CONTROL SYSTEMS (AEE009)

B.Tech -ECE-IV Sem

IARE-R16

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Unit-1
INTRODUCTION AND MODELING OF PHYSICAL SYSTEMS

What is Control System?

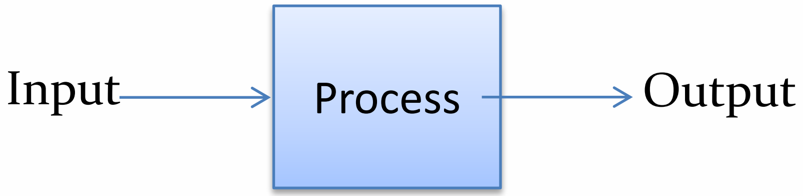
- A system Controlling the operation of another system.
- A system that can regulate itself and another system.
- A control System is a device, or set of devices to manage, command, direct or regulate the behaviour of other device(s) or system(s).

Definitions

❑ **System** – An interconnection of elements and devices for a desired purpose.

❑ **Control System** – An interconnection of components forming a system configuration that will provide a desired response.

❑ **Process** – The device, plant, or system under control. The input and output relationship represents the cause-and-effect relationship of the process.



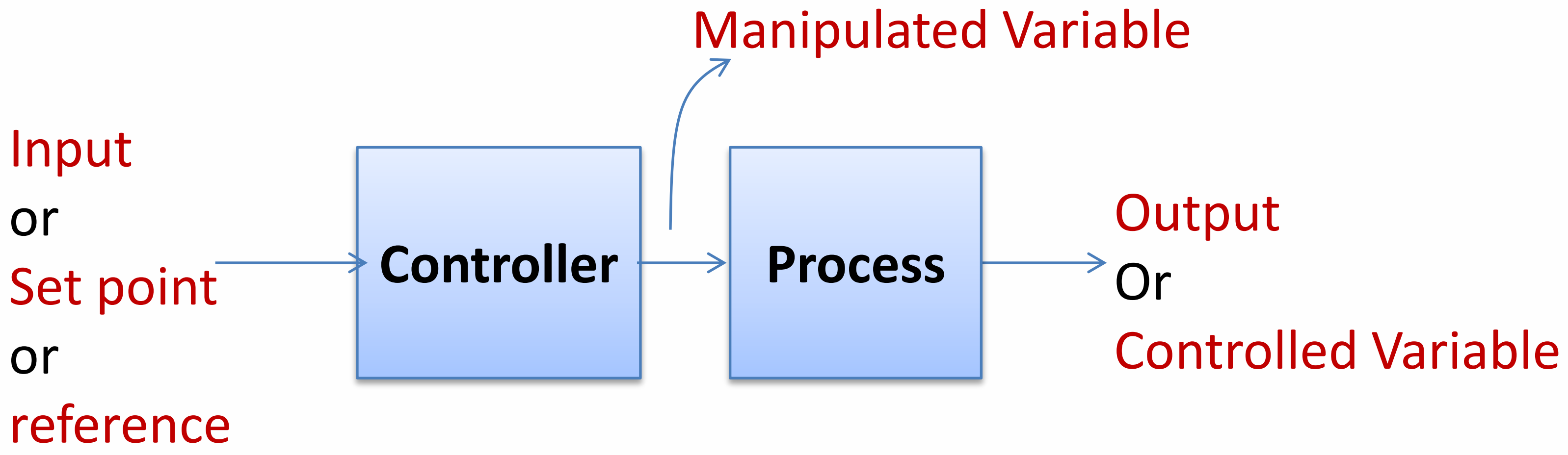
Definitions (Contd..)

- **Controlled Variable**– It is the quantity or condition that is measured and Controlled. Normally controlled variable is the output of the control system.

- **Manipulated Variable**– It is the quantity of the condition that is varied by the controller so as to affect the value of controlled variable.

- **Control** – Control means measuring the value of controlled variable of the system and applying the manipulated variable to the system to correct or limit the deviation of the measured value from a desired value.

Definitions (Contd..)



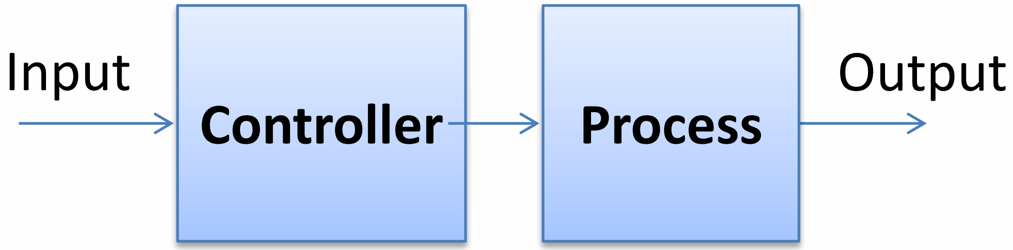
- **Disturbances**– A disturbance is a signal that tends to adversely affect the value of the system. It is an unwanted input of the system.

  - If a disturbance is generated within the system, it is called internal disturbance. While an external disturbance is generated outside the system.

Types of Control System

❑ **Open-Loop Control Systems** utilize a controller or control actuator to obtain the desired response.

- ❑Output has no effect on the control action. No feedback – no correction of disturbances
- ❑ In other words output is neither measured nor fed back.



Open-loop control system (without feedback).

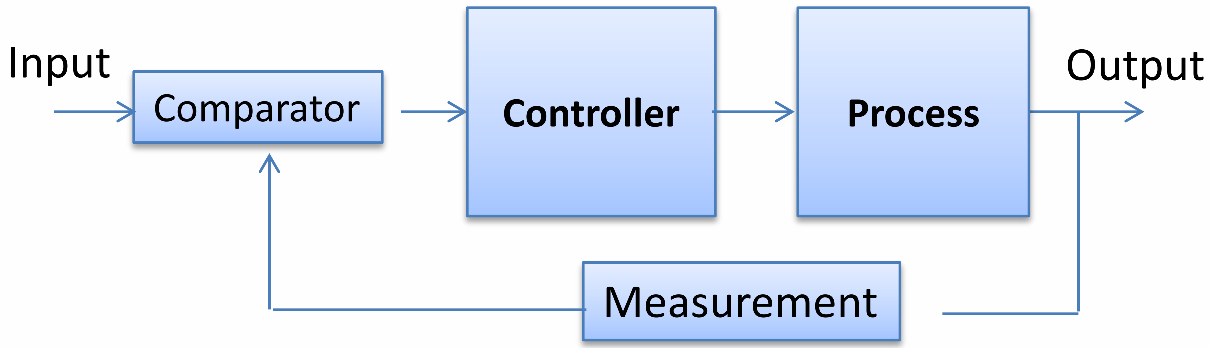
Examples:- Washing Machine, Toaster, Electric Fan

Types of Control System (Contd..)

- Since in open loop control systems reference input is not compared with measured output, for each reference input there is fixed operating condition.
- Therefore, the accuracy of the system depends on calibration.
- The performance of open loop system is severely affected by the presence of disturbances, or variation in operating/ environmental conditions.

Types of Control System (Contd..)

❑ **Closed-Loop Control Systems** utilizes feedback to compare the actual output to the desired output response.



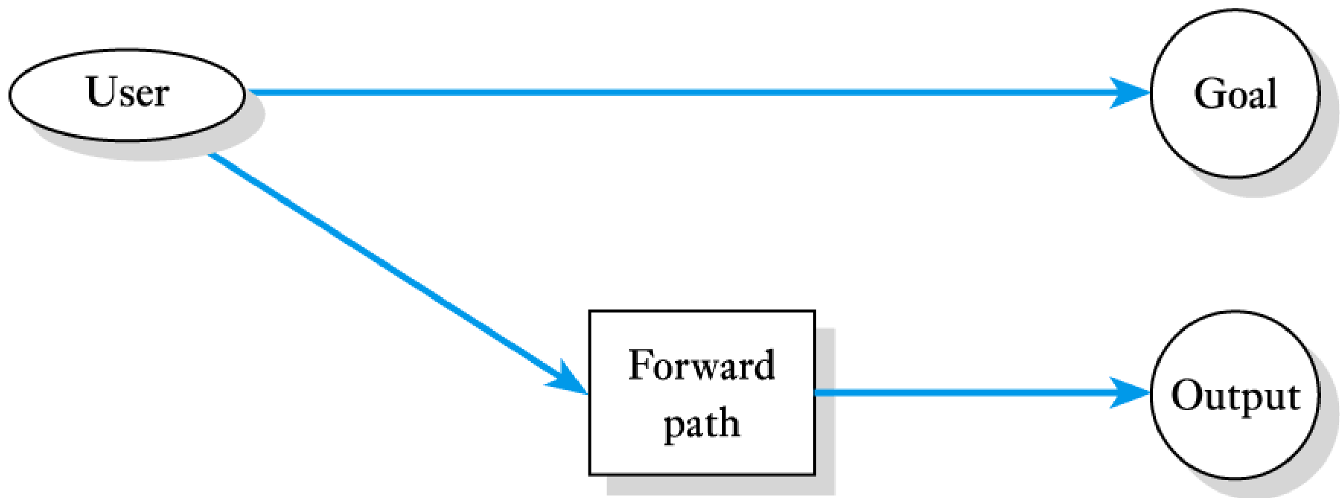
Closed-loop feedback control system (with feedback).

Examples:- Refrigerator, Iron

Types of Control System (Contd..)

- Simple control is often **open-loop**

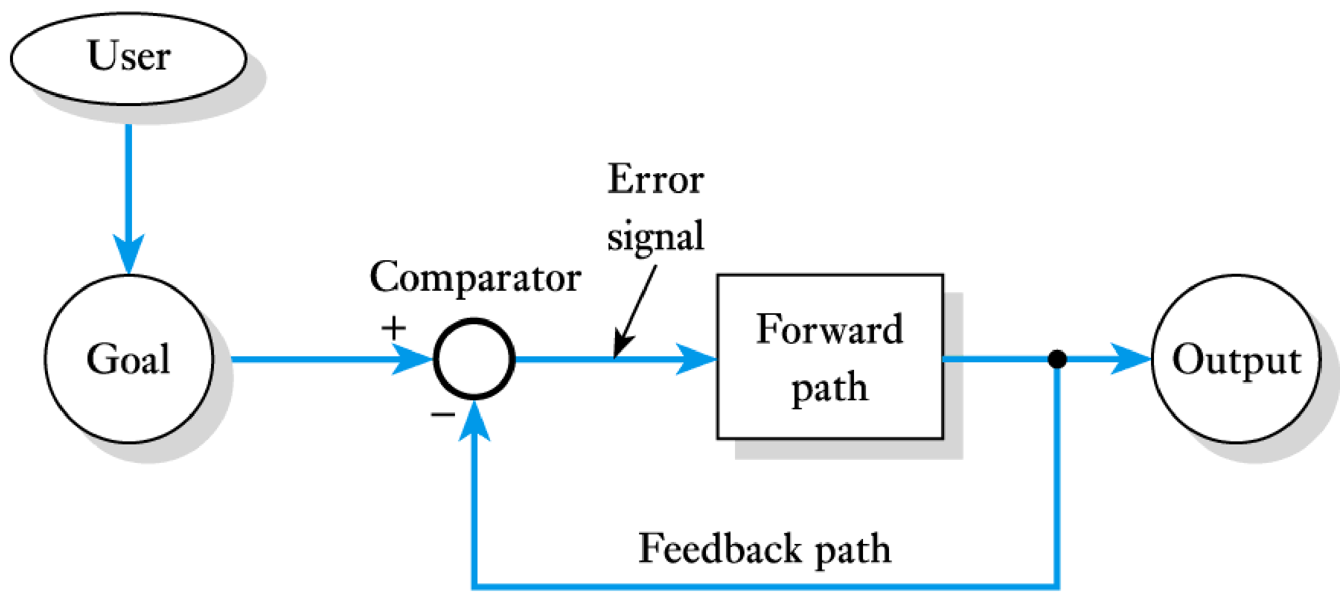
  - user has a **goal** and selects an input to a system to try to achieve this



(a) An open-loop system

Types of Control System (Contd..)

- More sophisticated arrangements are **closed-loop**
  - user inputs the goal to the system

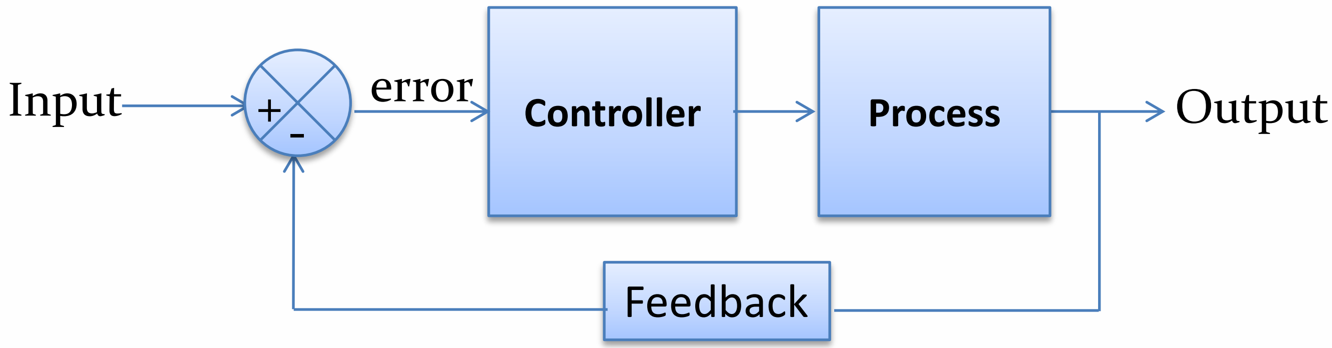


(b) A closed-loop system

Types of Control System (Contd..)

- **Feedback Control System**

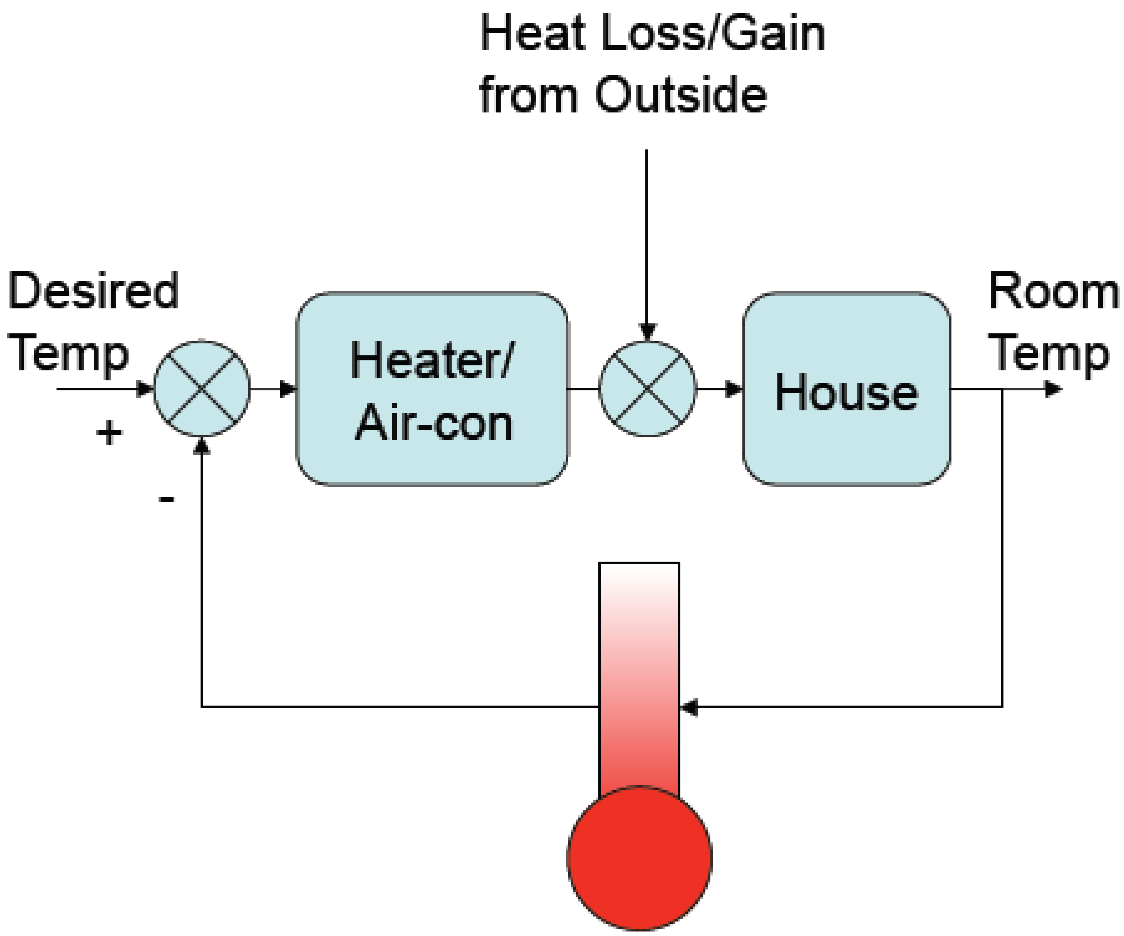
- A system that maintains a prescribed relationship between the output and some reference input by comparing them and using the difference (i.e. error) as a means of control is called a feedback control system.



- Feedback can be positive or negative.

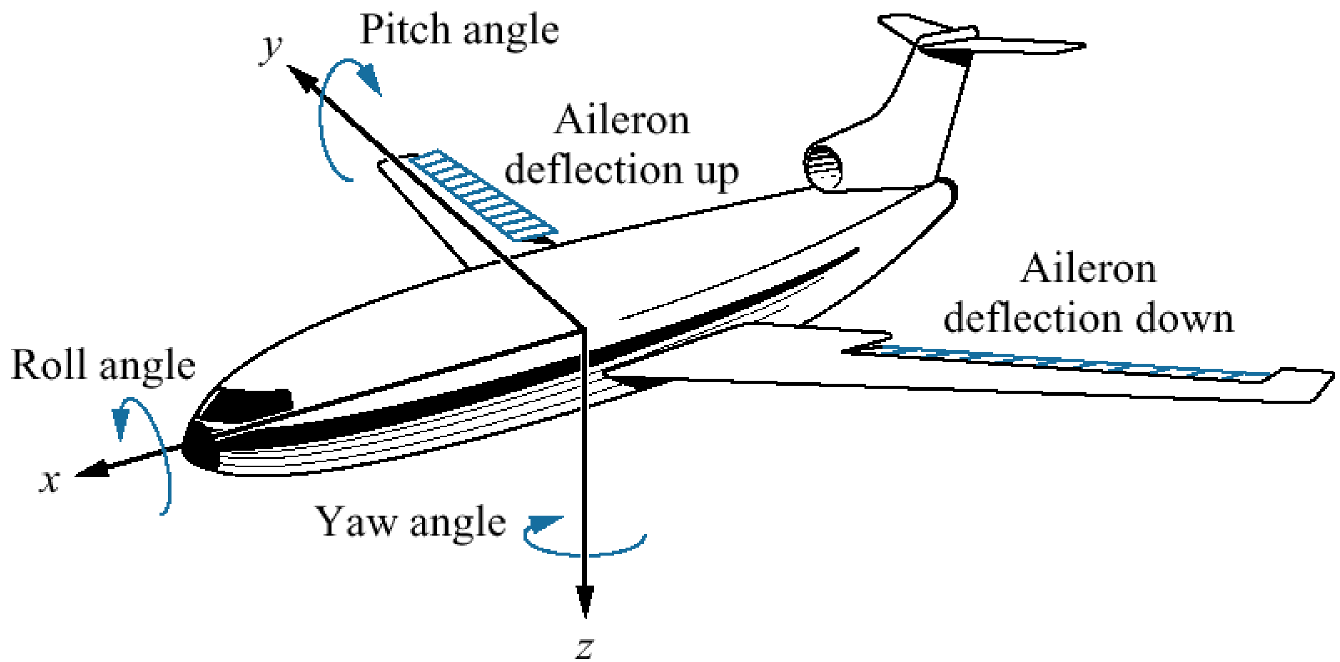
Examples of Control Systems

- Room temperature control

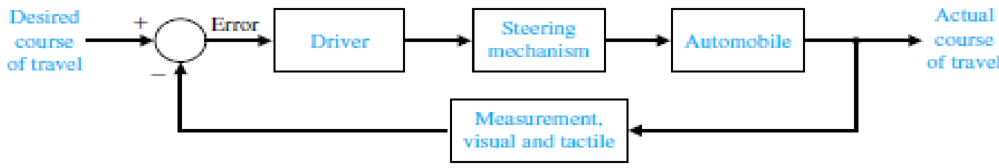


Examples of Modern Control Systems

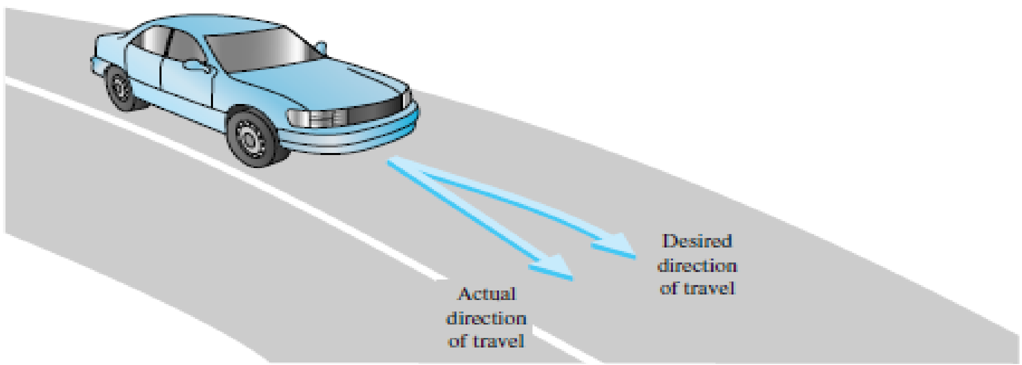
- ❑ **Aero plane landing system**



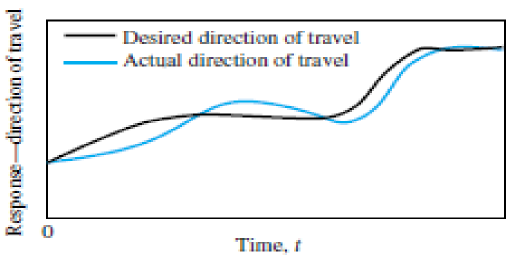
Examples of Modern Control Systems



(a)



(b)



(c)

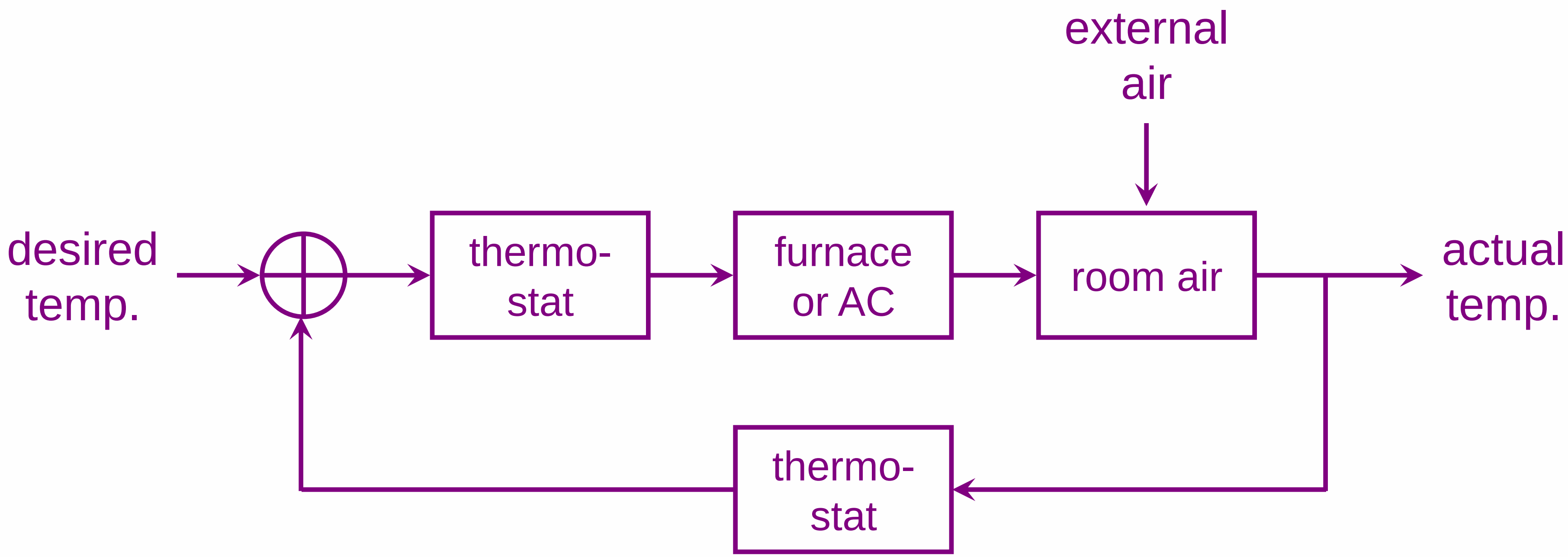
(a) Automobile steering control system.

(b) The driver uses the difference between the actual and the desired direction of travel to generate a controlled adjustment of the steering wheel.

(c) Typical direction-of-travel response.

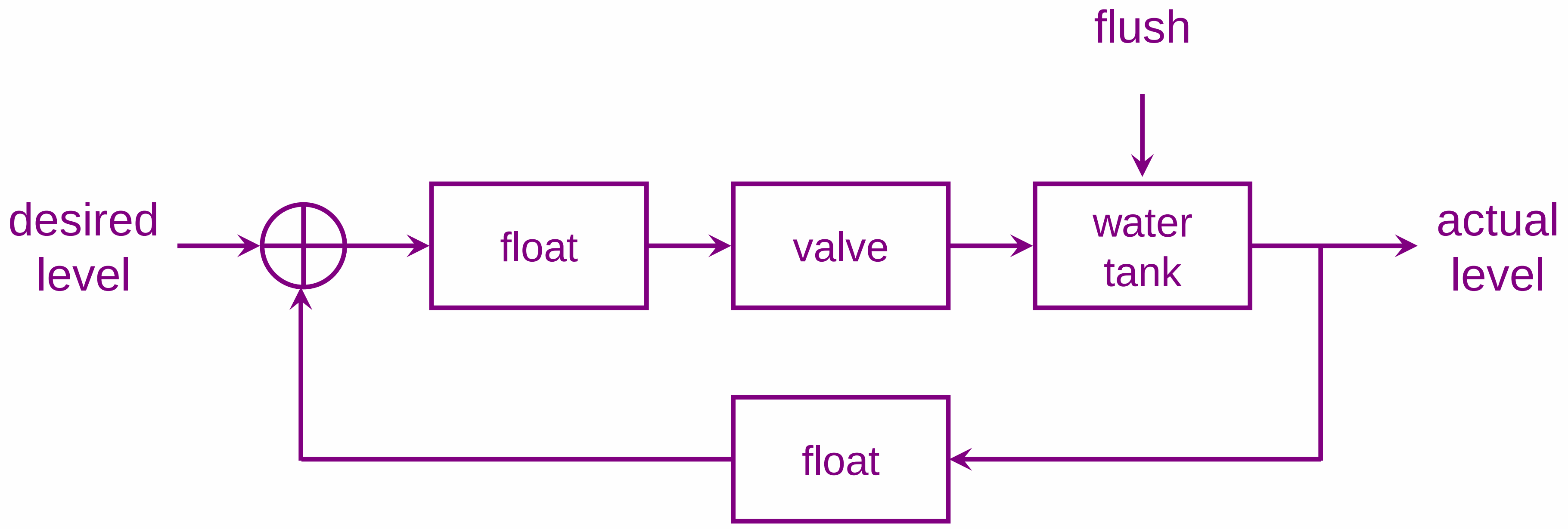
Thermostat Example

- Set thermostat to desired room temperature
- Thermostat measures room temperature
- Furnace or AC turn on if measured <> desired
- Air from furnace or AC changes room air temperature



Toilet Flush Example

- Float height determines desired water level
- Flush empties tank, float is lowered and valve opens
- Open valve allows water to enter tank
- Float returns to desired level and valve closes

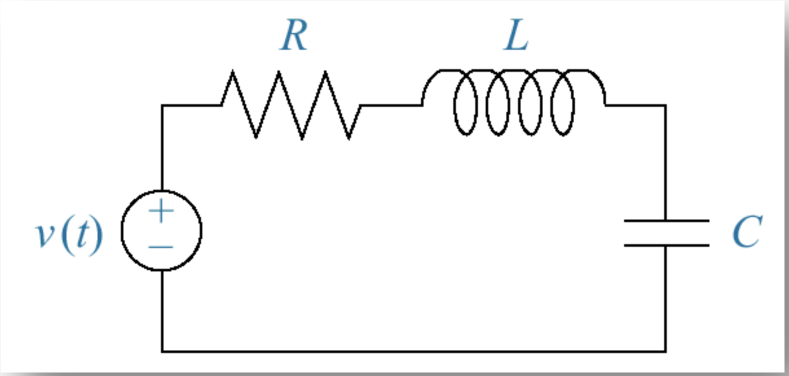


Mathematical Model

- A mathematical model is a set of equations (usually differential equations) that represents the dynamics of systems.
- In practice, the complexity of the system requires some assumptions in the determination model.
- How do we obtain the equations?
- Physical law of the process → Differential Equation
- Examples:
  - Mechanical system (Newton’s laws)
  - Electrical system (Kirchhoff’s laws)

Mathematical Model of RLC network

- Example: RLC Circuit



$$\boxed{-v(t) + V_R + V_L + V_c = 0}$$

$$\boxed{L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,d\tau = v(t)}$$

Laplace Transform

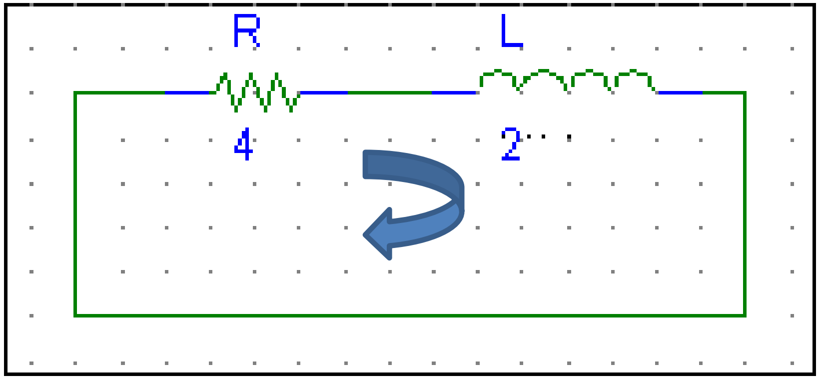
- The differential equations are transformed into algebraic equations, which are easier to solve.
- The Laplace transformation for a function of time, f(t) is:

$$\boxed{F(s) = \int_0^\infty f(t)e^{-st}\,dt = L\{f(t)\}}$$

- If, $f(t) = \frac{dy}{dt}$, then, $L\{f(t)\} = L\left\{\frac{dy}{dt}\right\} = sL\{y(t)\} - y(0)$
- Similarly, $L\left\{\frac{d^2y(t)}{dt^2}\right\} = sL\left\{\frac{dy(t)}{dt}\right\} - \frac{dy(0)}{dt}$
- Thus, $L\left\{\frac{d^2y(t)}{dt^2}\right\} = s^2L\{y(t)\} - sy(0) - \frac{dy(0)}{dt}$

Laplace Transform (contd..)

- Consider RL Network
- By applying KVL



$$4i(t) + 2\frac{d}{dt}i(t) = 0$$

- Assume i(0)=5A
- By applying Laplace transform

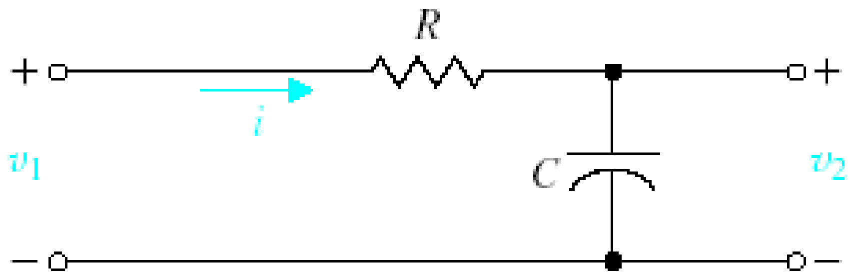
$$\int_0^\infty \left(4i(t) + 2\frac{d}{dt}i(t)\right)e^{-st}\,dt = 0$$

$$4I(s) + 2(sI(s) - i(0)) = 0$$

$$4I(s) + 2sI(s) - 10 = 0$$

$$I(s) = \frac{5}{s+2}$$

Transfer Function of Linear System



$$V_1(s) = \left(R + \frac{1}{Cs}\right) I(s)$$

$$V_2(s) = \left(\frac{1}{Cs}\right) I(s)$$

- Transfer function

$$\frac{V_2(s)}{V_1(s)} = \frac{\left(\frac{1}{Cs}\right)}{\left(R + \frac{1}{Cs}\right)} = \frac{1}{1 + sRC}$$

Transfer Function of Linear System

- Example

$$\frac{d^2}{dt^2}y(t) + 4\cdot\frac{d}{dt}y(t) + 3\cdot y(t) = 2\cdot r(t)$$

Initial Conditions: $Y(0) = 1$ $\quad \frac{d}{dt}y(0) = 0$ $\quad r(t) = 1$

The Laplace transform yields:

$$\left(s^2\cdot Y(s) - s\cdot y(0)\right) + 4\cdot(s\cdot Y(s) - y(0)) + 3\,Y(s) = 2\cdot R(s)$$

Since R(s)=1/s and y(0)=1, we obtain:

$$Y(s) = \frac{(s+4)}{\left(s^2 + 4s + 3\right)} + \frac{2}{s\cdot\left(s^2 + 4s + 3\right)}$$

The partial fraction expansion yields:

$$Y(s) = \left[\frac{\frac{3}{2}}{(s+1)} + \frac{\frac{-1}{2}}{(s+3)}\right] + \left[\frac{-1}{(s+1)} + \frac{\frac{1}{3}}{(s+3)}\right] + \frac{\frac{2}{3}}{s}$$

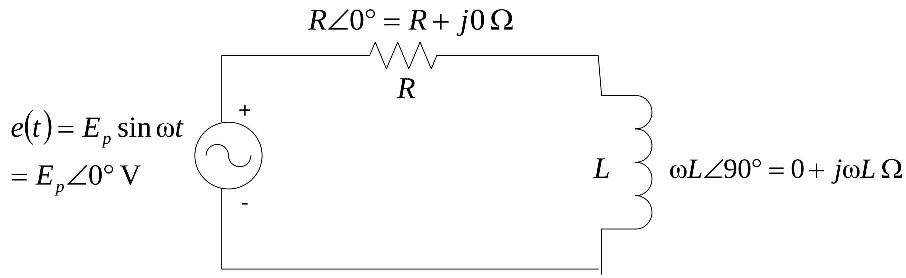
Therefore the transient response is:

$$y(t) = \left(\frac{3}{2}\cdot e^{-t} - \frac{1}{2}\cdot e^{-3\cdot t}\right) + \left(-1e^{-t} + \frac{1}{3}\cdot e^{-3t}\right) + \frac{2}{3}$$

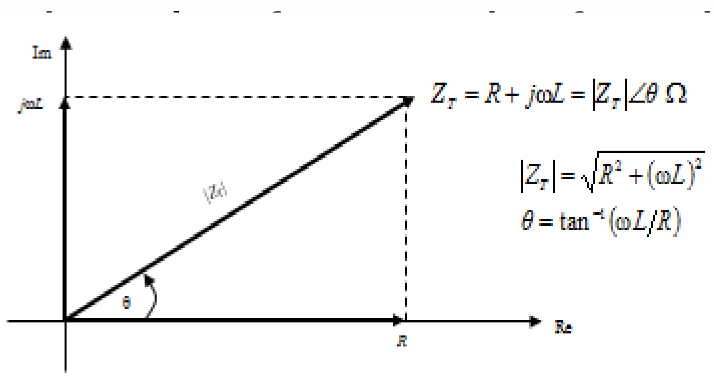
The steady-state response is:

$$\lim_{t\to\infty} y(t) = \frac{2}{3}$$

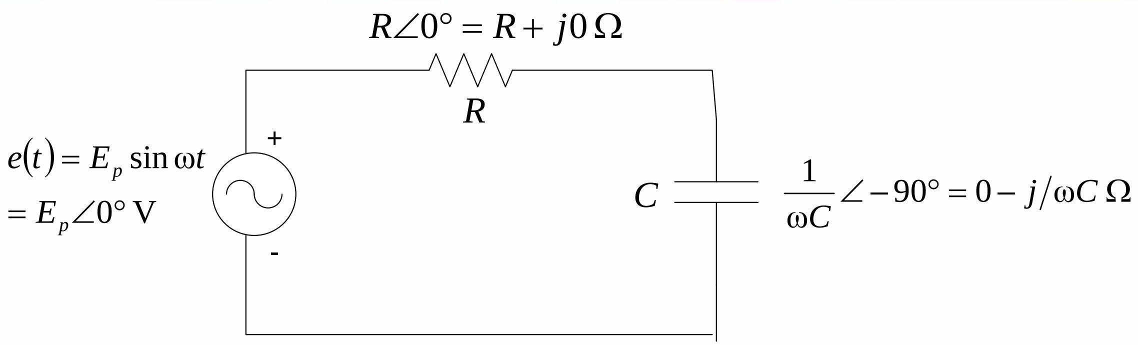
Series RL circuit



- ❑Circuit above is a series RL network connected to an ac voltage source
- ❑Need to find the phasor form of the total impedance of this combination $Z_T = (R + j0) + (0 + j\omega L) = R + j\omega L = R + j|X_L|$ ohms
- ❑The total impedance of this series combination is
- ❑The magnitude a by converting to polar form:
- ❑$|Z_T| = \sqrt{[R^2+(\omega L)^2}$
- ❑The plot of Z_T:



Series RC circuit



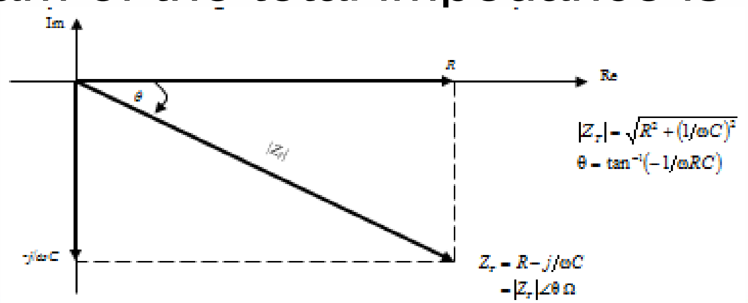
- Above is a series RC network connected to an ac voltage
- The total impedance of this combination is

$$|Z_T| = (R + j0) + (0 - j/\omega C) = R - j/\omega C = R - j|X_C|$$

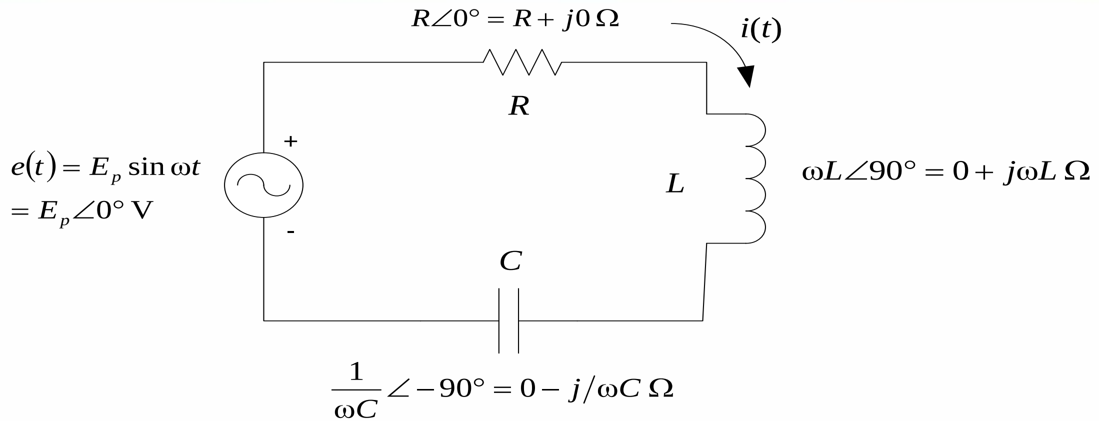
- The polar coordinates are $|Z_T| = \sqrt{R^2 + (1/\omega C)^2}$ ohms

$$\theta = \tan^{-1}\left(\frac{-1/\omega C}{R}\right) = \tan^{-1} - 1/\omega RC$$

- The phasor diagram of the total impedance is



Series RLC circuit



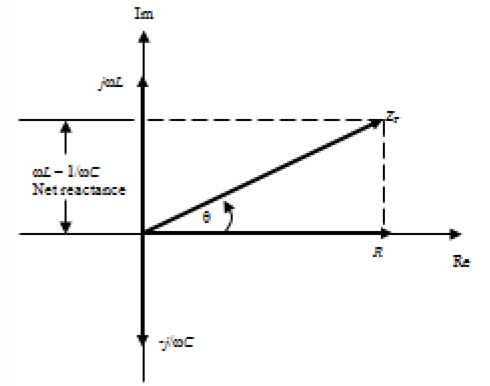
- ❑The total impedance of the RLC circuit is $Z_T = R + j(\omega L - 1/\omega C)$ ohms
- ❑In terms of magnitudes it is: $Z_T = R + j(|X_L| - |X_C|)\Omega$
- ❑Inductive and capacitive reactance have opposite signs
- ❑Thus **net reactance** may be either inductive or capacitive, depending which is larger
- ❑Polar coordinates are

$$|Z_T| = \sqrt{R^2 + (\omega L - 1/\omega C)^2} = \sqrt{R^2 + (|X_L| - |X_C|)^2}$$

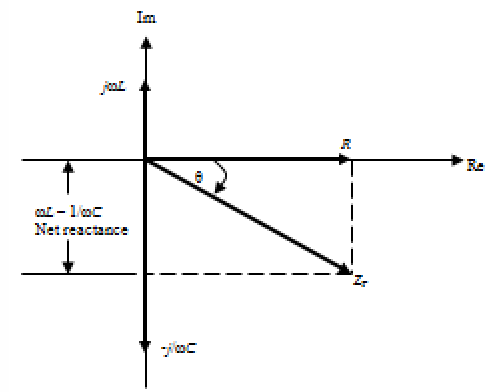
$$\theta = \tan^{-1}\frac{\omega L - 1/\omega C}{R} = \tan^{-1}\frac{|X_L| - |X_C|}{R}$$

Series RLC circuit

❑ The phasor diagram of the impedance when inductive reactance is greater than the capacitive reactance, i.e. when $|X_L|>|X_C|$



❑The phasor diagram of the impedance when capacitive reactance is greater than the inductive reactance, i.e. when $|X_C|>|X_L|$



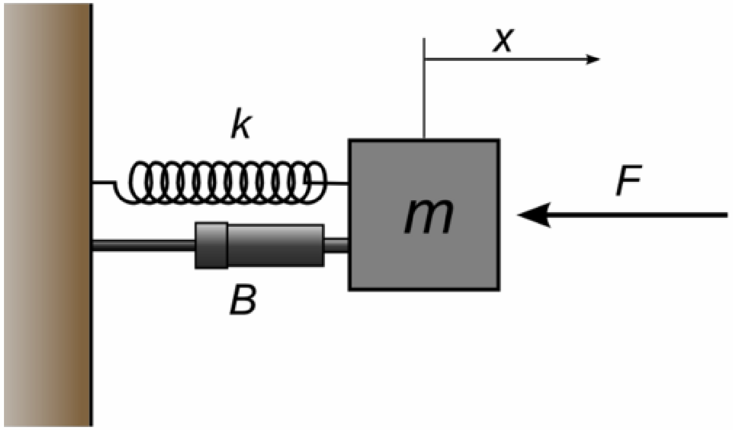
Newton's Second Law

- Newton's law of motion states that the algebraic sum of external forces acting on a rigid body in a given direction is equal to the product of the mass of the body and its acceleration in the same direction. The law can be expressed as

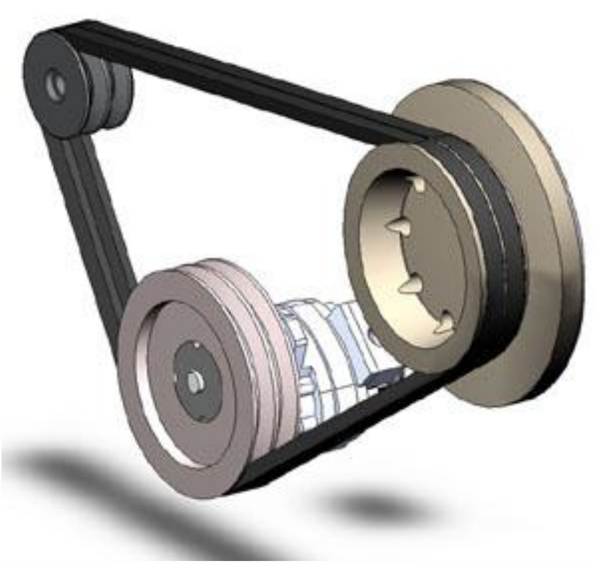
$$\sum F = Ma$$

Basic Types of Mechanical Systems

- Translational
  - Linear Motion



- Rotational
  - Rotational Motion

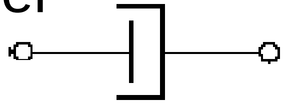


Elements of Translational Mechanical Systems

Translational Spring

Translational Mass

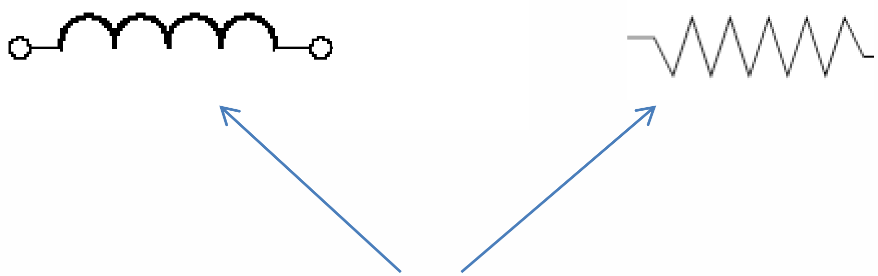
Translational Damper



Translational Spring

- A translational spring is a mechanical element that can be deformed by an external force such that the deformation is directly proportional to the force applied to it.

Translational Spring

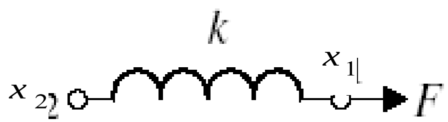


Circuit Symbols

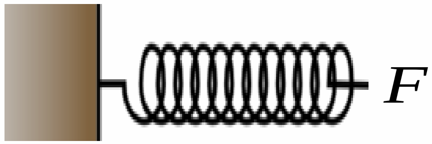
Translational Spring

Translational Spring (Contd..)

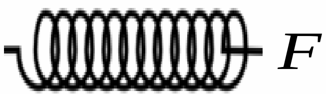
- If F is the applied force



- Then x_1 is the deformation if $x_2 = 0$



- Or $(x_1 - x_2)$ is the deformation.



- The equation of motion is given as

$$F = k(x_1 - x_2)$$

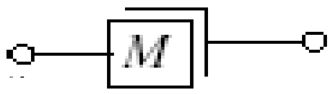
- Where k is stiffness of spring expressed in N/m

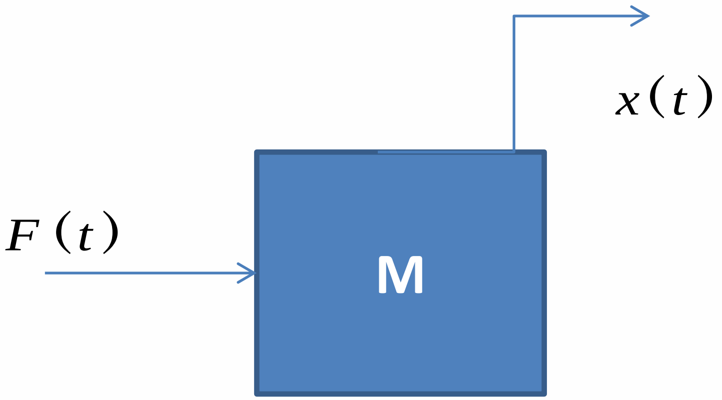
Translational Mass

- Translational Mass is an inertia element.
- A mechanical system without mass does not exist.
- If a force F is applied to a mass and it is displaced to x meters then the relation b/w force and displacements is given by Newton's law.

$$F = M \frac{d}{dt} x^2$$

Translational Mass

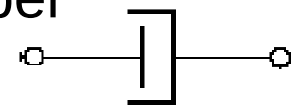




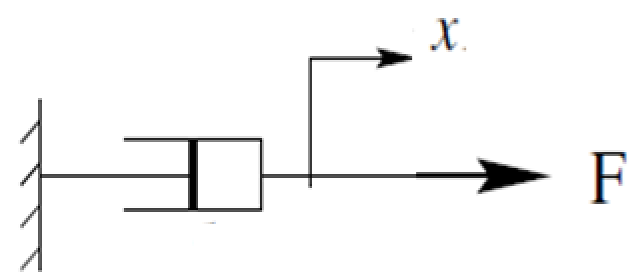
Translational Damper

- When the viscosity or drag is not negligible in a system, we often model them with the damping force.

Translational Damper



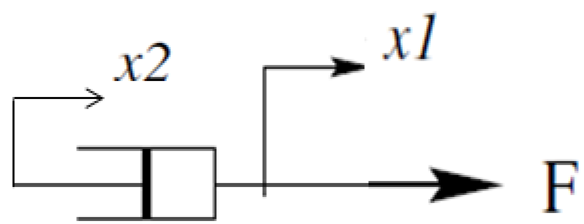
- All the materials exhibit the property of damping to some extent.



- If damping in the system is not enough then extra elements (e.g. Dashpot) are added to increase damping.

$$F = B\dot{x}$$

- Where ... fficient (*N/ms^{-1}*)



$$F = B(\dot{x}_1 - \dot{x}_2)$$

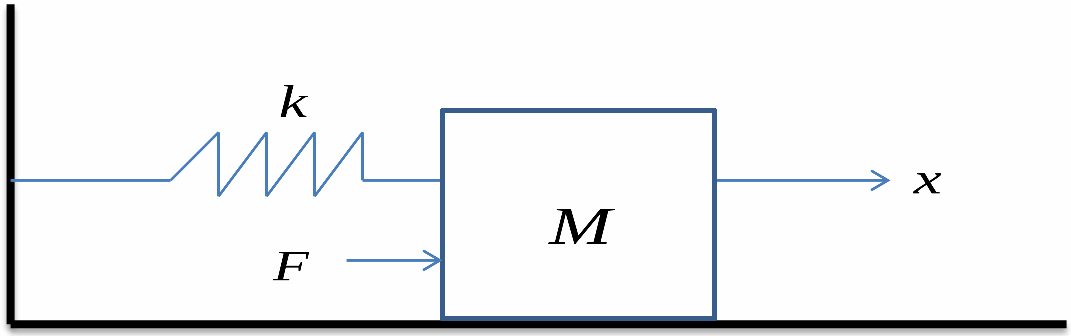
Transfer Function of Mechanical System

- The mechanical system requires just one differential equation, called the equation of motion, to describe it.
- Assume a positive direction of motion, for example, to the right.
- This assumed positive direction of motion is similar to assuming a current direction in an electrical loop.

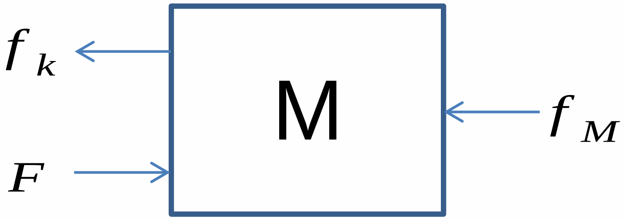
- **First,** draw a free-body diagram, placing on the body all forces that act on the body either in the direction of motion or opposite to it.
- **Second,** use Newton’s law to form a differential equation of motion by summing the forces and setting the sum equal to zero.
- **Finally,** assuming zero initial conditions, we take the Laplace transform of the differential equation, separate the variables, and arrive at the transfer function.

Example on MTS

- Consider the following system (friction is negligible)

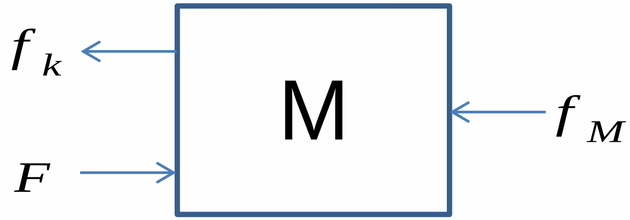


- Free Body Diagram



- Where f_k and f_M are force applied by the spring and inertial force respectively.

Example on MTS (Contd..)



$$F = f_k + f_M$$

- Then the differential equation of the system is:

$$F = M\ddot{x} + kx$$

- Taking the Laplace Transform of both sides and ignoring initial conditions we get

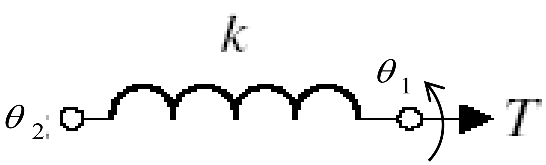
$$F(s) = Ms^2 X(s) + kX(s)$$

- The transfer function of the system is

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + k}$$

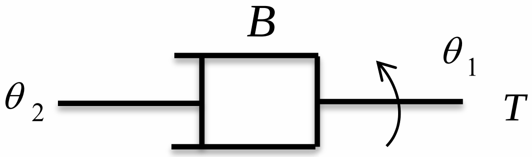
Elements of Rotational Mechanical Systems

- Rotational Spring



$$T = k(\theta_1 - \theta_2)$$

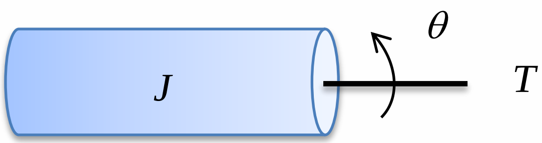
- Rotational Damper



$$T = B(\dot{\theta}_1 - \dot{\theta}_2)$$

Elements of Rotational Mechanical Systems

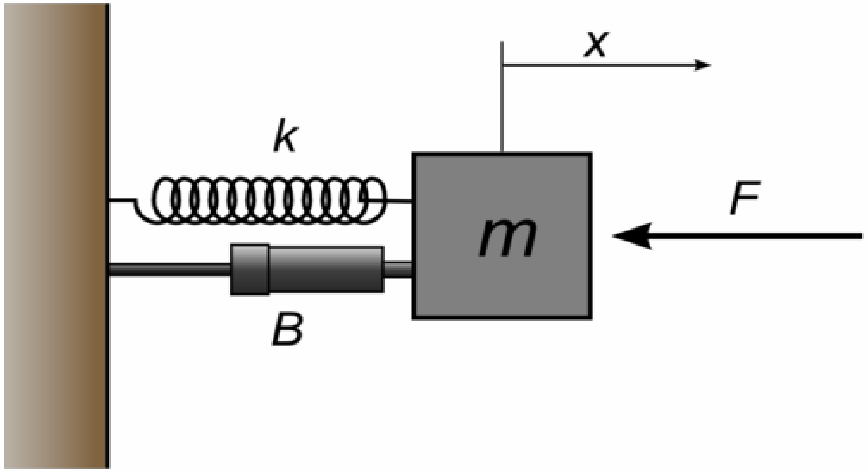
- Moment of Inertia



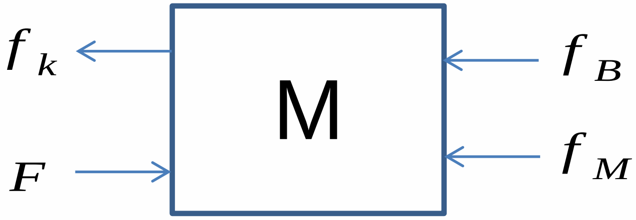
$$T = J\ddot{\theta}$$

Example#1 on MTS

- Consider the following system



- Free Body Diagram

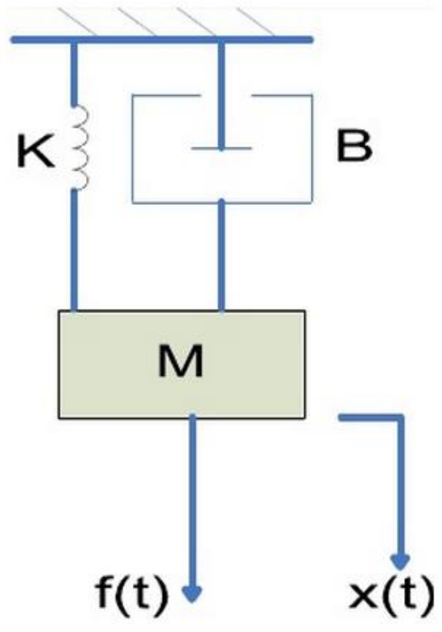


$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + k}$$

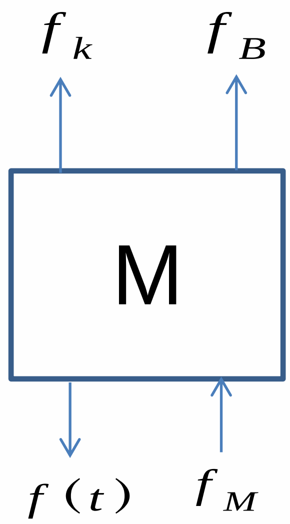
$$F = f_k + f_M + f_B$$

Example#2 on MTS

- Find the transfer function of the mechanical translational system given in Figure.



Free Body Diagram

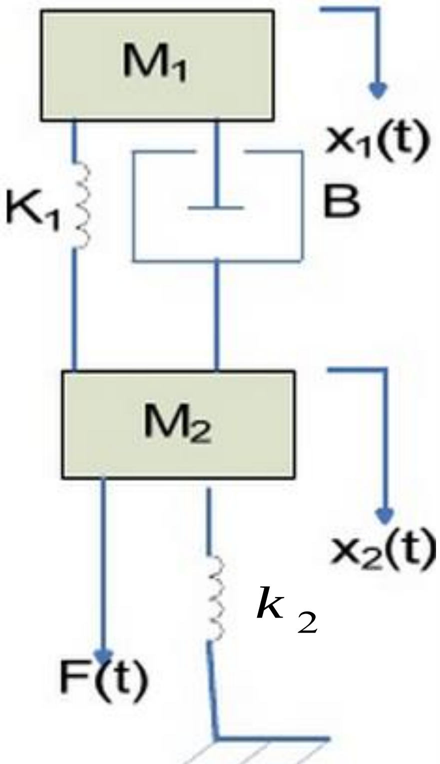


$$f(t) = f_k + f_M + f_B$$

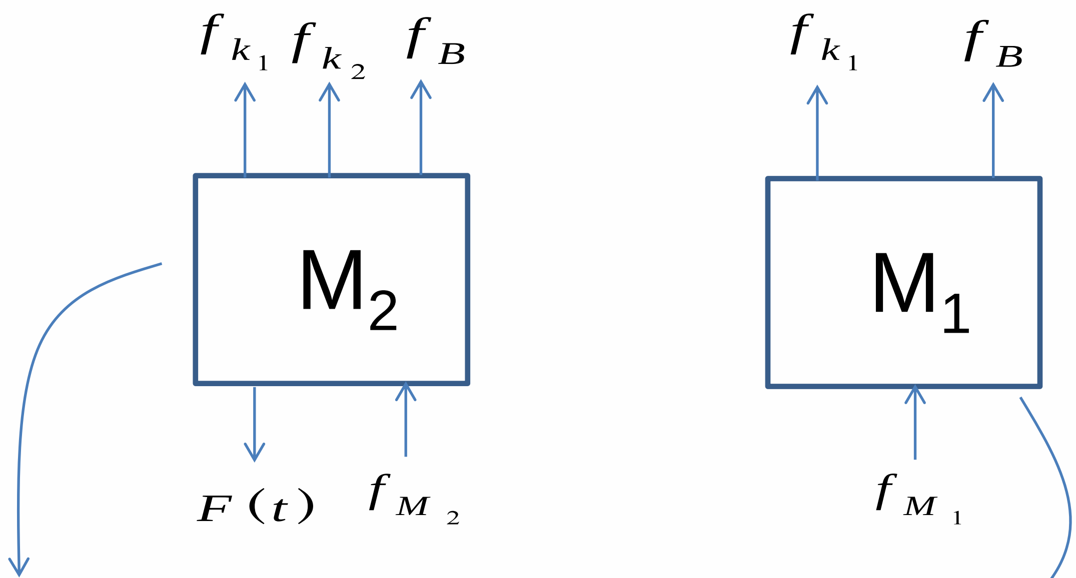
$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + k}$$

Example#3 on MTS

- Find the transfer function $X_2(s)/F(s)$ of the following system.



Free Body Diagram



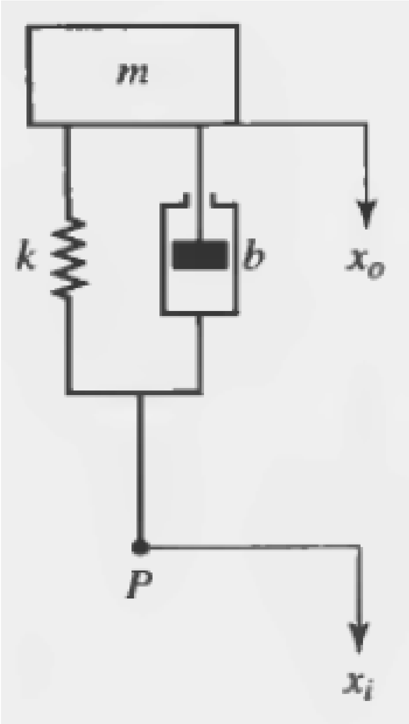
$$F(t) = f_{k_1} + f_{k_2} + f_{M_2} + f_B$$

$$0 = f_{k_1} + f_{M_1} + f_B$$

Automobile Suspension Example

$$m\ddot{x}_o + b(\dot{x}_o - \dot{x}_i) + k(x_o - x_i) = 0 \qquad (eq.1)$$

$$m\ddot{x}_o + b\dot{x}_o + kx_o = b\dot{x}_i + kx_i \qquad (eq.\ 2)$$

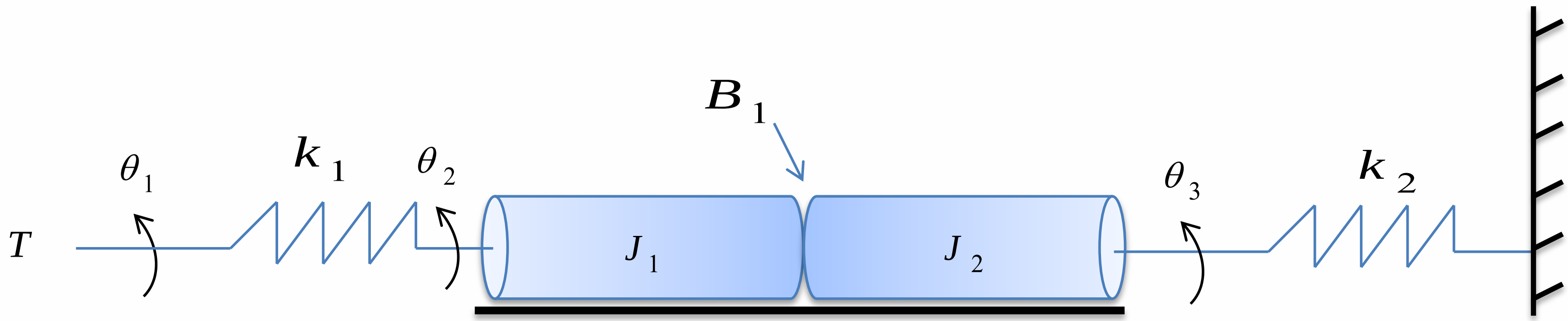


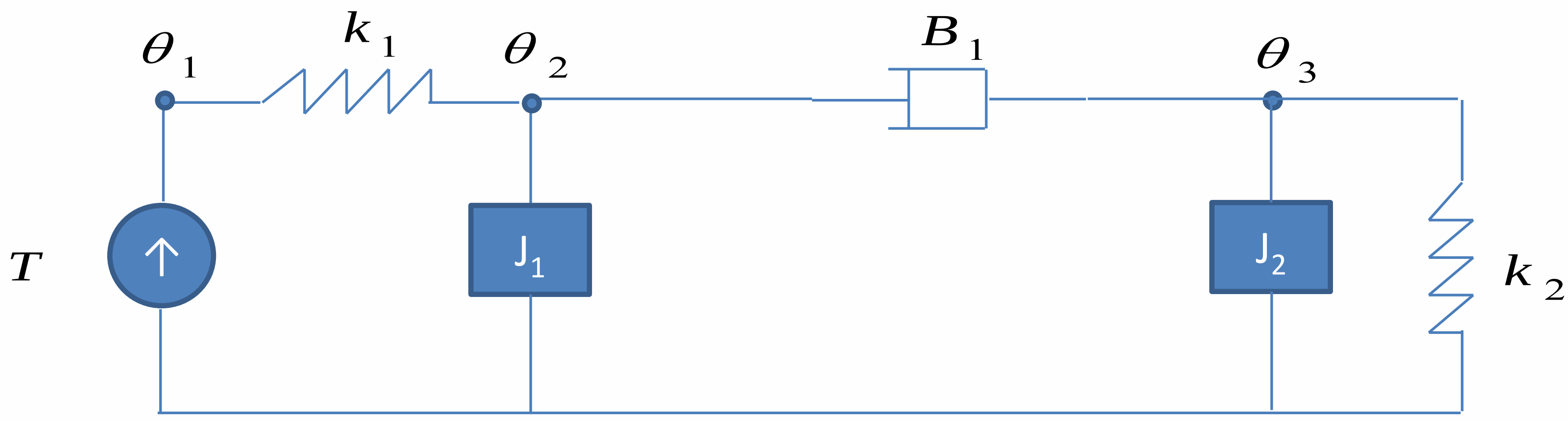
- Taking Laplace Transform of the equation (2)

$$ms^2X_o(s) + bsX_o(s) + kX_o(s) = bsX_i(s) + kX_i(s)$$

$$\frac{X_o(s)}{X_i(s)} = \frac{bs + k}{ms^2 + bs + k}$$

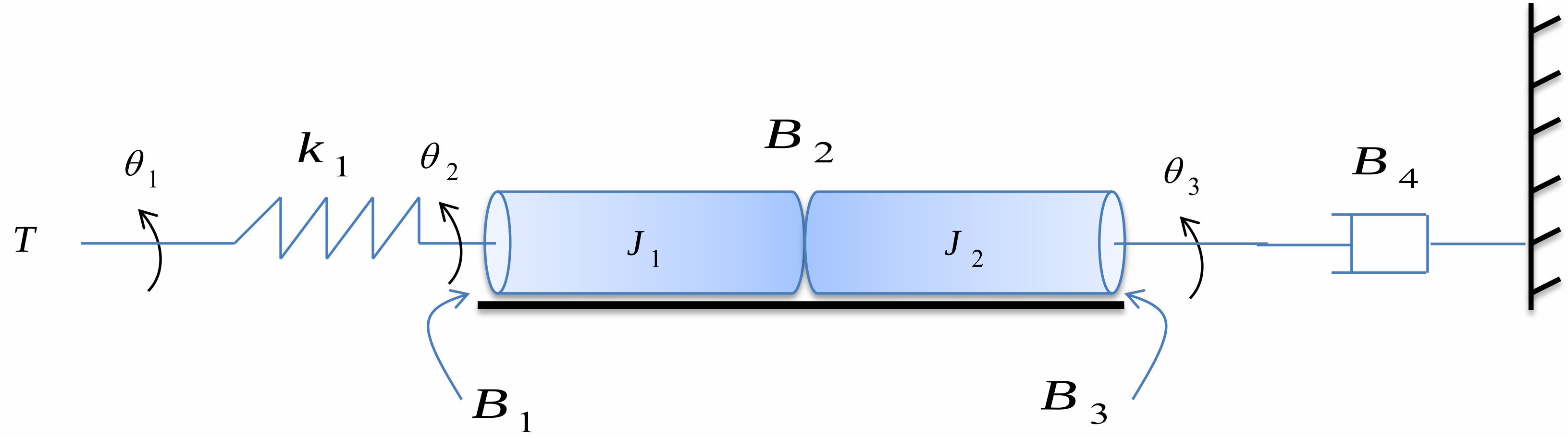
Example#1 on MRS

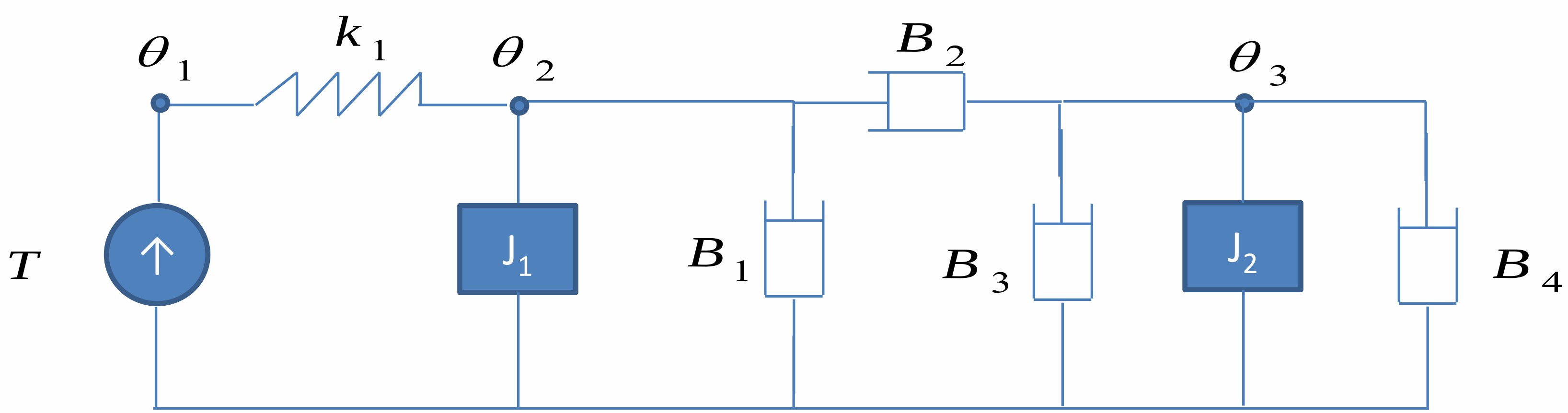




❑Write the equations similar to MTS and obtain transfer function

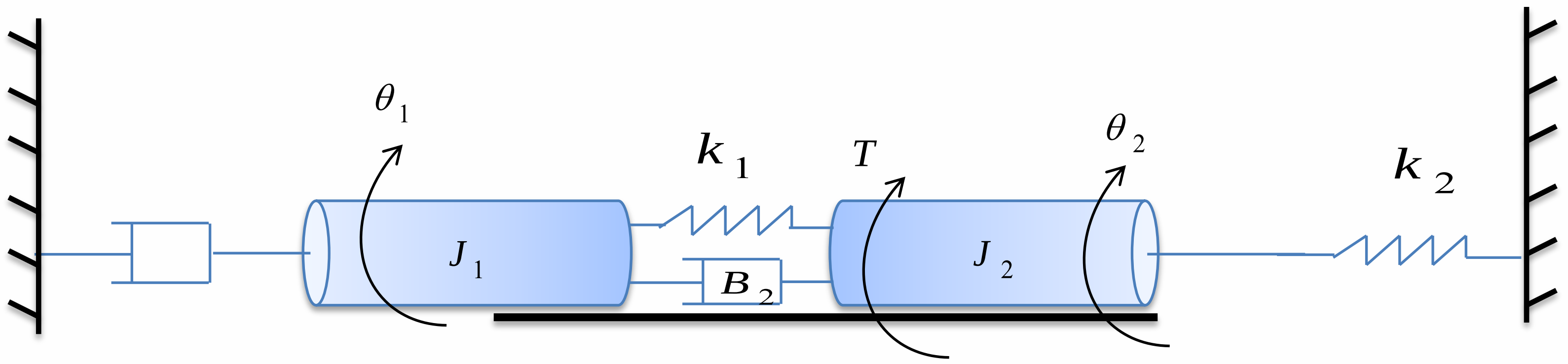
Example#2 on MRS





- Write the equations similar to MTS and obtain transfer function

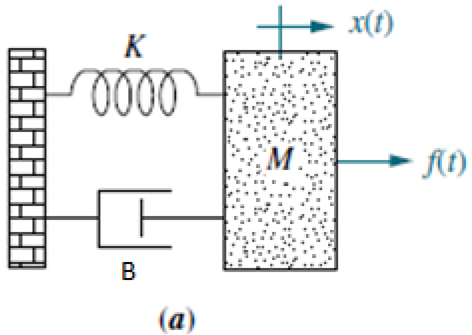
Exercise Problem on MRS



Electric Circuit Analogs

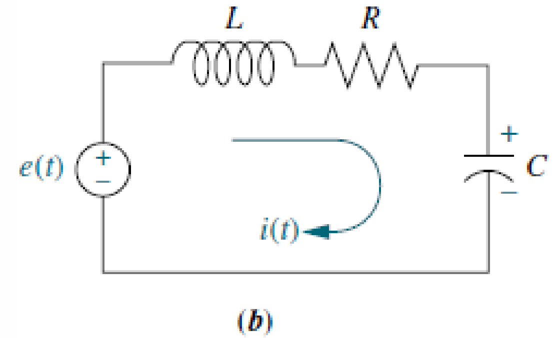
- An electric circuit that is analogous to a system from another discipline is called an electric circuit analog.
- The mechanical systems with which we worked can be represented by equivalent electric circuits.
- Analogs can be obtained by comparing the equations of motion of a mechanical system, with either electrical mesh or nodal equations.
- When compared with mesh equations, the resulting electrical circuit is called a **series analogy(Force voltage)**.
- When compared with nodal equations, the resulting electrical circuit is called a **parallel analogy(Force current)**.

Series (Force voltage) Analogy



(a)

1. Mass= M-> inductor =L
2. Damper=B-> resistor =R
3. Spring=K-> capacitor=1/C
4. Applied force=f(t)-> voltage source=e(t)
5. Velocity =v(t)-> mesh current=i(t)



(b)

Equation of motion of the above translational mechanical system is;

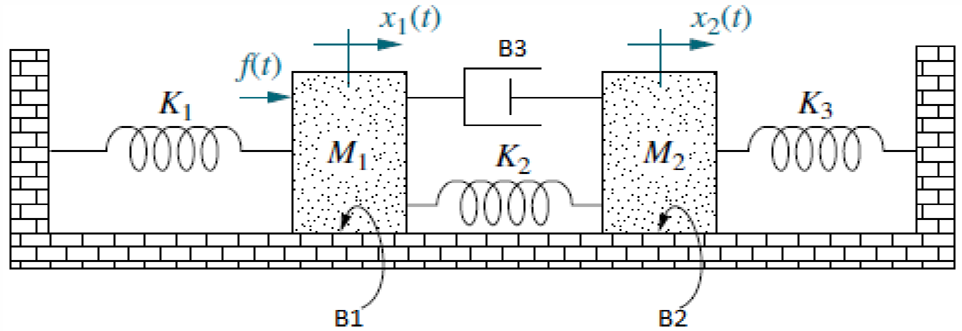
$$(Ms^2 + Bs + K)X(s) = F(s)$$

Kirchhoff's mesh equation for the above simple series RLC network is;

$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = E(s)$$

Mechanical System to a Series Analog

- ❑ Draw a series analog for the mechanical system.



- ❑ The equations of motion in the Laplace transform domain are;

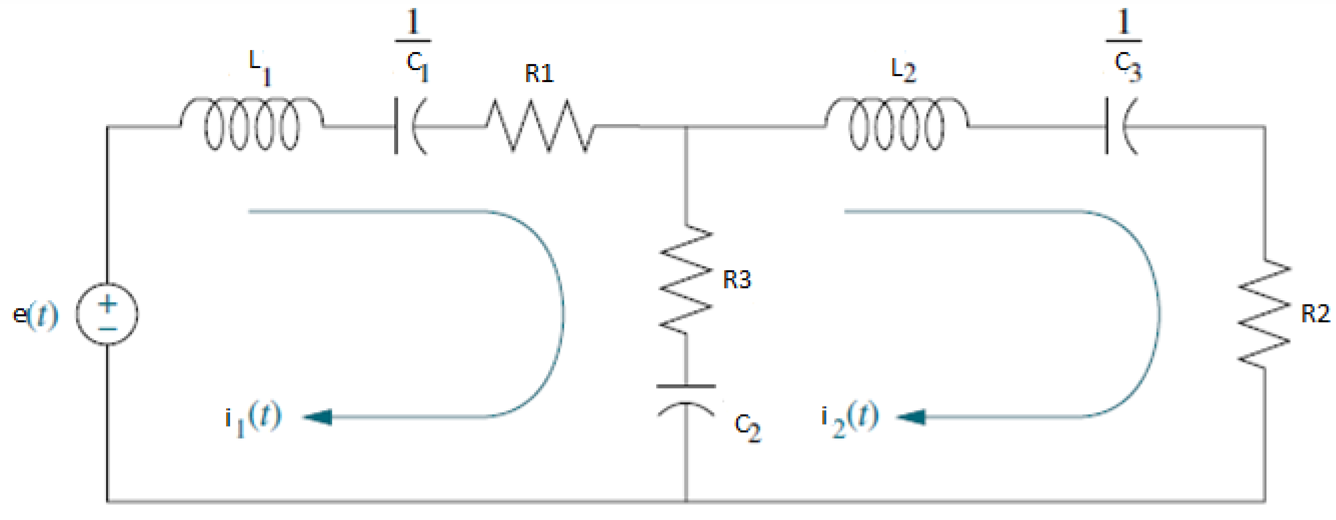
$$(M_1 s^2 + (B_1 + B_3)s + (K_1 + K_2)X_1(s) - (B_3 s + K_2)X_2(s) = F(s) \rightarrow (1)$$

$$-(B_3 s + K_2)X_1(s) + [M_2 s^2 + (B_2 + B_3)s + (K_2 + K_3)]X_2(s) = 0 \rightarrow (2)$$

- ❑ Coefficients represent sums of electrical impedance.
- ❑ Mechanical impedances associated withM1 form the first mesh,
- ❑ Where as impedances between the two masses are common to the two loops.

Mechanical System to a Series Analog

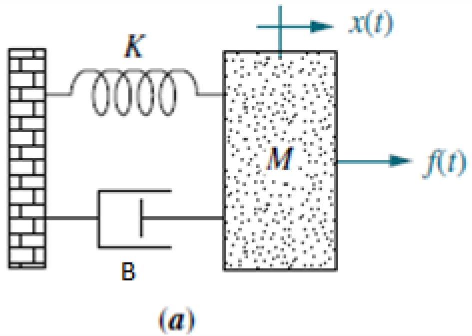
- Impedances associated with M2 form the second mesh.
- The result is shown in Figure below, where i1(t) and i2(t) are the currents of loop-1 and loop-2 respectively.



$$L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int i_1 dt + R_3(i_1 - i_2) + \frac{1}{C_2}\int (i_1 - i_2)dt = e(t) \rightarrow (1)$$

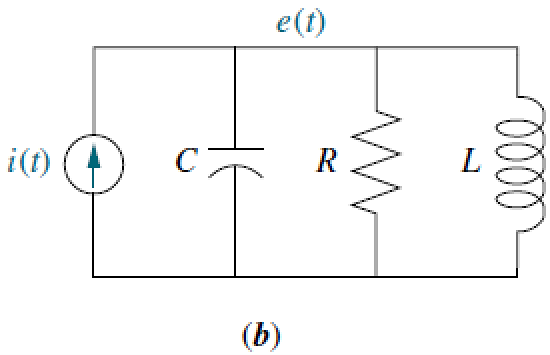
$$L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_3}\int i_2 dt + R_3(i_2 - i_1) + \frac{1}{C_2}\int (i_2 - i_1)dt = 0 \rightarrow (2)$$

Parallel (Force current) Analogy



(a)

1. Mass= M-> capacitor =C
2. Damper=B-> resistor =1/R
3. Spring=K-> inductor=1/L
4. Applied force=f(t)-> current source=i(t)
5. Velocity =v(t)-> node voltage =v(t)



(b)

- Equation of motion of the above translational mechanical system is;

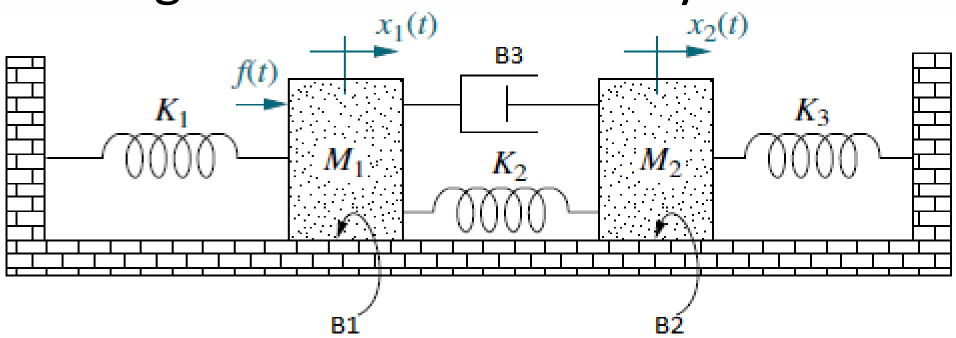
$$(Ms^2 + Bs + K)X(s) = F(s)$$

- Kirchhoff's nodal equation for the simple parallel RLC network shown above is;

$$\left(Cs + R + \frac{1}{Ls}\right)V(s) = I(s)$$

Mechanical System to a Parallel Analog

- Draw a parallel analog for the mechanical system.



- Equations of motion after conversion to velocity are;

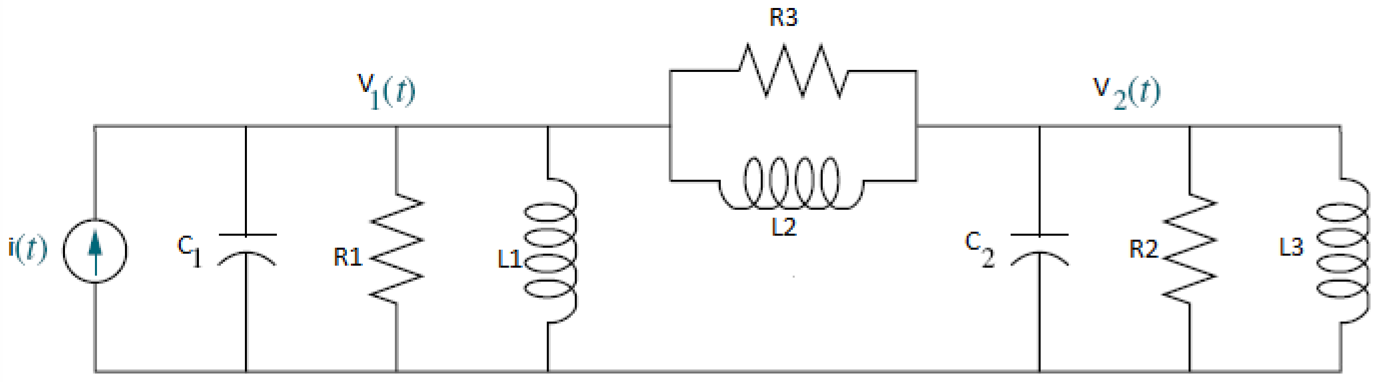
$$\left[(M_1 s + (B_1 + B_3) + \frac{(K_1 + K_2)}{s}\right] v_1(s) - \left(B_3 + \frac{K_2}{s}\right) v_2(s) = F(s) \rightarrow (1)$$

$$-\left(B_3 + \frac{K_2}{s}\right) v_1(s) + \left[M_2 s + (B_2 + B_3) + \frac{(K_2 + K_3)}{s}\right] v_2(s) = 0 \rightarrow (2)$$

- The Equation (1) and (2) are also analogous to electrical node equations.
- Coefficients represent sums of electrical admittances.
- Admittances associated with M1 form the elements connected to the first node

Mechanical System to a Parallel Analog

- whereas mechanical admittances b/w the two masses are common to the two nodes.
- Mechanical admittances associated with M2 form the elements connected to the second node.
- The result is shown in the Figure below, where V1(t) and V2(t) are the voltages of node-1 and node-2, respectively.



$$C_1 \frac{dV_1}{dt} + \frac{1}{R_1} V_1 + \frac{1}{L_1} \int V_1 dt + \frac{1}{R_3}(V_1 - V_2) + \frac{1}{L_2} \int (V_1 - V_2) dt = i(t) \rightarrow (1)$$

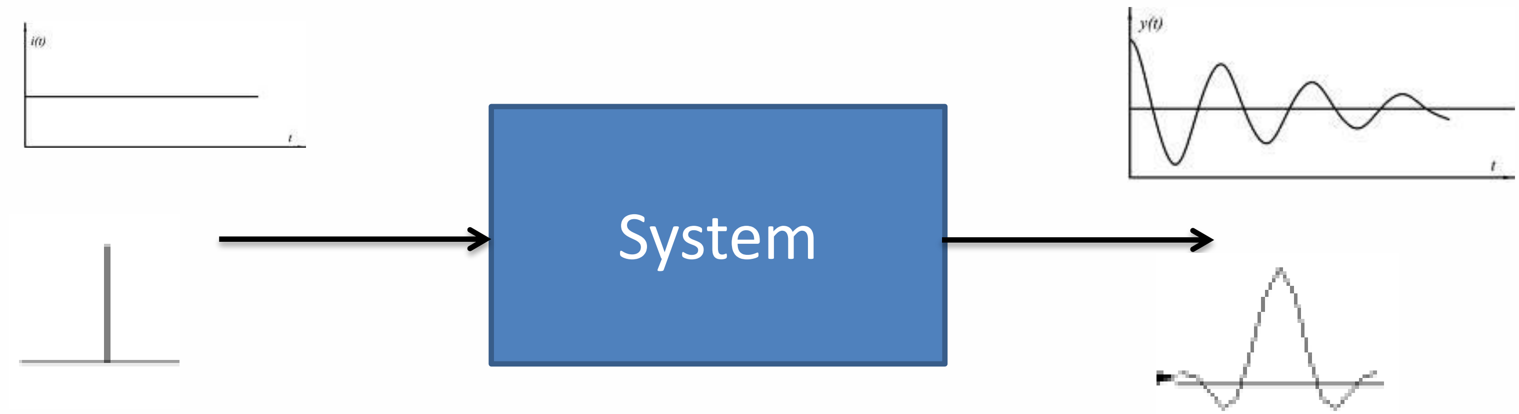
$$C_2 \frac{dV_2}{dt} + \frac{1}{R_2} V_2 + \frac{1}{L_3} \int V_2 dt + \frac{1}{R_3}(V_2 - V_1) + \frac{1}{L_2} \int (V_2 - V_1) dt = 0 \rightarrow (2)$$

Unit-2

BLOCK DIAGRAM REDUCTION AND TIME RESPONSE ANALYSIS

Introduction

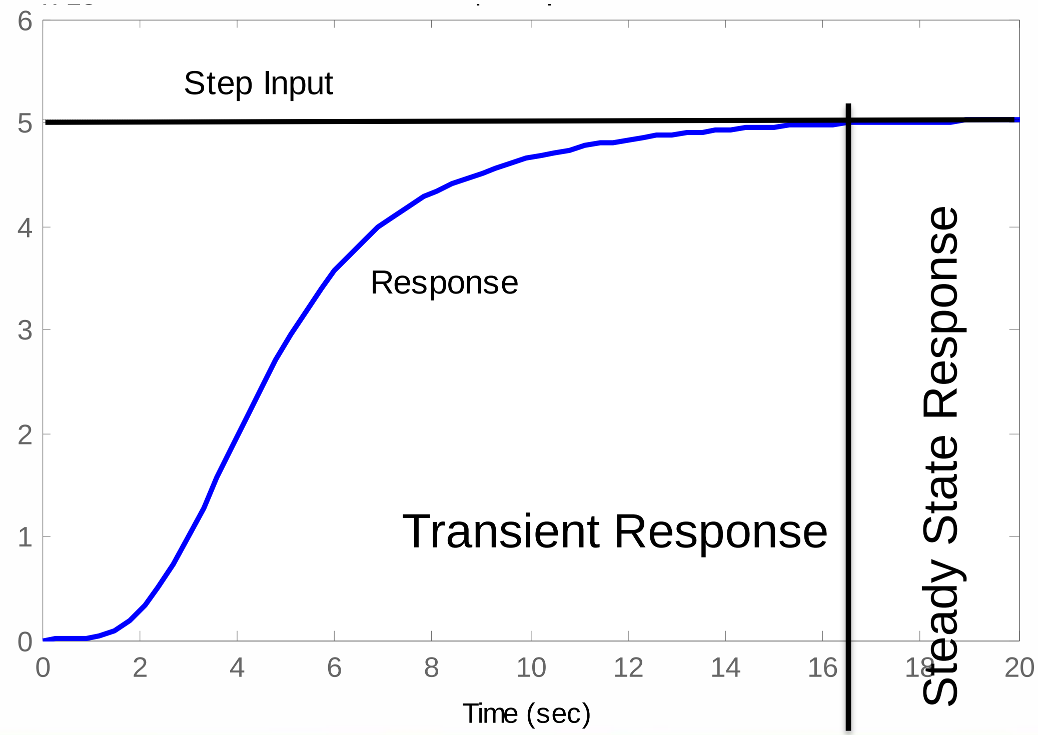
- Time response of a dynamic system response to an input expressed as a function of time.



- The time response of any system has two components
  - Transient response
  - Steady-state response

Introduction (Contd..)

- When the response of the system is changed from equilibrium it takes some time to settle down.
- This is called transient response.
- The response of the system after the transient response is called steady state response.



Introduction (Contd..)

- Transient response depend upon the system poles only and not on the type of input.

- It is therefore sufficient to analyze the transient response using a step input.

- The steady-state response depends on system dynamics and the input quantity.

- It is then examined using different test signals by final value theorem.

Introduction (Contd..)

- The first order system has only one pole.

$$\frac{C(s)}{R(s)} = \frac{K}{Ts + 1}$$

- Where ***K*** is the D.C gain and ***T*** is the time constant of the system.

- Time constant is a measure of how quickly a 1st order system responds to a unit step input.

- D.C Gain of the system is ratio between the input signal and the steady state value of output.

Introduction (Contd..)

- The first order system given below.

$$G(s) = \frac{10}{3s + 1}$$

- D.C gain is 10 and time constant is 3 seconds.

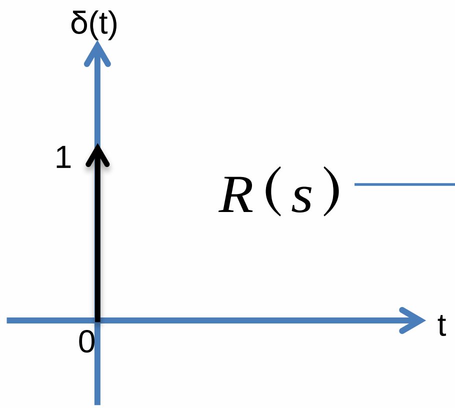
- For the following system

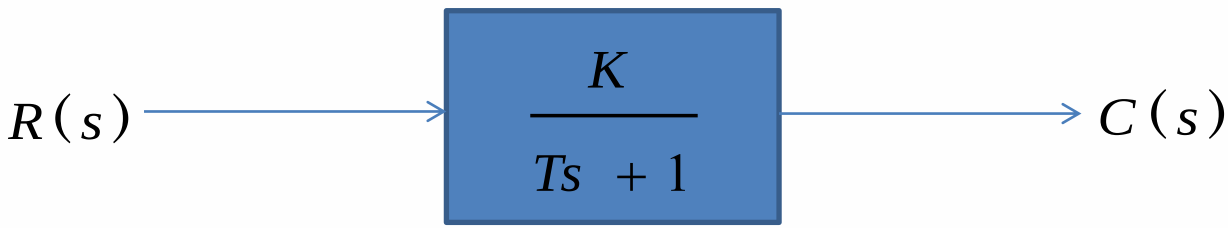
$$G(s) = \frac{3}{s + 5} = \frac{3/5}{1/5s + 1}$$

- D.C Gain of the system is 3/5 and time constant is 1/5 seconds.

Impulse Response of 1st Order System

- Consider the following 1st order system





$$R(s) = \delta(s) = 1$$

$$C(s) = \frac{K}{Ts+1}$$

Impulse Response of 1st Order System

$$C(s) = \frac{K}{Ts + 1}$$

- Re-arrange following equation as

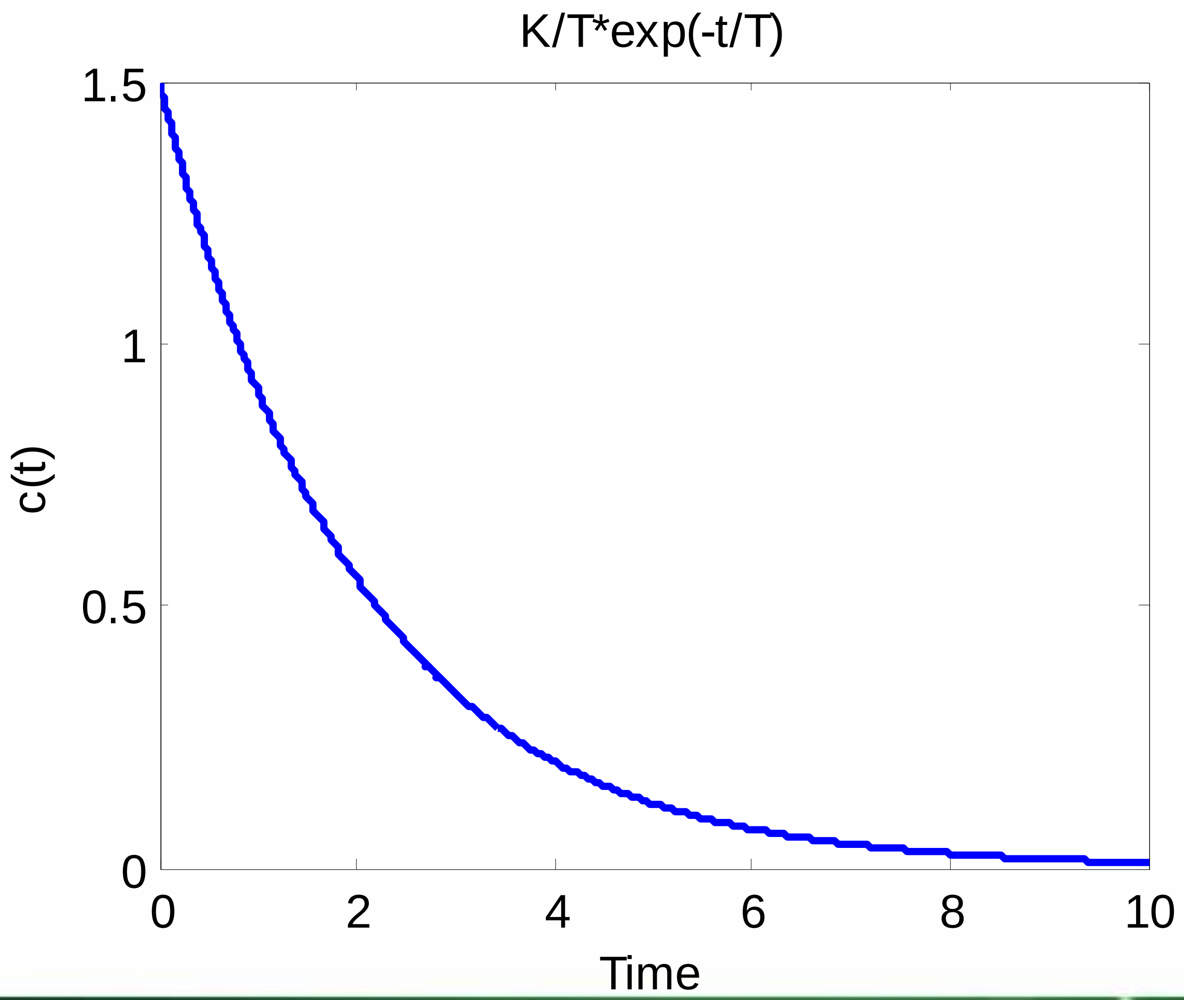
$$C(s) = \frac{K/T}{s + 1/T}$$

- In order to compute the response of the system in time domain we need to compute inverse Laplace transform of the above equation.

$$L^{-1}\left(\frac{C}{s+a}\right) = Ce^{-at} \qquad c(t) = \frac{K}{T}e^{-t/T}$$

Impulse Response of 1st Order System

- If K=3 and T=2s then $$c(t) = \frac{K}{T} e^{-t/T}$$



Step Response of 1st Order System

- Consider the following 1st order system

$$R(s) \longrightarrow \boxed{\frac{K}{Ts+1}} \longrightarrow C(s)$$

$$R(s) = U(s) = \frac{1}{s}$$

$$C(s) = \frac{K}{s(Ts+1)}$$

- In order to find out the inverse Laplace of the above equation, we need to break it into partial fraction expansion

$$C(s) = \frac{K}{s} - \frac{KT}{Ts+1}$$

Step Response of 1st Order System

$$C(s) = K\left(\frac{1}{s} - \frac{T}{Ts + 1}\right)$$

- Taking Inverse Laplace of above equation

$$c(t) = K\left(u(t) - e^{-t/T}\right)$$

- Where u(t)=1

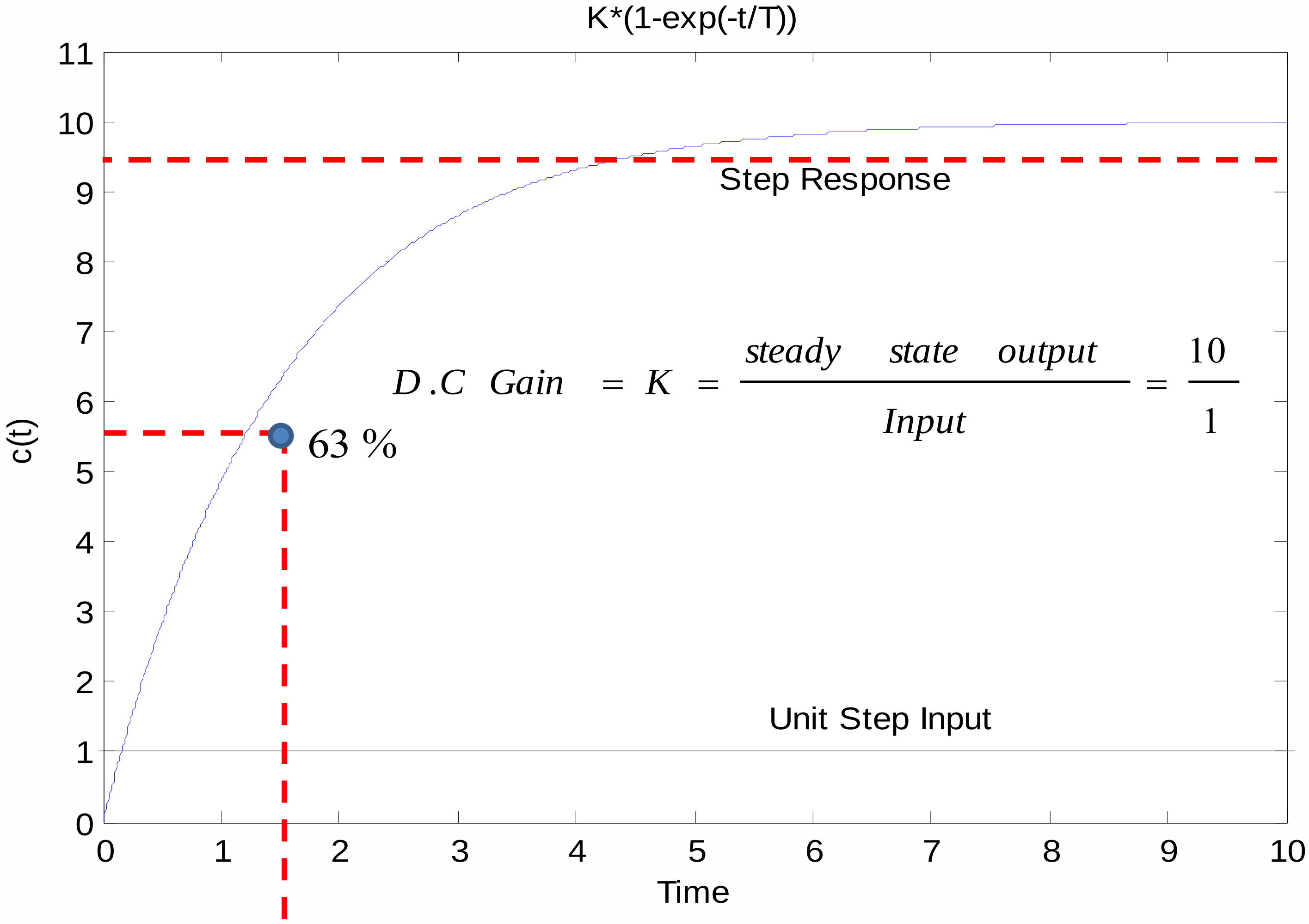
$$c(t) = K\left(1 - e^{-t/T}\right)$$

- When t=T (time constant)

$$c(t) = K\left(1 - e^{-1}\right) = 0.632\,K$$

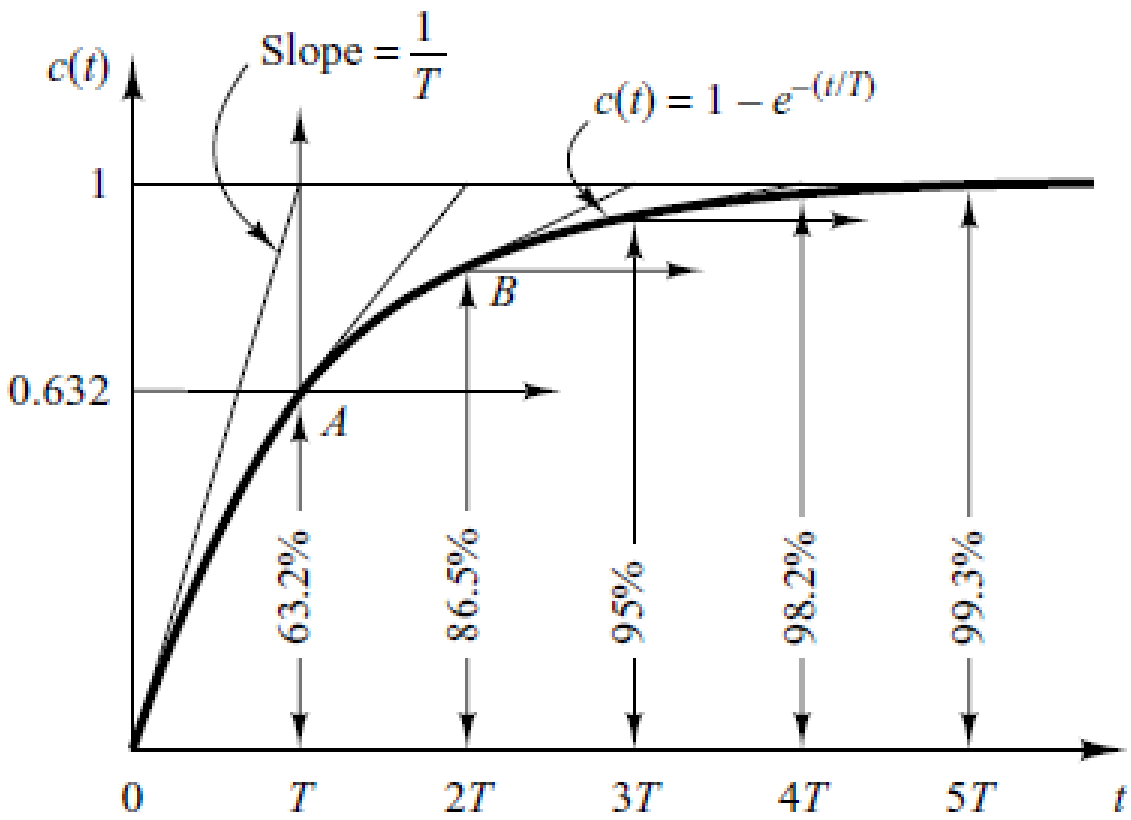
Step Response of 1st Order System

- If K=10 and T=1.5s then $c(t) = K\left(1 - e^{-t/T}\right)$



Step Response of 1st Order System

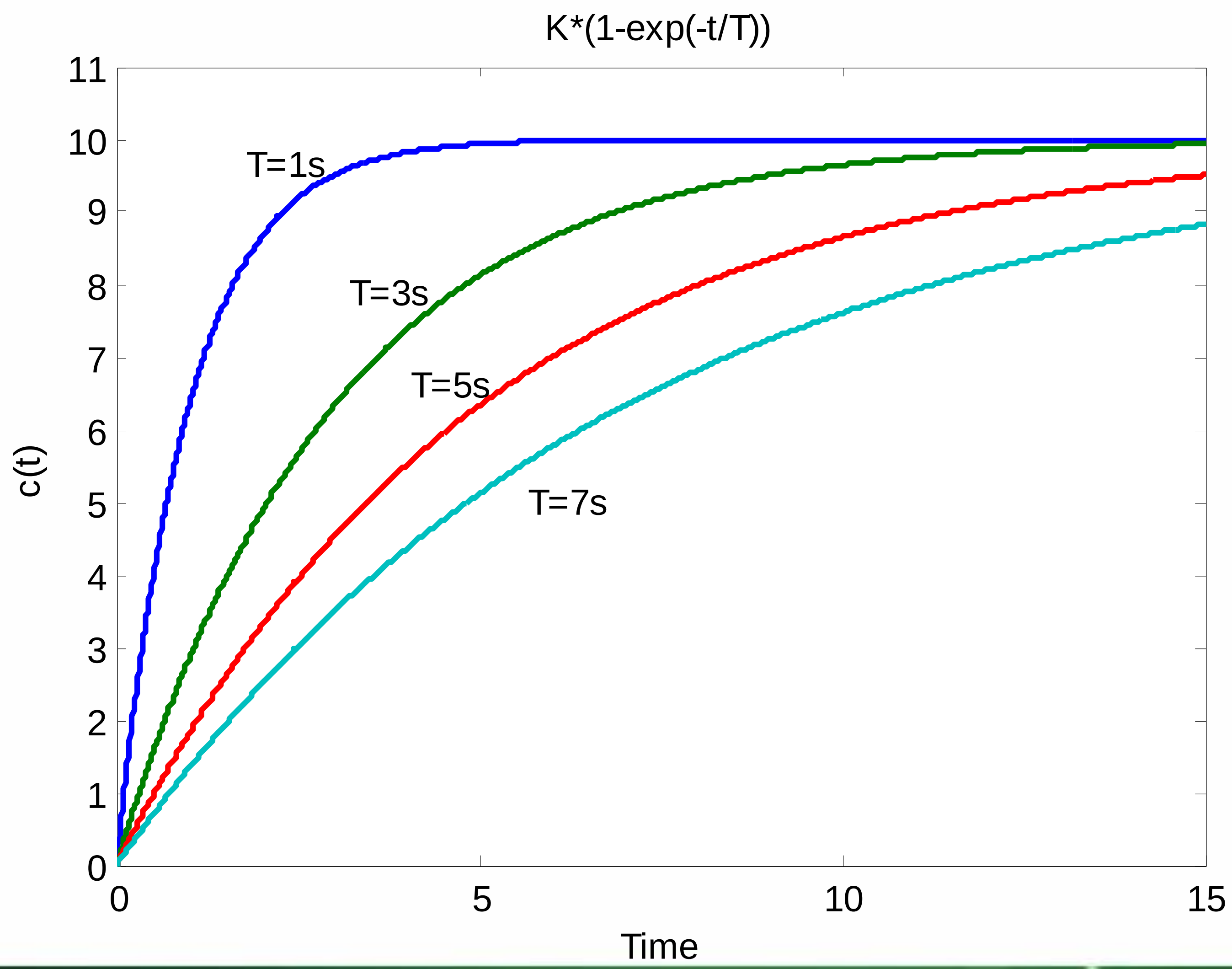
- System takes five time constants to reach its final value.



Step Response of 1st Order System

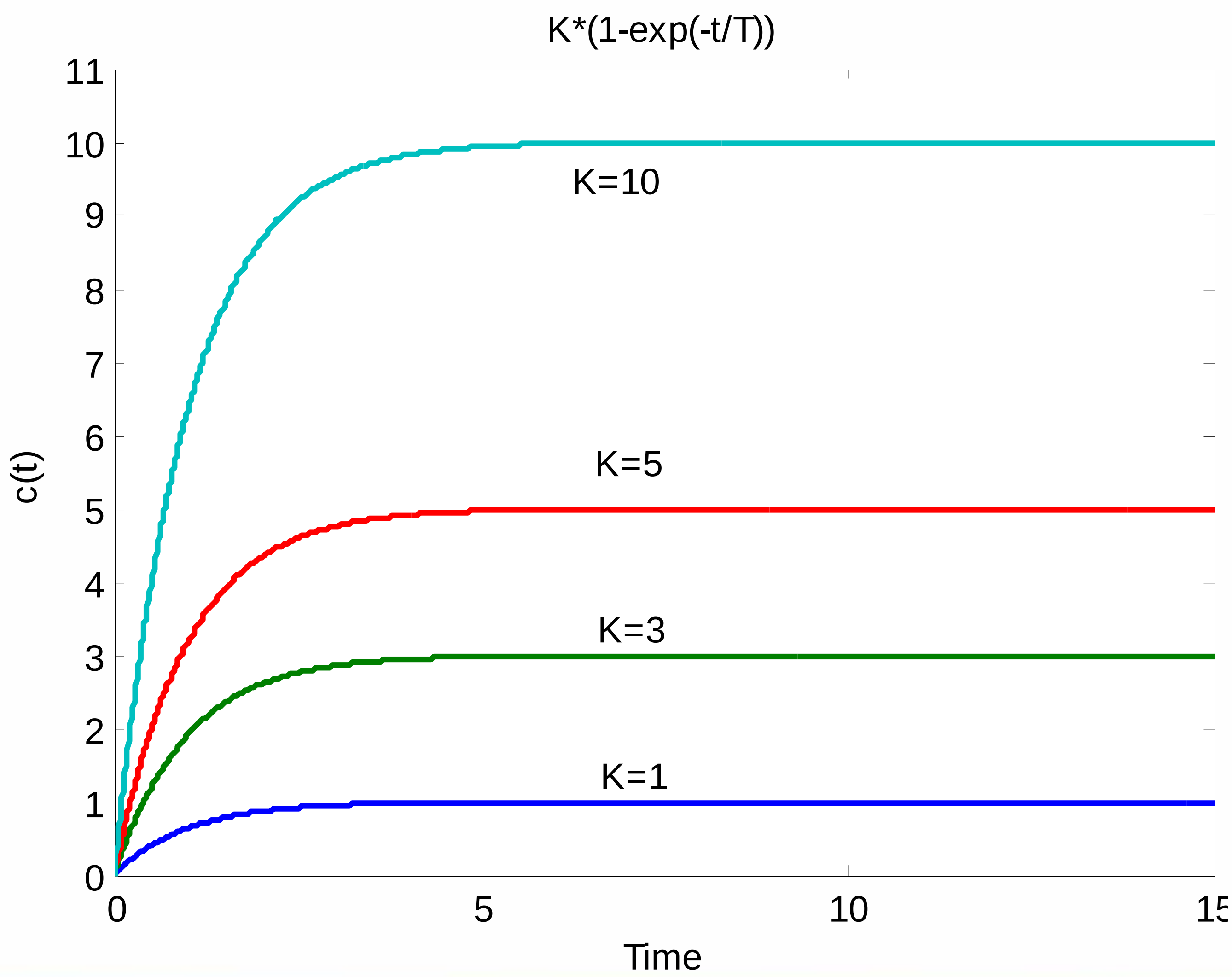
- If K=10 and T=1, 3, 5, 7

$$c(t) = K\left(1 - e^{-t/T}\right)$$



Step Response of 1st Order System

- If K=1, 3, 5, 10 and T=1 $c(t) = K\left(1 - e^{-t/T}\right)$



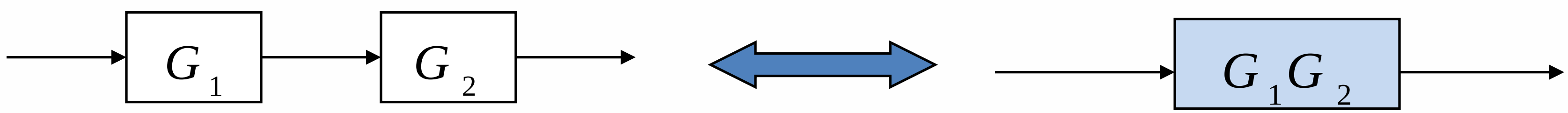
Block Diagram Algebra

- We often represent control systems using block diagrams. A block diagram consists of blocks that represent transfer functions of the different variables of interest.

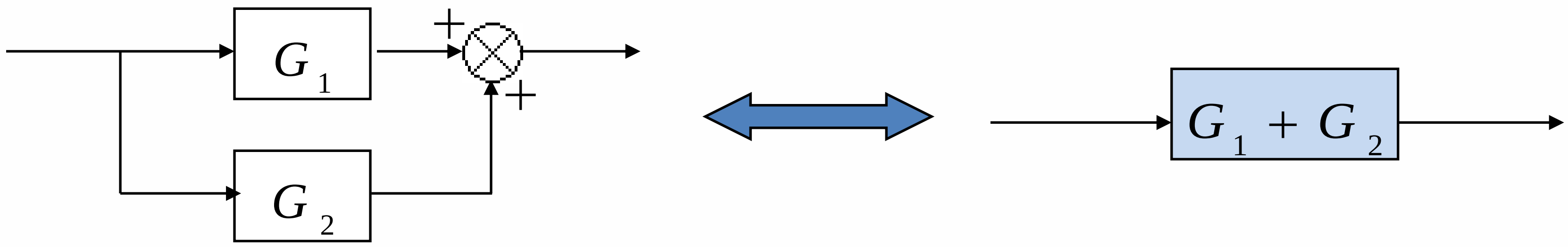
- If a block diagram has many blocks, not all of which are in cascade, then it is useful to have rules for rearranging the diagram such that you end up with only one block.

Reduction techniques

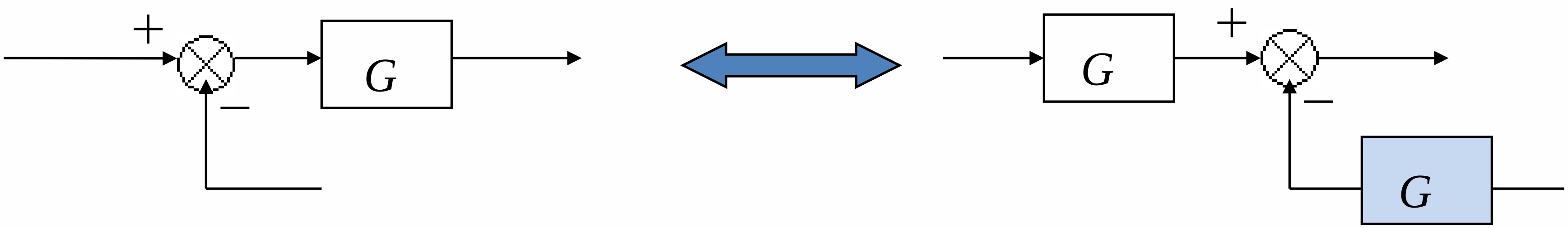
1. Combining blocks in cascade



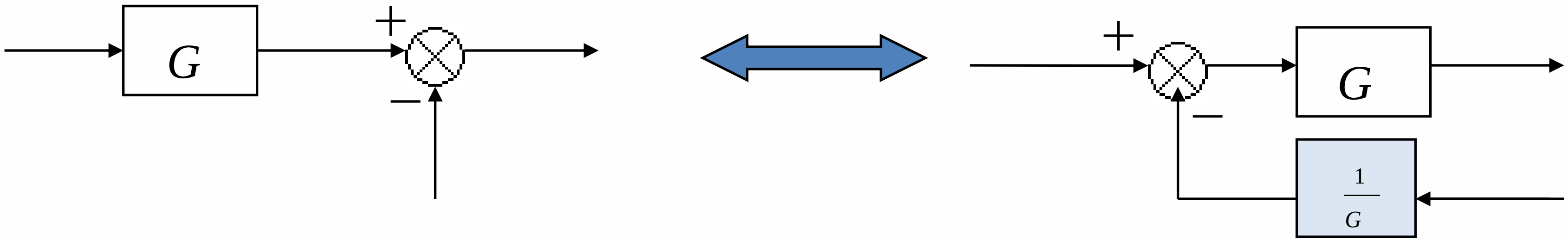
2. Combining blocks in parallel



3. Moving a summing point behind a block

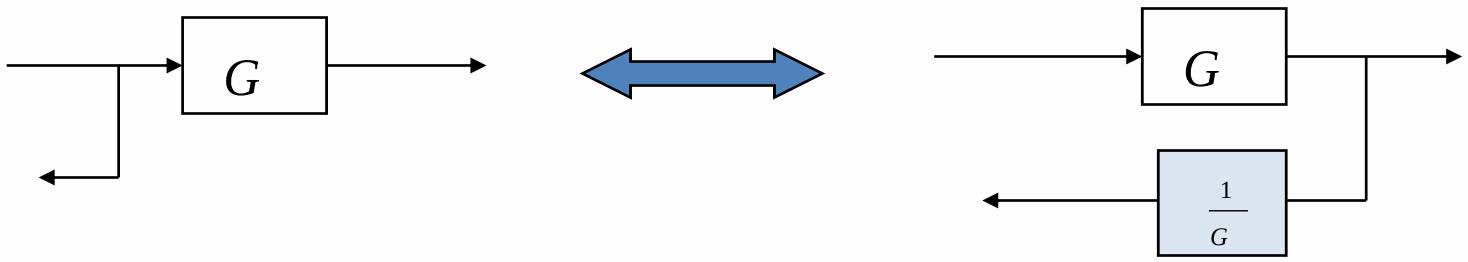


Moving a summing point ahead of a block

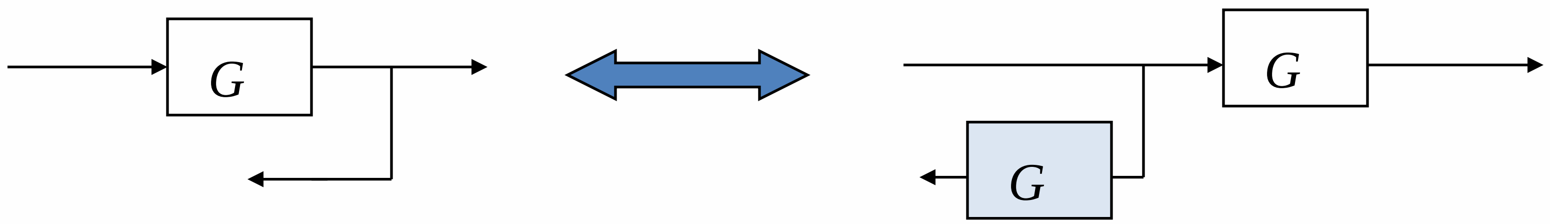


Reduction techniques

4. Moving a pickoff point behind a block

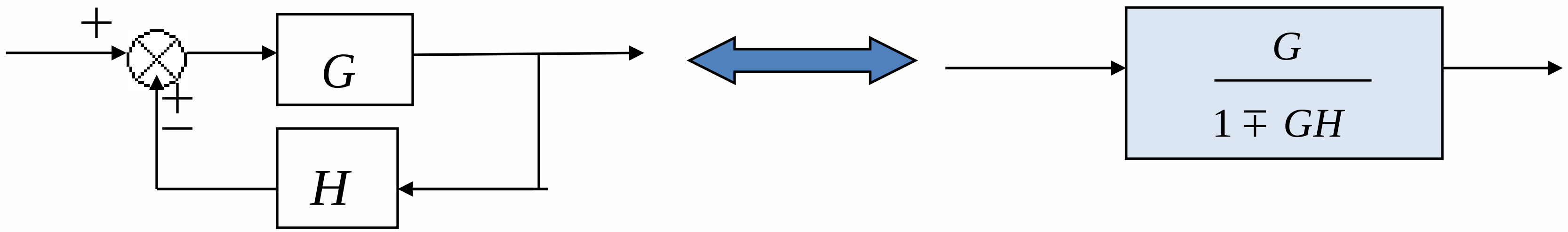


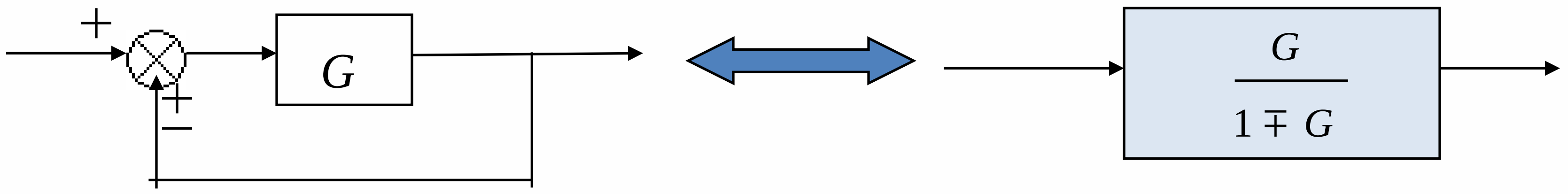
5. Moving a pickoff point ahead of a block



Reduction techniques

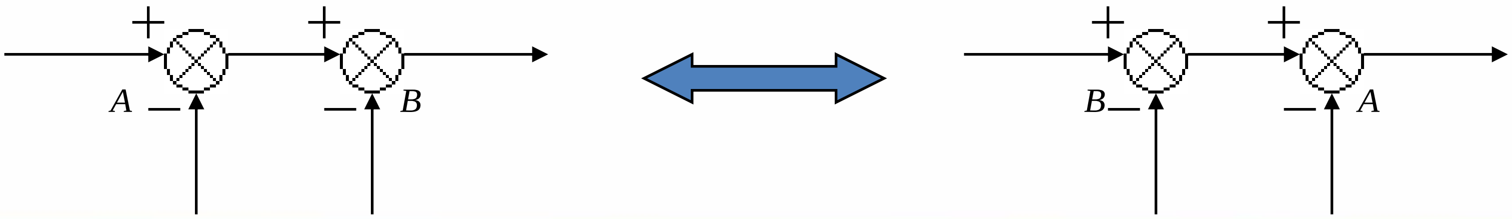
6. Eliminating a feedback loop





$H = 1$

7. Swap with two neighboring summing points



Signal flow graphs

- Alternative method to block diagram representation, developed by Samuel Jefferson Mason.

- Advantage: the availability of a flow graph gain formula, also called Mason's gain formula.

- A signal-flow graph consists of a network in which nodes are connected by directed branches.

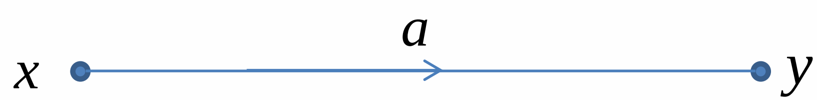
- It depicts the flow of signals from one point of a system to another and gives the relationships among the signals.

Fundamentals of Signal Flow Graphs

- Consider a simple equation below and draw its signal flow graph:

$$y = ax$$

- The signal flow graph of the equation is shown below;



- Every variable in a signal flow graph is designed by a **Node**.
- Every transmission function in a signal flow graph is designed by a **Branch**.
- Branches are always **unidirectional**.
- The arrow in the branch denotes the **direction** of the signal flow.

Terminologies

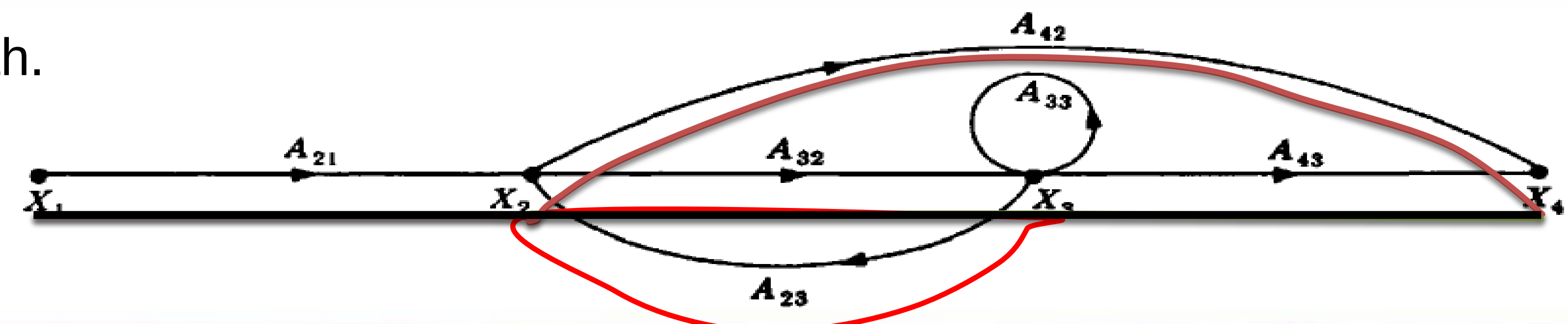
- An **input node** or source contain only the outgoing branches. i.e., X_1
- An **output node** or sink contain only the incoming branches. i.e., X_4
- A **path** is a continuous, unidirectional succession of branches along which no node is passed more than ones. i.e.,

X_1 *to* X_2 *to* X_4 X_2 *to* X_3 *to* X_4

- A **forward path** is a path from the input node to the output node. i.e.,

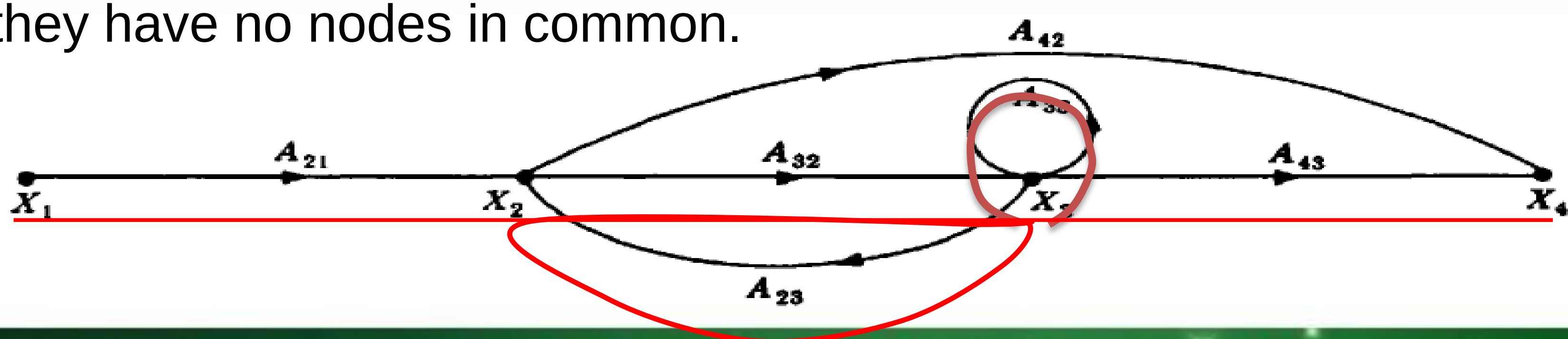
X_1 *to* X_2 *to* X_3 *to* X_4 , and X_1 *to* X_2 *to* X_4 , are forward paths.

- A **feedback path** or feedback loop is a path which originates and terminates on the same node. i.e.; X_2 *to* X_3 **and back to** X_2 is a feedback path.



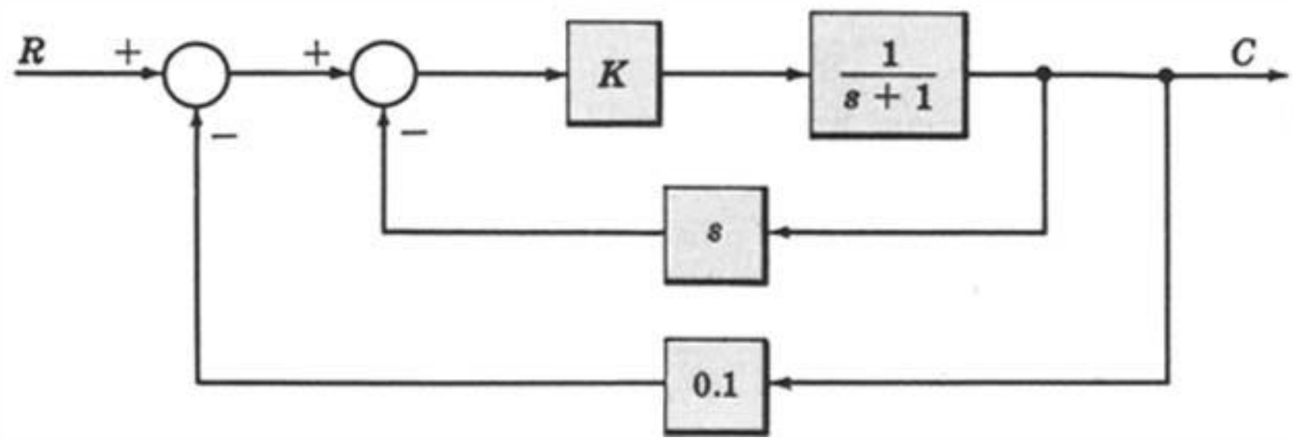
Terminologies

- A **self-loop** is a feedback loop consisting of a single branch. i.e.; A_{33} is a self loop.
- The **gain** of a branch is the transmission function of that branch.
- The **path gain** is the product of branch gains encountered in traversing a path. i.e. the gain of forwards path X_1 *to* X_2 *to* X_3 *to* X_4 is $A_{21}A_{32}A_{43}$
- The **loop gain** is the product of the branch gains of the loop. i.e., the loop gain of the feedback loop from X_2 to X_3 and back to X_2 is $A_{32}A_{23}$.
- Two loops, paths, or loop and a path are said to be **non-touching** if they have no nodes in common.



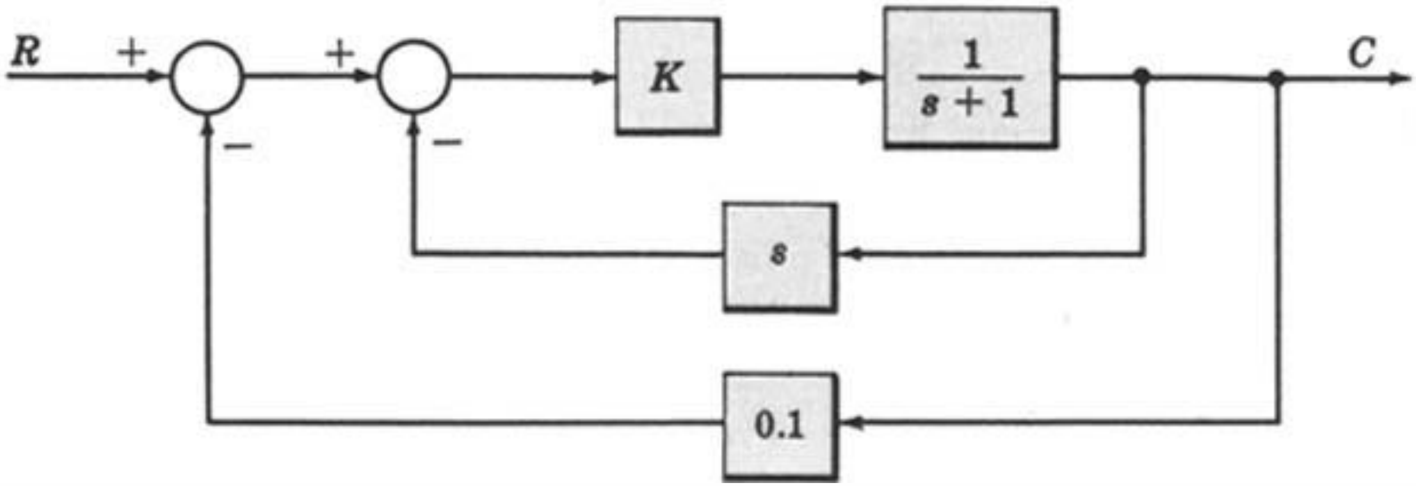
Block Diagram Reduction-Example-1

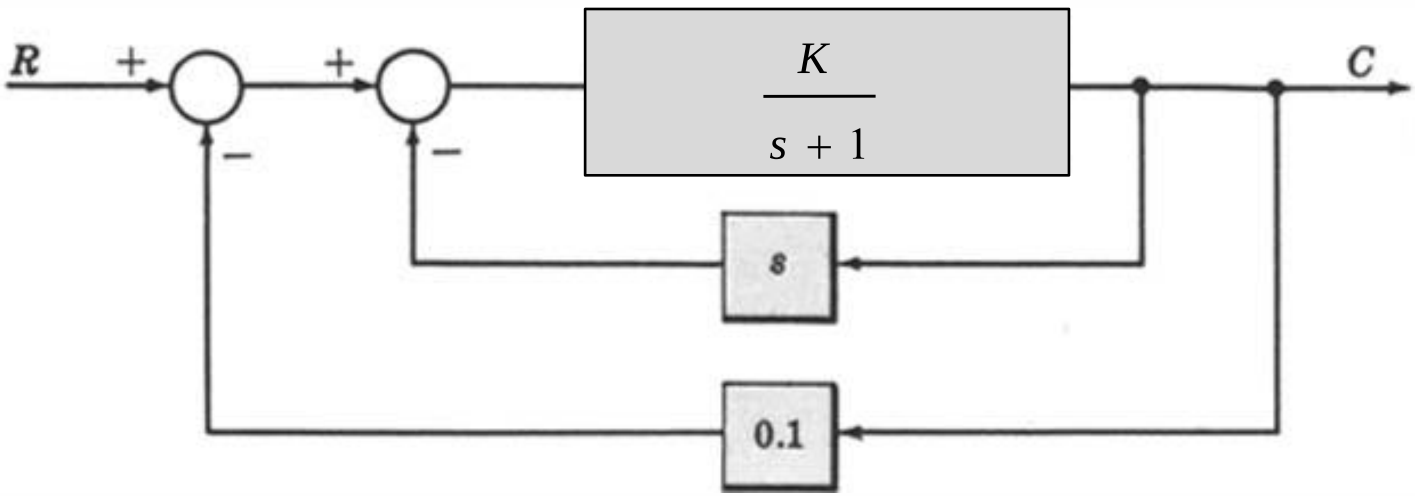
- For the system represented by the following block diagram determine:
  1. Open loop transfer function
  2. Feed Forward Transfer function
  3. control ratio
  4. feedback ratio
  5. error ratio
  6. closed loop transfer function
  7. characteristic equation



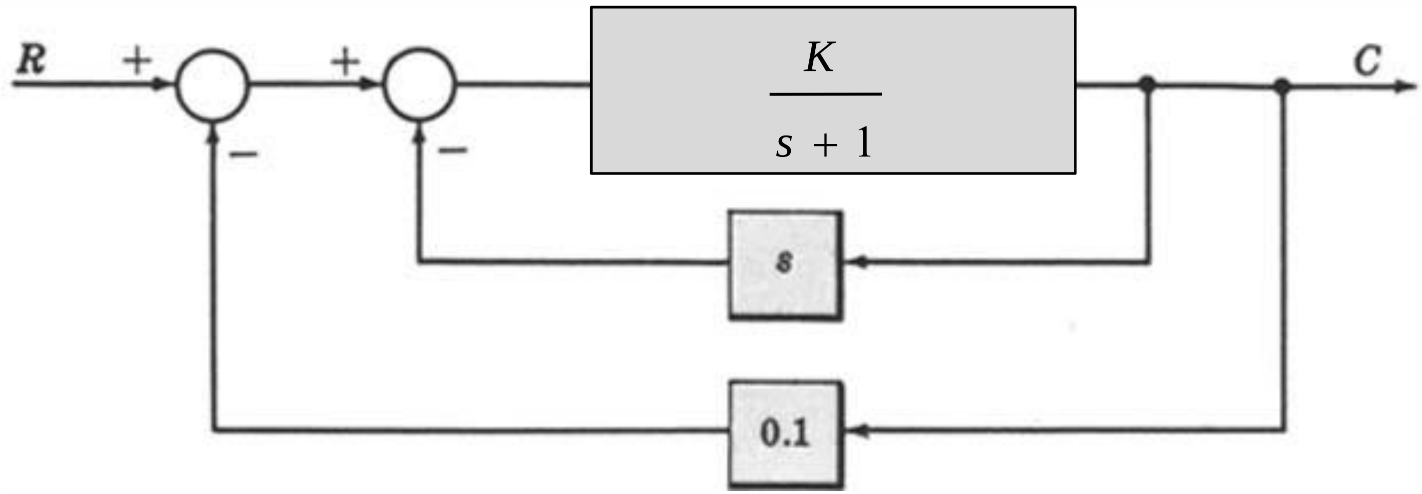
Block Diagram Reduction-Example-1 (Contd..)

- First we will reduce the given block diagram to canonical form

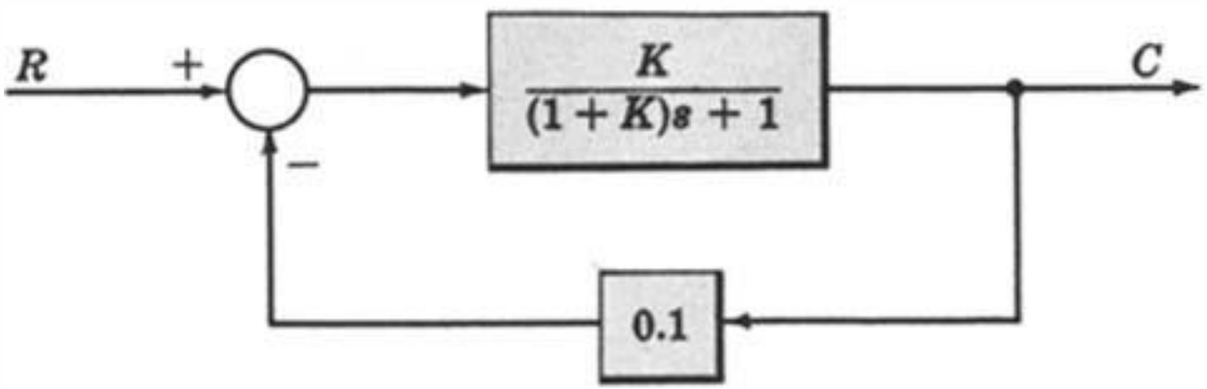




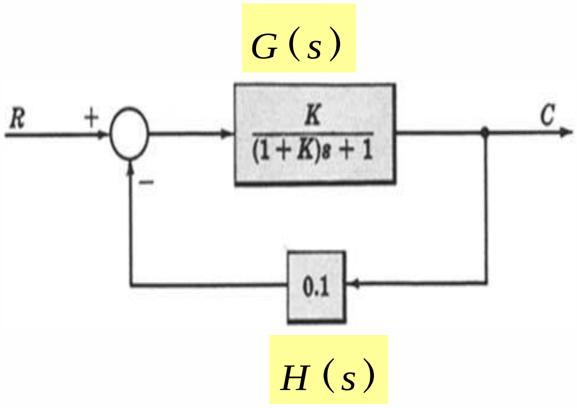
Block Diagram Reduction-Example-1 (Contd..)



$$\frac{G}{1 + GH} = \frac{\dfrac{K}{s+1}}{1 + \dfrac{K}{s+1}s}$$



Block Diagram Reduction-Example-1 (Contd..)



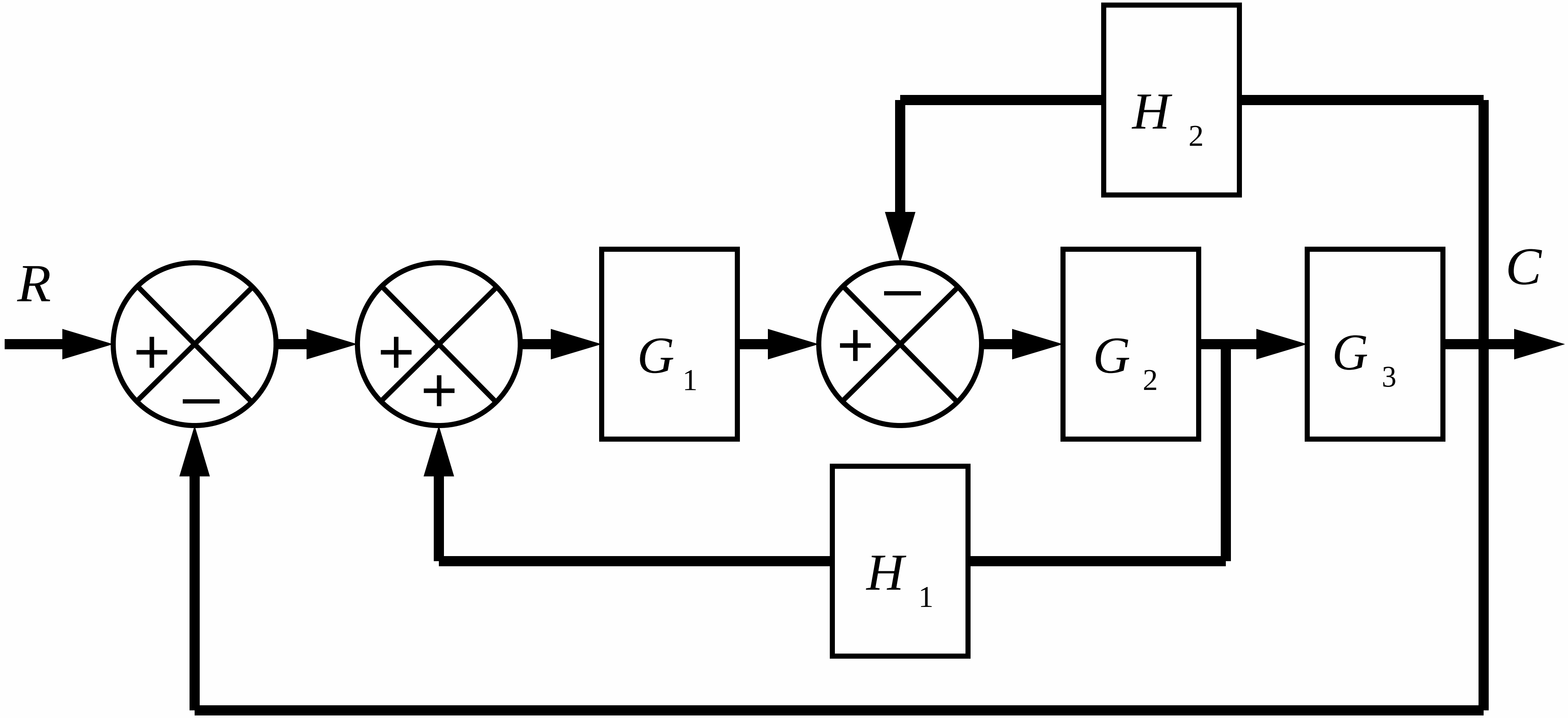
1. Open loop transfer functio $\frac{B(s)}{E(s)} = G(s)H(s)$
2. Feed Forward Transfe $\frac{C(s)}{E(s)} = G(s)$

   function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$
3. Control ratio $\frac{B(s)}{R(s)} = \frac{G(s)H(s)}{1+G(s)H(s)}$
4. Feedback $\frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$
5. Error ratio $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$
6. Closed loop transfer function

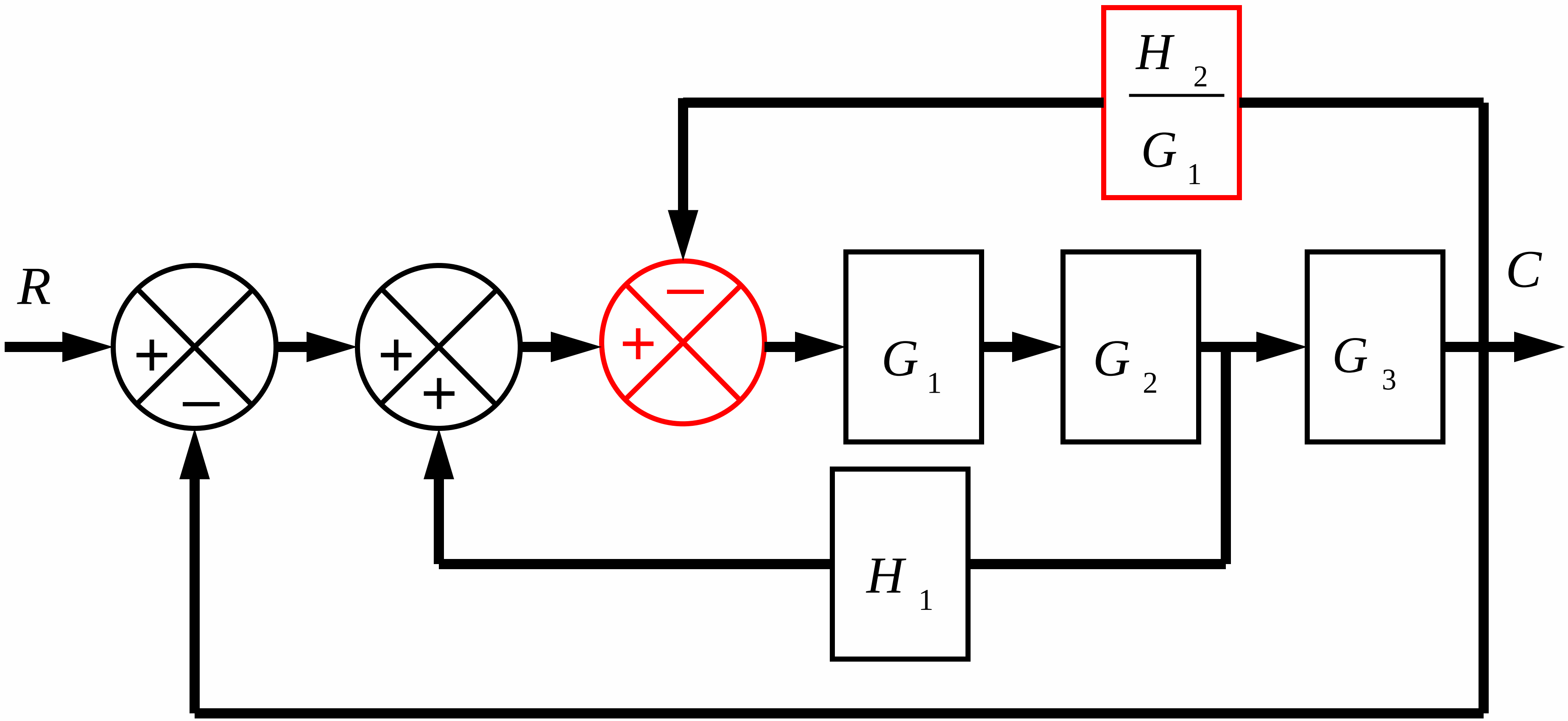
7. Characteristic equation

$$1 + G(s)H(s) = 0$$

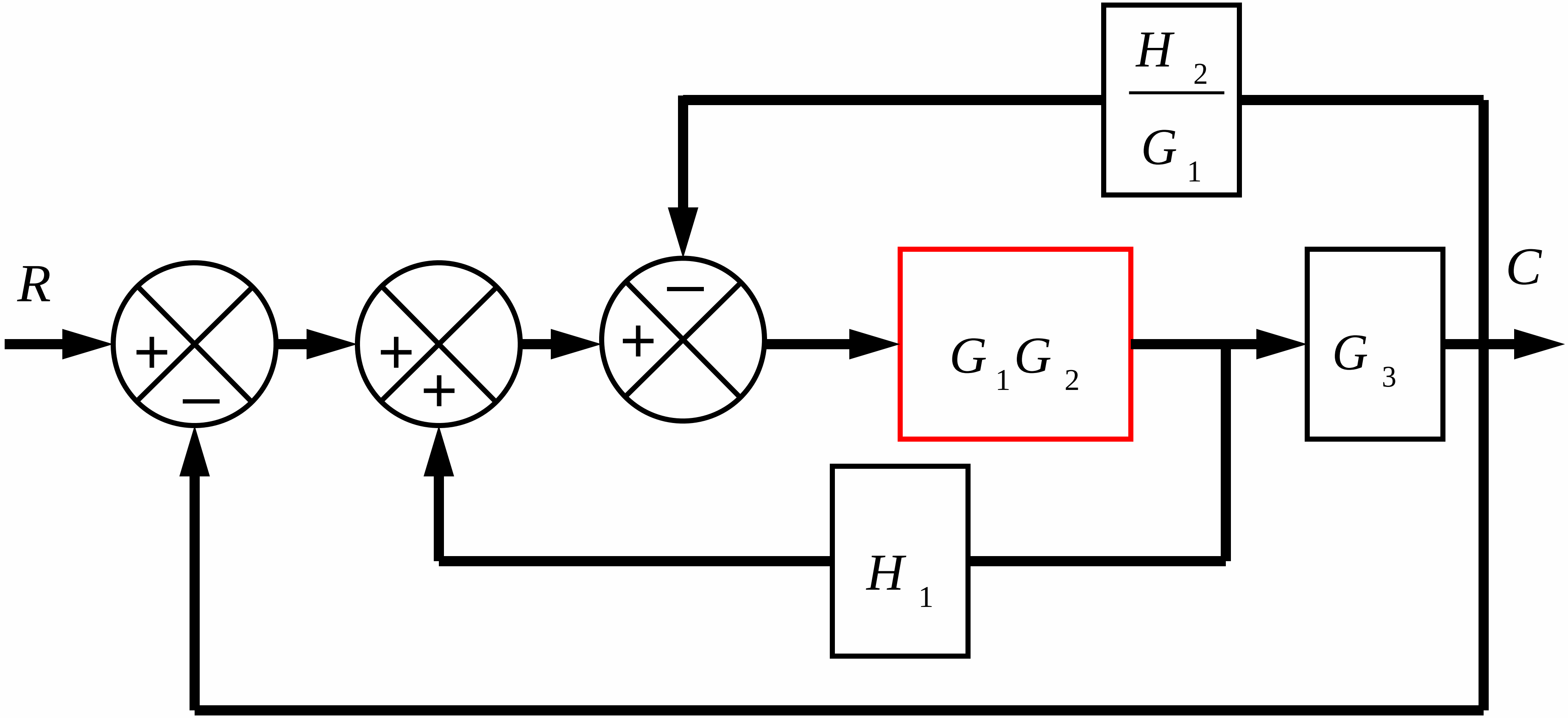
Block Diagram: Reduction Example-2



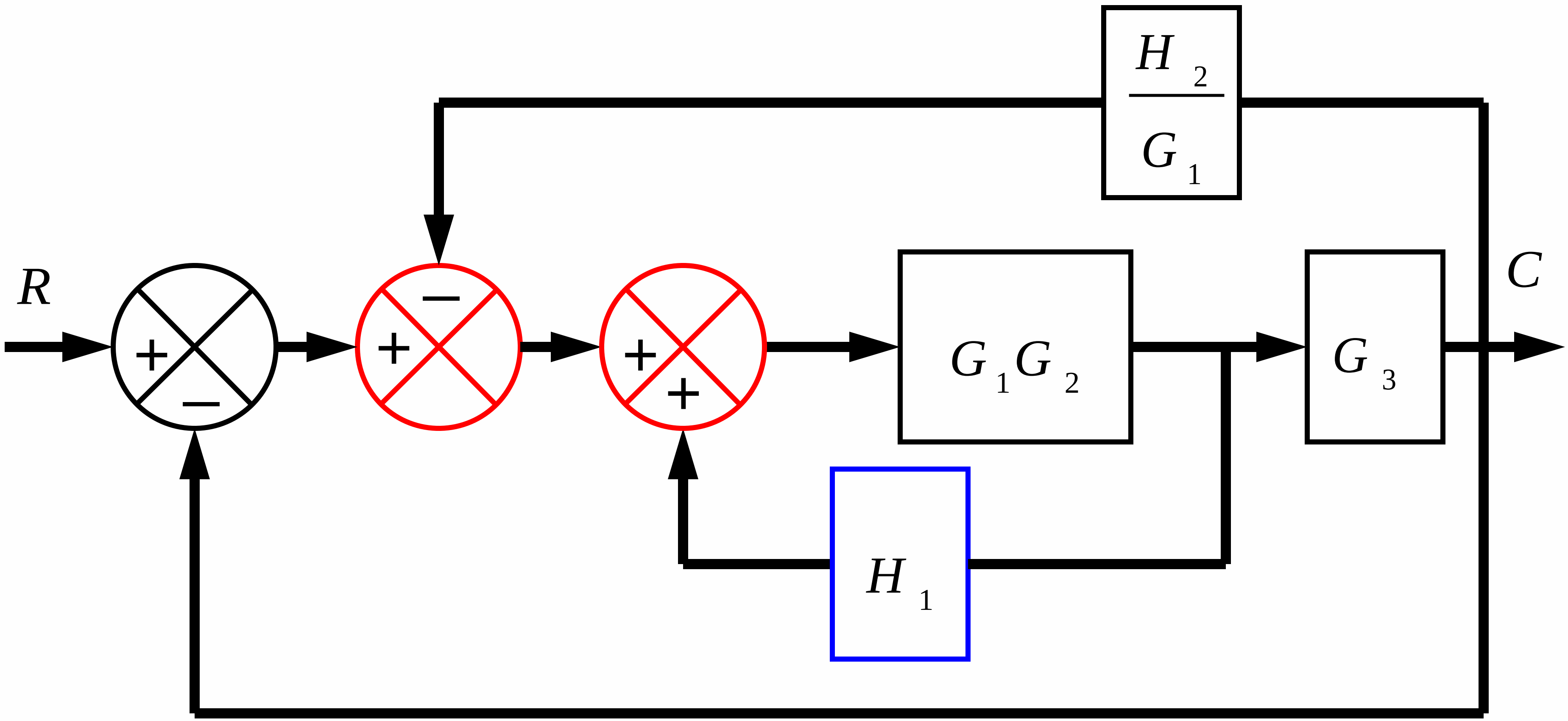
Block Diagram: Reduction Example-2 (Contd..)



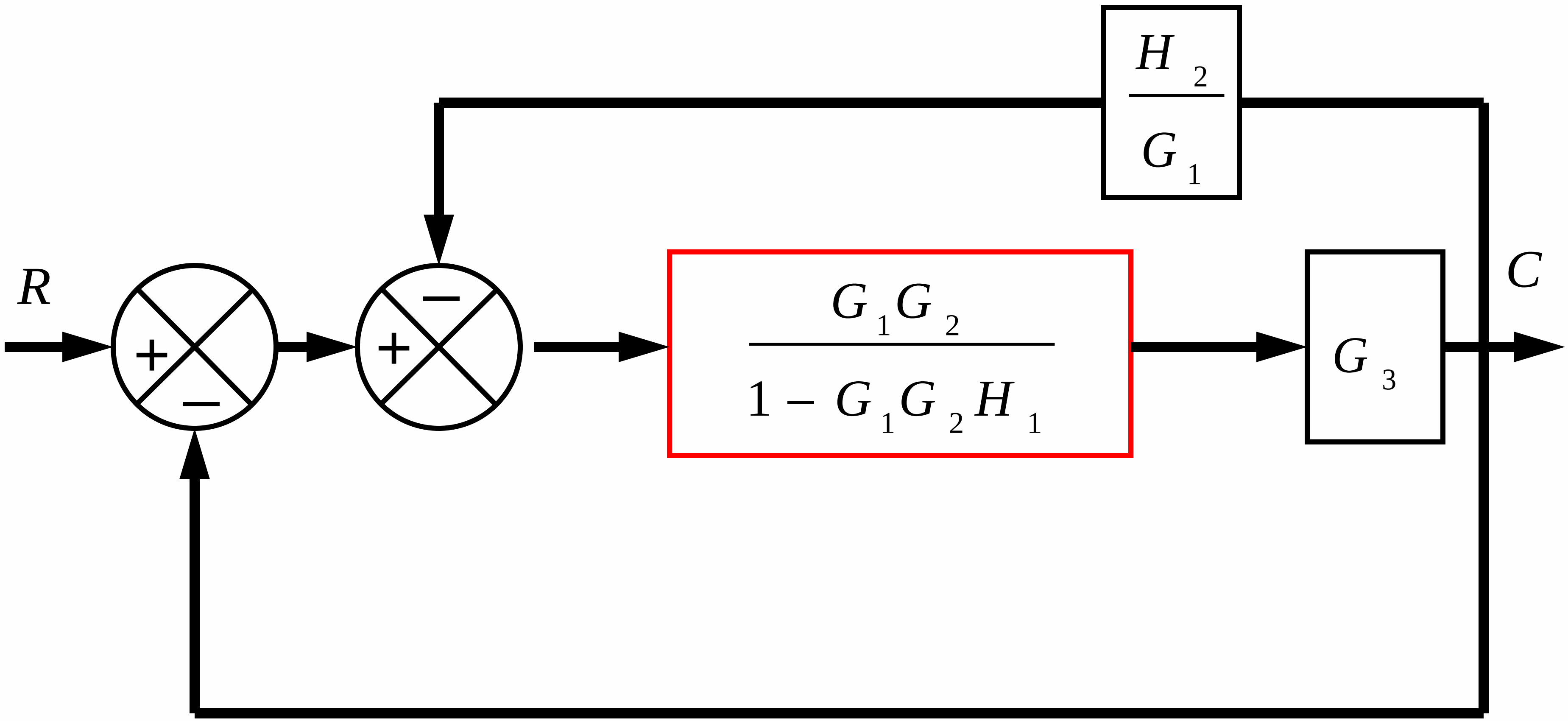
Block Diagram: Reduction Example-2 (Contd..)



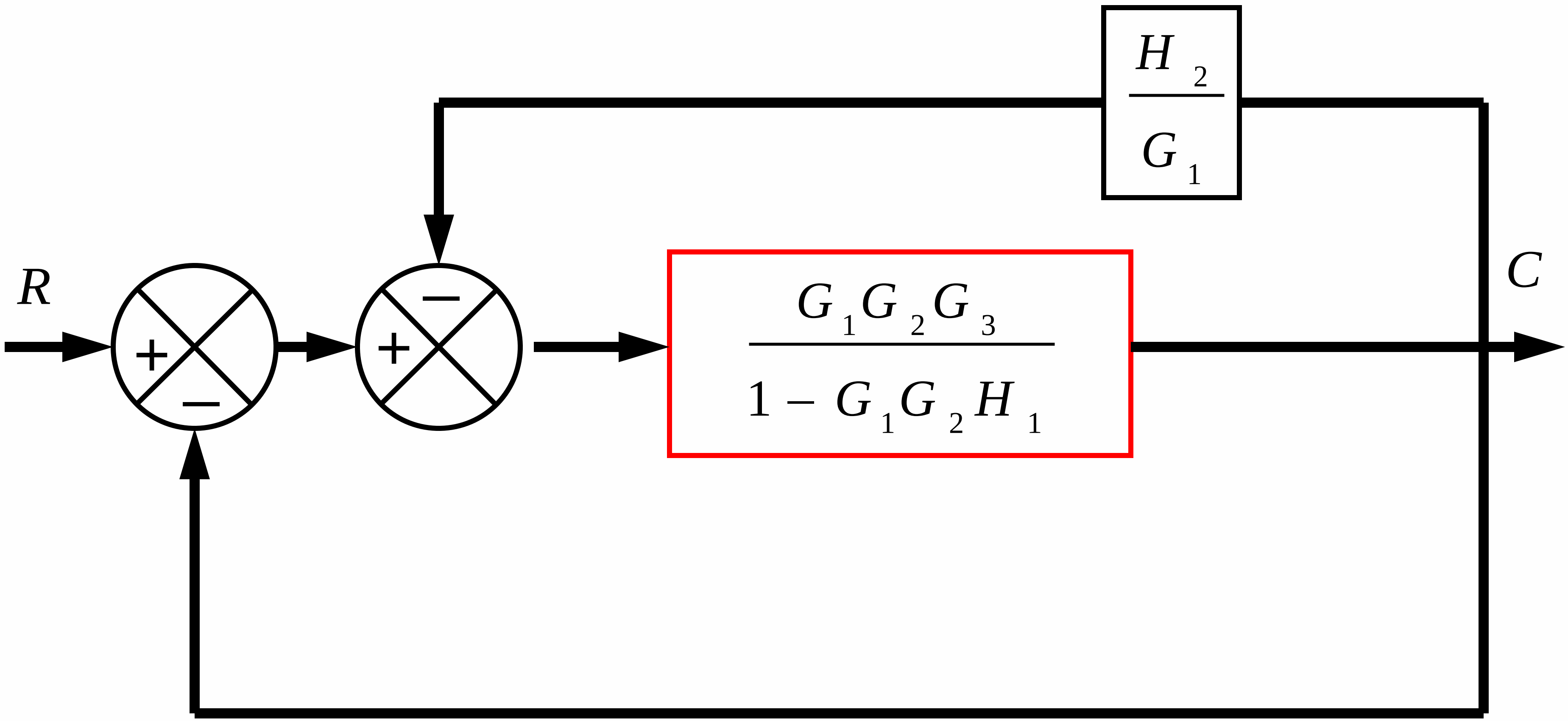
Block Diagram: Reduction Example-2 (Contd..)



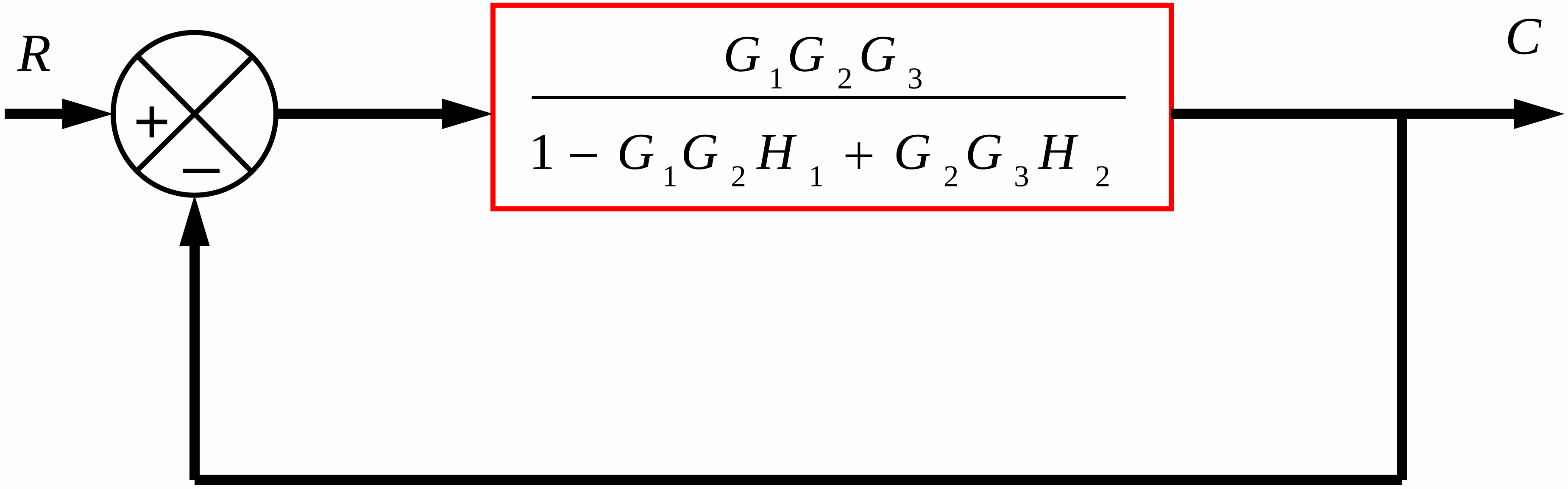
Block Diagram: Reduction Example-2 (Contd..)



Block Diagram: Reduction Example-2 (Contd..)



Block Diagram: Reduction Example-2 (Contd..)



Block Diagram: Reduction Example-2 (Contd..)

$$R \longrightarrow \boxed{\dfrac{G_1 G_2 G_3}{1 - G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3}} \longrightarrow C$$

Mason’s Rule

- The block diagram reduction technique requires successive application of fundamental relationships in order to arrive at the system transfer function.
- On the other hand, Mason’s rule for reducing a signal-flow graph to a single transfer function requires the application of one formula.
- The formula was derived by S. J. Mason when he related the signal-flow graph to the simultaneous equations that can be written from the graph.

Mason's Rule

- The transfer function, **$C(s)/R(s)$,** of a system represented by a signal-flow graph is;

$$\frac{C(s)}{R(s)} = \frac{\sum_{i=1}^{n} P_i \Delta_i}{\Delta}$$

- Where

- n = number of forward paths.
- P_i = the i^{th} forward-path gain.
- Δ = Determinant of the system
- Δ_i = Determinant of the i^{th} forward path

- Δ is called the signal flow graph determinant or characteristic function. Since Δ=0 is the system characteristic equation.

Mason's Rule

$$\frac{C(s)}{R(s)} = \frac{\sum_{i=1}^{n} P_i \Delta_i}{\Delta}$$

- Δ = 1- (sum of all individual loop gains) + (sum of the products of the gains of all possible two loops that do not touch each other) – (sum of the products of the gains of all possible three loops that do not touch each other) + … and so forth with sums of higher number of non-touching loop gains

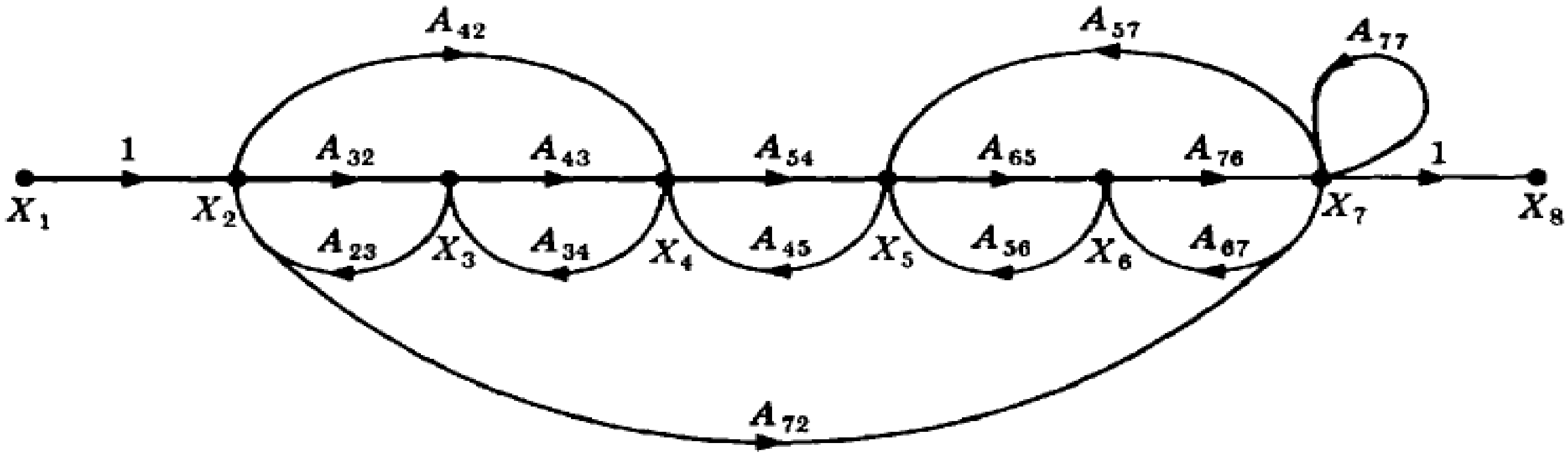
- Δ_i = value of Δ for the part of the block diagram that does not touch the i-th forward path (Δ_i = 1 if there are no non-touching loops to the i-th path.)

Systematic approach

1. Calculate forward path gain P_i for each forward path *i.*
2. Calculate all loop transfer functions
3. Consider non-touching loops 2 at a time
4. Consider non-touching loops 3 at a time
5. etc
6. Calculate Δ from steps 2,3,4 and 5
7. Calculate Δ_i as portion of Δ not touching forward path *i*

Example

❑ Apply Mason's Rule to calculate the transfer function of the system represented by following Signal Flow Graph

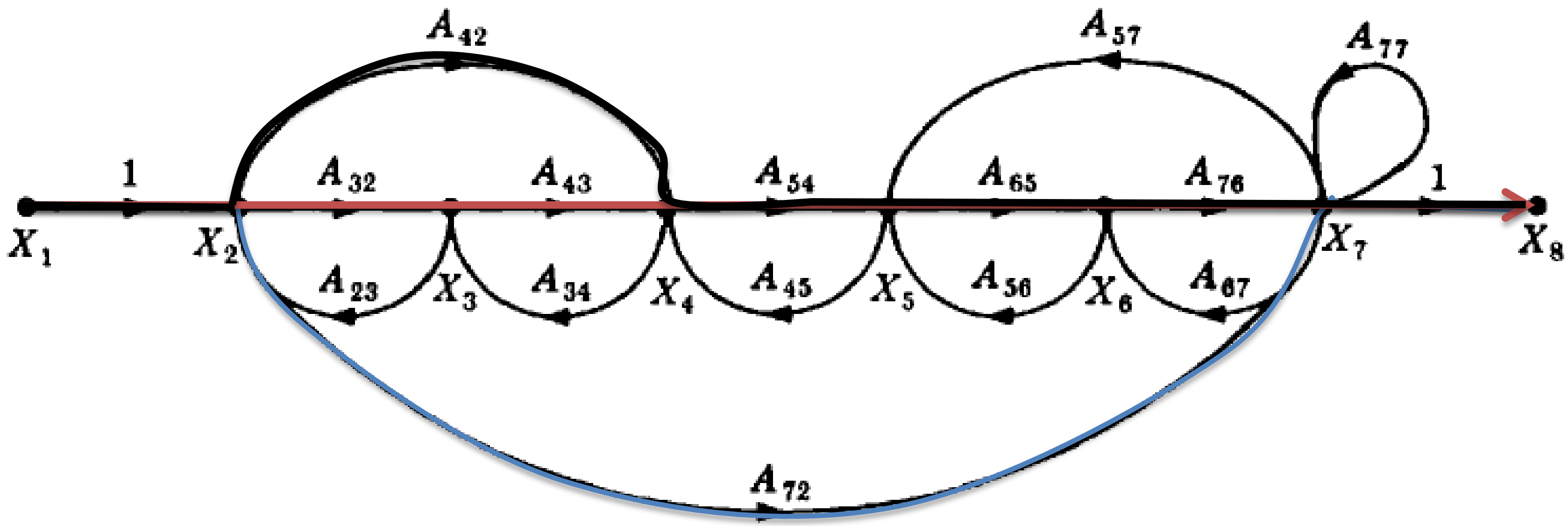


There are three forward paths, therefore n=3.

$$\frac{C(s)}{R(s)} = \frac{\sum_{i=1}^{3} P_i \Delta_i}{\Delta} = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3}{\Delta}$$

Example: Forward Paths

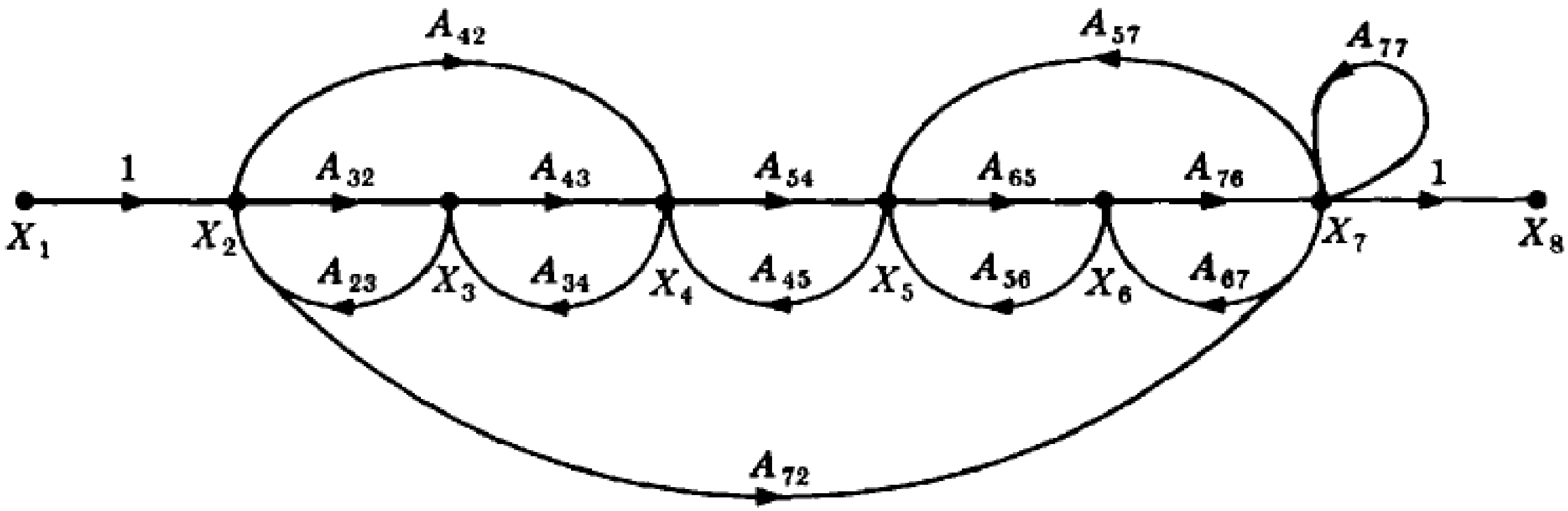
$P_3 = A_{42}\,A_{54}\,A_{65}\,A_{76}$



$P_1 = A_{32}\,A_{43}\,A_{54}\,A_{65}\,A_{76}$

$P_2 = A_{72}$

Example: Loop Gains of the Feedback Loops



$$L_1 = A_{32} A_{23}$$

$$L_2 = A_{43} A_{34}$$

$$L_3 = A_{54} A_{45}$$

$$L_4 = A_{65} A_{56}$$

$$L_5 = A_{76} A_{67}$$

$$L_6 = A_{77}$$

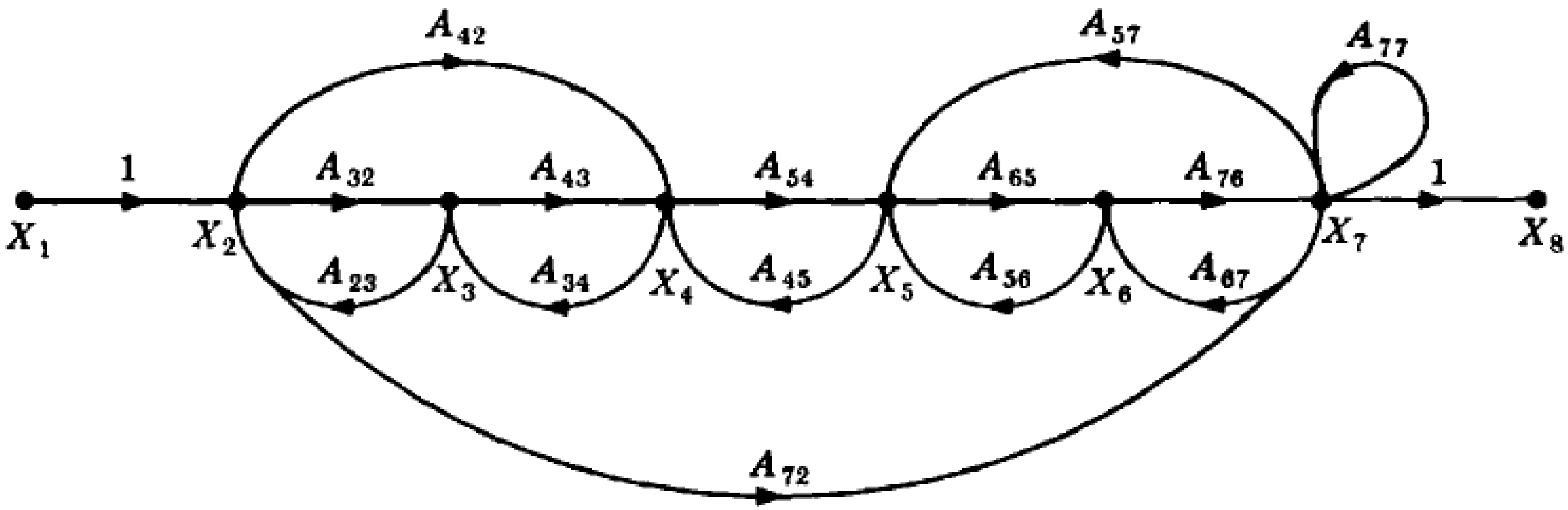
$$L_7 = A_{42} A_{34} A_{23}$$

$$L_8 = A_{65} A_{76} A_{67}$$

$$L_9 = A_{72} A_{57} A_{45} A_{34} A_{23}$$

$$L_{10} = A_{72} A_{67} A_{56} A_{45} A_{34} A_{23}$$

Example: two non-touching loops



$L_1 L_3$ $L_2 L_4$ $L_3 L_5$ $L_4 L_6$ $L_5 L_7$ $L_7 L_8$

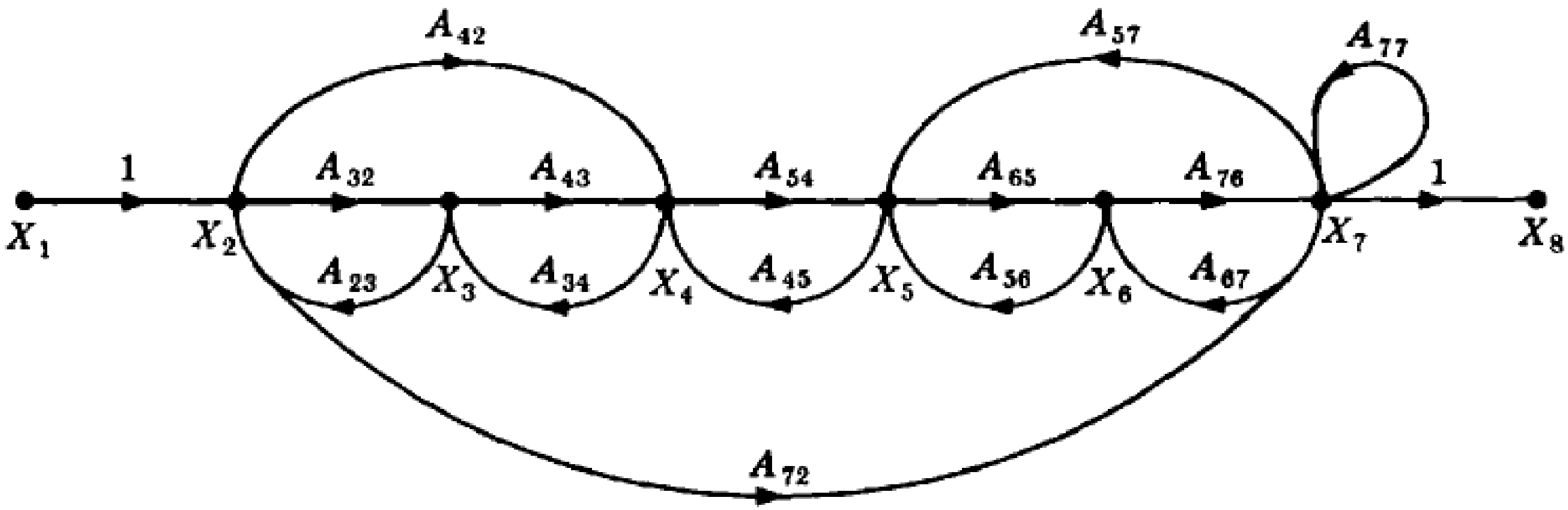
$L_1 L_4$ $L_2 L_5$ $L_3 L_6$ $L_4 L_7$

$L_1 L_5$ $L_2 L_6$

$L_1 L_6$ $L_2 L_8$

$L_1 L_8$

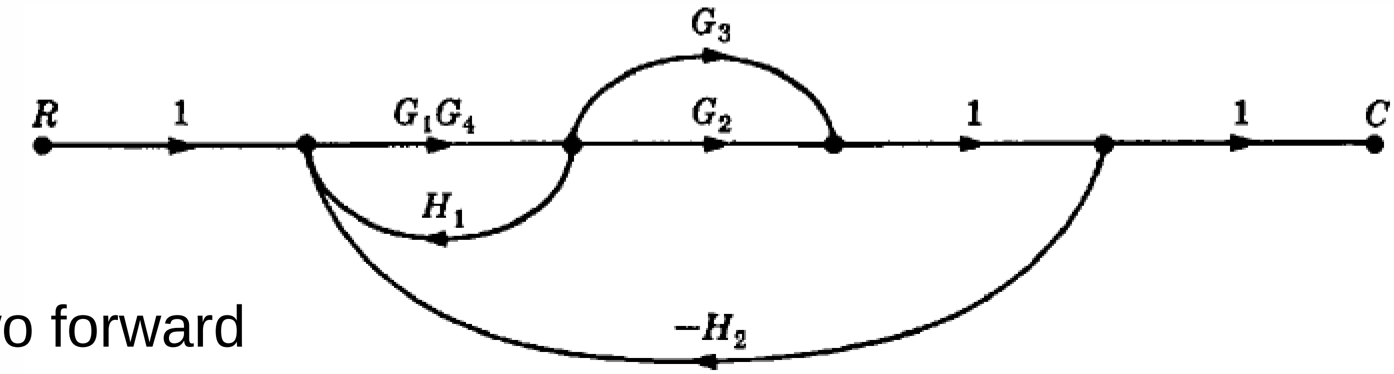
Example: Three non-touching loops



L_1L_3	L_2L_4	L_3L_5	L_4L_6	L_5L_7	L_7L_8
L_1L_4	L_2L_5	L_3L_6	L_4L_7		
L_1L_5	L_2L_6				
L_1L_6	L_2L_8				
L_1L_8					

Signal Flow Graph:Example#1

- Apply Mason's Rule to calculate the transfer function of the system represented by following Signal Flow Graph



- There are two forward pa

$$P_1 = G_1G_2G_4, \qquad P_2 = G_1G_3G_4$$

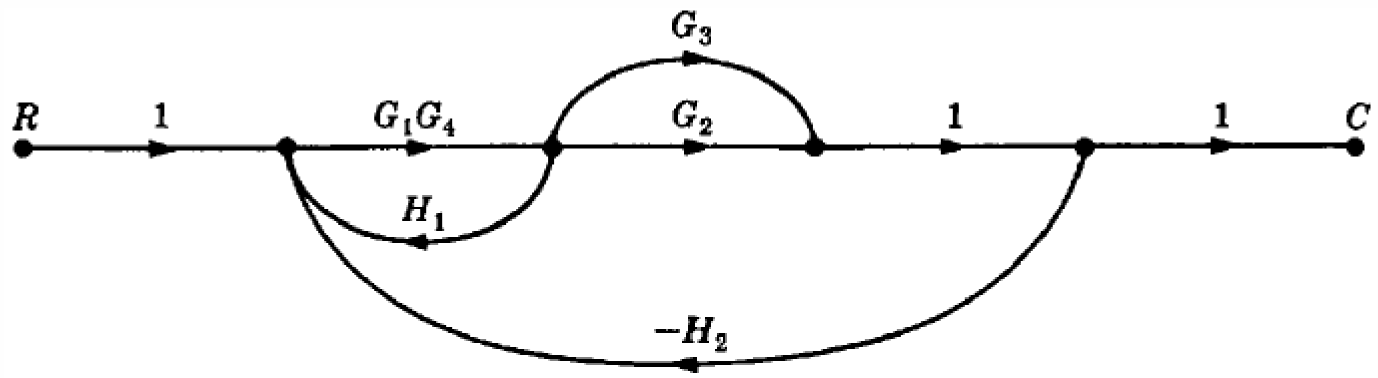
- Therefore,

$$\frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

- There are three feedback loops

$$L_1 = G_1G_4H_1, \qquad L_2 = -G_1G_2G_4H_2, \qquad L_3 = -G_1G_3G_4H_2$$

Signal Flow Graph:Example#1 (Contd..)



- There are no non-touching loops, therefore

Δ = 1- (sum of all individual loop gains)

$$\Delta = 1 - \left(L_1 + L_2 + L_3\right)$$

$$\Delta = 1 - \left(G_1 G_4 H_1 - G_1 G_2 G_4 H_2 - G_1 G_3 G_4 H_2\right)$$

Signal Flow Graph:Example#1 (Contd..)

- Eliminate forward path-1

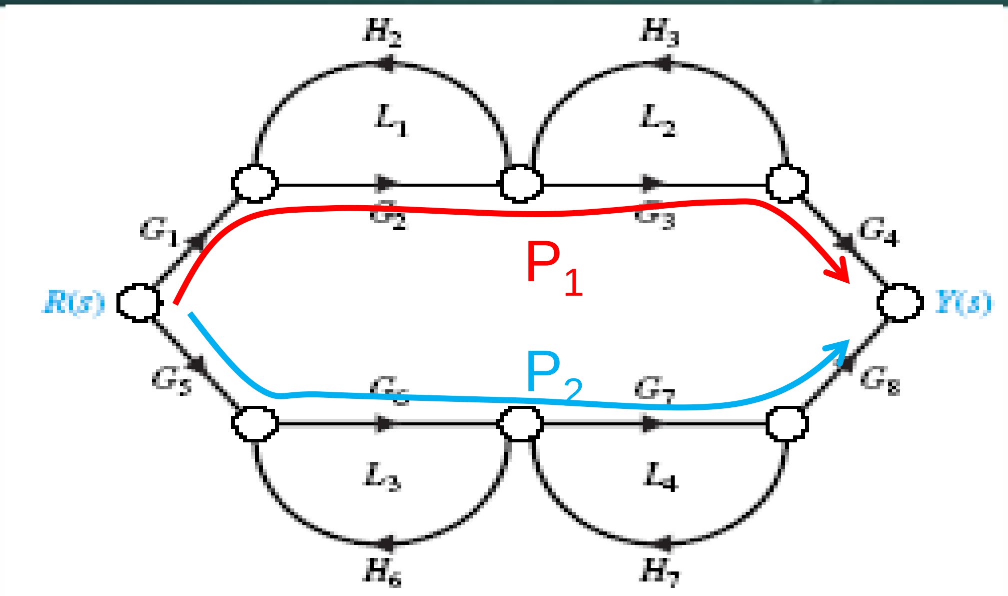
Δ_1 = 1- (sum of all individual loop gains)+...
Δ_1 = 1

- Eliminate forward path-2

Δ_2 = 1- (sum of all individual loop gains)+...
Δ_2 = 1

$$\frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_4 + G_1G_3G_4}{1 - G_1G_4H_1 + G_1G_2G_4H_2 + G_1G_3G_4H_2}$$

Signal Flow Graph:Example#2



1. Calculate forward path gains for each forward path.

$$P_1 = G_1 G_2 G_3 G_4 \text{ (path 1)} \quad \text{and} \quad P_2 = G_5 G_6 G_7 G_8 \text{ (path 2)}$$

2. Calculate all loop gains.

$$L_1 = G_2 H_2, \quad L_2 = H_3 G_3, \quad L_3 = G_6 H_6, \quad L_4 = G_7 H_7$$

3. Consider two non-touching loops.

$L_1 L_3$ $L_1 L_4$ $L_2 L_4$ $L_2 L_3$

Signal Flow Graph:Example#2 (Contd..)

4. Consider three non-touching loops.
   None.

5. Calculate Δ from steps 2,3,4.

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_1 L_3 + L_1 L_4 + L_2 L_3 + L_2 L_4)$$

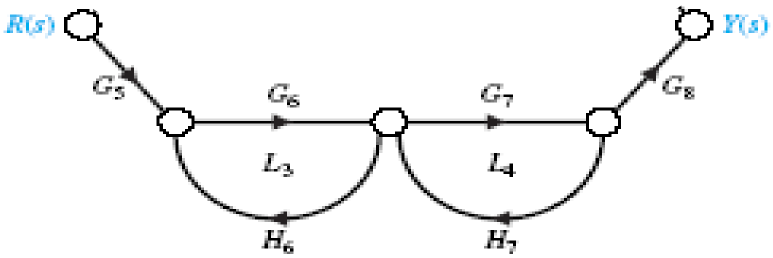
$$\Delta = 1 - (G_2 H_2 + H_3 G_3 + G_6 H_6 + G_7 H_7) + (G_2 H_2 G_6 H_6 + G_2 H_2 G_7 H_7 + H_3 G_3 G_6 H_6 + H_3 G_3 G_7 H_7)$$

Signal Flow Graph:Example#2 (Contd..)

- Eliminate forward path-1

$$\Delta_1 = 1 - (L_3 + L_4)$$

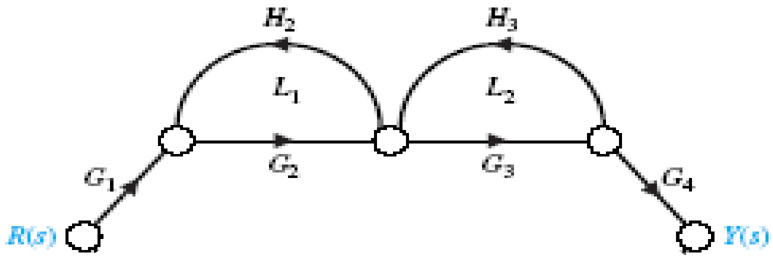
$$\Delta_1 = 1 - (G_6 H_6 + G_7 H_7)$$



- Eliminate forward path-2

$$\Delta_2 = 1 - (L_1 + L_2)$$

$$\Delta_2 = 1 - (G_2 H_2 + G_3 H_3)$$

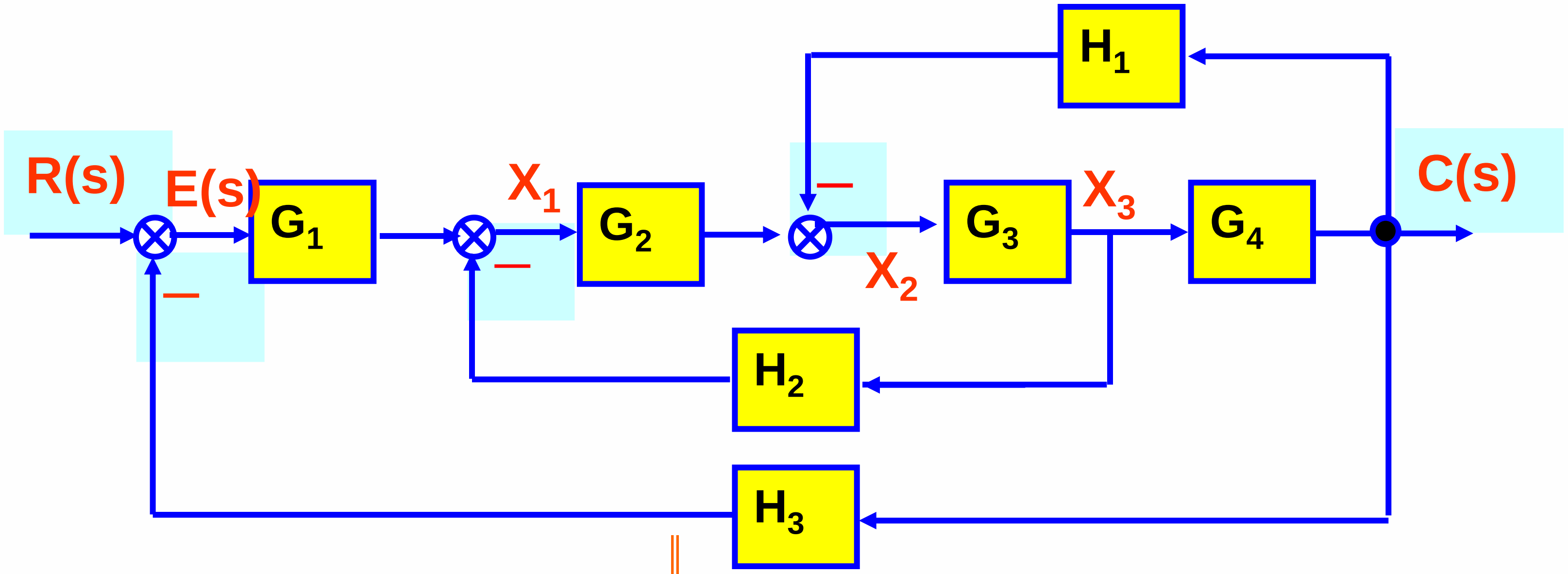


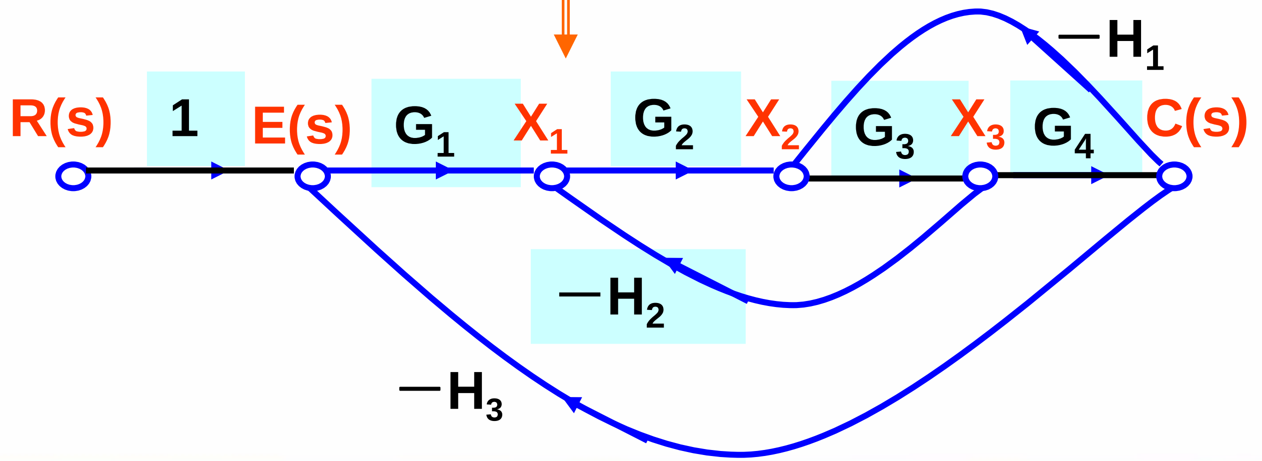
Signal Flow Graph:Example#2 (Contd..)

$$\frac{Y(s)}{R(s)} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

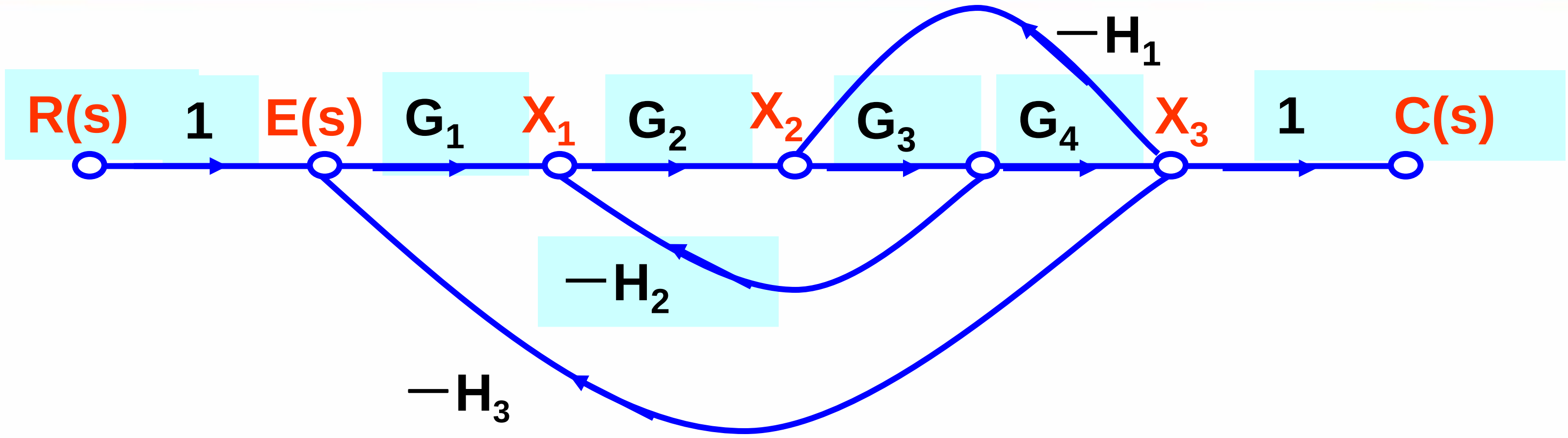
$$\frac{Y(s)}{R(s)} = \frac{G_1G_2G_3G_4[1-(G_6H_6+G_7H_7)] + G_5G_6G_7G_8[1-(G_2H_2+G_3H_3)]}{1-(G_2H_2+H_3G_3+G_6H_6+G_7H_7)+(G_2H_2G_6H_6+G_2H_2G_7H_7+H_3G_3G_6H_6+H_3G_3G_7H_7)}$$

Block Diagram to SFG:Example#3





Block Diagram to SFG:Example#3 (Contd..)

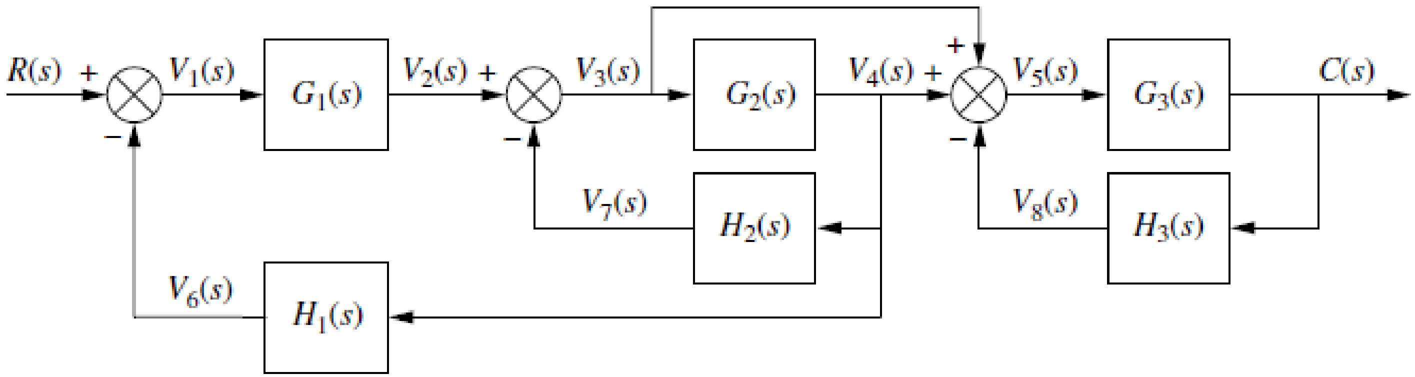


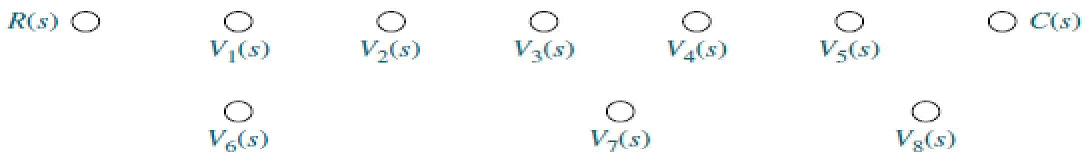
$$\Delta = 1 + (G_1G_2G_3G_4H_3 + G_2G_3H_2 + G_3G_4H_1)$$

$$P_1 = G_1G_2G_3G_4; \qquad \Delta_1 = 1$$

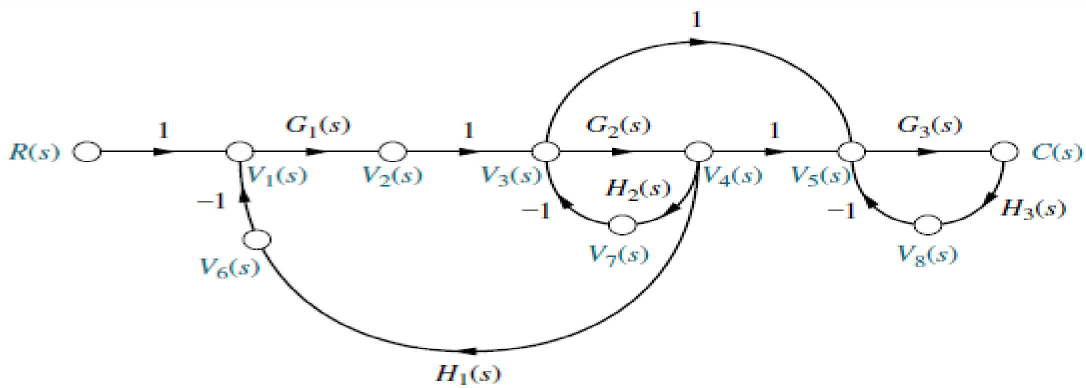
$$G = \frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4}{1 + G_1G_2G_3G_4H_3 + G_2G_3H_2 + G_3G_4H_1}$$

- Example-1: Convert the block diagram into a signal flow graph:

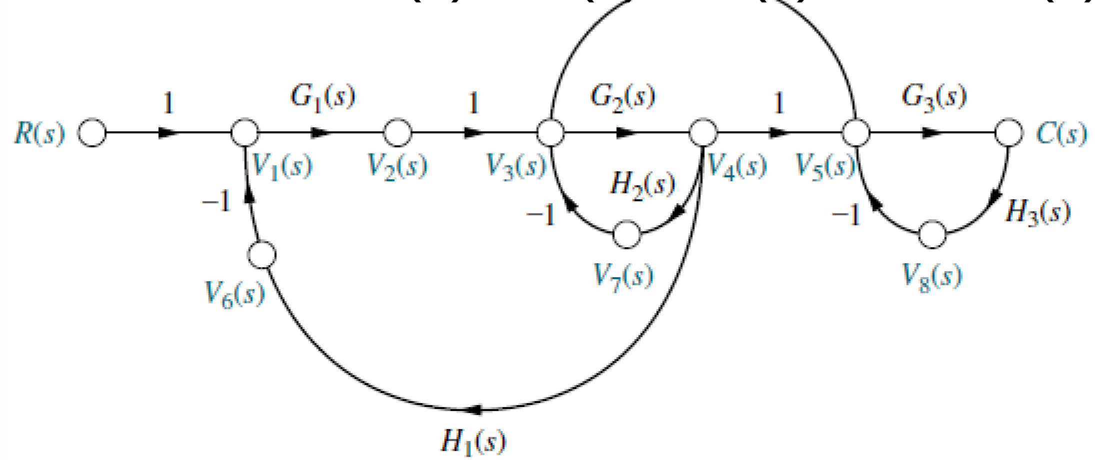


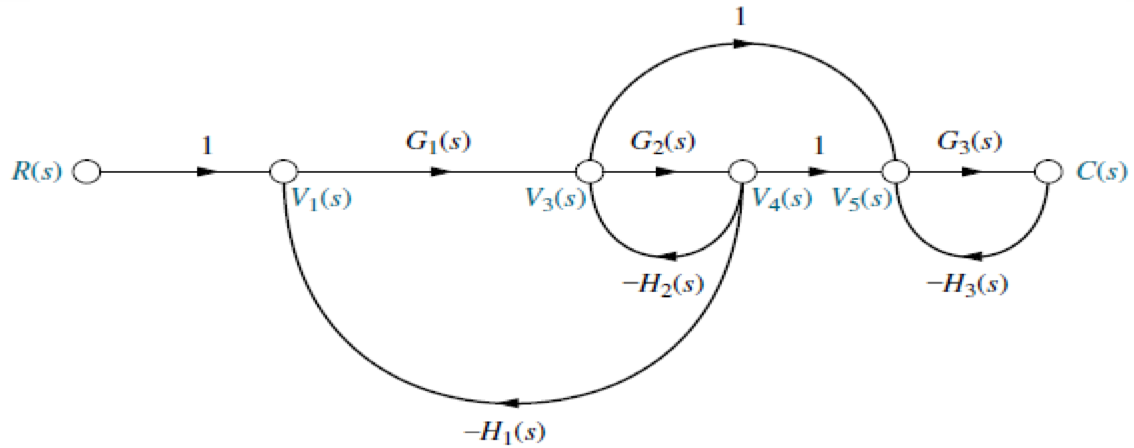


(a)



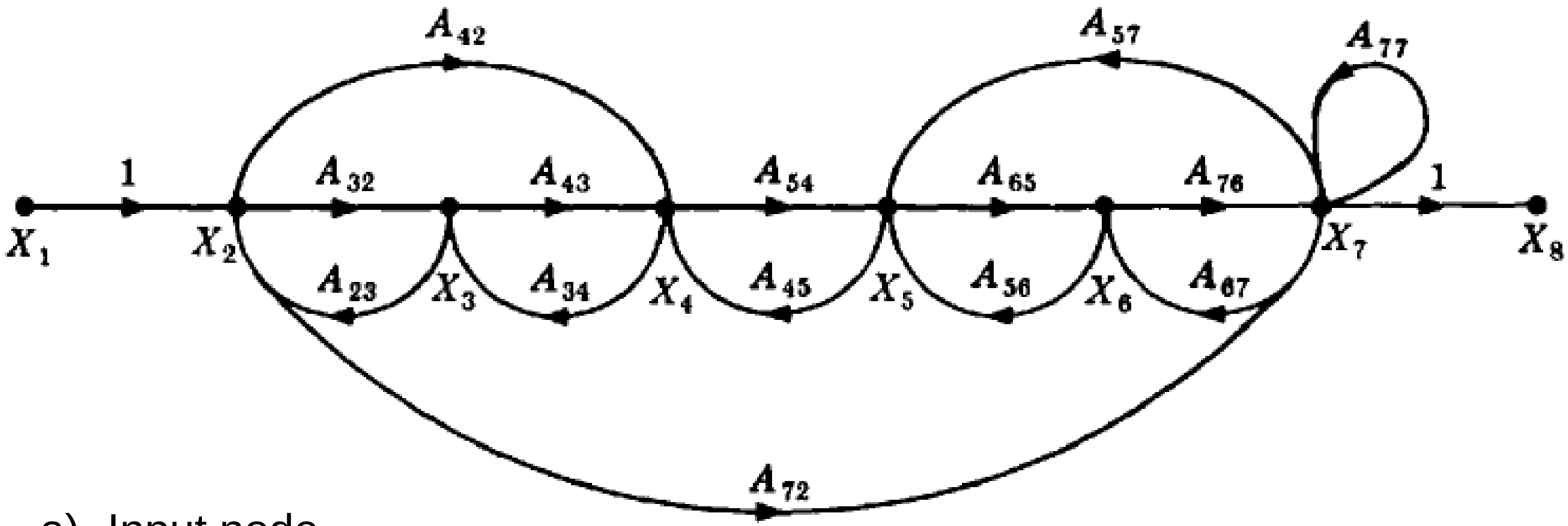
- If desired, simplify the signal-flow graph to the one shown in Figure (c) by eliminating signals that have a single flow in and a single flow out, such as V2(s), V6(s), V7(s), and V8(s).





(c)

Exaple-2: Consider the signal flow graph below and identify the following



a) Input node.
b) Output node.
c) Forward paths.
d) Feedback paths.
e) Self loop.
f) Determine the loop gains of the feedback loops.
g) Determine the path gains of the forward paths.

Example-2: Answers

(a) X_1

(b) X_8

(c) X_1 to X_2 to X_3 to X_4 to X_5 to X_6 to X_7 to X_8
X_1 to X_2 to X_7 to X_8
X_1 to X_2 to X_4 to X_5 to X_6 to X_7 to X_8

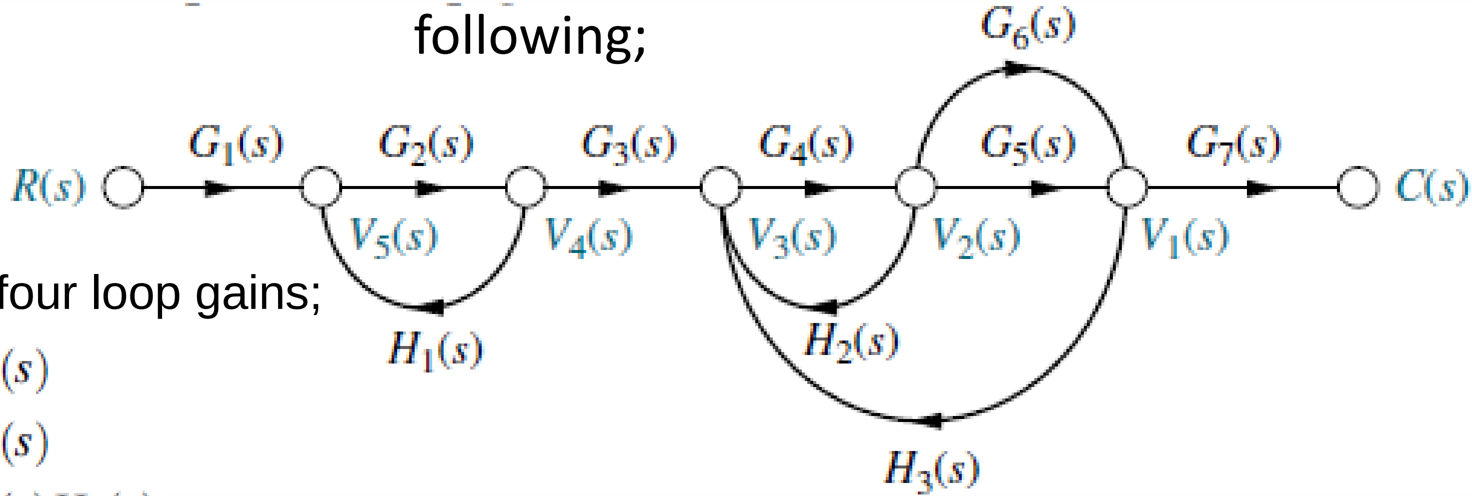
(d) X_2 to X_3 to X_2; X_3 to X_4 to X_3; X_4 to X_5 to X_4; X_2 to X_4 to X_3 to X_2;
X_2 to X_7 to X_5 to X_4 to X_3 to X_2; X_5 to X_6 to X_5; X_6 to X_7 to X_6;
X_5 to X_6 to X_7 to X_5; X_7 to X_7; X_2 to X_7 to X_6 to X_5 to X_4 to X_3 to X_2

(e) X_7 to X_7

(f) $A_{32}A_{23}$; $A_{43}A_{34}$; $A_{54}A_{45}$; $A_{65}A_{56}$; $A_{76}A_{67}$; $A_{65}A_{76}A_{57}$; A_{77}; $A_{42}A_{34}A_{23}$;
$A_{72}A_{57}A_{45}A_{34}A_{23}$; $A_{72}A_{67}A_{56}A_{45}A_{34}A_{23}$

(g) $A_{32}A_{43}A_{54}A_{65}A_{76}$; A_{72}; $A_{42}A_{54}A_{65}A_{76}$

❑Example-3: Consider the signal flow graph below and identify the following;



- There are four loop gains;

1. $G_2(s)H_1(s)$
2. $G_4(s)H_2(s)$
3. $G_4(s)G_5(s)H_3(s)$
4. $G_4(s)G_6(s)H_3(s)$

- There are two forward path gains;

1. $G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)G_7(s)$
2. $G_1(s)G_2(s)G_3(s)G_4(s)G_6(s)G_7(s)$

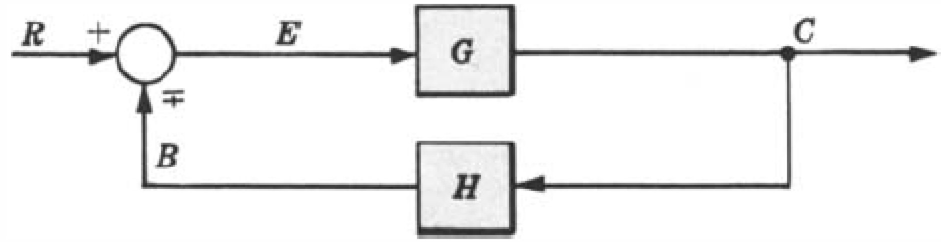
- Nontouching loops;

1 $G_2(s)H_1(s)$

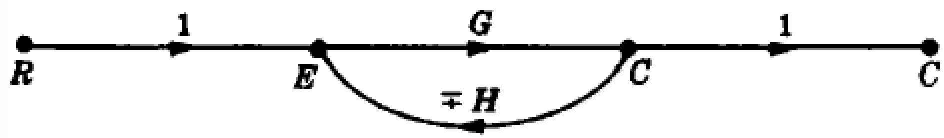
- Nontouching loop gains;

1. $[G_2(s)H_1(s)][G_4(s)H_2(s)]$
2. $[G_2(s)H_1(s)][G_4(s)G_5(s)H_3(s)]$
3. $[G_2(s)H_1(s)][G_4(s)G_6(s)H_3(s)]$

❑Example-4: Construct the signal flow graph of the block diagram of the canonical feedback control system and find the control ratio C/R.



The signal flow graph is easily constructed from Fig. Note that the $-$ or $+$ sign of the summing point is associated with H.



There is only one forward path; hence $P_1 = G$

There is only one (feedback) loop. Hence $P_{11} = \mp GH$

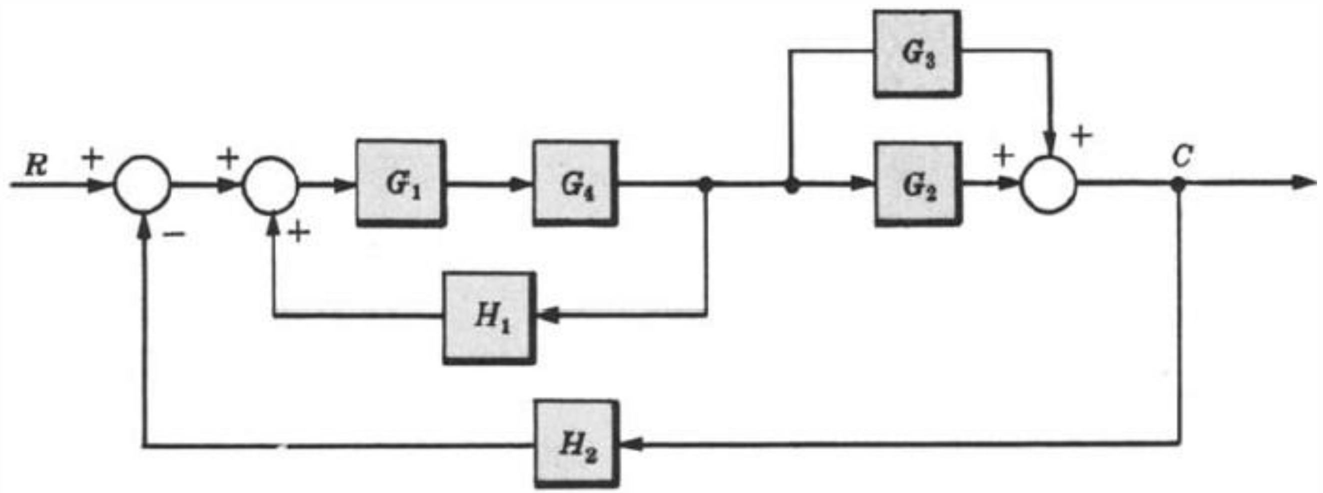
The characteristic function $\Delta = 1 - P_{11} = 1 \pm GH$

Since the loop touch the forward path $\Delta_1 = 1$

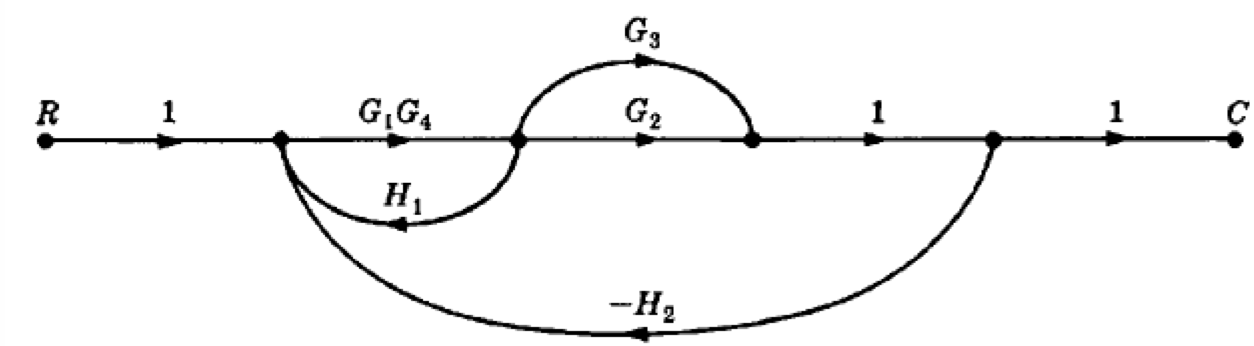
Finally, the control ratio is

$$\boxed{T = \frac{C}{R} = \frac{P_1\Delta_1}{\Delta} = \frac{G}{1 \pm GH}}$$

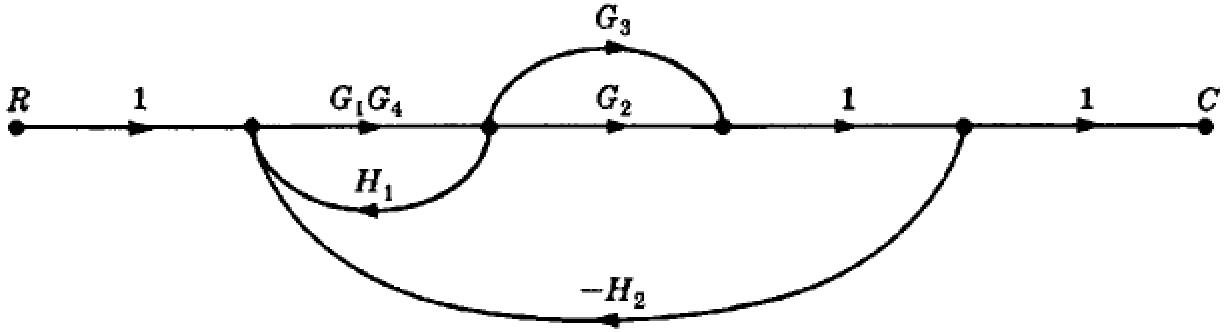
❑Example-5: Determine the control ratio C/R and the canonical block diagram of the feedback control system.



The signal flow graph is



❑Example-5:Continue. (finding the control ratio C/R)



There are two forward paths:

$P_1 = G_1G_2G_4 \qquad P_2 = G_1G_3G_4$

There are three feedback loops:

$P_{11} = G_1G_4H_1 \qquad P_{21} = -G_1G_2G_4H_2 \qquad P_{31} = -G_1G_3G_4H_2$

the **signal flow graph determinant** or **characteristic function**,

$\Delta = 1 - (P_{11} + P_{21} + P_{31})$

There are no nontouching loops, and all loops touch both forward paths; then

$\Delta_1 = 1 \qquad \Delta_2 = 1$

Therefore the control ratio is

$$T = \frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_4 + G_1G_3G_4}{1 - G_1G_4H_1 + G_1G_2G_4H_2 + G_1G_3G_4H_2}$$

$$= \frac{G_1G_4(G_2 + G_3)}{1 - G_1G_4H_1 + G_1G_2G_4H_2 + G_1G_3G_4H_2}$$

❑Example-5:Continue. (finding the canonical block diagram)

The direct transfer function is $G = \sum_i P_i \Delta_i$

$$G = P_1\Delta_1 + P_2\Delta_2$$

$$G = G_1G_4(G_2 + G_3)$$

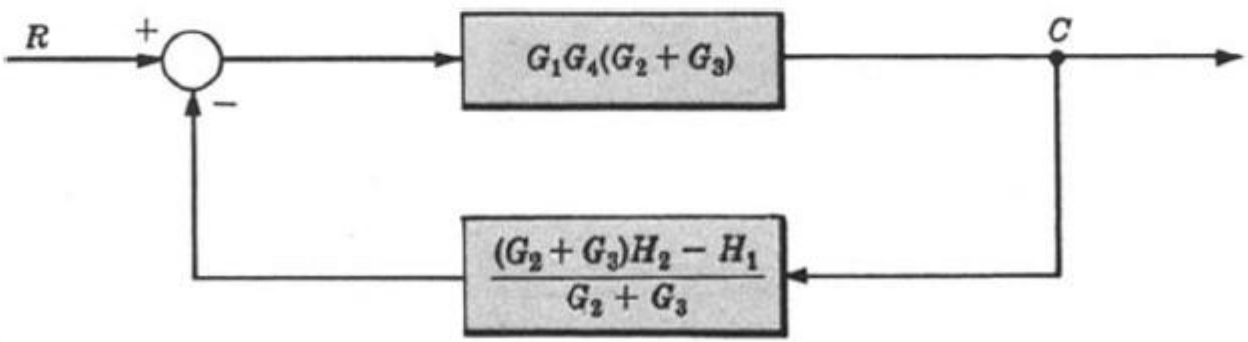
The loop transfer function is $GH = \Delta - 1$

$$GH = 1 - (P_{11} + P_{21} + P_{31}) - 1$$

$$GH = G_1G_4(G_3H_2 + G_2H_2 - H_1)$$

Therefore $\quad H = \dfrac{GH}{G} = \dfrac{(G_2 + G_3)H_2 - H_1}{G_2 + G_3}$

The canonical block diagram is



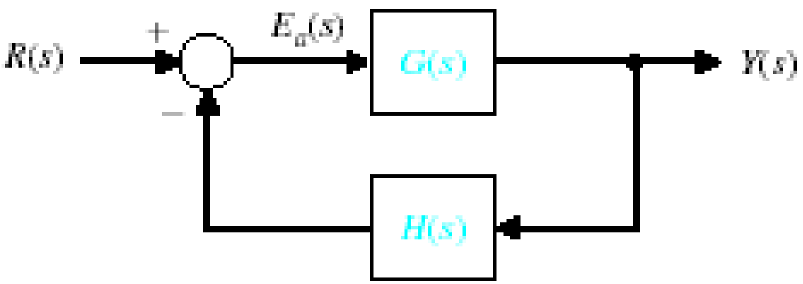
Sensitivity of Systems To Parameter Variations

For the closed-loop case if $GH(s) > 1$

$$Y(s) = \frac{1}{H(s)} \cdot R(s)$$

Output affected only by H(s)

$$G(s) + \Delta G(s)$$



Open Loop $\Delta Y(s) = \Delta G(s) \cdot R(s)$

Closed Loop

$$Y(s) + \Delta Y(s) = \frac{\left(G(s) + \Delta G(s)\right)}{1 + \left(G(s) + \Delta G(s)\right) \cdot H(s)} \cdot R(s)$$

$$\Delta Y(s) = \frac{\Delta G(s)}{\left(1 + GH(s) + \Delta GH(s)\right)(1 + GH(s))} \cdot R(s)$$

$$GH(s) > \Delta GH(s)$$

$$\Delta Y(s) = \frac{\Delta G(s)}{(1 + GH(s))^2} \cdot R(s)$$

The change in the output of the closed system is reduced by a factor of 1+GH(s)

Sensitivity of Systems To Parameter Variations

$$T(s) = \frac{Y(s)}{R(s)}$$

$$S = \frac{\dfrac{\Delta T(s)}{T(s)}}{\dfrac{\Delta G(s)}{G(s)}} \qquad S = \frac{\dfrac{dT}{T}}{\left(\dfrac{dG}{G}\right)} = \frac{\left(\dfrac{dT}{dT}\right)}{\left(\dfrac{dG}{dG}\right)} \times \frac{G}{T}$$

$$T(s) = \frac{1}{1 + H[(s) \times G(s)]}$$

$$S_G^T = \frac{\left(\dfrac{dT}{dT}\right)}{\left(\dfrac{dG}{dG}\right)} \times \frac{G}{T} = \frac{\left(\dfrac{dT}{dT}\right)}{\left(\dfrac{dG}{dG}\right)} \times \frac{G}{T} = \frac{1}{(1 + GH)^2} \times \frac{G}{\dfrac{G}{(1 + GH)}}$$

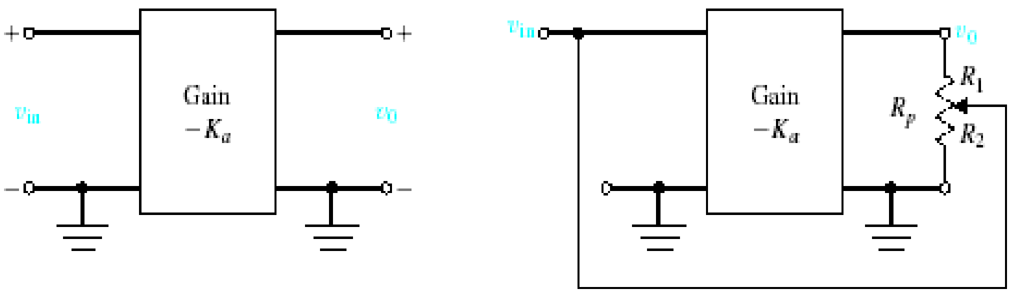
$$S_G^T = \frac{1}{(1 + GH)}$$

Sensitivity of the closed-loop to G variations reduced

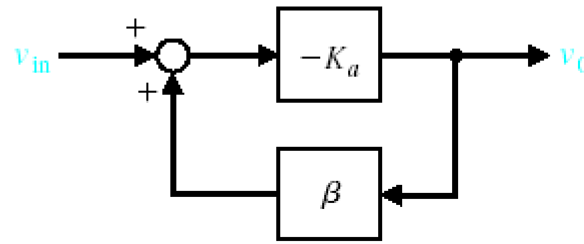
$$S_H^T = \frac{-GH}{(1 + GH)}$$

Sensitivity of the closed-loop to H variations
When GH is large sensitivity approaches 1
Changes in H directly affects the output response

(a) Open loop amplifier.
(b) Amplifier with feedback.



(a) (b)



Block diagram model of feedback amplifier assuming $R_p \gg R_0$ of the amplifier.

Open loop

$$v_o = -K_a \cdot v_{in}$$

$$T = -k_a$$

$$S_{Ka}^{T} = 1$$

Closed loop

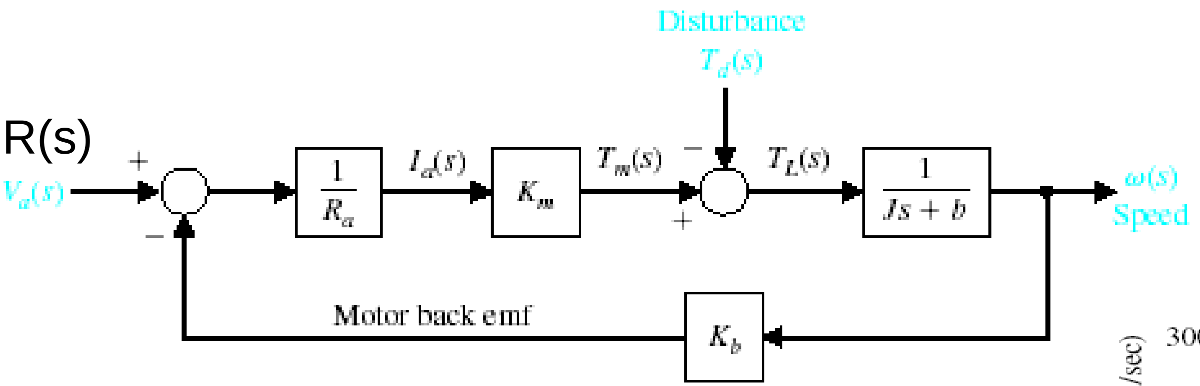
$$\beta = \frac{R2}{R1} \qquad R_p = R1 + R2$$

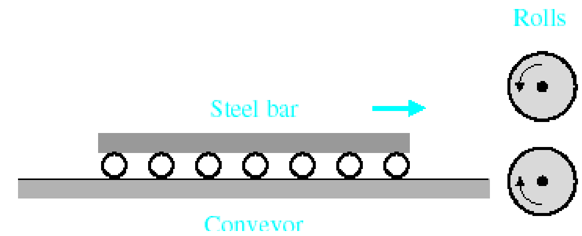
$$T = \frac{-K_a}{1 + K_a \cdot \beta} \qquad S_{Ka}^{T} = \frac{1}{1 + K_a \cdot \beta}$$

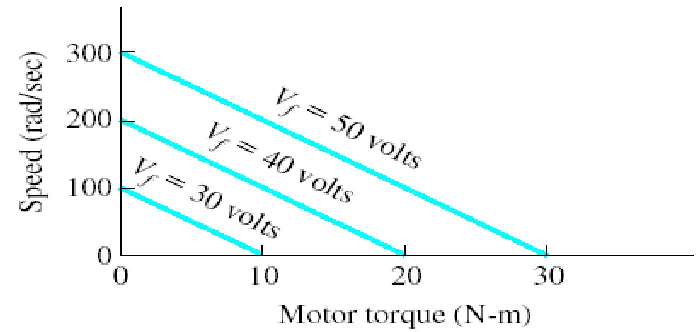
If Ka is large, the sensitivity is low.

$$K_a := 10^4 \qquad \beta := 0.1 \qquad S_{Ka}^{T} = \frac{1}{1 + 10^3} = 9.99 \times 10^{-4}$$

Disturbance Signals In a Feedback Systems







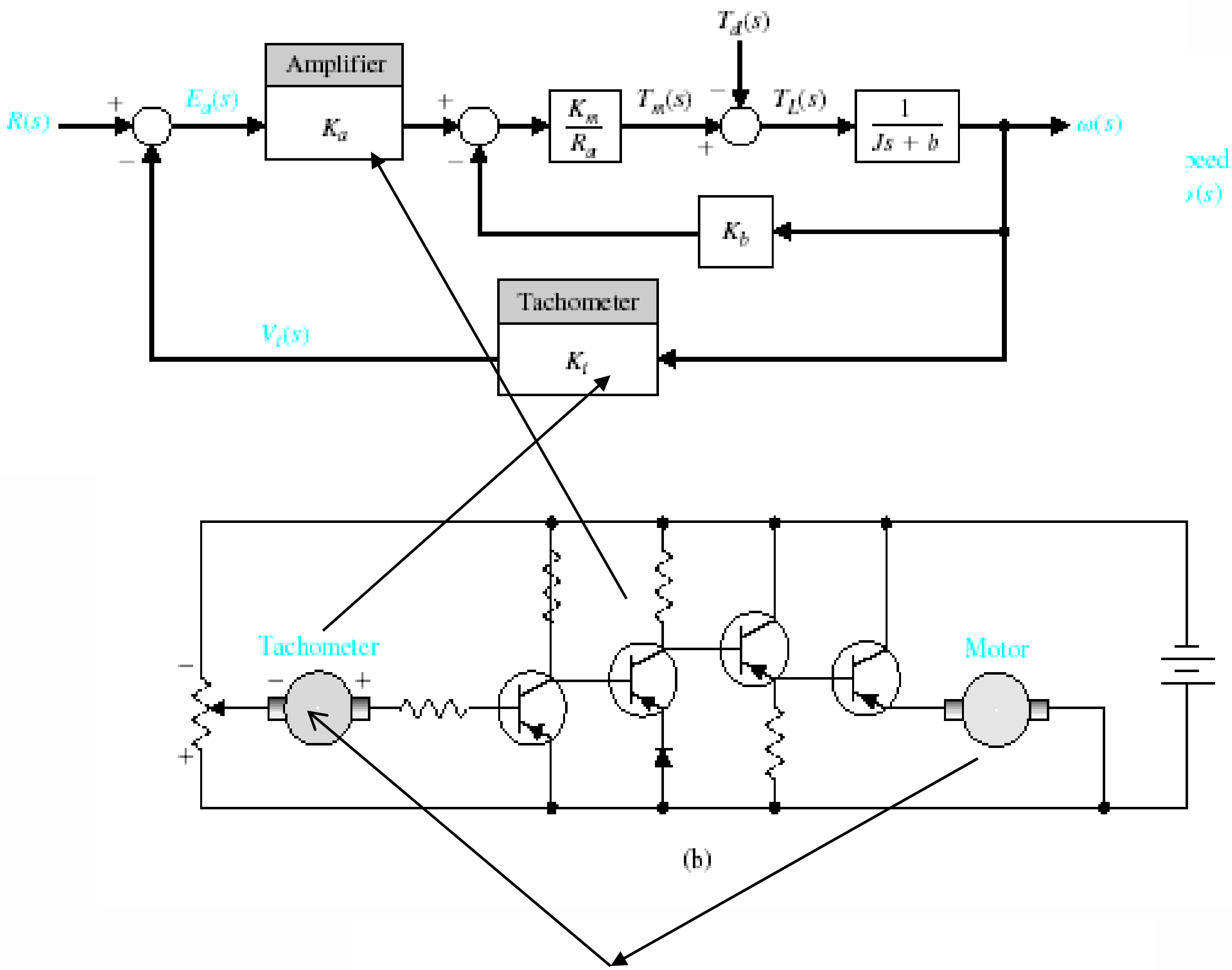
$$E(s) = R(s) - \omega(s) \qquad R(s) = 0$$

$$E(s) = -\omega(s)$$

$$E(s) = -\omega(s) = \frac{1}{\left[J\cdot s + b + \left(Km\cdot\frac{Kb}{Ra}\right)\right]}\cdot Td(s) \qquad Td(s) = \frac{D}{s}$$

$$\lim_{t \to \text{infinite}} E(t) = \lim_{s \to 0} s\cdot\left[\frac{1}{J\cdot s + b + \left(Km\cdot\frac{Kb}{Ra}\right)}\right]\cdot\left(\frac{D}{s}\right) = \frac{D}{b + \left(Km\cdot\frac{Kb}{Ra}\right)} = -\omega(\text{infinite})$$

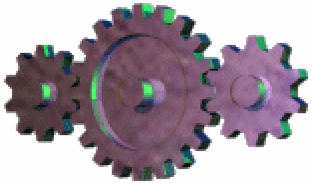
Disturbance Signals In a Feedback Systems

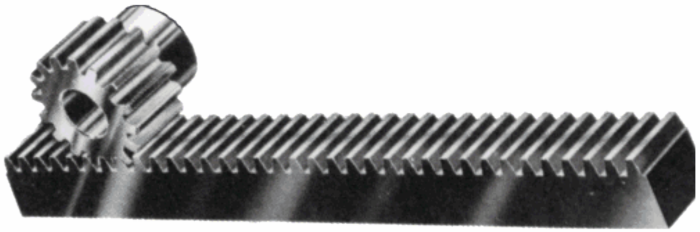


Closed-loop speed tachometer control system.

Gear

- **Gear** is a toothed machine part, such as a wheel or cylinder, that meshes with another toothed part to transmit motion or to change speed or direction.

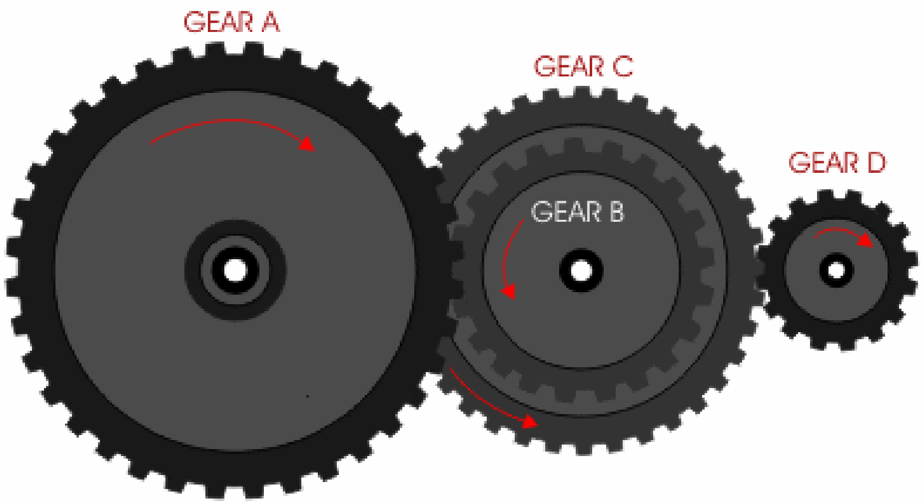




Fundamental Properties

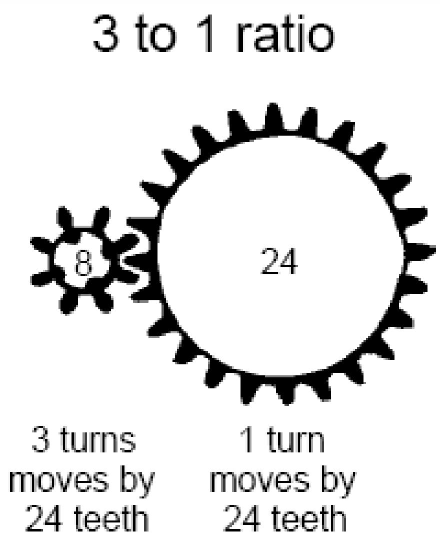
- The two gears turn in opposite directions: one clockwise and the other counterclockwise.

- Two gears revolve at different speeds when number of teeth on each gear are different.



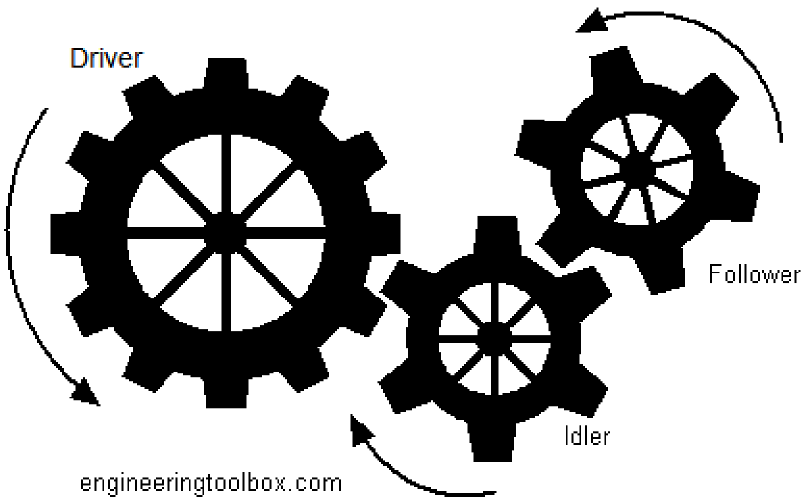
Gearing Up and Down

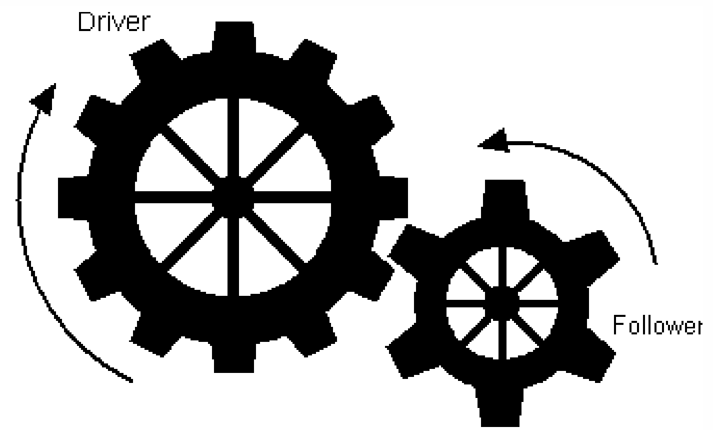
- Gearing up is able to convert torque to velocity.
- The more velocity gained, the more torque sacrifice.
- The ratio is exactly the same: if you get three times your original angular velocity, you reduce the resulting torque to one third.
- This conversion is symmetric: we can also convert velocity to torque at the same ratio.
- The price of the conversion is power loss due to friction.



Why Gearing is necessary? And gear train

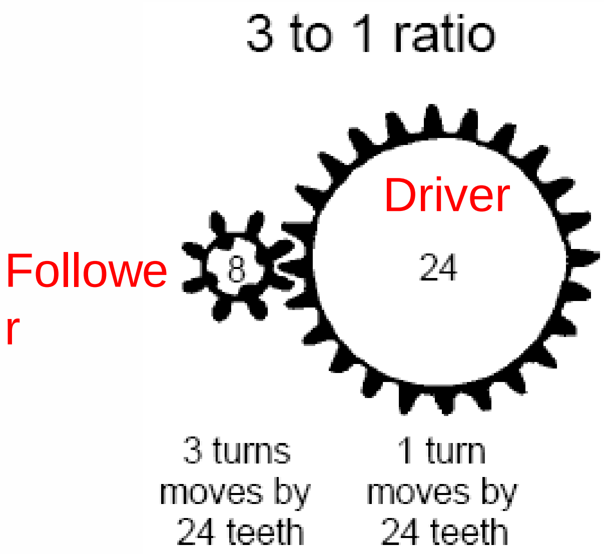
- A typical DC motor operates at speeds that are far too high to be useful, and at torques that are far too low.
- Gear reduction is the standard method by which a motor is made useful.





Gear Ratio

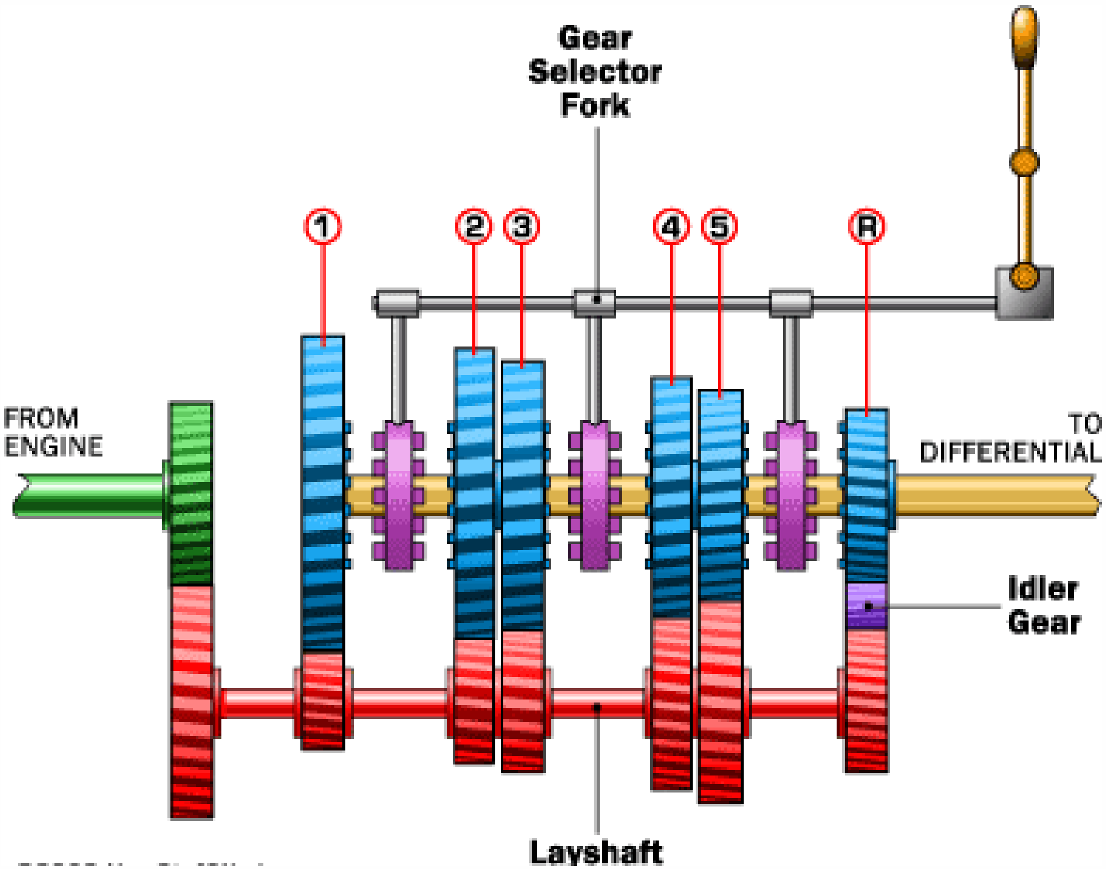
- You can calculate the **gear ratio** by using the number of teeth of the driver divided by the number of teeth of the follower.
- We gear up when we increase velocity and decrease torque.
  Ratio: 3:1
- We gear down when we increase torque and reduce velocity.
  Ratio: 1:3



Gear Ratio = # teeth input gear / # teeth output gear
= torque in / torque out = speed out / speed in

Example of Gear Trains

- A most commonly used example of gear trains is the gears of an automobile.

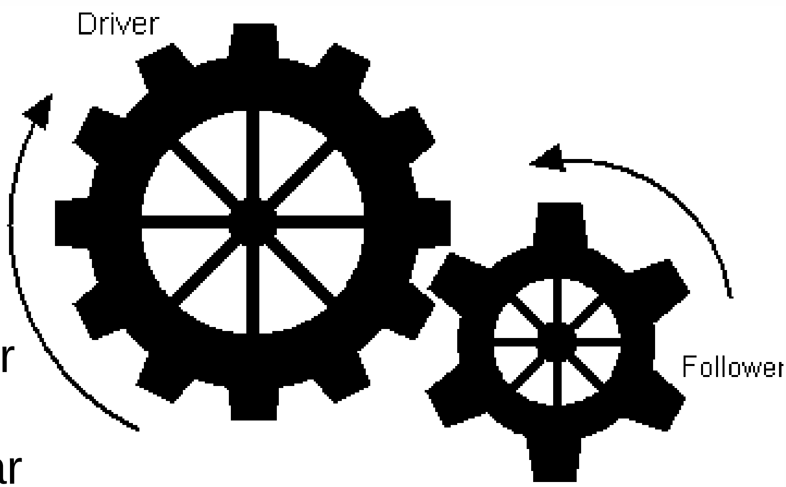


Mathematical Modelling of Gear Trains

- Gears increase or reduce angular velocity (while simultaneously decreasing or increasing torque, such that energy is conserved).

Energy of Driving Gear = Energy of Following Gear

$$N_1\theta_1 = N_2\theta_2$$



N_1 → Number of Teeth of Driving Gear

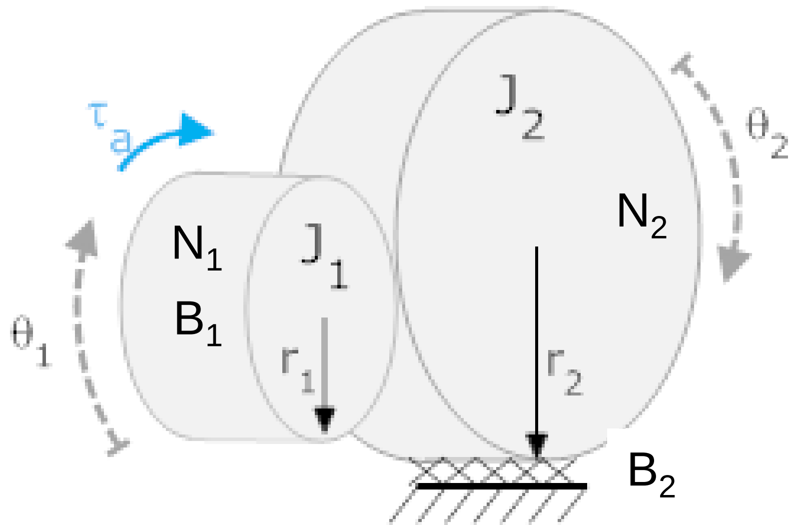
θ_1 → Angular Movement of Driving Gear

N_2 → Number of Teeth of Following Gear

θ_2 → Angular Movement of Following Gear

Mathematical Modelling of Gear Trains

- In the system below, a torque, τ_a, is applied to gear 1 (with number of teeth N_1, moment of inertia J_1 and a rotational friction B_1).
- It, in turn, is connected to gear 2 (with number of teeth N_2, moment of inertia J_2 and a rotational friction B_2).
- The angle θ_1 is defined positive clockwise, θ_2 is defined positive clockwise. The torque acts in the direction of θ_1.
- Assume that T_L is the load torque applied by the load connected to Gear-2.



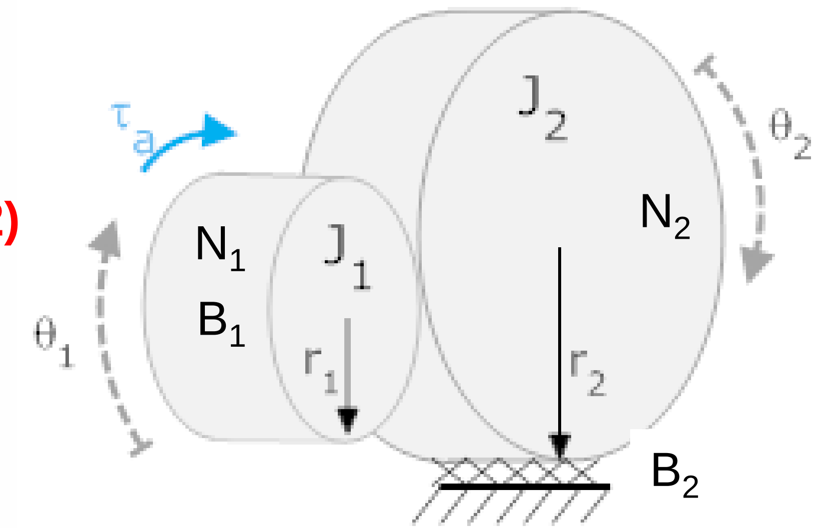
Mathematical Modelling of Gear Trains

- For Gear-1

$$\tau_a = J_1\ddot{\theta}_1 + B_1\dot{\theta}_1 + T_1 \longrightarrow \textbf{Eq (1)}$$

- For Gear-2

$$T_2 = J_2\ddot{\theta}_2 + B_2\dot{\theta}_2 + T_L \longrightarrow \textbf{Eq (2)}$$



- Since

$$N_1\theta_1 = N_2\theta_2$$

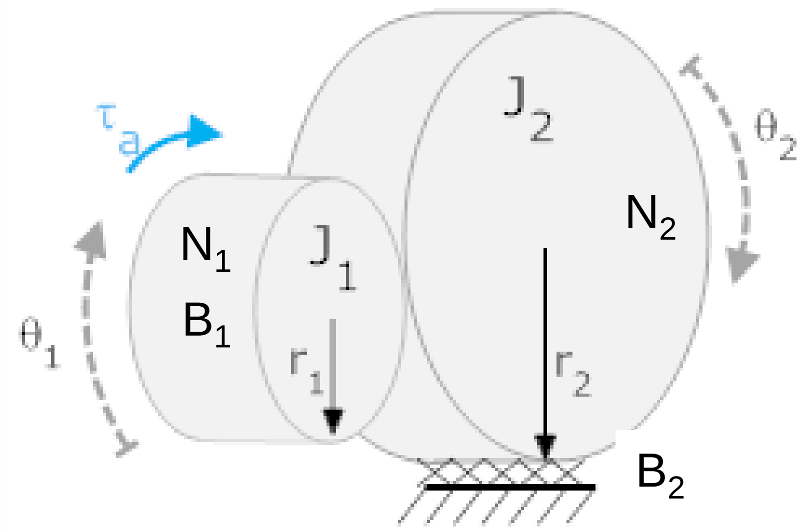
- therefore

$$\theta_2 = \frac{N_1}{N_2}\theta_1 \longrightarrow \textbf{Eq (3)}$$

Mathematical Modelling of Gear Trains

- Gear Ratio is calculated as

$$\frac{T_2}{T_1} = \frac{N_2}{N_1} \Rightarrow T_1 = \frac{N_1}{N_2} T_2$$



- Put this value in eq (1)

$$\tau_a = J_1\ddot{\theta}_1 + B_1\dot{\theta}_1 + \frac{N_1}{N_2} T_2$$

- Put T_2 from eq (2)

$$\tau_a = J_1\ddot{\theta}_1 + B_1\dot{\theta}_1 + \frac{N_1}{N_2}(J_2\ddot{\theta}_2 + B_2\dot{\theta}_2 + T_L)$$

- Substitute θ_2 from eq (3)

$$\tau_a = J_1\ddot{\theta}_1 + B_1\dot{\theta}_1 + \frac{N_1}{N_2}(J_2\frac{N_1}{N_2}\ddot{\theta}_1 + B_2\frac{N_1}{N_2}\dot{\theta}_2 + \frac{N_1}{N_2}T_L)$$

Mathematical Modelling of Gear Trains

$$\tau_a = J_1\ddot{\theta}_1 + B_1\dot{\theta}_1 + \frac{N_1}{N_2}\left(J_2\frac{N_1}{N_2}\ddot{\theta}_1 + B_2\frac{N_1}{N_2}\dot{\theta}_2 + \frac{N_1}{N_2}T_L\right)$$

- After simplification

$$\tau_a = J_1\ddot{\theta}_1 + \left(\frac{N_1}{N_2}\right)^2 J_2\ddot{\theta}_1 + B_1\dot{\theta}_1 + \left(\frac{N_1}{N_2}\right)^2 B_2\dot{\theta}_1 + \frac{N_1}{N_2}T_L$$

$$\tau_a = \left[J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2\right]\ddot{\theta}_1 + \left[B_1 + \left(\frac{N_1}{N_2}\right)^2 B_2\right]\dot{\theta}_1 + \frac{N_1}{N_2}T_L$$

$$\boxed{J_{eq} = J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2} \qquad \boxed{B_{eq} = B_1 + \left(\frac{N_1}{N_2}\right)^2 B_2}$$

$$\tau_a = J_{eq}\ddot{\theta}_1 + B_{eq}\dot{\theta}_1 + \frac{N_1}{N_2}T_L$$

Mathematical Modelling of Gear Trains

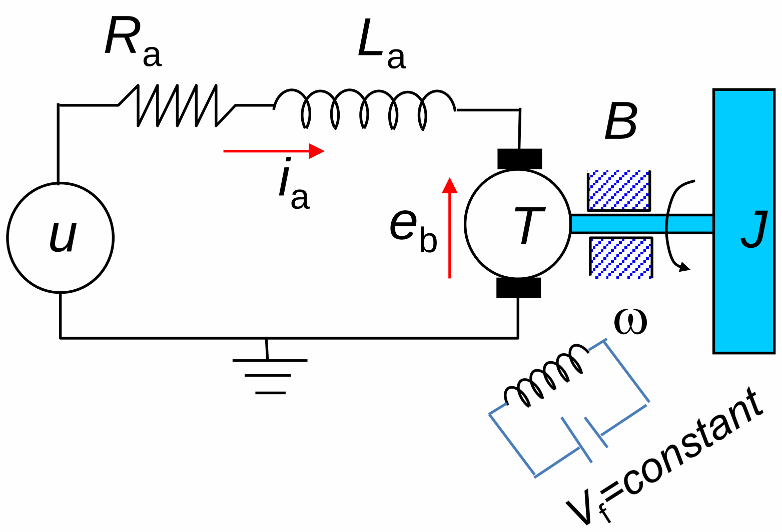
- For three gears connected together

$$J_{eq} = J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2 + \left(\frac{N_1}{N_2}\right)^2 \left(\frac{N_3}{N_4}\right)^2 J_3$$

$$B_{eq} = B_1 + \left(\frac{N_1}{N_2}\right)^2 B_2 + \left(\frac{N_1}{N_2}\right)^2 \left(\frac{N_3}{N_4}\right)^2 B_3$$

Armature Controlled D.C Motor

Input: voltage u
Output: Angular velocity ω



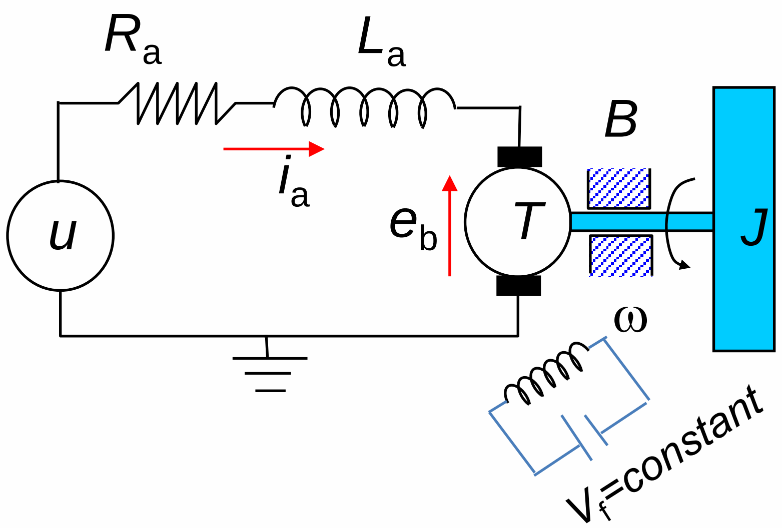
Elecrical Subsystem (loop method):

$$u = R_a i_a + L_a \frac{di_a}{dt} + e_b, \qquad \text{where } e_b = \text{back-emf voltage}$$

Mechanical Subsystem

$$T_{motor} = J\dot{\omega} + B\omega$$

Armature Controlled D.C Motor



Power Transformation:

Torque-Current: $T_{motor} = K_t i_a$

Voltage-Speed: $e_b = K_b \omega$

where K_t: torque constant, K_b: velocity constant For an ideal motor

$$K_t = K_b$$

Combing previous equations results in the following mathematical model:

$$\begin{cases} L_a \dfrac{di_a}{dt} + R_a i_a + K_b \omega = u \\ J\dot{\omega} + B\omega - K_t i_a = 0 \end{cases}$$

Armature Controlled D.C Motor

Taking Laplace transform of the system's differential equations with zero initial conditions gives:

$$\begin{cases} (L_a s + R_a) I_a(s) + K_b \Omega(s) = U(s) \\ (Js + B)\Omega(s) - K_t I_a(s) = 0 \end{cases}$$

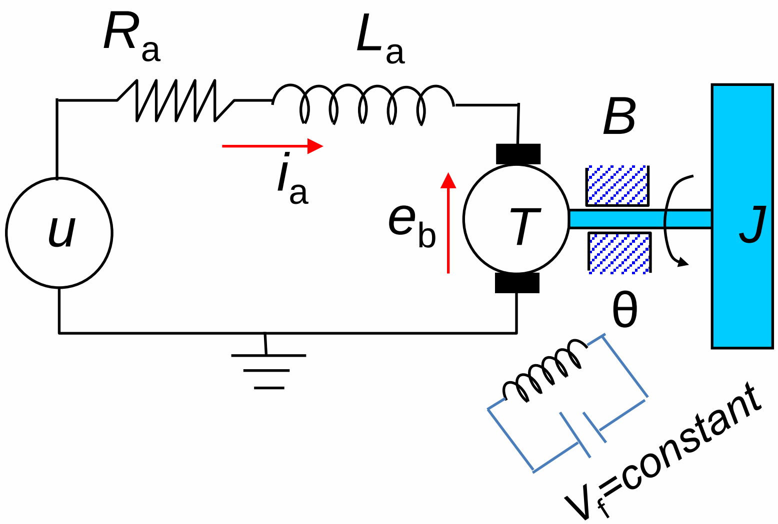
Eliminating I_a yields the input-output transfer function

$$\frac{\Omega(s)}{U(s)} = \frac{K_t}{L_a J s^2 + (JR_a + BL_a)s + BR_a + K_t K_b}$$

Armature Controlled D.C Motor

If output of the D.C motor is angular position θ then we know

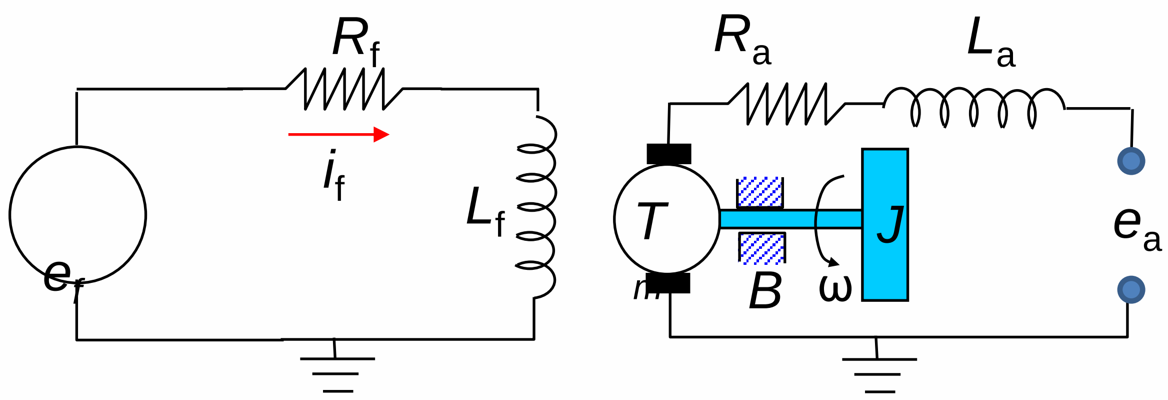
$$\omega = \frac{d\theta}{dt} \qquad or \qquad \Omega(s) = s\theta(s)$$



Which yields following transfer function

$$\frac{\theta(s)}{U(s)} = \frac{(K_t / R_a)}{s(Js + (B + K_t K_b / R_a))}$$

Field Controlled D.C Motor



Applying KVL at field circuit

$$e_f = i_f R_f + L_f \frac{di_f}{dt}$$

Mechanical Subsystem

$$T_m = J\dot{\omega} + B\omega$$

Field Controlled D.C Motor

Power Transformation:

Torque-Current: $T_m = K_f i_f$

where K_f: torque constant

Combing previous equations and taking Laplace transform (considering initial conditions to zero) results in the following mathematical model:

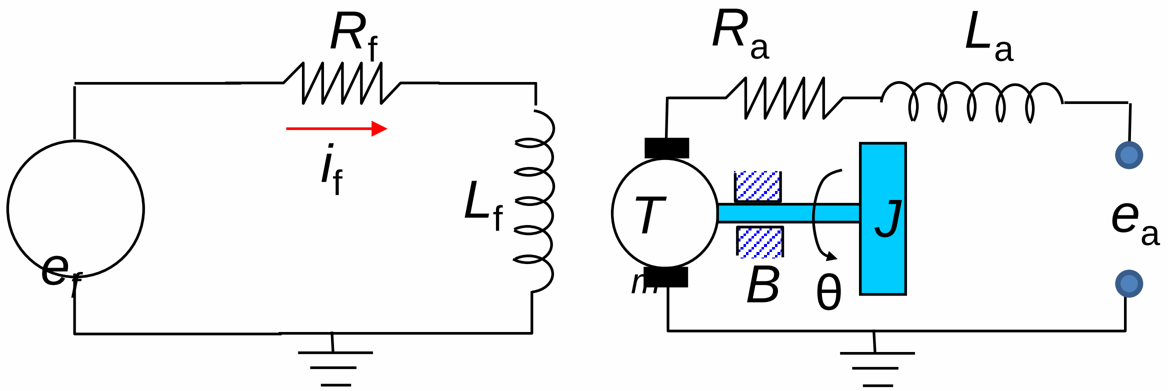
$$\begin{cases} E_f(s) = R_f I_f(s) + sL_f I_f(s) \\ Js\Omega(s) + B\Omega(s) = K_f I_f(s) \end{cases}$$

Field Controlled D.C Motor

Eliminating $I_f(S)$ yields

$$\frac{\Omega(s)}{E_f(s)} = \frac{K_f}{(Js+B)(L_f s + R_f)}$$

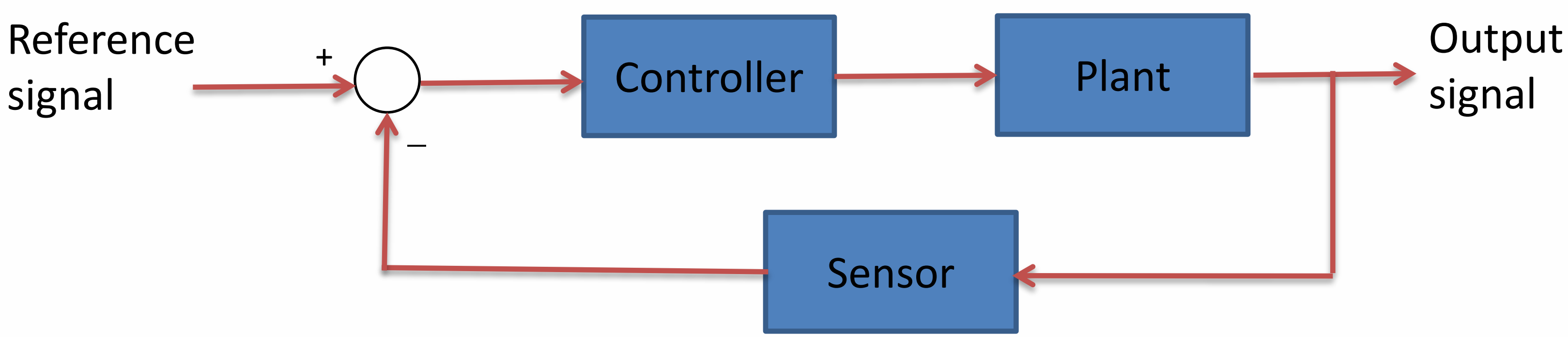
If angular position θ is output of the motor



$$\frac{\theta(s)}{E_f(s)} = \frac{K_f}{s(Js+B)(L_f s + R_f)}$$

Closed Loop Control of DC Drives

- **Closed loop control** is when the firing angle is varied automatically by a controller to achieve a reference speed or torque
- This requires the use of sensors to feed back the actual motor speed and torque to be compared with the reference values

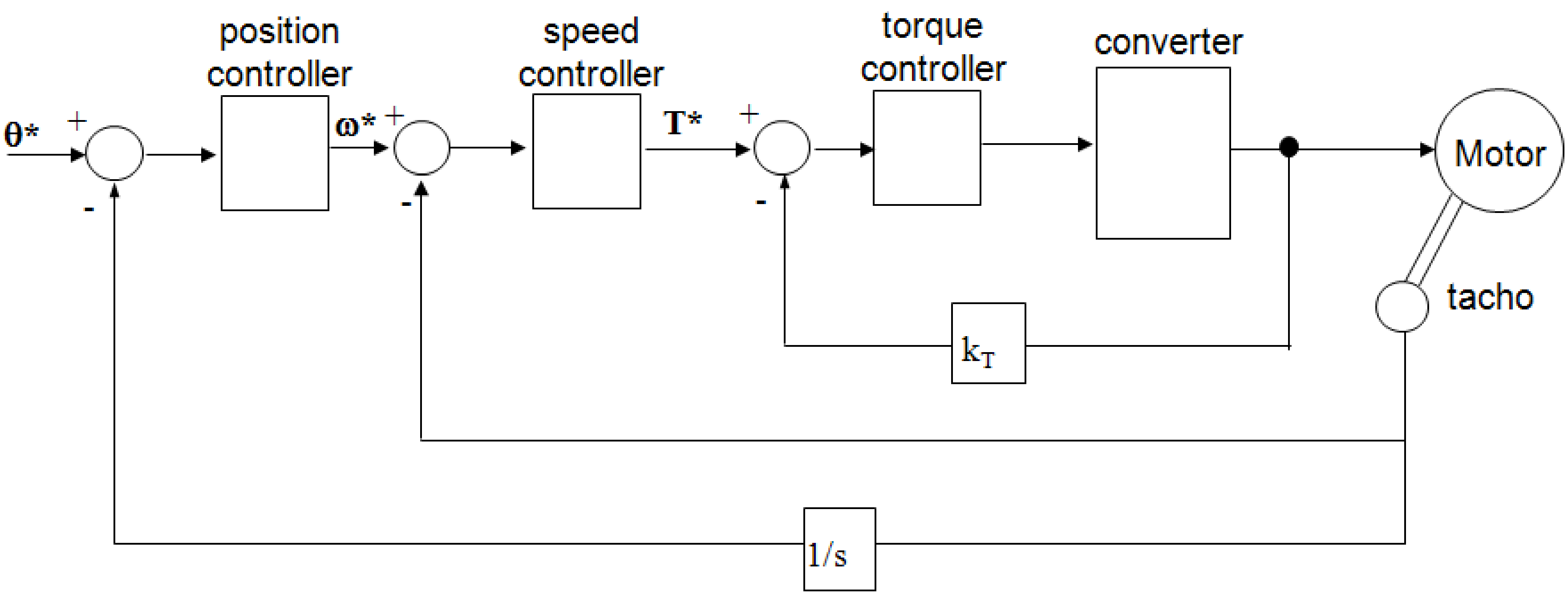


Closed Loop Control of DC Drives

- Feedback loops may be provided to satisfy one or more of the following:
  - Protection
  - Enhancement of response – fast response with small overshoot
  - Improve steady-state accuracy
- Variables to be controlled in drives:
  - Torque – achieved by controlling current
  - Speed
  - Position

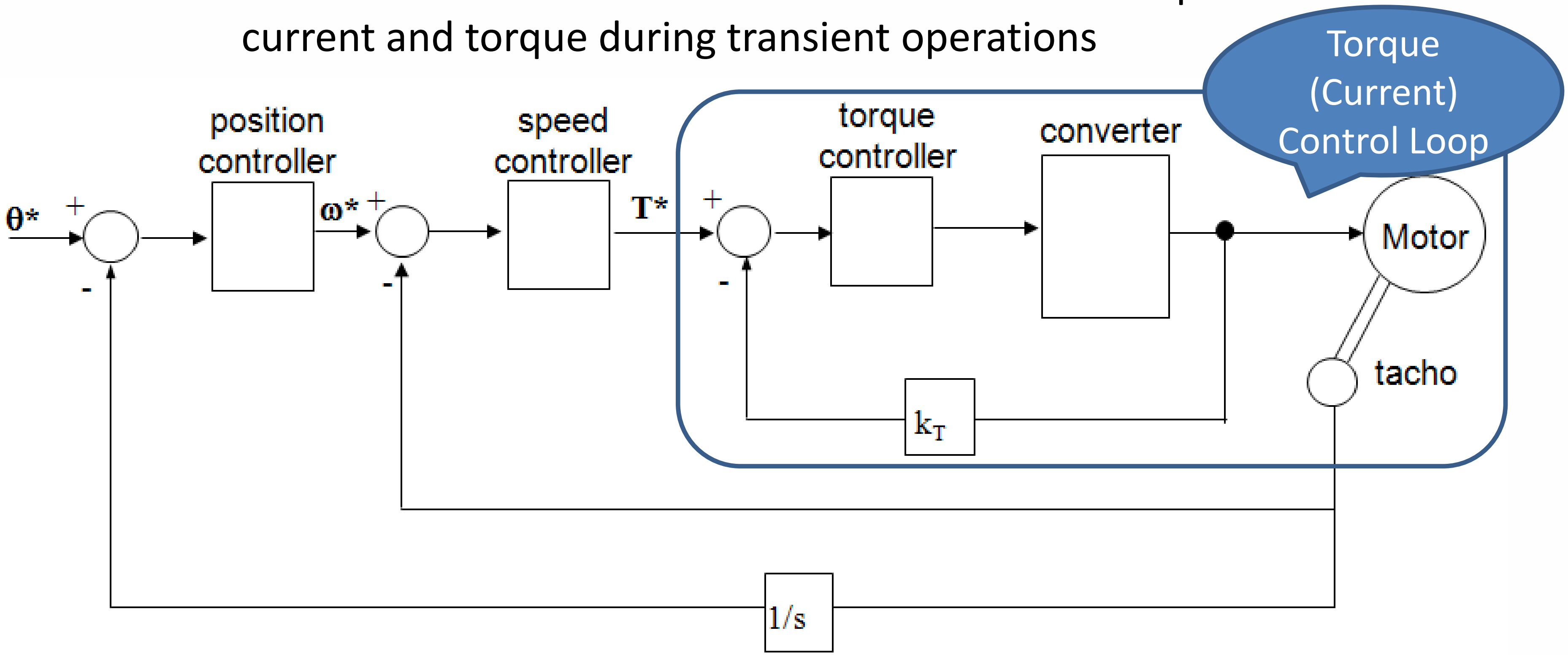
Closed Loop Control of DC Drives

- Cascade control structure
  - Flexible – outer loops can be added/removed depending on control requirements.
  - Control variable of inner loop (eg: speed, torque) can be limited by limiting its reference value
  - Torque loop is fastest, speed loop – slower and position loop - slowest



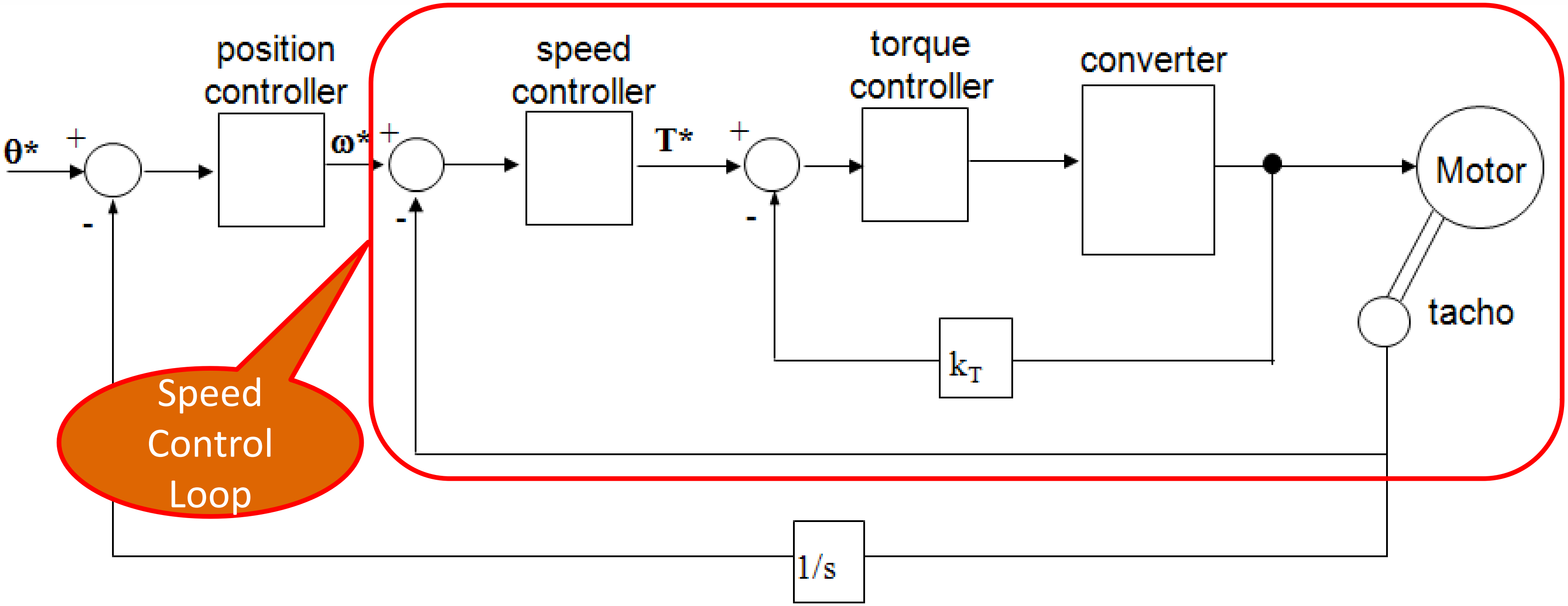
Closed Loop Control of DC Drives

- Cascade control structure:
  - <u>Inner Torque (Current) Control Loop:</u>
    - **Current control loop** is used to **control torque via armature current (i_a)** and **maintains current within a safe limit**
    - Accelerates and decelerates the drive at maximum permissible current and torque during transient operations



Closed Loop Control of DC Drives

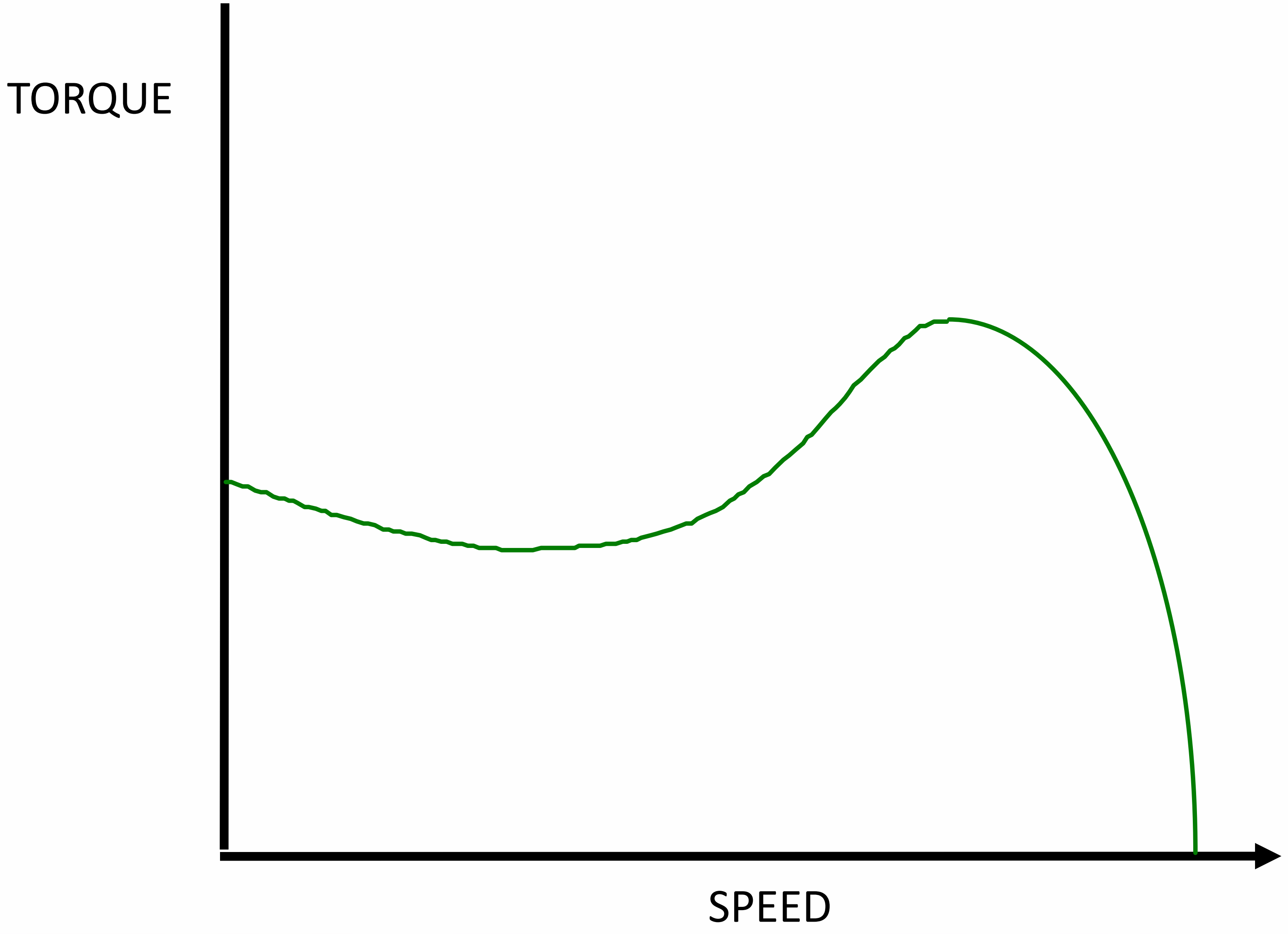
- Cascade control structure
  - <u>Speed Control Loop:</u>
    - **Ensures that the actual speed is always equal to reference speed ω***
    - Provides fast response to changes in ω*, T_L and supply voltage (i.e. any transients are overcome within the shortest feasible time) without exceeding motor and converter capability



A.C Servo Motor

- An AC servomotor is basically a two phase induction motor except for certain special design features. A two phase servomotor differs in the following two ways from a normal induction motor.

Speed - Torque Curve



Speed - Torque Curve

❑An induction motor produces different amounts of torque, (twisting force) at different speeds.

❑On the diagram, speed is on the horizontal axis and torque and current are on the vertical axis.

❑The motor will produce locked rotor torque, and draw 6-10 times the motor FLA amount.

❑As the motor increases speed, the torque will move through the Pull up torque region, and motor current will drop.

❑As the speed continues to increase, the motor then gets to the point where it produces maximum torque, called the breakdown torque, and the current continues to decrease.

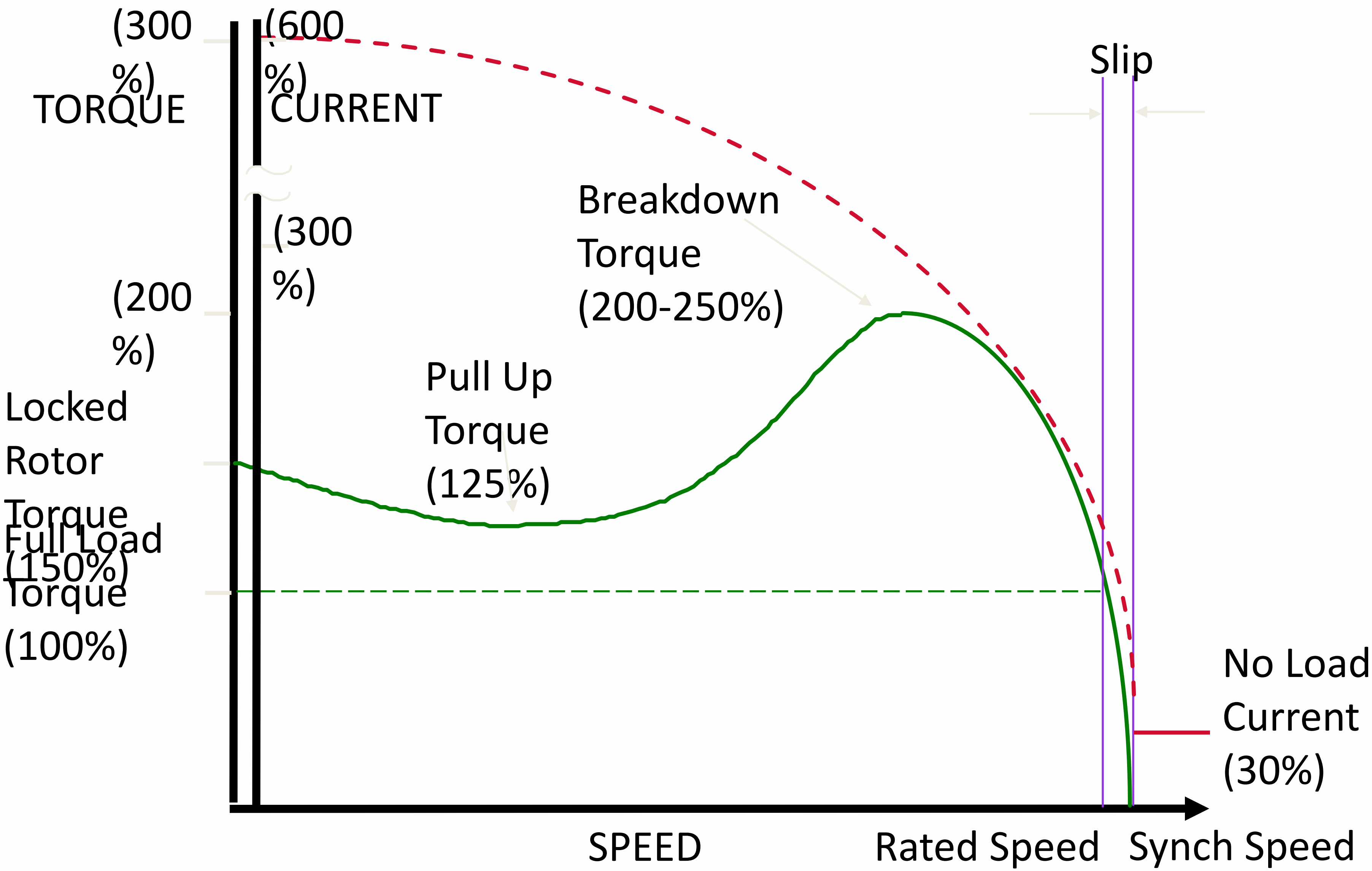
Speed - Torque Curve

- If this motor was running unloaded, the rotor speed would end up very close to the synchronous speed, and the current would end up about 30% of FLA.

- As the load was increased, the current would move back up the curve as well as the torque.

- And the motor speed would drop from very close to synchronous speed towards the rated speed.

Speed - Torque Curve

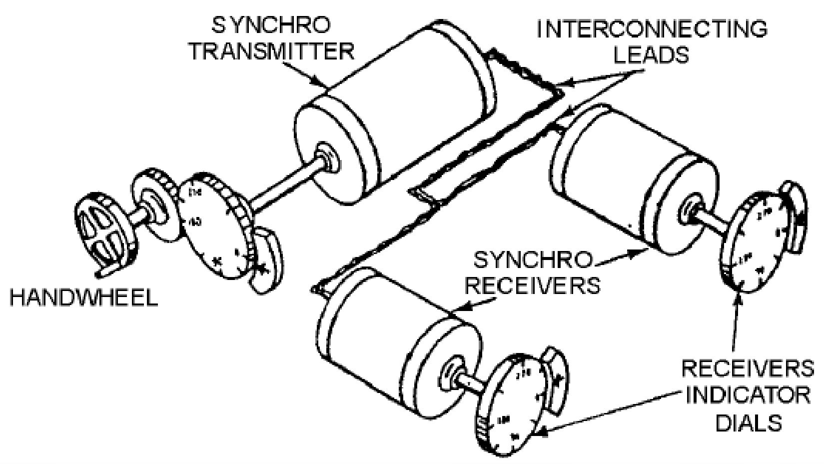


Synchros

- Synchros are used primarily for the rapid and accurate transmission of information between equipment and stations.

- Examples of such information are changes in course, speed, and range of targets or missiles; angular displacement (position) of the ship's rudder; and changes in the speed and depth of torpedoes.

- This information must be transmitted quickly and accurately. Synchros can provide this speed and accuracy. They are reliable, adaptable, and compact.



Synchros

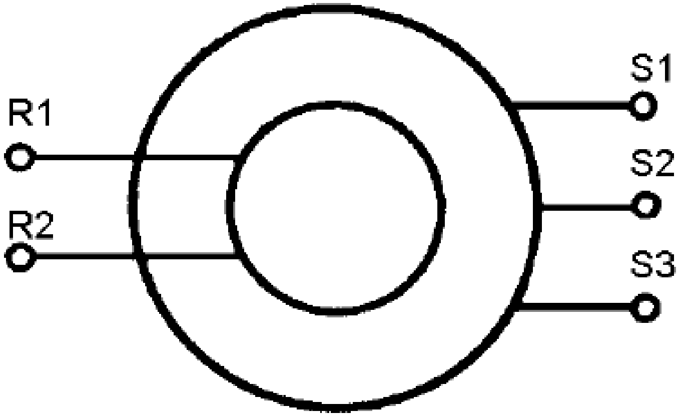
- Synchros work in teams. Two or more synchros interconnected electrically form a synchro system. There are two general classifications of synchro systems—TORQUE SYSTEMS AND CONTROL SYSTEMS.

- Torque-synchro systems use torque synchros and control-synchro systems use control synchros. The load dictates the type of synchro system, and thus the type of synchro.

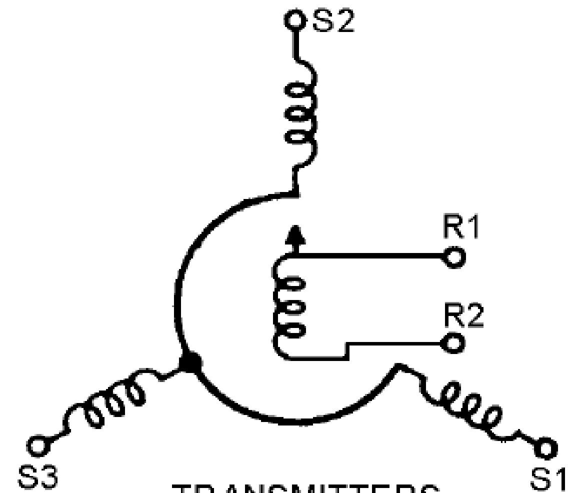
- Torque-synchro systems are classified "torque" because they are mainly concerned with the torque or turning force required to move light loads such as dials, pointers, or similar indicators.

- The positioning of these devices requires a relatively low amount of torque. Control synchros are used in systems that are designed to move heavy loads such as gun directors, radar antennas, and missile launchers.

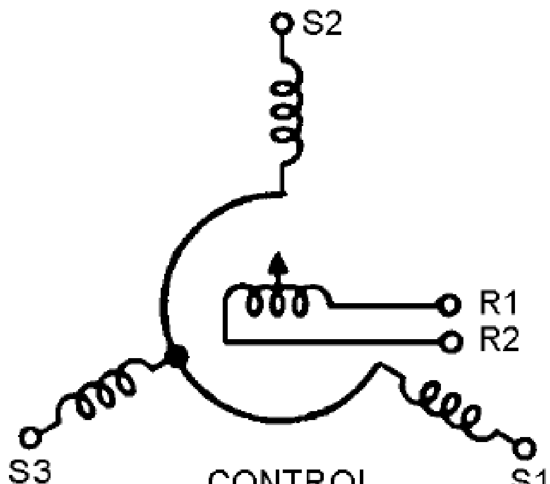
Schematic symbols for Synchros



TRANSMITTERS, RECEIVERS
CONTROLS TRANSFORMERS



TRANSMITTERS
AND RECEIVERS



CONTROL
TRANSFORMERS

Synchro Torque Transmitter

- The synchro transmitter converts the angular position of its rotor (mechanical input) into an electrical output signal.

- When a 115-volt ac excitation voltage is applied to the rotor of a synchro transmitter, the resultant current produces an ac magnetic field around the rotor winding.

- The lines of force cut through the turns of the three stator windings and, by transformer action, induce voltage into the stator coils.

- The effective voltage induced in any stator coil depends upon the angular position of that coil's axis with respect to the rotor axis.

- When the maximum effective coil voltage is known, the effective voltage induced into a stator coil at any angular displacement can be determined.

Control synchro systems

- When large amounts of power and a higher degree of accuracy are required, as in the movement of heavy radar antennas and gun turrets, torque synchro systems give way to the use of CONTROL SYNCHROS.

- Control synchros by themselves cannot move heavy loads. However, they are used to "control" servo systems, which in turn do the actual movement.

- There are three types of control synchros:
    - CONTROL TRANSMITTER (CX),
    - CONTROL TRANSFORMER (CT), and
    - CONTROL DIFFERENTIAL TRANSMITTER (CDX).

Control Transformer (CT)

- The CT compares two signals, the electrical signal applied to its stator and the mechanical signal applied to its rotor. Its output is a difference signal that controls a power amplifying device and thus the movement of heavy equipment.

- An interesting point about the rotor is that it is never connected to an ac supply and, therefore, induces no voltages in the stator coils. As a result, the CT stator currents are determined solely by the voltages applied to the high-impedance stator windings.

- The rotor itself is wound so that its position has very little effect on the stator currents. Also, there is never any appreciable current flowing in the rotor because its output voltage is always applied to a high-impedance load.
- As a result, the CT rotor does not try to follow the magnetic field of its stator and must be turned by some external force.

Introduction

- In time-domain analysis the response of a dynamic system to an input is expressed as a function of time.

- It is possible to compute the time response of a system if the nature of input and the mathematical model of the system are known.

- Usually, the input signals to control systems are not known fully ahead of time.

- It is therefore difficult to express the actual input signals mathematically by simple equations.

Standard Test Signals

- The characteristics of actual input signals are a sudden shock, a sudden change, a constant velocity, and constant acceleration.

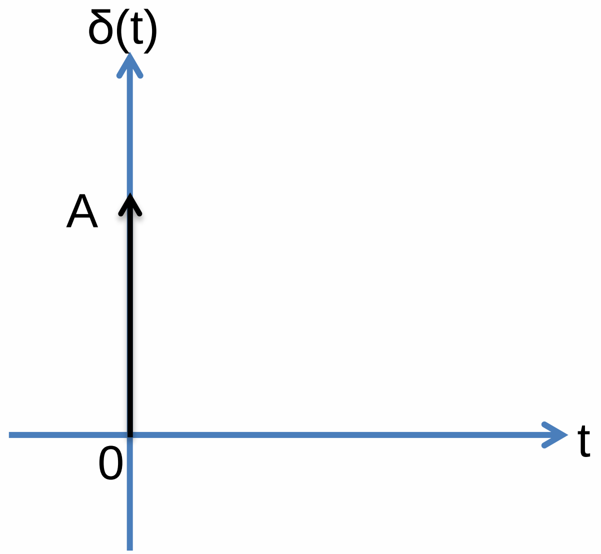
- The dynamic behavior of a system is therefore judged and compared under application of standard test signals – an impulse, a step, a constant velocity, and constant acceleration.

- The other standard signal of great importance is a sinusoidal signal.

Standard Test Signals

- **Impulse signal**
  - The impulse signal imitate the sudden shock characteristic of actual input signal.

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

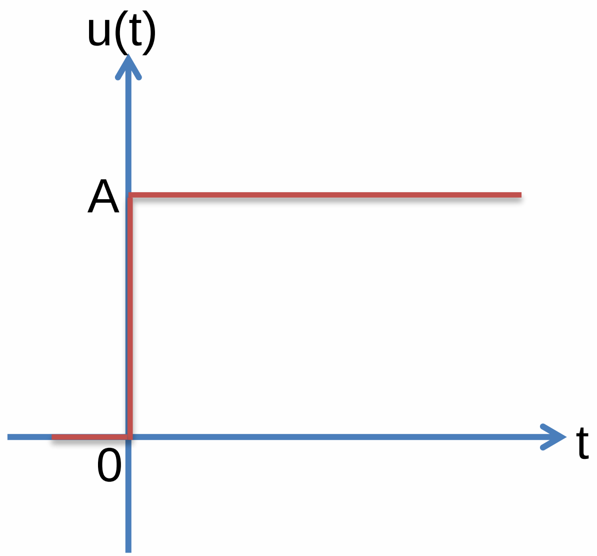


  - If A=1, the impulse signal is called unit impulse signal.

Standard Test Signals

- **Step signal**
  - The step signal imitate the sudden change characteristic of actual input signal.

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$



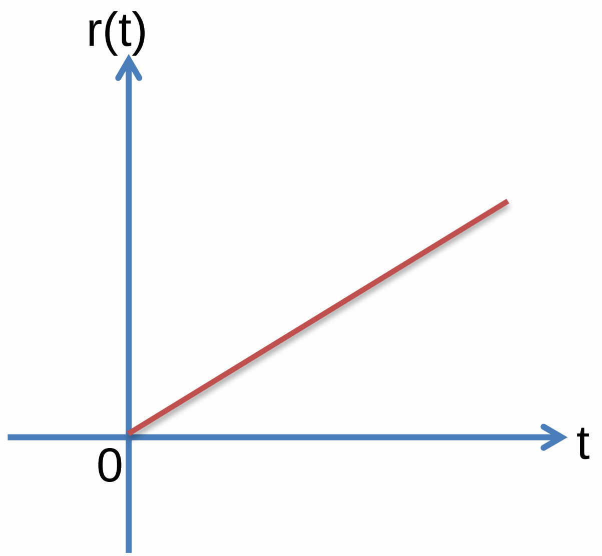
  - If A=1, the step signal is called unit step signal

Standard Test Signals

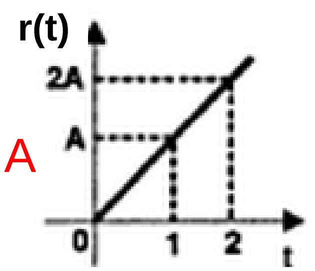
- **Ramp signal**
  - The ramp signal imitate the constant velocity characteristic of actual input signal.

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

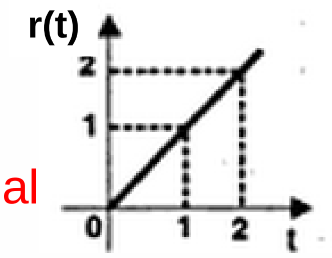
  - If *A=1*, the ramp signal is called unit ramp signal



ramp signal with slope A



unit ramp signal

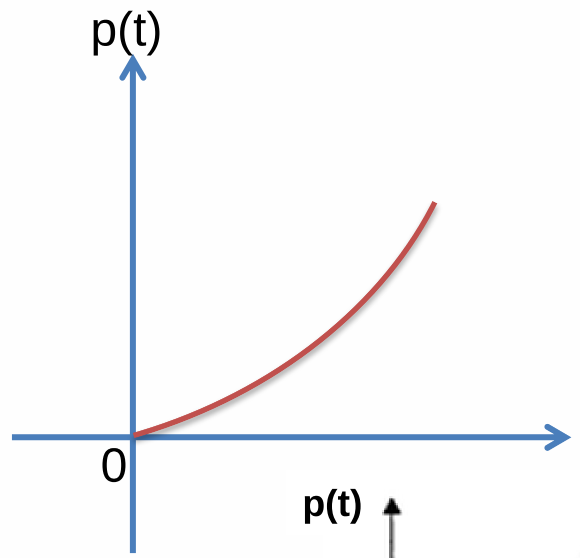


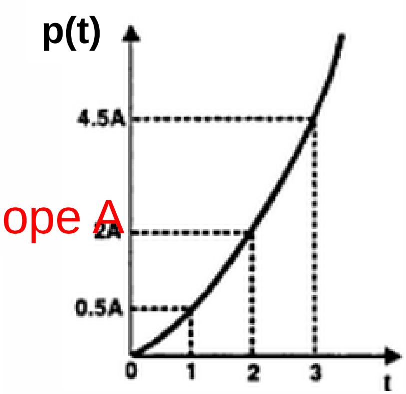
Standard Test Signals

- **Parabolic signal**
  - The parabolic signal imitate the constant acceleration characteristic of actual input signal.

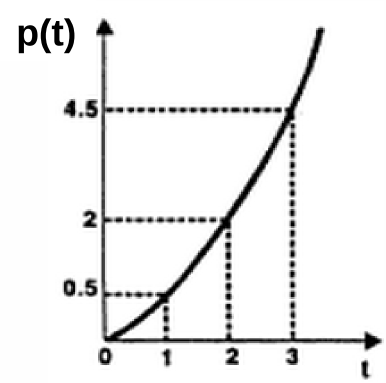
$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

  - If $A=1$, the parabolic signal is called unit parabolic signal.





parabolic signal with slope A



Unit parabolic signal

Relation between standard Test Signals

- Impulse

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

- Step

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- Ramp

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- Parabolic

$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$\int$ $\int$ $\int$

$\frac{d}{dt}$ $\frac{d}{dt}$ $\frac{d}{dt}$

Laplace Transform of Test Signals

- Impulse

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$L\{\delta(t)\} = \delta(s) = A$$

- Step

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{u(t)\} = U(s) = \frac{A}{S}$$

Laplace Transform of Test Signals

- Ramp

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{r(t)\} = R(s) = \frac{A}{s^2}$$

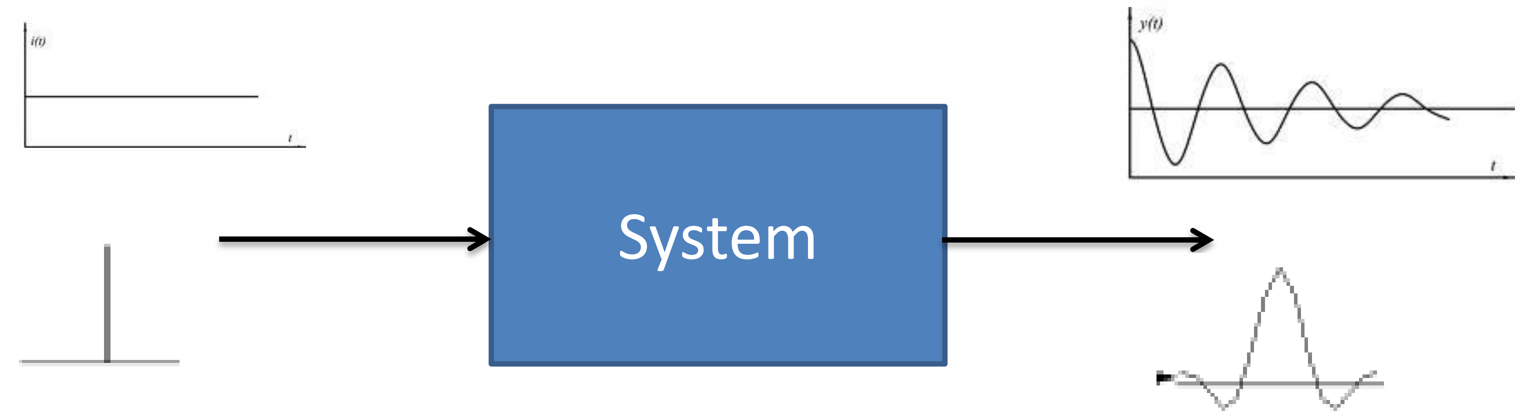
- Parabolic

$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{p(t)\} = P(s) = \frac{A}{S^3}$$

Time response

- ❑ Time response of a dynamic system response to an input expressed as a function of time.



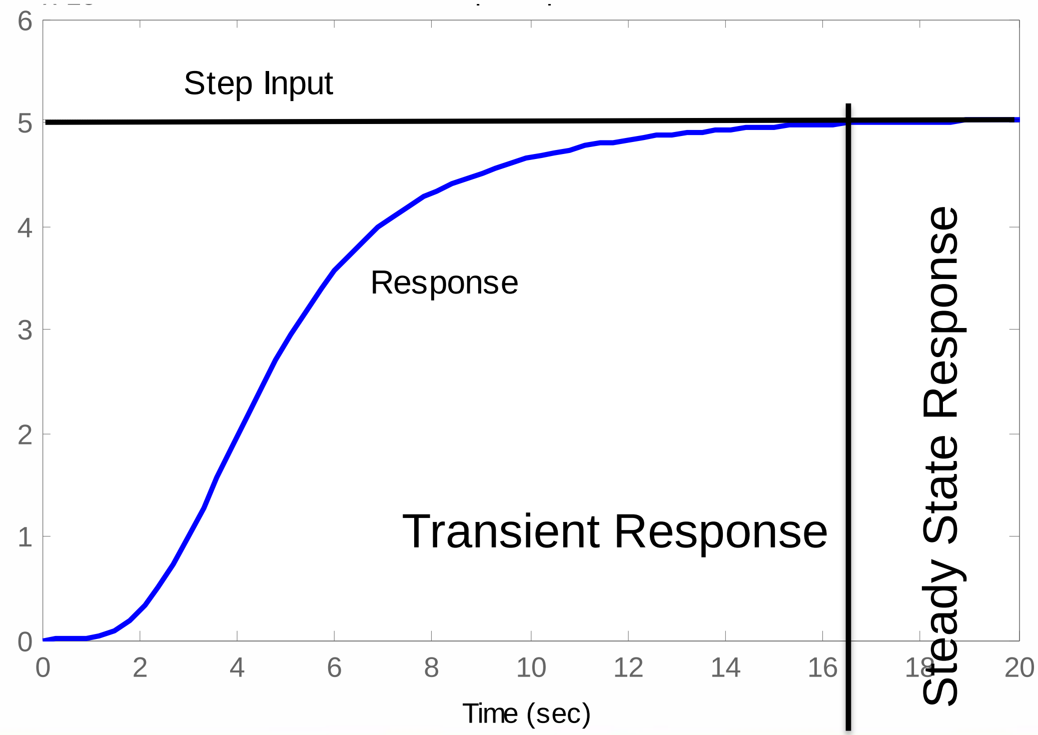
- ❑The time response of any system has two components
  - ❑Transient response
  - ❑Steady-state response

Time response (Contd..)

- When the response of the system is changed from equilibrium it takes some time to settle down.

- This is called transient response.

- The response of the system after the transient response is called steady state response.



Time response (Contd..)

- Transient response depend upon the system poles only and not on the type of input.
- It is therefore sufficient to analyze the transient response using a step input.
- The steady-state response depends on system dynamics and the input quantity.
- It is then examined using different test signals by final value theorem.

Time response (Contd..)

- The first order system has only one pole.

$$\frac{C(s)}{R(s)} = \frac{K}{Ts + 1}$$

- Where ***K*** is the D.C gain and ***T*** is the time constant of the system.

- Time constant is a measure of how quickly a 1st order system responds to a unit step input.

- D.C Gain of the system is ratio between the input signal and the steady state value of output.

Time response (Contd..)

- The first order system given below.

$$G(s) = \frac{10}{3s+1}$$

- D.C gain is 10 and time constant is 3 seconds.

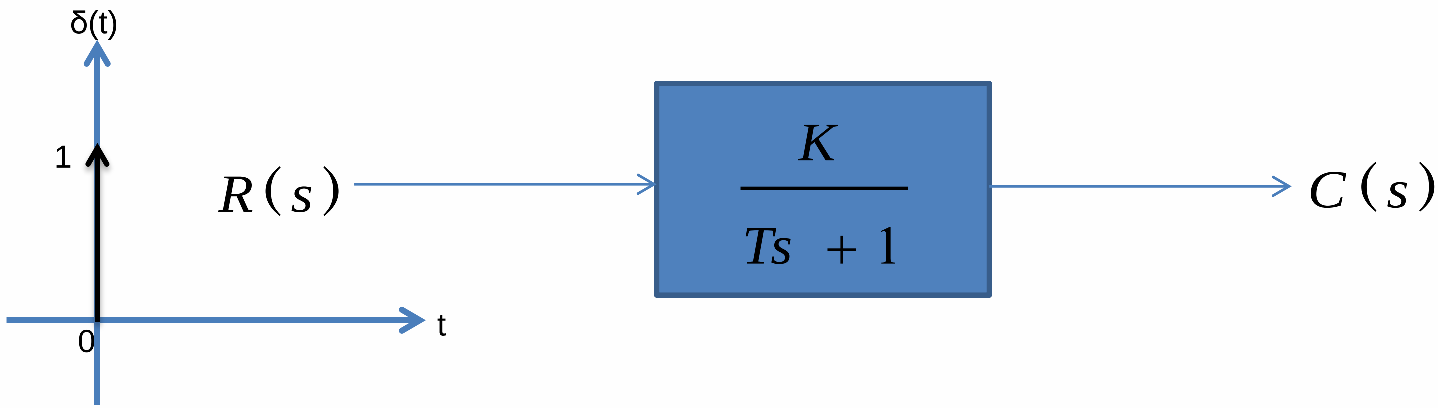
- For the following system

$$G(s) = \frac{3}{s+5} = \frac{3/5}{1/5s+1}$$

- D.C Gain of the system is 3/5 and time constant is 1/5 seconds.

Impulse Response of 1st Order System

- Consider the following 1st order system



$$R(s) = \delta(s) = 1$$

$$C(s) = \frac{K}{Ts + 1}$$

Impulse Response of 1st Order System

$$C(s) = \frac{K}{Ts + 1}$$

- Re-arrange following equation as

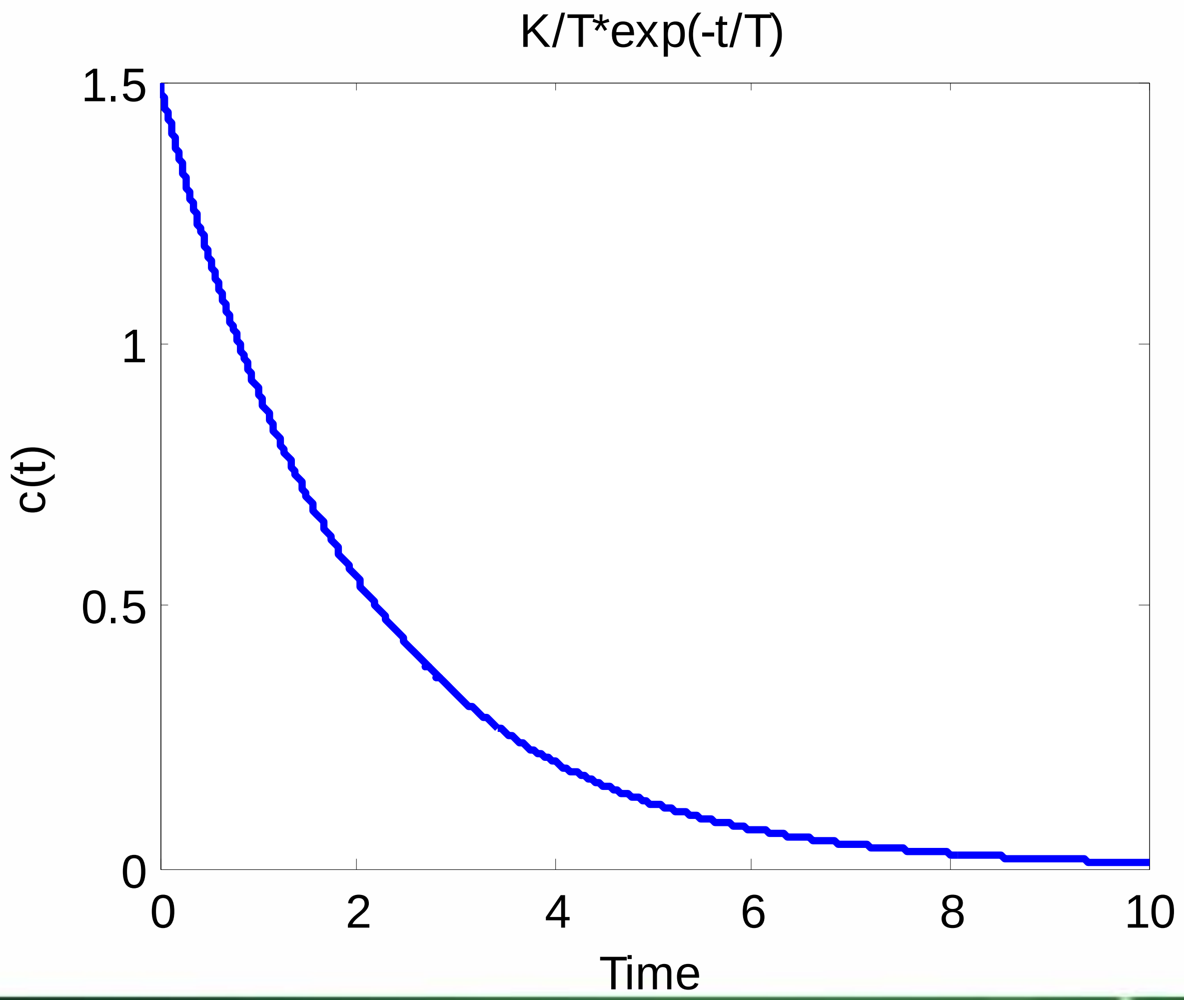
$$C(s) = \frac{K/T}{s + 1/T}$$

- In order to compute the response of the system in time domain we need to compute inverse Laplace transform of the above equation.

$$L^{-1}\left(\frac{C}{s + a}\right) = Ce^{-at} \qquad c(t) = \frac{K}{T}e^{-t/T}$$

Impulse Response of 1st Order System

- If K=3 and T=2s then $c(t) = \frac{K}{T} e^{-t/T}$



Step Response of 1st Order System

- Consider the following 1st order system

$$R(s) \longrightarrow \boxed{\frac{K}{Ts+1}} \longrightarrow C(s)$$

$$R(s) = U(s) = \frac{1}{s}$$

$$C(s) = \frac{K}{s(Ts+1)}$$

- In order to find out the inverse Laplace of the above equation, we need to break it into partial fraction expansion

$$C(s) = \frac{K}{s} - \frac{KT}{Ts+1}$$

Step Response of 1st Order System

$$C(s) = K\left(\frac{1}{s} - \frac{T}{Ts+1}\right)$$

- Taking Inverse Laplace of above equation

$$c(t) = K\left(u(t) - e^{-t/T}\right)$$

- Where u(t)=1

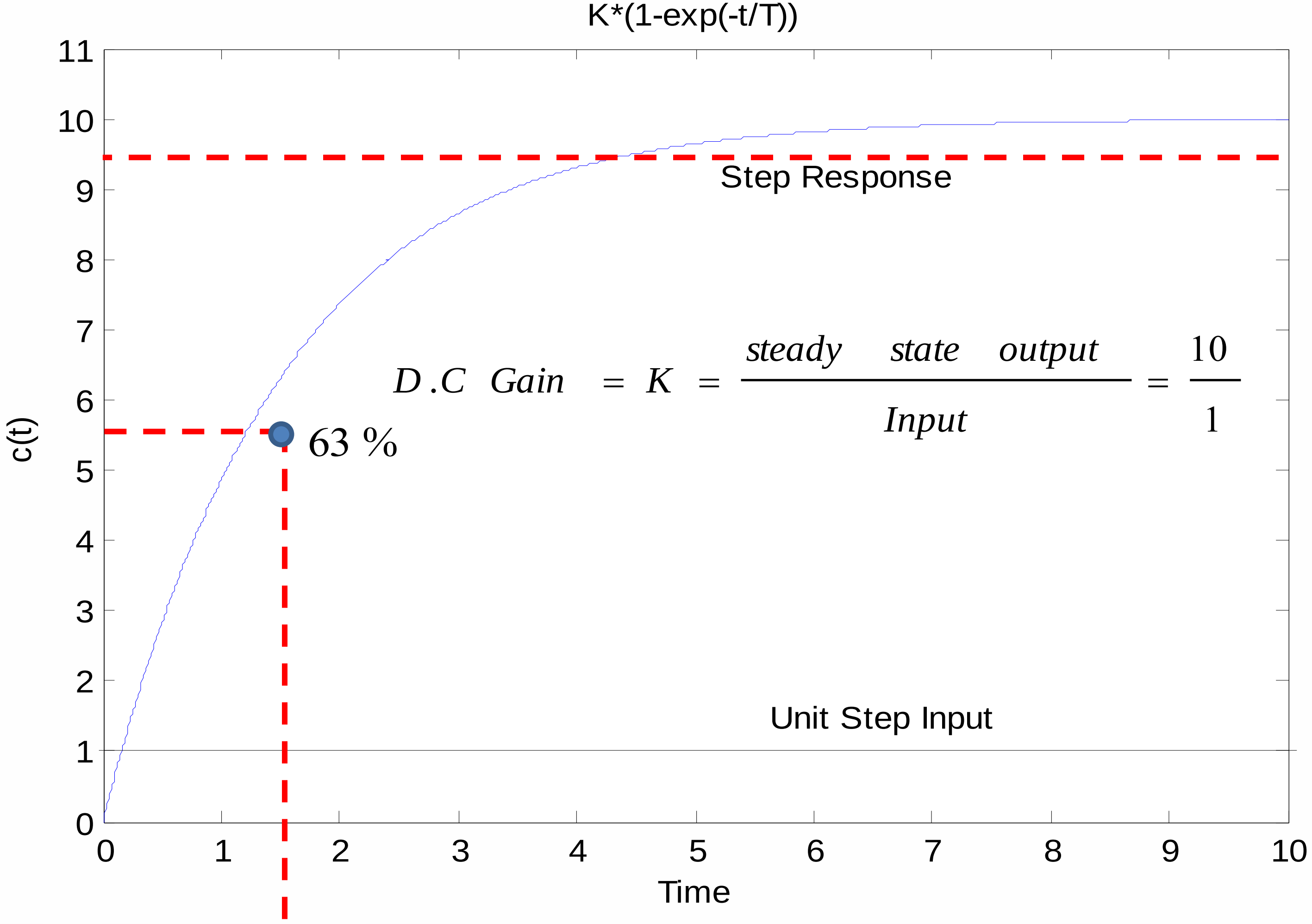
$$c(t) = K\left(1 - e^{-t/T}\right)$$

- When t=T (time constant)

$$c(t) = K\left(1 - e^{-1}\right) = 0.632K$$

Step Response of 1st Order System

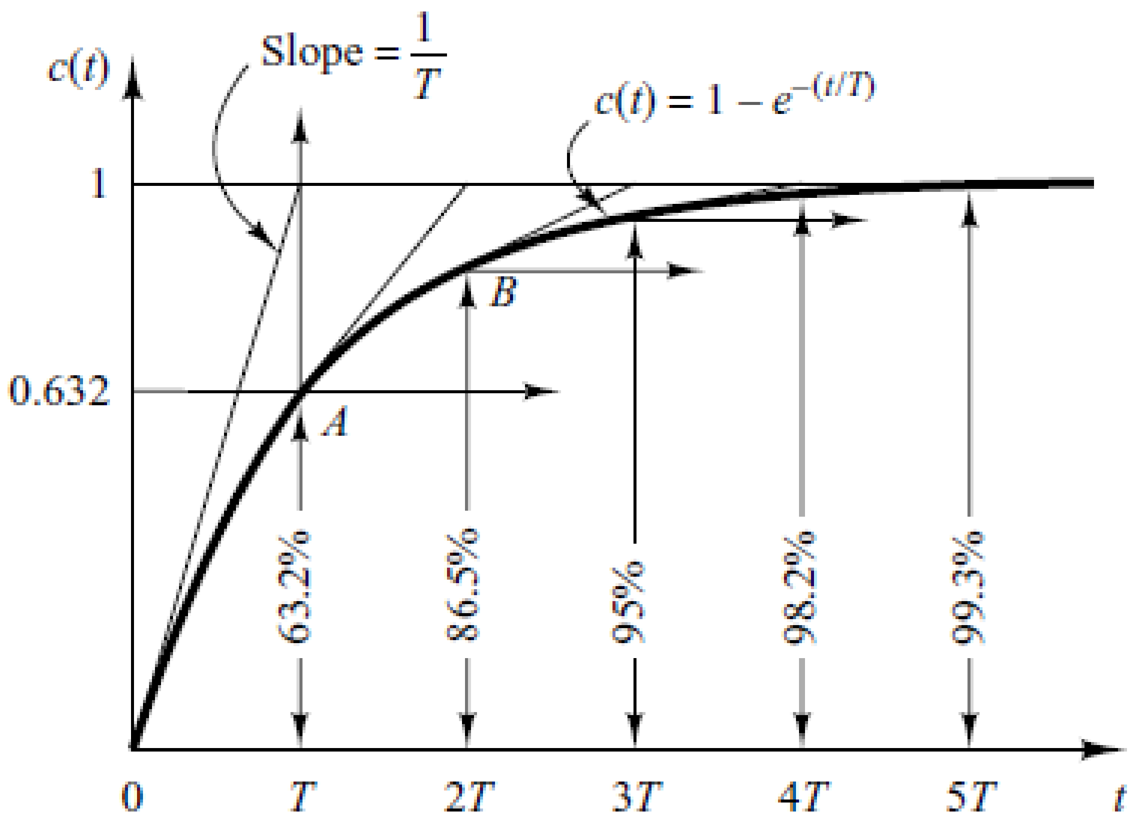
- If K=10 and T=1.5s then $$c(t) = K\left(1 - e^{-t/T}\right)$$



$$D.C\ \ Gain = K = \frac{steady\ \ state\ \ output}{Input} = \frac{10}{1}$$

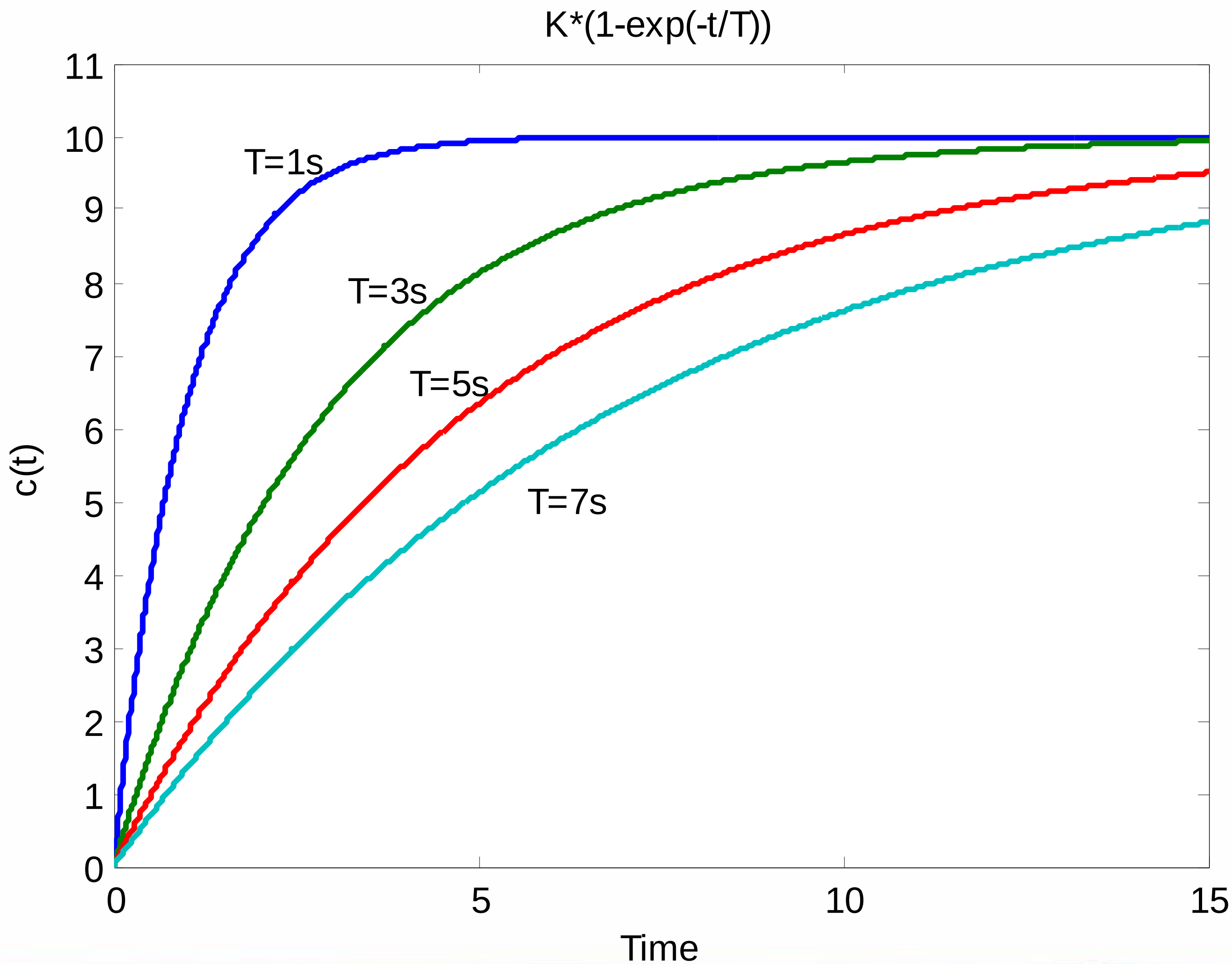
Step Response of 1st Order System

- System takes five time constants to reach its final value.



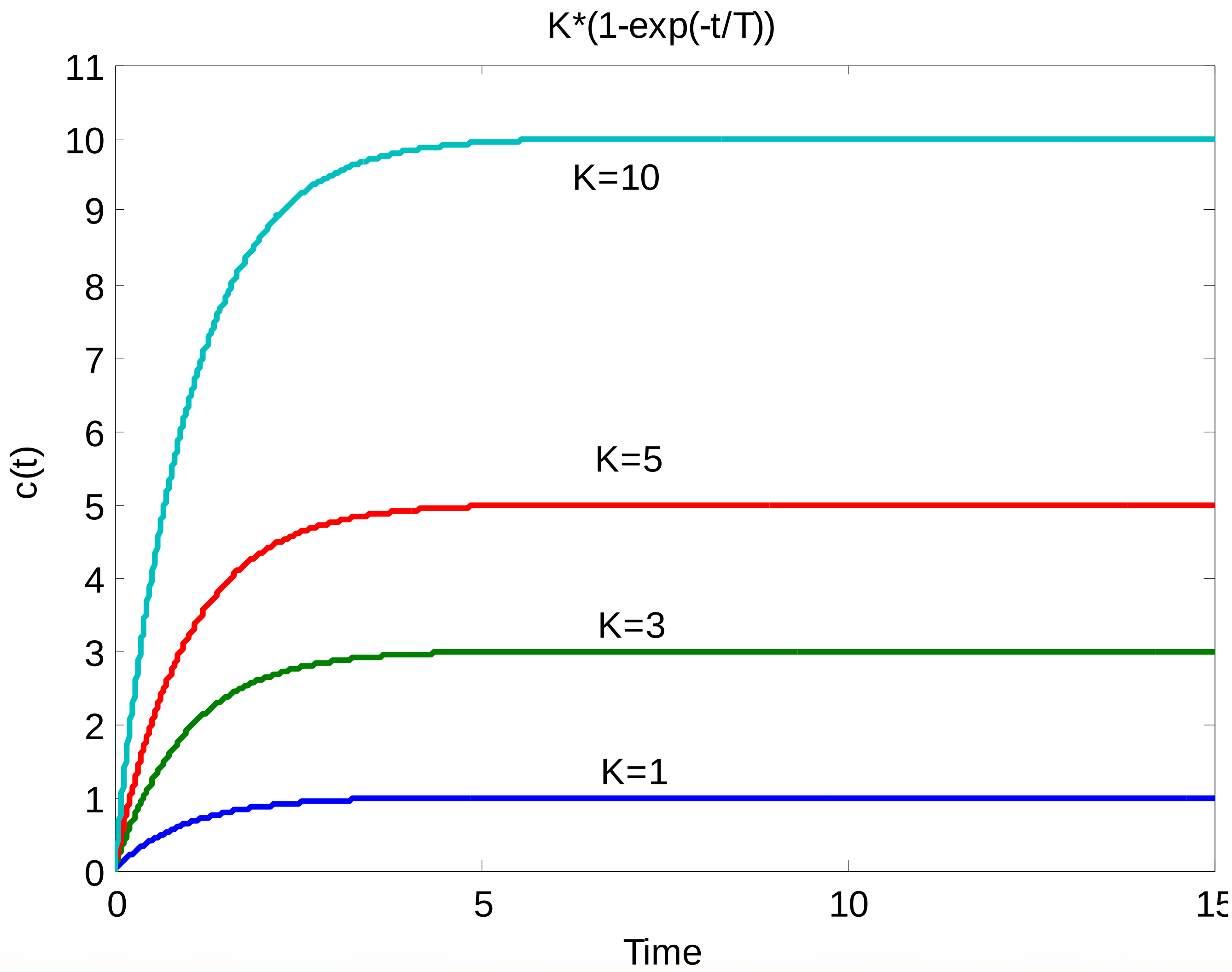
Step Response of 1st Order System

- If K=10 and T=1, 3, 5, 7 $$c(t) = K\left(1 - e^{-t/T}\right)$$



Step Response of 1st Order System

- If K=1, 3, 5, 10 and T=1 $$c(t) = K\left(1 - e^{-t/T}\right)$$



Steady State Error

- If the output of a control system at steady state does not exactly match with the input, the system is said to have steady state error

- Any physical control system inherently suffers steady-state error in response to certain types of inputs.

- A system may have no steady-state error to a step input, but the same system may exhibit nonzero steady-state error to a ramp input.

Classification of Control Systems

- Control systems may be classified according to their ability to follow step inputs, ramp inputs, parabolic inputs, and so on.

- The magnitudes of the steady-state errors due to these individual inputs are indicative of the goodness of the system.

Classification of Control Systems

- Consider the unity-feedback control system with the following open-loop transfer function

$$G(s) = \frac{K(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{s^N(T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)}$$

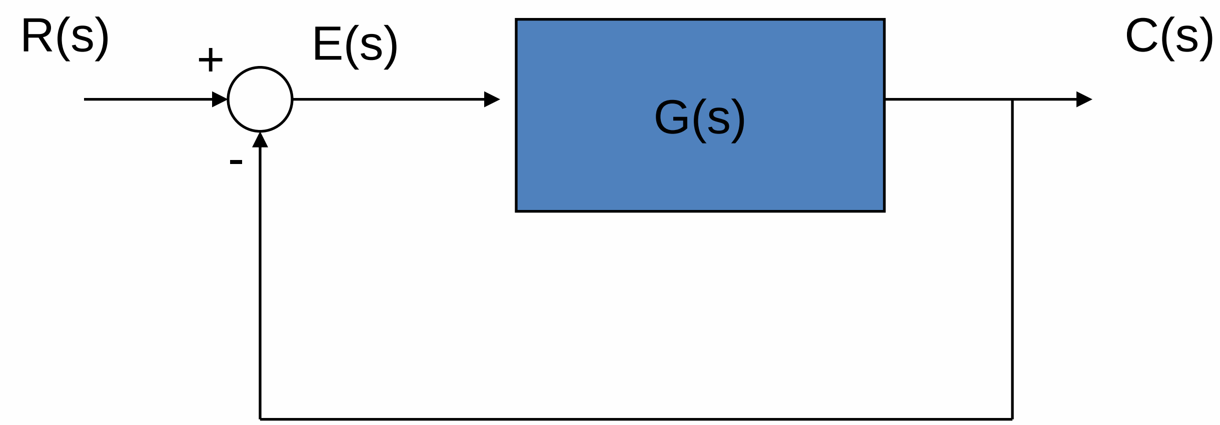
- It involves the term s^N in the denominator, representing N poles at the origin.

- A system is called type 0, type 1, type 2, ... , if N=0, N=1, N=2, ... , respectively.

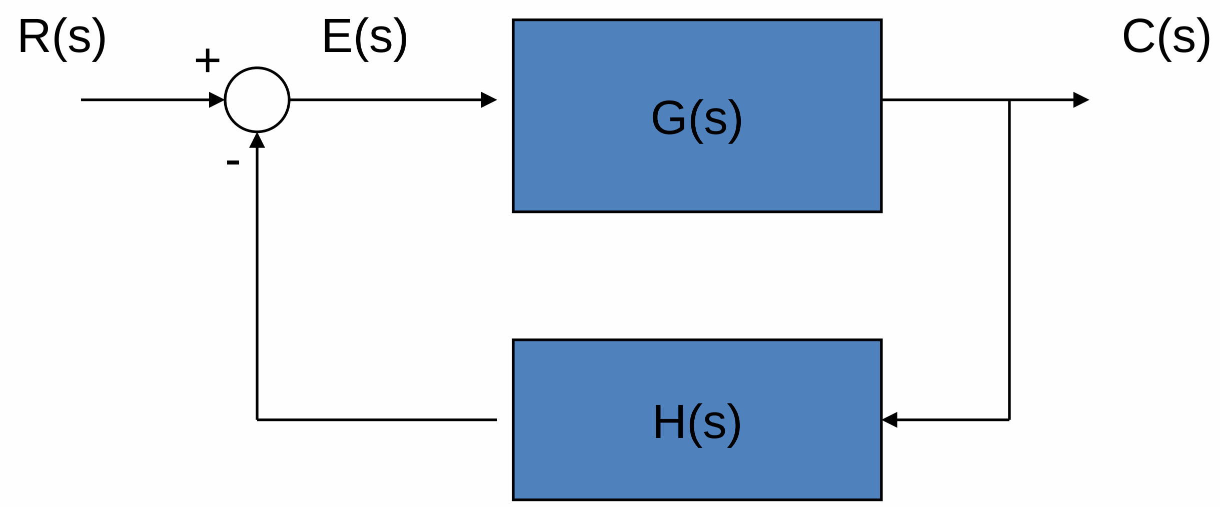
Classification of Control Systems

- As the type number is increased, accuracy is improved.
- However, increasing the type number aggravates the stability problem.
- A compromise between steady-state accuracy and relative stability is always necessary.

Steady-state error analysis



Unity feedback
H(s)=1



Non-unity feedback
H(s)≠1

Steady-state error analysis

For unity feedback system:

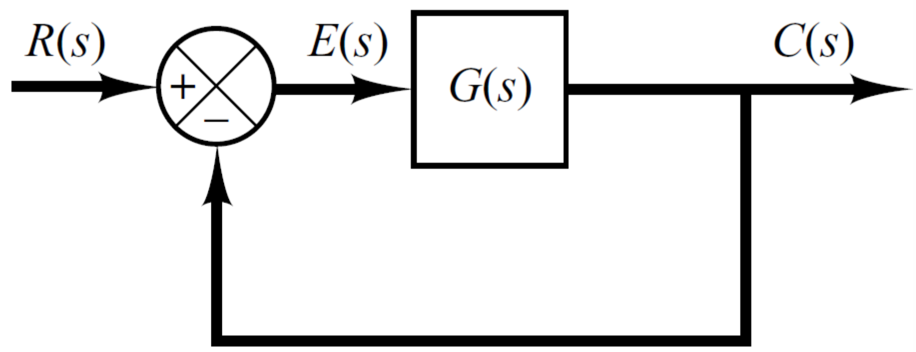
$$E(s) = R(s) - C(s) \rightarrow \quad \text{System error}$$

For a non-unity feedback system:

$$E(s) = R(s) - H(s)C(s) \rightarrow \quad \text{Actuating error}$$

Steady State Error of Unity Feedback Systems

- Consider the system shown in following figure.



- The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

$$G(s) = \frac{K(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{s^N(T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)}$$

Steady State Error of Unity Feedback Systems

- Steady state error is defined as the error between the input signal and the output signal when t-> infinity

- The transfer function between the error signal E(s) and the input signal R(s) is

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)}$$

- The final-value theorem provides a convenient way to find the steady-state performance of a stable system.

- Since E(s) is

$$E(s) = \frac{1}{1 + G(s)} R(s)$$

- The steady state error is

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

Second Order System

- We have already discussed the affect of location of poles and zeros on the transient response of 1^{st} order systems.

- Compared to the simplicity of a first-order system, a second-order system exhibits a wide range of responses that must be analyzed and described.

- Varying a first-order system's parameter (T, K) simply changes the speed and offset of the response

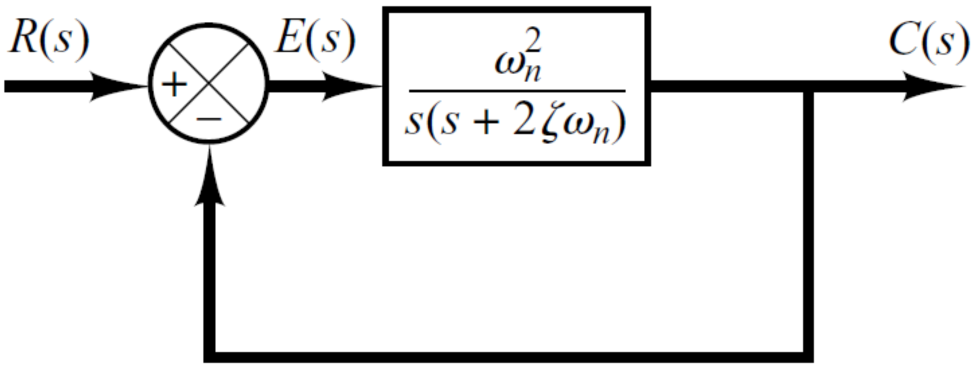
- Whereas, changes in the parameters of a second-order system can change the *form of* the response.

- A second-order system can display characteristics much like a first-order system or, depending on component values, display damped or pure oscillations for its *transient response*.

Introduction

- A general second-order system is characterized by the following transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



$\omega_n \longrightarrow$ un-damped natural frequency of the second order system, which is the frequency of oscillation of the system without damping.

$\zeta \longrightarrow$ damping ratio of the second order system, which is a measure of the degree of resistance to change in the system output.

Example

- Determine the un-damped natural frequency and damping ratio of the following second order system.

$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + 2s + 4}$$

- Compare the numerator and denominator of the given transfer function with the general 2nd order transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = 4 \qquad \Rightarrow \omega_n = 2$$

$$\cancel{s^2} + 2\zeta\omega_n s + \cancel{\omega_n^2} = \cancel{s^2} + 2s + \cancel{4}$$

$$\Rightarrow 2\zeta\omega_n s = 2s$$

$$\Rightarrow \zeta\omega_n = 1$$

$$\Rightarrow \zeta = 0.5$$

Introduction

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Two poles of the system are

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

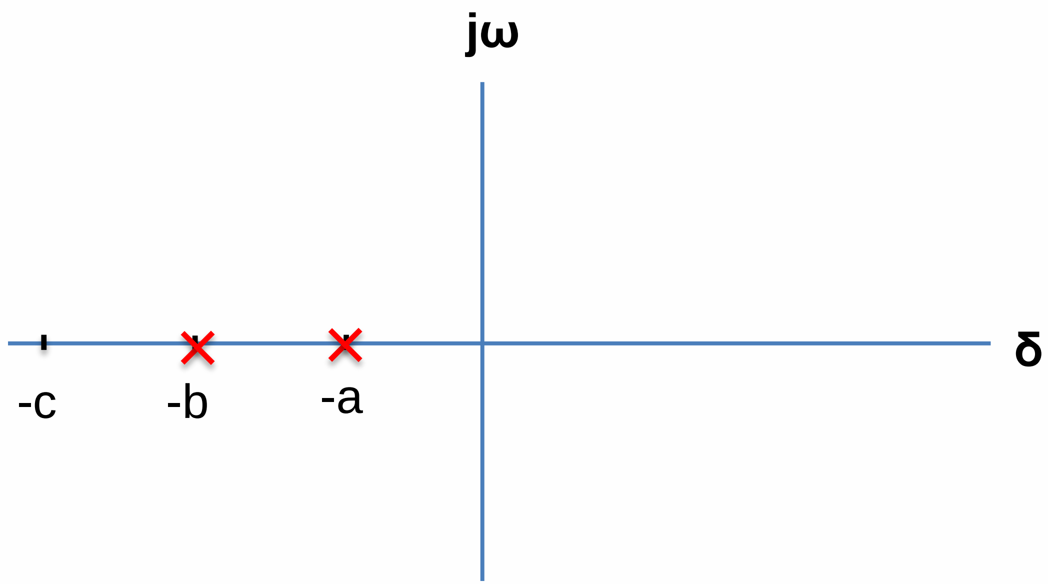
Introduction

$$-\omega_n \zeta + \omega_n \sqrt{\zeta^2 - 1}$$

$$-\omega_n \zeta - \omega_n \sqrt{\zeta^2 - 1}$$

- According the value of ζ , a second-order system can be set into one of the four categories

1. *Overdamped* - when the system has two real distinct poles (ζ >1).



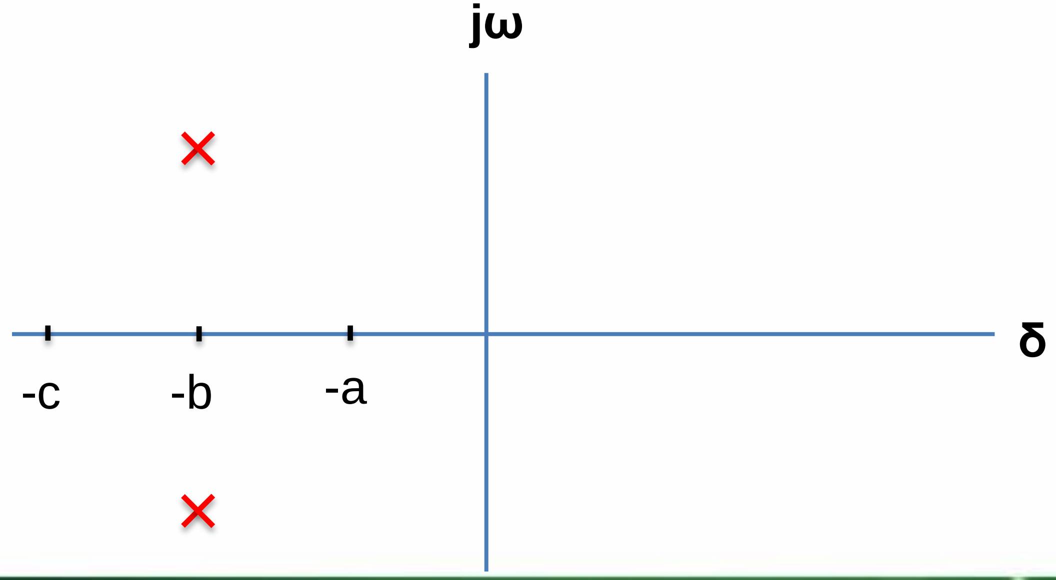
Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of ζ, a second-order system can be set into one of the four categories

2. *Underdamped* - when the system has two complex conjugate poles ($0 < \zeta < 1$)



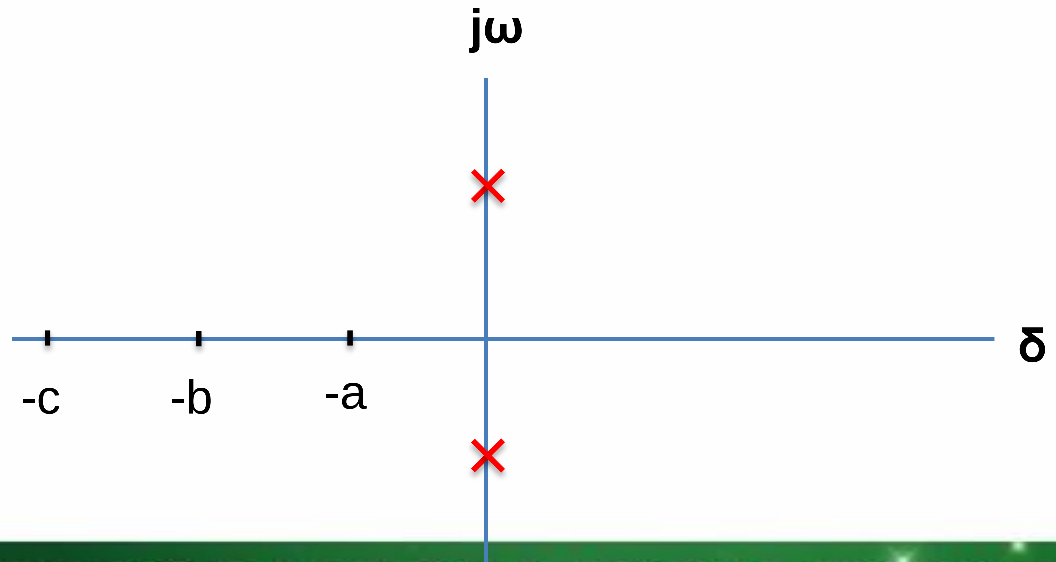
Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of ζ , a second-order system can be set into one of the four categories

3. *Undamped* - when the system has two imaginary poles (ζ = 0).



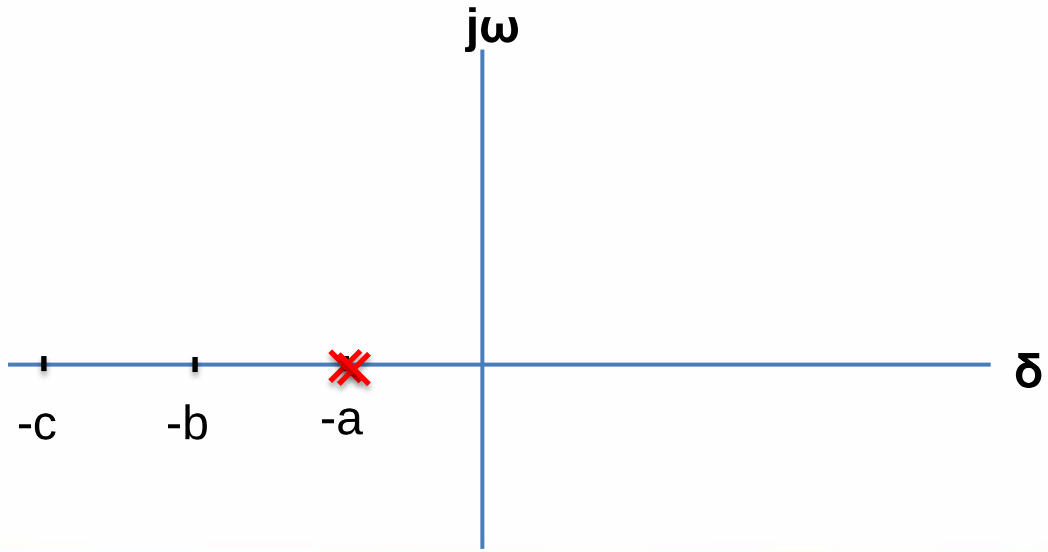
Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of ζ , a second-order system can be set into one of the four categories

4. *Critically damped* - when the system has two real but equal poles (ζ = 1).



Static Error Constants

- The static error constants are figures of merit of control systems. The higher the constants, the smaller the steady-state error.
- In a given system, the output may be the position, velocity, pressure, temperature, or the like.
- Therefore, in what follows, we shall call the output “position,” the rate of change of the output “velocity,” and so on.
- This means that in a temperature control system “position” represents the output temperature, “velocity” represents the rate of change of the output temperature, and so on.

Static Position Error Constant (K_p)

- The steady-state error of the system for a unit-step input is

$$e_{ss} = \lim_{s \to 0} \frac{\cancel{s}}{1 + G(s)} \frac{1}{\cancel{s}}$$

$$= \frac{1}{1 + G(0)}$$

- The static position error constant K_p is defined by

$$K_p = \lim_{s \to 0} G(s) = G(0)$$

- Thus, the steady-state error in terms of the static position error constant K_p is given by

$$e_{ss} = \frac{1}{1 + K_p}$$

Static Position Error Constant (K_p)

- For a Type 0 system

$$K_p = \lim_{s \to 0} \frac{K(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

- For Type 1 or higher order systems

$$K_p = \lim_{s \to 0} \frac{K(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 1$$

- For a unit step input the steady state error e_{ss} is

$$e_{ss} = \frac{1}{1 + K}, \qquad \text{for type 0 systems}$$

$$e_{ss} = 0, \qquad \text{for type 1 or higher systems}$$

Static Velocity Error Constant (K_v)

- The steady-state error of the system for a unit-ramp input is

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \frac{1}{s^2}$$

$$= \lim_{s \to 0} \frac{1}{sG(s)}$$

- The static velocity error constant K_v is defined by

$$K_v = \lim_{s \to 0} sG(s)$$

- Thus, the steady-state error in terms of the static velocity error constant K_v is given by

$$e_{ss} = \frac{1}{K_v}$$

Static Velocity Error Constant (K_v)

- For a Type 0 system

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

- For Type 1 systems

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{s(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

- For type 2 or higher order systems

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 2$$

Static Velocity Error Constant (K_v)

- For a ramp input the steady state error e_{ss} is

$$e_{ss} = \frac{1}{K_v} = \infty, \qquad \text{for type 0 systems}$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K}, \qquad \text{for type 1 systems}$$

$$e_{ss} = \frac{1}{K_v} = 0, \qquad \text{for type 2 or higher systems}$$

Static Acceleration Error Constant (K_a)

- The steady-state error of the system for parabolic input is

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \frac{1}{s^3}$$

$$= \frac{1}{\lim_{s \to 0} s^2 G(s)}$$

- The static acceleration error constant K_a is defined by

$$K_a = \lim_{s \to 0} s^2 G(s)$$

- Thus, the steady-state error in terms of the static acceleration error constant K_a is given by

$$e_{ss} = \frac{1}{K_a}$$

Static Acceleration Error Constant (K_a)

- For a Type 0 system

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

- For Type 1 systems

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

- For type 2 systems

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s^2(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

- For type 3 or higher order systems

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 3$$

Static Acceleration Error Constant (K_a)

- For a parabolic input the steady state error **e_{ss}** is

$$e_{ss} = \infty, \quad \text{for type 0 and type 1 systems}$$

$$e_{ss} = \frac{1}{K}, \quad \text{for type 2 systems}$$

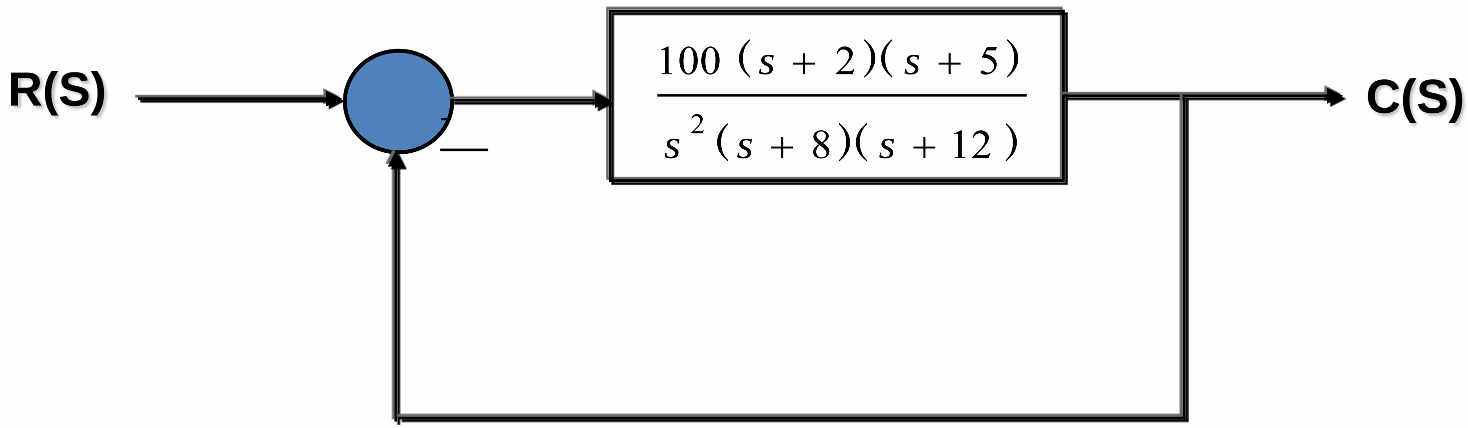
$$e_{ss} = 0, \quad \text{for type 3 or higher systems}$$

Summary

	Step Input $r(t) = 1$	Ramp Input $r(t) = t$	Acceleration Input $r(t) = \frac{1}{2}t^2$
Type 0 system	$\frac{1}{1 + K}$	∞	∞
Type 1 system	0	$\frac{1}{K}$	∞
Type 2 system	0	0	$\frac{1}{K}$

Example

- For the system shown in figure below evaluate the static error constants and find the expected steady state errors for the standard step, ramp and parabolic inputs.



Example

$$G(s) = \frac{100(s+2)(s+5)}{s^2(s+8)(s+12)}$$

$$K_p = \lim_{s \to 0} G(s)$$

$$K_p = \lim_{s \to 0} \left(\frac{100(s+2)(s+5)}{s^2(s+8)(s+12)} \right)$$

$$K_p = \infty$$

$$K_v = \lim_{s \to 0} sG(s)$$

$$K_v = \lim_{s \to 0} \left(\frac{100\,s(s+2)(s+5)}{s^2(s+8)(s+12)} \right)$$

$$K_v = \infty$$

$$K_a = \lim_{s \to 0} s^2 G(s)$$

$$K_a = \lim_{s \to 0} \left(\frac{100\,s^2(s+2)(s+5)}{s^2(s+8)(s+12)} \right)$$

$$K_a = \left(\frac{100(0+2)(0+5)}{(0+8)(0+12)} \right) = 10.4$$

Example

$K_p = \infty$ $K_v = \infty$ $K_a = 10.4$

$$e_{ss} = \frac{1}{1 + K_p} = 0$$

$$e_{ss} = \frac{1}{K_v} = 0$$

$$e_{ss} = \frac{1}{K_a} = 0.09$$

Step Response of underdamped System

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \xrightarrow{\text{Step Response}} C(s) = \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

- The partial fraction expansion of above equation is given as

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2\left(1 - \zeta^2\right)$$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2 + \omega_n^2 - \zeta^2\omega_n^2}$$

$$\left(s + 2\zeta\omega_n\right)^2$$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{\left(s + \zeta\omega_n\right)^2 + \omega_n^2\left(1 - \zeta^2\right)}$$

Step Response of underdamped System

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

- Above equation can be written as

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

- Where $\omega_d = \omega_n\sqrt{1 - \zeta^2}$, is the frequency of transient oscillations and is called damped natural frequency.

- The inverse Laplace transform of above equation can be obtained easily if C(s) is written in the following form:

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

Step Response of underdamped System

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\omega_d t$$

Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\omega_d t$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

- When $\zeta = 0$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

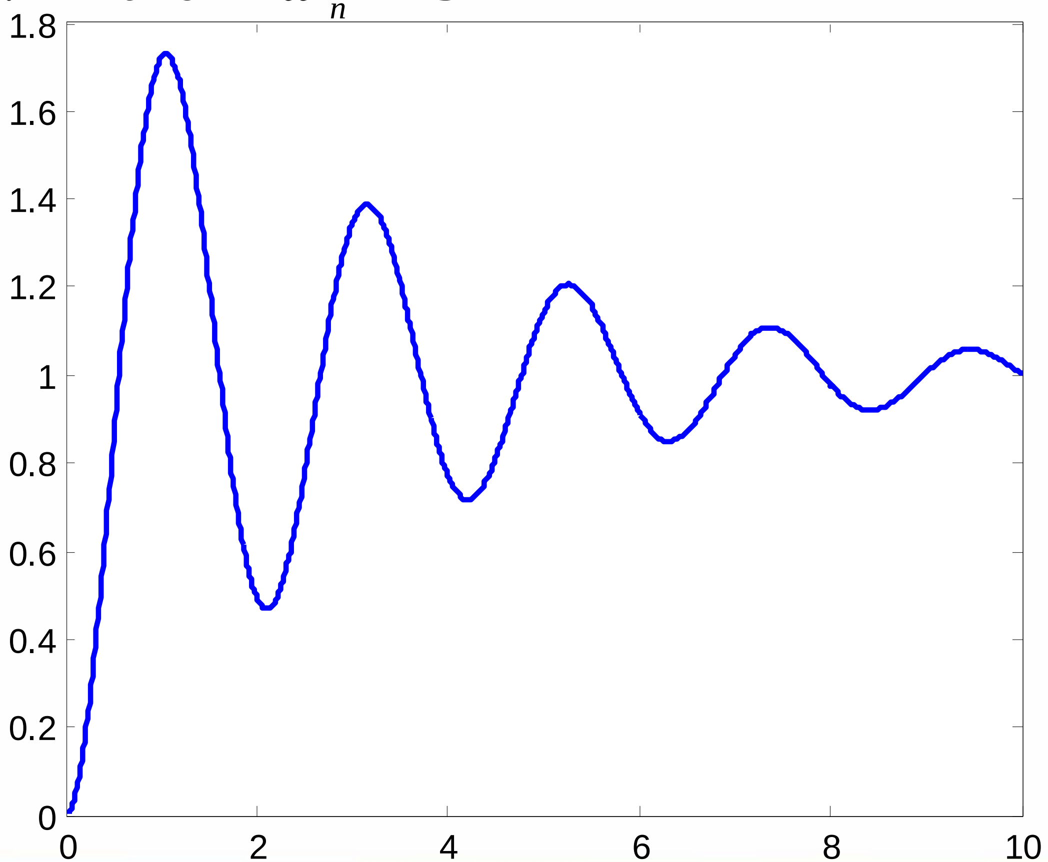
$$= \omega_n$$

$$c(t) = 1 - \cos\omega_n t$$

Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

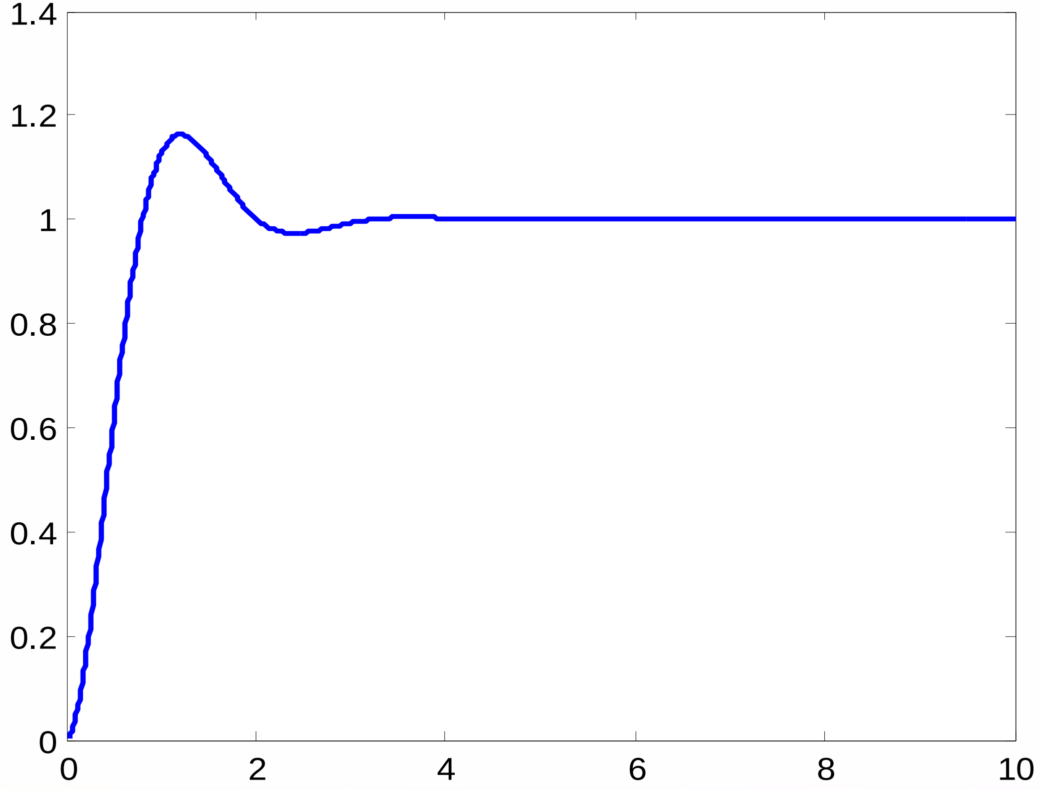
if $\zeta = 0.1$ and $\omega_n = 3$



Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin \omega_d t\right]$$

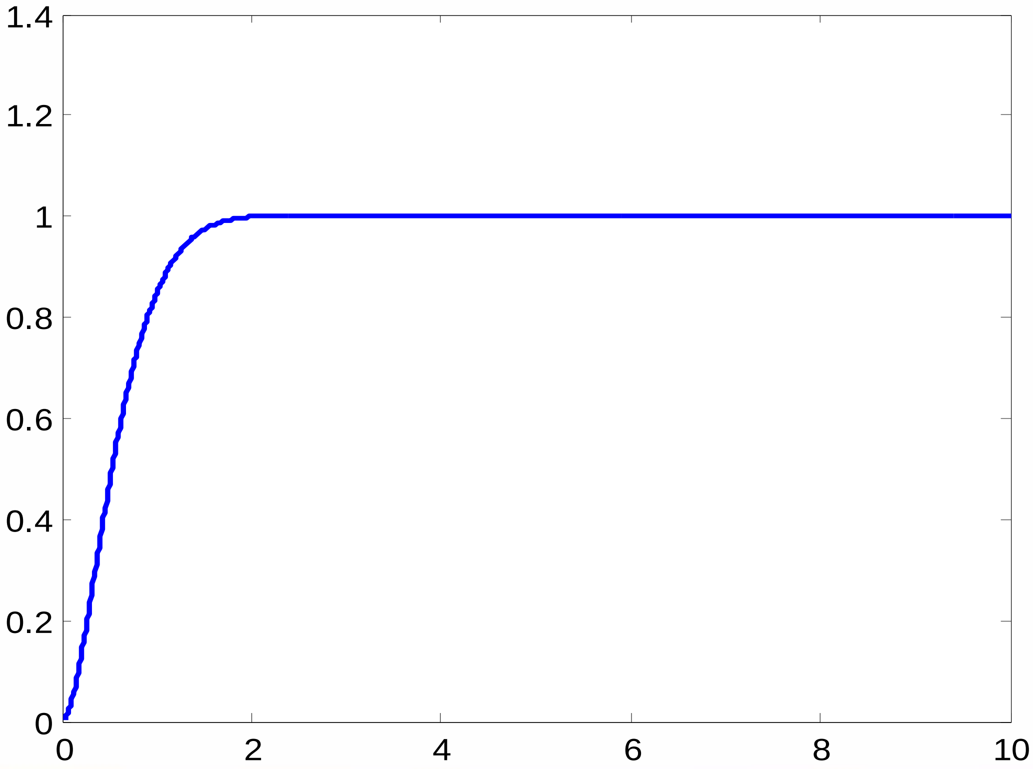
if $\zeta = 0.5$ and $\omega_n = 3$



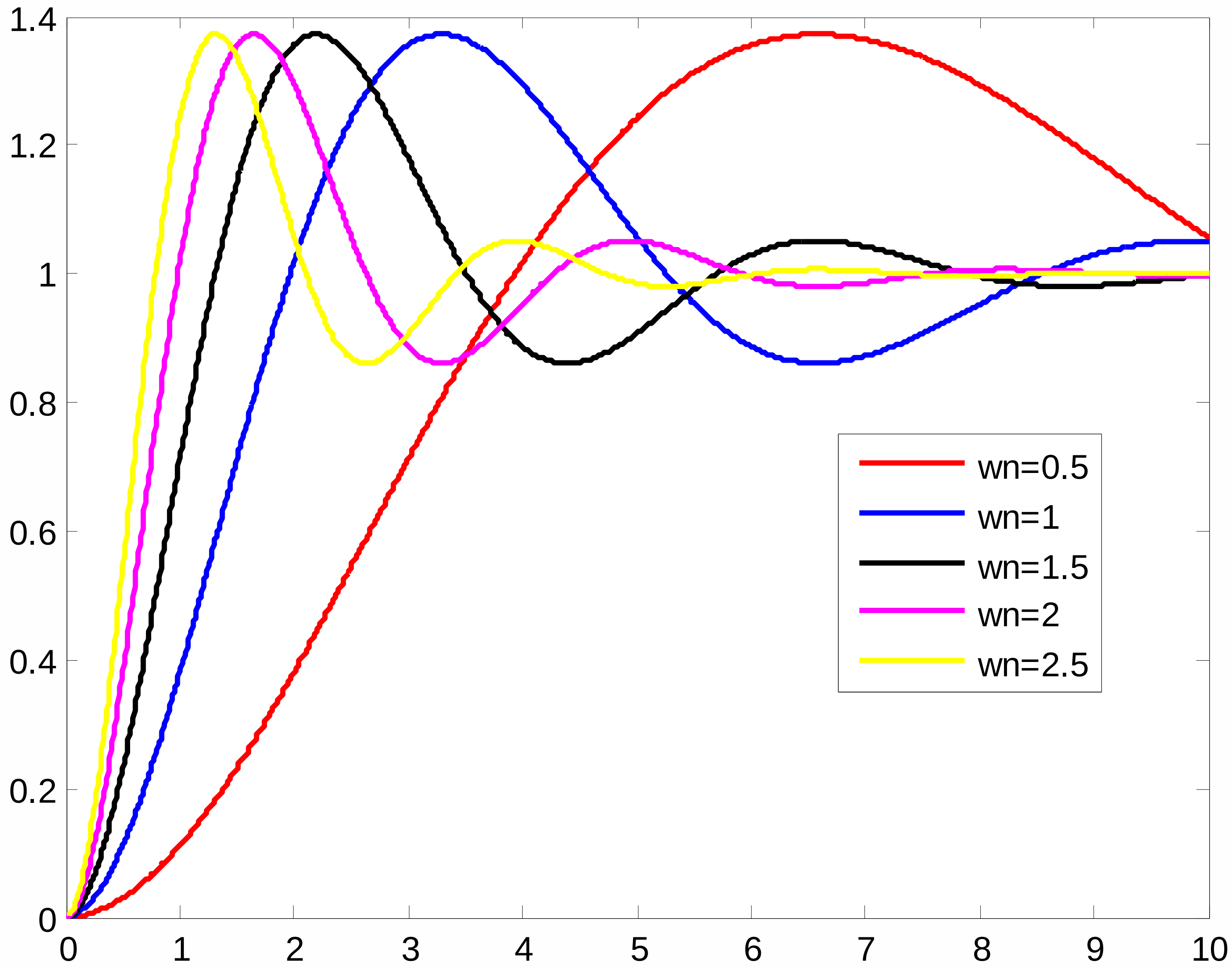
Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

if $\zeta = 0.9$ and $\omega_n = 3$



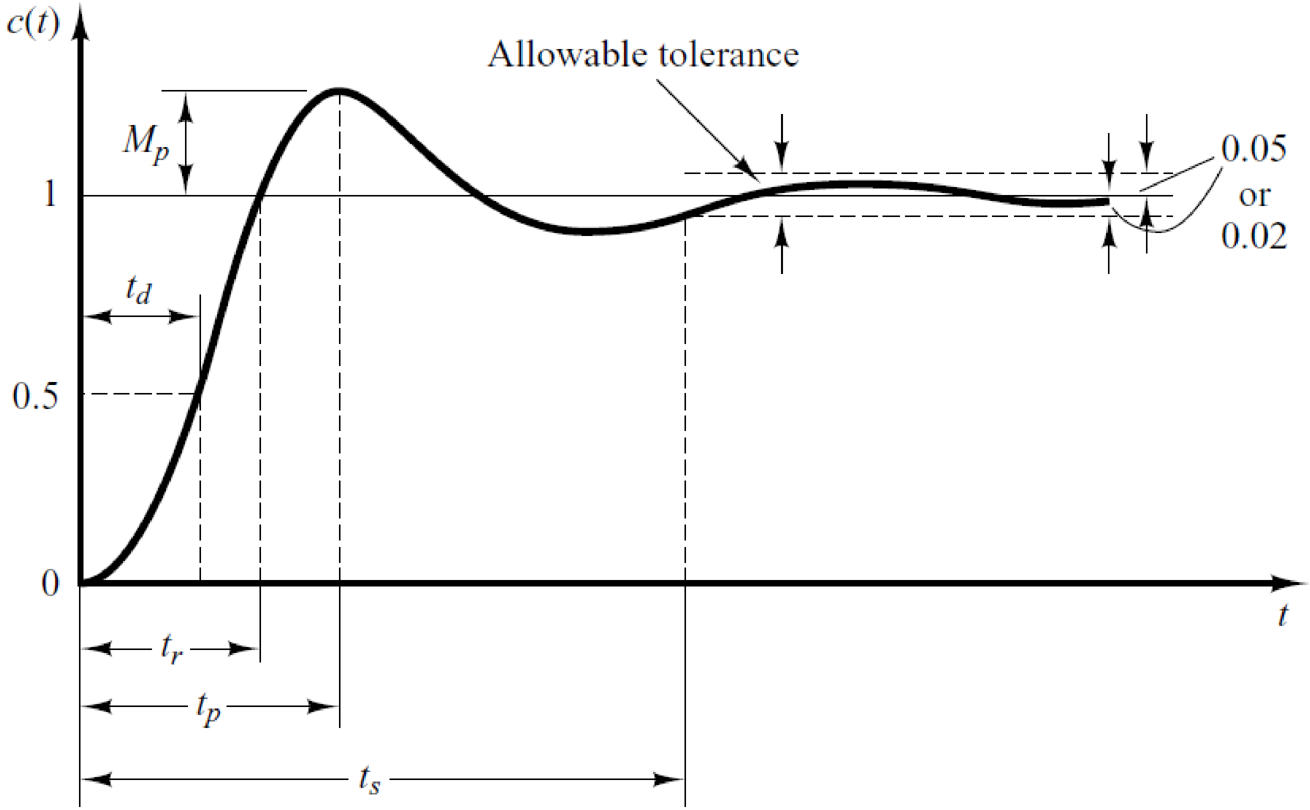
Step Response of underdamped System



Underdamped System

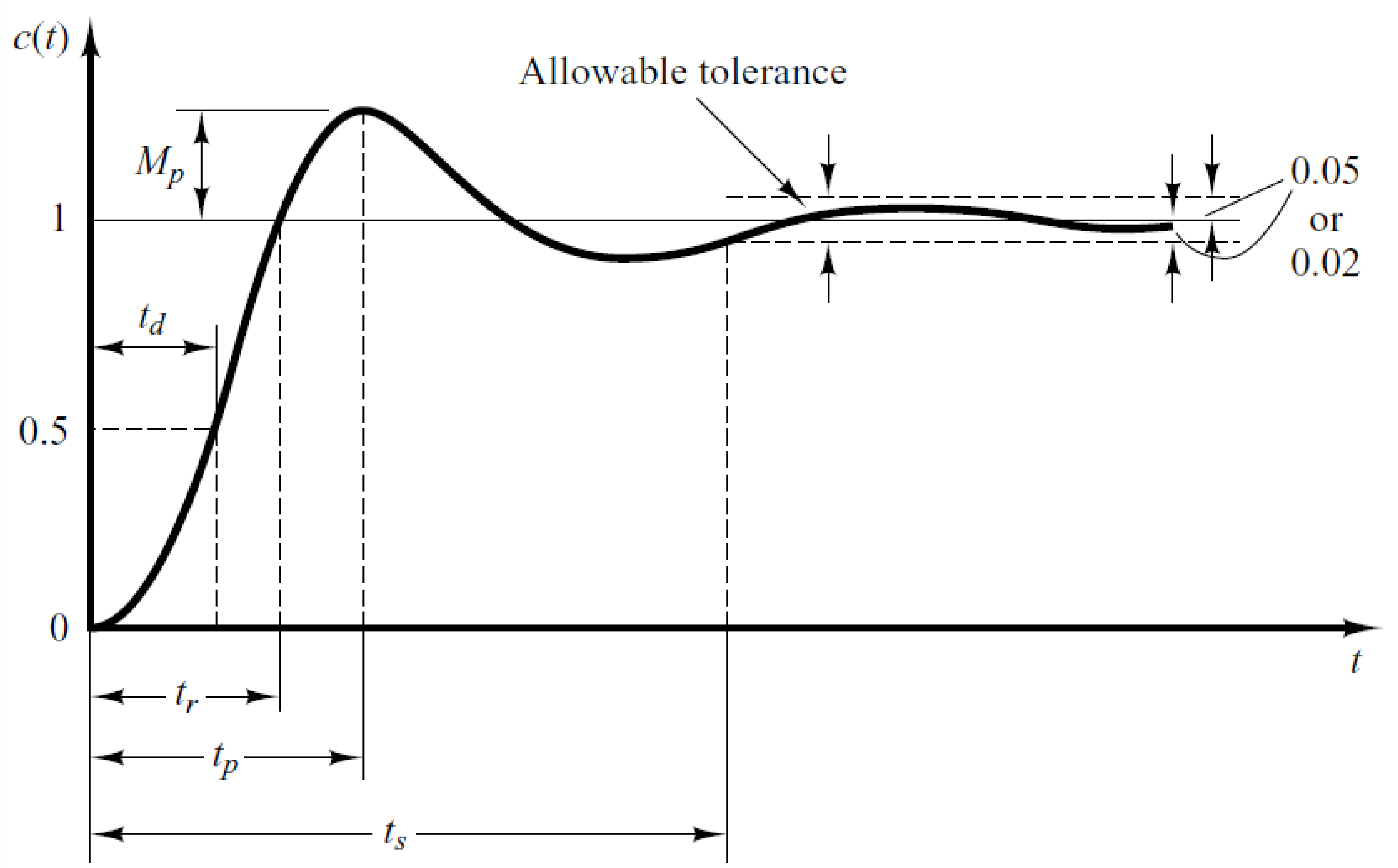
For $0<\zeta<1$ and $\omega_n > 0$, the 2nd order system's response due to a unit step input is as follows.

Important timing characteristics: delay time, rise time, peak time, maximum overshoot, and settling time.



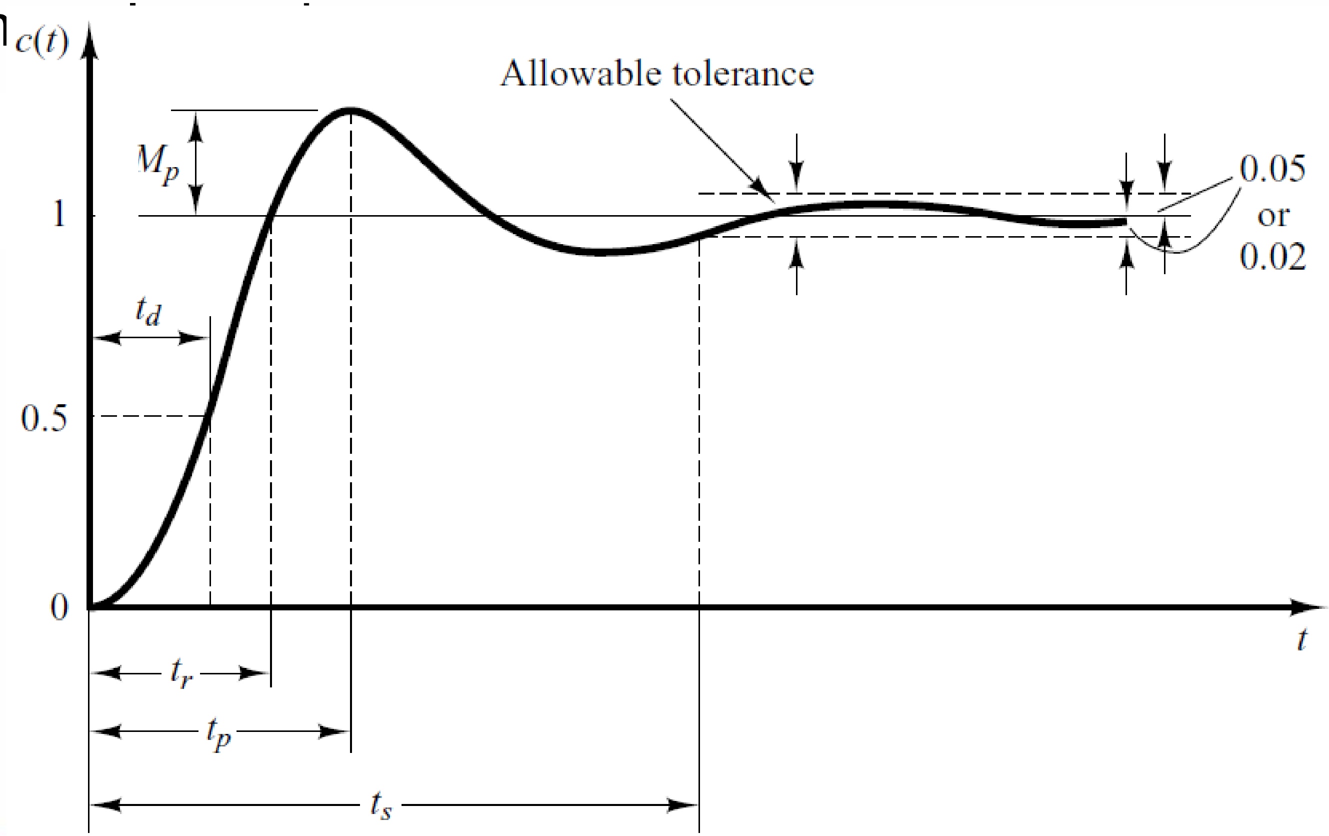
Delay Time

- The delay (t_d) time is the time required for the response to reach half the final value the very first time.



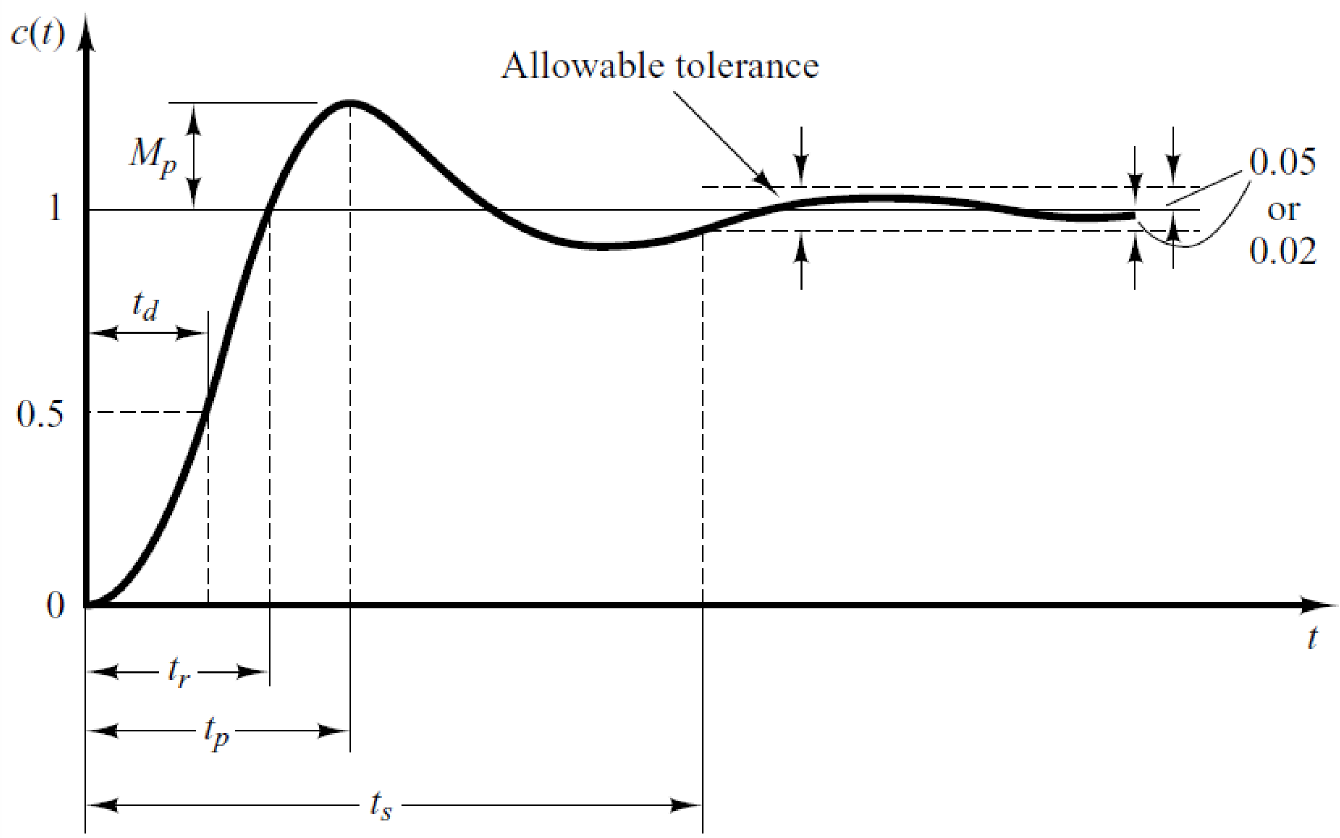
Rise Time

- The rise time is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value.
- For underdamped second order systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is com



Peak Time

- The peak time is the time required for the response to reach the first peak of the overshoot.



Time Domain Specifications (Rise Time)

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin \omega_d t\right]$$

Put $t = t_r$ in above equation

$$c(t_r) = 1 - e^{-\zeta\omega_n t_r}\left[\cos \omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin \omega_d t_r\right]$$

Where $c(t_r) = 1$

$$0 = -e^{-\zeta\omega_n t_r}\left[\cos \omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin \omega_d t_r\right]$$

$-e^{-\zeta\omega_n t_r} \neq 0$

$$0 = \left[\cos \omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin \omega_d t_r\right]$$

Time Domain Specifications (Rise Time)

$$\left[\cos \omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t_r\right] = 0$$

above equation can be re - writen as

$$\sin \omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta} \cos \omega_d t_r$$

$$\tan \omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta}$$

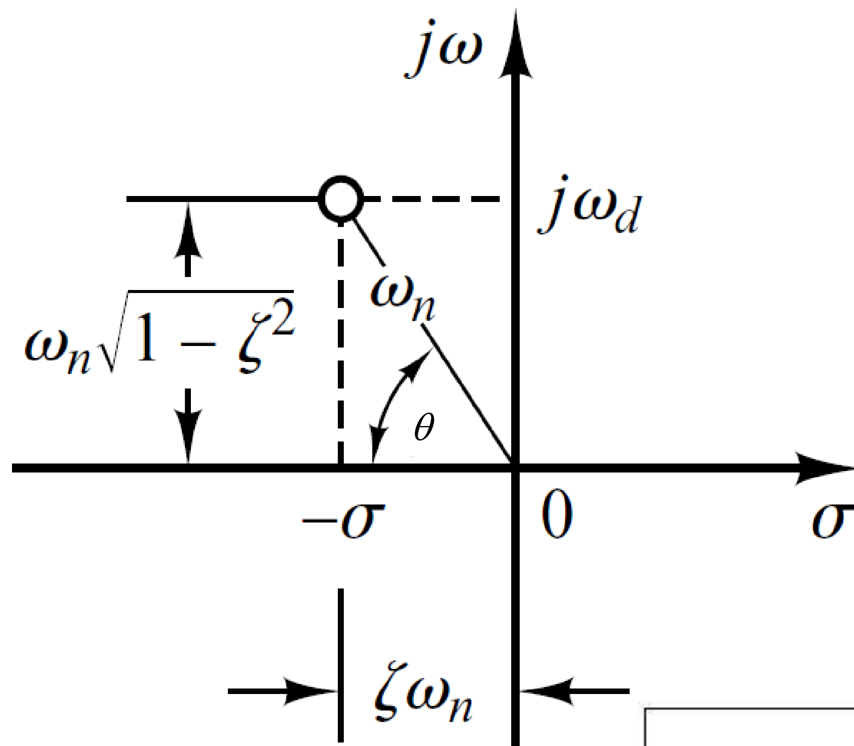
$$\omega_d t_r = \tan^{-1}\left(-\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

Time Domain Specifications (Rise Time)

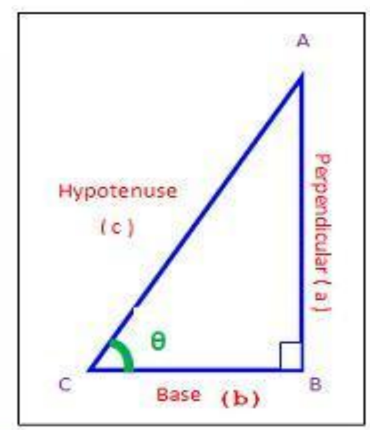
$$\omega_d t_r = \tan^{-1}\left(-\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(-\frac{\omega_n\sqrt{1-\zeta^2}}{\omega_n \zeta}\right)$$

$$\boxed{t_r = \frac{\pi - \theta}{\omega_d}}$$



$$\theta = \tan^{-1}\frac{a}{b}$$



Time Domain Specifications (Peak Time)

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\ \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\ \omega_d t\right]$$

- In order to find peak time let us differentiate above equation w.r.t *t*.

$$\frac{dc(t)}{dt} = \zeta\omega_n e^{-\zeta\omega_n t}\left[\cos\ \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\ \omega_d t\right] - e^{-\zeta\omega_n t}\left[-\omega_d \sin\ \omega_d t + \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\ \omega_d t\right]$$

$$0 = e^{-\zeta\omega_n t}\left[\zeta\omega_n \cos\ \omega_d t + \frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\sin\ \omega_d t + \omega_d \sin\ \omega_d t - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\ \omega_d t\right]$$

$$0 = e^{-\zeta\omega_n t}\left[\zeta\cancel{\omega_n} \cos\ \omega_d t + \frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\sin\ \omega_d t + \omega_d \sin\ \omega_d t - \frac{\zeta\omega_n\cancel{\sqrt{1-\zeta^2}}}{\cancel{\sqrt{1-\zeta^2}}}\cos\ \omega_d t\right]$$

Time Domain Specifications (Peak Time)

$$0 = e^{-\zeta\omega_n t}\left[\cancel{\zeta\omega_n \cos\omega_d t} + \frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\sin\omega_d t + \omega_d \sin\omega_d t - \cancel{\frac{\zeta\omega_n\sqrt{1-\zeta^2}}{\sqrt{1-\zeta^2}}\cos\omega_d t}\right]$$

$$e^{-\zeta\omega_n t}\left[\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\sin\omega_d t + \omega_d \sin\omega_d t\right] = 0$$

$$e^{-\zeta\omega_n t} \neq 0 \qquad \left[\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\sin\omega_d t + \omega_d \sin\omega_d t\right] = 0$$

$$\sin\omega_d t\left[\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}} + \omega_d\right] = 0$$

Time Domain Specifications (Peak Time)

$$\sin \omega_d t \left[\frac{\zeta^2 \omega_n}{\sqrt{1-\zeta^2}} + \omega_d \right] = 0$$

$$\left[\frac{\zeta^2 \omega_n}{\sqrt{1-\zeta^2}} + \omega_d \right] \neq 0$$

$$\sin \omega_d t = 0$$

$$\omega_d t = \sin^{-1} 0$$

$$t = \frac{0, \pi, 2\pi, \cdots}{\omega_d}$$

- Since for underdamped stable systems first peak is maximum peak therefore,

$$t_p = \frac{\pi}{\omega_d}$$

Maximum Overshoot

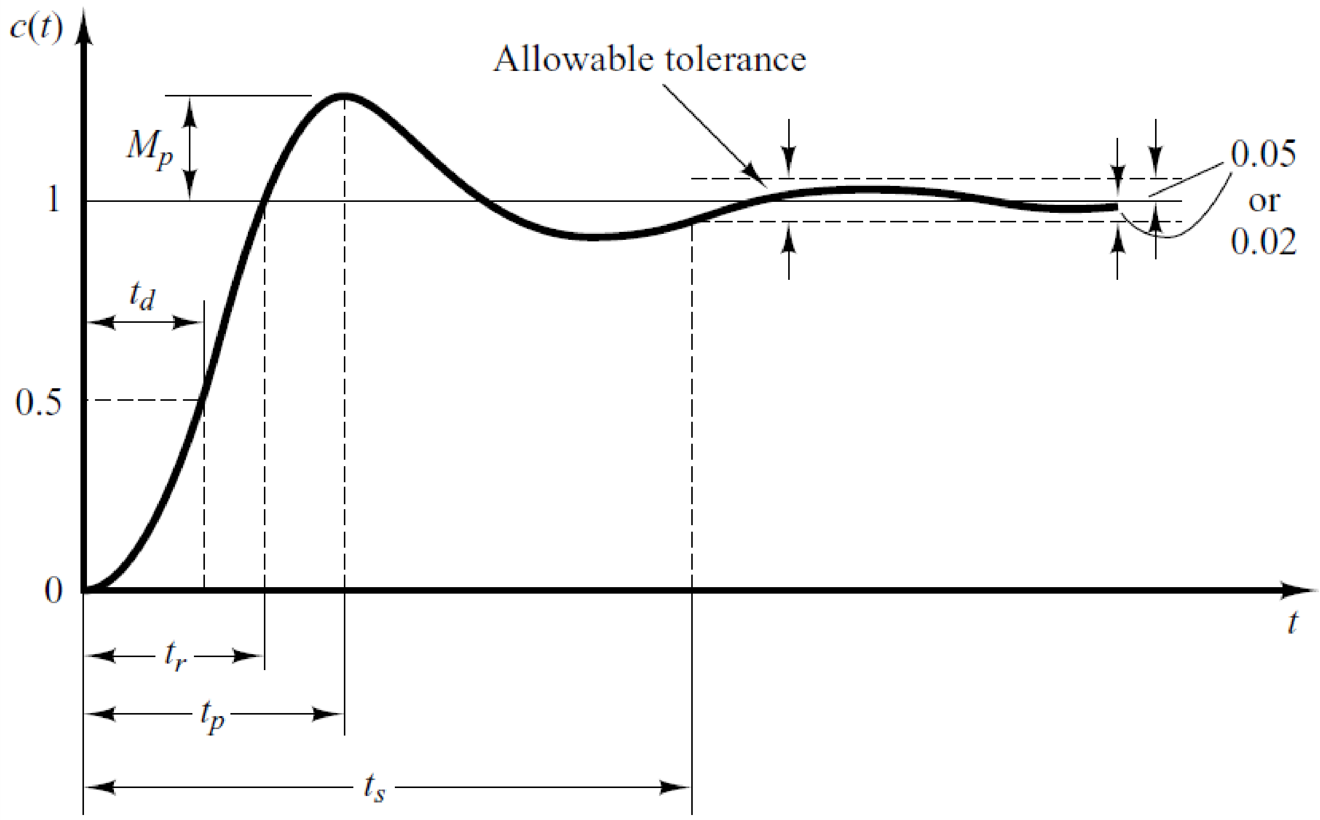
- The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

- The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.

Settling Time

- The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%).



Time Domain Specifications (Maximum Overshoot)

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

$$c(t_p) = 1 - e^{-\zeta\omega_n t_p}\left[\cos\omega_d t_p + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_p\right]$$

$$c(\infty) = 1$$

$$M_p = \left[\cancel{1} - e^{-\zeta\omega_n t_p}\left(\cos\omega_d t_p + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_p\right) - \cancel{1}\right] \times 100$$

Put $t_p = \frac{\pi}{\omega_d}$ in above equation

$$M_p = \left[- e^{-\zeta\omega_n \frac{\pi}{\omega_d}}\left(\cos\omega_d \frac{\pi}{\omega_d} + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d \frac{\pi}{\omega_d}\right)\right] \times 100$$

Time Domain Specifications (Maximum Overshoot)

$$M_p = \left[-e^{-\zeta\omega_n \frac{\pi}{\omega_d}} \left(\cos \cancel{\omega_d} \frac{\pi}{\cancel{\omega_d}} + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \cancel{\omega_d} \frac{\pi}{\cancel{\omega_d}} \right) \right] \times 100$$

Put $\omega_d = \omega_n \sqrt{1-\zeta^2}$ in above equation

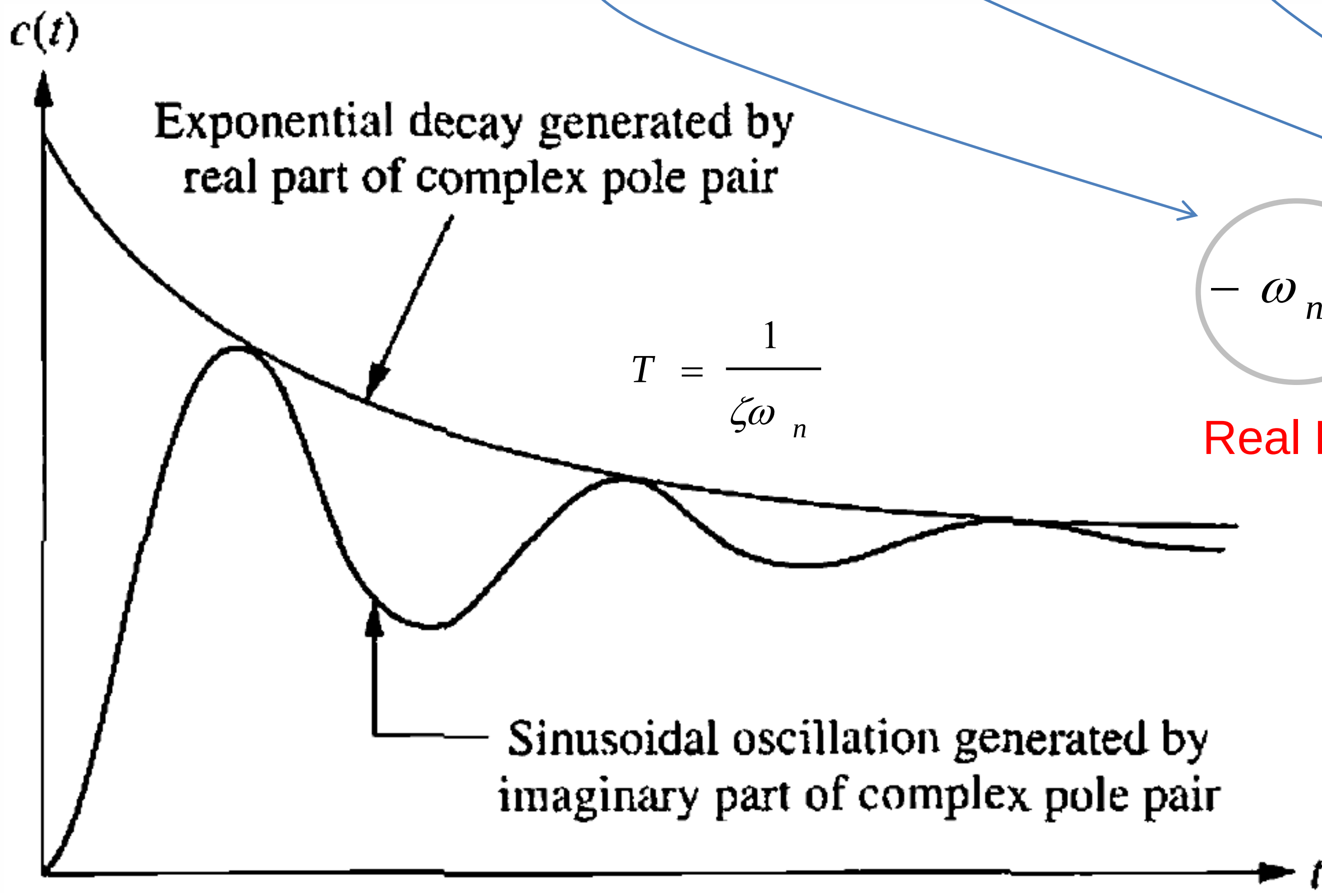
$$M_p = \left[-e^{-\zeta\cancel{\omega_n} \frac{\pi}{\cancel{\omega_n}\sqrt{1-\zeta^2}}} \left(\cos \pi + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \pi \right) \right] \times 100$$

$$M_p = \left[-e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} (-1+0) \right] \times 100$$

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

Time Domain Specifications (Settling Time)

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin \omega_d t\right]$$



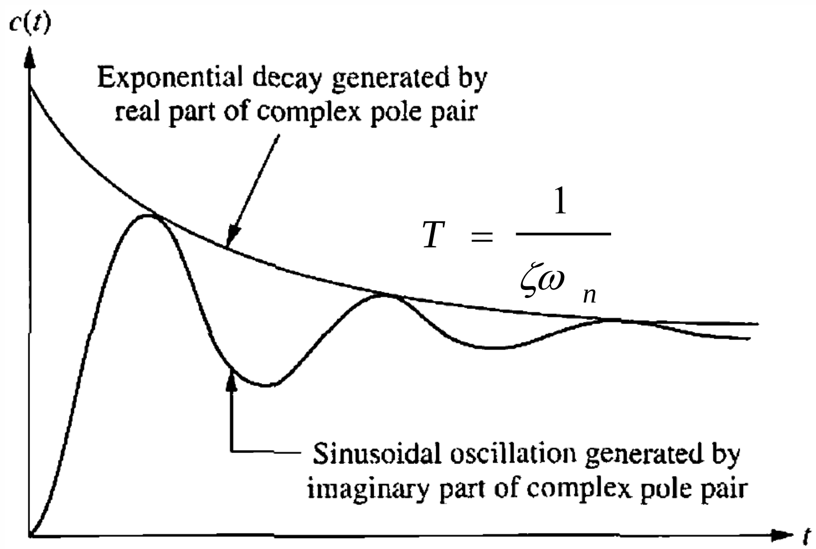
$$-\omega_n\zeta \pm \omega_n\sqrt{\zeta^2 - 1}$$

Real Part   Imaginary Part

Time Domain Specifications (Settling Time)

- Settling time (2%) criterion
  - Time consumed in exponential decay up to 98% of the input.

$$t_s = 4T = \frac{4}{\zeta\omega_n}$$



- Settling time (5%) criterion
  - Time consumed in exponential decay up to 95% of the input.

$$t_s = 3T = \frac{3}{\zeta\omega_n}$$

Summary of Time Domain Specifications

Rise Time

$$t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - \theta}{\omega_n \sqrt{1 - \zeta^2}}$$

Peak Time

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$$

Settling Time (2%)

$$t_s = 4T = \frac{4}{\zeta \omega_n}$$

$$t_s = 3T = \frac{3}{\zeta \omega_n}$$

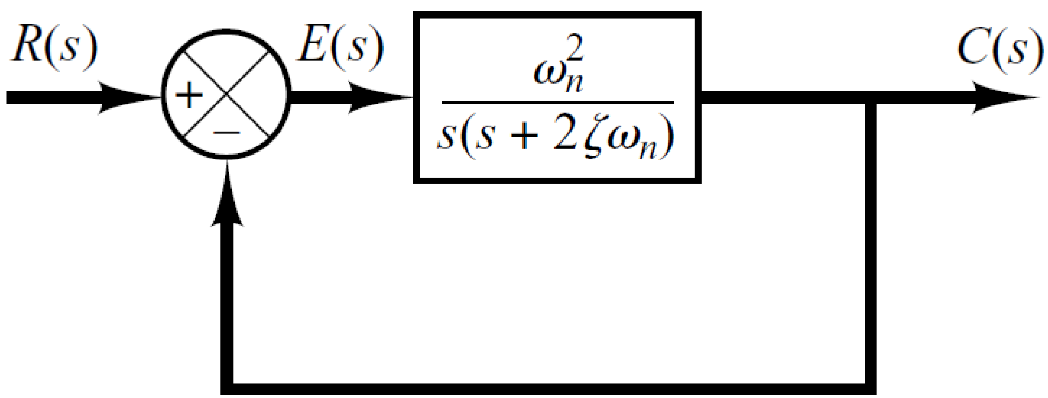
Settling Time (4%)

Maximum Overshoot

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

Example

- Consider the system shown in following figure, where damping ratio is 0.6 and natural undamped frequency is 5 rad/sec. Obtain the rise time t_r, peak time t_p, maximum overshoot M_p, and settling time 2% and 4% criterion t_s when the system is subjected to a unit-step input.



Example

Rise Time

$$t_r = \frac{\pi - \theta}{\omega_d}$$

Peak Time

$$t_p = \frac{\pi}{\omega_d}$$

Settling Time (2%)

$$t_s = 4T = \frac{4}{\zeta\omega_n}$$

$$t_s = 3T = \frac{3}{\zeta\omega_n}$$

Settling Time (4%)

Maximum Overshoot

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

Example

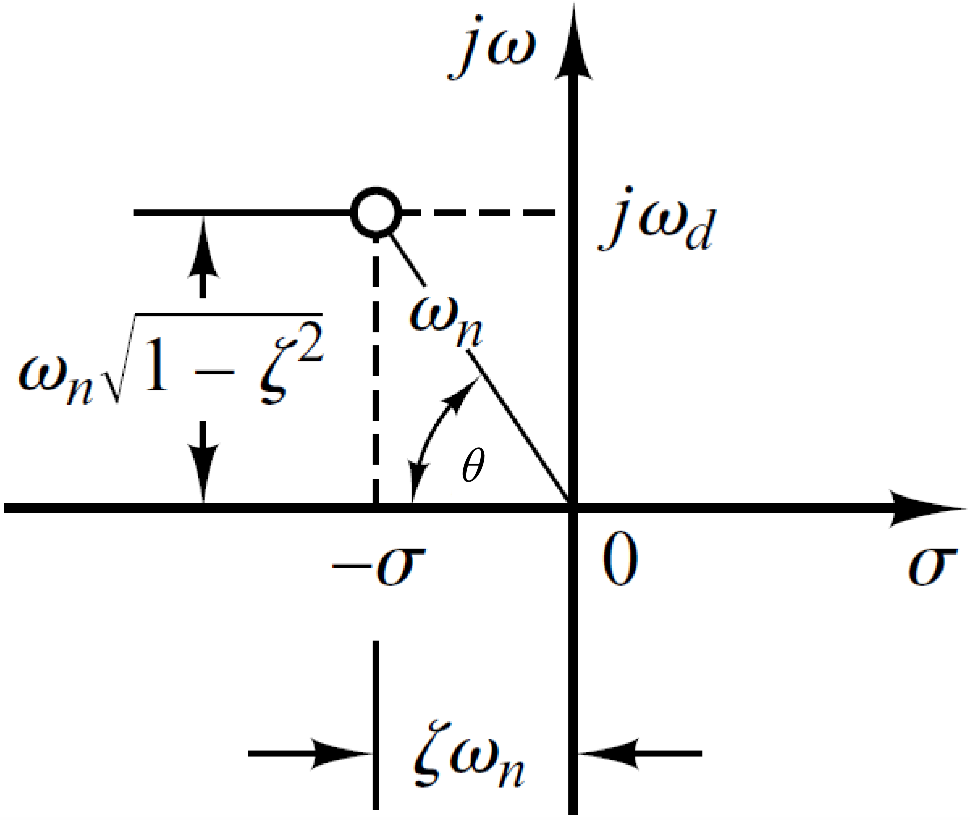
Rise Time

$$t_r = \frac{\pi - \theta}{\omega_d}$$

$$t_r = \frac{3.141 - \theta}{\omega_n \sqrt{1 - \zeta^2}}$$

$$\theta = \tan^{-1}\left(\frac{\omega_n \sqrt{1 - \zeta^2}}{\zeta\omega_n}\right) = 0.93 \text{ rad}$$

$$t_r = \frac{3.141 - 0.93}{5\sqrt{1 - 0.6^2}} = 0.55\ s$$



Example

Peak Time

$$t_p = \frac{\pi}{\omega_d}$$

$$t_p = \frac{3.141}{4} = 0.785\ s$$

Settling Time (2%)

$$t_s = \frac{4}{\zeta\omega_n}$$

$$t_s = \frac{4}{0.6 \times 5} = 1.33\ s$$

Settling Time (4%)

$$t_s = \frac{3}{\zeta\omega_n}$$

$$t_s = \frac{3}{0.6 \times 5} = 1s$$

Example

Maximum Overshoot

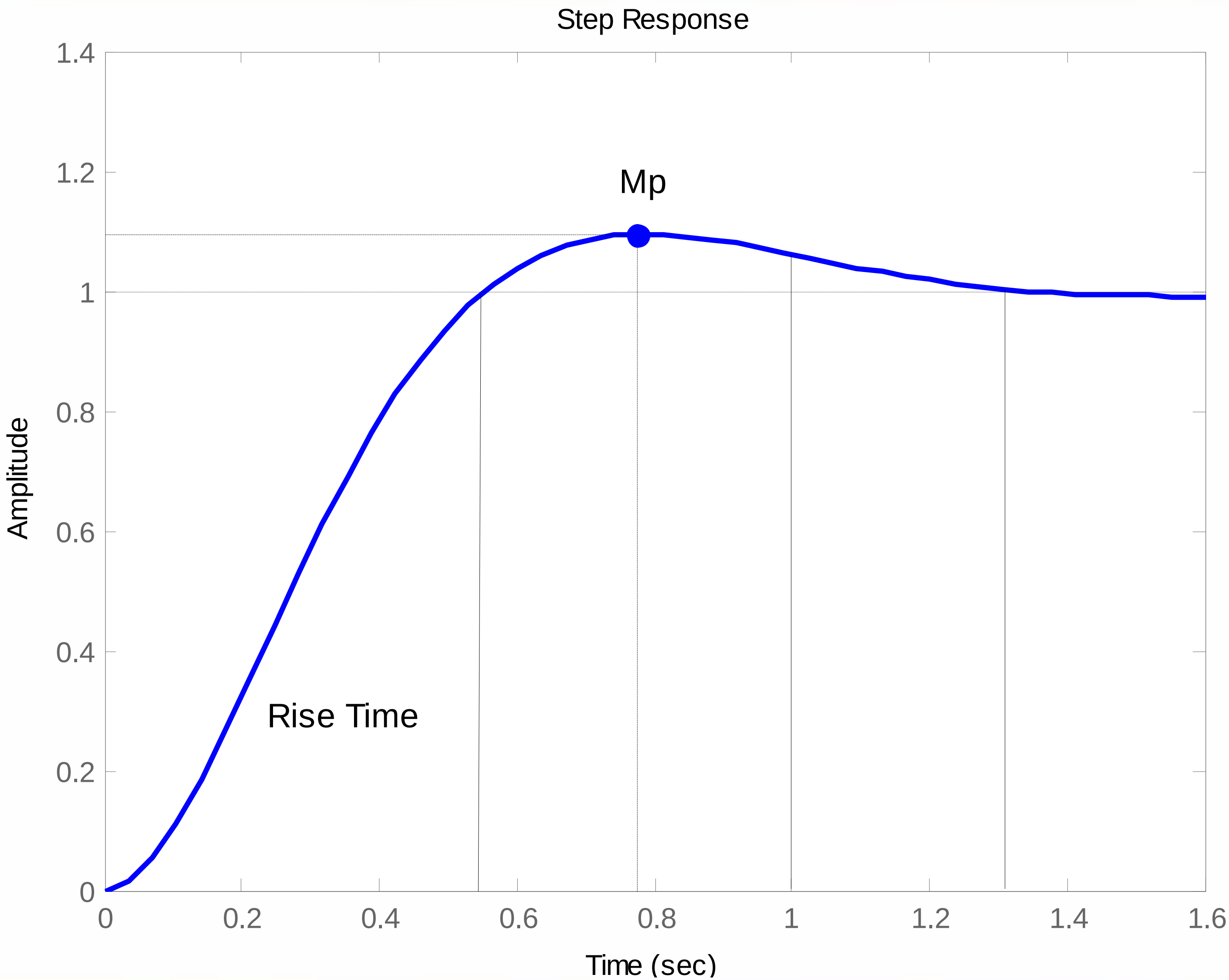
$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

$$M_p = e^{-\frac{3.141 \times 0.6}{\sqrt{1-0.6^2}}} \times 100$$

$$M_p = 0.095 \times 100$$

$$M_p = 9.5\%$$

Example


Step Response
Mp
Rise Time
Amplitude
Time (sec)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6

Example

- The Laplace Transform of Impulse response of a system is actually the transfer function of the system.
- Therefore taking Laplace Transform of the impulse response given by following equation.

$$c(t) = 3e^{-0.5t}$$

$$C(s) = \frac{3}{S+0.5} \times 1 = \frac{3}{S+0.5} \times \delta(s)$$

$$\frac{C(s)}{\delta(s)} = \frac{C(s)}{R(s)} = \frac{3}{S+0.5}$$

$$\frac{C(s)}{R(s)} = \frac{6}{2S+1}$$

Example

- Impulse response of a 1st order system is given below.

$$c(t) = 3e^{-0.5t}$$

- Find out
  - Time constant T=2
  - D.C Gain K=6
  - Transfer Function
  - Step Response

$$\frac{C(s)}{R(s)} = \frac{6}{2S+1}$$

Example

- For step response integrate impulse response

$$c(t) = 3e^{-0.5t}$$

$$\int c(t)dt = 3\int e^{-0.5t}dt$$

$$c_s(t) = -6e^{-0.5t} + C$$

- We can find out C if initial condition is known e.g. $c_s(0)=0$

$$0 = -6e^{-0.5\times 0} + C$$

$$C = 6$$

$$c_s(t) = 6 - 6e^{-0.5t}$$

Example

- If initial conditions are not known then partial fraction expansion is a better choice

$$\frac{C(s)}{R(s)} = \frac{6}{2S+1}$$

since $R(s)$ is a step input , $R(s) = \dfrac{1}{s}$

$$C(s) = \frac{6}{s(2S+1)}$$

$$\frac{6}{s(2S+1)} = \frac{A}{s} + \frac{B}{2s+1}$$

$$\frac{6}{s(2S+1)} = \frac{6}{s} - \frac{6}{s+0.5}$$

$$c(t) = 6 - 6e^{-0.5t}$$

Ramp Response of 1st Order System

- Consider the following 1st order system

$$R(s) \longrightarrow \boxed{\frac{K}{Ts+1}} \longrightarrow C(s)$$

$$R(s) = \frac{1}{s^2}$$

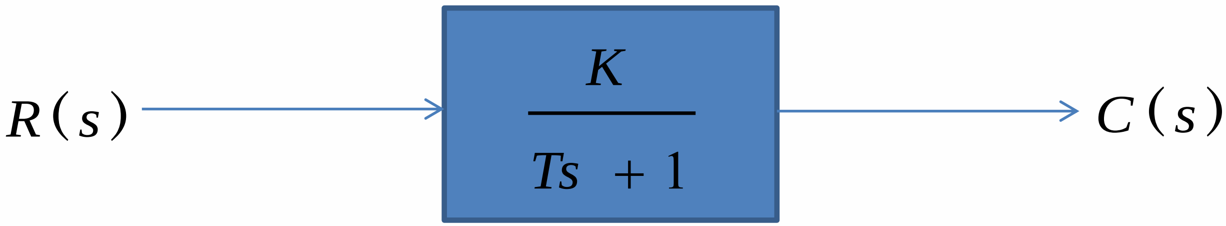
$$C(s) = \frac{K}{s^2(Ts+1)}$$

- The ramp response is given as

$$c(t) = K\left(t - T + Te^{-t/T}\right)$$

Parabolic Response of 1st Order System

- Consider the following 1st order system



$$R(s) = \frac{1}{s^3} \quad \text{Therefore,} \; C(s) = \frac{K}{s^3(Ts+1)}$$

Practical Determination of Transfer Function of 1^{st} Order Systems

- Often it is not possible or practical to obtain a system's transfer function analytically.
- Perhaps the system is closed, and the component parts are not easily identifiable.
- The system's step response can lead to a representation even though the inner construction is not known.
- With a step input, we can measure the time constant and the steady-state value, from which the transfer function can be calculated.

Practical Determination of Transfer Function of 1st Order Systems

- If we can identify *T* and *K* empirically we can obtain the transfer function of the system.

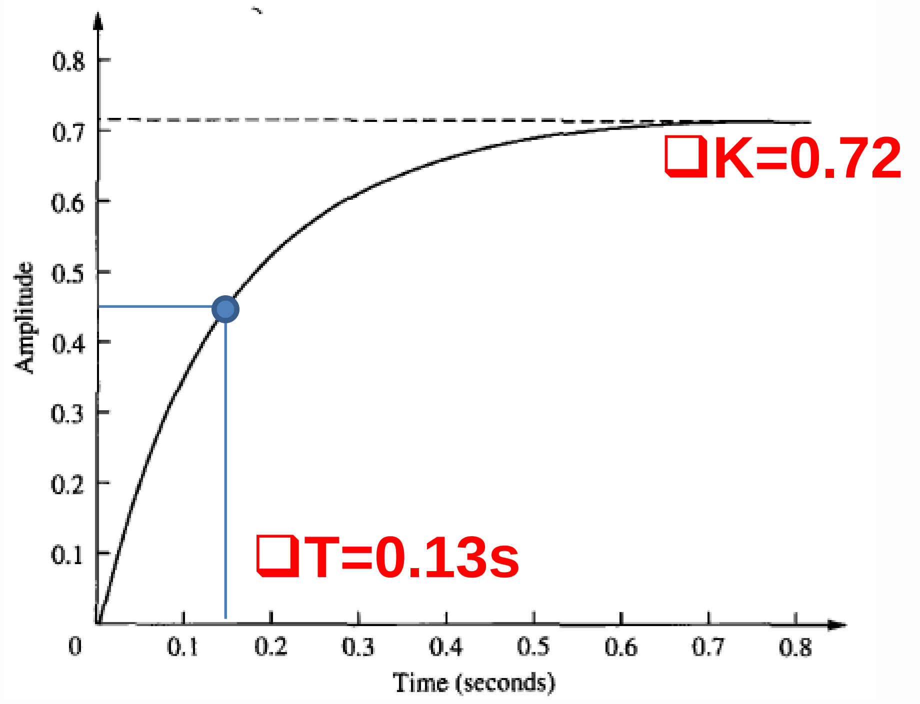
$$\frac{C(s)}{R(s)} = \frac{K}{Ts + 1}$$

Practical Determination of Transfer Function of 1^{st} Order Systems

- For example, assume the unit step response given in figure.

- From the response, we can measure the time constant, that is, the time for the amplitude to reach 63% of its final value.
- Since the final value is about 0.72 the time constant is evaluated where the curve reaches 0.63 x 0.72 = 0.45, or about 0.13 second.

- K is simply steady state value.



- Thus transfer function is obtained as:

$$\frac{C(s)}{R(s)} = \frac{0.72}{0.13s+1} = \frac{5.5}{s+7.7}$$

First Order System with a Zero

$$\frac{C(s)}{R(s)} = \frac{K(1+\alpha s)}{Ts+1}$$

- Zero of the system lie at *-1/α* and pole at *-1/T*.

- Step response of the system would be:

$$C(s) = \frac{K(1+\alpha s)}{s(Ts+1)}$$

$$C(s) = \frac{K}{s} + \frac{K(\alpha - T)}{(Ts+1)}$$

$$c(t) = K + \frac{K}{T}(\alpha - T)e^{-t/T}$$

First Order System With Delays

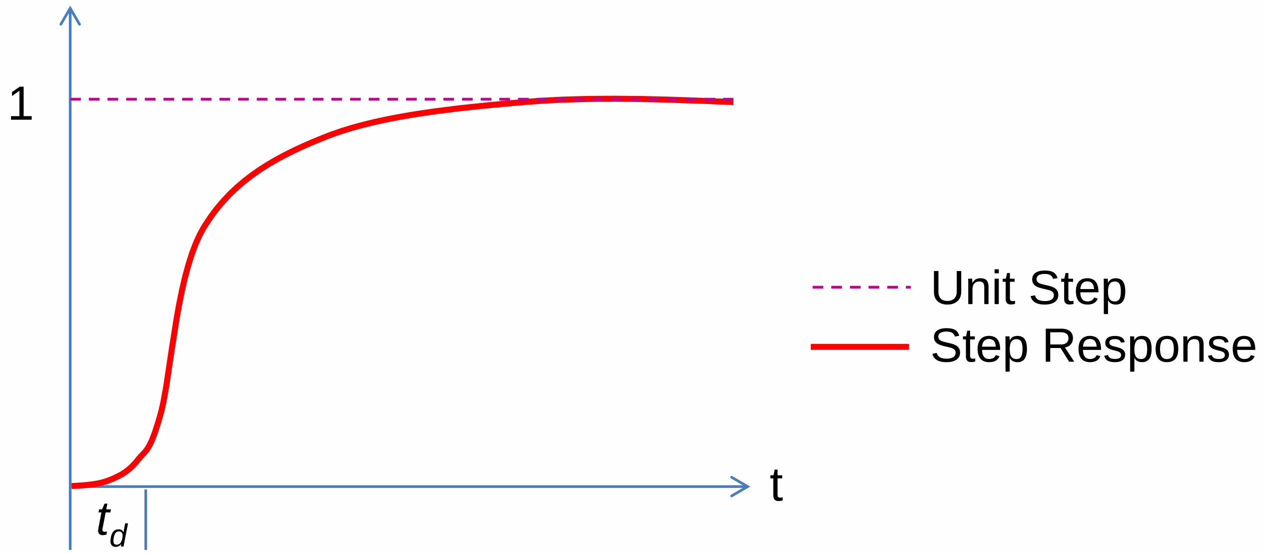
- Following transfer function is the generic representation of 1st order system with time lag.

$$\frac{C(s)}{R(s)} = \frac{K}{Ts + 1} e^{-st_d}$$

- Where t_d is the delay time.

First Order System With Delays

$$\frac{C(s)}{R(s)} = \frac{K}{Ts + 1} e^{-st_d}$$



First Order System With Delays

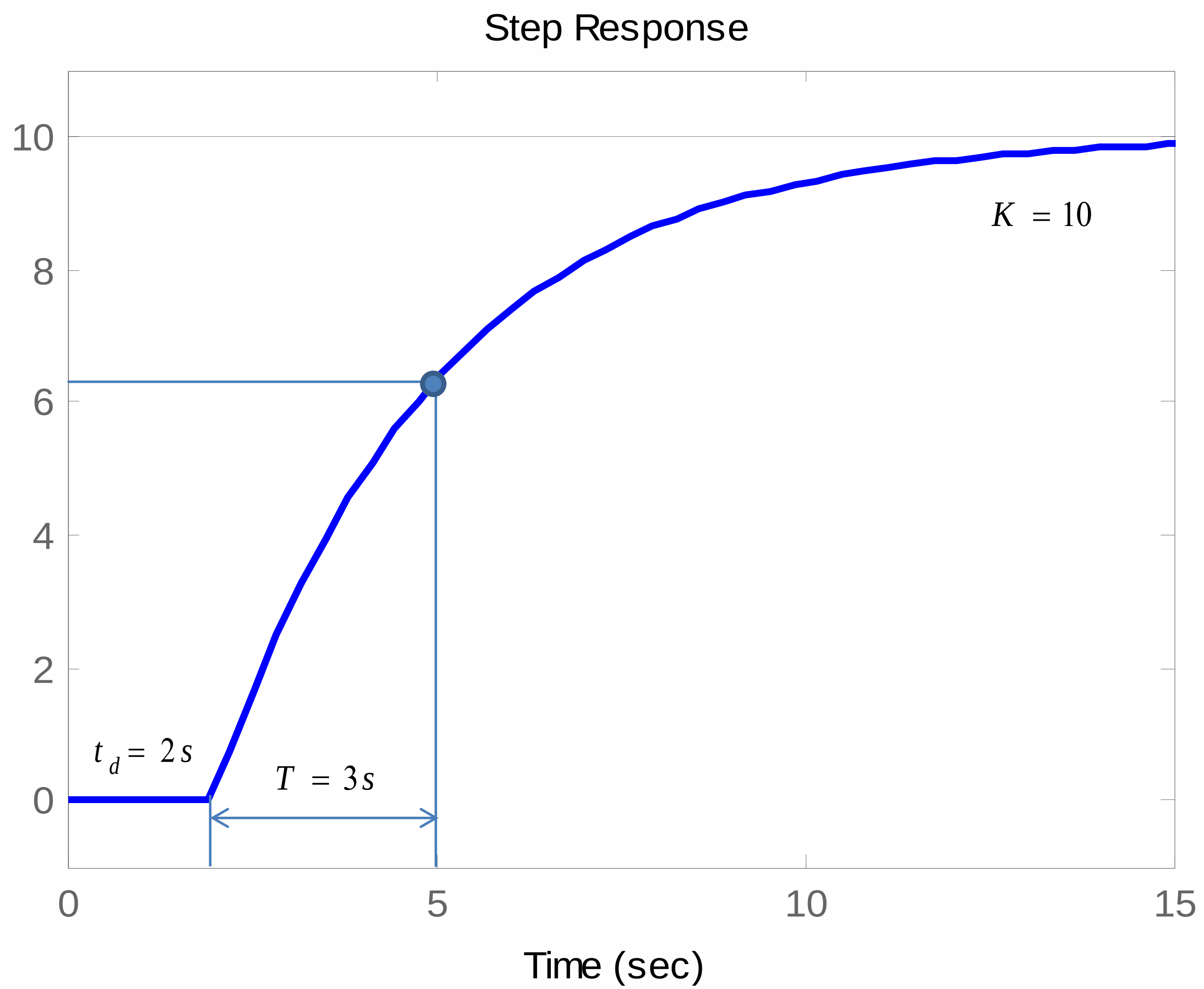
$$\frac{C(s)}{R(s)} = \frac{10}{3s+1} e^{-2s}$$

$$C(s) = \frac{10}{s(3s+1)} e^{-2s}$$

$$L^{-1}[e^{-\partial s} F(s)] = f(t-\partial)u(t-\partial)$$

$$L^{-1}[(\frac{10}{s} + \frac{-10}{s+1/3})e^{-2s}] =$$

$$[10(t-2) - 10e^{-1/3(t-2)}]u(t-2)$$



Extra Poles

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + \ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}$$

n>m for a real system

$$C(s) = \frac{1}{s} + \sum_{j=1}^{q} \frac{a_j}{s + p_j} + \sum_{k=1}^{r} \frac{b_k (s + \zeta_k \omega_k) + c_k \omega_k \sqrt{1 - \zeta_k}}{s^2 + 2\zeta_k \omega_k s + \omega_k^2} \qquad (q + 2r = n)$$

i.e. combination of first and second order systems

$$\frac{1}{s^3 + as^2 + bs + c} = \frac{1}{(s + f)(s^2 + ds + e)} \qquad s^3 + as^2 + bs + c = (s + f)(s^2 + ds + e) \Leftrightarrow$$

$$\Leftrightarrow s^3 + as^2 + bs + c = s^3 + (d + f)s^2 + (e + fd)s + fe \qquad \Leftrightarrow \begin{Bmatrix} 1 = 1 \\ a = d + f \\ b = e + fd \\ c = fe \end{Bmatrix}$$

Extra Poles

$$C(s) = \frac{1}{s} + \sum_{j=1}^{q} \frac{a_j}{s + p_j} + \sum_{k=1}^{r} \frac{b_k (s + \zeta_k \omega_k) + c_k \omega_k \sqrt{1 - \zeta_k}}{s^2 + 2\zeta_k \omega_k s + \omega_k^2}$$

$$c(t) = 1 + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{k=1}^{r} b_k e^{-\zeta_k \omega_k t} \cos\left(\omega_k \sqrt{1 - \zeta_k^2}\, t\right) + \sum_{k=1}^{r} c_k e^{-\zeta_k \omega_k t} \sin\left(\omega_k \sqrt{1 - \zeta_k^2}\, t\right)$$

The response of a higher order system is the sum of exponential and damped sinusoidal curves.

Assuming that all poles are at the left hand side then the final value of the output is “1” since all exponential terms will converge to 0.

Let’s assume that some poles have real parts that are far away from the imaginary axis=>

$$c(t) = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1 - \zeta^2}} \sin\left(\omega_d t + \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta}\right) \qquad e^{-\zeta \omega_n t} \to 0$$

Extra Poles

Overall performance is characterized by the isolated
(far away from zeros) poles that are close to the imaginary axis.

If we have only one pole (or a pair for complex roots) that is closed to the real axis then we say that this pole (or pair of poles) is (are) the DOMINANT pole(s) for the system.

A simple rule is that the dominant poles must be at least five to ten times closer to the imaginary axis than the other ones.

$$C(s) = \frac{1}{s} + \sum_{j=1}^{q} \frac{a_j}{s + p_j} + \sum_{k=1}^{r} \frac{b_k (s + \zeta_k \omega_k) + c_k \omega_k \sqrt{1 - \zeta_k}}{s^2 + 2\zeta_k \omega_k s + \omega_k^2}$$

$$c(t) = 1 + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{k=1}^{r} b_k e^{-\zeta_k \omega_k t} \cos\left(\omega_k \sqrt{1 - \zeta_k^2}\, t\right) +$$

$$+ \sum_{k=1}^{r} c_k e^{-\zeta_k \omega_k t} \sin\left(\omega_k \sqrt{1 - \zeta_k^2}\, t\right)$$

The values of b (numerator coefficients) determine the amplitude of the oscillations of the system but not its stability properties.

Unit-3
STABILITY ANALYSIS AND CONTROLLERS

Concept of Stability

- In order to know the location of the poles, we need to find the roots of the closed-loop characteristic equation.

- It turned out, however, that in order to judge a system's stability we don't need to know the actual location of the poles, just their sign. that is whether the poles are in the right-half or left-half plane.

- The Hurwitz criterion can be used to indicate that a characteristic polynomial with negative or missing coefficients is unstable.

- The Routh-Hurwitz Criterion is called a necessary and sufficient test of stability because a polynomial that satisfies the criterion is guaranteed to stable. The criterion can also tell us how many poles are in the right-half plane or on the imaginary axis.

Routh-Hurwitz Stability Criterion

- All the coefficients must be positive if all the roots are in the left half plane. Also it is necessary that all the coefficients for a stable system be nonzero.
- These requirements are necessary but not sufficient. That is we know the system is unstable if they are not satisfied; yet if they are satisfied, we must proceed further to ascertain the stability of the system.
- For example,

$$q(s) = s^3 + s^2 + 2s + 8 = (s+2)(s^2 - s + 4)$$

The system is unstable yet all coefficients are positive

- The Routh-Hurwitz is a necessary and sufficient criterion for the stability of linear systems.

Routh-Hurwitz Stability Criterion (Contd..)

- The Routh-Hurwitz criterion applies to a polynomial (characteristic equation) of the form:

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0$$

$$\text{assume} \quad a_0 \neq 0$$

- The Routh-Hurwitz array:

$$\begin{array}{llllll}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & .... \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & .... \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & .... \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & .... \\
. & . & . & & & \\
. & . & . & & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}$$

Routh-Hurwitz Stability Criterion (Contd..)

- Columns of s are only for accounting.
- The b row is calculated from the two rows above it.
- The c row is calculated from the two rows directly above it.
- Etc…
- The equations for the coefficients of the array are:

$$b_1 = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} \qquad b_2 = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}, \ldots\ldots.$$

$$c_1 = -\frac{1}{b_1}\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix} \qquad c_2 = -\frac{1}{b_1}\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix}, \ldots\ldots$$

- Note: the determinant in the expression for the ith coefficient in a row is formed from the first column and the (i+1)th column of the two preceding rows.

Routh-Hurwitz Stability Criterion

- The number of polynomial roots in the right half plane is equal to the number of sign changes in the first column of the array.
- Example:

$$P(s) = s^3 + s^2 + 2s + 8 = (s+2)(s^2 - s + 4)$$

The Routh array is :

s^3	1	2
s^2	1	8
s^1	-6	
s^0	8	

- Since there are two sign changes on the first column, there are two roots of the polynomial in the right half plane: system is unstable.

- Note: The Routh-Hurwitz criterion shows only the stability of the system, it does not give the locations of the roots, therefore no information about the transient response of a stable system is derived from the R-H criterion.

Routh-Hurwitz Stability Criterion (Contd..)

- From the equations, the array cannot be completed if the first element in a row is zero. Because the calculations require divisions by zero. We have 3 cases:

- **Case 1:** none of the elements in the first column of the array is zero. This is the simplest case. Follow the algorithm as shown in the previous slides.

- **Case 2:** The first element in a row is zero, with at least one nonzero element in the same row. In this case, replace the first element which is zero by a small number ε. All the elements that follow will be functions of ε. After all the elements are calculated, the signs of the elements in the first column are determined by letting ε approach zero.

Routh-Hurwitz Stability Criterion (Contd..)

- Example: $P(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$

$$
\begin{array}{llll}
s^5 & 1 & 2 & 11 \\
s^4 & 2 & 4 & 10 \\
s^3 & \varepsilon & 6 & \\
s^2 & -\dfrac{12}{\varepsilon} & 10 & \\
s^1 & 6 & & \\
s^0 & 10 & &
\end{array}
$$

- When we calculate the elements:

  b1=0, b2=6, there fore we put b1=ε and calculate the other coefficients.
- There are 2 sign changes regardless of ε is positive or negative. Therefore the system is unstable.

Routh-Hurwitz Stability Criterion (Contd..)

- **Case 3:** All elements in a row are zero.
- Example: $P(s) = s^2 + 1$

$$\begin{array}{lll} s^2 & 1 & 1 \\ s^1 & 0 & \\ s^0 & & \end{array}$$

- Here the array cannot be completed because of the zero element in the first column.
- Another example: $P(s) = s^3 + s^2 + 2s + 2$

The array is :

$$\begin{array}{lll} s^3 & 1 & 2 \\ s^2 & 1 & 2 \\ s^1 & 0 & \\ s^0 & & \end{array}$$

Routh-Hurwitz Stability Criterion (Contd..)

- Case 3 polynomial contains an even polynomial as a factor. It is called the auxiliary polynomial. In the first example, the auxiliary polynomial is $s^2 + 1$
- And in the second example, auxiliary polynomial is $s^2 + 2$
- Case 3 polynomial may be analyzed as follows:
- Suppose that the row of zeros is the s^i row, then the auxiliary polynomial is differentiated with respect to s, and the coefficients of the resulting polynomial used to replace the zeros in the s^i row. The calculation of the array then continues as in the case 1.

Routh-Hurwitz Stability Criterion (Contd..)

- Example: $P(s) = s^4 + s^3 + 3s^2 + 2s + 2$

The Routh array is :

$$\begin{array}{llll} s^4 & 1 & 3 & 2 \\ s^3 & 1 & 2 & \\ s^2 & 1 & 2 & \\ s^1 & 0 & & \\ s^0 & & & \end{array}$$

- Since the S1 row contains zeros, the auxiliary polynomial is obtained from the s2 row: $P_{aux}(s) = s^2 + 2$

- The derivative is 2s, therefore 2 replaces 0 in the s1 row, and the routh array is then completed.

Routh-Hurwitz Stability Criterion (Contd..)

- Example: $P(s) = s^4 + s^3 + 3s^2 + 2s + 2$

The Routh array now becomes :

$$\begin{array}{llll} s^4 & 1 & 3 & 2 \\ s^3 & 1 & 2 & \\ s^2 & 1 & 2 & \\ s^1 & \phi^2 & & \\ s^0 & 2 & & \end{array}$$

- Hence there are no roots in the right half plane.

- Note: When there is a row of zeros in the routh array, the systems is not stable. That is it will have roots either on the imaginary axis (as in this example), or it has roots on the right half plane.

Determination of range of gain K using RH Criterion

- Example: $P(s) = s^3 + 5s^2 + (9-K)s + K$

  The Routh array is :

$$\begin{array}{lll} s^3 & 1 & 9-K \\ s^2 & 5 & K \\ s^1 & 9-1.2K & \\ s^0 & K & \end{array}$$

- For the system to be stable there should not be any sign changes in the elements of 1st column
- Hence choose the value of K so that 1st column elements are positive
- From s0 row, system to be stable K>0
- From s1 row

$$9 - 1.2K > 0$$

$$9 > 1.2K$$

$$K < 7.5$$

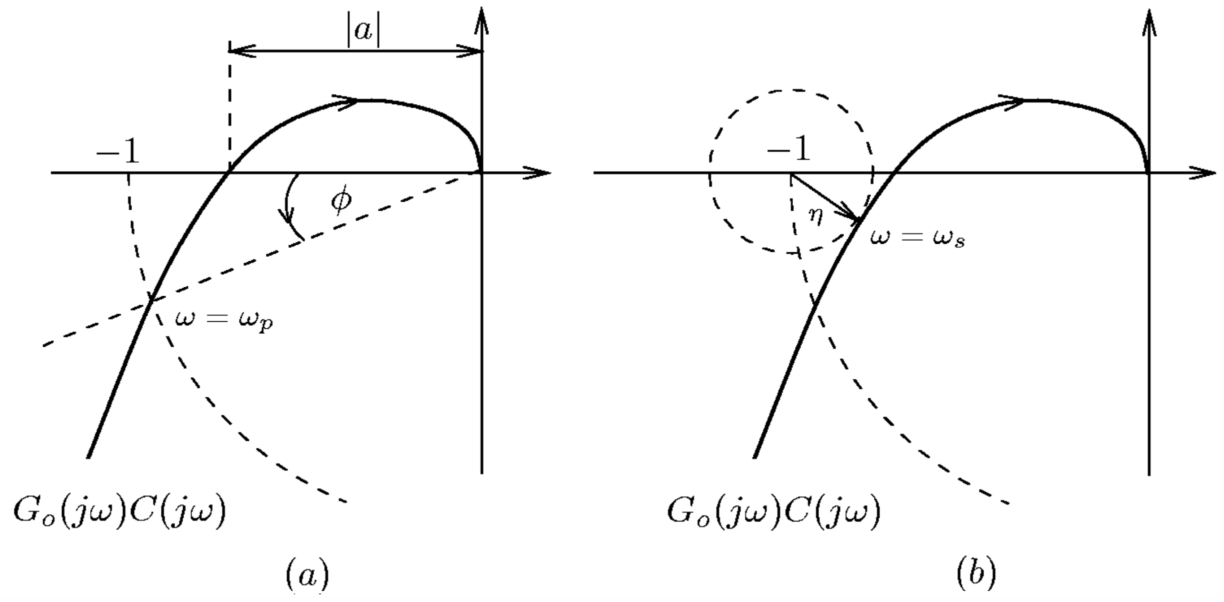
- Hence the range of K is 0<K<7.5

Stability of Control System

- There are several meanings of stability, in general there are two kinds of stability definitions in control system study.
  - Absolute Stability
  - Relative Stability

Stability Margins and Sensitivity Peaks

- In control system design, one often needs to go beyond the issue of closed loop stability. In particular, it is usually desirable to obtain some quantitative measures of how far from instability the nominal loop is, i.e. to quantify relative stability. This is achieved by introducing measures which describe the distance from the nominal open loop frequency response to the critical stability point (-1,0).



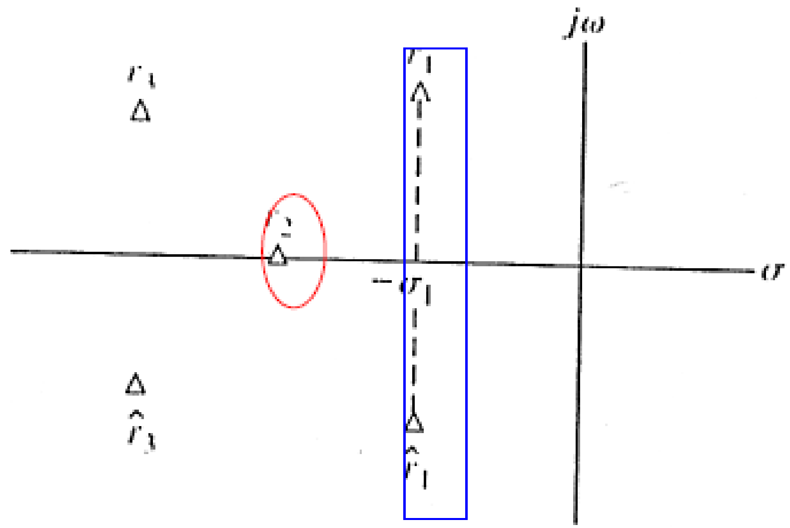
Gain and Phase Margins *Peak Sensitivity*

Relative Stability of Feedback Control Systems

- The verification of stability using the Routh-Hurwitz criterion provides only a partial answer to the question of stability----whether the system is absolutely stable.

- In practice, it is desired to determine the relative stability.

- The **relative stability of a system can be defined** as the property that is measured by the relative real part of each root or pair of roots.



Relative Stability of Feedback Control Systems

- Because the relative stability of a system is dictated by location of the roots of the characteristic equation, we can extend the Routh-Hurwitz criterion to ascertain relative stability.

- This can be accomplished by utilizing a change of variable, which shifts the s-plane vertical axis in order to utilize the Routh-Hurwitz criterion.

- The correct magnitude of shift the vertical axis must be obtained on a trial-and-error basis.

- One may determine the real part of the dominant roots without solving the high order polynomial $q(s)$.

Problems on RH Criterion

- **Example-1:** $P(s) = s^3 + 10s^2 + 31s + 1030$

  The Routh array is :

$$
\begin{array}{lll}
s^3 & 1 & 31 \\
s^2 & 1 & 103 \text{ (by dividing with 10)} \\
s^1 & -72 & \\
s^0 & 103 &
\end{array}
$$

- 1st Column of routh array has two sign changes (from 1 to -72 and from -72 to 103). Hence the system is unstable with two poles in the right-half plane.

Problems on RH Criterion (Contd..)

❑ **Example 2:**

❑ Construct a Routh table and determine the number of roots with positive real parts for the equation;

$$2s^3 + 4s^2 + 4s + 12 = 0$$

❑ Solution:

Since there are two changes of sign in the first column of Routh table, the equation above have two roots at right side (positive real parts).

Problems on RH Criterion (Contd..)

- **Example 3:**
- The characteristic equation of a given system is:

$$s^4 + 6s^3 + 11s^2 + 6s + K = 0$$

What restrictions must be placed upon the parameter K in order to ensure that the system is stable?

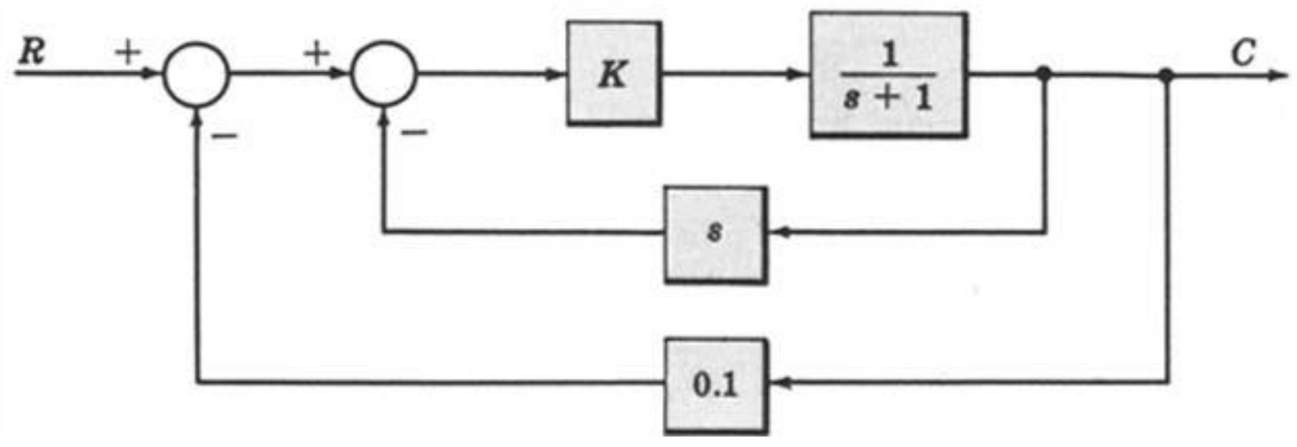
- Solution:

  For the system to be stable, 60 – 6K < 0, or k < 10, and K > 0.

  Thus 0 < K < 10

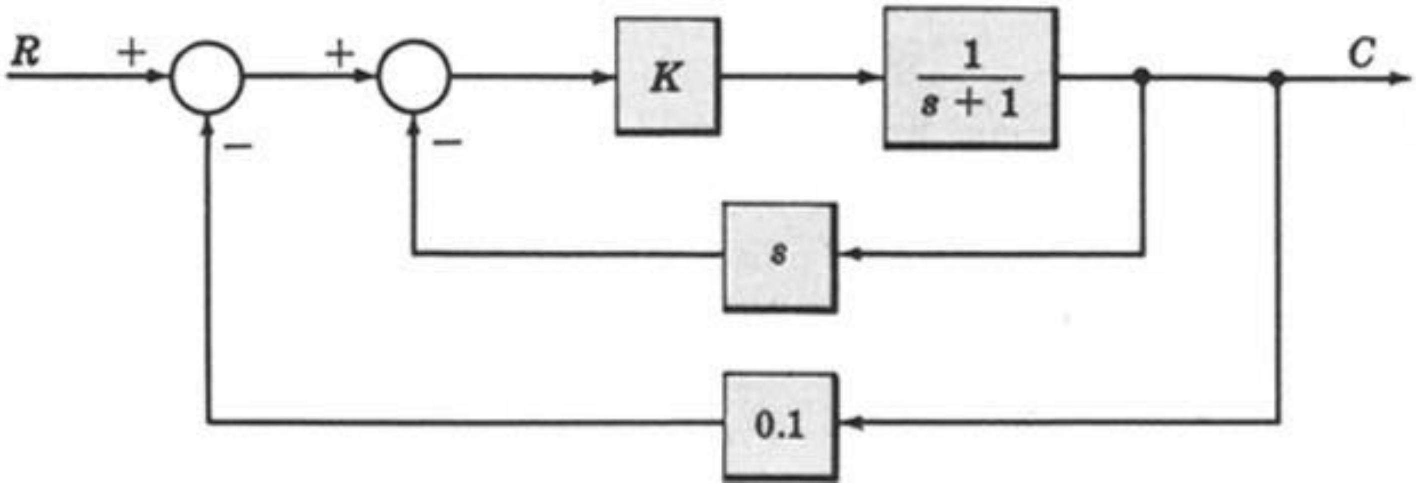
Block Diagram Reduction-Example-1

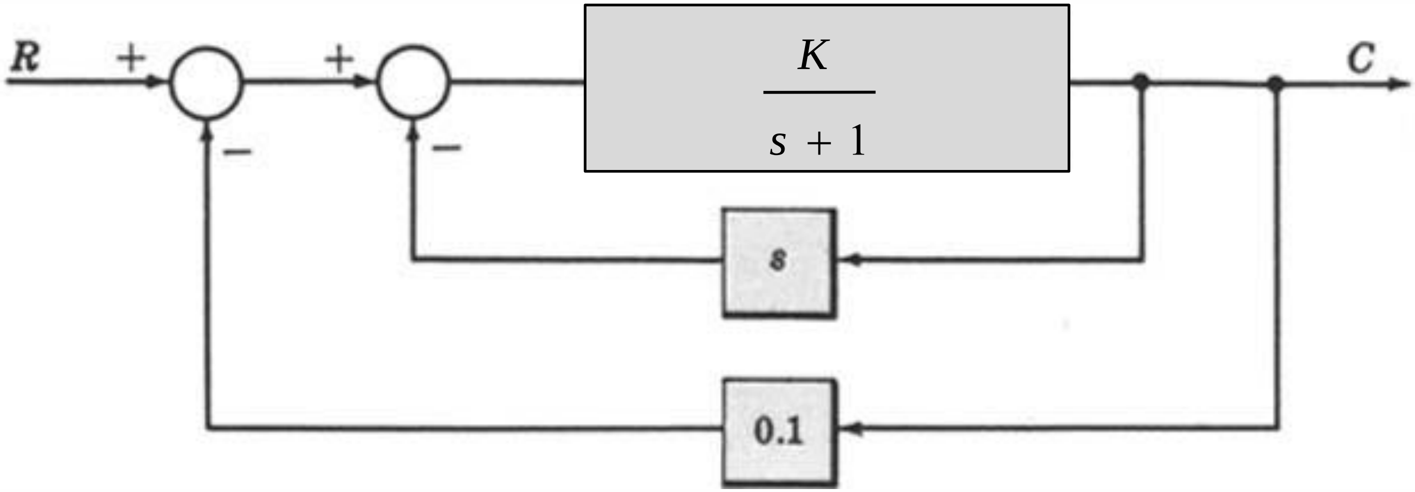
- For the system represented by the following block diagram determine:
  1. Open loop transfer function
  2. Feed Forward Transfer function
  3. control ratio
  4. feedback ratio
  5. error ratio
  6. closed loop transfer function
  7. characteristic equation



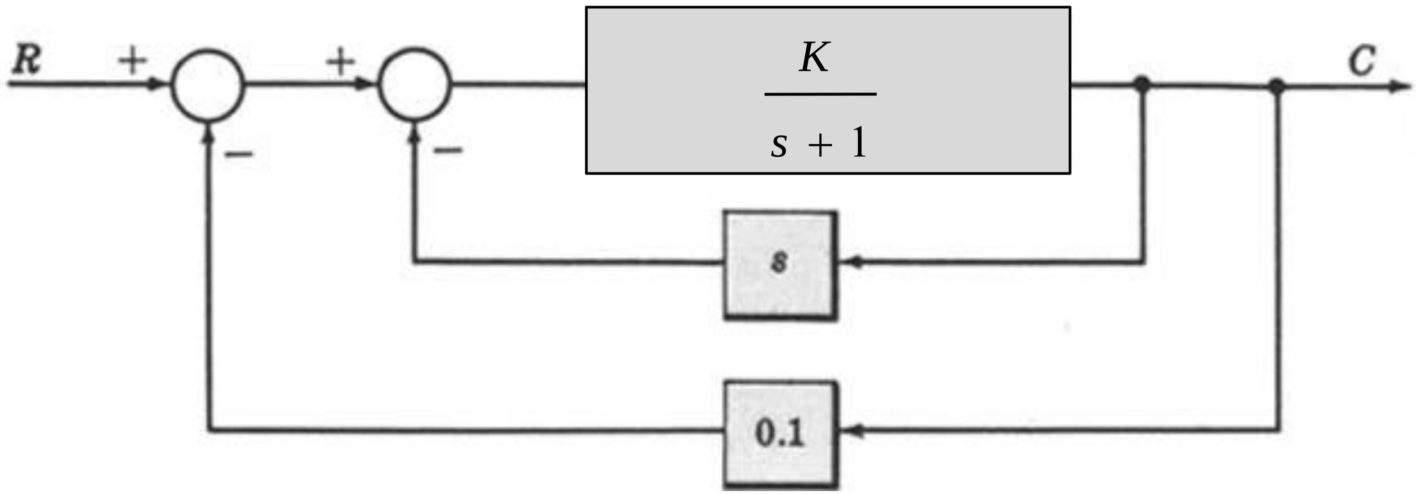
Block Diagram Reduction-Example-1 (Contd..)

- First we will reduce the given block diagram to canonical form

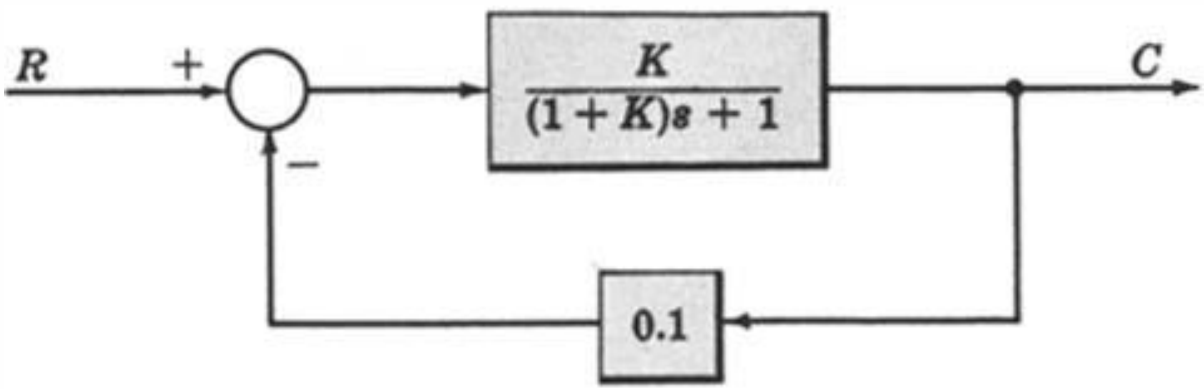




Block Diagram Reduction-Example-1 (Contd..)



$$\frac{G}{1+GH} = \frac{\dfrac{K}{s+1}}{1+\dfrac{K}{s+1}s}$$



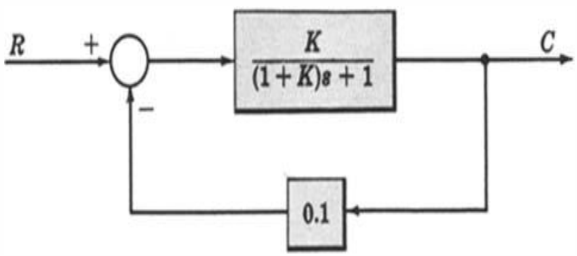
Block Diagram Reduction-Example-1 (Contd..)

1. Open loop transfer functio $\dfrac{B(s)}{E(s)} = G(s)H(s)$

2. Feed Forward Transfe $\dfrac{C(s)}{E(s)} = G(s)$

   function $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1 + G(s)H(s)}$

$G(s)$



$H(s)$

3. Control ratio $\dfrac{B(s)}{R(s)} = \dfrac{G(s)H(s)}{1 + G(s)H(s)}$

4. Feedback $\dfrac{E(s)}{R(s)} = \dfrac{1}{1 + G(s)H(s)}$

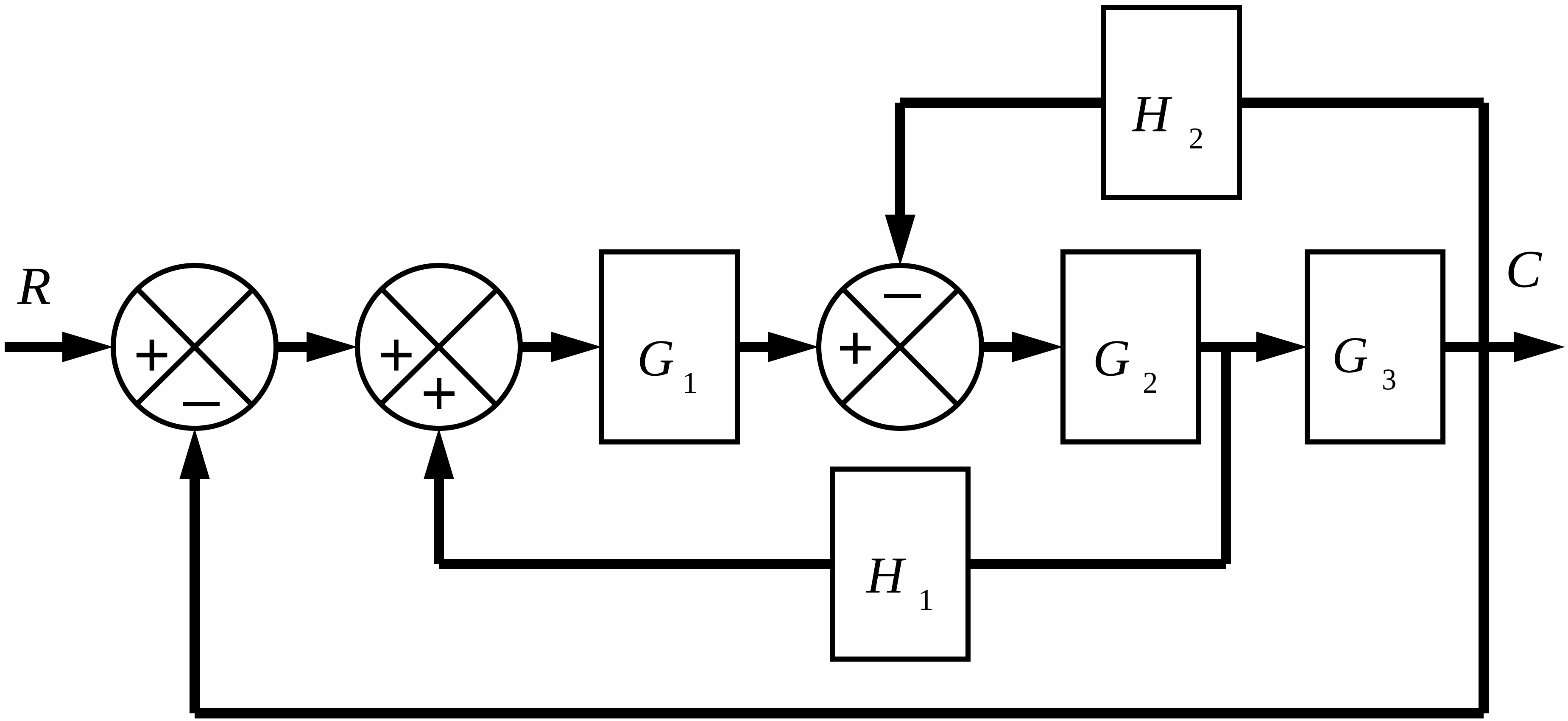
5. Error ratio $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1 + G(s)H(s)}$

6. Closed loop transfer function

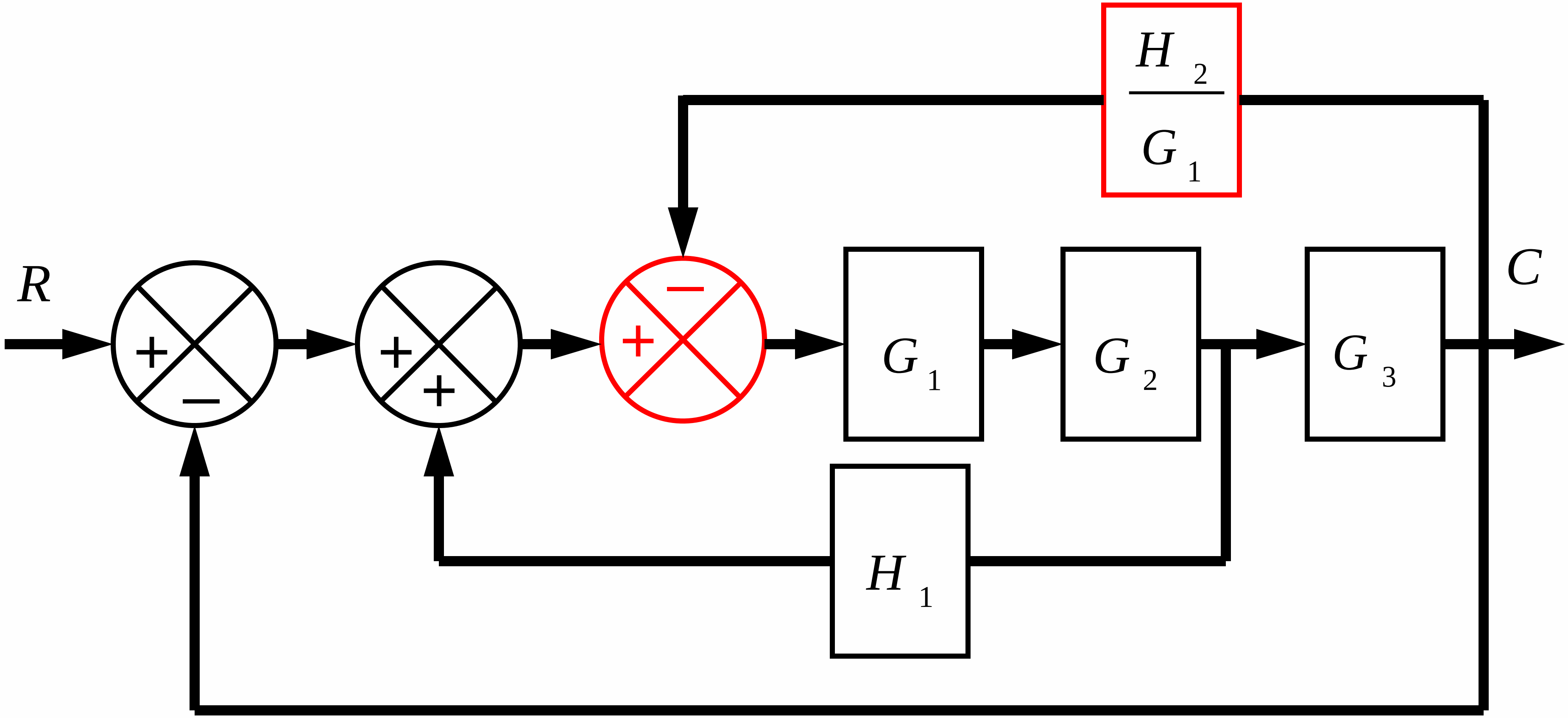
7. Characteristic equation

$$1 + G(s)H(s) = 0$$

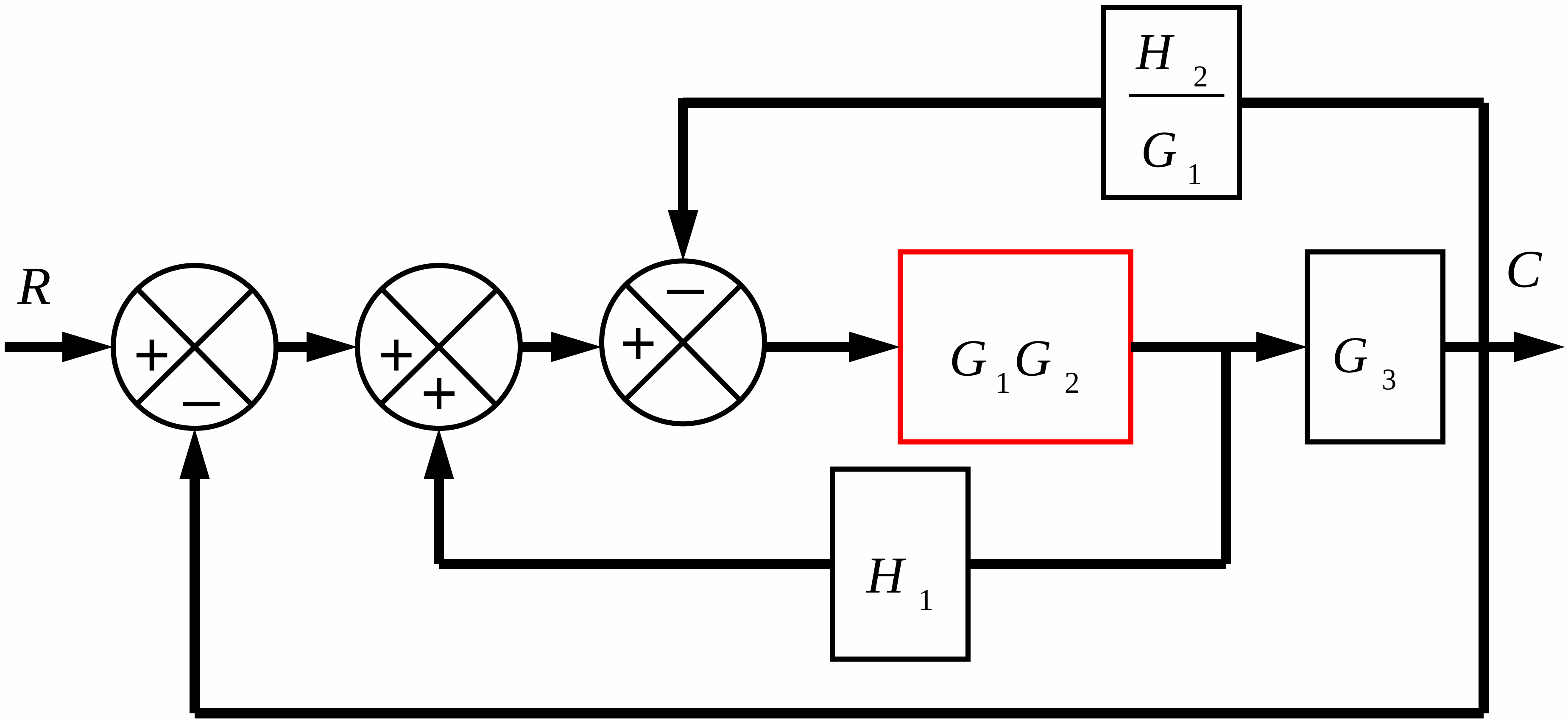
Block Diagram: Reduction Example-2



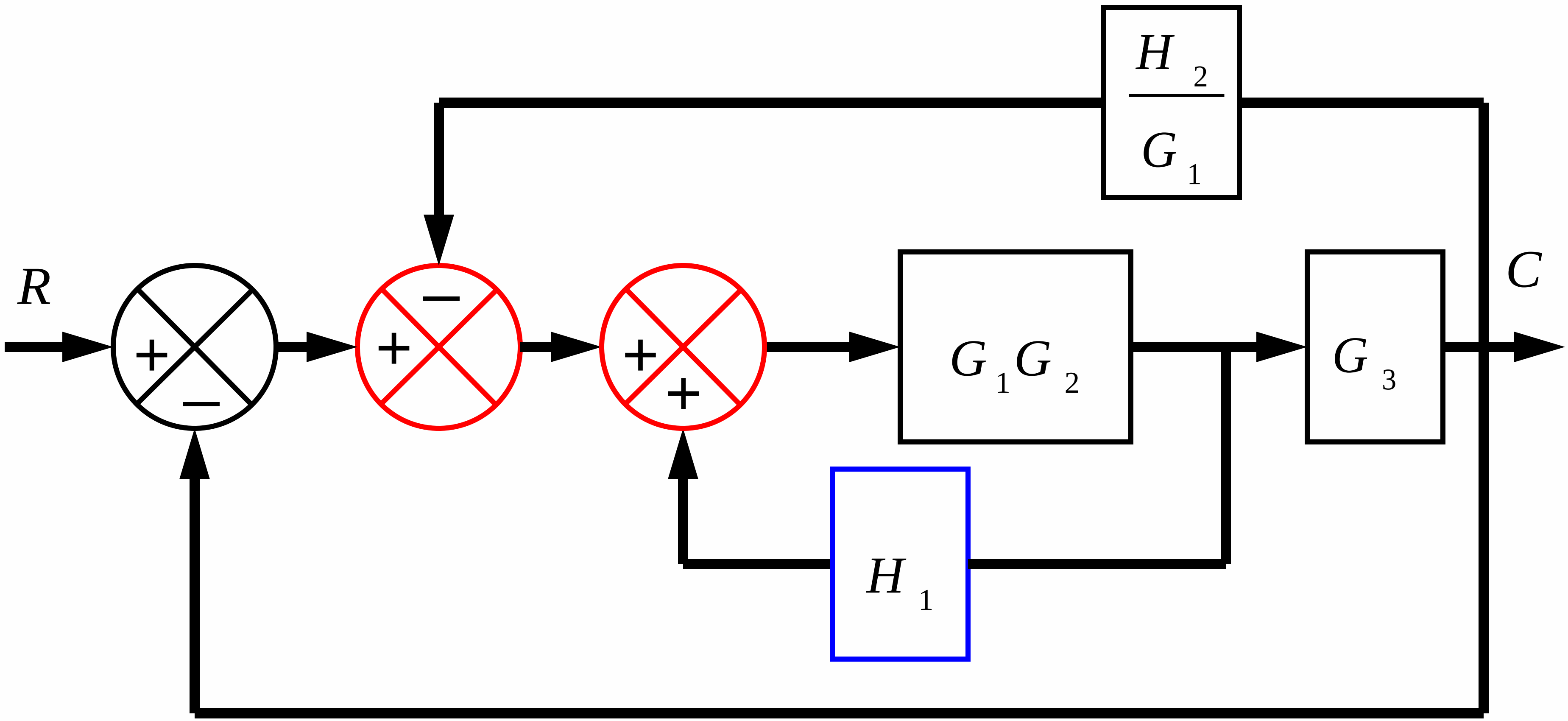
Block Diagram: Reduction Example-2 (Contd..)



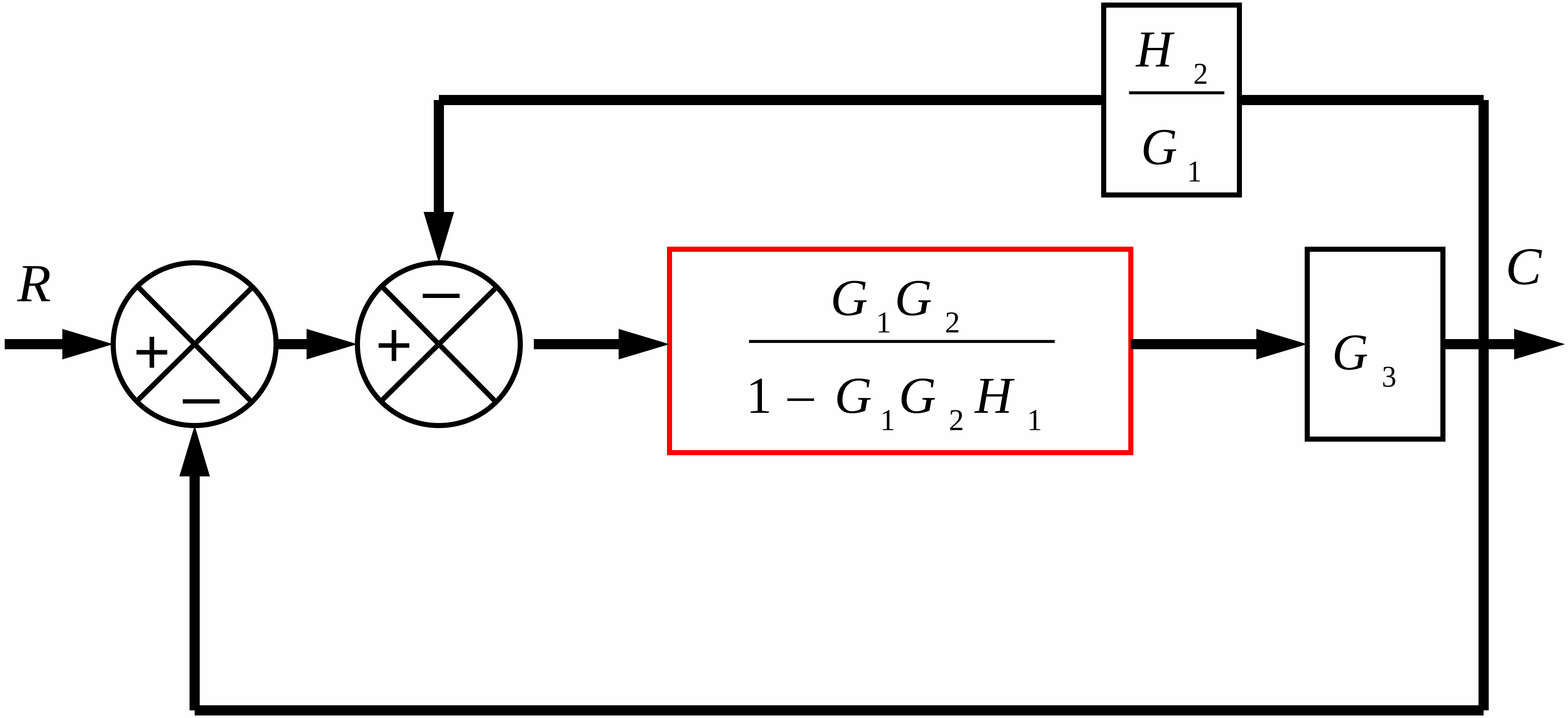
Block Diagram: Reduction Example-2 (Contd..)



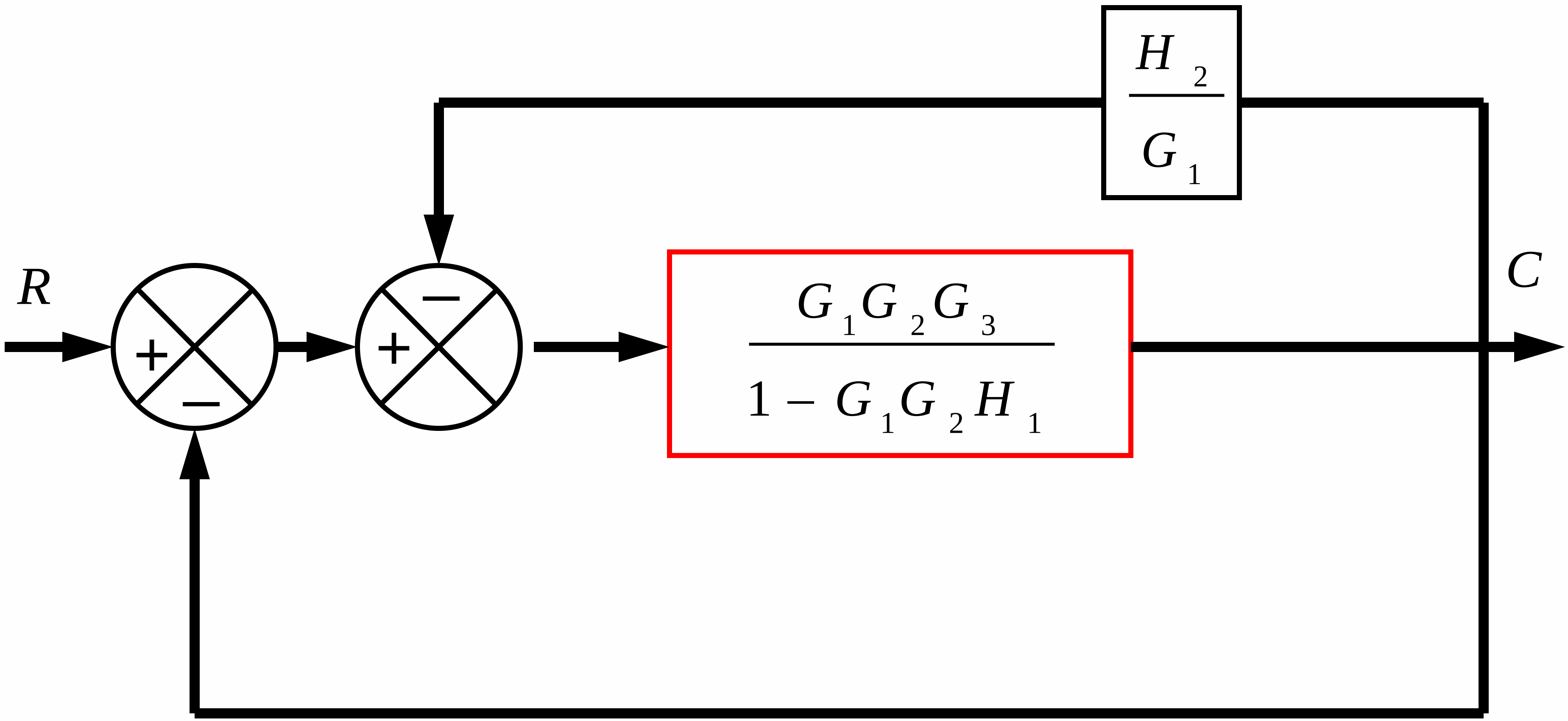
Block Diagram: Reduction Example-2 (Contd..)



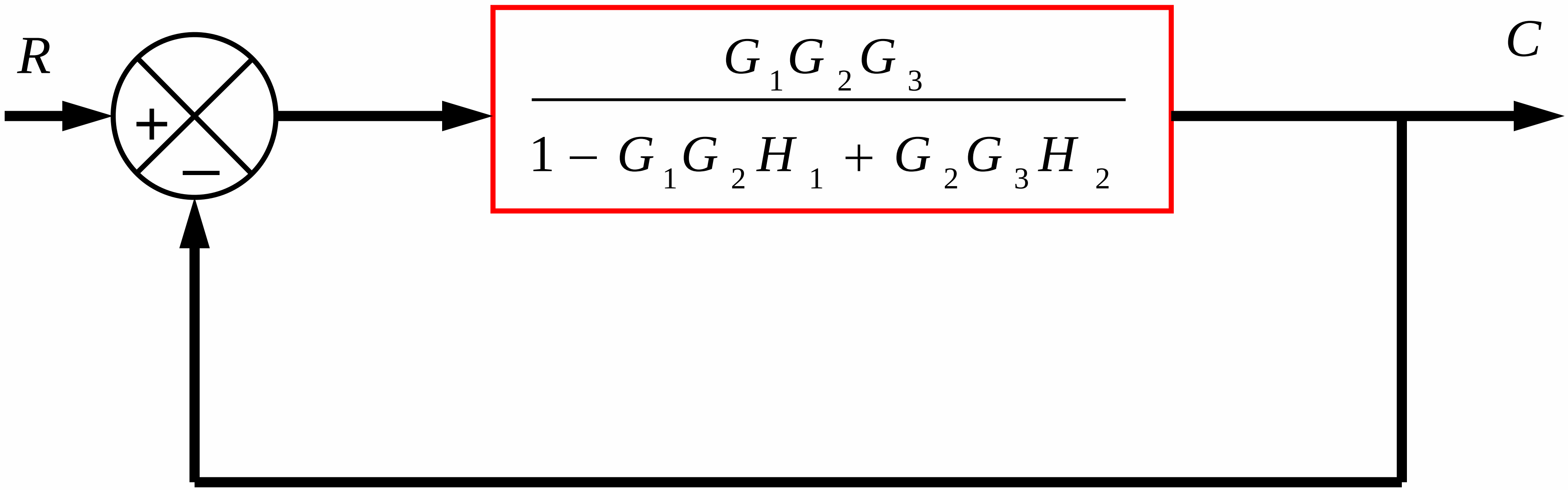
Block Diagram: Reduction Example-2 (Contd..)



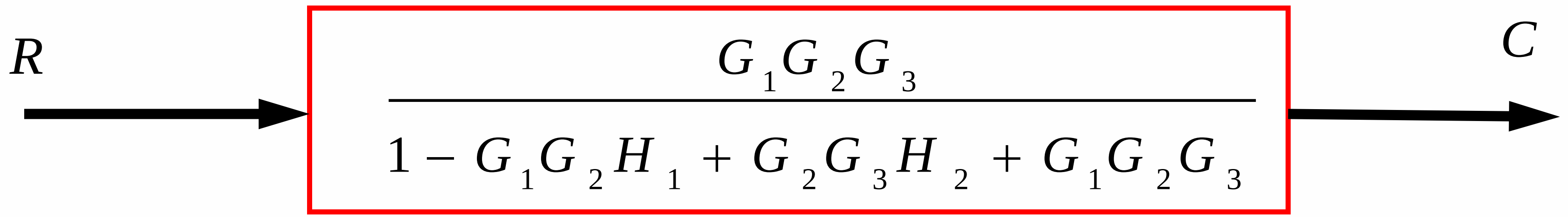
Block Diagram: Reduction Example-2 (Contd..)



Block Diagram: Reduction Example-2 (Contd..)

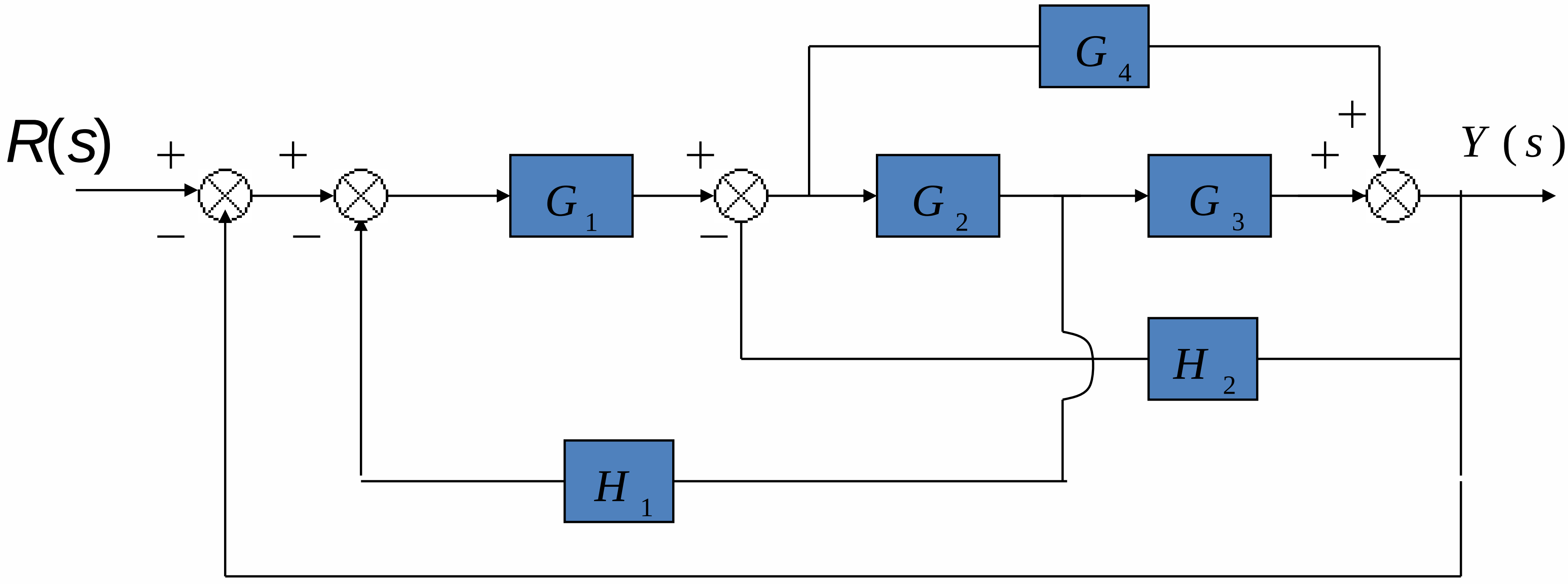


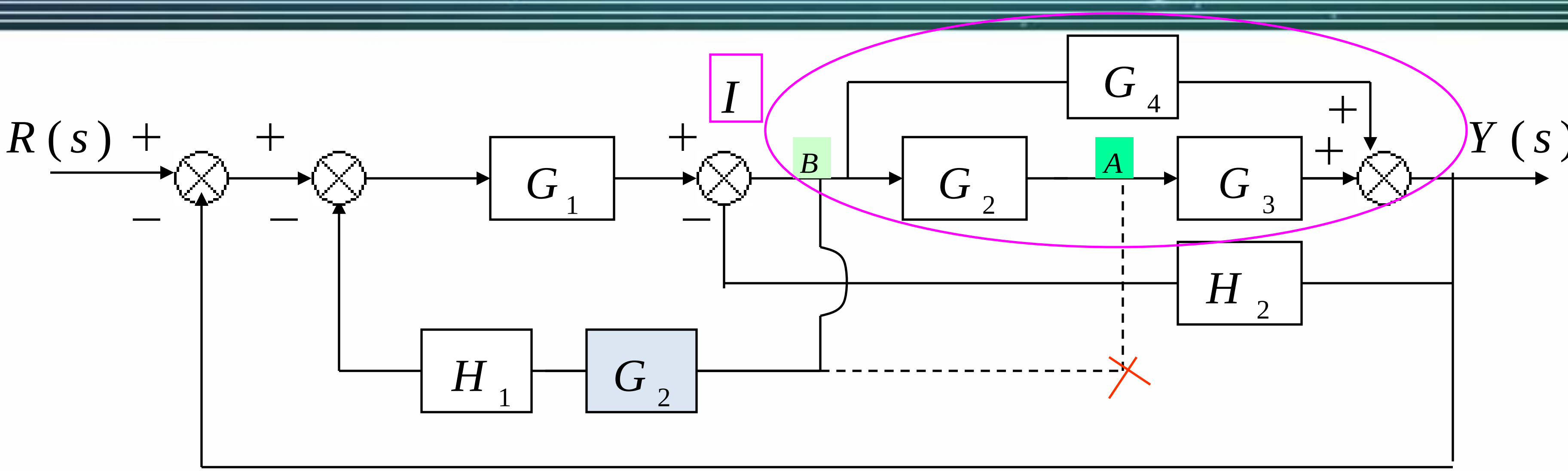
Block Diagram: Reduction Example-2 (Contd..)



Block Diagram: Reduction Example-1

- Find the transfer function of the following block diagram

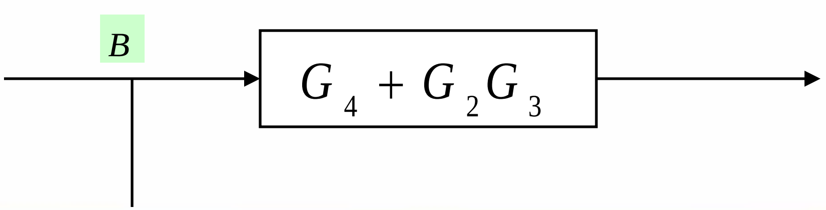




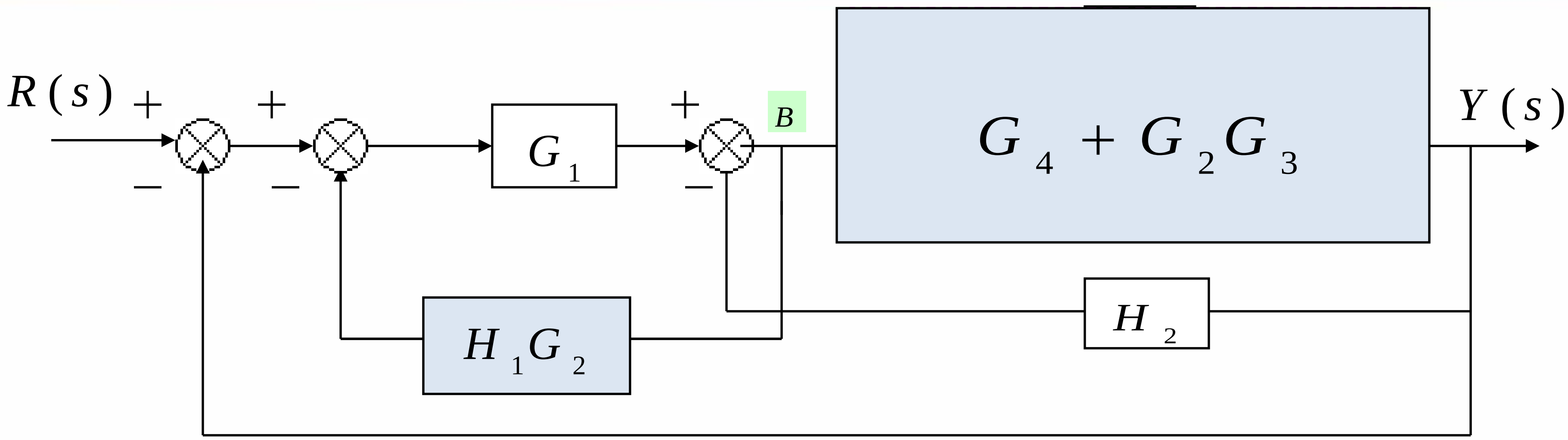
- Solution:

1. Moving pickoff point A ahead of blo $\boxed{G_2}$

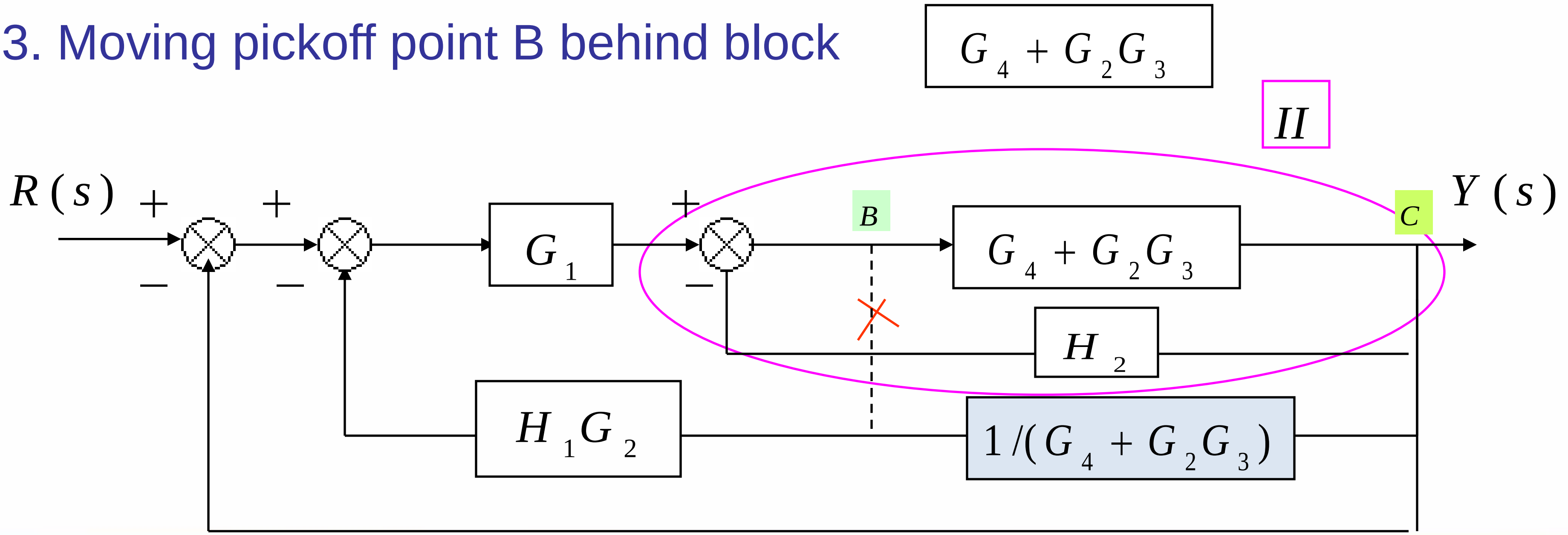
2. Eliminate loop I & simplify



Block Diagram: Reduction Example-1 (Contd..)

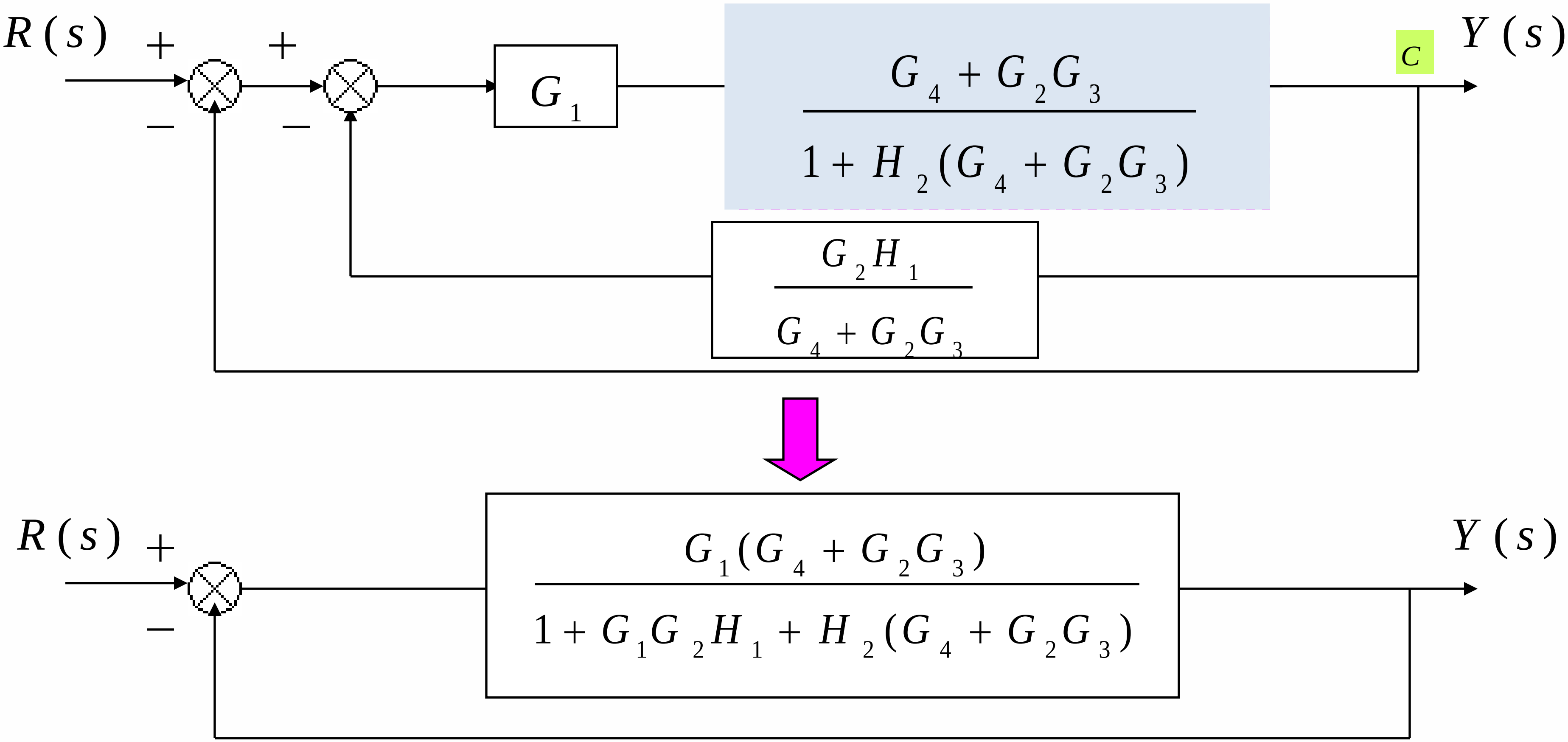


3. Moving pickoff point B behind block



Block Diagram: Reduction Example-1 (Contd..)

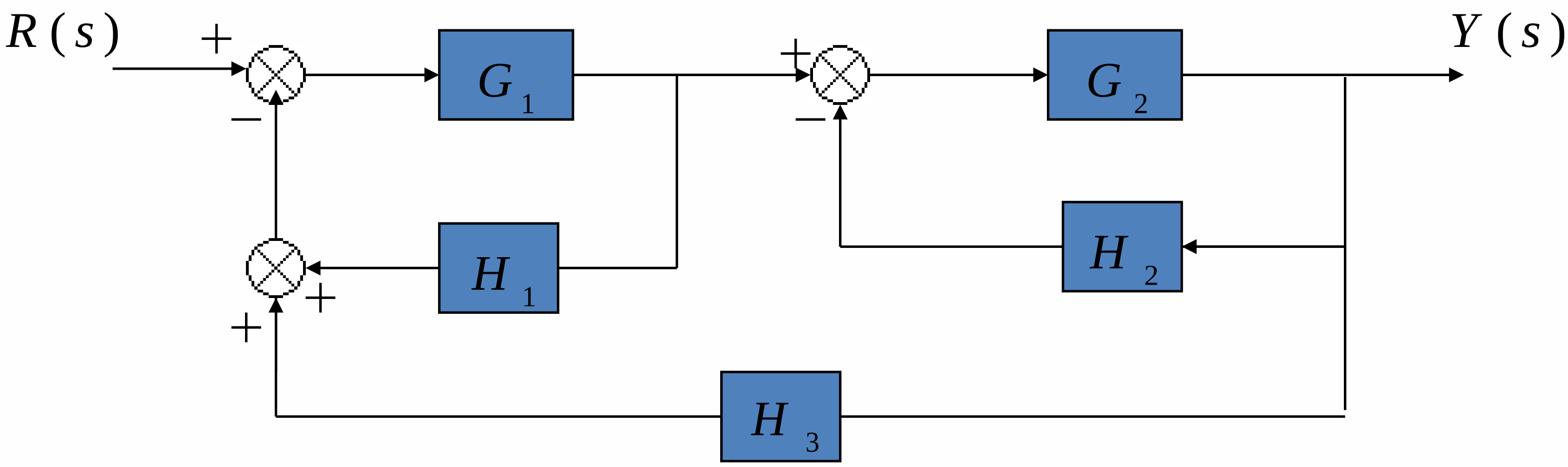
4. Eliminate loop III



$$T(s) = \frac{Y(s)}{R(s)} = \frac{G_1(G_4 + G_2G_3)}{1 + G_1G_2H_1 + H_2(G_4 + G_2G_3) + G_1(G_4 + G_2G_3)}$$

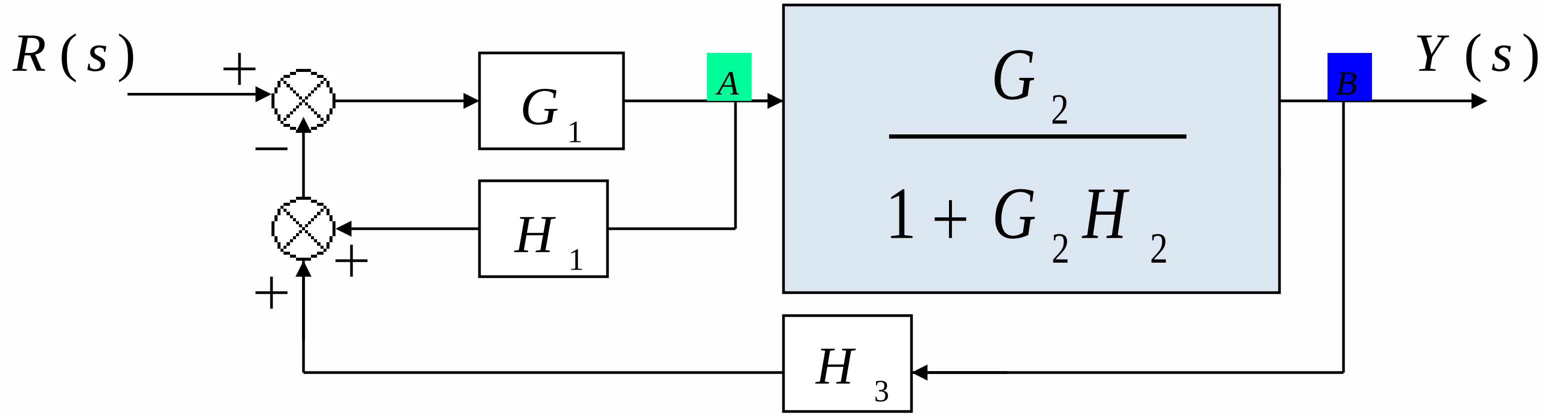
Block Diagram: Reduction Example-2

- Find the transfer function of the following block diagram

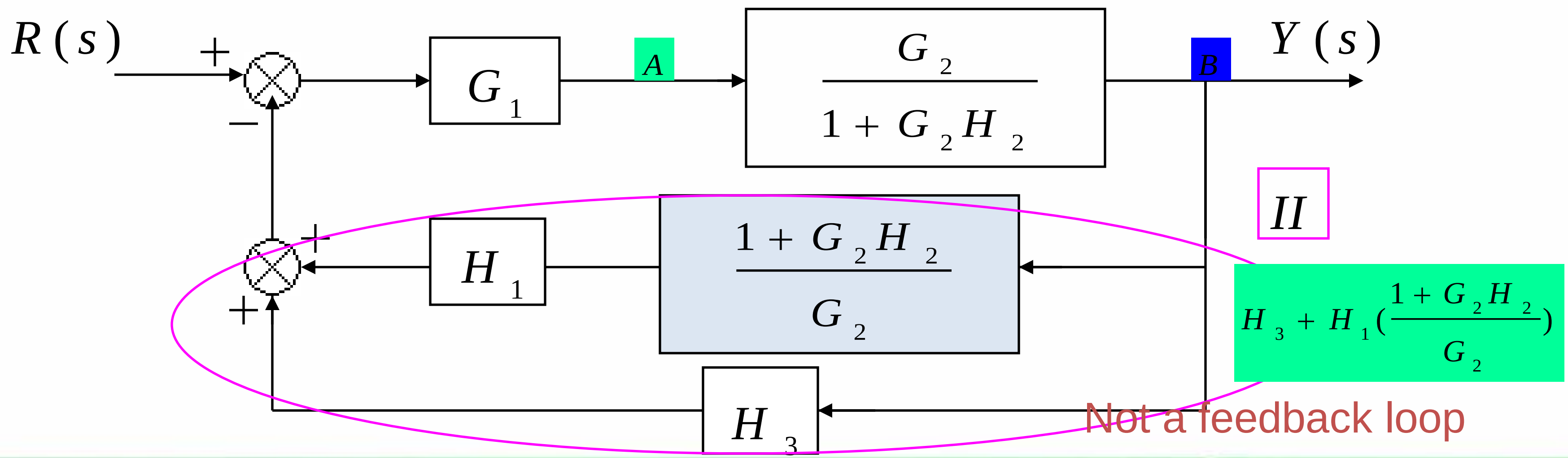


Block Diagram: Reduction Example-2 (Contd..)

1. Eliminate loop I

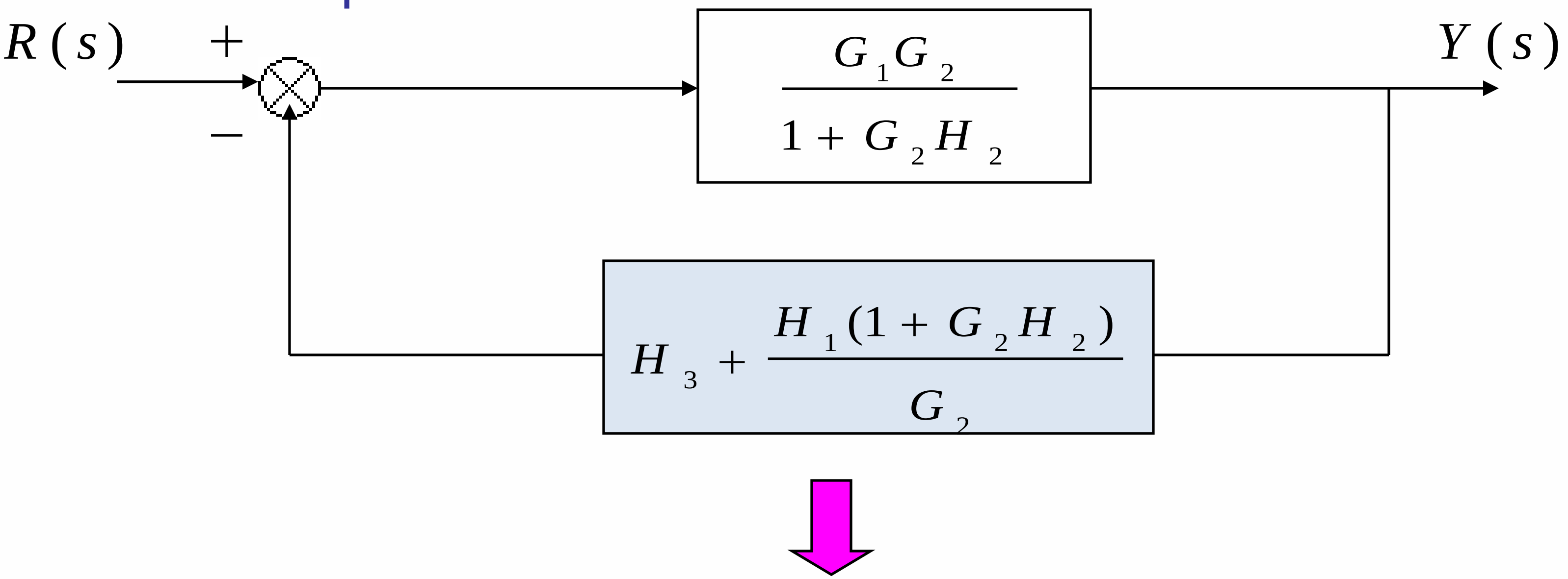


2. Moving pickoff point A behind block $\frac{G_2}{1+G_2H_2}$



Block Diagram: Reduction Example-2 (Contd..)

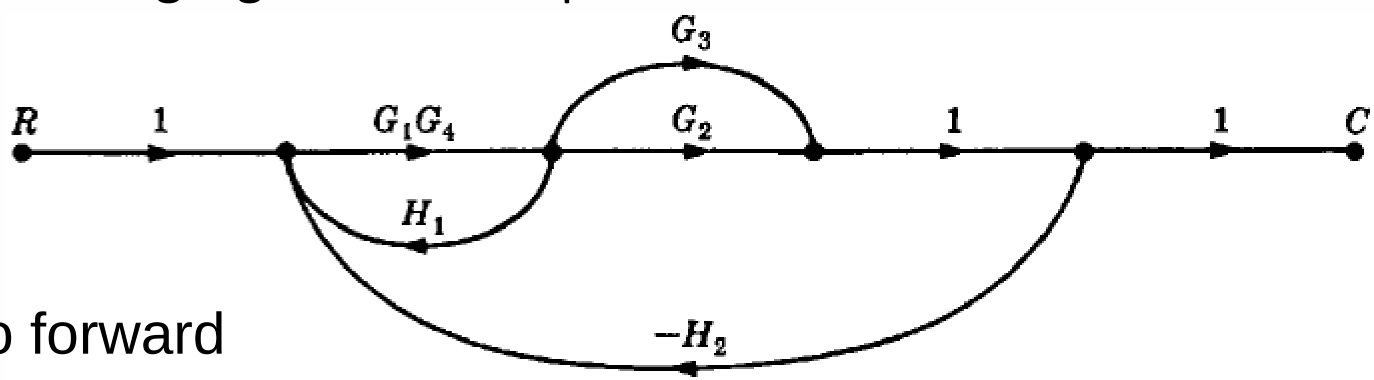
3. Eliminate loop II



$$T(s) = \frac{Y(s)}{R(s)} = \frac{G_1G_2}{1+G_2H_2+G_1G_2H_3+G_1H_1+G_1G_2H_1H_2}$$

Signal Flow Graph:Example#1

- Apply Mason's Rule to calculate the transfer function of the system represented by following Signal Flow Graph



- There are two forward pa

$$P_1 = G_1G_2G_4, \qquad P_2 = G_1G_3G_4$$

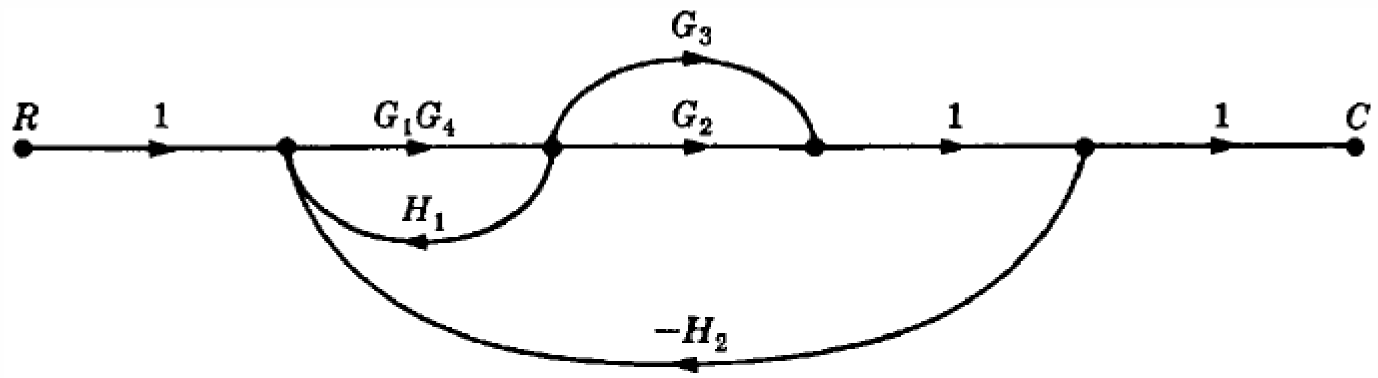
- Therefore,

$$\frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

- There are three feedback loops

$$L_1 = G_1G_4H_1, \qquad L_2 = -G_1G_2G_4H_2, \qquad L_3 = -G_1G_3G_4H_2$$

Signal Flow Graph:Example#1 (Contd..)



- There are no non-touching loops, therefore

Δ = 1- (sum of all individual loop gains)

$$\Delta = 1 - \left(L_1 + L_2 + L_3\right)$$

$$\Delta = 1 - \left(G_1 G_4 H_1 - G_1 G_2 G_4 H_2 - G_1 G_3 G_4 H_2\right)$$

Signal Flow Graph:Example#1 (Contd..)

- Eliminate forward path-1

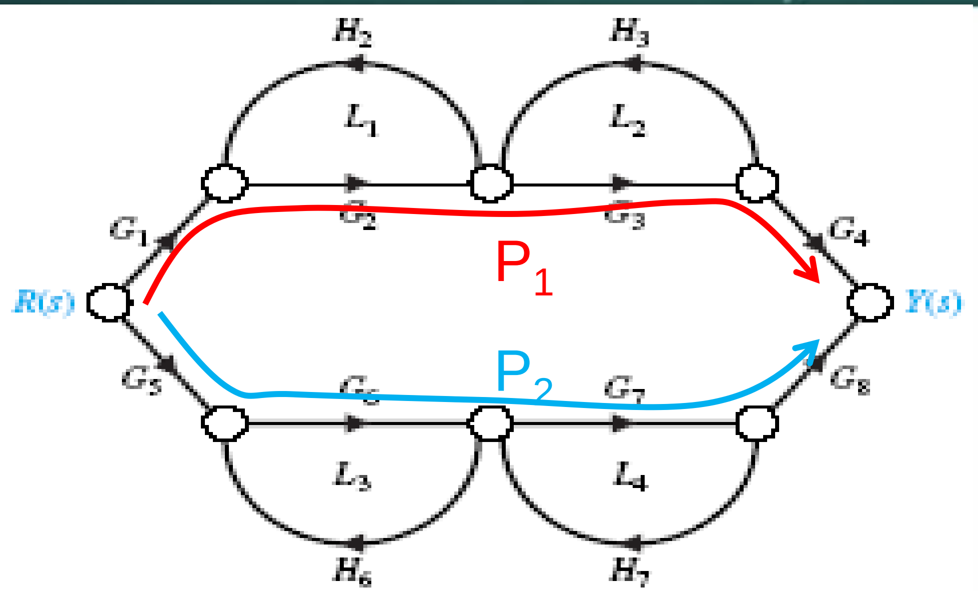
Δ_1 = 1- (sum of all individual loop gains)+...
Δ_1 = 1

- Eliminate forward path-2

Δ_2 = 1- (sum of all individual loop gains)+...
Δ_2 = 1

$$\frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_4 + G_1G_3G_4}{1 - G_1G_4H_1 + G_1G_2G_4H_2 + G_1G_3G_4H_2}$$

Signal Flow Graph:Example#2



1. Calculate forward path gains for each forward path.

$$P_1 = G_1G_2G_3G_4 \text{ (path } \quad 1) \quad \text{and} \quad P_2 = G_5G_6G_7G_8 \text{ (path } \quad 2)$$

2. Calculate all loop gains.

$$L_1 = G_2H_2, \quad L_2 = H_3G_3, \quad L_3 = G_6H_6, \quad L_4 = G_7H_7$$

3. Consider two non-touching loops.

L_1L_3 L_1L_4 L_2L_4 L_2L_3

Signal Flow Graph:Example#2 (Contd..)

4. Consider three non-touching loops.
   None.

5. Calculate Δ from steps 2,3,4.

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_1 L_3 + L_1 L_4 + L_2 L_3 + L_2 L_4)$$

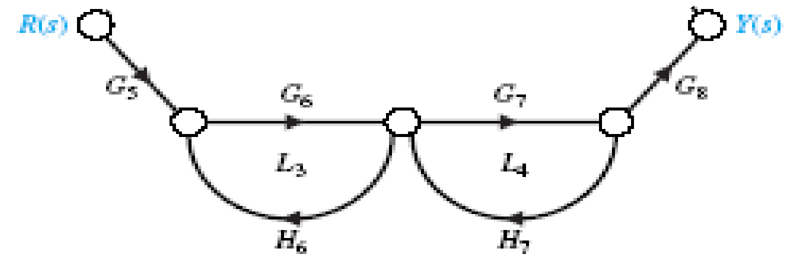
$$\Delta = 1 - (G_2 H_2 + H_3 G_3 + G_6 H_6 + G_7 H_7) + (G_2 H_2 G_6 H_6 + G_2 H_2 G_7 H_7 + H_3 G_3 G_6 H_6 + H_3 G_3 G_7 H_7)$$

Signal Flow Graph:Example#2 (Contd..)

- Eliminate forward path-1

$$\Delta_1 = 1 - (L_3 + L_4)$$

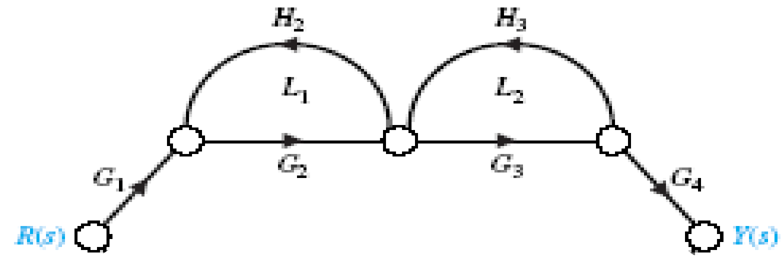
$$\Delta_1 = 1 - (G_6 H_6 + G_7 H_7)$$



- Eliminate forward path-2

$$\Delta_2 = 1 - (L_1 + L_2)$$

$$\Delta_2 = 1 - (G_2 H_2 + G_3 H_3)$$

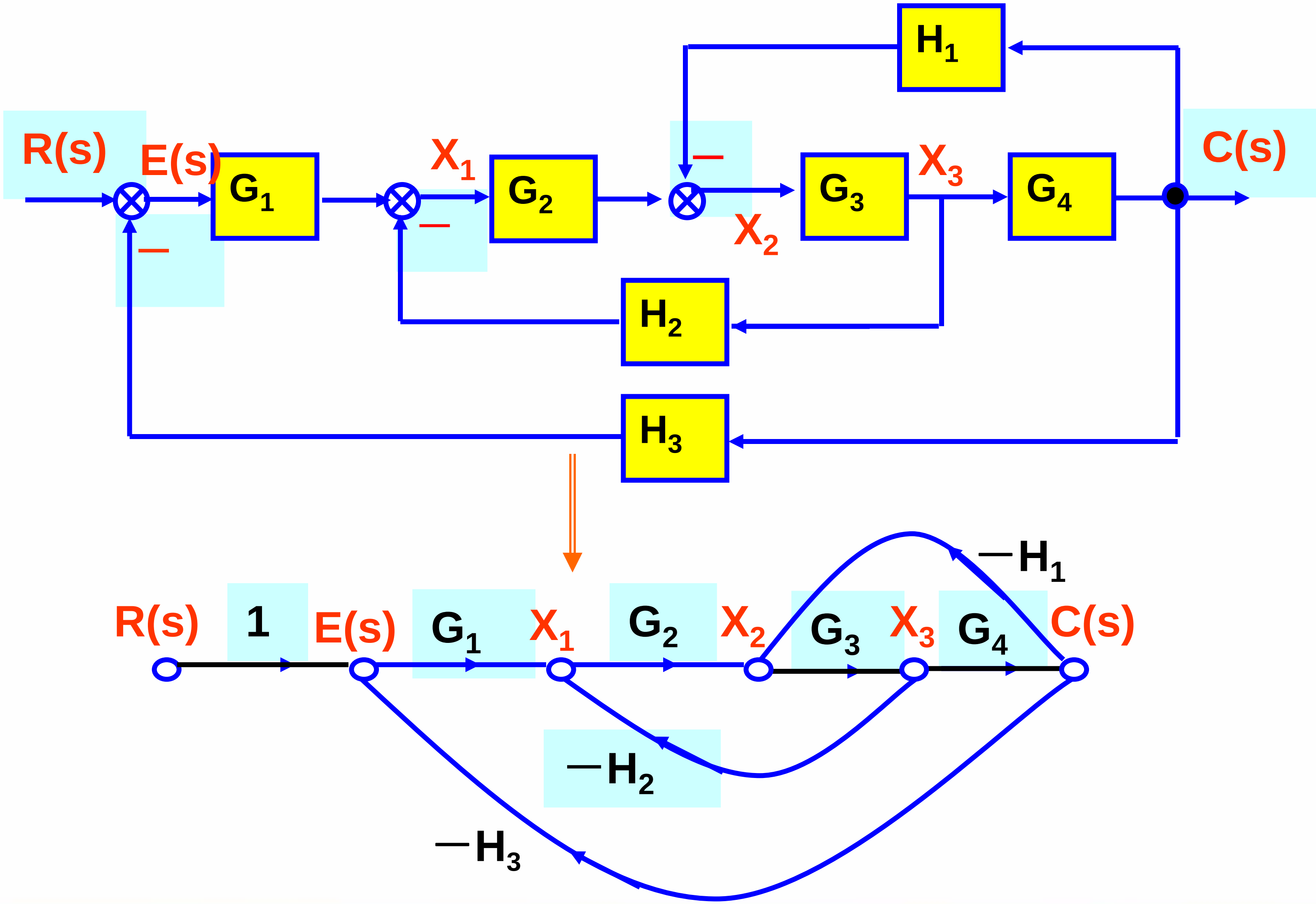


Signal Flow Graph:Example#2 (Contd..)

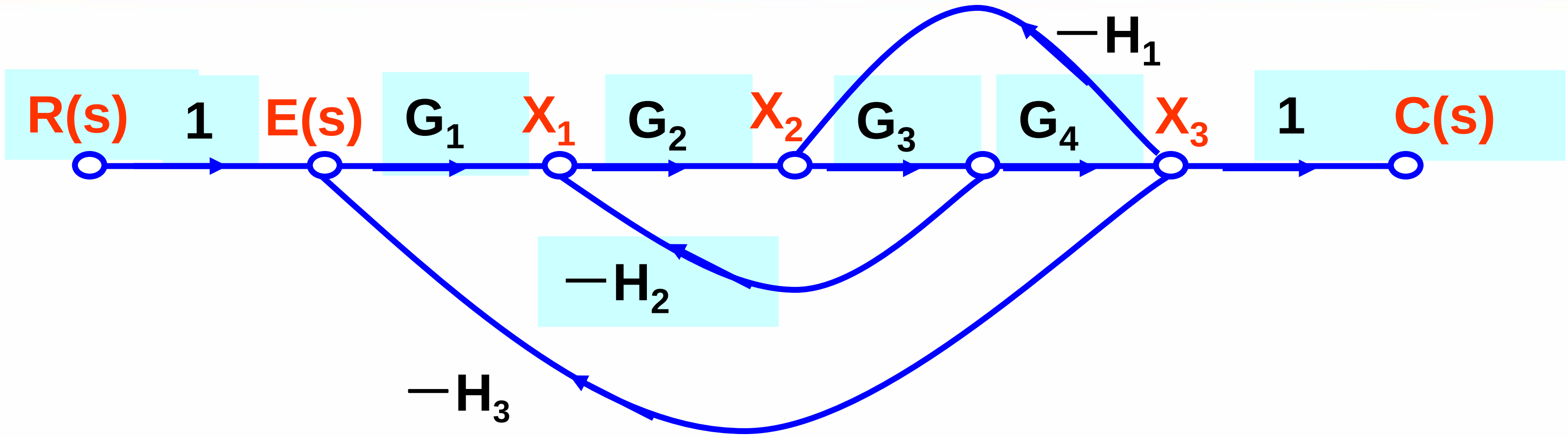
$$\frac{Y(s)}{R(s)} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$\frac{Y(s)}{R(s)} = \frac{G_1G_2G_3G_4[1-(G_6H_6+G_7H_7)] + G_5G_6G_7G_8[1-(G_2H_2+G_3H_3)]}{1-(G_2H_2+H_3G_3+G_6H_6+G_7H_7)+(G_2H_2G_6H_6+G_2H_2G_7H_7+H_3G_3G_6H_6+H_3G_3G_7H_7)}$$

Block Diagram to SFG:Example#3



Block Diagram to SFG:Example#3 (Contd..)



$$\Delta = 1 + (G_1G_2G_3G_4H_3 + G_2G_3H_2 + G_3G_4H_1)$$

$$P_1 = G_1G_2G_3G_4; \qquad \Delta_1 = 1$$

$$G = \frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4}{1 + G_1G_2G_3G_4H_3 + G_2G_3H_2 + G_3G_4H_1}$$

INTRODUCTION

Root Locus Technique:

- Root Locus → the locus of a single root (pole) of a closed loop system
- Root Loci → the locus of multiple roots (poles) of a closed loop system
- It is a graphical method for determining the location of the poles of a given closed loop system for some parameter values of the system. The parameter can be the system gain or time constant.
- Time constant being the design value of an open loop system is normally not varied; the only variable being the system gain.
- It is a time domain method.

INTRODUCTION (Contd)..

- We know that

  for a unity feedback system the characteristic equation is given by

  $$1 + G(S) = 0, \text{ and}$$

- For a non-unity feedback system the characteristic equation is given by

  $$1 + G(S)\,H(S) = 0$$

  where,

- G(S) : open loop transfer function of the system that is to be controlled for desired time domain specifications, and
- H(S) : feedback element (normally a transducer)

INTRODUCTION (Contd)..

- We know that for a closed loop system to be stable, its closed loop poles (roots of characteristic equation) should lie in the left half of the S-plane.
- We also know that a closed loop system is limitedly stable (on the verge of instability) if any of its roots lie on the imaginary axis of the S-plane and it is unstable if its poles lie in the right half of the S-plane.
- Using this method, we can exactly position the location of closed loop poles for a given value of system gain 'K' whereas Routh's method does not facilitate this.
- Using Routh's method we cannot determine relative stability of a system whereas this method allows us to do that.

Illustration by Example

- We know that for a second order closed loop system the general form is given by

$$M(S) = \omega_n^2 / (S^2 + 2\xi\omega_n S + \omega_n^2) = N(S)/D(S)$$

- Let

$$G(S) = K/S(S+1) \; ; \; M(S) = G(S)/1+G(S) = K/(S^2 + S + K)$$

$$M(S) = N(S)/D(S)$$

- For a unity feedback system, the characteristics equation is:

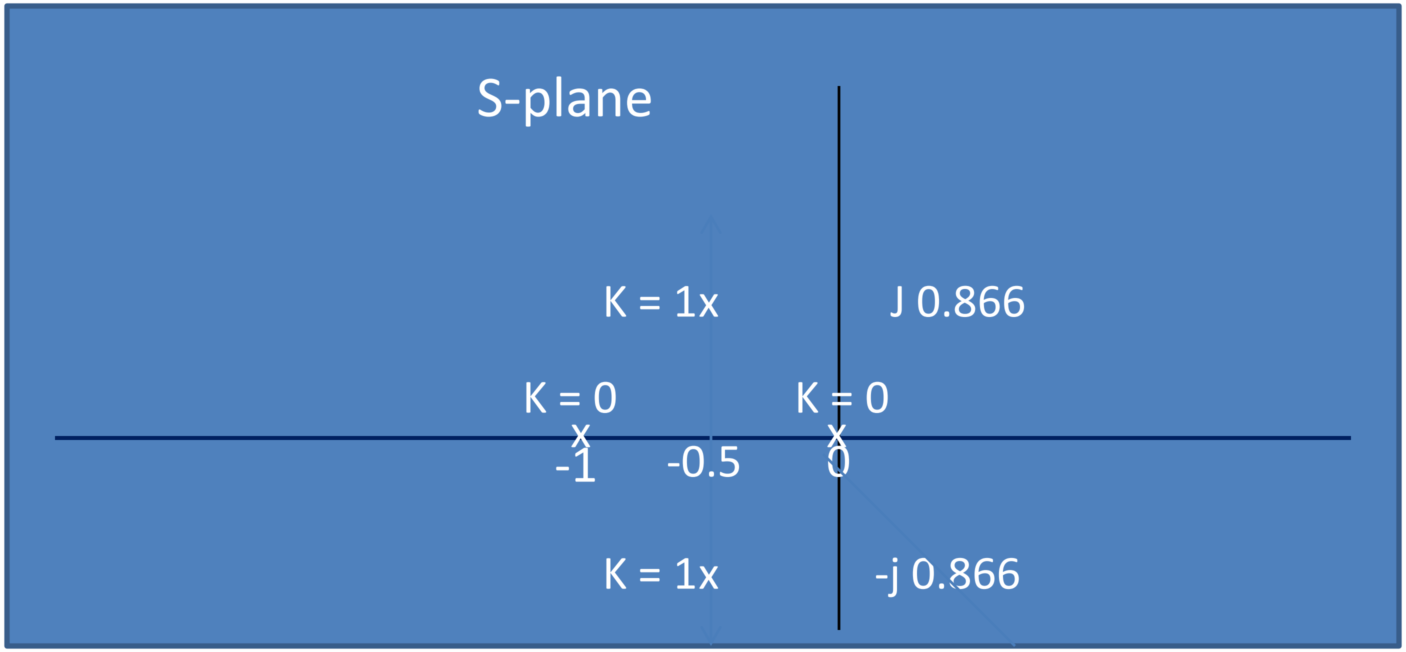
$$Q(S) = 1+G(S) = 0 \Rightarrow 1 + K/S(S+1) = 0$$

$$\Rightarrow S^2 + S + K = 0$$

- For K = 0; the roots of Q(S) are at S=0 & S=-1; which are the poles of the system.

Illustration by Example (Contd)..

- Looking at $Q(S) = S^2 + S + K = 0$ we conclude that,
- As we vary K from '0' to any higher value, the location of the roots of Q(S) will change (shift) in the S-plane.
- Thus the roots will chalk out a locus in the S-plane for a given range of 'K'. This is called Root Locus.



Why Requirement of Root Locus Method ?

- We know that we are interested in finding the roots of a characteristic equation for a range of a parameter of the system which generally is system gain ‘K’. Generally speaking we may be interested in determining the location of closed loop poles for a range of ‘K’

$$0 \leq K \leq \infty$$

- Now it is easy to factorize a second and third order characteristic equation for various values of ‘K’, but for higher order polynomials it is very difficult (near impossible) to factorize for determining their roots.

- Therefore we need a method to do so & that method is Root Locus.

The Method

- Though we are interested in determining the roots of the polynomial (characteristic equation), 1 + G(S) H(S) = 0; we do not start with this equation.
- Instead, we start with only G(S)H(S) or G(S) depending upon whether the closed loop system is non-unity or unity feedback.
- So, we rearrange the characteristic equation as:

$$G(S)H(S) = -1 \text{ (non-unity feedback), or}$$

$$G(S) = -1 \text{ (unity feedback)}$$

- The above rearrangement implies that

$$|G(S)H(S)| = 1 \;\;\&\;\; \arg\{G(S)H(S)\} = \pi$$

The Method (Contd)..

- The equations,

  $|G(S)H(S)| = 1$ & $\arg\{G(S)H(S)\} = л$, imply that

- For any point $S = S_1$ to be a root of the characteristic equation,

  $|G(S_1)H(S_1)| = 1$ & $\arg\{G(S_1)H(S_1)\} = л$ radians or 180 deg.

- Or, for a unity feedback system,

  $|G(S_1)| = 1$ & $\arg\{G(S_1)\} = л$ radians or 180 deg.

- The root locus is drawn on a graph sheet and every point on the locus is obtained by satisfying the angle condition. The value of 'K' for that point is then obtained graphically.

The Method (Contd)..

- Before going ahead with the method, it is necessary to define what is called ‘rational transfer function’.

- A rational transfer function is the one which has equal number of poles and zeros; that is Np = Nz

  Np: number of poles   Nz: number of zeros

- Consider the following transfer functions:

- $G_1(S)\ H_1(S)$ or $G_1(S) = K\ (S+1)/(S+2)$ -------- 1
- $G_2(S) = K\ (S+1)(S+2)/(S+3)(S+4)$ ---- 2
- $G_3(S) = K\ (S+1)/(S+2)(S+3)$ ------- 3
- $G_4(S) = K\ (S+1)/(S+2)(S+3)(S+4)$ ---- 4

The Method (Contd)..

- For, $G_1(S)$ = K (S+1)/(S+2), there is a finite pole at S = -2 & a finite zero at S = -1; Np= Nz = 1; hence it is a rational function
- $G_2(S)$ also has equal number of poles and zeros; Np = Nz = 2;
- $G_3(S)$ has 2 finite poles & 1 finite zero; Np ≠ Nz
- $G_4(S)$ has 3 finite poles and 1 finite zero; Np ≠ Nz
- Does it mean that $G_3(S)$ & $G_4(S)$ are not rational functions!!
- They both are, indeed, rational functions; the need is to find out the location of remaining zeros so that Np = Nz.

The Method (Contd)..

- In order to resolve the issue of 'how many zeros' a transfer function has, we need to understand what is zero of a transfer function.

- Let $G(S) = K (S+1)/(S+2)(S+3)$

- We all understand 'G(S)' as 'frequency dependent gain' offered by the system.

- Now, if we substitute S = -1 in G(S), its value = '0'; it means that gain offered at S= -1 equals '0'. Therefore S = -1 is a zero of the transfer function, G(S)

- Pole of a transfer function is a singularity because gain offered by G(S) at its pole = ∞. For example, S = -2 & -3 causes gain of G(S)= ∞

The Method (Contd)..

- Therefore, we say If the number of zeros are not equal to the number of finite poles of G(S), then number of zeros = Np – Nz shall lie at ∞.
- Let

  $G(S) = K (S+1)/(S+2)(S+3)$
- Lt. $S \longrightarrow \infty$ $G(S) \equiv$ lt. $S \longrightarrow \infty$ $K/S = 0$; the power of S is '1' therefore there is one zero at ∞. Thus we have one finite zero and another zero at ∞. Hence Np = Nz
- For, $G(S) = K (S+1)/(S+2)(S+3)(S+4)$
- we have one finite zero at S = -1 and two zeros at ∞
- Therefore both are rational functions

The Method (Contd)..

- Let

$$G(S)\,H(S) = K \frac{\prod_{j=1}^{m} (S + Zj)}{S^r \prod_{i=1}^{i=n} (S + Pi)}$$

- where, K: gain in the system

  r: number of poles at the origin of S-plane

  n & m: number of poles and zeros in the S-plane

The Method (Contd)..

$$|G(S)H(S)| = K \frac{\prod_{j=1}^{m} |(S + Zj)|}{|S^r| \prod_{i=1}^{i=n} |(S + Pi)|} = 1.0$$

$$K \prod_{j=1}^{j=m} |(S + Zj)| = |S^r| \prod_{i=1}^{I=n} |(S + Pi);$$ for K =0 we get poles of G(S)H(S)

The Method (Contd)..

$$\prod_{j=1}^{j=m} |(S + Zj)| = |S^r| \prod_{i=1}^{i=n} |(S + Pi)/K;$$

- For K $\longrightarrow$ ∞; we get zeros of G(S)H(S)
- We draw root locus for 0 ≤ K ≤ ∞

  Therefore,
- Starting points of root locus are poles of G(S)H(S), K=0
- End points of root locus are zeros of G(S)H(S), K = ∞

The Method (Contd)..The Angle Criteria

- **The Angle Criteria:**

$$G(S)H(S) = K \frac{\prod_{j=1}^{m} (S + Zj)}{\prod_{i=1}^{n} (S + Pi)}$$

The angle criteria is in degrees given by:

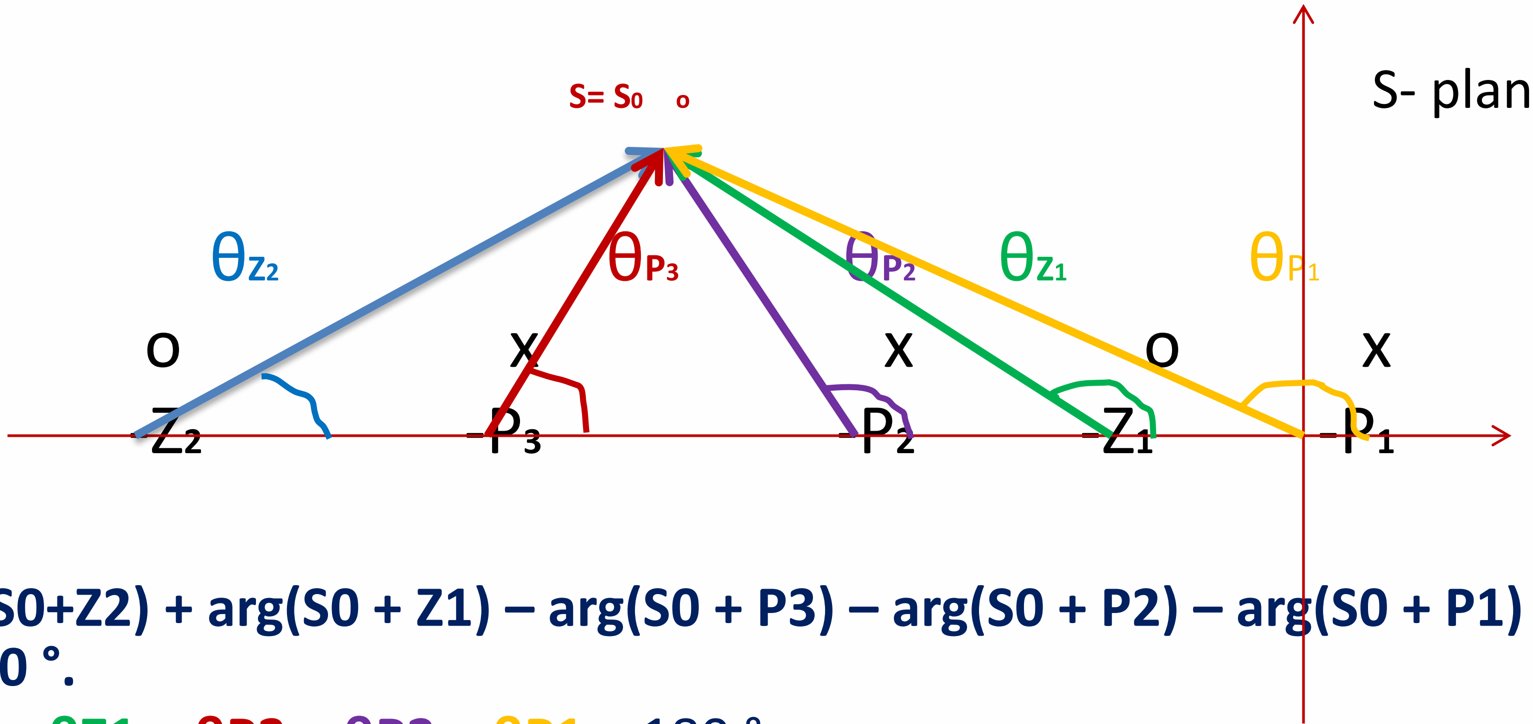
$$\sum_{j=1}^{m} \arg(S + Zj) - \sum_{i=1}^{n} \arg(S + Pj) = +/- (2q + 1)180; \quad q = 0,1,2,....$$

Implement Angle Criteria

- Since root locus is drawn satisfying angle criteria, now we explain how it is done.

1. Plot location of poles & zeros of G(S)H(S) in the S-plane
2. Choose any point S = S**0** in the S-plane.
3. From each pole & zero draw vectors to the chosen point, S**0**
4. Measure the angle subtended by each pole & zero at S**0**, in the CCW direction.
5. Remember that angle subtended by a pole is negative & that by a zero is positive
6. Algebraically add all the angles. If they sum up to 180 degrees, then S = S**0** is a point on the root locus.

❑ Graphical Illustration for Angle Criteria:



arg(S0+Z2) + arg(S0 + Z1) – arg(S0 + P3) – arg(S0 + P2) – arg(S0 + P1) = +/_ 180 °.

θZ2 + **θZ1** – **θP3** – **θP2** – **θP1** = 180 °

❑ If the above angle condition is satisfied then **S0** is on the locus.

Magnitude Criteria

- From the magnitude criteria, we calculate the value of gain 'K' at the point S = S0 which lies on the root locus (that is S=S0 satisfies angle criteria).

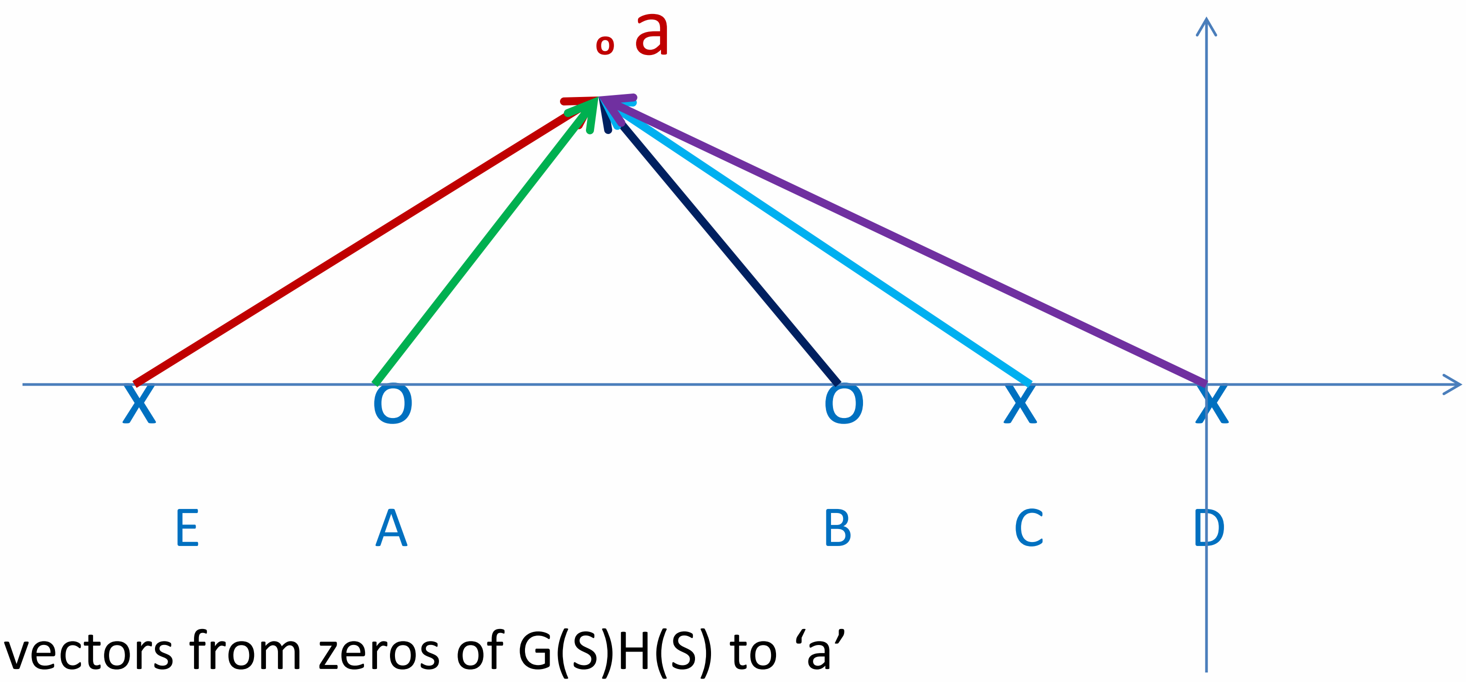
$$K \frac{\prod_{j=1}^{m} |(S_o + Z_j)|}{\prod_{i=1}^{n} |(S_o + P_i)|} = 1 \quad \text{or,} \quad K = \frac{\prod_{i=1}^{n} |(S_o + P_i)|}{\prod_{j=1}^{m} |(S_o + Z_j)|}$$

- K = product of vector lengths from poles of G(S)H(S) to S**0**/product of vector lengths from zeros of G(S)H(S) to S**0**.

Graphical Implementation of Magnitude Criteria

❑ Graphical method for determination of ‘K’:

Ea Ca Da : vectors from poles of G(S)H(S) to point ‘a’: S = S**0**



Aa Ba : vectors from zeros of G(S)H(S) to ‘a’

Gain K = (Ea)(Ca)(Da)/(Aa)(Ba)

We measure vector lengths, as per scale, and then calculate K

Construction Rules for Root Locus

- **Rule 1:**

  Root Locus is symmetrical about real axis of S-plane, because roots are either real or complex conjugate.

- **Rule 2:**

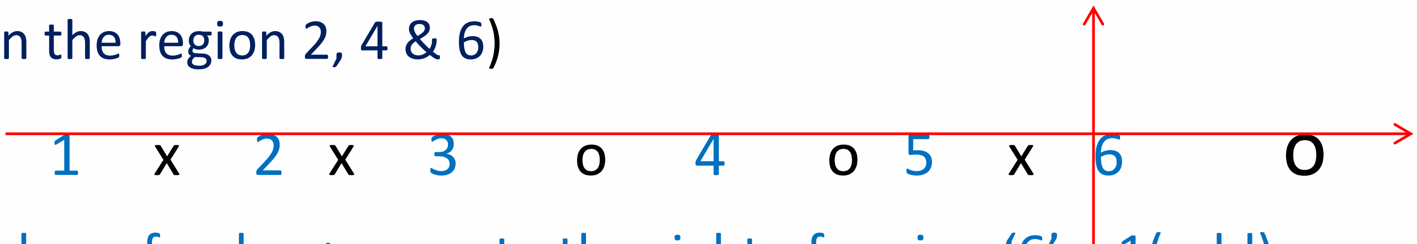
  As ‘K’ increases from ‘0’ to ‘∞’, the open loop poles of G(S)H(S) move (branch out) towards the zeros of G(S)H(S); some of the zeros may be at ‘∞’.

  The number of branches terminating on ‘∞’ equals Np – Nz; that is the difference between number of finite poles & zeros of G(S)H(S).

Construction Rules for Root Locus

Rule 3:

A point S = S0 on the real axis shall lie on the root locus iff the total number of open loop poles & zeros of G(S)H(S) to the right of S0 is odd. (Loci lie in the region 2, 4 & 6)



- The number of poles + zeros to the right of region ‘6’ = 1(odd)
- The number of poles + zeros to the right of region ‘5’ = 2(even)
- The number of poles + zeros to the right of region ‘4’ = 3(odd)
- The number of poles + zeros to the right of region ‘3’ = 4(even)
- The number of poles + zeros to the right of region ‘2’ = 5(odd)
- The number of poles + zeros to the right of region ‘1’ = 6(even)

Construction Rules for Root Locus

❑ Rule 3 (contd)..

❑ The poles are $K = 0$ points & the zeros are $K = \infty$ points. As we are interested in the range of K, $0 \leq K \leq \infty$, therefore the poles will start moving towards their respective zeros, in the region on the real axis, and terminate at zeros ($K = \infty$)

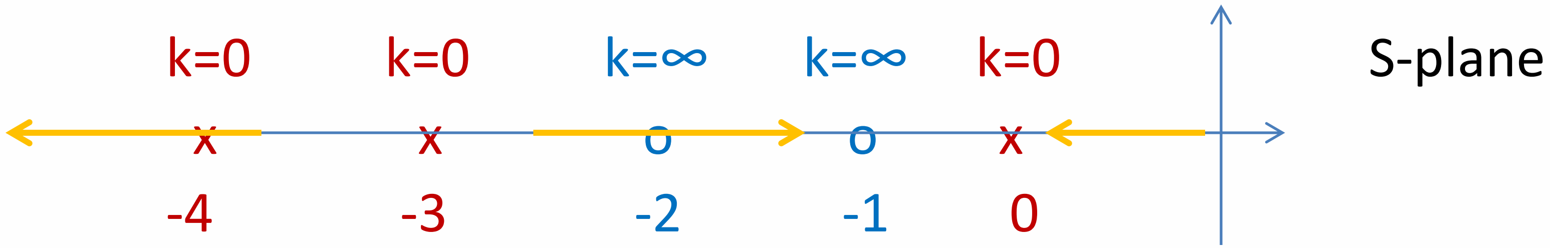
❑ Therefore, we can say that the loci of closed loop poles start at $K = 0$ (the location of the poles of G(S)H(S)) and terminate at $K = \infty$ (the location of the zeros of G(S)H(S))

Construction Rules for Root Locus

❑ Rule 3 (contd): Example for implementation

Let G(S)H(S) = K(S+1)(S+2)/s(S+3)(S+4)

1. Draw pole zero locations in the S-plane
2. Use angle criteria to mark the regions on the real axis of the S-plane where the root loci shall lie



❑ The regions where the loci shall lie are highlighted in yellow where the total angle subtended by poles & zeros = 180°

Construction rules for Root Locus

❑ Rule 3 (contd): Example for implementation

In the considered example:

1. No. of open loop poles = 3; root loci branches = 3 because each pole is a starting point.
2. Root Loci will start from S =0, -3 & -4 (K = 0 points)
3. As K increases, the loci moves from the poles to respective zeros (K = ∞ points)
4. The arrows show the direction of movement of poles
5. Np = 3 Nz = 2; no. of poles for which the loci shall terminate at ∞ = Np – Nz = 1
6. We observe that pole at S = -4 terminates at ∞

Construction rules for Root Locus

❑ Rule 4: (Angle of Asymptotes)

The ($N_p - N_z$) branches of the root locus asymptotically tend to ∞. The angles of asymptotes are given by:

$$\phi_{\mathbf{q}} = (2q+1)\, 180°/(N_p - N_z); \quad q = 0,1,2, \ldots, (N_p - N_z - 1)$$

1. $G(S) = K\,(S+1)(S+2)/S(S+3)(S+4)$

Np = no. of poles = 3; Nz = no. of zeros = 2; Np-Nz = 1

$q = 0;\ \phi = 180°$

2. $G(S) = K(S+2)/(S+1)(S+3)(S+5)(S+6)$

Np = no. of poles = 4; Nz = no. of zeros = 1; Np-Nz = 3

$q = 0,1,2;\ \phi_{\mathbf{0}} = 60°,\ \phi_{\mathbf{1}} = 180°,\ \phi_{\mathbf{2}} = 300°$

Construction rules for Root Locus

❑ Rule 5: (Centroid)

If no. of asymptotes are more than 1, they cross the real axis of the S-plane. Their point of intersection on the real axis is known as Centroid.

Centroid **σA** is given by:

$$\sigma_A = \frac{\sum_{i=1}^{n} P_i - \sum_{j=1}^{m} Z_j}{(N_p - N_z)} = \frac{(\text{Sum of real parts of poles} - \text{Sum of real parts of zeros})}{(\text{No. of poles} - \text{No. of zeros})}$$

Construction rules for Root Locus An Example

- **Example:**

  Determine 1) no. of loci on the real axis and their regions, 2) no. of asymptotes, 3) angle of asymptotes, 4) Centroid for a unity feedback system whose open-loop transfer function is given as: G(S) = K/S(S+1)(S+2)

- Solution Steps:

- Draw pole zero locations in the S-plane
- Determine no. of finite poles, Np, and zeros, Nz & Np-Nz
- Mark regions on the real axis where loci lie
- Find no. of asymptotes = Np – Nz & their respective angles
- If (Np-Nz) > 1 determine value of centroid
- Sketch root loci (free hand)

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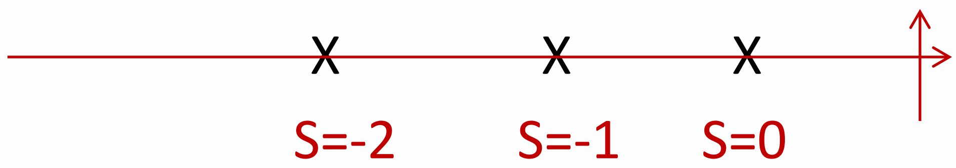
Construction rules for Root Locus An Example

1.



(poles are K=0 pts.)

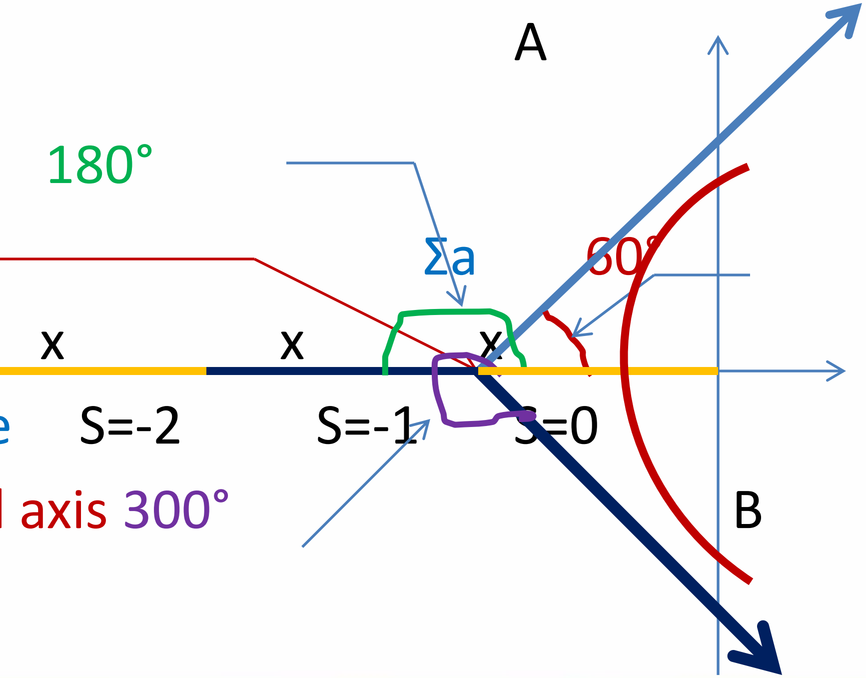
- Np = 3 Nz = 0 (no finite zero ; therefore all zeros at ∞)
- Np-Nz = 3
- Loci on the real axis will lie between S= 0 & S= -1; it will also lie in the region after S = -2 because total no. of poles & zeros to the right of the regions = odd.
- No. of asymptotes = Np-Nz = 3 & angles of asymptotes are given by $\phi_q = (2q+1)\,180°/(N_p - N_z)$; q = 0,1,2; $\phi_0 = 60°$, $\phi_1 = 180°$, $\phi_2 = 300°$
- Since (Np-Nz)>1 = 3 we will determine Centroid

Construction rules for Root Locus An Example

- **Centroid is given by:**

$$\sigma A = \frac{\text{(sum of real parts of poles – sum of real parts of zeros)}}{\text{(no. of finite poles – no. of finite zeros)}}$$

σA = {(0-1-2) – (0)}/(3-0) = -1.0



red loci is the loci in complex plane

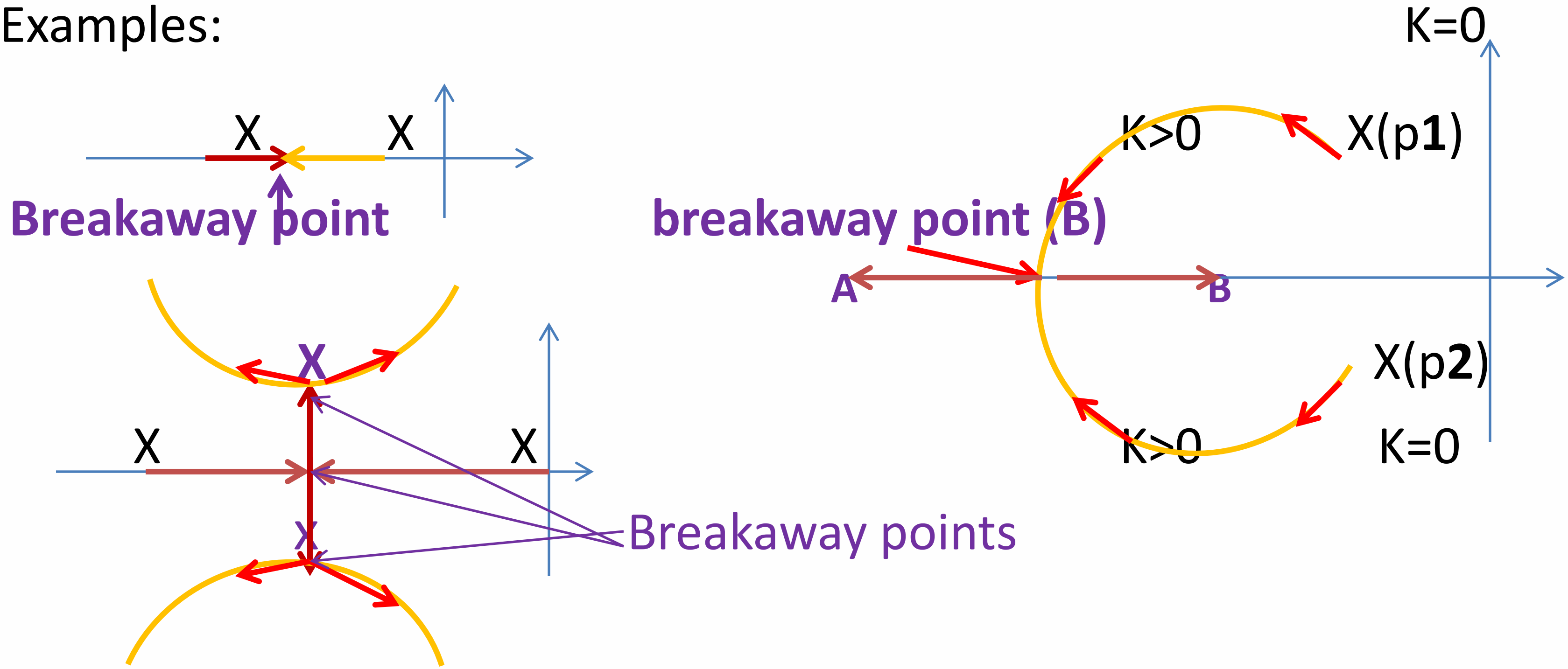
in yellow regions loci lie on the real axis 300°

Construction Rules for Root Locus (Breakaway points)

❑ Breakaway Points:

Multiple roots of the characteristic equation occur at these points. These are obtained using the formula dK / dS = 0. These points also satisfy the angle criteria.

Examples:



Construction Rules for Root Locus (Breakaway points Example)

- Example: Calculation for Breakaway points

$G(S) = K/S(S+1)(S+2)$

$1 + G(S)H(S) = 0 \Rightarrow K/S(S+1)(S+2) = -1$

$\Rightarrow K = -(S^3 + 3\,S^2 + 2S)$

$\Rightarrow dK/dS = -(3\,S^2 + 6S + 2) = 0$

We find the roots of the polynomial

$$3\,S^2 + 6S + 2 = 0$$

We get $S1 = -0.423$ & $S2 = -1.577$

We know that for the given G(S), the loci on the real axis will lie between ‘0’ & ‘-1’; therefore the breakaway point is = -0.423. S2 = -1.577 is not a breakaway point because between S=-1 & -2 no loci exists on the real axis of the S-plane.

Construction Rules for Root Locus (Breakaway points Example)

- **Example:**

  $G(S)H(S) = K/S(S+4)(S^2 + 4S + 20) = K/S(S+4)(S+2+j4)(S+2-j4)$

  To determine the breakaway points: $dK/dS = 0$. Substitute in $1+G(S)H(S) = 0$ to get $K = -S(S+4)(S^2 + 4S + 20)$

  $$dK/dS = S^3 + 6 S^2 + 18S + 20 = 0$$

  Factorize $dK/dS=0$, we get $S = -2$; $S = -2 \pm j\ 2.45$

- Now we find out that out of the roots of $dK/dS = 0$ which qualify to be breakaway points. To do this, we first draw the pole – zero locations of $G(S)H(S)$ in the S-plane

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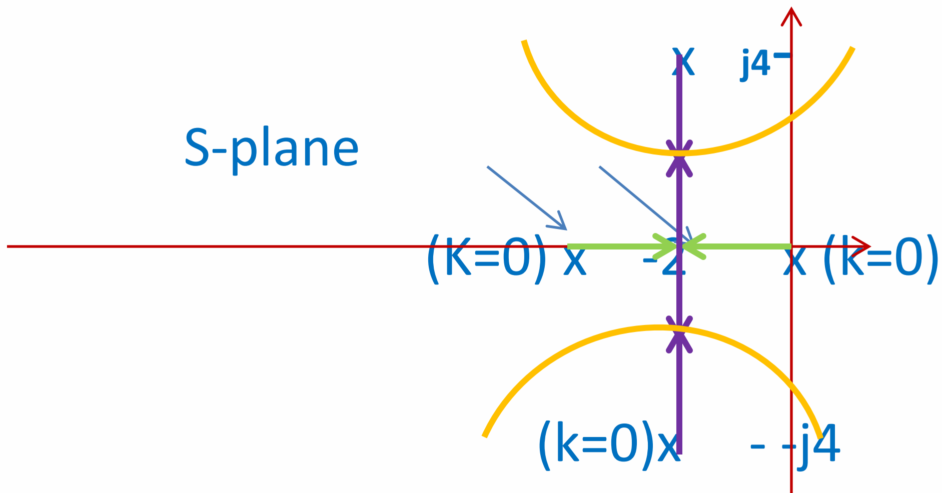
Construction Rules for Root Locus (Breakaway points Example)

❑ Example (contd):

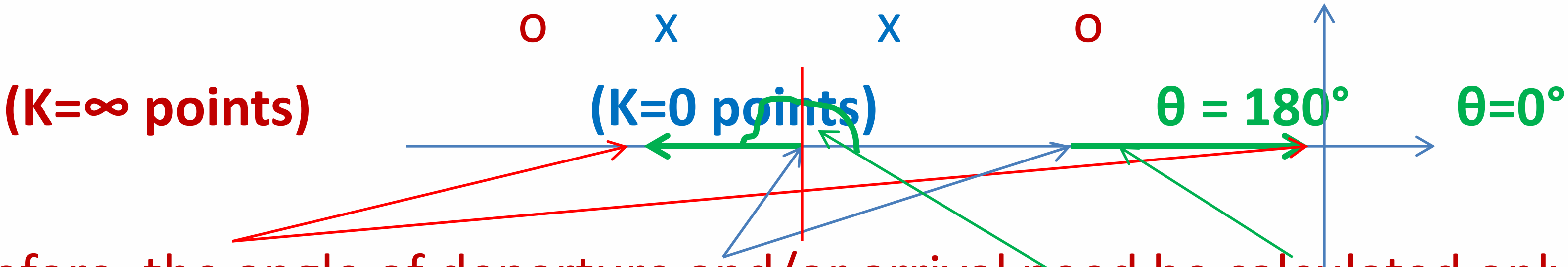


- ❑ Having plotted the location of poles, we know that the root locus on the real axis will lie between S = 0 (K=0) & S=-4(K=0).
- ❑ Now, one root of dK/dS = 0 lies at S = -2; therefore S=-2 is a breakaway point. Since, -2 is also real part of the complex pole (-2 +/- j4), therefore S= -2 +/- j2.45 (root of dK/dS =0) is also a breakaway point.

Construction Rules for Root Locus (Angle of Departure/ Arrival)

❑ Angle of Departure/Arrival:

For poles on the real axis: (either 0° or 180°)



Therefore, the angle of departure and/or arrival need be calculated only for complex poles & zeros.

Method:

1. choose a point S**0** very close to the pole ‘p’
2. Graphically determine the angle contributions due to other poles & zeros at the point S**0**.
3. determine angle of departure θ**p** from the pole ‘p’.

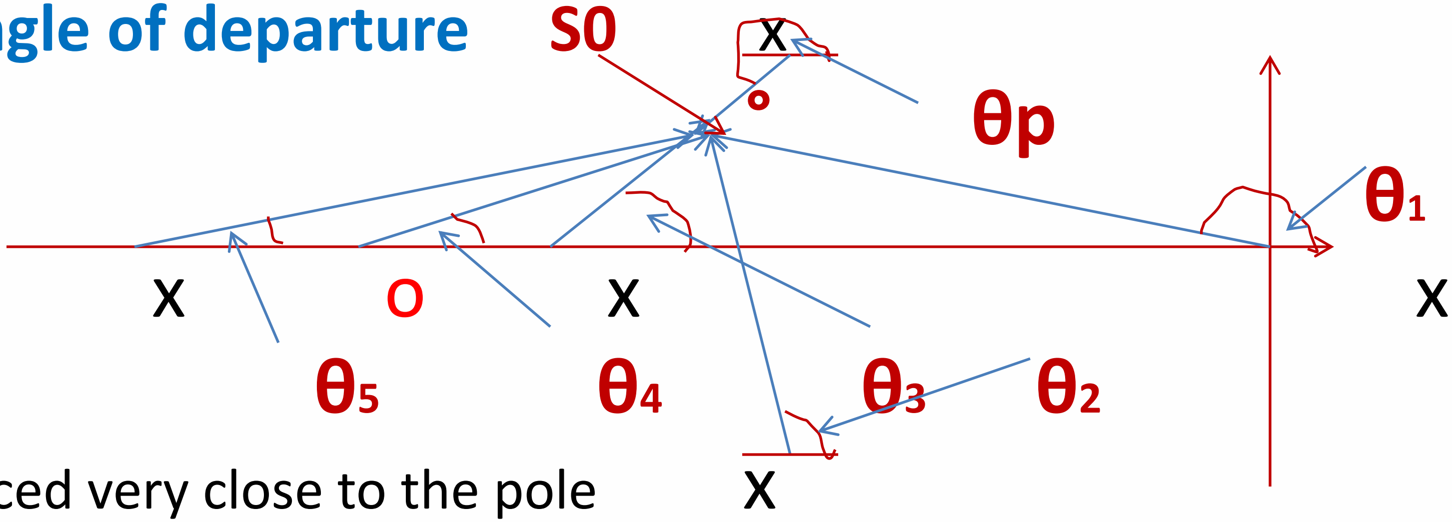
Construction Rules for Root Locus (Angle of Departure/ Arrival)

- Draw the pole-zero locations of G(S)H(S)
- Draw a point S**0** in the S-plane very close to the pole/zero for which departure angle is to be determined.
- Draw vectors to S**0** from each pole & zero of G(S)H(S).
- Calculate total angle, ϕ, subtended at S**0**.
- Angle of departure/arrival is given by $\phi - \theta p$/ $\phi + \theta z = (2q + 1)\ 180°$, or we have $\theta p = +/- (2q + 1)\ 180° + \phi$;

  $\theta z = +/- (2q+1)180° - \phi$
- $\theta p/\theta z$:the angle of departure/arrival for the pole/zero; θp is subtracted from ϕ because it is angle subtended by a pole.

Construction Rules for Root Locus (Determining Angle of Departure)

- θp: angle of departure



S_0 is placed very close to the pole for which angle of departure is to be calculated. For the sake of clarity, here, it is shown some distance from the pole.

- Angle subtended by other poles & zeros at S0, ϕ, is given by:

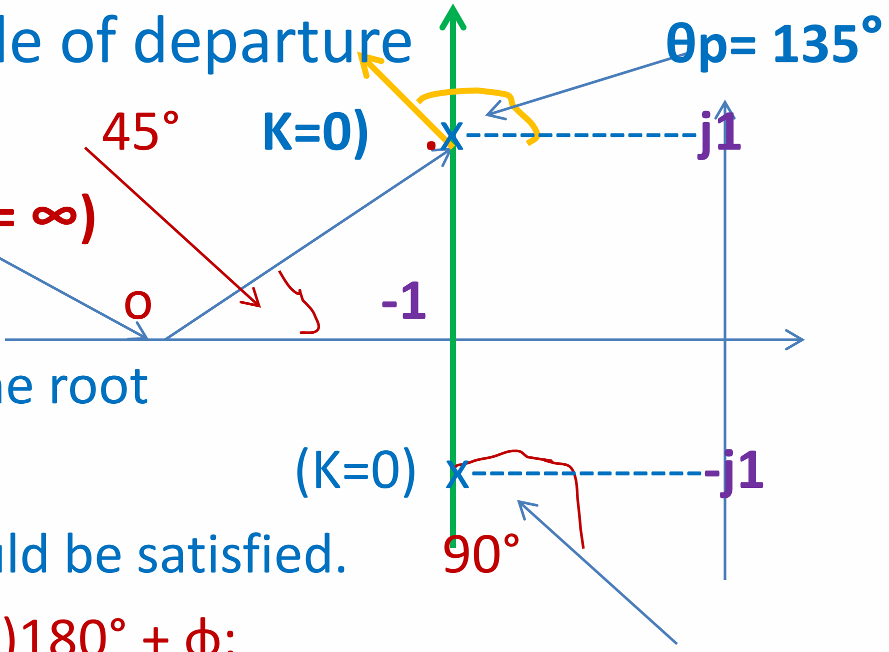
$$\phi = \theta 4 - (\theta 1 + \theta 2 + \theta 3 + \theta 5)$$

$\phi - \theta p = +/- (2q+1)180°$; $q = 0, 1, 2, \ldots$; $\theta p = +/-(2q+1)180° + \phi$

- Angle of arrival at a zero is calculated in a similar way.

Construction Rules for Root Locus (Example: Angle of Departure)

- Example: Calculation of angle of departure



Poles are at -1 +/- j1

Zero at S= -2

The K = 0 points are also points on the root locus; therefore at open loop pole location too, the angle criterion should be satisfied.

The angle $\phi = (45° - 90°)$; $\theta p = (2q+1)180° + \phi$;

$\theta p = 180° + (45° - 90°) = 135°$ is the angle of departure

❑ Example: Angle of Arrival (at zero located at -1+j1)

$\tan \theta\mathbf{1} = ½ = 0.5$

$\theta\mathbf{1} = 26.56°$

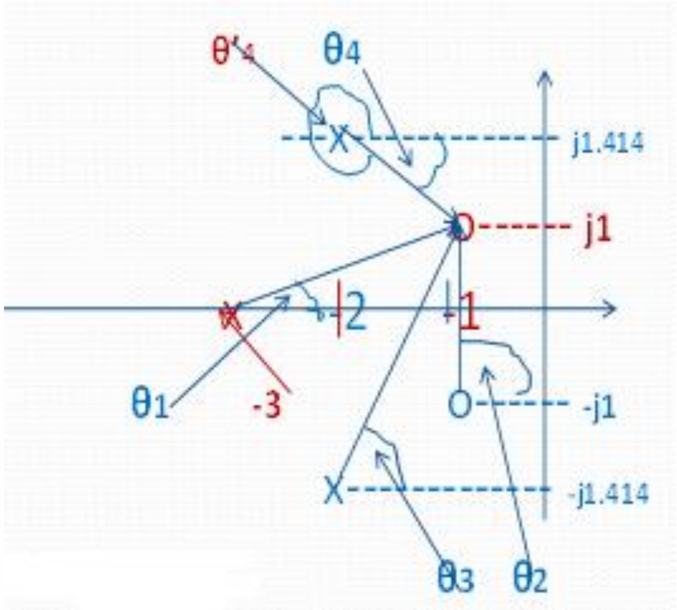
$\tan \theta\mathbf{3} = 2.414/1$

$\theta\mathbf{3} = 67.49°$

$\theta\mathbf{2} = 90°$

$\tan \theta\mathbf{4} = -0.414/1$

$\theta\mathbf{4} = -22.49°$ $\theta'4 = 360-22.49= 337.5°$

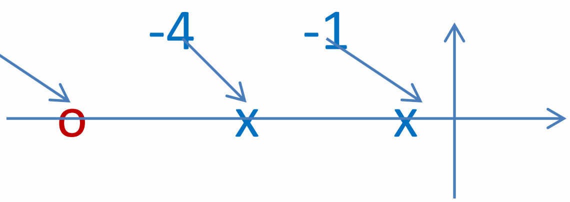


The total angle,ϕ, subtended at the zero= $\theta\mathbf{2} - \theta\mathbf{3} - \theta\mathbf{1} + \theta\mathbf{4} = 18.44°$.

Therefore angle of arrival $\theta z = 180° - \phi = 161.6°$

Graphical determination of 'K' for specified damping ratio

❑ Example:

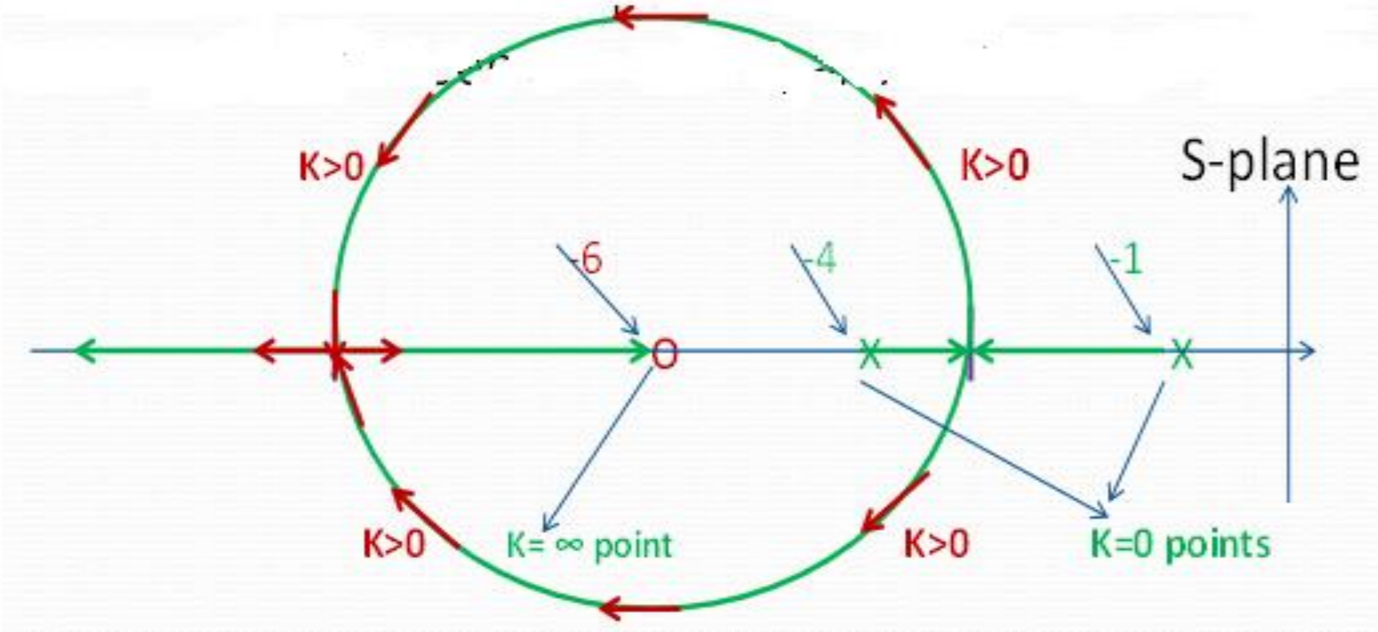


$$G(S) = K\,(S+6)/(S+1)(S+4)$$

1. K=0 points: S = -1 & S= -4 are poles of G(S)
2. K = ∞ points: S = -6 are zeros of G(S)
3. Loci on the real axis lies between S = -1 & S= -4; and between S = -6 & ∞
4. Since one zero is at ∞, therefore one closed loop pole will approach this zero asymptotically
5. Angle of asymptote: $\phi = 180°(2q+1)/Np\text{-}Nz = 180°$; q = 0
6. Since there is only one asymptote, there is no centroid

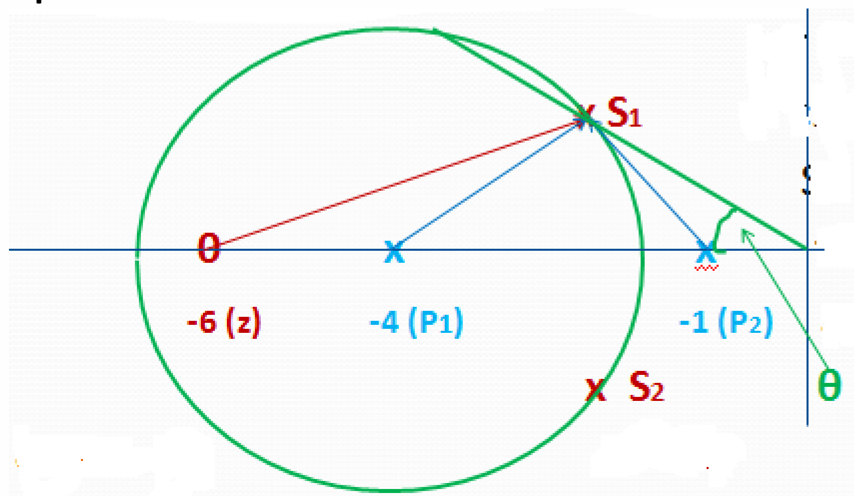
Graphical determination of 'K' for specified damping ratio (contd)..

- Breakaway points: $1 + G(S) = 0$; $1 + K (S+6)/(S+1)(S+4) = 0$; therefore, $K = - (S+1)(S+4)/(S+6)$
- $dK/dS = 0$; ⇒ $S^2 + 12 S + 26 = 0$ ⇒ $S\mathbf{1} = -9.16$, $S\mathbf{2} = -2.84$
- Both S1 & S2 are breakaway points because the root loci on the real axis lies between S = -1 & -4; and between S = -6 & ∞



Graphical determination of 'K' for specified damping ratio (contd)..

- Let us fix the location of closed poles at S1 & S2. Now we want to find K which yields S1 & S2. Let



S1 = -2 + j 1.5

ξ = Cos(θ)

- Draw vectors from each pole & zero of G(S) to S**1** or S**2** as shown.
- Then **K = product of the length of vectors from poles/ product of length of vectors from zeros**

$K = |S1 + 4|\,|S1 + 1| / |S1 + 6|$ **=** $|-2+j1.5+4|\,|-2+j1.5+1| / |-2+j1.5+6| = 1.05$

ξ = Cos(45°) = 0.707

Effect of adding Zeros on Stability of a Closed loop system

- $G(S) = K/(S+1)(S+2)(S+3)$
- The root loci is obtained as: As the root loci cross in to RH of S-plane,
- The closed loop system becomes unstable for a value of $K>K_1$. Let us now add a zero.
- Let us now add a zero at S= -4 the loci will be
- We observe that addition of a zero has stabilized the closed loop system for all values of K; $0 \leq K \leq \infty$

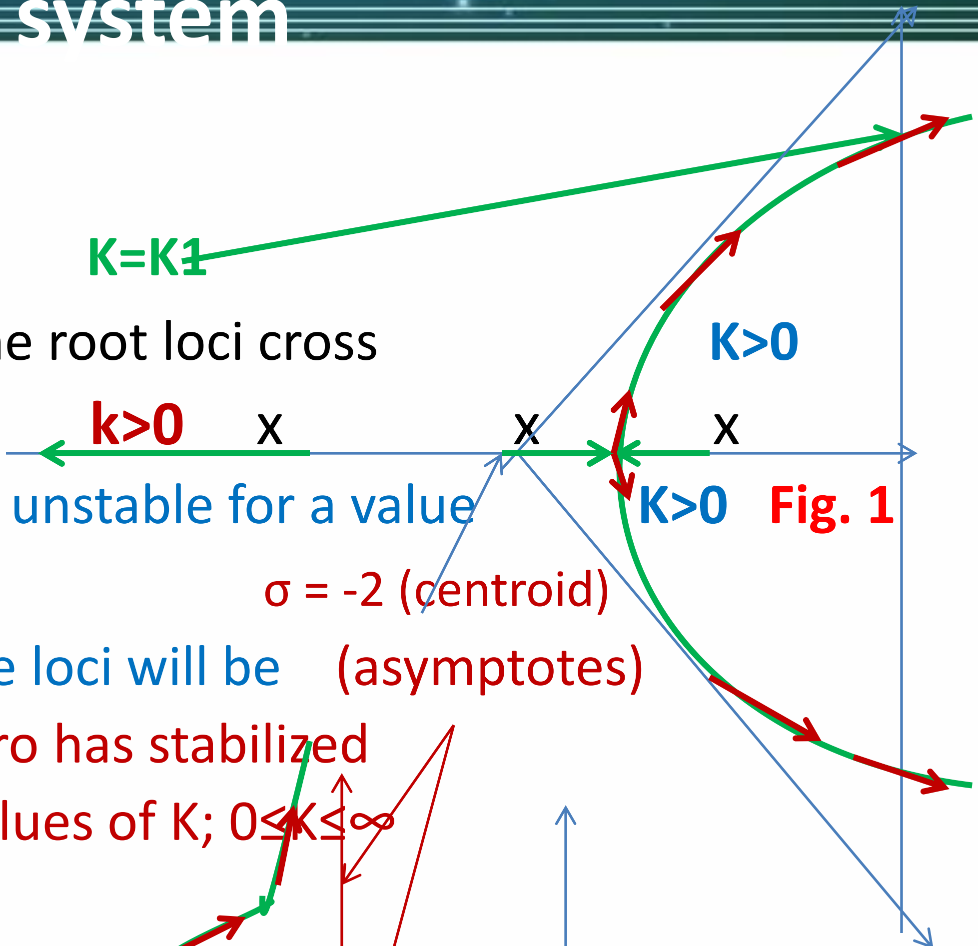
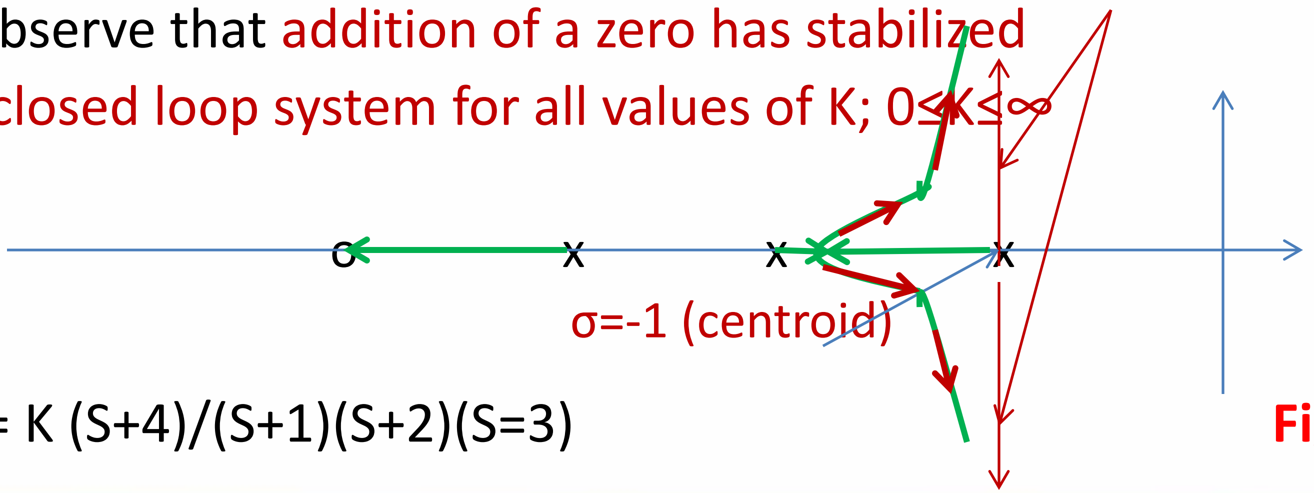


Fig. 1



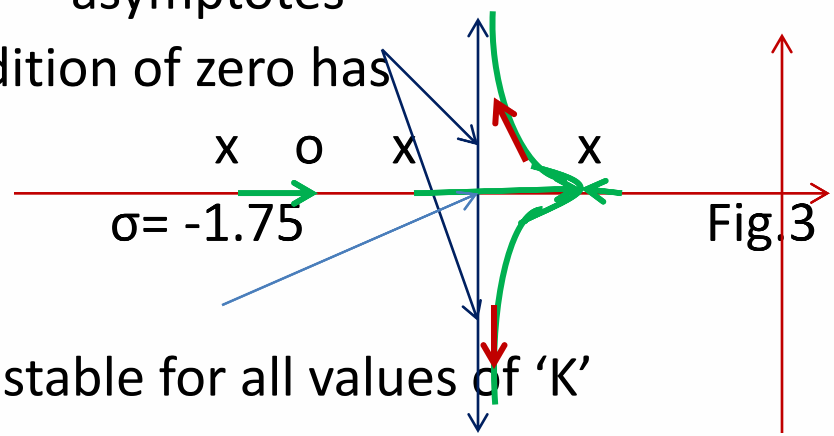
$G(S) = K(S+4)/(S+1)(S+2)(S=3)$

Fig. 2

Effect of adding Zeros on Stability of a Closed loop system (contd)..

- Let us now add a zero at S = -2.5 G(S) = K (S+2.5)/(S+1)(S+2)(S+3)

$\sigma = -1.75$



Looking at Figs. 1, 2 & 3 we see that addition of zero has

1. Reduced no. of asymptotes thereby preventing the locus from moving in to RH of the S-plane.
2. Therefore the CL system has become stable for all values of ‘K’
3. The location of zero also affects the locus.
4. Shifting zero location from S= -4 to -2.5 has moved centroid from -1 to -1.75 thereby shifting the starting point of asymptotes to further away from the Imaginary axis of the S-plane. In Fig.2 the breakaway point is to the left of σ; in Fig.3 it is to the right of σ.
5. Thus the system has become relatively more stable

Effect of adding Poles on Stability of a Closed loop system

- Adding a pole:

G(S) = K/(S=1)(S+2)

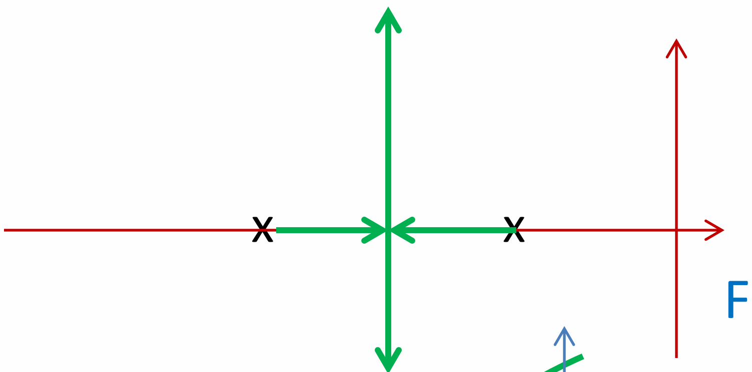


Fig. 1

G(S) = K/(S+1)(S+2)(S+3)

- We observe that addition of a pole affects stability of a CL system, as is seen from Fig.1 & 2

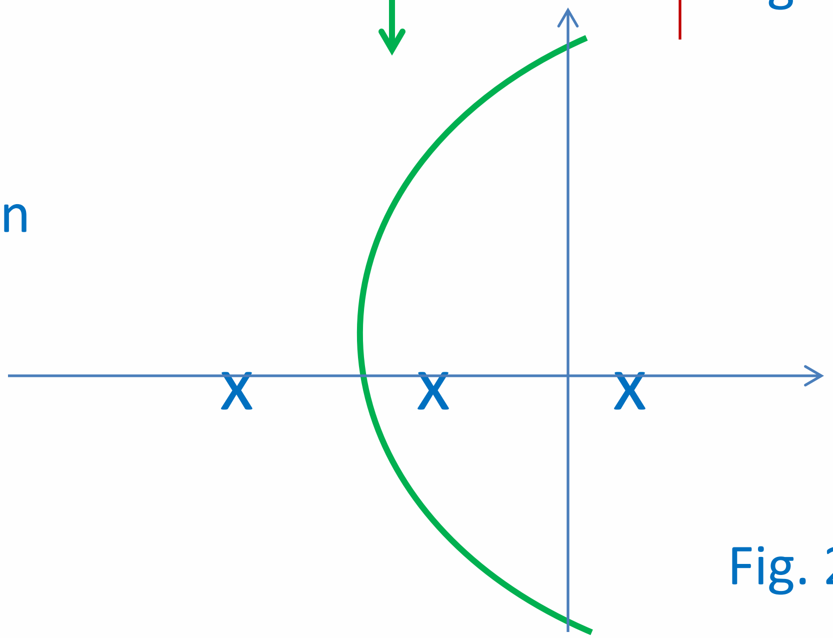


Fig. 2

Root Locus Problems

❑ Problem1:

For $G(S) = K(S + b)/S(S + a)$ & $H(S) = 1$ show that the loci of the complex roots are part of a circle with

center at $(-b,0)$,and

radius $= \sqrt{(b^2 - ab)}$

❑ Solution:

The angle criterion: $\arg\{(S + b)/S(S + a)\} = +/- 180°$

At, $S = \sigma + j\omega$ we have : $\arg\{(\sigma + j\omega + b)/(\sigma + j\omega)(\sigma + j\omega + a)\}$

or, $\tan^{-1}(\omega/\sigma + b) - \tan^{-1}(\omega/\sigma) - \tan^{-1}(\omega/\sigma + a) = -\pi$

$\tan^{-1}(\omega/\sigma) + \tan^{-1}(\omega/\sigma + a) = \pi + \tan^{-1}(\omega/\sigma + b)$

Take tan on both sides & simplify, to get

$(\sigma + b)(2\sigma + a) = \sigma(\sigma + a) - \omega^2$

$\sigma^2 + \omega^2 + 2b\sigma + ab = 0$

Root Locus Problems (contd)..

- Add & subtract b^2 term to get

  $(\sigma^2 + 2b\sigma + b^2) - b^2 + \omega^2 + ab = 0$

  $(\sigma + b)^2 + \omega^2 = b^2 - ab$ is the equation of the circle with

  center at (-b,0) & radius = $\sqrt{(b^2 - ab)}$

  For b = 1 & a = -1

  center = (-1,0) & radius = $\sqrt{2}$

- Problem 2:

  $H(S) = 1 \quad G(S) = 1/S(S + \alpha)$

Draw root locus as α varies between $0 \le \alpha \le \infty$

Solution:

'α' appears in the denominator polynomial of G(S). 'K' always appeared in the numerator of G(S). Therefore we manipulate to get 'α' in the numerator.

The Characteristic equation $Q(S) = 1 + G(S)H(S) = 0$

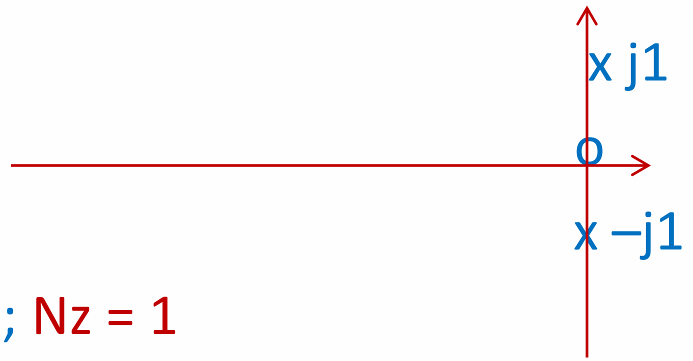
Root Locus Problems (contd)..

- ❑ $Q(S) = S^2 + \alpha S + 1 = 0$

From Q(S), we rewrite G(S) in a way that ‘α’ appears in the numerator

Therefore, we write

$$G(S) = \alpha S / S^2 + 1$$



The root locus for parameter ‘α’:

1. $\alpha = 0$ points: $S_1 = +j1$ & $S_2 = -j1$; $N_p = 2$
2. $\alpha = \infty$ points: $S = 0$; (another zero at ∞); $N_z = 1$
3. $N_p – N_z = 1$; No. of loci = 2
4. Locus on the real axis covers entire axis in the LH of S-plane
5. No. of asymptotes = 1
6. No Centroid (because only one asymptote)
7. Angle of asymptote (for $q = 0$) = 180°

Root Locus Problems (contd)..

- Breakaway point:

  $\alpha S/ S^2 + 1 = -1$; $\alpha = -(S^2 + 1)/S$; $d\alpha/dS = 0 \longrightarrow S^2 - 1 = 0$; $S = +/- 1$

  The breakaway point is S = -1 because it is a point on the loci

- Angle of Departure: (from pole at S = j1)

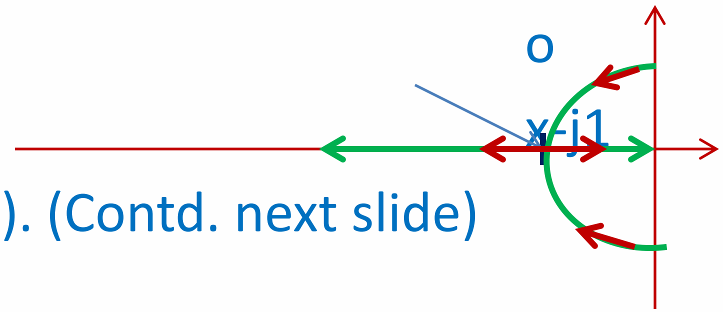
  Angle subtended at S= j1 by zero at S=0 is 90°

  Angle subtended at S = j1 by pole at S= -j1 = 90°

  Total angle subtended, $\phi = 90 - 90 = 0°$

  Angle of departure $\theta p = 180° + \phi = 180°$

- The Root Loci:



- It is a circle with radius = 1 & center (0,0). (Contd. next slide)

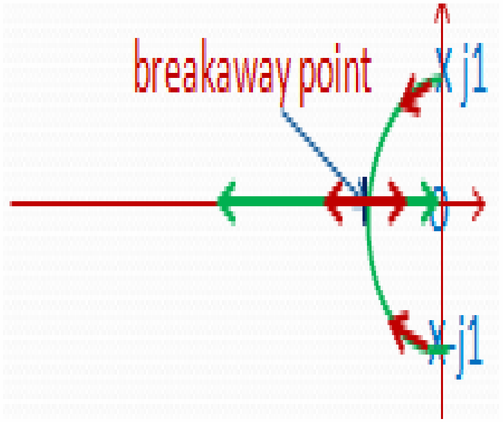
Root Locus Problems (contd)..

- Let us fix the location of closed loop poles for damping ratio $\xi = 0.5$ & determine time domain parameters. We redraw the locus.

  $\xi = \cos(\theta) = 0.5$; $\theta = 60°$. Draw a line at 60° from –ive real axis as shown.

  The intersections A & B on the locus define the location of the closed loop system.

  Since the locus is a circle with unity radius, the vector OA = 1 & therefore $\omega_n = 1$ rads/sec.

  $-\xi\omega_n = -0.5$; $\omega_d = \omega_n \sqrt{(1-\xi^2)} = 0.866$ rads/sec

- The CL poles are $-\xi\omega_n \pm j\,\omega_d = -0.5 \pm j\,0.866$
- The Characteristic equation is $(S + 0.5 + j\,0.866)(S + 0.5 - j\,0.866) = S^2 + S + 1 = 0$

  The derived Ch. Eq. is : $S^2 + \alpha S + 1 = 0$

  On comparing we get $\alpha = 1$.

Root Locus Problems (contd)..

❑ Problem 3:

Suppose that the Characteristic equation is given as:

$Q(S) = S^3 + K\,S^2 + 2S + 1 = 0$

You are asked to draw root locus for $0 \le K \le \infty$. How to draw?

Solution:

1. Collect all the terms containing ‘K’.
2. Divide terms containing ‘K’ by the balance terms
3. Write $Q(S) = 1 + N'(S)/D'(S)=0$
4. Write $G(S) = N'(s)/D'(S)$
5. Plot root locus
6. In the present case: $Q(S) = 1 + K\,S^2/\,S^3 + 2S +1 = 0$
7. $G(S) = K\,S^2/S^3 + 2S + 1$; Factorize denominator polynomial

PROBLEM: Construction of Root Locus

Draw the root locus for the open loop transfer function G(s) and settling time ts=4sec given, find the range of values of k and show that the loci of the complex roots are part of a circle with (-1,0) as centre and radius $=\sqrt{2}$

where $$G(s) = \frac{k(s+1)}{s(s-1)}$$

Step-1: The first step in constructing a root-locus plot is to locate the open-loop poles and zeros in s-plane.

- The k=0 points:

  s=0, s= 1

  no. of poles (n)= 2

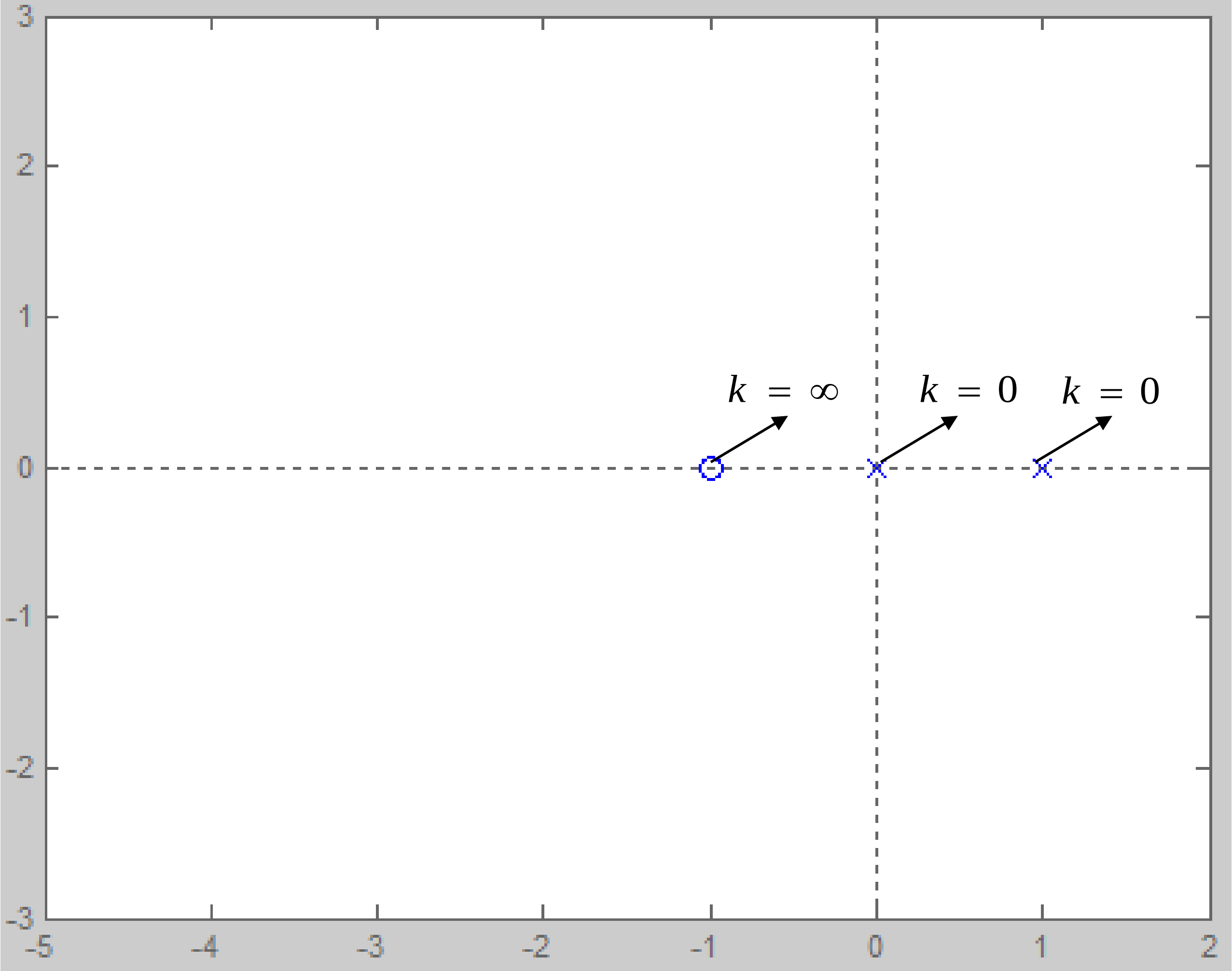
- The k=∞ points:

  s= -1

  no. of zeros (m)= 1

PROBLEM: Construction of Root Locus (contd)..

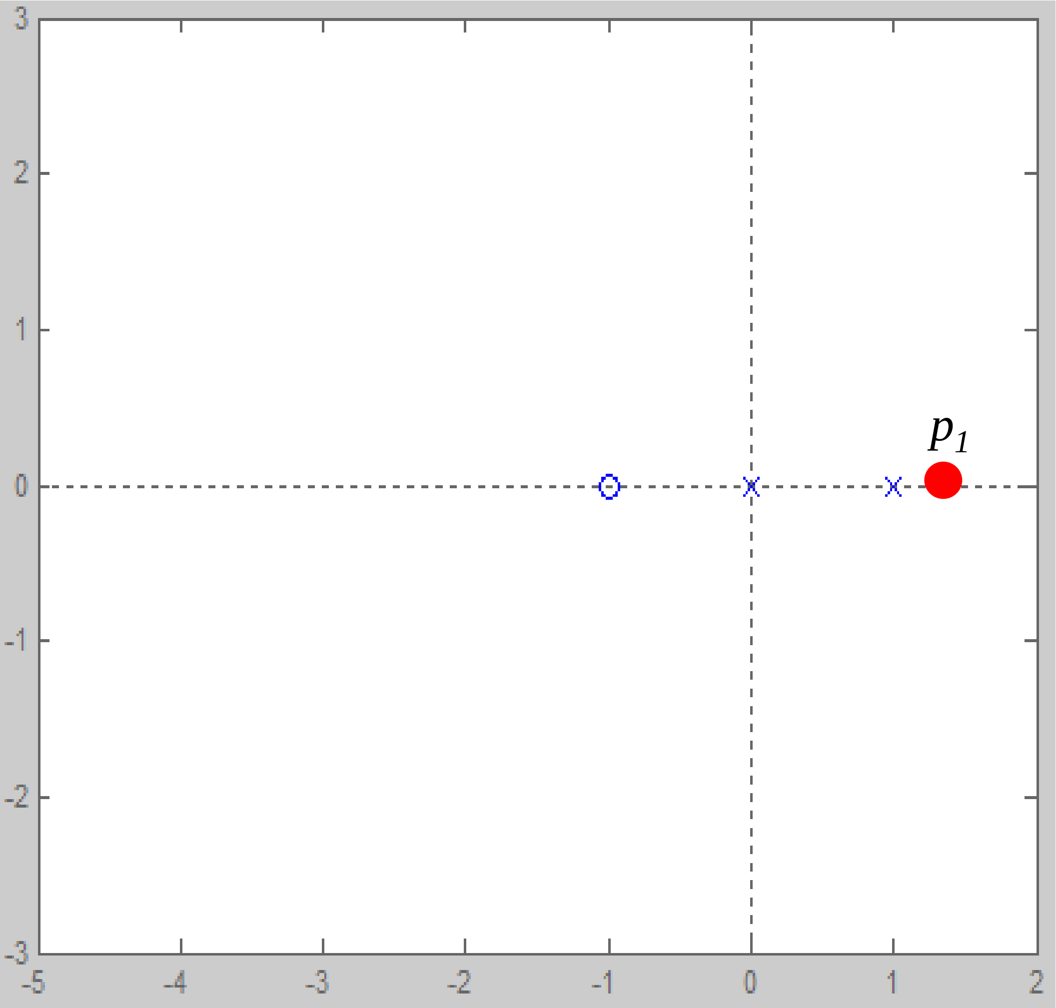
The poles and zeros in s-plane after step-1.



PROBLEM: Construction of Root Locus (contd)..

Step-2: Determine the root loci on the real axis.

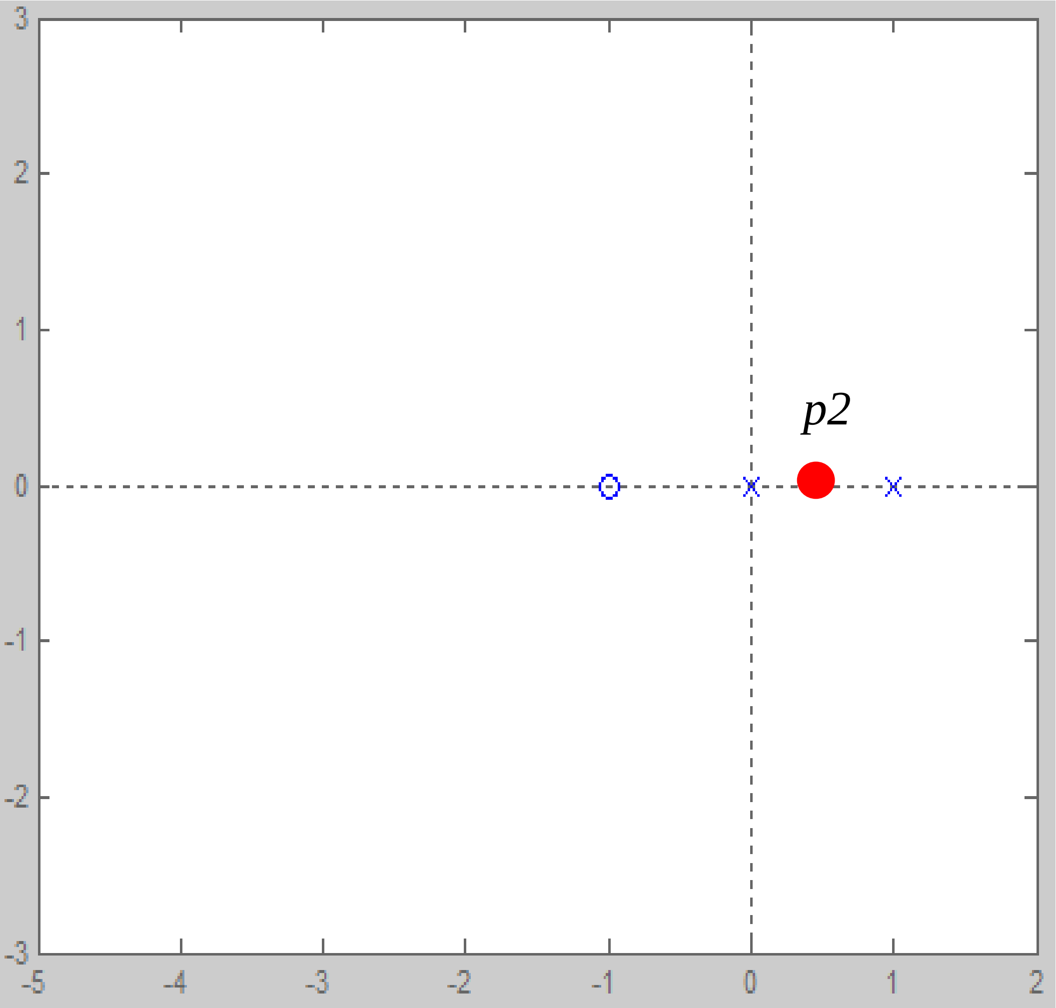
- To determine the root loci on real axis we select some test points.
- e.g: p_1 (on positive real axis).
- No. of real poles and zeros on the right of test point is zero (which is even)
- Hence, there is no root locus on the positive real axis.



PROBLEM: Construction of Root Locus (contd)..

Step-2: Determine the root loci on the real axis.

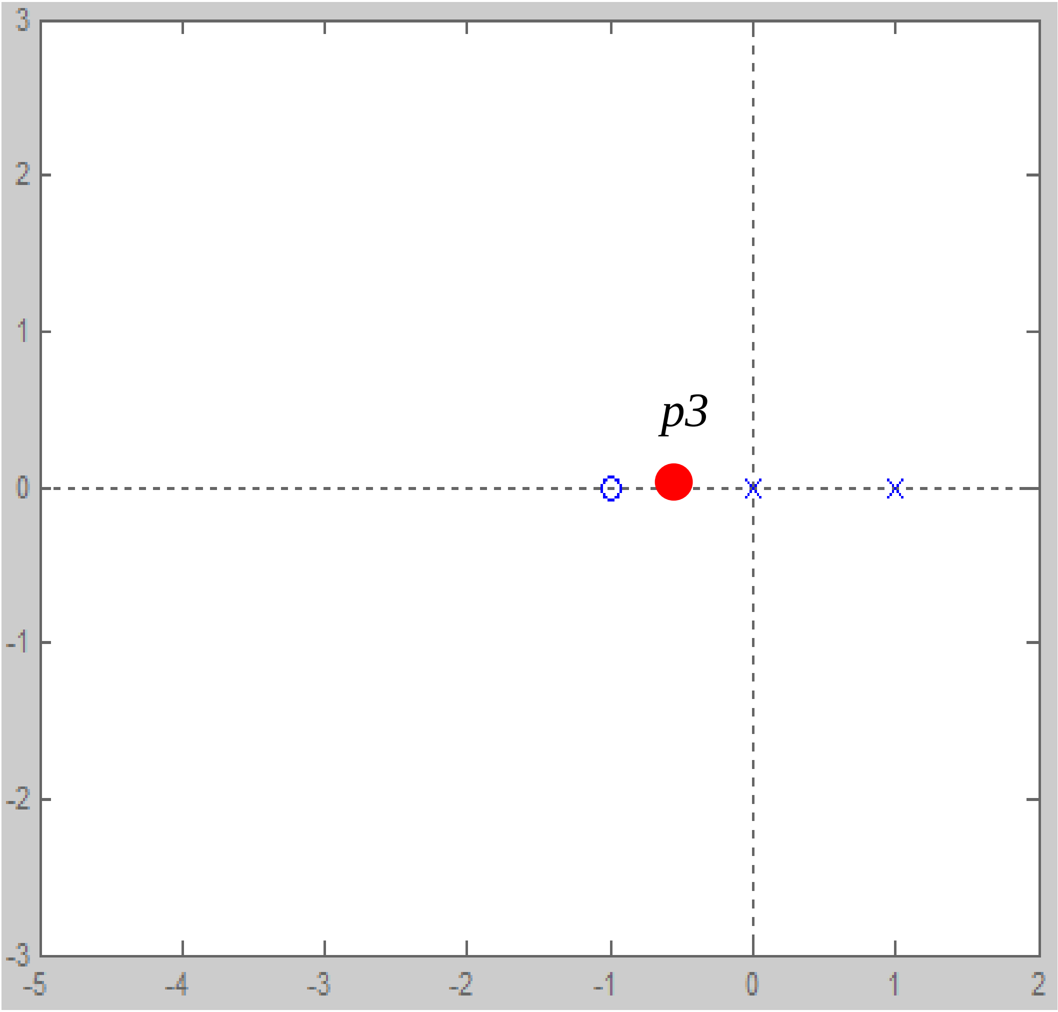
- Next, select a test point on the positive real axis between 1 and 0.
- No. of real poles and zeros on the right of test point is one (which is odd)
- Therefore, from 1 to 0 is part of the root locus.



PROBLEM: Construction of Root Locus (contd)..

Step-2: Determine the root loci on the real axis.

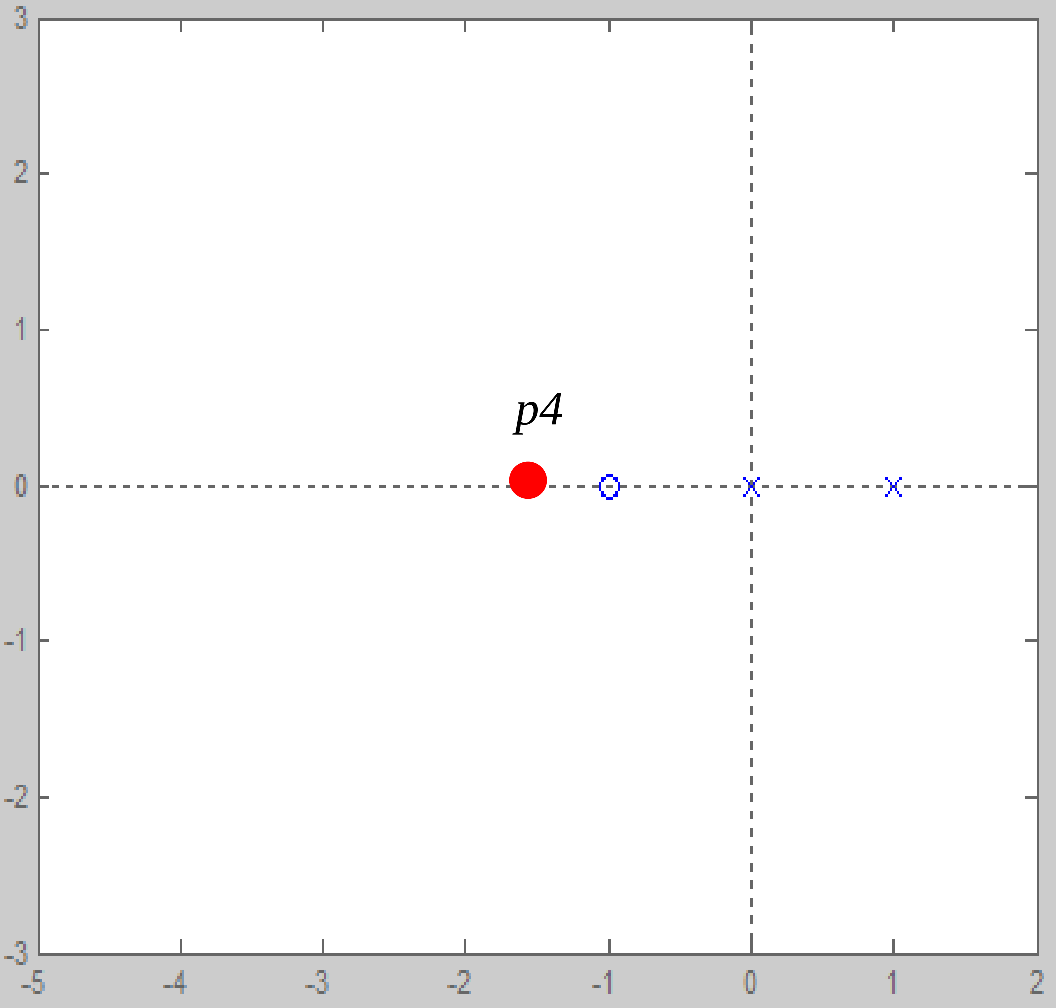
- Next, select a test point on the negative real axis between 0 and -1.
- No. of real poles and zeros on the right of test point is two (which is even)
- Therefore, from 0 to -1 is not part of the root locus.



PROBLEM: Construction of Root Locus (contd)..

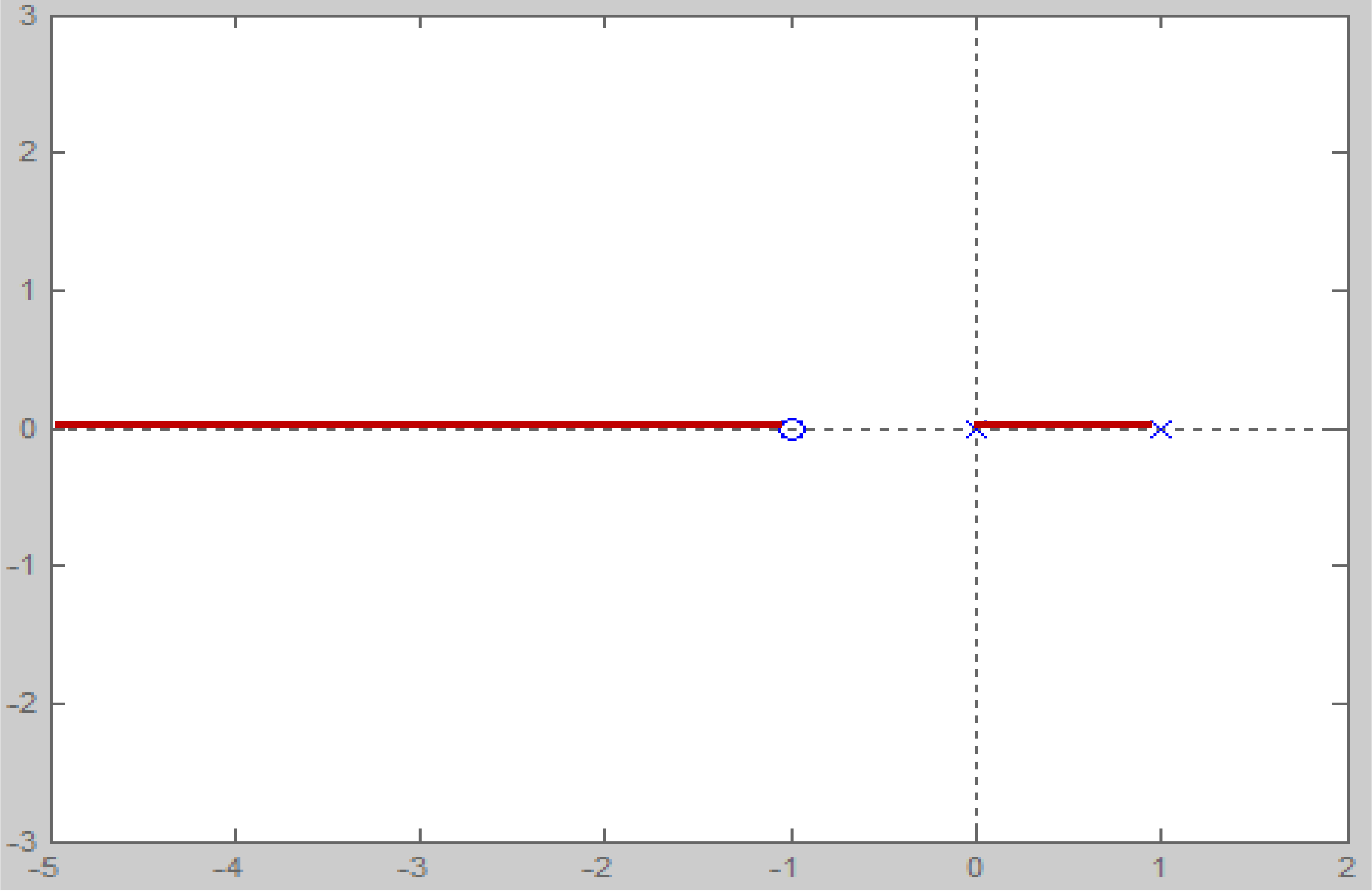
Step-2: Determine the root loci on the real axis.

- Next, select a test point on the negative real axis between -1 and - ∞.
- No. of real poles and zeros on the right of test point is three (which is odd)
- Therefore, from -1 to - ∞ is part of the root locus.



PROBLEM: Construction of Root Locus (contd)..

Step-2: Determine the root loci on the real axis.



PROBLEM: Construction of Root Locus (contd)..

Step-3: Determine the *asymptotes* of the root loci and angles.

$$Angle \quad of \quad asymptotes = \phi = \frac{\pm 180°(2q+1)}{n-m}$$

Where

n-----> number of poles (2)

m-----> number of zeros (1)

$$\phi = \frac{\pm 180°(2q+1)}{2-1}$$

$$\phi = \pm 180° \quad \text{when} \quad q = 0$$

- ❑ No. of asymptotes = n-m = 1
- ❑ The angle of asymptote is 180°.
- ❑ No centroid for this system

PROBLEM: Construction of Root Locus (contd)..

Step-4: Determine the *breakaway/break-in point.*

- The breakaway/break-in point is the point from which the root locus branches leaves/arrives real axis.
- The breakaway or break-in points can be determined from the roots of $dK/ds=0$
- It should be noted that not all the solutions of $dK/ds=0$ correspond to actual breakaway points.
- If a point at which $dK/ds=0$ is on a root locus, it is an actual breakaway or break-in point.
- The characteristic equation of the system is

$$1 + G(s)H(s) = 1 + \frac{K(s+1)}{s(s-1)} = 0$$

$$K = -\frac{s(s-1)}{s+1}$$

PROBLEM: Construction of Root Locus (contd)..

- The breakaway point can now be determined as

$$\frac{dK}{ds} = -\frac{d}{ds}\left[\frac{s(s-1)}{s+1}\right]$$

$$\frac{dK}{ds} = \frac{(s+1)(2s-1)-(s^2-s)(1)}{(s+1)^2}$$

- Set *dK/ds=0* in order to determine breakaway point.

$$\frac{(s+1)(2s-1)-(s^2-s)(1)}{(s+1)^2} = 0$$

$$s^2 + 2s - 1 = 0$$

- By solving the equation roots are at

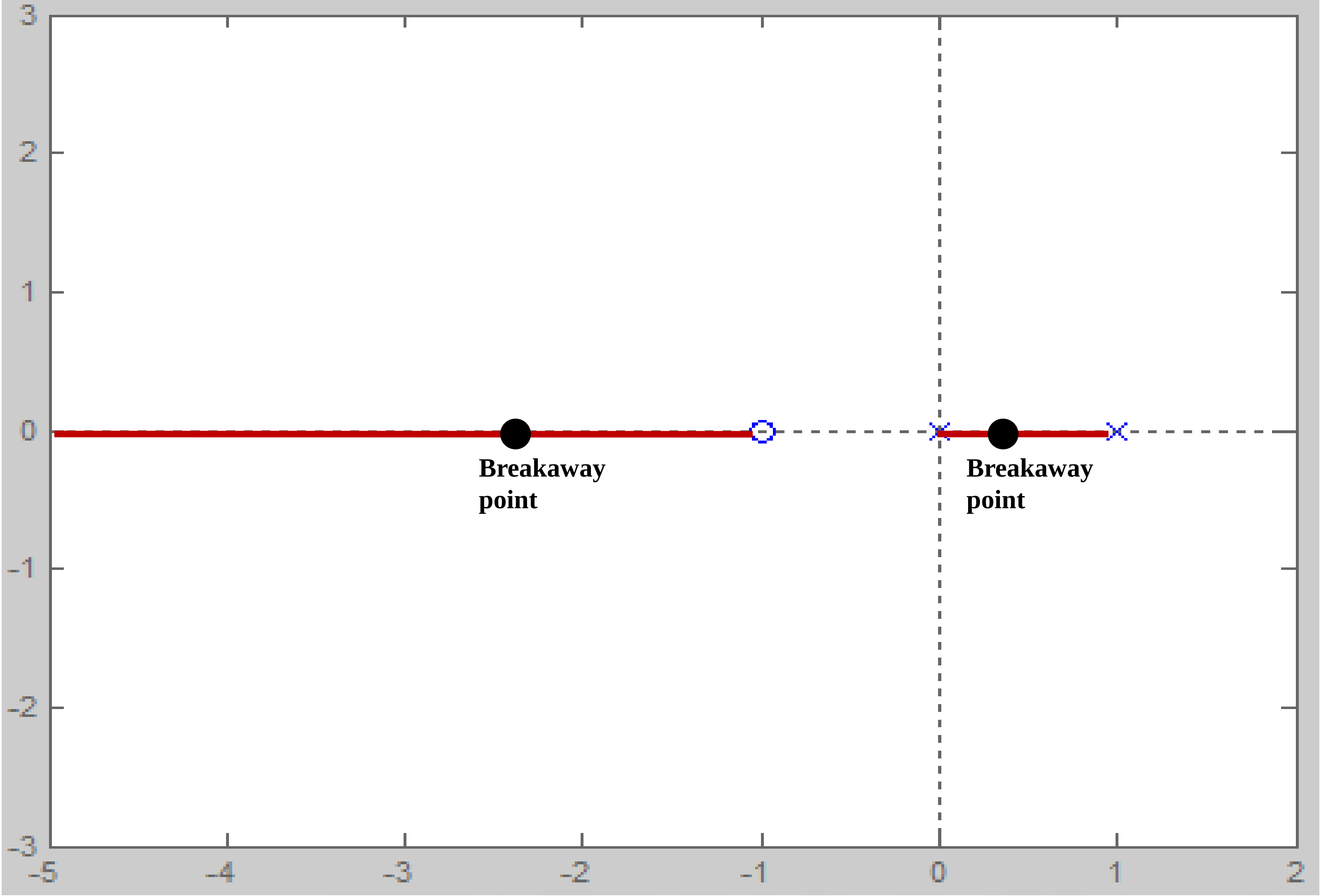
$$s = +0.414$$

$$= -2.414$$

- By substituting these s values in k equation, the value of k is positive real for s=0.414 (k=0.17), s=-2.414 (k=5.828). so these points are actual breakaway points.

PROBLEM: Construction of Root Locus (contd)..

Step-4: Determine the *breakaway/break-in point.*



PROBLEM: Construction of Root Locus (contd)..

Step-5: Determine the points where root loci cross the imaginary axis and range of K for stable operation

- The characteristic equation of closed loop system:

$$s(s-1)+k(s+1)=0$$

$$s^2+(k-1)s+k=0$$

$$\begin{array}{c|cc} s^2 & 1 & k \\ s^1 & k-1 & 0 \\ s^0 & k & \end{array}$$

$$k>0$$

$$k>1$$

$$s^2+1=0$$

$$s=\pm j1$$

- The root loci cuts the imaginary axis at $s=\pm j1$

PROBLEM: Construction of Root Locus (contd)..

Step-5: Determine the points where root loci cross the imaginary axis and range of K for stable operation

- The characteristic equation of closed loop system:

$$s(s-1)+k(s+1)=0$$

$$s^2+(k-1)s+k=0$$

$$s^2+2\xi\omega_n s+\omega_n^2=0$$

$$\omega_n=\sqrt{k}$$

$$\xi\omega_n=(\frac{k-1}{2})$$

$$t_s=4=\frac{4}{\xi\omega_n}$$

$$\xi\omega_n=1=(\frac{k-1}{2})$$

$$k=3$$

- The location of closed loop poles for k=3, ts=4 sec

$$s^2+2s+3=0$$

$$s=-1\pm j\sqrt{2}$$

PROBLEM: Construction of Root Locus (contd)..

- To show that the loci of the complex roots are part of a circle with (-1,0) as centre and radius = $\sqrt{2}$

- Apply the angle criterion:

$$\angle G(s) = \angle k \frac{(s+1)}{s(s-1)} = \pm\pi(2q+1)$$

$$s = \sigma + j\omega$$

$$\angle k + \angle \sigma + j\omega + 1 - \angle \sigma + j\omega - \angle \sigma + j\omega - 1 = -\pi$$

$$\pi + \tan^{-1}\left(\frac{\omega}{\sigma+1}\right) = \tan^{-1}\left(\frac{\omega}{\sigma}\right) + \tan^{-1}\left(\frac{\omega}{\sigma-1}\right)$$

- Apply the tan on both sides

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\frac{\frac{\omega}{\sigma} + \left(\frac{\omega}{\sigma-1}\right)}{1 - \frac{\omega}{\sigma}\left(\frac{\omega}{\sigma-1}\right)} = \frac{\tan(\pi) + \left(\frac{\omega}{\sigma+1}\right)}{1 - \tan(\pi)\left(\frac{\omega}{\sigma+1}\right)}$$

PROBLEM: Construction of Root Locus (contd)..

- By cross multiply and simplify:

$$\frac{\omega}{\sigma} + \frac{\omega}{\sigma - 1} = \frac{\omega}{\sigma + 1}\left[1 - \frac{\omega^2}{\sigma(\sigma - 1)}\right]$$

$$\sigma^2 + \omega^2 + 2\sigma - 1 = 0$$

- By add and subtract ‘1’ and rearrange

$$(\sigma^2 + 2\sigma + 1) - 1 + \omega^2 - 1 = 0$$

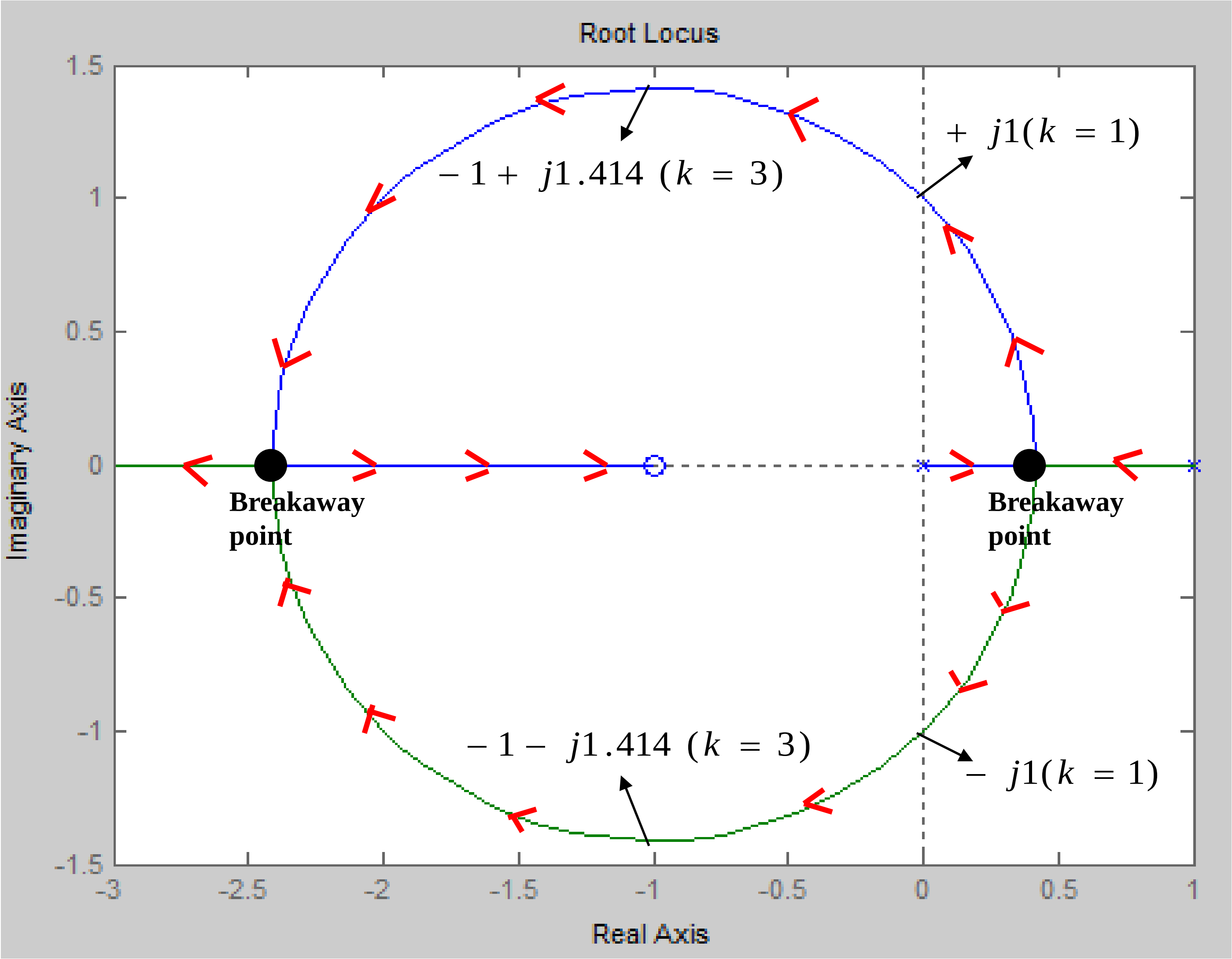
$$(\sigma + 1)^2 + \omega^2 = 2$$

- This is the equation of the circle with center at (-1,0) and radius $\sqrt{2}$

PROBLEM: Construction of Root Locus (contd)..

- Complete root locus for the given system

$$G(s) = \frac{k(s+1)}{s(s-1)}$$



PROBLEM: Construction of Root Locus

The characteristic equation of a feedback control system is

$$s^4 + 3s^3 + 12s^2 + (k - 16)s + k = 0$$

Sketch the root locus plot for $0<k<\infty$ and show that the system is conditionally stable (stable only for a range of gain k). Determine the range of gain for which the system is stable.

Solution:

To sketch the root locus, we require the open-loop transfer function G(s)H(s)

- $$1 + G(s)H(s) = s^4 + 3s^3 + 12s^2 - 16s + ks + k = 0$$

$$1 + G(s)H(s) = s(s^3 + 3s^2 + 12s - 16) + k(s+1) = 0$$

$$1 + \frac{k(s+1)}{s(s^3 + 3s^2 + 12s - 16)} = 1 + \frac{k(s+1)}{s(s-1)(s^2 + 4s + 16)} = 0$$

PROBLEM: Construction of Root Locus (contd)..

- $G(s)H(s) = \dfrac{k(s+1)}{s(s^3+3s^2+12s-16)} = \dfrac{k(s+1)}{s(s-1)(s+2+j3.42)(s+2-j3.42)}$
- The k=0 points: s=0, s= 1, s=-2+j3.42, s=-2-j3.42
  no. of poles (n)= 4
- The k=∞ points: s=-1
  no. of zeros (m)=1
- No. of root locus branches (n)=4
- Root locus exists on the real axis from s=1 to s=0 and to the left of s=-1
- No asymptotes (n-m)=3
- Angles of asymptotes $\phi = \pm 60^o, \pm 180^o$
- Centroid $\sigma = -0.66$
- The breakaway points are given by dk/ds=0.

  where $k = \dfrac{s(s-1)(s^2+4s+16)}{s+1}$

PROBLEM: Construction of Root Locus (contd)..

- $$\frac{dk}{ds} = (s+1)\frac{d}{ds}(s^4 + 3s^3 + 12s^2 - 16s) - (s^4 + 3s^3 + 12s^2 - 16s)\frac{d}{ds}(s+1) = 0$$

$$(s+1)(4s^3 + 9s^2 + 24s - 16) - s^4 - 3s^3 - 12s^2 + 16s = 0$$

$$3s^4 + 10s^3 + 21s^2 + 24s - 16 = 0$$

- By solving the above equation out of four roots only, s=0.45 and s= -2.26 are actual break points.

- Out of these s=0.45 is the breakaway point and s=-2.26 is the break-in point.

- Corresponding to these points k values are 2.64 and 77.66

- The angle departure of the root locus from the complex pole is $\theta_d = \pm 55.27^o$

PROBLEM: Construction of Root Locus (contd)..

Determine the points where root loci cross the imaginary axis and range of K for stable operation

- The characteristic equation of closed loop system:

$$s^4 + 3s^3 + 12s^2 + (k-16)s + k = 0$$

$$\begin{array}{l|ccc}
s^4 & 1 & 12 & k \\
s^3 & 3 & k-16 & \\
s^2 & \dfrac{36-k+16}{3} & k & \\
s^1 & \dfrac{\dfrac{52-k}{3}(k-16)-3k}{\dfrac{52-k}{3}} & & \\
s^0 & k & &
\end{array}$$

$$k > 0$$

$$52 - k > 0$$

$$k < 52$$

$$52k + 16k - k^2 - 832 - 9k > 0$$

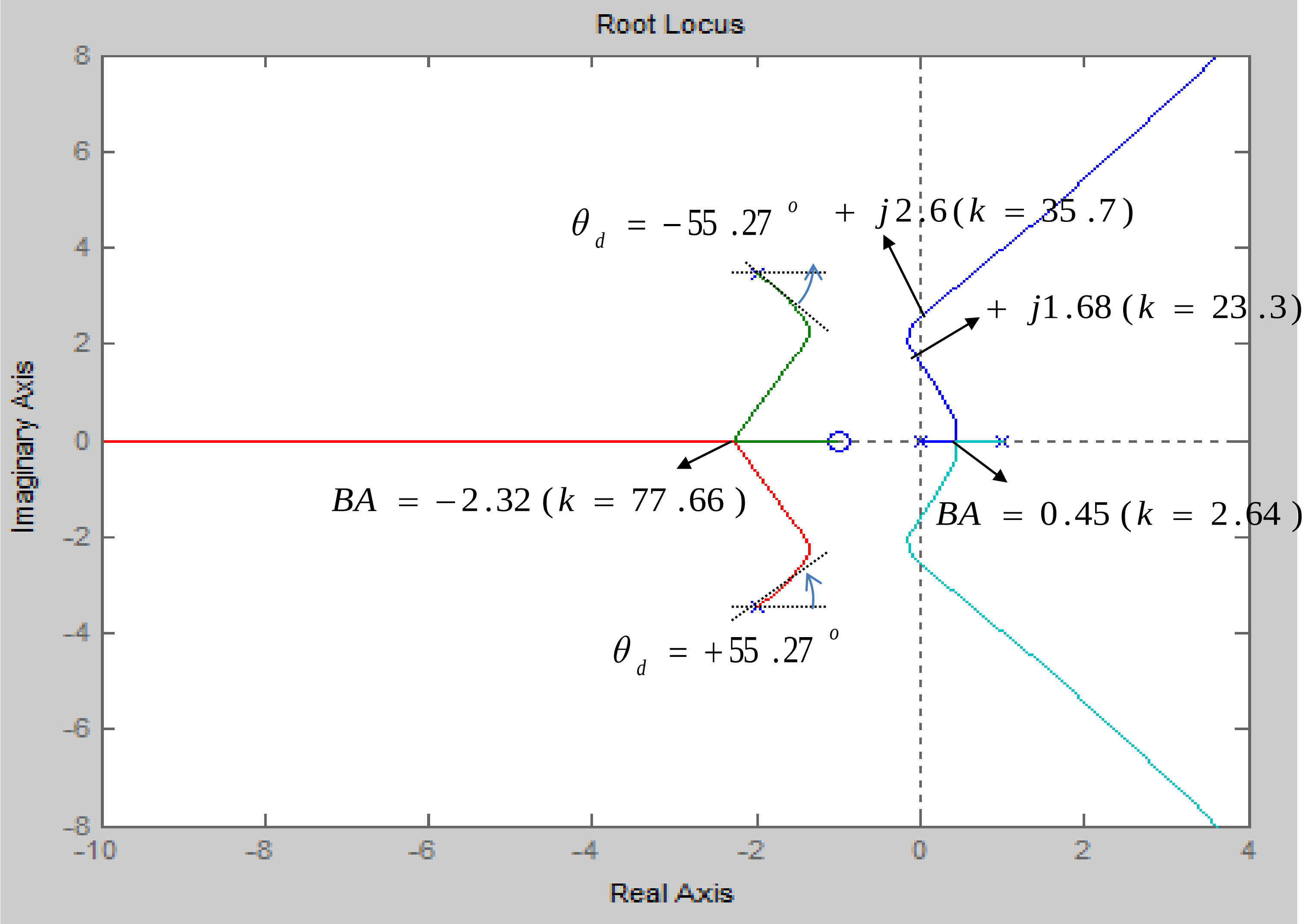
$$k^2 - 59k + 832 < 0$$

$$k > 23.3 \, and k < 35.7$$

- The range of values of k for stability is 23.3<k<35.7. The corresponding oscillation frequencies are 1.68 rad/sec and 2.6 rad/sec

PROBLEM: Construction of Root Locus (contd)..

- Complete root locus of the given system is



Why Controllers!

- If a closed loop system's response is not as desired then we make use of controllers.

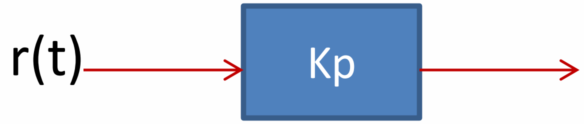
- Controllers are also needed because to improve the closed loop system's response we cannot alter / change /replace the system(plant) which is designed for certain steady state design specifications.

- System response has two components:
  transient response ,&
  steady state response
- There may be a requirement to either improve transient response or steady state response; or, both the responses may have to be improved.

- Different controllers are used for improving transient & steady state responses; a combination of controllers is used to improve both transient & steady state responses of a system.

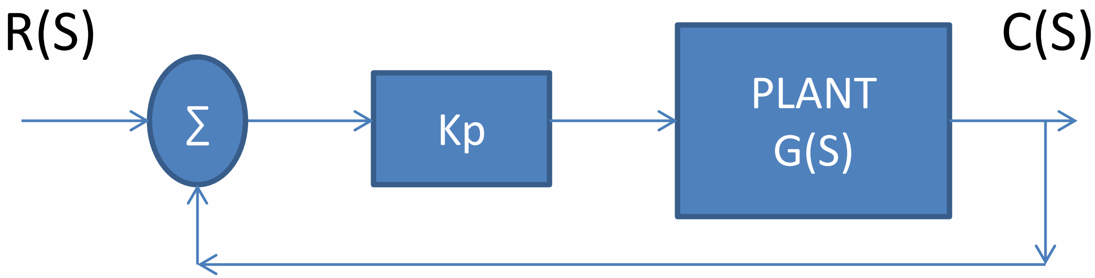
P: The Proportional Controller

- Proportional Controller:



$c(t) = K_P\, r(t)$

- P controller is a pure gain element. Generally put in cascade with the plant(system to be controlled)



- M(S) = C(S)/R(S) = Kp G(S)/1+ Kp G(S)
- Thus Kp provides additional gain to the loop; Kp can be <1 or >1
- The value of Kp determines the location of closed loop pole(s). It affects impulse response of the closed loop system.

P: The Proportional Controller (Example First order system)

❑ Example 1:

$G(S) = 1 /(S + 1)$; $Q(S) = 1 + G(S) = 0 \Rightarrow S+2 = 0 \Rightarrow S= -2$

$G\mathbf{1}(S) = Kp\ G(S) = Kp/(S+1)$; $Q(S) = 1 + G(S)=0 \Rightarrow S +(Kp+1)=0$

$\Rightarrow S = -(Kp +1)$

- ❑ Thus we observe that the location of CL pole varies with Kp. If Kp is increased then the pole moves farther away from the Imaginary Axis of the S plane.
- ❑ The Impulse response without Kp = e^{-t} ,and

  with Kp = $\exp\{ -(Kp +1)t\}$
- ❑ Thus we see that as Kp increases, impulse response decays faster to zero thus reducing settling time. We cannot increase Kp beyond a value as it may make higher order systems unstable.

P: The Proportional Controller (Example Second order system)

❑ Example 2:

$G(S) = 1/S^2 + 2S + 2$; $Q(S) = 1 + G(S) = 0$ ⇒ $S^2 + 2S + 3 = 0$

⇒ $S = -1 +/- j1$

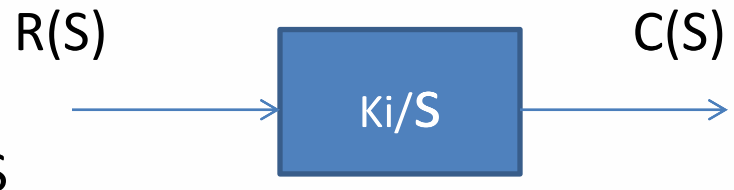
⇒ implies pole locations are fixed

$G_1(S) = Kp/S^2 + 2S + 2$; $Q(S) = S^2 + 2S + (2+Kp)$

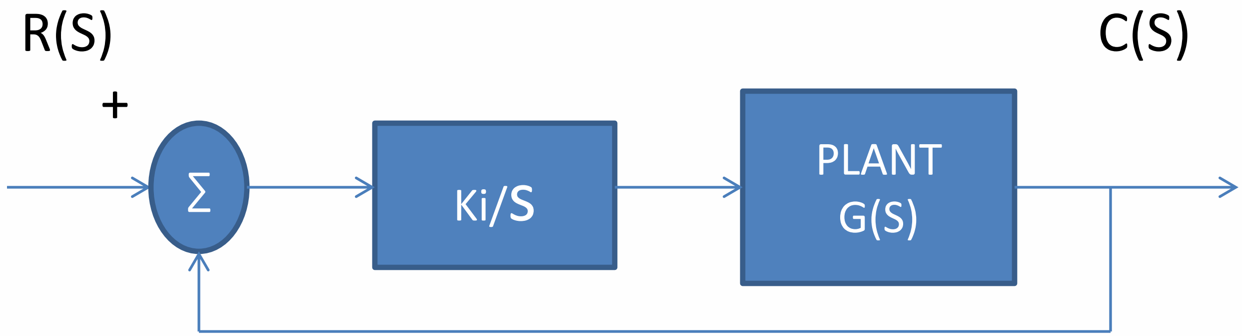
- ❑ We see that as Kp is increased the imaginary part of the roots increases thereby increasing ω**n** for the system, while maintaining intersection on the real axis = -1. Therefore it does not affect settling time.
- ❑ Thus we see that increasing Kp introduces high frequency oscillations in the system & it may not be appropriate to fix a high value for Kp as it reduces damping in the system.

I: The Integral Controller

❑ Integral Controller:



$C(S)/R(S) = G_c(S) = K_i/S$



$C(S)/R(S) = K_i\, G(S)/(S + K_i\, G(S))$; $G(S) = K/(S+1)$

$C(S)/R(S) = K_i\, K/(S^2 + S + K_i\, K)$ Therefore Integral control:

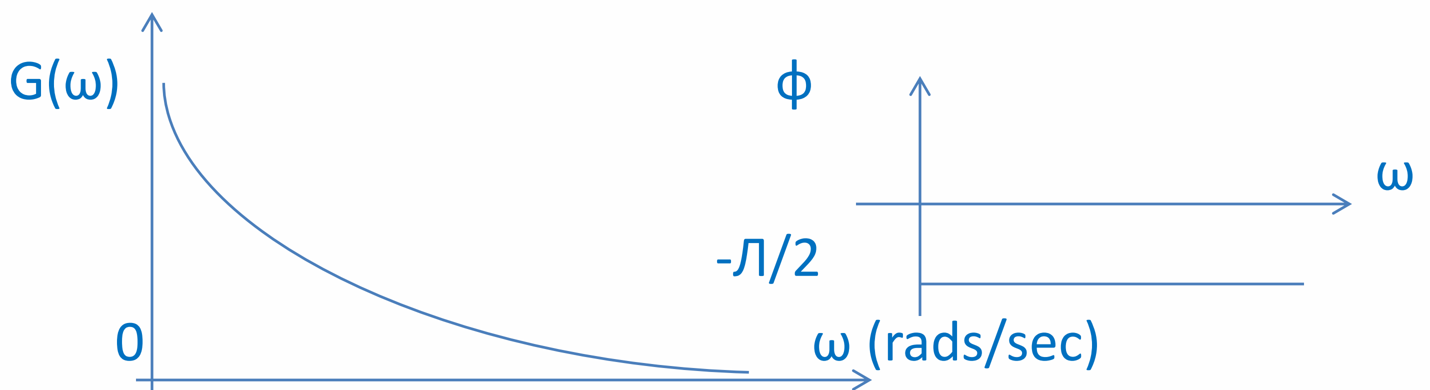
1. Increases the order of a system
2. Converts over-damped system in to an under damped one (governed by Ki)
3. As the integral gain Ki is varied, it varies ω_n of the system.
4. Reduces steady state error of a system & improves steady state response.

I: The Integral Controller: Frequency Response

- Magnitude Response:

$$G(S) = K\mathbf{i}/S \; ; \; S = j\omega \quad G(j\omega) = K\mathbf{i}/j\omega$$

$$|G(J\omega)| = G(\omega) = K\mathbf{i}/\omega \; ; \; \arg(G(j\omega)) = -\pi/2 \text{ rads} = \phi$$



- At very low frequencies the integrator provides very high gain and very high attenuation at high frequencies. It is a low pass filter.
- The phase lag : $\phi = -\pi/2$ rads (constant for all frequencies)

D: The Derivative Controller

❑ Derivative Controller:

$C(S) = (K_d\,S)\,R(S)$ 1

❑ Explanation:

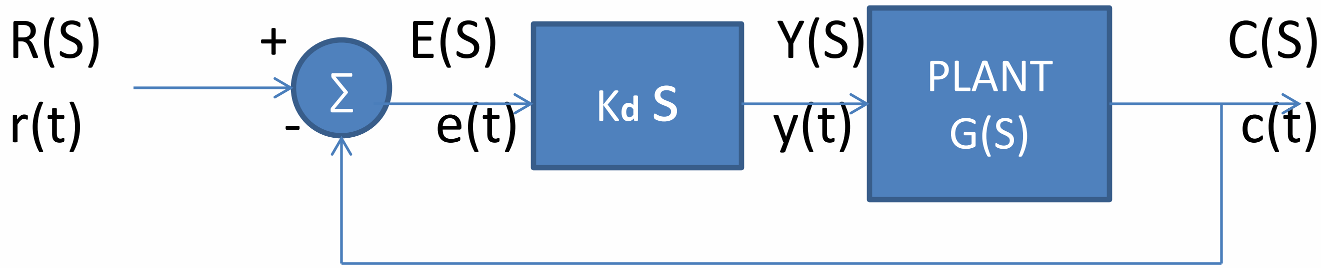
Take Inverse Laplace transform of (1); $c(t) = K_d\, r^{\circ}(t) = K_d\, dr/dt$

Thus we observe that derivative controller differentiates the input. It implies that if input is constant then the output of derivative block is equal to zero. Thus its output exists only if input is varying with time. Therefore, if this controller is in forward path, then we use a term like $(1 + K_d\, S)$ so that input to the plant does not become zero if the error signal has attained a constant value.

This is depicted in next slide.

D: The Derivative Controller

- **The Derivative Controller:**

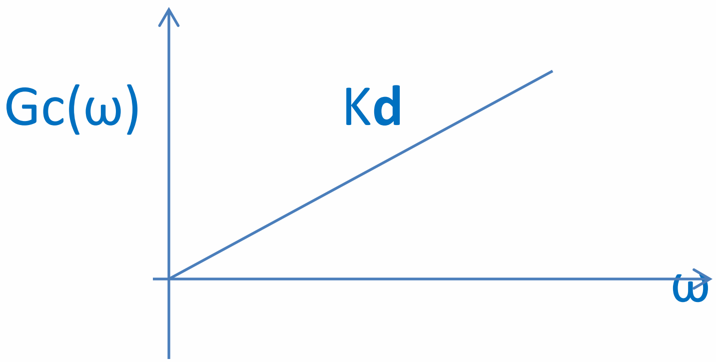


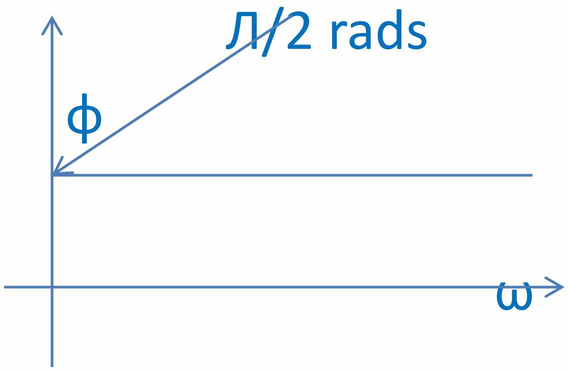
- If e(t), the error, attains a constant value then the output , y(t) =0 and the plant (system) will have zero input which is not acceptable.
- Therefore in the forward path, we use a term (1+ K**d** S) so that under steady state y(t) = e(t). Therefore the derivative block is replaced by the block 1+K**d** S in the forward path.
- In the feedback we retain it as K**d** S

D: The Derivative Controller: Frequency Response

$G_c(S) = K_d\, S$; $G_c(j\omega) = j\, K_d\, \omega$; $|G_c(j\omega)| = G_c(\omega) = K_d\, \omega$;

$\arg(G_c(j\omega)) = \pi/2 = \phi$

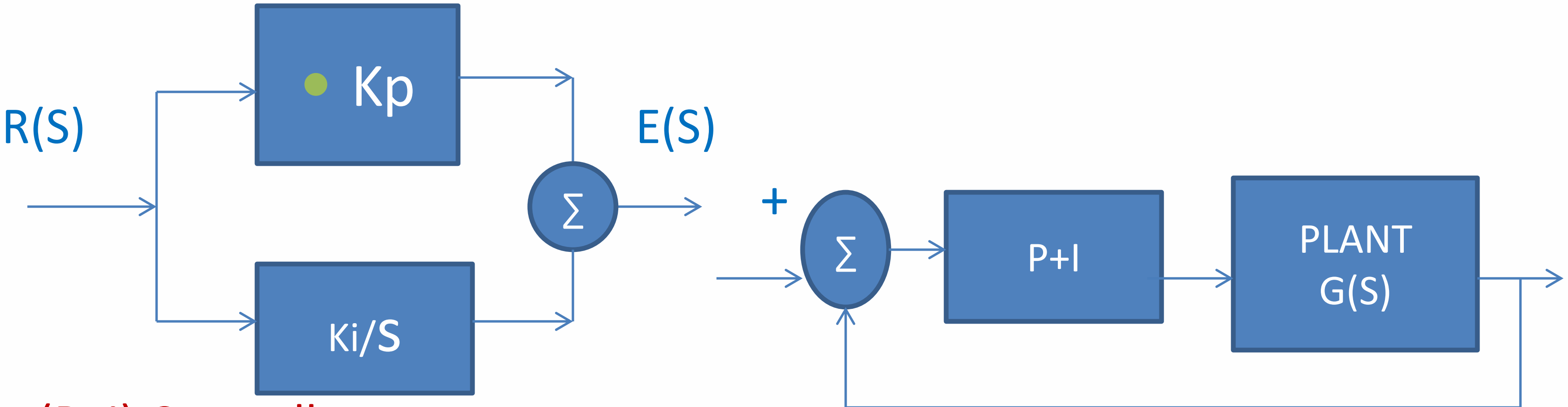




- We see from the above plots that derivative controller offer higher gain at higher frequencies , therefore it is a high pass filter.
- Phase introduced by it is positive.

Proportional plus Integral Control

- P+I Controller: $|G(j\omega)| = G(\omega) = \sqrt{(Ki^2 + (Kp\ \omega)^2)}/\omega$



(P+I) Controller

$E(S)/R(S) = Kp + Ki/S$; $G(S) = Kp + Ki/S$

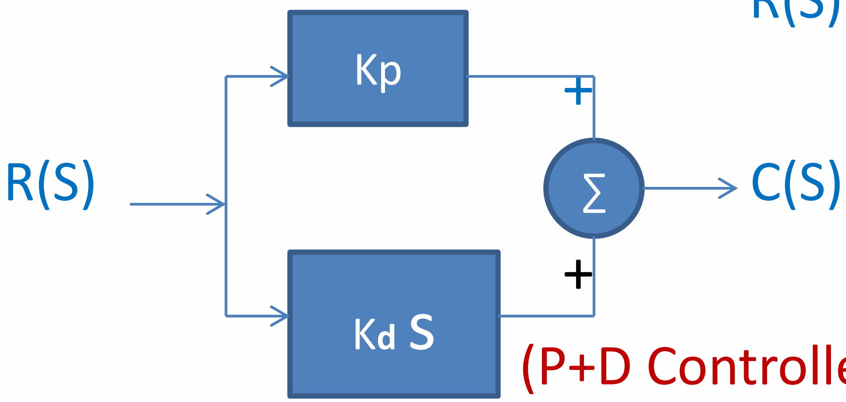
$G(j\omega) = Kp + Ki/j\omega = (Ki + j\ Kp\ \omega)/j\omega$

$|G(j\omega)| = G(\omega) = \sqrt{(Ki^2 + (Kp\ \omega)^2)}/\omega$

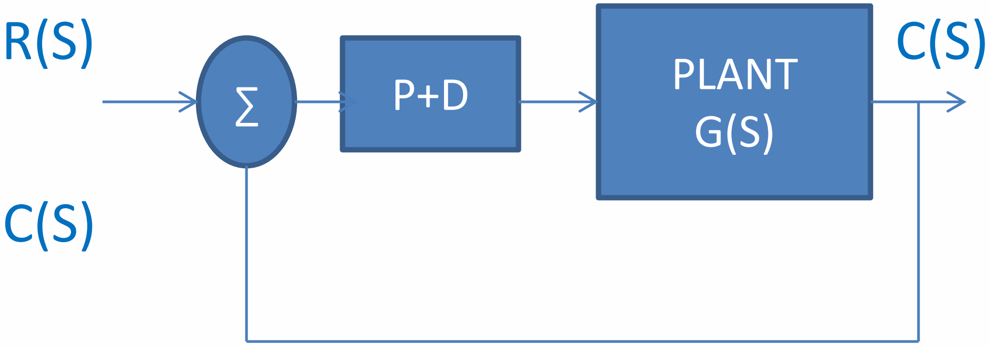
$\arg(G(j\omega)) = \tan^{-1}(Kp\ \omega/Ki) - \pi/2$

P+D Controller

- P+D Controller:



(P+D Controller)



$C(S)/R(S) = G(S) = Kp + Kd\,S$

$G(j\omega) = Kp + j\,Kd\,\omega$

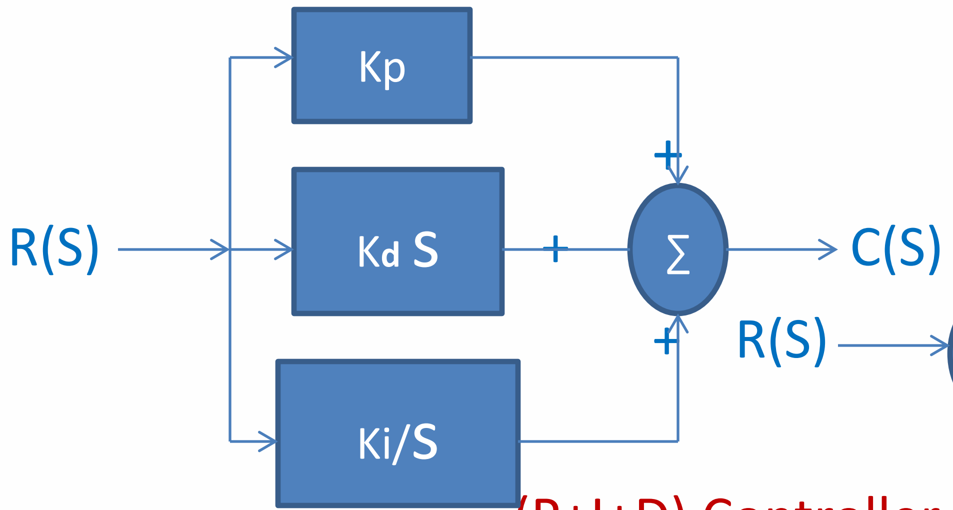
$|G(j\omega)| = G(\omega) = \sqrt{\{(Kp^2 + (Kd\,\omega)^2\}}$

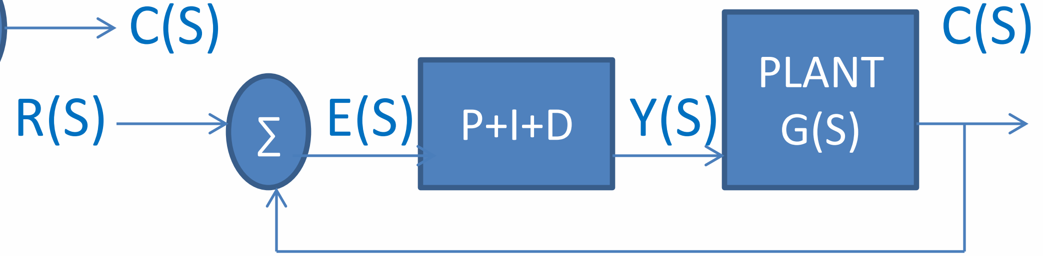
$\arg\{G(j\omega)\} = \tan^{-1}(Kd\,\omega/Kp)$

P+I+D Controller

- P+I+D Controller:

$$c(t) = \{Kp + Kd\, d/dt + ki \int dt\}\, r(t)$$





(P+I+D) Controller

$$C(S)/R(S) = G(S) = Kp + Kd\, S + Ki/S = (Kd\, S^2 + Kp\, S + Ki)/S$$

$$G(j\omega) = \{(Ki - Kd\, \omega) + j\, Kp\, \omega\}/j\omega$$

$$|G(j\omega)| = G(\omega) = \sqrt{\{(Ki - Kd\, \omega)^2 + (Kp\, \omega)^2\}}$$

$$\arg\{G(j\omega)\} = \tan^{-1}\{Kp\, \omega/(Ki - Kd\, \omega)\}$$

P+I+D Controller

- Observation:

  $K_p = 0$; I+D controller

  $K_i = 0$; P+D controller

  $K_d = 0$; P+I controller

- Thus we can choose a combination depending on the requirement.

  $y(t) = \{K_p + K_d\, d/dt + k_i \int dt\}\, e(t)$; e(t) is system error;

  y(t) is P+I+D output

- From the above equation, we observe that,

  If $e(t) > 0$ & is constant, then

  Output of 'D' block = 0

  Output of 'P' block = $K_p * e(t)$

  Output of 'I' block = $K_i \int e(t)\, dt$

- Thus the 'I' block output will keep increasing & can destabilize the CL system.

P+I+D Controller

- Therefore, if the closed loop system is to be stable then the error, e(t), should equal '0' under steady state.
- Zero steady state error implies,

  $K_p = \infty$, $K_v = \infty$ & $K_a = \infty$

  It means that all error constants should have a very high value.
- the forward path transfer function in the block diagram is equal to

  $H(S) = G_c(S)\, G(S) = \{(K_d S^2 + K_p S + K_i)/S\}\, G(S)$
- If G(S) is type '0': H(S) is type '1'; e(t) = 0 for step input
- If G(S) is type '1': H(S) is type '2'; e(t) = 0 for step & ramp inputs
- If G(S) is type '2': H(S) is type '3'; e(t) = 0 for step, ramp & parabolic inputs.
- The above observations are also valid for P+I controller & depend upon the location of zeros in G(S) . For an all pole G(S), order>2, the CL system may cease to be stable unless there are zeros associated with G(S). H(S) has 1 pair of CC zeros & pole at S=0.

P+I+D Controller

- The obvious concern is:

  If error, e(t) becomes'0' then will the controller output, y(t), become '0' !

  If y(t) attains a '0' value will the system, G(S), output also become '0'!

- The system output, c(t), will not become '0' because of the property of the I- controller , Ki/S.
- The I controller retains its output at its previous value, if input to it becomes '0' at $t = t_1$; that is the value y(t) attained at time $t = t_1 - \Delta t$.
- The I controller is known as pure integrator because of its linear (constant) slope.

P+I+D Controller

- The pure Integrator:

$$Y(S)/X(S) = 1/S$$

$$y°(t) = x(t) = dy(t);\ x(t) = U(t)=1.0$$

$$dy(t) = 1.0$$

- Rectangular Rule:

$y(t) = y(t\text{-}1) + dy(t) * \Delta t$; $y(t) = 0$ at $t=0$; Δt : time increment

let $\Delta t = 0.05$

$t = 1$; $y(1) = 0 + 1 * 0.05 = 0.05$

$t = 2$; $y(2) = y(1) + 1 * 0.05 = .05 + .05 = 0.1$

$t = 3$; $y(3) = y(2) + 1 * 0.05 = 0.1 + 0.05 = 0.15$

- Let us now make $x(t) = 0$; therefore $dy(t) = 0$

$t = 4$; $y(4) = y(3) + 0 * 0.05 = 0.15$

$t = 5$; $y(5) = y(4) + 0 = 0.15$

Thus we see that even after input $x(t) = 0$; the output is retained.

Unit-IV
FREQUENCY DOMAIN ANALYSIS

Frequency Domain Specifications

- We have studied about time domain specifications like, rise time ,t**r**; peak time, t**p**; settling time, t**s**; peak overshoot, M**p**.
- Now, we define frequency domain specifications for a given system and determine their correlation with the time domain specifications.
- This correlation between time & frequency domain is necessary as it enables us to derive time domain specifications from frequency domain ones & vice-versa.
- Further, we may like to analyze a given system either in time domain or frequency domain & hence we need to have a set of specification in each domain for evaluating a given system's response.
- Like in time domain, here too we consider a second order system for deriving frequency domain specifications.

Frequency Domain Specifications (contd)..

- Given, a closed loop transfer function, T(S) = C(S)/R(S), as

$$T(S) = C(S)/R(S) = \omega_n^2 / (S^2 + 2\xi\, \omega_n S + \omega_n^2)$$

- For determining frequency response, we let $S = j\omega$ in T(S) because we are interested in real frequencies which lie on the Imaginary axis of the S-plane.

$$T(j\omega) = \omega_n^2 / (-\omega^2 + j2\xi\, \omega_n\, \omega + \omega_n^2)$$

$$T(j\omega) = \omega_n^2 / \omega_n^2 \{ (1-(\omega/ \omega_n))^2 + j2\xi\, \omega/ \omega_n \}$$

- Let $u = \omega/ \omega_n$; u: normalized frequency

  ω_n: natural frequency of oscillation of the system

  ω : input signal frequency

- Thus, $T(j\omega) = 1/\{ (1-u^2) + j\, 2\xi\, u\}$ .... (1)

$$|T(j\omega)| = M(u) = 1/\sqrt{ (1-u^2)^2 + 4\xi^2 u^2} \quad ..... (2)$$

$$\arg\{T(j\omega)\} = \phi = -\tan^{-1} \{2\xi\, u/(1-u^2)\} \quad ....(3)$$

Frequency Domain Specifications (contd)..

- The magnitude & phase response are part of frequency response. Equations(2) & (3) corresponding to magnitude & phase response tell us that,
- if we feed an input signal $r(t) = A \sin(\omega t)$ to the system, the output signal will have

$$\text{magnitude} = A/\sqrt{(1-u^2)^2 + 4\xi^2 u^2}\text{, and the}$$

$$\text{phase introduced} = -\tan^{-1}\{2\xi u/(1-u^2)\}$$

- Thus the output signal, under steady state, will be

$$c(t) = A/[\sqrt{(1-u^2)^2 + 4\xi^2 u^2}] \sin(\omega t - \tan^{-1}\{2\xi u/(1-u^2)\})$$

- We observe that the output amplitude is dependent on the input frequency, and so is the phase lag introduced in the output signal.

Frequency Domain Specifications (contd)..

- Reproducing equations (2) & (3), we have

$M(u) = 1/\sqrt{(1-u^2)^2 + 4\xi^2 u^2}$ ..... (2)

$\phi = -\tan^{-1}\{2\xi u/(1-u^2)\}$ ....(3)

Plotting M & ϕ vs. u, $u = \omega/\omega_n$

u	M	ϕ
0.0	1.0	0 $(\omega=0)$
1.0	$1/(2\xi)$	$-\pi/2$ $(\omega=\omega_n)$
∞	0	$-\pi$ $(\omega_n \rightarrow \infty)$

- Observation:

At $\omega=\omega_n$, the value of 'M' is inversely proportional to ξ.

The lower the ξ higher the 'M' implies higher peak in the magnitude response.

Frequency Domain Specifications (contd)..

Resonant Frequency:

The frequency where 'M' has a peak value is called resonant frequency. At this frequency, the slope of the magnitude curve, M, is zero. Differentiate 'M' w.r.t 'u' in equation (1)

Therefore, $\left.\dfrac{dM}{du}\right|_{u = u_r} = 0 \Rightarrow u_r^2 = 1 - 2\xi^2 \Rightarrow u_r = \sqrt{(1-2\xi^2)}$

$\Rightarrow \omega_r = \omega_n \sqrt{(1-2\xi^2)}$

Resonant frequency : $\omega_r = \omega_n \sqrt{(1-2\xi^2)}$ .... (4)

Resonant Peak, Mr:

The maximum value of magnitude is known as 'Resonant peak'

$M(u) = 1/\sqrt{\{(1-u^2)^2 + 4\xi^2 u^2\}}$; at resonant frequency $u=u_r$, we get M_r.

Substitute for $u= u_r$ in $M(u)$, to get $M_r = 1/\{2\xi \sqrt{(1-\xi^2)}\}$ ..... (5)

Frequency Domain Specifications (contd)..

- **Phase angle, ϕr at Resonant Frequency:**

Phase angle: $\phi = -\tan^{-1}\{2\xi\, u/(1-u^2)\}$

Substitute for u = u**r** in ϕ, to get

$$\phi \mathbf{r} = -\tan^{-1}\{\sqrt{(1-2\,\xi^2)}/\xi\} \quad \ldots\ldots (6)$$

From equations (4) & (5), as reproduced below

$$\omega r = \omega \mathbf{n} \sqrt{(1-2\,\xi^2)} \quad \ldots. (4)$$

$$M\mathbf{r} = 1/\{2\xi \sqrt{(1-\xi^2)}\} \quad \ldots.. (5)$$

It is seen that as ξ approaches ‘0’

ωr approaches ω**n**, and

M**r** approaches ∞

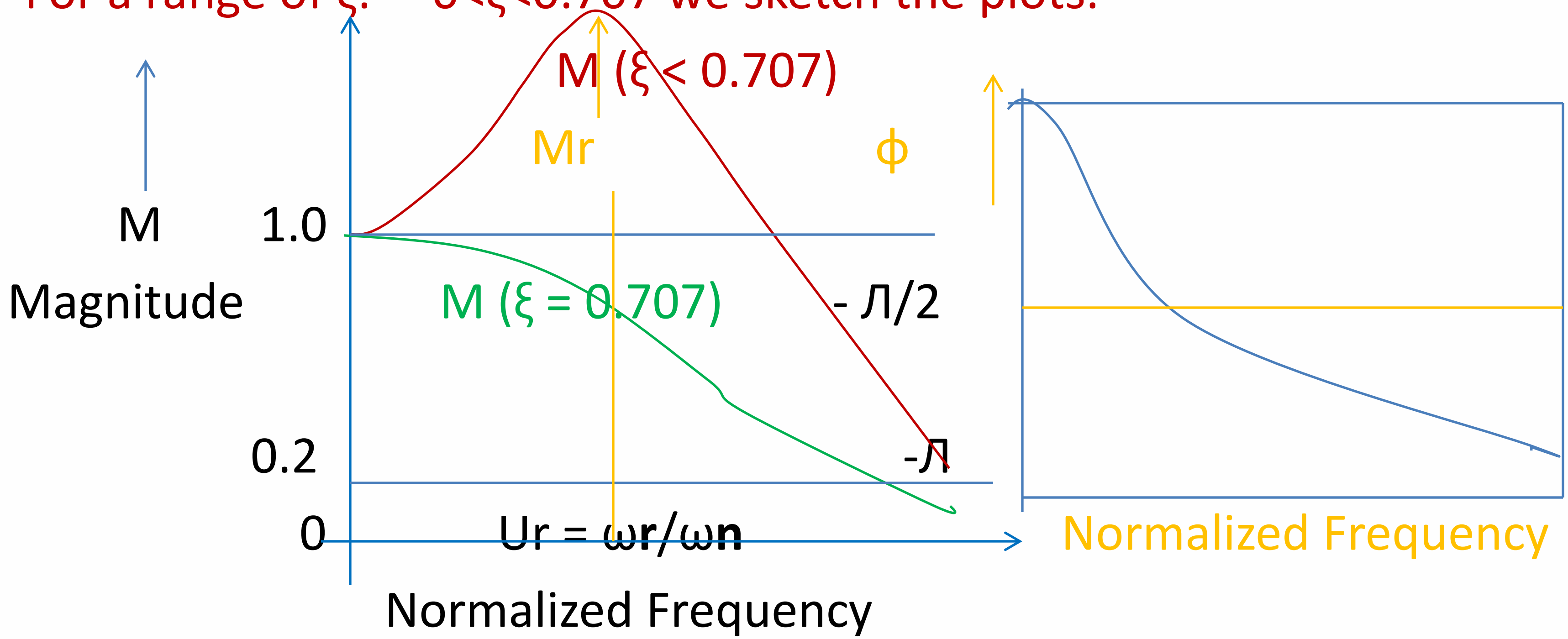
At $\xi = 0.707$; M**r** = 1 & ωr = 0

Therefore there is no resonant peak & hence no resonant frequency.

Frequency Domain Specifications

❑ The magnitude & phase plot:

For a range of ξ: 0<ξ<0.707 we sketch the plots.



- We observe that for ξ≥ 0.707, the magnitude plot decreases monotonically from M=1 at u=0. Thus there is no resonant peak for ξ≥ 0.707 & the greatest value of M = 1.0

Frequency Domain Specifications (contd)..

- **Bandwidth, ωb:**

  The frequency at which M = 0.707 ($1/\sqrt{2}$) is called cut off frequency, ω**c**.

- The range of frequencies for which $M \geq 1/\sqrt{2}$ is defined as bandwidth, ω**b** of a system. Since control systems are low pass filters, ω**b** = ω**c** .
- At u = u**b** = ω**b** / ω**n**; (the normalized bandwidth), the expression for M is

$$M(u_b) = 1/\sqrt{\{(1-u_b^{\,2})^2 + 4\xi^2 u_b^{\,2}\}} = 1/\sqrt{2}$$

  Solving the above equation, we get

  $ub^4 - 2(1-2\xi^2)ub^2 - 1 = 0$ Let $ub^2 = x$; solve for x & then for ub. $Ub = \sqrt{x}$

- Solving for ub we get: $ub = \sqrt{[1-2\xi^2 + \sqrt{(2-4\xi^2+4\xi^4)}]}$

Frequency Domain Specifications (contd)..

❑ Bandwidth:

The denormalized bandwidth is given by,

$$\omega b = \omega \mathbf{n} \sqrt{[1-2\xi^2 + \sqrt{(2-4\xi^2+4\xi^4)}]}$$

Thus, we observe that bandwidth is a function of damping, ξ only.

ξ	ωb
0.2	1.51 ω**n**
0.5	1.272 ω**n**
0.707	0.999 ω**n**

Thus we observe that as damping increases the bandwidth reduces.

Frequency Domain Specifications (contd)..

- Correlation between time and frequency domain parameters:

Time Domain:

$$M_p = \exp(-\pi\xi/\sqrt{1-\xi^2})$$

$$t_p = \pi/\omega_n \sqrt{1-\xi^2}; \qquad \omega_d = \omega_n \sqrt{1-\xi^2}$$

Frequency Domain:

$$M_r = 1/\{2\xi \sqrt{1-\xi^2}\}; \qquad \omega_r = \omega_n \sqrt{1-2\xi^2}$$

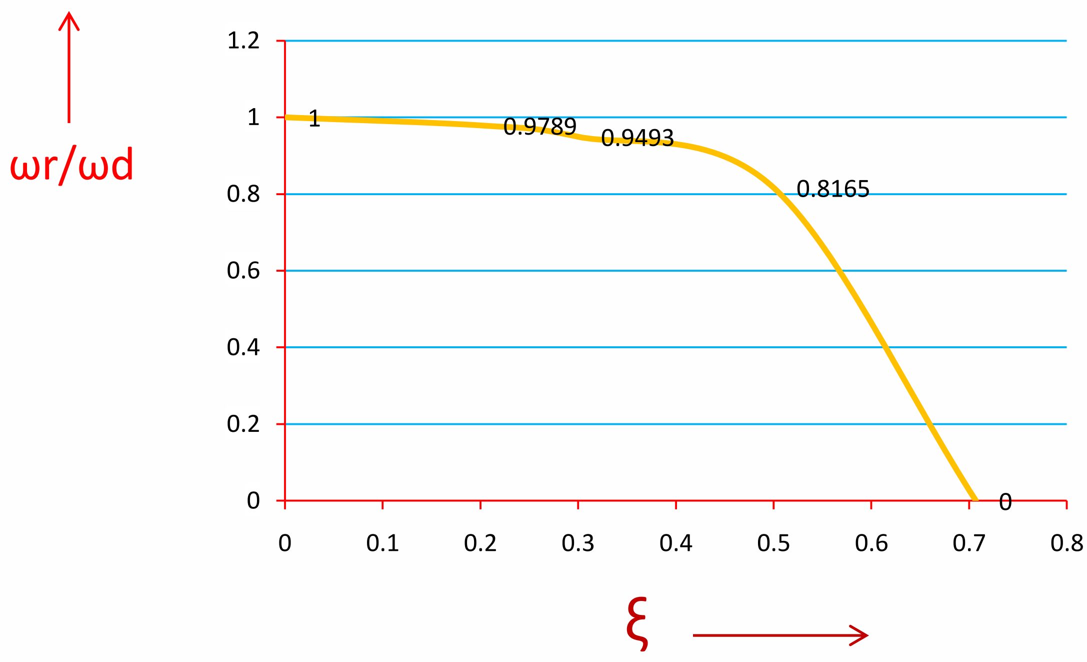
- From the above equations we understand that no matter in which domain (frequency or time) we are analyzing a system performance, the other domain (time or frequency) parameters can be easily estimated using the above set of relationships.
- For example, working in time domain from the root locus we can fix ξ, ω_n, for a desired location of closed loop poles and then we can determine frequency domain parameters using above equations.

Frequency Domain Specifications (contd)..

- Correlation between time & frequency domain parameters:

$$\omega_r / \omega_d = \sqrt{(1-2\xi^2)} / \sqrt{(1-\xi^2)}$$

ξ	ω_r / ω_d
0.707	0.0
0.5	0.8165
0.3	0.9493
0.2	0.9789
0.0	1.0



POLAR PLOT

❑ Polar Plot:

Magnitude and phase of $G(j\omega)$ is plotted in X-Y plane (graph sheet)

$$G(j\omega) = Re[G(j\omega)] + Img\,[G(j\omega)]$$

$$G(j\omega) = |G(j\omega)|\ \arg\{G(j\omega)\} = M\exp(-j\phi)$$

As ω is varied from ‘0’ to ‘∞’; the ‘$M(\omega = \omega 1)$’ value is marked on the graph sheet at an angle of $\phi(\omega = \omega 1)$

❑ Example 1:

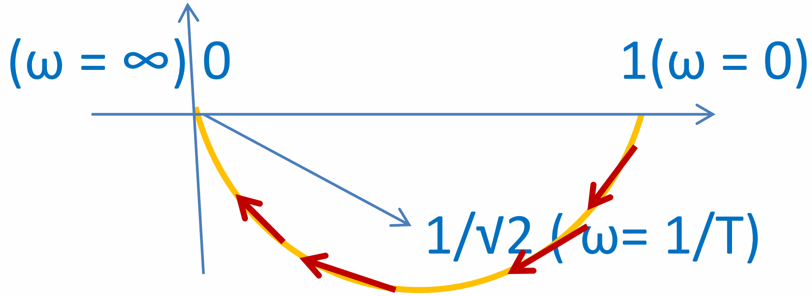
$$G(S) = 1/(1 + TS) \Rightarrow G(j\omega) = 1/(1 + j\,\omega T)$$

$$\Rightarrow M(\omega) = 1/\sqrt{(1 + (\omega T)^2)}; \qquad \phi(\omega) = -\tan^{-1}(\omega T)$$

$\omega \rightarrow 0;\ M = 1 \qquad \phi = 0°$

$\omega \rightarrow \infty;\ M = 0 \qquad \phi = -\pi/2$

$\omega = 1/T;\ \ M = 1/\sqrt{2} \quad \phi = -\pi/4$



POLAR PLOT (Contd..)

❑ Observations:

1. The $\omega = 0$ & $\omega = \infty$ are important points in a polar plot.
2. The angle subtended by $G(j\omega)$ or $G(j\omega)\ H(j\omega)$ at these frequencies indicate the number of quadrants the polar plot is going to traverse in the $G(j\omega)$ or $G(j\omega)\ H(j\omega)$ plane.
3. As we shall see later the intersection of the polar plot with the negative real axis of the $G(j\omega)$ or $G(j\omega)\ H(j\omega)$ plane is a very important information because it allows us to determine the stability of a CL system, as also its relative stability.
4. Polar plot need not be drawn for all the frequencies from 0 to ∞; the necessary points are $\omega = 0$ & $\omega = \infty$ and those values of ω at which the polar plot intersects with the negative real axis of the $G(j\omega)$ or $G(j\omega)$ $H(j\omega)$ plane.

POLAR PLOT (Contd..)

- Example 2:

$$G(S) \text{ or } G(S)H(S) = 1/S(1+TS)$$

$$G(j\omega) = 1/j\omega\,(1 + j\,T\omega); \quad M(\omega) = 1/\omega\,\sqrt{(1 + T^2\omega^2)};$$

$$\phi(\omega) = -\pi/2 - \tan^{-1}(T\omega)$$

$\omega = 0$; $M = \infty$; $\phi = -\pi/2$ Angle measured in CW direction: -

$\omega = \infty$; $M = 0$; $\phi = -\pi$ Angle measured in CCW direction: +

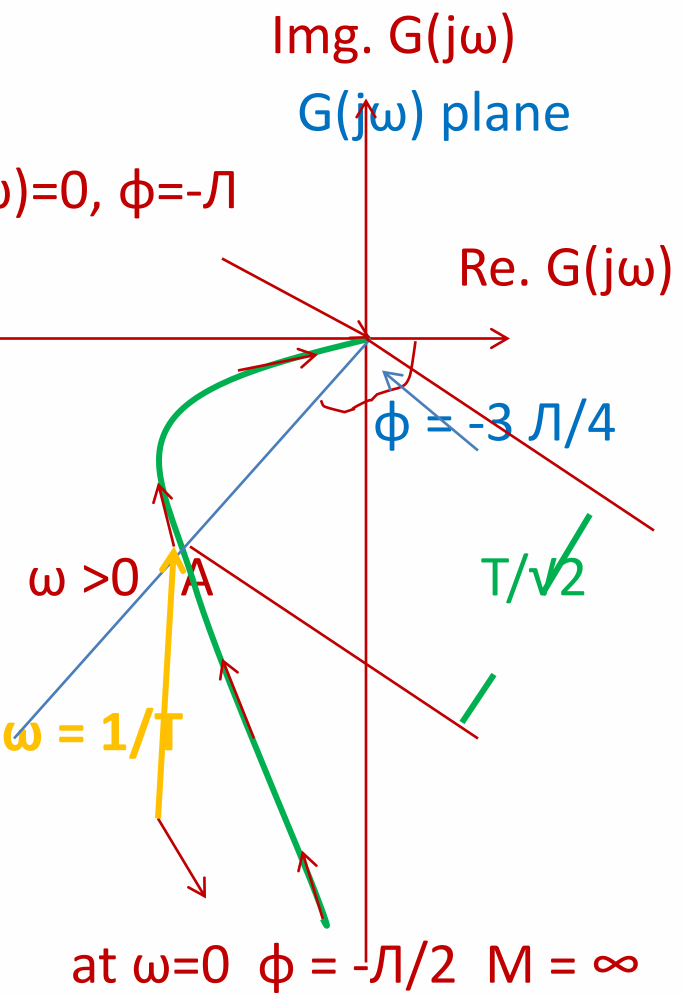
$\omega = 1/T$; $M = T/\sqrt{2}$ $\phi = -3\pi/4$

- Note: we observe that between $\omega = 0$ & $\omega = \infty$ the angle changes by $\pi/2$; therefore the polar plot will traverse only in one quadrant.

The polar plot is shown in the next slide

POLAR PLOT (Contd..)

- Polar plot:



- At ω =1/T; M = T/√2 ϕ = -3Л/4
  In order to plot this point, we draw an angle ϕ = -3Л/4 and then mark the point M = T/√2
  At point A,
  M = T/√2, ϕ = -3Л/4, ω = 1/T

POLAR PLOT (Contd..)

- Example 3:

  $G(S) = 1/(1+T_1 S)(1+T_2 S)$; $G(j\omega) = 1/(1 + j\,\omega T_1)\,(1 + j\,\omega T_2)$

  $$M(\omega) = 1/\sqrt{(1+ \omega^2 T_1^2)}\,\sqrt{(1+ \omega^2 T_2^2)}$$

  $$\phi(\omega) = -\tan^{-1}(T_1\omega) - \tan^{-1}(T_2\omega)$$

  $\omega = 0$; $M = 1$; $\phi = 0$ Angle measured in CW direction: -

  $\omega = \infty$; $M = 0$; $\phi = -\pi$ Angle measured in CCW direction: +

- We observe that ϕ changes from 0 to $-\pi$ as ω changes from 0 to ∞.
- Therefore, the polar plot will traverse two quadrants in the $G(j\omega)$ or $G(j\omega)\,H(j\omega)$ plane.
- Since the polar plot traverses two quadrants, we need to determine point(s) of intersection between polar plot & the Imaginary & negative real axis of the $G(j\omega)$ plane.

POLAR PLOT (Contd..)

- Intersection with real & imaginary axis of the G(jω) plane:
- Procedure:

1. Rationalize G(jω) or G(jω) H(jω)
2. Separate in to real & imaginary parts of G(jω) or G(jω) H(jω)
3. For intersection on real axis; imaginary part = 0. Make imaginary part = 0 by making its numerator = 0. We get value of ω at point of intersection. Calculate the value of real part at this value of ω. Draw a vector of this length from the origin to get intersection on the real axis.
4. For intersection on imaginary axis; real part = 0. Make real part = 0 by making its numerator = 0. We get value of ω at point of intersection. Calculate the value of imaginary part at this value of ω. Draw a vector of this length from the origin to get intersection on the real axis.

POLAR PLOT (Contd..)

- Determination of Intersection point(s):
- $G(j\omega)$ can be written as, $G(j\omega) = 1/[(1-\omega^2T_1T_2) + j\,\omega(T_1 + T_2)]$

  Rationalize: multiply & divide $G(j\omega)$ by $[(1-\omega^2T_1T_2) - j\,\omega(T_1 + T_2)]$; that is conjugate of the denominator.

  We get,

  $G(j\omega) = [(1-\omega^2T_1T_2) + j\,\omega(T_1 + T_2)]/[(1-\omega^2T_1T_2)^2 + \omega^2(T_1 + T_2)^2]$

  Real part $= (1-\omega^2T_1T_2)/[(1-\omega^2T_1T_2)^2 + \omega^2(T_1 + T_2)^2]$

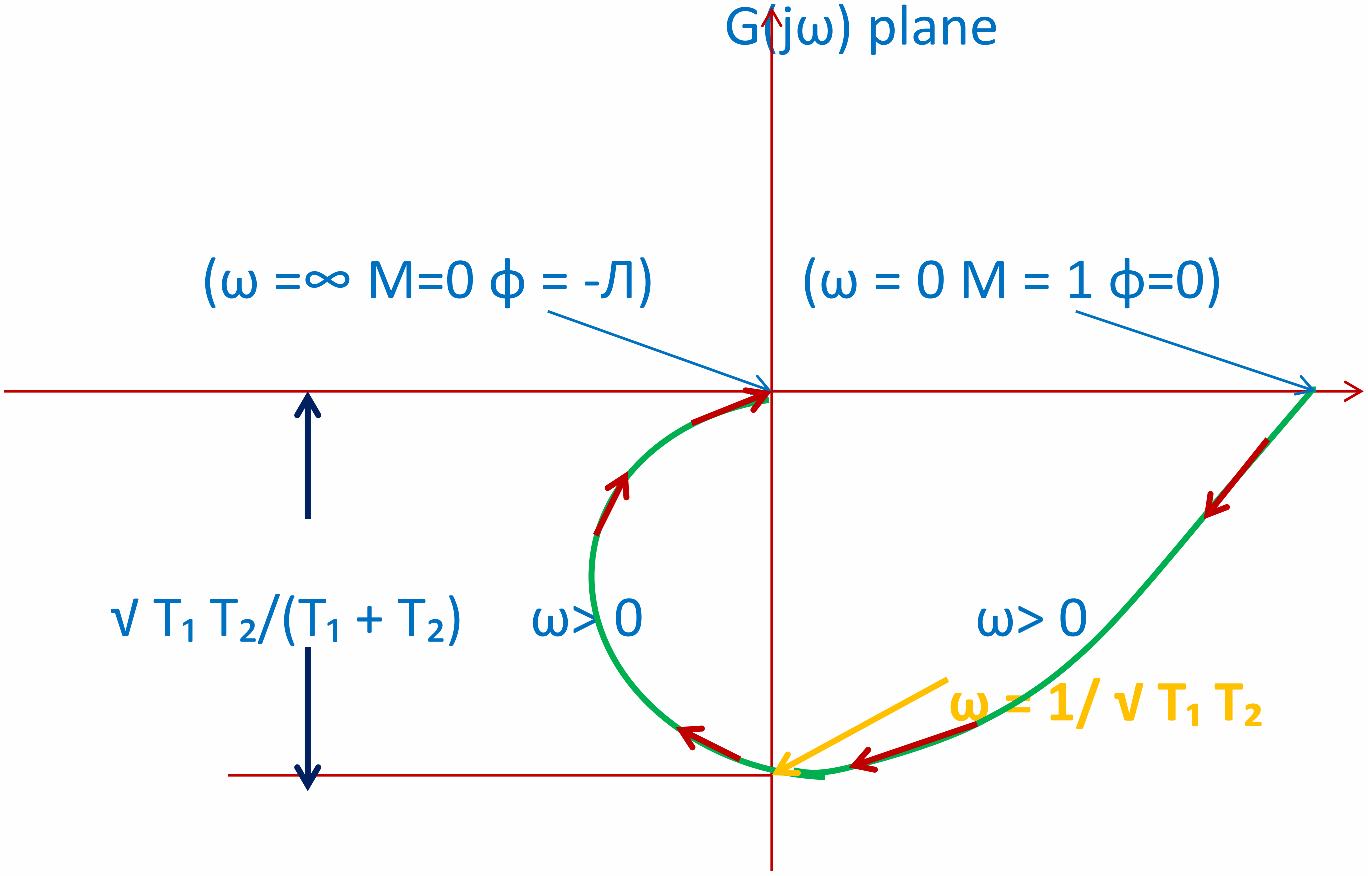
  Imaginary part $= \omega(T_1 + T_2)/[(1-\omega^2T_1T_2)^2 + \omega^2(T_1 + T_2)^2]$
- We see from the above that

  Imag. Part cannot be zero, &

  Real part = 0 for $1-\omega^2T_1T_2 = 0$; $\omega^2 = 1/T_1T_2$

  at intersection on imaginary axis, the frequency $\omega = 1/\sqrt{T_1T_2}$

POLAR PLOT (Contd..)



POLAR PLOT

- Example 4:

  $G(S) = 1/(1+T_1 S)(1+T_2 S)(1+T_3 S)$;

  $G(j\omega) = 1/(1 + j\,\omega T_1)(1 + j\,\omega T_2)(1 + j\,\omega T_3)$

  $M(\omega) = 1/\sqrt{(1+ \omega^2 T_1^2)}\sqrt{(1+ \omega^2 T_2^2)}\sqrt{(1+ \omega^2 T_3^2)}$

  $\phi(\omega) = -\tan^{-1}(T_1\omega) - \tan^{-1}(T_2\omega) - \tan^{-1}(T_3\omega)$

  $\omega = 0$; $M = 1$; $\phi = 0$ Angle measured in CW direction: -

  $\omega = \infty$; $M = 0$; $\phi = -3\pi/2$ Angle measured in CCW direction: +

- We observe that ϕ changes from 0 to $-3\pi/2$ as ω changes from 0 to ∞.
- Therefore, the polar plot will traverse three quadrants in the $G(j\omega)$ or $G(j\omega)\,H(j\omega)$ plane.
- Since the polar plot traverses three quadrants, we need to determine point(s) of intersection between polar plot & the Imaginary & negative real axis of the $G(j\omega)$ plane.

POLAR PLOT (Contd..)

- Intersection on the Real & Imaginary axis of $G(j\,\omega)$ plane:

  Following the procedure as explained earlier, we have:

- For intersection on Imaginary Axis:

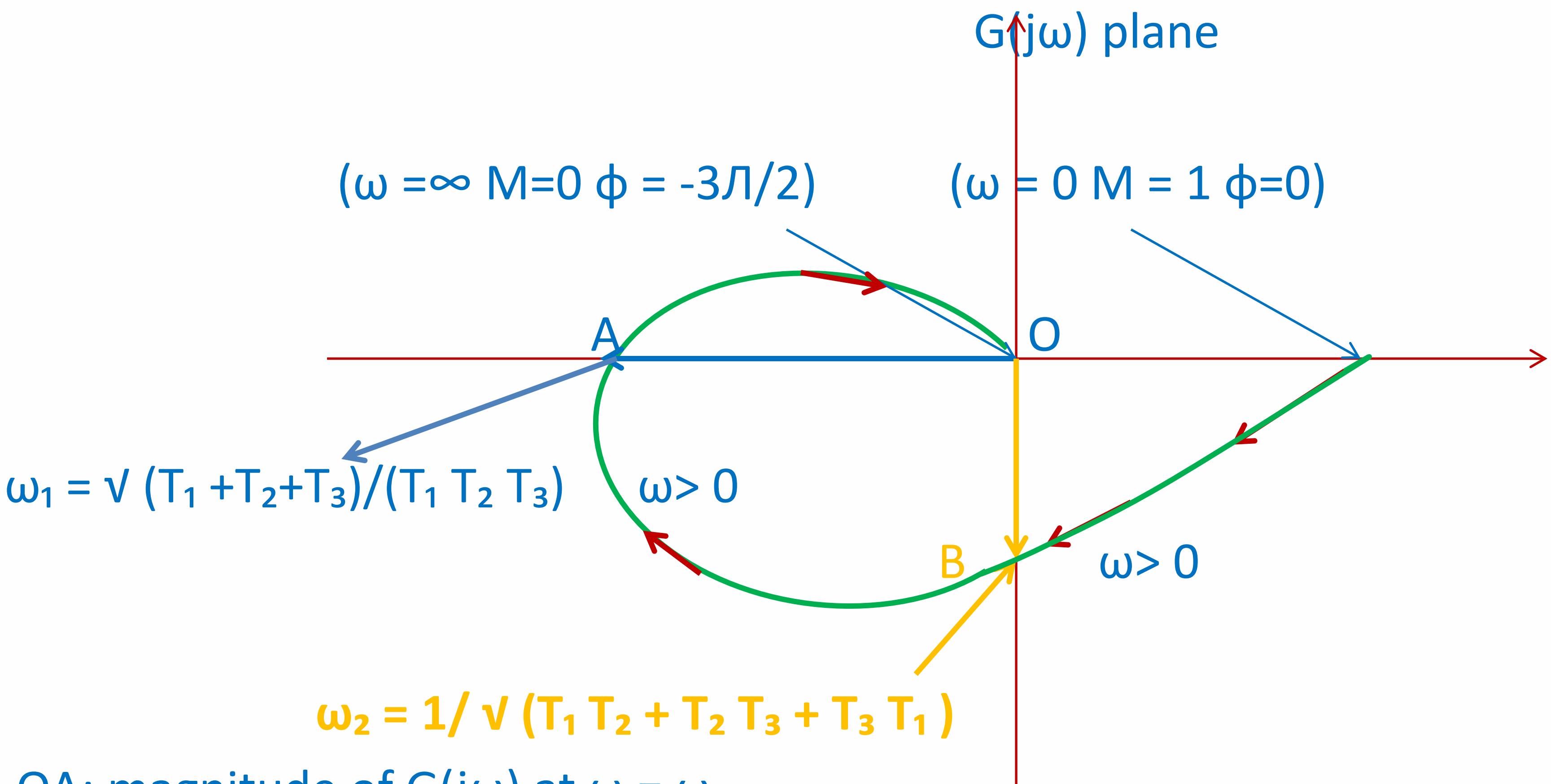
$$\omega = 1/\sqrt{(T_1 T_2 + T_3 T_1 + T_2 T_3)}$$

- For intersection on real Axis:

$$\omega = \sqrt{[T_1 + T_2 + T_3 / T_1 T_2 T_3]}$$

  For the above values of ω, determine the magnitude of the points with imaginary intersection.

POLAR PLOT (Contd..)



OA: magnitude of $G(j\omega)$ at $\omega = \omega_1$

OB : magnitude of $G(j\omega)$ at $\omega = \omega_2$

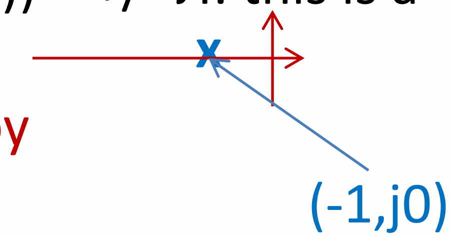
POLAR PLOT (Relative Stability)

Relative Stability:

1. It is defined for systems that are open loop stable.
2. We have the Characteristic equation $Q(S) = 1 + G(S)H(S) = 0$
3. For real frequencies (frequency response) $S = j\omega$
4. Therefore, $Q(j\omega) = 1 + G(j\omega) H(j\omega) = 0$
5. Or, $G(j\omega) H(j\omega) = -1$
6. therefore, $|G(j\omega) H(j\omega)| = 1$ & $\arg(G(j\omega) H(j\omega)) = \phi = \pm \pi$
7. When loop gain = $|G(j\omega) H(j\omega)| = 1$ & $\arg(G(j\omega) H(j\omega)) = \pm \pi$
8. Phase introduced due to error detector = 180°
9. Therefore, total phase in the loop = 360° & $|G(j\omega) H(j\omega)| = 1$
10. The CL system response is oscillatory & it is on the verge of instability

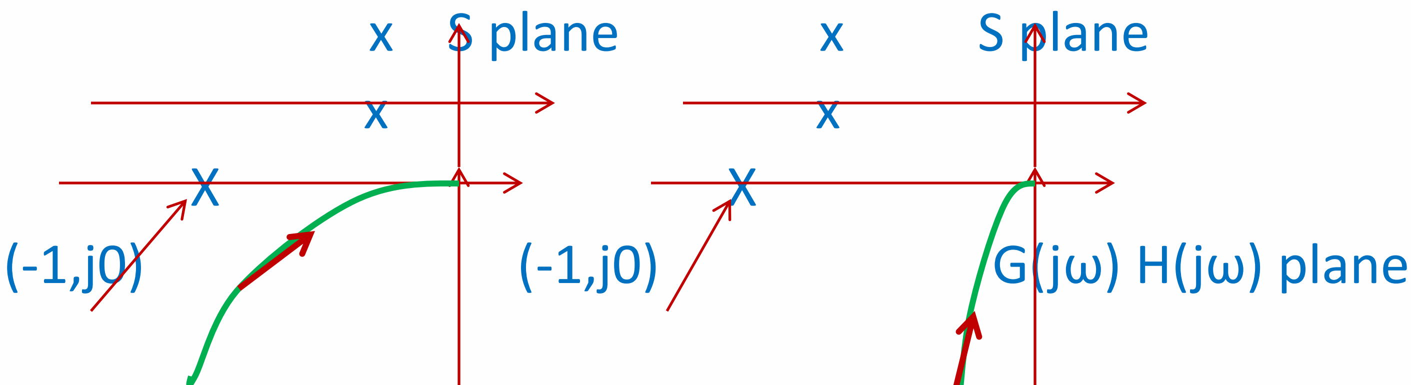
POLAR PLOT (Relative Stability Contd..)

11. loop gain = |G(jω) H(j ω)|= 1 & arg(G(jω) H(j ω)) = +/- Л: this is a point (-1, j0) in the G(jω) H(j ω) plane.



12. Stability of a closed loop system is determined by non-encirclement of (-1,j0) point. As the polar plot gets closer to (-1,j0) point, the CL system tends towards instability.

Polar plot & Location of closed loop poles:



We observe that polar plot closer to (-1,j0) point implies CL poles are closer to the Imaginary axis of the S-plane

POLAR PLOT (Relative Stability Contd..)

- As the CL poles move closer to the Imaginary axis of the S plane, the system takes more time to settle down (reach steady state) & is therefore relatively less stable than the one which has CL poles far removed from the Imaginary axis of the S plane.
- In frequency domain it implies that as the polar plot moves closer to the (-1,j0) in the G(jω) H(jω) plane, the CL system becomes relatively less & less stable.
- Therefore proximity of the polar plot to the (-1,j0) point determines CL system's relative stability.
- If the polar plot passes through (-1,j0) point then the CL system is on the verge of instability
- If the polar plot encircles the (-1,j0) point then the CL system is unstable.

POLAR PLOT (Relative Stability Contd..)

❑ Example of Relative stability:

Plot 1:

Intersects negative real axis at 'b'

Plot 2:

Intersects negative real axis at 'c'

Plot 3:

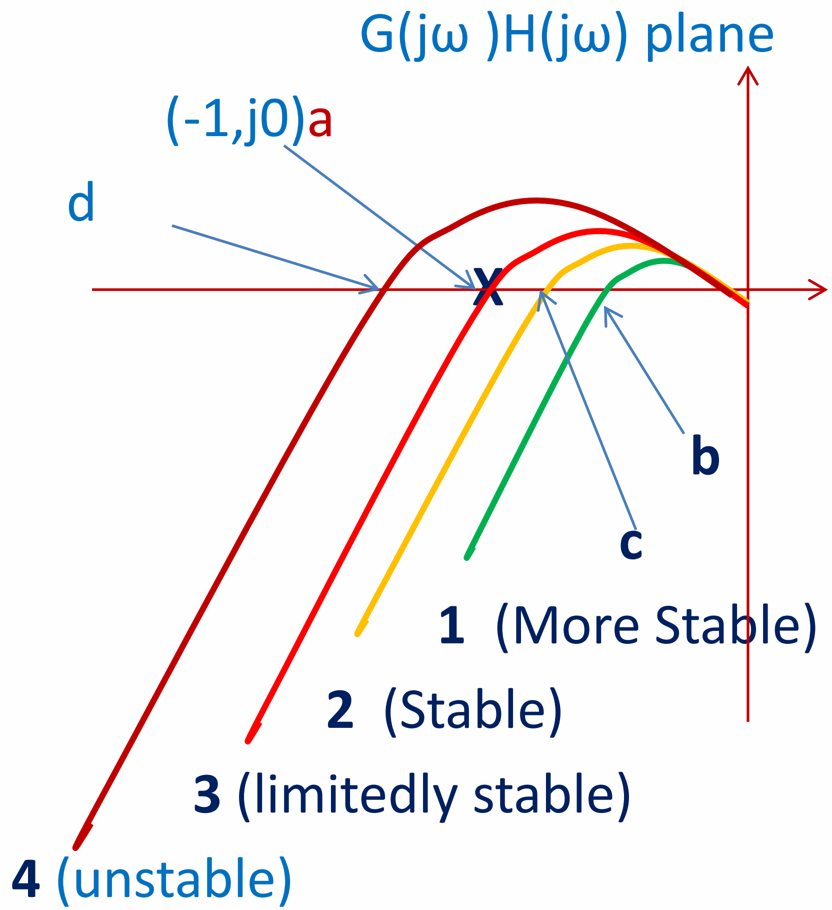
Passes through (-1,j0) point

Plot 4:

Encircles (-1,j0) point &

intersects negative real axis

at 'd'

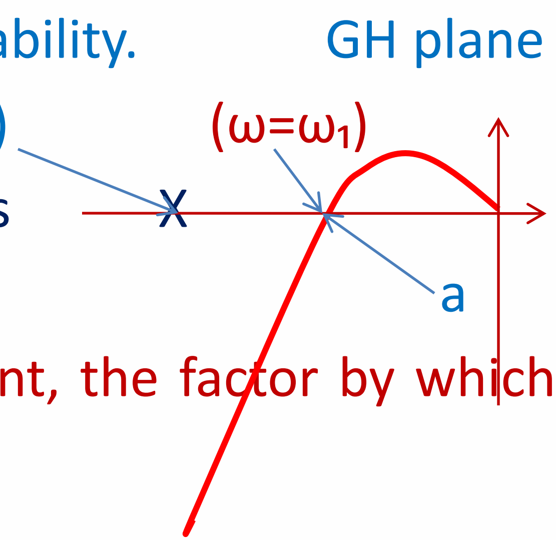


|b|<|c|<|a|<|d|

Relative Stability Index: Gain & Phase Margin

❑ Gain Margin:

1. The margin between actual gain ‘K’ (of the system) and the critical gain causing oscillations (in the system output) is called Gain Margin (GM)
2. Critical gain: the value of ‘K’ at which the Polar plot- { $G(j\omega)H(j\,\omega)$} plot - passes through (-1,j0) point.
3. Definition of GM: It is the factor by which the system gain can be increased to drive it to the verge of instability.
4. At $\omega = \omega_1$, the magnitude of intersection with the negative real axis is ‘a’; the phase angle = Л



5. For the plot to pass through (-1,j0) point, the factor by which the gain is to be increased = 1/a. GM = 1/a

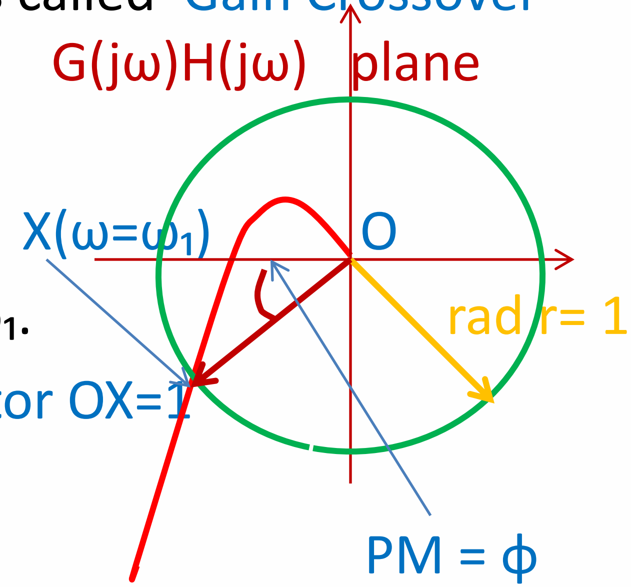
Relative Stability Index: Gain & Phase Margin

1. $|G(j\omega)H(j\omega)| = a$, at $\omega = \omega_1$
2. $\arg\{G(j\omega)H(j\omega)\} = \phi = \pi$, at $\omega = \omega_1$
3. $\omega = \omega_1$ is the frequency at which $\phi = 180°$.
4. $\omega = \omega_1$ is called 'Phase Crossover Frequency'
5. Phase crossover frequency: is defined as the frequency at which the phase offered by the system is π
6. Gain Margin is now defined in terms of phase crossover frequency as
7. 'reciprocal of the gain at the frequency at which phase angle becomes 180'
8. Thus GM value is obtained at phase crossover frequency.
9. GM = 1/a; In decibels: GM = 20 Log(1/a) = - 20 Log(a)

Relative Stability Index: Gain & Phase Margin

❑ Phase Margin:

1. It is calculated at 'Gain Crossover Frequency'
2. The frequency at which $|G(j\omega)H(j\omega)| = 1$ is called 'Gain Crossover frequency'
3. Draw a **unit circle** as shown.
4. The point of intersection of unit circle
5. with polar plot is X , say, the frequency is ω_1.
6. The $|G(j\omega)H(j\omega)|$ (at $\omega=\omega_1$) = length of vector OX=1



7. Therefore $\omega=\omega_1$ is the gain cross over frequency.
8. The angle made by OX with the negative real axis of the G(jω)H(jω) plane is Phase Margin (PM), ϕ, of the system.

Relative Stability Index: Gain & Phase Margin

Phase Margin & Stability of CL system:

1. It is defined as the amount of additional phase lag at the gain cross over frequency required to bring the system to the verge of instability.
2. It is measured in the CCW direction from the negative real axis of the $G(j\omega)\,H(j\omega)$ plane.
3. If $\omega = \omega_1$ is the gain cross over frequency, then phase margin (PM) is computed as:
4. $PM = \phi = \arg\{G(j\omega_1)\,H(j\omega_1)\} + 180°$
5. Since systems introduce phase lag , $\arg\{G(j\omega_1)\,H(j\omega_1)\}$ is always negative.
6. If PM is positive, the CL system is stable
7. If PM is negative the CL system is unstable
8. If PM = 0 the CL system is on the verge of instability

Relative Stability Index: Gain & Phase Margin

❑ GM & Stability of CL system:

GM is calculated as the inverse of the $|G(j\omega)H(j\omega)|$ = 'a' at the point of its intersection with negative real axis of the GH plane.

GM = 1/a ; or, GM = - 20 Log (a) in dB.

1. If GM is positive, CL system is stable
2. If GM is negative, CL system is unstable
3. If GM = '0', CL system is on the verge of instability

❑ Interpretation of Relative Stability from GM & PM Values:

1. Large GM or large PM imply sluggish CL system
2. GM close to '1' or PM close to '0°' imply highly oscillatory system
3. GM of about 6 dB or PM of 30-35° imply reasonably good degree of relative stability
4. Generally a good GM automatically guarantees a good PM & vice-versa.

Relative Stability Index: Gain & Phase Margin

- Special Cases:

  We have said that generally a good GM yields good PM & vice versa. In certain cases, it may not hold.

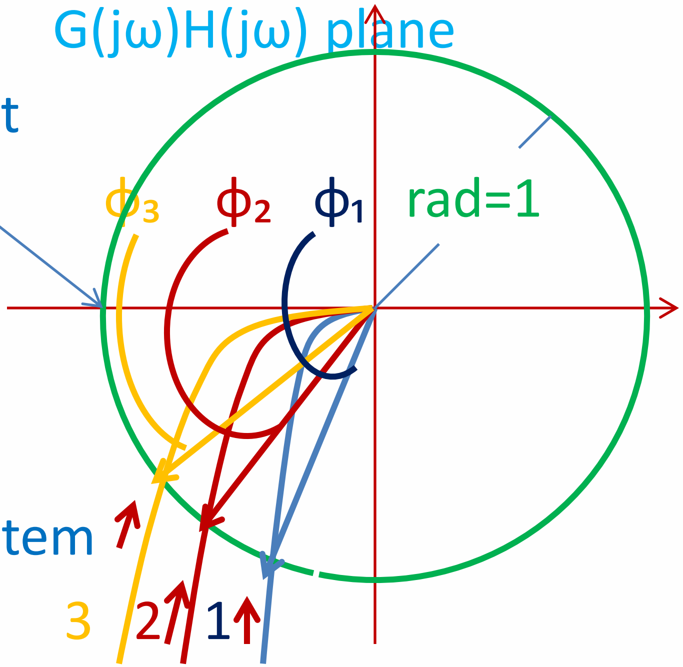
- Case 1:

  Plot 1: gain K_1 ;PM = ϕ_1; GM = ∞

  Plot 2: gain K_2; PM = ϕ_2; GM = ∞

  Plot 3: gain K_3; PM = ϕ_3; GM = ∞

  $K_3 > K_2 > K_1$; $\phi_3 < \phi_2 < \phi_1$



- We see that as we increase gain in the system the Phase Margin reduces whereas the
- Gain Margin does not change. Therefore in such cases we need to focus only on PM because GM is not adjustable.

Relative Stability Index: Gain & Phase Margin

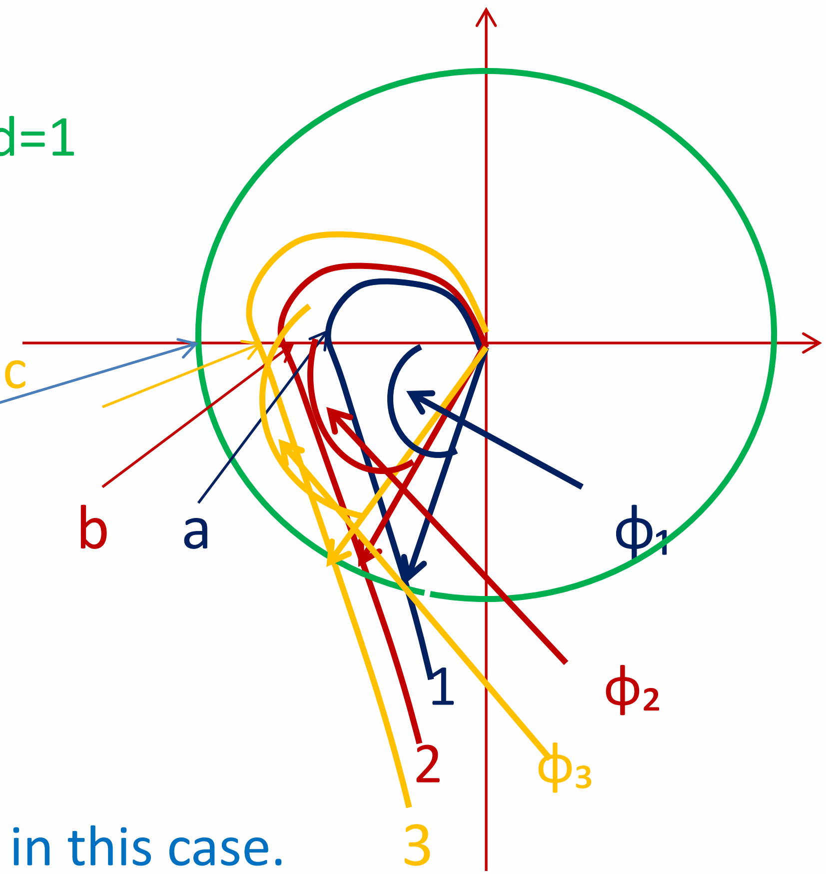
- Case 2:

  Plot 1: gain K_1 ;PM = ϕ_1; GM = 1/a rad=1

  Plot 2: gain K_2; PM = ϕ_2; GM = 1/b

  Plot 3: gain K_3; PM = ϕ_3; GM = 1/c

  $K_3 > K_2 > K_1$; $\phi_3 < \phi_2 < \phi_1$



- We see that as we increase gain the GM reduces appreciably , but the PM does not vary much.
- Therefore, we need to monitor GM in this case.

Polar Plot: Correlation between PM & ξ

- Correlation between Phase Margin & Damping ξ:

  Let $G(S) = \omega \mathbf{n}^2 / S(S + 2\xi\omega\mathbf{n})$; for a unity feedback system

- At the gain cross over frequency, $\omega = \omega_1$

$$|G(j\,\omega)H(j\omega)| = 1.0$$

$$\text{or,}\quad \omega\mathbf{n}^2 / \omega_1\sqrt{(\omega_1^2 + 4\,\xi^2\,\omega\mathbf{n}^2)} = 1.0$$

$$\text{or,}\quad \omega_1^2(\omega_1^2 + 4\,\xi^2\,\omega\mathbf{n}^2) = \omega\mathbf{n}^4$$

$$\text{or,}\quad (\omega_1/\,\omega\mathbf{n})^4 + 4\,\xi^2\,(\omega_1/\,\omega\mathbf{n}\,)^2 - 1 = 0;\ \text{let } (\omega_1/\omega\mathbf{n}\,)^2 = x$$

$$\text{or,}\quad x^2 + 4\,\xi^2\,x - 1 = 0$$

$$\text{or,}\quad x = -2\,\xi^2 +/- \sqrt{(1 + 4\,\xi^4)}$$

$$\text{or,}\quad (\omega_1/\,\omega\mathbf{n}\,)^2 = \sqrt{(1 + 4\,\xi^4)} - 2\,\xi^2$$

$$\text{or,}\quad \omega_1 = \omega\mathbf{n}\,\sqrt{(\sqrt{(1 + 4\,\xi^4)} - 2\,\xi^2)}$$

- The above equation relates ξ with gain cross over frequency, ω_1

Polar Plot: Correlation between PM & ξ

$$\arg\{G(j\omega)H(j\omega)\} = -90° - \tan^{-1}(\omega/2\xi\omega_n)$$

at $\omega = \omega_1$, $\phi_1 = -90° - \tan^{-1}(\omega_1/2\xi\omega_n)$

$$PM = \phi = 180° + \phi_1 = 180° - 90° - \tan^{-1}(\omega_1/2\xi\omega_n)$$

$$\phi = 90° - \tan^{-1}(\omega_1/2\xi\omega_n)$$

- Substitute for ω_1 to get,

$$\phi = 90° - \tan^{-1}[\sqrt{\sqrt{1+4\xi^4} - 2\xi^2}/2\xi]$$

or, $[\sqrt{\sqrt{1+4\xi^4} - 2\xi^2}/2\xi] = \tan(90° - \phi) = \cot\phi$

or, $\tan\phi = 2\xi / [\sqrt{\sqrt{1+4\xi^4} - 2\xi^2}]$

or, $\phi = \tan^{-1}\{2\xi / [\sqrt{\sqrt{1+4\xi^4} - 2\xi^2}]\}$

- The above equation gives a relationship between ξ & ϕ for an under damped system.
- In the range $\xi \le 0.707$, a reasonably good approximation is given by

$$\xi = 0.01\,\phi$$

Polar plot Examples: Computation of GM & PM

$$G(j\omega) = K/j\omega\,(1+j\,0.2\omega)(1+j\,0.05\,\omega)$$

- For $K = 1$:

  PM = $\phi = 76°$;

- Intersection on negative real axis, $a = -0.04$

  GM = 20 Log $|1/a|$ = 28 dB

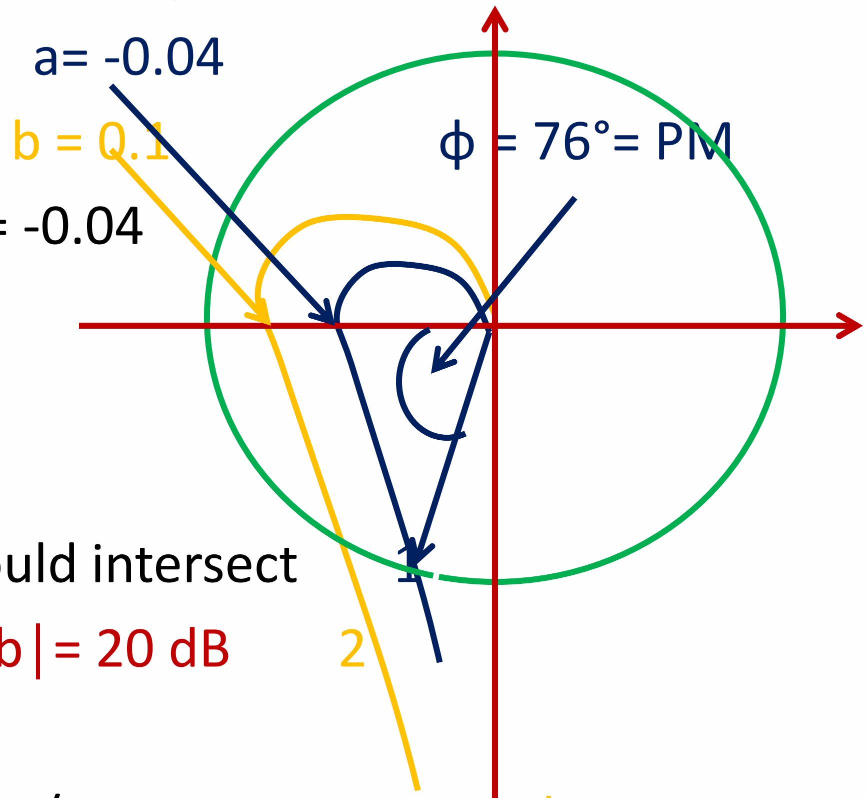
- Suppose we desire a GM = 20 dB, &

  PM = 40°

- For a GM = 20 dB, the polar plot should intersect the negative real axis at : 20 Log $|1/b|$ = 20 dB

  therefore, $b = 0.1$

- This is achieved if K is increased by 0.1/0.04 = 2.5; K = 2.5. Plot 2

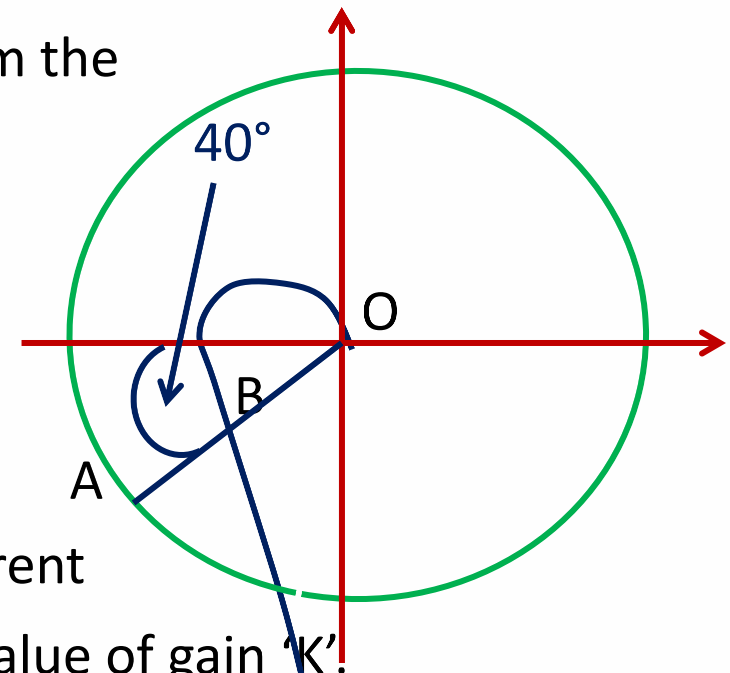


Polar plot Examples: Computation of GM & PM

- To achieve PM = 40°, we have:
- Draw an angle of 40° in CCW direction from the negative real axis of GH plane, as shown
- We see that for PM = 40° , gain ‘K’ is to be increased by the ratio OA/OB

$$OA/OB = 1/0.191 = 5.24$$

$$K = 5.24$$



- Thus we note that GM & PM are two different
- Specifications not achievable for a single value of gain ‘K’.

Analytical Method: Gain & Phase Margin

❑ Example:

❑ $G(S) = K/S(1+0.2S)(1+.05S)$ ⟹ $G(j\omega) = K/j\omega(1+j0.2\omega)(1+j0.05\omega)$

❑ We know that for determining GM, we need to find intersection on negative real axis (Imaginary part = 0).

❑ Determine value of ω for which Imaginary part = 0.

Simplify $G(j\omega)$ to get $G(j\omega) = K/[-0.25\,\omega^2 + j\omega\,(1-0.01\,\omega^2)]$

Rationalize $G(j\omega)$ to get,

$$G(j\omega) = -0.25K\,\omega^2/Den \; - j\,\omega(1-0.01\,\omega^2)/Den$$

Where, $Den = [(-0.25\,\omega^2)^2 + (\omega(1-0.01\,\omega^2))^2]$

For Imaginary part = 0, ⟹ $1-0.01\,\omega^2 = 0$; ⟹ $\omega = 10 = \omega_1$

ω_1: phase cross over frequency. Magnitude of $G(j\omega)$ at $\omega = \omega_1$

$|G(j\omega)| = K/0.25(\omega_1)^2 = K/25 = a$ (Contd.)

Analytical Method: Gain & Phase Margin

- For a desired GM = 20 dB, we have

$$20 \text{ Log } (1/a) = 20 \quad , \text{or, } a = 1/10 = 0.1$$

$$K/25 = a; \ K = 2.5$$

- Calculation of PM:

Let $\omega = \omega_2$ be the gain crossover frequency;

$PM = 180° + \arg\{G(j\omega)\}$; Desired PM = 40°

$\arg\{G(j\omega)\} = -90° - \tan^{-1}(0.2\ \omega_2) - \tan^{-1}(0.05\ \omega_2)$

$PM = -90° - \tan^{-1}(0.2\ \omega_2) - \tan^{-1}(0.05\ \omega_2) + 180° = 40°$

$\tan^{-1}(0.2\ \omega_2) - \tan^{-1}(0.05\ \omega_2) = 50°$; Apply tan on

$0.25\ \omega_2/[1-0.01\ \omega_2^2] = \tan 50° = 1.2$ rads; $\omega_2 = 4$ rads/sec

$|G(j\omega)|$ at $\omega = \omega_2$ is $= K/[\omega_2 \sqrt{\{1+(0.2\ \omega_2)^2\}} \sqrt{\{1+(0.05\ \omega_2)^2\}} = 1$

For $\omega_2 = 4$, $K = 5.2$

BODE PLOT

- From the frequency response of open loop transfer function G(S) or G(S)H(S), closed loop system stability & relative stability is determined; as in polar plots & root locus methods.

1. We draw two plots for each transfer function
2. Magnitude plot in dB
3. Phase plot
4. Both the plots are drawn on semi log paper
5. Magnitude in dB is given by 20 Log $|G(j\omega)|$

   or 20 Log $|G(j\omega)H(j\omega)|$

   Angle $\phi(\omega)$ is plotted in degrees

BODE PLOT (Contd..)

- Note on Log Scale:

  The advantage of Log scale is that we can handle a very large data size
- Linear Scale:

  -30 -20 -10 0 10 20

- In linear scale each segment is incremented equally.
- Log Scale:
- In log scale, we decide the multiplication factor ‘x’. Let x = 10

  -2 -1 0 1 2 3 (linear scale) ω

  0.01 0.1 1 10 100 1000 (Log scale) Log ω

BODE PLOT (Contd..)

- Conversion to Log scale:

  Log **10** ω = 0 (on linear scale) ⇨ ω = 1 (on log scale)

  Log **10** ω = 1 (on linear scale) ⇨ ω = 10 (on log scale)

  Log **10** ω = 2 (on linear scale) ⇨ ω = 100 (on log scale)

  Log **10** ω = -1 (on linear scale) ⇨ ω = 0.1 (on log scale)

  Log **10** ω = -2 (on linear scale) ⇨ ω = 0.01 (on log scale)

- We observe from the above that

  1. on the positive side increment by '1' on linear scale corresponds to multiplication by '10' on the Log scale ,and
  2. on the negative side increment by '-1' on linear scale corresponds to division by '10' on the Log scale
  3. We also observe that on the Log scale we cannot start with a value of ω = 0, but it can assume a very small value

BODE PLOT (Contd..)

- Thus, we observe that increment by ‘1’ on linear scale causes multiplication by ‘10’ on Log scale and hence enabling data compression and thus facilitating usage of large chunks of data.
- Further observations on Log scale:
  1. Between ω = 1 & ω = 10 on the log scale, if we want to mark ω = 2 then we write: Log **10** ^{2} = 0.301 (which is 30.1% of the segment length between ‘1’ & ‘10’ on the Log scale
  2. Between ω = 1 & ω = 10 on the log scale, if we want to mark ω = 3 then we write: Log **10** ^{3} = 0.477 (which is 47.7% of the segment length between ‘1’ & ‘10’ on the Log scale
  3. Between ω = 1 & ω = 10 on the log scale, if we want to mark ω = 5 then we write: Log **10** ^{5} = 0.699 (which is 69.9% of the segment length between ‘1’ & ‘10’ on the Log scale

  Thus we see that the marking is not linear.

BODE PLOT (Contd..)

- Representation of Transfer Functions:
- We have two ways of representing a transfer function:
- Pole-Zero Form:

$$G(S) = K \left[\prod_{j=1}^{m} (S + Zj)\right] / \left[\prod_{i=1}^{n} (S + Pi)\right] \;;\; m \le n$$

- Time – Constant Form:

$$G(S) = \{K \textstyle\prod Zj/\prod Pi\} \left\{\left[\prod_{j=1}^{m} (1+S/Zj)\right] / \left[\prod_{i=1}^{n} (1+S/Pi)\right]\right\}$$

- Let $K_1 = K \prod Zj/\prod Pi$; $Tzj = 1/Zj$; $Tpi = 1/Pi$; Tzj & Tpi are time constants

$$G(S) = K_1 \left[\prod_{j=1}^{m} (1+ Tzj\, S)\right] / \left[\prod_{i=1}^{n} (1+ Tpi\, S)\right]$$

BODE PLOT (Contd..)

- Example:

  Given, $G(S) = 10\,(S + 2)\,(S+4)/(S + 5)\,(S + 10)$ in pole- zero form

  Convert in to time constant form

- Solution:

  $G(S) = (10)(2)(4)(1 + S/2)(1+ S/4) \,/\, (5)(10)(1 + S/5)(1 + S/10)$

  $K_1 = (10)(2)(4)/(5)(10) = 8/5$

  $G(S) = (8/5)\ (1+0.5\,S)(1+0.25S)/(1+0.2S)(1+0.1S)$

- Where, $Tz1 = 0.5$; $Tz2 = 0.25$; $Tp1 = 0.2$; $Tp2 = 0.1$ are time constants

- Convert Time constant form in to Pole-Zero form:

  $G(S) = (8/5)(.5)(.25)(S + 1/.5)(S + 1/.25)/[(.2)(.1)(S+1/.2)\ (S+1/.1)]$

  $G(S) = K\,(S + 2)(S + 4)/(S + 5)(S + 10)$

  $K = (8/5)(.5)(.25)/(.2)(.1) = 10$

- In Bode & Polar plots we use Time Constant form

BODE PLOT (Method for Drawing)

- Example:

$$G(S) = 1/(1+TS) \Rightarrow G(j\omega) = 1/(1 + j\,T\omega)$$

$$\Rightarrow |G(j\omega)| = 1/\sqrt{(1 + (T\omega)^2)}\ ; \qquad \arg[G(j\omega)] = -\tan^{-1}(\omega T)$$

- The Log – magnitude in dB is given by:

$$20 \text{ Log } \mathbf{10}\ |G(j\omega)| = M(\omega) = 20 \text{ Log } \mathbf{10}\ [1/\sqrt{(1 + (T\omega)^2)}]$$

$$M(\omega) = -10 \text{ Log } \mathbf{10}\ (1 + (T\omega)^2) \text{ ---------- } 1$$

- Two cases are considered:

1. For $T\omega <<< 1$ (low frequency asymptote); $M(\omega) = 0.0$ because $(T\omega)^2$ can be neglected as compared to '1'
2. For $T\omega >>> 1$ (high frequency asymptote); $M(\omega) = -20 \text{ Log } \mathbf{10}\ (T\omega)$....... 2; '1' can be neglected

ωT (rads)	M(ω) in dB
1	0
10	-20

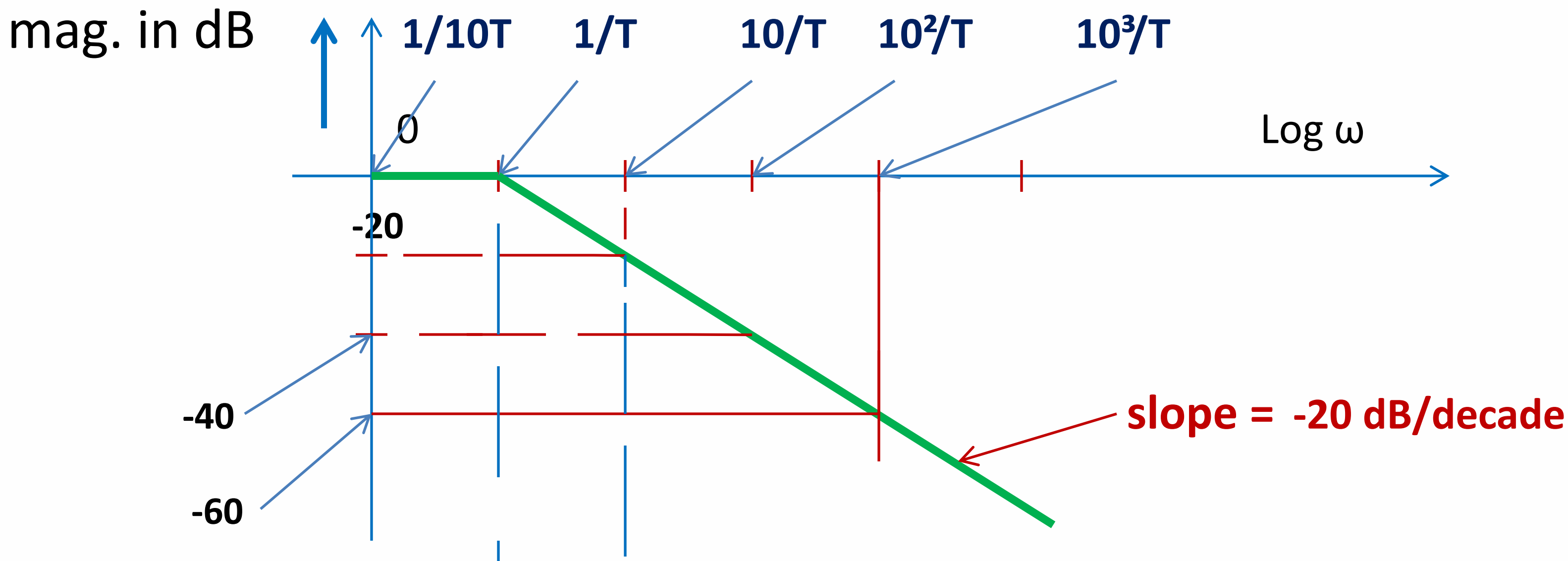
ωT (rads)	M(ω) in dB
100	-40
1000	-60 (cont)

BODE PLOT (Method for Drawing) Contd..

- We observe from the table in the previous slide that,
  1. For a decade change in frequency (1 to 10, 10 to 100, & so on) the magnitude changes by -20 dB.
  2. Therefore the slope of the magnitude plot is -20 dB/decade change in frequency.
- We have two plots: for $\omega T <<< 1$ & $\omega T >>> 1$
- For $\omega T <<< 1$; $M(\omega) = 0$ & for $\omega T >>> 1$; $M(\omega)$ has slope of -20 dB/decade
- At $\omega T = 1$; $M(\omega)$ in equation (2) = 0 dB & $M(\omega)$ in equation (1) =0 therefore the two meet at $\omega T = 1$, if we extend the low frequency asymptote; (as they are both = 0)
- This meeting point is called 'Corner Frequency' & is derived from $\omega T = 1$; or, $\omega = 1/T$ is the corner frequency.

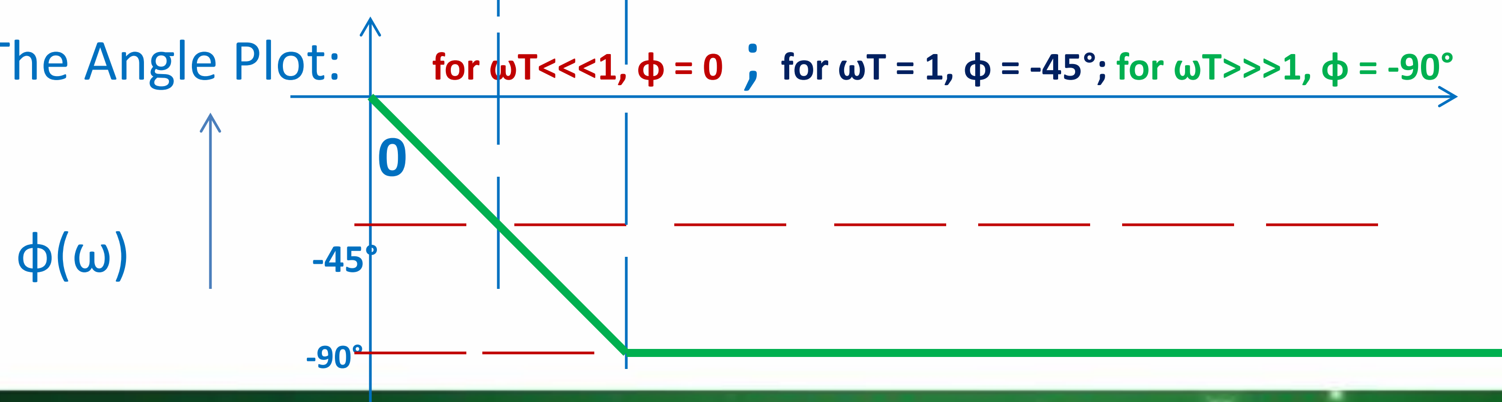
BODE PLOT (Method for Drawing) Contd..

- The Log-magnitude in dB is plotted as:



(Log mag. Plot – semi log graph paper)

- The Angle Plot:



Bode Plots: Different types of Transfer Functions

- Example: First order 'zero'

$G(S) = (1+TS)$ $G(j\omega) = (1 + j\,T\omega)$

$|G(j\omega)| = \sqrt{(1 + (T\omega)^2)}$; $\arg[G(j\omega)] = \tan^{-1}(\omega T)$

- The Log – magnitude in dB is given by:

$20 \text{ Log } \mathbf{10}\, |G(j\omega)| = M(\omega) = 20 \text{ Log } \mathbf{10}\, [\sqrt{(1 + (T\omega)^2)}]$

$M(\omega) = 10 \text{ Log } \mathbf{10}\, (1 + (T\omega)^2)$ ---------- 1

- Two cases are considered:

1. For $T\omega <<< 1$ (low frequency asymptote); $M(\omega) = 0.0$ because $(T\omega)^2$ can be neglected as compared to '1'
2. For $T\omega >>> 1$ (high frequency asymptote); $M(\omega) = 20 \text{ Log } \mathbf{10}\, (T\omega)$… 2; '1' can be neglected

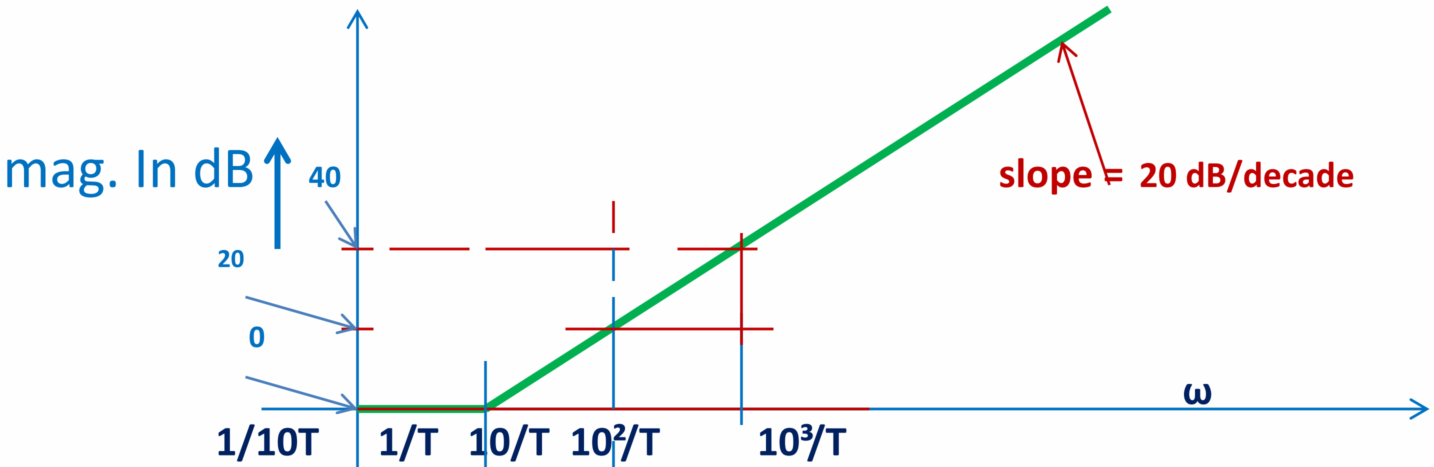
ωT (rads)	M(ω) in dB	ωT (rads)	M(ω) in dB
1	0	100	40
10	20	1000	60

Bode Plots: Different types of Transfer Functions

- We observe from the table in the previous slide that,
  - For a decade change in frequency (1 to 10, 10 to 100, & so on) the magnitude changes by 20 dB.
  - Therefore the slope of the magnitude plot is 20 dB/decade change in frequency.
  - We have two plots: for $\omega T<<<1$ & $\omega T>>>1$
  - For $\omega T<<<1$; $M(\omega) =0$ & for $\omega T >>>1$; $M(\omega)$ has slope of 20 dB/decade
  - At $\omega T=1$; $M(\omega)$ in equation (2) = 0 dB & $M(\omega)$ in equation (1) =0 therefore the two meet at $\omega T=1$, if we extend the low frequency asymptote; (as they are both = 0)
  - This meeting point is called 'Corner Frequency' & is derived from $\omega T=1$; or, $\omega = 1/T$ is the corner frequency.

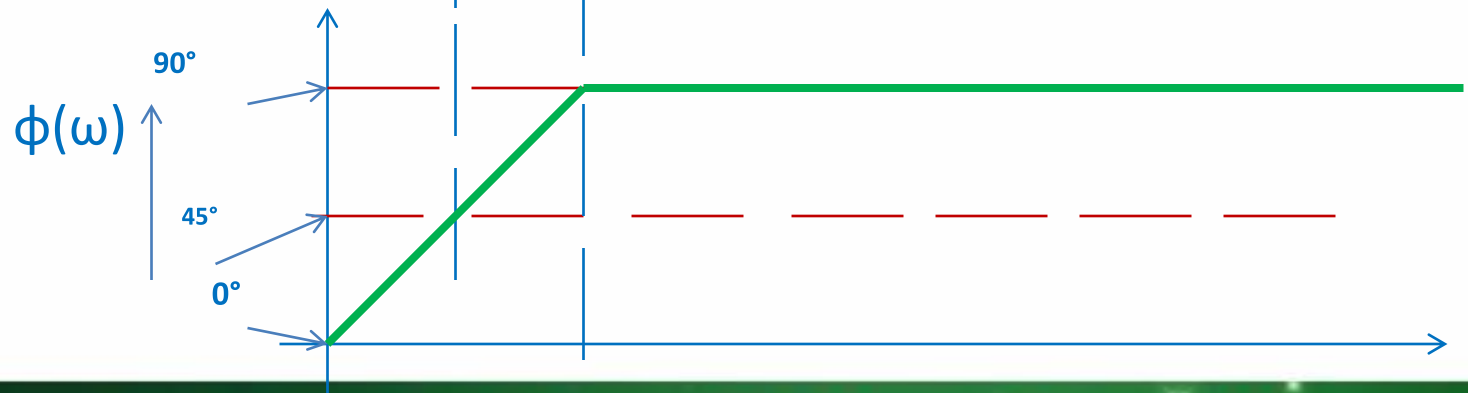
Bode Plots: Different types of Transfer Functions

- The Log-magnitude in dB is plotted as:



(Log mag. Plot – semi log graph paper)

- The Angle Plot: for ωT<<1, ϕ = 0 ; for ωT = 1, ϕ = 45°; for ωT>>1, ϕ = 90°



Bode Plots: Different types of Transfer Functions

- Example:

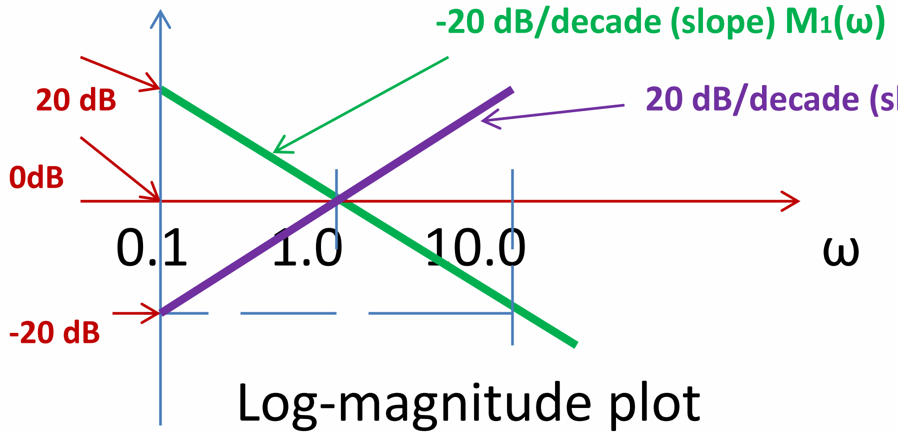
  Consider 1) $G\mathbf{1}(S) = 1/S$ & 2) $G\mathbf{2}(S) = S$

  1) $G\mathbf{1}(j\omega) = 1/j\omega$; $|G\mathbf{1}(j\omega)| = 1/\omega$ & $G\mathbf{2}(j\omega) = j\omega$; $|G\mathbf{2}(j\omega)| = \omega$
  2) The Log – magnitude in dB is given by:

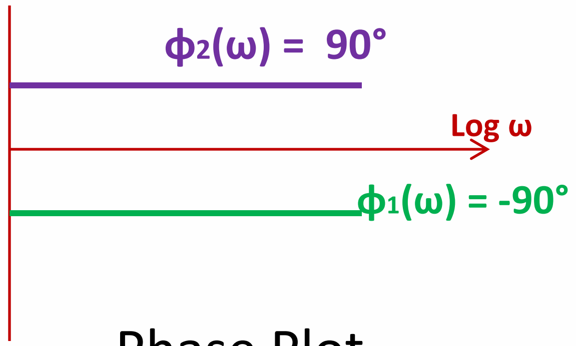
  $20 \text{ Log } \mathbf{10}\ |G\mathbf{1}(j\omega)| = M\mathbf{1}(\omega) = 20 \text{ Log } \mathbf{10}\ [1/\omega] = -20 \text{ Log } \mathbf{10}\ (\omega)$

  $20 \text{ Log } \mathbf{10}\ |G\mathbf{2}(j\omega)| = M\mathbf{2}(\omega) = 20 \text{ Log } \mathbf{10}\ [\omega] = 20 \text{ Log } \mathbf{10}\ (\omega)$

  Angle : $\phi\mathbf{1}(\omega) = -90°$ Angle : $\phi\mathbf{2}(\omega) = 90°$



Log-magnitude plot



Phase Plot

Bode Plots: Different types of Transfer Functions

- We have drawn Bode plots for first order transfer functions having a simple (order 1) pole or a simple (order 1)zero. We now generalize it to multiple order poles & zeros which may be present in a given transfer function.

$G1(S) = 1/(1 + TS)^m$ (pole of order 'm'), &

$G2(S) = (1 + TS)^m$ (zero of order 'm')

$G1(j\omega) = 1/(1 + j T\omega)^m$; $|G1(j\omega)| = 1/[\sqrt{(1+(\omega T)^2}]^m$

Log-magnitude (in dB) $= 20 \text{ Log}10 \{1/[\sqrt{(1+(\omega T)^2}]^m\}$

$= -10 \text{ m Log}10 \{(1+(\omega T)^2]$ 1

Angle $= -\text{ m } \tan^{-1}(\omega T)$

$G2(j\omega) = (1 + j T\omega)^m$; $|G2(j\omega)| = [\sqrt{(1+(\omega T)^2}]^m$

Log-magnitude (in dB) $= 20 \text{ Log}10 [\sqrt{(1+(\omega T)^2}]^m$

$= 10 \text{ m Log}10 \{(1+(\omega T)^2]$ 2

Angle $= \text{ m } \tan^{-1}(\omega T)$

Bode Plots: Different types of Transfer Functions

- For G**1**(S) :

  Log-magnitude (in dB) = $-10\,m\,\mathrm{Log}\mathbf{10}\,\{(1+(\omega T)^2]$

- For G**2**(S) :

  Log-magnitude (in dB) = $10\,m\,\mathrm{Log}\mathbf{10}\,\{(1+(\omega T)^2]$

- Thus we observe that, for $\omega T >>> 1$, the

  slope of log-mag. plot for pole of order 'm' = -20 m dB/decade

  slope of log-mag. plot for zero of order 'm' = 20 m dB/decade

- While the respective angles are given by

$$-/+\ m \tan^{-1}(\omega T)$$

where m = 1,2,3 … is the order of the pole & zero. So as 'm' increases the slopes and the angle increase.

Bode Plots: Different types of Transfer Functions

- Multiple Poles & Zeros at the Origin of the S plane:

  Consider 1) $G\mathbf{1}(S) = 1/S^m$ & 2) $G\mathbf{2}(S) = S^m$

  1) $G\mathbf{1}(j\omega) = 1/(j\omega)^m$; $|G\mathbf{1}(j\omega)| = 1/\omega^m$ & $G\mathbf{2}(j\omega) = (j\omega)^m$; $|G\mathbf{2}(j\omega)| = \omega^m$
  2) The Log – magnitude in dB is given by:

  $20 \text{ Log } \mathbf{10}\ |G\mathbf{1}(j\omega)| = M\mathbf{1}(\omega) = 20 \text{ Log } \mathbf{10}\ [1/\omega^m] = -20\ m \text{ Log } \mathbf{10}\ (\omega)$

  $20 \text{ Log } \mathbf{10}\ |G\mathbf{2}(j\omega)| = M\mathbf{2}(\omega) = 20 \text{ Log } \mathbf{10}\ [\omega^m] = 20\ m \text{ Log } \mathbf{10}\ (\omega)$

  Angle : $\phi\mathbf{1}(\omega) = -\ m\ 90°$ Angle : $\phi\mathbf{2}(\omega) = m\ 90°$

- Here again we observe that the slope for log-magnitude plot of

  G**1**(S) is -20m dB/decade & angle is –m 90°, &

  G**2**(S) is 20m dB/decade & angle is m 90°

- where, m = 1,2,3 …. Is the order of the pole and zero

  As ‘m’ increases, slopes & angle increase

Bode Plots: Different types of Transfer Functions

- $G(S) = K\,(1+T_1 S)(1+T_2 S)/S^m(1 + T_3 S)(1 + T_4 S)$

  We have a combination of poles & zeros. There can be any number of poles & zeros in a transfer function. We need to plot Log-magnitude plot in dB & Angle plot in degrees

- Log-magnitude plot:

  $G(j\omega) = K\,(1 + j\,T_1\omega)(1 + j\,T_2\omega)/(j\omega)^m(1 + j\,T_3\omega)(1 + j\,T_4\omega)$

  $20 \log |G(j\omega)| = 20 \log |K\,(1+j\,T_1\omega)(1+j\,T_2\omega)/(j\omega)^m(1+j\,T_3\omega)(1+j\,T_4\omega)|$

  $20 \log K + 20 \log \sqrt{(1+ (T_1\omega)^2} + 20 \log \sqrt{(1 + (T_2\omega)^2}$

  $-20\,m \log \omega - 20 \log \sqrt{(1 + (T_3\omega)^2} - 20 \log \sqrt{(1+(T_4\omega)^2}$ ….. 1

- From equation (1) we make out that log-magnitude plot in dB, for a given G(S), is obtained by algebraically adding asymptotic plot of each pole & zero including the constant gain term 'K'

Bode Plots: Different types of Transfer Functions

- Example:

$$G(S) = 10(1+S)(1+10S)/S(1+5S)(1+20S)$$

- Bode Plot:

$$G(j\omega) = 10(1+j\,1\omega)(1+j\,10\omega)/j\omega(1+j\,5\omega)(1+j\,20\omega)$$

1. $K = 10$; magnitude in dB = 20 log 10 = 20 dB
2. $(1+j1\omega)$; corner frequency $\omega T = 1$; $\omega = 1/T$; $\omega = 1$; up to $\omega = 1$, magnitude = 0; for $\omega \geq 1$, magnitude plot has a slope of 20 dB/decade
3. $(1+j\,10\,\omega)$; corner frequency $\omega T = 1$; $\omega = 1/10$; $\omega = 0.1$; up to $\omega = 0.1$, magnitude = 0; for $\omega \geq 0.1$, magnitude plot has a slope of 20 dB/decade
4. ω; corresponds to pole at origin; magnitude plot has a slope of -20 dB /decade

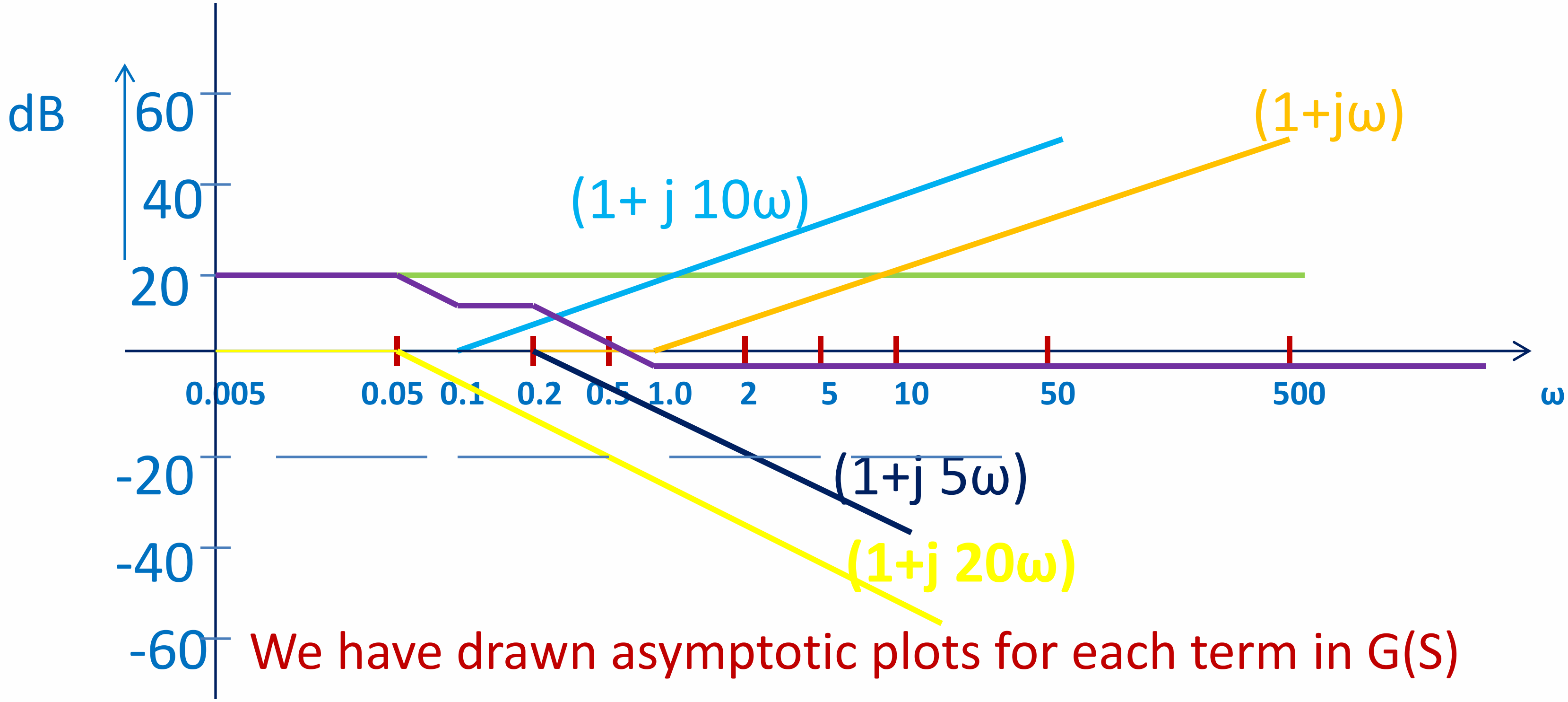
Bode Plots: Different types of Transfer Functions

- $(1+j5\omega)$; corner frequency $\omega T = 1$; $\omega = 1/5$; $\omega = 0.2$; up to $\omega = 0.2$, magnitude = 0; for $\omega \geq 0.2$, magnitude plot has a slope of -20 dB/decade
- $(1+j\ 20\ \omega)$; corner frequency $\omega T = 1$; $\omega = 1/20$; $\omega = 0.05$; up to $\omega = 0.05$, magnitude = 0; for $\omega \geq 0.05$, magnitude plot has a slope of -20 dB/decade.
- The lowest corner frequency is 0.05; therefore we take lowest frequency in log ω scale as 0.005

The complete log- magnitude plot is shown in the next slide

Bode Plots: Different types of Transfer Functions

- Complete log-magnitude plot: —— **complete log-magnitude plot**

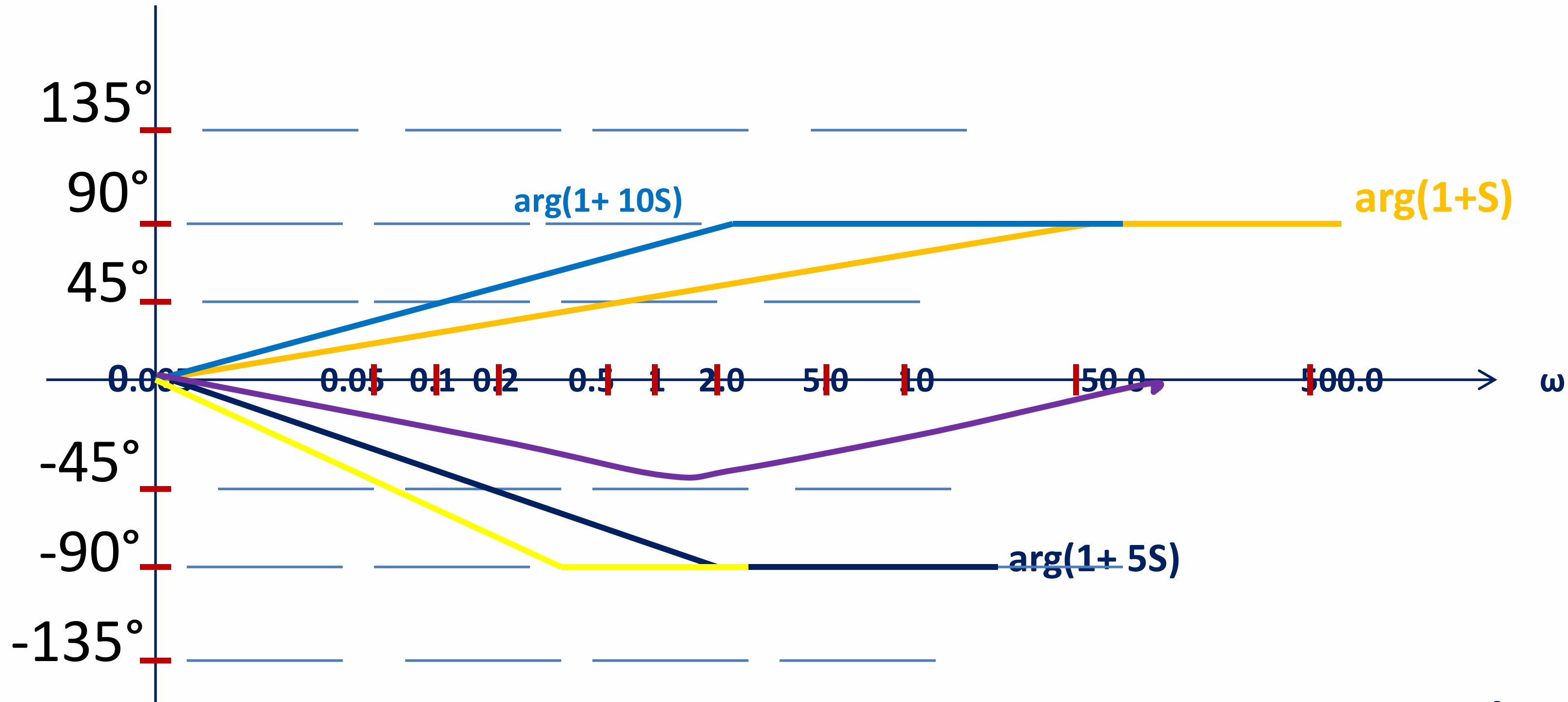


We have drawn asymptotic plots for each term in G(S)

- Now, we algebraically add all the plots keeping in mind that slope change occurs at corner frequency only; corner frequencies are 0.05, 0.1; slope change begins at these frequencies.

Bode Plots: Different types of Transfer Functions

- Complete Angle plot: ___ complete Angle plot



- **Constant term introduces '0' phase. At corner frequency angle is +/- 45°. At ten times the corner frequency angle can be taken as +/- 90°. These are asymptotic plots for angle of each term in G(S).**
- **Complete Angle plot is obtained by algebraically adding all the individual plots.**

BODE PLOT: For 2^{nd} order Under damped Transfer Functions

- Under damped systems have complex conjugate poles. Let us consider normalized form of a second order system, given by

$$G(j\,u) = 1/(1 + j2\xi u - u^2);$$

$$|G(j\,u)| = 1/\sqrt{[(1-u^2)^2 + (2\xi u)^2]}$$

- The log-magnitude plot is given by

$$20 \log |G(j\,u)| = M(u) = -10 \log[(1-u^2)^2 + (2\xi u)^2]$$

For $u <<< 1$; higher order terms in u are neglected to obtain

$$M(u) = 0 \text{ dB}$$

For $u >>> 1$; $M(u) = -10 \log u^4 = -40 \log u$; $(2\xi u)^2 << u^4$ because $\xi < 1$

- Therefore, log magnitude plot consists of 2 straight line asymptotes

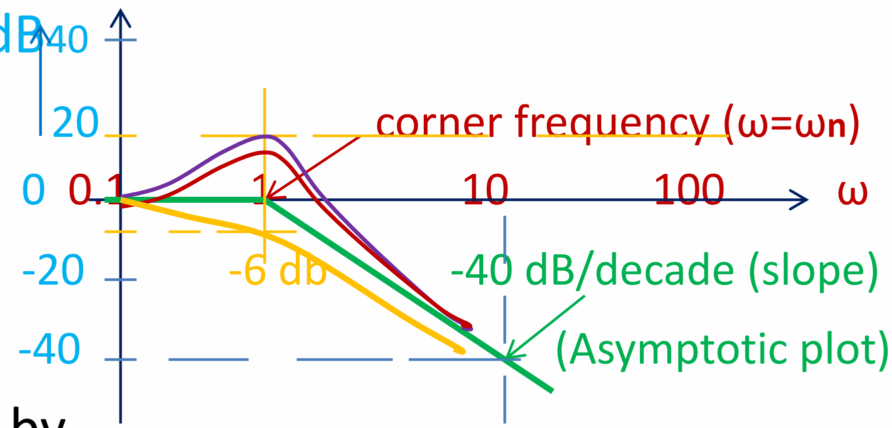
  - one horizontal line at '0' dB for $u<<<1$

  - the other, a line with a slope of -40 dB/decade for $u>>>1$

- These 2 asymptotes meet on '0'dB line at $u = 1$; i.e. at $\omega = \omega_n$.

BODE PLOT: For 2nd order Under damped Transfer Functions

- The asymptotic plot for 2nd order system is:
- Asymptotic plots are approx. plots; error at u = 1.



- Exact Plot:

The log-magnitude plot is given by

$M(u) = -10 \log[(1-u^2)^2 + (2\xi u)^2]$; Actual plots are drawn around Asymptotic plot.

We directly substitute for u = 1 & determine M(u) for different ξ values. M(u), u=1, is function of ξ.

u=1	M(u)
$\xi = 0.05$	20 dB
$\xi = 0.1$	14 dB
$\xi = 1.0$	-6 dB

BODE PLOT: For 2nd order Under damped Transfer Functions

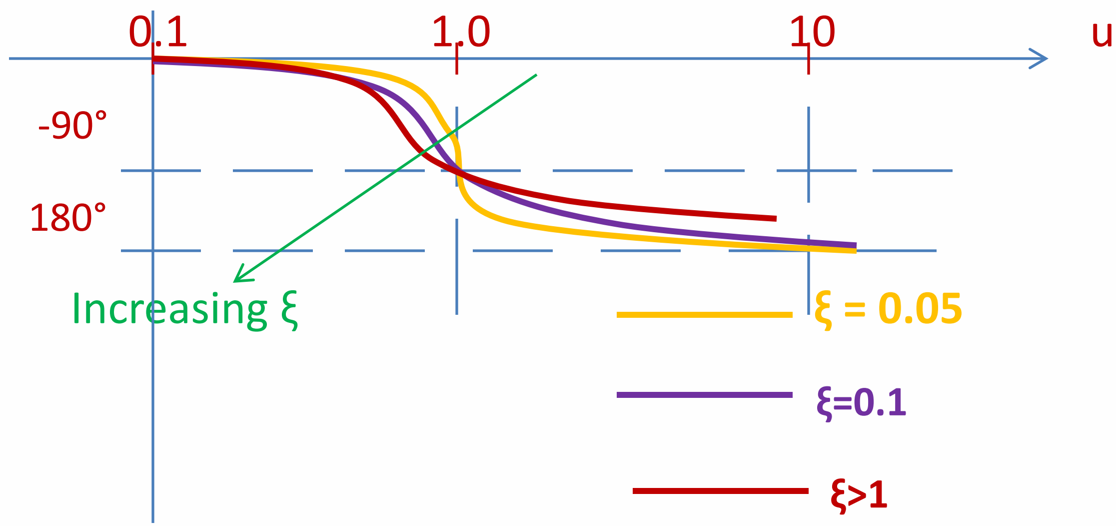
- The Phase Plot:

  The phase angle is given by: $\phi(u) = -\tan^{-1}(2\xi u/1-u^2)$;

  We observe that $\phi(u)$ is a function of u & ξ. However, at u=1, for any value of ξ, $\phi(u) = -90°$.

  for $u = 0$; $\phi(u) = 0$ & for $u = \infty$, $\phi(u) = -180°$

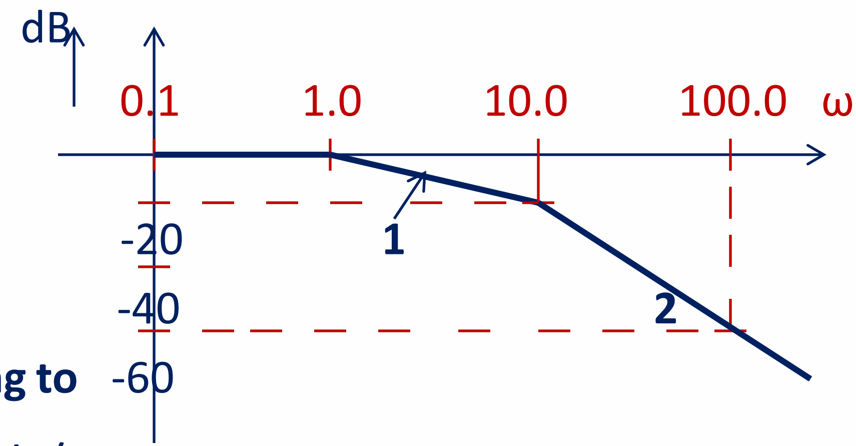
- For $0<u<1$ & $1<u<\infty$, $\phi(u)$ is dependent on ξ value.



Determination of Transfer Function from Bode Plot

- The problem of Synthesis:
- Given a transfer function, we know how to draw Bode plot.
- Now we will have the reverse problem:

Given the Bode (log-magnitude) plot how to determine the transfer function. This is the process of system identification from a given frequency response.



Solution:

Slope of plot '1' = -20 dB/decade

Slope of plot '2' = -40 dB/decade

Corner frequency ($\omega T = 1$) corresponding to

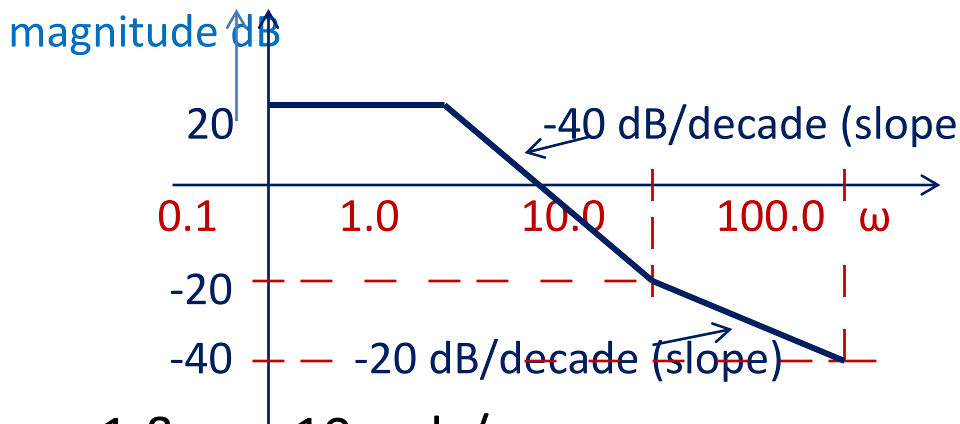
plot '1' = 1 rad/sec & plot '2' = 0.1 rads/sec

- The gain up to 1st corner frequency (= 1 rad/sec) = 0 dB; therefore K = 1

The transfer function, G(S) = 1/(1 + S)(1 + 0.1S)

Determination of Transfer Function from Bode Plot

- Determine G(S)



- Corner frequencies are at ω = 1 & ω = 10 rads/sec

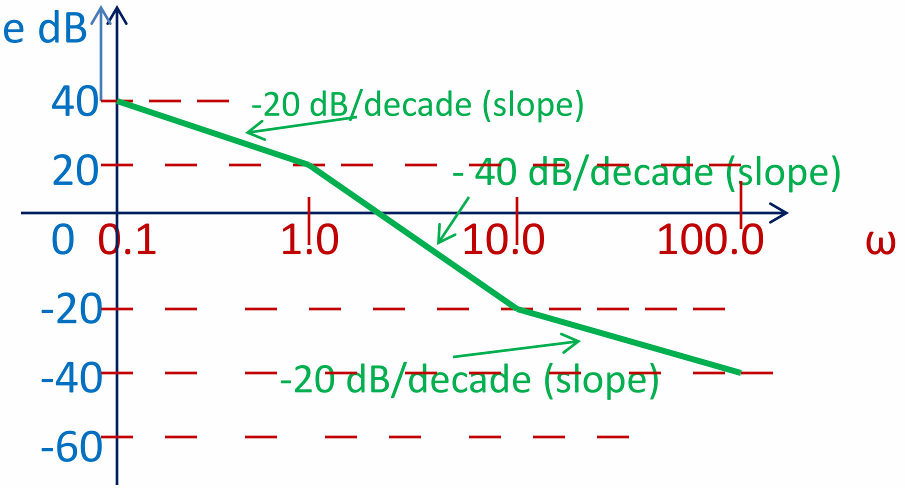
  Up to ω = 1 rads/sec, the gain(magnitude) = 20 dB. We determine 'K' from it. 20 Log **10** K = 20 dB; therefore K = 10.

- At ω = 1 rads/sec, magnitude plot falls with a slope of -40 dB/decade. This corresponds to a double pole term like,$1/(1+S)^2$ in G(S). From ω = 10 rads/sec, the slope changes to -20 dB/decade, therefore there is a zero term like (1 + 0.1S) in G(S).

- Therefore $G(S) = K(1 + 0.1S)/(1 + S)^2$

Determination of Transfer Function from Bode Plot

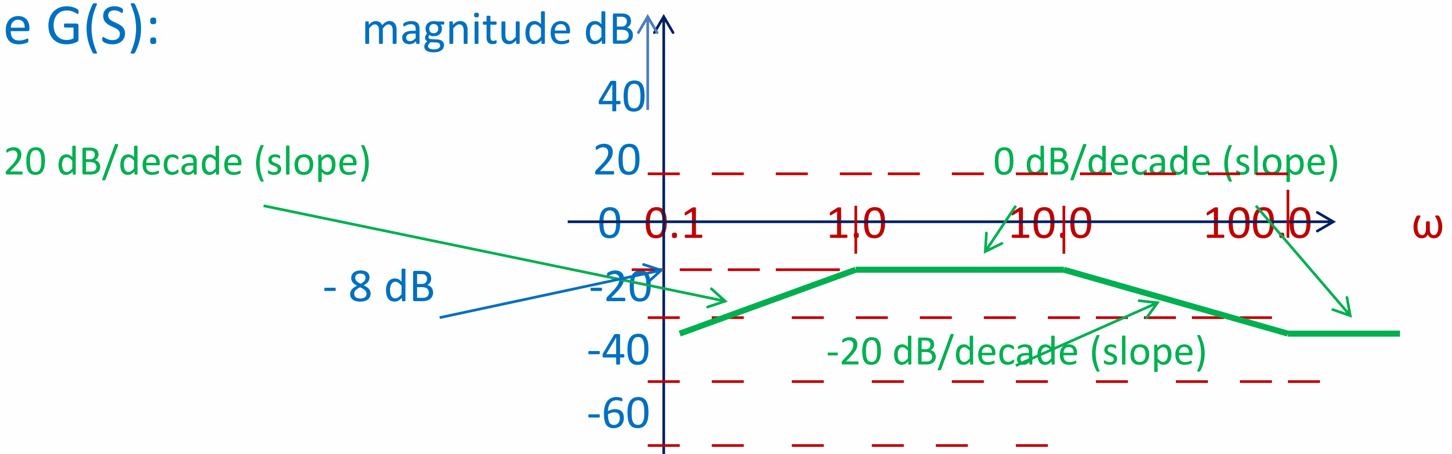
- Determine G(S):



- There is a ramp with a slope: -20 dB/decade, starting at ω = 0.1 r/s. It implies a term 1/S in G(S). At ω = 1 r/s; its magnitude should be '0' dB, but it is 20 dB. It implies 'K' = 10 in G(S). From ω = 1 r/s to ω= 10 r/s, the slope is -40 dB/decade. It implies a term 1/(1 + S) in G(S). From ω = 10 r/s to ω= 100 r/s, the slope is -20 dB/decade. It implies a term (1 + 0.1 S) in G(S).
- Therefore, the transfer function is: G(S) = 10 (1+ 0.1 S)/S(S + 1)

Determination of Transfer Function from Bode Plot

- Determine G(S):



- Starting, there is a ramp slope= 20 dB/decade; it implies a S term in G(S); its magnitude should = 0 at ω = 1 r/s, but it is not so. It implies a gain term 'K' in G(S). To determine 'K' we write
- 20 Log K + 20 log ω = -8 at ω = 1 r/s; or, 20 log K = -8; K = 0.3981
- From ω = 1 to 10 r/s ; slope is '0'; implies a term 1/(S +1) in G(S). From ω = 10 to 100 r/s ; slope is -20 dB/decade; implies a term 1/(1+ 0.1S) in G(S). From ω=1000 r/s onwards, the slope is '0'; implies a term (1 + 0.01 S) in G(S).
- Therefore, G(S) = 0.3981 (1 + 0.01 S)/(S + 1)(1 + 0.01 S)

Nyquist Method for finding Stability of CL System

- Stability study is carried out graphically from the open loop frequency response.
- Nyquist Stability Criterion:
- The characteristic equation: $Q(S) = 1 + G(S)H(S) = 0$

  $G(S)H(S) = K\,(S+Z_1)(S+Z_2)\ \ldots\ldots(S + Z\mathbf{m})/(S+P_1)(S+P_2)\ldots\ (S + P\mathbf{n});\ m \leq n$

  $Q(S) = 1+ K\,(S+Z_1)(S+Z_2)\ \ldots\ldots(S + Z\mathbf{m})/(S+P_1)(S+P_2)\ldots\ (S + P\mathbf{n})$

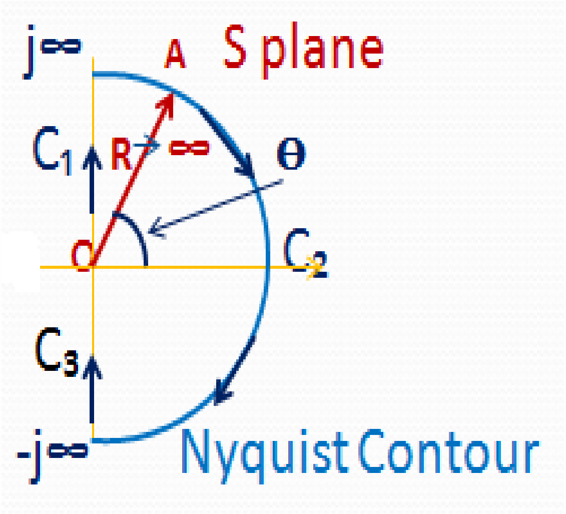
  On simplification, we write:

  $Q(S) = (S+Z_1')(S+Z_2')\ \ldots\ldots(S + Z\mathbf{n}')/(S+P_1)(S+P_2)\ldots\ (S + P\mathbf{n})$
- We observe that
  - Zeros of Q(S) at $S = -Z_1'$, $S = -Z_2'$, ……$S = -Z\mathbf{n}'$ are the roots of the characteristic equation
  - Poles of Q(S) at $S = -P_1$, $S = -P_2$, … $S = -P\mathbf{n}$ are the same as open loop poles of the system
  - For stable system, zeros of Q(S), roots of characteristic equation, must be in the LH of the S-plane.

Nyquist Method for finding Stability of CL System

- Even if some open loop poles lie in the RH of the S plane, all the zeros of Q(S), poles of CL system, must lie in the LH of the S plane. It means that an unstable open loop system can be made stable with an appropriate design of CL system.
- The Nyquist Contour:

  Since we interested in finding out whether there are any zeros of Q(S) in the RH of the S plane, we choose a contour that completely encloses RH of the S plane. This is called Nyquist Contour.

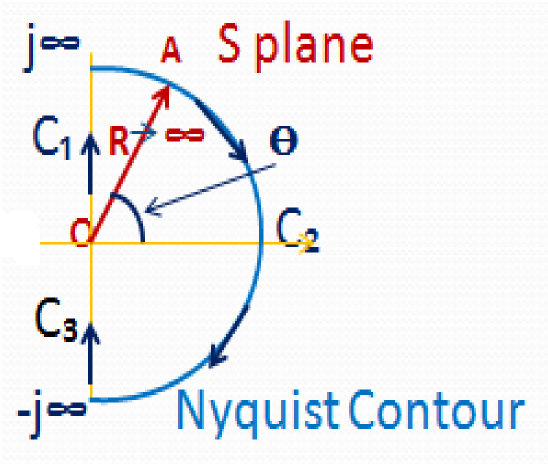


- In CW direction, starting from the origin of the S plane, we traverse Nyquist Contour. along the paths C_1 C_2 and C_3.

  Since R⟶ ∞, entire RH is enclosed

Nyquist Method for finding Stability of CL System

- From the Nyquist Contour we observe that for $S = j\omega$, along path C_1 frequency, ω, varies from '0' to ∞ along path C_3 frequency, ω, varies from $-\infty$ to '0'.



- The path C_2 is a circle of infinite radius ($R \rightarrow \infty$). Any point on C_2 can be represented in polar form as: $S = R \exp(+/- j\theta)$. Along C_2, while traversing in the direction of arrows, the angle Θ varies from 90° to -90°.
- The Nyquist Contour as defined in the aforesaid lines, encloses all the right half S plane zeros & poles of 1 + G(S)H(S).

Nyquist Method for finding Stability of CL System

- The Stability Criterion & Nyquist Theorem:

  Let,

  Z: be the number of zeros of Q(S) in RH of the S plane

  P: be the number of poles of Q(S) in RH of the S plane

- Nyquist Theorem:

  As point $S = S_0$ moves along the Nyquist contour in the S plane, in the Q(S) plane a closed contour $\Gamma\mathbf{q}$ is traversed which encloses the origin 'N' times in CCW direction; where N = P-Z.

- For every point $S = S_0$ on the Nyquist contour, Q(S) has a value. If we plot the values of Q(S) in the plane called 'Q(S) plane', then, according to Nyquist theorem, we will obtain a closed path, $\Gamma\mathbf{q}$, which will enclose the origin of 'Q(S) plane' 'N' times.

- Stability Criterion:

  We know that zeros of Q(S), Z, are the closed loop system poles & therefore should lie in the LH of the S plane for system stability.

Nyquist Method for finding Stability of CL System

- Stability Criterion (contd.):

  Therefore, $Z = 0$ (for stable CL system).

- So for a stable CL system, we have two situations:

  for $P \neq 0$:

  $\Rightarrow N = P-Z = P$

  that the CCW encirclements of the origin of 'Q(S) plane' should be equal to the number of poles, P, of Q(S) (open loop poles of G(S)H(S)) in the RH of the S plane.

- The above assertion implies that even if the open loop system is unstable, the CL system can be stable.

- For $P = 0$: (no poles of G(S)H(S) in RH of the S plane) the number of encirclements $N = 0$ for a stable CL system

Nyquist Method for finding Stability of CL System

- Modified Stability Criterion:

  We know that, $Q(S) = 1 + G(S)H(S)$

  $\Rightarrow G(S)H(S) = Q(S) - 1$

- Therefore, we say that while,

  Γq encircles the origin in Q(S) plane

  ΓGH will encircle (-1,j0) point in the GH plane

- In G(S)H(S) plane, we state the Nyquist Stability Criterion as:

  For P ≠0:

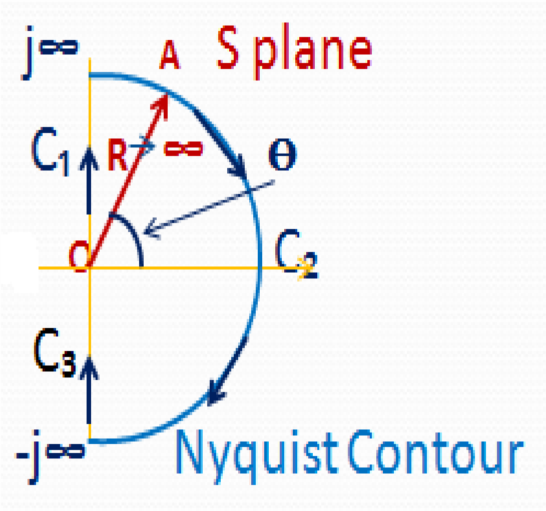
  If the contour **ΓGH** of the open loop transfer function G(S)H(S), corresponding to the Nyquist contour in the S plane, encircles the point (-1,j0) in the CCW direction as many times as the number of right of S-plane poles of G(S)H(S), the CL system is stable.

  For P = 0: The CL system is stable if no encirclements of (-1,j0) point.

Nyquist Method for finding Stability of CL System

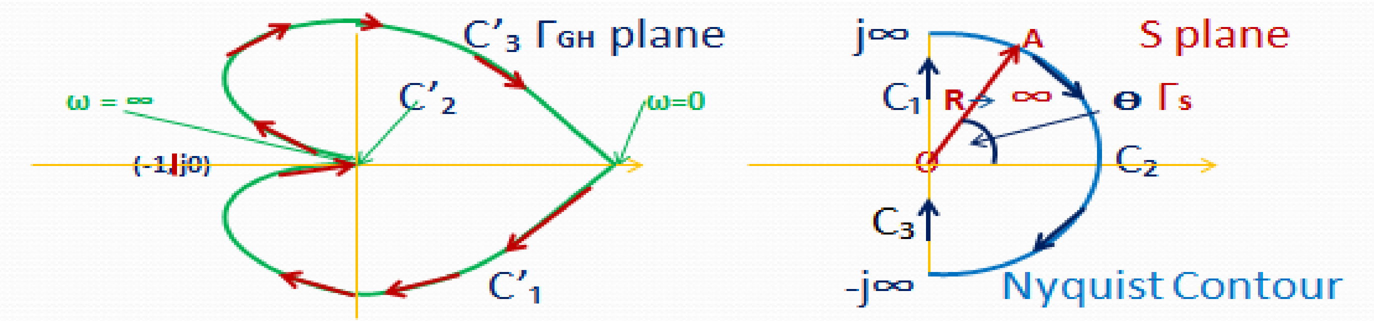
- **Mapping of Nyquist contour in to ΓGH contour:**

  Following steps are followed:



1. Convert G(S)H(S) in to G(jω) H(jω)
2. For S = jω; $0 \le \omega \le \infty$ (segment C_1) draw polar (Nyquist) plot in GH plane
3. For contour C_2: S = R exp(jΘ); R $\longrightarrow$ ∞. Substitute S = R exp(jΘ) in G(S)H(S) and let R $\longrightarrow$ ∞ for $\infty \le S \le -\infty$. The entire segment maps to ‘0’ in the GH plane
4. For $-\infty \le \omega \le 0$ (segment C_3) draw polar plot for negative frequencies; which is mirror image of plot for C_1.

Nyquist Method: Examples



- $G(S)H(S) = K/(1+T_1 S)(1+T_2 S)$; C'_3 is mirror image of C'_1

1. Corresponding to C_1 in Γ**s** plane we have the Nyquist plot in Γ**GH** plane as C'_1.
2. Corresponding to C_2 in Γ**s** plane we have; $S = R\exp(j\theta)$ in $G(S)H(S)$; $R \longrightarrow \infty$

   $G(S)H(S) = K/(T_1 R e^{j\theta} + 1)(T_2 R e^{j\theta} + 1)$ as $R \longrightarrow \infty$ therefore

   $G(S)H(S) = 0\, e^{-j2\theta}$; $|G(S)H(S)| = 0$; $\arg\{G(S)H(S)\} = -2\theta$

   On C_2 ; θ varies from +90° to -90° as we move from $+j\infty$ to $-j\infty$

   arg {G(S)H(S)} varies from -180° to + 180° . This is C'_2 in Γ**GH** plane.
3. C_3 in Γ**S** plane is mapped as C'_3 (Nyquist plot) in Γ**GH** plane. (Contd.)

Nyquist Method: Examples (Contd..)

- For the example in previous slide:

  We have drawn the Nyquist plot for a given G(S)H(S). Now we need to determine the stability of its closed loop system.

- The number of encirclements, N, of (-1,j0) point is given by:

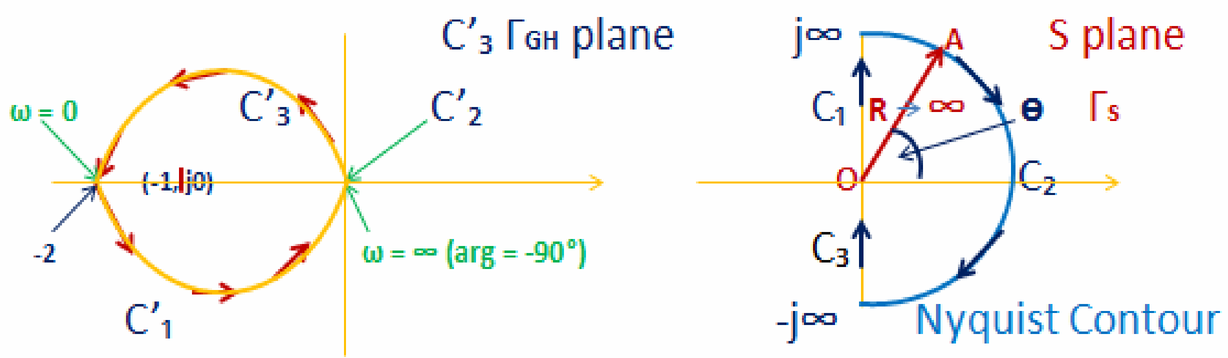
$$N = P - Z$$

- For closed loop system to be stable, Z = 0

  In this example, P = 0 because no poles of G(S)H(S) are in the RH of S plane.

- Therefore N should be equal to ‘0’, i.e. that there should be no encirclement of (-1,j0) point. We see from the Nyquist diagram that it does not encircle (-1,j0) point & hence the closed loop system is stable.

Nyquist Method: Examples (Contd..)



- G(S)H(S) = (S+2)/(S + 1)) (S -1); C'$_3$ is mirror image of C'$_1$
  1. Corresponding to C_1 in Γ**s** plane we have the Nyquist plot in Γ**GH** plane as C'$_1$.
  2. Corresponding to C_2 in Γ**s** plane we have; S = R exp(jΘ) in G(S)H(S);

     $R \longrightarrow \infty$

     $G(S)H(S) = (2+ R e^{j\theta})/(1+ R e^{j\theta})(R e^{j\theta} -1)$ as, $R \longrightarrow \infty$ therefore

     $G(S)H(S) = 0\, e^{-j\theta}$; $|G(S)H(S)| = 0$; $\arg\{G(S)H(S)\} = -\Theta$

     on C_2 ; Θ varies from +90° to -90° as we move from +j∞ to -j∞

     arg{G(S)H(S)} varies from -90° to +90° . This is C'$_2$ in Γ**GH** plane.
  3. C_3 in Γ**S** plane is mapped as C'$_3$ (Nyquist plot) in Γ**GH** plane. (Contd. .)

Nyquist Method: Examples (Contd..)

- Having drawn the Nyquist diagram, we need to determine the stability of related CL system.
- Observation:

G(S) H(S) has a pole in the RH of the S plane; therefore P = 1

$$N = P - Z$$

Z = 0 for stable CL system

Therefore, $N = P = 1$

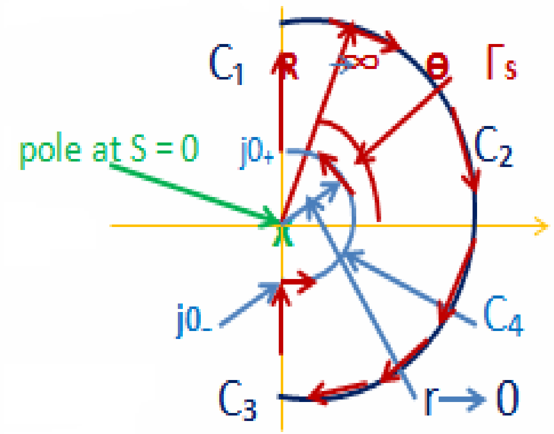
⇒ that the Nyquist plot should encircle (-1,j0) point once in the CCW direction for the CL system to be stable.

- From the Nyquist diagram we that it is encircling (-1,j0) point once in CCW direction. Therefore, the CL system is stable

Nyquist Method: Examples (Contd..)

- Case: G(S)H(S) has a pole at the origin of the S plane:

- Since there is a pole at the origin in the S plane, while drawing the Nyquist contour we bypass the origin because pole is a singularity. Bypassing is done by drawing a circle of very small radius 'r'; as $r \rightarrow 0$. A point on the semi circle, C_4, is represented by

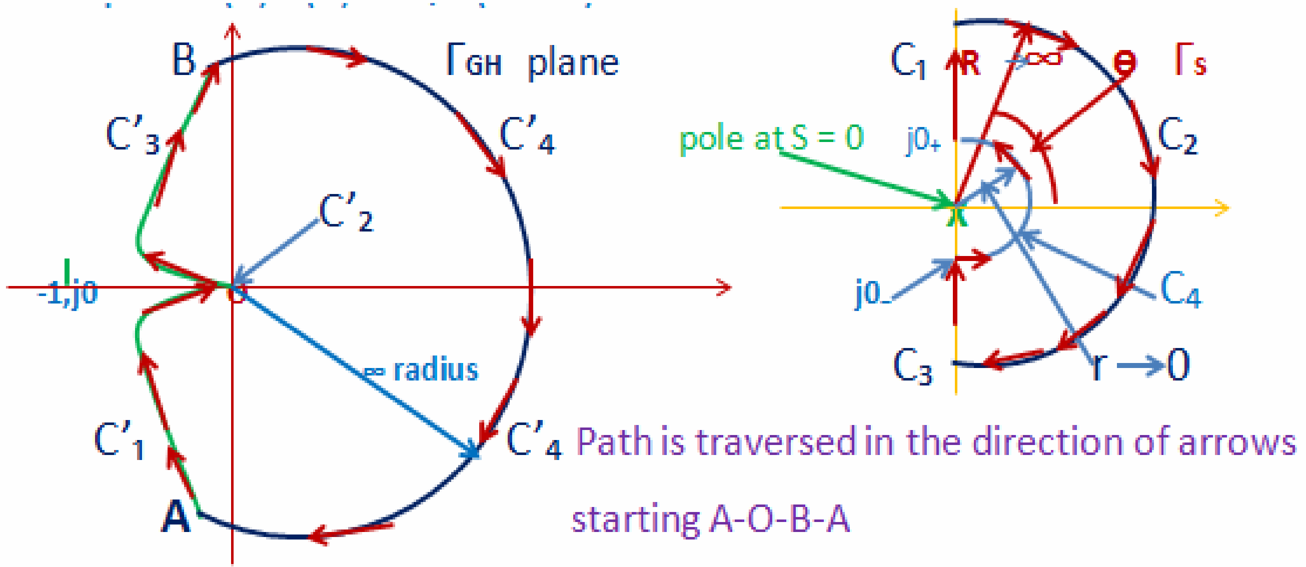


$$S = r\, e^{j\phi}$$

- The Nyquist contour is traversed starting 1) $s = j0_+$ to $j\infty$ (C_1) 2) $S = j\infty$ to $-j\infty$ (C_2), 3) $S = -j\infty$ to $j0_-$ (C_3) and 4) $S = j0_-$ to $j0_+$ (C_4)

Nyquist Method: Examples (Contd..)

- Example: $G(S)H(S) = K/S(1+TS)$



- A : $\omega = j0_+$; $|G(j\omega)\ H(j\omega)| = \infty$; arg = -90° B: $\omega = j0_+$; $|G(j\omega)\ H(j\omega)| = \infty$; arg = -90°

  o: $\omega = j\infty$ to $-j\infty$; $|G(j\omega)\ H(j\omega)| = 0$; arg = -180° to 180°

  C_1 is mapped in to C'_1 & C_3 is mapped in to C'_3 (Nyquist/polar plot)

  C_2 is mapped in to C'_2(origin); C_4 is mapped in to C'_4. (Contd.)

Nyquist Method: Examples (Contd..)

- $G(j\omega)H(j\omega) = K/j\omega(1+j\omega T)$

1. C_1: mapping in to Γ_{GH} plane: polar plot, C'_1
2. C_2: mapping in to Γ_{GH} plane: point C'_2 for $S = R e^{j\theta}$
3. $G(S)H(S) = K/ R e^{j\theta} (1+T R e^{j\theta})$ as $R \rightarrow \infty$
4. $G(S)H(S) = |G(S)H(S)| e^{j\theta}$; $0 e^{-j2\theta}$; $\arg(G(S)H(S)) = -2\theta$
5. Since θ changes from +90 to -90 ; $\arg(G(S)H(S))$ changes from -180° to + 180°. So we get point 'O' in Γ_{GH} plane.
6. C_4 mapping in to C'_4 in Γ_{GH} plane for $S = r e^{j\phi}$ as $r \rightarrow 0$
7. $G(S)H(S) = K/ r e^{j\phi} (1+T r e^{j\phi})$ as $r \rightarrow 0$
8. $G(S)H(S) = |G(S)H(S)| e^{j\phi}$; $\infty e^{-j\phi}$; $\arg(G(S)H(S)) = -\phi$
9. Since ϕ changes from -90 to +90 ; $\arg(G(S)H(S))$ changes from 90° to -90°. So we get C'_4 Γ_{GH} plane.

Nyquist Method: Examples

- For the example in previous Lecture:

  We have drawn the Nyquist plot for a given G(S)H(S). Now we need to determine the stability of its closed loop system.

- The number of encirclements, N, of (-1,j0) point is given by:

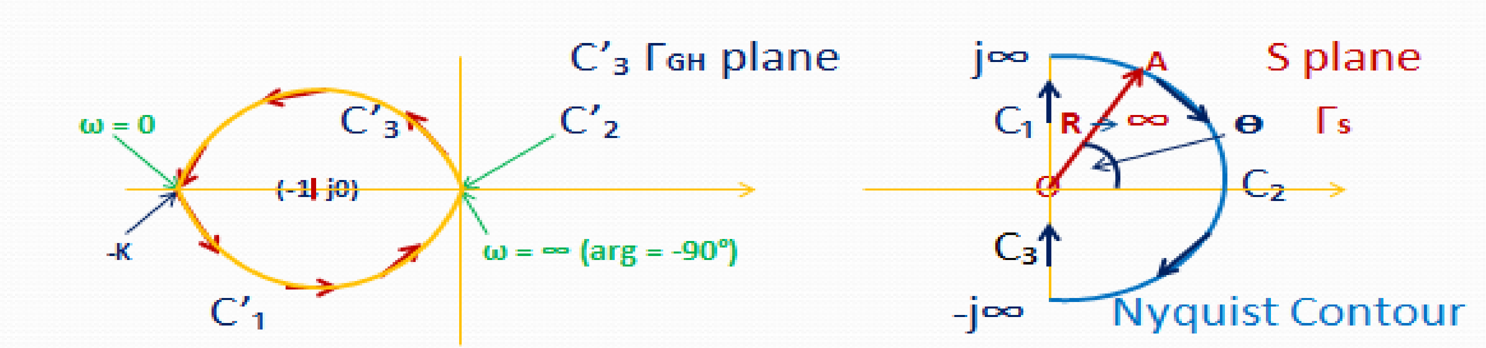
$$N = P - Z$$

- For closed loop system to be stable, Z = 0

  In this example, P = 0 because no poles of G(S)H(S) are in the RH of S plane.

- Therefore N should be equal to '0', i.e. that there should be no encirclement of (-1,j0) point. We see from the Nyquist diagram that it does not encircle (-1,j0) point & hence the closed loop system is stable.

Nyquist Method: Examples (Contd..)



G(S)H(S) = K/(S -1); C'₃ is mirror image of C'₁

1. Corresponding to C_1 in Γ**s** plane we have the Nyquist plot in Γ**GH** plane as C'_1.
2. Corresponding to C_2 in Γ**s** plane we have; $S = R \exp(j\theta)$ in G(S)H(S);

   $R \longrightarrow \infty$

   $G(S)H(S) = K/(R e^{j\theta} - 1)$ as, $R \longrightarrow \infty$ therefore

   $G(S)H(S) = 0 e^{-j\theta}$; $|G(S)H(S)| = 0$; $\arg\{G(S)H(S)\} = -\theta$

   On C_2; θ varies from +90° to -90° as we move from +j∞ to -j∞

   arg{G(S)H(S)} varies from -90° to +90° . This is C'_2 in Γ**GH** plane.
3. C_3 in Γ**S** plane is mapped as C'_3 (Nyquist plot) in Γ**GH** plane. (Contd. .)

Nyquist Method: Examples (Contd..)

- Having drawn the Nyquist diagram, we need to determine the stability of related CL system.
- Observation:

G(S) H(S) has a pole in the RH of the S plane; therefore P = 1

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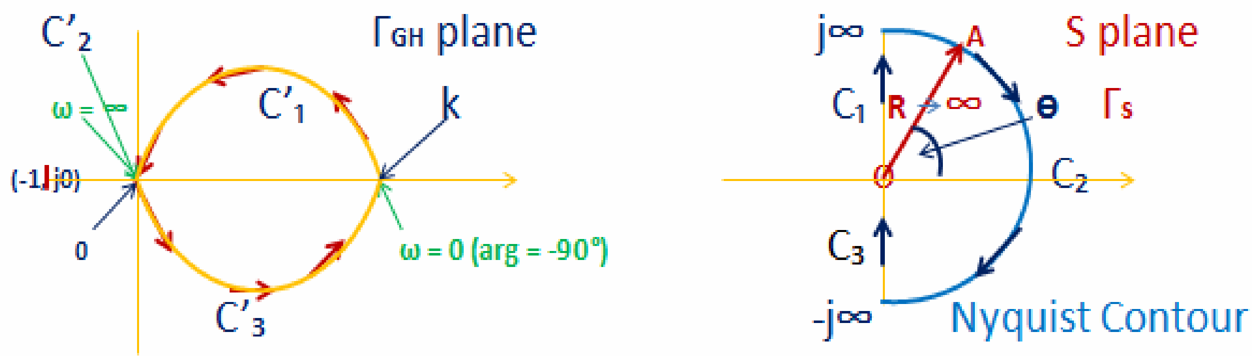
Nyquist Method: Examples (Contd..)



$G(S)H(S) = K/(1-S)$; C'_3 is mirror image of C'_1

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   $G(S)H(S) = K/(R\, e^{j\Theta} - 1)$ as, $R \rightarrow \infty$ therefore

   $G(S)H(S) = 0\, e^{-j\Theta}$; $|G(S)H(S)| = 0$; $\arg\{G(S)H(S)\} = -\Theta$

   On C_2 ; Θ varies from +90° to -90° as we move from $+j\infty$ to $-j\infty$

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Z = 0 for stable CL system

Therefore, $N = P = 1$

⇒ that the Nyquist plot should encircle (-1,j0) point once in the CCW direction for the CL system to be stable.

- From the Nyquist diagram we that it is not encircling (-1,j0) point once in CCW direction. Therefore, the CL system is unstable.

Unit-V
STATE SPACE ANALYSIS

State-Space Modeling

- Alternative method of modeling a system than
  - Differential / difference equations
  - Transfer functions
- Uses matrices and vectors to represent the system parameters and variables
- In control engineering, a **state space representation** is a mathematical model of a physical system as a set of input, output and state variables related by first-order differential equations.
- To abstract from the number of inputs, outputs and states, the variables are expressed as vectors.

Motivation for State-Space Modeling

- Easier for computers to perform matrix algebra
  - e.g. MATLAB does all computations as matrix math
- Handles multiple inputs and outputs
- Provides more information about the system
  - Provides knowledge of internal variables (states)

Definitions

State

- The state of a dynamic system is the smallest set of variables (called state variables) such that knowledge of these variables at t=t0 , together with knowledge of the input for t ≥ t0 , completely determines the behavior of the system for any time t to t0 .

State Variables

- The state variables of a dynamic system are the variables making up the smallest set of variables that determine the state of the dynamic system.

- If at least n variables x1, x2, …… , xn are needed to completely describe the behavior of a dynamic system (so that once the input is given for t ≥ t0 and the initial state at t=t0 is specified, the future state of the system is completely determined), then such n variables are a set of state variables.

Definitions (Contd..)

State Vector

- A state vector is thus a vector that determines uniquely the system state x(t) for any time t≥ t0, once the state at t=t0 is given and the input u(t) for t ≥ t0 is specified.

State Space

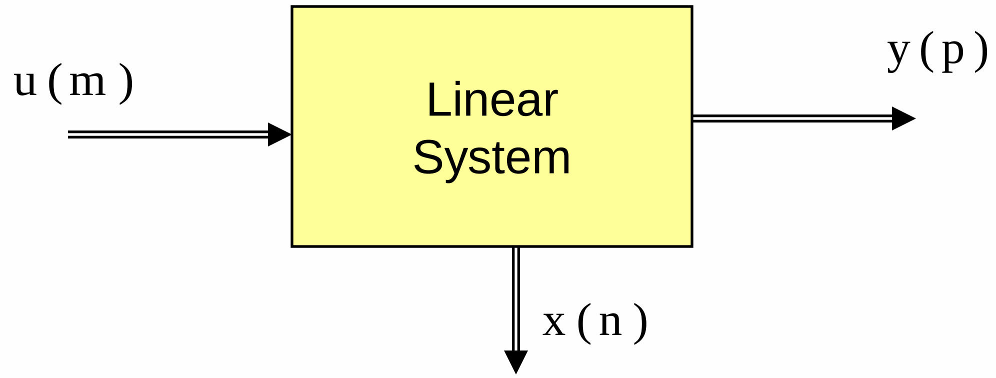
- The n-dimensional space whose coordinate axes consist of the x1 axis, x2 axis, ….., xn axis, where x1, x2,…… , xn are state variables, is called a state space.
- "State space" refers to the space whose axes are the state variables. The state of the system can be represented as a vector within that space.

Definitions (Contd..)

State-Space Equations

- In state-space analysis we are concerned with three types of variables that are involved in the modeling of dynamic systems: input variables, output variables, and state variables.
- The number of state variables to completely define the dynamics of the system is equal to the number of integrators involved in the system.
- Assume that a multiple-input, multiple-output system involves **n** integrators. Assume also that there are **m** inputs $\mathbf{u_1(t), u_2(t), \ldots\ldots\ u_m(t)}$ and **p** outputs $\mathbf{y_1(t), y_2(t), \ldots\ldots\ y_p(t)}$.

State variable technique



$$u(m) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_i(t) \\ u_m(t) \end{bmatrix}$$

Input vector

$$x(n) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_i(t) \\ x_n(t) \end{bmatrix}$$

State vector

$$y(p) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_i(t) \\ y_p(t) \end{bmatrix}$$

Output vector

State Model of LTI System

State Differential Equation

- The state of a system is described by the set of first-order differential equations written in terms of the state variables [x_1 x_2 ... x_n]. These first-order differential equations can be written in general form as

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n + b_{11}u_1 + \cdots b_{1m}u_m$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n + b_{21}u_1 + \cdots b_{2m}u_m$$

$$\vdots$$

$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \ldots a_{nn}x_n + b_{n1}u_1 + \cdots b_{nm}u_m$$

State Model of LTI System (Contd..)

❑ Thus, this set of simultaneous differential equations can be written in matrix form as follows:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \cdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \cdots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

n: number of state variables, m: number of inputs.

❑ The column matrix consisting of the state variables is called the **state vector** and is written as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Dorf and Bishop, Modern Control Systems

State Model of LTI System (Contd..)

- The vector of input signals is defined as u. Then the system can be represented by the compact notation of the state differential equation as

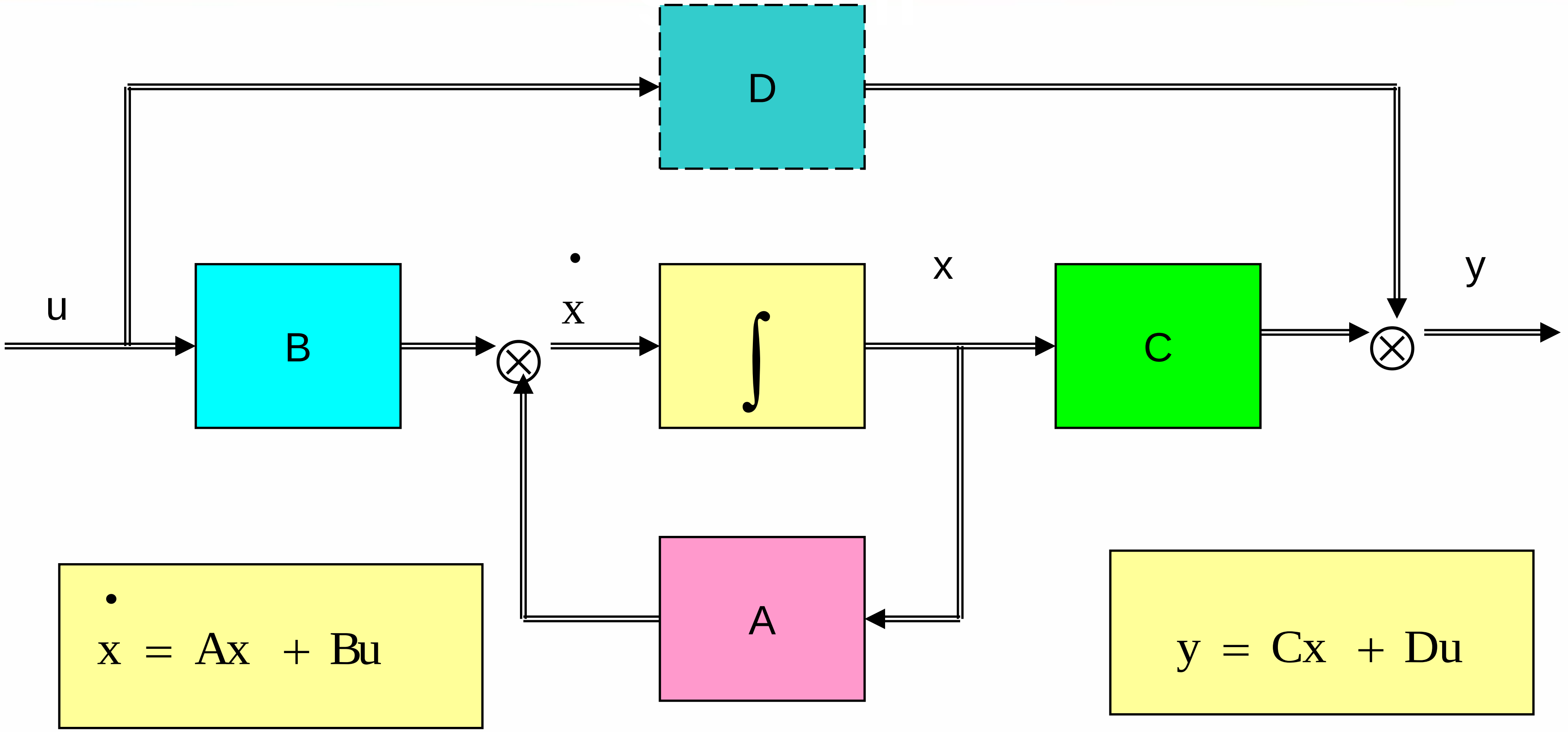
$$\dot{x} = A\,x + B\,u$$

- This differential equation is also commonly called the state equation. The matrix **A** is an nxn square matrix, and **B** is an nxm matrix. The state differential equation relates the rate of change of the state of the system to the state of the system and the input signals. In general, the outputs of a linear system can be related to the state variables and the input signals by the output equation

$$y = C\,x + D\,u$$

- Where **y** is the set of output signals expressed in column vector form. The state-space representation (or state-variable representation) is comprised of the state variable differential equation and the output equation.

Block diagram representation of a LTI system

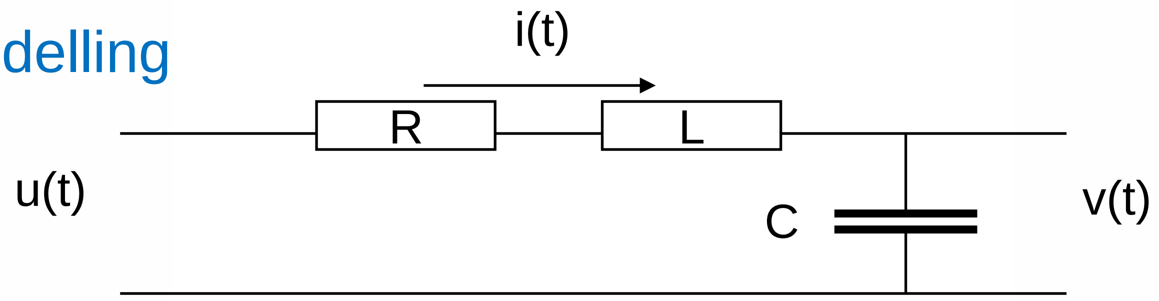


A= System Matrix(n,n)
B= Input Matrix (n,m)
x= State Vector (n,1)
u= Input Vector (m,1)

C= Output Matrix (p,n)
D= Direct Transmission Matrix (p,m)
y= Output Vector (p,1)

State Space Representation Example

- RLC circuit: modelling



- Writing differential equations

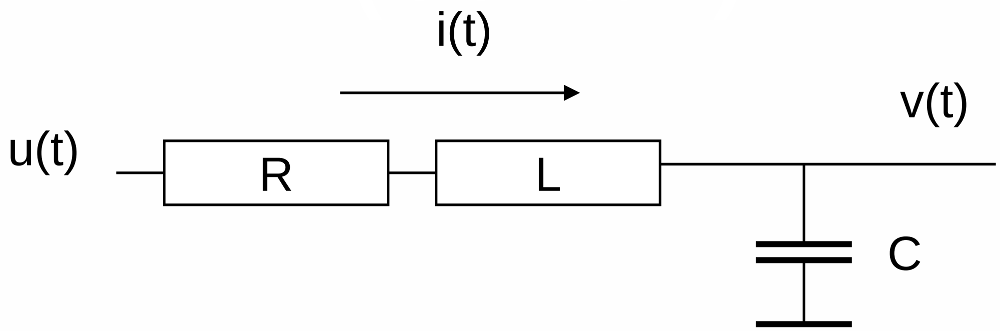
$$u(t) = L\frac{di}{dt} + Ri(t) + v(t)$$

$$i(t) = C\frac{dv}{dt}$$

$$\boxed{u(t) = LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v(t)}$$

Constant coefficient
Second order
Differential equation

State Space Representation Example (Contd..)



$$u(t) = L\frac{di}{dt} + Ri + v = LC\frac{d^2v}{dt} + RC\frac{dv}{dt} + v$$

Can be written

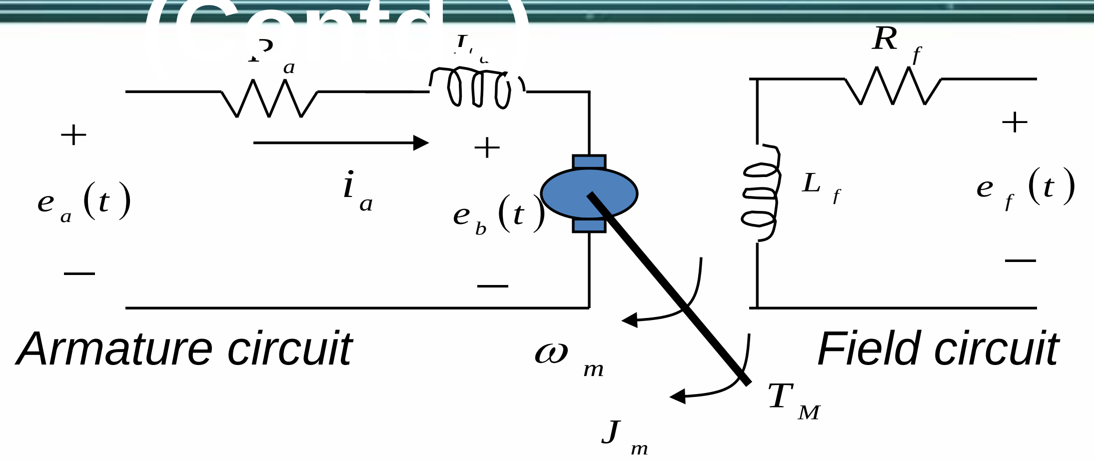
$$\frac{\mathrm{di}}{\mathrm{dt}} = \dot{\mathrm{i}} = -\frac{\mathrm{Ri}}{\mathrm{L}} - \frac{\mathrm{v}}{\mathrm{L}} + \frac{\mathrm{u}}{\mathrm{L}}$$

$$\frac{\mathrm{dv}}{\mathrm{dt}} = \dot{\mathrm{v}} = \frac{\mathrm{i}}{\mathrm{c}}$$

$$\boxed{\begin{bmatrix} \dot{i} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \frac{-R}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i \\ v \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u}$$

State Space Representation Example (Contd..)

- **Armature controlled DC motor**



$$\begin{cases} e_a(t) = R_a i_a + e_b + L_a \dfrac{di_a}{dt} \\ e_b = k_b \omega_m \end{cases}$$

$$\frac{di_a}{dt} = \frac{-R_a}{L_a} i_a - \frac{k_b}{L_a}\omega_m + \frac{1}{L_a} e_a$$

$$T_M = k_i i_a$$

$$T_M = J_m \frac{d^2\theta_m}{dt^2} + B_m \frac{d\theta_m}{dt}$$

$$\frac{d\omega_m}{dt} = \frac{k_i}{J_m} i_a - \frac{B_m}{J_m}\frac{d\theta_m}{dt}$$

$$\frac{d\theta_m}{dt} = \omega_m$$

State Space Representation Example (Contd..)

- Selecting the armature current i(t) and angular displacement of the shaft θ(t), and the angular velocity of the shaft ω(t) as the state variables.

$$\begin{bmatrix} \dot{i}_a \\ \dot{\omega}_m \\ \dot{\theta}_m \end{bmatrix} = \begin{bmatrix} -\frac{R_a}{L_a} & \frac{-K_b}{L_a} & 0 \\ \frac{K_i}{J_m} & \frac{-B_m}{J_m} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_a \\ \omega_m \\ \theta_m \end{bmatrix} + \begin{bmatrix} \frac{1}{L_a} \\ 0 \\ 0 \end{bmatrix} [e_a]$$

- The state equations are as shown in the previous slide.

- The state matrix form as shown beside

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \theta_m(t) \\ \omega_m(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_a \\ \omega_m \\ \theta_m \end{bmatrix}$$

State Model Example-1

- Transfer function of the system is $T(s) = \dfrac{Y(s)}{U(s)} = \dfrac{1}{s^3 + 6s^2 + 10s + 5}$

- The differential equation will be obtained by taking inverse Laplace tran

$$\dddot{y} + 6\ddot{y} + 10\dot{y} + 5y = u$$

- The derivatives of the inputs are not present in the differential equation,

  phase variables can be selected as the state variables

$$x_1 = y \qquad y = x_1$$

$$x_2 = \dot{y} = \dot{x}_1 \qquad \dot{x}_1 = x_2$$

$$x_3 = \ddot{y} = \dot{x}_2 \qquad \dot{x}_2 = x_3$$

$$\dddot{y} = -6\ddot{y} - 10\dot{y} - 5y + u \qquad \dot{x}_3 = -5x_1 - 10x_2 - 6x_3 + u$$

State Model Example-1 (Contd..)

- State Model in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -10 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

State Model Example-2

- The system in integral-differential form

$$\ddot{y} + 7\dot{y} + 5y + 6\int_0^t y\,dt = \dot{u} + 3u + 2\int_0^t u\,dt$$

- By differentiating the system equation will be obtained as follows

$$\dddot{y} + 7\ddot{y} + 5\dot{y} + 6y = \ddot{u} + 3\dot{u} + 2u$$

- Comparing the above equation with standard 3rd order differential equation

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

$$a_1 = 7, a_2 = 5, a_3 = 6$$

- Therefore,

$$b_0 = 0, b_1 = 1, b_2 = 3, b_3 = 2$$

$$\beta_0 = b_0 = 0$$

$$\beta_1 = b_1 - a_1\beta_0 = 1 - 7x0 = 1$$

$$\beta_2 = b_2 - a_2\beta_0 - a_1\beta_1 = 3 - 5x0 - 7x0 = -4$$

$$\beta_3 = b_3 - a_3\beta_0 - a_2\beta_1 - a_1\beta_2 = 2 - 6x0 - 5x1 - 7x(-4) = 25$$

State Model Example-2 (Contd..)

- The state variables are defined as

$$x_1 = y - \beta_0 u$$

$$x_2 = \dot{x}_1 - \beta_1 u$$

$$x_3 = \dot{x}_2 - \beta_2 u$$

$$\dot{x}_3 = -a_3 x_1 - a_2 x_2 - a_1 x_3 + \beta_3 u$$

- The state and output equations are as follows

$$y = x_1 + \beta_0 u$$

$$\dot{x}_1 = x_2 + \beta_1 u$$

$$\dot{x}_2 = x_3 + \beta_2 u$$

$$\dot{x}_3 = -6 x_1 - 5 x_2 - 7 x_3 + 25 u$$

State Model Example-2 (Contd..)

- State Model in vector matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -5 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ -4 \\ 25 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

State Transition matrix

❑ Assuming that the system is continuous and linear that A and B are time-invariant and using Laplace transform

$$\dot{x} = Ax + Bu$$

$$sX(s) - x(0) = AX(s) + BU(s)$$

$$(sI - A)X(s) = x(0) + BU(s)$$

$$X(s) = (sI - A)^{-1}[x(0) + BU(s)]$$

❑ Taking the inverse Laplace transform of resolvent matrix

❑ State Transition matrix $\boxed{\Phi(t) = L^{-1}[(sI - A)^{-1}]}$

State Transition matrix (Contd..)

- The state vector will take the following form (convolution)

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

- The **matrix exponential function** is defined as

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^kt^k}{k!} + \cdots$$

- Which converges for all finite t and any A.
- Then the solution of the state differential equation is found to be

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$X(s) = [sI - A]^{-1}x(0) + [sI - A]^{-1}BU(s)$$

State Transition matrix (Contd..)

- where we note that $[sI-A]^{-1}=\phi(s)$, which is the Laplace transform of $\phi(t)=e^{At}$.

- The matrix exponential function $\phi(t)$ describes the unforced response of the system and is called the fundamental or state transition matrix.

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

- Properties of the transition matrix

$$\boxed{\Phi(t) = L^{-1}[(sI - A)^{-1}]}$$

$$\Phi^{k}(t) = (e^{At})^{k} = e^{Akt} = \Phi(kt)$$

$$\Phi(0) = I_n$$

$$\Phi^{-1}(t) = \Phi(-t)$$

$$\Phi(t_2 - t_1)\Phi(t_1 - t_0) = \Phi(t_2 - t_0)$$

$$\Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2) = \Phi(t_2)\Phi(t_1)$$

State Transition matrix (Contd..)

- Obtain the STM for the state model

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

- **Solution:**

$$[sI - A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s-1 & -1 \\ 0 & s-1 \end{bmatrix}$$

$$\phi(s) = [sI - A]^{-1} = \frac{adj\,[sI - A]}{sI - A} = \frac{\begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}^T}{(s-1)^2} = \frac{\begin{bmatrix} s-1 & 1 \\ 0 & s-1 \end{bmatrix}}{(s-1)^2}$$

$$STM = \phi(t) = L^{-1}[\phi(s)] = L^{-1}[sI - A]^{-1} = L^{-1}\begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} \\ 0 & \frac{1}{s-1} \end{bmatrix} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

Controllability

Full-state feedback design commonly relies on **pole-placement techniques**. It is important to note that a system must be completely controllable and completely observable to allow the flexibility to place all the closed-loop system poles arbitrarily. The concepts of controllability and observability were introduced by Kalman in the 1960s.

- Controllability:

A system is completely controllable if there exists an unconstrained control $u(t)$ that can transfer any initial state $x(t_0)$ to any other desired location $x(t)$ in a finite time, $t_0 \leq t \leq T$.

Proof of controllability matrix

$$x_{k+1} = Ax_k + Bu_k$$

$$x_{k+2} = Ax_{k+1} + Bu_{k+1}$$

$$x_{k+2} = A(Ax_k + Bu_k) + Bu_{k+1} = A^2 x_k + ABu_k + Bu_{k+1}$$

$$x_{k+n} = A^n x_k + A^{n-1}Bu_k + A^{n-2}Bu_{k+1} + \cdots + ABu_{k+(n-2)} + Bu_{k+(n-1)}$$

$$x_{k+n} - A^n x_k = A^{n-1}Bu_k + A^{n-2}Bu_{k+1} + \cdots + ABu_{k+(n-2)} + Bu_{k+(n-1)}$$

$$x_{k+n} - A^n x_k = \begin{bmatrix} A^{n-1}B & \cdots & AB & B \end{bmatrix} \begin{bmatrix} u_k \\ \vdots \\ u_{k+(n-2)} \\ u_{k+(n-1)} \end{bmatrix}$$

Initial condition

Controllability (Contd..)

- For the system

$$\dot{x} = Ax + Bu$$

- We can determine whether the system is controllable by examining the algebraic condition

$$rank \begin{bmatrix} B & AB & A^2B \cdots & A^{n-1}B \end{bmatrix} = n$$

- The matrix A is an nxn matrix an B is an nx1 matrix. For multi input systems, B can be nxm, where m is the number of inputs.

- For a single-input, single-output system, the controllability matrix P_c is described in terms of A and B as

$$P_c = \begin{bmatrix} B & AB & A^2B \cdots & A^{n-1}B \end{bmatrix}$$

- Which is nxn matrix. Therefore, if the determinant of P_c is nonzero, the system is controllable.

Controllability with Example

- Example-1: Consider the system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ AB = \begin{bmatrix} 0 \\ 1 \\ -a_2 \end{bmatrix}, \ A^2B = \begin{bmatrix} 1 \\ -a_2 \\ (a_2^2 - a_1) \end{bmatrix}$$

$$\mathrm{P_c} = \begin{bmatrix} \mathrm{B} & \mathrm{AB} & \mathrm{A^2B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -\mathrm{a}_2 \\ 1 & -\mathrm{a}_2 & (\mathrm{a}_2^2 - \mathrm{a}_1) \end{bmatrix}$$

- The determinant of P_c =1 and ≠0 , hence this system is controllable.

Controllability with Example

- Example-2: Consider a system represented by the two state equations

$$\dot{x}_1 = -2x_1 + u \ , \ \dot{x}_2 = -3x_2 + d\, x_1$$

The output of the system is $y=x_2$. Determine the condition of controllability.

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ d & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ and \ AB = \begin{bmatrix} -2 & 0 \\ d & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ d \end{bmatrix}$$

$$P_c = \begin{bmatrix} 1 & -2 \\ 0 & d \end{bmatrix}$$

- The determinant of pc is equal to d, which is nonzero only when d is nonzero.

Observability

All the poles of the closed-loop system can be placed arbitrarily in the complex plane if and only if the system is **observable**. Observability refers to the ability to estimate a state variable.

- Observability:

A system is completely observable if and only if there exists a finite time T such that the initial state x(0) can be determined from the observation history y(t) given the control u(t).

Proof of observability matrix

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k + Du_k \cdots (1)$$

$$y_{k+1} = Cx_{k+1} + Du_{k+1}$$

$$y_{k+1} = C(Ax_k + Bu_k) + Du_{k+1} = CAx_k + CBu_k + Du_{k+1} \cdots (2)$$

$$y_{k+(n-1)} = CA^{n-1}x_k + CA^{n-2}Bu_k + CA^{n-3}Bu_{k+1} + \cdots + CBu_{k+(n-2)} + Du_{k+(n-1)} \cdots (n)$$

$$(1), (2), \cdots (n) \Rightarrow \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x_k$$

$$= \underline{\underline{\begin{bmatrix} y_k - Du_k & y_{k+1} - CBu_k - Du_{k+1} & \cdots & \cdots & CABu_{k+(n-3)} - CBu_{k+(n-2)} - Du_{k+(n-1)} \end{bmatrix}}}$$

Inputs & outputs

Observability (Contd..)

❑ Consider the single-input, single-output system

$$\dot{x} = Ax + Bu \quad and \quad y = Cx + Du$$

❑ Where C is a 1xn row vector, and x is an nx1 column vector. This system is completely observable when the determinant of the **observability matrix P_0** is nonzero.

$$P_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

❑ Rank of Po is n

Observability (Contd..)

- **Example 1:** $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$Po = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \qquad \boxed{rank\ [Po] = 2}$$

observable

- The **rank** of a matrix is defined by the number of linearly independent rows and/or the number of linearly independent columns

Observability (Contd..)

- **Example 2:**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$Po = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}$$

$$\boxed{rank\ [Po] = 1}$$

unobservable

Role of Compensators

- Compensators are used in cascade or feedback to achieve desired response from a closed loop system.
- Desired response is measured in terms of time domain parameters (specifications) like, rise time, peak time, settling time and peak overshoot.
- In terms of frequency response, desired response is measured in terms of frequency domain specifications like, resonant peak, resonant frequency and phase at resonant frequency.
- We have studied the relationship between frequency & time domain parameters and know that one set can be derived from the other.
- If the closed loop system does not meet with the time domain and/or frequency domain specifications; a compensator is used to achieve the same.

Lag Compensator

- **Lag Compensator:**
  - It introduces phase lag between its input and output.
  - It basically is an integrator (Low Pass Filter)
  - It can be of any order, having 'n' number of time constants, but it should yield phase lag between its input & output.
  - It is designed using simple RC networks. Operational Amplifiers are also used to design it.
  - Its attenuated output can be appropriately amplified

- **Different Types of Lag Compensators:**
  - $G(S) = K/S$; phase = $-90°$
  - $G(S) = K/(1+TS)$; phase = $-\tan^{-1}(\omega T)$
  - $G(S) = K(1+T_1 S)/(1+T_2 S) = \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$; $T_1 < T_2$

    Negative angle implies phase lag

Realization of Basic Compensators: Lag

- **Lag Compensator:**

$$G(S) = (S+Z)/(S+P) = (Z/P)\ [1+(1/Z)\ S]/[1+(1/P)S]$$

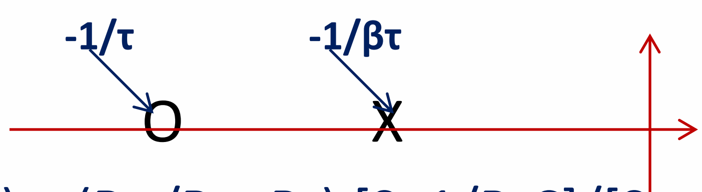
Let, $\beta = Z/P \Rightarrow P = Z/\beta$; Let $\tau = 1/Z$; $\tau > 0 \Rightarrow P = 1/(\beta\tau)$

Therefore, $G(S) = (S + 1/\tau)/[S + 1/(\beta\tau)]$

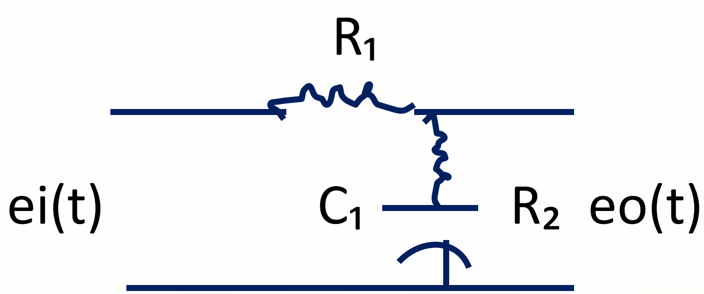
$\Rightarrow G(S) = \beta\ (1 + \tau S)/(1 + \beta\tau S)$; $\beta = Z/P > 1$ 1

- Equation 1 gives the transfer function of a lead compensator. Since $\beta>1$, it will introduce phase lead between its output & input.

- **Pole-Zero Location:**



- Lead Network: $Eo(S)/Ei(S) = (R_2 /R_1+ R_2)\ [S+1/R_2C]/[S + 1/R_2C(R_1+ R_2)/R_2]$



On simplification, we get

$$Eo(S)/Ei(S) = (1/\beta)\ (S + 1/\tau)/[S + 1/(\beta\tau)]$$

Realization Lag Compensators (Contd..)

- Lag Compensator (Contd.):

  $G(S) = Eo(S)/Ei(S) = (1/\beta)(S + 1/\tau)/[S + 1/(\beta\tau)]$

  For drawing Bode plot we convert G(S) in to time constant form as:

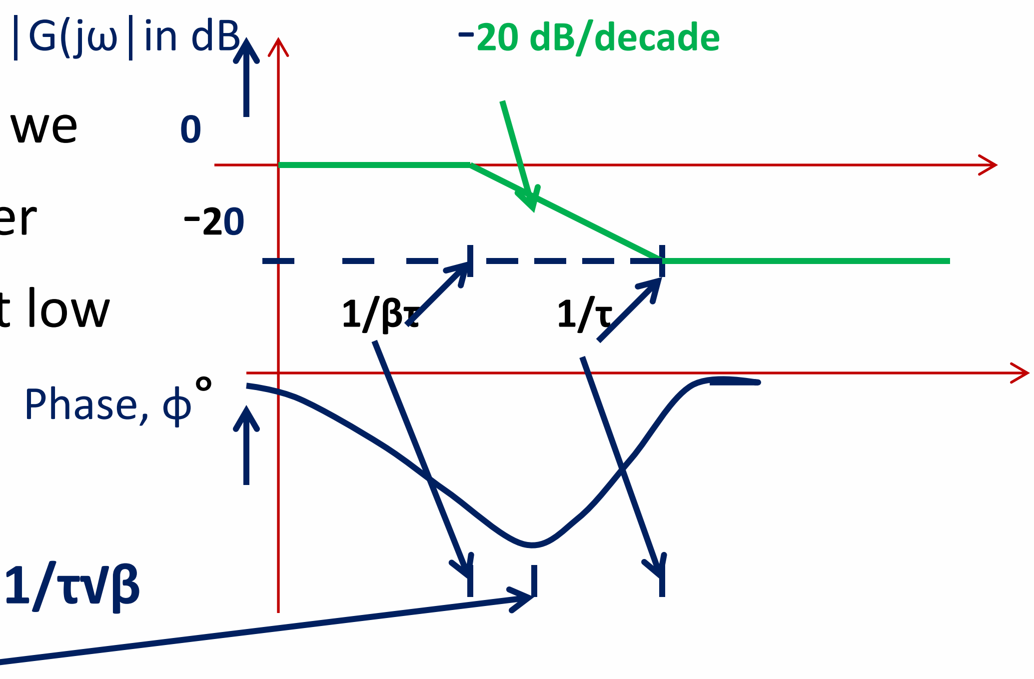
  $G(S) = (1 + \tau S)/(1 + \beta\tau S)$; τ : time constant & α is attenuation

  $\tau = R_2 C_1$ & $\beta = (R_1 + R_2)/ R_2$

- **Bode plot:**

From the magnitude plot we observe that gain at higher frequencies is less than at low frequencies.

(Contd. Next slide)



$\omega m = 1/\sqrt{\tau(\beta\tau)} = 1/\tau\sqrt{\beta}$

Realization Lag Compensators (Contd..)

- Hence Signal to Noise (S/N) ratio at the output of the lag compensator is better than at its input.

  Typically β is normally chosen to be 10.0

- Phase Response:

  The phase lead is given by $\phi = \tan^{-1}(\omega\tau) - \tan^{-1}(\beta\omega\tau)$

$$\tan\phi = \omega\tau(1-\beta)/[1+\beta\,\omega^2\tau^2]$$

- To determine the frequency at which maximum phase lead occurs, we have $d\phi/d\omega = 0$

$$d\phi/d\omega = \tau/[1+\omega^2\tau^2] - \beta\tau/[1+\beta^2\omega^2\tau^2] = 0$$

- On simplification, we get $\omega = \omega m = 1/\tau\sqrt{\beta} = \sqrt{(1/\tau)(1/\beta\tau)}$

  which is geometric mean of two corner frequencies. So at $\omega = \omega m$, we get maximum phase lag, ϕm

Realization Lag Compensators (Contd..)

- Maximum Phase Lead, ϕm:

  $\phi = \tan^{-1}(\omega\tau) - \tan^{-1}(\beta\omega\tau)$; Substitute for $\omega = \omega m = 1/\tau\sqrt{\beta}$

  $\phi m = \tan^{-1}(1/\sqrt{\beta}) - \tan^{-1}(\sqrt{\beta})$

  $\tan \phi m = (1-\beta)/2\sqrt{\beta}$

  $\sin \phi m = (1-\beta)/(1+\beta)$

  $\beta = (1- \sin \phi m)/(1+ \sin \phi m)$ 3

  From (3) β can be determined for maximum phase lead desired.

- For phase lead > 60° the network attenuation increases sharply, therefore for phase lead ≥ 60° it is advisable to use 2 cascaded lead networks.

Lead Compensator

- **Lead Compensator:**
  - It introduces phase lead between its input and output.
  - It basically is a differentiator (High Pass Filter)
  - It can be of any order, having 'n' number of time constants, but it should yield phase lead between its input & output.
  - It is designed using simple RC networks. Operational Amplifiers are also used to design it.
  - Its attenuated output can be appropriately amplified

- **Different Types of Lead Compensators:**
  - $G(S) = K\,S$; phase = $90°$
  - $G(S) = K(1+TS)$; phase = $\tan^{-1}(\omega T)$
  - $G(S) = K\,(1+T_1 S)/(1+T_2 S) = \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$; $T_1 > T_2$

Positive angle implies phase lead

Realization of Basic Compensators: Lead

- **Lead Compensator:**

  $G(S) = (S+Z)/(S+P) = (Z/P)\ [1+(1/Z)S]/[1+(1/P)S]$

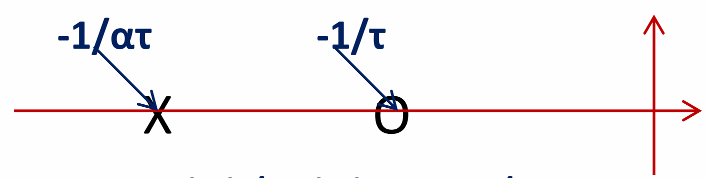
  Let, $\alpha = Z/P \Rightarrow P = Z/\alpha$; Let $\tau = 1/Z$; $\tau > 0 \Rightarrow P = 1/(\alpha\tau)$

  Therefore, $G(S) = (S + 1/\tau)/[S + 1/(\alpha\tau)]$

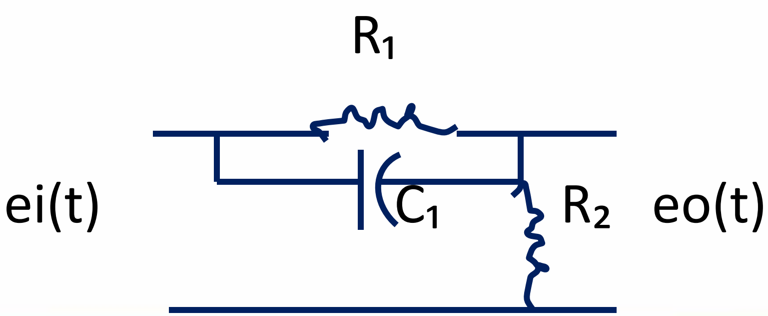
  $\Rightarrow G(S) = \alpha\,(1 + \tau S)/(1 + \alpha\tau S);\ \alpha = Z/P < 1 \quad \ldots\ 1$

- Equation 1 gives the transfer function of a lead compensator. Since $\alpha<1$, it will introduce phase lead between its output & input.

- **Pole-Zero Location:**



Lead Network:



$Eo(S)/Ei(S) = R_2/[R_2 + R_1/(1 + C_1 R_1 S)]$

On simplification, we get

$Eo(S)/Ei(S) = (S + 1/\tau)/[S + 1/(\alpha\tau)]$

Realization Lead Compensators (Contd..)

- **Lead Compensator (Contd.):**

  $G(S) = Eo(S)/Ei(S) = (S + 1/\tau)/[S + 1/(\alpha\tau)]$

  For drawing Bode plot we convert G(S) in to time constant form as:

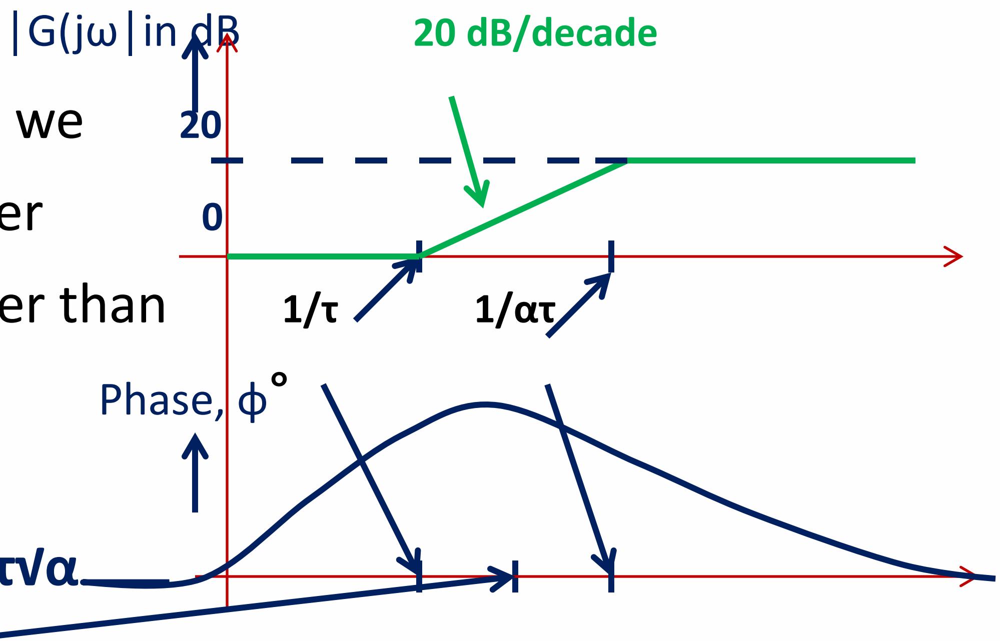
  $G(S) = \alpha\,(1 + \tau S)/(1 + \alpha\tau S)$; τ : time constant & α is attenuation

  $\tau = R_1 C_1$ & $\alpha = R_2/(R_1 + R_2)$

- **Bode plot:**

  From the magnitude plot we observe that gain at higher frequencies is much higher than at low frequencies.

  (Contd. Next slide)



$\omega m = 1/\sqrt{\tau(\alpha\tau)} = 1/\tau\sqrt{\alpha}$

Realization Lead Compensators (Contd..)

- Higher frequencies normally correspond to noise, hence Signal to Noise (S/N) ratio at the output of the lead compensator is poorer than its input.

  To improve S/N ratio α is normally chosen to be ≥ 0.1

- **Phase Response:**

  The phase lead is given by $\phi = \tan^{-1}(\omega\tau) - \tan^{-1}(\alpha\omega\tau)$

  $$\tan\phi = \omega\tau\,(1-\alpha)/[1 + \alpha\,\omega^2\,\tau^2]$$

- To determine the frequency at which maximum phase lead occurs, we have $d\phi/d\omega = 0$

  $$d\phi/d\omega = \tau/[1 + \omega^2\,\tau^2] - \alpha\tau/[1 + \alpha^2\omega^2\,\tau^2] = 0$$

  On simplification, we get $\omega = \omega m = 1/\tau\sqrt{\alpha} = \sqrt{(1/\tau)(1/\alpha\tau)}$

  which is geometric mean of two corner frequencies. So at $\omega = \omega m$, we get maximum phase lead, ϕm

Realization Lead Compensators (Contd..)

- **Maximum Phase Lead, ϕm:**

  $\phi = \tan^{-1}(\omega\tau) - \tan^{-1}(\alpha\omega\tau)$; Substitute for $\omega = \omega m = 1/\tau\sqrt{\alpha}$

  $\phi m = \tan^{-1}(1/\sqrt{\alpha}) - \tan^{-1}(\sqrt{\alpha})$

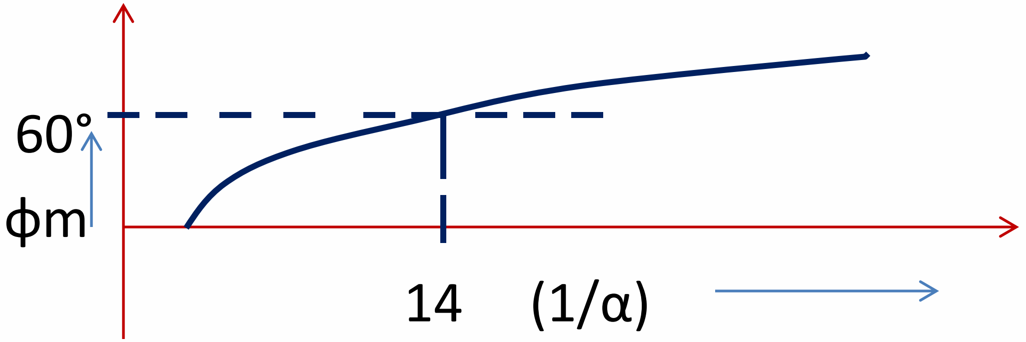
  $\tan \phi m = (1-\alpha)/2\sqrt{\alpha}$

  $\sin \phi m = (1-\alpha)/(1+\alpha)$

  $\alpha = (1- \sin \phi m)/(1+ \sin \phi m)$ 2

- From (2) α can be determined for maximum phase lead desired.

  For phase lead > 60° the network attenuation increases sharply, therefore for phase lead ≥ 60° it is advisable to use 2 cascaded lead networks.



Realization of Basic Compensators: Lag-Lead

- **Lag Lead Compensator:**

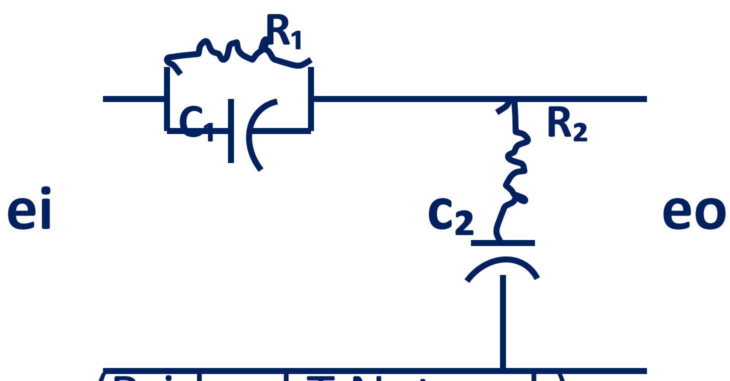
  $Gc(S) = \{(S + 1/\tau_1)/[S + 1/(\beta\tau_1)]\}\ \{(S + 1/\tau_2)/[S + 1/(\alpha\tau_2)]\};\ \beta>1;\ \alpha<1$

  ………… LAG…………… .. ………………LEAD………….

- Lag & Lead networks are in cascade.

  $Gc(S) = (S + 1/\tau_1)\ (S + 1/\tau_2)/[S + (1/\beta\tau_1 + 1/\alpha\tau_2)\ S + 1/\alpha\beta\tau_1\tau_2]$

  Network:



(Bridged T Network)

When forward path transfer function has complex poles close to jω axis, **phase lead or lag networks are not effective.**

In such cases Bridged T network is used.

Realization of Basic Compensators: Lag-Lead

- The transfer function of Bridged T network is given by:

  $E_o(S)/E_i(S) = [\,(S+1/R_1C_1)(S+1/R_2C_2)/\{S + (1/R_1C_1+1/R_2C_1+1/R_2C_2)S+1/R_1R_2C_1C_2\}]$

- Comparing with G(S), we get

$$R_1C_1 = \tau_1\,;\; R_2C_2 = \tau_2\,; \quad .... \;\; 1$$

$$R_1R_2C_2C_1 = \alpha\beta\tau_1\tau_2 \qquad ..... \; 2$$

$$1/R_1C_1+1/R_2C_1+1/R_2C_2 = 1/\beta\tau_1 + 1/\alpha\tau_2$$

- From 1 & 2, we get: $\alpha\beta = 1$ 3

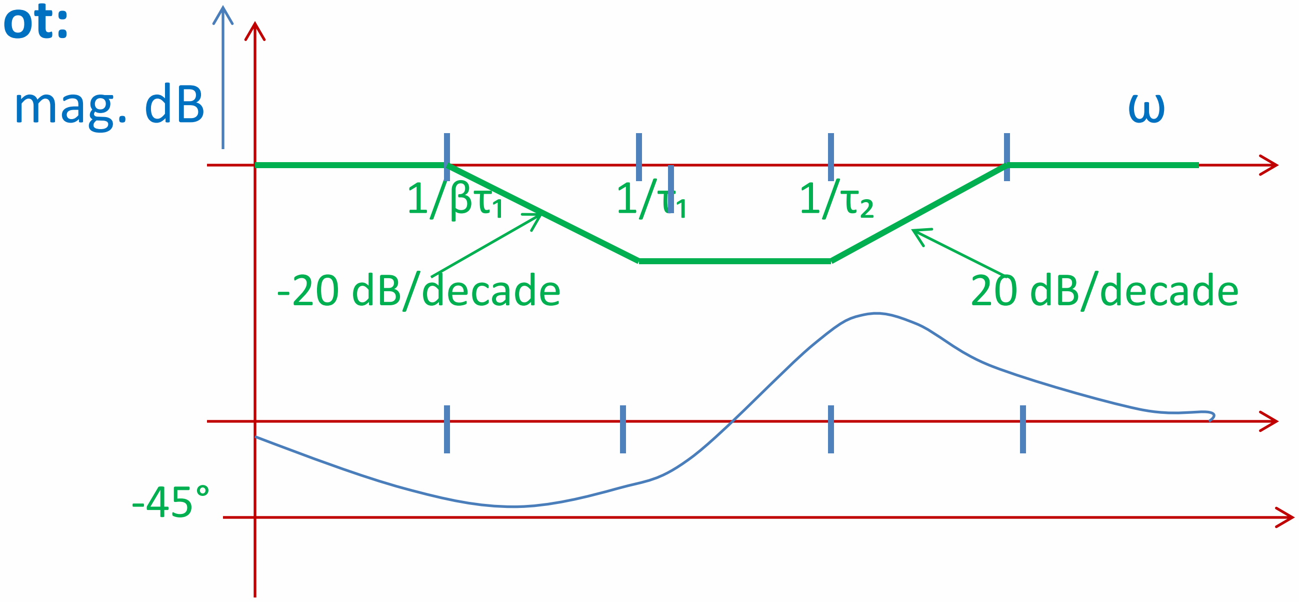
  From 3 we see that a single lag lead network does not permit us an independent choice of α & β. Therefore we write Gc(S) as:

  $Gc(S) = (S + 1/\tau_1)(S + 1/\tau_2)/[S + 1/\beta\tau_1][S + \beta/\tau_2]$; for $\alpha = 1/\beta$

  and, $1/R_1C_1+1/R_2C_1+1/R_2C_2 = 1/\beta\tau_1 + \beta/\tau_2$

Realization of Basic Compensators: Lag-Lead

- **Bode Plot:**



Diagonalization

- Multiplying the diagonal matrices are easy comparing to normal matrices

$$D = \begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix}$$

$$D^2 = \begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix} x \begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 10^2 & 0 \\ 0 & -1^2 \end{bmatrix}$$

- So A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix.
- Consider a state equation $\dot{x} = Ax + Bu$
- It's characteristic equation is $|\lambda I - A| = 0$

$$\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots\ldots + a_n = 0$$

- The roots of the characteristic equation are called eigenvalues of the matrix A

Diagonalization (Contd..)

- For the system matrix A all its n eigenvalues are distinct then the model matrix will be special matrix called Vander Monde matrix

$$V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

Diagonalization (Contd..)

- Diagonalize the system matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

- Eigen values of the system matrix A are the roots of the characteristic equation

$$|\lambda I - A| = \left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \right| = \left| \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda + 6 \end{bmatrix} \right|$$

$$|\lambda I - A| = \lambda^3 + 6\lambda^2 + 11\lambda + 6 = (\lambda + 1)(\lambda + 2)(\lambda + 3) = 0$$

- The eigen values are λ1=-1, λ2=-2, λ3=-3
- The matrix A has distinct eigen values, hence the modal matrix can be written directly in vander monde form as

$$V = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

Diagonalization (Contd..)

- The inverse of the modal matrix

$$V^{-1} = \frac{adj\ (V)}{\Delta} = \frac{1}{-2}\begin{bmatrix} -6 & 6 & -2 \\ -5 & 8 & -3 \\ -1 & 2 & -1 \end{bmatrix}^T = \frac{1}{2}\begin{bmatrix} 6 & 5 & 1 \\ -6 & -8 & -2 \\ 2 & 3 & 1 \end{bmatrix}$$

- The diagonal matrix is given by

$$V^{-1}AV = \frac{1}{2}\begin{bmatrix} 6 & 5 & 1 \\ -6 & -8 & -2 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$