COMPLEX ANALYSIS AND PROBABILITY DISTRIBUTION

CONTENTS

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UNIT-I

Complex Functions And Differentiation

Derivative of a complex function

$$f(z) = u(x, y) + iv(x, y) \text{ for } z = x + iy$$

$$f'(z) = \lim_{\Delta z \to 0} \left[\frac{f(z + \Delta z) - f(z)}{\Delta z}\right] \text{ exists}$$

Its value does not depend on the direction.

Ex : Show that the function $f(z) = x^2 - y^2 + i2xy$ is differentiable for all values of z.

for $\Delta z = \Delta x + i\Delta y$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \frac{(x+\Delta x)^2 - (y+\Delta y)^2 + 2i(x+\Delta x)(y+\Delta y) - x^2 + y^2 - 2ixy}{\Delta x + i\Delta y}$$

$$= 2x + i2y + \frac{(\Delta x)^2 - (\Delta y)^2 + 2i\Delta x\Delta y}{\Delta x + i\Delta y}$$

(1) choose $\Delta y = 0, \Delta x \to 0 \Rightarrow f'(z) = 2x + i2y$

(2) choose $\Delta x = 0, \Delta y \to 0 \Rightarrow f'(z) = 2x + i2y$

** Another method :

$$f(z) = (x + iy)^2 = z^2$$

$$f'(z) = \lim_{\Delta z \to 0} \left[\frac{(z + \Delta z)^2 - z^2}{\Delta z}\right] = \lim_{\Delta z \to 0} \left[\frac{(\Delta z)^2 + 2z\Delta z}{\Delta z}\right]$$

$$= \lim_{\Delta z \to 0} \Delta z + 2z = 2z$$

$\text{Ex : Show that the function } f(z) = 2y + ix \text{ is not}$

$\text{differentiable anywhere in the complex plane.}$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{2y + 2\Delta y + ix + i\Delta x - 2y - ix}{\Delta x + i\Delta y} = \frac{2\Delta y + i\Delta x}{\Delta x + i\Delta y}$$

$\text{if } \Delta z \to 0 \text{ along a line thriugh } z \text{ of slope } m \Rightarrow \Delta y = m\Delta x$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta x, \Delta y \to 0} [\frac{2\Delta y + i\Delta x}{\Delta x + i\Delta y}] = \frac{2m + i}{1 + im}$$

$\text{The limit depends on } m \text{ (the direction), so } f(z)$

$\text{is nowhere differentiable.}$

ANALYTIC FUNCTION

Ex : Show that the function $f(z) = 1/(1-z)$ **is analytic everywhere except at** $z = 1$**.**

$$f^{'}(z) = \lim_{\Delta z \to 0}\left[\frac{f(z+\Delta z) - f(z)}{\Delta z}\right] = \lim_{\Delta z \to 0}\left[\frac{1}{\Delta z}\left(\frac{1}{1-z-\Delta z} - \frac{1}{1-z}\right)\right]$$

$$= \lim_{\Delta z \to 0}\left[\frac{1}{(1-z-\Delta z)(1-z)}\right] = \frac{1}{(1-z)^2}$$

Provided $z \neq 1$, $f(z)$ **is analytic everywhere such that** $f^{'}(z)$ **is independent of the direction.**

Cauchy-Riemann relation

A function f(z)=u(x,y)+iv(x,y) is differentiable and analytic, there must be particular connection between u(x,y) and v(x,y)

$$L = \lim_{\Delta z \to 0} [\frac{f(z+\Delta z) - f(z)}{\Delta z}]$$

$$f(z) = u(x,y) + iv(x,y) \qquad \Delta z = \Delta x + i\Delta y$$

$$f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)$$

$$\Rightarrow L = \lim_{\Delta x, \Delta y \to 0} [\frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)}{\Delta x + i\Delta y}]$$

(1) if suppose Δz is real $\Rightarrow \Delta y = 0$

$$\Rightarrow L = \lim_{\Delta x \to 0} [\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}] = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

(2) if suppose Δz is imaginary $\Rightarrow \Delta x = 0$

$$\Rightarrow L = \lim_{\Delta y \to 0} [\frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i\frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}] = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{Cauchy - Riemann relations}$$

Ex : In which domain of the complex plane is $f(z) = |x| - i|y|$ an analytic function?

$$u(x,y) = |x|,\ v(x,y) = -|y|$$

$$(1)\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial}{\partial x}|x| = \frac{\partial}{\partial y}[-|y|] \Rightarrow \text{(a) } x>0,\ y<0 \text{ the fouth quatrant}$$

$$\text{(b) } x<0,\ y>0 \text{ the second quatrant}$$

$$(2)\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow \frac{\partial}{\partial x}[-|y|] = -\frac{\partial}{\partial y}|x|$$

$z = x + iy$ and complex conjugate of z is $z^* = x - iy$

$\Rightarrow x = (z + z^*)/2$ and $y = (z - z^*)/2i$

$$\Rightarrow \frac{\partial f}{\partial z^*} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z^*} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

If $f(z)$ is analytic, then the Cauchy - Riemann relations are satisfied. $\Rightarrow \partial f / \partial z^* = 0$ implies an analytic fonction of z contains the combination of $x + iy$, not $x - iy$

If Cauchy - Riemann relations are satisfied

$$(1)\ \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)=-\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\Rightarrow\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial^2 y^2}=0$$

$$(2)\ \text{the same result for function } v(x,y)\Rightarrow\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial^2 y^2}=0$$

$\Rightarrow u(x,y)$ and $v(x,y)$ are solutions of Laplace's equation in two dimension.

For two families of curves $u(x,y)=$ conctant and $v(x,y)=$ constant, the normal vectors correspond ing the two curves, respectively, are

$$\vec{\nabla}u(x,y)=\frac{\partial u}{\partial x}\hat{i}+\frac{\partial u}{\partial y}\hat{j}\ \text{ and }\ \vec{\nabla}v(x,y)=\frac{\partial v}{\partial x}\hat{i}+\frac{\partial v}{\partial y}\hat{j}$$

$$\vec{\nabla}u\cdot\vec{\nabla}v=\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}=-\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}=0\ \text{ orthogonal}$$

UNIT-II

COMPLEX INTEGRATION

Isolated singularity (pole): $f(z) = \dfrac{g(z)}{(z - z_0)^n}$

n **is a positive integer,** $g(z)$ **is analytic at all points in some neighborhood containing** $z = z_0$ **and** $g(z_0) \neq 0$**, the** $f(z)$ **has a pole of order** n **at** $z = z_0$**.**

** **An alternate definition for that** $f(z)$ **has a pole of order** n **at** $z = z_0$ **is**

$$\lim_{z \to z_0} [(z - z_0)^n f(z)] = a$$

$f(z)$ **is analytic and** a **is a finite, non-zero complex number**

(1) if $a = 0$**, then** $z = z_0$ **is a pole of order less than** n**.**

(2) if a **is infinite, then** $z = z_0$ **is a pole of order greater than** n**.**

(3) if $z = z_0$ **is a pole of** $f(z) \Rightarrow |f(z)| \to \infty$ **as** $z \to z_0$

(4) from any direction, if no finite n **satisfies the limit** $\Rightarrow$ **essential singularity**

Ex : Find the singularities of the function

$$(1)\ f(z)=\frac{1}{1-z}-\frac{1}{1+z}$$

$$\Rightarrow f(z)=\frac{2z}{(1-z)(1+z)} \text{ poles of order 1 at } z=1 \text{ and } z=-1$$

$$(2)\ f(z)=\tanh z$$

$$=\frac{\sinh z}{\cosh z}=\frac{\exp z-\exp(-z)}{\exp z+\exp(-z)}$$

$f(z)$ has a singularity when $\exp z=-\exp(-z)$

$$\Rightarrow \exp z=\exp[i(2n+1)\pi]=\exp(-z) \text{ n is any integer}$$

$$\Rightarrow 2z=i(2n+1)\pi \Rightarrow z=(n+\frac{1}{2})\pi i$$

Using l'Hospital's rule

$$\lim_{z\to(n+1/2)\pi i}\{\frac{[z-(n+1/2)\pi i]\sinh z}{\cosh z}\}=\lim_{z\to(n+1/2)\pi i}\{\frac{[z-(n+1/2)\pi i]\cosh z+\sinh z}{\sinh z}\}=1$$

each singularity is a simple pole (n = 1)

Remove singularities:

Singularity makes the value of $f(z)$ undetermined, but $\lim_{z \to z_0} f(z)$ exists and independent of the direction from which z_0 is approached.

Ex: Show that $f(z) = \sin z / z$ is a removable singularity at $z = 0$

***Sol*: $\lim_{z \to 0} f(z) = 0/0$ undetermined**

$$f(z) = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\ldots..\right) = 1 - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots.$$

$\lim_{z \to 0} f(z) = 1$ is independent of the way $z \to 0$, so

$f(z)$ has a removable singularity at $z = 0$.

The behavior of $f(z)$ at infinity is given by that of $f(1/\xi)$ at $\xi = 0$, where $\xi = 1/z$

Ex : Find the behavior at infinity of (i) $f(z) = a + bz^{-2}$
(ii) $f(z) = z(1+z^2)$ and (iii) $f(z) = \exp z$

(i) $f(z) = a + bz^{-2} \Rightarrow$ **set** $z = 1/\xi \Rightarrow f(1/\xi) = a + b\xi^2$
is analytic at $\xi = 0 \Rightarrow f(z)$ **is analytic at** $z = \infty$

(ii) $f(z) = z(1-z^2) \Rightarrow f(1/\xi) = 1/\xi + 1/\xi^3$ **has a pole of order 3 at** $z = \infty$

(iii) $f(z) = \exp z \Rightarrow f(1/\xi) = \sum_{n=0}^{\infty} (n!)^{-1} \xi^{-n}$

$f(z)$ **has an essential singularity at** $z = \infty$

If $f(z_0) = 0$ and $f(z) = (z - z_0)^n g(z)$, if n is a positive integer, and $g(z_0) \neq 0$

(i) $z = z_0$ is called a zero of order n.

(ii) if $n = 1$, $z = z_0$ is called a simple zero.

(iii) $z = z_0$ is also a pole of order n of $1/f(z)$

Ex : Show that $f(z) = \sin z / z$ is a removable singularity at $z = 0$

Sol : $\lim_{z \to 0} f(z) = 0/0$ undetermined

$$f(z) = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\dots\dots\right) = 1 - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots.$$

$\lim_{z \to 0} f(z) = 1$ is independent of the way $z \to 0$, so

$f(z)$ has a removable singularity at $z = 0$.

Ex : Evaluate the complex integral of $f(z) = 1/z$, along the circle $|z| = R$, starting and finishing at $z = R$.

$$z(t) = R\cos t + iR\sin t,\ 0 \le t \le 2\pi$$

$$\frac{dx}{dt} = -R\sin t,\ \frac{dy}{dt} = R\cos t,\ f(z) = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = u+iv,$$

$$u = \frac{x}{x^2+y^2} = \frac{\cos t}{R},\ v = \frac{-y}{x^2+y^2} = \frac{-\sin t}{R}$$

$$\int_{C_1} \frac{1}{z}dz = \int_0^{2\pi} \frac{\cos t}{R}(-R\sin t)dt - \int_0^{2\pi} \left(\frac{-\sin t}{R}\right)R\cos t\,dt$$

$$+ i\int_0^{2\pi} \frac{\cos t}{R}R\cos t\,dt + i\int_0^{2\pi} \left(\frac{-\sin t}{R}\right)(-R\sin t)dt$$

$$= 0 + 0 + i\pi + i\pi = 2\pi i$$

**** The integral is also calculated by**

$$\int_{C_1} \frac{dz}{z} = \int_0^{2\pi} \frac{-R\sin t + iR\cos t}{R\cos t + iR\sin t}dt = \int_0^{2\pi} i\,dt = 2\pi i$$

The calculated result is independent of R.

Ex : Evaluate the complex integral of $f(z) = \mathrm{Re}(z)$ along the path C_1, C_2 and C_3 as shown in the previous examples.

(i) $C_1 : \int_0^{2\pi} R\cos t(-R\sin t + iR\cos t)dt = i\pi R^2$

(ii) $C_2 : \int_0^{\pi} R\cos t(-R\sin t + iR\cos t)dt = \frac{i\pi}{2}R^2$

(iii) $C_3 = C_{3a} + C_{3b}$:

$$\int_0^1 (1-t)R(-R+iR)dt + \int_0^1 (-sR)(-R-iR)ds$$

$$= R^2\int_0^1 (1-t)(-1+i)dt + R^2\int_0^1 s(1+i)ds$$

$$= \frac{1}{2}R^2(-1+i) + \frac{1}{2}R^2(1+i) = iR^2$$

The integral depends on the different path.

Ex : Consider two closed contour C and γ in the Argand diagram, γ being sufficiently small that it lies completely with C. Show that if the function $f(z)$ is analytic in the region between the two contours then $\oint_C f(z)dz = \oint_\gamma f(z)dz$

the area is bounded by Γ, and

$f(z)$ is analytic

$$\oint_\Gamma f(z)dz = 0$$

$$= \oint_C f(z)dz + \oint_\gamma f(z)dz + \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz$$

If take the direction of contour γ as that of contour $C \Rightarrow \oint_C f(z)dz = \oint_\gamma f(z)dz$

Morera's theorem :

if $f(z)$ is a continuous function of z in a closed domain R bounded by a curve C, for $\oint_C f(z)dz = 0 \Rightarrow f(z)$ is analytic.

Cauchy's integral formula

If $f(z)$ is analytic within and on a closed contour C

and z_0 is a point within C then $f(z_0) = \dfrac{1}{2\pi i}\oint_C \dfrac{f(z)}{z - z_0}\,dz$

$$I = \oint_C \frac{f(z)}{z - z_0}dz = \oint_\gamma \frac{f(z)}{z - z_0}dz$$

for $z = z_0 + \rho\exp(i\theta),\; dz = i\rho\exp(i\theta)d\theta$

$$I = \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{\rho e^{i\theta}} i\rho e^{i\theta} d\theta$$

$$= i\int_0^{2\pi} f(z_0 + \rho e^{i\theta})d\theta \overset{\rho\to 0}{=} 2\pi i f(z_0)$$

The integral form of the derivative of a complex function :

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz$$

$$f'(z_0) = \lim_{h\to 0}\frac{f(z_0+h) - f(z_0)}{h}$$

$$= \lim_{h\to 0}\left[\frac{1}{2\pi i}\oint_C \frac{f(z)}{h}\left(\frac{1}{z-z_0-h} - \frac{1}{z-z_0}\right)dz\right]$$

$$= \lim_{h\to 0}\left[\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0-h)(z-z_0)}dz\right]$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz$$

For nth derivative $f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$

UNIT-III

POWER SERIES EXPANSION OF COMPLEX FUNCTION

Taylor and Laurent series

If $f(z)$ is analytic inside and on a circle C of radius R centered on the point $z = z_0$, and z is a point inside C, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

$f(z)$ is analytic inside and on C, so $f(z) = \dfrac{1}{2\pi i}\oint_C \dfrac{f(\xi)}{\xi - z} d\xi$ **where ξ lies on C**

expand $\dfrac{1}{\xi - z}$ **as a geometric series in** $\dfrac{z - z_0}{\xi - z_0} \Rightarrow \dfrac{1}{\xi - z} = \dfrac{1}{\xi - z_0}\sum_{n=0}^{\infty}\left(\dfrac{z - z_0}{\xi - z_0}\right)^n$

$$\Rightarrow f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{\xi - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{\xi - z_0}\right)^n d\xi = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z - z_0)^n \oint_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z - z_0)^n \frac{2\pi i f^{(n)}(z_0)}{n!} = \sum_{n=0}^{\infty}(z - z_0)^n \frac{f^{(n)}(z_0)}{n!}$$

If $f(z)$ has a pole of order p at $z = z_0$ but is analytic at every other point inside and on C. Then $g(z) = (z - z_0)^p f(z)$ is analytic at $z = z_0$ and expanded as a Taylor series $g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$.

Thus, for all z inside C $f(z)$ can be expanded as a Laurent series

$$f(z) = \frac{a_{-p}}{(z - z_0)^p} + \frac{a_{-p+1}}{(z - z_0)^{p-1}} + \dots\dots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2$$

$$a_n = b_{n+p} \quad \text{and} \quad b_n = \frac{g^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\oint \frac{g(z)}{(z - z_0)^{n+1}} dz$$

$$\Rightarrow a_n = \frac{1}{2\pi i}\oint \frac{g(z)}{(z - z_0)^{n+1+p}} dz = \frac{1}{2\pi i}\oint \frac{f(z)}{(z - z_0)^{n+1}} dz$$

$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ is analytic in a region R between two circles C_1 and C_2 centered on $z = z_0$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

(1) If $f(z)$ is analytic at $z = z_0$, then all $a_n = 0$ for $n < 0$.

It may happen $a_n = 0$ for $n \geq 0$, the first non-vanishing term is $a_m (z\text{-}z_0)^m$ with $m > 0$, $f(z)$ is said to have a zero of order m at $z = z_0$.

(2) If $f(z)$ is not analytic at $z = z_0$

(i) possible to find $a_{-p} \neq 0$ but $a_{-p-k} = 0$ for all $k > 0$

$f(z)$ has a pole of order p at $z = z_0$, a_{-1} is called the residue of $f(z)$

(ii) impossible to find a lowest value of $-p \Rightarrow$ essential singularity

Ex : Find the Laurent series of $f(z)=\dfrac{1}{z(z-2)^3}$ about the singularities $z=0$ and $z=2$. Hence verify that $z=0$ is a pole of order 1 and $z=2$ is a pole of order 3, and find the residue of $f(z)$ at each pole.

(1) point $z=0$

$$f(z)=\frac{-1}{8z(1-z/2)^3}=\frac{-1}{8z}[1+(-3)(\frac{-z}{2})+\frac{(-3)(-4)}{2!}(\frac{-z}{2})^2+\frac{(-3)(-4)(-5)}{3!}(\frac{-z}{2})^3+\ldots]$$

$$=-\frac{1}{8z}-\frac{3}{16}-\frac{3}{16}z-\frac{5z^2}{32}-\ldots \qquad z=0 \text{ is a pole of order } 1$$

(2) point $z=2 \Rightarrow$ set $z-2=\xi \Rightarrow z(z-2)^3=(2+\xi)\xi^3=2\xi^3(1+\xi/2)$

$$f(z)=\frac{1}{2\xi^3(1+\xi/2)}=\frac{1}{2\xi^3}[1-(\frac{\xi}{2})+(\frac{\xi}{2})^2-(\frac{\xi}{2})^3+(\frac{\xi}{2})^4-\ldots]$$

$$=\frac{1}{2\xi^3}-\frac{1}{4\xi^2}+\frac{1}{8\xi}-\frac{1}{16}+\frac{\xi}{32}-..=\frac{1}{2(z-2)^3}-\frac{1}{4(z-2)^2}+\frac{1}{8(z-2)}-\frac{1}{16}+\frac{z-2}{32}-$$

$z=2$ is a pole of order 3, the residue of $f(z)$ at $z=2$ is $1/8$.

How to obtain the residue ?

$$f(z)=\frac{a_{-m}}{(z-z_0)^m}+......+\frac{a_{-1}}{(z-z_0)}+a_0+a_1(z-z_0)+a_2(z-z_0)^2+...$$

$$\Rightarrow (z-z_0)^m f(z)=a_{-m}+a_{-m+1}(z-z_0)+.......+a_{-1}(z-z_0)^{m-1}+...$$

$$\Rightarrow \frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]=(m-1)!a_{-1}+\sum_{n=1}^{\infty}b_n(z-z_0)^n$$

Take the limit $z \to z_0$

$$R(z_0)=a_{-1}=\lim_{z\to z_0}\{\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\} \text{ residue at } z=z_0$$

(1) For a simple pole $m=1 \Rightarrow R(z_0)=\lim_{z\to z_0}[(z-z_0)f(z)]$

(2) If $f(z)$ has a simple at $z=z_0$ and $f(z)=\dfrac{g(z)}{h(z)}$, $g(z)$ is analytic and

non - zero at z_0 and $h(z_0)=0$

$$\Rightarrow R(z_0)=\lim_{z\to z_0}\frac{(z-z_0)g(z)}{h(z)}=g(z_0)\lim_{z\to z_0}\frac{(z-z_0)}{h(z)}=g(z_0)\lim_{z\to z_0}\frac{1}{h'(z)}=\frac{g(z_0)}{h'(z_0)}$$

Ex : Suppose that $f(z)$ has a pole of order m at the point $z = z_0$. By considering the Laurent series of $f(z)$ about z_0, deriving a general expression for the residue $R(z_0)$ of $f(z)$ at $z = z_0$. Hence evaluate the residue of the function $f(z) = \dfrac{\exp iz}{(z^2+1)^2}$ at the point $z = i$.

$$f(z) = \frac{\exp iz}{(z^2+1)^2} = \frac{\exp iz}{(z+i)^2(z-i)^2}$$ **poles of order 2 at $z = i$ and $z = -i$**

for pole at $z = i$:

$$\frac{d}{dz}[(z-i)^2 f(z)] = \frac{d}{dz}\left[\frac{\exp iz}{(z+i)^2}\right] = \frac{i}{(z+i)^2}\exp iz - \frac{2}{(z+i)^3}\exp iz$$

$$R(i) = \frac{1}{1!}\left[\frac{i}{(2i)^2}e^{-1} - \frac{2}{(2i)^3}e^{-1}\right] = \frac{-i}{2e}$$

Residue theorem

$f(z)$ has a pole of order m at $z = z_0$

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n$$

$$I = \oint_C f(z)dz = \oint_\gamma f(z)dz$$

set $z = z_0 + \rho e^{i\theta} \Rightarrow dz = i\rho e^{i\theta} d\theta$

$$I = \sum_{n=-m}^{\infty} a_n \oint_C (z - z_0)^n dz = \sum_{n=-m}^{\infty} a_n \int_0^{2\pi} i\rho^{n+1} e^{i(n+1)\theta} d\theta$$

for $n \neq -1 \Rightarrow \int_0^{2\pi} i\rho^{n+1} e^{i(n+1)\theta} d\theta = \dfrac{i\rho^{n+1} e^{i(n+1)\theta}}{i(n+1)} \Big|_0^{2\pi} = 0$

for $n = 1 \Rightarrow \int_0^{2\pi} i d\theta = 2\pi i$

$$I = \oint_C f(z)dz = 2\pi i a_{-1}$$

$f(z)$ is continuous within and on a closed contour C

and analytic, except for a finite number of poles within C

$$\oint_C f(z)dz = 2\pi i \sum_j R_j$$

$\sum_j R_j$ is the sum of the residues of $f(z)$ at its poles within C

The integral I of $f(z)$ along an open contour C

if $f(z)$ has a simple pole at $z = z_0$

$\Rightarrow f(z) = \phi(z) + a_{-1}(z - z_0)^{-1}$

$\phi(z)$ is analytic within some neighbour surroundin g z_0

$|z - z_0| = \rho$ and $\theta_1 \le \arg(z - z_0) \le \theta_2$

ρ is chosen small enough that no singularit y of $f(z)$ except $z = z_0$

$$I = \int_C f(z)dz = \int_C \phi(z)dz + a_{-1}\int_C (z - z_0)^{-1} dz$$

$$\lim_{\rho\to 0}\int_C \phi(z)dz = 0$$

$$I = \lim_{\rho\to 0}\int_C f(z)dz = \lim_{\rho\to 0}\left(a_{-1}\int_{\theta_1}^{\theta_2}\frac{1}{\rho e^{i\theta}} i\rho e^{i\theta} d\theta\right) = ia_{-1}(\theta_2 - \theta_1)$$

for a closed contour $\theta_2 = \theta_1 + 2\pi \Rightarrow I = 2\pi i a_{-1}$

Integrals of sinusoidal functions

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta \quad \text{set } z = \exp i\theta \text{ in unit circle}$$

$$\Rightarrow \cos\theta = \frac{1}{2}(z+\frac{1}{z}), \quad \sin\theta = \frac{1}{2i}(z-\frac{1}{z}), \quad d\theta = -iz^{-1}dz$$

$$\text{Ex: Evaluate } I = \int_0^{2\pi} \frac{\cos 2\theta}{a^2+b^2-2ab\cos\theta}d\theta \quad \text{for } b > a > 0$$

$$\cos n\theta = \frac{1}{2}(z^n + z^{-n}) \Rightarrow \cos 2\theta = \frac{1}{2}(z^2+z^{-2})$$

$$\frac{\cos 2\theta}{a^2+b^2-2ab\cos\theta}d\theta = \frac{\frac{1}{2}(z^2+z^{-2})(-iz^{-1})dz}{a^2+b^2-2ab\cdot\frac{1}{2}(z+z^{-1})} = \frac{-\frac{1}{2}(z^4+1)idz}{z^2(za^2+zb^2-abz^2-ab)}$$

$$= \frac{i}{2ab}\frac{(z^4+1)dz}{z^2(z^2-z(\frac{a}{b}-+\frac{b}{a})+1)} = \frac{i}{2ab}\frac{(z^4+1)}{z^2(z-\frac{a}{b})(z-\frac{b}{a})}dz$$

$$I = \frac{i}{2ab}\oint_C \frac{z^4+1}{z^2(z-\frac{a}{b})(z-\frac{b}{a})}dz$$ double poles at $z=0$ and $z=a/b$ within the unit circle

Residue: $R(z_0) = \lim_{z\to z_0}\{\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]$

(1) pole at $z=0, m=2$

$$R(0) = \lim_{z\to 0}\{\frac{1}{1!}\frac{d}{dz}[z^2\frac{z^4+1}{z^2(z-a/b)(z-b/a)}]\}$$

$$= \lim_{z\to 0}\{\frac{4z^3}{(z-a/b)(z-b/a)} + \frac{(z^4+1)(-1)[2z-(a/b+b/a)]}{(z-a/b)^2(z-b/a)^2}\} = a/b+b/a$$

(2) pole at $z=a/b, m=1$

$$R(a/b) = \lim_{z\to a/b}[(z-a/b)\frac{z^4+1}{z^2(z-a/b)(z-b/a)}] = \frac{(a/b)^4+1}{(a/b)^2(a/b-b/a)} = \frac{-(a^4+b^4)}{ab(b^2-a^2)}$$

$$I = 2\pi i\times\frac{i}{2ab}[\frac{a^2+b^2}{ab}-\frac{a^4+b^4}{ab(b^2-a^2)}] = \frac{2\pi a^2}{b^2(b^2-a^2)}$$

Some infinite integrals

$$\int_{-\infty}^{\infty} f(x)dx$$

$f(z)$ has the following properties :

(1) $f(z)$ is analytic in the upper half - plane, $\operatorname{Im} z \geq 0$, except for a finite number of poles, none of which is on the real axis.

(2) on a semicircle Γ of radius R, R times the maximum of $|f|$ on Γ tends to zero as $R \to \infty$ (a sufficient condition is that $zf(z) \to 0$ as $|z| \to \infty$).

(3) $\int_{-\infty}^{0} f(x)dx$ and $\int_{0}^{\infty} f(x)dx$ both exist

$$\Rightarrow \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{j} R_j$$

for $\left|\int_{\Gamma} f(z)dz\right| \leq 2\pi R \times$ (maximum of $|f|$ on Γ), the integral along Γ tends to zero as $R \to \infty$.

$$\text{Ex : Evaluate } I=\int_0^{\infty}\frac{dx}{(x^2+a^2)^4}\quad a \text{ is real}$$

$$\oint_C \frac{dz}{(z^2+a^2)^4}=\int_{-R}^{R}\frac{dx}{(x^2+a^2)^4}+\int_{\Gamma}\frac{dz}{(z^2+a^2)^4}\ \text{ as } R\to\infty$$

$$\Rightarrow\int_{\Gamma}\frac{dz}{(z^2+a^2)^4}\to 0\Rightarrow\oint_C\frac{dz}{(z^2+a^2)^4}=\int_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)^4}$$

$(z^2+a^2)^4=0\Rightarrow$ poles of order 4 at $z=\pm ai$,

only $z=ai$ at the upper half - plane

$$\text{set } z=ai+\xi,\ \xi\to 0\Rightarrow\frac{1}{(z^2+a^2)^4}=\frac{1}{(2ai\xi+\xi^2)^4}=\frac{1}{(2ai\xi)^4}\left(1-\frac{i\xi}{2a}\right)^{-4}$$

$$\text{the coefficient of } \xi^{-1} \text{ is } \frac{1}{(2a)^4}\frac{(-4)(-5)(-6)}{3!}\left(\frac{-i}{2a}\right)^3=\frac{-5i}{32a^7}$$

$$\int_0^{\infty}\frac{dx}{(x^2+a^2)^4}=2\pi i\left(\frac{-5i}{32a^7}\right)=\frac{10\pi}{32a^7}\Rightarrow I=\frac{1}{2}\times\frac{10\pi}{32a^7}=\frac{5\pi}{32a^7}$$

For poles on the real axis:

Principal value of the integral, defined as $\rho \to 0$

$$P\int_{-R}^{R} f(x)dx = \int_{-R}^{z_0-\rho} f(x)dx + \int_{z_0+\rho}^{R} f(x)dx$$

for a closed contour C

$$\oint_C f(z)dz = \int_{-R}^{z_0-\rho} f(x)dx + \int_\gamma f(z)dz + \int_{z_0+\rho}^{R} f(x)dx + \int_\Gamma f(z)dz$$

$$= P\int_{-R}^{R} f(x)dx + \int_\gamma f(z)dz + \int_\Gamma f(z)dz$$

(1) for $\int_\gamma f(z)dz$ has a pole at $z = z_0 \Rightarrow \int_\gamma f(z)dz = -\pi i a_1$

(2) for $\int_\Gamma f(z)dz$ set $z = \mathrm{R}e^{i\theta}$ $dz = i\,\mathrm{R}e^{i\theta}\,d\theta$

$$\Rightarrow \int_\Gamma f(z)dz = \int_\Gamma f(\mathrm{R}e^{i\theta})i\,\mathrm{R}e^{i\theta}\,d\theta$$

If $f(z)$ vanishes faster than $1/R^2$ as $R \to \infty$, the integral is zero

Ex : Find the principal value of $\int_{-\infty}^{\infty} \frac{\cos mx}{x-a} dx$ a real, $m > 0$

Consider the integral $I = \oint_C \frac{e^{imz}}{z-a} dz = 0$ no pole in the

upper half - plane, and $|(z-a)^{-1}| \to 0$ as $|z| \to \infty$

$$I = \oint_C \frac{e^{imz}}{z-a} dz$$

$$= \int_{-R}^{a-\rho} \frac{e^{imx}}{x-a} dx + \oint_{\gamma} \frac{e^{imz}}{z-a} dz + \int_{a+\rho}^{R} \frac{e^{imx}}{x-a} dx + \int_{\Gamma} \frac{e^{imz}}{z-a} dz = 0$$

As $R \to \infty$ and $\rho \to 0 \Rightarrow \int_{\Gamma} \frac{e^{imz}}{z-a} dz \to 0$

$$\Rightarrow P\int_{-\infty}^{\infty} \frac{e^{imx}}{x-a} dx - i\pi a_{-1} = 0 \text{ and } a_{-1} = e^{ima}$$

$$\Rightarrow P\int_{-\infty}^{\infty} \frac{\cos mx}{x-a} dx = -\pi \sin ma \text{ and } P\int_{-\infty}^{\infty} \frac{\sin mx}{x-a} dx = \pi \cos ma$$

Integral of multivalued functions

Multivalue d functions such as $z^{1/2}, Lnz$

Single branch point is at the otigin. We let $R \to \infty$

and $\rho \to 0$. The integrand is multivalue d, its values

along two lines AB and CD joining $z = \rho$ to $z = R$

are not equal and opposite.

$$\text{Ex}: I = \int_0^\infty \frac{dx}{(x+a)^3 x^{1/2}} \quad \text{for } a > 0$$

(1) the integrand $f(z) = (z+a)^{-3} z^{-1/2}, |zf(z)| \to 0$ as $\rho \to 0$ and $R \to \infty$

the two circles make no contributi on to the contour integral

(2) pole at $z = -a$, and $(-a)^{1/2} = a^{1/2} e^{i\pi/2} = ia^{1/2}$

$$R(-a) = \lim_{z \to -a} \frac{1}{(3-1)!} \frac{d^{3-1}}{dz^{3-1}} [(z+a)^3 \frac{1}{(z+a)^3 z^{1/2}}]$$

$$= \lim_{z \to -a} \frac{1}{2!} \frac{d^2}{dz^2} z^{-1/2} = \frac{-3i}{8a^{5/2}}$$

$$\int_{AB} dz + \int_{\Gamma} dz + \int_{DC} dz + \int_{\gamma} dz = 2\pi i\left(\frac{-3i}{8a^{5/2}}\right)$$

and $\int_{\Gamma} dz = 0$ and $\int_{\gamma} dz = 0$

along line AB $\Rightarrow z = xe^{i0}$, along line CD $\Rightarrow z = xe^{i2\pi}$

$$\int_{0, A\to B}^{\infty} \frac{dx}{(x+a)^3 x^{1/2}} + \int_{\infty, C\to D}^{0} \frac{dx}{(xe^{i2\pi} + a)^3 x^{1/2} e^{(1/2\times 2\pi i)}} = \frac{3\pi}{4a^{5/2}}$$

$$\Rightarrow \left(1 - \frac{1}{e^{i\pi}}\right)\int_{0}^{\infty} \frac{dx}{(x+a)^3 x^{1/2}} = \frac{3\pi}{4a^{5/2}}$$

$$\Rightarrow \int_{0}^{\infty} \frac{dx}{(x+a)^3 x^{1/2}} = \frac{3\pi}{8a^{5/2}}$$

UNIT-IV

SINGLE RANDOM VARIABLES

Basic Concepts

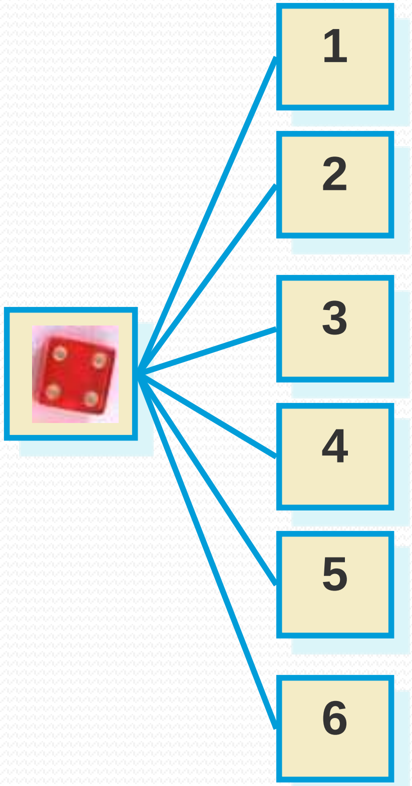
- An **experiment** is the process by which an observation (or measurement) is obtained.
- **Experiment: Record an age**
- **Experiment: Toss a die**
- **Experiment: Record an opinion (yes, no)**
- **Experiment: Toss two coins**

- A **simple event** is the outcome that is observed on a single repetition of the experiment.
  - The basic element to which probability is applied.
  - One and only one simple event can occur when the experiment is performed.
- A **simple event** is denoted by E with a subscript.

- Each simple event will be assigned a probability, measuring “how often” it occurs.
- The set of all simple events of an experiment is called the **sample space, S.**

Example

- **The die toss:**
- Simple events:



E_1

E_2

E_3

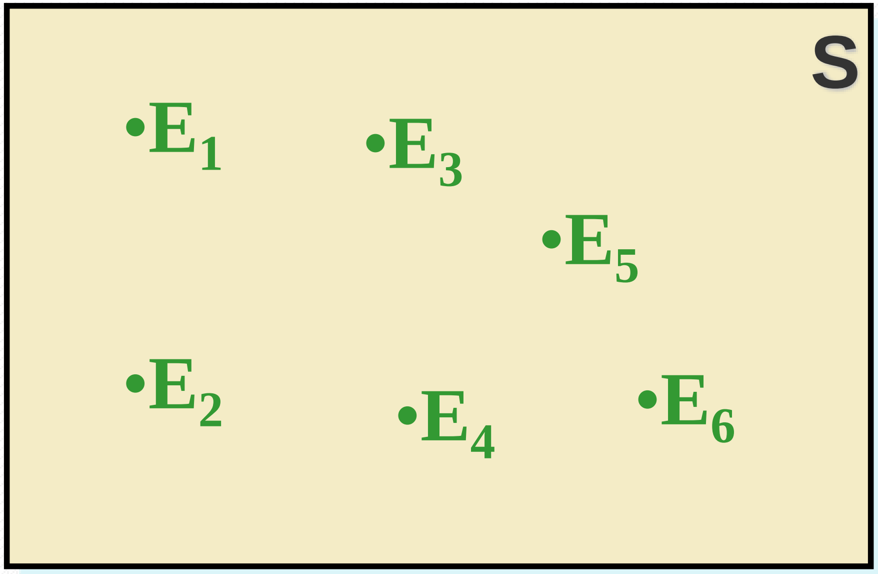
E_4

E_5

E_6

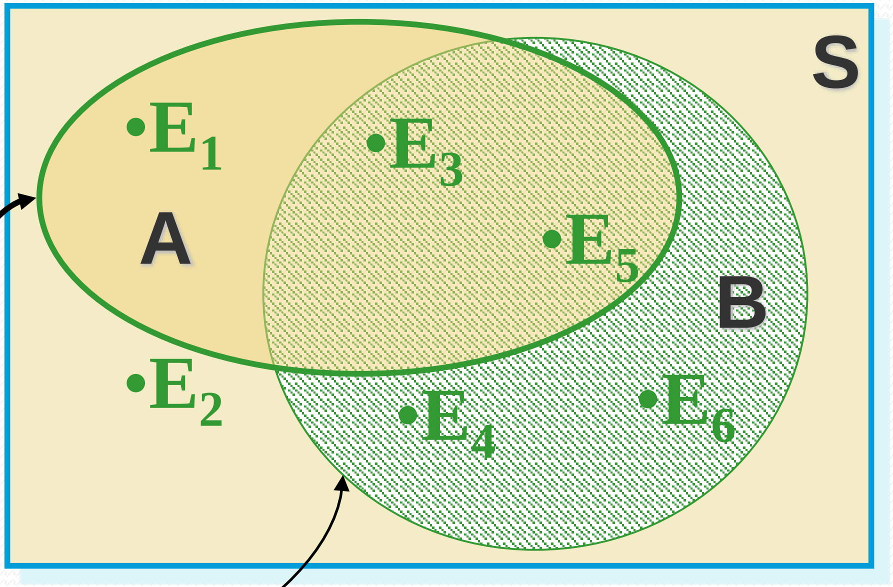
Sample space:

$$S = \{E_1, E_2, E_3, E_4, E_5, E_6\}$$



- An **event** is a collection of one or more **simple events.**

- **The die toss:**
  - –A: an odd number
  - –B: a number > 2



A ={E_1, E_3, E_5}

B ={E_3, E_4, E_5, E_6}

- Two events are **mutually exclusive** if, when one event occurs, the other cannot, and vice versa.

- **Experiment: Toss a die**
  - A: observe an odd number
  - B: observe a number greater than 2
  - C: observe a 6
  - D: observe a 3

Not Mutually Exclusive

Mutually Exclusive

B and C?
B and D?

- The probability of an event A measures "how often" we think A will occur. We write **P(A)**.
- Suppose that an experiment is performed n times. The relative frequency for an event A is

$$\frac{\text{Number of times A occurs}}{n} = \frac{f}{n}$$

- If we let n get infinitely large,

$$\boxed{P(A) = \lim_{n \to \infty} \frac{f}{n}}$$

- P(A) must be between 0 and 1.
  - If event A can never occur, P(A) = 0. If event A always occurs when the experiment is performed, P(A) =1.
- The sum of the probabilities for all simple events in S equals 1.

•The **probability of an event A** is found by adding the probabilities of all the simple events contained in A.

Finding Probabilities

- Probabilities can be found using
  - Estimates from empirical studies
  - Common sense estimates based on equally likely events.

- **Examples:**
  - –Toss a fair coin. P(Head) = 1/2
  - –10% of the U.S. population has red hair. Select a person at random. P(Red hair) = .10

Example

- Toss a fair coin twice. What is the probability of observing at least one head?

1st Coin	2nd Coin	E_i	$P(E_i)$
H	H	HH	1/4
	T	HT	1/4
T	H	TH	1/4
	T	TT	1/4

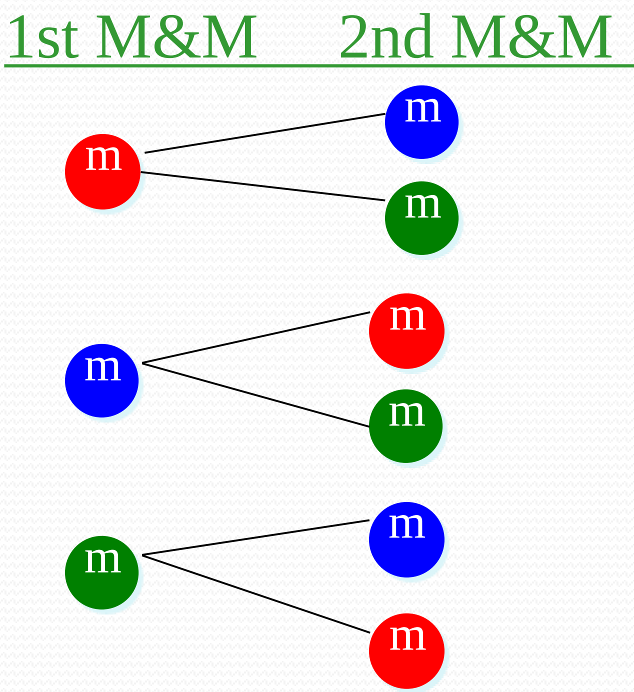
P(at least 1 head)

$= P(E_1) + P(E_2) + P(E_3)$

$= 1/4 + 1/4 + 1/4 = 3/4$

Example

- A bowl contains three M&Ms®, one red, one blue and one green. A child selects two M&Ms at random. What is the probability that at least one is red?



E_i	$P(E_i)$
RB	1/6
RG	1/6
BR	1/6
BG	1/6
GB	1/6
GR	1/6

P(at least 1 red)

= P(RB) + P(BR)+ P(RG) + P(GR)

= 4/6 = 2/3

Counting Rules

- If the simple events in an experiment are **equally likely**, you can calculate

$$P(A) = \frac{n_A}{N} = \frac{\text{number of simple events in A}}{\text{total number of simple events}}$$

- You can use **counting rules** to find n_A and N.

The *mn* Rule

- If an experiment is performed in two stages, with ***m*** ways to accomplish the first stage and ***n*** ways to accomplish the second stage, then there are ***mn*** ways to accomplish the experiment.
- This rule is easily extended to ***k*** stages, with the number of ways equal to

$$n_1 n_2 n_3 \dots n_k$$

Example: Toss two coins. The total number of simple events is:

$$2 \times 2 = 4$$

Examples

Example: Toss three coins. The total number of simple events is

$2 \times 2 \times 2 = 8$

Example: Toss two dice. The total number of simple events is:

$6 \times 6 = 36$

Example: Two M&Ms are drawn from a dish containing two red and two blue candies. The total number of simple eve

$4 \times 3 = 12$

Permutations

- The number of ways you can arrange ***n*** distinct objects, taking them ***r*** at a time is

$$P_r^n = \frac{n!}{(n-r)!}$$

where $n! \equiv n(n-1)(n-2)\ldots(2)(1)$ and $0! \equiv 1.$

Example: How many 3-digit lock combinations can we make from the numbers 1, 2, 3, and 4?

The order of the choice is important!

$$P_3^4 = \frac{4!}{1!} = 4(3)(2) = 24$$

Combinations

- The number of distinct combinations of ***n*** distinct objects that can be formed, taking them ***r*** at a time is

$$C_r^n = \frac{n!}{r!(n-r)!}$$

Example: Three members of a 5-person committee must be chosen to form a subcommittee. How many different subcommittees could be formed?

The order of the choice is not important!

$$C_3^5 = \frac{5!}{3!(5-3)!} = \frac{5(4)(3)(2)1}{3(2)(1)(2)1} = \frac{5(4)}{(2)1} = 10$$

Example

- A box contains six M&Ms®, four red
- and two green. A child selects two M&Ms at random. What is the probability that exactly one is red?

The order of the choice is not important!

$$C_2^6 = \frac{6!}{2!4!} = \frac{6(5)}{2(1)} = 15$$

ways to choose 2 M & Ms.

$$C_1^2 = \frac{2!}{1!1!} = 2$$

ways to choose 1 green M & M.

$$C_1^4 = \frac{4!}{1!3!} = 4$$

ways to choose 1 red M & M.

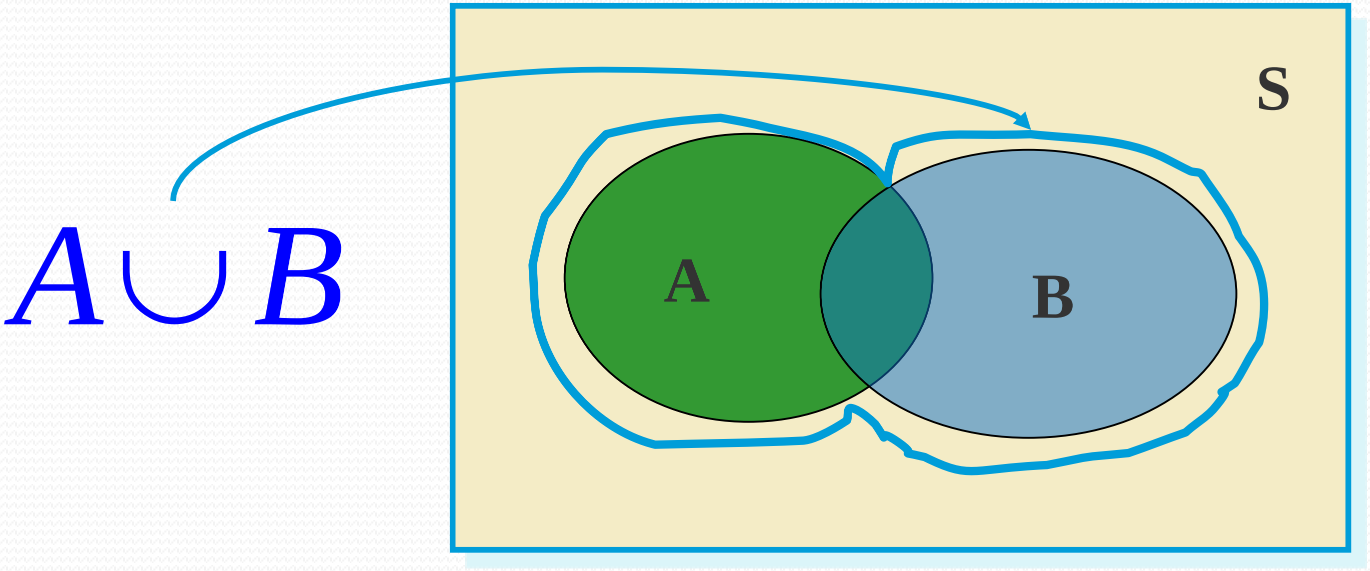
4 × 2 = 8 ways to choose 1 red and 1 green M&M.

P(exactly one red) = 8/15

Event Relations

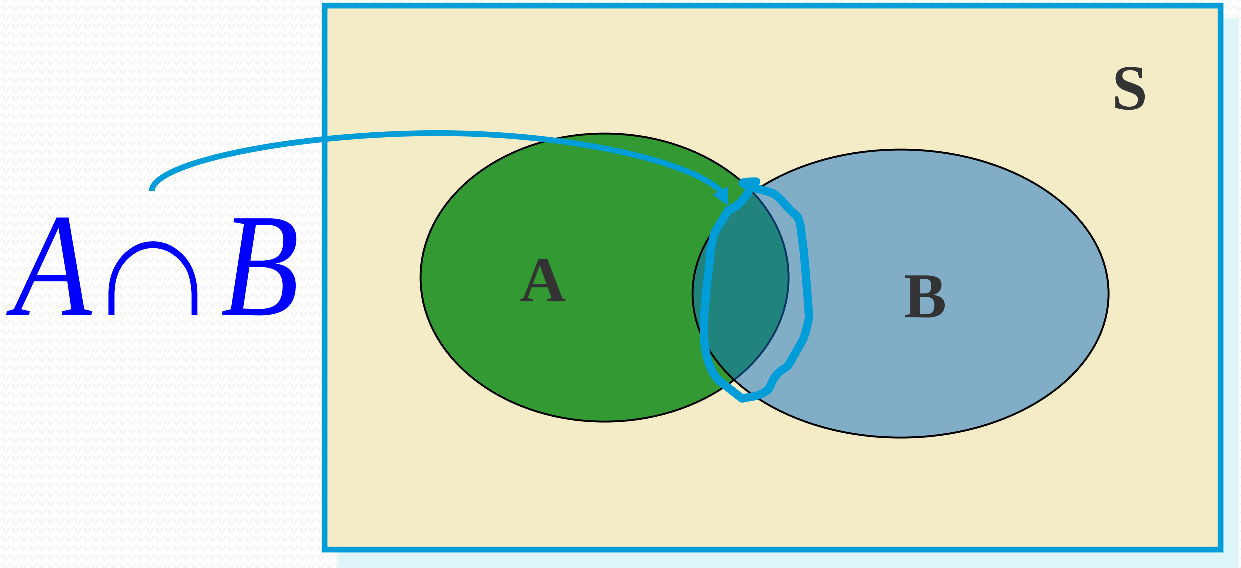
- The **union** of two events, A and B, is the event that either A **or** B **or both** occur when the experiment is performed. We write

$$A \cup B$$



Event Relations

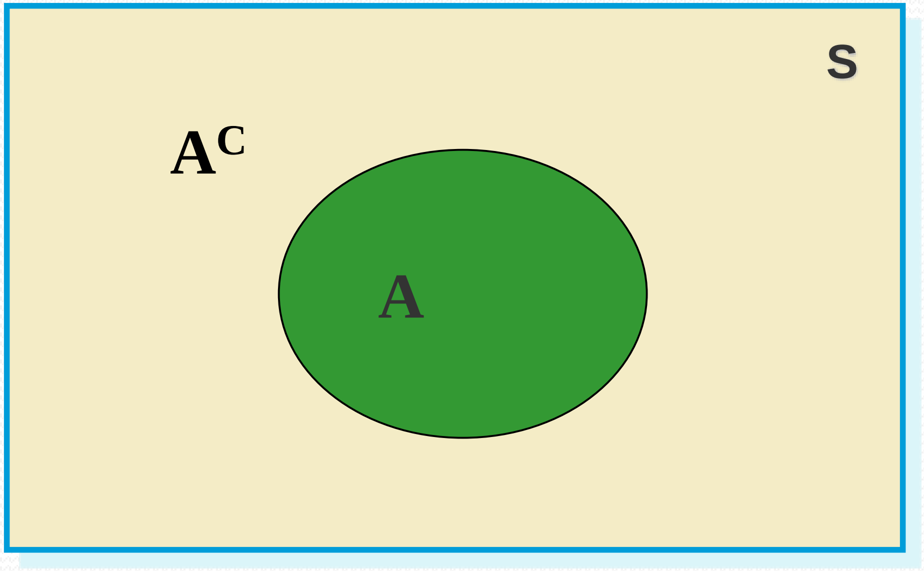
- The **intersection** of two events, **A** and **B**, is the event that both A **and** B occur when the experiment is performed. We write **A ∩ B**.



- If two events A and B are **mutually exclusive,** then P(**A ∩ B) = 0.**

Event Relations

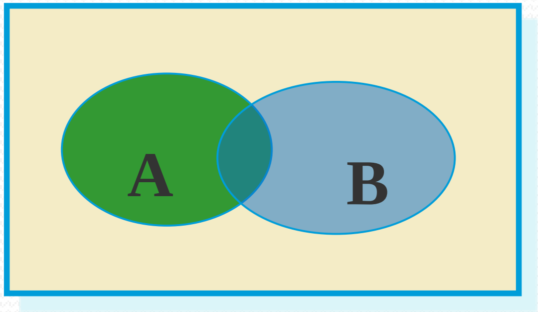
- The **complement** of an event **A** consists of all outcomes of the experiment that do not result in event A. We write A^C.



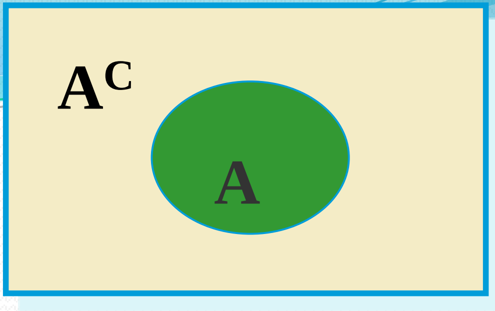
Calculating Probabilities for Unions and Complements

- There are special rules that will allow you to calculate probabilities for composite events.
- **The Additive Rule for Unions:**
- For any two events, **A** and **B**, the probability of their union, **P(A ∪ B)**, is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Calculating Probabilities for Complements



- We know that for any event **A**:
  - $P(A \cap A^C) = 0$
- Since either A or A^C must occur,

  $P(A \cup A^C) = 1$
- so that $P(A \cup A^C) = P(A) + P(A^C) = 1$

$$\boxed{P(A^C) = 1 - P(A)}$$

Calculating Probabilities for Intersections

- In the previous example, we found $P(A \cap B)$ directly from the table. Sometimes this is impractical or impossible. The rule for calculating $P(A \cap B)$ depends on the idea of **independent and dependent events.**

Two events, **A** and **B**, are said to be **independent** if and only if the probability that event **A** occurs does not change, depending on whether or not event **B** has occurred.

Conditional Probabilities

- The probability that A occurs, given that event B has occurred is called the **conditional probability** of A given B and is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \text{ if } P(B) \neq 0$$

"given"

Defining Independence

- We can redefine independence in terms of conditional probabilities:

Two events A and B are **independent** if and only if

$$P(A|B) = P(A) \quad \text{or} \quad P(B|A) = P(B)$$

Otherwise, they are **dependent**.

- Once you've decided whether or not two events are independent, you can use the following rule to calculate their intersection.

The Multiplicative Rule for Intersections

- For any two events, **A** and **B**, the probability that both **A** and **B** occur is

$$P(A \cap B) = P(A)\, P(B \text{ given that } A \text{ occurred}) = P(A)P(B|A)$$

- If the events **A** and **B** are independent, then the probability that both **A** and **B** occur is

$$P(A \cap B) = P(A)\, P(B)$$

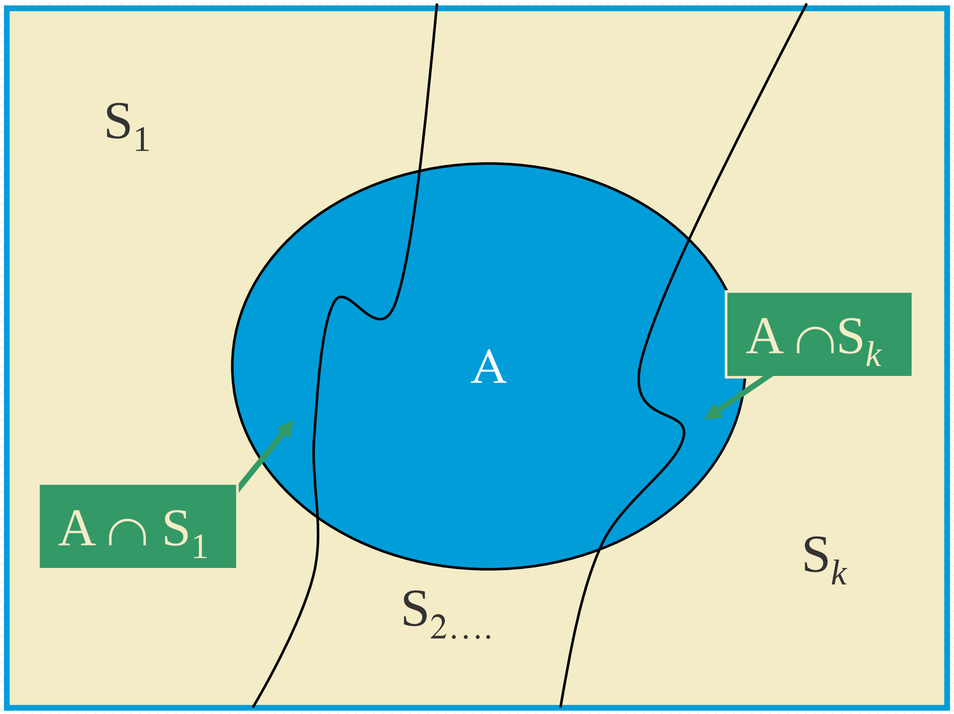
The Law of Total Probability

- Let S_1 , S_2 , S_3 ,...., S_k be mutually exclusive and exhaustive events (that is, one and only one must happen). Then the probability of another event A can be written as

$$P(A) = P(A \cap S_1) + P(A \cap S_2) + \ldots + P(A \cap S_k)$$

$$= P(S_1)P(A|S_1) + P(S_2)P(A|S_2) + \ldots + P(S_k)P(A|S_k)$$

The Law of Total Probability



$P(A) = P(A \cap S_1) + P(A \cap S_2) + \ldots + P(A \cap S_k)$

$= P(S_1)P(A|S_1) + P(S_2)P(A|S_2) + \ldots + P(S_k)P(A|S_k)$

Bayes' Rule

- Let S_1 , S_2 , S_3 ,…., S_k be mutually exclusive and exhaustive events with prior probabilities $P(S_1)$, $P(S_2)$,…,$P(S_k)$. If an event A occurs, the posterior probability of S_i, given that A occurred is

$$P(S_i \mid A) = \frac{P(S_i)P(A \mid S_i)}{\sum P(S_i)P(A \mid S_i)} \text{ for } i = 1, 2, \ldots k$$

Random Variables

- A quantitative variable *x* is a **random variable** if the value that it assumes, corresponding to the outcome of an experiment is a chance or random event.
- Random variables can be **discrete** or **continuous.**

- **Examples:**
  - ✓ *x* = SAT score for a randomly selected student
  - ✓ *x* = number of people in a room at a randomly selected time of day
  - ✓ *x* = number on the upper face of a randomly tossed die

UNIT-V

PROBABILITY DISTRIBUTIONS

Probability Distributions for Discrete Random Variables

- The **probability distribution for a discrete random variable *x*** resembles the relative frequency distributions we constructed in Chapter 1. It is a graph, table or formula that gives the possible values of *x* and the probability *p(x)* associated with each value.

We must have

$$0 \le p(x) \le 1 \text{ and } \sum p(x) = 1$$

Probability Distributions

- Probability distributions can be used to describe the population, just as we described samples in Chapter 1.
  - **Shape:** Symmetric, skewed, mound-shaped…
  - **Outliers:** unusual or unlikely measurements
  - **Center and spread:** mean and standard deviation. A population mean is called μ and a population standard deviation is called σ.

The Mean and Standard Deviation

- Let x be a discrete random variable with probability distribution $p(x)$. Then the mean, variance and standard deviation of x are given as

$$\boxed{\begin{aligned}&\text{Mean}: \mu = \sum x p(x)\\ &\text{Variance}: \sigma^2 = \sum (x-\mu)^2 p(x)\\ &\text{Standard deviation}: \sigma = \sqrt{\sigma^2}\end{aligned}}$$

Example

- Toss a fair coin 3 times and record x the number of heads.

x	$p(x)$	$xp(x)$	$(x-\mu)^2 p(x)$
0	1/8	0	$(-1.5)^2(1/8)$
1	3/8	3/8	$(-0.5)^2(3/8)$
2	3/8	6/8	$(0.5)^2(3/8)$
3	1/8	3/8	$(1.5)^2(1/8)$

$$\mu = \sum xp(x) = \frac{12}{8} = 1.5$$

$$\sigma^2 = \sum (x-\mu)^2 p(x)$$

$$\sigma^2 = .28125 + .09375 + .09375 + .28125 = .75$$

$$\sigma = \sqrt{.75} = .688$$

Introduction

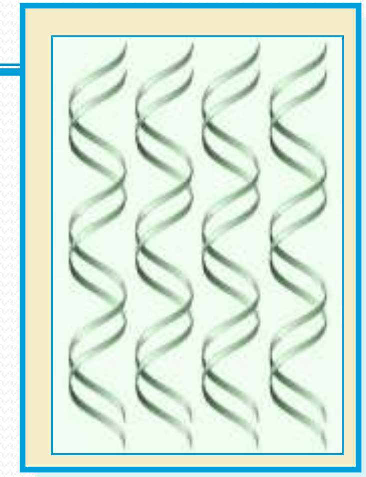
- Discrete random variables take on only a finite or countably number of values.
- Three discrete probability distributions serve as models for a large number of practical applications:

✓The **binomial** random variable

✓The **Poisson** random variable

The Binomial Random Variable

- Many situations in real life resemble the coin toss, but the coin is not necessarily fair, so that $P(H) \neq 1/2$.



- Example: A geneticist samples 10 people and counts the number who have a gene linked to Alzheimer's disease.

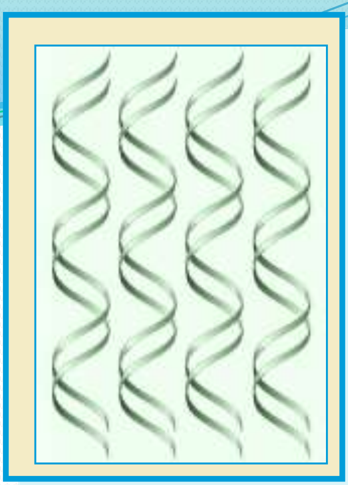
- Coin: Person
- Head: Has gene
- Tail: Doesn't have gene
- Number of tosses: $n = 10$
- P(H): P(has gene) = proportion in the population who have the gene.

The Binomial Experiment

1. The experiment consists of ***n* identical trials.**
2. Each trial results in **one of two outcomes**, success (S) or failure (F).
3. The probability of success on a single trial is p and **remains constant** from trial to trial. The probability of failure is $q = 1 - p$.
4. The trials are **independent**.
5. We are interested in ***x*, the number of successes in *n* trials.**

Binomial or Not?

- Very few real life applications satisfy these requirements exactly.



- Select two people from the U.S. population, and suppose that 15% of the population has the Alzheimer's gene.
  - For the first person, p = P(gene) = .15
  - For the second person, $p \approx$ P(gene) = .15, even though one person has been removed from the population.

The Binomial Probability Distribution

- For a binomial experiment with *n* trials and probability *p* of success on a given trial, the probability of *k* successes in *n* trials is

$$P(x=k)=C_k^n p^k q^{n-k} = \frac{n!}{k!(n-k)!} p^k q^{n-k} \text{ for } k = 0,1,2,\ldots n.$$

$$\text{Recall} \quad C_k^n = \frac{n!}{k!(n-k)!}$$

$$\text{with } n! = n(n-1)(n-2)\ldots(2)1 \text{ and } 0! \equiv 1.$$

The Mean and Standard Deviation

- For a binomial experiment with *n* trials and probability *p* of success on a given trial, the measures of center and spread are:

$$\text{Mean: } \mu = np$$

$$\text{Variance: } \sigma^2 = npq$$

$$\text{Standard deviation: } \sigma = \sqrt{npq}$$

Cumulative Probability Tables

You can use the cumulative probability tables to find probabilities for selected binomial distributions.

✓Find the table for the correct value of n.

✓Find the column for the correct value of p.

✓The row marked "k" gives the cumulative probability, $P(x \leq k) = P(x = 0) + \ldots + P(x = k)$

The Poisson Random Variable

- The Poisson random variable *x* is a model for data that represent the number of occurrences of a specified event in a given unit of time or space.

- Examples:
  - The number of calls received by a switchboard during a given period of time.
  - The number of machine breakdowns in a day
  - The number of traffic accidents at a given intersection during a given time period.

The Poisson Probability Distribution

- ***x*** is the number of events that occur in a period of time or space during which an average of **μ** such events can be expected to occur. The probability of *k* occurrences of this event is

$$P(x=k)=\frac{\mu^k e^{-\mu}}{k!}$$

For values of k = 0, 1, 2, … The mean and standard deviation of the Poisson random variable are

Mean: μ

Standard deviation:

$$\sigma=\sqrt{\mu}$$

Cumulative Probability Tables

You can use the cumulative probability tables to find probabilities for selected Poisson distributions.

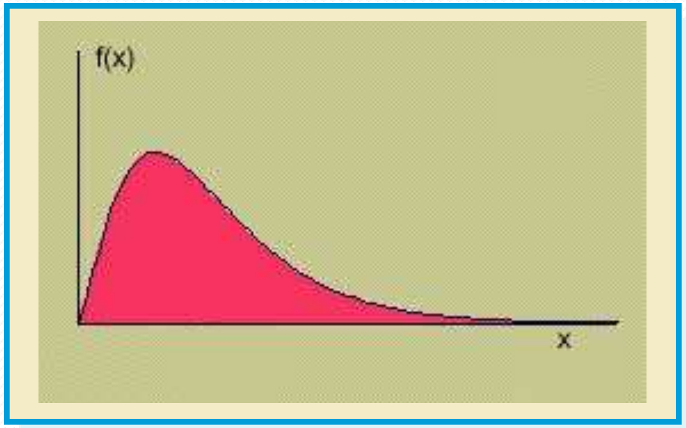
- ✓Find the column for the correct value of μ.
- ✓The row marked "k" gives the cumulative probability, $P(x \leq k) = P(x = 0) + \ldots + P(x = k)$

Continuous Random Variables

- Continuous random variables can assume the infinitely many values corresponding to points on a line interval.
- **Examples:**
  - Heights, weights
  - length of life of a particular product
  - experimental laboratory error

Continuous Random Variables

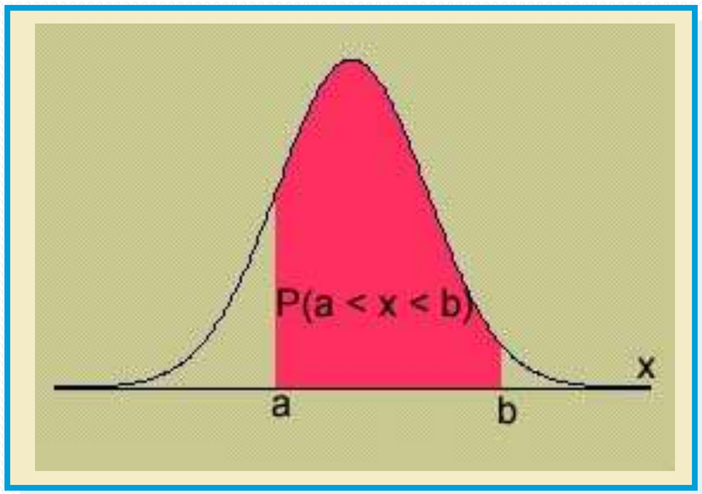
- A **smooth curve** describes the probability distribution of a continuous random variable.



•The depth or density of the probability, which varies with *x*, may be described by a mathematical formula *f* (*x*), called the probability distribution or probability density function for the random variable *x*.

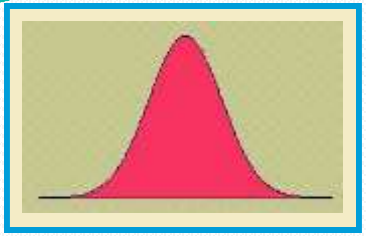
Properties of Continuous Probability Distributions

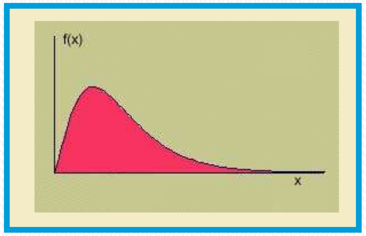
- The area under the curve is equal to **1.**
- P(a ≤ x ≤ b) = **area under the curve** between a and b.



•There is no probability attached to any single value of x. That is, **P(x = a) = 0.**

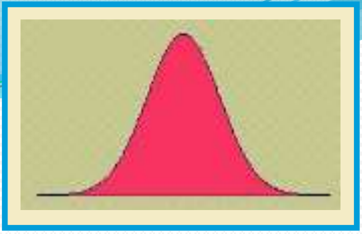
Continuous Probability Distributions





- There are many different types of continuous random variables
- We try to pick a model that
  - Fits the data well
  - Allows us to make the best possible inferences using the data.
- One important continuous random variable is the **normal random variable**.

The Normal Distribution



- The formula that generates the normal probability distribution is:

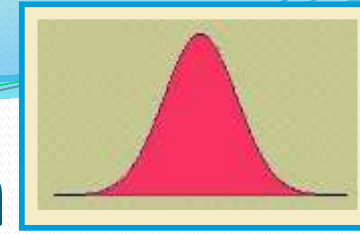
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{for } -\infty < x < \infty$$

$$e = 2.7183 \qquad \pi = 3.1416$$

μ and σ are the population mean and standard deviation.

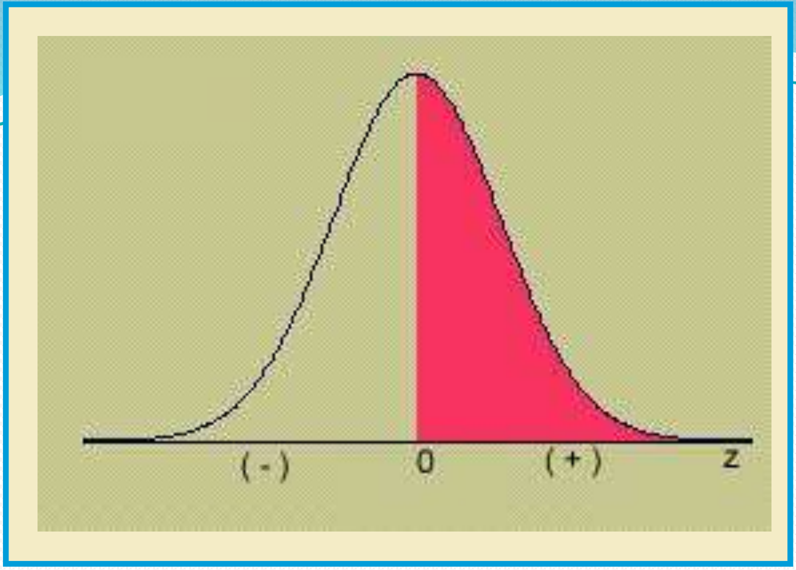
- The shape and location of the normal curve changes as the mean and standard deviation change.

The Standard Normal Distribution



- To find $P(a < x < b)$, we need to find the area under the appropriate normal curve.
- To simplify the tabulation of these areas, we **standardize** each value of x by expressing it as a z-score, the number of standard deviations σ it lies from the mean μ.

$$\boxed{z = \frac{x - \mu}{\sigma}}$$



The Standard Normal (z) Distribution

- Mean = 0; Standard deviation = 1
- When $x = \mu$, $z = 0$
- Symmetric about $z = 0$
- Values of z to the left of center are negative
- Values of z to the right of center are positive
- Total area under the curve is 1.

Finding Probabilities for the General Normal Random Variable

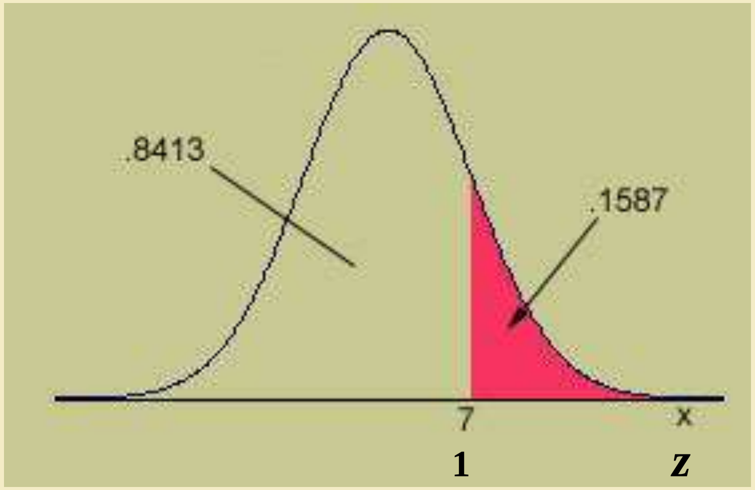
✓To find an area for a normal random variable *x* with mean μ and standard deviation σ, *standardize or rescale* the interval in terms of *z.*

✓Find the appropriate area using Table 3.

Example: *x* has a normal distribution with $\mu = 5$ and $\sigma = 2$. Find P(*x* > 7).

$$P(x > 7) = P(z > \frac{7-5}{2})$$

$$= P(z > 1) = 1 - .8413 = .1587$$



The Normal Approximation to the Binomial

- We can calculate binomial probabilities using
  - The binomial formula
  - The cumulative binomial tables
  - Java applets
- When n is large, and p is not too close to zero or one, areas under the normal curve with mean np and variance npq can be used to approximate binomial probabilities.