \iff y'' + \frac{1}{x} y' + \left(1 - \frac{v^2}{x^2}\right) y = 0$$

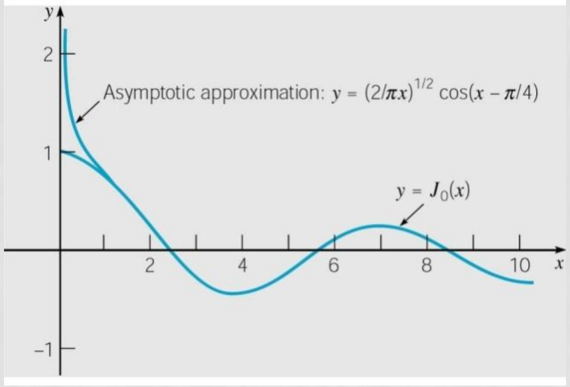
- Thus, for large x, our equation can be approximated by

$$y'' + y = 0,$$

whose solns are sin x and cos x. Indeed, it can be shown that

$$J_0(x) \cong \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{\pi}{4}\right), \text{ as } x \to \infty$$

$$Y_0(x) \cong \left(\frac{2}{\pi x}\right)^{1/2} \sin\left(x - \frac{\pi}{4}\right), \text{ as } x \to \infty$$



BESSEL EQUATION OF ORDER ONE-HALF

- The Bessel Equation of order one-half is

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

- We assume solutions have the form

$$y(x) = \phi(r, x) = \sum_{n=0}^{\infty} a_n x^{r+n}, \text{ for } a_0 \neq 0,\ x > 0$$

- Substituting these into the differential equation, we obtain

$$\sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n} + \sum_{n=0}^{\infty} a_n (r+n) x^{r+n}$$

$$+ \sum_{n=0}^{\infty} a_n x^{r+n+2} - \frac{1}{4} \sum_{n=0}^{\infty} a_n x^{r+n} = 0$$

RECURRENCE RELATION

- Using results of previous slide, we obtain

$$\sum_{n=0}^{\infty}\left[(r+n)(r+n-1)+(r+n)-\frac{1}{4}\right]a_n x^{r+n}+\sum_{n=0}^{\infty}a_n x^{r+n+2}=0$$

or

$$\left(r^2-\frac{1}{4}\right)a_0x^r+\left[(r+1)^2-\frac{1}{4}\right]a_1x^{r+1}+\sum_{n=2}^{\infty}\left\{\left[(r+n)^2-\frac{1}{4}\right]a_n+a_{n-2}\right\}x^{r+n}=0$$

- The roots of indicial equation are r_1 = ½, r_2 = - ½ , and note that they differ by a positive integer.
- The recurrence relation is

$$a_n(r)=-\frac{a_{n-2}(r)}{(r+n)^2-1/4},\ \ n=2,3,\ldots$$

FIRST SOLUTION: COEFFICIENTS

- Consider first the case r_1 = ½. From the previous slide,

$$\left(r^2-1/4\right)a_0x^r+\left[(r+1)^2-\frac{1}{4}\right]a_1x^{r+1}+\sum_{n=2}^{\infty}\left\{\left[\left(r+n\right)^2-\frac{1}{4}\right]a_n+a_{n-2}\right\}x^{r+n}=0$$

- Since r_1 = ½, a_1 = 0, and hence from the recurrence relation, $a_1 = a_3 = a_5 = \ldots = 0$. For the even coefficients, we have

$$a_{2m}=-\frac{a_{2m-2}}{\left(1/2+2m\right)^2-1/4}=-\frac{a_{2m-2}}{2m\left(2m+1\right)},\ \ m=1,2,\ldots$$

- It follows that

$$a_2=-\frac{a_0}{3!},\ a_4=-\frac{a_2}{5\cdot 4}=\frac{a_0}{5!},\ldots$$

and

$$a_{2m}=\frac{(-1)^m a_0}{(2m+1)!},\ \ m=1,2,\ldots$$

BESSEL FUNCTION OF FIRST KIND, ORDER ONE-HALF

- It follows that the first solution of our equation is, for $a_0 = 1$,

$$y_1(x) = x^{1/2}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m}\right], \quad x > 0$$

$$= x^{-1/2}\left[\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}\right], \quad x > 0$$

$$= x^{-1/2} \sin x, \quad x > 0$$

- The **Bessel function of the first kind of order one-half**, $J_{½}$, is defined as

$$J_{1/2}(x) = \left(\frac{2}{\pi}\right)^{1/2} y_1(x) \quad = \quad \left(\frac{2}{\pi x}\right)^{1/2} \sin x, \quad x > 0$$

SECOND SOLUTION: EVEN COEFFICIENTS

- Now consider the case r_2 = - ½. We know that

$$\left(r^2 - 1/4\right)a_0 x^r + \left[(r+1)^2 - \frac{1}{4}\right]a_1 x^{r+1} + \sum_{n=2}^{\infty}\left\{\left[\left(r+n\right)^2 - \frac{1}{4}\right]a_n + a_{n-2}\right\}x^{r+n} = 0$$

- Since r_2 = - ½ , a_1 = arbitrary. For the even coefficients

$$a_{2m} = -\frac{a_{2m-2}}{\left(-1/2+2m\right)^2 - 1/4} = -\frac{a_{2m-2}}{2m\left(2m-1\right)}, \quad m = 1, 2, \ldots$$

- It follows that

$$a_2 = -\frac{a_0}{2!}, \; a_4 = -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4!}, \ldots$$

and

$$a_{2m} = \frac{(-1)^m a_0}{(2m)!}, \quad m = 1, 2, \ldots$$

SECOND SOLUTION: ODD COEFFICIENTS

- For the odd coefficients,

$$a_{2m+1} = -\frac{a_{2m-1}}{(-1/2+2m+1)^2 - 1/4} = -\frac{a_{2m-1}}{2m(2m+1)}, \quad m = 1, 2, \ldots$$

- It follows that

$$a_3 = -\frac{a_1}{3!}, \; a_5 = -\frac{a_3}{5\cdot 4} = \frac{a_1}{5!}, \ldots$$

and

$$a_{2m+1} = \frac{(-1)^m a_1}{(2m+1)!}, \quad m = 1, 2, \ldots$$

SECOND SOLUTION

- Therefore

$$y_2(x) = x^{-1/2}\left[a_0 \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} + a_1 \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}\right], \quad x > 0$$

$$= x^{-1/2}\left[a_0 \cos x + a_1 \sin x\right], \quad x > 0$$

- The second solution is usually taken to be the function

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x, \quad x > 0$$

where $a_0 = (2/\pi)^{½}$ and $a_1 = 0$.

- The general solution of Bessel's equation of order one-half is

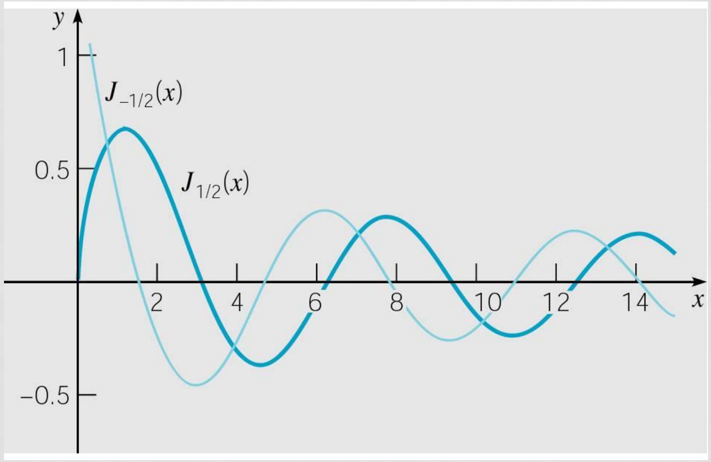
$$y(x) = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$$

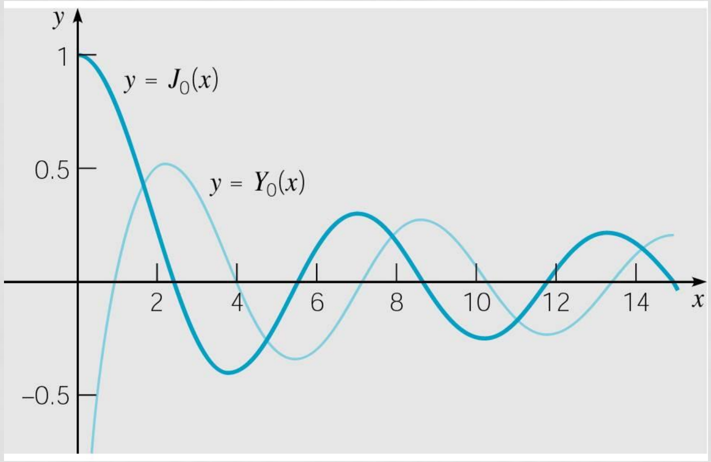
GRAPHS OF BESSEL FUNCTIONS, ORDER ONE-HALF

- Graphs of $J_{½}$, $J_{-½}$ are given below. Note behavior of $J_{½}$, $J_{-½}$ similar to J_0 , Y_0 for large x, with phase shift of π/4.

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x, \qquad J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x$$

$$J_0(x) \cong \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{\pi}{4}\right), \qquad Y_0(x) \cong \left(\frac{2}{\pi x}\right)^{1/2} \sin\left(x - \frac{\pi}{4}\right), \text{ as } x \to \infty$$





BESSEL EQUATION OF ORDER ONE

- The Bessel Equation of order one is

$$x^2 y'' + xy' + \left(x^2 - 1\right)y = 0$$

- We assume solutions have the form

$$y(x) = \phi(r, x) = \sum_{n=0}^{\infty} a_n x^{r+n}, \text{ for } a_0 \neq 0, \ x > 0$$

- Substituting these into the differential equation, we obtain

$$\sum_{n=0}^{\infty} a_n (r+n)(r+n-1)x^{r+n} + \sum_{n=0}^{\infty} a_n (r+n)x^{r+n}$$

$$+ \sum_{n=0}^{\infty} a_n x^{r+n+2} - \sum_{n=0}^{\infty} a_n x^{r+n} = 0$$

RECURRENCE RELATION

- Using the results of the previous slide, we obtain

$$\sum_{n=0}^{\infty}\left[(r+n)(r+n-1)+(r+n)-1\right]a_n x^{r+n}+\sum_{n=0}^{\infty}a_n x^{r+n+2}=0$$

or

$$\left(r^2-1\right)a_0 x^r+\left[(r+1)^2-1\right]a_1 x^{r+1}+\sum_{n=2}^{\infty}\left\{\left[(r+n)^2-1\right]a_n+a_{n-2}\right\}x^{r+n}=0$$

- The roots of indicial equation are $r_1 = 1$, $r_2 = -1$, and note that they differ by a positive integer.
- The recurrence relation is

$$a_n(r)=-\frac{a_{n-2}(r)}{(r+n)^2-1},\quad n=2,3,\ldots$$

FIRST SOLUTION: COEFFICIENTS

- Consider first the case $r_1 = 1$. From previous slide,

$$\left(r^2-1\right)a_0x^r + \left[(r+1)^2-1\right]a_1x^{r+1} + \sum_{n=2}^{\infty}\left\{\left[(r+n)^2-1\right]a_n + a_{n-2}\right\}x^{r+n} = 0$$

- Since $r_1 = 1$, $a_1 = 0$, and hence from the recurrence relation, $a_1 = a_3 = a_5 = \ldots = 0$. For the even coefficients, we have

$$a_{2m} = -\frac{a_{2m-2}}{(1+2m)^2-1} = -\frac{a_{2m-2}}{2^2(m+1)m}, \quad m = 1, 2, \ldots$$

- It follows that

$$a_2 = -\frac{a_0}{2^2\cdot 2\cdot 1},\ a_4 = -\frac{a_2}{2^2\cdot 3\cdot 2} = \frac{a_0}{2^4\,3!2!}, \ldots$$

and $$a_{2m} = \frac{(-1)^m a_0}{2^{2m}(m+1)!m!}, \quad m = 1, 2, \ldots$$

BESSEL FUNCTION OF FIRST KIND, ORDER ONE

- It follows that the first solution of our differential equation is

$$y_1(x) = a_0 x \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^{2m}(m+1)!m!} x^{2m} \right], \quad x > 0$$

- Taking a_0 = ½, the **Bessel function of the first kind of order one**, J_1, is defined as

$$J_1(x) = \frac{x}{2} \left[\sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m}(m+1)!m!} x^{2m} \right], \quad x > 0$$

- The series converges for all x and hence J_1 is analytic everywhere.

SECOND SOLUTION

- For the case r_1 = -1, a solution of the form

$$y_2(x) = a\,J_1(x)\ln x + x^{-1}\left[1 + \sum_{n=1}^{\infty} c_n x^{2n}\right], \quad x > 0$$

is guaranteed by Theorem 5.7.1.
- The coefficients c_n are determined by substituting y_2 into the ODE and obtaining a recurrence relation, etc. The result is:

$$y_2(x) = -J_1(x)\ln x + x^{-1}\left[1 - \sum_{m=1}^{\infty} \frac{(-1)^m\left(H_m + H_{m-1}\right)}{2^{2m} m!(m-1)!} x^{2n}\right], \quad x > 0$$

where H_k is as defined previously. See text for more details.
- Note that $J_1 \to 0$ as $x \to 0$ and is analytic at $x = 0$, while y_2 is unbounded at $x = 0$ in the same manner as $1/x$.

BESSEL FUNCTION OF SECOND KIND, ORDER ONE

- The second solution, the **Bessel function of the second kind of order one**, is usually taken to be the function

$$Y_1(x) = \frac{2}{\pi}\left[-y_2(x) + (\gamma - \ln 2)J_1(x)\right], \quad x > 0$$

where γ is the Euler-Mascheroni constant.

- The general solution of Bessel's equation of order one is

$$y(x) = c_1 J_1(x) + c_2 Y_1(x), \quad x > 0$$

- Note that J_1, Y_1 have same behavior at x = 0 as observed on previous slide for J_1 and y_2