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CONTROL SYSTEMS III B.Tech I semester (JNTUH-R13)

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Control Systems

Chapter 1: Introduction to Control Systems Objectives

In this chapter we describe a general process for designing a control system.

A control system consisting of interconnected components is designed to achieve a desired purpose. To understand the purpose of a control system, it is useful to examine examples of control systems through the course of history. These early systems incorporated many of the same ideas of feedback that are in use today.

Modern control engineering practice includes the use of control design strategies for improving manufacturing processes, the efficiency of energy use, advanced automobile control, including rapid transit, among others.

We also discuss the notion of a design gap. The gap exists between the complex physical system under investigation and the model used in the control system synthesis.

The iterative nature of design allows us to handle the design gap effectively while accomplishing necessary tradeoffs in complexity, performance, and cost in order to meet the design specifications.

Introduction

System – An interconnection of elements and devices for a desired purpose.

Control System – An interconnection of components forming a system configuration that will provide a desired response.

Process – The device, plant, or system under control. The input and output relationship represents the cause-andeffect relationship of the process.



Introduction

Open-Loop Control Systems

utilize a controller or control actuator to obtain the desired response.



Open-loop control system (without feedback).

Closed-Loop Control Systems

utilizes feedback to compare the actual output to the desired output response.



Closed-loop feedback control system (with feedback).



Multivariable Control System

Greece (BC) – Float regulator mechanism Holland (16th Century)– Temperature regulator



Water-level float regulator





Closed-loop feedback system.

18th Century James Watt's centrifugal governor for the speed control of a steam engine.

1920s Minorsky worked on automatic controllers for steering ships.

1930s Nyquist developed a method for analyzing the stability of controlled systems

1940s Frequency response methods made it possible to design linear closed-loop control systems

1950s Root-locus method due to Evans was fully developed

1960s State space methods, optimal control, adaptive control and

1980s Learning controls are begun to investigated and developed.

Present and on-going research fields. Recent application of modern control theory includes such non-engineering systems such as biological, biomedical, economic and socio-economic systems



to generate a controlled adjustment of the steering wheel.

(a) Automobile

steering control

the difference

and the desired

direction of travel

(b) The driver uses

between the actual

system.

(c) Typical directionof-travel response.

Examples of Modern Control Systems



Time, r

direction

of travel

(b)

Desired direction of travel Actual direction of travel

direction of travel

Responseп



A negative feedback system block diagram depicting a basic closed-loop control system. The control device is often called a "controller."



A manual control system for regulating the level of fluid in a tank by adjusting the output valve. The operator views the level of fluid through a port in the side of the tank.





Coordinated control system for a boiler-generator.



A computer control system.



The Utah/MIT Dextrous Robotic Hand: A dextrous robotic hand having 18 degrees of freedom, developed as a research tool by the Center for Engineering Design at the University of Utah and the Artificial Intelligence Laboratory at MIT. It is controlled by five Motorola 68000 microprocessors and actuated by 36 high-performance electropneumatic actuators via high-strength polymeric tendons. The hand has three fingers and a thumb. It uses touch sensors and tendons for control. (Photograph by Michael Milochik. Courtesy of University of Utah.)



A feedback control system model of the national income.



A laboratory robot used for sample preparation. The robot manipulates small objects, such as test tubes, and probes in and out of tight places at relatively high speeds [41]. (© Copyright 1993 Hewlett-Packard Company. Reproduced with permission.)

The Future of Control Systems



The Honda P3 humanoid robot. P3 walks, climbs stairs and turns corners. Photo courtesy of American Honda Motor, Inc.

The Future of Control Systems



Future evolution of control systems and robotics.

Control System Design



specifications, then finalize the design.

The control system design process.

then iterate the configuration and the actuator.





ELECTRIC SHIP CONCEPT



CVN(X) FUTURE AIRCRAFT CARRIER











(b)

(a) Open-loop (without feedback) control of the speed of a turntable.(b) Block diagram model.





(b)

(a) Closed-loop control of the speed of a turntable.
(b) Block diagram model.







The blood glucose and insulin levels for a healthy person.



(b)

(a) Open-loop (without feedback) control and(b) closed-loop control of blood glucose.

Sequential Design Example



(a) A disk drive ©1999 Quantum Corporation. All rights reserved.
(b) Diagram of a disk drive.



Sequential Design Example



Closed-loop control system for disk drive.
Mathematical Models of Systems Objectives

We use quantitative mathematical models of physical systems to design and analyze control systems. The dynamic behavior is generally described by ordinary differential equations. We will consider a wide range of systems, including mechanical, hydraulic, and electrical. Since most physical systems are nonlinear, we will discuss linearization approximations, which allow us to use Laplace transform methods.

We will then proceed to obtain the input–output relationship for components and subsystems in the form of transfer functions. The transfer function blocks can be organized into block diagrams or signal-flow graphs to graphically depict the interconnections. Block diagrams (and signal-flow graphs) are very convenient and natural tools for designing and analyzing complicated control systems

Introduction Six Step Approach to Dynamic System Problems

Define the system and its components Formulate the mathematical model and list the necessary assumptions Write the differential equations describing the model Solve the equations for the desired output variables Examine the solutions and the assumptions If necessary, reanalyze or redesign the system



$$T_{a}(t) - T_{s}(t) = 0$$

 $T_{a}(t) = T_{s}(t)$

$$\omega(t) = \omega_{\rm s}(t) - \omega_{\rm a}(t)$$

 $T_a(t)$ = through - variable

angular rate difference = across-variable

(a) Torsional spring-mass system.(b) Spring element.



Electrical Capacitance

$$v_2 \circ \underbrace{i}_{l} = C \circ v_1$$
 $i = C \cdot \frac{d}{dt} v_{21}$ $E = \frac{1}{2} \cdot M \cdot v_{21}^2$

Translational Mass

$$F \longrightarrow M = W_1 = F = M \cdot \frac{d}{dt} v_2 \qquad E = \frac{1}{2} \cdot M \cdot v_2^2$$

Rotational Mass

$$T \longrightarrow J \longrightarrow \omega_1 = Constant$$

$$T = J \cdot \frac{d}{dt} \omega_2$$

$$E = \frac{1}{2} \cdot J \cdot \omega_2^2$$

Fluid Capacitance

$$Q \longrightarrow C_f OP_1$$

Thermal Capacitance

$$q \xrightarrow{\mathbf{C}_t} \underbrace{\mathcal{T}_1}_{t} = constant$$

$$Q = C_{f} \cdot \frac{d}{dt} P_{21}$$
 $E = \frac{1}{2} \cdot C_{f} \cdot P_{21}^{2}$

$$q = C_t \cdot \frac{d}{dt} T_2 \qquad \qquad E = C_t \cdot T_2$$

Electrical Resistance

$$v_2 \circ \underbrace{R}_{i} \circ v_1$$
 $i = \frac{1}{R} \cdot v_{21}$ $P = \frac{1}{R} \cdot v_{21}^2$

Translational Damper

$$F \longrightarrow v_1 \qquad F = b \cdot v_{21} \qquad P = b \cdot v_{21}^2$$

Rotational Damper

$$T \longrightarrow \omega_{1} \qquad T = b \cdot \omega_{21} \qquad P = b \cdot \omega_{21}^{2}$$

Fluid Resistance

$$P_{2} \circ \underbrace{R_{f}}_{P_{2}} \circ \underbrace{Q}_{P_{1}} \qquad Q = \frac{1}{R_{f}} \cdot P_{21} \qquad P = \frac{1}{R_{f}} \cdot P_{21}^{2}$$

Thermal Resistance

$$\mathcal{T}_{2} \circ \underbrace{R_{t}}_{R_{t}} q \circ \mathcal{T}_{1}$$
 $q = \frac{1}{R_{t}} \cdot T_{21}$ $P = \frac{1}{R_{t}} \cdot T_{21}$



(a) Spring-mass-damper system.(b) Free-body diagram.



$$\frac{\mathbf{v}(t)}{\mathbf{R}} + \mathbf{C} \cdot \frac{\mathbf{d}}{\mathbf{d}t} \mathbf{v}(t) + \frac{1}{\mathbf{L}} \cdot \int_{0}^{t} \mathbf{v}(t) \, \mathbf{d}t = \mathbf{r}(t)$$

RLC circuit.

$$y(t) = K_1 \cdot e^{-\alpha_1 \cdot t} \cdot \sin(\beta_1 \cdot t + \theta_1)$$



Typical voltage response for underdamped RLC circuit.

$$K_2 := 1$$
 $\alpha_2 := .5$ $\beta_2 := 10$ $\theta_2 := 2$

$$\mathbf{y}(\mathbf{t}) := \mathbf{K}_2 \cdot \mathbf{e}^{-\alpha_2 \cdot \mathbf{t}} \cdot \sin(\beta_2 \cdot \mathbf{t} + \theta_2)$$

$$y_1(t) := K_2 \cdot e^{-\alpha_2 \cdot t}$$
 $y_2(t) := -K_2 \cdot e^{-\alpha_2 \cdot t}$



Linear Approximations



(a) A mass sitting on a nonlinear spring.(b) The spring force versus y.

Linear Approximations

Linear Systems - Necessary condition

Principle of Superposition

Property of Homogeneity

Taylor Series <u>http://www.maths.abdn.ac.uk/%7Eigc/tch/ma2001/notes/node46.ht</u> <u>ml</u>

Linear Approximations – Example 2.1



Students are encouraged to investigate linear approximation accuracy for different value

Historical Perspective - Heaviside's Operators

Origin of Operational Calculus (1887)

Historical Perspective - Heaviside's Operators Origin of Operational Calculus (1887)



(*) Oliver Heaviside: Sage in Solitude, Paul J. Nahin, IEEE Press 1987.

Definition

$$L(f(t)) = \int_0^\infty f(t) \cdot e^{-s \cdot t} dt \qquad = F(s)$$

Here the complex frequency is $s = \rho + j \cdot w$

The Laplace Transform exists when

$$\int_0^\infty f(t) \cdot e^{-s \cdot t} dt < \infty$$

this means that the integral converç

Determine the Laplace transform for the functions

a)
$$f_1(t) := 1$$
 for $t \ge 0$

$$F_1(s) := \int_0^\infty e^{-s \cdot t} dt = -\frac{1}{s} \cdot e^{-(s \cdot t)} = \frac{1}{s}$$

b)
$$f_2(t) = e^{-(a \cdot t)}$$

$$F_{2}(s) = \int_{0}^{\infty} e^{-(a \cdot t)} \cdot e^{-(s \cdot t)} dt = -\frac{1}{s+1} \cdot e^{-[(s+a) \cdot t]} \qquad F_{2}(s) = \frac{1}{s+a}$$

Evaluate the laplace transform of the derivative of a function

$$\begin{split} L\left(\frac{d}{dt}f(t)\right) &= \int_{0}^{\infty} \frac{d}{dt}f(t) \cdot e^{-(s \cdot t)} dt \\ \text{by the use of } \int u \, dv &= u \cdot v - \int v \, du \\ \text{where } u &= e^{-(s \cdot t)} \quad dv &= df(t) \\ \text{and, from which} \\ du &= -s \cdot e^{-(s \cdot t)} \cdot dt \quad \text{and } v &= f(t) \\ \text{we obtain} \\ \int_{0}^{\infty} u \, dv &= f(t) \cdot e^{-(s \cdot t)} - \int_{0}^{\infty} f(t) \cdot \left[-s \cdot e^{-(s \cdot t)}\right] dt \\ &= -f(0+) + s \cdot \int_{0}^{\infty} f(t) \cdot e^{-(s \cdot t)} dt \\ &\longrightarrow L\left(\frac{d}{dt}f(t)\right) &= sF(s) - f(0+) \text{ note that the initial condition is included in the transform} \end{split}$$

Practical Example - Consider the circuit.

The KVL equation is

 $4 \cdot i(t) + 2 \cdot \frac{d}{dt}i(t) = 0$ assume i(0+) = 5 A



Applying the Laplace Transform, we have

$$\int_{0}^{\infty} \left(4 \cdot i(t) + 2 \cdot \frac{d}{dt} i(t) \right) \cdot e^{-(s \cdot t)} dt = 0 \qquad 4 \cdot \int_{0}^{\infty} i(t) \cdot e^{-(s \cdot t)} dt + 2 \cdot \int_{0}^{\infty} \frac{d}{dt} i(t) \cdot e^{-(s \cdot t)} dt = 0$$

$$4 \cdot I(s) + 2 \cdot (s \cdot I(s) - i(0)) = 0 \qquad 4 \cdot I(s) + 2 \cdot s \cdot I(s) - 10 = 0$$

 $I(s) := \frac{5}{s+2}$ transforming back to the time domain, with our present knowledge of Laplace transform, we may say that

 $t \equiv (0, 0.01..2)$





The Partial-Fraction Expansion (or Heaviside expansion theorem)

Suppose that

$$F(s) = \frac{s + z1}{(s + p1) \cdot (s + p2)}$$

or

| - () | | К1 | К2 |
|---------|---|-------------|----------|
| F (S) | = | + | <u> </u> |
| | | $s + p_{I}$ | s + pz |

The partial fraction expansion indicates that F(s) consists of a sum of terms, each of which is a factor of the denominator. The values of K1 and K2 are determined by combining the individual fractions by means of the lowest common denominator and comparing the resultant numerator coefficients with those of the coefficients of the numerator before separation in different terms.

Evaluation of Ki in the manner just described requires the simultaneous solution of n equations. An alternative method is to multiply both sides of the equation by (s + pi) then setting s = -pi, the right-hand side is zero except for Ki so that

$$Ki = \frac{(s + pi) \cdot (s + z1)}{(s + p1) + (s + p2)} \qquad s = -pi$$

| Property | Time Domain | Frequency Doma |
|-----------------------------|-----------------------------------|---|
| 1. Time delay | $f(t - T) \cdot u(t - T)$ | $e^{-(s\cdot T)}\cdot F(s)$ |
| 2. Time scaling | f(at) | $\frac{1}{a} \cdot F\left(\frac{s}{a}\right)$ |
| 3. Frequency differentiatio | $n t \cdot f(t)$ | $-\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{F}(\mathrm{s})$ |
| 4. Frequency shifting | $f(t) \cdot e^{-(a \cdot t)}$ | F(s + a) |
| 5. Frequency Integration | $\frac{f(t)}{t}$ | $\int_0^\infty F(s)\mathrm{d}s$ |
| 6. Initial-value Theorem | $\operatorname{Lim}(f(t)) = f(0)$ | Lim(s·F(s)) |
| 7 Final value Theorem | $t \rightarrow 0$ | $s \rightarrow infinite$ |
| 7. Fillal-value Theolem | t -> infinite | $s \rightarrow 0$ |

Useful Transform Pairs

An s-plane plot of the poles and zeros of Y(s).

The Laplace Transform





Response of the spring-mass-damper system.



An RC network.

$$V_{1}(s) = \left(R + \frac{1}{Cs}\right) \cdot I(s) \qquad Z_{1}(s) = R$$
$$V_{2}(s) = \left(\frac{1}{Cs}\right) \cdot I(s) \qquad Z_{2}(s) = \frac{1}{Cs}$$

$$\frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}$$

Example 2.2

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} y(t) + 4 \cdot \frac{\mathrm{d}}{\mathrm{d}t} y(t) + 3 \cdot y(t) = 2 \cdot \mathbf{r}(t)$$

nitial Conditions:
$$Y(0) = 1$$
 $\frac{d}{dt}y(0) = 0$

The Laplace transform yields:

$$(s^2 \cdot Y(s) - s \cdot y(0)) + 4 \cdot (s \cdot Y(s) - y(0)) + 3Y(s) = 2 \cdot R(s)$$

Since R(s)=1/s and y(0)=1, we obtain: $Y(s) = \frac{(s+4)}{(s^2+4s+3)} + \frac{2}{s \cdot (s^2+4s+3)}$ The partial fraction expansion yields:

$$Y(s) = \left[\frac{\frac{3}{2}}{(s+1)} + \frac{\frac{-1}{2}}{(s+3)}\right] + \left[\frac{-1}{(s+1)} + \frac{\frac{1}{3}}{(s+3)}\right] + \frac{\frac{2}{3}}{s}$$

Therefore the transient response is:

$$\mathbf{y}(t) = \left(\frac{3}{2} \cdot e^{-t} - \frac{1}{2} \cdot e^{-3 \cdot t}\right) + \left(-1e^{-t} + \frac{1}{3} \cdot e^{-3t}\right) + \frac{2}{3}$$

The steady-state response is:

$$\lim_{t\to\infty} y(t) = \frac{2}{3}$$

r(t) = 1



An inverting amplifier operating with ideal conditions.



(a) Two-mass mechanical system. (b) Two-node electric circuit analog C1 = M1, C2 = M2, L = 1/k, R1 = 1/b1, R2 = 1/b2.



A dc motor (a) wiring diagram and (b) sketch.



A pancake dc motor with a flat-wound armature and a permanent magnet rotor. These motors are capable of providing high torque with a low rotor inertia. A typical mechanical time constant is in the range of 15 ms. (Courtesy of Mavilor Motors.)





Block diagram model of field-controlled dc motor.



Armature-controlled dc motor.





Range of control response time and power to load for electromechanical and electrohydraulic devices.



$$\frac{V_2(s)}{V_1(s)} = \frac{-1}{RCs}$$



$$\frac{V_2(s)}{V_1(s)} = -RCs$$


$$\frac{V_2(s)}{V_1(s)} = \frac{R_2(R_1 \cdot C \cdot s + 1)}{R_1}$$



$$\frac{V_{2}(s)}{V_{1}(s)} = \frac{-(R_{1} \cdot C_{1} \cdot s + 1)(R_{2} \cdot C_{2} \cdot s + 1)}{R_{1} \cdot C_{2} \cdot s}$$



$$\frac{\theta(s)}{V_{f}(s)} = \frac{K_{m}}{s \cdot (J \cdot s + b) \left(L_{f} \cdot s + R_{f}\right)}$$



$$\frac{\theta(s)}{V_{a}(s)} = \frac{K_{m}}{s \cdot \left[\left(R_{a} + L_{a} \cdot s \right) \left(J \cdot s + b \right) + K_{b} \cdot K_{m} \right]}$$



$$\frac{\theta(s)}{V_{c}(s)} = \frac{K_{m}}{s(\tau \cdot s + 1)}$$
$$\tau = \frac{J}{(b - m)}$$

m = slope of linearizec
torque-speed curve
(normally negative)



$$\begin{split} \frac{V_{o}(s)}{V_{c}(s)} &= \frac{\left(\frac{K}{R_{c} \cdot R_{q}}\right)}{\left(s \cdot \tau_{c} + 1\right) \cdot \left(s \cdot \tau_{q} + 1\right)} \\ \tau_{c} &= \frac{L_{c}}{R_{c}} \qquad \tau_{q} = \frac{L_{q}}{R_{q}} \\ \text{For the unloaded case:} \\ i_{d} &= 0 \qquad \tau_{c} = \tau_{q} \\ 0.05s < \tau_{c} < 0.5s \\ V_{12} &= V_{q} \qquad V_{34} = V_{d} \end{split}$$







Gear Ratio = n = N1/N2 $N_2 \cdot \theta_L = N_1 \cdot \theta_m$ $\theta_L = n \cdot \theta_m$ $\omega_L = n \cdot \omega_m$



$$\frac{V_2(s)}{V_1(s)} = \frac{R_2}{R} = \frac{R_2}{R_1 + R_2}$$
$$\frac{R_2}{R} = \frac{\theta}{\theta_{\text{max}}}$$



$$V_{2}(s) = k_{s} (\theta_{1}(s) - \theta_{2}(s))$$
$$V_{2}(s) = k_{s} \cdot \theta_{erro}(s)$$

$$k_{s} = \frac{V_{battery}}{\theta_{max}}$$



$$V_2(s) = K_t \cdot \omega(s) = K_t \cdot s \cdot \theta(s)$$

 $K_t = constant$



$$\frac{V_2(s)}{V_1(s)} = \frac{k_a}{s \cdot \tau + 1}$$

Ro = output resistance Co = output capacitance

 $\tau = R_0 \cdot C_0 \qquad \tau < 1s$

and is often negligibl for controller amplifie



$$x_{o}(t) = y(t) - x_{in}(t)$$
$$\frac{X_{o}(s)}{X_{in}(s)} = \frac{-s^{2}}{s^{2} + \left(\frac{b}{M}\right) \cdot s + \frac{1}{M}}$$

For low frequency oscillations, where $\omega < \omega_n$

Л

$$\frac{X_{o}(j \cdot \omega)}{X_{in}(j \cdot \omega)} = \frac{\omega^{2}}{\frac{k}{M}}$$

$$\frac{\mathrm{T}(\mathrm{s})}{\mathrm{q}(\mathrm{s})} = \frac{1}{\mathrm{C}_{\mathrm{t}} \cdot \mathrm{s} + \left(\mathrm{Q} \cdot \mathrm{S} + \frac{1}{\mathrm{R}}\right)}$$

 $T = T_o - T_e$ = temperature difference due to thermal proc

- C_t = thermal capacitance
- Q = fluid flow rate = constant
- S = specific heat of water
- R_t = thermal resistance of insulation
- $q(\boldsymbol{s})$ = rate of heat flow of heating element



 $\mathbf{x} = \mathbf{r} \cdot \mathbf{\theta}$

converts radial motion to linear mo

$$V_f(s)$$
 \longrightarrow $G(s) = \frac{K_m}{s(Js+b)(L_fs+R_f)}$ $\xrightarrow{\text{Output}} \theta(s)$

Block diagram of dc motor.



General block representation of two-input, two-output system.



Block diagram of interconnected system.



Original Diagram

Equivalent Diagram



Original Diagram

Equivalent Diagram



Original Diagram

Equivalent Diagram





Original Diagram

Equivalent Diagram







Negative feedback control system.

Example 2.7







Signal-flow graph of the dc motor.

For complex systems, the block diagram method can become difficult to complete. By using the signal-flow graph model, the reduction procedure (used in the block diagram method) is not necessary to determine the relationship between system variables.



Signal-flow graph of interconnected system.

$$Y_1(s) = G_{11}(s) \cdot R_1(s) + G_{12}(s) \cdot R_2(s)$$

 $Y_{2}(s) = G_{21}(s) \cdot R_{1}(s) + G_{22}(s) \cdot R_{2}(s)$



Signal-flow graph of two algebraic equations.

 $a_{11} \cdot x_1 + a_{12} \cdot x_2 + r_1 = x_1$

$$a_{21} \cdot x_1 + a_{22} \cdot x_2 + r_2 = x_2$$





Two-path interacting system.

$$\frac{\mathbf{Y}(s)}{\mathbf{R}(s)} = \frac{\left[\mathbf{G}_{1} \cdot \mathbf{G}_{2} \cdot \mathbf{G}_{3} \cdot \mathbf{G}_{4} \cdot \left(1 - \mathbf{L}_{3} - \mathbf{L}_{4}\right)\right] + \left[\mathbf{G}_{5} \cdot \mathbf{G}_{6} \cdot \mathbf{G}_{7} \cdot \mathbf{G}_{8} \cdot \left(1 - \mathbf{L}_{1} - \mathbf{L}_{2}\right)\right]}{1 - \mathbf{L}_{1} - \mathbf{L}_{2} - \mathbf{L}_{3} - \mathbf{L}_{4} + \mathbf{L}_{1} \cdot \mathbf{L}_{3} + \mathbf{L}_{1} \cdot \mathbf{L}_{4} + \mathbf{L}_{2} \cdot \mathbf{L}_{3} + \mathbf{L}_{2} \cdot \mathbf{L}_{4}}$$

Example 2.10



The signal-flow graph of the armature-controlled dc motor.

$$\frac{Y(s)}{R(s)} = \frac{G_1 \cdot G_2 \cdot G_3 \cdot G_4}{1 + G_2 \cdot G_3 \cdot H_2 - G_3 \cdot G_4 \cdot H_1 + G_1 \cdot G_2 \cdot G_3 \cdot G_4 \cdot H_3}$$



Multiple-loop system.

$$\frac{\mathbf{Y}(\mathbf{s})}{\mathbf{R}(\mathbf{s})} = \frac{\mathbf{P}_1 + \mathbf{P}_2 \cdot \boldsymbol{\Delta}_2 + \mathbf{P}_3}{\Delta}$$

$$\begin{split} P_1 &= G_1 \cdot G_2 \cdot G_3 \cdot G_4 \cdot G_5 \cdot G_6 & P_2 &= G_1 \cdot G_2 \cdot G_7 \cdot G_6 & P_3 &= G_1 \cdot G_2 \cdot G_3 \cdot G_4 \cdot G_8 \\ \Delta &= 1 - \left(L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8 \right) + \left(L_5 \cdot L_7 + L_5 \cdot L_4 + L_3 \cdot L_4 \right) \end{split}$$

 $\Delta_1 = \Delta_3 = 1 \qquad \qquad \Delta_2 = 1 - L_5 = 1 + G_4 \cdot H_4$

Design Examples



Analysis and design using a system model.







(b)



(c)



Speed control of an electric traction motor.

Design Examples



An accelerometer mounted on a jet-engine test sled.



Accelerometer response.



Exploded view of the ORCA robot showing the components [15]. (Source: © Copyright 1993 Hewlett-Packard Company. Reproduced with permission.)

Design Examples



⁽b)

(a) Ladder network and (b) its signal-flow graph.

BLOCK DIAGRAM REDUCTION OF MULTIPLE SYSTEMS

Components of a block diagram for a linear, time-invariant system



a. Cascaded subsystems;b. equivalent transfer function



a. Parallel subsystems;b. equivalent transfer function



(b)

- a. Feedback control system;
 - b. simplified model;

c. equivalent transfer function



Block diagram algebra for summing junctions

equivalent forms for moving a block a. to the left past a summing junction; b. to the right past a summing junction





Block diagram algebra for pickoff points

equivalent forms for moving a blocka. to the left past a pickoff point;b. to the right past a pickoff point



(*a***)**



Block diagram reduction via familiar forms for Example

Problem: Reduce the block diagram shown in figure to a single transfer function


Block diagram reduction via familiar forms for Example Cont.



- a. collapse summing junctions;
- b. form equivalent cascaded system in the forward path
- c. form equivalent parallel system in the feedback path;
- d. form equivalent feedback system and multiply by cascadedG₁(s)







Problem: Reduce the block diagram shown in figure to a single transfer function



Steps in the block diagram reduction for Example

a)Move $G_2(s)$ to the left past of pickoff point to create parallel subsystems, and reduce the feedback system of $G_3(s)$ and $H_3(s)$

b)Reduce parallel pair of $1/G_2(s)$ and unity, and push $G_1(s)$ to the right past summing junction

c)Collapse the summing junctions, add the 2 feedback elements, and combine the last 2 cascade blocks

d)Reduce the feedback system to the left

e) finally, Multiple the 2 cascade blocks and obtain final result.



Second-order feedback control system



The closed loop transfer function is $T(s) = \frac{K}{s^2 + as + K}$

Note K is the amplifier gain, As K varies, the poles move through the three ranges of operations OD, CD, and UD $0 < K < a^2/4$ system is over damped $K = a^2/4$ system is critically damped $K > a^2/4$ system is under damped

Finding transient response Example



Problem: For the system shown, find peak time, percent overshot, and settling time. Solution: The closed loop transfer function is $T(s) = \frac{25}{s^2 + 5s + 25}$

And

$$\omega_n = \sqrt{25} = 5$$

2 \xi \alpha_n = 5 so \xi = 0.5

using values for ξ and ω_n and equation in chapter 4 we find

$$T_{p} = \frac{\pi}{\omega_{n} \sqrt{1 - \xi^{2}}} = 0.726 \text{ sec}$$

% $OS = e^{-\xi \pi / \sqrt{1 - \xi^{2}}} X \ 100 = 16.303$
 $T_{s} = \frac{4}{\xi \omega_{n}} = 1.6 \text{ sec}$

Gain design for transient response Example



Problem: Design the value of gain K, so that the system will respond with a 10% overshot. $T(s) = \frac{K}{s^2 + 5s + K}$

Solution: The closed loop transfer function is

$$\omega_n = \sqrt{K}$$
 and $2\xi\omega_n = 5$ thus $\xi = \frac{5}{2\sqrt{K}}$

For 10% OS we find $\xi = 0.591$

We substitute this value in previous equation to find K = 17.9

Signal-flow graph components:

a. system;b. signal;c. interconnection of systems and signals



Building signal-flow graphs

| а. | cascaded system | $R(s)$ \bigcirc | $O_{V_2(s)}$ | $O_{V_1(s)}$ | $\bigcirc C(s)$ | $R(s) \bigcirc \begin{array}{c} G_1(s) & G_2(s) & G_3(s) \\ & & & & \\ V_2(s) & V_1(s) \end{array} \bigcirc \begin{array}{c} C(s) \\ C(s) \end{array}$ |
|--------------------------------------|---|-------------------|-----------------------|--------------|-----------------|---|
| b. | nodes cascaded system signal-flow graph; | | <i>(a)</i> | | | (b) |
| C. parallel system nodes | | | $\bigcup_{V_1(s)}$ | | | $G_1(s) = V_1(s)$ 1 |
| d. e. feedb | parallel system signal-flow graph; ack system nodes | $R(s)$ \bigcirc | С _{V2} (|) (s) | $\bigcirc C(s)$ | $R(s) \bigcirc \begin{array}{c} G_2(s) \\ V_2(s) \\ \end{array} \bigcirc \begin{array}{c} C(s) \\ \end{array} $ |
| f. feedback system signal-flow graph | | | C V ₃ (|) (s) | | $V_3(s)$ |
| | | | (<i>c</i> |) | | (d) |
| | | R(s) | C E(s |) 5) | $\bigcirc C(s)$ | $R(s) \bigcirc \frac{1}{E(s)} \bigcirc G(s) \\ -H(s) \bigcirc C(s)$ |

(**e**)

(f)

Converting a block diagram to a signal-flow graph

Problem: Convert the block diagram to a signal-flow graph.



Converting a block diagram to a signal-flow graph

- Signal-flow graph development:
- a. signal nodes;
- b. signal-flow graph;
- c. simplified signal-flow graph



Mason's dule - Definitions



Loop gain: The product of branch gains found by traversing a path that starts at a node and ends at the same node, following the direction of the signal flow, without passing through any other node more than once. $G_2(s)H_2(s)$, $G_4(s)H_2(s)$, $G_4(s)G_5(s)H_3(s)$, $G_4(s)G_6(s)H_3(s)$ *Forward-path gain:* The product of gains found by traversing a path from input node to output node in the direction of signal flow. $G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)G_7(s)$, $G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)G_7(s)$ *Nontouching loops:* loops that do not have any nodes in common. $G_2(s)H_1(s)$ does not touch $G_4(s)H_2(s)$, $G_4(s)G_5(s)H_3(s)$, and $G_4(s)G_6(s)H_3(s)$ *Nontouching-loop gain:* The product of loop gains from nontouching loops taken 2, 3,4, or more at a time.

 $[G_{2}(s)H_{1}(s)][G_{4}(s)H_{2}(s)], \ [G_{2}(s)H_{1}(s)][G_{4}(s)G_{5}(s)H_{3}(s)], \ [G_{2}(s)H_{1}(s)][G_{4}(s)G_{6}(s)H_{3}(s)]$

Mason's Rule

The Transfer function. C(s)/ R(s), of a system represented by a signal-flow graph is

$$G(s) = \frac{C(s)}{R(s)} = \frac{\sum_{k} T_{k} \Delta_{k}}{\Delta}$$

Where

K = number of forward paths

 T_k = the kth forward-path gain

= 1 - loop gains + nontouching-loop gains taken 2 at a time - nontouching-loop gains taken 3 at a time + nontouching-loop gains taken 4 at a $\frac{1}{2}$ me -

= - loop gain terms in that touch the kth forward path. In other words, Δ_k is for the by eliminating from Δ those loop gains that touch the kth forward path. Δ_k Δ

Transfer function via Mason's rule



Problem: Find the transfer function for the signal flow graph

Now

 $\Delta = 1 - [G2(s)H1(s) + G_4(s)H_2(s) + G_7(s)H_4(s) + G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)] + [G_2(s)H_1(s)G_4(s)H_2(s) + G_2(s)H_1(s)G_7(s)H_4(s) + G_4(s)H_2(s)G_7(s)H_4(s)] - [G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)]$

$$\Delta_{1} = \mathbf{1} - \mathbf{G7(s)H4(s)}$$

$$G(s) = \frac{T_{1}\Delta_{1}}{\Delta} \quad \underline{[G_{1}(s)G_{2}(s)G_{3}(s)G_{4}(s)G_{5}(s)][\mathbf{1}-G_{7}(s)H_{4}(s)]}$$

$$\Delta$$



Signal-Flow Graphs of State Equations

(continued) e. form dx_3/dt ; f. form output



Alternate Representation: Cascade Form

$$\frac{R(s)}{24} = \frac{1}{s+2} + \frac{1}{x_3(s)} + \frac{1}{s+3} + \frac{1}{x_2(s)} + \frac{1}{s+4} + \frac{C(s)}{x_1(s)} + \frac{C(s)}{x_1(s)} + \frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)}$$





Alternate Representation: Cascade Form







$$\frac{C(s)}{R(s)} = \frac{\text{Alternate Representation Barallel Form}}{(s+2)(s+3)(s+4)} = \frac{12}{(s+2)} - \frac{12}{(s+3)} + \frac{12}{(s+4)}$$



Alternate Representation: Parallel Form Repeated roots $\frac{C(s)}{R(s)} = \frac{(s+3)}{(s+1)^2(s+2)} = \frac{2}{(s+1)^2} \frac{1}{(s+1)} \frac{1}{(s+2)}$ $x_{1} = -x_{1} + x_{2}$ $x_{2} = x_{2} + 2r$ $x_{3} = -2x_{3} + r$ $y = c(t) = x_{1} - 1 / 2x_{2} + x_{3}$ $X_2(s)$ $X_1(s)$ 2 R(s)C(s) $X = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} X + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} r$ $X_3(s)$ $y = [1 - 1/2 \ 1]X$

Alternate Representation: controller canonical form

 $G(s) = C(s)/R(s) = (s^2 + 7s + 2)/(s^3 + 9s^2 + 26s + 24)$

This form is obtained from the phase-variable form simply by ordering the phase variable in reverse order



Alternate Representation: controller canonical form

System matrices that contain the coefficients of the characteristic polynomial are called *companion matrices* to the characteristic polynomial.

Phase-variable form result in lower companion matrix

Controller canonical form results in upper companion matrix

Alternate Representation: observer canonical form

Observer canonical form so named for its use in the design of observers $G(s) = C(s)/R(s) = (s^2 + 7s + 2)/(s^3 + 9s^2 + 26s + 24)$ $= (1/s+7/s^2+2/s^3)/(1+9/s+26/s^2+24/s^3)$

Cross multiplying

 $(1/s+7/s^{2}+2/s^{3})R(s) = (1+9/s+26/s^{2}+24/s^{3})C(s)$ And C(s) = $1/s[R(s)-9C(s)] + 1/s^{2}[7R(s)-26C(s)] + 1/s^{3}[2R(s)-24C(s)]$

= 1/s{ [R(s)-9C(s)] + 1/s {[7R(s)-26C(s)]+1/s [2R(s)-24C(s)]}}



Alternate Representation: observer canonical form

$$X_{1} = -9x_{1} + x_{2} + r$$

$$x_{2} = -26x_{1} + x_{3} + 7r$$

$$X_{3} = -24x_{1} + 2r$$

$$y = c(t) = x_{1}$$

$$X = \begin{bmatrix} -9 & 1 & 0 \\ -26 & 0 & 1 \\ -24 & 0 & 0 \end{bmatrix} X + \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} X$$

Note that the observer form has A matrix that is transpose of the controller canonical form, B vector is the transpose of the controller C vector, and C vector is the transpose of the controller B vector. The 2 forms are called duals.

Feedback control system for Example



Problem Represent the feedback control system shown in state space. Model the forward transfer function in cascade form.

Solution first we model the forward transfer function as in (a), Second we add the feedback and input paths as shown in (b) complete system. Write state equations

$$E(s) \bigcirc \frac{100}{-2} \xrightarrow{\frac{1}{s}} 1 \xrightarrow{\frac{1}{s}} 5 \xrightarrow{5} C(s)$$
(a)

$$R(s) \bigcirc \frac{1}{E(s)} \bigcirc \frac{1}{100} \bigcirc \frac{1}{s} & 1 & 0 \\ -2 & -2 & -3 & -1 \\ (b) & & & \\ (b) & & \\ (b) & & & \\ (b) & & \\ (b) & & &$$

$$x_{1} = -3x_{1} + x_{2}$$

$$x_{2} = -2x_{2} + 100(r - c)$$

but $c = 5x_1 + (x_2 - 3x_1) = 2x_1 + x_2$

Feedback control system for Example

$$x_{1}^{-1} = -3x_{1} + x_{2}$$

$$x_{2} = -200x_{1} - 102x_{2} + 100r$$

$$y = c(t) = 2x_{1} + x_{2}$$

$$\begin{array}{l} \ ^{\square} \\ X \end{array} = \begin{bmatrix} -3 & 1 \\ -200 & -102 \end{bmatrix} X + \begin{bmatrix} 0 \\ 100 \end{bmatrix} r \\ y = \begin{bmatrix} 2 & 1 \end{bmatrix} X$$



UNIT-II TIME RESPONSE ANALYSIS

Feedback Control System Characteristics Objectives

In this chapter we extend the ideas of modeling to include control system characteristics, such as sensitivity to model uncertainties, steady-state errors, transient response characteristics to input test signals, and disturbance rejection. We investigate the important role of the system error signal which we generally try to minimize.

We will also develop the concept of the sensitivity of a system to a parameter change, since it is desirable to minimize the effects of unwanted parameter variation. We then describe the transient performance of a feedback system and show how this performance can be readily improved. We will also investigate a design that reduces the impact of disturbance signals.

Open-And Closed-Loop Control Systems



A closed-loop system.

An **open-loop** (direct) system operates without feedback and directly generates the output in response to an input signal.

A **closed-loop** system uses a measurement of the output signal and a comparison with the desired output to generate an error signal that is applied to the actuator.

Open-And Closed-Loop Control Systems

$$H(s) = 1$$

$$Y(s) = \frac{G(s)}{1 + G(s)} \cdot R(s)$$

$$E(s) = \frac{1}{1 + G(s)} \cdot R(s)$$
Error Signal



Thus, to reduce the error, the magnitude of $|1 + G(s)| \ge 1 \ge 1$

 $H(s) \neq 1$

- - / ``

$$Y(s) = \frac{G(s)}{1 + H(s)G(s)} \cdot R(s)$$

$$E(s) = \frac{1}{1 + H[(s) \cdot G(s)]} \cdot R(s)$$

Thus, to reduce the error, the magnitude of $|1 + G(s)H(s)| \ge 1 \ge 1$

Sensitivity of Control Systems To Parameter Variations

For the closed-loop case if

 $Y(s) = \frac{1}{H(s)} \cdot R(s)$ Output affected only by H(s)

 $G(s) + \Delta G(s)$

Open Loop $\Delta Y(s) = \Delta G(s) \cdot R(s)$

Closed Loop

$$Y(s) + \Delta Y(s) = \frac{(G(s) + \Delta G(s))}{1 + (G(s) + \Delta G(s)) \cdot H(s)} \cdot R(s)$$

$$\Delta Y(s) = \frac{\Delta G(s)}{(1 + GH(s) + \Delta GH(s))(1 + GH(s))} \cdot R(s)$$

 $GH(s) > \Delta GH(s)$ $\Delta Y(s) = \frac{\Delta G(s)}{(1 + GH(s))^2} \cdot R(s)$

The change in the output of the closed system is reduced by a factor of 1+GH(s)



Sensitivity of Control Systems To Parameter Variations



 $s_H^T = \frac{-GH}{(1 + GH)}$

Sensitivity of the closed-loop to H variations When GH is large sensitivity approaches 1 Changes in H directly affects the output response Example 4.1



Open loop

Closed loop

Block diagram model of feedback amplifier assuming $R_p \gg R_0$ of the amplifier.

$$v_{0} = -K_{a} \cdot v_{in} \qquad \beta = \frac{R2}{R1} \qquad R_{p} = R1 + R2$$

$$T = -k_{a} \qquad T = \frac{-K_{a}}{1 + K_{a} \cdot \beta} \qquad S_{Ka}^{T} = \frac{1}{1 + K_{a} \cdot \beta}$$

If Ka is large, the sensitivity is low.

$$K_a := 10^4$$
 $\beta := 0.1$ $S_{Ka}^T = \frac{1}{1+10^3} = 9.99 \times 10^{-4}$

Control of the Transient Response of Control Systems



Open-loop speed control system (without feedback).

$$\frac{\omega(s)}{V_a(s)} = G(s) = \frac{K_1}{\tau_1 \cdot s + 1}$$

where,

$$K_1 = \frac{K_m}{R_a \cdot b + K_b \cdot K_m} \qquad \tau_1 = \frac{R_a \cdot J}{R_a \cdot b + K_b \cdot K_m}$$

Control of the Transient Response of Control Systems

(a) Closed-loop speed control system.(b) Transistorized closed-loop speed control system.



Control of the Transient Response of Control Systems



The response of the open-loop and closed-loop speed control system when τ 5 10 and $K_1K_aK_t = 100$. The time to reach 98% of the final value for the open-loop and closed-loop system is 40 seconds and 0.4 second, respectively.
Disturbance Signals In a Feedback Control Systems



Disturbance Signals In a Feedback Control Systems



Closed-loop speed tachometer control system.

Disturbance Signals In a Feedback Control Systems



Strive to maintain Ka large and Ra < 2 ohms





$$E_{c}(s) = \frac{1}{1 + G(s)} \cdot R(s)$$
 $H(s) = 1$

Steady State Error

 $\lim_{t \to 0} e(t) = \lim_{s \to 0} s \cdot E(s)$

For a step unit input

$$e_{O}(\text{infinite}) = \lim_{s \to 0} s \cdot (1 - G(s)) \cdot \frac{1}{s} = \lim_{s \to 0} (1 - G(0))$$

$$e_{c}(\text{infinite}) = \lim_{s \to 0} s \cdot \left(\frac{1}{1 + G(s)}\right) \cdot \frac{1}{s} = \lim_{s \to 0} \left(\frac{1}{1 + G(0)}\right)$$

Increased Number of components and Complexity

Loss of Gain

Instability

Design Example: English Channel Boring Machines



 $Y(s) = T(s) \cdot R(s) + Td(s) \cdot D(s)$

$$Y(s) = \frac{K + 11 \cdot s}{s^2 + 12 \cdot s + K} \cdot R(s) + \frac{1}{s^2 + 12 \cdot s + K} \cdot D(s)$$

Design Example: English Channel Boring Machines



Steady state error for R(s)=1/s and D(s)=0

$$\lim_{t \to \text{ infinite}} e(t) = \lim_{s \to 0} s \cdot \left(\frac{1}{1 + \frac{K + 11 \cdot s}{s} + s} \right) \cdot \frac{1}{s} = 0$$

Steady state error for R(s)=0 D(s)=1/s

$$\lim_{t \to \text{ infinite}} y(t) = \lim_{s \to 0} s \cdot \left(\frac{1}{s^2 + 12 \cdot s + K} \right) \cdot \frac{1}{s} = \frac{1}{K}$$

Study system for different Values of gain K



disturbance step input D(s) 5 1/s for K = 100.

Transient vs Steady-State

The output of any differential equation can be broken up into two parts,
a transient part (which decays to zero as t goes to infinity) and
a steady-state part (which does not decay to zero as t goes to infinity).

$$y(t) = y_{tr}(t) + y_{ss}(t)$$
$$\lim_{t \to 0} y_{tr}(t) = 0$$

Either part might be zero in any particular case.

Prototype systems $c(t) + \frac{1}{\tau}c(t) = kr(t)$ $\overline{\tau}$

1st Order system

2nd order system

$$\ddot{c}(t) + 2\zeta \omega_n \dot{c}(t) + \omega_n^2 c(t) = kr(t)$$

Agenda:

transfer function response to test signals impulse step ramp parabolic sinusoidal

1st order system

 $G(s) = \frac{C(s)}{R(s)} = \frac{1/T}{s + 1/T}$

Impulse responseO(s) =Step responseR(s) =Ramp responseRelationship between impulse, step and rampRelationship between impulse, step and ramp responses

$$r(t) = \delta(t), R(s) = 1, \quad c_{\delta}(t) = \frac{1}{T} e^{-t/T} \mathbf{1}(t)$$

$$r(t) = \mathbf{1}(t), R(s) = \frac{1}{s}, \quad c_{step}(t) = \left[1 - e^{-t/T}\right] \mathbf{1}(t)$$

$$r(t) = t\mathbf{1}(t), R(s) = \frac{1}{s^{2}}, \quad c_{ramp}(t) = \left[t - T + Te^{-t/T}\right] \mathbf{1}(t)$$

1st Order system

Prototype parameter: Time constant

Relate problem specific parameter to prototype parameter.

Parameters: problem specific constants. Numbers that do not change with time, but do change from problem to problem.

We learn that the time constant defines a problem specific time scale that is more convenient than the arbitrary time scale of seconds, minutes, hours, days, etc, or fractions thereof.

Transient vs Steady state

Consider the impulse, step, ramp responses computed earlier. Identify the steady state and the transient parts.

1st order system

Step

Consider the impulse, step, ramp responses computed earlier. Identify the steady state and the transient parts.

Impulse response
$$G(s) = \frac{C(s)}{Ramp response} = \frac{1/T}{s+1/T}, \quad T > Relationship between impulse, step and ramp responses$$

$$r(t) = \delta(t), R(s) = 1, \quad c_{\delta}(t) = \frac{1}{T} e^{-t/T} 1(t)$$

$$r(t) = 1(t), R(s) = \frac{1}{S}, \quad c_{step}(t) = \left[1 - e^{-t/T}\right] 1(t)$$

$$r(t) = t 1(t), R(s) = \frac{1}{S^2}, \quad c_{ramp}(t) = \left[t - T + Te^{-t/T}\right] 1(t)$$

Compare steady-state part to input function, transient part to TF.

2nd order system $G(s) = \frac{C(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

Over damped

- •(two real distinct roots = two 1st order systems with real poles)
- **Critically damped**
- (a single pole of multiplicity two, highly unlikely, requires exact matching)
 Underdamped
- (complex conjugate pair of poles, oscillatory behavior, most common) step response

$$c_{step}(t) = K \left[1 - \frac{e^{-\zeta\omega_{n}t}}{\sqrt{1 - \zeta^{2}}} \sin\left(\omega_{d}t + \tan^{-1}\left(\sqrt{1 - \zeta^{2}}/\zeta\right)\right) \right] 1(t)$$
$$c_{\delta}(t) = K \left[\frac{\omega_{n}}{\sqrt{1 - \zeta^{2}}} e^{-\zeta\omega_{n}t} \sin\left(\omega_{d}t\right) \right] 1(t)$$

2nd Order System

Prototype parameters: undamped natural frequency, damping ratio

Relating problem specific parameters to prototype parameters

Transient vs Steady state

Consider the step, responses computed earlier. Identify the steady state and the transient parts.

2nd order system $G(s) = \frac{C(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

Over damped

- •(two real distinct roots = two 1st order systems with real poles)
- **Critically damped**
- (a single pole of multiplicity two, highly unlikely, requires exact matching)
 Underdamped
- (complex conjugate pair of poles, oscillatory behavior, most common) step response

$$c_{step}(t) = K \left[1 - \frac{e^{-\zeta\omega_{n}t}}{\sqrt{1 - \zeta^{2}}} \sin\left(\omega_{d}t + \tan^{-1}\left(\sqrt{1 - \zeta^{2}}/\zeta\right)\right) \right] 1(t)$$
$$c_{\delta}(t) = K \left[\frac{\omega_{n}}{\sqrt{1 - \zeta^{2}}} e^{-\zeta\omega_{n}t} \sin\left(\omega_{d}t\right) \right] 1(t)$$

Use of Prototypes

Too many examples to cover them all We cover important prototypes We develop intuition on the prototypes We cover how to convert specific examples to prototypes We transfer our insight, based on the study of the prototypes to the specific situations.

Transient-Response Spedifications

- 1. Delay time, t_d : The time required for the response to reach half the final value the very first time.
- Rise time, t_r: the time required for the response to rise from 10% to 90% (common for overdamped and 1st order systems); 5% to 95%;
 or 0% to 100% (common for underdamped systems);
 of its final value
- 1. Peak time, t_p:
- 2. Maximum (percent) overshoot, M_p:
- 3. Settling time, t_s



Allowable M_p determines damping ratio.

Settling time then determines undamped natural frequency.

Theory is used to derive relationships between design specifications and prototype parameters.

Which are related to problem parameters.

Higher order system

PFEs have linear denominators.

• each term with a real pole has a time constant

•each complex conjugate pair of poles has a damping ratio and an undamped natural frequency.

Proportional control of plant w integrator

$$G_C(s) = K_p, \quad G(s) = \frac{1}{s(Js+b)}$$

Integral control of Plant w disturbance

$$G_{C}(s) = \frac{K}{s}, \quad G(s) = \frac{1}{s(Js+b)}$$

Proportional Control of plant w/o integrator

$$G_C(s) = K, \quad G(s) = \frac{1}{Ts+1}$$

Integral control of plant w/o integrator

$$G_{C}(s) = \frac{K}{s}, \quad G(s) = \frac{1}{Ts+1}$$

UNIT-III STABILITY ANALYSIS IN S- DOMAIN

The issue of ensuring the stability of a closed-loop feedback system is central to control system design. Knowing that an unstable closed-loop system is generally of no practical value, we seek methods to help us analyze and design stable systems. A stable system should exhibit a bounded output if the corresponding input is bounded. This is known as bounded-input, bounded-output stability and is one of the main topics of this chapter.

The stability of a feedback system is directly related to the location of the roots of the characteristic equation of the system transfer function. The Routh– Hurwitz method is introduced as a useful tool for assessing system stability. The technique allows us to compute the number of roots of the characteristic equation in the right half-plane without actually computing the values of the roots. Thus we can determine stability without the added computational burden of determining characteristic root locations. This gives us a design method for determining values of certain system parameters that will lead to closed-loop stability. For stable systems we will introduce the notion of relative stability, which allows us to characterize the degree of stability. A stable system is a dynamic system with a bounded response to a bounded input.

Absolute stability is a stable/not stable characterization for a closed-loop feedback system. Given that a system is stable we can further characterize the degree of stability, or the relative stability.

The Concept of Stability



The concept of stability can be illustrated by a cone placed on a plane horizontal surface.

> A necessary and sufficient condition for a feedback system to be stable is that all the poles of the system transfer function have negative real parts.

A system is considered marginally stable if only certain bounded inputs will result in a bounded output.

It was discovered that all coefficients of the characteristic polynomial must have the same sign and non-zero if all the roots are in the left-hand plane.

These requirements are necessary but not sufficient. If the above requirements are not met, it is known that the system is unstable. But, if the requirements are met, we still must investigate the system further to determine the stability of the system.

The Routh-Hurwitz criterion is a necessary and sufficient criterion for the stability of linear systems.



$$\begin{aligned} b_{n-1} &= \frac{1}{a_{n-1}} = \frac{1}{a_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \end{vmatrix} \\ b_{n-3} &= \frac{-1}{a_{n-1}} \begin{vmatrix} a_{n-2} & a_{n-4} \\ a_{n-1} & a_{n-3} \end{vmatrix} \\ c_{n-1} &= \frac{-1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix} \end{aligned}$$

Case One: No element in the first column is zero.

Example 6.1 Second-order system

The Characteristic polynomial of a second-order system is:

$$q(s) = a_2 \cdot s^2 + a_1 \cdot s + a_0$$

The Routh array is written as:

$$\begin{vmatrix} s^2 \\ s^1 \\ s^0 \end{vmatrix} \begin{vmatrix} a_2 \\ a_1 \\ a_1 \\ 0 \\ 0 \end{vmatrix}$$

- I

 \mathbf{a}

w here:

$$b_1 = \frac{a_1 \cdot a_0 - (0) \cdot a_2}{a_1} = a_0$$

Therefore the requirement for a stable second-order system is simply that all coefficients be positive or all the coefficients be negative.

Case Two: Zeros in the first column while some elements of the row containing a zero in the first column are nonzero.

If only one element in the array is zero, it may be replaced with a small positive number ϵ that is allow ed to approach zero after completing the array.

 $q(s) = s^{5} + 2s^{4} + 2s^{3} + 4s^{2} + 11s + 10$

The Routh array is then:

| s^{5} | 1 | 2 | 11 |
|---------|-----------------------|----|----|
| s^4 | 2 | 4 | 10 |
| s^{3} | b_1 | 6 | 0 |
| s^2 | <i>C</i> ₁ | 10 | 0 |
| s^1 | d_1 | 0 | 0 |
| s^{0} | 10 | 0 | 0 |

w here:

$$b_1 = \frac{2 \cdot 2 - 1 \cdot 4}{2} = 0 = \varepsilon$$
 $c_1 = \frac{4\varepsilon - 2 \cdot 6}{\varepsilon} = \frac{-12}{\varepsilon}$ $d_1 = \frac{6 \cdot c_1 - 10\varepsilon}{c_1} = 6$

There are two sign changes in the first column due to the large negative number calculated for c1. Thus, the system is unstable because two roots lie in the right half of the plane.

Case Three: Zeros in the first column, and the other elements of the row containing the zero are also zero.

This case occurs when the polynomial q(s) has zeros located symetrically about the origin of the s-plane, such as $(s+\sigma)(s-\sigma)$ or $(s+j\omega)(s-j\omega)$. This case is solved using the auxiliary polynomial, U(s), which is located in the row above the row containing the zero entry in the Routh array.

$$q(s) = s^{3} + 2 \cdot s^{2} + 4s + K$$

Routh array:
 $s^{3} | 1 - 4$
 $s^{2} | 2 - K$
 $s^{1} | \frac{8-K}{2} = 0$
 $s^{0} | K = 0$

For a stable system we require that 0 < s < 8

For the marginally stable case, K=8, the s^1 row of the Routh array contains all zeros. The auxiliary plynomial comes from the s^2 row.

$$U(s) = 2s^{2} + Ks^{0} = 2 \cdot s^{2} + 8 = 2(s^{2} + 4) = 2(s + j \cdot 2)(s - j \cdot 2)$$

It can be proven that U(s) is a factor of the characteristic polynomial:

$$\frac{q(s)}{U(s)} = \frac{s+2}{2}$$
 Thus, when K=8, the factors of the characteristic polynomial are:

$$q(s) = (s + 2)(s + j \cdot 2)(s - j \cdot 2)$$

Case Four: Repeated roots of the characteristic equation on the jw-axis.

With simple roots on the jw-axis, the system will have a marginally stable behavior. This is not the case if the roots are repeated. Repeated roots on the jw-axis will cause the system to be unstable. Unfortunately, the routh-array will fail to reveal this instability.

Example 6.4



A completely integrated, six-legged, micro robot system. The six-legged design provides maximum dexterity. Legs also provide a unique sensory system for environmental interaction. It is equipped with a sensor network that includes 150 sensors of 12 different types. The legs are instrumented so that the robot can determine the lay of the terrain, the surface texture, hardness, and even color. The gyro-stabilized camera and range finder can be used for gathering data beyond the robot's immediate reach. This high-performance system is able to walk quickly, climb over obstacles, and perform dynamic motions. (Courtesy of IS Robotics Corporation.)
Example 6.5 Welding control



Welding head position control.

Using block diagram reduction we find that: $q(s) = s^4 + 6s^3 + 11s^2 + (K + 6)s + Ka$

The Routh array is then:

For the system to be stable both b_3 and c_3 must be positive.

Using these equations a relationship can be determined for K ar

The Relative Stability of Feedback Control Systems

It is often necessary to know the relative damping of each root to the characteristic equation. Relative system stability can be measured by observing the relative real part of each root. In this diagram r2 is relatively more stable than the pair of roots labeled r1.



One method of determining the relative stability of each root is to use an axis shift in the s-domain and then use the Routh array as shown in Example 6.6 of the text. Problem statement: Design the turning control for a tracked vehicle. Select K and a so that the system is stable. The system is modeled below.



Design Example: Tracked Vehicle Turning Control

The characteristic equation of this system is:

 $1 + G_c \cdot G(s) = 0$

or

$$1 + \frac{K(s + a)}{s(s + 1)(s + 2)(s + 5)} = 0$$

Thus,

$$s(s + 1)(s + 2)(s + 5) + K(s + a) = 0$$

or

$$s^4 + 8s^3 + 17s^2 + (K + 10)s + Ka = 0$$

To determine a stable region for the system, we establish the Routh arra

where

$$b_3 = \frac{126 - K}{8}$$
 and $c_3 = \frac{b_3(K + 10) - 8Ka}{b_3}$

Design Example: Tracked Vehicle Turning Control



In the preceding chapters we discussed how the performance of a feedback system can be described in terms of the location of the roots of the characteristic equation in the s-plane. We know that the response of a closed-loop feedback system can be adjusted to achieve the desired performance by judicious selection of one or more system parameters. It is very useful to determine how the roots of the characteristic equation move around the s-plane as we change one parameter.

The locus of roots in the s-plane can be determined by a graphical method. A graph of the locus of roots as one system parameter varies is known as a root locus plot. The root locus is a powerful tool for designing and analyzing feedback control systems and is the main topic of this chapter. We will discuss practical techniques for obtaining a sketch of a root locus plot by hand. We also consider computer-generated root locus plots and illustrate their effectiveness in the design process. The popular PID controller is introduced as a practical controller structure.

We will show that it is possible to use root locus methods for design when two or three parameters vary. This provides us with the opportunity to design feedback systems with two or three adjustable parameters. For example the PID controller has three adjustable parameters. We will also define a measure of sensitivity of a specified root to a small incremental change in a system parameter.





The root locus is a graphical procedure for determining the poles of a closed-loop system given the poles and zeros of a forward-loop system. Graphically, the locus is the set of paths in the complex plane traced by the closed-loop poles as the root locus gain is varied from zero to infinity.

In mathematical terms, given a forward-loop transfer function, KG(s) where K is the root locus gain, and the corresponding closed-loop transfer function

$$\frac{\mathrm{KG}(\mathbf{s})}{1 + \mathrm{KG}(\mathbf{s})} \xrightarrow{R(s)} \xrightarrow{K} \xrightarrow{G(s)} \xrightarrow{Y(s)}$$

the root locus is the set of paths traced by the roots of

 $1 + \mathbf{KG}(\mathbf{s}) = 0$

77.000 C X

as K varies from zero to infinity. As K changes, the solution to this equation changes. This equation is called the characteristic equation. This equation defines where the poles will be located for any value of the root locus gain, K. In other words, it defines the characteristics of the system behavior for various values of controller gain.









No matter what we pick K to be, the closed-loop system must always have n poles, where n is the number of poles of G(s).

The root locus must have n branches, each branch starts at a pole of G(s) and goes to a zero of G(s).

If G(s) has more poles than zeros (as is often the case), m < n and we say that G(s) has zeros at infinity. In this case, the limit of G(s) as $s \rightarrow$ infinity is zero.

The number of zeros at infinity is n-m, the number of poles minus the number of zeros, and is the number of branches of the root locus that go to infinity (asymptotes).

Since the root locus is actually the locations of all possible closed loop poles, from the root locus we can select a gain such that our closed-loop system will perform the way we want. If any of the selected poles are on the right half plane, the closed-loop system will be unstable. The poles that are closest to the imaginary axis have the greatest influence on the closed-loop response, so even though the system has three or four poles, it may still act like a second or even first order system depending on the location(s) of the dominant pole(s).

Example



First, we must develop a transfer function for the entire control system.

$$G_{\mathcal{S}}(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{\left(\frac{K}{s}\right)}{1 + \left(\frac{K}{s}\right)(1)} = \frac{K}{s + K}$$

Next, we use the characteristic equation of the denominator to find the roots as the value of K varies. These can then be plotted on a complex plane. Note: the value of gain 'K' is normally found from 0 to +infinity.



Note: because all of the roots for all values of K are real negative this system will always be stable, and it will always tend to have a damped response. The large the value of K, the more stable the system becomes. Example

MATLAB Example - Plotting the root locus of a transfer function

Consider an open loop system which has a transfer function of

G(s) = (s+7)/s(s+5)(s+15)(s+20)

How do we design a feedback controller for the system by using the root locus method? Enter the transfer function, and the command to plot the root locus:

```
num=[1 7];
den=conv(conv([1 0],[1 5]),conv([1 15],[1 20]));
rlocus(num,den)
axis([-22 3 -15 15])
```



Graphical Method

Given the system elements (you should assume negative feedback),

$$G(s) = \frac{K}{s^2 + 3s + 2}$$
 $H(s) = 1$

Step 1: (put equation in standard form)

$$1 + G(s)H(s) = 1 + \left(\frac{K}{s^2 + 3s + 2}\right)(1) = 1 + K\frac{1}{(s+1)(s+2)}$$

Step 2: (find loci ending at infinity)

m = 0 n = 2 (from the poles and zeros of the previous step) n-m = 2 (loci end at infinity)

Step 3: (plot roots)



Step 4: (find asymptotes angles and real axis intersection)

Graphical Method

Step 5: (find the breakout points for the roots)



Note: because the loci do not intersect the imaginary axis, we know the system will be stable, so step 6 is not necessary, but we it will be done for illustrative purposes.

Step 6: (find the imaginary intercepts)

$$1 + G(s)H(s) = 0$$

$$1 + K \frac{1}{s^2 + 3s + 2} = 0$$

$$s^2 + 3s + 2 + K = 0$$

$$(j\omega)^2 + 3(j\omega) + 2 + K = 0$$

$$-\omega^2 + 3j\omega + 2 + K = 0$$

$$\omega^2 + \omega(-3j) + (-2 - K) = 0$$

$$\omega = \frac{3j \pm \sqrt{(-3j)^2 - 4(-2 - K)}}{2} = \frac{3j \pm \sqrt{-9 + 8 + 4K}}{2} = \frac{3j \pm \sqrt{4K - 1}}{2}$$

In this case the frequency has an imaginary value. This means that there will be no frequency that will intercept the imaginary axis.

Root Locus Design GUI (rltool)

The Root Locus Design GUI is an interactive graphical tool to design compensators using the root locus method. This GUI plots the locus of the closed-loop poles as a function of the compensator gains. You can use this GUI to add compensator poles and zeros and analyze how their location affects the root locus and various time and frequency domain responses. Click on the various controls on the GUI to see what they do.



UNIT-IV FREQUENCY RESPONSE ANALYSIS

Frequency Response Methods and Stability

In previous chapters we examined the use of test signals such as a step and a ramp signal. In this chapter we consider the steady-state response of a system to a sinusoidal input test signal. We will see that the response of a linear constant coefficient system to a sinusoidal input signal is an output sinusoidal signal at the same frequency as the input. However, the magnitude and phase of the output signal differ from those of the input sinusoidal signal, and the amount of difference is a function of the input frequency. Thus we will be investigating the steady-state response of the system to a sinusoidal input as the frequency varies.

We will examine the transfer function G(s) when s =jw and develop methods for graphically displaying the complex number G(j)as w varies. The Bode plot is one of the most powerful graphical tools for analyzing and designing control systems, and we will cover that subject in this chapter. We will also consider polar plots and log magnitude and phase diagrams. We will develop several time-domain performance measures in terms of the frequency response of the system as well as introduce the concept of system bandwidth.

Introduction

The frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal. The sinusoid is a unique input signal, and the resulting output signal for a linear system, as well as signals throughout the system, is sinusoidal in the steady-state; it differs form the input waveform only in amplitude and phase.

Polar Plots





Polar Plots



Polar plot for $G(j\omega) = K/j\omega(j\omega\tau + 1)$. Note that $\omega = \infty$ at the origin.

Polar Plots



Polar plot for $G(j\omega) = K/j\omega(j\omega\tau + 1)$. Note that $\omega = \infty$ at the origin.

Bode Plots – Real Poles



Bode diagram for $G(j\omega) = 1/(j\omega\tau + 1)$: (a) magnitude plot and (b) phase plot.

Bode Plots – Real Poles

$$\omega := \frac{0.1}{\tau}, \frac{0.11}{\tau} \dots 1000 \qquad j := \sqrt{-1} \qquad R := 1 \qquad C := 0.01 \qquad \tau := R \cdot C$$

$$G(\omega) := \frac{1}{j \cdot \omega \cdot \tau + 1} \qquad \omega I := \frac{1}{\tau} \qquad \omega I = 100 \quad (\text{break frequency or corner frequency})$$

$$= \frac{0}{100} \left(\frac{-3dB}{-10} - \frac{-3dB}{-10} - \frac{-3dB}{-10} - \frac{-3dB}{-10} - \frac{100}{-10} - \frac{100}$$

Bode Plots – Real Poles

 $\phi(\omega) := -atan(\omega \cdot \tau)$



(break frequency or corner frequency)

Bode Plots – Real Poles (Graphical Construction)



Asymptotic curve for $(j\omega\tau + 1)^{-1}$.

Bode Plots – Real Poles



Bode diagram for $(j\omega)^{\pm N}$.





Bode diagram for $(1 + j\omega\tau)^{-1}$.

Bode Plots – Real Poles

$$db(G,\omega) := 20 \cdot log(|G(j \cdot \omega)|)$$

Phase shift:

Magnitude:

$$ps(G,\omega) := \frac{180}{\pi} \cdot arg(G(j \cdot \omega)) - 360 \cdot (if(arg(G(j \cdot \omega)) \ge 0, 1, 0))$$

Assume

K := 2
$$G(s) := \frac{K}{s \cdot (1+s) \cdot \left(1+\frac{s}{3}\right)}$$

Next, choose a frequency rangefor the plots (use powers of 10 for convenient plotting):

lowest frequency (in Hz): $\omega_{start} := .01$ number of points:N := 50highest frequency (in Hz): $\omega_{end} := 100$ step size: $r := log \left(\frac{\omega_{start}}{\omega_{end}} \right) \cdot \frac{1}{N}$ range for plot:i := 0..Nrange variable: $\omega_i := \omega_{end} \cdot 10^{i \cdot r}$ $s_i := j \cdot \omega_i$

Bode Plots – Real Poles



 ω_{i}

Bode Plots – Complex Poles



Bode diagram for $G(j\omega) = [1 + (2\zeta/\omega_n)j\omega + (j\omega/\omega_n)^2]^{-1}$.

Bode Plots – Complex Poles



Bode diagram for $G(j\omega) = [1 + (2\zeta/\omega_n)j\omega + (j\omega/\omega_n)^2]^{-1}$.

Bode Plots – Complex Poles

$$\omega_{\rm r} = \omega_{\rm n} \cdot \sqrt{1 - 2 \cdot \zeta^2} \qquad \zeta < 0.707$$

$$M_{p\omega} = |G(\omega_r)| = \frac{1}{(2 \cdot \zeta \cdot \sqrt{1 - \zeta^2})}$$

$$\zeta < 0.707$$
Frequency Response Plots 3.25 1.0**Bode Plots – Complex Poles** 0.90 3.0 $\omega_r | \omega_n$ 2.75 0.802.5 0.702.250.60 $\omega_{\rm r} = \omega_{\rm n} \cdot \sqrt{1 - 2 \cdot \zeta^2} \qquad \zeta < 0.707 \qquad 2.0$ $M_{\rm p\omega} = \left| G(\omega_{\rm r}) \right| = \frac{1}{\left(2 \cdot \zeta \cdot \sqrt{1 - \zeta^2} \right)} \qquad \zeta < 0.707^{1.75}$ $M_{P_{44}}$ $\omega_r | \omega_n$ 0.50 0.40 $M_{p_{\mu}}$ 1.5 0.30 1.25 0.201.0 L 0.20 0.10 0.30 0.500.400.600.70ζ

Frequency Response Plots

Bode Plots – Complex Poles



Frequency Response Plots

Bode Plots – Complex Poles







w := 4

Given

Finding the Resonance Frequency

 $20\log(|T(w)|) = 5.282$

wr := Find(w) wr = 0.813

Mpw := 1

Given

 $20\log(Mpw) = 5.282$ Finding Maximum value of the frequency response.

Mpw := Find(Mpw) Mpw = 1.837 Mpw = 1.837 10 10 $Bode(x_0)$ -10 -20 Closed-Loop Bode Diagram 0 0

Assume that the system has dominant second-order roo



 $wn := Find(wn) \qquad wn = 0.888$

















```
% Bode plot script for Figure 8.22
%
num=5*[0.1 1];
f1=[1 0]; f2=[0.5 1]; f3=[1/2500 .6/50 1];
den=conv(f1,conv(f2,f3));
%
sys=tf(num,den);
bode(sys)
```





(a) The relationship between $(M_{p_{\omega}}, \omega_r)$ and (ζ, ω_n) for a second-order system. (b) MATLAB script.





Bode Plots

Bode plot is the representation of the magnitude and phase of G(j*w) (where the frequency vector w contains only positive frequencies). To see the Bode plot of a transfer function, you can use the MATLAB bode

command.

For example,

bode(50,[193040])

displays the Bode plots for the transfer function:

 $50 / (s^3 + 9 s^2 + 30 s + 40)$





Let's say that we have the following system:

where K is a variable (constant) gain and G(s) is the plant under consideration.

<u>The gain margin is defined as the change in open loop gain required to make the</u> <u>system unstable.</u> Systems with greater gain margins can withstand greater changes in system parameters before becoming unstable in closed loop. Keep in mind that unity gain in magnitude is equal to a gain of zero in dB.

The phase margin is defined as the change in open loop phase shift required to make a closed loop system unstable.

The phase margin is the difference in phase between the phase curve and -180 deg at the point corresponding to the frequency that gives us a gain of 0dB (the gain cross over frequency, Wgc).

Likewise, the gain margin is the difference between the magnitude curve and OdB at the point corresponding to the frequency that gives us a phase of -180 deg (the phase cross over frequency, Wpc).



We can find the gain and phase margins for a system directly, by using MATLAB. Just enter the margin command.

This command returns the gain and phase margins, the gain and phase cross over frequencies, and a graphical representation of these on the Bode plot.

margin(50,[193040])



Magnitude:

 $db(G,\omega) \coloneqq 20 \cdot log(|G(j \cdot \omega)|)$

Phase shift:

$$ps(G,\omega) := \frac{180}{\pi} \cdot arg(G(j \cdot \omega)) - 360 \cdot (if(arg(G(j \cdot \omega)) \ge 0, 1, 0))$$

Assume

K := 2

$$G(s) := \frac{K}{s \cdot (1+s) \cdot \left(1 + \frac{s}{3}\right)}$$

Next, choose a frequency rangefor the plots (use powers of 10 for convenient plotting):

lowest frequency (in Hz): $\omega_{start} := .01$ number of points:N := 50highest frequency (in Hz): $\omega_{end} := 100$ step size: $r := log \left(\frac{\omega_{start}}{\omega_{end}} \right) \cdot \frac{1}{N}$ range for plot:i := 0..Nrange variable: $\omega_i := \omega_{end} \cdot 10^{i \cdot r}$ $s_i := j \cdot \omega_i$

Guess forcrossover frequency: $\omega_c := 1$

Solve for the gain crossover frequency:

$$\omega_{c} \coloneqq \operatorname{root}(db(G, \omega_{c}), \omega_{c}) \qquad \omega_{c} = 1.193$$

Calculate thephase margin

 $pm := ps(G, \omega_c) + 180$ pm = 18.265 degrees

Gain Margin

Now using the phase angle plot, estimate the frequency at which the phase shift crosses 180 degr

$$\omega_{\rm gm} \coloneqq 1.8$$

Solve for $\boldsymbol{\omega}$ at the phase shift point of 180 degrees:

$$\omega_{\rm gm} := \operatorname{root}(\operatorname{ps}(G, \omega_{\rm gm}) + 180, \omega_{\rm gm})$$

$$\omega_{\rm gm} = 1.732$$

Calculate thegain margin

$$\operatorname{gm} := -\operatorname{db}(G, \omega_{\operatorname{gm}}) \qquad \operatorname{gm} = 6.021$$

The Nyquist Stability Criterion

The Nyquist plot allows us also to predict the stability and performance of a closed-loop system by observing its open-loop behavior. The Nyquist criterion can be used for design purposes regardless of open-loop stability (Bode design methods assume that the system is stable in open loop). Therefore, we use this criterion to determine closed-loop stability when the Bode plots display confusing information.

The Nyquist diagram is basically a plot of G(j* w) where G(s) is the open-loop transfer function and w is a vector of frequencies which encloses the entire right-half plane. In drawing the Nyquist diagram, both positive and negative frequencies (from zero to infinity) are taken into account. In the illustration below we represent positive frequencies in red and negative frequencies in green. The frequency vector used in plotting the Nyquist diagram usually looks like this (if you can imagine the plot stretching out to infinity):

However, if we have open-loop poles or zeros on the jw axis, G(s) will not be defined at those points, and we must loop around them when we are plotting the contour. Such a contour would look as follows:





The Cauchy criterion

The Cauchy criterion (from complex analysis) states that when taking a closed contour in the complex plane, and mapping it through a complex function G(s), <u>the number of times that the plot of G(s) encircles the origin is equal to the number of zeros of G(s) enclosed by the frequency contour minus the number of poles of G(s) enclosed by the frequency contour minus the origin are counted as positive if they are in the same direction as the original closed contour or negative if they are in the opposite direction.</u>

When studying feedback controls, we are not as interested in G(s) as in the closed-loop transfer function:

G(s)

1 + G(s)

<u>If 1+ G(s) encircles the origin, then G(s) will enclose the point -1.</u> <u>Since we are interested in the closed-loop stability, we want to know if there are any</u> <u>closed-loop poles (zeros of 1 + G(s)) in the right-half plane.</u>

Therefore, the behavior of the Nyquist diagram around the -1 point in the real axis is very important; however, the axis on the standard nyquist diagram might make it hard to see what's happening around this point

Gain Margin is defined as the change in open-loop gain expressed in decibels (dB), required at 180 degrees of phase shift to make the system unstable. First of all, let's say that we have a system that is stable if there are no Nyquist encirclements of -1, such as :

50 ______ s^3 + 9 s^2 + 30 s + 40

Looking at the roots, we find that we have no open loop poles in the right half plane and therefore no closed-loop poles in the right half plane if there are no Nyquist encirclements of -1. Now, how much can we vary the gain before this system becomes unstable in closed loop?

The open-loop system represented by this plot will become unstable in closed loop if the gain is increased past a certain boundary.



The Nyquist Stability Criterion

and that the Nyquist diagram can be viewed by typing: nyquist (50, [1 9 30 40])



Phase margin as the change in open-loop phase shift required at unity gain to make a closed-loop system unstable.

From our previous example we know that this particular system will be unstable in closed loop if the Nyquist diagram encircles the -1 point. However, we must also realize that if the diagram is shifted by theta degrees, it will then touch the -1 point at the negative real axis, making the system marginally stable in closed loop. Therefore, the angle required to make this system marginally stable in closed loop is called the phase margin (measured in degrees). In order to find the point we measure this angle from, we draw a circle with radius of 1, find the point in the Nyquist diagram with a magnitude of 1 (gain of zero dB), and measure the phase shift needed for this point to be at an angle of 180 deg.



The Nyquist Stability Criterion



Consider the Negative Feedback System

Remember from the Cauchy criterion that the number <u>N of times that the plot of G(s)H(s) encircles -1 is</u> equal to the number Z of zeros of 1 + G(s)H(s) enclosed by the frequency contour minus the number <u>P</u> of poles of 1 + G(s)H(s) enclosed by the frequency contour (N = Z - P).

Keeping careful track of open- and closed-loop transfer functions, as well as numerators and denominators, you should convince yourself that:

- the zeros of 1 + G(s)H(s) are the poles of the closed-loop transfer function
- the poles of 1 + G(s)H(s) are the poles of the open-loop transfer function.

The Nyquist criterion then states that:

- P = the number of open-loop (unstable) poles of G(s)H(s)
- N = the number of times the Nyquist diagram encircles -1
- clockwise encirclements of -1 count as positive encirclements
- counter-clockwise (or anti-clockwise) encirclements of -1 count as negative encirclements
- Z = the number of right half-plane (positive, real) poles of the closed-loop system

The important equation which relates these three quantities is:

$$Z = P + N$$

The Nyquist Stability Criterion - Application

Knowing the number of right-half plane (unstable) poles in open loop (P), and the number of encirclements of -1 made by the Nyquist diagram (N), we can determine the closed-loop stability of the system.

If Z = P + N is a positive, nonzero number, the closed-loop system is unstable.

We can also use the Nyquist diagram to find the range of gains for a closed-loop unity feedback system to be stable. The system we will test looks like this:



where G(s) is :



The Nyquist Stability Criterion

This system has a gain K which can be varied in order to modify the response of the closed-loop system. However, we will see that we can only vary this gain within certain limits, since we have to make sure that our closed-loop system will be stable. This is what we will be looking for: the range of gains that will make this system stable in the closed loop.

The first thing we need to do is find the number of positive real poles in our open-loop transfer function:

```
roots([1 -8 15])

ans =

5

3
```

The poles of the open-loop transfer function are both positive. Therefore, we need two anticlockwise (N = -2) encirclements of the Nyquist diagram in order to have a stable closed-loop system (Z = P + N). If the number of encirclements is less than two or the encirclements are not anti-clockwise, our system will be unstable.

Let's look at our Nyquist diagram for a gain of 1:

```
nyquist([11024], [1-815])
```

There are two anti-clockwise encirclements of -1. Therefore, the system is stable for a gain of 1.



The Nyquist Stability Criterion

MathCAD Implementation

$$w := -100, -99.9.100 \qquad j := \sqrt{-1} \qquad s(w) := j \cdot w$$

$$G(w) := \frac{s(w)^2 + 10 s(w) + 24}{s(w)^2 - 8 \cdot s(w) + 15}$$

$$\frac{Im(G(w))}{0} = 0$$

$$-2 \frac{1}{-2} \frac{1}{-1} = 0$$

$$Re(G(w))$$

There are two anticlockwise encirclements of -1. Therefore, the system is

Therefore, the system is stable for a gain of 1.


Time-Domain Performance Criteria Specified In The Frequency Domain



Squaring and rearrenging

$$\left(u - \frac{M^2}{1 - M^2}\right)^2 + v^2 = \left(\frac{M}{1 - M^2}\right)^2$$

which is the equation of a circle on u-v planwe with a center at $u = \frac{M^2}{1 - M^2} \qquad v = 0$

Time-Domain Performance Criteria Specified In The Frequency Domain



Polar plot of $G(j\omega)$ for two values of a gain $(K_2 > K_1)$.

Closed-loop frequency response of $T(j\omega) = G(j\omega)/1 + G(j\omega)$. Note that $K_2 > K_1$.

Polar Stability Plot - Nichol Mathcad Implementation

This example makes a polar plot of a transfer function and draws one contour of constant closed-loop magnitude. To draw the plot, enter a definition for the transfer fun**G(s)**

$$\mathbf{G}(\mathbf{s}) \coloneqq \frac{45000}{\mathbf{s} \cdot (\mathbf{s} + 2) \cdot (\mathbf{s} + 30)}$$

The frequency range defined by the next two equations provides a logarithmic frequency scale running from 1 to 100. You can change this range by editing the definition **mfand** ω_m :

$$\mathbf{m} \coloneqq \mathbf{0} \dots \mathbf{100} \qquad \boldsymbol{\omega}_{\mathbf{m}} \coloneqq \mathbf{10}^{\mathbf{.02 \cdot m}}$$

Now enter a value for M to define the closed-loop magnitude contour that will be plotted.

Calculate the points on the M-circle:

$$MC_{m} := \left(\frac{-M^{2}}{M^{2}-1} + \left|\frac{M}{M^{2}-1}\right| \cdot \exp\left(2\cdot\pi\cdot j\cdot.01\cdot m\right)\right)$$

The first plot show, the contour of constant closed-loop magnitude

The first plot show, the contour of constant closed-loop magnitude, and the Nyquist of the open loop system





Nichols chart. The phase curves for the closed-loop system are shown in color.

$$G(\omega) := \frac{1}{j \cdot \omega \cdot (j \cdot \omega + 1) \cdot (0.2 \cdot j \cdot \omega + 1)}$$

 $M_{pw} \coloneqq 2.5$ dB

-3dB

The closed-loop phase angle at ωr is equal to -72 degrees and $\omega b = 1.33$ The closed-loop phase angle atob is equal to -142 degrees



$$G(\omega) := \frac{0.64}{j \cdot \omega \cdot \left[(j \cdot \omega)^2 + j \cdot \omega + 1 \right]}$$

Phase Margin = 30 degrees

On the basis of the phase we estimate := 0.30 $M_{pw} := 9$ dB $M_{pw} := 2.8$ $\omega_r := 0.88$ From equation $M_{pw} = \frac{1}{\zeta} := 0.18$

$$M_{pw} = \frac{1}{2 \cdot \zeta \cdot \sqrt{1 - \zeta^2}} \qquad \zeta := 0.18$$

We are confronted with comflectings

The app arent conflict is caused by the nature of $G(j\omega)$ which slopes rapidally toward 180 degrees line from the 0-dB axis. The designer must use the frequency-domain-time-domain correlation with caution





Examples – Bode and Nyquist



A closed-loop control system example for Nyquist and Bode with relative stability.



Examples - Bode





Examples – Bode and Nyquist





Examples - Nichols



Examples - Nichols



The Design of Feedback Control Systems

PID

Compensation Networks

Different Types of Feedback Control

On-Off Control

This is the simplest form of control.



Proportional Control

A proportional controller attempts to perform better than the On-off type by applying power in proportion to the difference in temperature between the measured and the set-point. As the gain is increased the system responds faster to changes in set-point but becomes progressively underdamped and eventually unstable. The final temperature lies below the set-point for this system because some difference is required to keep the heater supplying power.



Proportional, Derivative Control

The stability and overshoot problems that arise when a proportional controller is used at high gain can be mitigated by adding a term proportional to the time-derivative of the error signal. The value of the damping can be adjusted to achieve a critically damped response.



Proportional+Integral+Derivative Control

Although PD control deals neatly with the overshoot and ringing problems associated with proportional control it does not cure the problem with the steady-state error. Fortunately it is possible to eliminate this while using relatively low gain by adding an integral term to the control function which becomes



The Characteristics of P, I, and D controllers

A proportional controller (Kp) will have the effect of reducing the rise time and will reduce, but never eliminate, the steady-state error.

An integral control (Ki) will have the effect of eliminating the steady-state error, but it may make the transient response worse.

A derivative control (Kd) will have the effect of increasing the stability of the system, reducing the overshoot, and improving the transient response.

Proportional Control

By only employing proportional control, a steady state error occurs.

Proportional and Integral Control

The response becomes more oscillatory and needs longer to settle, the error disappears.

Proportional, Integral and Derivative Control

All design specifications can be reached.

The Characteristics of P, I, and D controllers

| CL RESPONSE | RISE TIME | OVERSHOOT | SETTLING TIME | S-S ERROR |
|-------------|--------------|-----------|---------------|--------------|
| Кр | Decrease | Increase | Small Change | Decrease |
| Ki | Decrease | Increase | Increase | Eliminate |
| Kd | Small Change | Decrease | Decrease | Small Change |

Tips for Designing a PID Controller

- 1. Obtain an open-loop response and determine what needs to be improved
- 2. Add a proportional control to improve the rise time
- 3. Add a derivative control to improve the overshoot
- 4. Add an integral control to eliminate the steady-state error
- 5. Adjust each of Kp, Ki, and Kd until you obtain a desired overall response.

Lastly, please keep in mind that you do not need to implement all three controllers (proportional, derivative, and integral) into a single system, if not necessary. For example, if a PI controller gives a good enough response (like the above example), then you don't need to implement derivative controller to the system. Keep the controller as simple as possible.

Open-Loop Control - Example



num=1; den=[1 10 20]; <u>step</u>(num,den)



Proportional Control - Example

The proportional controller (Kp) reduces the rise time, increases the overshoot, and reduces the steady-state error.

MATLAB Example

 $T(s) = \frac{Kp}{s^2 + 10s + (20 + Kp)}$

Kp=300; num=[Kp]; den=[1 10 20+Kp]; t=0:0.01:2; step(num,den,t)



Proportional - Derivative - Example

The derivative controller (Kd) reduces both the overshoot and the settling time.

MATLAB Example



Proportional - Integral - Example

The integral controller (Ki) decreases the rise time, increases both the overshoot and the settling time, and eliminates the steady-state error

MATLAB Example



Syntax

rltool

rltool(sys)

rltool(sys,comp)





🛃 Root Locus Design: sys File Tools Window Help Current Compensator Gain(s+15.00) K = 1 +/-Grid R ۶Ľ \square Gain: 8.6155 DDDA R Axes settings: =/z Zoom: 🔲 Impulse 🔲 Bode 🔲 Nyquist Nichols Step Click the Pole/Zero text to see the Denominator/Numerator. Check the Delete box to remove the associated pole/zero





Example - Practice

Consider the following configuration:



Example - Practice

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The design a system for the following specifications:

Zero steady state error Settling time within 5 seconds Rise time within 2 seconds Only some overshoot permitted

Lead or Phase-Lead Compensator Using Root Locus

A first-order lead compensator can be designed using the root locus. A lead compensator in root locus form is given by

$$G_{c}(s) = \frac{(s+z)}{(s+p)}$$

where the magnitude of z is less than the magnitude of p. A phase-lead compensator tends to shift the root locus toward the left half plane. This results in an improvement in the system's stability and an increase in the response speed.

When a lead compensator is added to a system, the value of this intersection will be a larger negative number than it was before. The net number of zeros and poles will be the same (one zero and one pole are added), but the added pole is a larger negative number than the added zero. Thus, the result of a lead compensator is that the asymptotes' intersection is moved further into the left half plane, and the entire root locus will be shifted to the left. This can increase the region of stability as well as the response speed.
Lead or Phase-Lead Compensator Using Root Locus

In Matlab a phase lead compensator in root locus form is implemented by using the transfer function in the form

```
numlead=kc*[1 z];
denlead=[1 p];
```

and using the ${\tt conv}$ () function to implement it with the numerator and denominator of the plant

```
newnum=conv(num,numlead);
newden=conv(den,denlead);
```

Lead or Phase-Lead Compensator Using Frequency Response

A first-order phase-lead compensator can be designed using the frequency response. A lead compensator in frequency response form is given by

$$G_{c}(s) = \frac{(1 + \alpha \cdot \tau \cdot s)}{\alpha \cdot (1 + \tau \cdot s)} \qquad p = \frac{1}{\tau} \qquad z = \frac{1}{\alpha \tau} \qquad \omega_{m} = \sqrt{z \cdot p} \qquad \sin(\phi_{m}) = \frac{\alpha - 1}{\alpha + 1}$$

In frequency response design, <u>the phase-lead compensator adds positive phase to the</u> <u>system over the frequency range</u>. A bode plot of a phase-lead compensator looks like the following



Lead or Phase-Lead Compensator Using Frequency Response

Additional positive phase increases the phase margin and thus increases the stability of the system. This type of compensator is designed by determining **alfa** from the amount of phase needed to satisfy the phase margin requirements, and determining **tal** to place the added phase at the new gain-crossover frequency.

Another effect of the lead compensator can be seen in the magnitude plot. The lead compensator increases the gain of the system at high frequencies (the amount of this gain is equal to alfa. This can increase the crossover frequency, which will help to decrease the rise time and settling time of the system.

Lead or Phase-Lead Compensator Using Frequency Response

In Matlab, a phase lead compensator in frequency response form is implemented by using the transfer function in the form

```
numlead=[aT 1];
```

```
denlead=[T 1];
```

and using the conv() function to multiply it by the numerator and denominator of the plant

```
newnum=conv(num,numlead);
newden=conv(den,denlead);
```

Lag or Phase-Lag Compensator Using Root Locus

A first-order lag compensator can be designed using the root locus. A lag compensator in root locus form is given by

$$G_{c}(s) = \frac{(s+z)}{(s+p)}$$

where the magnitude of z is greater than the magnitude of p. A phase-lag compensator tends to shift the root locus to the right, which is undesirable. For this reason, the pole and zero of a lag compensator must be placed close together (usually near the origin) so they do not appreciably change the transient response or stability characteristics of the system.

When a lag compensator is added to a system, the value of this intersection will be a smaller negative number than it was before. The net number of zeros and poles will be the same (one zero and one pole are added), but the added pole is a smaller negative number than the added zero. Thus, the result of a lag compensator is that the asymptotes' intersection is moved closer to the right half plane, and the entire root locus will be shifted to the right.

Lag or Phase-Lag Compensator Using Root Locus

It was previously stated that that lag controller should only minimally change the transient response because of its negative effect. If the phase-lag compensator is not supposed to change the transient response noticeably, what is it good for? The answer is that a phase-lag compensator can improve the system's steady-state response. It works in the following manner. At high frequencies, the lag controller will have unity gain. At low frequencies, the gain will be z0/p0 which is greater than 1. This factor z/p will multiply the position, velocity, or acceleration constant (Kp, Kv, or Ka), and the steady-state error will thus decrease by the factor z0/p0. In Matlab, a phase lead compensator in root locus form is implemented by using the transfer function in the form

```
numlag=[1 z];
```

```
denlag=[1 p];
```

and using the conv() function to implement it with the numerator and denominator of the plant

```
newnum=conv(num,numlag);
newden=conv(den,denlag);
```

Lag or Phase-Lag Compensator using Frequency Response

A first-order phase-lag compensator can be designed using the frequency response. A lag compensator in frequency response form is given by

$$G_{c}(s) = \frac{(1 + \alpha \cdot \tau \cdot s)}{\alpha \cdot (1 + \tau \cdot s)}$$

The phase-lag compensator looks similar to a phase-lead compensator, except that a is now less than 1. The main difference is that the lag compensator adds negative phase to the system over the specified frequency range, while a lead compensator adds positive phase over the specified frequency. A bode plot of a phase-lag compensator looks like the following



Lag or Phase-Lag Compensator using Frequency Response

In Matlab, a phase-lag compensator in frequency response form is implemented by using the transfer function in the form

```
numlead=[a*T 1];
denlead=a*[T 1];
```

and using the ${\tt conv}$ () function to implement it with the numerator and denominator of the plant

```
newnum=conv(num,numlead);
newden=conv(den,denlead);
```

Lead-lag Compensator using either Root Locus or Frequency Response

A lead-lag compensator combines the effects of a lead compensator with those of a lag compensator. The result is a system with improved transient response, stability and steady-state error. To implement a lead-lag compensator, first design the lead compensator to achieve the desired transient response and stability, and then add on a lag compensator to improve the steady-state response

Exercise - Dominant Pole-Zero Approximations and Compensations

The influence of a particular pole (or pair of complex poles) on the response is mainly determir by two factors: the real part of the pole and the relative magnitude of the residue at the pole. I real part determines the rate at which the transient term due to the pole decays; the larger the r part, the faster the decay. The relative magnitude of the residue determines the percentage of total response due to a particular pole.

Investigate (using Simulink) the impact of a closed-loop negative real pole on the overshoot of system having complex poles.

$$T(s) = \frac{pr \cdot \omega n^2}{(s + pr) \cdot \left[s^2 + (2 \cdot \zeta \cdot \omega n \cdot s) + \omega n^2\right]}$$

Make pr to vary (2, 3, 5) times the real part of the complex pole for different val ψ (2, 3, 5) times the real part of the complex pole for different val ψ (2, 3, 5) times the real part of the complex pole for different val ψ (2, 3, 5) times the real part of the complex pole for different val ψ (2, 3, 5) times the real part of the complex pole for different val ψ (2, 3, 5) times the real part of the complex pole for different val ψ (2, 3, 5) times the real part of the complex pole for different val ψ (2, 3, 5) times the real part of the complex pole for different val ψ (2, 3, 5) times the real part of the complex pole for different val ψ (2, 3, 5) times the real part of the complex pole for different val ψ (2, 3, 5) times the real part of the complex pole for different val ψ (2, 3, 5) times the real part of the complex pole for different val ψ (2, 3, 5) times the real part of the complex pole for different val ψ (2, 3, 5) times the real part of the complex pole for different val ψ (2, 3, 5) times the real part of the complex pole for different val ψ (3, 5) times the real part of the complex pole for different val ψ (3, 5) times the real part of the complex pole for different val ψ (3, 5) times the real part of the complex pole for different val ψ (3, 5) times the complex pole for different val ψ (3, 5) times the complex pole for different val ψ (4, 5) times the complex pole for different val ψ (4, 5) times the complex pole for different val ψ (4, 5) times the complex pole for different val ψ (4, 5) times the complex pole for different val ψ (4, 5) times the complex pole for different val ψ (4, 5) times the complex pole for different val ψ (4, 5) times the complex pole for different val ψ (4, 5) times the complex pole for different val) times the complex pole for different val ψ (4, 5) times the complex pole for different val ψ (4, 5) times the complex pole for different val ψ (4, 5) times the complex pole for different val ψ (4, 5) times the complex pole for different val ψ

Investigate (using Simulink) the impact of a closed-loop negative real zero on the overshoot of system having complex poles.

$$T(s) = \frac{(s + zr)}{\left[s^2 + \left(2 \cdot \zeta \cdot \omega n \cdot s\right) + \omega n^2\right]}$$

Make zr to vary (2, 3, 5) times the real part of the complex pole for different valű (3.6 f 0.5, 0.7).

Exercise - Lead and Lag Compensation

Investigate (using Matlab and Simulink) the effect of lead and lag compensations on the systems indicated below. Summarize your observations. Plot the root-locus, bode diagra and output for a step input before and after the compensations.

Remember

lead compensation: z<p (place zero below the desired root location or to the left of the fi real poles)

lag compensation: z>p (locate the pole and zero near the origin of the s-plane)

Lead Compensation (use z=1.33, p=20 and K=15).



Lag Compensation (use z=0.09, and p=0.015, K=1/6)



Summarize your findings

Problem 10.36

Determine a compensator so that the percent overshoot is less than 20% and Kv (velocity constant) is greater than 8.







Poles and Zeros and Transfer Functions

Transfer Function:

A transfer function is defined as the ratio of the Laplace transform of the output to the input with all initial conditions equal to zero. Transfer functions are defined only for linear time invariant systems.

Considerations:

Transfer functions can usually be expressed as the ratio of two polynomials in the complex variable, s.

Factorization:

A transfer function can be factored into the following form.

$$G(s) = \frac{K(s+z_1)(s+z_2)...(s+z_m)}{(s+p_1)(s+p_2)...(s+p_m)}$$

The roots of the numerator polynomial are called zeros.

The roots of the denominator polynomial are called poles.

Poles, Zeros and the S-Plane

An Example:

You are given the following transfer function. Show the poles and zeros in the s-plane.

$$G(s) = \frac{(s+8)(s+14)}{s(s+4)(s+10)}$$



Characterization:

Considering the transfer function of the previous slide. We note that we have 4 different types of terms in the previous general form: These are:

$$K_{_{B}}, \frac{1}{s}, \frac{1}{(s/p + 1)}, (s/z + 1)$$

Expressing in dB:

Given the tranfer function:

 $G(jw) = \frac{K_{B}(jw/z+1)}{(jw)(jw/p+1)}$

 $20\log |G(jw|=20\log K_{B}+20\log |(jw/z+1)|-20\log |jw|-20\log |jw/p+1|$

<u>Mechanics</u>: We have 4 distinct terms to consider:

20logK_B

20log|(jw/z +1)|

-20log|jw|

-20log|(jw/p + 1)|



Mechanics:

The gain term, 20logK_B, is just so many dB and this is a straight line on Bode paper, independent of omega (radian frequency).

The term, - 20log|jw| = - 20logw, when plotted on semi-log paper is a straight line sloping at -20dB/decade. It has a magnitude of 0 at w = 1.



Mechanics:

The term, $-20\log|(jw/p + 1)$, is drawn with the following approximation: If w -20\log|(jw/p + 1)| = 0 dB, a flat line on the Bode. If w > p we use the approximation of $-20\log(w/p)$, which slopes at -20dB/dec starting at w = p. Illustrated below. It is easy to show that the plot has an error of -3dB at w = p and -1 dB at w = p/2 and w = 2p. One can easily make these corrections if it is appropriate.



 $\omega = p$

Mechanics:

When we have a term of $20\log|(jw/z + 1)|$ we approximate it be a straight line of slop 0 dB/dec when w < z. We approximate it as $20\log(w/z)$ when w > z, which is a straight line on Bode paper with a slope of + 20dB/dec. Illustrated below.



 $\omega = z$

Example 1:

Given:

$$G(jw) = \frac{50,000(jw+10)}{(jw+1)(jw+500)}$$

<u>First</u>: <u>Always</u>, <u>always</u>, <u>always</u> get the poles and zeros in a form such that the constants are associated with the jw terms. In the above example we do this by factoring out the 10 in the numerator and the 500 in the denominator.

$$G(jw) = \frac{50,000x10(jw/10+1)}{500(jw+1)(jw/500+1)} = \frac{100(jw/10+1)}{(jw+1)(jw/500+1)}$$

<u>Second</u>: When you have neither poles nor zeros at 0, start the Bode at $20\log_{10}K = 20\log_{10}100 = 40$ dB in this case.

Example 1:

(continued)

<u>Third:</u> Observe the order in which the poles and zeros occur.
<u>This is the secret of being able to quickly sketch the Bode.</u>
In this example we first have a pole occurring at 1 which causes the Bode to break at 1 and slope – 20 dB/dec.
Next, we see a zero occurs at 10 and this causes a slope of +20 dB/dec which cancels out the – 20 dB/dec, resulting in a flat line (0 db/dec). Finally, we have a pole that occurs at w = 500 which causes the Bode to slope down at – 20 dB/dec.

We are now ready to draw the Bode.

Before we draw the Bode we should observe the range over which the transfer function has active poles and zeros. This determines the scale we pick for the w (rad/sec) at the bottom of the Bode.

The dB scale depends on the magnitude of the plot and experience is the best teacher here.

Bode Plot Magnitude for 100(1 + jw/10)/(1 + jw/1)(1 + jw/500)



Using Matlab For Frequency Response

Instruction: We can use Matlab to run the frequency response for the previous example. We place the transfer function in the form:

 $\frac{5000(s+10)}{(s+1)(s+500)} = \frac{[5000s+50000]}{[s^2+501s+500]}$

The Matlab Program

num = [5000 50000]; den = [1 501 500]; Bode (num,den)

In the following slide, the resulting magnitude and phase plots (exact) are shown in light color (blue). The approximate plot for the magnitude (Bode) is shown in heavy lines (red). We see the 3 dB errors at the corner frequencies.

Bode Diagrams



Phase for Bode Plots

Comment:

Generally, the phase for a Bode plot is not as easy to draw or approximate as the magnitude. In this course we will use an analytical method for determining the phase if we want to make a sketch of the phase.

Illustration: Consider the transfer function of the previous example. We express the angle as follows:

 $\angle G(jw) = \tan^{-1}(w/10) - \tan^{-1}(w/1) - \tan^{-1}(w/500)$

We are essentially taking the angle of each pole and zero. Each of these are expressed as the $tan^{-1}(j part/real part)$

Usually, about 10 to 15 calculations are sufficient to determine a good idea of what is happening to the phase.

Bode Plots

Example 2:

dB

Given the transfer function. Plot the Bode magnitude.

 $G(s) = \frac{100(1+s/10)}{s(1+s/100)^2}$

Consider first only the two terms of



Which, when expressed in dB, are; 20log100 – 20 logw. This is plotted below.

40 40 20 -20 -20 1 ω (rad/sec)

The is a tentative line we use until we encounter the first pole(s) or zero(s) not at the origin. Bode Plots







 ω (rad/sec)







Bode Plots

Design Problem:

Design a G(s) that has the following Bode plot.

Example 7



wlg
Bode Plots

Procedure: The two break frequencies need to be found. Recall:

 $#dec = log_{10}[w_2/w_1]$

Then we have:

(#dec)(40dB/dec) = 30 dB

 $\log_{10}[w_1/30] = 0.75 \longrightarrow w_1 = 5.33 \text{ rad/sec}$

Also:

 $\log_{10}[w_2/900]$ (-40dB/dec) = - 30dB

This gives $w_2 = 5060 \text{ rad/sec}$

Bode Plots

Procedure:

$$G(s) = \frac{(1+s/5.3)^2 (1+s/5060)^2}{(1+s/30)^2 (1+s/900)^2}$$

<u>Clearing:</u>

$$G(s) = \frac{(s+5.3)^2(s+5060)^2}{(s+30)^2(s+900)^2}$$

Use Matlab and conv:

$$N1 = (s^{2} + 10.6s + 28.1) \qquad N2 = (s^{2} + 10120s + 2.56xe^{7})$$

$$N1 = [1\ 10.6\ 28.1] \qquad N2 = [1\ 10120\ 2.56e+7]$$

$$N = conv(N1,N2)$$

$$1 \quad 1.86e+3 \quad 2.58e+7 \quad 2.73e+8 \quad 7.222e+8$$

$$s^{4} \quad s^{3} \qquad s^{2} \qquad s^{1} \qquad s^{0}$$



Procedure:

The final G(s) is given by;

$$G(s) = \frac{(s^4 + 10130.6s^3 + 2.571e^8s^2 + 2.716e^8s + 7.194e^8)}{(s^4 + 1860s^3 + 9.189e^2s^2 + 5.022e^7s + 7.29e^8)}$$

Testing:

We now want to test the filter. We will check it at $\omega = 5.3$ rad/sec And $\omega = 164$. At $\omega = 5.3$ the filter has a gain of 6 dB or about 2. At $\omega = 164$ the filter has a gain of 30 dB or about 31.6.

We will check this out using MATLAB and particularly, Simulink.

Matlab (Simulink) Model:



Filter Output at ω = 5.3 rad/sec



Produced from Matlab Simulink

Filter Output at ω = 70 rad/sec



Produced from Matlab Simulink

Reverse Bode Plot







Polar Plot

Introduction

The polar plot of sinusoidal transfer function $G(j\omega)$ is a plot of the magnitude of $G(j\omega)$ verses the phase angle of $G(j\omega)$ on polar coordinates as ω is varied from zero to infinity. Therefore it is the locus of as ω is varied from zero to infinity. As So it is the plot of vector $G(j\omega)$ as ω is varied from zero to infinity.

 $|G(j\omega)| \angle G(j\omega) = Me^{j\phi(\omega)}$

 $Me^{j\phi(\omega)}$

Introduction conti...

In the polar plot the magnitude of $G(j\omega)$ is plotted as the distance from the origin while phase angle is measured from positive real axis. + angle is taken for anticlockwise direction. Polar plot is also known as Nyquist Plot.

Steps to draw Polar Plot

Step 1: Determine the T.F G(s) Step 2: Put s=j ω in the G(s) Step 3: At $\omega=0$ & $\omega=\infty$ find by & & Step 4: At $\omega=0$ & $\omega=\infty$ find by & & Step 5: Rationalize the function G(j ω) and separate of the real and imaginary parts $\lim_{\omega \to \infty} |G(j\omega)|$ Step 6: Put Re [G(j ω)]=0, determine the frequency at which plot intersects the Im axis and calculate intersection value by putting the above calculated frequency in G(j ω) $\lim_{\omega \to \infty} \angle G(j\omega)$ Steps to draw Polar Plot conti...

Step 7: Put Im $[G(j\omega)]=0$, determine the frequency at which plot intersects the real axis and calculate intersection value by putting the above calculated frequency in $G(j\omega)$ Step 8: Sketch the Polar Plot with the help of above information

Polar Plot for Type 0 System

Let
$$G(s) = \frac{K}{(1+sT_1)(1+sT_2)}$$

Step 1: Put s=j ω

Step 2: Faking the limit for magnitude of G(j
$$\omega$$
)
= $\frac{K}{\sqrt{1 + (\omega T_1)^2}} \sqrt{1 + (\omega T_2)^2} \angle -\tan^{-1} \omega T_1 - \tan^{-1} \omega T_2$

$$\lim_{\omega \to 0} |G(j\omega)| = \frac{K}{\sqrt{1 + (\omega T_1)^2} \sqrt{1 + j(\omega T_2)^2}} = K$$

Stepling aligning the limit of the Phase Angle of $G(j\omega)$
 $\sqrt{1 + (\omega T_1)^2} \sqrt{1 + j(\omega T_2)^2}$

$$\lim_{\omega \to 0} \angle G(j\omega) = \angle -\tan^{-1} \omega T_1 - \tan^{-1} \omega T_2 = 0$$
$$\lim_{\omega \to \infty} \angle G(j\omega) = \angle -\tan^{-1} \omega T_1 - \tan^{-1} \omega T_2 = -180$$

Step 4: Separate the real and Im part of $G(j\omega)$

Step 5:
$$\mathbb{G}(t_j \mathbb{R}) = 0 \frac{K(1 - \omega^2 T_1 T_2)}{1 + \omega^2 T_1^2 + \omega^2 T_2^2 + \omega^4 T_1 T_2} - j \frac{K\omega(T_1 + T_2)}{1 + \omega^2 T_1^2 + \omega^2 T_2^2 + \omega^4 T_1 T_2}$$

$$\frac{K(1-\omega^2 T_1 T_2)}{1+\omega^2 T_1^2+\omega^2 T_2^2+\omega^4 T_1 T_2} = 0 \Longrightarrow \omega = \frac{1}{\sqrt{T_1 T_2}} \& \omega = \infty$$

So When

$$\omega = \frac{1}{\sqrt{T_1 T_2}} \Rightarrow G(j\omega) = \frac{K\sqrt{T_1 T_2}}{T_1 + T_2} \angle -90^{\circ}$$

& $\omega = \infty \qquad \Rightarrow G(j\omega) = 0 \angle -180^{\circ}$

Step 6: Put Im [G(j ω)]=0 $\frac{K\omega(T_1 + T_2)}{1 + \omega^2 T_1^2 + \omega^2 T_2^2 + \omega^4 T_1 T_2} = 0 \Rightarrow \omega = 0 \& \pm \infty$ So When $\omega = 0 \Rightarrow G(j\omega) = K \angle 0^0$

 $\omega = \infty \Longrightarrow G(j\omega) = 0 \angle 180^{\circ}$



Figure 1: Polar Plot of Type 0 System

Polar Plot for Type 1 System

Let
$$G(s) = \frac{K}{s(1+sT_1)(1+sT_2)}$$

Step 1: Put s=j ω

$$G(j\omega) = \frac{K}{j\omega(1+j\omega T_1)(1+j\omega T_2)}$$
$$= \frac{K}{\omega\sqrt{1+(\omega T_1)^2}\sqrt{1+j(\omega T_2)^2}} \angle -90^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

Step 2: Taking the limit for magnitude of G(j
$$\omega$$
)

$$\lim_{\omega \to 0} |G(j\omega)| = \frac{K}{\omega\sqrt{1 + (\omega T_1)^2}\sqrt{1 + j(\omega T_2)^2}} = \infty$$

Step 3: Taking the limit of the Phase Angle of
$$G(j\omega)$$

 $\omega \rightarrow \infty$ $\omega \sqrt{1 + (\omega T_1)^2} \sqrt{1 + j(\omega T_2)^2}$

$$\lim_{\omega \to 0} \angle G(j\omega) = \angle -90^{\circ} - \tan^{-1} \omega T_1 - \tan^{-1} \omega T_2 = -90^{\circ}$$
$$\lim_{\omega \to \infty} \angle G(j\omega) = \angle -90^{\circ} - \tan^{-1} \omega T_1 - \tan^{-1} \omega T_2 = -270^{\circ}$$

Step 4: Separate the real and Im part of $G(j\omega)$

Step 5: Put
$$\mathbf{R}(\mathbf{g}[\mathbf{G}(\mathbf{j}\omega)] = 0 - \omega K(T_1 + T_2) + j \frac{j(K\omega^2 T_1 T_2 - K)}{\omega + \omega^3 (T_1^2 + T_2^2 + \omega^2 T_1^2 T_2^2)} + j \frac{j(K\omega^2 T_1 T_2 - K)}{\omega + \omega^3 (T_1^2 + T_2^2 + \omega^2 T_1^2 T_2^2)}$$

$$\frac{-\omega K(T_1 + T_2)}{\omega + \omega^3 (T_1^2 + T_2^2 + \omega^2 T_1^2 T_2^2)} = 0 \Longrightarrow \omega = \infty$$

So at
$$\omega = \infty \qquad \Rightarrow G(j\omega) = 0 \angle -270^0$$

Step 6: Put Im [G(j
$$\omega$$
)]=0

$$\frac{j(K\omega^2 T_1 T_2 - K)}{\omega + \omega^3 (T_1^2 + T_2^2 + \omega^2 T_1^2 T_2^2)} = 0 \implies \omega = \frac{1}{\sqrt{T_1 T_2}} \& \omega = \pm \infty$$

So When

$$\omega = \frac{1}{\sqrt{T_1 T_2}} \Rightarrow \quad G(j\omega) = -\frac{K\sqrt{T_1 T_2}}{T_1 + T_2} \angle 0^0$$
$$\omega = \infty \Rightarrow \qquad G(j\omega) = \infty \angle 0^0$$



Polar Plot for Type 2 System

Let

Similar to above

$$G(s) = \frac{K}{s^2 (1 + sT_1)(1 + sT_2)}$$



Figure 3: Polar Plot of Type 2 System

Note: Introduction of additional pole in denominator contributes a constant - 180° to the angle of G(j ω) for all frequencies. See the figure 1, 2 & 3 Figure 1+(-180^o Rotation)=figure 2 Figure 2+(-180^o Rotation)=figure 3 Ex: Sketch the polar plot for G(s)=20/s(s+1)(s+2) Solution: Step 1: Put s=j ω

$$G(j\omega) = \frac{20}{j\omega(j\omega+1)(j\omega+2)}$$
$$= \frac{20}{\omega\sqrt{\omega^2+1}\sqrt{\omega^2+4}} \angle -90^0 - \tan^{-1}\omega - \tan^{-1}\omega/2$$

Step 2: Taking the limit for magnitude of $G(j\omega)$

$$\lim_{\omega \to 0} |G(j\omega)| = \frac{20}{\omega\sqrt{\omega^2 + 1}\sqrt{\omega^2 + 4}} = \infty$$
$$\lim_{\omega \to \infty} |G(j\omega)| = \frac{20}{\omega\sqrt{\omega^2 + 1}\sqrt{\omega^2 + 4}} = 0$$

Step 3: Taking the limit of the Phase Angle of $G(j\omega)$

$$\lim_{\omega \to 0} \angle G(j\omega) = \angle -90^{\circ} - \tan^{-1}\omega - \tan^{-1}\omega/2 = -90^{\circ}$$
$$\lim_{\omega \to \infty} \angle G(j\omega) = \angle -90^{\circ} - \tan^{-1}\omega - \tan^{-1}\omega/2 = -270^{\circ}$$

Step 4: Separate the real and Im part of $G(j\omega)$

$$G(j\omega) = \frac{-60\omega^2}{(\omega^4 + \omega^2)(4 + \omega^2)} + j\frac{j20(\omega^3 - 2\omega)}{(\omega^4 + \omega^2)(4 + \omega^2)}$$

$$\frac{-60\omega^2}{(\omega^4 + \omega^2)(4 + \omega^2)} = 0 \Rightarrow \omega = \infty$$

So at
$$\omega = \infty \qquad \Rightarrow G(j\omega) = 0 \angle -270^0$$

Step 6: Put Im $[G(j\omega)]=0$

$$\frac{j20(\omega^{3} - 2\omega)}{(\omega^{4} + \omega^{2})(4 + \omega^{2})} = 0 \Rightarrow \omega = \pm\sqrt{2} \& \omega = \pm\infty$$

So for positive value of ω
 $\omega = \sqrt{2} \Rightarrow \quad G(j\omega) = -\frac{10}{3} \angle 0^{0}$
 $\omega = \infty \Rightarrow \qquad G(j\omega) = 0 \angle 0^{0}$



Figure 4: Polar Plot G(s)=20/s(s+1)(s+2)

Gain Margin, Phase Margin & Stability



Phase Crossover Frequency (ω_p) : The frequency where a polar plot intersects the –ve real axis is called phase crossover frequency Gain Crossover Frequency (ω_g) : The frequency where a polar plot intersects the unit circle is called gain crossover frequency

So at $\omega_{\sf g}$

$$|G(j\omega)| = Unity$$

Phase Margin (PM):

Phase margin is that amount of additional phase lag at the gain crossover frequency required to bring the system to the verge of instability (marginally stabile) Φ_m =180⁰+ Φ

Where

if if $\begin{array}{l} \Phi= \angle G(j\omega_g) \\ \Phi_m > 0 => + \mathsf{PM} \\ \Phi_m < 0 => - \mathsf{PM} \end{array}$

(Stable System) (Unstable System)

Gain Margin (GM): The gain margin is the reciprocal of magnitude $G(j\omega)$ the frequency at which the phase angle is -180°.

In terms of dB

$$GM = \frac{1}{|G(jwc)|} = \frac{1}{x}$$

$$GM in \ dB = 20\log_{10} \frac{1}{|G(jwc)|} = -20\log_{10} |G(jwc)| = -20\log_{10}(x)$$

Stable: If critical point (-1+j0) is within the plot as shown, Both GM & PM are +ve



Stable system: +GM, +PM

GM=20log₁₀(1
Unstable: If critical point (-1+j0) is outside the plot as shown, Both GM & PM are -ve



GM=20log₁₀(1

Marginally Stable System: If critical point (-1+j0) is on the plot as shown, Both GM & PM are ZERO

GM=20log₁₀(1



Marginally stable system: GM=0 dB, PM=0^o

```
>> [Gm,Pm,Wcg,Wcp] = margin(g)
Gm =
  2.5000
                   G(s) = \frac{10}{s(s/5+1)(s/20+1)}
Pm =
 22.5359
Wcg =
  10.0000 (phase crossover freq.)
Wcp =
  6.0783 (gain crossover freq.)
```

The inverse polar plot of G(j ω) is a graph of 1/G(j ω) as a function of ω . **Ex:** if G(j ω) =1/j ω then 1/G(j ω)=j ω

 $\lim_{\omega \to 0} |G(j\omega)^{-1}| = 0$ $\lim_{\omega \to \infty} |G(j\omega)^{-1}| = \infty$ $-270^{\circ} \quad \text{Im}$ $(\omega \text{ increasing})$ $(\omega = 0) \quad (\omega = 0) \quad \text{Re}$ $-180^{\circ} \quad (-90)^{\circ}$ Inverse polar Plot of 1/j(ω)

Knowledge Before Studying Nyquist Criterion



unstable if there is any pole on RHP (right half plane)

$$G(s) = \frac{N_G(s)}{D_G(s)} \qquad \qquad H(s) = \frac{N_H(s)}{D_H(s)}$$

Open-loop system:

$$G(s)H(s) = \frac{N_G(s) N_H(s)}{D_G(s) D_H(s)}$$

Characteristic equation:

$$1 + G(s)H(s) = 1 + \frac{N_G N_H}{D_G D_H} = \frac{D_G D_H + N_G N_H}{D_G D_H}$$

poles of G(s)H(s) and 1+G(s)H(s) are the same

Closed-loop system:

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{N_G(s)D_H(s)}{D_G(s)D_H(s) + N_G(s)N_H(s)}$$

zero of 1+G(s)H(s) is pole of T(s)

$$G(s)H(s) = \frac{(s-1)(s-2)(s-3)(s-4)}{(s-5)(s-6)(s-7)(s-8)}$$



To know stability, we have to know a,b,c,d

Stability from Nyquist plot

From a Nyquist plot, we can tell a number of closed-loop poles on the right half plane.

If there is any closed-loop pole on the right half plane, the system goes unstable.

If there is no closed-loop pole on the right half plane, the system is stable.

Nyquist Criterion

Nyquist plot is a plot used to verify stability of the system.

mapping contour

function
$$F(s) = \frac{(s-z_1)(s-z_2)}{(s-p_1)(s-p_2)}$$

mapping all points (contour) from one plane to another by function F(s).

$$F(s) = \frac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)}$$





(e)

Pole/zero inside the contour has 360 deg. angular change. $R = \frac{1}{V}$ Pole/zero outside contour has 0 deg. angular change. Move clockwise around contour, zero inside yields rotation in clockwise, pole inside yields rotation in counterclockwise

Characteristic equation





N = # of counterclockwise direction about the origin

P = # of poles of characteristic equation inside contour

= # of poles of open-loop system

- z = # of zeros of characteristic equation inside contour
 - = # of poles of closed-loop system

Z = P-N



Characteristic equation

Increase size of the contour to cover the right half plane



Nyquist diagram of G(s)H(s)

'Open-loop system'

Mapping from characteristic equ. to open-loop system by shifting to the left one step

Z = # of closed-loop poles inside the right half planeP = # of open-loop poles inside the right half planeN = # of counterclockwise revolutions around -1





Location not known

Location is known

If there is a gain, K, in front of open-loop transfer function, the Nyquist plot will expand by a factor of K.



Nyquist plot example



Open loop system has pole at 2

$$G(s) = \frac{1}{\operatorname{as-pg}}$$

If we multiply the open-loop with a gain, K, the<u>n ຜູ້ເຈີ can_move</u> the closedloop pole's ອອອາໂເວິກ to the left-half plane Nyquist plot example (cont.)

New look of open-loop system:

$$G(s) = \frac{K}{s-2}$$

Corresponding closed-loop system:

Evaluate value of K for stability
$$(K-2)$$

$$K \ge 2$$

Adjusting an open-loop gain to guarantee stability



Step I: sketch a Nyquist Diagram

Step II: find a range of K that makes the system stable!

How to make a Nyquist plot?

Easy way by Matlab Nyquist: 'nyquist' Bode: 'bode'

Step I: make a Nyquist plot

Starts from an open-loop transfer function (set K=1) Set and find frequency response At dc, $s = j\omega$ Find at which the imaginary part equals zero $\omega = 0 \rightarrow s = 0$ ω

$$G(s)H(s) = \frac{(s+3)(s+5)}{(s-2)(s-4)} = \frac{s^2 + 8s + 15}{s^2 - 6s + 8}$$

$$G(j\omega)H(j\omega) = \frac{-\omega^2 + 8j\omega + 15}{-\omega^2 - 6j\omega + 8} = \frac{(15 - \omega^2) + 8j\omega}{(8 - \omega^2) - 6j\omega}$$

$$= \frac{(15 - \omega^2) + 8j\omega}{(8 - \omega^2) - 6j\omega} \times \frac{(8 - \omega^2) + 6j\omega}{(8 - \omega^2) + 6j\omega}$$

$$= \frac{(15 - \omega^2)(8 - \omega^2) - 48\omega^2 + j(154\omega - 14\omega^3)}{(8 - \omega^2)^2 + 6^2\omega^2}$$

Need the imaginary term = 0,

And get

 $\omega = 0, \sqrt{11}$

back in to the transfer function Substitute G(s) = -1.33

$$\frac{(15-11)(8-11)-48(11)}{(8-11)^2+6^2(11)} = \frac{-540}{412} = -1.31$$



Step II: satisfying stability condition

P = 2, N has to be 2 to guarantee stability Marginally stable if the plot intersects -1 For stability, 1.33K has to be greater than 1

K > 1/1.33

Example

Evaluate a range of K that makes the system stable



Step I: find frequency at which imaginary part = 0

Set
$$S = j\omega$$

$$G(j\omega) = \frac{K}{((j\omega)^2 + 2j\omega + 2)(j\omega + 2)}$$

$$= \frac{4(1 - \omega^2) - j\omega(6 - \omega^2)}{16(1 - \omega^2)^2 + \omega^2(6 - \omega^2)^2}$$

At $\omega = 0, \sqrt{6}$ the imaginary part = 0 Plug $\omega = b\sqrt[3]{6}k$ in the transfer function and get G = -0.05 **Step II: consider stability condition**

P = 0, N has to be 0 to guarantee stability Marginally stable if the plot intersects -1 For stability, 0.05K has to be less than 1

K < 1/0.05

Gain margin is the change in open-loop gain (in dB), required at 180 of phase shift to make the closed-loop system unstable.

Phase margin is the change in open-loop phase shift, required at unity gain to make the closed-loop system unstable.

GM/PM tells how much system can tolerate before going unstable!!!

GM and PM via Nyquist plot



GM and PM via Bode Plot



Example

Find Bode Plot and evaluate a value of K that makes the system stable The system has a unity feedback with an open-loop transfer function

$$G(s) = \frac{K}{(s+2)(s+4)(s+5)}$$

First, let's find Bode Plot of G(s) by assuming that K=40 (the value at which magnitude plot starts from 0 dB)



At phase = -180, ω = 7 rad/sec, magnitude = -20 dB

GM>0, system is stable!!! Can increase gain up 20 dB without causing instability (20dB = 10) Start from K = 40 with K < 400, system is stable

Closed-loop transient and closed-loop frequency responses

'2nd system'



$$\frac{C(s)}{R(s)} = T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Damping ratio and closed-loop frequency response



Magnitude Plot of closed-loop system

Response speed and closed-loop frequency response

$$\begin{split} \omega_{BW} &= \omega_n \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \\ \omega_{BW} &= \frac{4}{T_s \zeta} \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \\ \omega_{BW} &= \frac{\pi}{T_p \sqrt{1 - \zeta^2}} \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \end{split}$$

 \mathcal{O}_{BW} frequency at which magnitude is 3dB down

from value at dc (0 rad/sec), or

$$M = \frac{1}{\sqrt{2}}$$
Findfrom ω_{BW} Open-loopFrequencyResponse



From open-loop frequency response, we can find ω_{BW} at the open-loop frequency that the magnitude lies between -6dB to -7.5dB (phase between -135 to -225)

Relationship between damping ratio and phase margin of open-loop frequency response

Phase margin of open-loop frequency response Can be written in terms of damping ratio as following

$$\phi_{M} = \tan^{-1} \frac{2\zeta}{\sqrt{-2\zeta^{2} + \sqrt{1 + 4\zeta^{4}}}}$$

Open-loop system with a unity feedback has a bode plot below, approximate settling time and peak time



$$\phi_M = \tan^{-1} \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}}$$
$$\zeta = 0.32$$

Solve for PM = 35

$$T_{s} = \frac{4}{\omega_{BW}\zeta} \sqrt{(1 - 2\zeta^{2}) + \sqrt{4\zeta^{4} - 4\zeta^{2} + 2}}$$

= 5.5
$$T_{p} = \frac{\pi}{\omega_{BW}\sqrt{1 - \zeta^{2}}} \sqrt{(1 - 2\zeta^{2}) + \sqrt{4\zeta^{4} - 4\zeta^{2} + 2}}$$

= 1.43



STATE SPACE REPRESENTATION

Objectives

How to find mathematical model, called a state-space representation, for a linear, time-invariant system How to convert between transfer function and state space models How to find the solution of state equations for homogeneous &non homogeneous systems



Two approaches for analysis and design of control system

- 1. Classical Technique or Frequency Domain Technique
- 2. Modern Technique or Time Domain Technique

Some definitions

•System variable : any variable that responds to an input or initial conditions in a system

State variables : the smallest set of linearly independent system variables such that the values of the members of the set at time t0 along with known forcing functions completely determine the value of all system variables for all t ≥ t0
State vector : a vector whose elements are the state variables
State space : the n-dimensional space whose axes are the state variables
State equations : a set of first-order differential equations with b variables, where the n variables to be solved are the state variables

•Output equation : the algebraic equation that expresses the output variables of a system as linear combination of the state variables and the inputs.

•For *n*th-order, write *n simultaneous, first-order differential equations* in terms of the state variables (*state equations*).

•If we know the initial condition of all of the state variables at as well as the system input for , we can solve the equations

Graphic representation of state space and a state vector



For a dynamic system, the state of a system is described in terms of a set of state variables

$$[x_1(t) \ x_2(t) \ \dots \ x_n(t)]$$

The state variables are those variables that determine the future behavior of a system when the present state of the system and the excitation signals are known. Consider the system shown in Figure 1, where $y_1(t)$ and $y_2(t)$ are the output signals and $u_1(t)$ and $u_2(t)$ are the input signals. A set of state variables $[x_1 x_2 ... x_n]$ for the system shown in the figure is a set such that knowledge of the initial values of the state variables $[x_1(t_0) x_2(t_0) ... x_n(t_0)]$ at the initial time t_0 , and of the input signals $u_1(t)$ and $u_2(t)$ for t>= t_0 , suffices to determine the future values of the outputs and state variables.



Figure 1. Dynamic system.

In an actual system, there are several choices of a set of state variables that specify the *energy stored in a system* and therefore adequately describe the dynamics of the system.

The state variables of a system characterize the dynamic behavior of a system. The engineer's interest is primarily in physical, where the variables are voltages, currents, velocities, positions, pressures, temperatures, and similar physical variables.

The State Differential Equation:

The state of a system is described by the set of first-order differential equations written in terms of the state variables $[x_1 \ x_2 \ \dots \ x_n]$. These first-order differential equations can be written in general form as

$$\dot{x}_{1} = a_{11}x_{1} + a_{12}x_{2} + \dots + a_{1n}x_{n} + b_{11}u_{1} + \dots + b_{1m}u_{m}$$
$$\dot{x}_{2} = a_{21}x_{1} + a_{22}x_{2} + \dots + a_{2n}x_{n} + b_{21}u_{1} + \dots + b_{2m}u_{m}$$
$$\vdots$$
$$\dot{x}_{n} = a_{n1}x_{1} + a_{n2}x_{2} + \dots + a_{nn}x_{n} + b_{n1}u_{1} + \dots + b_{nm}u_{m}$$

Thus, this set of simultaneous differential equations can be written in matrix form as follows:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \cdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \cdots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

n: number of state variables, m: number of inputs.

The column matrix consisting of the state variables is called the **state vector** and is written as

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}$$

The vector of input signals is defined as u. Then the system can be represented by the compact notation of the state differential equation as

$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u}$

This differential equation is also commonly called the state equation. The matrix **A** is an nxn square matrix, and **B** is an nxm matrix. The state differential equation relates the rate of change of the state of the system to the state of the system and the input signals. In general, the outputs of a linear system can be related to the state variables and the input signals by the output equation



Where **y** is the set of output signals expressed in column vector form. The state-space representation (or state-variable representation) is comprised of the state variable differential equation and the output equation.

General State Representation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

State equation

output equation

- \mathcal{X} = state vector
- \mathcal{X} = derivative of the state vector with respect to time
- *Y* = output vector
- \mathcal{U} = input or control vector
- A = system matrix
- $B_{= \text{ input matrix}}$
- *C* = output matrix
- *D* = feedforward matrix

AN EXAMPLE OF THE STATE VARIABLE CHARACTERIZATION OF A SYSTEM



The state of the system can be described in terms of a set of variables [x₁ x₂], where x₁ is the capacitor voltage v_c(t) and x₂ is equal to the inductor current i_L(t). This choice of state variables is intuitively satisfactory because the stored energy of the network can be described in terms of these variables.

Therefore $x_1(t_0)$ and $x_2(t_0)$ represent the total initial energy of the network and thus the state of the system at $t=t_0$.

Utilizing Kirchhoff's current low at the junction, we obtain a first order differential equation by describing the rate of change of capacitor voltage

$$\mathbf{i}_{c} = \mathbf{C} \frac{\mathrm{d}\mathbf{v}_{c}}{\mathrm{d}t} = \mathbf{u}(t) - \mathbf{i}_{L}$$

Kirchhoff's voltage low for the right-hand loop provides the equation describing the rate of change of inducator current as

$$L\frac{di_{L}}{dt} = -Ri_{L} + v_{c}$$

The output of the system is represented by the linear algebraic equation

$$\mathbf{v}_0 = \mathbf{R}\,\mathbf{i}_{\rm L}(t)$$

We can write the equations as a set of two first order differential equations in terms of the state variables $x_1 [v_c(t)]$ and $x_2 [i_l(t)]$ as follows:



The output signal is then

$$y_1(t) = v_0(t) = R x_2$$

Utilizing the first-order differential equations and the initial conditions of the network represented by $[x_1(t_0) x_2(t_0)]$, we can determine the system's future and its output.

The state variables that describe a system are not a unique set, and several alternative sets of state variables can be chosen. For the RLC circuit, we might choose the set of state variables as the two voltages, $v_c(t)$ and $v_l(t)$.

We can write the state variable differential equation for the RLC circuit as

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} \mathbf{u}(t)$$

and the output as

$$\mathbf{y} = \begin{bmatrix} \mathbf{0} & \mathbf{R} \end{bmatrix} \mathbf{x}$$

RLC network



1. State variables i(t) q(t)

- 3. q(t) i(t) an be solved using Laplace Transform
- 4. Other network variables can be obtained

$$v_L(t) = -\frac{1}{C}q(t) - Ri(t) + v(t)$$

(1)

(2)

5. (1),(2) : state-space representation

Other variables $v_R(t) v_C(t)$

$$\frac{dv_R}{dt} = -\frac{R}{L}v_R - \frac{R}{L}v_C + \frac{R}{L}v(t)$$
$$\frac{dv_C}{dt} = \frac{1}{RC}v_R$$

Each variables : linearly independent

In vector-matrix form

$$\dot{x} = Ax + Bu \tag{1}$$

where

$$\dot{x} = \begin{bmatrix} dq/dt \\ di/dt \end{bmatrix} A = \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix}$$
$$x = \begin{bmatrix} q \\ i \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1/L \end{bmatrix} \qquad u = v(t)$$

$$y = Cx + Du \tag{2}$$

where

$$y = v_L(t)$$
 $C = [-1/C - R]$ $D = 1$

State space representation using phase variable form

$$\frac{d^{n} y}{dt^{n}} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_{1} \frac{dy}{dt} + a_{0} y = b_{0} u$$

choose

$$x_1 = y$$
 $x_2 = \frac{dy}{dt}$ $x_3 = \frac{d^2 y}{dt^2}$ • • • $x_n = \frac{d^{n-1} y}{dt^{n-1}}$





$$y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Example : TF to State Space

1. Inverse Laplace

 $\ddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$

2. Select state variables



Decomposing a transfer function







Example



$$y(t) = 2x_1 + 7x_2 + x_3$$

$$\downarrow$$

$$y = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

State Space to TF

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

→ Laplace Transform sX(s) = AX(s) + BU(s)Y(s) = CX(s) + DU(s)

$$\longrightarrow X(s) = (sI - A)^{-1} BU(s)$$
$$\longrightarrow Y(s) = \left[C(sI - A)^{-1} B + D\right] U(s)$$
$$T(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1} B + D$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$
$$\longrightarrow (sI - A) = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s + 3 \end{bmatrix}$$
$$(sI - A)^{-1} = \frac{adj(sI - A)}{det(sI - A)}$$

$$=\frac{\begin{bmatrix} (s^2+3s+2) & s+3 & 1\\ -1 & s(s+3) & s\\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3+3s^2+2s+1}$$

$$T(s) = C(sI - A)^{-1}B + D$$

$$\longrightarrow T(s) = \frac{10(s^{2} + 3s + 2)}{s^{3} + 3s^{2} + 2s + 1}$$

Solution of homogeneous state equation

The solution of the state differential equation can be obtained in a manner similar to the approach we utilize for solving a first order differential equation. Consider the first-order differential equation

$$\dot{x} = ax; x(0) = 0$$

$$dx = axdt$$

Where x(t) and u(t) are scalar functions of time. We expect an exponential solution of the form e^{at.} Taking the Laplace transform of both sides, we have

 \dot{x} on integrating above equation

$$\log x = at + c$$

 $X = e^{at.} e^{c}$
 $x = x(0) = e^{c}$

on substituting the intial condition ,the solution of homogeneous state equation of first order differential equation is

$$x = e^{at} x(0)$$
$$\dot{x} = AX(t), x(0) = 0$$
$$e^{At} = \mathbf{I} + At + \frac{A^2 t^2}{2!} + \dots + \frac{A^k t^k}{k!} + \dots$$

Solution of homogeneous state equation

The solution of the state differential equation can be obtained in a manner similar to the approach we utilize for solving a first order differential equation. Consider the first-order differential equation

$$\dot{x} = ax + bu; x(0) = 0$$

Where x(t) and u(t) are scalar functions of time. By taking laplace transform

$$s X(s) - x_0 = a X(s) + b U(s)$$

The inverse Laplace transform of X(s) results in the solution

$$x(t) = e^{at}x(0) + \int_{0}^{t} e^{a(t-\tau)}bu(\tau)d\tau$$

We expect the solution of the state differential equation to be similar to x(t) and to be of differential form. The **matrix exponential function** is defined as

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \dots + \frac{A^kt^k}{k!} + \dots$$
which converges for all finite t and any A. Then the solution of the state differential equation is found to be

$$x(t) = e^{At}x(0) + \int_{0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$
$$X(s) = [sI - A]^{-1}x(0) + [sI - A]^{-1}BU(s)$$

where we note that $[sI-A]^{-1}=\varphi(s)$, which is the Laplace transform of $\varphi(t)=e^{At}$. The matrix exponential function $\varphi(t)$ describes the unforced response of the system and is called the fundamental or state transition matrix.

$$\mathbf{x}(t) = \phi(t) \mathbf{x}(0) + \int_{0}^{t} \phi(t-\tau) \mathbf{B} \mathbf{u}(\tau) d\tau$$

THE TRANSFER FUNCTION FROM THE STATE EQUATION

The transfer function of a single input-single output (SISO) system can be obtained from the state variable equations.

 $\dot{\mathbf{x}} = \mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{u}$ $\mathbf{y} = \mathbf{C} \mathbf{x}$

where y is the single output and u is the single input. The Laplace transform of the equations

$$sX(s) = AX(s) + BU(s)$$

 $Y(s) = CX(s)$

where B is an nx1 matrix, since u is a single input. We do not include initial conditions, since we seek the transfer function. Reordering the equation

$$[sI-A]X(s) = BU(s)$$
$$X(s) = [sI-A]^{-1}BU(s) = \phi(s)BU(s)$$
$$Y(s) = C\phi(s)BU(s)$$

Therefore, the transfer function G(s)=Y(s)/U(s) is

$$\mathbf{G}(\mathbf{s}) = \mathbf{C}\phi(\mathbf{s})\mathbf{B}$$

Example:

Determine the transfer function G(s)=Y(s)/U(s) for the RLC circuit as described by the state differential function

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} \mathbf{u} \quad , \quad \mathbf{y} = \begin{bmatrix} 0 & R \end{bmatrix} \mathbf{x}$$

$$[sI-A] = \begin{bmatrix} s & \frac{1}{C} \\ -\frac{1}{L} & s + \frac{R}{L} \end{bmatrix} \qquad \qquad \varphi(s) = [sI-A]^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} s + \frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & s \end{bmatrix}$$
$$\Delta(s) = s^2 + \frac{R}{L}s + \frac{1}{LC}$$

Then the transfer function is

$$G(s) = \begin{bmatrix} 0 & R \end{bmatrix} \begin{bmatrix} s + \frac{R}{L} & -\frac{1}{C\Delta(s)} \\ \frac{1}{\Delta(s)} & \frac{-1}{C\Delta(s)} \\ \frac{1}{L\Delta(s)} & \frac{-1}{\Delta(s)} \end{bmatrix} \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}$$
$$G(s) = \frac{R/LC}{\Delta(s)} = \frac{R/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

CONSIDER THE SYSTEM

$$\frac{C(s)}{R(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24}$$

$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s)$$

$$\ddot{c} + 9\ddot{c} + 26\dot{c} + 24\underline{c} = 24r$$
State variables
$$x_1 = c$$

$$x_2 = \dot{c}$$

$$x_3 = \ddot{c}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r$$

$$y = c = x_1$$
Output equation

$$\dot{x}_{1} = x_{2}$$

$$\dot{x}_{2} = x_{3}$$

$$\dot{x}_{2} = -24x_{1} - 26x_{2} - 9x_{3} + 24r$$

$$y = c = x_{1}$$

$$\begin{bmatrix} \dot{x}_{1} \\ \dot{x}_{2} \\ \dot{x}_{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_{1} \\ x_{2} \\ x_{3} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{1} \\ x_{2} \\ x_{3} \end{bmatrix}$$

THANK YOU