Power Point Presentation

on

ELECTROMAGNETIC FIELD THEORY

B.Tech IV Semester (R16)

Prepared

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ELECTRICAL AND ELECTRONICS ENGINEERING

UNIT-1
INTRODUCTION TO ELECTRO-STATICS

The Electrostatic Field

1.1 Introduction

1.2 Coulomb's Law

1.3 The Electric Field

1.4 Continuous Charge Distribution

1.1 Introduction

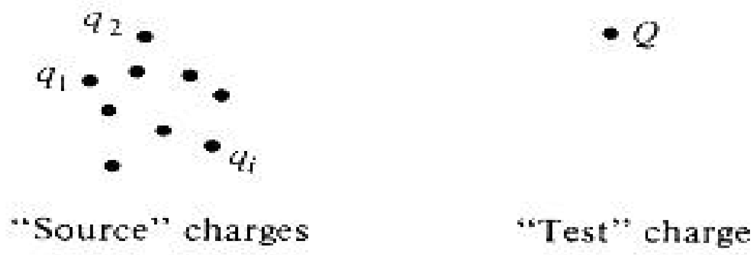


Figure 2.1

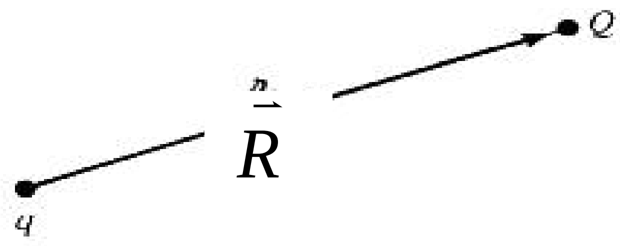


Figure 2.2

The fundamental problem for electromagnetic to solve is to calculate the interaction of charges in a given configuration.

That is, what force do they exert on another charge Q ?
The simplest case is that the source charges are stationary.

Principle of Superposition:
The interaction between any two charges is completely unaffected by the presence of other charges.

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots$$

$\vec{F}_i$ is the force on Q due to q_i

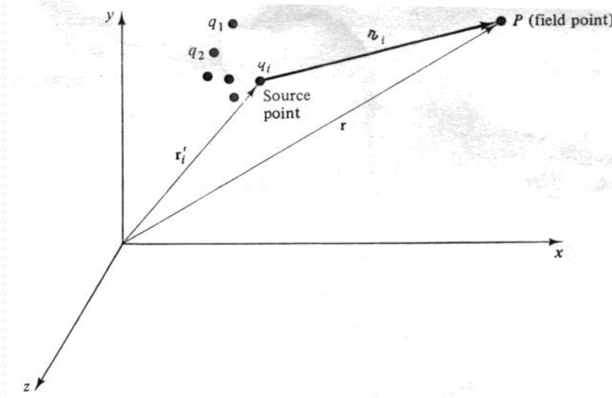
1.2 Coulomb's Law

The force on a charge Q due to a single point charge q is given by Coulomb`s law.

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{R^2}\hat{R} \qquad \vec{R} = \vec{r}_Q - \vec{r}_q = R\,\hat{R}$$

$$\varepsilon_0 = 8.85\times 10^{-12}\frac{c^2}{N\cdot m^2}$$

1.3 The Electric Field



$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdots$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 Q}{R_1^2}\hat{R}_1 + \frac{q_2 Q}{R_2^2}\hat{R}_2 + \cdots \right)$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{q_1}{R_1^2}\hat{R}_1 + \frac{q_2}{R_2^2}\hat{R}_2 + \frac{q_3}{R_3^2}\hat{R}_3 + \cdots \right)$$

$$\vec{F} = Q\vec{E}$$

$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{R_1^2}\hat{R}_i$$

the electric field of the source charge

1.4 Continuous Charge Distributions

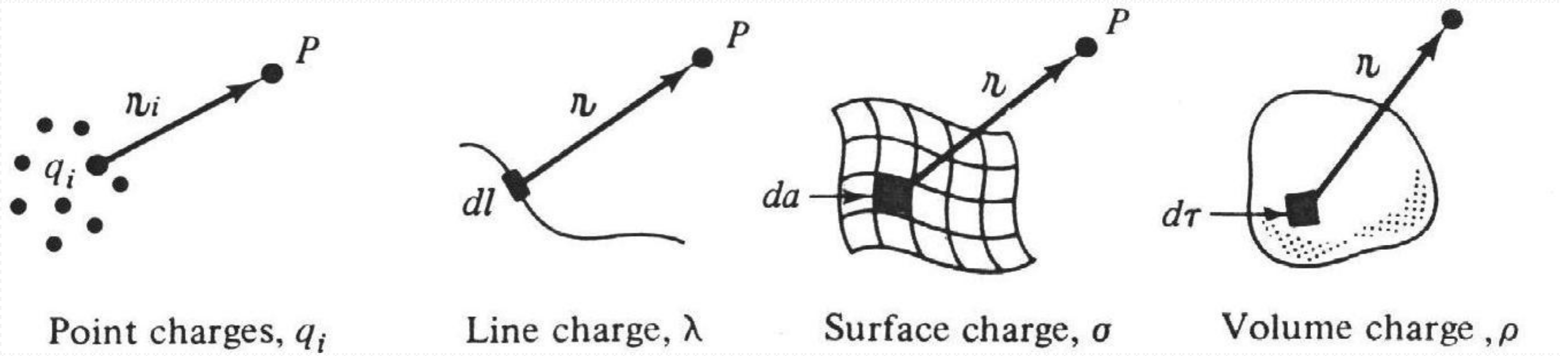


Figure 2.5

$$\sum_{i=1}^{n} (\quad)\, q_i \sim \int_{line} (\quad)\, \lambda\, dl \sim \int_{surface} (\quad)\, \sigma\, da \sim \int_{volume} (\quad)\, \rho\, d\tau$$

$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{R_i^2} \hat{R}_i \qquad \vec{E}(P) = \frac{1}{4\pi\varepsilon_0} \int_{line} \frac{\hat{R}}{R^2} \lambda\, d\ell$$

$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0} \int_{surface} \frac{\hat{R}}{R^2} \sigma\, da \qquad \vec{E}(P) = \frac{1}{4\pi\varepsilon_0} \int_{volume} \frac{\hat{R}}{R^2} \rho\, d\tau$$

1.4

Example:

the test point is at $\vec{P} = (x, y, z)$ the source point is at (x', y', z')

$$\vec{R} = (x - x')\hat{i} + (y - y')\hat{j} + (z - z')\hat{k}$$

$$\vec{R} = \left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{1/2} \qquad \hat{R} = \frac{\vec{R}}{|R|}$$

$$\vec{E}(x, y, z) = \frac{1}{4\pi\varepsilon_0}\int_{volume}\frac{(x - x')\hat{i} + (y - y')\hat{j} + (z - z')\hat{k}}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}}\rho(x', y', z')\,dx'dy'dz'$$

$$\because \quad \vec{E}(x, y, z) = \frac{1}{4\pi\varepsilon_0}\int_{volume}\frac{\hat{R}}{R^2}\rho(x', y', z')dx'dy'dz'$$

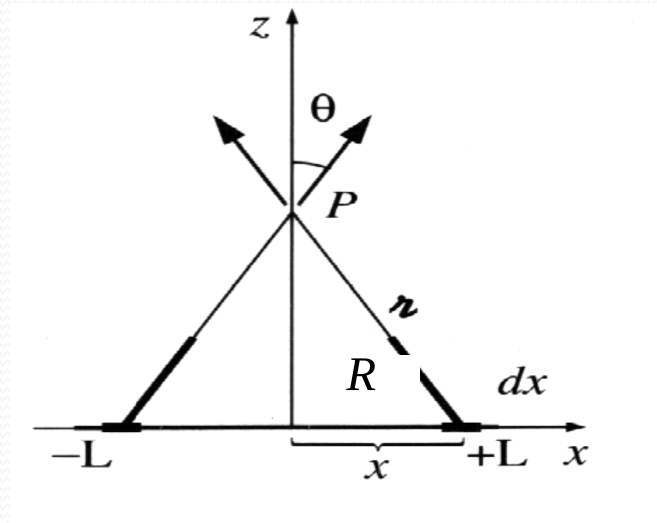
$$= \frac{1}{4\pi\varepsilon_0}\int_{volume}\frac{\hat{R}}{R^2}\cdot\frac{R}{R}\rho(x', y', z')dx'dy'dz'$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{volume}\frac{\vec{R}}{R^3}\rho(x', y', z')dx'dy'dz'$$

Example 2.1 Find the electric field a distance z above the midpoint of a straight of length 2L, which carries a uniform line charge λ

Solution:

$$d\vec{E} = 2\frac{1}{4\pi\varepsilon_0}\left(\frac{\lambda dx}{R^2}\right)\cos\theta\ \hat{z}$$



$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_0^L \frac{2\lambda z}{(z^2+x^2)^{3/2}}dx$$

$$= \frac{2\lambda z}{4\pi\varepsilon_0}\left[\frac{x}{z^2\sqrt{z^2+x^2}}\right]\Bigg|_0^L$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}}$$

(1) z>>*L* $\vec{E} \cong \frac{1}{4\pi\varepsilon_0}\frac{2\lambda L}{z^2}$

(2) $L \to \infty$ $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{z}$

UNIT – II
CONDUCTORS AND DIELECTRICS

2 Divergence and Curl of Electrostatic Fields

2.1 Field Lines and Gauss's Law

2.2 The Divergence of E

2.3 Application of Gauss's Law

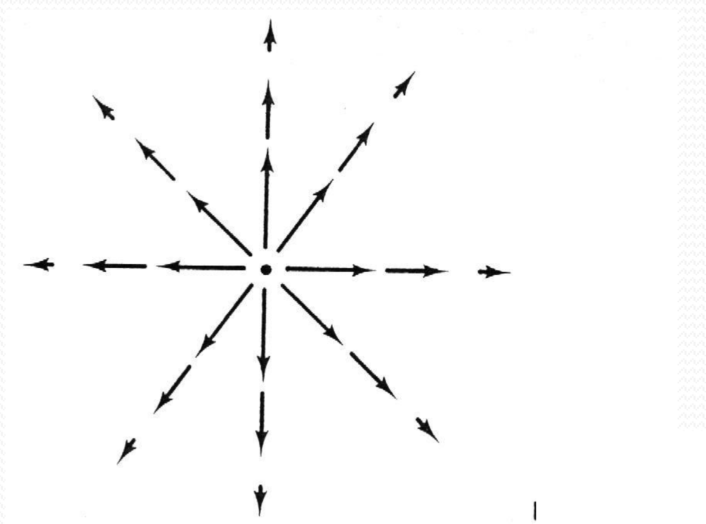
2.4 The Curl of E

2.1 Fields lines and Gauss's law

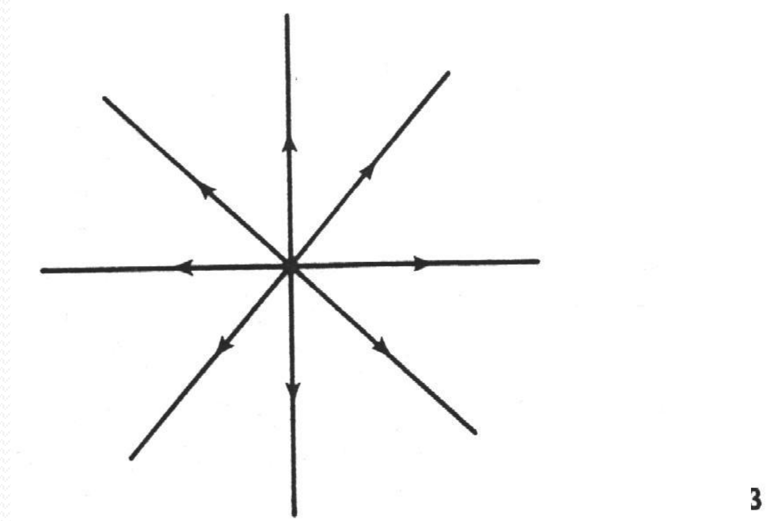
A single point charge q, situated at the origin

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

Because the field falls off like $1/r^2$, the vectors get shorter as I go father away from the origin, and they always point radially outward. This vectors can be connect up the arrows to form the *field lines.* The magnitude of the field is indicated by the density of the lines.



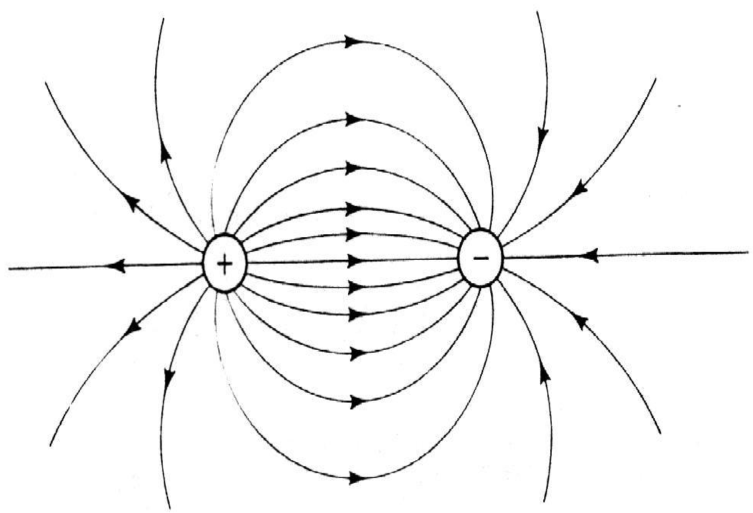
strength length of arrow



strength density of field line

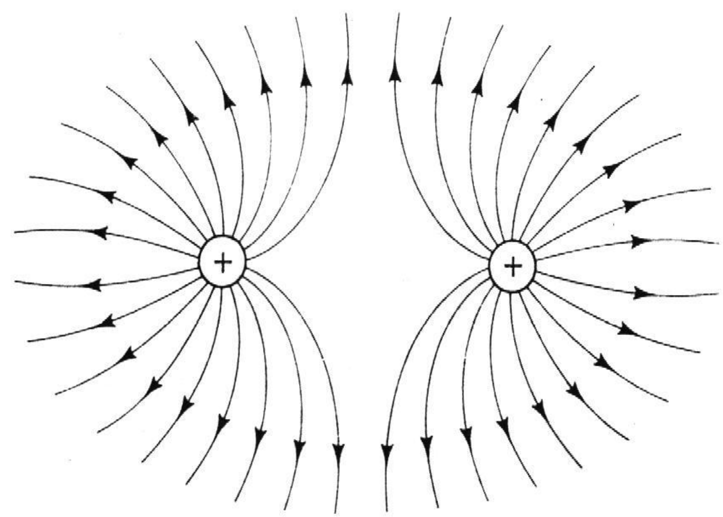
2.1

1. Field lines emanate from a point charge symmetrically in all directions.
2. Field lines originate on positive charges and terminate on negative ones.
3. They cannot simply stop in midair, though they may extend out to infinity.
4. Field lines can never cross.



Equal but opposite charges

Figure 2.14



Equal charges

Figure 2.15

2.1

Since in this model the fields strength is proportional to the number of lines per unit area, the *flux* of $\vec{E}$ ($\int \vec{E} \cdot d\vec{a}$) is proportional to the the number of field lines passing through any surface .

The flux of E through a sphere of radius r is:

$$\oint \vec{E} \cdot d\vec{a} = \int \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r^2}\hat{r}\right) \cdot (r^2 \sin\theta \, d\theta \, d\varphi \, \hat{r}) = \frac{1}{\varepsilon_0} q$$

The flux through any surface enclosing the charge is q/ε_0

According to the principle of superposition, the total field is the sum of all the individual fields:

$$\vec{E} = \sum_{i=1}^{n} \vec{E}_i \quad , \quad \oint \vec{E} \cdot d\vec{a} = \sum_{i=1}^{n} \left(\oint \vec{E}_i \cdot d\vec{a}\right) = \sum_{i=1}^{n} \left(\frac{1}{\varepsilon_0} q_i\right)$$

A charge outside the surface would contribute nothing to the total flux,since its field lines go in one side and out other.

2.1

Gauss's Law in integral form $\oint \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$

Turn integral form into a differential one , by applying the divergence theorem

$$\oint_{surface} \vec{E} \cdot d\vec{a} = \int_{volume} (\nabla \cdot \vec{E}) d\tau$$

$$= \frac{1}{\varepsilon_0} Q_{enc} = \int_{volume} (\frac{1}{\varepsilon_0} \rho) d\tau$$

Gauss's law in differential form $\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho$

2.2 The Divergence of E

Calculate the divergence of E directly

$$\nabla \cdot \vec{E} = \frac{1}{4\pi\varepsilon_0}\int \nabla \cdot \left(\frac{\hat{R}}{R^2}\right)\rho(r')d\tau, \qquad \vec{R} = \vec{r} - \vec{r}'$$

The r-dependence is contained in R

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\int_{all\,space} \frac{\hat{R}}{R^2}\rho(\vec{r}')d\tau'$$

From

$$\nabla \cdot \left(\frac{\hat{R}}{R^2}\right) = 4\pi\delta^3(R)$$

Thus

$$\nabla \cdot \vec{E} = \frac{1}{4\pi\varepsilon_0}\int 4\pi\delta^3(\vec{r} - \vec{r}')\rho(\vec{r}')d\tau' = \frac{1}{\varepsilon_0}\rho(\vec{r})$$

2.3 Application of Gauss's Law

Example 2.2 Find the field outside a uniformly charged sphere of radius a

Sol:

$$\oint_{surface} \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

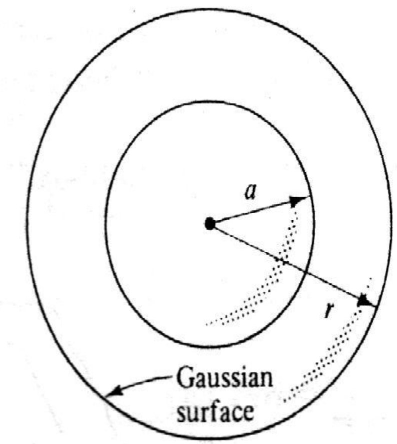


Figure 2.18

(a) E point radially outward ,as does $d\vec{a}$

$$\oint_{surface} \vec{E} \cdot d\vec{a} = \int |E| d\vec{a}$$

(b)E is constant over the Gaussian surface

$$\oint_{surface} |E| d\vec{a} = |E| \oint_{surface} d\vec{a} = |E| 4\pi r^2$$

Thus $|E| 4\pi r^2 = \frac{1}{\varepsilon_0} q \Longrightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$

2.3

1. Spherical symmetry. *Make your Gaussian surface a concentric sphere (Fig 2.18)*
2. Cylindrical symmetry. *Make your Gaussian surface a coaxial cylinder (Fig 2.19)*
3. Plane symmetry. *Use a Gaussian surface a coaxial the surface (Fig 2.20)*

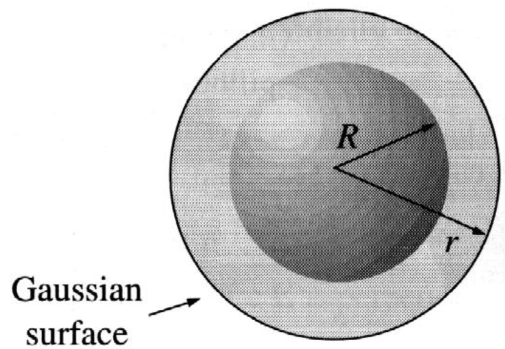


Figure 2.18

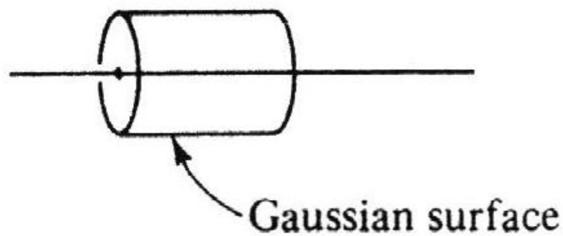


Figure 2.19

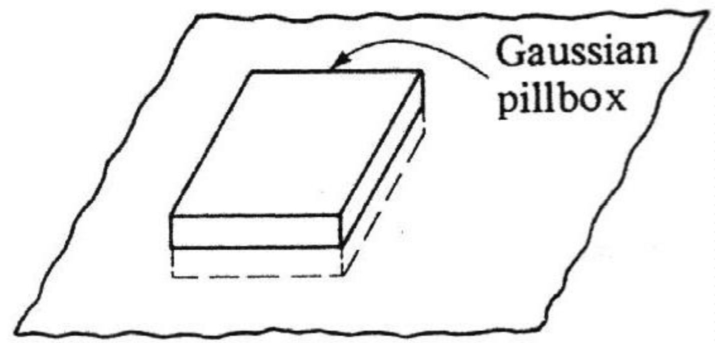


Figure 2.20

2.3

Example 2.3 Find the electric field inside the cylinder which contains charge density as $\rho = kr$

Solution:

$$\oint_{surface} \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

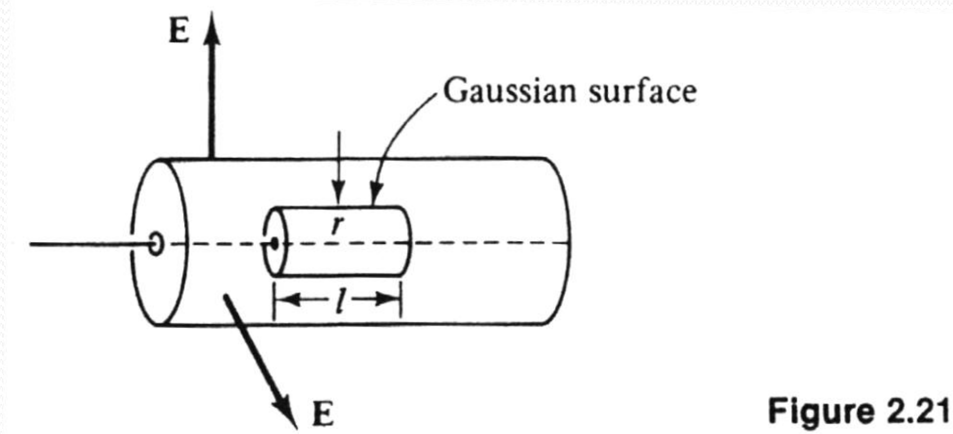


Figure 2.21

The enclosed charge is

$$Q_{enc} = \int \rho \, d\tau = \int (kr')(r'dr'd\varphi\, dz) = 2\pi k l \int_0^r r'^2 dr' = \frac{2}{3}\pi k l r^3$$

$$\oint \vec{E} \cdot d\vec{a} = \int |E| da = |E| \int da = |E| 2\pi r l$$ (by symmetry)

thus

$$|E| 2\pi r l = \frac{1}{\varepsilon_0} \frac{2}{3} \pi k l r^3 \Rightarrow \boxed{\vec{E} = \frac{1}{3\varepsilon_0} k r^2 \hat{r}}$$

Example 2.4 An infinite plane carries a uniform surface charge σ.
Find its electric field.

Solution: Draw a "Gaussian pillbox
Apply Gauss's law to this surface

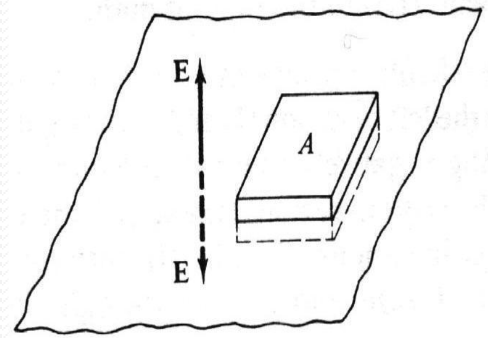


Figure 2.22

$$\oint_{surface} \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

By symmetry, E points away from the plane
thus, the top and bottom surfaces yields

$$\int \vec{E} \cdot d\vec{a} = 2A\left|E\right|$$

(the sides contribute nothing)

$$2A\left|E\right| = \frac{1}{\varepsilon_0}\sigma A \quad \Longrightarrow \quad \boxed{\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}}$$

2.3

Example 2.5 Two infinite parallel planes carry equal but opposite uniform charge densities $\pm\sigma$ .Find the field in each of the three regions.

Solution:

The field is (σ/ε_0), and points to the right, between the plane elsewhere it is zero.

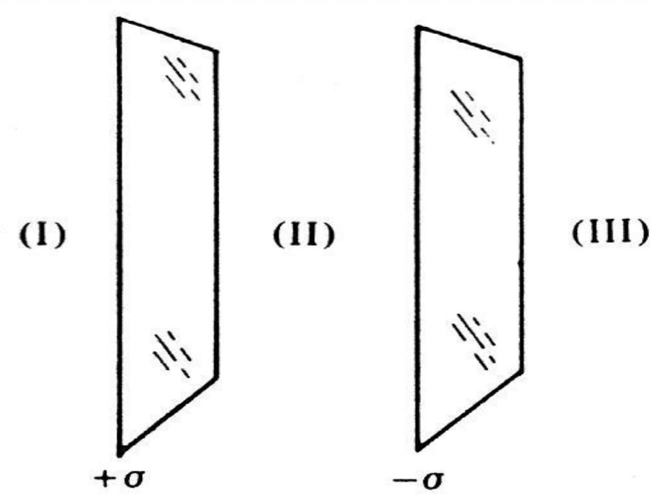


Figure 2.23

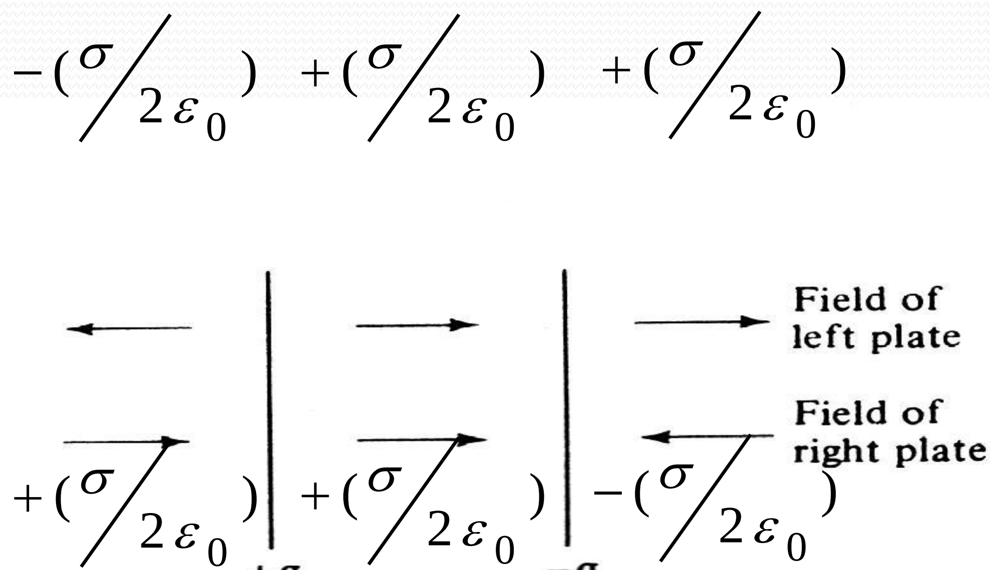


Figure 2.24

2.4 The Curl of E

A point charge at the origin. If we calculate the line integral of this field from some point a to some other point b :

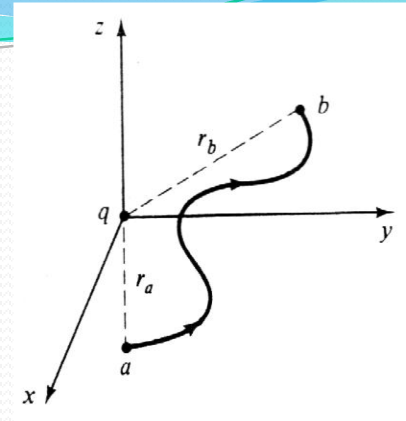


Figure 2.29

$$\int_a^b \vec{E} \cdot d\vec{l}$$

In spherical coordinate

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\varphi\,\hat{\varphi} \qquad \vec{E}\cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}d\vec{r}$$

$$\int_a^b \vec{E}\cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0}\int_a^b \frac{q}{r^2}dr = \frac{-1}{4\pi\varepsilon_0}\frac{q}{r}\bigg|_{r_a}^{r_b} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

This integral is independent of path. It depends on the two end points

$$\oint \vec{E}\cdot d\vec{l} = 0 \qquad (\text{ if } r_a = r_b)$$

Apply by Stokes' theorem, $\nabla \times \vec{E} = 0$

The principle of superposition

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots$$

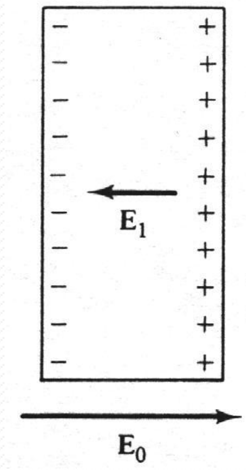
so

$$\nabla \times \vec{E} = \nabla \times (\vec{E}_1 + \vec{E}_2 + \cdots) = (\nabla \times \vec{E}_1) + (\nabla \times \vec{E}_2) + \cdots = 0$$

$$\nabla \times \vec{E} = 0$$

must hold for any static charge distribution whatever.

2.5 Basic Properties of Conductors



e^- are free to move in a conductor

(1) $\vec{E} = 0$ inside a conductor

otherwise, the free charges that produce $\vec{E}$ will move to make $\vec{E} = 0$ inside a conductor

(2) $\rho = 0$ inside a conductor

$$\because \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{E} = 0 \qquad \Rightarrow \qquad \rho = 0$$

(3) Any net charge resides on the surface

(4) V is constant, throughout a conductor.

$$V(b) - V(a) = -\int_a^b \vec{E} \cdot d\vec{l} = 0$$

$$\therefore V(b) = V(a)$$

(5) $\vec{E}$ is perpendicular to the surface, just outside a conductor. Otherwise, $\vec{E}_{\parallel}$ will move the free charge to make $\vec{E}_{\parallel} = 0$ in terms of energy, free charges staying on the surface have a minimum energy.

σ

$$Energy = \frac{1}{8\pi\varepsilon_0}\frac{q^2}{R}$$

$E_{in} = 0 \quad W_{in} = 0$

ρ

$$Energy = \frac{3}{20\pi\varepsilon_0}\frac{q^2}{R}$$

$E_{in} \neq 0 \quad W_{in} \neq 0$

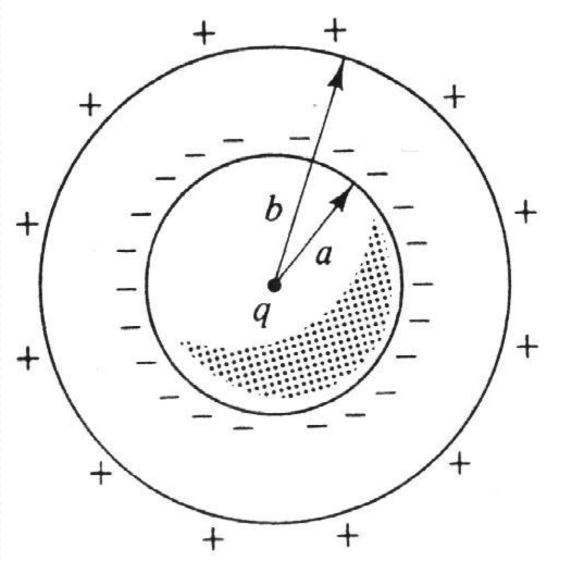
Example : A point charge q at the center of a spherical conducting shell. How much induced charge will accumulate there?

Solution :

$\because E_{in} = 0$ $\quad Q$ induced

$\therefore 4\pi a^2 \cdot \sigma_a = -q$

$$\sigma_a = -\frac{1}{4\pi}\frac{q}{a^2}$$



charge conservation

$4\pi b^2 \cdot \sigma_b = -4\pi a^2 \sigma_a$

$$\sigma_b = \frac{1}{4\pi}\frac{q}{b^2}$$

$E_{in} = 0$

$Q_{enc} = q + q_{induced} = 0$

$q_{induecd} = -q$

2.7 Induced Charge

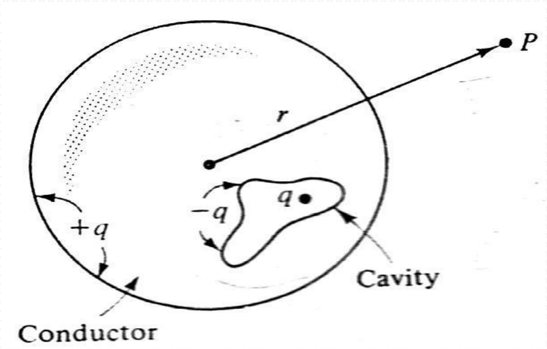
Example 2.9 Within the cavity is a charge $+q$. What is the field outside the sphere?

$-q$ distributes to shield q and to make $E_{in} = 0$

i.e., $V_{surface} = constant$

from charge conservation and symmetry.
$+q$ uniformly distributes at the surface

$$\therefore \vec{E}(P) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$



$$\oint \vec{E}\cdot d\vec{l} = 0$$

$$\int_a^b \vec{E}\cdot d\vec{l} = 0$$ a,b are arbitrary chosen

$$\vec{E}_{incavity} = 0$$

$\therefore$ a Faraday cage can shield out stray $\vec{E}$

2.8 The Surface Charge on a Conductor

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

$$\because \; E_{in} = 0 \;\; \text{or} \;\; \vec{E}_{below} = 0$$

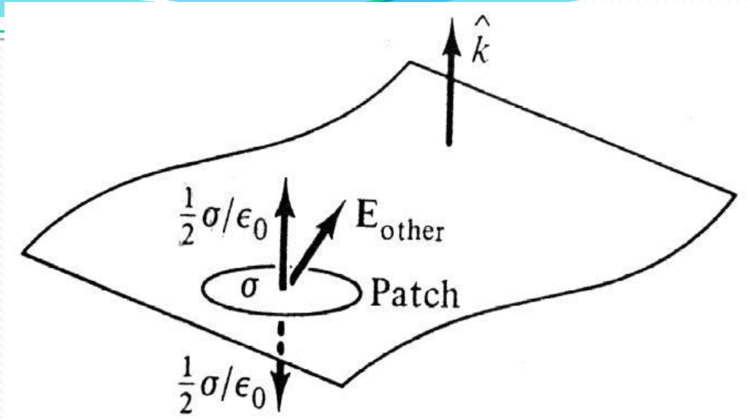
$$\therefore \; -\nabla V = \vec{E}_{above} = \frac{\sigma}{\varepsilon_0}\hat{n} \qquad \text{or} \quad \frac{\partial V}{\partial n} = -\frac{\sigma}{\varepsilon_0}$$

if we know *V*, we can get σ.

2.8

Force on a surface charge,

$$\vec{f} = \sigma \vec{E}_{average} = \frac{1}{2}\sigma(\vec{E}_{above} + \vec{E}_{below})$$



why the average?

$$\vec{E}_{above} = \vec{E}_{other} + \frac{\sigma}{2\varepsilon_0}\hat{k} \qquad \vec{E}_{below} = \vec{E}_{other} - \frac{\sigma}{2\varepsilon_0}\hat{k}$$

$$\vec{E}_{other} = \frac{1}{2}(\vec{E}_{above} + \vec{E}_{below}) \overset{\sigma \to 0}{=} \vec{E}_{average}$$

In case of a conductor

$$f = \frac{1}{2}\sigma\vec{E}_{above} = \frac{\sigma^2}{2\varepsilon_0}\hat{n}$$

electrostatic pressure

$$P = \frac{\sigma^2}{2\varepsilon_0} = \frac{\varepsilon_0}{2}E^2$$

2.9 Capacitors

Consider 2 conductors (Fig 2.53)

The potential difference

$$V = V_+ - V_- = -\int_{(-)}^{(+)} \vec{E} \cdot d\vec{l}$$

(V is constant.)

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{r}}{r^2}\rho\, d\tau$$

double ρ → double Q → double $\vec{E}$ → double V

Define the ratio between Q and V to be capacitance

$$C = \frac{Q}{V}$$

a geometrical quantity

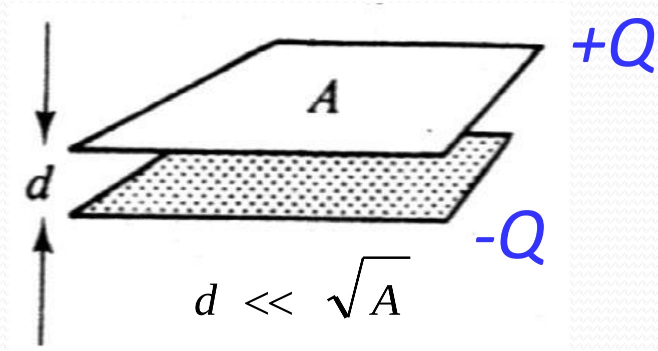
in mks 1 farad(F)= 1 Coulomb / volt

↑ inconveniently large ;

$10^{-6} F$: *microfarad*

$10^{-12} F$: *picofarad*

Example 2.10 Find the capacitance of a "parallel-plate capacitor"?



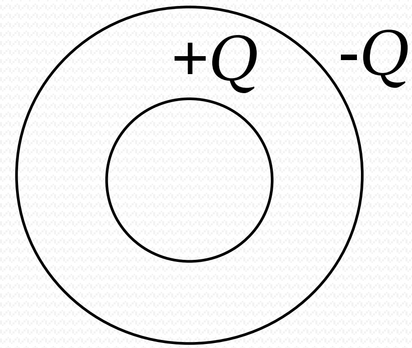
Solution:

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A\varepsilon_0}$$

$$V = E \cdot d = \frac{Q}{A\varepsilon_0} d$$

$$C = \frac{A\varepsilon_0}{d}$$

Example 2.11 Find capacitance of two concentric spherical shells with radii a and b .



Solution:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$$

$$V = -\int_b^a \vec{E}\cdot d\vec{l} = -\frac{Q}{4\pi\varepsilon_0}\int_b^a \frac{1}{r^2}dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$C = \frac{Q}{V} = 4\pi\varepsilon_0 \frac{ab}{(b-a)}$$

2.9

The work to charge up a capacitor

$$dW = V\,dq = \left(\frac{q}{C}\right)dq$$

$$W = \int_0^Q \left(\frac{q}{C}\right)dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV = \frac{1}{2}CV^2$$

2.10 Conductors and Dielectrics

- Conductors
  - Current, current density, drift velocity, continuity
  - Energy bands in materials
  - Mobility, micro/macro Ohm's Law
  - Boundary conditions on conductors
  - Methods of Images
- Dielectrics
  - Polarization, displacement, electric field
  - Permittivity, susceptibility, relative permittivity
  - Dielectrics research
  - Boundary conditions on dielectrics

Conductors and Dielectrics

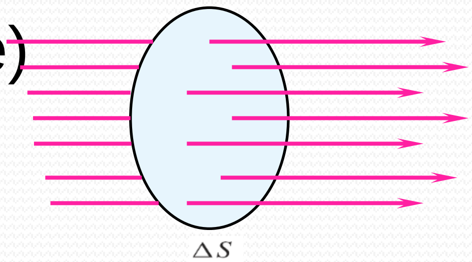
- Polarization
  - Static alignment of charge in material
  - Charge aligns when voltage applied, moves no further
  - Charge proportional to voltage
- Conduction
  - Continuous motion of charge through material
  - Enters one side, exits another
  - Current proportional to voltage

2.11 Current and current density

- Basic definition of current C/s = Amps

$$I = \frac{dQ}{dt}$$

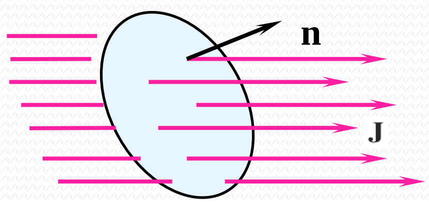
- Basic current density (J perp. surface)



$$\Delta I = J_N \Delta S$$

- Vector current density

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$



Current density and charge velocity

- Basic definition of current

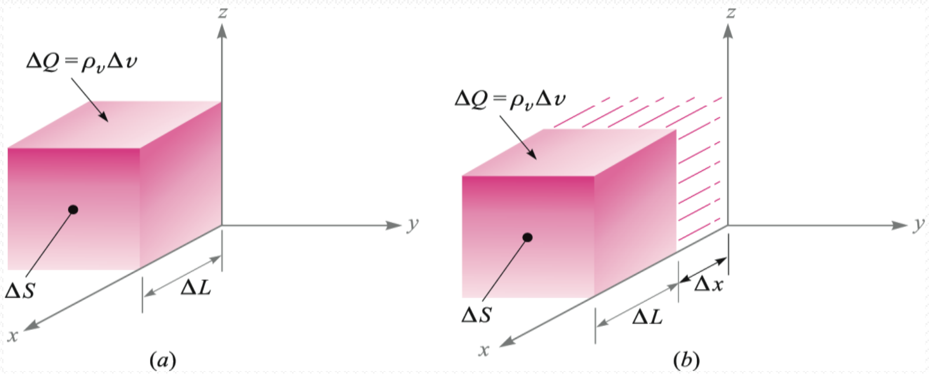
$$I = \frac{\Delta Q}{\Delta t} = \frac{\rho_v \, \Delta v}{\Delta t} = \frac{\rho_v \, dS \, \Delta x}{\Delta t} = \rho_v \, dS \, v$$

- Combining with earlier expression

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

- Gives current density

$$\boldsymbol{J} = \rho_v \, \boldsymbol{v}$$



Charge and current continuity

- Current leaving any closed surface is time rate of change of charge within that surface

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S} \quad = -\frac{dQ_i}{dt} \quad = -\frac{d}{dt}\int_{\text{vol}} \rho_\nu \, dv$$

- Using divergence theorem on left

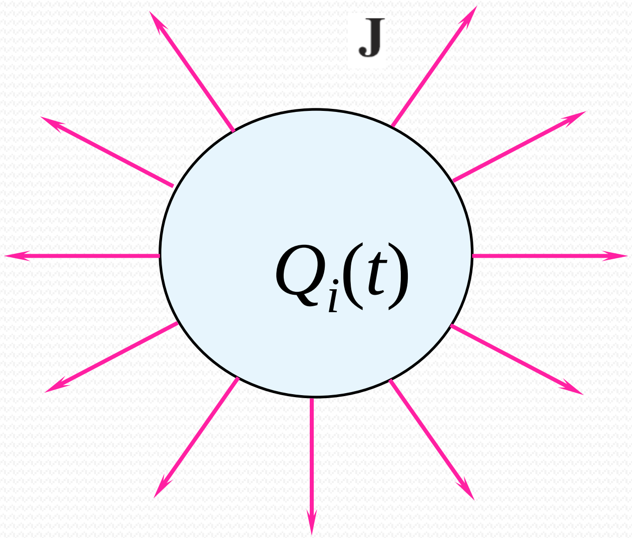
$$\int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv = -\frac{d}{dt}\int_{\text{vol}} \rho_\nu \, dv$$

- Taking time derivative inside integral

$$\int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv = \int_{\text{vol}} -\frac{\partial \rho_\nu}{\partial t}\, dv$$

- Equating integrands

$$(\nabla \cdot \mathbf{J}) = -\frac{\partial \rho_\nu}{\partial t}$$



Example – charge and current continuity

- Given spherically symmetric current density

$$\boldsymbol{J} = \frac{1}{r} e^{-t} \boldsymbol{a_r}$$

- Current increasing from r = 5m to r= 6m at t=1s

$$I = J_r S \;=\; \frac{1}{5} e^{-1} 4\pi 5^2 = 23.1A \text{ @ } 5m, \qquad \frac{1}{6} e^{-1} 4\pi 6^2 = 27.7 \text{@ } 6m$$

- Current density from continuity equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot J = -\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{1}{r} e^{-t} \right) = -\frac{1}{r^2} e^{-t}$$

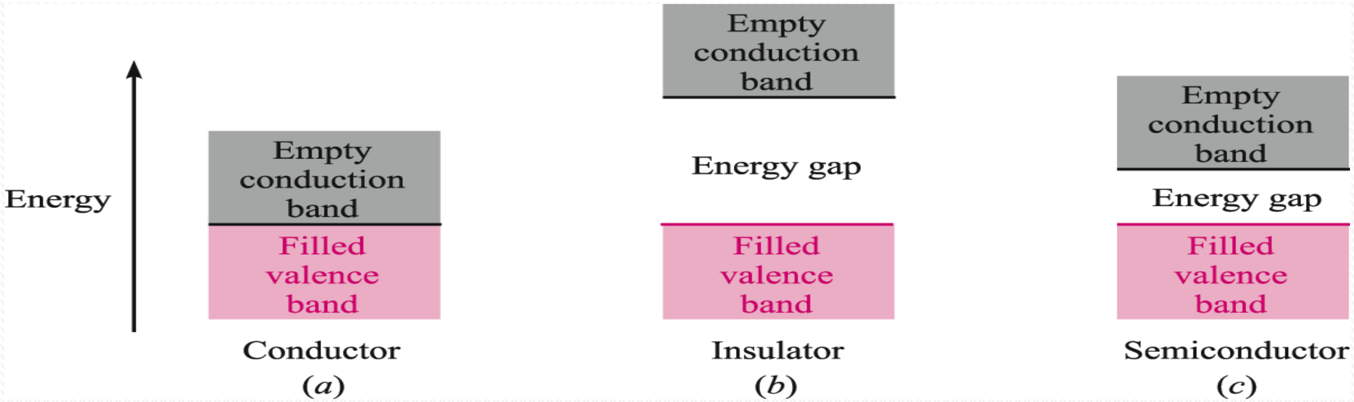
- Charge density ρ integral w.r.t. time

$$\rho = -\frac{1}{r^2} \int e^{-t} dt = \frac{1}{r^2} e^{-t} + const(r)$$

- Drift velocity is thus

$$v_r = \frac{J_r}{\rho_v} = \frac{\frac{1}{r} e^{-t}}{\frac{1}{r^2} e^{-t}} = r \;\; m/s$$

2.12 Energy Band Structure in Three Material Types



Discrete quantum states broaden into energy bands in condensed materials with overlapping potentials

- Valence band – outermost filled band
- Conduction band – higher energy unfilled band

Band structure determines type of material

a) Insulators show large energy gaps, requiring large amounts of energy to lift electrons into the conduction band. When this occurs, the dielectric breaks down.

b) Conductors exhibit no energy gap between valence and conduction bands so electrons move freely

c) Semiconductors have a relatively small energy gap, so modest amounts of energy (applied through heat, light,or an electric field) may lift electrons from valence to conduction bands.

2.13 Ohm's Law (microscopic form)

Free electrons are accelerated by an electric field. The applied force on an electron of charge $Q = -e$ *is*

$$\mathbf{F} = -e\mathbf{E}$$

But in reality the electrons are constantly bumping into things (like a terminal velocity) so they attain an equilibrium or *drift velocity*:

$$\mathbf{v}_d = -\mu_e \mathbf{E}$$

where μ_e is the electron *mobility,* expressed in units of m^2/V-s. The drift velocity is used in the current density through:

$$\mathbf{J} = \rho_v \mathbf{v}_d = -\rho_e \mu_e \mathbf{E} = \sigma \mathbf{E}$$

So Ohm's Law in point form (material property)

$$\mathbf{J} = \sigma \mathbf{E}$$

With the conductivity given as:

$$\sigma = -\rho_e \mu_e$$

$$\sigma = -\rho_e \mu_e + \rho_h \mu_h$$

S/m (electrons/holes)

Ohm's Law (macroscopic form)

- For constant electric field

$$|V_{ab}| = \int_b^a \boldsymbol{E} \cdot d\boldsymbol{L} = EL$$

- Ohm's Law becomes

$$\frac{I}{S} = J = \sigma E = \sigma \frac{V}{L}$$

- Rearranging gives

$$I = \frac{\sigma S}{L} V \qquad G = \frac{\sigma S}{L} \quad siemens \quad (conductance)$$

- Or

$$V = \frac{L}{\sigma S} I \qquad R = \frac{L}{\sigma S} = \frac{\rho L}{S} \quad ohms \quad (resistance)$$

- Variation with geometry
- Conductance vs. Resistance

2.14 Boundary conditions for conductors

- No electric field in interior
  - Otherwise charges repel to the surface
- No tangential electric field at surface
  - $E_t = 0$
  - Otherwise charges redistribute along surface
- Normal electric field at surface
  - $\varepsilon_o E_n = D_n = \rho_s$
  - Displacement Normal equals Charge Density (Gauss's Law)

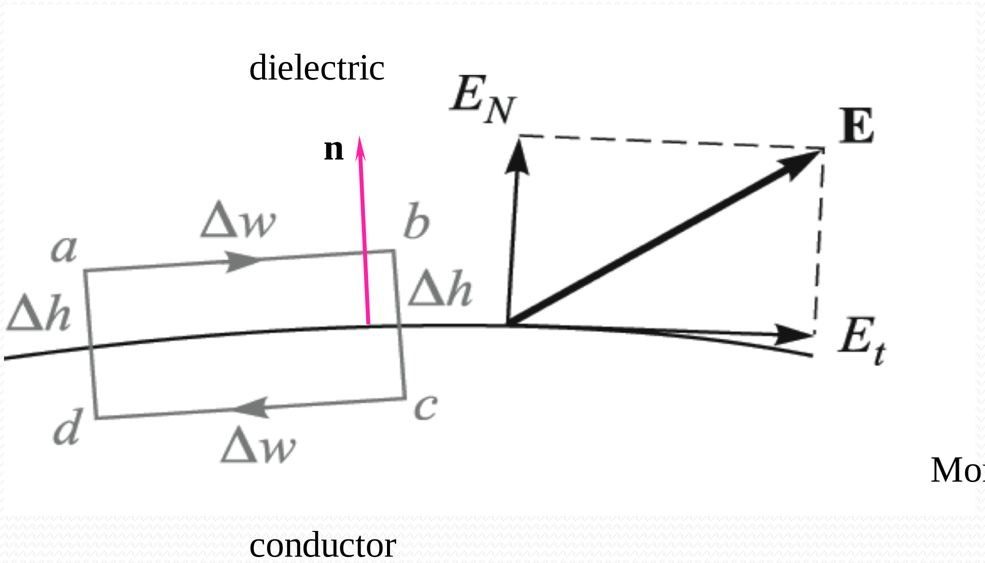
Boundary Condition for Tangential Electric Field E

Over the rectangular integration path, we use $\oint \mathbf{E} \cdot d\mathbf{L} = 0$ or $\int_a^b + \int_b^c + \int_c^d + \int_d^a = 0$

To find:

$$E_t \Delta w - \underbrace{E_{N,\text{at}\,b} \tfrac{1}{2} \Delta h + E_{N,\text{at}\,a} \tfrac{1}{2} \Delta h}_{} = 0$$

These become negligible as Δh approaches zero.



Therefore

$$E_t = 0$$

More formally:

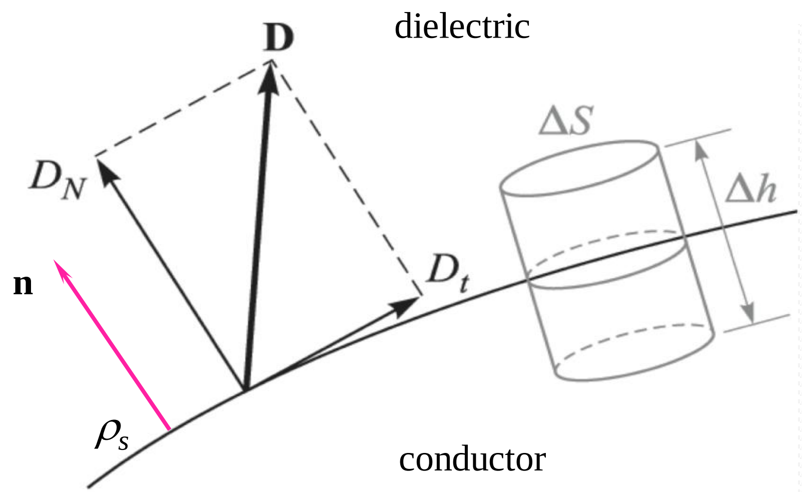
$$\mathbf{E} \times \mathbf{n}\Big|_s = 0$$

Boundary Condition for the Normal Displacement **D**

Gauss' Law is applied to the cylindrical surface shown below:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{top}} + \int_{\text{bottom}} + \int_{\text{sides}} = Q$$

This reduces to: $D_N \Delta S = Q = \rho_S \Delta S$ as Δh approaches zero



Therefore

$$D_N = \rho_S$$

More formally:

$$\mathbf{D} \cdot \mathbf{n}\big|_s = \rho_s$$

2.15 Summary

1. The static electric field intensity inside a conductor is zero.
2. The static electric field intensity at the surface of a conductor is everywhere directed normal to that surface.
3. The conductor surface is an equipotential surface.

$$\mathbf{E} \times \mathbf{n}\Big|_s = 0$$

Tangential E is zero

At the surface:

$$\mathbf{D} \cdot \mathbf{n}\Big|_s = \rho_s$$

Normal D is equal to the surface charge density

Example - Boundary Conditions for Conductors

- Potential given by

$$V = 100(x^2 - y^2)$$

- Potential at (2,-1,3) is 300 V. Also 300 V along entire surface where

$$300 = 100(x^2 - y^2)$$

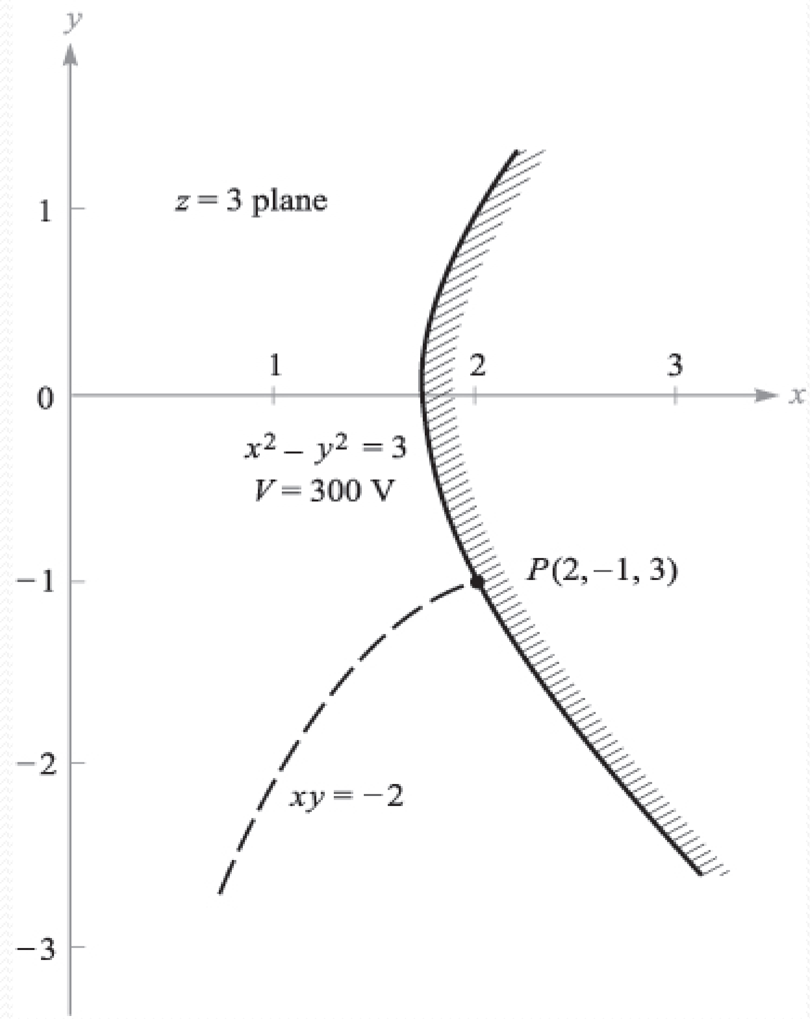
- Thus we can "insert" conductor in region provided the conductor follow hyperbola

$$3 = x^2 - y^2$$

- The Electric Field is at all times normal to conducting surface

$$E = -\nabla V = -200x\boldsymbol{a_x} + 200y\ \boldsymbol{a_y}$$

- Electric field at point 2,-1,3)
  - Ex = -400 V/m, Ey = -200 V/m
  - Down and to left



Example – Streamlines of Electric Field

- Slope of line equals electric field ratio

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{200\,y}{-200\,x} = -\frac{y}{x}$$

- Rearranging

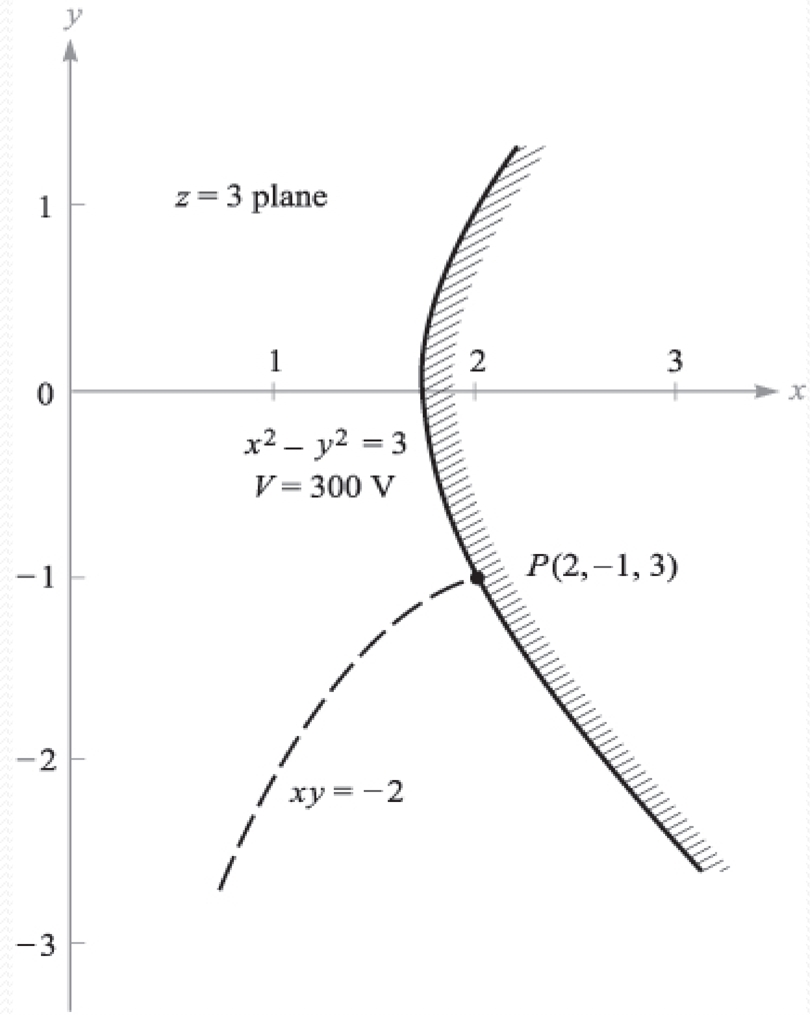
$$\frac{dx}{x} + \frac{dy}{y} = 0$$

$$\ln x + \ln y = C_1$$

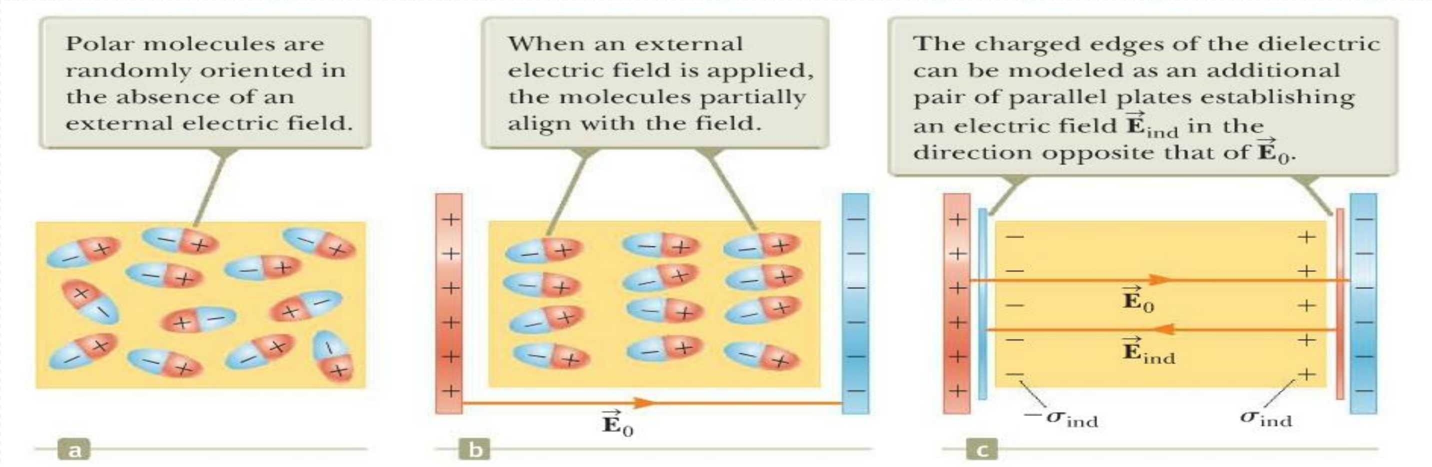
$$xy = C_2$$

- Evaluate at P(2,-1,3)

xy =-2



2.16 Dielectrics



- Material has random oriented dipoles
- Applied field aligns dipoles (negative at (+) terminal, positive at (-) terminal
- Effect is to cancel applied field, lower voltage
- OR, increase charge to maintain voltage
- Either increases capacitance C= Q/V

Review Dipole Moment

- Define dipole moment

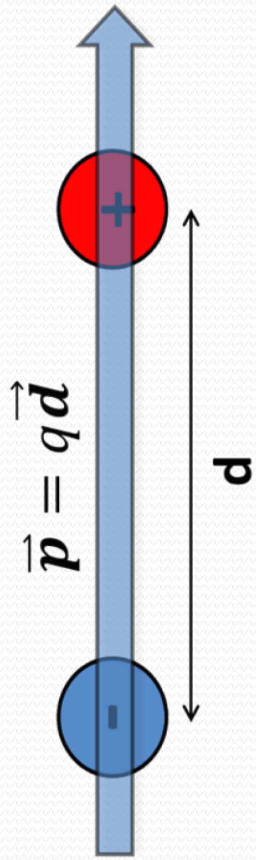
$$\boldsymbol{p} = Q\boldsymbol{d}$$

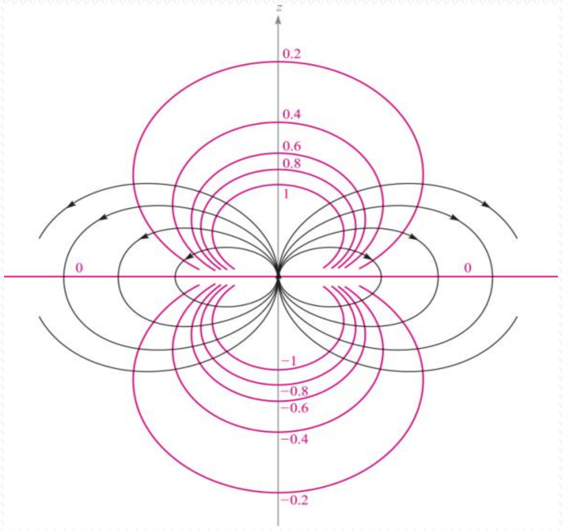
- Potential for dipole

$$V = \frac{Qd\cos\theta}{4\pi\epsilon_0 r^2}$$

- Written in terms of dipole moment and position

$$V = \frac{\mathbf{p}\cdot\mathbf{a}_r}{4\pi\epsilon_0 r^2}$$

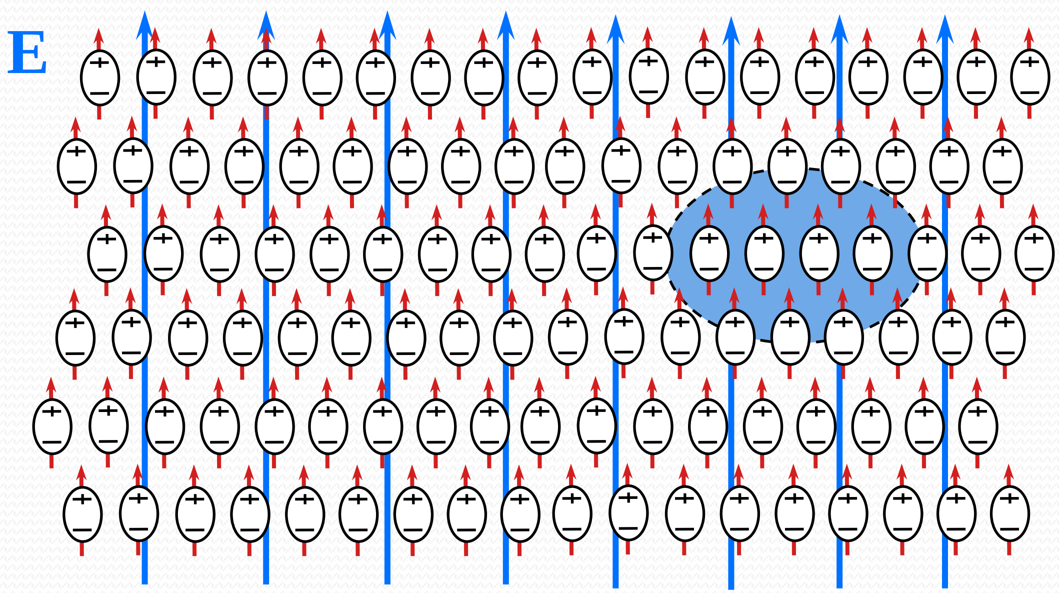




- Dipole moment determines “strength” of polar molecule amount of charge (Q) and offset (d) of charge

Polarization as sum of dipole moments (per volume)

Introducing an electric field may increase the charge separation in each dipole, and possibly re-orient dipoles so that there is some aggregate alignment, as shown here. The effect is small, and is greatly exaggerated here!

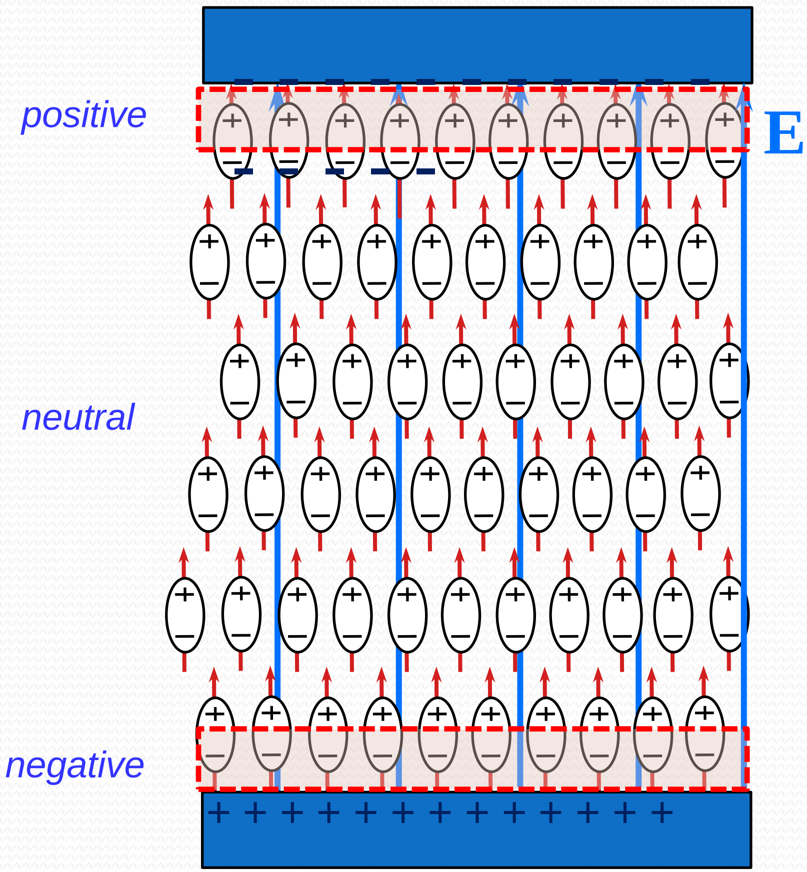


The effect is to increase **P**.

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n \Delta v} \mathbf{p}_i = n\boldsymbol{p} = nQ\boldsymbol{d}$$

n = charge/volume
p = polarization of individual dipole
P = polarization/volume

Polarization near electrodes



- From diagram
  - Excess positive bound charge near top negative electrode
  - Excess negative bound charge near bottom positive electrode
  - Rest of material neutral
- Excess charge in bound (red) volumes

$$\Delta Q = nQd\,\Delta S$$

- Writing in terms of polarization

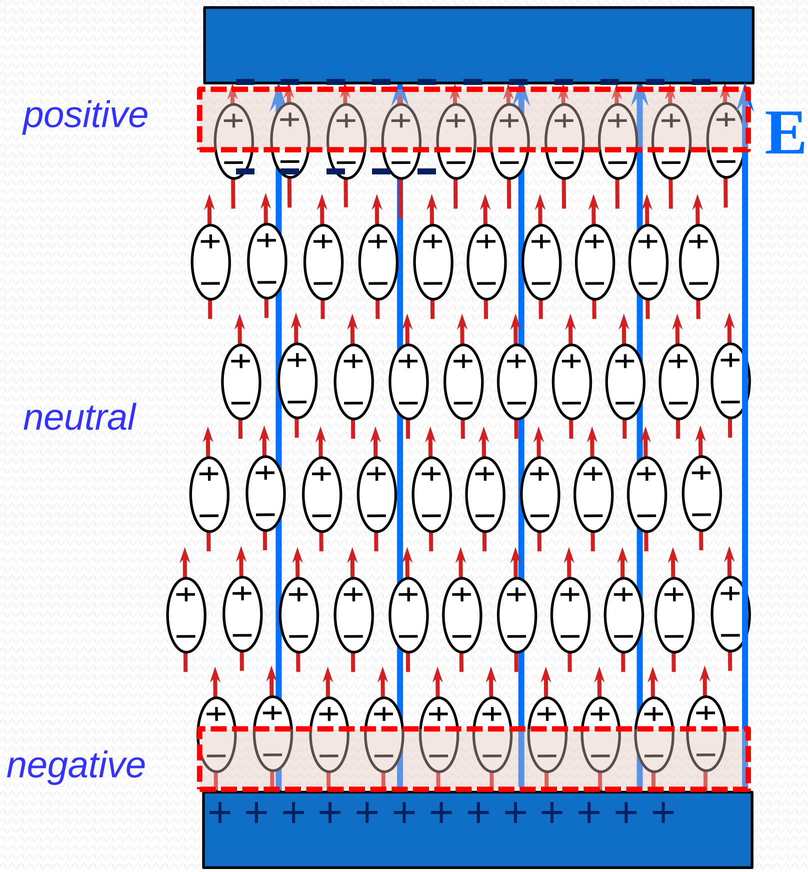
$$\Delta Q = \boldsymbol{P}\,\Delta S$$

- Writing similar to Gauss's law

$$Q_b = -\oint P \cdot ds$$

(*Note dot product sign, outward normal leaves opposite charge enclosed)*

Combining total, free, and bound charge



- Total, free, and bound charge

$$Q_t = Q_f + Q_b$$

- Total

$$Q_t = \oint \varepsilon_o E \cdot ds$$

- Free

$$Q_f = \oint D \cdot ds$$

- Bound

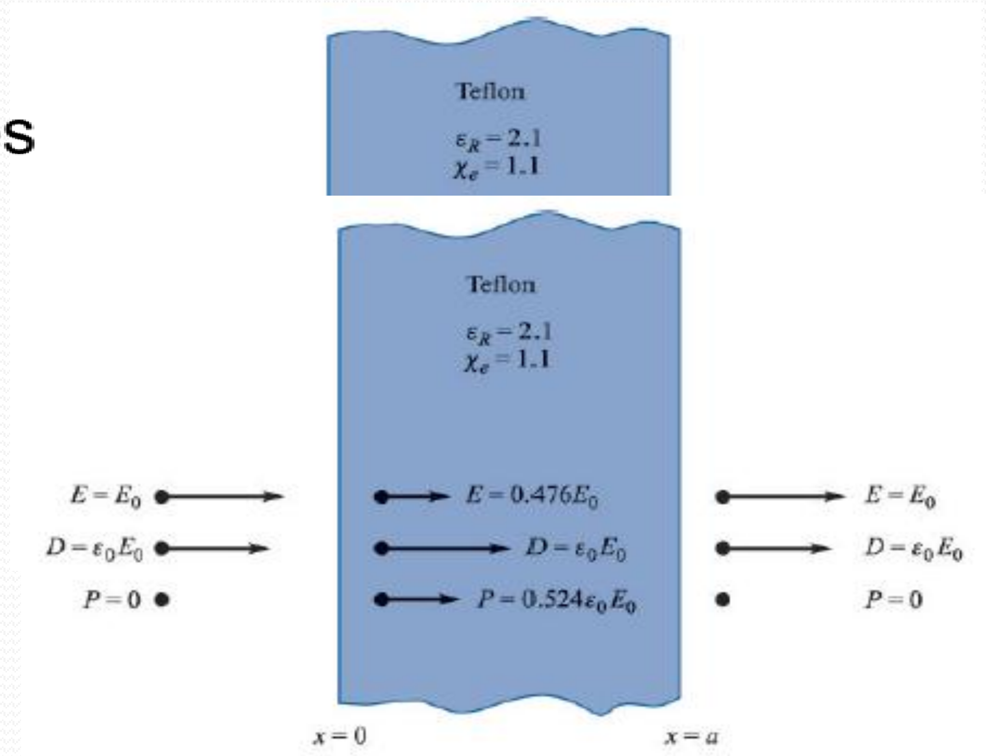
$$Q_b = -\oint P \cdot ds$$

- Combining

$$D = \varepsilon_o E + P$$

D, P, and E in Dielectric

- D continuous
- Polarization increases
- E decreases
- $\boldsymbol{D} = \varepsilon_o \boldsymbol{E} + \boldsymbol{P}$
- C/m^2



Charge Densities

Taking the previous results and using the divergence theorem, we find the point form expressions:

Bound Charge: $$Q_b = \int_v \rho_b \, dv = -\oint_S \mathbf{P} \cdot d\mathbf{S} \quad \Longrightarrow \quad \nabla \cdot \mathbf{P} = -\rho_b$$

Total Charge: $$Q_T = \int_v \rho_T \, dv = \oint_S \epsilon_0 \mathbf{E} \cdot d\mathbf{S} \quad \Longrightarrow \quad \nabla \cdot \epsilon_0 \mathbf{E} = \rho_T$$

Free Charge: $$Q = \int_v \rho_v \, dv = \oint_S \mathbf{D} \cdot d\mathbf{S} \quad \Longrightarrow \quad \nabla \cdot \mathbf{D} = \rho_v$$

2.17 Electric Susceptibility and the Dielectric Constant

A stronger electric field results in a larger polarization in the medium. In a *linear* medium, the relation between **P** and **E** is linear, and is given $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$

where χ_e is the electric *susceptibility* of the medium.

We may now write: $\mathbf{D} = \epsilon_0 \mathbf{E} + \chi_e \epsilon_0 \mathbf{E} = (\chi_e + 1)\epsilon_0 \mathbf{E}$

where the *dielectric constant*, or *relative permittivity* is defined as:

$$\epsilon_r = \chi_e + 1$$

$$\epsilon = \epsilon_r \epsilon_0$$

Leading to the overall permittivity of the medium: $\mathbf{D} = \epsilon \mathbf{E}$

Permittivity of Materials

- Typical permittivity for various solids and liquids.
  - Teflon – 2
  - Plastics - 3-6
  - Ceramics 8-10
  - Titanates>100
  - Acetone 2 1
  - Water 78
- Actual dielectric “constant” varies with:
  - Temperature
  - Direction
  - Field Strength
  - Frequency
  - Real & Imaginary components

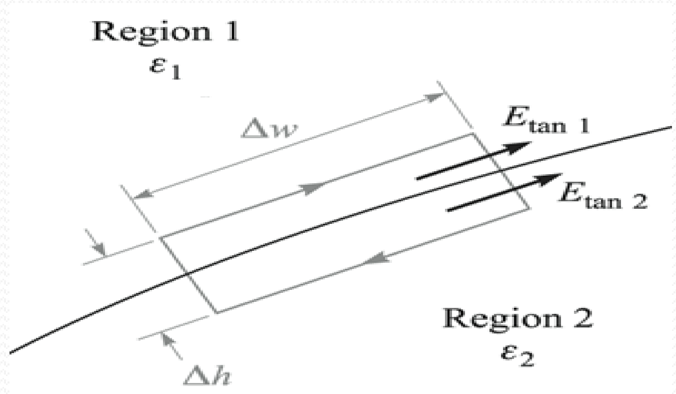
Other applications

- Other Applications
  - Bio
  - Liquid Crystal
  - Composite polymers
  - Titanates
  - Wireless characterization
  - MRI dyes
  - Ground water monitoring
  - Oil Drilling fluid characterization (GPR)

2.18 Boundary Condition for Tangential Electric Field **E**

Since **E** is conservative, we setup line integral straddling both dielectrics:

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$



Left and right sides cancel, so

$$E_{\tan 1}\,\Delta w - E_{\tan 2}\,\Delta w = 0$$

Leading to Continuity for tangential E

$$E_{\tan 1} = E_{\tan 2}$$

And **Discontinuity for tangential D**

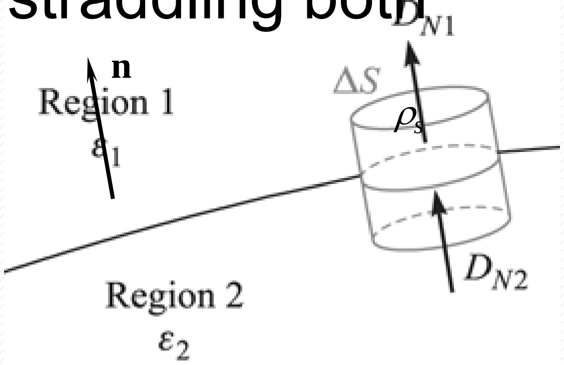
$$\frac{D_{\tan 1}}{\epsilon_1} = E_{\tan 1} = E_{\tan 2} = \frac{D_{\tan 2}}{\epsilon_2}$$

E same, D higher in high permittivity material

Boundary Condition for Normal Displacement **D**

Apply Gauss' Law to the cylindrical volume straddling both dielectrics

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S}$$



Flux enters and exits only through top and bottom surfaces, zero on sides

$$D_{N1}\Delta S - D_{N2}\Delta S = \Delta Q = \rho_S \Delta S$$

Leading to **Continuity for normal D (for $\rho_S = 0$)**

$$D_{N1} - D_{N2} = \rho_S$$

$$D_{N1} = D_{N2}$$

And **Discontinuity for normal E**

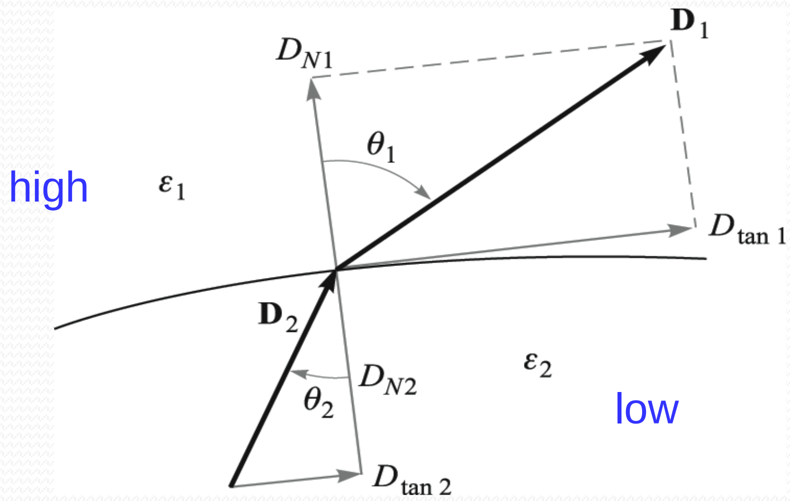
$$\varepsilon_1 E_{N1} = \varepsilon_2 E_{N2}$$

D same. E lower in high permittivity material

Bending of D at boundary

- Boundary conditions
  - D_N continuous
  - $\frac{D_{T1}}{D_{T2}} = \frac{\varepsilon_1}{\varepsilon_2}$
- Trigonometry
  - $\tan(\theta_1) = \frac{D_{T1}}{D_N}$
  - $\tan(\theta_2) = \frac{D_{T2}}{D_N}$
- Eliminating D_N

$$\frac{\tan(\theta_1)}{\tan(\theta_2)} = \frac{D_{T1}}{D_{T2}} = \frac{\varepsilon_1}{\varepsilon_2}$$



UNIT – III
MAGNETOSTATICS

3 Electric potential

3.1 Introduction to Potential

3.2 Comments on Potential

3.3 Poisson's Equation and Laplace's Equation

3.4 The Potential of a Localized Charge Distribution

3.5 Electrostatic Boundary Conditions

3.1 introduction to potential

Any vector whose curl is zero is equal to the gradient of some scalar. We define a function:

$$V(p) = -\int_{\vartheta}^{p} \vec{E} \cdot d\vec{l}$$

Where is some standard reference point ; *V* depends only on the point *P*. *V* is called the *electric potential*.

$$V(b) - V(a) = -\int_{\vartheta}^{b} \vec{E} \cdot d\vec{l} - \left(-\int_{\vartheta}^{a} \vec{E} \cdot d\vec{l} \right) = -\int_{a}^{b} \vec{E} \cdot d\vec{l}$$

The fundamental theorem for gradients

$$V(b) - V(a) = \int_{a}^{b} (\nabla V) \cdot d\vec{l}$$

so $$\int_{a}^{b} (\nabla V) \cdot d\vec{l} = -\int_{a}^{b} \vec{E} \cdot d\vec{l} \Longrightarrow \boxed{\vec{E} = -\nabla V}$$

3.2 Comments on potential

(1)The name Potential is not potential energy

$$\vec{F} = q\vec{E} = -q\nabla V \qquad \Delta U = \vec{F}\cdot\vec{X}$$

V : Joule/coulomb U : Joule

3.2

(2) Advantage of the potential formulation V is a scalar function, but E is a vector quantity

$$V(r) \qquad\qquad \vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$$

If you know *V*, you can easily get *E:* $\vec{E} = -\nabla V$.

E_x, E_y, E_z are not independent functions

$$\nabla \times \vec{E} = 0 \quad \text{so} \quad \frac{\partial E_x}{\partial y} \overset{①}{=} \frac{\partial E_y}{\partial x}, \quad \frac{\partial E_z}{\partial y} \overset{②}{=} \frac{\partial E_y}{\partial z}, \quad \frac{\partial E_x}{\partial z} \overset{③}{=} \frac{\partial E_z}{\partial x}$$

$$\partial_y \partial_z E_x = \partial_z \partial_y E_x \overset{①}{=} \partial_z \partial_x E_y = \partial_x \partial_z E_y \overset{②}{=} \partial_x \partial_y E_z = \partial_y \partial_x E_z \Rightarrow \partial_z E_x \overset{③}{=} \partial_x E_z$$

(3)The reference point ϑ

Changing the reference point amounts to adds a constant to the potential

$$V'(p) = -\int_{\vartheta'}^{p} \vec{E} \cdot d\vec{l} = -\int_{\vartheta'}^{\vartheta} \vec{E} \cdot d\vec{l} - \int_{\vartheta}^{p} \vec{E} \cdot d\vec{l} = K + V(p)$$

(Where K is a constant)

Adding a constant to *V* will not affect the potential difference between two point:

$$V'(b) - V'(a) = V(b) - V(a)$$

Since the derivative of a constant is zero: $\nabla V = \nabla V'$
For the different *V*, the field *E* remains the same.

Ordinarily we set $V(\infty) = 0$

(4) Potential obeys the superposition principle

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdots \quad , \quad \vec{F} = Q\vec{E}$$

Dividing through by Q

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots$$

Integrating from the common reference point to p ,

$$V = V_1 + V_2 + \cdots$$

(5) *Unit of potential*

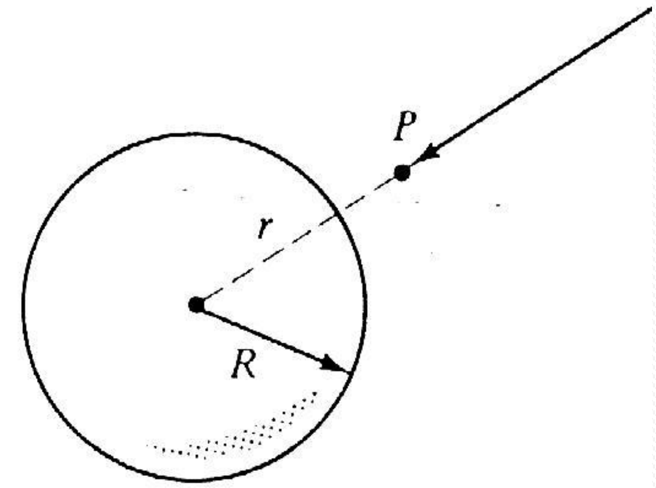
Volt=Joule/Coulomb

$$F : newton \qquad F \cdot x : Joule$$

$$F = qE = q\nabla V \rightarrow \frac{qV}{X}$$

$$V : \frac{F \cdot X}{q} \rightarrow$$ Joule/Coulomb

Example 2.6 Find the potential inside and outside a spherical shell of radius R, which carries a uniform surface charge (the total charge is q).



solution**:**

$$\vec{E}_{in} = 0 \qquad \vec{E}_{out} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

for r>R:

$$V(\vec{r}) = -\int_{\infty}^{r} \vec{E}\cdot d\vec{\ell} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r'}\Bigg|_{\infty}^{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

for r≤R:

$$V(R) = \frac{1}{4\pi\varepsilon_0}\frac{q}{R} \qquad V(\vec{r}) \neq V(R) = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}$$

↑

$$\vec{E}_{in} = 0$$

3.3 Poisson's Eq. & Laplace's Eq.

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} = \nabla \cdot (-\nabla V) = -\nabla^2 V$$

$$\uparrow$$

$$\vec{E} = -\nabla V$$

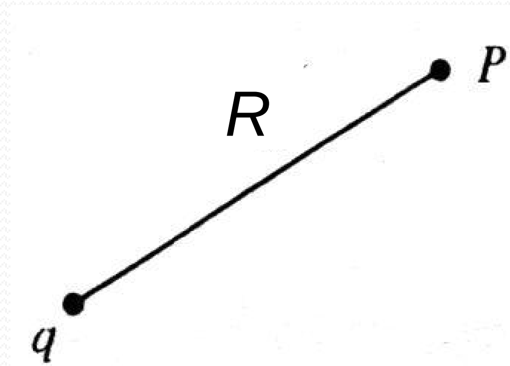
Poisson's Eq. $\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0}}$

$\rho = 0$ $\boxed{\nabla^2 V = 0}$ Laplace's eq.

3.4 The Potential of a Localized Charge Distribution

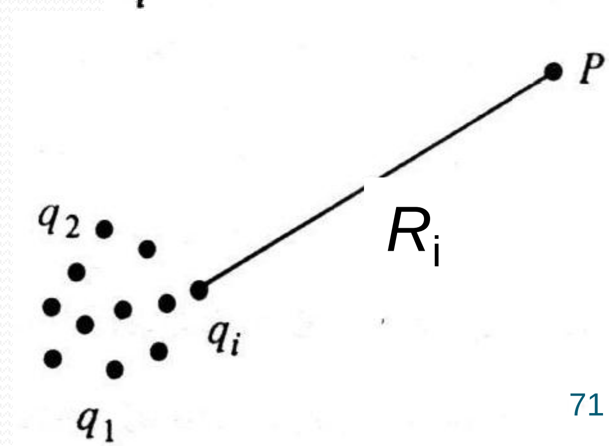
$$\vec{E} = -\nabla V \qquad V - V_\infty = -\int_\infty^r E\,dr' \qquad V_\infty = 0$$

$$V(r) = \frac{-1}{4\pi\varepsilon_0}\int_\infty^r \frac{q}{r'^2}\,dr' = \frac{1}{4\pi\varepsilon_0}\frac{q}{r'}\bigg|_\infty^r = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$



- Potential for a point charge

$$V(P) = \frac{1}{4\pi\varepsilon_0}\frac{q}{R} \qquad R = \left|\vec{r} - \vec{r}_p\right|$$



- Potential for a collection of charge

$$V(P) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{R_i} \qquad R_i = \left|\vec{r}_i - \vec{r}_p\right|$$

3.4

- Potential of a continuous distribution

for volume charge | for a line charge | for a surface charge

$$\delta q = \rho \, d\tau \qquad \delta q = \lambda \, d\ell \qquad \delta q = \sigma \, da$$

$$V(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho}{R} d\tau \qquad V(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda}{R} d\ell \qquad V(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma}{R} da$$

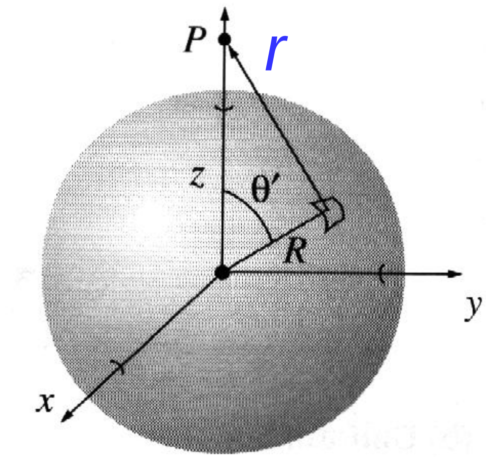
- **Corresponding electric field** $\left[\ -\nabla \frac{1}{R} = \frac{\hat{R}}{R^2}\ \right]$

$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{r}}{R^2}\rho \, d\tau \qquad \vec{E}(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{r}}{R^2}\lambda \, d\ell \qquad \vec{E}(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{r}}{R^2}\sigma \, da$$

Example 2.7 Find the potential of a uniformly charged spherical shell of radius *R*.

Solution:

$$V(r)=\frac{1}{4\pi\varepsilon_0}\int\frac{\sigma}{r}da', \qquad r^2=R^2+z^2-2Rz\cos\theta'$$



$$4\pi V(z)=\sigma\int\frac{R^2\sin\theta' d\theta' d\varphi'}{\sqrt{R^2+z^2-2Rz\cos\theta'}}$$

$$=2\pi R^2\sigma\int_0^{\pi}\frac{\sin\theta'}{\sqrt{R^2+z^2-2Rz\cos\theta'}}d\theta'$$

$$=2\pi R^2\sigma\left(\frac{1}{Rz}\sqrt{R^2+z^2-2Rz\cos\theta'}\right)\Bigg|_0^{\pi}$$

$$=\frac{2\pi R\sigma}{z}\left(\sqrt{R^2+z^2+2Rz}-\sqrt{R^2+z^2-2Rz}\right)$$

$$=\frac{2\pi R\sigma}{z}\left[\sqrt{(R+z)^2}-\sqrt{(R-z)^2}\right]$$

3.4

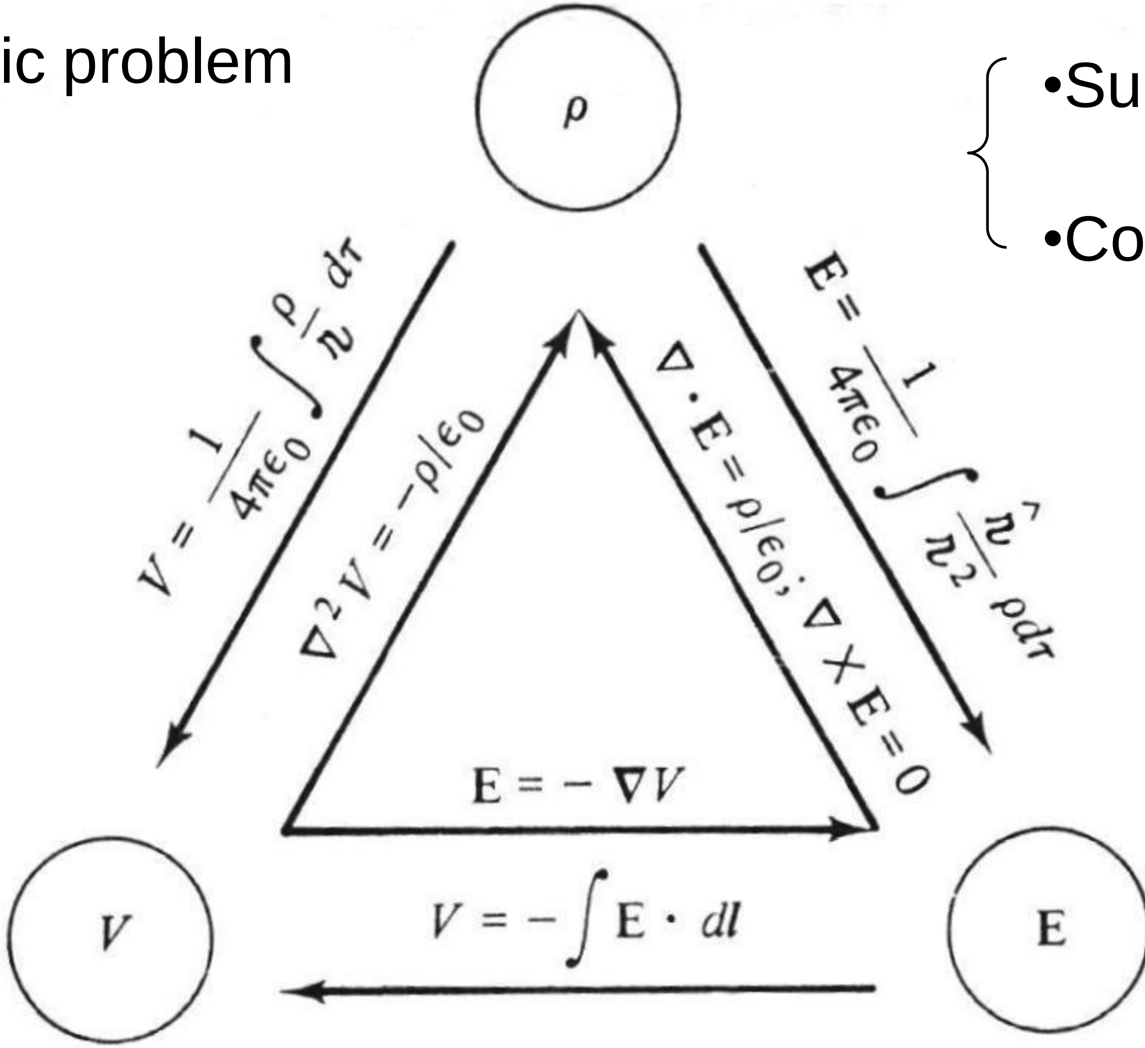
$$V(z) = \frac{R\sigma}{2\varepsilon_0 z}[(R+z)-(z-R)] = \frac{R^2\sigma}{\varepsilon_0 z}, \quad outside$$

$$V(z) = \frac{R\sigma}{2\varepsilon_0 z}[(R+z)-(R-z)] = \frac{R\sigma}{\varepsilon_0}, \quad inside$$

3.5 Electrostatic Boundary Condition

Electrostatic problem

- •Superposition
- •Coulomb law



The above equations are differential or integral.

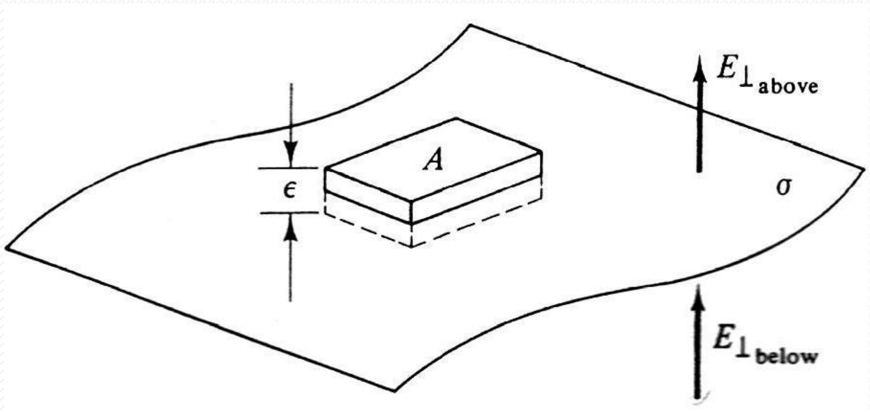
For a unique solution, we need boundary conditions. (e.q. , $V(\infty)=0$)

(boundary value problem. Dynamics: *initial value problem*.)

3.5

B.C. at surface with charge

$\perp$:



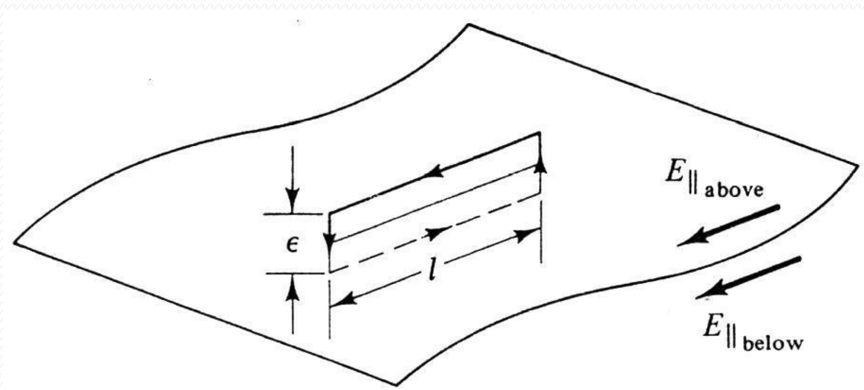
$$\oint_{surface} \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

||

$$E_{\perp\,above} \cdot A - E_{\perp\,below} \cdot A$$

$$E_{\perp\,above} - E_{\perp\,below} = \frac{\sigma}{\varepsilon_0}$$

||:



$$\oint \vec{E} \cdot d\vec{l} = 0 \qquad (\because \nabla \times \vec{E} = 0)$$

$$E_{\parallel\,above} = E_{\parallel\,below}$$

$\perp$ +||

$$\boxed{\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}}$$

3.5

Potential B.C.

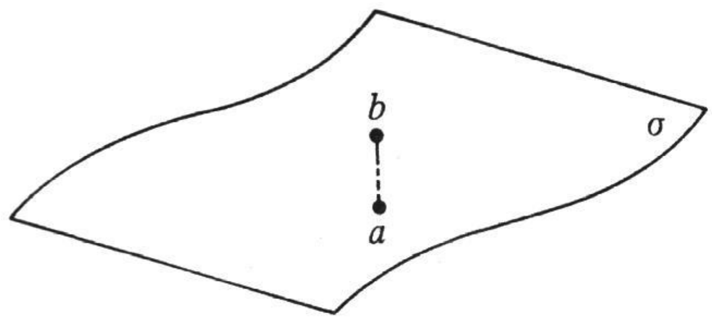


Figure 2.38

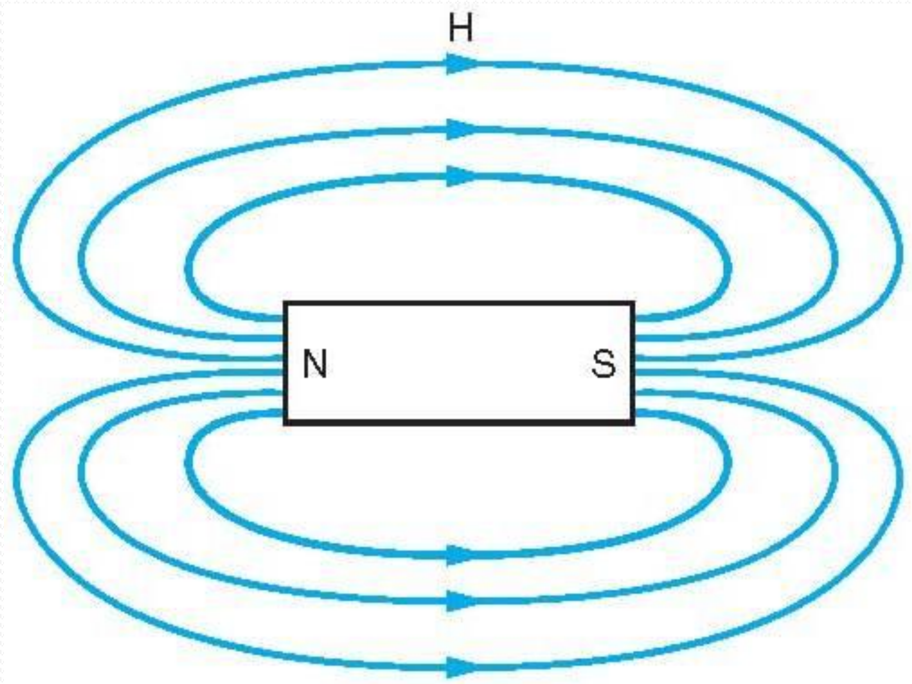
$$\because V_{above} - V_{below} = -\int_a^b \vec{E} \cdot d\vec{\ell} \xrightarrow{\overline{ab} \to 0} 0$$

$$\therefore \boxed{V_{above} = V_{below}}$$

$$\because \vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n} \qquad \vec{E} = -\nabla V \qquad \frac{\partial}{\partial n}V = \nabla V \cdot \hat{n}$$

$$\therefore \boxed{\frac{\partial}{\partial n}V_{above} - \frac{\partial}{\partial n}V_{below} = -\frac{\sigma}{\varepsilon_0}}$$

3.6 Magnetism and electricity have not been considered distinct phenomena until Hans Christian Oersted conducted an experiment that showed a compass deflecting in proximity to a current carrying wire

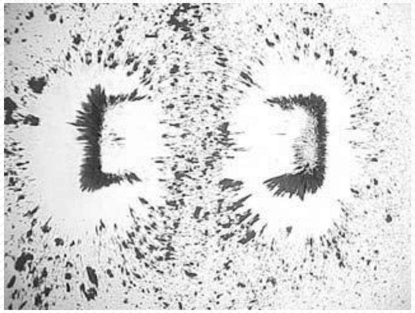


Produced by

-time varying electric fields

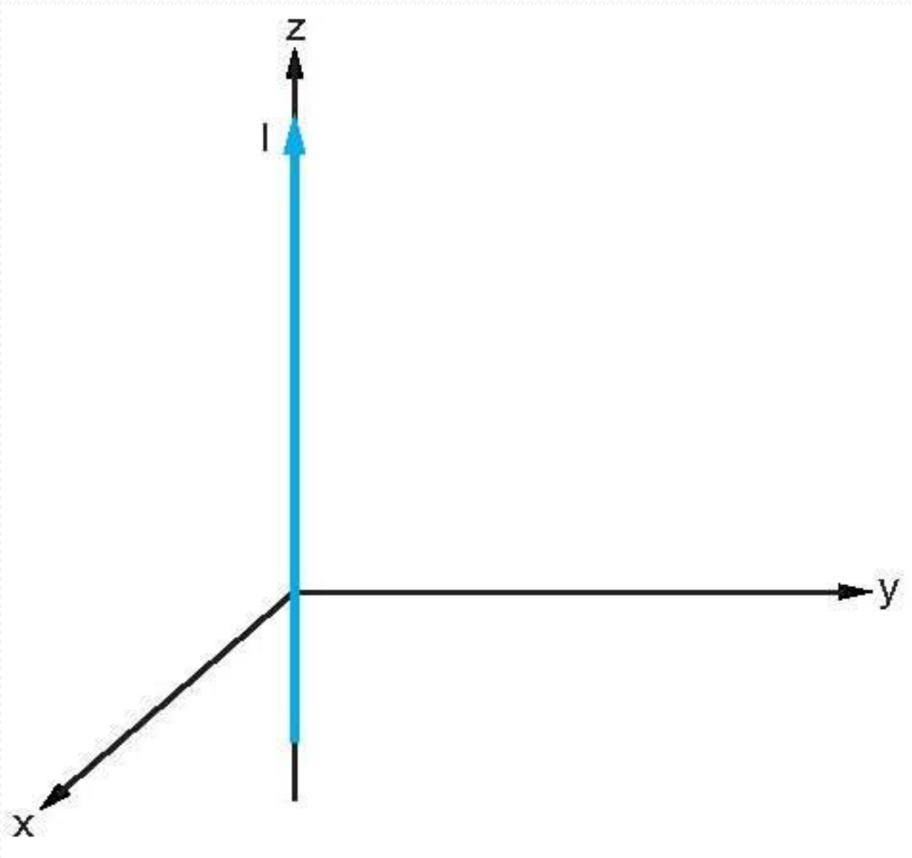
-permanent magnet (arises from quantum mechanical electron spin/ can be considered charge in motion=current)

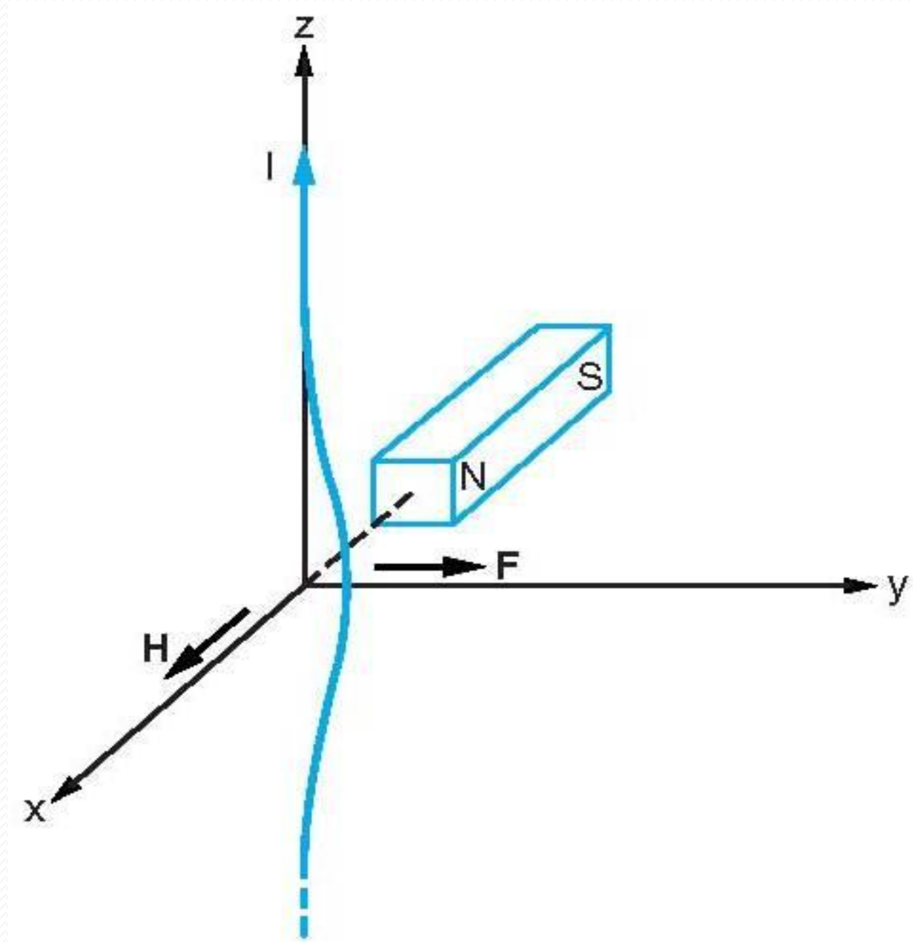
-steady electric currents



$$[H]_{SI} = \frac{A}{m}$$

If we place a wire with current I in the presence of a magnetic field, the charges in the conductor experience another force F_m





F_m ~ q, u, B

$$\overline{H} = \frac{1}{\mu}\overline{B}$$

q –charge

u –velocity vector

B -strength of the field (magnetic flux density)

μ_r –relative permeability

μ –absolute permeability

μ_0-permeability of the free space

$M=\chi_m H$- is the magnetization for linear and homogeneous medium

3.7 Relative permeabilities for a variety of materials

Material	**μ/(H m^{-1})**	**μ_r**	**Application**
Ferrite U 60	1.00E-05	8	UHF chokes
Ferrite M33	9.42E-04	750	Resonant circuit RM cores
Nickel (99% pure)	7.54E-04	600	-
Ferrite N41	3.77E-03	3000	Power circuits
Iron (99.8% pure)	6.28E-03	5000	-
Ferrite T38	1.26E-02	10000	Broadband transformers
Silicon GO steel	5.03E-02	40000	Dynamos, mains transformers
supermalloy	1.26	1000000	Recording heads

3.8 Lorentz's Force equation

$$\overline{F_e} = q\overline{E}$$

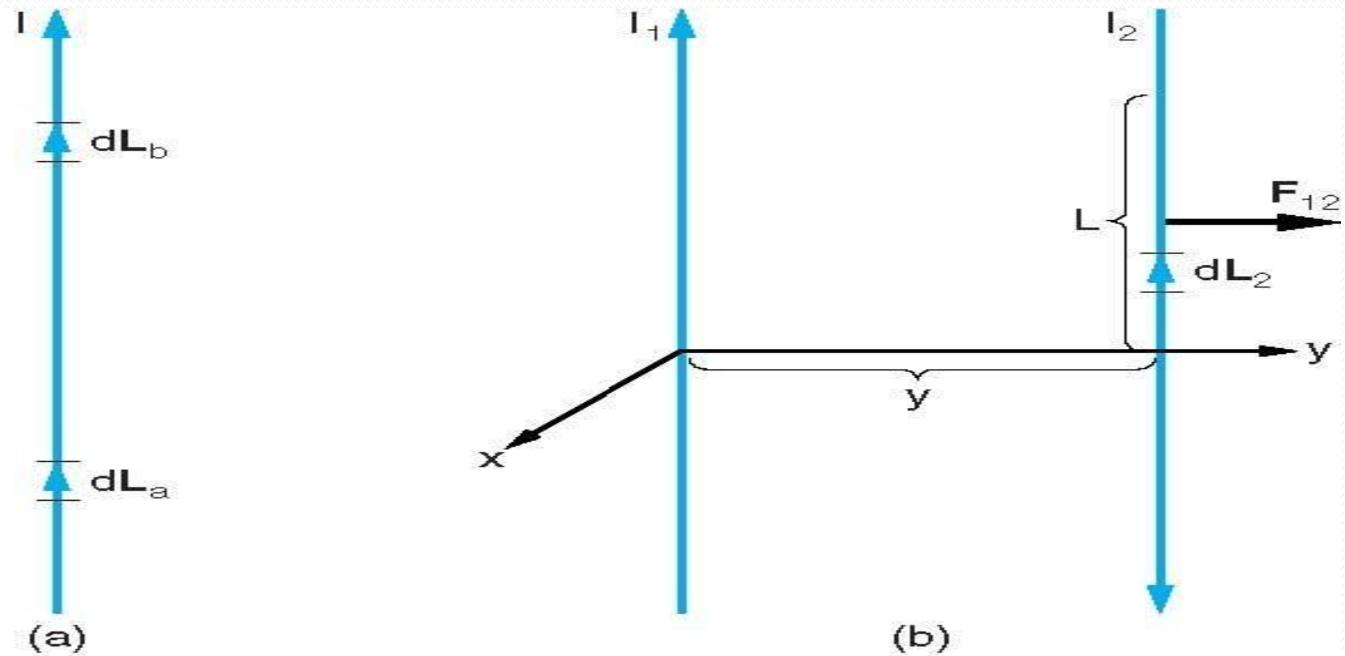
$$\overline{F_m} = q\overline{u} \times \overline{B}$$

$$\Rightarrow \quad \overline{F_e} + \overline{F_m} = F$$

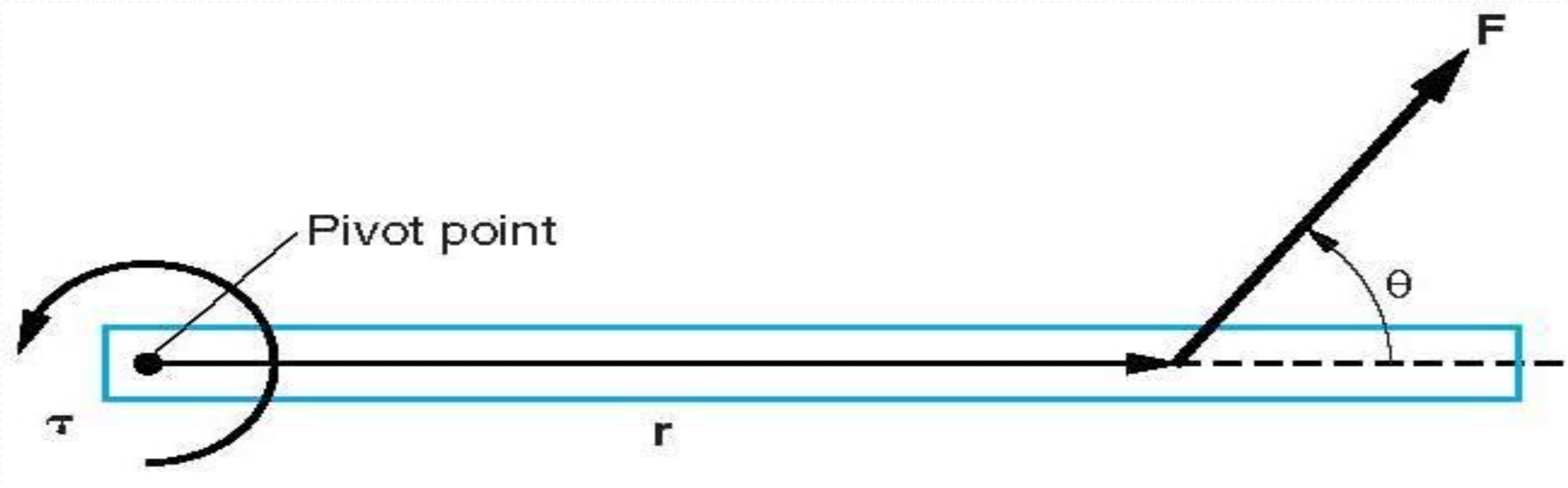
$$\overline{F} = q(\overline{E} + \overline{u} \times \overline{B})$$

Note: Magnetic force is zero for q moving in the direction of the magnetic field (sin0=0)

When electric current is passed through a magnetic field a force is exerted on the wire normal to both the magnetic field and the current direction. This force is actually acting on the individual charges moving in the conductor.

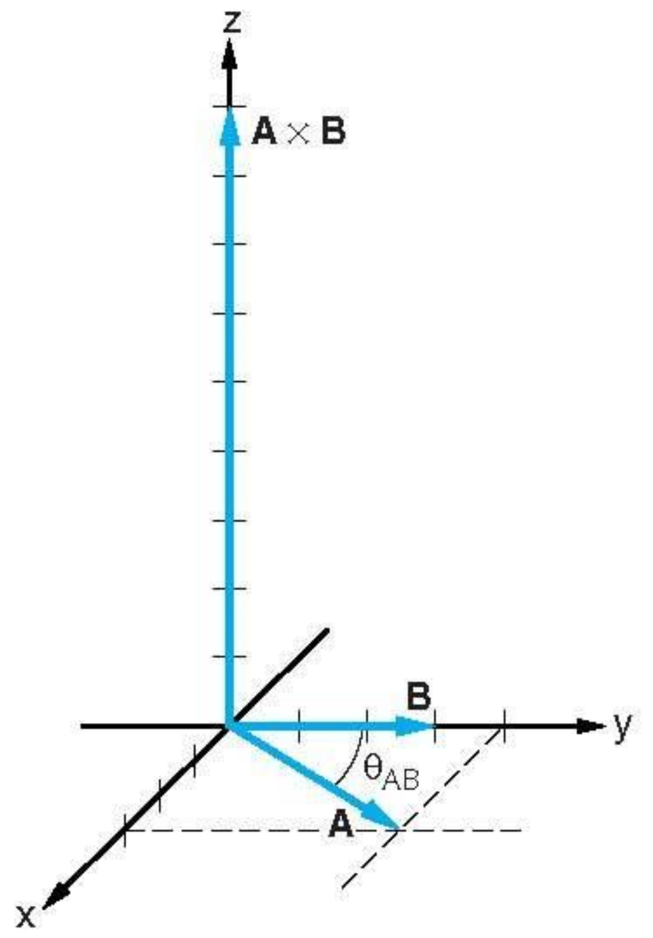


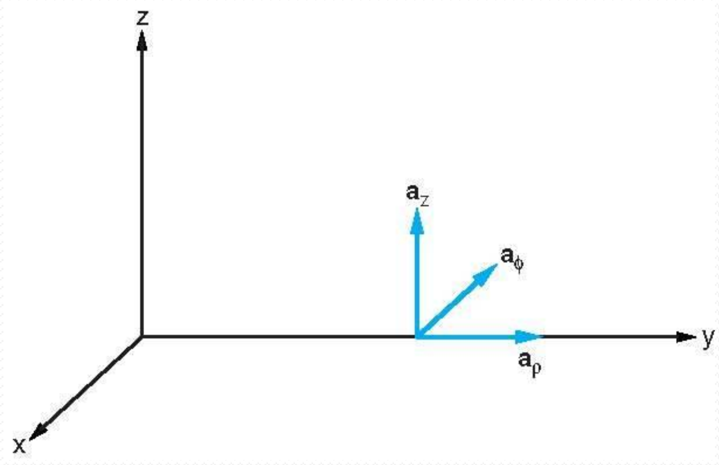
The magnetic force is exerting a torque on the current carrying coil

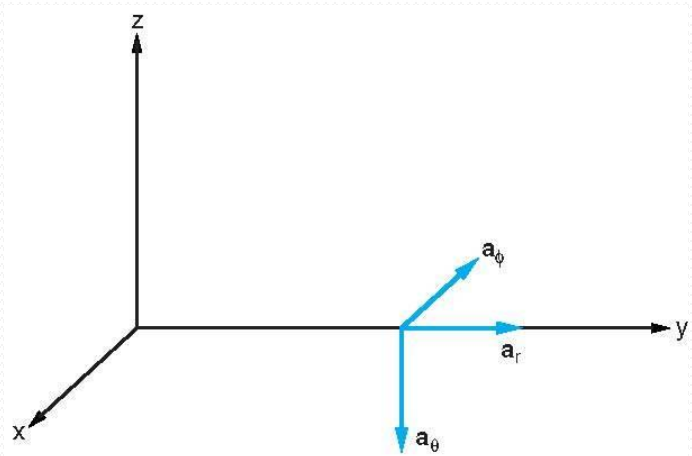


$$\tau = \overline{r} \times \overline{F_m} = |r| \times |F_m| \sin \theta \, \overline{a_n}$$

3.9 Cross product





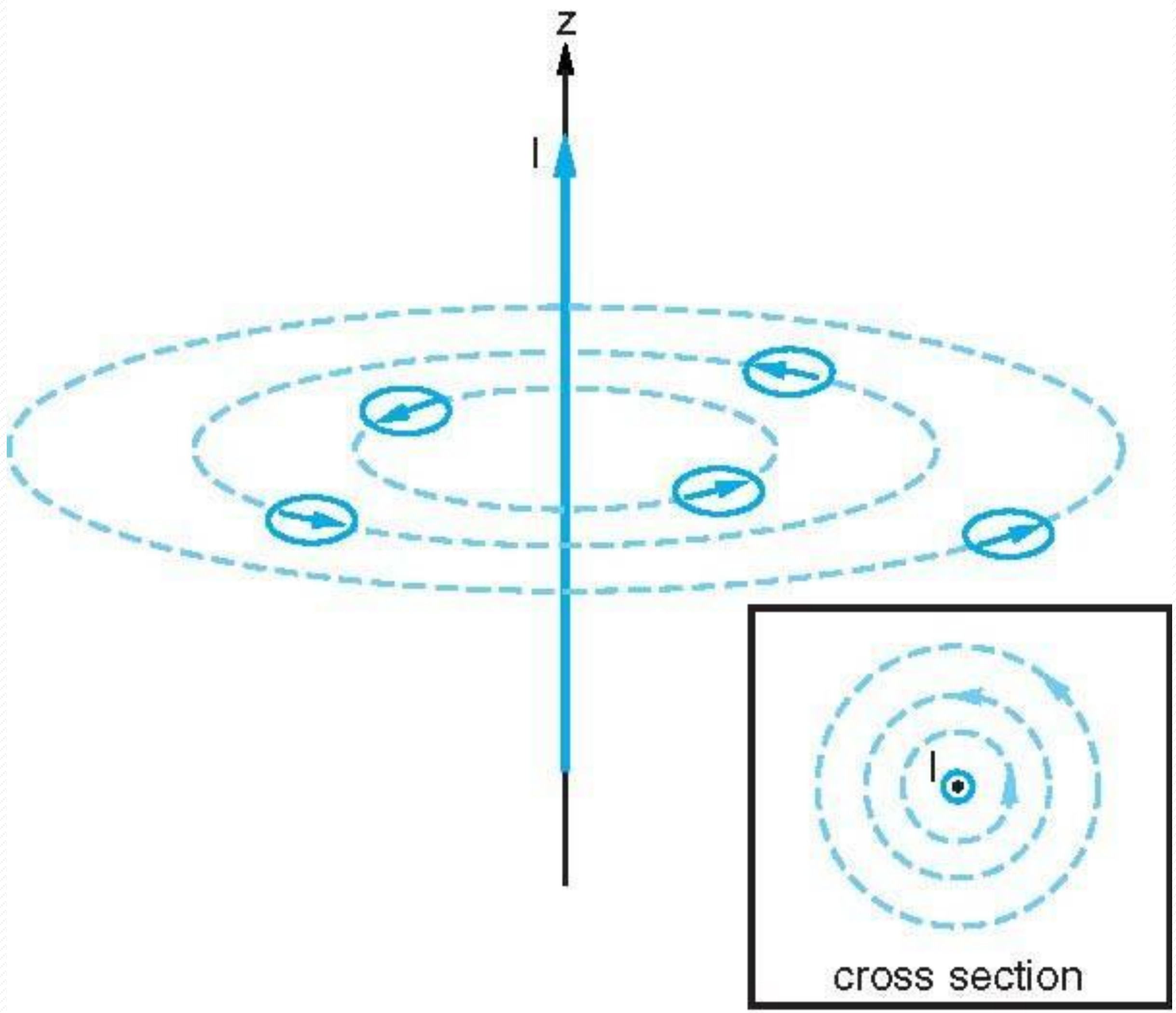


Fundamental Postulates of Magnetostatics in Free Space

$$\nabla \bullet \overline{B} = 0$$

$$\nabla \times \overline{B} = \mu_o \overline{J}$$

$$\vec{\nabla} \bullet \overline{J} = 0$$


z
I
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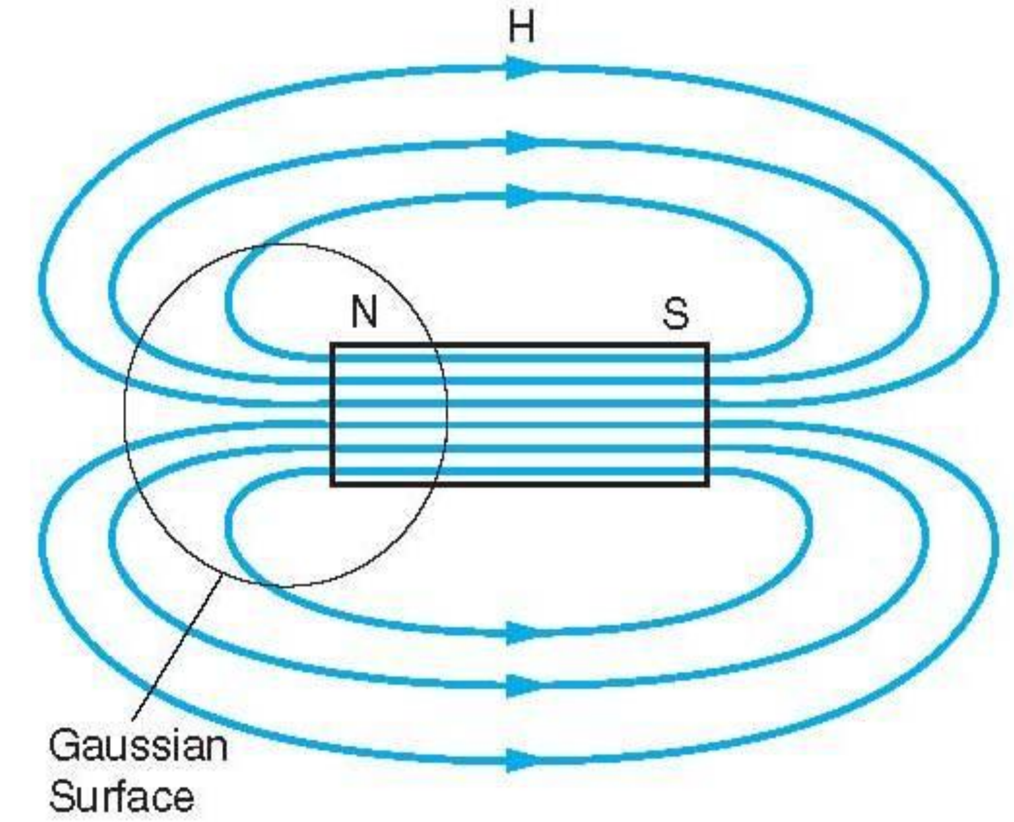
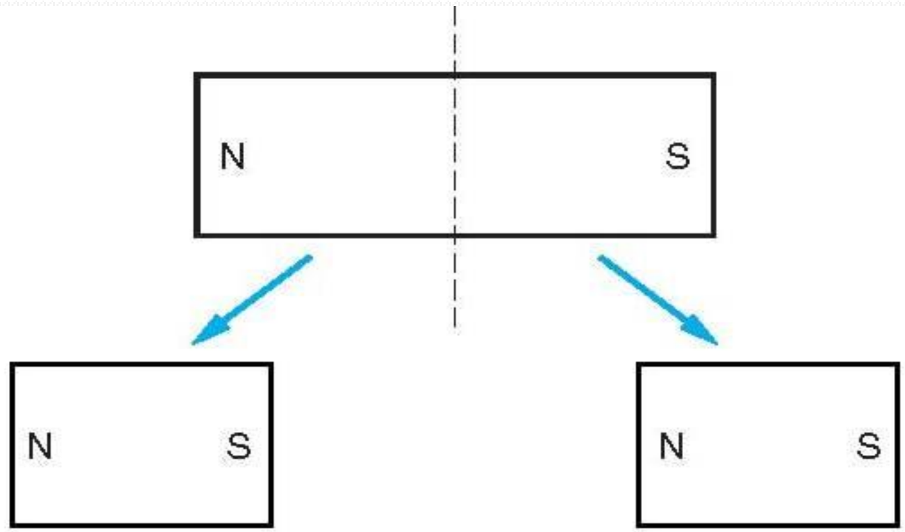
cross section

3.10 Law of conservation of magnetic flux



$$\oint_s \overline{B} \bullet d\,\overline{s} = 0$$

There are no magnetic flow sources, and the magnetic flux lines always close upon themselves



3.11 Ampere's circuital law

$$\int_s \left(\nabla \times \overline{B}\right) \bullet d\,\overline{s} = \mu_0 \int_s \overline{J} \bullet d\,\overline{s}$$

$$\oint_c \overline{B} \bullet d\,\overline{l} = \mu_0 I$$

The circulation of the magnetic flux density in free space around any closed path is equal to μ_0 times the total current flowing through the surface bounded by the path

Postulates of Magnetostatics in Free Space	
Differential Form	Integral Form
$\nabla \bullet \overline{B} = 0$	$\oint_s \overline{B} \bullet d\overline{s} = 0$
$\nabla \times \overline{B} = \mu_0 J$	$\oint_c \overline{B} \bullet d\overline{l} = \mu_0 I$

UNIT – IV
FORCE IN MAGNETIC FIELD AND MAGNETIC POTENTIAL

4 Work and Energy in Electrostatics

4.1 The Work Done in Moving a Charge

4.2 The Energy of a Point Charge Distribution

4.3 The Energy of a Continuous Charge Distribution

4.4 Comments on Electrostatic Energy

4.1 The Work Done in Moving a charge

A test charge Q feels a force Q $\vec{E}$ (from the stationary source charges).

To move this test charge, we have to apply a force

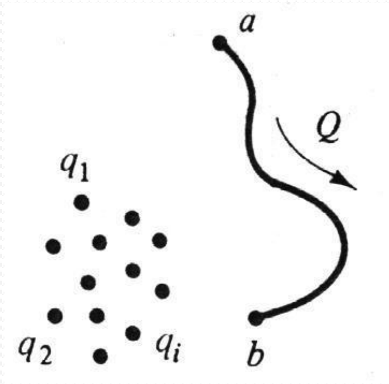
$$\vec{F} = -Q\vec{E}$$

↑ conservative

The total work we do is

$$W = \int_a^b \vec{F} \cdot d\vec{\ell} = -Q\int_a^b \vec{E} \cdot d\vec{\ell} = Q\left[V(b) - V(a)\right]$$

$$\boxed{V(b) - V(a) = \frac{W}{Q}}$$



So, bring a charge from ∞ to P, the work we do is

$$W = Q\left[V(P) - V(\infty)\right] = QV(P)$$

↑ $V(\infty) = 0$

4.2 The Energy of a Point Charge Distribution

It takes no work to bring in first charges

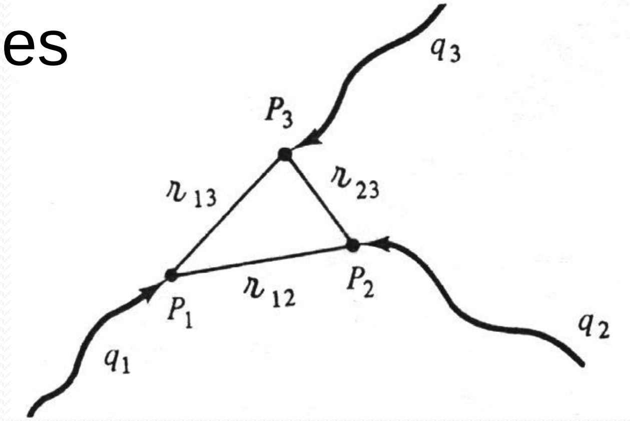


Figure 2.40

$$W_1 = 0$$ for q_1

Work needed to bring in q_2 is :

$$W_2 = q_2[\frac{1}{4\pi\varepsilon_0}\frac{q_1}{R_{12}}] = \frac{1}{4\pi\varepsilon_0}q_2(\frac{q_1}{R_{12}})$$

Work needed to bring in q_3 is :

$$W_3 = q_3[\frac{1}{4\pi\varepsilon_0}\frac{q_1}{R_{13}} + \frac{1}{4\pi\varepsilon_0}\frac{q_2}{R_{23}}] = \frac{1}{4\pi\varepsilon_0}q_3(\frac{q_1}{R_{13}} + \frac{q_2}{R_{23}})$$

Work needed to bring in q_4 is :

$$W_4 = \frac{1}{4\pi\varepsilon_0}q_4[\frac{q_1}{R_{14}} + \frac{q_2}{R_{24}} + \frac{q_3}{R_{34}}]$$

4.2

Total work

$$W = W_1 + W_2 + W_3 + W_4$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{R_{12}} + \frac{q_1 q_3}{R_{13}} + \frac{q_2 q_3}{R_{23}} + \frac{q_1 q_4}{R_{14}} + \frac{q_2 q_4}{R_{24}} + \frac{q_3 q_4}{R_{34}}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{q_i q_j}{R_{ij}}$$

$$\underline{\underline{R_{ij} = R_{ji}}} \quad \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n}\frac{q_i q_j}{R_{ij}} \quad = \frac{1}{2}\sum_{i=1}^{n} q_i\left(\sum_{\substack{j=1 \\ j\neq i}}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_j}{R_{ij}}\right)$$

$$\boxed{W = \frac{1}{2}\sum_{i=1}^{n} q_i V(P_i)} \qquad V(P_i) = \sum_{\substack{j=1 \\ j\neq i}}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_j}{R_{ij}}$$

Dose not include the first charge

4.3 The Energy of a Continuous Charge Distribution

$$\delta q = \rho \, d\tau$$

↑ Volume charge density

$$W = \frac{1}{2}\int \rho V d\tau$$

$$\rho = \varepsilon_0 \nabla \cdot \vec{E} \qquad \nabla \cdot (\vec{E} V) = (\nabla \cdot \vec{E}) V + \vec{E} \cdot (\nabla V)$$

$$\frac{\varepsilon_0}{2}\int (\nabla \cdot \vec{E}) V d\tau = \frac{\varepsilon_0}{2}\left[\int \nabla \cdot (\vec{E} V) d\tau + \int \vec{E} \cdot (-\nabla V) d\tau\right]$$

‖ F.T.for $\nabla \cdot$ ‖

$$\oint_{surface} V\vec{E} \cdot d\vec{a} \qquad \vec{E}$$

$$= \frac{\varepsilon_0}{2}\left(\oint_{surface} V\vec{E} \cdot d\vec{a} + \oint_{volume} E^2 d\tau\right)$$

$$surface \to \infty \quad \Rightarrow \quad \boxed{W = \frac{\varepsilon_0}{2}\int_{all\ space} E^2 d\tau}$$

Example 2.8 Find the energy of a uniformly charged spherical shell of total charge q and radius R

Sol.1： $q = 4\pi R^2 \sigma$ $\qquad V = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{R}$

$$W = \frac{1}{2}\int \rho V d\tau = \frac{1}{2}\int \sigma V da = \frac{1}{2}\int \frac{q}{A}\frac{1}{4\pi\varepsilon_0}\frac{q}{R} da = \frac{1}{8\pi\varepsilon_0}\frac{q^2}{R}$$

Sol.2： $\vec{E} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r^2}\hat{r}$ $\qquad E^2 = \dfrac{q^2}{(4\pi\varepsilon_0)^2 r^4}$

$$W = \frac{\varepsilon_0}{2}\int E^2 d\tau = \frac{\varepsilon_0}{2}\int_{space}^{outer} \frac{q^2}{(4\pi\varepsilon_o)^2}\frac{1}{r^4}\left(r^2 \sin\theta \, d\theta \, d\varphi \, dr\right)$$

$$= \frac{q^2}{32\pi^2\varepsilon_0}\left[2\cdot 2\pi\int_R^\infty \frac{1}{r^2} dr\right] = \frac{1}{8\pi\varepsilon_0}\frac{q^2}{R}$$

4.4 Comments on Electrostatic Energy

(1) A perplexing inconsistent

$$W = \frac{\varepsilon_0}{2}\int_{all\ space} E^2 d\tau \quad \Rightarrow \quad \text{Energy} > 0$$

or

$$\left.\begin{aligned} W &= \frac{1}{2}\sum_{i=1}^{n} q_i V(P_i) \\ W &= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ j>i}}^{n} \frac{q_i q_j}{r_{ij}} \end{aligned}\right\} \Rightarrow \text{Energy} > 0 \quad \text{or} \quad < 0$$

Only for point charge

$$q_1 = q\ , \quad q_2 = -q \Rightarrow W = \frac{1}{4\pi\varepsilon_0}\frac{\left(-q^2\right)}{r_{12}} < 0$$ ($\because q_1$ and q_2) are attractive

$$W = \frac{\varepsilon_0}{2}\int_{all\ space} E^2 d\tau$$ is more complete

the energy of a point charge itself,

$$W = \frac{\varepsilon_0}{2(4\pi\varepsilon_0)^2}\int (\frac{q^2}{r^4})\left(r^2 \sin\theta d\theta d\varphi dr\right) = \frac{q^2}{8\pi\varepsilon_0}\int_0^\infty \frac{1}{r^2}dr = \infty$$

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(P_i)$$, $V(P_i)$ does not include q_i

$$W = \frac{1}{2}\int \rho V d\tau$$, $V(P)$ is the full potential

There is no distinction for a continuous distribution, because

$$\rho(d\tau)_p \xrightarrow{d\tau \to 0} 0$$

(2)Where is the energy stored? In charge or in field ?

Both are fine in ES. But,it is useful to regard the energy as being stored in the field at a density

$$\varepsilon_0 \frac{E^2}{2} = \text{Energy per unit volume}$$

(3)The superposition principle,not for ES energy

$$W_1 = \frac{\varepsilon_0}{2}\int E_1^2 d\tau \qquad W_2 = \frac{\varepsilon_0}{2}\int E_2^2 d\tau$$

$$W_{tot} = \frac{\varepsilon_0}{2}\int (\vec{E}_1 + \vec{E}_2)^2 d\tau$$

$$= \frac{\varepsilon_0}{2}\int (E_1^2 + E_2^2 + 2\vec{E}_1 \cdot \vec{E}_2) d\tau$$

$$= W_1 + W_2 + \varepsilon_0 \int (\vec{E}_1 \cdot \vec{E}_2) d\tau$$

4.5 Magnetic Fields in analogy with Electric Fields

Electric Field:

- Distribution of charge creates an electric field ***E(r)*** in the surrounding space.
- Field exerts a force $\boldsymbol{F}=q\,\boldsymbol{E}(\boldsymbol{r})$ on a charge q at $\boldsymbol{r}$

Magnetic Field:

- <u>Moving</u> charge or current creates a magnetic field ***B(r)*** in the surrounding space.
- Field exerts a force $\boldsymbol{F}$ on a charge <u>moving</u> q at $\boldsymbol{r}$
- (emphasis this chapter is on force law)

4.6 Magnetic Fields and Magnetic Forces

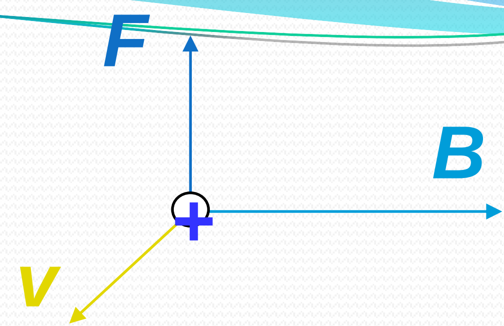
Magnetic Force on a moving charge

- proportional to electric charge
- perpendicular to velocity $\boldsymbol{v}$
- proportional to speed v (for a given geometry)
- perpendicular to Magnetic Field $\boldsymbol{B}$
- proportional to field strength B (for a given geometry)

$$\boldsymbol{F} = q\,\boldsymbol{v} \times \boldsymbol{B}$$

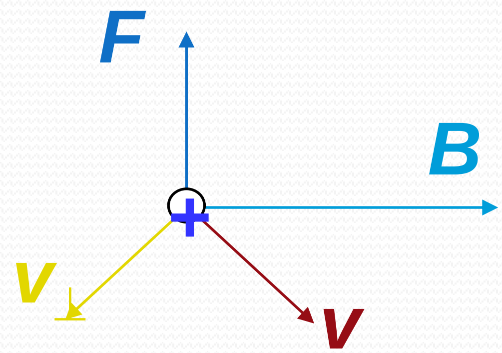
$$\mathbf{F} = q\,\mathbf{v} \times \mathbf{B}$$

$$F = |q|\, v\, B \sin\theta$$
$$\quad = |q|\, v\, B \qquad (\mathbf{v} \perp B)$$



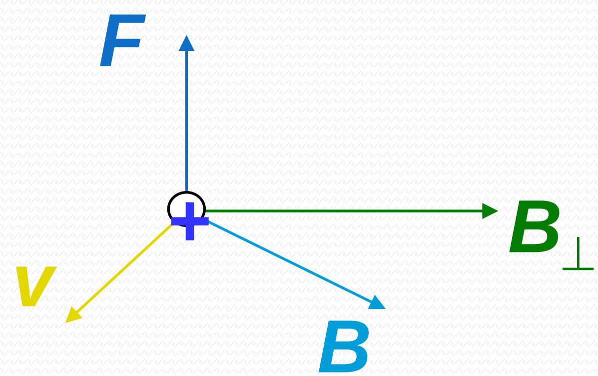
$$\mathbf{F} = q\,\mathbf{v} \times \mathbf{B}$$

$$F = |q|\, v_{\perp}\, B$$



$$\mathbf{F} = q\,\mathbf{v} \times \mathbf{B}$$

$$F = |q|\, v\, B_{\perp}$$



4.7 Magnetic Fields

Units of Magnetic Field Strength:

$$[B] = [F]/([q][v])$$

$$= N/(C\ m\ s^{-1})$$

$$= \text{Tesla}$$

Defined in terms of force on standard current

CGS Unit 1 Gauss = 10^{-4} Tesla

Earth's field strength ~ 1 Gauss

Direction = direction of velocity which generates <u>no</u> force

Electromagnetic Force:

$$\boldsymbol{F} = q\,(\boldsymbol{E} + \boldsymbol{v}\ ´\ \boldsymbol{B})$$

= *Lorentz Force Law*

4.8 Magnetic Field Lines and Magnetic Flux

Magnetic Field Lines

- Mapped out with compass
- Are not lines of force (***F*** is not parallel to ***B***)
- Field Lines never intersect

Magnetic Flux

- $dF_B = \mathbf{B} \cdot d\mathbf{A}$

$$d\Phi_B = \vec{B} \cdot d\vec{A}$$

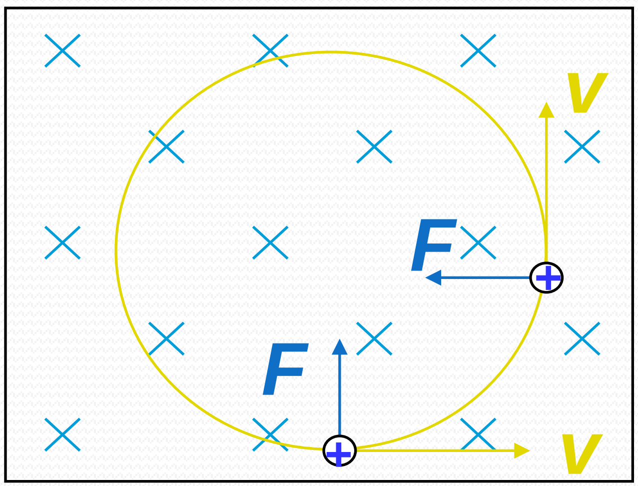
$$\Phi_B = \int \vec{B} \cdot d\vec{A}$$

$$\oint \vec{B} \cdot d\vec{A} = 0 \quad \text{no magneti} \quad \text{c charge!} \quad \text{(no monopo} \quad \text{les)}$$

4.9 Motion of Charged Particles in a Magnetic Field

Charged Particle moving perpendicular to the Magnetic Field

- Circular Motion!
- (simulations)



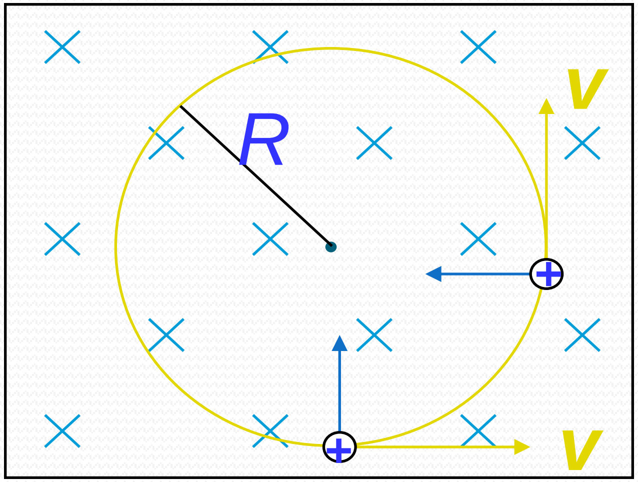
Charged Particle moving perpendicular to a uniform Magnetic Field

$$F = |q|vB = \frac{mv^2}{R}$$

$$R = \frac{mv}{|q|B}$$

$$\omega = \frac{v}{R} = \frac{|q|B}{m}$$

$$= \text{cyclotron} \quad \text{frequency}$$



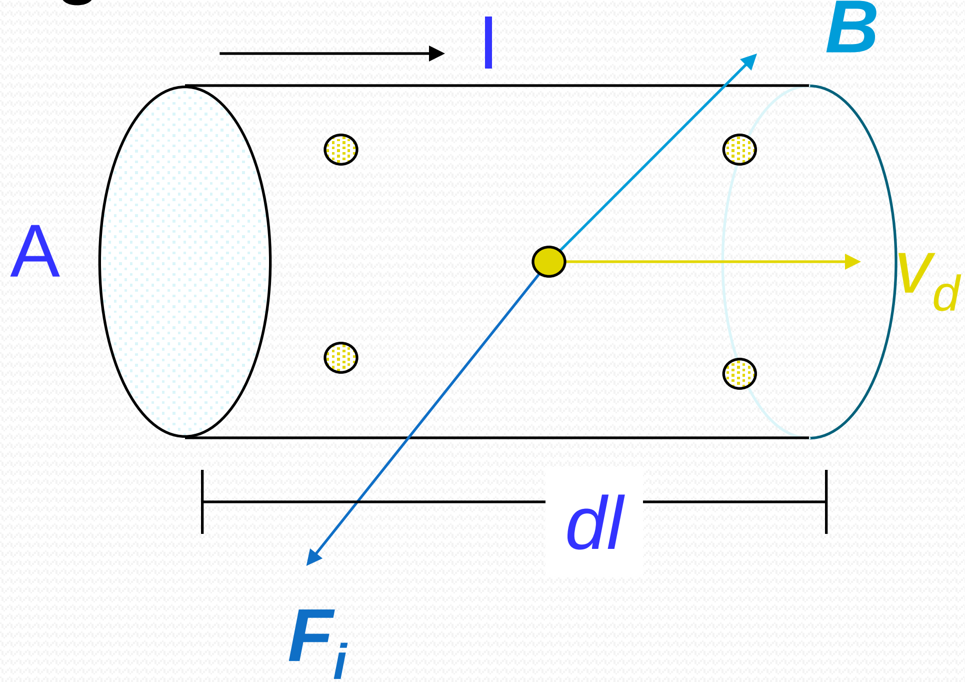
Work done by the Magnetic Field on a free particle:

$$\begin{aligned} dW &= \vec{F} \cdot d\vec{x} \\ &= \left(q\vec{v} \times \vec{B}\right) \cdot \vec{v}\,dt \\ &= 0! \end{aligned}$$

=> no change in Kinetic Energy!

Motion of a free charged particle in **<u>any</u>** magnetic field has constant speed.

4.11 Magnetic Force on a Current Carrying Wire



$$F = \sum F_i = \sum q_i \vec{v}_i \times \vec{B}$$

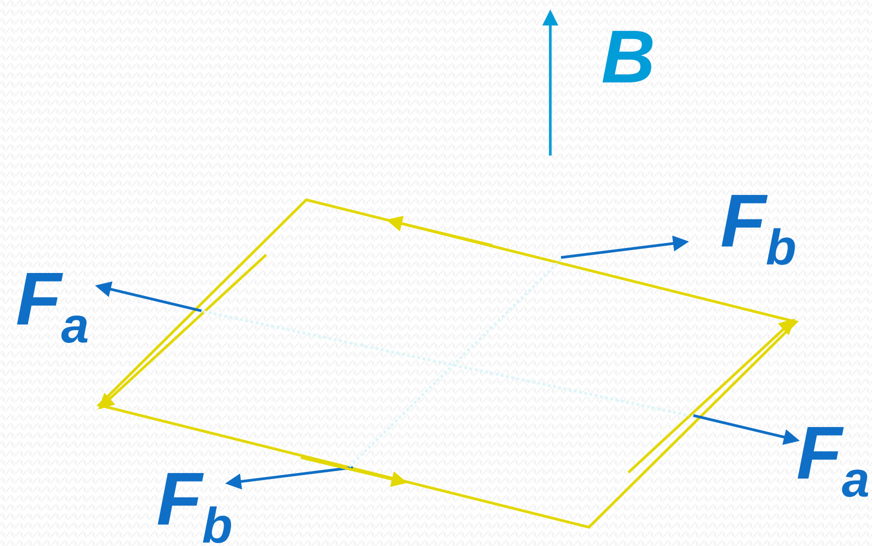
$$= Nq\vec{v}_d \times \vec{B} = n \cdot volume \cdot q\vec{v}_d \times \vec{B}$$

$$= nAdlq\vec{v}_d \times \vec{B} = \vec{J}Adl \times \vec{B}$$

$$= Id\vec{l} \times \vec{B} \quad (RHR)$$

4.12 Torque on a Current Loop (from **F** = *I* ***l*** x **B**)

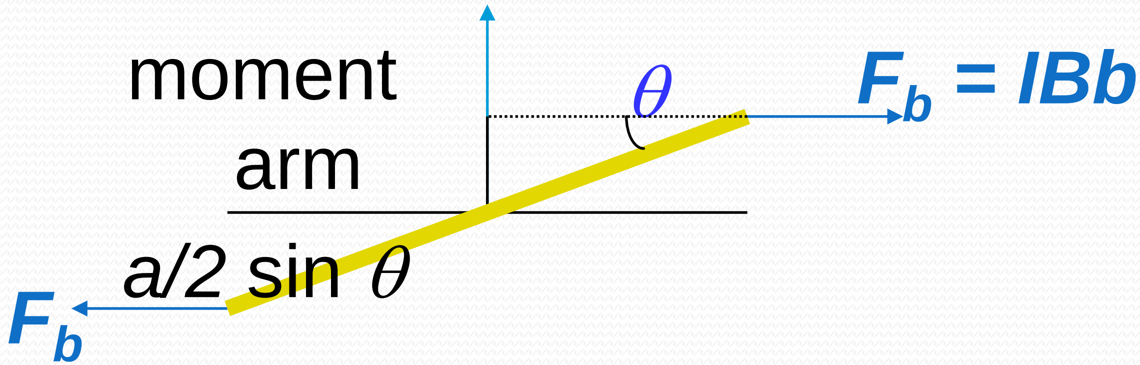
Rectangular loop in a magnetic field (directed along z axis) short side length a, long side length b, tilted with short sides at an angle with respect to ***B***, long sides still perpendicular to ***B***.



Forces on short sides cancel: no net force or torque.

Forces on long sides cancel for no net force but there is a net torque.

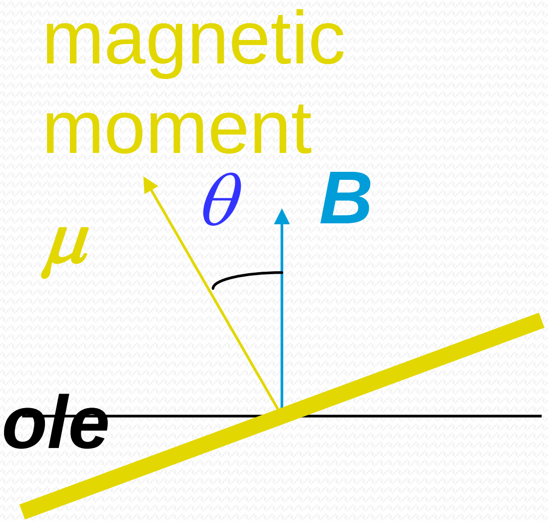
Torque calculation: Side view



$$\tau = \boldsymbol{F_b}\, a/2 \sin\theta + \boldsymbol{F_b}\, a/2 \sin\theta$$

$$= Iab\, B \sin\theta = I\, A\, B \sin\theta$$

$$= I\, \boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{\mu} \times \boldsymbol{B}$$

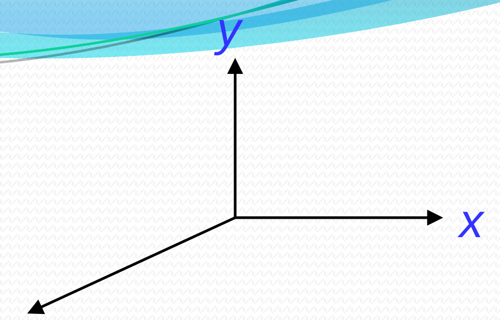


Magnetic Dipole ~ Electric Dipole

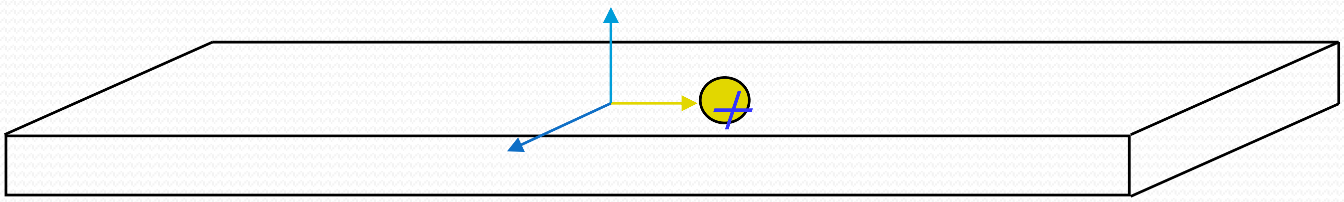
$\boldsymbol{U} = -\boldsymbol{\mu}\cdot\boldsymbol{B}$

Switch current *direction* every 1/2 rotation => DC motor

4.13 Hall Effect

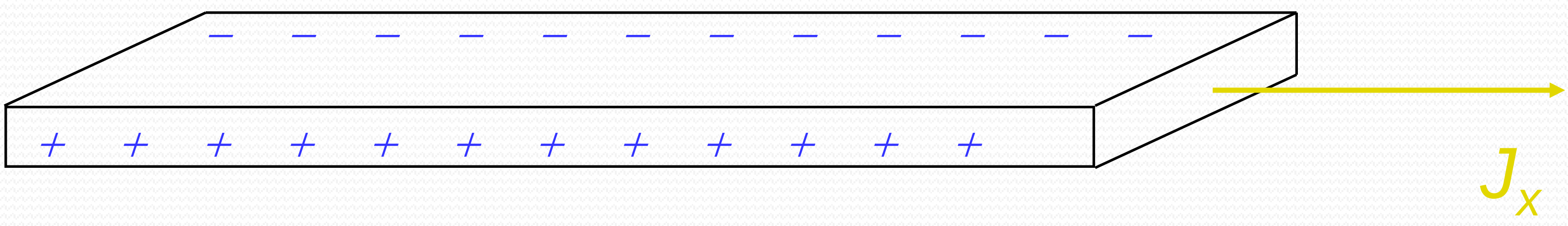


Conductor in a uniform magnetic field

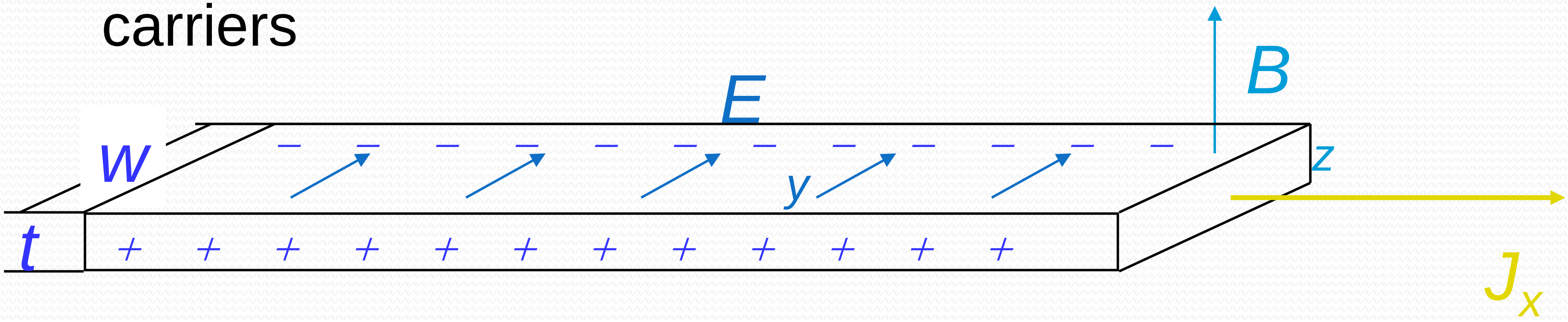


Magnetic force on charge carriers $\boldsymbol{F} = q\,\boldsymbol{v}_d$ ´ $\boldsymbol{B}$

$F_z = qv_dB$ Charge accumulates on edges



Equilibrium: Magnetic Force = Electric Force on bulk charge carriers



Charge accumulates on edges $F_z = 0 = qv_dB_y + q\,E_z$

$$v_d = -\frac{E_z}{B_y}$$

$$J_x = nqv_d = -nq\frac{E_z}{B_y}$$

$$nq = \frac{-J_x B_y}{E_z}$$

Hall EMF $V_H = E_z w$

$$I = J_x tw$$

$$nq = \frac{-IB_y}{V_H t}$$

UNIT – V
TIME VARYING FIELDS AND FINITE ELEMENT METHOD

5.1 Time-Varying Fields

Stationary charges → electrostatic fields

Steady currents → magnetostatic fields

Time-varying currents → electromagnetic fields

Only in a non-time-varying case can electric and magnetic fields be considered as independent of each other. In a time-varying (dynamic) case the two fields are interdependent. A changing magnetic field induces an electric field, and vice versa.

The Continuity Equation

Electric charges may not be created or destroyed (the principle of conservation of charge).

Consider an arbitrary volume *V* bounded by surface *S*. A net charge *Q* exists within this region. If a net current *I* flows across the surface out of this region, the charge in the volume must decrease at a rate that equals the current:

ρ_V

$$I = \int_S \bar{J} \cdot \overline{dS} = -\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho_V \, dv$$

Divergence theorem

$$\int_V \nabla \cdot \bar{J} dv = -\int_V \frac{\partial \rho_V}{\partial t} dv$$

Partial derivative because may be a function of both time and space

This equation must hold regardless of the choice of *V*, therefore the integrands must be equal:

$$\nabla \cdot \bar{J} = -\frac{\partial \rho_V}{\partial t} \qquad (A/m^3)$$

the equation of continuity

For steady currents

$$\nabla \cdot \bar{J} = 0$$

Kirchhoff' s current law follows from this

that is, steady electric currents are divergences or solenoidal.

Displacement Current

For magnetostatic field, we recall that $\nabla \times \overline{H} = \overline{J}$

Taking the divergence of this equation we have

$$\nabla \cdot \left(\nabla \times \overline{H} \right) = 0 = \nabla \cdot \overline{J}$$

However the continuity equation requires that

$$\nabla \cdot \overline{J} = -\frac{\partial \rho_V}{\partial t} \neq 0$$

Thus we must modify the magnetostatic curl equation to agree with the continuity equation. Let us add a term to the former so that it becomes

$$\nabla \times \overline{H} = \overline{J} + \overline{J}_d$$

where $\overline{J}$ is the conduction current density $\overline{J} = \sigma_E \overline{E}$, and $\overline{J}_d$ is to be determined and defined.

Displacement Current continued

Taking the divergence we have

$$\nabla\cdot\left(\nabla\times\overline{H}\right)=0=\nabla\cdot\overline{J}+\nabla\cdot\overline{J}_d \longrightarrow \nabla\cdot\overline{J}_d=-\nabla\cdot\overline{J}$$

In order for this equation to agree with the continuity equation,

Gauss' law

$$\nabla\cdot\overline{J}_d=-\nabla\cdot\overline{J}=\frac{\partial\rho_V}{\partial t}=\frac{\partial}{\partial t}\left(\nabla\cdot\overline{D}\right)=\nabla\cdot\frac{\partial\overline{D}}{\partial t}$$

$$\overline{J}_d=\frac{\partial\overline{D}}{\partial t}$$ ← displacement current density

$$\boxed{\nabla\times\overline{H}=\overline{J}+\frac{\partial\overline{D}}{\partial t}}$$

$$\int_S\nabla x\overline{H}\cdot\overline{ds}=\int_S\left(\overline{J}+\frac{\partial\overline{D}}{\partial t}\right)\cdot\overline{ds} \longrightarrow \boxed{\oint_L\overline{H}\cdot\overline{dl}=I+\oint_S\frac{\partial\overline{D}}{\partial t}\cdot\overline{ds}}$$

Stokes' theorem

Displacement Current continued

A typical example of displacement current is the current through a capacitor when an alternating voltage source is applied to its plates. The following example illustrates the need for the displacement current.

Using an <u>unmodified</u> form of Ampere's law

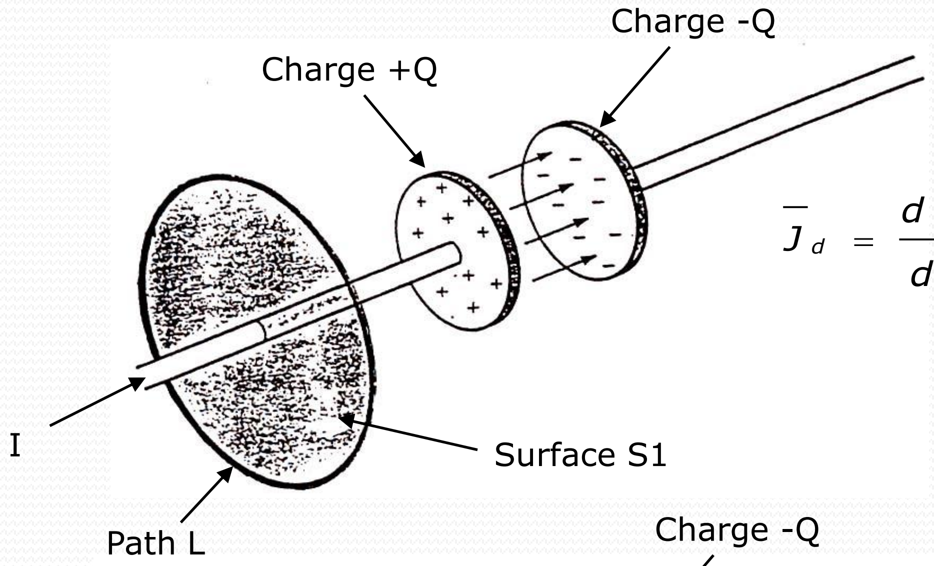
$$\begin{cases} \oint_L \overline{H} \cdot \overline{dl} = \int_{S_1} \overline{J} \cdot \overline{dS} = I_{enc} = I = \dfrac{dQ}{dt} \\ \oint_L \overline{H} \cdot \overline{dl} = \int_{S_2} \overline{J} \cdot \overline{dS} = I_{enc} = 0 \end{cases}$$

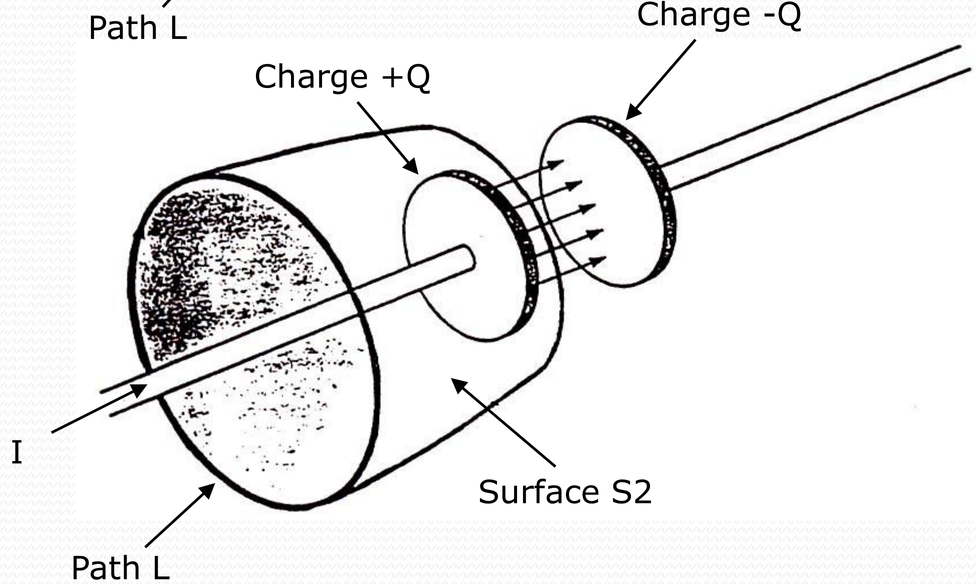
$\overline{J}_d = 0$

(no conduction current flows through s_2 ($\overline{J} = 0$))

To resolve the conflict we need to include $\overline{J}_d$ in Ampere's law.

Displacement Current continued





The total current density is $\bar{J} + \bar{J}_d$. In the first equation $\bar{J}_d = 0$ so it remains valid. In the second equation $\bar{J} = 0$ so that

$$\bar{J}_d = \frac{d\bar{D}}{dt}$$

$$\oint_L \bar{H}\cdot d\bar{l} = \int_{S_2} \bar{J}_d \cdot d\bar{S} = \frac{d}{dt}\overbrace{\int_{S_2} \bar{D}\cdot d\bar{S}}^{Q} \leftarrow \boxed{\bar{J} = 0}$$

$$= \frac{dQ}{dt} = I$$

$$\oint_{S_1+S_2} \bar{D}\cdot d\bar{S} = Q$$

$$\int_{S_1} \bar{D}\cdot d\bar{S} = 0$$

So we obtain the same current for either surface though it is conduction current in S_1 and displacement current in S_2.

Faraday's Law

Faraday discovered experimentally that a current was induced in a conducting loop when the magnetic flux linking the loop changed. In differential (or point) form this experimental fact is described by the following equation

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}$$

Taking the surface integral of both sides over an open surface and applying Stokes' theorem, we obtain

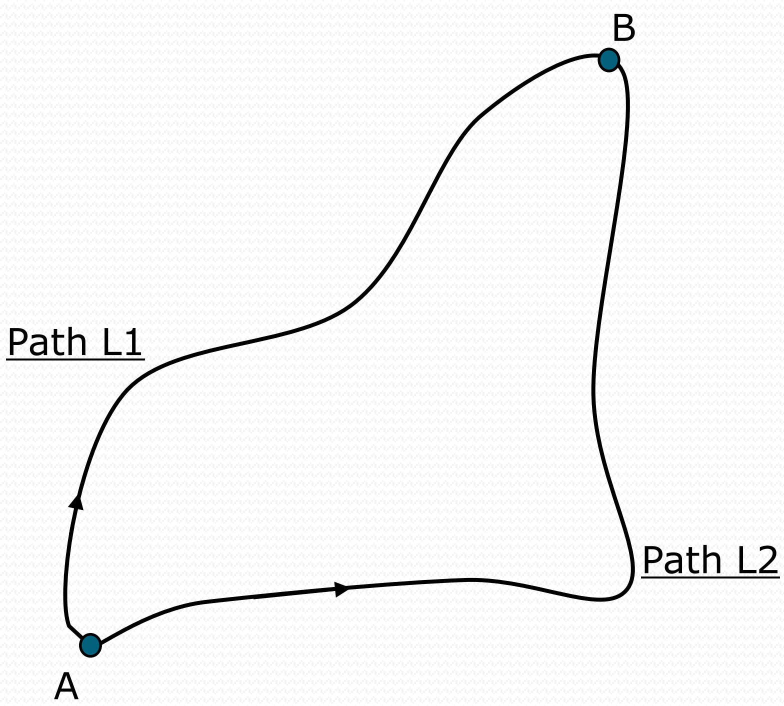
Integral form →

$$\oint_L \overline{E} \cdot \overline{dl} = -\frac{\partial}{\partial t}\int_S \overline{B} \cdot \overline{dS} = -\frac{\partial \psi}{\partial t}$$

where ψ is the magnetic flux through the surface S.

Faraday's Law continued

Time-varying electric field is not conservative.



The effect of electromagnetic induction. When time-varying magnetic fields are present, the value of the line integral of $\overline{E}$ from A to B may depend on the path one chooses.

Suppose that there is only one unique voltage $V_{AB} = V_A - V_B$. Then

$$V_{AB} = \oint_{L_1} \overline{E} \cdot d\overline{l} = \oint_{L_2} \overline{E} \cdot d\overline{l}$$

However,

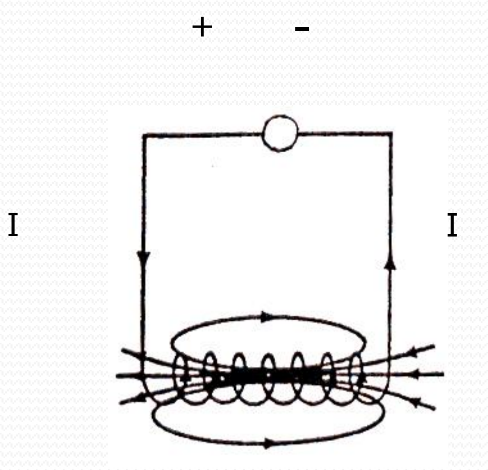
$$\oint_L \overline{E} \cdot \overline{dl} = \underbrace{\int_{L_1} \overline{E} \cdot \overline{dl}}_{V_{AB}} - \underbrace{\int_{L_2} \overline{E} \cdot \overline{dl}}_{V_{AB}} = -\frac{\partial \psi}{\partial t}$$

$$V_{AB} - V_{AB} \neq 0$$

Thus V_{AB} can be unambiguously defined only if $\partial \psi / \partial t = 0$.(in practice, if $\lambda >>$ than the dimensions of system in question)

Inductance

A circuit carrying current *I* produces a magnetic field $\bar{B}$ which causes a flux $\psi = \int \bar{B} \cdot d\bar{S}$ to pass through each turn of the circuit. If the medium surrounding the circuit is linear, the flux ψ is proportional to the current *I* producing it.



Magnetic field **B** produced by a circuit.

$$V = N \frac{\partial \psi}{\partial t}$$ (voltage induced across coil)

$$= Nk \frac{\partial I}{\partial t} = L \frac{\partial I}{\partial t}$$

$$L = Nk = \frac{N\psi}{I} = \frac{\lambda}{I}$$, H (henry)

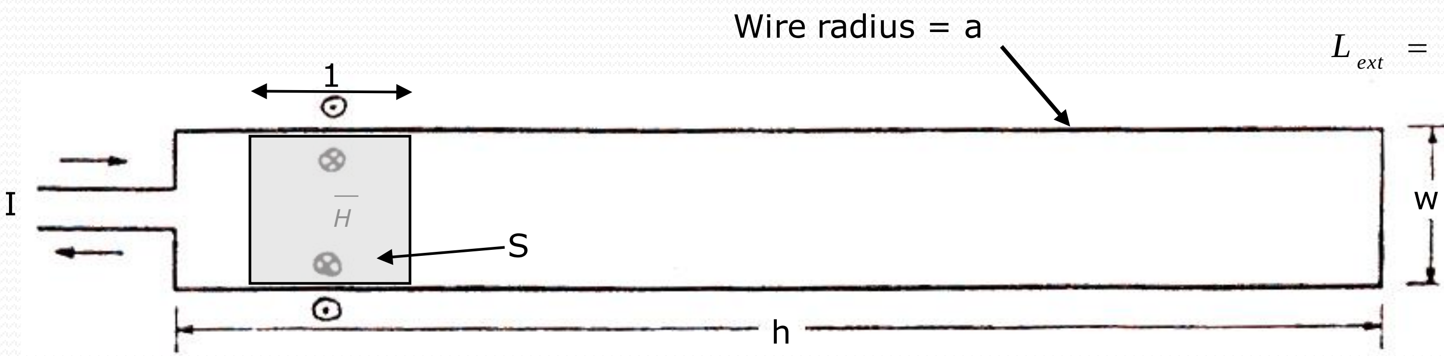
<u>Self-inductance</u> *L* is defined as the ratio of the magnetic <u>flux linkage</u> λ to the current *I*.

In an inductor such as a coaxial or a parallel-wire transmission line, the inductance produced by the flux internal to the conductor is called the internal inductance L_{in} while that produced by the flux external to it is called external inductance

$$L_{ext} \quad (L = L_{in} + L_{ext})$$

Let us find L_{ext} of a very long rectangular loop of wire for which $R = 0$ $a \ll w$ and $w \ll h$. This geometry represents a parallel-conductor transmission line. Transmission lines are usually characterized by per unit length parameters.

$$L_{ext} = \frac{\Psi}{I}$$



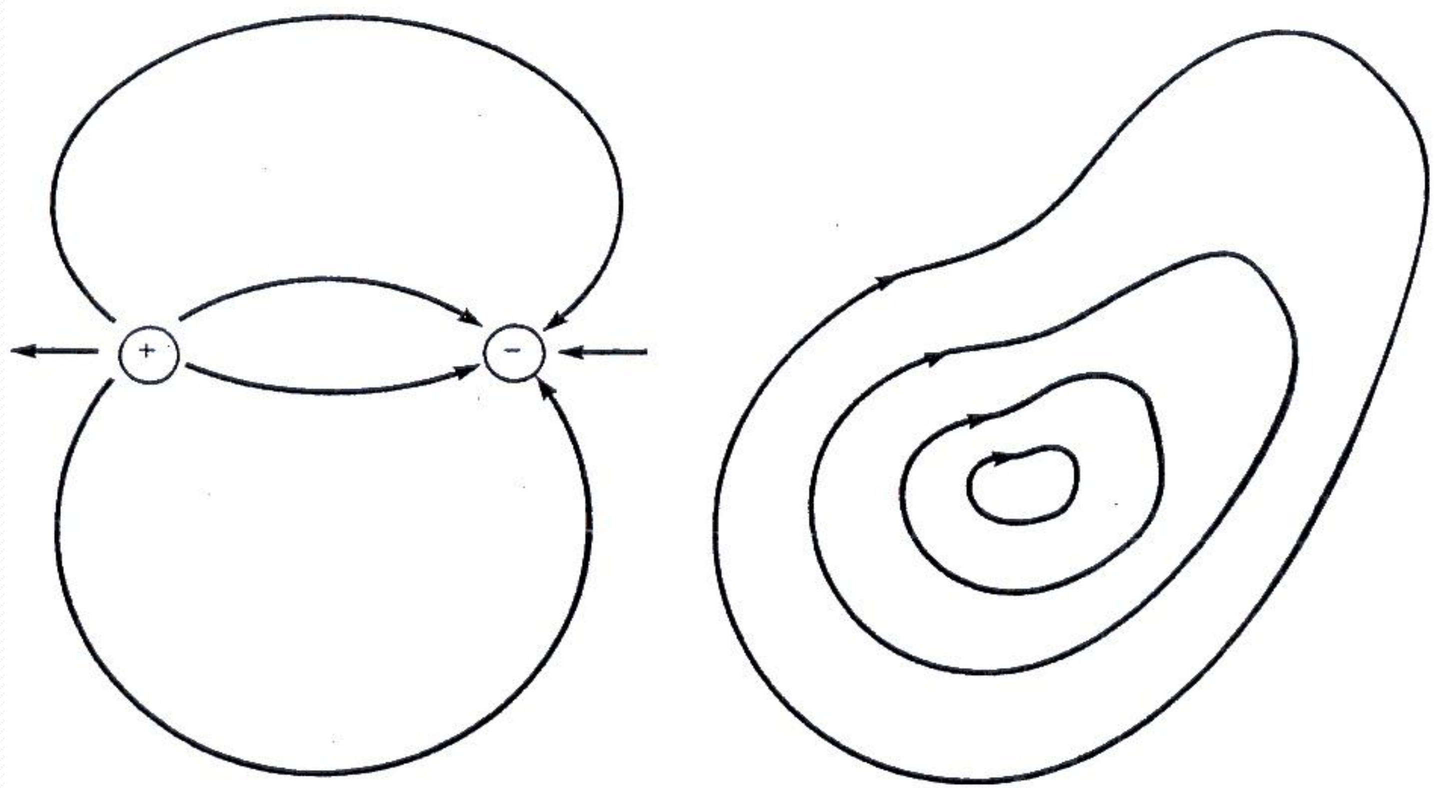
Finding the inductance per unit length of a parallel-conductor transmission line.

General Forms of Maxwell’ s Equations

	Differential	**Integral**	**Remarks**
1	$\nabla \cdot \overline{D} = \rho_V$	$\oint_S \overline{D} \cdot d\overline{s} = \int_V \rho_V \, dv$	Gauss’ Law
2	$\nabla \cdot \overline{B} = 0$	$\oint_S \overline{B} \cdot d\overline{s} = 0$	Nonexistence of isolated magnetic charge
3	$\nabla x \overline{E} = -\frac{\partial \overline{B}}{\partial t}$	$\oint_L \overline{E} \cdot d\overline{l} = -\frac{\partial}{\partial t} \int_s \overline{B} \cdot d\overline{s}$	Faraday’ s Law
4	$\nabla x \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t}$	$\oint_L \overline{H} \cdot d\overline{l} = \int_s \left(\overline{J} + \frac{\partial \overline{D}}{\partial t} \right) \cdot d\overline{s}$	Ampere’ s circuital law

In 1 and 2, S is a closed surface enclosing the volume V

In 2 and 3, L is a closed path that bounds the surface S



Electric fields can originate on positive charges and can end on negative charges. But since nature has neglected to supply us with magnetic charges, magnetic fields cannot begin or end; they can only form closed loops.

Sinusoidal Fields

In electromagnetics, information is usually transmitted by imposing amplitude, frequency, or phase modulation on a sinusoidal carrier. Sinusoidal (or time-harmonic) analysis can be extended to most waveforms by Fourier and Laplace transform techniques.

Sinusoids are easily expressed in phasors, which are more convenient to work with. Let us consider the "curl *H*" equation.

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t} \qquad \overline{H} = f(x, y, z, t)$$

Its phasor representation is

$$\nabla \times \underline{\overline{H}} = \underline{\overline{J}} + j\omega \underline{\overline{D}} = \underline{\overline{J}} + j\omega\varepsilon \underline{\overline{E}}$$

$\underline{H}$ is a vector function of position, but it is independent of time. The three scalar components of $\overline{H}$ are complex numbers; that is, if

$$\overline{H}(x, y, z, t) = f_1(x, y, z)\cos(\omega t + \phi_1)\overline{e}_x + f_2(x, y, z)\cos(\omega t + \phi_2)\overline{e}_y$$

then

$$\underline{\overline{H}}(x, y, z) = f_1(x, y, z)e^{j\phi_1}\overline{e}_x + f_2(x, y, z)e^{j\phi_2}\overline{e}_y$$

Sinusoidal Fields continued

$$e^{j\omega t}$$

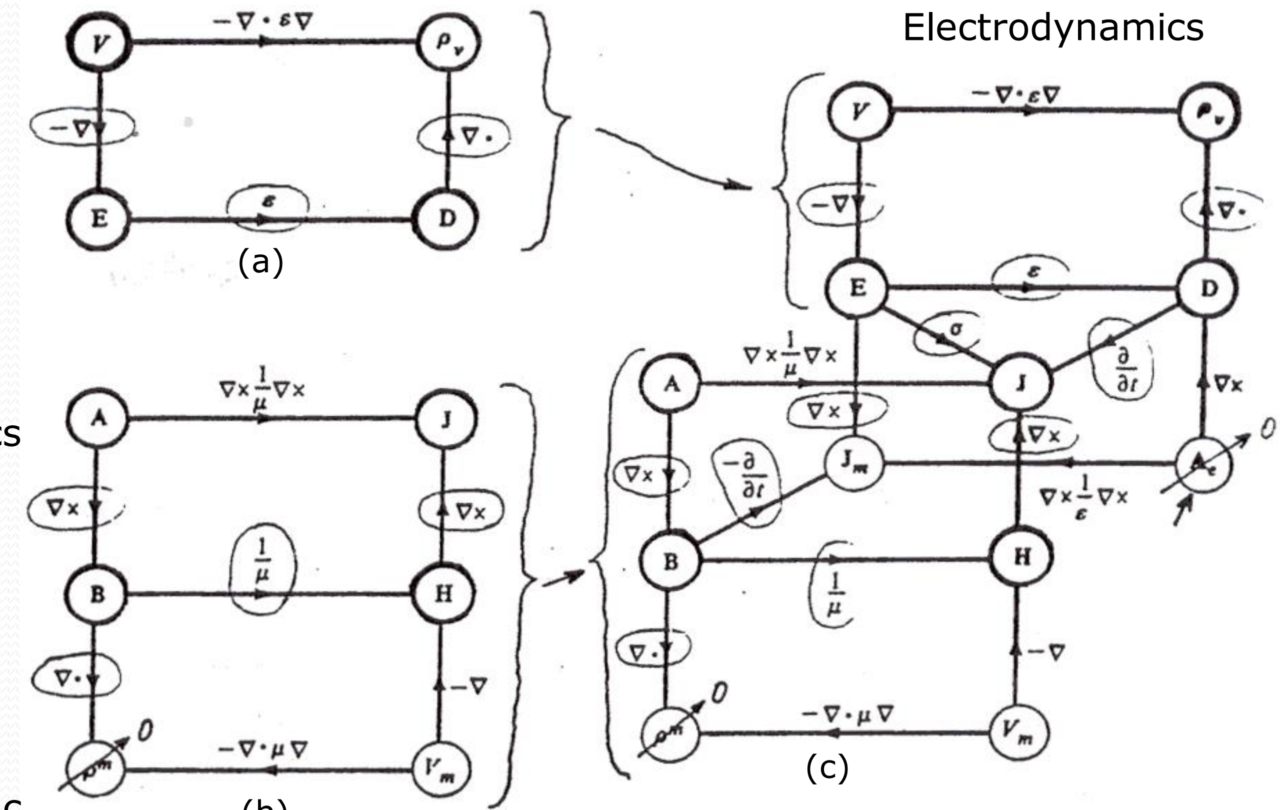
Point Form	**Integral Form**
$\nabla \cdot \underline{\overline{D}} = \underline{\rho_V}$	$\oint \underline{\overline{D}} \cdot d\overline{s} = \int \underline{\rho_V}\, dv$
$\nabla \cdot \underline{\overline{B}} = 0$	$\oint \underline{\overline{B}} \cdot d\overline{s} = 0$
$\nabla \times \underline{\overline{E}} = -j\omega \underline{\overline{B}}$	$\oint \underline{\overline{E}} \cdot d\overline{l} = -j\omega \int \underline{\overline{B}} \cdot d\overline{s}$
$\nabla \times \underline{\overline{H}} = \underline{\overline{J}} + j\omega \underline{\overline{D}}$	$\oint \underline{\overline{H}} \cdot d\overline{l} = \int \left(\underline{\overline{J}} + j\omega \underline{\overline{D}} \right) \cdot d\overline{s}$

$$\overline{H}(x,y,z,t) = \mathrm{Re}\left[\underline{\overline{H}}(x,y,z)\, e^{j\omega t} \right]$$

Electrostatics

Electrodynamics

Magnetostatics



Free magnetic charge density ( ) $\rho^m = 0$

Electromagnetic flow diagram showing the relationship between the potentials and vector fields: (a) electrostatic system, (b) magnetostatic system, (c) electromagnetic system. [Adapted with permission from IEE Publishing Department]

The Skin Effect

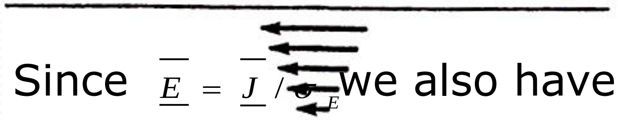
When time-varying fields are present in a material that has high conductivity, the fields and currents tend to be confined to a region near the surface of the material. This is known as the "skin effect". Skin effect increases the effective resistance of conductors at high frequencies.

Let us consider the vector wave equation (Helmholtz's equation) for the electric field:

$$\nabla^2 \overline{E} = j\omega\mu(\sigma_E + j\omega\varepsilon)\overline{E} \qquad \left(\nabla^2 \overline{E} = \nabla^2 E_x \overline{e}_x + \nabla^2 E_y \overline{e}_y + \nabla^2 E_z \overline{e}_z\right)$$

If the material in question is a very good conductor, so that $\sigma_E \gg \omega\varepsilon$ we can write

$$\nabla^2 \overline{E} = j\omega\mu\sigma_E \overline{E}$$



Since $\overline{E} = \overline{J}/\sigma_E$ we also have

$$\nabla^2 \overline{J} = j\omega\mu\sigma_E \overline{J}$$

Consider current flowing in the +*x* direction through a conductive material filling the half-space *z*<0. The current density is independent of *y* and *x*, so that we have

$$\frac{\partial^2 J_x}{\partial z^2} = \overbrace{j\omega\mu\sigma_E}^{\gamma^2} J_x \longrightarrow J_x = Ae^{-\frac{(1+j)z}{\delta}} + Be^{\frac{(1+j)z}{\delta}} = \underbrace{Ae^{-\frac{(1+j)z}{\delta}}}_{\to 0} + Be^{\frac{(1+j)z}{\delta}}$$

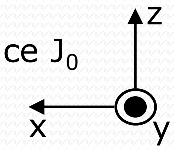
$$\gamma = \alpha + j\beta$$

$$\alpha = \beta = \frac{1}{\delta}$$

Current density at surface J_0

(Air) Z>0

(Metal) Z<0



Skin depth

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma_E}} = \frac{1}{\sqrt{\pi\mu\sigma_E f}}$$

Magnitude of current density decreases exponentially with depth

$$\underline{J}_x = Ae^{-(1+j)z/\delta} + Be^{(1+j)z/\delta} = Ae^{-\frac{z}{\delta}}e^{-j\frac{z}{\delta}} + Be^{\frac{z}{\delta}}e^{j\frac{z}{\delta}}$$

As $z \to -\infty$, the first term increases and would give rise to infinitely large currents. Since this is physically unreasonable, the constant *A* must vanish. From the condition when *z*=0 we have $\underline{J}_x = J_o$

$$J_o = Be^{(0)}e^{(0)} = B$$

and

$$\boxed{\underline{J}_x(z) = J_o e^{\frac{z}{\delta}} e^{j\frac{z}{\delta}}}$$

$$z \le 0$$

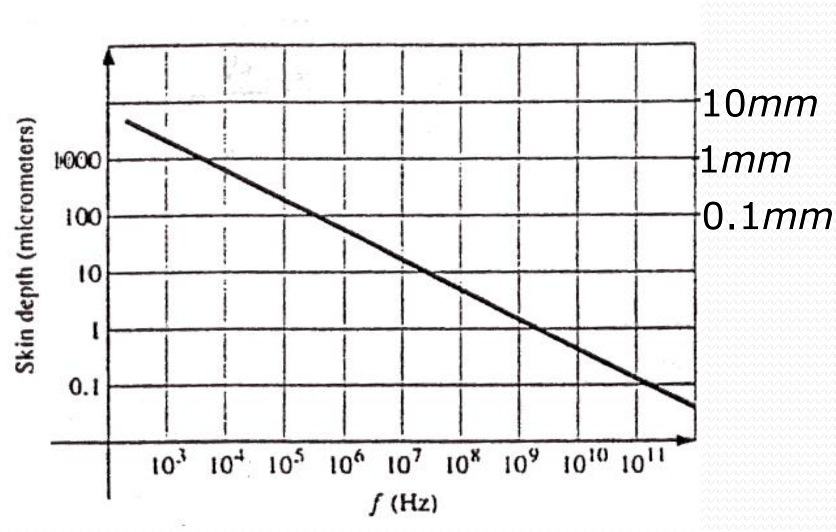
Current density magnitude decreases exponentially with depth. Its phase changes as well.

At $|z| = \delta$

$$|\underline{J}_x| = \frac{1}{e} J_o = 0.37\, J_o$$

At $|z| = \pi\delta$

$$\underline{J}_x = -0.043\, J_o$$



Skin depth δ versus frequency for copper

$$\begin{cases} 10\ kHz \to \delta = 0.6\,mm \\ 60\ Hz \to \delta = 8.6\,mm \end{cases}$$

Skin Depth and Surface Resistance for Metals

$$\delta = \frac{1}{\sqrt{\pi\mu\sigma_E f}} = \frac{A}{\sqrt{f}}$$

$$R_S = \sqrt{\frac{\pi\mu f}{\sigma_E}} = B\sqrt{f}$$

Metal	$\sigma_E\,(\Omega m)^{-1}$	A	B
Silver	$6.81 \cdot 10^{7}$	$6.10 \cdot 10^{-2}$	$2.41 \cdot 10^{-7}$
Copper	$5.91 \cdot 10^{7}$	$6.55 \cdot 10^{-2}$	$2.59 \cdot 10^{-7}$
Gold	$4.10 \cdot 10^{7}$	$7.86 \cdot 10^{-2}$	$3.10 \cdot 10^{-7}$
Aluminum	$3.54 \cdot 10^{7}$	$8.46 \cdot 10^{-2}$	$3.34 \cdot 10^{-7}$
Iron	$1.02 \cdot 10^{7}$	$1.58 \cdot 10^{-1}$	$6.22 \cdot 10^{-7}$

Surface Impedance

If $\underline{J}_x(z) = J_o e^{\frac{z}{\delta}} e^{j\frac{z}{\delta}}$, the total current per unit width is

$$\frac{1}{1+j} = \frac{1}{\sqrt{2}} e^{-j45^\circ}$$

$$\underline{I}_w = \int_{-\infty}^{0} \underline{J}_x dz = \int_{-\infty}^{0} J_o e^{\frac{(1+j)z}{\delta}} dz = \frac{J_o\delta}{1+j}$$

← 45° out of phase with J_o

Since $J_o = \delta_E E_o$ we can write

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma_E}}$$

$$\underline{I}_w = \frac{\sigma_E\delta}{1+j} E_o = \frac{1}{Z_S} E_o$$

← Voltage per unit length

← Surface impedance

$$Z_S = \frac{E_o}{\underline{I}_w} = \frac{1+j}{\sigma_E\delta} = (1+j)\sqrt{\frac{\omega\mu}{2\sigma_E}} = R_S + jX_S$$

$$(R_S = X_S)$$

$[Z_S] = \Omega$ or

"ohms per square"

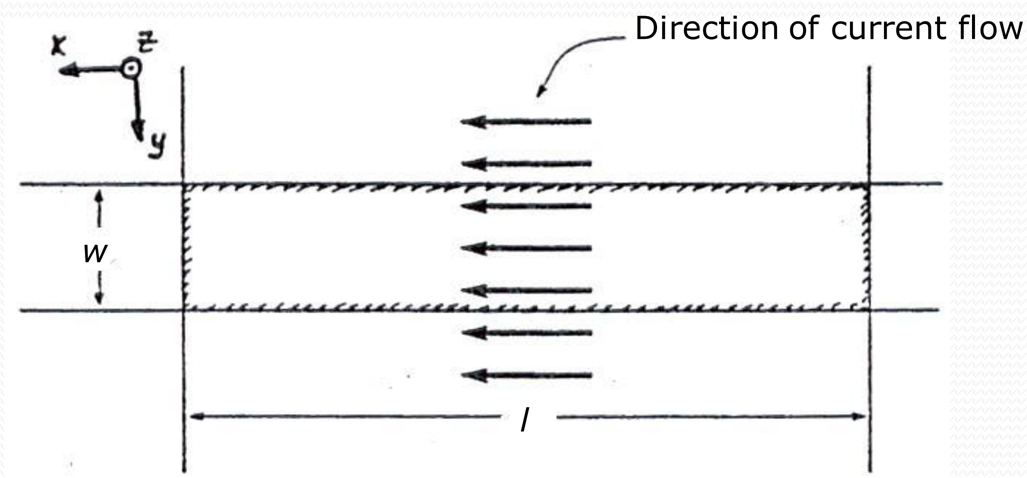
$$\underline{I} = \underline{I}_w w$$

$$\underline{V} = \underline{E}_o l$$

$$\frac{\underline{V}}{\underline{I}} = \frac{\underline{E}_o l}{\underline{I}_w w} = Z_S\left(\frac{l}{w}\right)$$

If $l=w$ (a square surface)

$$\frac{\underline{V}}{\underline{I}} = Z_S$$



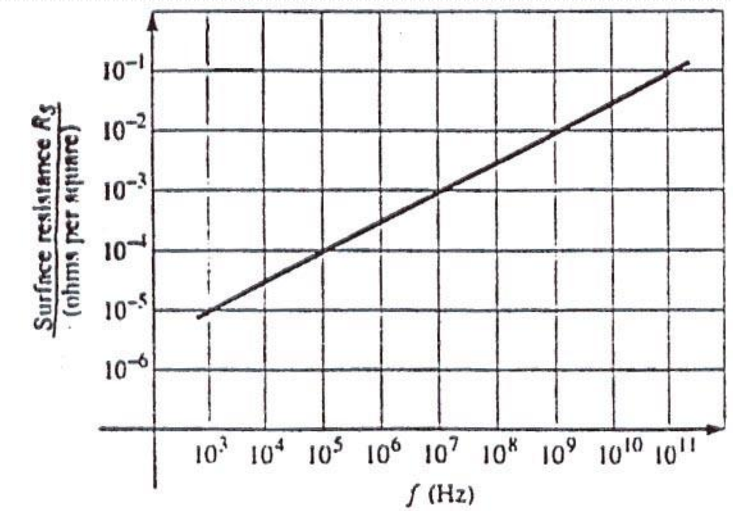
Illustrating the concept of "ohms per square"

Surface Impedance continued

$$\left(R_{dc} = \frac{l}{\sigma_E A} \right)$$

$$R_S = \frac{1}{\sigma_E \delta} = \sqrt{\frac{\pi f \mu}{\sigma_E}}$$

Copper



R_S is equivalent to the dc resistance per unit length of the conductor having cross-sectional area $1 \times \delta$

$$R_{ac} = \frac{l}{\underbrace{\sigma\delta}_{A} \; w} = \frac{Rsl}{w}$$

For a wire of radius *a*,

$$w = 2\pi a$$

$$\ell = 1cm = 10^4 \mu m$$

$$2r = d = 0.1mm = 100 \mu m$$

$$f = 10 GHz = 10^{10} Hz$$

$$\sigma_E = 5.91 \cdot 10^7 (\Omega \cdot m)^{-1}$$

$$\mu = \mu_o = 4\pi \cdot 10^{-7} H/m$$

$$\delta = \frac{1}{\sqrt{\pi \mu \sigma_E f}} = 0.6 \mu m << \frac{d}{2} = 50 \mu m$$

$$Z_S = \frac{(1+j)}{\sigma_E \delta} = (1+j) x 0.026 \Omega$$

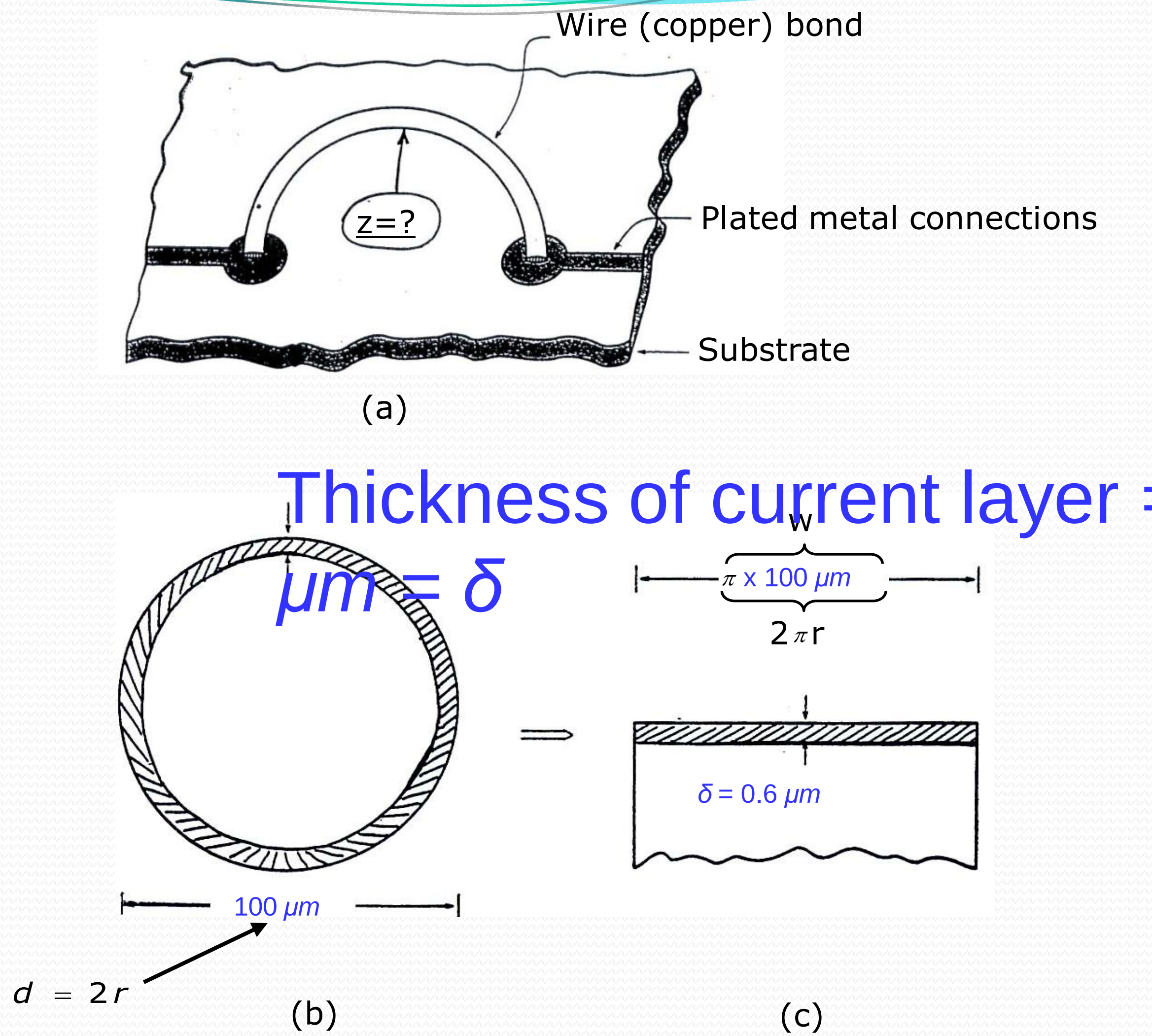
$$Z = Z_S \left(\frac{l}{w} \right) = Z_S \left(\frac{10^4 \mu m}{\pi \cdot 100 \mu m} \right)$$

$$= (0.86 + j\underbrace{0.86}) \Omega$$

$$X = \omega L_{int}$$

$$Z = R + jX$$

$$L_{int} = X / \omega \text{ (Internal inductance)}$$



Finding the resistance and internal inductance of a wire bond. The bond, a length of wire connecting two pads on an IC, is shown in (a). (b) is a cross-sectional view showing skin depth. In (c) we imagine the conducting layer unfolded into a plane.

THANK YOU