Control Systems
In this chapter we describe a general process for designing a control system.

A control system consisting of interconnected components is designed to achieve a desired purpose. To understand the purpose of a control system, it is useful to examine examples of control systems through the course of history. These early systems incorporated many of the same ideas of feedback that are in use today.

Modern control engineering practice includes the use of control design strategies for improving manufacturing processes, the efficiency of energy use, advanced automobile control, including rapid transit, among others.

We also discuss the notion of a design gap. The gap exists between the complex physical system under investigation and the model used in the control system synthesis.

The iterative nature of design allows us to handle the design gap effectively while accomplishing necessary tradeoffs in complexity, performance, and cost in order to meet the design specifications.
Introduction

**System** – An interconnection of elements and devices for a desired purpose.

**Control System** – An interconnection of components forming a system configuration that will provide a desired response.

**Process** – The device, plant, or system under control. The input and output relationship represents the cause-and-effect relationship of the process.
Introduction

**Open-Loop Control Systems** utilize a controller or control actuator to obtain the desired response.

**Closed-Loop Control Systems** utilizes feedback to compare the actual output to the desired output response.

**Multivariable Control System**
History

Greece (BC) – Float regulator mechanism
Holland (16th Century) – Temperature regulator

Watt’s Flyball Governor (18th century)
History

Water-level float regulator
Closed-loop feedback system.
History

**18th Century** James Watt’s centrifugal governor for the speed control of a steam engine.

**1920s** Minorsky worked on automatic controllers for steering ships.

**1930s** Nyquist developed a method for analyzing the stability of controlled systems.

**1940s** Frequency response methods made it possible to design linear closed-loop control systems.

**1950s** Root-locus method due to Evans was fully developed.

**1960s** State space methods, optimal control, adaptive control and

**1980s** Learning controls are begun to investigated and developed.

**Present** and on-going research fields. Recent application of modern control theory includes such non-engineering systems such as biological, biomedical, economic and socio-economic systems.

?????????????????????????????????????????????????????
Examples of Modern Control Systems

(a) Automobile steering control system.

(b) The driver uses the difference between the actual and the desired direction of travel to generate a controlled adjustment of the steering wheel.

(c) Typical direction-of-travel response.
Examples of Modern Control Systems

A negative feedback system block diagram depicting a basic closed-loop control system. The control device is often called a “controller.”
Examples of Modern Control Systems

A manual control system for regulating the level of fluid in a tank by adjusting the output valve. The operator views the level of fluid through a port in the side of the tank.
Examples of Modern Control Systems

A three-axis control system for inspecting individual semiconductor wafers with a highly sensitive camera.
Examples of Modern Control Systems

Coordinated control system for a boiler-generator.
Examples of Modern Control Systems

A computer control system.
Examples of Modern Control Systems

The Utah/MIT Dextrous Robotic Hand: A dextrous robotic hand having 18 degrees of freedom, developed as a research tool by the Center for Engineering Design at the University of Utah and the Artificial Intelligence Laboratory at MIT. It is controlled by five Motorola 68000 microprocessors and actuated by 36 high-performance electropneumatic actuators via high-strength polymeric tendons. The hand has three fingers and a thumb. It uses touch sensors and tendons for control.
(Photograph by Michael Milochik. Courtesy of University of Utah.)
Examples of Modern Control Systems

A feedback control system model of the national income.
Examples of Modern Control Systems

A laboratory robot used for sample preparation. The robot manipulates small objects, such as test tubes, and probes in and out of tight places at relatively high speeds [41].

(© Copyright 1993 Hewlett-Packard Company. Reproduced with permission.)
The Future of Control Systems

The Honda P3 humanoid robot. P3 walks, climbs stairs and turns corners.
Photo courtesy of American Honda Motor, Inc.
The Future of Control Systems

Future evolution of control systems and robotics.

- Fixed Automation
- NC Machines
- Robotics
- Power tools
- Hand tools
- Extended tools
- Unilateral manipulators
- Mechanical master/slave manipulators
- Control systems (programmable)
- Digital control systems

Intelligent Systems

- Extensive flexibility and autonomy
- Improvements:
  - Sensors
  - Vision
  - Languages
  - Artificial intelligence
  - Vision
  - Man-machine interface
  - Supervisory control
Control System Design

1. Establish control goals
2. Identify the variables to control
3. Write the specifications for the variables
4. Establish the system configuration and identify the actuator
5. Obtain a model of the process, the actuator, and the sensor
6. Describe a controller and select key parameters to be adjusted
7. Optimize the parameters and analyze the performance

If the performance does not meet the specifications, then iterate the configuration and the actuator.

If the performance meets the specifications, then finalize the design.

The control system design process.
Design Example

Input angle, $\theta_i(t)$  + 50 volts

Output voltage, $v_o(t)$
Design Example

ELECTRIC SHIP CONCEPT

Vision

Integrated Power System
- Electric Drive
- Reduce # of Prime Movers
- Fuel savings
- Reduced maintenance

All Electric Ship
- Reduced Manning
- Automation
- Eliminate auxiliary systems (steam, hydraulics, compressed air)

Electrically Reconfigurable Ship
- Technology Insertion
- Warfighting Capabilities

Increasing Affordability and Military Capability

Design Example

Propulsion Motor

Motor Drive

Generator

Prime Mover

Main Power Distribution

Ship Service Power

Power Conversion Module
Design Example

CVN(X) FUTURE AIRCRAFT CARRIER
Design Example
Design Example
Design Example

(a) Open-loop (without feedback) control of the speed of a turntable.
(b) Block diagram model.
Design Example

(a) Closed-loop control of the speed of a turntable.
(b) Block diagram model.
Initial wages + Actual wages → Industry → Prices

Wage increase + Automatic cost of living increase → Cost of living → $K_1$
The blood glucose and insulin levels for a healthy person.
Design Example

(a) Open-loop (without feedback) control and
(b) closed-loop control of blood glucose.
Sequential Design Example

(a) A disk drive ©1999 Quantum Corporation. All rights reserved.
(b) Diagram of a disk drive.
Sequential Design Example

Closed-loop control system for disk drive.
We use quantitative mathematical models of physical systems to design and analyze control systems. The dynamic behavior is generally described by ordinary differential equations. We will consider a wide range of systems, including mechanical, hydraulic, and electrical. Since most physical systems are nonlinear, we will discuss linearization approximations, which allow us to use Laplace transform methods.

We will then proceed to obtain the input–output relationship for components and subsystems in the form of transfer functions. The transfer function blocks can be organized into block diagrams or signal-flow graphs to graphically depict the interconnections. Block diagrams (and signal-flow graphs) are very convenient and natural tools for designing and analyzing complicated control systems.
Introduction

Six Step Approach to Dynamic System Problems

Define the system and its components
Formulate the mathematical model and list the necessary assumptions
Write the differential equations describing the model
Solve the equations for the desired output variables
Examine the solutions and the assumptions
If necessary, reanalyze or redesign the system
Differential Equation of Physical Systems

\[ T_a(t) - T_s(t) = 0 \]

\[ T_a(t) = T_s(t) \]

\[ \omega(t) = \omega_s(t) - \omega_a(t) \]

\[ T_a(t) \text{ through-variable} \]

 angular rate difference = across-variable

(a) Torsional spring-mass system.
(b) Spring element.
Differential Equation of Physical Systems

**Electrical Inductance**
\[ v_{21} = L \cdot \frac{di}{dt} \]

**Translational Spring**
\[ v_{21} = \frac{1}{k} \cdot \frac{dF}{dt} \]

**Rotational Spring**
\[ \omega_{21} = \frac{1}{k} \cdot \frac{dT}{dt} \]

**Fluid Inertia**
\[ P_{21} = I \cdot \frac{dQ}{dt} \]
Differential Equation of Physical Systems

Electrical Capacitance

\[ v_2 \rightarrow i \rightarrow C \rightarrow v_1 \]

\[ i = C \frac{d}{dt} v_{21} \]

\[ E = \frac{1}{2} \cdot M \cdot v_{21}^2 \]

Translational Mass

\[ F \rightarrow v_2 \rightarrow M \rightarrow v_1 = \text{constant} \]

\[ F = M \frac{d}{dt} v_2 \]

\[ E = \frac{1}{2} \cdot M \cdot v_2^2 \]

Rotational Mass

\[ T \rightarrow \omega_2 \rightarrow J \rightarrow \omega_1 = \text{constant} \]

\[ T = J \frac{d}{dt} \omega_2 \]

\[ E = \frac{1}{2} \cdot J \cdot \omega_2^2 \]

Fluid Capacitance

\[ Q \rightarrow C_f \rightarrow P_1 \rightarrow P_2 \]

\[ Q = C_f \frac{d}{dt} P_{21} \]

\[ E = \frac{1}{2} \cdot C_f \cdot P_{21}^2 \]

Thermal Capacitance

\[ q \rightarrow C_t \rightarrow T_1 \rightarrow T_2 \]

\[ q = C_t \frac{d}{dt} T_2 \]

\[ E = C_t \cdot T_2 \]
Differential Equation of Physical Systems

Electrical Resistance
\[ i = \frac{1}{R} \cdot v_{21} \]
\[ P = \frac{1}{R} \cdot v_{21}^2 \]

Translational Damper
\[ F = b \cdot v_{21} \]
\[ P = b \cdot v_{21}^2 \]

Rotational Damper
\[ T = b \cdot \omega_{21} \]
\[ P = b \cdot \omega_{21}^2 \]

Fluid Resistance
\[ Q = \frac{1}{R_f} \cdot P_{21} \]
\[ P = \frac{1}{R_f} \cdot P_{21}^2 \]

Thermal Resistance
\[ q = \frac{1}{R_t} \cdot T_{21} \]
\[ P = \frac{1}{R_t} \cdot T_{21} \]
Differential Equation of Physical Systems

(a) Spring-mass-damper system.
   (b) Free-body diagram.

\[ M \cdot \frac{d^2 y(t)}{dt^2} + b \cdot \frac{dy(t)}{dt} + k \cdot y(t) = r(t) \]
Differential Equation of Physical Systems

\[
\frac{v(t)}{R} + C \cdot \frac{d}{dt} v(t) + \frac{1}{L} \cdot \int_0^t v(t) \, dt = r(t)
\]

\[
y(t) = K_1 \cdot e^{-\alpha_1 \cdot t} \cdot \sin(\beta_1 \cdot t + \theta_1)
\]
Differential Equation of Physical Systems

Typical voltage response for underdamped RLC circuit.
**Differential Equation of Physical Systems**

\[ K_2 := 1 \quad \alpha_2 := 0.5 \quad \beta_2 := 10 \quad \theta_2 := 2 \]

\[ y(t) := K_2 \cdot e^{-\alpha_2 \cdot t} \cdot \sin(\beta_2 \cdot t + \theta_2) \]

\[ y_1(t) := K_2 \cdot e^{-\alpha_2 \cdot t} \quad y_2(t) := -K_2 \cdot e^{-\alpha_2 \cdot t} \]
Linear Approximations

(a) A mass sitting on a nonlinear spring.
(b) The spring force versus $y$. 
Linear Approximations

Linear Systems - Necessary condition

- Principle of Superposition
- Property of Homogeneity

Taylor Series

http://www.maths.abdn.ac.uk/~igc/tch/ma2001/notes/node46.html
Linear Approximations – Example 2.1

Students are encouraged to investigate linear approximation accuracy for different values of $\theta_0$. 

\[
M := 200\text{gm} \quad g := 9.8\frac{\text{m}}{\text{s}^2} \quad L := 100\text{cm} \quad \theta_0 := 0\text{rad} \quad \theta := -\pi, -\frac{15\pi}{16} \ldots \pi
\]

Length $L$

\[
T_0 := M \cdot g \cdot L \sin(\theta_0)
\]

\[
T_1(\theta) := M \cdot g \cdot L \sin(\theta)
\]

\[
T_2(\theta) := M \cdot g \cdot L \cos(\theta_0)(\theta - \theta_0) + T_0
\]
The Laplace Transform

Historical Perspective - Heaviside’s Operators

Origin of Operational Calculus (1887)
Historical Perspective - Heaviside's Operators
Origin of Operational Calculus (1887)

\[ p = \frac{d}{dt} \]

\[ \frac{1}{p} = \int_{0}^{t} 1 \, du \]

\[ \nu = \frac{v}{Z(p)} \]

\[ Z(p) = R + L \cdot p \]

\[ i = \frac{1}{R + L \cdot p} \cdot H(t) = \frac{1}{L \cdot p \left(1 + \frac{R}{L \cdot p}\right)} \cdot H(t) = \frac{1}{R} \left[ \frac{R}{L} \cdot \frac{1}{p} - \left(\frac{R}{L}\right)^2 \cdot \frac{1}{p^2} + \left(\frac{R}{L}\right)^3 \cdot \frac{1}{p^3} \ldots \right] \cdot H(t) \]

\[ i = \frac{1}{p} \cdot H(t) = \frac{t^n}{n!} \]

\[ i = \frac{1}{R} \left[ \frac{R}{L} \cdot t - \left(\frac{R}{L}\right)^2 \cdot \frac{t^2}{2!} + \left(\frac{R}{L}\right)^3 \cdot \frac{t^3}{3!} \ldots \right] \]

\[ i = \frac{1}{R} \left[ 1 - e^{-\left(\frac{R}{L}\right) \cdot t} \right] \]

The Laplace Transform

Definition

\[ L(f(t)) = \int_0^\infty f(t) \cdot e^{-s \cdot t} \, dt = F(s) \]

Here the complex frequency is \( s = \rho + j \cdot w \)

The Laplace Transform exists when

\[ \int_0^\infty f(t) \cdot e^{-s \cdot t} \, dt < \infty \]

this means that the integral converges
The Laplace Transform

Determine the Laplace transform for the functions

a) \( f_1(t) := 1 \) for \( t \geq 0 \)

\[
F_1(s) := \int_0^\infty e^{-st} \, dt = \frac{1}{s} \cdot e^{-(s \cdot t)} \quad \frac{1}{s}
\]

b) \( f_2(t) = e^{-(a \cdot t)} \)

\[
F_2(s) = \int_0^\infty e^{-(a \cdot t)} \cdot e^{-(s \cdot t)} \, dt = \frac{1}{s + 1} \cdot e^{-(s+a \cdot t)} \quad F_2(s) = \frac{1}{s + a}
\]
The Laplace Transform

Evaluate the laplace transform of the derivative of a function

\[ \mathcal{L}\left( \frac{d}{dt} f(t) \right) = \int_{0}^{\infty} \frac{df(t)}{dt} e^{-st} \, dt \]

by the use of

\[ \int u \, dv = uv - \int v \, du \]

where

\[ u = e^{-(s \cdot t)} \quad dv = df(t) \]

and, from which

\[ du = -s \cdot e^{-(s \cdot t)} \, dt \]

we obtain

\[ \int_{0}^{\infty} u \, dv = f(t) \cdot e^{-(s \cdot t)} - \int_{0}^{\infty} f(t) \left[ -s \cdot e^{-(s \cdot t)} \right] dt \]

\[ = -f(0+) + s \int_{0}^{\infty} f(t) e^{-(s \cdot t)} \, dt \]

\[ \mathcal{L}\left( \frac{d}{dt} f(t) \right) = sF(s) - f(0+) \quad \text{note that the initial condition is included in the transform} \]
The Laplace Transform

Practical Example - Consider the circuit.

The KVL equation is

\[ 4 \cdot i(t) + 2 \cdot \frac{d}{dt} i(t) = 0 \quad \text{assume } i(0^+) = 5 \, \text{A} \]

Applying the Laplace Transform, we have

\[
\int_0^\infty \left( 4 \cdot i(t) + 2 \cdot \frac{d}{dt} i(t) \right) e^{-s \cdot t} \, dt = 0 \quad \int_0^\infty i(t) \cdot e^{-s \cdot t} \, dt + 2 \int_0^\infty \frac{d}{dt} i(t) \cdot e^{-s \cdot t} \, dt = 0
\]

\[ 4 \cdot I(s) + 2 \cdot (s \cdot I(s) - i(0)) = 0 \quad 4 \cdot I(s) + 2 \cdot s \cdot I(s) - 10 = 0 \]

\( I(s) := \frac{5}{s + 2} \)  

transforming back to the time domain, with our present knowledge of Laplace transform, we may say that

\[ i(t) \equiv 5 \cdot e^{-2 \cdot t} \]

\[ t \equiv (0, 0.01\ldots 2) \]
The Partial-Fraction Expansion (or Heaviside expansion theorem)

Suppose that

\[ F(s) = \frac{s + z_1}{(s + p_1)(s + p_2)} \]

or

\[ F(s) = \frac{K_1}{s + p_1} + \frac{K_2}{s + p_2} \]

The partial fraction expansion indicates that \( F(s) \) consists of a sum of terms, each of which is a factor of the denominator. The values of \( K_1 \) and \( K_2 \) are determined by combining the individual fractions by means of the lowest common denominator and comparing the resultant numerator coefficients with those of the coefficients of the numerator before separation in different terms.

Evaluation of \( K_i \) in the manner just described requires the simultaneous solution of \( n \) equations. An alternative method is to multiply both sides of the equation by \((s + p_i)\) then setting \( s = -p_i \), the right-hand side is zero except for \( K_i \) so that

\[ K_i = \frac{(s + p_i)(s + z_1)}{(s + p_1) + (s + p_2)} \quad \text{for} \quad s = -p_i \]
# The Laplace Transform

<table>
<thead>
<tr>
<th>Property</th>
<th>Time Domain</th>
<th>Frequency Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Time delay</td>
<td>(f(t - T) \cdot u(t - T))</td>
<td>(e^{-(s \cdot T)} \cdot F(s))</td>
</tr>
<tr>
<td>2. Time scaling</td>
<td>(f(at))</td>
<td>(\frac{1}{a} \cdot F\left(\frac{s}{a}\right))</td>
</tr>
<tr>
<td>3. Frequency differentiation</td>
<td>(t \cdot f(t))</td>
<td>(\frac{d}{ds} F(s))</td>
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<td>4. Frequency shifting</td>
<td>(f(t) \cdot e^{-(a \cdot t)})</td>
<td>(F(s + a))</td>
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<td>5. Frequency Integration</td>
<td>(\frac{f(t)}{t})</td>
<td>(\int_0^\infty F(s) , ds)</td>
</tr>
<tr>
<td>6. Initial-value Theorem</td>
<td>(\text{Lin}(f(t)) = f(0))</td>
<td>(\text{Lin}(s \cdot F(s)))</td>
</tr>
<tr>
<td></td>
<td>(t \to 0)</td>
<td>(s \to \text{infinite})</td>
</tr>
<tr>
<td>7. Final-value Theorem</td>
<td>(\text{Lin}(f(t)))</td>
<td>(\text{Lin}(s \cdot F(s)))</td>
</tr>
<tr>
<td></td>
<td>(t \to \text{infinite})</td>
<td>(s \to 0)</td>
</tr>
</tbody>
</table>
The Laplace Transform

Useful Transform Pairs
The Laplace Transform

Consider the mass-spring-damper system

\[ Y(s) = \frac{(Ms + b)\cdot yo}{Ms^2 + bs + k} \]

\[ y(s) = \frac{\left( s + \frac{b}{M} \right) \cdot (yo)}{s^2 + \left( \frac{b}{M} \right) \cdot s + \frac{k}{M}} = \frac{(s + 2\zeta\omega_n)}{s^2 + 2\zeta\omega_n + \omega_n^2} \]

\[ s1 = -(\zeta\omega_n) + \omega_n\sqrt{\zeta^2 - 1} \]

\[ s2 = -(\zeta\omega_n) - \omega_n\sqrt{\zeta^2 - 1} \]

Roots

Real
Real repeated
Imaginary (conjugates)
Complex (conjugates)
The Laplace Transform

Response of the spring-mass-damper system.
The Transfer Function of Linear Systems

An RC network.

\[ V_1(s) = \left( R + \frac{1}{Cs} \right) \cdot I(s) \]

\[ Z_1(s) = R \]

\[ V_2(s) = \left( \frac{1}{Cs} \right) \cdot I(s) \]

\[ Z_2(s) = \frac{1}{Cs} \]

\[ \frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} \]
The Transfer Function of Linear Systems

Example 2.2

\[
\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 3y(t) = 2r(t)
\]

Initial Conditions: \(Y(0) = 1\), \(\frac{dy(0)}{dt} = 0\)

\(r(t) = 1\)

The Laplace transform yields:

\[
(s^2 \cdot Y(s) - s \cdot y(0)) + 4(s \cdot Y(s) - y(0)) + 3Y(s) = 2R(s)
\]

Since \(R(s) = \frac{1}{s}\) and \(y(0) = 1\), we obtain:

\[
Y(s) = \frac{(s + 4)}{(s^2 + 4s + 3)} + \frac{2}{s \cdot (s^2 + 4s + 3)}
\]

The partial fraction expansion yields:

\[
Y(s) = \left[ \frac{3}{2(s + 1)} + \frac{-1}{2(s + 3)} \right] + \left[ -\frac{1}{(s + 1)} + \frac{1}{3(s + 3)} \right] + \frac{2}{s}
\]

Therefore the transient response is:

\[
y(t) = \left( \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} \right) + \left( -e^{-t} + \frac{1}{3}e^{-3t} \right) + \frac{2}{3}
\]

The steady-state response is:

\[
\lim_{{t \to \infty}} y(t) = \frac{2}{3}
\]
The Transfer Function of Linear Systems

An inverting amplifier operating with ideal conditions.
The Transfer Function of Linear Systems

(a) Two-mass mechanical system. (b) Two-node electric circuit analog $C_1 = M_1$, $C_2 = M_2$, $L = 1/k$, $R_1 = 1/b_1$, $R_2 = 1/b_2$. 
A dc motor (a) wiring diagram and (b) sketch.
A pancake dc motor with a flat-wound armature and a permanent magnet rotor. These motors are capable of providing high torque with a low rotor inertia. A typical mechanical time constant is in the range of 15 ms. (Courtesy of Mavilor Motors.)
The Transfer Function of Linear Systems

\[ \phi = K_f \cdot i_f \]

\[ T_m = K_1 \cdot K_f \cdot i_f(t) \cdot i_a(t) \]

Field controlled motor - Laplace Transform

\[ T_m(s) = (K_1 \cdot K_f \cdot I_a) \cdot I_f(s) \]

\[ V_f(s) = (R_f + L_f \cdot s) \cdot I_f(s) \]

\[ T_m(s) = T_L(s) + T_d(s) \]

\[ T_L(s) = J \cdot s^2 \cdot \theta(s) + b \cdot s \cdot \theta(s) \]

Rearranging equations

\[ T_L(s) = T_m(s) - T_d(s) \]

\[ T_m(s) = K_m \cdot I_f(s) \]

\[ I_f(s) = \frac{V_f(s)}{R_f + L_f \cdot s} \]

\[ T_d(s) = 0 \]

\[ \frac{\theta(s)}{V_f(s)} = \frac{K_m}{s \cdot (J \cdot s + b) \cdot (L_f \cdot s + R_f)} \]
The Transfer Function of Linear Systems

Block diagram model of field-controlled dc motor.
The Transfer Function of Linear Systems

Armature-controlled dc motor.
The Transfer Function of Linear Systems
The Transfer Function of Linear Systems

Range of control response time and power to load for electromechanical and electrohydraulic devices.
The Transfer Function of Linear Systems

\[
\frac{V_2(s)}{V_1(s)} = \frac{-1}{RCs}
\]

\[
\frac{V_2(s)}{V_1(s)} = -RCs
\]
The Transfer Function of Linear Systems

\[ \frac{V_2(s)}{V_1(s)} = \frac{R_2 \left( R_1 \cdot C_1 \cdot s + 1 \right)}{R_1} \]

\[ \frac{V_2(s)}{V_1(s)} = \frac{-(R_1 \cdot C_1 \cdot s + 1)(R_2 \cdot C_2 \cdot s + 1)}{R_1 \cdot C_2 \cdot s} \]
The Transfer Function of Linear Systems

\[
\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s \cdot (J \cdot s + b) \left( L_f \cdot s + R_f \right)}
\]

\[
\frac{\theta(s)}{V_a(s)} = \frac{K_m}{s \cdot \left[ (R_a + L_a \cdot s) (J \cdot s + b) + K_b \cdot K_m \right]}
\]
The Transfer Function of Linear Systems

\[
\frac{\theta(s)}{V_c(s)} = \frac{K_m}{s(\tau_s + 1)}
\]

\[
\tau = \frac{J}{(b - m)}
\]

\[ m = \text{slope of linearized torque-speed curve (normally negative)} \]

\[
\frac{V_o(s)}{V_c(s)} = \left(\frac{K}{R_c \cdot R_q}\right)
\]

\[
\tau_c = \frac{L_c}{R_c}, \quad \tau_q = \frac{L_q}{R_q}
\]

For the unloaded case:
\[ i_d = 0 \quad \tau_c = \tau_q \]
\[ 0.05s < \tau_c < 0.5s \]
\[ V_{12} = V_q \quad V_{34} = V_d \]
The Transfer Function of Linear Systems

\[
\frac{Y(s)}{X(s)} = \frac{K}{s(Ms + B)}
\]

\[
K = \frac{A \cdot k_x}{k_p} \quad B = \left( b + \frac{A^2}{k_p} \right)
\]

\[
k_x = \frac{d}{dx} g \\
k_p = \frac{d}{dP} g
\]

\[
g = g(x, P) = \text{flow}
\]

\[
A = \text{area of piston}
\]

Gear Ratio = \( n = \frac{N_1}{N_2} \)

\[
N_2 \cdot \theta_L = N_1 \cdot \theta_m \\
\theta_L = n \cdot \theta_m \\
\omega_L = n \cdot \omega_m
\]
The Transfer Function of Linear Systems

\[ \frac{V_2(s)}{V_1(s)} = \frac{R_2}{R} = \frac{R_2}{R_1 + R_2} \]

\[ \frac{R_2}{R} = \frac{\theta}{\theta_{\text{max}}} \]

\[ V_2(s) = k_s \left( \theta_1(s) - \theta_2(s) \right) \]

\[ V_2(s) = k_s \cdot \theta_{\text{error}}(s) \]

\[ k_s = \frac{V_{\text{battery}}}{\theta_{\text{max}}} \]
The Transfer Function of Linear Systems

\[ V_2(s) = K_t \cdot \omega(s) = K_t \cdot s \cdot \theta(s) \]

\[ K_t = \text{constant} \]

\[ \frac{V_2(s)}{V_1(s)} = \frac{k_a}{s \cdot \tau + 1} \]

Ro = output resistance
Co = output capacitance

\( \tau = R_o \cdot C_o \quad \tau < 1s \)

and is often negligible for controller amplifie
The Transfer Function of Linear Systems

\[ x_0(t) = y(t) - x_{in}(t) \]

\[
\frac{X_0(s)}{X_{in}(s)} = \frac{-s^2}{s^2 + \left(\frac{b}{M}\right)s + \frac{k}{M}}
\]

For low frequency oscillations, where \( \omega < \omega_n \)

\[
\frac{X_0(j\cdot\omega)}{X_{in}(j\cdot\omega)} = \frac{\omega^2}{k} \frac{1}{M}
\]

\[
T(s) = \frac{1}{C_t \cdot s + \left(Q \cdot S + \frac{1}{R}\right)}
\]

\( T = T_0 - T_e \) = temperature difference due to thermal proc

C_t = thermal capacitance
Q = fluid flow rate = constant
S = specific heat of water
R_t = thermal resistance of insulation
q(s) = rate of heat flow of heating element
The Transfer Function of Linear Systems

\[ x = r \cdot \theta \]

converts radial motion to linear motion.
Block Diagram Models

\[ G(s) = \frac{K_m}{s(Js + b)(L_f s + R_f)} \]

Block diagram of dc motor.

General block representation of two-input, two-output system.
Block Diagram Models

Block diagram of interconnected system.
Block Diagram Models

Original Diagram

Equivalent Diagram

Original Diagram

Equivalent Diagram
Block Diagram Models

Original Diagram

Equivalent Diagram

Original Diagram

Equivalent Diagram
Block Diagram Models

Original Diagram

Equivalent Diagram

Original Diagram

Equivalent Diagram

\[ \frac{G}{1 + GH} \]
Block Diagram Models

Negative feedback control system.
Block Diagram Models

Example 2.7
Block Diagram Models

Example 2.7
For complex systems, the block diagram method can become difficult to complete. By using the signal-flow graph model, the reduction procedure (used in the block diagram method) is not necessary to determine the relationship between system variables.
Signal-Flow Graph Models

Signal-flow graph of interconnected system.

\[ Y_1(s) = G_{11}(s)R_1(s) + G_{12}(s)R_2(s) \]

\[ Y_2(s) = G_{21}(s)R_1(s) + G_{22}(s)R_2(s) \]
Signal-Flow Graph Models

\[ a_{11}x_1 + a_{12}x_2 + r_1 = x_1 \]

\[ a_{21}x_1 + a_{22}x_2 + r_2 = x_2 \]
Signal-Flow Graph Models

Example 2.8

Two-path interacting system.

\[
\frac{Y(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4 (1 - L_3 - L_4) + G_5 G_6 G_7 G_8 (1 - L_1 - L_2)}{1 - L_1 - L_2 - L_3 - L_4 + L_1 L_3 + L_1 L_4 + L_2 L_3 + L_2 L_4}
\]
Signal-Flow Graph Models

Example 2.10

\[ G_1(s) = \frac{K_m}{(R_a + L_\alpha s)} \]

The signal-flow graph of the armature-controlled dc motor.

\[
\frac{Y(s)}{R(s)} = \frac{G_1 \cdot G_2 \cdot G_3 \cdot G_4}{1 + G_2 \cdot G_3 \cdot H_2 - G_3 \cdot G_4 \cdot H_1 + G_1 \cdot G_2 \cdot G_3 \cdot G_4 \cdot H_3}
\]
Signal-Flow Graph Models

\[
\frac{Y(s)}{R(s)} = \frac{P_1 + P_2 \cdot \Delta_2 + P_3}{\Delta}
\]

\[
P_1 = G_1 \cdot G_2 \cdot G_3 \cdot G_4 \cdot G_5 \cdot G_6 \quad P_2 = G_1 \cdot G_2 \cdot G_7 \cdot G_6 \quad P_3 = G_1 \cdot G_2 \cdot G_3 \cdot G_4 \cdot G_8
\]

\[
\Delta = 1 - \left( L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8 \right) + \left( L_5 \cdot L_7 + L_5 \cdot L_4 + L_3 \cdot L_4 \right)
\]

\[
\Delta_1 = \Delta_3 = 1 \quad \Delta_2 = 1 - L_5 = 1 + G_4 \cdot H_4
\]

Multiple-loop system.
Analysis and design using a system model.
Design Examples

Speed control of an electric traction motor.
Design Examples

An accelerometer mounted on a jet-engine test sled.
Design Examples

![Graph showing accelerometer response.](image)

Accelerometer response.
Design Examples

Exploded view of the ORCA robot showing the components [15].
(Source: © Copyright 1993 Hewlett-Packard Company. Reproduced with permission.)
Design Examples

(a) Ladder network
and (b) its signal-flow graph.
BLOCK DIAGRAM REDUCTION OF MULTIPLE SYSTEMS
Components of a block diagram for a linear, time-invariant system

(a) Signals

\[ C(s) = R_1(s) + R_2(s) - R_3(s) \]

(b) System

(c) Summing junction

(d) Pickoff point
a. Cascaded subsystems;
b. equivalent transfer function
a. Parallel subsystems;  
b. equivalent transfer function

\[ X_1(s) = R(s) G_1(s) \]
\[ X_2(s) = R(s) G_2(s) \]
\[ X_3(s) = R(s) G_3(s) \]

\[ C(s) = [\pm G_1(s) \pm G_2(s) \pm G_3(s)] R(s) \]
a. Feedback control system;
b. simplified model;
c. equivalent transfer function
Block diagram algebra for summing junctions

Equivalent forms for moving a block
a. to the left past a summing junction;
b. to the right past a summing junction
Block diagram algebra for pickoff points

equivalent forms for moving a block
a. to the left past a pickoff point;
b. to the right past a pickoff point

(a)

(b)
Block diagram reduction via familiar forms for Example

**Problem:** Reduce the block diagram shown in figure to a single transfer function
Block diagram reduction via familiar forms for Example Cont.

**Steps in solving Example**

a. collapse summing junctions;  

b. form equivalent cascaded system in the forward path;  

c. form equivalent parallel system in the feedback path;  

d. form equivalent feedback system and multiply by cascaded $G_1(s)$
Block diagram reduction by moving blocks Example

Problem: Reduce the block diagram shown in figure to a single transfer function
Steps in the block diagram reduction for Example

a) Move $G_2(s)$ to the left past of pickoff point to create parallel subsystems, and reduce the feedback system of $G_3(s)$ and $H_3(s)$

b) Reduce parallel pair of $1/G_2(s)$ and unity, and push $G_1(s)$ to the right past summing junction

c) Collapse the summing junctions, add the 2 feedback elements, and combine the last 2 cascade blocks

d) Reduce the feedback system to the left

e) Finally, Multiple the 2 cascade blocks and obtain final result.
Second-order feedback control system

The closed loop transfer function is

\[ T(s) = \frac{K}{s^2 + as + K} \]

Note K is the amplifier gain, As K varies, the poles move through the three ranges of operations OD, CD, and UD

\[ 0 < K < \frac{a^2}{4} \] system is over damped

\[ K = \frac{a^2}{4} \] system is critically damped

\[ K > \frac{a^2}{4} \] system is under damped
Finding transient response Example

![Block Diagram](image)

**Problem:** For the system shown, find peak time, percent overshoot, and settling time.

**Solution:** The closed loop transfer function is 

\[ T(s) = \frac{25}{s^2 + 5s + 25} \]

And

\[ \omega_n = \sqrt{25} = 5 \]

\[ 2 \xi \omega_n = 5 \quad \text{so} \quad \xi = 0.5 \]

using values for \( \xi \) and \( \omega_n \) and equation in chapter 4 we find

\[ T_p = \frac{\pi}{\omega_n \sqrt{1 - \xi^2}} = 0.726 \quad \text{sec} \]

\[ \% OS = e^{-\xi \pi \sqrt{1 - \xi^2}} \times 100 = 16.303 \]

\[ T_s = \frac{4}{\xi \omega_n} = 1.6 \quad \text{sec} \]
Gain design for transient response Example

Problem: Design the value of gain $K$, so that the system will respond with a 10% overshot.

Solution: The closed loop transfer function is

$$T(s) = \frac{K}{s^2 + 5s + K}$$

$$\omega_n = \sqrt{K} \quad \text{and} \quad 2\xi\omega_n = 5 \quad \text{thus} \quad \xi = \frac{5}{2\sqrt{K}}$$

For 10% OS we find \( \xi = 0.591 \)

We substitute this value in previous equation to find $K = 17.9$
Signal-flow graph components:

a. system;
b. signal;
c. interconnection of systems and signals

(a) $G(s)$

(b) $V(s)$

(c) Diagram with nodes and arrows labeled $R_1(s)$, $G_1(s)$, $G_2(s)$, $G_3(s)$, $G_4(s)$, $G_5(s)$, $G_6(s)$, $R_2(s)$, $C_1(s)$, $C_2(s)$, $C_3(s)$.
Building signal-flow graphs

a. cascaded system nodes

b. cascaded system signal-flow graph;

c. parallel system nodes
d. parallel system signal-flow graph;
e. feedback system nodes
f. feedback system signal-flow graph
**Problem:** Convert the block diagram to a signal-flow graph.
Converting a block diagram to a signal-flow graph.

Signal-flow graph development:

a. signal nodes;
b. signal-flow graph;
c. simplified signal-flow graph
Mason’s ūle - Definitions

**Loop gain:** The product of branch gains found by traversing a path that starts at a node and ends at the same node, following the direction of the signal flow, without passing through any other node more than once. \( G_2(s)H_2(s), \ G_4(s)H_2(s), \ G_4(s)G_5(s)H_3(s), \ G_4(s)G_6(s)H_3(s) \)

**Forward-path gain:** The product of gains found by traversing a path from input node to output node in the direction of signal flow. \( G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)G_7(s), \ G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)G_7(s) \)

**Nontouching loops:** loops that do not have any nodes in common. \( G_2(s)H_1(s) \) does not touch \( G_4(s)H_2(s), \ G_4(s)G_5(s)H_3(s), \) and \( G_4(s)G_6(s)H_3(s) \)

**Nontouching-loop gain:** The product of loop gains from nontouching loops taken 2, 3,4, or more at a time.

\[ [G_2(s)H_1(s)][G_4(s)H_2(s)], \ [G_2(s)H_1(s)][G_4(s)G_5(s)H_3(s)], \ [G_2(s)H_1(s)][G_4(s)G_6(s)H_3(s)] \]
Mason’s Rule

The Transfer function, \( \frac{C(s)}{R(s)} \), of a system represented by a signal-flow graph is

\[
G(s) = \frac{C(s)}{R(s)} = \frac{\sum T_k \Delta_k}{\Delta}
\]

Where

\( K \) = number of forward paths

\( T_k \) = the \( kth \) forward-path gain

\[
\Delta_k = 1 - \sum \text{ loop gains} + \sum \text{ nontouching-loop gains taken 2 at a time} - \sum \text{ nontouching-loop gains taken 3 at a time} + \sum \text{ nontouching-loop gains taken 4 at a time} - \ldots \ldots
\]

\[\Delta_k \] is formed by eliminating from \( \Delta \) those loop gains that touch the \( kth \) forward path. In other words, \( \Delta_k \) is formed by eliminating from \( \Delta \) those loop gains that touch the \( kth \) forward path.
Transfer function via Mason’s rule

Problem: Find the transfer function for the signal flow graph

Solution:

forward path
\( G_1(s)G_2(s)G_3(s)G_4(s)G_5(s) \)

Loop gains
\( G_2(s)H_1(s), G_4(s)H_2(s), G_7(s)H_4(s), G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s) \)

Nontouching loops

2 at a time
\( G_2(s)H_1(s)G_4(s)H_2(s) \)
\( G_2(s)H_1(s)G_7(s)H_4(s) \)
\( G_4(s)H_2(s)G_7(s)H_4(s) \)

3 at a time \( G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s) \)

Now
\[ \Delta = 1 - \left[ G_2(s)H_1(s) + G_4(s)H_2(s) + G_7(s)H_4(s) + G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s) \right] + \left[ G_2(s)H_1(s)G_4(s)H_2(s) + G_2(s)H_1(s)G_7(s)H_4(s) + G_4(s)H_2(s)G_7(s)H_4(s) \right] - \left[ G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s) \right] \]

\[ \Delta_1 = 1 - G_7(s)H_4(s) \]

\[ G(s) = \frac{T_1\Delta_1}{\Delta} = \frac{[G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)][1 - G_7(s)H_4(s)]}{\Delta} \]
Signal-Flow Graphs of State Equations

Problem: draw signal-flow graph for:

\[ x_1 = 2x_1 - 5x_2 + 3x_3 + 2r \]
\[ x_2 = -6x_1 - 2x_2 + 2x_3 + 5r \]
\[ x_3 = x_1 - 3x_2 - 4x_3 + 7r \]
\[ y = -4x_1 + 6x_2 + 9x_3 \]

a. place nodes;

b. interconnect state variables and derivatives;

c. form \( \frac{dx_1}{dt} \);

d. form \( \frac{dx_2}{dt} \)
Signal-Flow Graphs of State Equations

(continued)

e. form $dx_3/dt$;

f. form output
Alternate Representation: Cascade Form

\[
\frac{C(s)}{R(s)} = \frac{24}{(s + 2)(s + 3)(s + 4)}
\]
Alternate Representation: Cascade Form

\[ R(s) \xrightarrow{\frac{1}{s}} C(s) \]

(a)

\[ R(s) \xrightarrow{\frac{1}{s}} X_3(s) \xrightarrow{-2} X_2(s) \xrightarrow{-3} X_1(s) \xrightarrow{-4} C(s) \]

(b)

\[
\begin{align*}
\dot{x}_1 &= -4x_1 + x_2 \\
x_2 &= -3x_2 + x_3 \\
x_3 &= -2x_3 + 24r \\
y &= c(t) = x_1 \\
y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} X \\
X &= \begin{bmatrix} -4 & 1 & 0 \\
0 & -3 & 1 \\
0 & 0 & -2 \end{bmatrix} X + \begin{bmatrix} 0 \\
0 \\
24 \end{bmatrix} r
\end{align*}
\]
\[
\frac{C(s)}{R(s)} = \frac{24}{(s + 2)(s + 3)(s + 4)} = \frac{12}{(s + 2)} - \frac{24}{(s + 3)} + \frac{12}{(s + 4)}
\]

\[x_1 = -2x_1 + 12r\]
\[x_2 = -3x_2 - 24r\]
\[x_3 = -4x_3 + 12r\]
\[y = c(t) = x_1 + x_2 + x_3\]

\[
\begin{bmatrix}
-2 & 0 & 0 \\
0 & -3 & 0 \\
0 & 0 & -4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} + 
\begin{bmatrix}
12 \\
-24 \\
12
\end{bmatrix}
\begin{bmatrix}
r
\end{bmatrix}
\]

\[y = \begin{bmatrix}1 & 1 & 1\end{bmatrix}X\]
Alternate Representation: Parallel Form Repeated roots

\[
\frac{C(s)}{R(s)} = \frac{(s + 3)}{(s + 1)^2 (s + 2)} = \frac{2}{(s + 1)^2} \quad \frac{1}{(s + 1)} \quad \frac{1}{(s + 2)}
\]

\[
x_1 = -x_1 + x_2
\]

\[
x_2 = x_2 + 2r
\]

\[
x_3 = -2x_3 + r
\]

\[
y = c(t) = x_1 - 1/2x_2 + x_3
\]

\[
X = \begin{bmatrix}
-1 & 1 & 0 \\
0 & -1 & 0 \\
0 & 0 & -2
\end{bmatrix} X + \begin{bmatrix}
0 \\
2 \\
1
\end{bmatrix} r
\]

\[
y = \begin{bmatrix}
1 & -1/2 & 1
\end{bmatrix} X
\]
Alternate Representation: controller canonical form

\[ G(s) = C(s)/R(s) = (s^2 + 7s + 2)/(s^3 + 9s^2 + 26s + 24) \]

This form is obtained from the phase-variable form simply by ordering the phase variable in reverse order.

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
\end{bmatrix}
= \begin{bmatrix}
    0 & 1 & 0 \\
    0 & 0 & 1 \\
    -24 & -26 & -9 \\
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
\end{bmatrix}
+ \begin{bmatrix}
    0 \\
    0 \\
    1 \\
\end{bmatrix} r
\]

\[
y = \begin{bmatrix}
    2 & 7 & 1 \\
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
\end{bmatrix}
= \begin{bmatrix}
    -9 & -26 & -24 \\
    1 & 0 & 0 \\
    0 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
\end{bmatrix}
+ \begin{bmatrix}
    0 \\
    1 \\
    0 \\
\end{bmatrix} r
\]

\[
y = \begin{bmatrix}
    1 & 7 & 2 \\
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
\end{bmatrix}
\]
Alternate Representation: controller canonical form

System matrices that contain the coefficients of the characteristic polynomial are called *companion matrices* to the characteristic polynomial.

Phase-variable form result in lower companion matrix

Controller canonical form results in upper companion matrix
Alternate Representation: observer canonical form

Observer canonical form so named for its use in the design of observers

\[ G(s) = \frac{C(s)}{R(s)} = \frac{(s^2 + 7s + 2)}{(s^3 + 9s^2 + 26s + 24)} \]
\[ = \frac{(1/s+7s^2+2/s^3)}{(1+9/s+26/s^2+24/s^3)} \]

Cross multiplying

\[ (1/s+7/s^2+2/s^3)R(s) = (1+9/s+26/s^2+24/s^3)C(s) \]
And \[ C(s) = 1/s[R(s)-9C(s)] + 1/s^2[7R(s)-26C(s)] + 1/s^3[2R(s)-24C(s)] \]
\[ = 1/s\{ [R(s)-9C(s)] + 1/s \{[7R(s)-26C(s)]+1/s \ [2R(s)-24C(s)]\} \} \]
Alternate Representation: observer canonical form

\[
\begin{align*}
    x_1 &= -9x_1 + x_2 + r \\
    x_2 &= -26x_1 + x_3 + 7r \\
    x_3 &= -24x_1 + 2r \\
    y &= c(t) = x_1
\end{align*}
\]

\[
X = \begin{bmatrix}
    -9 & 1 & 0 \\
    -26 & 0 & 1 \\
    -24 & 0 & 0 \\
\end{bmatrix}
\]

\[
y = \begin{bmatrix}
    1 & 0 & 0
\end{bmatrix}X
\]

Note that the observer form has A matrix that is transpose of the controller canonical form, B vector is the transpose of the controller C vector, and C vector is the transpose of the controller B vector. The 2 forms are called duals.
Problem Represent the feedback control system shown in state space. Model the forward transfer function in cascade form.

Solution first we model the forward transfer function as in (a), Second we add the feedback and input paths as shown in (b) complete system. Write state equations

\[
\begin{align*}
    x_1 &= -3x_1 + x_2 \\
    x_2 &= -2x_2 + 100(r - c)
\end{align*}
\]

but \(c = 5x_1 + (x_2 - 3x_1) = 2x_1 + x_2\)
Feedback control system for Example

\[ x_1 = -3x_1 + x_2 \]
\[ x_2 = -200x_1 - 102x_2 + 100r \]
\[ y = c(t) = 2x_1 + x_2 \]

\[ X = \begin{bmatrix} -3 & 1 \\ -200 & -102 \end{bmatrix} X + \begin{bmatrix} 0 \\ 100 \end{bmatrix} r \]
\[ y = \begin{bmatrix} 2 & 1 \end{bmatrix} X \]
State-space forms for

\[ \frac{C(s)}{R(s)} = \frac{s + 3}{[(s + 4)(s + 6)]}. \]

Note: \( y = c(t) \)
UNIT-II
TIME RESPONSE ANALYSIS
In this chapter we extend the ideas of modeling to include control system characteristics, such as sensitivity to model uncertainties, steady-state errors, transient response characteristics to input test signals, and disturbance rejection. We investigate the important role of the system error signal which we generally try to minimize.

We will also develop the concept of the sensitivity of a system to a parameter change, since it is desirable to minimize the effects of unwanted parameter variation. We then describe the transient performance of a feedback system and show how this performance can be readily improved. We will also investigate a design that reduces the impact of disturbance signals.
Open-And Closed-Loop Control Systems

An **open-loop** (direct) system operates without feedback and directly generates the output in response to an input signal.

A **closed-loop** system uses a measurement of the output signal and a comparison with the desired output to generate an error signal that is applied to the actuator.
Open-And Closed-Loop Control Systems

\[ H(s) = 1 \]

\[ Y(s) = \frac{G(s)}{1 + G(s)} \cdot R(s) \]

\[ E(s) = \frac{1}{1 + G(s)} \cdot R(s) \quad \text{Error Signal} \]

Thus, to reduce the error, the magnitude of

\[ |1 + G(s)| \geq \frac{1}{\eta} \geq 1 \]

\[ H(s) \neq 1 \]

\[ Y(s) = \frac{G(s)}{1 + H(s)G(s)} \cdot R(s) \]

\[ E(s) = \frac{1}{1 + H[(s)G(s)]} \cdot R(s) \]

Thus, to reduce the error, the magnitude of

\[ |1 + G(s)H(s)| \geq \frac{1}{\eta} \geq 1 \]
Sensitivity of Control Systems To Parameter Variations

For the closed-loop case if \( GH(s) > 1 \)

\[
Y(s) = \frac{1}{H(s)} \cdot R(s) \quad \text{Output affected only by } H(s)
\]

\( G(s) + \Delta G(s) \)

Open Loop \( \Delta Y(s) = \Delta G(s) \cdot R(s) \)

Closed Loop

\[
Y(s) + \Delta Y(s) = \frac{(G(s) + \Delta G(s))}{1 + (G(s) + \Delta G(s)) \cdot H(s)} \cdot R(s)
\]

\[
\Delta Y(s) = \frac{\Delta G(s)}{(1 + GH(s) + \Delta GH(s))(1 + GH(s))} \cdot R(s)
\]

\( GH(s) > \Delta GH(s) \)

\[
\Delta Y(s) = \frac{\Delta G(s)}{(1 + GH(s))^2} \cdot R(s) \quad \text{The change in the output of the closed system is reduced by a factor of } 1 + GH(s)
\]
Sensitivity of Control Systems To Parameter Variations

\[ T(s) = \frac{Y(s)}{R(s)} \]

\[ S = \frac{\Delta T(s)}{\Delta G(s)} \cdot \frac{T(s)}{G(s)} \]

\[ T(s) = \frac{1}{1 + H \left[ (s) \cdot G(s) \right]} \]

\[ S^T_G = \frac{\dfrac{d}{d G} T}{\dfrac{d}{d G} G} \cdot \frac{G}{T} = \frac{1}{1 + GH} \cdot \frac{G}{G} \]

\[ S^T_H = -\frac{GH}{1 + GH} \]

Sensitivity of the closed-loop to \( G \) variations reduced

Sensitivity of the closed-loop to \( H \) variations

When \( GH \) is large sensitivity approaches 1

Changes in \( H \) directly affects the output response
Example 4.1

Open loop
\[ v_o = -K_a \cdot v_{in} \]
\[ T = -K_a \]
\[ S_{K_a} T = 1 \]

Closed loop
\[ \beta = \frac{R_2}{R_1} \]
\[ R_p = R_1 + R_2 \]
\[ T = \frac{-K_a}{1 + K_a \cdot \beta} \]
\[ S_{K_a} T = \frac{1}{1 + K_a \cdot \beta} \]

If \( K_a \) is large, the sensitivity is low.

\[ K_a := 10^4 \quad \beta := 0.1 \quad S_{K_a} T = \frac{1}{1 + 10^3} = 9.99 \times 10^{-4} \]

(a) Open loop amplifier.
(b) Amplifier with feedback.

Block diagram model of feedback amplifier assuming \( R_p \gg R_0 \) of the amplifier.
Control of the Transient Response of Control Systems

\[
\frac{\omega(s)}{V_a(s)} = G(s) = \frac{K_1}{\tau_1 s + 1}
\]

where,

\[
K_1 = \frac{K_m}{R_a b + K_b K_m} \quad \tau_1 = \frac{R_a J}{R_a b + K_b K_m}
\]
Control of the Transient Response of Control Systems

\[
\omega(s) = \frac{K_a \cdot G(s)}{R(s)} = \frac{K_a \cdot K_1}{\tau_1 s + 1 + K_a \cdot K_t \cdot K_1} = \frac{K_a \cdot \frac{K_1}{\tau_1}}{s + \left( \frac{1 + K_a \cdot K_t \cdot K_1}{\tau_1} \right)}
\]
Control of the Transient Response of Control Systems

The response of the open-loop and closed-loop speed control system when $\tau = 5$ and $K_1K_aK_t = 100$. The time to reach 98% of the final value for the open-loop and closed-loop system is 40 seconds and 0.4 second, respectively.
Disturbance Signals In a Feedback Control Systems

\[ R(s) \]

\[ V_d(s) \]

\[ \frac{1}{R_a} \]

\[ I_d(s) \]

\[ T_m(s) \]

\[ T_L(s) \]

\[ \frac{1}{J s + b} \]

\[ \omega(s) \]

Motor back emf

\[ K_b \]

\[ E(s) = R(s) - \omega(s) \quad R(s) = 0 \]

\[ E(s) = -\omega(s) \]

\[ E(s) = -\omega(s) = \frac{1}{J \cdot s + b + \left( K_m \cdot \frac{K_b}{R_a} \right)} \cdot T_d(s) \]

\[ T_d(s) = \frac{D}{s} \]

\[ \lim_{t \to \infty} E(t) = \lim_{s \to 0} \left[ \frac{1}{J \cdot s + b + \left( K_m \cdot \frac{K_b}{R_a} \right)} \right] \cdot \left( \frac{D}{s} \right) = \frac{D}{b + \left( K_m \cdot \frac{K_b}{R_a} \right)} = -\omega(\text{infinite}) \]
Disturbance Signals In a Feedback Control Systems

Closed-loop speed tachometer control system.
Disturbance Signals In a Feedback Control Systems

\[ G_1(s) = K_a \frac{K_m}{R_a} \quad G_2(s) = \frac{1}{(J \cdot s + b)} \]

\[ E(s) = -\omega(s) = \frac{G(s)}{1 + G_1(s) \cdot G_2(s) \cdot H(s)} \cdot T_d(s) \]

\[ G_1 G_2 H(s) > 1 \]

\[ E(s) = \frac{1}{G_1(s) \cdot H(s)} \cdot T_d(s) \]

If \( G_1(s) H(s) \) very large the effect of the disturbance can be minimized

\[ G_1(s) H(s) = \frac{K_a \cdot K_m}{R_a} \left( K_t + \frac{K_b}{K_a} \right) \approx \frac{K_a \cdot K_m \cdot K_t}{R_a} \] since \( K_a \gg K_b \)

Strive to maintain \( K_a \) large and \( R_a < 2 \) ohms
Steady-State Error

\[ E_o(s) = R(s) - Y(s) = (1 - G(s)) \cdot R(s) \]

\[ E_c(s) = \frac{1}{1 + G(s)} \cdot R(s) \]

\[ H(s) = 1 \]

Steady State Error

\[ \lim_{t \to 0} e(t) = \lim_{s \to 0} s \cdot E(s) \]

For a step unit input

\[ e_o(\text{infinite}) = \lim_{s \to 0} s \cdot (1 - G(s)) \cdot \frac{1}{s} = \lim_{s \to 0} (1 - G(0)) \]

\[ e_c(\text{infinite}) = \lim_{s \to 0} s \cdot \left( \frac{1}{1 + G(s)} \right) \cdot \frac{1}{s} = \lim_{s \to 0} \left( \frac{1}{1 + G(0)} \right) \]
The Cost of Feedback

Increased Number of components and Complexity

Loss of Gain

Instability
Design Example: English Channel Boring Machines

\[ Y(s) = T(s) \cdot R(s) + Td(s) \cdot D(s) \]

\[ Y(s) = \frac{K + 11s}{s^2 + 12s + K} \cdot R(s) + \frac{1}{s^2 + 12s + K} \cdot D(s) \]
Design Example: English Channel Boring Machines

Study system for different Values of gain $K$

Steady state error for $R(s)=1/s$ and $D(s)=0$

$$\lim_{t \to \text{infinite}} e(t) = \lim_{s \to 0} s \cdot \left( \frac{1}{1 + \frac{K + 11s}{s^2 + s}} \right) \cdot \frac{1}{s} = 0$$

Steady state error for $R(s)=0$ $D(s)=1/s$

$$\lim_{t \to \text{infinite}} y(t) = \lim_{s \to 0} s \cdot \left( \frac{1}{s^2 + 12s + K} \right) \cdot \frac{1}{s} = \frac{1}{K}$$

The response $y(t)$ to (a) a unit input step $r(t)$ and (b) a unit disturbance step input $D(s) 5 1/s$ for $K = 100$. 
Transient vs Steady-State

The output of any differential equation can be broken up into two parts,
• a transient part (which decays to zero as $t$ goes to infinity) and
• a steady-state part (which does not decay to zero as $t$ goes to infinity).

\[ y(t) = y_{tr}(t) + y_{ss}(t) \]
\[ \lim_{t \to 0} y_{tr}(t) = 0 \]

Either part might be zero in any particular case.
Prototype systems

1\textsuperscript{st} Order system

\[ \dot{c}(t) + \frac{1}{\tau} c(t) = kr(t) \]

2\textsuperscript{nd} order system

\[ \ddot{c}(t) + 2\zeta \omega_n \dot{c}(t) + \omega_n^2 c(t) = kr(t) \]

Agenda:
- transfer function
- response to test signals
  - impulse
  - step ramp
  - parabolic
  - sinusoidal
1\textsuperscript{st} order system

Impulse response
Step response
Ramp response

Relationship between impulse, step and ramp
Relationship between impulse, step and ramp responses

\[ G(s) = \frac{C(s)}{R(s)} = \frac{1/T}{s + 1/T} \]

\[ r(t) = \delta(t), \quad R(s) = 1, \quad c_{\delta}(t) = \frac{1}{T} e^{-t/T} 1(t) \]

\[ r(t) = 1(t), \quad R(s) = \frac{1}{s}, \quad c_{\text{step}}(t) = \left[ 1 - e^{-t/T} \right] 1(t) \]

\[ r(t) = t1(t), \quad R(s) = \frac{1}{s^2}, \quad c_{\text{ramp}}(t) = \left[ t - T + Te^{-t/T} \right] 1(t) \]
1st Order system

Prototype parameter: Time constant

Relate problem specific parameter to prototype parameter.

Parameters: problem specific constants. Numbers that do not change with time, but do change from problem to problem.

We learn that the time constant defines a problem specific time scale that is more convenient than the arbitrary time scale of seconds, minutes, hours, days, etc, or fractions thereof.
Transient vs Steady state

Consider the impulse, step, ramp responses computed earlier. Identify the steady state and the transient parts.
Consider the impulse, step, ramp responses computed earlier. Identify the steady state and the transient parts.

\[ G(s) = \frac{C(s)}{R(s)} = \frac{1/T}{s + 1/T}, \quad T > 0 \]

**Impulse response**
- \( r(t) = \delta(t), R(s) = 1 \)
- \( r(t) = 1(t), R(s) = \frac{1}{s} \)
- \( r(t) = t1(t), R(s) = \frac{1}{s^2} \)

**Step response**
- \( c(t) = \frac{1}{\delta} e^{-t/T} 1(t) \)
- \( c_{step}(t) = \left[ 1 - e^{-t/T} \right] 1(t) \)

**Ramp response**
- \( c_{ramp}(t) = \left[ t - T + Te^{-t/T} \right] 1(t) \)

Compare steady-state part to input function, transient part to TF.
2\textsuperscript{nd} order system \quad G(s) = \frac{C(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}

Over damped
• (two real distinct roots = two 1\textsuperscript{st} order systems with real poles)

Critically damped
• (a single pole of multiplicity two, highly unlikely, requires exact matching)

Underdamped
• (complex conjugate pair of poles, oscillatory behavior, most common)

step response

\[ c_{\text{step}}(t) = K \left[ 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \sin \left( \omega_d t + \tan^{-1} \left( \frac{\sqrt{1 - \zeta^2}}{\zeta} \right) \right) \right] 1(t) \]

\[ c_\delta(t) = K \left[ \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t) \right] 1(t) \]
2nd Order System

Prototype parameters:
undamped natural frequency,
damping ratio

Relating problem specific parameters to prototype parameters
Transient vs Steady state

Consider the step, responses computed earlier. Identify the steady state and the transient parts.
2\textsuperscript{nd} order system \[ G(s) = \frac{C(s)}{R(s)} = \frac{K \omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

**Over damped**
- (two real distinct roots = two 1\textsuperscript{st} order systems with real poles)

**Critically damped**
- (a single pole of multiplicity two, highly unlikely, requires exact matching)

**Underdamped**
- (complex conjugate pair of poles, oscillatory behavior, most common)

  step response

\[ c_{\text{step}}(t) = K \left[ 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1 - \zeta^2}} \sin \left( \omega_d t + \tan^{-1} \left( \sqrt{1 - \zeta^2 / \zeta} \right) \right) \right] 1(t) \]

\[ c_\delta(t) = K \left[ \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin(\omega_d t) \right] 1(t) \]
Use of Prototypes

Too many examples to cover them all
We cover important prototypes
We develop intuition on the prototypes
We cover how to convert specific examples to prototypes
We transfer our insight, based on the study of the prototypes to the specific situations.
Transient-Response Specifications

1. Delay time, $t_d$: The time required for the response to reach half the final value the very first time.
2. Rise time, $t_r$: the time required for the response to rise from 10% to 90% (common for overdamped and 1st order systems); 5% to 95%; or 0% to 100% (common for underdamped systems); of its final value
1. Peak time, $t_p$:
2. Maximum (percent) overshoot, $M_p$:
3. Settling time, $t_s$
Derived relations for 2nd Order Systems

\[ \omega_d = \omega_n \sqrt{1 - \zeta^2} \]
\[ \sigma = \zeta \omega_n \]

\[ t_r = \frac{\pi - \beta}{\omega_d} \quad t_p = \frac{\pi}{\omega_d} \]
\[ \beta = \tan^{-1} \left( \frac{\omega_d}{\sigma} \right) \]

\[ M_p = e^{-\frac{\zeta \pi}{\sqrt{1 - \zeta^2}}} \times 100\% \]

\[ t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta \omega_n} \quad 2\% \]
\[ t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta \omega_n} \quad 5\% \]

Allowable \( M_p \) determines damping ratio.
Settling time then determines undamped natural frequency.
Theory is used to derive relationships between design specifications and prototype parameters.
Which are related to problem parameters.

See book for details. (Pg. 232)
Higher order system

PFEs have linear denominators.

- each term with a real pole has a time constant

- each complex conjugate pair of poles has a damping ratio and an undamped natural frequency.
Proportional control of plant with integrator

\[ G_C(s) = K_p, \quad G(s) = \frac{1}{s(Js + b)} \]
Integral control of Plant w disturbance

\[ G_c(s) = \frac{K}{s}, \quad G(s) = \frac{1}{s(Js + b)} \]
Proportional Control of plant w/o integrator

\[ G_C(s) = K, \quad G(s) = \frac{1}{Ts + 1} \]
Integral control of plant w/o integrator

\[ G_c(s) = \frac{K}{s}, \quad G(s) = \frac{1}{Ts + 1} \]
UNIT-III
STABILITY ANALYSIS IN S- DOMAIN
Chapter 6 – The Stability of Linear Feedback Systems

The issue of ensuring the stability of a closed-loop feedback system is central to control system design. Knowing that an unstable closed-loop system is generally of no practical value, we seek methods to help us analyze and design stable systems. A stable system should exhibit a bounded output if the corresponding input is bounded. This is known as bounded-input, bounded-output stability and is one of the main topics of this chapter.

The stability of a feedback system is directly related to the location of the roots of the characteristic equation of the system transfer function. The Routh–Hurwitz method is introduced as a useful tool for assessing system stability. The technique allows us to compute the number of roots of the characteristic equation in the right half-plane without actually computing the values of the roots. Thus we can determine stability without the added computational burden of determining characteristic root locations. This gives us a design method for determining values of certain system parameters that will lead to closed-loop stability. For stable systems we will introduce the notion of relative stability, which allows us to characterize the degree of stability.
The Concept of Stability

A stable system is a dynamic system with a bounded response to a bounded input.

Absolute stability is a stable/not stable characterization for a closed-loop feedback system. Given that a system is stable we can further characterize the degree of stability, or the relative stability.
The Concept of Stability

A necessary and sufficient condition for a feedback system to be stable is that all the poles of the system transfer function have negative real parts.

A system is considered marginally stable if only certain bounded inputs will result in a bounded output.
The Routh-Hurwitz Stability Criterion

It was discovered that all coefficients of the characteristic polynomial must have the same sign and non-zero if all the roots are in the left-hand plane.

These requirements are necessary but not sufficient. If the above requirements are not met, it is known that the system is unstable. But, if the requirements are met, we still must investigate the system further to determine the stability of the system.

The Routh-Hurwitz criterion is a necessary and sufficient criterion for the stability of linear systems.
The Routh-Hurwitz Stability Criterion

The Routh-Hurwitz criterion states that the number of roots of \( q(s) \) with positive real parts is equal to the number of changes in sign of the first column of the Routh array.

The characteristic equation, \( q(s) \), is given by:

\[
a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_1 s + a_0 = 0
\]

The Routh array is:

\[
\begin{array}{cccc}
  s^n & a_n & a_{n-2} & a_{n-4} \\
  s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} \\
  s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} \\
  s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} \\
  s^0 & h_{n-1} & \\
\end{array}
\]

The elements of the Routh array are calculated as follows:

- \( b_{n-1} = \frac{(a_{n-1})(a_{n-2}) - a_n(a_{n-3})}{a_{n-1}} = -1 \)
- \( b_{n-3} = -1 \frac{a_{n-2}}{a_{n-1}} \)
- \( c_{n-1} = -1 \frac{a_{n-1}}{b_{n-1}} \)
The Routh-Hurwitz Stability Criterion

Case One: No element in the first column is zero.

Example 6.1 Second-order system

The Characteristic polynomial of a second-order system is:

\[ q(s) = a_2 s^2 + a_1 s + a_0 \]

The Routh array is written as:

\[
\begin{array}{c|cc}
  & s^2 & a_2 & a_0 \\
  & s^1 & a_1 & 0 \\
  & s^0 & b_1 & 0 \\
\end{array}
\]

Where:

\[ b_1 = \frac{a_1 a_0 - (0) a_2}{a_1} = a_0 \]

Therefore the requirement for a stable second-order system is simply that all coefficients be positive or all the coefficients be negative.
The Routh-Hurwitz Stability Criterion

Case Two: Zeros in the first column while some elements of the row containing a zero in the first column are nonzero.

If only one element in the array is zero, it may be replaced with a small positive number $\varepsilon$ that is allowed to approach zero after completing the array.

$$q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

The Routh array is then:

$$\begin{array}{c|ccc}
  s^5 & 1 & 2 & 11 \\
  s^4 & 2 & 4 & 10 \\
  s^3 & b_1 & 6 & 0 \\
  s^2 & c_1 & 10 & 0 \\
  s^1 & d_1 & 0 & 0 \\
  s^0 & 10 & 0 & 0 \\
\end{array}$$

where:

$$b_1 = \frac{2 \cdot 2 - 1 \cdot 4}{2} = 0 = \varepsilon \quad c_1 = \frac{4\varepsilon - 2 \cdot 6}{\varepsilon} = \frac{-12}{\varepsilon} \quad d_1 = \frac{6 \cdot c_1 - 10\varepsilon}{c_1} = 6$$

There are two sign changes in the first column due to the large negative number calculated for $c_1$. Thus, the system is unstable because two roots lie in the right half of the plane.
The Routh-Hurwitz Stability Criterion

**Case Three:** Zeros in the first column, and the other elements of the row containing the zero are also zero.

This case occurs when the polynomial $q(s)$ has zeros located symmetrically about the origin of the s-plane, such as $(s+\sigma)(s-\sigma)$ or $(s+j\omega)(s-j\omega)$. This case is solved using the auxiliary polynomial, $U(s)$, which is located in the row above the row containing the zero entry in the Routh array.

$$q(s) = s^3 + 2s^2 + 4s + K$$

Routh array:

<table>
<thead>
<tr>
<th>$s^3$</th>
<th>1</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^2$</td>
<td>2</td>
<td>$K$</td>
</tr>
<tr>
<td>$s^1$</td>
<td>$\frac{8-K}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$s^0$</td>
<td>$K$</td>
<td>0</td>
</tr>
</tbody>
</table>

For a stable system we require that $0 < s < 8$

For the marginally stable case, $K=8$, the $s^1$ row of the Routh array contains all zeros. The auxiliary polynomial comes from the $s^2$ row.

$$U(s) = 2s^2 + Ks^0 = 2s^2 + 8 = 2(s^2 + 4) = 2(s + j2)(s - j2)$$

It can be proven that $U(s)$ is a factor of the characteristic polynomial:

$$\frac{q(s)}{U(s)} = \frac{s + 2}{2}$$

Thus, when $K=8$, the factors of the characteristic polynomial are:

$$q(s) = (s + 2)(s + j2)(s - j2)$$
The Routh-Hurwitz Stability Criterion

Case Four: Repeated roots of the characteristic equation on the jw-axis.

With simple roots on the jw-axis, the system will have a marginally stable behavior. This is not the case if the roots are repeated. Repeated roots on the jw-axis will cause the system to be unstable. Unfortunately, the routh-array will fail to reveal this instability.
A completely integrated, six-legged, micro robot system. The six-legged design provides maximum dexterity. Legs also provide a unique sensory system for environmental interaction. It is equipped with a sensor network that includes 150 sensors of 12 different types. The legs are instrumented so that the robot can determine the lay of the terrain, the surface texture, hardness, and even color. The gyro-stabilized camera and range finder can be used for gathering data beyond the robot’s immediate reach. This high-performance system is able to walk quickly, climb over obstacles, and perform dynamic motions. (Courtesy of IS Robotics Corporation.)
Example 6.5 Welding control

Using block diagram reduction we find that:

The Routh array is then:

\[
\begin{array}{ccc}
   & 1 & 11 & Ka \\
 s^4 & 6 & (K + 6) \\
 s^3 & b_3 & Ka \\
 b_3 & c_3 & \ \\
 s^1 & Ka & \ \\
 s^0 & \ \\
\end{array}
\]

where: \( b_3 = \frac{60 - K}{6} \) and \( c_3 = \frac{b_3(K + 6) - 6 \cdot Ka}{b_3} \)

For the system to be stable both \( b_3 \) and \( c_3 \) must be positive.

Using these equations a relationship can be determined for \( K \) at
The Relative Stability of Feedback Control Systems

It is often necessary to know the relative damping of each root to the characteristic equation. Relative system stability can be measured by observing the relative real part of each root. In this diagram r2 is relatively more stable than the pair of roots labeled r1.

One method of determining the relative stability of each root is to use an axis shift in the s-domain and then use the Routh array as shown in Example 6.6 of the text.
Problem statement: Design the turning control for a tracked vehicle. Select $K$ and $\alpha$ so that the system is stable. The system is modeled below.
Design Example: Tracked Vehicle Turning Control

The characteristic equation of this system is:

\[ 1 + G_c \cdot G(s) = 0 \]

or

\[ 1 + \frac{K(s + a)}{s(s + 1)(s + 2)(s + 5)} = 0 \]

Thus,

\[ s(s + 1)(s + 2)(s + 5) + K(s + a) = 0 \]

or

\[ s^4 + 8s^3 + 17s^2 + (K + 10)s + Ka = 0 \]

To determine a stable region for the system, we establish the Routh array

\[
\begin{array}{c|ccc}
 s^4 & 1 & 17 & Ka \\
 s^3 & 8 & (K + 10) & 0 \\
 s^2 & b_3 & Ka \\
 s^1 & c_3 \\
 s^0 & Ka \\
\end{array}
\]

where

\[ b_3 = \frac{126 - K}{8} \quad \text{and} \quad c_3 = \frac{b_3(K + 10) - 8Ka}{b_3} \]
### Design Example: Tracked Vehicle Turning Control

\[
\begin{array}{c|ccc}
  s^4 & 1 & 17 & Ka \\
  s^3 & 8 & (K+10) & 0 \\
  s^2 & b_3 & Ka & \\
  s^1 & c_3 & \\
  s^0 & Ka & \\
\end{array}
\]

where

\[
b_3 = \frac{126 - K}{8} \quad \text{and} \quad c_3 = \frac{b_3(K + 10) - 8Ka}{b_3}
\]

Therefore,

\[
K < 126 \\
K \cdot a > 0 \\
(K + 10)(126 - K) - 64Ka > 0
\]
In the preceding chapters we discussed how the performance of a feedback system can be described in terms of the location of the roots of the characteristic equation in the s-plane. We know that the response of a closed-loop feedback system can be adjusted to achieve the desired performance by judicious selection of one or more system parameters. It is very useful to determine how the roots of the characteristic equation move around the s-plane as we change one parameter.

The locus of roots in the s-plane can be determined by a graphical method. A graph of the locus of roots as one system parameter varies is known as a root locus plot. The root locus is a powerful tool for designing and analyzing feedback control systems and is the main topic of this chapter. We will discuss practical techniques for obtaining a sketch of a root locus plot by hand. We also consider computer-generated root locus plots and illustrate their effectiveness in the design process. The popular PID controller is introduced as a practical controller structure.

We will show that it is possible to use root locus methods for design when two or three parameters vary. This provides us with the opportunity to design feedback systems with two or three adjustable parameters. For example the PID controller has three adjustable parameters. We will also define a measure of sensitivity of a specified root to a small incremental change in a system parameter.
The Root Locus Method

\[
\frac{s^2 + 30s + 625}{s^8 + 100s^5 + 6400s^4 + 176000s^3 + 2280000s^2 + 136000000s}
\]
The root locus is a graphical procedure for determining the poles of a closed-loop system given the poles and zeros of a forward-loop system. Graphically, the locus is the set of paths in the complex plane traced by the closed-loop poles as the root locus gain is varied from zero to infinity.

In mathematical terms, given a forward-loop transfer function, \( KG(s) \) where \( K \) is the root locus gain, and the corresponding closed-loop transfer function

\[
\frac{KG(s)}{1 + KG(s)}
\]

the root locus is the set of paths traced by the roots of

\[
1 + KG(s) = 0
\]

as \( K \) varies from zero to infinity. As \( K \) changes, the solution to this equation changes. This equation is called the characteristic equation. This equation defines where the poles will be located for any value of the root locus gain, \( K \). In other words, it defines the characteristics of the system behavior for various values of controller gain.
The Root Locus Method
The Root Locus Method

\[ r = \text{complex root locations} \]
\[ K = \text{gain vector} \]
\[ 1 + KG(s) = 0 \]
\[ [r,K] = \text{rlocus(sys)} \]
The Root Locus Method

Root locus: common method

Root locus: alternate method
The Root Locus Method

Cross-hair placement at selected gain
Other two-pole locations for the same gain

Select a point in the graphics window

selected_point =
-2.0509 + 4.3228i
ans =
20.5775
No matter what we pick $K$ to be, the closed-loop system must always have $n$ poles, where $n$ is the number of poles of $G(s)$. The root locus must have $n$ branches, each branch starts at a pole of $G(s)$ and goes to a zero of $G(s)$.

If $G(s)$ has more poles than zeros (as is often the case), $m < n$ and we say that $G(s)$ has zeros at infinity. In this case, the limit of $G(s)$ as $s \to \infty$ is zero.

The number of zeros at infinity is $n-m$, the number of poles minus the number of zeros, and is the number of branches of the root locus that go to infinity (asymptotes).

Since the root locus is actually the locations of all possible closed loop poles, from the root locus we can select a gain such that our closed-loop system will perform the way we want. If any of the selected poles are on the right half plane, the closed-loop system will be unstable. The poles that are closest to the imaginary axis have the greatest influence on the closed-loop response, so even though the system has three or four poles, it may still act like a second or even first order system depending on the location(s) of the dominant pole(s).
Example

Note: This controller has adjustable gain. After this design is built we must anticipate that all values of K will be used. It is our responsibility to make sure that none of the possible K values will lead to instability.

\[ G(s) = \frac{K}{s} \quad H(s) = 1 \]

First, we must develop a transfer function for the entire control system.

\[ G_g(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{\left(\frac{K}{s}\right)}{1 + \left(\frac{K}{s}\right)(1)} = \frac{K}{s + K} \]

Next, we use the characteristic equation of the denominator to find the roots as the value of K varies. These can then be plotted on a complex plane. Note: the value of gain 'K' is normally found from 0 to +infinity.

\[ s + K = 0 \]

<table>
<thead>
<tr>
<th>K</th>
<th>root</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>etc..</td>
<td></td>
</tr>
</tbody>
</table>

Note: because all of the roots for all values of K are real negative this system will always be stable, and it will always tend to have a damped response. The larger the value of K, the more stable the system becomes.
MATLAB Example - Plotting the root locus of a transfer function

Consider an open loop system which has a transfer function of

\[ G(s) = \frac{s+7}{s(s+5)(s+15)(s+20)} \]

How do we design a feedback controller for the system by using the root locus method? Enter the transfer function, and the command to plot the root locus:

```matlab
num=[1 7];
den=conv(conv([1 0],[1 5]),conv([1 15],[1 20]));
rlocus(num,den)
axis([-22 3 -15 15])
```
Graphical Method

Given the system elements (you should assume negative feedback),

\[ G(s) = \frac{K}{s^2 + 3s + 2} \quad H(s) = 1 \]

Step 1: (put equation in standard form)

\[ 1 + G(s)H(s) = 1 + \left( \frac{K}{s^2 + 3s + 2} \right) (1) = 1 + K \frac{1}{(s+1)(s+2)} \]

Step 2: (find loci ending at infinity)

\[ m = 0 \quad n = 2 \quad \text{(from the poles and zeros of the previous step)} \]
\[ n - m = 2 \quad \text{(loci end at infinity)} \]

Step 3: (plot roots)

![Root Plot]

Step 4: (find asymptotes angles and real axis intersection)

\[ \beta(k) = \frac{180^\circ (2k+1)}{2} \quad k \in J[0, 1] \]

\[ \beta(0) = \frac{180^\circ (2(0)+1)}{2} = 90^\circ \]

\[ \beta(1) = \frac{180^\circ (2(1)+1)}{2} = 270^\circ \]

\[ \sigma = \frac{(0)(-1-2)}{2} = 0 \]

![Asymptotes Plot]
Graphical Method

Step 5: (find the breakout points for the roots)

\[ A = 1 \quad B = s^2 + 3s + 2 \]
\[ \frac{d}{ds}A = 0 \quad \frac{d}{ds}B = 2s + 3 \]
\[ A\left(\frac{d}{ds}B\right) - B\left(\frac{d}{ds}A\right) = 0 \]
\[ 1(2s + 3) - (s^2 + 3s + 2)(0) = 0 \]
\[ 2s + 3 = 0 \]
\[ s = -1.5 \]

Note: because the loci do not intersect the imaginary axis, we know the system will be stable, so step 6 is not necessary, but we it will be done for illustrative purposes.

Step 6: (find the imaginary intercepts)

\[ 1 + G(s)H(s) = 0 \]
\[ 1 + K\frac{1}{s^2 + 3s + 2} = 0 \]
\[ s^2 + 3s + 2 + K = 0 \]
\[ (j\omega)^2 + 3(j\omega) + 2 + K = 0 \]
\[ -\omega^2 + 3j\omega + 2 + K = 0 \]
\[ \omega^2 + \omega(-3j) + (-2 - K) = 0 \]
\[ \omega = \frac{3j \pm \sqrt{(-3j)^2 - 4(-2 - K)}}{2} = \frac{3j \pm \sqrt{-9 + 8 + 4K}}{2} = \frac{3j \pm \sqrt{4K - 1}}{2} \]

In this case the frequency has an imaginary value. This means that there will be no frequency that will intercept the imaginary axis.
Root Locus Design GUI (rltool)

The Root Locus Design GUI is an interactive graphical tool to design compensators using the root locus method. This GUI plots the locus of the closed-loop poles as a function of the compensator gains. You can use this GUI to add compensator poles and zeros and analyze how their location affects the root locus and various time and frequency domain responses. Click on the various controls on the GUI to see what they do.
UNIT-IV
FREQUENCY RESPONSE ANALYSIS
Frequency Response Methods and Stability

In previous chapters we examined the use of test signals such as a step and a ramp signal. In this chapter we consider the steady-state response of a system to a sinusoidal input test signal. We will see that the response of a linear constant coefficient system to a sinusoidal input signal is an output sinusoidal signal at the same frequency as the input. However, the magnitude and phase of the output signal differ from those of the input sinusoidal signal, and the amount of difference is a function of the input frequency. Thus we will be investigating the steady-state response of the system to a sinusoidal input as the frequency varies.

We will examine the transfer function $G(s)$ when $s = jw$ and develop methods for graphically displaying the complex number $G(jw)$ as $w$ varies. The Bode plot is one of the most powerful graphical tools for analyzing and designing control systems, and we will cover that subject in this chapter. We will also consider polar plots and log magnitude and phase diagrams. We will develop several time-domain performance measures in terms of the frequency response of the system as well as introduce the concept of system bandwidth.
Introduction

The frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal. The sinusoid is a unique input signal, and the resulting output signal for a linear system, as well as signals throughout the system, is sinusoidal in the steady-state; it differs from the input waveform only in amplitude and phase.
Frequency Response Plots

Polar Plots

\[ \text{Im}(G) = X(\omega) \]

\[ \text{Re}(G) = R(\omega) \]
Frequency Response Plots

Polar Plots

\[ \omega := -1000, -999, \ldots, 1000 \]
\[ j := \sqrt{-1} \]
\[ R := 1 \]
\[ C := 0.01 \]
\[ \omega_1 := \frac{1}{R \cdot C} \]

\[ G(\omega) := \frac{1}{(j \cdot \frac{\omega}{\omega_1}) + 1} \]
Frequency Response Plots

Polar Plots

\[ \omega := 0, 1 \ldots 1000 \quad \tau := 0.5 \quad K := 10C \]

\[ G_1(\omega) := \frac{K}{\tau} \frac{1}{j\omega + \frac{1}{\tau}} \]

Polar plot for \( G(j\omega) = Klj\omega(j\omega\tau + 1) \). Note that \( \omega = \infty \) at the origin.
Polar plot for $G(j\omega) = K / j\omega(j\omega\tau + 1)$. Note that $\omega = \infty$ at the origin.
Frequency Response Plots

Bode Plots – Real Poles

Bode diagram for $G(j\omega) = 1/(j\omega\tau + 1)$: (a) magnitude plot and (b) phase plot.
Frequency Response Plots

Bode Plots – Real Poles

\[ \omega := \frac{0.1}{\tau}, \frac{0.11}{\tau} \ldots 1000 \]
\[ j := \sqrt{-1} \quad R := 1 \quad C := 0.01 \quad \tau := R \cdot C \]

\[ G(\omega) := \frac{1}{j \cdot \omega \cdot \tau + 1} \]
\[ \omega_1 := \frac{1}{\tau} \quad \omega_1 = 100 \quad \text{(break frequency or corner frequency)} \]

\[ 20 \cdot \log(|G(\omega)|) \]

(break frequency or corner frequency)
Frequency Response Plots

Bode Plots – Real Poles

\[ \phi(\omega) := -\tan(\omega \cdot \tau) \]
Frequency Response Plots

Bode Plots – Real Poles (Graphical Construction)

Asymptotic curve for $( j \omega \tau + 1)^{-1}$. 
Frequency Response Plots

Bode Plots – Real Poles

Bode diagram for \((j\omega)^\pm N\).
Bode diagram for \((1 + j\omega \tau)^{-1}\).
Frequency Response Plots

Bode Plots – Real Poles

Magnitude:

\[ \text{db}(G, \omega) := 20 \cdot \log(|G(j \cdot \omega)|) \]

Phase shift:

\[ \text{ps}(G, \omega) := \frac{180}{\pi} \cdot \arg(G(j \cdot \omega)) - 360 \cdot (\text{if}(\arg(G(j \cdot \omega)) \geq 0, 1, 0)) \]

Assume

\[ K := 2 \quad G(s) := \frac{K}{s \cdot (1 + s) \cdot \left(1 + \frac{s}{3}\right)} \]

Next, choose a frequency range for the plots (use powers of 10 for convenient plotting):

lowest frequency (in Hz): \( \omega_{\text{start}} := .01 \) number of points: \( N := 50 \)

highest frequency (in Hz): \( \omega_{\text{end}} := 100 \)

step size: \( r := \log\left(\frac{\omega_{\text{start}}}{\omega_{\text{end}}}\right) \cdot \frac{1}{N} \)

range for plot: \( i := 0 \ldots N \) range variable: \( \omega_i := \omega_{\text{end}} \cdot 10^{i \cdot r} \quad s_i := j \cdot \omega_i \)
Frequency Response Plots

Bode Plots – Real Poles

range for plot: \( i := 0 \ldots N \) range variable: \( \omega_i := \omega_{\text{end}} \cdot 10^{i \cdot r} \) \( s_i := j \cdot \omega_i \)
Bode diagram for $G(j\omega) = \left[1 + \left(2\frac{\zeta}{\omega_n}\right)j\omega + \left(j\omega/\omega_n\right)^2\right]^{-1}$. 
Frequency Response Plots

Bode Plots – Complex Poles

Bode diagram for $G(j\omega) = \left[ 1 + (2\zeta/\omega_n) j\omega + (j\omega/\omega_n)^2 \right]^{-1}$. 
Frequency Response Plots

Bode Plots – Complex Poles

\[ \omega_r = \omega_n \cdot \sqrt{1 - 2 \cdot \zeta^2} \quad \zeta < 0.707 \]

\[ M_p \omega = \left| G(\omega_r) \right| = \frac{1}{\left(2 \cdot \zeta \cdot \sqrt{1 - \zeta^2}\right)} \quad \zeta < 0.707 \]
Frequency Response Plots

Bode Plots – Complex Poles

\[ \omega_r = \omega_n \cdot \sqrt{1 - 2 \cdot \zeta^2} \quad \zeta < 0.707 \]

\[ M_{p\omega} = \left| G(\omega_r) \right| = \frac{1}{\left( 2 \cdot \zeta \cdot \sqrt{1 - \zeta^2} \right)} \quad \zeta < 0.707^{1.75} \]
Frequency Response Plots

Bode Plots – Complex Poles
Frequency Response Plots

Bode Plots – Complex Poles
Performance Specification In the Frequency Domain

\[ R(s) \xrightarrow{+} \frac{\omega_n^2}{s(s + 2\zeta\omega_n)} \xrightarrow{-} Y(s) \]

Graph:
- \(20 \log M_{p_\omega}\) vs. \(\omega\)
- \(20 \log |T|, \text{dB}\) vs. \(\omega\)
- \(\omega_r\) and \(\omega_B\) as significant frequencies
Performance Specification In the Frequency Domain

\( \omega := .1, .11, .2 \) \hspace{1cm} K := 2 \hspace{1cm} j := \sqrt{-1} \n
\[ G(\omega) := \frac{K}{j\cdot\omega \cdot (j\cdot\omega + 1)(j\cdot\omega + 2)} \] \hspace{1cm} \text{Bode1}(\omega) := 20 \cdot \log(|G(\omega)|) \n
\[ T(\omega) := \frac{G(\omega)}{1 + G(\omega)} \] \hspace{1cm} \text{Bode2}(\omega) := 20 \cdot \log(|T(\omega)|)
Performance Specification In the Frequency Domain

\[ w := 4 \]

Finding the Resonance Frequency

Given

\[ 20 \log(|T(w)|) = 5.282 \]

\[ wr := \text{Find}(w) \quad wr = 0.813 \]

\[ Mpw := 1 \]

Finding Maximum value of the frequency response

Given

\[ 20 \log(Mpw) = 5.282 \]

\[ Mpw := \text{Find}(Mpw) \quad Mpw = 1.837 \]

Closed-Loop Bode Diagram
Assume that the system has dominant second-order roots

\[ \zeta = 0.1 \]

Finding the damping factor

Given

\[ M_{pw} = \left[ 2 \cdot \zeta \left( \sqrt{1 - \zeta^2} \right) \right]^{-1} \]

\[ \zeta := \text{Find}(\zeta) \]

\[ \zeta = 0.284 \]

Finding the damping factor

\[ \omega_n = 0.1 \]

Finding the natural frequency

Given

\[ w_r = \omega_n \sqrt{1 - 2 \cdot \zeta^2} \]

\[ \omega_n := \text{Find}(\omega_n) \]

\[ \omega_n = 0.888 \]
Performance Specification In the Frequency Domain

\[ \frac{\omega_B}{\omega_n} \approx -1.19\zeta + 1.85 \]
Performance Specification
In the Frequency Domain

\[ GH_1(\omega) = \frac{5}{j \cdot \omega \cdot (0.5 \cdot j \cdot \omega + 1) \cdot \left( j \cdot \frac{\omega}{6} + 1 \right)} \]
Performance Specification In the Frequency Domain
Example
Performance Specification In the Frequency Domain
Example
Performance Specification In the Frequency Domain
Example
Performance Specification In the Frequency Domain
Example
Frequency Response Methods Using MATLAB
Frequency Response Methods Using MATLAB

\[ G(s) = \text{sys} \]

\[ [\text{mag}, \text{phase}, w] = \text{bode}(\text{sys}, w) \]

User-supplied frequency (optional)
Frequency Response Methods Using MATLAB

$n$ points between $10^a$ and $10^b$

$$w = \text{logspace}(a, b, n)$$

Logarithmically spaced vector

Example

Generate 200 points between 0.1 and 1000.

```matlab
>> w = logspace(-1, 3, 200);
>> bode(sys, w);
```

![Magnitude vs. Frequency Graph](image)
% Bode plot script for Figure 8.22
num=5*[0.1 1];
f1=[1 0]; f2=[0.5 1]; f3=[1/2500 .6/50 1];
den=conv(f1,conv(f2,f3));
sys=tf(num,den);
bode(sys)
Frequency Response Methods Using MATLAB

(a) The relationship between \( M_{p\omega} \) and \( \zeta \) for a second-order system.

```
zeta=[0.15:0.01:0.7];
w_r_over_w_n=sqrt(1-2*zeta.^2);
M_p=(2*zeta .* sqrt(1-zeta.^2)).^(-1);

% Generate plots
subplot(211), plot(zeta, M_p), grid
xlabel('\zeta'), ylabel('M_{p\omega}')
subplot(212), plot(zeta, w_r_over_w_n), grid
xlabel('\zeta'), ylabel('\omega_r/\omega_n')
```

(a) The relationship between \( (M_{p\omega}, \omega_r) \) and \( (\zeta, \omega_n) \) for a second-order system. (b) MATLAB script.
Frequency Response Methods Using MATLAB

- **Initial gain $K$**

- **Update $K$**

- Compute closed-loop transfer function:
  
  $$T(s) = \frac{K}{s(s + 1)(s + 2) + K}.$$ 

- **Closed-loop Bode diagram**

- **Check time domain specs:**
  
  $$T_s = \frac{4}{\xi \omega_n},$$
  
  $$M_p = 1 + e^{-\xi \pi/\sqrt{1-\xi^2}}.$$ 

  If satisfied, then exit and continue analysis.

- **Establish relationship between frequency domain specs and time domain specs.**

- Determine $M_{P_{\omega}}$ and $\omega_r$.

- Determine $\omega_n$ and $\xi$. 
Frequency Response Methods Using MATLAB

```
>> K=2; engrave1
mp =
  1.8371
wr =
  0.8171
>>
>>
>> zeta=0.29; wn=0.88; engrave2
ts =
  15.6740
po =
  38.5979
```

- `num=[K]; den=[1 3 2 K];`  
- `sys=tf(num,den);`  
- `w=logspace(-1,1,400);`  
- `[mag,phase,w]=bode(sys,w);`  
- `[mp,l]=max(mag);wr=w(l);`  
- `mp,wr`

- Closed-loop transfer function
- Closed-loop Bode diagram

Determine $\omega_n$ and $\zeta$ from Fig. 8.11 using $M_p$ and $\omega_r$.

```
ts=4/zeta/wn
po=100*exp(-zeta*pi/sqrt(1-zeta^2))
```

Check specs and iterate, if necessary.
Bode Plots

Bode plot is the representation of the magnitude and phase of \( G(j*\omega) \) (where the frequency vector \( \omega \) contains only positive frequencies). To see the Bode plot of a transfer function, you can use the MATLAB `bode` command.

For example,

```
bode(50,[1 9 30 40])
```

displays the Bode plots for the transfer function:

\[
50 / (s^3 + 9 s^2 + 30 s + 40)
\]
Gain and Phase Margin

Let's say that we have the following system:

where $K$ is a variable (constant) gain and $G(s)$ is the plant under consideration.

The gain margin is defined as the change in open loop gain required to make the system unstable. Systems with greater gain margins can withstand greater changes in system parameters before becoming unstable in closed loop. Keep in mind that unity gain in magnitude is equal to a gain of zero in dB.

The phase margin is defined as the change in open loop phase shift required to make a closed loop system unstable.

The phase margin is the difference in phase between the phase curve and $-180$ deg at the point corresponding to the frequency that gives us a gain of $0$dB (the gain cross over frequency, $W_{gc}$).

Likewise, the gain margin is the difference between the magnitude curve and $0$dB at the point corresponding to the frequency that gives us a phase of $-180$ deg (the phase cross over frequency, $W_{pc}$).
Gain and Phase Margin

Graph showing gain margin and phase margin with Wgc and Wpc labels.
Gain and Phase Margin

We can find the gain and phase margins for a system directly, by using MATLAB. Just enter the `margin` command. This command returns the gain and phase margins, the gain and phase cross over frequencies, and a graphical representation of these on the Bode plot.

```
margin(50,[1 9 30 40])
```
Gain and Phase Margin

Magnitude:

$$\text{db}(G, \omega) := 20 \cdot \log(|G(j \cdot \omega)|)$$

Phase shift:

$$\text{ps}(G, \omega) := \frac{180}{\pi} \cdot \arg(G(j \cdot \omega)) - 360 \cdot (\text{if}(\arg(G(j \cdot \omega)) \geq 0, 1, 0))$$

Assume

$$K := 2 \quad G(s) := \frac{K}{s \cdot (1 + s) \cdot \left(1 + \frac{s}{3}\right)}$$

Next, choose a frequency range for the plots (use powers of 10 for convenient plotting):

lowest frequency (in Hz): \( \omega_{\text{start}} := .01 \)  
number of points: \( N := 50 \)

highest frequency (in Hz): \( \omega_{\text{end}} := 100 \)

step size: \( r := \log\left(\frac{\omega_{\text{start}}}{\omega_{\text{end}}}\right) \cdot \frac{1}{N} \)

range for plot: \( i := 0..N \)  
range variable: \( \omega_i := \omega_{\text{end}} \cdot 10^{i \cdot r} \) \( s_i := j \cdot \omega_i \)
Gain and Phase Margin

Guess for crossover frequency: $\omega_c := 1$

Solve for the gain crossover frequency:

$$\omega_c := \text{root}\left(\text{db}\left( G, \omega_c \right), \omega_c \right) \quad \omega_c = 1.193$$

Calculate the phase margin

$$\text{pm} := \text{ps}\left( G, \omega_c \right) + 180 \quad \text{pm} = 18.265 \text{ degrees}$$

Gain Margin

Now using the phase angle plot, estimate the frequency at which the phase shift crosses 180 degrees:

$$\omega_{gm} := 1.8$$

Solve for $\omega$ at the phase shift point of 180 degrees:

$$\omega_{gm} := \text{root}\left( \text{ps}\left( G, \omega_{gm} \right) + 180, \omega_{gm} \right)$$

$$\omega_{gm} = 1.732$$

Calculate the gain margin

$$\text{gm} := -\text{db}\left( G, \omega_{gm} \right) \quad \text{gm} = 6.021$$
The Nyquist Stability Criterion

The Nyquist plot allows us also to predict the stability and performance of a closed-loop system by observing its open-loop behavior. The Nyquist criterion can be used for design purposes regardless of open-loop stability (Bode design methods assume that the system is stable in open loop). Therefore, we use this criterion to determine closed-loop stability when the Bode plots display confusing information.

The Nyquist diagram is basically a plot of $G(j\omega)$ where $G(s)$ is the open-loop transfer function and $\omega$ is a vector of frequencies which encloses the entire right-half plane. In drawing the Nyquist diagram, both positive and negative frequencies (from zero to infinity) are taken into account. In the illustration below we represent positive frequencies in red and negative frequencies in green. The frequency vector used in plotting the Nyquist diagram usually looks like this (if you can imagine the plot stretching out to infinity):

However, if we have open-loop poles or zeros on the jw axis, $G(s)$ will not be defined at those points, and we must loop around them when we are plotting the contour. Such a contour would look as follows:
The Cauchy criterion

The Cauchy criterion (from complex analysis) states that when taking a closed contour in the complex plane, and mapping it through a complex function $G(s)$, the number of times that the plot of $G(s)$ encircles the origin is equal to the number of zeros of $G(s)$ enclosed by the frequency contour minus the number of poles of $G(s)$ enclosed by the frequency contour. Encirclements of the origin are counted as positive if they are in the same direction as the original closed contour or negative if they are in the opposite direction.

When studying feedback controls, we are not as interested in $G(s)$ as in the closed-loop transfer function:

$$
\frac{G(s)}{1 + G(s)}
$$

If $1+ G(s)$ encircles the origin, then $G(s)$ will enclose the point -1. Since we are interested in the closed-loop stability, we want to know if there are any closed-loop poles (zeros of $1 + G(s)$) in the right-half plane.

Therefore, the behavior of the Nyquist diagram around the -1 point in the real axis is very important; however, the axis on the standard *nyquist* diagram might make it hard to see what's happening around this point.
Gain and Phase Margin

Gain Margin is defined as the change in open-loop gain expressed in decibels (dB), required at 180 degrees of phase shift to make the system unstable. First of all, let's say that we have a system that is stable if there are no Nyquist encirclements of -1, such as:

\[
\frac{50}{s^3 + 9s^2 + 30s + 40}
\]

Looking at the roots, we find that we have no open loop poles in the right half plane and therefore no closed-loop poles in the right half plane if there are no Nyquist encirclements of -1. Now, how much can we vary the gain before this system becomes unstable in closed loop?

The open-loop system represented by this plot will become unstable in closed loop if the gain is increased past a certain boundary.
The Nyquist Stability Criterion

and that the Nyquist diagram can be viewed by typing:

```plaintext
nyquist (50, [1 9 30 40 ])
```
Phase margin as the change in open-loop phase shift required at unity gain to make a closed-loop system unstable.

From our previous example we know that this particular system will be unstable in closed loop if the Nyquist diagram encircles the -1 point. However, we must also realize that if the diagram is shifted by theta degrees, it will then touch the -1 point at the negative real axis, making the system marginally stable in closed loop. Therefore, the angle required to make this system marginally stable in closed loop is called the phase margin (measured in degrees). In order to find the point we measure this angle from, we draw a circle with radius of 1, find the point in the Nyquist diagram with a magnitude of 1 (gain of zero dB), and measure the phase shift needed for this point to be at an angle of 180 deg.
w := -100, -99.9..100 \quad j := \sqrt{-1} \quad s(w) := j \cdot w \quad f(w) := -1

G(w) := \frac{504.6}{s(w)^3 + 9 \cdot s(w)^2 + 30 \cdot s(w) + 40}
Consider the Negative Feedback System

Remember from the Cauchy criterion that the number $N$ of times that the plot of $G(s)H(s)$ encircles $-1$ is equal to the number $Z$ of zeros of $1 + G(s)H(s)$ enclosed by the frequency contour minus the number $P$ of poles of $1 + G(s)H(s)$ enclosed by the frequency contour ($N = Z - P$).

Keeping careful track of open- and closed-loop transfer functions, as well as numerators and denominators, you should convince yourself that:

- the zeros of $1 + G(s)H(s)$ are the poles of the closed-loop transfer function
- the poles of $1 + G(s)H(s)$ are the poles of the open-loop transfer function.

The Nyquist criterion then states that:

- $P =$ the number of open-loop (unstable) poles of $G(s)H(s)$
- $N =$ the number of times the Nyquist diagram encircles $-1$
- clockwise encirclements of $-1$ count as positive encirclements
- counter-clockwise (or anti-clockwise) encirclements of $-1$ count as negative encirclements
- $Z =$ the number of right half-plane (positive, real) poles of the closed-loop system

The important equation which relates these three quantities is:

$$Z = P + N$$
The Nyquist Stability Criterion - Application

Knowing the number of right-half plane (unstable) poles in open loop (P), and the number of encirclements of -1 made by the Nyquist diagram (N), we can determine the closed-loop stability of the system.

If \( Z = P + N \) is a positive, nonzero number, the closed-loop system is unstable.

We can also use the Nyquist diagram to find the range of gains for a closed-loop unity feedback system to be stable. The system we will test looks like this:

where \( G(s) \) is :

\[
\frac{s^2 + 10s + 24}{s^2 - 8s + 15}
\]
This system has a gain $K$ which can be varied in order to modify the response of the closed-loop system. However, we will see that we can only vary this gain within certain limits, since we have to make sure that our closed-loop system will be stable. This is what we will be looking for: the range of gains that will make this system stable in the closed loop.

The first thing we need to do is find the number of positive real poles in our open-loop transfer function:

```matlab
c roots([1 -8 15])
an s =
5
3
```

The poles of the open-loop transfer function are both positive. Therefore, we need two anti-clockwise ($N = -2$) encirclements of the Nyquist diagram in order to have a stable closed-loop system ($Z = P + N$). If the number of encirclements is less than two or the encirclements are not anti-clockwise, our system will be unstable.

Let's look at our Nyquist diagram for a gain of 1:

```matlab
cyquist([1 10 24], [1 -8 15])
```

There are two anti-clockwise encirclements of -1. Therefore, the system is stable for a gain of 1.
The Nyquist Stability Criterion

MathCAD Implementation

\( w := -100, -99.9..100 \quad j := \sqrt{-1} \quad s(w) := j \cdot w \)

\[ G(w) := \frac{s(w)^2 + 10 \cdot s(w) + 24}{s(w)^2 - 8 \cdot s(w) + 15} \]

There are two anti-clockwise encirclements of -1.
Therefore, the system is stable for a gain of 1.
The Nyquist Stability Criterion

```matlab
>> num = [0.5]; den = [1 2 1 0.5];
>> sys = tf(num, den);
>> nyquist(sys)
```

![Nyquist Plot](image)
Open and closed-loop frequency responses are related by:

\[ T(j\omega) = \frac{G(j\omega)}{1 + G(j\omega)} \]

\[ M_{pw} = \frac{1}{2\cdot\zeta\cdot\sqrt{1 - \zeta^2}} \quad \zeta < 0.707 \]

\[ G(\omega) = u + j \cdot v \quad M = M(\omega) \]

\[ M(\omega) = \left| \frac{G(j\omega)}{1 + G(j\omega)} \right| = \left| \frac{u + jv}{1 + u + jv} \right| = \frac{\sqrt{u^2 + v^2}}{\sqrt{(1 + u)^2 + v^2}} \]

Squaring and rearranging

\[ \left( u - \frac{M^2}{1 - M^2} \right)^2 + v^2 = \left( \frac{M}{1 - M^2} \right)^2 \]

which is the equation of a circle on u-v plane with a center at

\[ u = \frac{M^2}{1 - M^2} \quad v = 0 \]
Time-Domain Performance Criteria Specified In The Frequency Domain

Polar plot of $G(j\omega)$ for two values of a gain ($K_2 > K_1$).

Closed-loop frequency response of $T(j\omega) = G(j\omega)/1 + G(j\omega)$. Note that $K_2 > K_1$. 
The Nichols Stability Method

Polar Stability Plot - Nichols Mathcad Implementation

This example makes a polar plot of a transfer function and draws one contour of constant closed-loop magnitude. To draw the plot, enter a definition for the transfer function \( G(s) \):

\[
G(s) := \frac{45000}{s \cdot (s + 2) \cdot (s + 30)}
\]

The frequency range defined by the next two equations provides a logarithmic frequency scale running from 1 to 100. You can change this range by editing the definitions for \( m \) and \( \omega_m \):

\[
m := 0..100 \quad \omega_m := 10^{0.02 \cdot m}
\]

Now enter a value for \( M \) to define the closed-loop magnitude contour that will be plotted.

\[
M := 1.1
\]

Calculate the points on the \( M \)-circle:

\[
MC_m := \left( \frac{-M^2}{M^2 - 1} + \left| \frac{M}{M^2 - 1} \right| \cdot \exp(2\pi j \cdot 01 \cdot m) \right)
\]

The first plot shows \( G \), the contour of constant closed-loop magnitude.
The Nichols Stability Method

The first plot shows $G$, the contour of constant closed-loop magnitude $M$, and the Nyquist of the open loop system.
Nichols chart. The phase curves for the closed-loop system are shown in color.
The Nichols Stability Method

\[ G(\omega) := \frac{1}{j\omega(1 + j\omega)(0.2j\omega + 1)} \]

\[ M_{pw} := 2.5 \text{ dB} \quad \omega_r := 0.8 \]

The closed-loop phase angle at \( \omega_r \) is equal to -72 degrees and \( \omega_b = 1.33 \).
The closed-loop phase angle at \( \omega_b \) is equal to -142 degrees.

Nichols diagram for \( G(j\omega) = \frac{1}{j\omega(j\omega + 1)(0.2j\omega + 1)} \). Three points on curve are shown for \( \omega = 0.5, 0.8, \) and 1.35, respectively.
The Nichols Stability Method

\[ G(\omega) := \frac{0.64}{j\omega \left[ (j\omega)^2 + j\omega + 1 \right]} \]

Phase Margin = 30 degrees

On the basis of the phase we estimate \( \zeta := 0.30 \)

\( M_{pw} := 9 \text{ dB} \quad M_{pw} := 2.8 \quad \omega_r := 0.88 \)

From equation

\[ M_{pw} = \frac{1}{2\zeta \sqrt{1 - \zeta^2}} \quad \zeta := 0.18 \]

We are confronted with conflicting

The apparent conflict is caused by the nature of \( G(j\omega) \) which slopes rapidly toward 180 degrees line from the 0-dB axis. The designer must use the frequency-domain-time-domain correlation with caution
The Nichols Stability Method

Magnitude of $\frac{G}{1+G} = -18 \text{ dB}$
A closed-loop control system example for Nyquist and Bode with relative stability.
Examples - Bode

\[ \frac{0.5}{s^3 + 2s^2 + s + 0.5} \]

[Mag, phase, w] = bode(sys);

or

[Gm, Pm, Wcg, Wcp] = margin(sys);

Example

num = [0.5]; den = [1 2 1 0.5];
sys = tf(num, den);
margin(sys);

Gm = gain margin (dB)
Pm = phase margin (deg)
Wcg = freq. for phase = -180
Wcp = freq. for gain = 0 dB
Examples - Bode

```
num=[0.5];
den=[1 2 1 0.5];
sys=tf(num,den);
% w=logspace(-1,1,200);
% [mag,phase,w]=bode(sys,w);
% margin(mag,phase,w);
```
Examples – Bode and Nyquist

Open-loop system

Specify frequency range

Gain margin

Phase margin

The Nyquist plot of
\[ G(s) = \frac{0.5}{s^3 + 2s^2 + s + 0.5} \]

Compute gain and phase margins.

Nyquist plot

Label gain and phase margins on plot.
Examples - Nichols

\[ G(s) = \text{sys} \]

\[ [\text{mag}, \text{phase}, \text{w}] = \text{nichols(sys, w)} \]

User-supplied frequency (optional)
Examples - Nichols

num=[1]; den=[0.2 1.2 1 0 ];
sys=tf(num,den);
w=logspace(-1,1,400);
nichols(sys,w);
ngrid

Set up to generate Fig. 9.27
Plot Nichols chart and add grid lines.
The Design of Feedback Control Systems

PID

Compensation Networks
Different Types of Feedback Control

On-Off Control

This is the simplest form of control.
Proportional Control
A proportional controller attempts to perform better than the On-off type by applying power in proportion to the difference in temperature between the measured and the set-point. As the gain is increased the system responds faster to changes in set-point but becomes progressively underdamped and eventually unstable. The final temperature lies below the set-point for this system because some difference is required to keep the heater supplying power.
Proportional, Derivative Control

The stability and overshoot problems that arise when a proportional controller is used at high gain can be mitigated by adding a term proportional to the time-derivative of the error signal. The value of the damping can be adjusted to achieve a critically damped response.
Proportional+Integral+Derivative Control

Although PD control deals neatly with the overshoot and ringing problems associated with proportional control it does not cure the problem with the steady-state error. Fortunately it is possible to eliminate this while using relatively low gain by adding an integral term to the control function which becomes
The Characteristics of P, I, and D controllers

A proportional controller (Kp) will have the effect of reducing the rise time and will reduce, but never eliminate, the steady-state error.

An integral control (Ki) will have the effect of eliminating the steady-state error, but it may make the transient response worse.

A derivative control (Kd) will have the effect of increasing the stability of the system, reducing the overshoot, and improving the transient response.
Proportional Control
By only employing proportional control, a steady state error occurs.

Proportional and Integral Control
The response becomes more oscillatory and needs longer to settle, the error disappears.

Proportional, Integral and Derivative Control
All design specifications can be reached.
The Characteristics of P, I, and D controllers

<table>
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<tr>
<th>CL RESPONSE</th>
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<th>OVERSHOOT</th>
<th>SETTLING TIME</th>
<th>S-S ERROR</th>
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<tr>
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Tips for Designing a PID Controller

1. Obtain an open-loop response and determine what needs to be improved
2. Add a proportional control to improve the rise time
3. Add a derivative control to improve the overshoot
4. Add an integral control to eliminate the steady-state error
5. Adjust each of Kp, Ki, and Kd until you obtain a desired overall response.

Lastly, please keep in mind that you do not need to implement all three controllers (proportional, derivative, and integral) into a single system, if not necessary. For example, if a PI controller gives a good enough response (like the above example), then you don't need to implement derivative controller to the system. Keep the controller as simple as possible.
Open-Loop Control - Example

\[ G(s) = \frac{1}{s^2 + 10s + 20} \]

num=1;
den=[1 10 20];
step(num,den)
Proportional Control - Example

The proportional controller (Kp) reduces the rise time, increases the overshoot, and reduces the steady-state error.

MATLAB Example

\[ T(s) = \frac{Kp}{s^2 + 10s + (20 + Kp)} \]

Kp=300;
num=[Kp];
den=[1 10 20+Kp];
t=0:0.01:2;
step(num,den,t)
Proportional - Derivative - Example

The derivative controller (Kd) reduces both the overshoot and the settling time.

MATLAB Example

\[ T(s) = \frac{Kd \cdot s + Kp}{s^2 + (10 + Kd) \cdot s + (20 + Kp)} \]

Kp=300;
Kd=10;
um=[Kd Kp];
den=[1 10+Kd 20+Kp];
t=0:0.01:2;
step(num,den,t)
Proportional - Integral - Example

The integral controller (Ki) decreases the rise time, increases both the overshoot and the settling time, and eliminates the steady-state error.

MATLAB Example

\[ T(s) = \frac{K_p \cdot s + Ki}{s^3 + 10s^2 + (20 + K_p)\cdot s + Ki} \]

\begin{align*}
K_p &= 30; \\
Ki &= 70; \\
num &= [K_p \ Ki]; \\
den &= [1 \ 10 \ 20+K_p \ Ki]; \\
t &= 0:0.01:2; \\
step(num,den,t)
\end{align*}

Ki=70

Ki=100
Syntax

rltool
rltool(sys)
rltool(sys,comp)
RLTOOL
 RLTOOL

[Image of Root Locus Design interface with a circle and Gain set to 0.6155]
RLTOOL

LTI Viewer configuration changed from 4 to 5.
Right-click on any response plot axes to access the LTI Viewer controls.
Example - Practice

Consider the following configuration:
Example - Practice

The design a system for the following specifications:

- Zero steady state error
- Settling time within 5 seconds
- Rise time within 2 seconds
- Only some overshoot permitted
Lead or Phase-Lead Compensator Using Root Locus

A first-order lead compensator can be designed using the root locus. A lead compensator in root locus form is given by

\[ G_c(s) = \frac{(s + z)}{(s + p)} \]

where the magnitude of \( z \) is less than the magnitude of \( p \). A phase-lead compensator tends to shift the root locus toward the left half plane. This results in an improvement in the system's stability and an increase in the response speed.

When a lead compensator is added to a system, the value of this intersection will be a larger negative number than it was before. The net number of zeros and poles will be the same (one zero and one pole are added), but the added pole is a larger negative number than the added zero. Thus, the result of a lead compensator is that the asymptotes' intersection is moved further into the left half plane, and the entire root locus will be shifted to the left. This can increase the region of stability as well as the response speed.
In Matlab a phase lead compensator in root locus form is implemented by using the transfer function in the form
\[
\text{numlead}=\text{kc}*[1 \; z]; \\
\text{denlead}=[1 \; p];
\]
and using the conv() function to implement it with the numerator and denominator of the plant
\[
\text{newnum}=\text{conv(num,numlead)}; \\
\text{newden}=\text{conv(den,denlead)};
\]
A first-order phase-lead compensator can be designed using the frequency response. A lead compensator in frequency response form is given by

$$G_c(s) = \frac{(1 + \alpha \cdot \tau \cdot s)}{\alpha \cdot (1 + \tau \cdot s)}$$

$p = \frac{1}{\tau}$  
$z = \frac{1}{\alpha \tau}$  
$\omega_m = \sqrt{z \cdot p}$  
$\sin(\phi_m) = \frac{\alpha - 1}{\alpha + 1}$

In frequency response design, the phase-lead compensator adds positive phase to the system over the frequency range. A bode plot of a phase-lead compensator looks like the following.
Additional positive phase increases the phase margin and thus increases the stability of the system. This type of compensator is designed by determining \( \alpha \) from the amount of phase needed to satisfy the phase margin requirements, and determining \( \tau \) to place the added phase at the new gain-crossover frequency.

Another effect of the lead compensator can be seen in the magnitude plot. The lead compensator increases the gain of the system at high frequencies (the amount of this gain is equal to \( \alpha \)). This can increase the crossover frequency, which will help to decrease the rise time and settling time of the system.
In Matlab, a phase lead compensator in frequency response form is implemented by using the transfer function in the form

```matlab
numlead=[aT 1];
denlead=[T 1];
```
and using the `conv()` function to multiply it by the numerator and denominator of the plant

```matlab
newnum=conv(num,numlead);
newden=conv(den,denlead);
```
Lag or Phase-Lag Compensator Using Root Locus

A first-order lag compensator can be designed using the root locus. A lag compensator in root locus form is given by

\[ G_c(s) = \frac{(s + z)}{(s + p)} \]

where the magnitude of \( z \) is greater than the magnitude of \( p \). A phase-lag compensator tends to shift the root locus to the right, which is undesirable. For this reason, the pole and zero of a lag compensator must be placed close together (usually near the origin) so they do not appreciably change the transient response or stability characteristics of the system.

When a lag compensator is added to a system, the value of this intersection will be a smaller negative number than it was before. The net number of zeros and poles will be the same (one zero and one pole are added), but the added pole is a smaller negative number than the added zero. Thus, the result of a lag compensator is that the asymptotes' intersection is moved closer to the right half plane, and the entire root locus will be shifted to the right.
Lag or Phase-Lag Compensator Using Root Locus

It was previously stated that that lag controller should only minimally change the transient response because of its negative effect. If the phase-lag compensator is not supposed to change the transient response noticeably, what is it good for? The answer is that a phase-lag compensator can improve the system's steady-state response. It works in the following manner. At high frequencies, the lag controller will have unity gain. At low frequencies, the gain will be $z_0/p_0$ which is greater than 1. This factor $z/p$ will multiply the position, velocity, or acceleration constant ($K_p$, $K_v$, or $K_a$), and the steady-state error will thus decrease by the factor $z_0/p_0$.

In Matlab, a phase lead compensator in root locus form is implemented by using the transfer function in the form

```
numlag=[1  z];
denlag=[1  p];
```

and using the `conv()` function to implement it with the numerator and denominator of the plant

```matlab
newnum=conv(num,numlag);
newden=conv(den,denlag);
```
Lag or Phase-Lag Compensator using Frequency Response

A first-order phase-lag compensator can be designed using the frequency response. A lag compensator in frequency response form is given by

\[ G_C(s) = \frac{(1 + \alpha \cdot \tau \cdot s)}{\alpha \cdot (1 + \tau \cdot s)} \]

The phase-lag compensator looks similar to a phase-lead compensator, except that \( \alpha \) is now less than 1. The main difference is that the lag compensator adds negative phase to the system over the specified frequency range, while a lead compensator adds positive phase over the specified frequency. A bode plot of a phase-lag compensator looks like the following.
In Matlab, a phase-lag compensator in frequency response form is implemented by using the transfer function in the form

\[
\text{numlead} = [a \times T, 1] ; \\
\text{denlead} = a \times [T, 1] ; \\
\]

and using the \texttt{conv()} function to implement it with the numerator and denominator of the plant

\[
\text{newnum} = \texttt{conv(num, numlead)} ; \\
\text{newden} = \texttt{conv(den, denlead)} ;
\]
A lead-lag compensator combines the effects of a lead compensator with those of a lag compensator. The result is a system with improved transient response, stability and steady-state error. To implement a lead-lag compensator, first design the lead compensator to achieve the desired transient response and stability, and then add on a lag compensator to improve the steady-state response.
Exercise - Dominant Pole-Zero Approximations and Compensations

The influence of a particular pole (or pair of complex poles) on the response is mainly determined by two factors: the real part of the pole and the relative magnitude of the residue at the pole. The real part determines the rate at which the transient term due to the pole decays; the larger the real part, the faster the decay. The relative magnitude of the residue determines the percentage of total response due to a particular pole.

Investigate (using Simulink) the impact of a closed-loop negative real pole on the overshoot of a system having complex poles.

\[ T(s) = \frac{pr \cdot \omega_n^2}{(s + pr) \left[ s^2 + (2 \cdot \zeta \cdot \omega_n \cdot s) + \omega_n^2 \right]} \]

Make \( pr \) to vary \((2, 3, 5)\) times the real part of the complex pole for different values of \( \zeta \) \((0.3, 0.5, 0.7)\).

Investigate (using Simulink) the impact of a closed-loop negative real zero on the overshoot of a system having complex poles.

\[ T(s) = \frac{(s + zr)}{\left[ s^2 + (2 \cdot \zeta \cdot \omega_n \cdot s) + \omega_n^2 \right]} \]

Make \( zr \) to vary \((2, 3, 5)\) times the real part of the complex pole for different values of \( \zeta \) \((0.3, 0.5, 0.7)\).
Exercise - Lead and Lag Compensation

Investigate (using Matlab and Simulink) the effect of lead and lag compensations on the systems indicated below. Summarize your observations. Plot the root-locus, bode diagram and output for a step input before and after the compensations.

Remember
lead compensation: \( z < p \) (place zero below the desired root location or to the left of the first real poles)
lag compensation: \( z > p \) (locate the pole and zero near the origin of the \( s \)-plane)

Lead Compensation (use \( z = 1.33 \), \( p = 20 \) and \( K = 15 \)).
Lag Compensation (use $z=0.09$, and $p=0.015$, $K=1/6$)

Summarize your findings
Problem 10.36

Determine a compensator so that the percent overshoot is less than 20% and $K_v$ (velocity constant) is greater than 8.
**Transfer Function:**

A transfer function is defined as the ratio of the Laplace transform of the output to the input with all initial conditions equal to zero. Transfer functions are defined only for linear time invariant systems.

**Considerations:**

Transfer functions can usually be expressed as the ratio of two polynomials in the complex variable, \( s \).

**Factorization:**

A transfer function can be factored into the following form.

\[
G(s) = \frac{K(s + z_1)(s + z_2)\ldots(s + z_m)}{(s + p_1)(s + p_2)\ldots(s + p_n)}
\]

*The roots of the numerator polynomial are called zeros.*

*The roots of the denominator polynomial are called poles.*
An Example:

You are given the following transfer function. Show the poles and zeros in the s-plane.

\[ G(s) = \frac{(s + 8)(s + 14)}{s(s + 4)(s + 10)} \]
Characterization: Considering the transfer function of the previous slide. We note that we have 4 different types of terms in the previous general form: These are:

\[ K_B, \quad \frac{1}{s}, \quad \frac{1}{(s/p + 1)}, \quad (s/z + 1) \]

Expressing in dB: Given the transfer function:

\[ G(jw) = \frac{K_B (jw/z + 1)}{(jw)/(jw/p + 1)} \]

\[ 20\log |G(jw)| = 20\log K_B + 20\log |(jw/z + 1)| - 20\log |jw| - 20\log |jw/p + 1| \]
Mechanics: We have 4 distinct terms to consider:

\[ 20\log K_B \]

\[ 20\log |(jw/z + 1)| \]

\[ -20\log |jw| \]

\[ -20\log |(jw/p + 1)| \]
This is a sheet of 5 cycle, semi-log paper. This is the type of paper usually used for preparing Bode plots.
The gain term, $20\log K_B$, is just so many dB and this is a straight line on Bode paper, independent of omega (radian frequency).

The term, $-20\log|j\omega| = -20\log w$, when plotted on semi-log paper is a straight line sloping at -20dB/decade. It has a magnitude of 0 at $w = 1$. 

![Graph showing the behavior of $-20\log|j\omega|$ on semi-log paper]
The term, \(-20\log|j\omega/p + 1|\), is drawn with the following approximation: If \(\omega < p\) we use the approximation that \(-20\log|j\omega/p + 1| = 0\) dB, a flat line on the Bode. If \(\omega > p\) we use the approximation of \(-20\log(\omega/p)\), which slopes at \(-20\) dB/dec starting at \(\omega = p\). Illustrated below. It is easy to show that the plot has an error of \(-3\) dB at \(\omega = p\) and \(-1\) dB at \(\omega = p/2\) and \(\omega = 2p\). One can easily make these corrections if it is appropriate.
When we have a term of $20\log|\frac{j\omega}{z} + 1|$ we approximate it be a straight line of slope 0 dB/dec when $\omega < z$. We approximate it as $20\log(\frac{\omega}{z})$ when $\omega > z$, which is a straight line on Bode paper with a slope of + 20dB/dec. Illustrated below.
Example 1:

**Given:**

\[ G(jw) = \frac{50,000(jw + 10)}{(jw + 1)(jw + 500)} \]

**First:** Always, always, always get the poles and zeros in a form such that the constants are associated with the \( jw \) terms. In the above example we do this by factoring out the 10 in the numerator and the 500 in the denominator.

\[ G(jw) = \frac{50,000 \times 10(jw/10 + 1)}{500(jw + 1)(jw/500 + 1)} = \frac{100(jw/10 + 1)}{(jw + 1)(jw/500 + 1)} \]

**Second:** When you have neither poles nor zeros at 0, start the Bode at \( 20\log_{10}K = 20\log_{10}100 = 40 \text{ dB} \) in this case.
Third: Observe the order in which the poles and zeros occur. This is the secret of being able to quickly sketch the Bode. In this example we first have a pole occurring at 1 which causes the Bode to break at 1 and slope $-20 \text{ dB/dec}$. Next, we see a zero occurs at 10 and this causes a slope of $+20 \text{ dB/dec}$ which cancels out the $-20 \text{ dB/dec}$, resulting in a flat line ($0 \text{ db/dec}$). Finally, we have a pole that occurs at $w = 500$ which causes the Bode to slope down at $-20 \text{ dB/dec}$.

We are now ready to draw the Bode.

Before we draw the Bode we should observe the range over which the transfer function has active poles and zeros. This determines the scale we pick for the $w$ (rad/sec) at the bottom of the Bode.

The dB scale depends on the magnitude of the plot and experience is the best teacher here.
Bode Plot Magnitude for \(100(1 + j\omega/10)/(1 + j\omega/1)(1 + j\omega/500)\)
**Instruction:** We can use Matlab to run the frequency response for the previous example. We place the transfer function in the form:

\[
\frac{5000(s+10)}{(s+1)(s+500)} = \frac{5000s + 50000}{s^2 + 501s + 500}
\]

**The Matlab Program**

```matlab
num = [5000 50000];
den = [1 501 500];
Bode (num,den)
```

In the following slide, the resulting magnitude and phase plots (exact) are shown in light color (blue). The approximate plot for the magnitude (Bode) is shown in heavy lines (red). We see the 3 dB errors at the corner frequencies.
Bode Diagrams

From: U(1)

To: Y(1)

Bode for: \[ G(jw) = \frac{100(1 + jw/10)}{(1 + jw)(1 + jw/500)} \]
Comment: Generally, the phase for a Bode plot is not as easy to draw or approximate as the magnitude. In this course we will use an analytical method for determining the phase if we want to make a sketch of the phase.

Illustration: Consider the transfer function of the previous example. We express the angle as follows:

$$\angle G(j\omega) = \tan^{-1}(w/10) - \tan^{-1}(w/1) - \tan^{-1}(w/500)$$

We are essentially taking the angle of each pole and zero. Each of these are expressed as the $\tan^{-1}(j\text{ part/real part})$

Usually, about 10 to 15 calculations are sufficient to determine a good idea of what is happening to the phase.
Example 2: Given the transfer function. Plot the Bode magnitude.

\[ G(s) = \frac{100(1 + s/10)}{s(1 + s/100)^2} \]

Consider first only the two terms of

\[ \frac{100}{jw} \]

Which, when expressed in dB, are; \( 20\log100 - 20 \log w \).
This is plotted below.

The \( \ldots \ldots \ldots \) is a tentative line we use until we encounter the first pole(s) or zero(s) not at the origin.
Example 2: (continued) The completed plot is shown below.

\[ G(s) = \frac{100(1 + s/10)}{s(1 + s/100)^2} \]

The completed plot is shown below.
Example 3:

Given:

\[ G(s) = \frac{80(1 + j\omega)^3}{(j\omega)^3(1 + j\omega/20)^2} \]

\[ 20 \log 80 = 38 \text{ dB} \]
Example 4:

Given:

\[ G(j\omega) = \frac{10(1 - j\omega/2)}{(1 + j0.025\omega)(1 + j\omega/500)^2} \]

Sort of a low pass filter
Example 5

Given:

A high pass filter

\[ G(j\omega) = \frac{(1 + j\omega/30)^2 (1 + j\omega/100)^2}{(1 + j\omega/2)^2 (1 + j\omega/1700)^2} \]
Given:

Example 6

\[ H(jw) = \frac{640(jw + 1)(0.01jw + 1)}{(jw)^2(jw + 10)} = \frac{64(jw + 1)(0.01jw + 1)}{(jw)^2(0.1jw + 1)} \]
Design Problem: Design a G(s) that has the following Bode plot.
Bode Plots

Procedure: The two break frequencies need to be found.
Recall:
#dec = \log_{10}[w_2/w_1]

Then we have:

\((#\text{dec})(40\text{dB/dec}) = 30 \text{ dB}\)

\(\log_{10}[w_1/30] = 0.75 \rightarrow w_1 = 5.33 \text{ rad/sec}\)

Also:

\(\log_{10}[w_2/900] (-40\text{dB/dec}) = -30\text{dB}\)

This gives \(w_2 = 5060 \text{ rad/sec}\)
Bode Plots

Procedure:

\[ G(s) = \frac{(1 + s / 5.3)^2 (1 + s / 5060)^2}{(1 + s / 30)^2 (1 + s / 900)^2} \]

Clearing:

\[ G(s) = \frac{(s + 5.3)^2 (s + 5060)^2}{(s + 30)^2 (s + 900)^2} \]

Use Matlab and conv:

\[ N = \text{conv}(N1, N2) \]

\[ N1 = (s^2 + 10.6s + 28.1) \quad \quad N2 = (s^2 + 10120s + 2.56 \times 10^7) \]

\[ \begin{align*}
N1 &= [1 \\ 10.6 \\ 28.1] \\
N2 &= [1 \\ 10120 \\ 2.56e+7]
\end{align*} \]

\[ \begin{align*}
N &= \text{conv}(N1, N2) \\
&= [1 \\ 1.86e+3 \\ 2.58e+7 \\ 2.73e+8 \\ 7.222e+8]
\end{align*} \]

\[ \begin{align*}
s^4 & \quad s^3 & \quad s^2 & \quad s^1 & \quad s^0
\end{align*} \]
Bode Plots

Procedure: The final $G(s)$ is given by:

$$G(s) = \frac{(s^4 + 10130.6s^3 + 2.571e^8s^2 + 2.716e^8s + 7.194e^8)}{(s^4 + 1860s^3 + 9.189e^2s^2 + 5.022e^7s + 7.29e^8)}$$

Testing: We now want to test the filter. We will check it at $\omega = 5.3$ rad/sec and $\omega = 164$. At $\omega = 5.3$ the filter has a gain of 6 dB or about 2. At $\omega = 164$ the filter has a gain of 30 dB or about 31.6.

We will check this out using MATLAB and particularly, Simulink.
Matlab (Simulink) Model:

\[
\frac{s^4 + 10103.6s^3 + 2.571e+7s^2 + 2.716e8s + 7.194e8}{s^4 + 1860s^3 + 9.189e5s^2 + 5.022e7s + 7.29e8}
\]

Sine Wave → Scope1 → Transfer Fcn → in → To Workspace2 → Clock → t → To Workspace1 → out → To Workspace
Filter Output at $\omega = 5.3$ rad/sec

Produced from Matlab Simulink
Filter Output at $\omega = 70 \text{ rad/sec}$

Produced from Matlab Simulink
Required:

From the partial Bode diagram, determine the transfer function
(Assume a minimum phase system)

Example 8
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From the partial Bode diagram, determine the transfer function

(Assume a minimum phase system)

Example 9
Polar Plot
The polar plot of sinusoidal transfer function $G(j\omega)$ is a plot of the magnitude of $G(j\omega)$ verses the phase angle of $G(j\omega)$ on polar coordinates as $\omega$ is varied from zero to infinity. Therefore it is the locus of $G(j\omega)$ as $\omega$ is varied from zero to infinity.

As

So it is the plot of vector $|G(j\omega)|\angle G(j\omega)$ as $\omega$ is varied from zero to infinity.

$$|G(j\omega)|\angle G(j\omega) = Me^{j\phi(\omega)}$$

$$Me^{j\phi(\omega)}$$
In the polar plot the magnitude of $G(j\omega)$ is plotted as the distance from the origin while phase angle is measured from positive real axis. + angle is taken for anticlockwise direction. Polar plot is also known as Nyquist Plot.
Steps to draw Polar Plot

Step 1: Determine the T.F G(s)
Step 2: Put s=jω in the G(s)
Step 3: At ω=0 & ω=∞ find by &
Step 4: At ω=0 & ω=∞ find by &
Step 5: Rationalize the function G(jω) and separate the real and imaginary parts
Step 6: Put Re [G(jω )]=0, determine the frequency at which plot intersects the Im axis, and calculate intersection value by putting the above calculated frequency in G(jω)
Steps to draw Polar Plot conti…

Step 7: Put $\text{Im} \left[ G(j\omega) \right] = 0$, determine the frequency at which plot intersects the real axis and calculate intersection value by putting the above calculated frequency in $G(j\omega)$

Step 8: Sketch the Polar Plot with the help of above information
Polar Plot for Type 0 System

Let \( G(s) = \frac{K}{(1+sT_1)(1+sT_2)} \)

Step 1: Put \( s=j\omega \)

Step 2: Taking the limit for magnitude of \( G(j\omega) \)

\[
|G(j\omega)| = \frac{K}{\sqrt{1+(\omega T_1)^2} \sqrt{1+(\omega T_2)^2}} \angle -\tan^{-1} \omega T_1 - \tan^{-1} \omega T_2
\]
Type 0 system conti...

\[
\lim_{\omega \to 0} |G(j\omega)| = \frac{K}{\sqrt{1 + (\omega T_1)^2} \sqrt{1 + j(\omega T_2)^2}} = K
\]

Step 3: Taking the limit of the Phase Angle of \( G(j\omega) \)

\[
\lim_{\omega \to \infty} |G(j\omega)| = \frac{K}{\sqrt{1 + (\omega T_1)^2} \sqrt{1 + j(\omega T_2)^2}}
\]

\[
\lim_{\omega \to 0} \angle G(j\omega) = \angle - \tan^{-1} \omega T_1 - \tan^{-1} \omega T_2 = 0
\]

\[
\lim_{\omega \to \infty} \angle G(j\omega) = \angle - \tan^{-1} \omega T_1 - \tan^{-1} \omega T_2 = -180
\]
Step 4: Separate the real and Im part of $G(j\omega)$

Step 5: When \[ G(j\omega) = \frac{K(1 - \omega^2T_1T_2)}{1 + \omega^2T_1^2 + \omega^2T_2^2 + \omega^4T_1T_2} - j \frac{K\omega(T_1 + T_2)}{1 + \omega^2T_1^2 + \omega^2T_2^2 + \omega^4T_1T_2} \]

\[ \frac{K(1 - \omega^2T_1T_2)}{1 + \omega^2T_1^2 + \omega^2T_2^2 + \omega^4T_1T_2} = 0 \Rightarrow \omega = \frac{1}{\sqrt{T_1T_2}} \& \omega = \infty \]

So When

\[ \omega = \frac{1}{\sqrt{T_1T_2}} \Rightarrow G(j\omega) = \frac{K\sqrt{T_1T_2}}{T_1 + T_2} \angle -90^0 \]

& \omega = \infty \Rightarrow G(j\omega) = 0 \angle -180^0
Type 0 system conti...

Step 6: Put $\text{Im} \left[ G(j\omega) \right] = 0$

\[
\frac{K\omega(T_1 + T_2)}{1 + \omega^2 T_1^2 + \omega^2 T_2^2 + \omega^4 T_1 T_2} = 0 \implies \omega = 0 \& \pm \infty
\]

So When

\[
\omega = 0 \implies G(j\omega) = K \angle 0^0
\]

\[
\omega = \infty \implies G(j\omega) = 0 \angle 180^0
\]
Figure 1: Polar Plot of Type 0 System
Polar Plot for Type 1 System

Let \( G(s) = \frac{K}{s(1+sT_1)(1+sT_2)} \)

Step 1: Put \( s=j\omega \)

\[
G(j\omega) = \frac{K}{j\omega(1+j\omega T_1)(1+j\omega T_2)}
= \frac{K}{\omega \sqrt{1+(\omega T_1)^2} \sqrt{1+j(\omega T_2)^2}} \angle -90^\circ - \tan^{-1} \omega T_1 - \tan^{-1} \omega T_2
\]
Type 1 system conti...

Step 2: Taking the limit for magnitude of $G(j\omega)$

$$\lim_{\omega \to 0} |G(j\omega)| = \frac{K}{\omega \sqrt{1 + (\omega T_1)^2} \sqrt{1 + j(\omega T_2)^2}} = \infty$$

Step 3: Taking the limit of the Phase Angle of $G(j\omega)$

$$\lim_{\omega \to \infty} \angle G(j\omega) = \angle -90^0 - \tan^{-1} \omega T_1 - \tan^{-1} \omega T_2 = -90^0$$

$$\lim_{\omega \to \infty} \angle G(j\omega) = \angle -90^0 - \tan^{-1} \omega T_1 - \tan^{-1} \omega T_2 = -270^0$$
Type 1 system conti...

Step 4: Separate the real and Im part of $G(j\omega)$

Step 5: Put $\text{Re} [G(j\omega)] = 0$

\[
-\frac{\omega K (T_1 + T_2)}{\omega + \omega^3 (T_1^2 + T_2^2 + \omega^2 T_1^2 T_2^2)} + j \frac{j (K \omega^2 T_1 T_2 - K)}{\omega + \omega^3 (T_1^2 + T_2^2 + \omega^2 T_1^2 T_2^2)}
\]

\[
-\frac{\omega K (T_1 + T_2)}{\omega + \omega^3 (T_1^2 + T_2^2 + \omega^2 T_1^2 T_2^2)} = 0 \implies \omega = \infty
\]

So at

$\omega = \infty \implies G(j\omega) = 0 \angle -270^0$
Type 1 system conti...

Step 6: Put $\text{Im} \ [G(j\omega)] = 0$

\[
\frac{j(K\omega^2T_1T_2 - K)}{\omega + \omega^3(T_1^2 + T_2^2 + \omega^2T_1^2T_2^2)} = 0 \Rightarrow \omega = \frac{1}{\sqrt{T_1T_2}} \& \omega = \pm \infty
\]

So When

\[
\omega = \frac{1}{\sqrt{T_1T_2}} \Rightarrow G(j\omega) = -\frac{K\sqrt{T_1T_2}}{T_1 + T_2} \angle 0^\circ
\]

\[
\omega = \infty \Rightarrow G(j\omega) = \infty \angle 0^\circ
\]
Type 1 system conti...

\[ \omega = \frac{1}{\sqrt{T_1 T_2}} \]

Figure 2: Polar Plot of Type 1 System
Polar Plot for Type 2 System

Let

Similar to above

\[ G(s) = \frac{K}{s^2 (1 + sT_1)(1 + sT_2)} \]
Type 2 system conti…

Figure 3: Polar Plot of Type 2 System
Note: Introduction of additional pole in denominator contributes a constant $-180^0$ to the angle of $G(j\omega)$ for all frequencies. See the figure 1, 2 & 3

Figure 1+(-180$^0$ Rotation)=figure 2
Figure 2+(-180$^0$ Rotation)=figure 3
Ex: Sketch the polar plot for \( G(s) = \frac{20}{s(s+1)(s+2)} \)

Solution:
Step 1: Put \( s = j\omega \)

\[
G(j\omega) = \frac{20}{j\omega(j\omega + 1)(j\omega + 2)}
= \frac{20}{\omega\sqrt{\omega^2 + 1}\sqrt{\omega^2 + 4}} \angle -90^\circ - \tan^{-1}\omega - \tan^{-1}\omega/2
\]
Step 2: Taking the limit for magnitude of $G(j\omega)$

$$
\lim_{\omega \to 0} |G(j\omega)| = \frac{20}{\omega \sqrt{\omega^2 + 1} \sqrt{\omega^2 + 4}} = \infty
$$

$$
\lim_{\omega \to \infty} |G(j\omega)| = \frac{20}{\omega \sqrt{\omega^2 + 1} \sqrt{\omega^2 + 4}} = 0
$$

Step 3: Taking the limit of the Phase Angle of $G(j\omega)$

$$
\lim_{\omega \to 0} \angle G(j\omega) = \angle -90^0 - \tan^{-1} \omega - \tan^{-1} \omega / 2 = -90^0
$$

$$
\lim_{\omega \to \infty} \angle G(j\omega) = \angle -90^0 - \tan^{-1} \omega - \tan^{-1} \omega / 2 = -270^0
$$
Step 4: Separate the real and Im part of \( G(j\omega) \)

\[
G(j\omega) = \frac{-60\omega^2}{(\omega^4 + \omega^2)(4 + \omega^2)} + j\frac{j20(\omega^3 - 2\omega)}{(\omega^4 + \omega^2)(4 + \omega^2)}
\]

\[
\frac{-60\omega^2}{(\omega^4 + \omega^2)(4 + \omega^2)} = 0 \Rightarrow \omega = \infty
\]

So at

\[
\omega = \infty \quad \Rightarrow G(j\omega) = 0 \angle -270^0
\]
Step 6: Put \( \text{Im} [G(j\omega)] = 0 \)

\[
\frac{j20(\omega^3 - 2\omega)}{(\omega^4 + \omega^2)(4 + \omega^2)} = 0 \implies \omega = \pm \sqrt{2} \quad \& \quad \omega = \pm \infty
\]

So for positive value of \( \omega \)

\[
\omega = \sqrt{2} \implies G(j\omega) = -\frac{10}{3} \angle 0^0
\]

\[
\omega = \infty \implies G(j\omega) = 0 \angle 0^0
\]
Figure 4: Polar Plot \( G(s) = \frac{20}{s(s+1)(s+2)} \)
Gain Margin, Phase Margin & Stability
Phase Crossover Frequency ($\omega_p$) : The frequency where a polar plot intersects the –ve real axis is called phase crossover frequency

Gain Crossover Frequency ($\omega_g$) : The frequency where a polar plot intersects the unit circle is called gain crossover frequency

So at $\omega_g$

$$|G(j\omega)| = Unity$$
Phase Margin (PM):
Phase margin is that amount of additional phase lag at the gain crossover frequency required to bring the system to the verge of instability (marginally stable)
\[ \Phi_m = 180^\circ + \Phi \]

Where
\[ \Phi = \angle G(j\omega_g) \]
if \[ \Phi_m > 0 \Rightarrow +PM \] (Stable System)
if \[ \Phi_m < 0 \Rightarrow -PM \] (Unstable System)
Gain Margin (GM):
The gain margin is the reciprocal of magnitude $|G(j\omega)|$ at the frequency at which the phase angle is $-180^0$.

In terms of dB

$$GM = \frac{1}{|G(j\omega)|} = \frac{1}{x}$$

$$GM\text{ in } dB = 20\log_{10} \frac{1}{|G(j\omega)|} = -20\log_{10} |G(j\omega)| = -20\log_{10}(x)$$
Stability

**Stable:** If critical point \((-1+j0)\) is within the plot as shown, both GM & PM are +ve

\[ GM = 20\log_{10}(1) \]

Stable system: +GM, +PM
**Unstable:** If critical point (-1+j0) is outside the plot as shown, both GM & PM are -ve.

\[ GM = 20 \log_{10}(1 + \frac{1}{x}) \text{ dB} \]
**Marginally Stable System:** If critical point \((-1+j0)\) is on the plot as shown, Both GM & PM are ZERO

\[
GM = 20 \log_{10}(1)
\]
>> [Gm,Pm,Wcg,Wcp] = margin(g)

\[
G(s) = \frac{10}{s(s/5 + 1)(s/20 + 1)}
\]

Gm = 
\[
2.5000
\]

Pm = 
\[
22.5359
\]

Wcg = 
\[
10.0000 \text{ (phase crossover freq.)}
\]

Wcp = 
\[
6.0783 \text{ (gain crossover freq.)}
\]
The inverse polar plot of $G(j\omega)$ is a graph of $1/G(j\omega)$ as a function of $\omega$.

**Ex:** if $G(j\omega) = 1/j\omega$ then $1/G(j\omega) = j\omega$

\[
\lim_{\omega \to 0} |G(j\omega)^{-1}| = 0
\]

\[
\lim_{\omega \to \infty} |G(j\omega)^{-1}| = \infty
\]
Knowledge Before Studying Nyquist Criterion

\[ T(s) = \frac{G(s)}{1 + G(s)H(s)} \]

unstable if there is any pole on RHP (right half plane)

\[ G(s) = \frac{N_G(s)}{D_G(s)} \quad \text{and} \quad H(s) = \frac{N_H(s)}{D_H(s)} \]
Open-loop system:

\[ G(s)H(s) = \frac{N_G(s) \cdot N_H(s)}{D_G(s) \cdot D_H(s)} \]

Characteristic equation:

\[ 1 + G(s)H(s) = 1 + \frac{N_G N_H}{D_G D_H} = \frac{D_G D_H + N_G N_H}{D_G D_H} \]

poles of \( G(s)H(s) \) and \( 1 + G(s)H(s) \) are the same

Closed-loop system:

\[ T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{N_G(s) \cdot D_H(s)}{D_G(s) \cdot D_H(s) + N_G(s)N_H(s)} \]

zero of \( 1 + G(s)H(s) \) is pole of \( T(s) \)
To know stability, we have to know a, b, c, d.
From a Nyquist plot, we can tell a number of closed-loop poles on the right half plane. If there is any closed-loop pole on the right half plane, the system goes unstable. If there is no closed-loop pole on the right half plane, the system is stable.
Nyquist plot is a plot used to verify stability of the system.

Function $F(s) = \frac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)}$

mapping all points (contour) from one plane to another by function $F(s)$. 
\[ F(s) = \frac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)} \]
Pole/zero inside the contour has 360 deg. angular change. Pole/zero outside contour has 0 deg. angular change. Move clockwise around contour, zero inside yields rotation in clockwise, pole inside yields rotation in counterclockwise.
Characteristic equation

\[ F(s) = 1 + G(s)H(s) \]

N = \# of counterclockwise direction about the origin
P = \# of poles of characteristic equation inside contour
\( = \# \) of poles of open-loop system
z = \# of zeros of characteristic equation inside contour
\( = \# \) of poles of closed-loop system

Z = P - N
Characteristic equation

Increase size of the contour to cover the right half plane

More convenient to consider the open-loop system (with known pole/zero)
Nyquist diagram of \( G(s)H(s) \)

‘Open-loop system’

Mapping from characteristic equ. to open-loop system by shifting to the left one step

\[ Z = P - N \]

\( Z \) = # of closed-loop poles inside the right half plane
\( P \) = # of open-loop poles inside the right half plane
\( N \) = # of counterclockwise revolutions around -1
\( \bigcirc = \text{zeros of } 1 + G(s)H(s) \)
= poles of closed-loop system
Location not known

\( \bigcirc = \text{poles of } G(s)H(s) \)
\( \times = \text{poles of } 1 + G(s)H(s) \)
Location is known
Properties of Nyquist plot

If there is a gain, $K$, in front of open-loop transfer function, the Nyquist plot will expand by a factor of $K$. 
Nyquist plot example

Open loop system has pole at 2

Closed-loop system has pole at 1

If we multiply the open-loop with a gain, $K$, then we can move the closed-loop pole’s position to the left-half plane.

$$G(s) = \frac{1}{s-2}$$
New look of open-loop system:

$$G(s) = \frac{K}{s - 2}$$

Corresponding closed-loop system:

$$\frac{G(s)}{1 + G(s)} = \frac{K}{s + (K - 2)}$$

Evaluate value of $K$ for stability

$$K \geq 2$$
Adjusting an open-loop gain to guarantee stability

Step I: sketch a Nyquist Diagram
Step II: find a range of $K$ that makes the system stable!
How to make a Nyquist plot?

Easy way by Matlab
Nyquist: ‘nyquist’
Bode: ‘bode’
Step I: make a Nyquist plot

Starts from an open-loop transfer function (set $K=1$)

Set and find frequency response

At dc, $s = j\omega$

Find at which the imaginary part equals zero

$\omega = 0 \rightarrow s = 0$

$\omega$
\[
G(s)H(s) = \frac{(s + 3)(s + 5)}{(s - 2)(s - 4)} = \frac{s^2 + 8s + 15}{s^2 - 6s + 8}
\]

\[
G(j\omega)H(j\omega) = \frac{-\omega^2 + 8j\omega + 15}{-\omega^2 - 6j\omega + 8} = \frac{(15 - \omega^2) + 8j\omega}{(8 - \omega^2) - 6j\omega}
\]

\[
= \frac{(15 - \omega^2) + 8j\omega}{(8 - \omega^2) - 6j\omega} \times \frac{(8 - \omega^2) + 6j\omega}{(8 - \omega^2) + 6j\omega}
\]

\[
= \frac{(15 - \omega^2)(8 - \omega^2) - 48\omega^2 + j(154\omega - 14\omega^3)}{(8 - \omega^2)^2 + 6^2\omega^2}
\]

Need the imaginary term = 0,

\[
\omega = 0, \sqrt{11}
\]

Substitute \(\omega = \sqrt{11}\) back into the transfer function

And get

\[
G(s) = -1.33
\]

\[
\frac{(15 - 11)(8 - 11) - 48(11)}{(8 - 11)^2 + 6^2(11)} = \frac{-540}{412} = -1.31
\]
At dc, $s=0$, imaginary part=0.

At imaginary part=0, $\omega = \sqrt{11}$. $-1.33 \frac{1}{K}$.
Step II: satisfying stability condition

\[ P = 2, \ \text{N has to be 2 to guarantee stability} \]

Marginally stable if the plot intersects -1

For stability, \( 1.33K \) has to be greater than 1

\[ K > \frac{1}{1.33} \]

or \[ K > 0.75 \]
Example

Evaluate a range of $K$ that makes the system stable

$$G(s) = \frac{K}{(s^2 + 2s + 2)(s + 2)}$$
Step I: find frequency at which imaginary part = 0

Set \( s = j\omega \)

\[
G(j\omega) = \frac{K}{((j\omega)^2 + 2j\omega + 2)(j\omega + 2)}
= \frac{4(1-\omega^2) - j\omega(6-\omega^2)}{16(1-\omega^2)^2 + \omega^2(6-\omega^2)^2}
\]

At \( \omega = 0, \sqrt{6} \) the imaginary part = 0

Plug \( \omega = \sqrt{6} \) back in the transfer function and get \( G = -0.05 \)
Step II: consider stability condition

\( P = 0, N \) has to be 0 to guarantee stability

Marginally stable if the plot intersects -1

For stability, \( 0.05K \) has to be less than 1

\[
K < \frac{1}{0.05}
\]

or \( K < 20 \)
Gain Margin and Phase Margin

Gain margin is the change in open-loop gain (in dB), required at 180° of phase shift to make the closed-loop system unstable.

Phase margin is the change in open-loop phase shift, required at unity gain to make the closed-loop system unstable.

GM/PM tells how much system can tolerate before going unstable!!!
GM and PM via Nyquist plot

\[ GM = \frac{K_{\text{max}}}{K_{\text{current}}} = \frac{1}{|G(j\omega)H(j\omega)|} \]

\[ PM = 180^\circ + \angle G(j\omega_x)H(j\omega_x) \]

Gain margin = \( G_M = 20 \log a \)

Phase margin = \( \Phi_M = \alpha \)
The frequency at which the phase equals 180 degrees is called the **phase crossover frequency** \( \omega_{\Phi_M} \).

The frequency at which the magnitude equals 1 is called the **gain crossover frequency** \( \omega_{G_M} \).
Find Bode Plot and evaluate a value of $K$ that makes the system stable.
The system has a unity feedback with an open-loop transfer function

$$G(s) = \frac{K}{(s + 2)(s + 4)(s + 5)}$$

First, let’s find Bode Plot of $G(s)$ by assuming that $K=40$ (the value at which magnitude plot starts from 0 dB)
At phase = -180, $\omega = 7$ rad/sec, magnitude = -20 dB
GM>0, system is stable!!!
Can increase gain up 20 dB without causing instability (20dB = 10)
Start from $K = 40$
with $K < 400$, system is stable
Closed-loop transient and closed-loop frequency responses

‘2nd system’

\[
\frac{C(s)}{R(s)} = T(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]
Damping ratio and closed-loop frequency response

Magnitude Plot of closed-loop system

\[ M_p = \frac{1}{2\zeta \sqrt{1 - \zeta^2}} \]

\[ \omega_p = \omega_n \sqrt{1 - 2\zeta^2} \]

Log frequency (rad/s)
Response speed and closed-loop frequency response

\[ \omega_{BW} = \omega_n \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \]

\[ \omega_{BW} = \frac{4}{T_s \zeta} \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \]

\[ \omega_{BW} = \frac{\pi}{T_p \sqrt{1-\zeta^2}} \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \]

\( \omega_{BW} \) frequency at which magnitude is 3dB down from value at dc (0 rad/sec), or \( M = \frac{1}{\sqrt{2}} \)
From open-loop frequency response, we can find \( \omega_{BW} \) at the open-loop frequency that the magnitude lies between -6dB to -7.5dB (phase between -135 to -225).
Relationship between damping ratio and phase margin of open-loop frequency response

Phase margin of open-loop frequency response can be written in terms of damping ratio as following:

\[ \phi_M = \tan^{-1} \left( \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}} \right) \]
Open-loop system with a unity feedback has a bode plot below, approximate settling time and peak time

\[ \omega_{BW} = 3.7 \]

\[ \text{PM} = 35 \]
\[ \phi_M = \tan^{-1} \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}} \]

Solve for PM = 35

\[ \zeta = 0.32 \]

\[ T_s = \frac{4}{\omega_{BW} \zeta} \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \]

\[ = 5.5 \]

\[ T_p = \frac{\pi}{\omega_{BW} \sqrt{1 - \zeta^2}} \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \]

\[ = 1.43 \]
UNIT-5

STATE SPACE REPRESENTATION
Objectives

How to find mathematical model, called a state-space representation, for a linear, time-invariant system
How to convert between transfer function and state space models
How to find the solution of state equations for homogeneous & non-homogeneous systems
Plant

Mathematical Model: Differential equation

- Linear, time invariant
- Frequency Domain Technique
- Time Domain Technique
Two approaches for analysis and design of control system

1. Classical Technique or Frequency Domain Technique
2. Modern Technique or Time Domain Technique
• System variable: any variable that responds to an input or initial conditions in a system
• State variables: the smallest set of linearly independent system variables such that the values of the members of the set at time \( t_0 \) along with known forcing functions completely determine the value of all system variables for all \( t \geq t_0 \)
• State vector: a vector whose elements are the state variables
• State space: the \( n \)-dimensional space whose axes are the state variables
• State equations: a set of first-order differential equations with \( b \) variables, where the \( n \) variables to be solved are the state variables
• Output equation: the algebraic equation that expresses the output variables of a system as linear combination of the state variables and the inputs.
• For \( n \)th-order, write \( n \) simultaneous, first-order differential equations in terms of the state variables (state equations).
• If we know the initial condition of all of the state variables at \( t_0 \) as well as the system input for \( t \geq t_0 \), we can solve the equations
Graphic representation of state space and a state vector
For a dynamic system, the state of a system is described in terms of a set of state variables

\[
\begin{bmatrix}
  x_1(t) & x_2(t) & \ldots & x_n(t)
\end{bmatrix}
\]

The state variables are those variables that determine the future behavior of a system when the present state of the system and the excitation signals are known. Consider the system shown in Figure 1, where \( y_1(t) \) and \( y_2(t) \) are the output signals and \( u_1(t) \) and \( u_2(t) \) are the input signals. A set of state variables \([x_1 \ x_2 \ \ldots \ x_n]\) for the system shown in the figure is a set such that knowledge of the initial values of the state variables \([x_1(t_0) \ x_2(t_0) \ \ldots \ x_n(t_0)]\) at the initial time \( t_0 \), and of the input signals \( u_1(t) \) and \( u_2(t) \) for \( t \geq t_0 \), suffices to determine the future values of the outputs and state variables.

Figure 1. Dynamic system.
In an actual system, there are several choices of a set of state variables that specify the *energy stored in a system* and therefore adequately describe the dynamics of the system.

The state variables of a system characterize the dynamic behavior of a system. The engineer’s interest is primarily in physical, where the variables are voltages, currents, velocities, positions, pressures, temperatures, and similar physical variables.

**The State Differential Equation:**

The state of a system is described by the set of first-order differential equations written in terms of the state variables \([x_1 \ x_2 \ \ldots \ x_n]\). These first-order differential equations can be written in general form as

\[
\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_{11}u_1 + \ldots + b_{1m}u_m \\
\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + b_{21}u_1 + \ldots + b_{2m}u_m \\
\vdots \\
\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + b_{n1}u_1 + \ldots + b_{nm}u_m
\]
Thus, this set of simultaneous differential equations can be written in matrix form as follows:

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}
\]

n: number of state variables, m: number of inputs.

The column matrix consisting of the state variables is called the \textbf{state vector} and is written as

\[
X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
\]
The vector of input signals is defined as $u$. Then the system can be represented by the compact notation of the state differential equation as

$$\dot{x} = Ax + Bu$$

This differential equation is also commonly called the state equation. The matrix $A$ is an $n \times n$ square matrix, and $B$ is an $n \times m$ matrix. The state differential equation relates the rate of change of the state of the system to the state of the system and the input signals. In general, the outputs of a linear system can be related to the state variables and the input signals by the output equation

$$y = Cx + Du$$

Where $y$ is the set of output signals expressed in column vector form. The state-space representation (or state-variable representation) is comprised of the state variable differential equation and the output equation.
General State Representation

\[ \dot{x} = Ax + Bu \quad \text{State equation} \]
\[ y = Cx + Du \quad \text{output equation} \]

\[ x \quad = \text{state vector} \]
\[ \dot{x} \quad = \text{derivative of the state vector with respect to time} \]
\[ y \quad = \text{output vector} \]
\[ u \quad = \text{input or control vector} \]
\[ A \quad = \text{system matrix} \]
\[ B \quad = \text{input matrix} \]
\[ C \quad = \text{output matrix} \]
\[ D \quad = \text{feedforward matrix} \]
• The state of the system can be described in terms of a set of variables $[x_1, x_2]$, where $x_1$ is the capacitor voltage $v_c(t)$ and $x_2$ is equal to the inductor current $i_L(t)$. This choice of state variables is intuitively satisfactory because the stored energy of the network can be described in terms of these variables.
Therefore $x_1(t_0)$ and $x_2(t_0)$ represent the total initial energy of the network and thus the state of the system at $t=t_0$.

Utilizing Kirchhoff’s current low at the junction, we obtain a first order differential equation by describing the rate of change of capacitor voltage

$$i_c = C\frac{dv_c}{dt} = u(t) - i_L$$

Kirchhoff’s voltage low for the right-hand loop provides the equation describing the rate of change of inductor current as

$$L\frac{di_L}{dt} = -Ri_L + v_c$$

The output of the system is represented by the linear algebraic equation

$$v_0 = Ri_L(t)$$
We can write the equations as a set of two first order differential equations in terms of the state variables $x_1 [v_C(t)]$ and $x_2 [i_L(t)]$ as follows:

$$C \frac{dv_C}{dt} = u(t) - i_L$$  \quad \Rightarrow \quad \frac{dx_1}{dt} = -\frac{1}{C} x_2 + \frac{1}{C} u(t)$$

$$L \frac{di_L}{dt} = -R i_L + v_C$$  \quad \Rightarrow \quad \frac{dx_2}{dt} = \frac{1}{L} x_1 - \frac{R}{L} x_2$$

The output signal is then  \quad y_1(t) = v_0(t) = R x_2

Utilizing the first-order differential equations and the initial conditions of the network represented by $[x_1(t_0) \ x_2(t_0)]$, we can determine the system's future and its output.

The state variables that describe a system are not a unique set, and several alternative sets of state variables can be chosen. For the RLC circuit, we might choose the set of state variables as the two voltages, $v_C(t)$ and $v_L(t)$. 
We can write the state variable differential equation for the RLC circuit as

\[
\dot{x} = \begin{bmatrix}
0 & -\frac{1}{C} \\
1 & -\frac{R}{L} \\
\frac{1}{L} & -\frac{1}{L}
\end{bmatrix} x + \begin{bmatrix}
1 \\
0
\end{bmatrix} u(t)
\]

and the output as

\[
y = \begin{bmatrix}
0 & R
\end{bmatrix} x
\]
1. State variables \( i(t), q(t) \)
2. \[ L \frac{di}{dt} + Ri + \frac{1}{C} \int idt = v(t) \]

Using \( i(t) = \frac{dq}{dt} \)

\[ L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = v(t) \]

\[ \frac{dq}{dt} = i \]

\[ \frac{di}{dt} = -\frac{1}{LC} q - \frac{R}{L} i + \frac{1}{L} v(t) \] \hspace{1cm} (1)

3. \( q(t) \) and \( i(t) \) can be solved using Laplace Transform

4. Other network variables can be obtained

\[ v_L(t) = -\frac{1}{C} q(t) - Ri(t) + v(t) \] \hspace{1cm} (2)

5. (1),(2) : state-space representation
Other variables \( v_R(t) \) \( v_C(t) \)

\[
\frac{dv_R}{dt} = -\frac{R}{L} v_R - \frac{R}{L} v_C + \frac{R}{L} v(t)
\]

\[
\frac{dv_C}{dt} = \frac{1}{RC} v_R
\]

Each variables : linearly independent
In vector-matrix form

\[ \dot{x} = Ax + Bu \]  \hspace{1cm} (1)

where

\[ \dot{x} = \begin{bmatrix} dq/dt \\ di/dt \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix} \]

\[ x = \begin{bmatrix} q \\ i \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1/L \end{bmatrix} \quad u = v(t) \]

\[ y = Cx + Du \]  \hspace{1cm} (2)

where

\[ y = v_L(t) \quad C = \begin{bmatrix} -1/C & -R \end{bmatrix} \quad D = 1 \]
State space representation using phase variable form

\[
\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_0 u
\]

choose

\[x_1 = y \quad x_2 = \frac{dy}{dt} \quad x_3 = \frac{d^2 y}{dt^2} \quad \cdots \quad x_n = \frac{d^{n-1} y}{dt^{n-1}}\]

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\[
\begin{align*}
\dot{x}_1 &= \frac{dy}{dt} \\
\dot{x}_2 &= \frac{d^2 y}{dt^2} \\
\dot{x}_3 &= \frac{d^3 y}{dt^3} \\
\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= -a_0 x_1 - a_1 x_2 - \cdots - a_{n-1} x_n + b_0 u
\end{align*}
\]
\[
\begin{aligned}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\vdots \\
\dot{x}_{n-1} \\
\dot{x}_n
\end{bmatrix}
&= 
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
\vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\
-a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \cdots & -a_{n-1}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_{n-1} \\
x_n
\end{bmatrix}
&+ 
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_{n-1} \\
x_n
\end{bmatrix} + 
\begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
0 \\
b_0
\end{bmatrix}
\end{aligned}
\]

\[
y = 
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_{n-1} \\
x_n
\end{bmatrix}
\]
Example : TF to State Space

1. Inverse Laplace

   \[ \dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r \]

2. Select state variables

   \[ \begin{align*}
   x_1 &= c \\
   x_2 &= \dot{c} \\
   x_3 &= \ddot{c}
   \end{align*} \]

   \[ \begin{align*}
   \dot{x}_1 &= x_2 \\
   \dot{x}_2 &= x_3 \\
   \dot{x}_3 &= -24x_1 - 26x_2 - 9x_3 + 24r
   \end{align*} \]

   \[ y = c = x_1 \]

\[ G(s) = \frac{N(s)}{D(s)} \]

- \( N(s) \) numerator
- \( D(s) \) denominator
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-24 & -26 & -9
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
24
\end{bmatrix} r
\]

\[
y = \begin{bmatrix}
1 & 0 & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\]
Decomposing a transfer function

(a) 

\[
\frac{b_2s^2 + b_1s + b_0}{a_3s^3 + a_2s^2 + a_1s + a_0}
\]

(b) 

\[
\frac{1}{a_3s^3 + a_2s^2 + a_1s + a_0}
\]

Internal variables: \(X_2(s), X_3(s)\)

\[
\frac{b_2s^2 + b_1s + b_0}{a_3s^3 + a_2s^2 + a_1s + a_0}
\]
\[ Y(s) = C(s) = \left( b_2 s^2 + b_1 s + b_0 \right) X_1(s) \]

\[ y(t) = b_2 \frac{d^2 x_1}{dt^2} + b_1 \frac{dx_1}{dt} + b_0 x_1 \]

\[ y(t) = b_0 x_1 + b_1 x_2 + b_2 x_3 \]
Example

(a) \[ \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24} \]

(b) \[ \frac{1}{s^3 + 9s^2 + 26s + 24} \]
Internals: \( X_2(s), X_3(s) \)

(c) Diagram with blocks and equations.
\[ y(t) = 2x_1 + 7x_2 + x_3 \]

\[ y = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \]
\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

\[ \text{Laplace Transform} \]
\[ sX(s) = AX(s) + BU(s) \]
\[ Y(s) = CX(s) + DU(s) \]

\[ X(s) = (sI - A)^{-1}BU(s) \]

\[ Y(s) = \left[ C(sI - A)^{-1}B + D \right]U(s) \]

\[ T(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D \]
\[
\begin{align*}
\dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x
\end{align*}
\]

\[
(sI - A) = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}
\]

\[
(sI - A)^{-1} = \frac{\text{adj}(sI - A)}{\det(sI - A)}
\]

\[
= \begin{bmatrix} (s^2 + 3s + 2) & s + 3 & 1 \\ -1 & s(s + 3) & s \\ -s & -(2s + 1) & s^2 \end{bmatrix}
\]

\[
= \frac{s^3 + 3s^2 + 2s + 1}{s^3 + 3s^2 + 2s + 1}
\]
\[ T(s) = C(sI - A)^{-1} B + D \]

\[ \rightarrow T(s) = \frac{10(s^2 + 3s + 2)}{s^3 + 3s^2 + 2s + 1} \]
The solution of the state differential equation can be obtained in a manner similar to the approach we utilize for solving a first order differential equation. Consider the first-order differential equation

\[ \dot{x} = ax; \quad x(0) = 0 \]

\[ dx = ax \, dt \]

Where \( x(t) \) and \( u(t) \) are scalar functions of time. We expect an exponential solution of the form \( e^{at} \). Taking the Laplace transform of both sides, we have

\[ \hat{x} \]

on integrating above equation

\[ \log x = at + c \]

\[ X = e^{at} \cdot e^c \]

\[ x = x(0) = e^c \]

on substituting the intial condition , the solution of homogeneous state equation of first order differential equation is

\[ \dot{x} = e^{at} \, x(0) \]

\[ \dot{x} = AX(t), \, x(0) = 0 \]

\[ e^{At} = I + At + \frac{A^2 t^2}{2!} + \cdots + \frac{A^k t^k}{k!} + \cdots \]
The solution of the state differential equation can be obtained in a manner similar to the approach we utilize for solving a first order differential equation. Consider the first-order differential equation

$$\dot{x} = ax + bu; \ x(0) = 0$$

Where \(x(t)\) and \(u(t)\) are scalar functions of time. By taking Laplace transform

$$s \ X(s) - x_0 = a \ X(s) + bU(s)$$

The inverse Laplace transform of \(X(s)\) results in the solution

$$x(t) = e^{at}x(0) + \int_{0}^{t} e^{a(t-\tau)}bu(\tau)d\tau$$

We expect the solution of the state differential equation to be similar to \(x(t)\) and to be of differential form. The **matrix exponential function** is defined as

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^kt^k}{k!} + \cdots$$
which converges for all finite $t$ and any $A$. Then the solution of the state differential equation is found to be

$$x(t) = e^{At}x(0) + \int_{0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$X(s) = [sI - A]^{-1}x(0) + [sI - A]^{-1}Bu(s)$$

where we note that $[sI-A]^{-1}=\varphi(s)$, which is the Laplace transform of $\varphi(t)=e^{At}$. The matrix exponential function $\varphi(t)$ describes the unforced response of the system and is called the fundamental or state transition matrix.

$$x(t) = \phi(t)x(0) + \int_{0}^{t} \phi(t-\tau)Bu(\tau)\,d\tau$$
THE TRANSFER FUNCTION FROM THE STATE EQUATION

The transfer function of a single input-single output (SISO) system can be obtained from the state variable equations.

\[ \dot{x} = Ax + Bu \]
\[ y = Cx \]

where \( y \) is the single output and \( u \) is the single input. The Laplace transform of the equations

\[ sX(s) = AX(s) + BU(s) \]
\[ Y(s) = CX(s) \]

where \( B \) is an nx1 matrix, since \( u \) is a single input. We do not include initial conditions, since we seek the transfer function. Reordering the equation
\[
[sI - A]X(s) = BU(s)
\]
\[
X(s) = [sI - A]^{-1}BU(s) = \phi(s)BU(s)
\]
\[
Y(s) = C\phi(s)BU(s)
\]

Therefore, the transfer function \(G(s) = \frac{Y(s)}{U(s)}\) is

\[
G(s) = C\phi(s)B
\]

**Example:**

Determine the transfer function \(G(s) = \frac{Y(s)}{U(s)}\) for the RLC circuit as described by the state differential function

\[
\dot{x} = \begin{bmatrix}
0 & -\frac{1}{C} \\
1 & \frac{1}{CR}
\end{bmatrix} x + \begin{bmatrix}
\frac{1}{C} \\
0
\end{bmatrix} u \quad , \quad y = \begin{bmatrix}
0 & R
\end{bmatrix} x
\]
\[
[sI - A] = \begin{bmatrix}
s & \frac{1}{C} \\
\frac{1}{L} & s + \frac{R}{L}
\end{bmatrix}
\]

\[
\phi(s) = [sI - A]^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix}
\frac{R}{L} & -\frac{1}{C} \\
\frac{1}{L} & s
\end{bmatrix}
\]

\[
\Delta(s) = s^2 + \frac{R}{L}s + \frac{1}{LC}
\]

Then the transfer function is

\[
G(s) = \begin{bmatrix}
0 & R \\
\frac{s + \frac{R}{L}}{\Delta(s)} & -\frac{1}{C\Delta(s)}
\end{bmatrix} \begin{bmatrix}
\frac{1}{C} \\
0
\end{bmatrix}
\]

\[
G(s) = \frac{R/\text{LC}}{\Delta(s)} = \frac{R/\text{LC}}{s^2 + \frac{R}{L}s + \frac{1}{\text{LC}}}
\]
CONSIDER THE SYSTEM

\[
\frac{C(s)}{R(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24}
\]

\[
\left(s^3 + 9s^2 + 26s + 24\right)C(s) = 24R(s)
\]

\[
\ddot{c} + 9\dot{c} + 26\dot{c} + 24c = 24r
\]

State variables

\[
x_1 = c
\]

\[
x_2 = \dot{c}
\]

\[
x_3 = \ddot{c}
\]

System equations

\[
\dot{x}_1 = x_2
\]

\[
\dot{x}_2 = x_3
\]

\[
\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r
\]

Output equation

\[
y = c = x_1
\]
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_2 &= -24x_1 - 26x_2 - 9x_3 + 24r \\
y &= c = x_1 \\
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_2
\end{bmatrix} &=
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-24 & -26 & -9
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
24
\end{bmatrix} r \\
y &= \begin{bmatrix}
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\]
THANK YOU