ELECTROMAGNETIC THEORY AND TRANSMISSION LINES
B.TECH IV SEM-ECE

Prepared By

Dr.P.Ashok Babu,Professor

Mrs.A.Usha Rani,Assistant Professor

Mr.Murali Krishna, Assistant Professor

Dr.S.Pedda Krishna, Professor.

UNIT -I

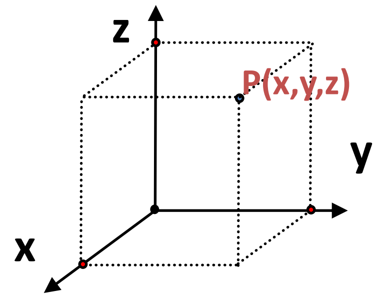
Introduction to Co-Ordinate System and Electrostatics

Cartesian Coordinates

Or

Rectangular Coordinates

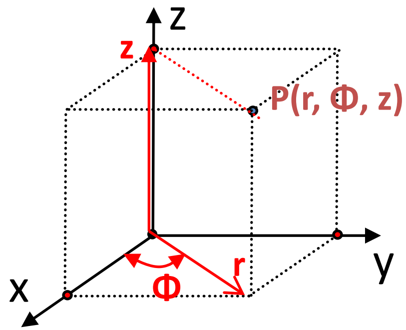
P (x, y, z)



Cylindrical Coordinates

P (r, Φ, z)

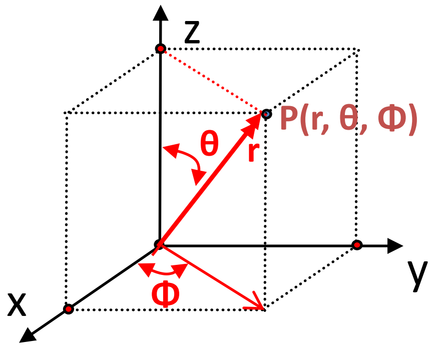
$X = r \cos \Phi$,
$Y = r \sin \Phi$,
$Z = z$



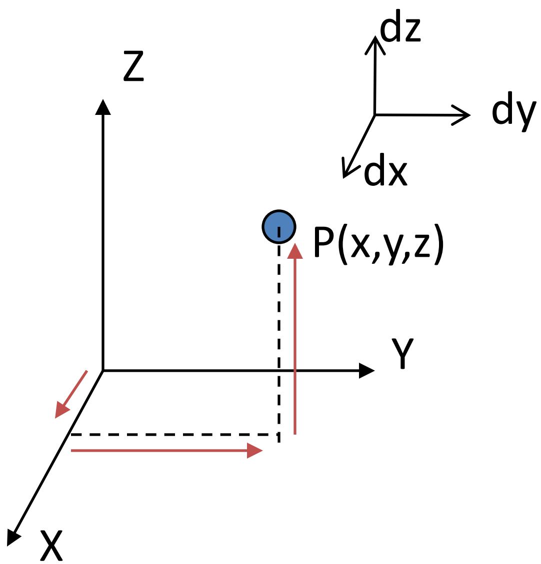
_Spherical Coordinates

P (r, θ, Φ)

$X = r \sin \theta \cos \Phi$,
$Y = r \sin \theta \sin \Phi$,
$Z = z \cos \theta$



Cartesian Coordinates



- dx, dy, dz are infinitesimal displacements along X,Y,Z.
- Volume element is given by

  dv = dx dy dz
- Area element is

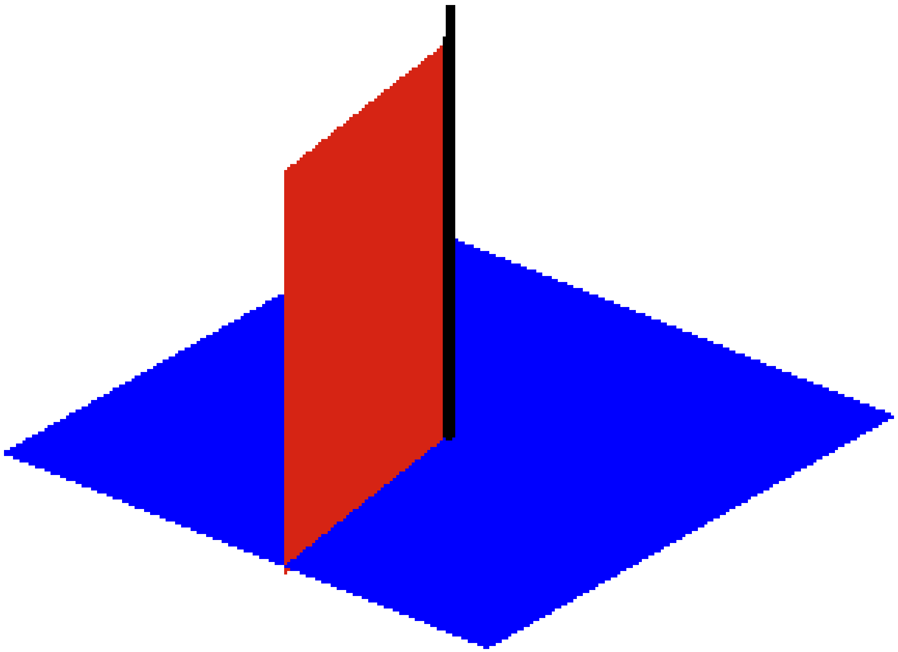
  da = dx dy **or** dy dz **or** dxdz
- Line element is

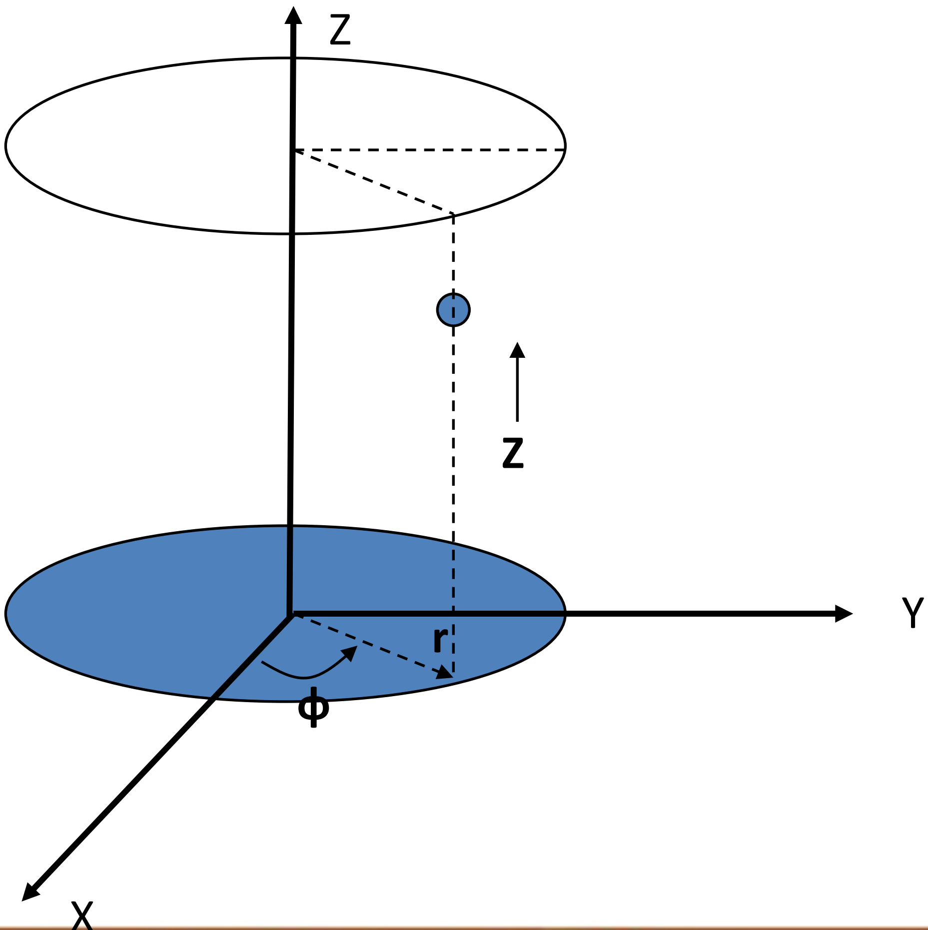
  dx **or** dy **or** dz

Ex: Show that volume of a cube of edge ***a*** is a^3.

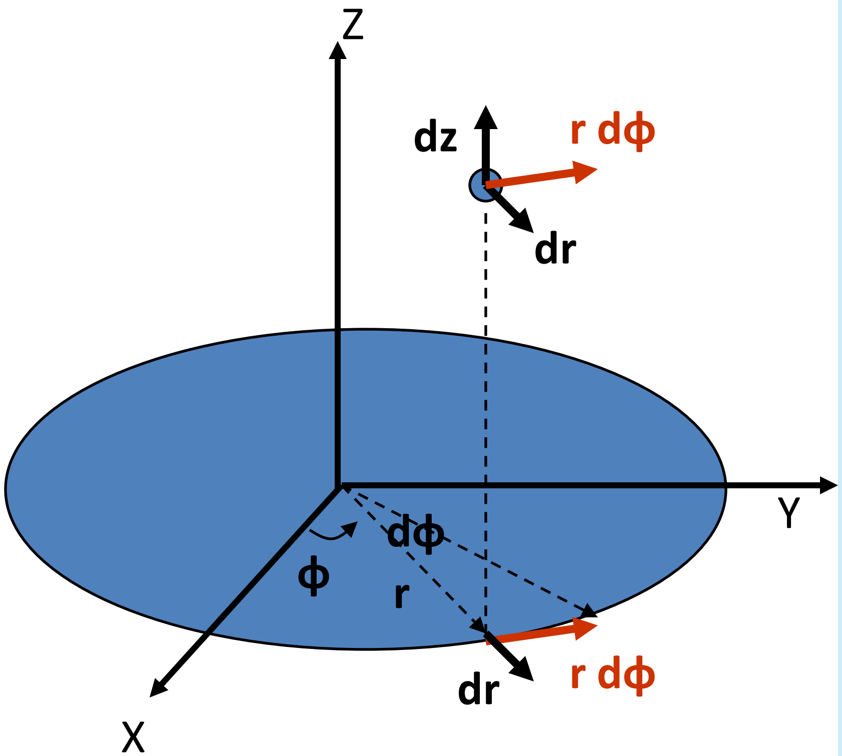
$$V = \int_v dv = \int_0^a dx \int_0^a dy \int_0^a dz = a^3$$

Cylindrical coordinate system





Cylindrical coordinate system



ϕ is azimuth angle

- **dr** is infinitesimal displacement along **r**, **r dφ** is along **φ** and **dz** is along **z** direction.
- **Volume element** is given by

  dv = dr r dφ dz

- **Limits of integration** of **r, θ, φ** are

  0<r<∞ , 0<z <∞ , o<φ <2π

Ex: Show that Volume of a Cylinder of radius 'R' and height 'H' is π R^2H .

Differential quantities:

Length element:

$$d\vec{l} = \hat{a}_r dr + \hat{a}_\phi r d\Phi + \hat{a}_z dz$$

Area element:

$$d\vec{s}_r = \hat{a}_r r d\phi dz$$

$$d\vec{s}_\phi = \hat{a}_\phi dr dz$$

$$d\vec{s}_z = \hat{a}_z r dr d\phi$$

Volume element:

$$dv = r\ dr\ d\phi\ dz$$

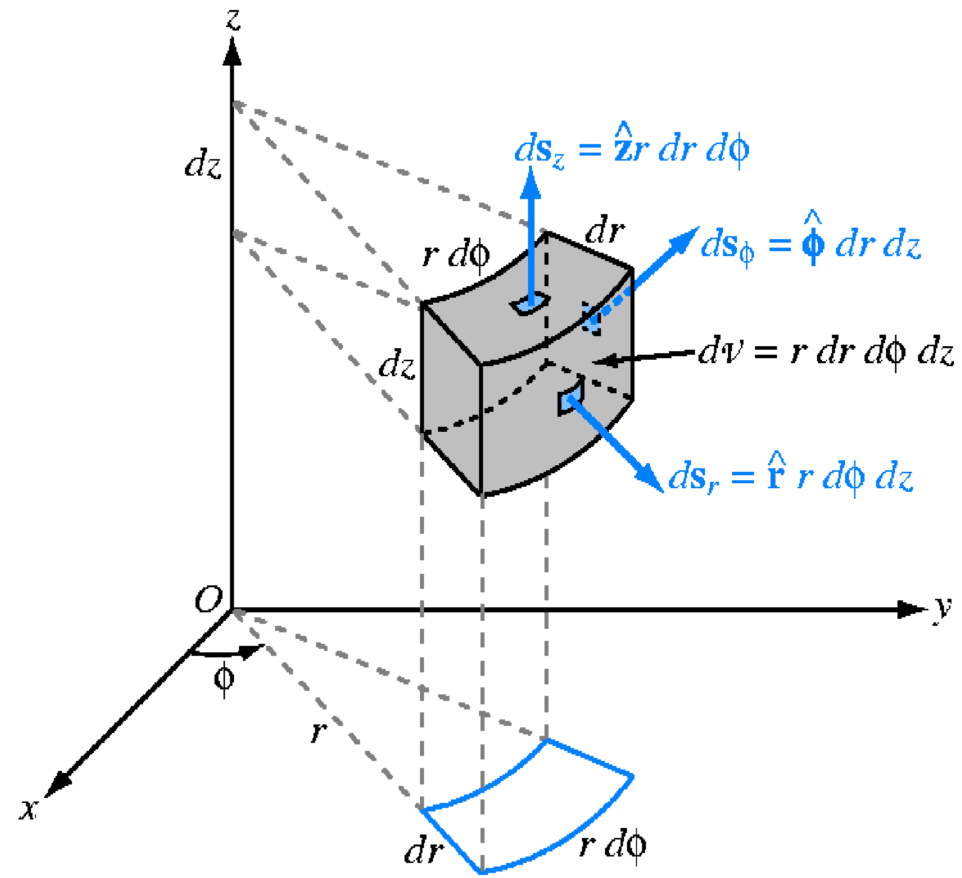
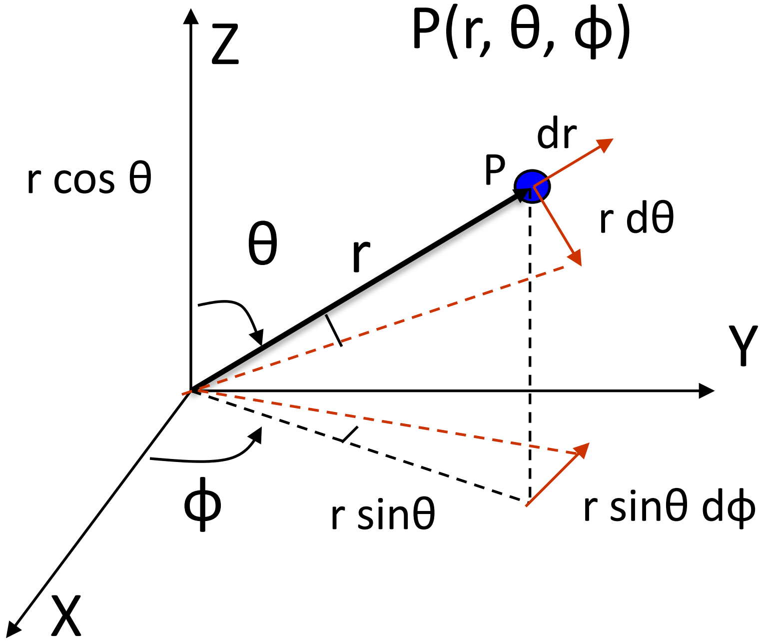


Figure 3-10

Limits of integration of **r, θ, φ** are **0<r<∞ , 0<z <∞ , o<φ <2π**

Spherical Coordinate System



- **dr** is infinitesimal displacement along **r**, **r dθ** is along **θ** and **r sinθ dφ** is along **φ** direction.
- **Volume element** is given by

  $dv = dr\ r\ d\theta\ r \sin\theta\ d\varphi$

- **Limits of integration** of **r, θ, φ** are

$0<r<\infty$, $0<\theta<\pi$, $o<\varphi<2\pi$

Ex: Show that Volume of a sphere of radius R is $4/3\ \pi\ R^3$.

Volume of a sphere of radius 'R'

$$V = \int dv = \int_v r^2 dr \sin\theta \, d\theta \, d\varphi$$

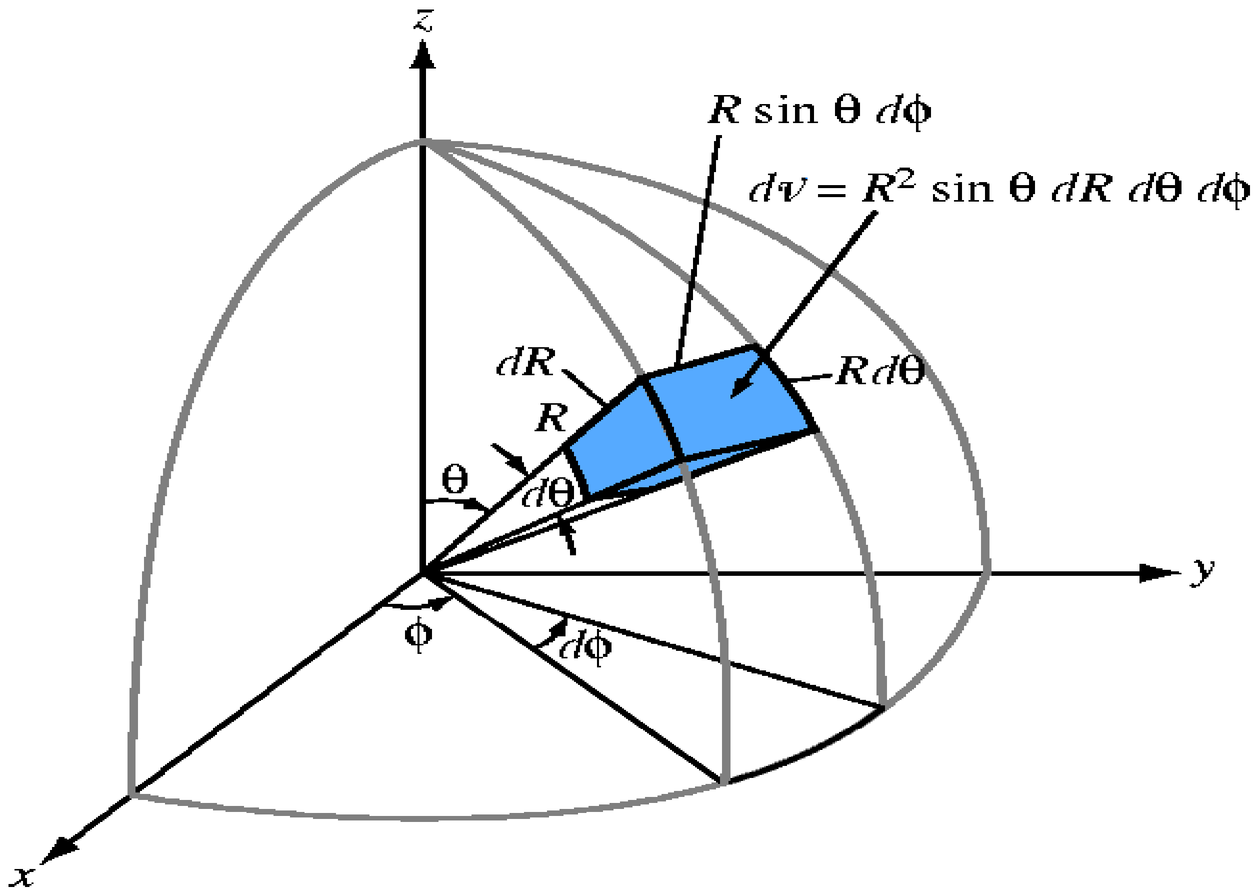
$$= \int_0^R r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\varphi$$

$$= \frac{R^3}{3} . 2 . 2\pi = \frac{4}{3}\pi R^3$$

Try Yourself:

1)Surface area of the sphere= $4\pi R^2$.

Spherical Coordinates: Volume element in space



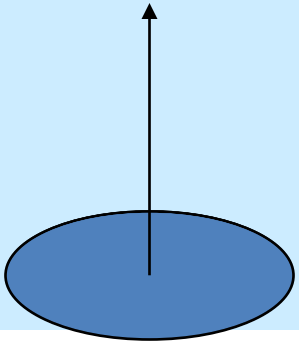
Points to remember

System	Coordinates	dl_1	dl_2	dl_3
Cartesian	x,y,z	dx	dy	dz
Cylindrical	r, φ,z	dr	rdφ	dz
Spherical	r,θ, φ	dr	rdθ	r sinθdφ

- **Volume element** : $dv = dl_1\, dl_2\, dl_3$
- **If Volume charge density 'ρ' depends only on 'r':**

$$Q = \int_v \rho\, dv = \int_l \rho\, 4\pi r^2\, dr$$

Ex: For Circular plate: NOTE
Area element da=r dr dφ in both the coordinate systems (because $\theta=90^0$)



The Gradient operator

The gradient is the closest thing to an ordinary derivative, taking a scalar-valued function into a vector field.

The simplest geometric definition is "the derivative of a function with respect to distance along the direction in which the function changes most rapidly," and the direction of the gradient vector is along that most-rapidly changing direction.

$$\textbf{grad } f(\vec{r}) = \nabla f(\vec{r})$$

$$\textbf{rectangular:}\ \nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

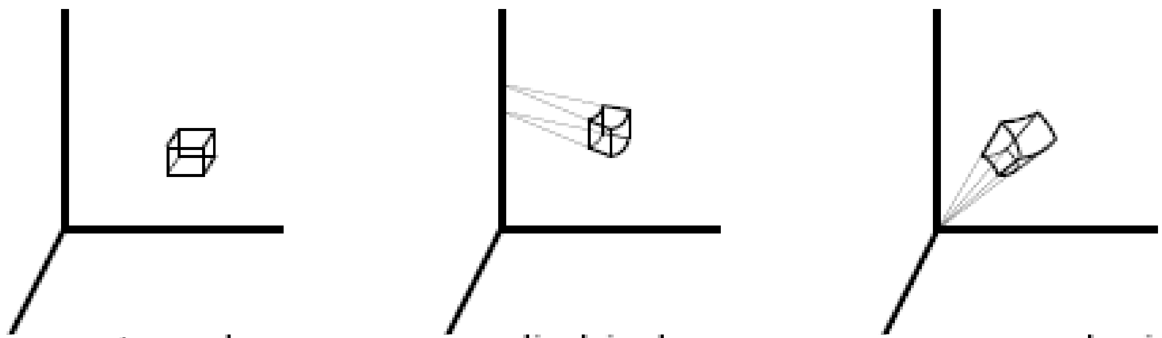
$$\textbf{cylindrical:}\ \nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{z}\frac{\partial}{\partial z}$$

$$\textbf{spherical:}\ \nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Divergence in spherical coordinates

The coordinate system is orthogonal if the surfaces made by setting the value of the respective coordinates to a constant intersect at right angles. In the spherical example this means that a surface of constant $\boldsymbol{r}$ is a sphere. A surface of constant $\boldsymbol{\theta}$ is a half-plane starting from the z-axis. These intersect perpendicular to each other. If you set the third coordinate, $\boldsymbol{\varphi}$, to a constant you have a cone that intersects the other two at right angles.

$$div\,\vec{v} = \nabla \cdot \vec{v} = \frac{1}{r^2}\frac{\partial (r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\sin\theta\, v_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

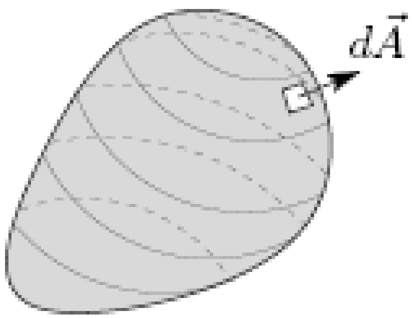


		rectangular	cylindrical	spherical
volume	$d^3r =$	$dx\,dy\,dz$	$r\,dr\,d\theta\,dz$	$r^2\sin\theta\,dr\,d\theta\,d\phi$
area	$d^2r =$	$dx\,dy$	$r\,d\theta\,dz$ or $r\,d\theta\,dr$	$r^2\sin\theta\,d\theta\,d\phi$

Gauss's Theorem

Recall the original definition of the divergence of a vector field:

$$div\,\vec{v} = \lim_{V\to 0}\frac{1}{V}\frac{dV}{dt} = \lim_{V\to 0}\frac{1}{V}\oint \vec{v}\cdot d\vec{A}$$



Fix a surface and evaluate the surface integral of $\vec{v}$ over the surface:

$$\oint_S \vec{v}\cdot d\vec{A}$$

Now divide this volume into a lot of little volumes, ΔV_k with individual bounding surfaces S_k. If you do the surface integrals of $\vec{v}\cdot d\vec{A}$ over each of these pieces and add all of them, the result is the original surface integral.

Stokes' Theorem

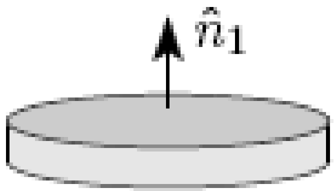
The expression for the curl in terms of integrals is

$$\text{curl}\,\vec{v} = \lim_{V\to 0} \frac{1}{V} \oint d\vec{A} \times \vec{v}$$

Use exactly the same reasoning as that was used in the case of the Gauss's theorem, this leads to

$$\oint_S d\vec{A} \times \vec{v} = \int_V \text{curl}\,\vec{v}\,dV$$

Let us first apply it to a particular volume, one that is very thin and small. Take a tiny disk of height Δh , with top and bottom area ΔA_1 . Let $\hat{n}_1$ be the unit normal vector out of the top area. For small enough values of these dimensions, $d\vec{A} \times \vec{v}$ is simply the value of the vector $\nabla \times \vec{v}$ inside the volume times the volume $\Delta h \Delta A_1$

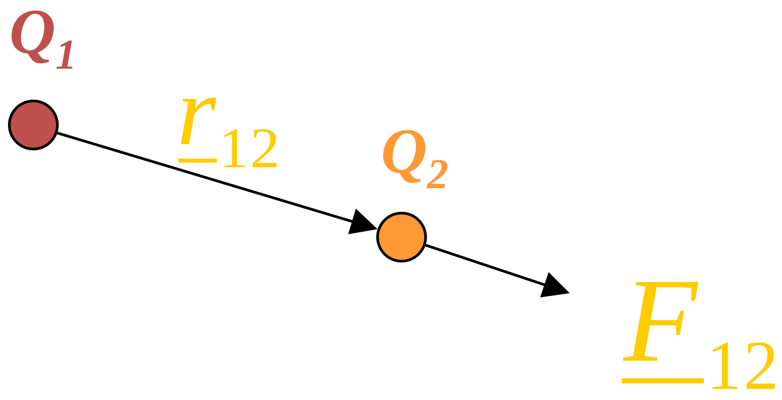


$$\oint_S d\vec{A} \times \vec{v} = \int_V \text{curl}\,\vec{v}\,dV = \text{curl}\,\vec{v}\,\Delta A_1 \Delta h$$

Coulomb’s Law

- Coulomb’s law is the “law of action” between charged bodies.
- Coulomb’s law gives the electric force between two point charges in an otherwise empty universe.
- A point charge is a charge that occupies a region of space which is negligibly small compared to the distance between the point charge and any other object.

Coulomb's Law



Unit vector in direction of R_{12}

Force due to Q_1 acting on Q_2

$$\underline{F}_{12} = \hat{a}_{R_{12}} \frac{Q_1 Q_2}{4 \pi \varepsilon_0 r_{12}^2}$$

Coulomb's Law

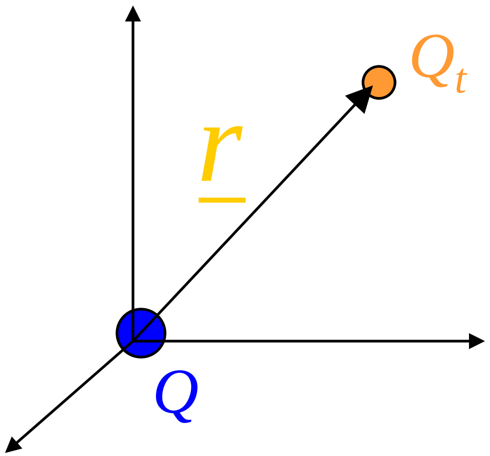
- The force on Q_1 due to Q_2 is equal in magnitude but opposite in direction to the force on Q_2 due to Q_1.

$$\overline{F}_{21} = -\overline{F}_{12}$$

Electric Field

- Consider a point charge Q placed at the origin of a coordinate system in an otherwise empty universe.
- A test charge Q_t brought near Q experiences a force:

$$\underline{F}_{Q_t} = \hat{a}_r \frac{QQ_t}{4\pi\varepsilon_0 r^2}$$



Electric Field

- The existence of the force on Q_t can be attributed to an electric field produced by Q.
- The electric field produced by Q at a point in space can be defined as the force per unit charge acting on a test charge Q_t placed at that point.

$$\overline{E} = \lim_{Q_t \to 0} \frac{\overline{F}_{Q_t}}{Q_t}$$

Electric Field

- The electric field describes the effect of a stationary charge on other charges and is an abstract "action-at-a-distance" concept, very similar to the concept of a gravity field.
- The basic units of electric field are *newtons per coulomb*.
- In practice, we usually use *volts per meter*.

Electric Field

- For a point charge at the origin, the electric field at any point is given by

$$\overline{E}(r) = \hat{a}_r \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{Q\,\underline{r}}{4\pi\varepsilon_0 r^3}$$

Electric Field

- For a point charge located at a point *P'* described by a position vector $\underline{r}'$

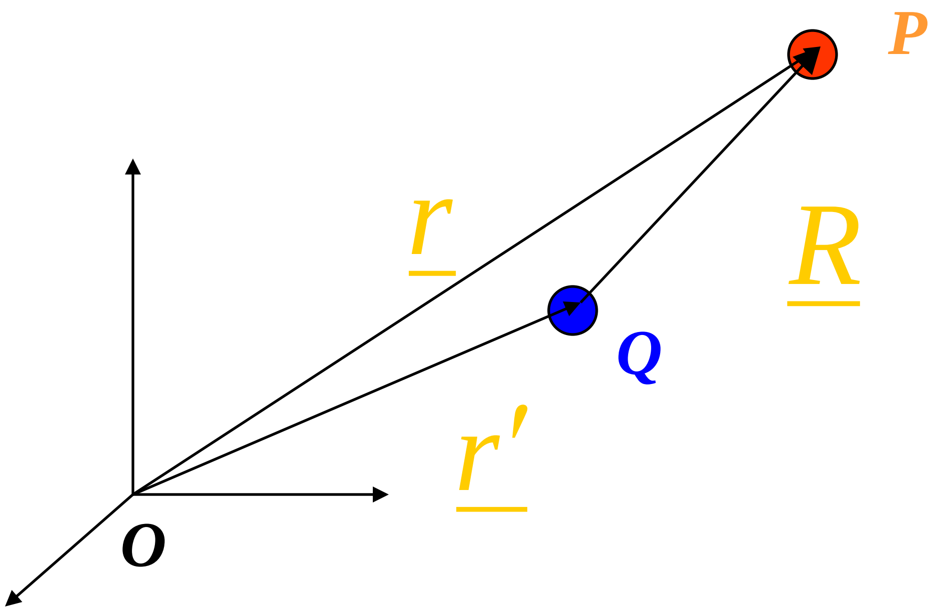
the electric field at *P* is given by

$$\underline{E}(\underline{r}) = \frac{Q\,\underline{R}}{4\pi\varepsilon_0 R^3}$$

where

$$\underline{R} = \underline{r} - \underline{r}'$$

$$R = \left|\underline{r} - \underline{r}'\right|$$

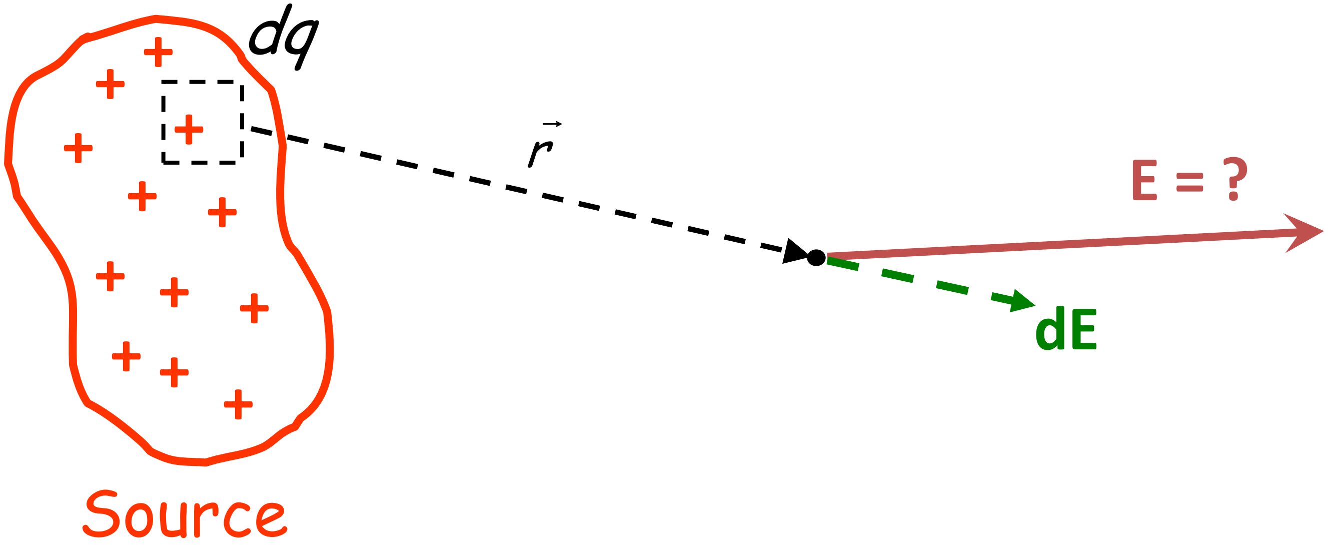


Electric Field

- In electromagnetics, it is very popular to describe the source in terms of *primed coordinates*, and the observation point in terms of *unprimed coordinates*.
- As we shall see, for continuous source distributions we shall need to integrate over the source coordinates.

Field due to Different Types of Charges

Continuous Charge Distributions



- Cut source into small ("infinitesimal") charges dq
- Each produces

$$\overrightarrow{dE} = k_e \frac{(dq)}{r^2}\hat{\mathbf{r}}$$

or

$$\left|\overrightarrow{dE}\right| = k_e \frac{dq}{r^2}$$

Steps:

- Draw a coordinate system on the diagram
- Choose an integration variable (*e.g.*, x)
- Draw an infinitesimal element dx
- Write r and any other variables in terms of x
- Write dq in terms of dx
- Put limits on the integral
- Do the integral or look it up in tables.

Continuous Charge Distributions

Charge distributed along a line:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \hat{r}' \frac{\lambda\, dx}{r'^2}.$$

Charge distributed over a surface:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_S \hat{r}' \frac{\sigma\, dS}{r'^2}.$$

Charge distributed inside a volume:

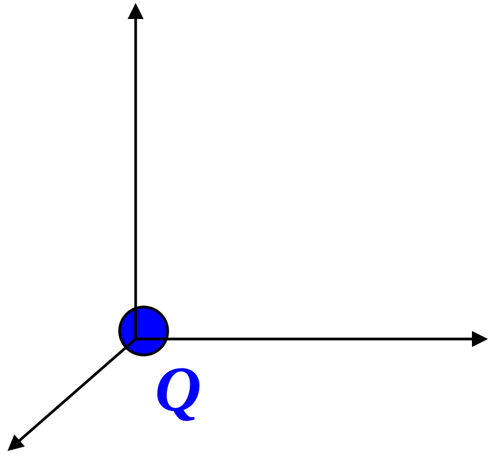
$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_V \hat{r}' \frac{\rho\, dV}{r'^2}.$$

If the charge distribution is uniform, then λ, σ, and ρ can be taken outside the integrals.

Electric Flux Density

Electric Flux Density

Consider a point charge at the origin:



Electric Flux Density of a Point Charge

(1) Assume from symmetry the form of the field

$$\underline{D} = \hat{a}_r \, D_r(r)$$

spherical symmetry

(2) Construct a family of Gaussian surfaces

spheres of radius *r* where

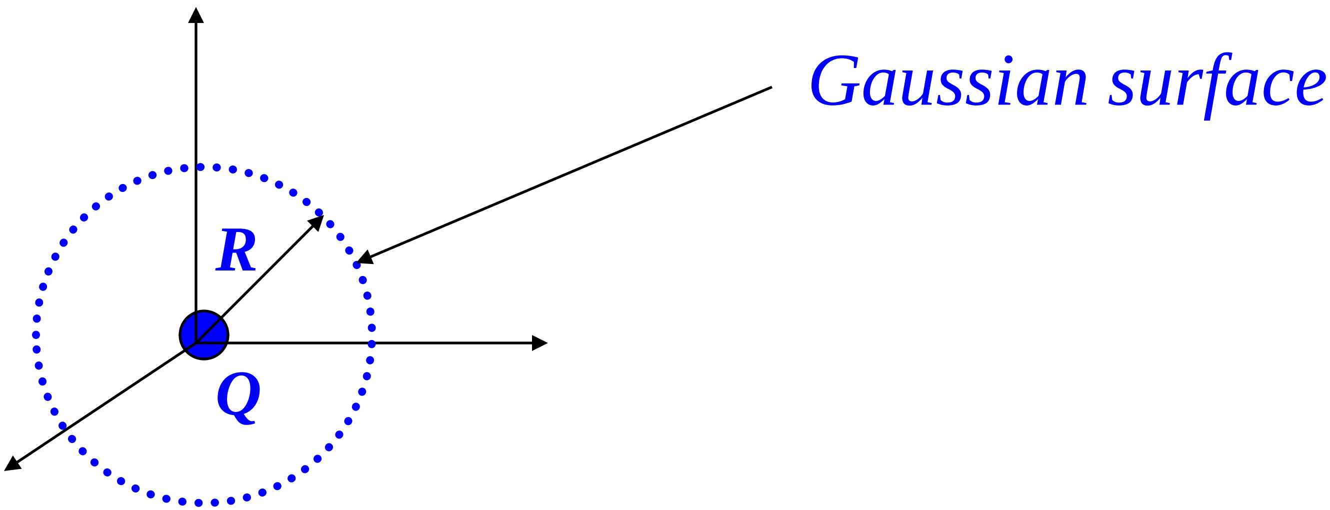
$$0 \le r \le \infty$$

Electric Flux Density of a Point Charge

(3) Evaluate the total charge within the volume enclosed by each Gaussian surface

$$Q_{encl} = \int_V q_{ev}\, dv$$

Electric Flux Density of a Point Charge



$$Q_{encl} = Q$$

Electric Flux Density of a Point Charge

(4) For each Gaussian surface, evaluate the integral

$$\oint_S \underline{D} \cdot d\,\underline{s} = DS$$

surface area of Gaussian surface.

magnitude of ***D*** on Gaussian surface.

$$\oint_S \underline{D} \cdot d\,\underline{s} = D_r(r)\; 4\pi\, r^2$$

Electric Flux Density of a Point Charge

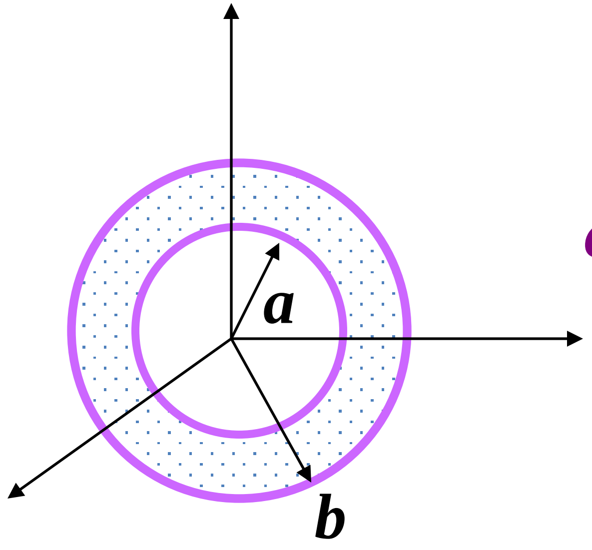
(5) Solve for ***D*** on each Gaussian surface

$$D = \frac{Q_{encl}}{S}$$

$$\boxed{\underline{D} = \hat{a}_r \frac{Q}{4\pi r^2}} \quad \Rightarrow \quad \underline{E} = \frac{\underline{D}}{\varepsilon_0} = \hat{a}_r \frac{Q}{4\pi\varepsilon_0 r^2}$$

Electric Flux Density of a Spherical Shell of Charge

Consider a spherical shell of uniform charge density:



$$q_{ev} = \begin{cases} q_0, & a \le r \le b \\ 0, & \text{otherwise} \end{cases}$$

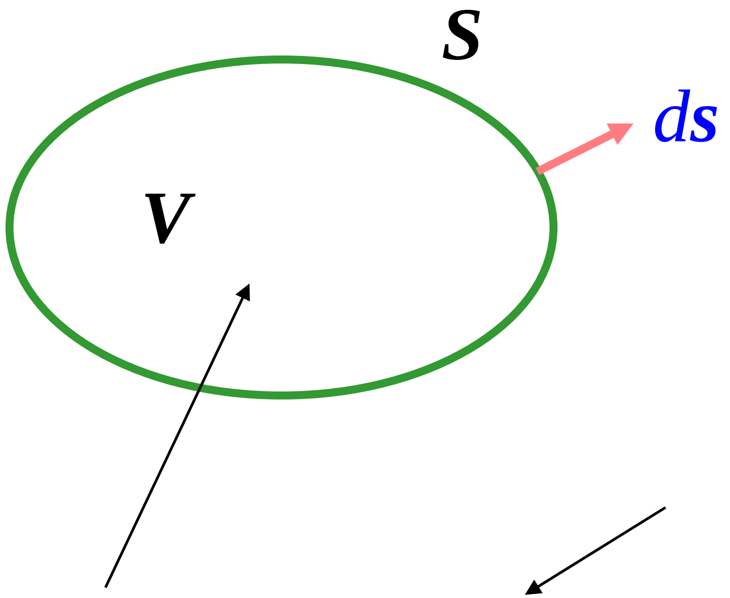
Gauss Law, It's Applications to Symmetrical Charge Distributions

Gauss's Law

- Gauss's law states that "the net electric flux emanating from a close surface *S* is equal to the total charge contained within the volume *V* bounded by that surface."

$$\oint_S \underline{D} \cdot d\underline{s} = Q_{encl}$$

Gauss's Law (Cont'd)



By convention, ***ds*** is taken to be outward from the volume ***V***.

$$Q_{encl} = \int_V q_{ev}\, dv$$

Since volume charge density is the most general, we can always write Q_{encl} in this way.

Applications of Gauss's Law

- Gauss's law is an integral equation for the unknown electric flux density resulting from a given charge distribution.

$$\oint_S \underline{D} \cdot d\,\underline{s} = Q_{encl}$$

known (Q_{encl})

unknown ($\underline{D}$)

Applications of Gauss's Law (Cont'd)

- In general, solutions to *integral equations* must be obtained using numerical techniques.

- However, for certain symmetric charge distributions closed form solutions to Gauss's law can be obtained.

Applications of Gauss's Law (Cont'd)

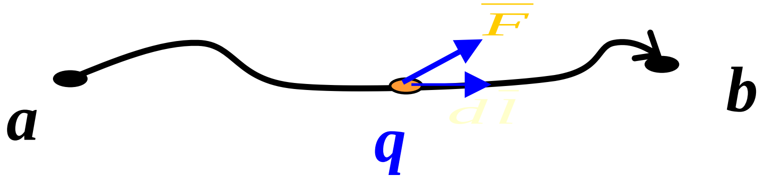
- Closed form solution to Gauss's law relies on our ability to construct a suitable family of *Gaussian surfaces*.

- A *Gaussian surface* is a surface to which the electric flux density is normal and over which equal to a constant value.

Electric Potential: Potential Field Due To Different Types of Charges

Electrostatic Potential

- An electric field is a force field.
- If a body being acted on by a force is moved from one point to another, then work is done.
- The concept of scalar electric potential provides a measure of the work done in moving charged bodies in an electrostatic field.
- The work done in moving a test charge from one point to another in a region of electric field:



$$W_{a \to b} = -\int_a^b \underline{F} \cdot d\underline{l} = -q\int_a^b \underline{E} \cdot d\underline{l}$$

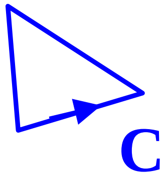
Electrostatic Potential

- In evaluating line integrals, it is customary to take the ***dl*** in the direction of increasing coordinate value so that the manner in which the path of integration is traversed is unambiguously determined by the limits of integration.

$$W_{a \to b} = -q \int_{5}^{3} \underline{E} \bullet \hat{a}_x \, dx$$

Electrostatic Potential

- The electrostatic field is conservative:
  - The value of the line integral depends only on the end points and is independent of the path taken.
  - The value of the line integral around any closed path is zero.



$$\oint_C \underline{E} \cdot d\,\underline{l} = 0$$

Electrostatic Potential

- The work done per unit charge in moving a test charge from point *a* to point *b* is the *electrostatic potential difference* between the two points:

$$V_{ab} \equiv \frac{W_{a \to b}}{q} = -\int_a^b \underline{E} \cdot d\,\underline{l}$$

electrostatic potential difference

Units are volts.

Electrostatic Potential

- Since the electrostatic field is conservative we can write

$$V_{ab} = -\int_a^b \underline{E} \bullet d\,\underline{l} = -\int_a^{P_0} \underline{E} \bullet d\,\underline{l} - \int_{P_0}^b \underline{E} \bullet d\,\underline{l}$$

$$= -\int_{P_0}^b \underline{E} \bullet d\,\underline{l} - \left(-\int_{P_0}^a \underline{E} \bullet d\,\underline{l} \right)$$

$$= V(b) - V(a)$$

Electrostatic Potential

- Thus the ***electrostatic potential*** *V* is a scalar field that is defined at every point in space.
- In particular the value of the *electrostatic potential* at any point *P* is given by

$$V(\underline{r}) = -\int_{P_0}^{P} \underline{E} \bullet d\,\underline{l}$$

reference point

Electrostatic Potential

- The ***reference point*** (P_0) is where the potential is zero (analogous to ***ground*** in a circuit).
- Often the reference is taken to be at infinity so that the potential of a point in space is defined as

$$V(\underline{r}) = -\int_{\infty}^{P} \underline{E} \bullet d\,\underline{l}$$

Electrostatic Potential and Electric Field

- The work done in moving a point charge from point *a* to point *b* can be written as

$$W_{a \to b} = Q V_{ab} = Q \{V(b) - V(a)\}$$

$$= -Q \int_{a}^{b} \underline{E} \bullet d\underline{l}$$

Electrostatic Potential and Electric Field

- Along a short path of length *Dl* we have

$$\Delta W = Q \Delta V = -Q \underline{E} \cdot \Delta \underline{l}$$

or

$$\Delta V = -\underline{E} \cdot \Delta \underline{l}$$

Electrostatic Potential and Electric Field

- Along an incremental path of length *dl* we have

$$dV = -\underline{E} \cdot d\,\underline{l}$$

- Recall from the definition of *directional derivative*:

$$dV = \nabla V \cdot d\,\underline{l}$$

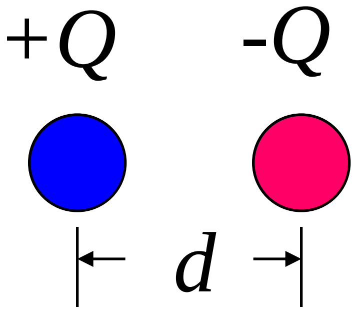
>Thus

$$\boxed{\underline{E} = -\nabla V}$$

Potential Gradient and the Dipole field due to Dipole, Maxwell's Two Equations for Electrostatic Field

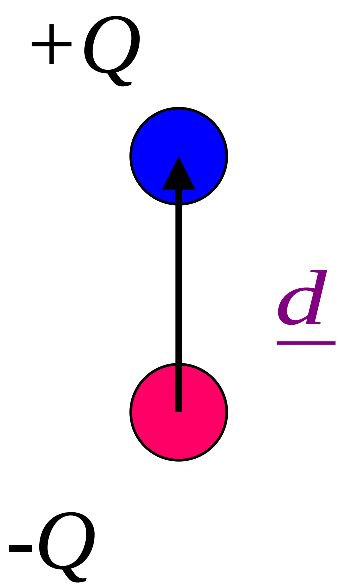
Charge Dipole

- An electric charge dipole consists of a pair of equal and opposite point charges separated by a small distance (i.e., much smaller than the distance at which we observe the resulting field).



Dipole Moment

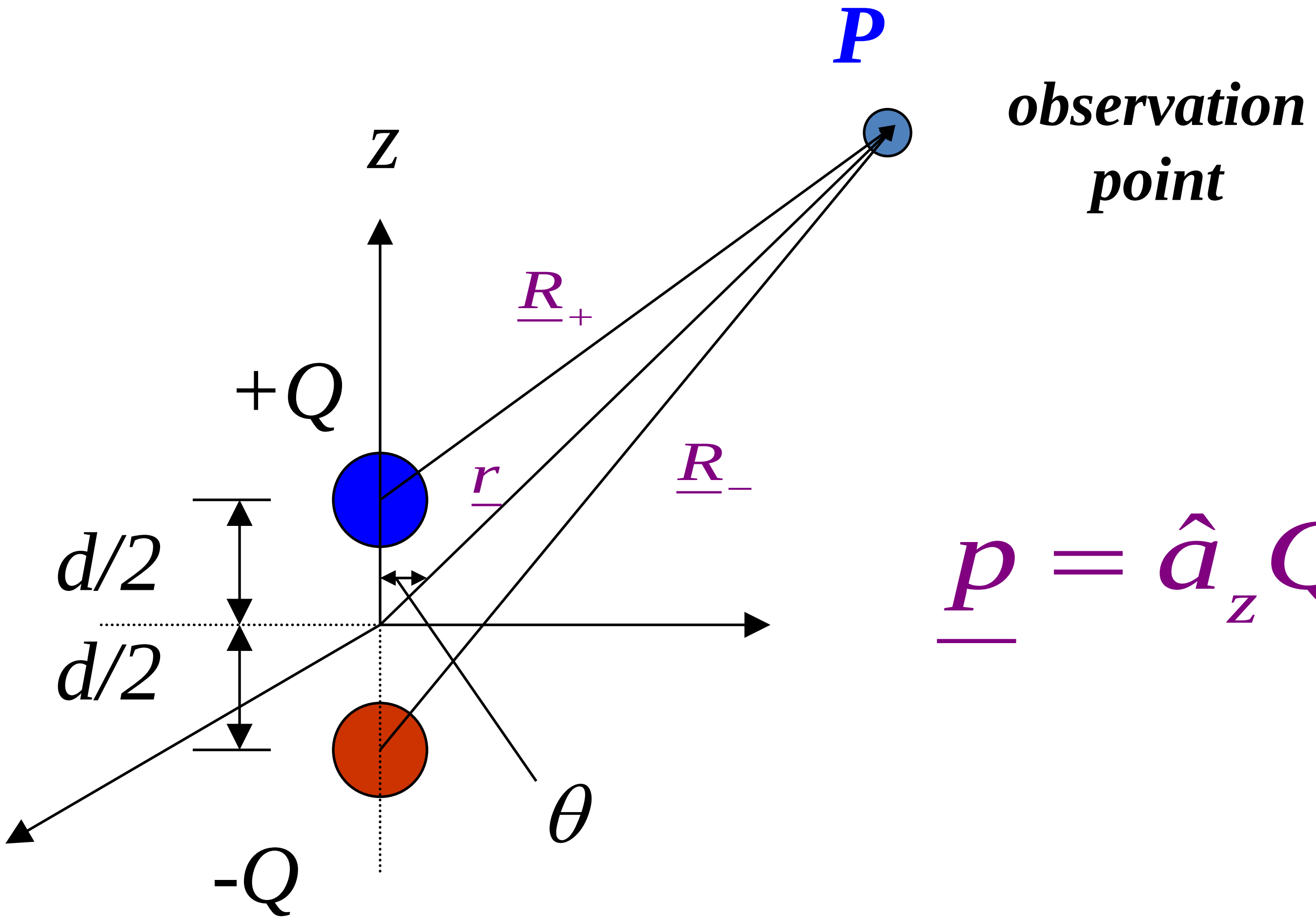
- Dipole moment p is a measure of the strength of the dipole and indicates its direction



$$\underline{p} = Q\underline{d}$$

p is in the direction from the negative point charge to the positive point charge

Electrostatic Potential Due to Charge Dipole



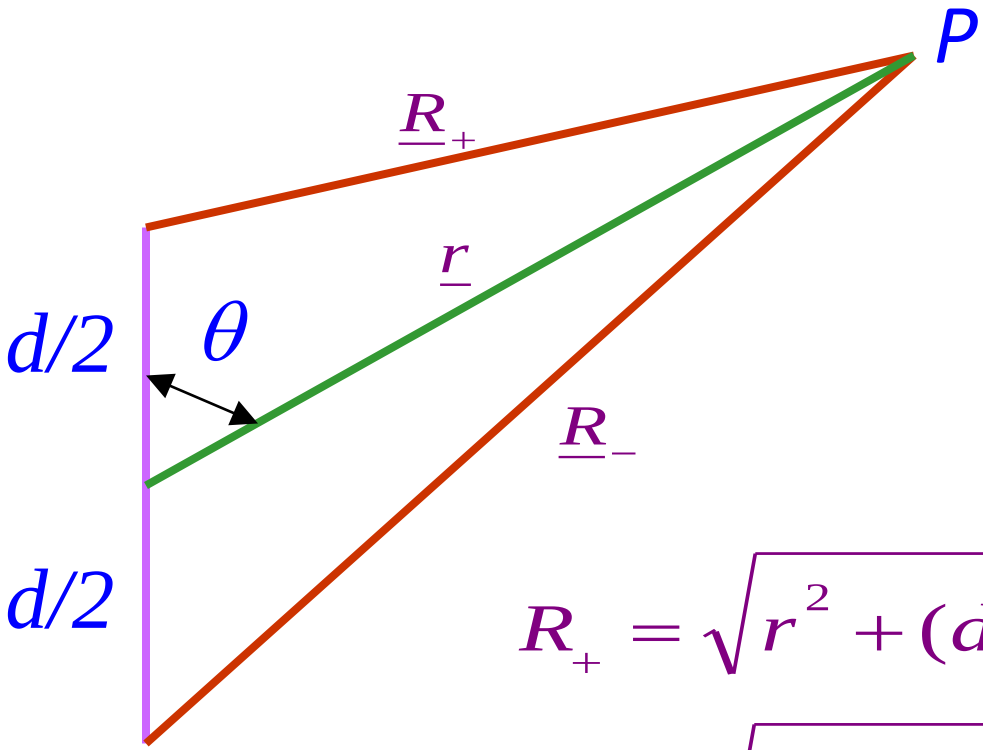
$$\underline{p} = \hat{a}_z Qd$$

Electrostatic Potential Due to Charge Dipole (Cont'd)

$$V(\underline{r}) = V(r,\theta) = \frac{Q}{4\pi\varepsilon_0 R_+} - \frac{Q}{4\pi\varepsilon_0 R_-}$$

cylindrical symmetry

Electrostatic Potential Due to Charge Dipole (Cont'd)



$$R_+ = \sqrt{r^2 + (d/2)^2 - rd\cos\theta}$$

$$R_- = \sqrt{r^2 + (d/2)^2 + rd\cos\theta}$$

Electrostatic Potential Due to Charge Dipole in the Far-Field

- assume $R >> d$

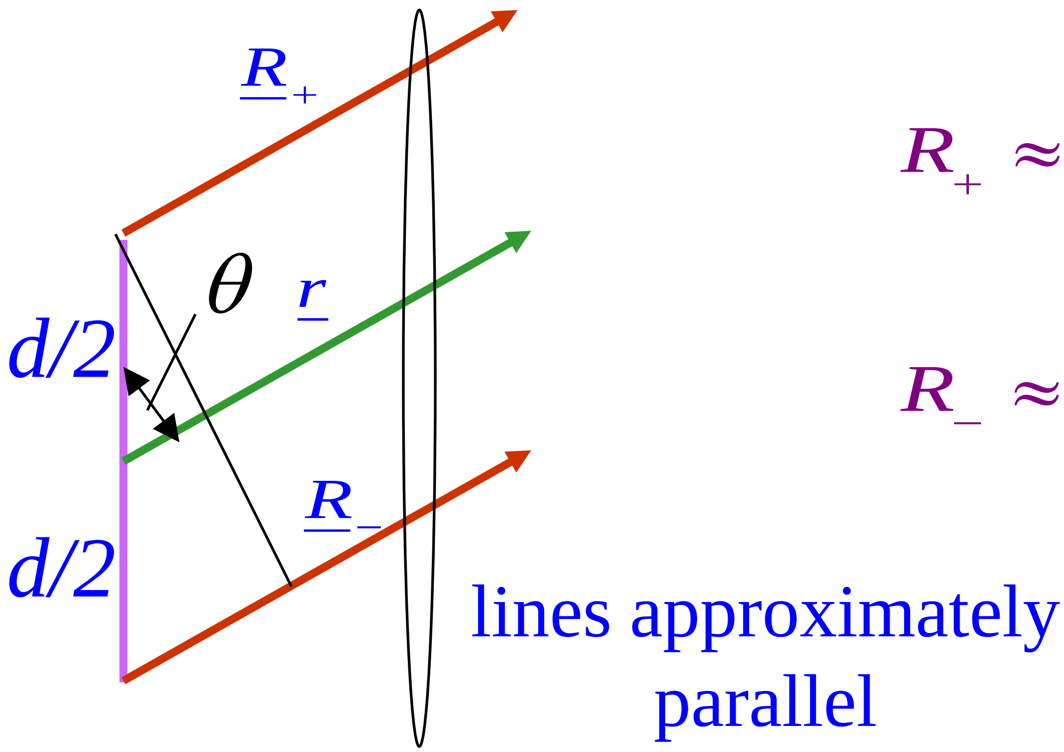
- *zeroth order* approximation:

$$R_+ \approx R \qquad R_- \approx R \quad \longrightarrow \quad V \approx 0$$

not good enough!

Electrostatic Potential Due to Charge Dipole in the Far-Field (Cont'd)

- *first order* approximation from geometry:



$$R_+ \approx r - \frac{d}{2}\cos\theta$$

$$R_- \approx r + \frac{d}{2}\cos\theta$$

Electrostatic Potential Due to Charge Dipole in the Far-Field (Cont'd)

- *Taylor series* approximation:

$$\frac{1}{R_+} = \left\{ r - \frac{d}{2}\cos\theta \right\}^{-1} = \frac{1}{r}\left\{ 1 - \frac{d}{2r}\cos\theta \right\}^{-1}$$

$$\approx \frac{1}{r}\left(1 + \frac{d}{2r}\cos\theta \right)$$

$$\frac{1}{R_-} \approx \frac{1}{r}\left(1 - \frac{d}{2r}\cos\theta \right)$$

Recall :

$$(1+x)^n \approx 1 + nx, \quad x << 1$$

Electrostatic Potential Due to Charge Dipole in the Far-Field (Cont'd)

$$V(r,\theta) \approx \frac{Q}{4\pi\varepsilon_0 r}\left[\left(1+\frac{d\cos\theta}{2r}\right)-\left(1-\frac{d\cos\theta}{2r}\right)\right]$$

$$= \frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2}$$

Electrostatic Potential Due to Charge Dipole in the Far-Field (Cont'd)

- In terms of the *dipole moment*:

$$V \approx \frac{1}{4\pi\varepsilon_0}\frac{\underline{p}\cdot\hat{a}_r}{r^2}$$

Energy Density in Electrostatic Field

Energy Density in Electrostatic Field

- To determine the energy that is present in an assembly of charges
- let us first determine the amount of work required to assemble them.
- Let us consider a number of discrete charges Q_1, Q_2,......., Q_N are brought from infinity to their present position one by one.

Energy Density in Electrostatic Field

- Since initially there is no field present, the amount of work done in bring Q_1 is zero.
- Q_2 is brought in the presence of the field of Q_1, the work done $W_1 = Q_2 V_{21}$ where V_{21} is the potential at the location of Q_2 due to Q_1.

It takes no work to bring in first charges

$W_1 = 0$ for q_1

Work needed to bring in q_2 is :

$$W_2 = q_2[\frac{1}{4\pi\varepsilon_0}\frac{q_1}{R_{12}}] = \frac{1}{4\pi\varepsilon_0}q_2(\frac{q_1}{R_{12}})$$

Work needed to bring in q_3 is :

$$W_3 = q_3[\frac{1}{4\pi\varepsilon_0}\frac{q_1}{R_{13}} + \frac{1}{4\pi\varepsilon_0}\frac{q_2}{R_{23}}] = \frac{1}{4\pi\varepsilon_0}q_3(\frac{q_1}{R_{13}} + \frac{q_2}{R_{23}})$$

Work needed to bring in q_4 is :

$$W_4 = \frac{1}{4\pi\varepsilon_0}q_4[\frac{q_1}{R_{14}} + \frac{q_2}{R_{24}} + \frac{q_3}{R_{34}}]$$

Total work

$W = W_1 + W_2 + W_3 + W_4$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{R_{12}} + \frac{q_1 q_3}{R_{13}} + \frac{q_2 q_3}{R_{23}} + \frac{q_1 q_4}{R_{14}} + \frac{q_2 q_4}{R_{24}} + \frac{q_3 q_4}{R_{34}}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{q_i q_j}{R_{ij}}$$

$$\underline{\underline{R_{ij} = R_{ji}}} \quad \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n}\frac{q_i q_j}{R_{ij}} \quad = \frac{1}{2}\sum_{i=1}^{n} q_i \left(\sum_{\substack{j=1 \\ j\neq i}}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_j}{R_{ij}}\right)$$

$$\boxed{W = \frac{1}{2}\sum_{i=1}^{n} q_i V\left(P_i\right)} \qquad V\left(P_i\right) = \sum_{\substack{j=1 \\ j\neq i}}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_j}{R_{ij}}$$

Where is the energy stored? In charge or in field ?

Both are fine in ES. But,it is useful to regard the energy as being stored in the field at a density

$$\varepsilon_0 \frac{E^2}{2} = \text{Energy per unit volume}$$

The superposition principle,not for ES energy

$$W_1 = \frac{\varepsilon_0}{2}\int E_1^2 d\tau \qquad W_2 = \frac{\varepsilon_0}{2}\int E_2^2 d\tau$$

$$W_{tot} = \frac{\varepsilon_0}{2}\int (\vec{E}_1 + \vec{E}_2)^2 d\tau$$

$$= \frac{\varepsilon_0}{2}\int (E_1^2 + E_2^2 + 2\vec{E}_1 \cdot \vec{E}_2) d\tau$$

$$= W_1 + W_2 + \varepsilon_0 \int (\vec{E}_1 \cdot \vec{E}_2) d\tau$$

Convection and Conduction Currents, Continuity Equation and Relaxation Time

Convection and Conduction Currents

- Current (in amperes) through a given area is the electric charge passing through the area per unit time

Current $$I = \frac{dQ}{dt}$$

- Current density is the amount of current flowing through a surface, A/m2, or the current through a unit normal area at that point
- Current density

$$J = \frac{\Delta I}{\Delta S}$$

- Where

$$I = \int_S J . dS$$

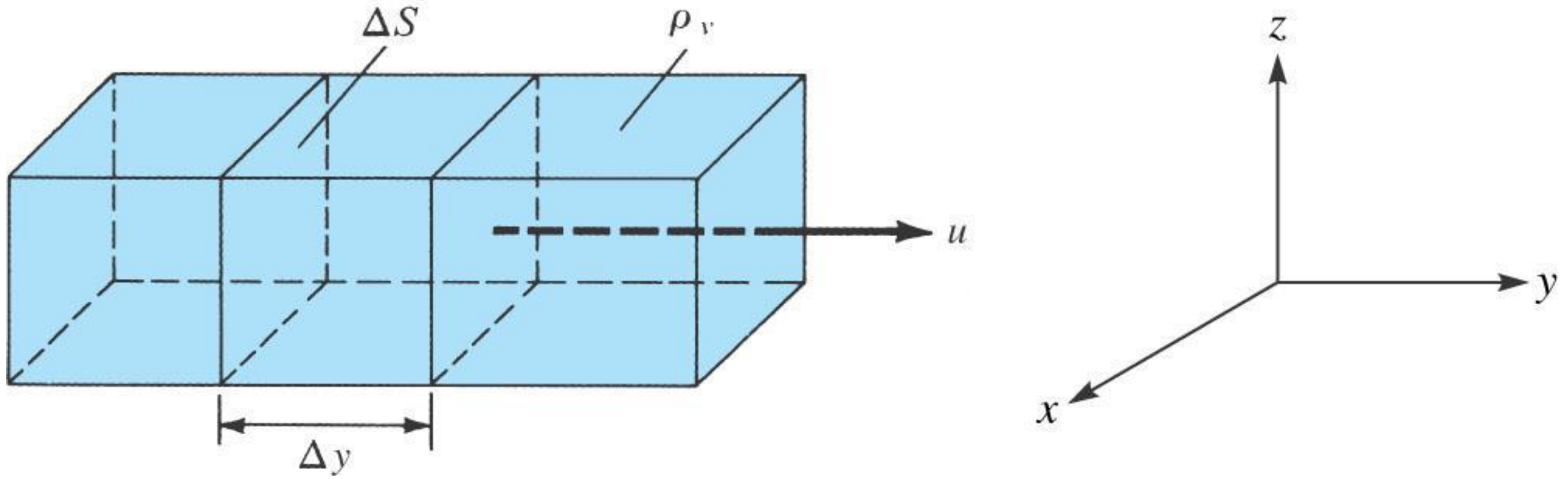
Convection and Conduction Currents

- Depending on how the current is produced, there are different types of current density
  - Convection current density
  - Conduction current density
  - Displacement current density
    - Current generated by a magnetic field

Convection current density

- Does not involve conductors and does not obey Ohm’s law
- Occurs when current flows through an insulating medium such as liquid, gas, or vacuum

Convection and Conduction Currents



$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \Delta S \frac{\Delta y}{\Delta t} = \rho_v \Delta S v_y$$

- Where v is the velocity vector of the fluid

$$J_y = \frac{\Delta I}{\Delta S} = \rho_v v_y$$

Convection and Conduction Currents

➢ Conduction current density

- Current in a conductor
- Obeys Ohm's law
- Consider a large number of free electrons travelling in a metal with mass (m), velocity (v), and scattering time (time between electron collisions), τ

$$F = -qE = \frac{mv}{\tau}$$

- The carrier density is determined by the number of electrons, n, with charge, e

$$\rho_v = -ne$$

- Conduction current density can then be calculated as

$$J = \rho_v v = \frac{ne^2\tau}{m} E = \sigma E$$

- This relationship between current concentration and electric field is known as Ohm's Law.

Continuity Equation and Relaxation Time

Continuity Equation

- Due to the principle of charge conservation, the time rate of decrease of charge within a given volume must be equal to the net outward current flow through the closed surface of the volume.

- Thus, the current coming out of the closed surface is

$$I_{out} = \int_S J.dS = -\frac{dQ_{in}}{dt}$$

- Where Q_{in} is the total charge enclosed by the closed surface. Invoking divergence theorem

$$\oint_S J.dS = \int_v \nabla .J dv =$$

- But,

$$-\frac{dQ_{in}}{dt} = -\frac{d}{dt}\int_v \rho_v dv = -\int_v \frac{d\rho_v}{dt} dv$$

Continuity Equation and Relaxation Time

- From the above three equations, we can write as

$$\int_v \nabla \bullet J dv = -\int_v \frac{\partial \rho_v}{\partial t} dv$$

$$\nabla \bullet J = \frac{\partial \rho_v}{\partial t}$$

- which is called the continuity of current equation.
- The continuity equation is derived from the principle of conservation of charge
- It states that there can be no accumulation of charge at any point
- For steady currents, $\frac{\partial \rho_v}{\partial t} = 0$
- Hence, $\nabla \bullet J = 0$
- The total charge leaving a volume is the same as the total charge entering it.

Continuity Equation and Relaxation Time

Relaxation Time

- Utilizing the continuity equation and material properties such as permittivity and conductivity, one can derive a time constant

- We start with Ohm's and Gauss' Laws

$$J = \sigma E$$

$$\nabla \bullet E = \frac{\rho_v}{\varepsilon}$$

$$\nabla \bullet \mathrm{J} = \nabla \bullet \sigma E = \frac{\sigma \rho_v}{\varepsilon} = -\frac{\partial \rho_v}{\partial t}$$

$$\frac{\sigma \rho_v}{\varepsilon} + \frac{\partial \rho_v}{\partial t} = 0$$

$$\frac{\partial \rho_v}{\rho_v} = -\frac{\sigma \partial t}{\varepsilon}$$

$$\ln \rho_v = -\frac{\sigma t}{\varepsilon} + \ln \rho_{v0}$$

Continuity Equation and Relaxation Time

$$\rho_v = \rho_{v0} e^{-\frac{t}{T_r}}$$

$$T_r = \frac{\varepsilon}{\sigma}$$

- ρ_{v0} is the initial charge density. The relaxation time(Tr) is the time it takes a charge placed in the interior of a material to drop by e-1 (=36.8%) of its initial value.
- For good conductors T_r is approx. 2*10-19 s.
- For good insulators T_r can be days

Capacitance- Parallel plate, Co-axial and Spherical Capacitor

Capacitance

- Capacitance is an intuitive characterization of a capacitor. It tells you, how much charge a capacitor can hold for a given voltage
- The property of a capacitor to 'store electricity' may be called its capacitance
- Generally speaking, to have a capacitor we must have two (or more) conductors carrying equal but opposite charges

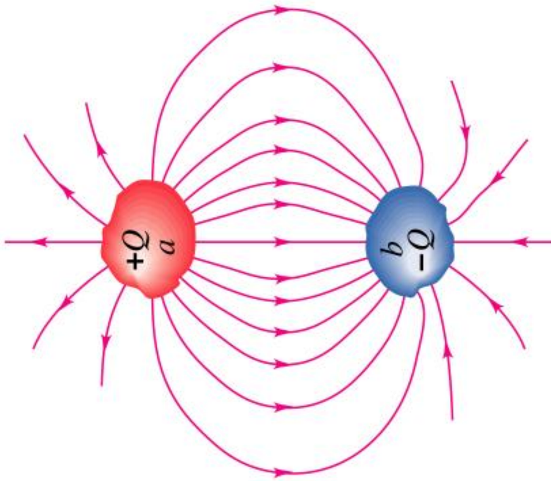


Figure 1. Charge carriers of conductor with opposite polarity

Capacitance

- Suppose we give Q coulomb of charge to one of the two plates of capacitor, the potential difference V is established between the two plates, then its capacitance is $C = \frac{Q}{V_{ab}}$
- The capacitance C is a physical property of the capacitor and in measured in farads (F)
- The charge Q on the surface of the plate and the potential difference Vab between the plates can be represented in terms of electric field

$$\oint \varepsilon E.\ dS = Q$$

$$V_{ab} = V_a - V_b = \int_a^b E.dl$$

- Therefore, the capacitance C can be written as

$$C = \frac{Q}{V_{ab}} = \frac{\oint \varepsilon E.\ dS}{\int_a^b E.dl}$$

Capacitance

Procedure for Obtaining Capacitance

- Capacitance can be obtained for any given two-conductor capacitance by following either of these methods:
  - Assuming Q and determining V in terms of Q (involving Gauss's law)
  - Assuming V and determining Q in terms of V (involving solving Laplace's equation)
- Capacitance can be determined using first method are as follows
  - Choose a suitable coordinate system.
  - Let the two conducting plates carry charges + Q and - Q
  - Determine E using Coulomb's or Gauss's law and find V
  - Finally, obtain C from $C = \frac{Q}{V}$

Parallel-Plate Capacitor

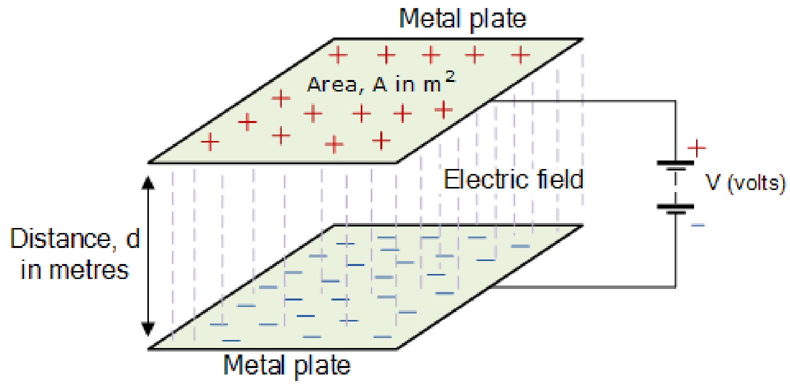


Figure 2. Parallel plate conductors

- the charge density is given by

$$\rho_S = \frac{Q}{A}$$

- The flux passing through the medium and flux density is given by

$$\Psi = Q$$

$$\psi = \int_S D_n . dA = Q = \int_S \rho_S . dA$$

$$D_n = \rho_S$$

Parallel-Plate Capacitor

- But, we know,

$$D = \varepsilon E$$

- The charge density in terms of electric field as

$$E = \frac{\rho_S}{\varepsilon}$$

- Where,

$$\rho_S = \frac{Q}{A}$$

- The above equation modifies to

$$E = \frac{Q}{A\varepsilon}$$

- Also, the relation between electric field and electric potential can be written as

$$V = \int_0^d E \,.\, dl = Ed = \frac{Q}{A\varepsilon} d$$

- Thus, the parallel plate capacitor $C = \frac{Q}{V}$ can be written as

$$C = \frac{Q}{\frac{Q}{A\varepsilon} d} = \frac{A\varepsilon}{d}$$

Coaxial(Cylindrical) Capacitor

- Consider length l of two coaxial conductors of inner radius a and outer radius b (b > a) as shown in Figure 3

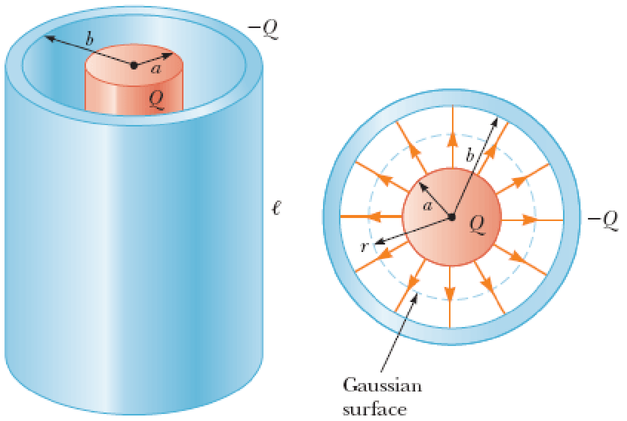


Figure 3 Cylindrical conductors

- By applying Gauss's law to an arbitrary Gaussian cylindrical surface of radius ρ (a < ρ < b), we obtain

$$Q = \Psi = \oiint_S D.dA = \oiint_S \varepsilon E.dA$$

$$Q = \oiint_S \varepsilon E.dA = \varepsilon E \int_0^{2\pi} \rho \, d\phi \int_0^l dl$$

$$Q = \varepsilon E(2\pi\rho l)$$

Coaxial(Cylindrical) Capacitor

- The potential difference between the inner and outer conductors can be written as

$$V = -\int_{-}^{+} E.dr = -\frac{Q}{2\pi\varepsilon l}\int_{b}^{a}\frac{d\rho}{\rho}$$

$$= -\frac{Q}{2\pi\varepsilon l}\left[\ln(\rho)\right]_{b}^{a}$$

$$V = -\frac{Q}{2\pi\varepsilon l}\left[\ln(a) - \ln(b)\right] = \frac{Q}{2\pi\varepsilon l}\left[\ln(b) - \ln(a)\right]$$

$$V = \frac{Q}{2\pi\varepsilon l}\ln\left(\frac{b}{a}\right)$$

- Thus the capacitance of a coaxial cylinder is given by

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{2\pi\varepsilon l}\ln\left(\frac{b}{a}\right)} = \frac{2\pi\varepsilon l}{\ln\left(\frac{b}{a}\right)}$$

Spherical Capacitor

- Consider the inner sphere of radius a and outer sphere of radius b(b> a) separated by a dielectric medium with permittivity ε as shown in Figure 4

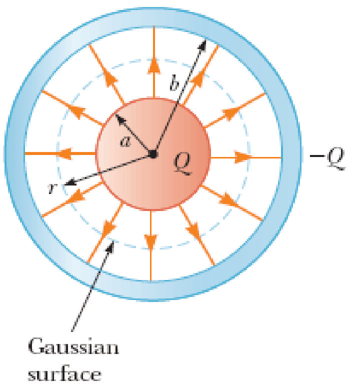


Figure 4. Spherical conductor

- By applying Gauss's law to an arbitrary Gaussian spherical surface of radius r(a<r<b), we obtain

$$Q = \Psi = \oint_S D.dA = \oint_S \varepsilon E.dA$$

$$Q = \oint_S \varepsilon E.dA = \varepsilon E r^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta$$

$$Q = \varepsilon E (4\pi r^2)$$

Spherical Capacitor

- Therefore, the potential difference between the inner and outer sphere can be written as

$$V = -\int_{-}^{+} E.dl = -\frac{Q}{4\pi\varepsilon}\int_{b}^{a}\frac{dr}{r^2}$$

$$= -\frac{Q}{4\pi\varepsilon}\left[-\frac{1}{r}\right]_{b}^{a}$$

$$\rho_S = \frac{Q}{A}$$

$$V = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{a}-\frac{1}{b}\right)$$

- Thus, the capacitance of a spherical capacitor is given by

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi\varepsilon}\left(\frac{1}{a}-\frac{1}{b}\right)} = \frac{4\pi\varepsilon}{\frac{1}{a}-\frac{1}{b}}$$

$$C = \frac{Q}{V}$$

- By letting b→ ∞, C→4πεa, which is the capacitance of a spherical capacitor whose outer plate is infinitely large

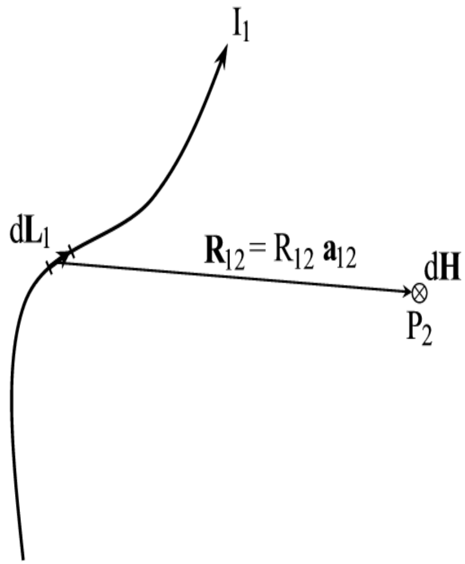
UNIT II

Magnetostatics and Time varying fields

Biot-Savart Law

The *Law of Biot-Savart* is

$$d\mathbf{H}_2 = \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{12}}{4\pi R_{12}^2} \quad \text{(A/m)}$$



To get the total field resulting from a current, you can sum the contributions from each segment by integrating

$$\mathbf{H} = \int \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}. \quad \text{(A/m)}$$

Note: The Biot-Savart law is analogous to the Coulomb's law equation for the electric field resulting from a differential charge

Vector Magnetic Potential

- Vector identity: "the divergence of the curl of any vector field is identically zero."

$$\nabla \cdot (\nabla \times \underline{A}) = 0$$

- "If the divergence of a vector field is identically zero, then that vector field can be written as the curl of some vector potential field."

Vector Magnetic Potential (Cont'd)

- Since the magnetic flux density is solenoidal, it can be written as the curl of a vector field called the vector magnetic potential.

$$\nabla \cdot \underline{B} = 0 \quad \Rightarrow \quad \underline{B} = \nabla \times \underline{A}$$

Vector Magnetic Potential (Cont'd)

- The general form of the B-S law is

$$\underline{B}(\underline{r}) = \int_{V'} \frac{\mu_0 \underline{J}(\underline{r}') \times \underline{R}}{4\pi R^3} dv'$$

- Note that

$$\nabla \left(\frac{1}{R} \right) = -\frac{\underline{R}}{R^3}$$

Vector Magnetic Potential (Cont’d)

- Furthermore, note that the *del* operator operates only on the unprimed coordinates so that

$$\frac{\underline{J}(\underline{r}')\times \overline{R}}{R^3} = -\underline{J}(\underline{r}')\times \nabla\left(\frac{1}{R}\right)$$

$$= \nabla\left(\frac{1}{R}\right)\times \underline{J}(\underline{r}')$$

$$= \nabla \times \left(\frac{\underline{J}(\underline{r}')}{R}\right)$$

Vector Magnetic Potential (Cont'd)

- Hence, we have

$$\underline{B}(\underline{r}) = \nabla \times \frac{\mu_0}{4\pi} \int_{V'} \frac{\underline{J}(\underline{r}')}{R} dv'$$

$\underline{A}(\underline{r})$

Vector Magnetic Potential (Cont'd)

- For a surface distribution of current, the vector magnetic potential is given by

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \int_{S'} \frac{\underline{J}_s(\underline{r}')}{R} \, ds'$$

- For a line current, the vector magnetic potential is given by

$$\underline{A}(\underline{r}) = \frac{\mu_0 I}{4\pi} \int_{L'} \frac{d\overline{l}'}{R}$$

Vector Magnetic Potential (Cont'd)

- In some cases, it is easier to evaluate the vector magnetic potential and then use $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, rather than to use the B-S law to directly find $\boldsymbol{B}$.
- In some ways, the vector magnetic potential $\boldsymbol{A}$ is analogous to the scalar electric potential V.

Vector Magnetic Potential (Cont'd)

- In classical physics, the vector magnetic potential is viewed as an auxiliary function with no physical meaning.
- However, there are phenomena in quantum mechanics that suggest that the vector magnetic potential is a real (i.e., measurable) field.

Forces due to Magnetic Fields, Ampere's Force Law

Ampere’s Law of Force

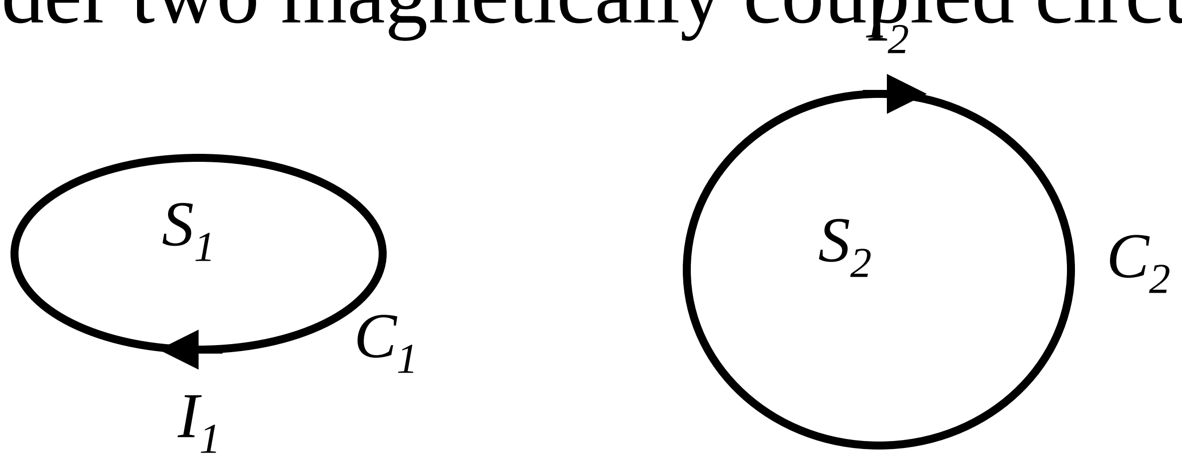
- *Ampere’s law of force* is the “law of action” between current carrying circuits.
- *Ampere’s law of force* gives the magnetic force between two *current carrying circuits* in an otherwise empty universe.
- Ampere’s law of force involves complete circuits since current must flow in closed loops.

Inductances and Magnetic Energy

Flux Linkage

- To discuss about inductance, first we have to know the flux linkage

- Consider two magnetically coupled circuits



Flux Linkage (Cont'd)

- The magnetic flux produced I_1 linking the surface S_2 is given by

$$\Psi_{12} = \int_{S_2} \underline{B}_1 \cdot d\,\underline{s}_2$$

- If the circuit C_2 comprises N_2 turns and the circuit C_1 comprises N_1 turns, then the total flux linkage is given by

$$\Lambda_{12} = N_1 N_2 \Psi_{12} = N_1 N_2 \int_{S_2} \underline{B}_1 \cdot d\,\underline{s}_2$$

Mutual Inductance

- The *mutual inductance* between two circuits is the magnetic flux linkage to one circuit per unit current in the other circuit:

$$L_{12} = \frac{\Lambda_{12}}{I_1} = \frac{N_1 N_2 \Psi_{12}}{I_1}$$

Mutual Inductance

$$L_{12} = \frac{\Lambda_{12}}{I_1} = \frac{N_1 N_2 \Psi_{12}}{I_1} = \frac{N_1 N_2}{I_1} \int_{S_2} \underline{B}_1 \cdot d\underline{s}_2$$

$$= \frac{N_1 N_2}{I_1} \oint_{C_2} \underline{A}_1 \cdot d\underline{l}_2$$

Mutual Inductance (Cont'd)

$$L_{12} = \frac{N_1 N_2}{I_1} \oint_{C_2} \underline{A}_1 \cdot d\,\underline{l}_2$$

$$= \frac{\mu_0 N_1 N_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\,\underline{l}_1 \cdot d\,\underline{l}_2}{R_{12}}$$

$$\underline{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint_{C_1} \frac{d\,\underline{l}_1}{R_{12}}$$

Mutual Inductance (Cont'd)

- The Neumann formula for mutual inductance tells us that
  - $L_{12} = L_{21}$
  - the mutual inductance depends only on the geometry of the conductors and not on the current

Self Inductance

- Self inductance is a special case of mutual inductance.
- The self inductance of a circuit is the ratio of the self magnetic flux linkage to the current producing it:

$$L_{11} = \frac{\Lambda_{11}}{I_1} = \frac{N_1^2 \Psi_{11}}{I_1}$$

Self Inductance (Cont'd)

- For an isolated circuit, we call the self inductance, inductance, and evaluate it using

$$L = \frac{\Lambda}{I} = \frac{N^2 \Psi}{I}$$

Energy Stored in Magnetic Field

- The magnetic energy stored in a region permeated by a magnetic field is given by

$$W_m = \frac{1}{2}\int_V \underline{B} \cdot \underline{H}\, dv = \frac{1}{2}\int_V \mu H^2\, dv$$

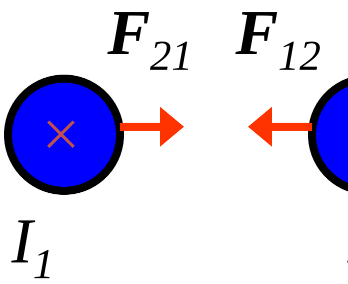
Energy Stored in an Inductor

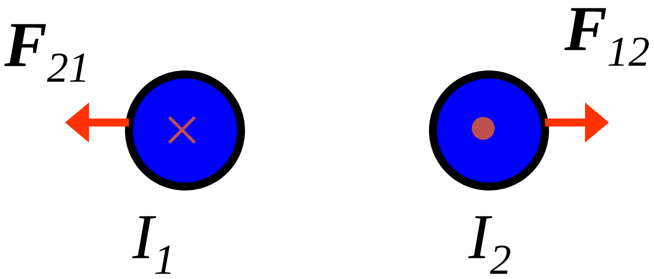
- The magnetic energy stored in an inductor is given by

$$W_m = \frac{1}{2} L I^2$$

Ampere’s Law of Force (Cont’d)

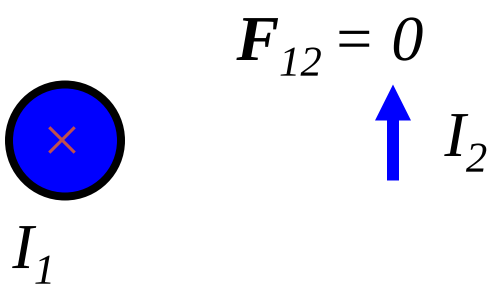
- Two parallel wires carrying current in the same direction attract.
- Two parallel wires carrying current in the opposite directions repel.





Ampere's Law of Force (Cont'd)

- A short current-carrying wire oriented perpendicular to a long current-carrying wire experiences no force.



Ampere's Law of Force (Cont'd)

The magnitude of the force is inversely proportional to the distance squared.

The magnitude of the force is proportional to the product of the currents carried by the two wires.

Ampere's Law of Force (Cont'd)

- The direction of the force established by the experimental facts can be mathematically represented by

$$\hat{a}_{F_{12}} = \hat{a}_2 \times \left(\hat{a}_1 \times \hat{a}_{R_{12}}\right)$$

- $\hat{a}_{F_{12}}$: unit vector in direction of force on I_2 due to I_1
- $\hat{a}_2$: unit vector in direction of current I_2
- $\hat{a}_1$: unit vector in direction of current I_1
- $\hat{a}_{R_{12}}$: unit vector in direction of I_2 from I_1

Ampere's Law of Force (Cont'd)

- The force acting on a current element $I_2 d\mathbf{l}_2$ by a current element $I_1 d\mathbf{l}_1$ is given by

$$\underline{F}_{12} = \frac{\mu_0}{4\pi} \frac{I_2 d\underline{l}_2 \times \left(I_1 d\underline{l}_1 \times \hat{a}_{R_{12}} \right)}{R_{12}^2}$$

Permeability of free space
$\mu_0 = 4\pi \times 10^{-7}$ F/m

Ampere's Law of Force (Cont'd)

- The total force acting on a circuit C_2 having a current I_2 by a circuit C_1 having current I_1 is given by

$$\underline{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\,\underline{l}_2 \times \left(d\,\underline{l}_1 \times \hat{a}_{R_{12}} \right)}{R_{12}^2}$$

Ampere's Law of Force (Cont'd)

- The force on C_1 due to C_2 is equal in magnitude but opposite in direction to the force on C_2 due to C_1.

$$\underline{F}_{21} = -\underline{F}_{12}$$

Force on a Moving Charge

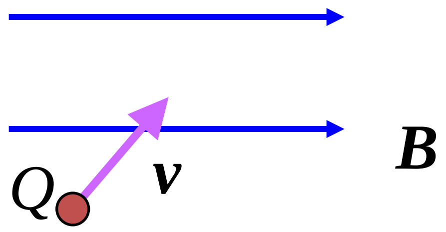
- The total force exerted on a circuit *C* carrying current *I* that is immersed in a magnetic flux density ***B*** is given by

$$\underline{F} = I \oint_C d\,\underline{l} \times \underline{B}$$

Force on a Moving Charge

- A moving point charge placed in a magnetic field experiences a force given by

$$\underline{F}_m = Q\,\underline{v} \times \underline{B} \qquad Id\,\underline{l} \Leftrightarrow Q\,\underline{v}$$



The force experienced by the point charge is in the direction into the paper.

Lorentz Force

- If a point charge is moving in a region where both electric and magnetic fields exist, then it experiences a total force given by

$$\underline{F} = \underline{F}_e + \underline{F}_m = q\left(\underline{E} + \underline{v} \times \underline{B}\right)$$

- The Lorentz force equation is useful for determining the equation of motion for electrons in electromagnetic deflection systems such as CRTs.

Faraday's Law and Transformer EMF, Motional EMF

Faraday's Law of electromagnetic Induction

- Steady current produces a magnetic field
- In 1831, Michael Faraday discovery that a time-varying magnetic field would produce an electric current
- According to Faraday's experiments, a static magnetic field produces no current flow, but a time-varying field produces an induced voltage (electromotive force)
- induced emf (in volts), in any closed circuit is equal to the time rale of change of the magnetic flux linkage by the circuit(Faraday’s law)
- According to Faraday’s, It can be expressed as

$$V_{emf} = -\frac{d\lambda}{dt} = -N\frac{d\psi}{dt}$$

- Where,N is the number of turns in the circuit ψ is the flux through each turn.
- The negative sign shows to oppose the flux producing it(Lenz's law)

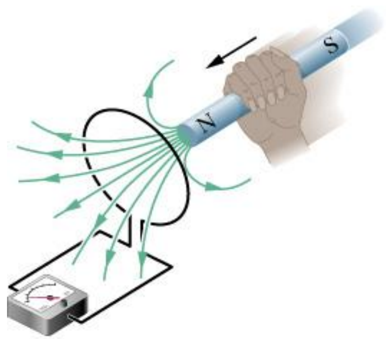
Faraday's Law of electromagnetic Induction

- Induced emf (For a single turn N=1) in terms of E and B can be written as

$$V_{emf} = -\frac{d\psi}{dt} = \oint_L E \bullet dl = -\frac{d}{dt}\int_S B \bullet ds$$

- The variation of flux with time may be caused in three ways
  - By having a stationary loop in a time-varying B field
  - By having a time-varying loop area in a static B field
  - By having a time-varying loop area in a time-varying B field

- Induced emf (For a single turn N=1) in terms of E and B can be written as



Faraday's Law of electromagnetic Induction

- This emf induced by the time-varying current (producing the time-varying B field) in a stationary loop is often referred to as transformer emf

- A stationary conducting loop is in a time varying magnetic B field

$$V_{emf} = -\frac{d\psi}{dt} = \oint_L E \bullet dl = -\frac{d}{dt}\int_S B \bullet ds$$

- By applying Stokes's theorem to the middle term

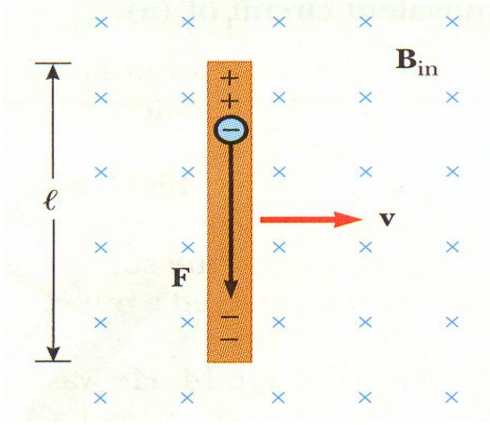
$$\int_S (\Delta \times E) \bullet dS = -\frac{\partial}{\partial t}\int_S B \bullet ds$$

$$\Delta \times E = -\frac{\partial B}{\partial t}$$

Faraday's Law of electromagnetic Induction

➢ Moving Loop in Static B Field (Motional emf)

- When a conducting loop is moving in a static B field, an emf is induced in the loop



$$F = qvB$$

- We define the motional electric field Em as

$$E_m = \frac{F_m}{Q} = v \times B$$

- Moving with uniform velocity u as consisting of a large number of free electrons, the emf induced in the loop is

$$V_{emf} = \oint_L E \bullet dl = -\frac{d}{dt}\int_L v \times B \bullet dl$$

- By applying Stokes's theorem

$$\Delta \times E_m = \Delta \times (v \times B)$$

Maxwell's equations in integral form

➢ Faraday's law of Induction:

- This describes the creation of an electric field by a changing magnetic flux
- The law states that the emf, which is the line integral of the electric field around any closed path, equals the rate of change of the magnetic flux through any surface bounded by that path
- One consequence is the current induced in a conducting loop placed in a time-varying B

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt}$$

➢ Modified Ampere's law:

- It describes the creation of a magnetic field by an electric field and electric currents
- The line integral of the magnetic field around any closed path is the given sum

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_o I + \varepsilon_o \mu_o \frac{d\Phi_E}{dt}$$

Maxwell’s Equations in Point form and Integral form for Time-Varying Fields

Maxwell's equations in point form

- Maxwell equations in differential form

$$\nabla \cdot \vec{D} = \rho_v,$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},$$

$$\nabla \cdot \vec{B} = 0,$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t},$$

Maxwell's equations in integral form

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{q}{\varepsilon_o} \quad \text{Gauss's law (electric)}$$

$$\oint_S \mathbf{B} \cdot d\mathbf{A} = 0 \quad \text{Gauss's law in magnetism}$$

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt} \quad \text{Faraday's law}$$

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_o I + \varepsilon_o \mu_o \frac{d\Phi_E}{dt} \quad \text{Ampere-Maxwell law}$$

- The two Gauss's laws are symmetrical, apart from the absence of the term for magnetic monopoles in Gauss's law for magnetism
- Faraday's law and the Ampere-Maxwell law are symmetrical in that the line integrals of E and B around a closed path are related to the rate of change of the respective fluxes

Maxwell's equations in integral form

➢ Gauss's law (electrical):

- The total electric flux through any closed surface equals the net charge inside that surface divided by eo
- This relates an electric field to the charge distribution that creates it

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{q}{\varepsilon_o}$$

➢ Gauss's law (magnetism):

- The total magnetic flux through any closed surface is zero
- This says the number of field lines that enter a closed volume must equal the number that leave that volume
- This implies the magnetic field lines cannot begin or end at any point
- Isolated magnetic monopoles have not been observed in nature

$$\oint_S \mathbf{B} \cdot d\mathbf{A} = 0$$

Maxwell’s equations in integral form

- **Faraday’s law of Induction:**
  - This describes the creation of an electric field by a changing magnetic flux
  - The law states that the emf, which is the line integral of the electric field around any closed path, equals the rate of change of the magnetic flux through any surface bounded by that path
  - One consequence is the current induced in a conducting loop placed in a time-varying B

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt}$$

- **Modified Ampere’s law:**
  - It describes the creation of a magnetic field by an electric field and electric currents
  - The line integral of the magnetic field around any closed path is the given sum

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_o I + \varepsilon_o \mu_o \frac{d\Phi_E}{dt}$$

UNIT-III

Uniform Plane Waves

Reflection and Refraction of Uniform Plane Wave

Reflection and Refraction of Uniform Plane Wave

- Till now, we have studied plane waves in various medium
- Let us try to explore how plane waves will behave at a media interface
- Electromagnetic waves are often at the interface of boundary may be reflected or refracted or changes direction at the interface
- When a radio wave reflects from a surface, the ratio of the two (reflected wave/incident wave) is known as the 'reflection coefficient' of the surface
- How much of the incident wave has been transmitted through the material is given by another ratio(transmitted wave /incident wave) known as 'transmission coefficient'

Reflection and Refraction of Plane Wave contd..,

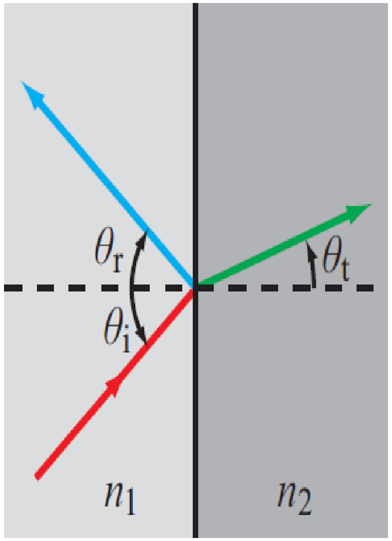
- Transmission and reflection ratio depends on the
  - conductivity (σ),
  - permittivity (ε) and
  - permittivity (ε) of the material as well as material properties of the air which the radio wave is incident
- Also some part of the wave will be transmitted through the material
  - how much of the incident wave has been transmitted through the material is also dependent on the material parameters mentioned above
  - It is given by another ratio known as ‘transmission coefficient’
  - It is the ratio of the transmitted wave divided by the incident wave

Reflection and Refraction of Plane Wave contd..,

- In plane wave incident from media interface, consider two cases
  - Normal incidence
  - Obliquely incidence
- The electric and magnetic field expressions
  - in all the regions of interest and apply the boundary conditions to get the values of the transmission and reflection coefficients
  - Type of boundary interfaces for the solutions of transmission and reflection coefficients
- dielectric –conductor interface (both normal and oblique incidence)
  - dielectric –dielectric interface (both normal and oblique incidence)

Reflection and Refraction of Plane Wave contd..,

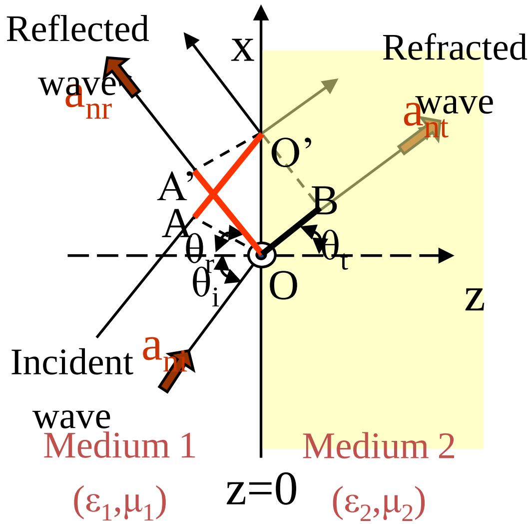
- At an interface between two media, the angles of incidence, reflection, and refraction are all measured with respect to the normal.
  - Incidence angle (θ_i): angle at which the incident wave makes with the normal to the interface
  - Reflection angle (θ_r): angle at which the reflected wave makes with the normal to the interface
  - Transmission or refraction angle (θ_t): angle at which the transmitted (refracted) wave makes with the normal to the interface



Reflection and Refraction of Plane Wave contd..,

- If a plane wave is incident obliquely upon a surface that is not a conductor boundary, part of the wave is transmitted and part of it reflected.
- In this case the transmitted wave will be refracted; that is the direction of propagation will be altered.
- Consider a planar interface between two dielectric media. A plane wave is incident at an angle from medium 1.

- In the diagram incident ray 2 travels a distance AO' where as the transmitted ray 1 travels a distance OB, and reflected ray 1 travels from O to A'



Reflection and Refraction of Plane Wave contd..,

- Snell's Law of refraction
  - The angle of reflection is equal to the angle of incident

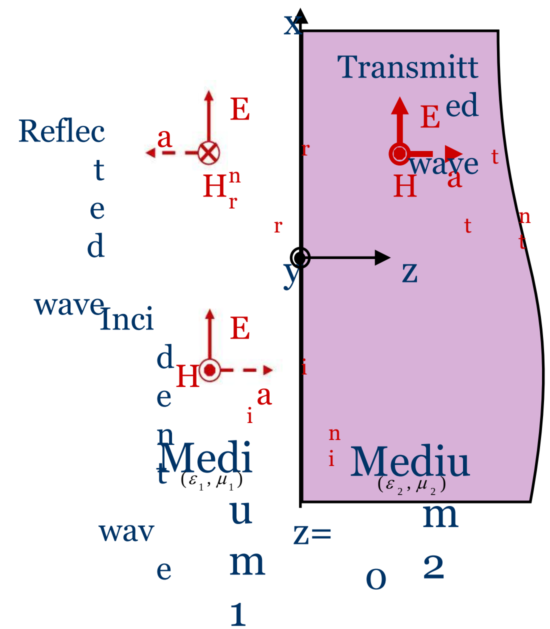
$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{u_2}{u_1} = \frac{\beta_1}{\beta_2} = \frac{\omega\sqrt{\mu\varepsilon_1}}{\omega\sqrt{\mu\varepsilon_2}} = \frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}} = \frac{\sqrt{\varepsilon_o\varepsilon_{r1}}}{\sqrt{\varepsilon_o\varepsilon_{r2}}} = \frac{\sqrt{\varepsilon_{r1}}}{\sqrt{\varepsilon_{r2}}} = \frac{n_1}{n_2} = \frac{\sqrt{\frac{\mu_o}{\varepsilon_2}}}{\sqrt{\frac{\mu_o}{\varepsilon_1}}} = \frac{\eta_2}{\eta_1}$$

n : refractive index

Reflection of a Plane Wave at Normal Incidence
-Dielectric Boundary

Normal Incidence -Dielectric Boundary

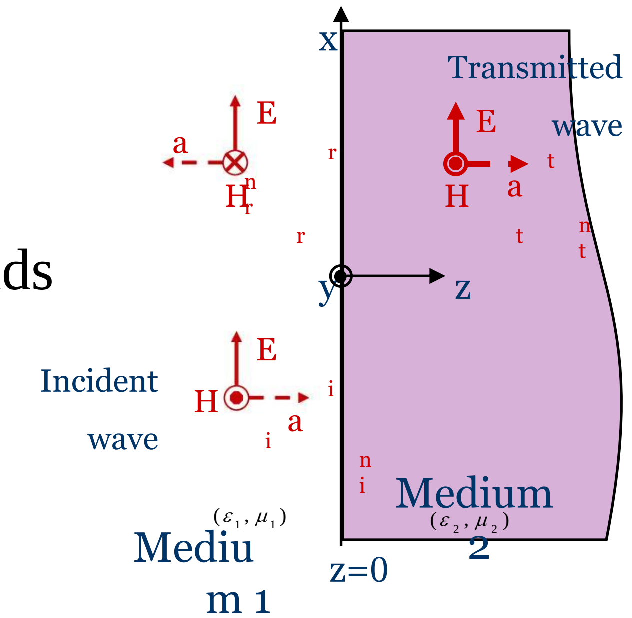
- Consider the boundary is an interface between two dielectrics
- We will consider the case of normal incidence, when the incident wave propagation vector is along the normal to the interface between two media
- Generally, consider a time harmonic x-polarized electric field incident from medium 1 (μ_1, ε_1, σ_1) to medium 2 (μ_2, ε_2, σ_2)
- For the case of perfect dielectric
  - $\sigma_1 = \sigma_2 = 0$, $\varepsilon_1 \neq \varepsilon_2$
  - No loss or absorption of power



Normal Incidence -Dielectric Boundary contd..,

- We will assume plane waves with electric field vector oriented along the x-axis and propagating along the positive z-axis without loss of generality
- For z<0 (we will refer this region as region I and it is assumed to be a medium1)
- When a plane electromagnetic wave incident on the surface of perfect dielectric
  - part of energy is transmitted
  - part of it is reflected
- Let the wave incident on medium 1

the incident, reflected and transmitted fields shown in Figure



Normal Incidence -Dielectric Boundary contd..,

- Incident wave (inside medium 1)

$$\vec{E}_i(z) = \hat{a}_x E_{i0} e^{-j\beta_1 z}$$

$$\vec{H}_i(z) = \hat{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z}$$

- Reflected wave (inside medium 1)

$$\vec{E}_r(z) = \hat{a}_x E_{r0} e^{-j\beta_1 z}$$

$$\vec{H}_r(z) = (-\hat{a}_z) \times \frac{1}{\eta_1} \vec{E}_r(z) = -\hat{a}_y \frac{E_{r0}}{\eta_1} e^{j\beta_1 z}$$

- Transmitted wave (inside medium 2)

$$\vec{E}_t(z) = \hat{a}_x E_{t0} e^{-j\beta_2 z}$$

$$\vec{H}_t(z) = \hat{a}_z \times \frac{1}{\eta_2} \vec{E}_t(z) = \hat{a}_y \frac{E_{t0}}{\eta_2} e^{-j\beta_2 z}$$

Normal Incidence -Dielectric Boundary contd..,

- The tangential components (the x-components) of the electric and magnetic field intensities must be continuous. (at interface z=0)

$$E_{1\tan} = E_{2\tan} \qquad H_{1\tan} = H_{2\tan}$$

- The continuity of tangential components of the electric and magnetic fields require that

$$\vec{E}_i(0) + \vec{E}_r(0) = \vec{E}_t(0) \quad \Rightarrow \quad E_{io} + E_{ro} = E_{to}$$

$$\vec{H}_i(0) + \vec{H}_r(0) = \vec{H}_t(0) \quad \Rightarrow \quad \frac{1}{\eta_1}(E_{io} - E_{ro}) = \frac{E_{to}}{\eta_2}$$

- Rearranging these two equations

$$\therefore \quad E_{ro} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{io} \qquad E_{to} = \frac{2\eta_2}{\eta_2 + \eta_1} E_{io}$$

Normal Incidence -Dielectric Boundary contd..,

- The reflection coefficient

$$\Gamma = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

Reflection coefficient (+ or -) $\leq$ 1

- The transmission coefficient

$$\tau = \frac{E_{to}}{E_{io}} = \frac{2\eta_2}{\eta_2 + \eta_1}$$

$$1 + \Gamma = \tau$$

which is related to reflection coefficient

- Special cases

$\eta_1 = \eta_2 \; : \; \Gamma = 0, \quad \tau = 1$

$\eta_2 = 0\,(short)\; \Gamma = -1$ E/H, E=0 perfect conductor !!

$\eta_2 = \infty\,(open)\; \Gamma = 1$ H(I)=0 No current !!

Normal Incidence - Dielectric Boundary contd..,

- Further more, the reflection and transmission coefficient

$$\therefore \quad \frac{H_{ro}}{H_{io}} = -\frac{E_{ro}}{E_{io}} = \frac{\eta_1 - \eta_2}{\eta_2 + \eta_1} \qquad \frac{H_{to}}{H_{io}} = \frac{\eta_1 E_{to}}{\eta_2 E_{io}} = \frac{2\eta_1}{\eta_2 + \eta_1}$$

- The permeabilities of all known dielectrics do not differ appreciably from free space, so that $\mu_1 = \mu_2 = \mu_0$

$$\frac{E_{ro}}{E_{io}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\sqrt{\frac{\mu_0}{\varepsilon_2}} - \sqrt{\frac{\mu_0}{\varepsilon_1}}}{\sqrt{\frac{\mu_0}{\varepsilon_2}} + \sqrt{\frac{\mu_0}{\varepsilon_1}}} = \frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}, \qquad \frac{E_{to}}{E_{io}} = \frac{2\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}$$

- Similarly,

$$\frac{H_{ro}}{H_{io}} == \frac{\sqrt{\varepsilon_2} - \sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}, \qquad \frac{H_{to}}{H_{io}} == \frac{2\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}$$

Reflection of a Plane Wave Normal Incidence
-Conducting Boundary

Oblique Incidence - Conducting Boundary

- In z-direction (x=constant)

$$E_{1y} = jA\sin(\beta_1 \cos\theta_i z)e^{j\omega t}$$

$$H_{1x} = B\cos(\beta_1 \cos\theta_i z)e^{j\omega t}$$

$\Big\}$ 90° out of phase

no P_{av} is propagate.

- In x-direction (z=constant)

$$E_{1y} = Ce^{j(\omega t - \beta_1 x \sin\theta_i)}$$

$$H_{1z} = De^{j(\omega t - \beta_1 x \sin\theta_i)}$$

$\Big\}$ traveling wave

C = f(z) , *D = g(z)*

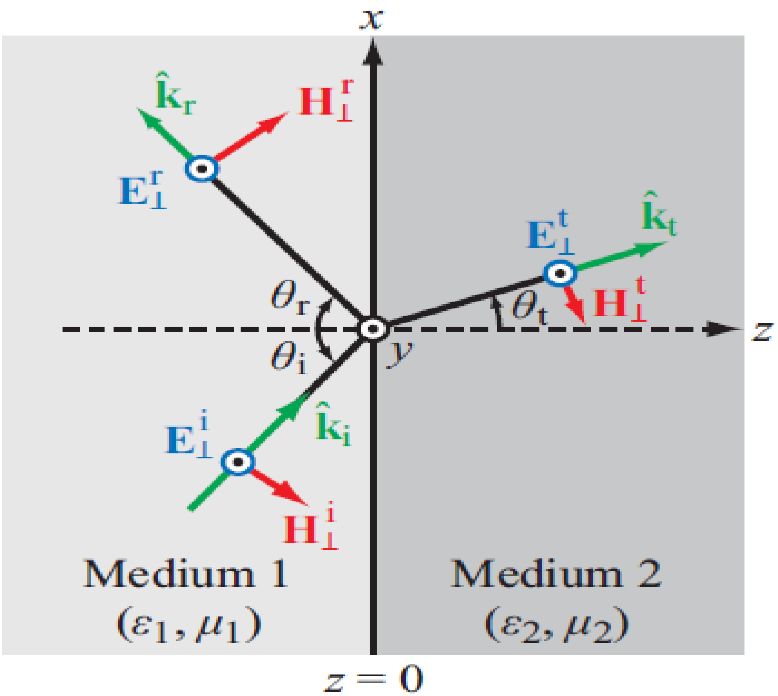
$$u_{1x} = \frac{\omega}{\beta_1 \sin\theta_i} = \frac{u_1}{\sin\theta_i} \qquad \lambda_{1x} = \frac{\lambda_1}{\sin\theta_i}$$

Oblique Incidence - Conducting Boundary

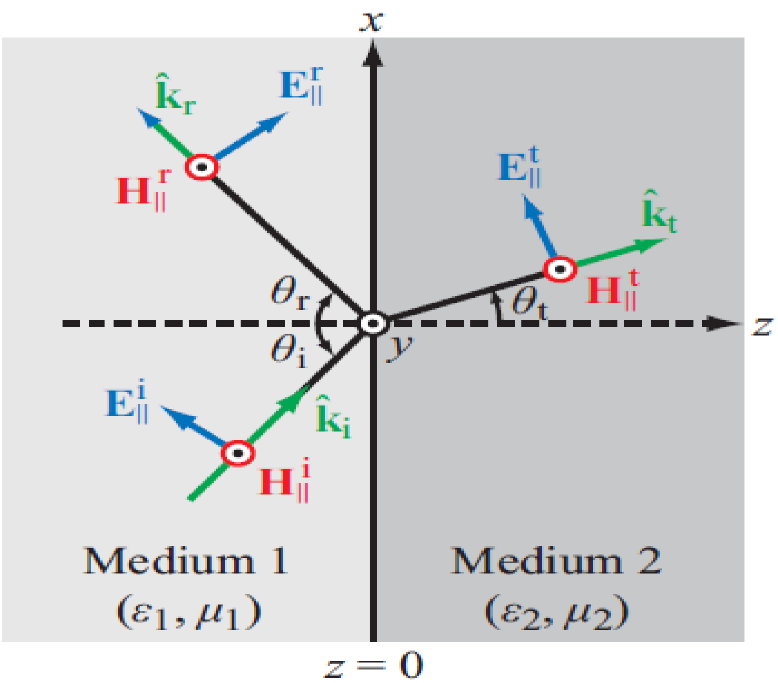
Oblique Incidence - Conducting Boundary

- When EM Wave incident obliquely on the interface between two media boundary, it can be decomposed into:
  - Horizontal Polarization
  - Vertical Polarization
- Horizontal Polarization, or **transverse electric (TE) polarization** → The **E** Field vector is parallel to the boundary surface or perpendicular to the plane of incidence
- Vertical Polarization, or **transverse magnetic (TM) polarization** → The **H** Field vector is parallel to the boundary surface and the electric vector is parallel to the plane of incidence

Oblique Incidence - Conducting Boundary



(a) Perpendicular polarization

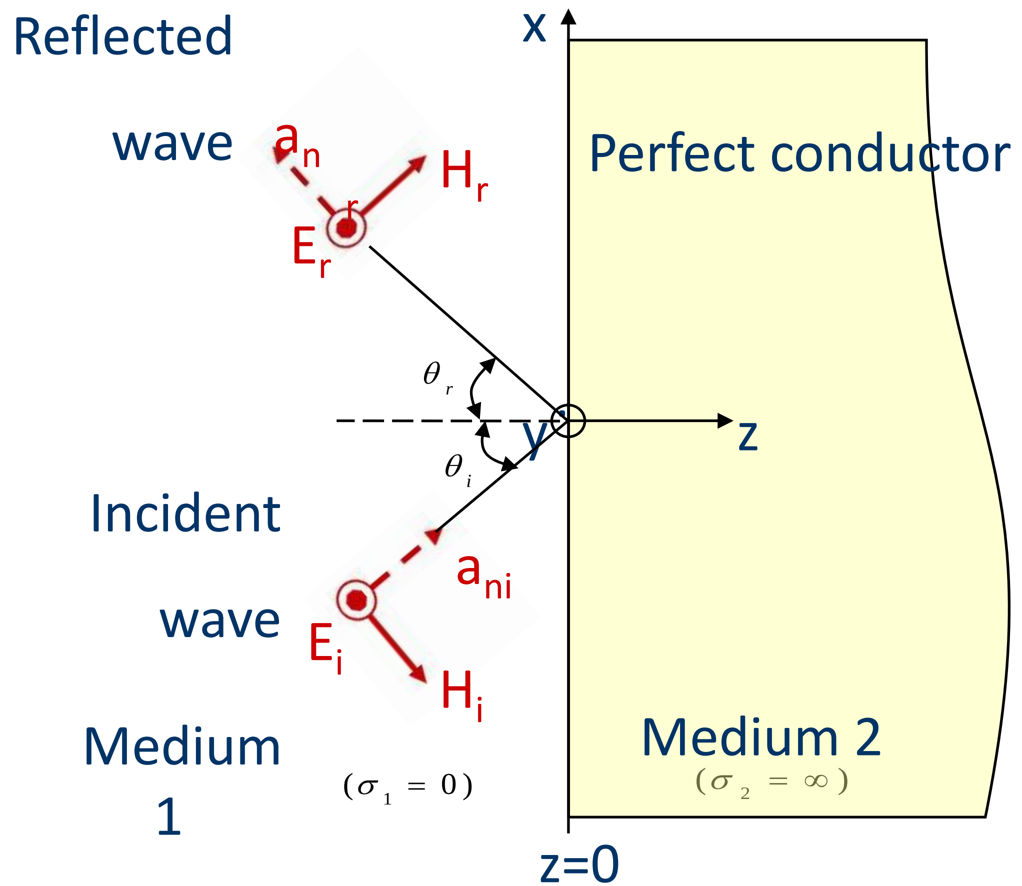


(b) Parallel polarization

- Let us consider the wave incident on a perfect conductor
  - The wave is totally reflected with the angle of incidence equal to the angle of reflection

Oblique Incidence - Conducting Boundary

- Perpendicular Polarization



Oblique Incidence - Conducting Boundary

Perpendicular Polarization

- Assume the wave is propagating obliquely along an arbitrary direction a_r and the electric field vector is along y-direction(normal to the plane of incidence x-z plane) and direction of propagation
- Using direction cosines the direction of propagation of incidence wave can be written as

$$\hat{a}_{ni} = \hat{a}_x \sin\theta_i + \hat{a}_z \cos\theta_i$$

- Similarly the direction of propagation of reflected wave (-z direction)can be written as

$$\hat{a}_{ni} = \hat{a}_x \sin\theta_i - \hat{a}_z \cos\theta_i$$

Oblique Incidence - Conducting Boundary

- Incident wave (inside medium 1)

$$\vec{E}_i(x,z) = \hat{a}_y E_{i0} e^{-j\beta_1 \hat{a}_{ni}\cdot\vec{R}} = \hat{a}_y E_{i0} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_i(x,z) = \frac{1}{\eta_1}\left[\hat{a}_{ni} \times \vec{E}_i(x,z)\right]$$

$$= \frac{E_{i0}}{\eta_1}(-\hat{a}_x \cos\theta_i + \hat{a}_z \sin\theta_i) e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

- Reflected wave (inside medium 1)

$$\vec{E}_r(x,z) = \hat{a}_y E_{r0} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

- Boundary condition, z = 0

$$\vec{E}_1(x,0) = \vec{E}_i(x,0) + \vec{E}_r(x,0)$$

Oblique Incidence - Conducting Boundary

$$= \hat{a}_y (E_{i0} e^{-j\beta_1 x \sin\theta_i} + E_{r0} e^{-j\beta_1 x \sin\theta_r}) = 0$$, for all x

$$\therefore \quad E_{r0} = -E_{i0} \quad \& \quad \theta_i = \theta_r$$

- Snell's law of reflection the angle of reflection equals the angle of incidence

$$\vec{E}_r(x,z) = \hat{a}_y E_{r0} e^{-j\beta_1 (x \sin\theta_r - z\cos\theta_r)}$$

$$= -\hat{a}_y E_{i0} e^{-j\beta_1 (x \sin\theta_i - z\cos\theta_i)}$$

$$(\because \text{ snell's law of reflection })$$

Oblique Incidence - Conducting Boundary

$$\vec{E}_r(x,z) = \hat{a}_y E_{r0} e^{-j\beta_1 (x\sin\theta_r - z\cos\theta_r)}$$
$$= -\hat{a}_y E_{i0} e^{-j\beta_1 (x\sin\theta_i - z\cos\theta_i)}$$

$$(\because \text{ snell's law of reflection })$$

- Similarly, Magnetic field of the reflected wave can be written as

$$\vec{H}_r(x,z) = \frac{1}{\eta_1}\left[\hat{a}_{nr} \times \vec{E}_r(x,z)\right]$$

$$= \frac{E_{i0}}{\eta_1}\left(-\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i\right) e^{-j\beta_1 (x\sin\theta_i - z\cos\theta_i)}$$

Oblique Incidence - Conducting Boundary

- Total field in medium 1 can be written as(standing wave in terms of electric and magnetic field)

$$\vec{E}_1(x,z) = \vec{E}_i(x,z) + \vec{E}_r(x,z)$$

$$= \hat{a}_y E_{i0}\left(e^{-j\beta_1 z\cos\theta_i} - e^{j\beta_1 z\cos\theta_i}\right)e^{-j\beta_1 x\sin\theta_i}$$

$$= -\hat{a}_y\, j2E_{i0}\sin(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}$$

$$\vec{H}_1(x,y) = -2\frac{E_{i0}}{\eta_1}\Big[\hat{a}_x\cos\theta_i\cos(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}$$

$$+\hat{a}_z\, j\sin\theta_i\sin(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}\Big]$$

Oblique Incidence - Conducting Boundary

- In z-direction (x=constant)

$$E_{1y} = jA\sin(\beta_1 \cos\theta_i z)e^{j\omega t}$$

$$H_{1x} = B\cos(\beta_1 \cos\theta_i z)e^{j\omega t}$$

$\Big\}$ 90° out of phase

no P_{av} is propagate.

- In x-direction (z=constant)

$$E_{1y} = Ce^{j(\omega t - \beta_1 x \sin\theta_i)}$$

$$H_{1z} = De^{j(\omega t - \beta_1 x \sin\theta_i)}$$

$\Big\}$ traveling wave

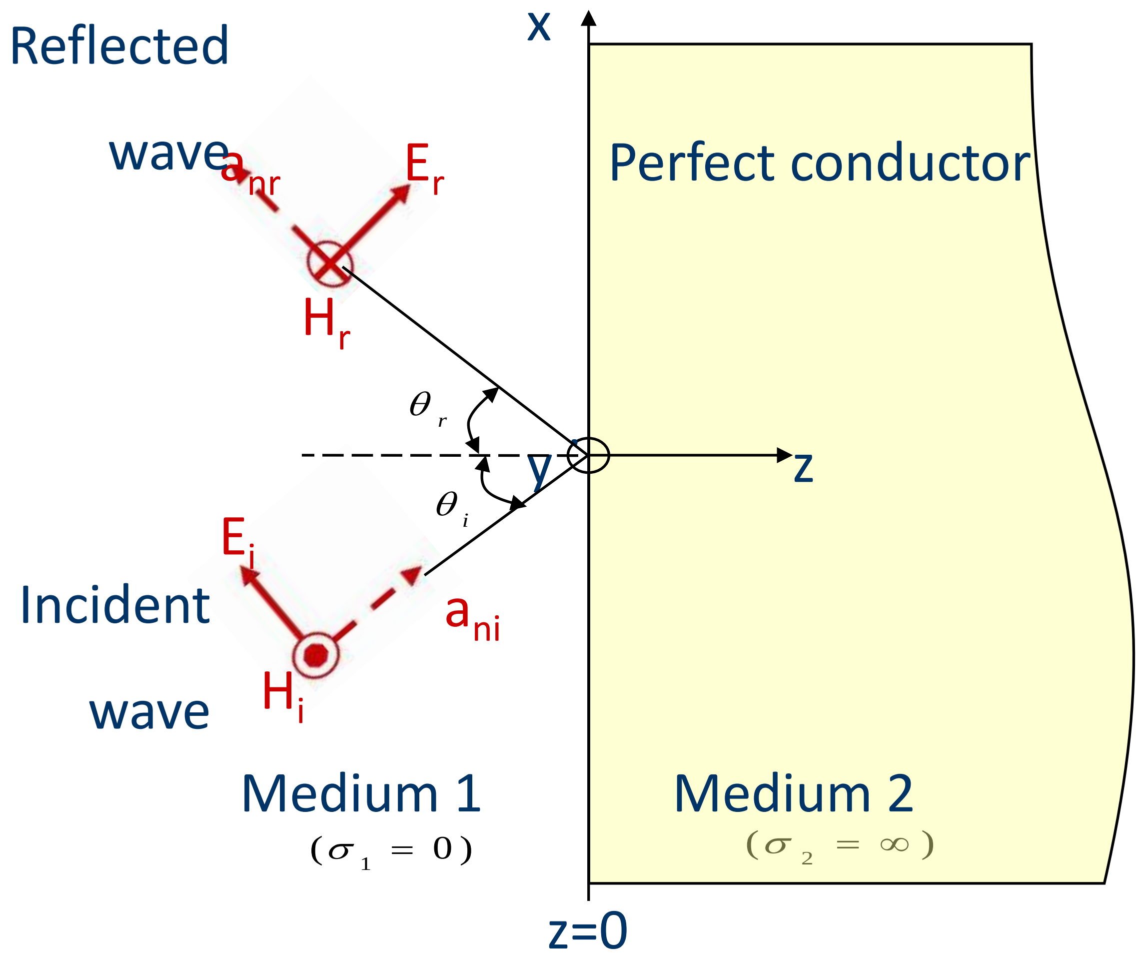
C = f(z) , *D = g(z)*

$$u_{1x} = \frac{\omega}{\beta_1 \sin\theta_i} = \frac{u_1}{\sin\theta_i} \qquad \lambda_{1x} = \frac{\lambda_1}{\sin\theta_i}$$

Oblique Incidence - Conducting Boundary

Oblique Incidence - Conducting Boundary

- Parallel Polarization



Oblique Incidence - Conducting Boundary

Parallel Polarization

- Assume the wave is propagating obliquely along an arbitrary direction a_r and the electric field vector is parallel to the plane of incidence (x-z plane) and the magnetic field vector is normal to the plane of incidence

- Using direction cosines the direction of propagation of incidence wave can be written as

$$\hat{a}_{ni} = \hat{a}_x \sin\theta_i + \hat{a}_z \cos\theta_i$$

- Similarly the direction of propagation of reflected wave (-z direction) can be written as

$$\hat{a}_{ni} = \hat{a}_x \sin\theta_i - \hat{a}_z \cos\theta_i$$

Oblique Incidence - Conducting Boundary

- Incident wave (inside medium 1)

$$\vec{E}_i(x,z) = E_{i0}(\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i)e^{-j\beta_1 \hat{a}_{ni}\cdot\vec{R}}$$

$$= E_{i0}(\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_i(x,z) = \hat{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

- Reflected wave (inside medium 1)

$$\vec{E}_r(x,z) = E_{r0}(\hat{a}_x \cos\theta_i + \hat{a}_z \sin\theta_r)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r(x,z) = -\hat{a}_y \frac{E_{r0}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

- Boundary condition, z = 0

Oblique Incidence - Conducting Boundary

$$\rightarrow \quad (E_{io}\cos\theta_i)e^{-j\beta_1 x\sin\theta_i} + (E_{ro}\cos\theta_r)e^{-j\beta_1 x\sin\theta_r} = 0$$

(for all x)

$$\therefore \quad E_{ro} = -E_{io} \qquad \theta_r = \theta_i$$

- Total field in medium 1 can be written as(standing wave in terms of electric and magnetic field)

$$\vec{E}_1(x,z) = \vec{E}_i(x,z) + \vec{E}_r(x,z)$$
$$= -2E_{i0}[\hat{a}_x\, j\cos\theta_i \sin(\beta_1 z\cos\theta_i)$$
$$+ \hat{a}_z \sin\theta_i \cos(\beta_1 z\cos\theta_i)]e^{-j\beta_1 x\sin\theta_i}$$

$$\vec{H}_1(x,z) = \vec{H}_i(x,z) + \vec{H}_r(x,z)$$
$$= \hat{a}_y\, 2\frac{E_{i0}}{\eta_1}\cos(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}$$

Oblique Incidence - Conducting Boundary

- In z-direction E_{1x} , H_{1y} : standing-wave

  E_{1z} , H_{1y} $u_{1x} = u_1 / \sin\theta_i$

- The wave is non uniform plane wave

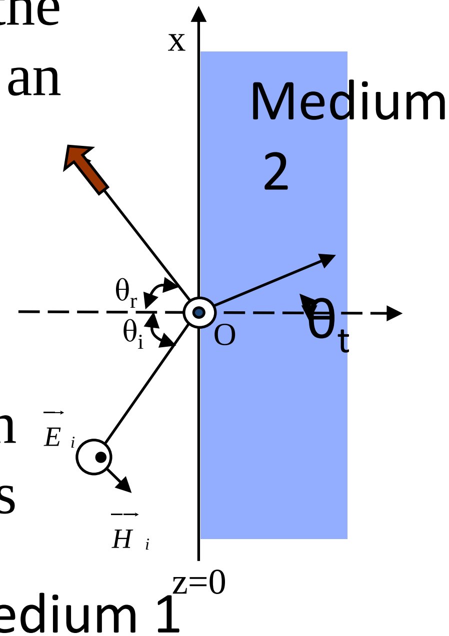
Reflection of a Plane Wave Oblique Incidence -Interface between dielectric media

➢ Perpendicular Polarization

- The electric field phasors for the perpendicular polarization, with reference to the system of coordinates in the figure
- Let us assume that the incident wave propagates in the first quadrant of x-z plane and propagation makes an angle θ_i with the normal

$$\vec{E}_i(x,z) = \bar{a}_y E_{io} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_i(x,z) = \frac{E_{io}}{\eta_1}(-\bar{a}_x \cos\theta_i + \bar{a}_z \sin\theta_i) e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$



- Let us assume that the reflected wave propagates in the second quadrant of xz plane and reflection makes an angle θ_r with the normal

$$\vec{E}_r(x,z) = \bar{a}_y E_{ro} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r(x,z) = \frac{E_{ro}}{\eta_1}(\bar{a}_x \cos\theta_r + \bar{a}_z \sin\theta_r) e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

Oblique Incidence -Interface between dielectric media

- Similarly the transmitted fields are

$$\vec{E}_t(x,z) = \bar{a}_y E_{to} e^{-j\beta_2 (x\sin\theta_t + z\cos\theta_t)}$$

$$\vec{H}_t(x,z) = \frac{E_{to}}{\eta_2}(-\bar{a}_x\cos\theta_i + \bar{a}_z\sin\theta_i)e^{-j\beta_2 (x\sin\theta_i + z\cos\theta_i)}$$

- Equating the tangential components of electric field
  - electric field has only E_y component and it is tangential at the interface z=0)

$$\vec{E}_{iy}(x,o) + \vec{E}_{ry}(x,o) = \vec{E}_{ty}(x,o)$$

- Similarly the magnetic field
  - magnetic field has two components: H_x and H_z , but only H_x is tangential at the interface z=0)

$$\vec{H}_{ix}(x,o) + \vec{H}_{rx}(x,o) = \vec{H}_{tx}(x,o)$$

Oblique Incidence -Interface between dielectric media

- Boundary at z=0 gives

$$e^{-j\beta_1 x \sin\theta_i} + \Gamma e^{-j\beta_1 x \sin\theta_r} = \tau e^{-j\beta_2 x \sin\theta_t}$$

$$-\frac{1}{\eta_1}\cos\theta_i e^{-j\beta_1 x \sin\theta_i} + \frac{\Gamma}{\eta_1}\cos\theta_r e^{-j\beta_1 x \sin\theta_r} = \frac{\tau}{\eta_2}\cos\theta_t e^{-j\beta_2 x \sin\theta_t}$$

- If E_x and H_y are to be continuous at the interface z = 0 for all x, then, this x variation must be the same on both sides of the equations (also known as phase matching condition)

$$\beta_1 \sin\theta_i = \beta_1 \sin\theta_r = \beta_2 \sin\theta_t$$

- Follows Snell's law $\theta_i = \theta_r$

Oblique Incidence -Interface between dielectric media

- Now we can simplify above two equations by applying Snell's

$$1 + \Gamma = \tau$$

$$-\frac{1}{\eta_1}\cos\theta_i + \frac{\Gamma}{\eta_1}\cos\theta_r = -\frac{\tau}{\eta_2}\cos\theta_t$$

- The above two equations has two unknowns τ and Γ and it can be solved easily as follows

$$\left(\frac{1}{\eta_1}\cos\theta_i - \frac{\Gamma}{\eta_1}\cos\theta_r\right)\frac{\eta_2}{\cos\theta_t} = \tau \qquad\qquad 1 + \Gamma = \tau$$

Oblique Incidence -Interface between dielectric media

- Solving the equations and rearranging, we get

$$\Gamma_{\perp} = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

$$\tau_{\perp} = \frac{E_{to}}{E_{io}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

$$1 + \Gamma_{\perp} = \tau_{\perp}$$

- For normal incidence, it is a particular case and put

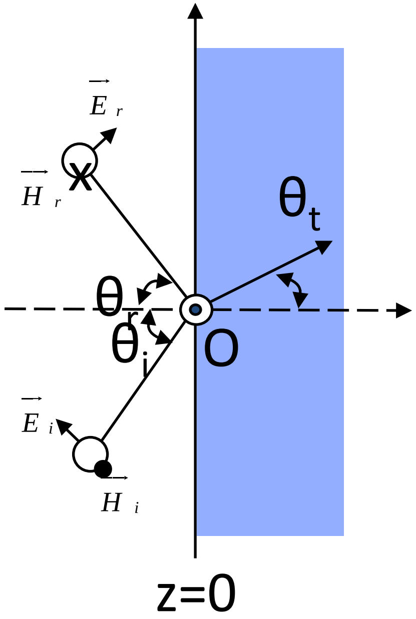
$$\theta_i = \theta_r = \theta_t = 0$$

➢ Parallel Polarization

- In this case, electric field vector lies in the x-z plane
- Since the magnetic field is transversal to the plane of incidence such waves are also called as transverse magnetic (TM) waves
- So let us start with H vector which will have only y component

$$\vec{H}_i(x,z) = \bar{a}_y \frac{E_{io}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{E}_i(x,z) = E_{io}(\bar{a}_x \cos\theta_i - \bar{a}_z \sin\theta_i)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$



- Similar to the previous case of perpendicular polarization, we can write the reflected magnetic field and electric field vectors as

$$\vec{H}_r(x,z) = -\bar{a}_y \frac{E_{ro}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{E}_r(x,z) = E_{ro}(\bar{a}_x \cos\theta_r + \bar{a}_z \sin\theta_i)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

Oblique Incidence -Interface between dielectric media

- Similarly the transmitted fields are

$$\overrightarrow{H}_t(x,z) = \overline{a}_y \frac{E_{to}}{\eta_2} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$\overrightarrow{E}_t(x,z) = E_{to}(\overline{a}_x cos\theta_t - \overline{a}_z sin\theta_t)e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

- Equating the tangential components of magnetic field
  - electric field has only H_y component and it is tangential at the interface z=0

$$\overrightarrow{H}_{ix}(x,o) + \overrightarrow{H}_{rx}(x,o) = \overrightarrow{H}_{tx}(x,o)$$

- Similarly the magnetic field
  - magnetic field has two components: E_x and E_z , but only E_x is tangential at the interface z=0)

$$\vec{E}_{iy}(x,o) + \vec{E}_{ry}(x,o) = \vec{E}_{ty}(x,o)$$

Oblique Incidence -Interface between dielectric media

- Boundary at z=0 gives

$$\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + \Gamma\cos\theta_r e^{-j\beta_1 x\sin\theta_r} = \tau\cos\theta_t e^{-j\beta_2 x\sin\theta_t}$$

$$\frac{1}{\eta_1} e^{-j\beta_1 x\sin\theta_i} - \frac{\Gamma}{\eta_1} e^{-j\beta_1 x\sin\theta_r} = \frac{\tau}{\eta_2} e^{-j\beta_2 x\sin\theta_t}$$

- If E_x and H_y are to be continuous at the interface z = 0 for all x, then, this x variation must be the same on both sides of the equations (also known as phase matching condition)

$$\beta_1\sin\theta_i = \beta_1\sin\theta_r = \beta_2\sin\theta_t$$

- Follows Snell's law $\theta_i = \theta_r$

Oblique Incidence -Interface between dielectric media

- Now we can simplify above two equations by applying Snell's

$$\cos\theta_i + \Gamma\cos\theta_r = \tau\cos\theta_t \qquad \frac{1}{\eta_1}(1-\Gamma) = \frac{\tau}{\eta_2}$$

- The above two equations has two unknowns τ and Γ and it can be solved easily as follows

$$\left(\frac{\cos\theta_i + \Gamma\cos\theta_r}{\cos\theta_t}\right) = \tau \qquad \frac{\eta_2}{\eta_1}(1-\Gamma) = \tau$$

- Therefore,

$$\Gamma_{\parallel} = \frac{E_{ro}}{E_{io}} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

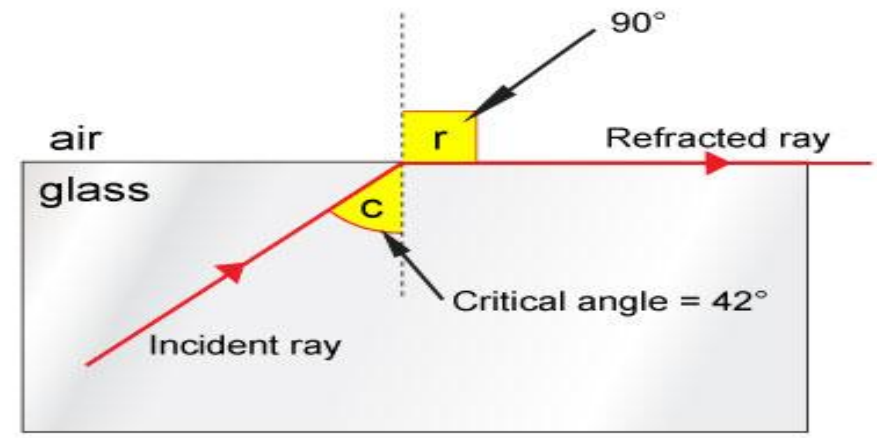
$$\tau_{\parallel} = \frac{E_{to}}{E_{io}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

- Note that cosine terms multiplication in the above equations is different from the previous expression for perpendicular polarization

Brewster Angle, Critical Angle

Critical Angle

When light travels from one medium to another it changes speed and is refracted.
As the angle of incidence increases so does the angle of refraction.
When the angle of incidence reaches a value known as the **critical angle** the refracted rays travel along the surface of the medium or in other words are refracted to an angle of 90°.
The critical angle for the angle of incidence in glass is 42°.



Surface Impedance

It is defined as the ratio of the tangential component of the electric field to the surface current density at the conductor surface.

It is given by

$$Z_s = \frac{E_{\tan}}{J_s} \, \Omega$$

where E_{tan} is the tangential component parallel to the surface of the conductor.

And J_s is the surface current density.

Critical Angle

> When the angle of incidence of the light ray reaches the critical angle (42^0) the angle of refraction is 90^0. The refracted ray travels along the surface of the denser medium in this case the glass.

> According to the law of refraction

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

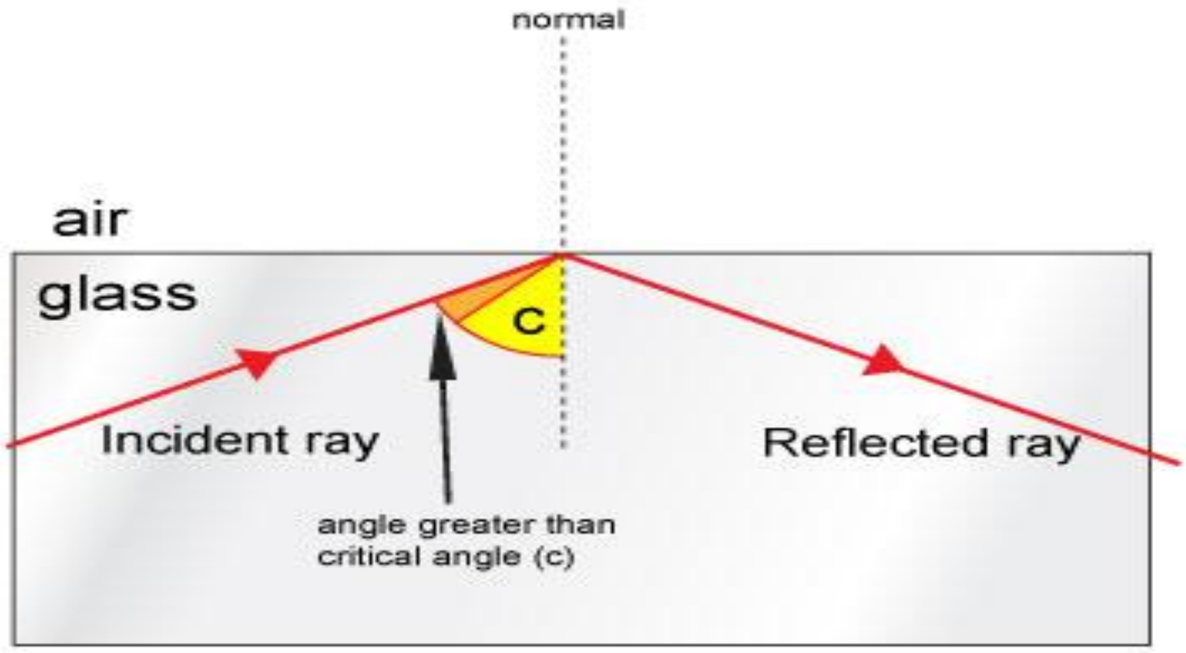
$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{n_1}{n_2} = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}$$

$$if\ \theta_i = \theta_c\ then\ \theta_t = \frac{\pi}{2}$$

$$\theta_c = \sin^{-1}\frac{n_2}{n_1}\ or\ \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

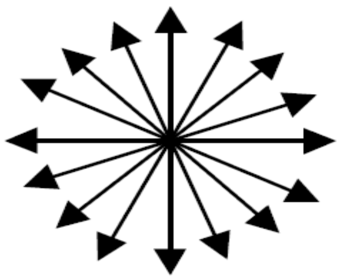
Total Internal Reflection

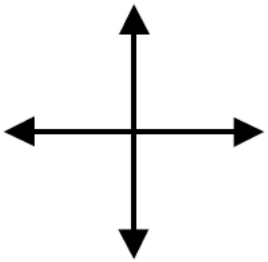
>When the angle of incidence of the light ray is greater than the critical angle then no refraction takes place. Instead, all the light is reflected back into the denser material in this case the glass. This is called **total internal reflection.**



Brewster Angle

For unpolarized waves the electric fields are in many directions as shown in figure, e.g. unpolarized light

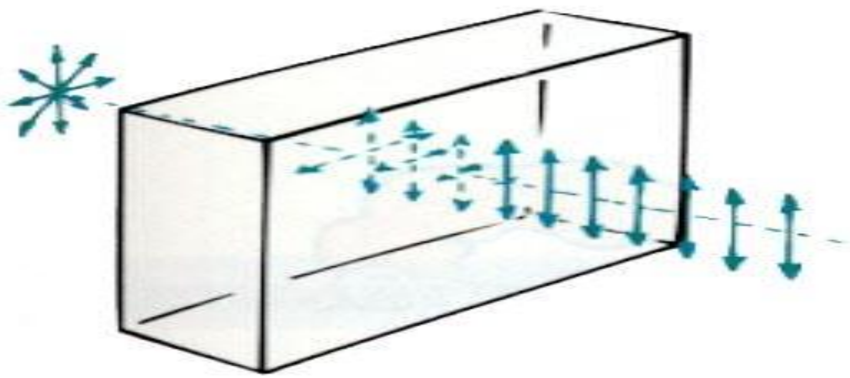




Where as in polarized waves the electric field vector is either vertical or horizontal as shown in figure below.

Brewster Angle

We can convert unpolarized light into polarized light by passing the light through a polarizing filter.

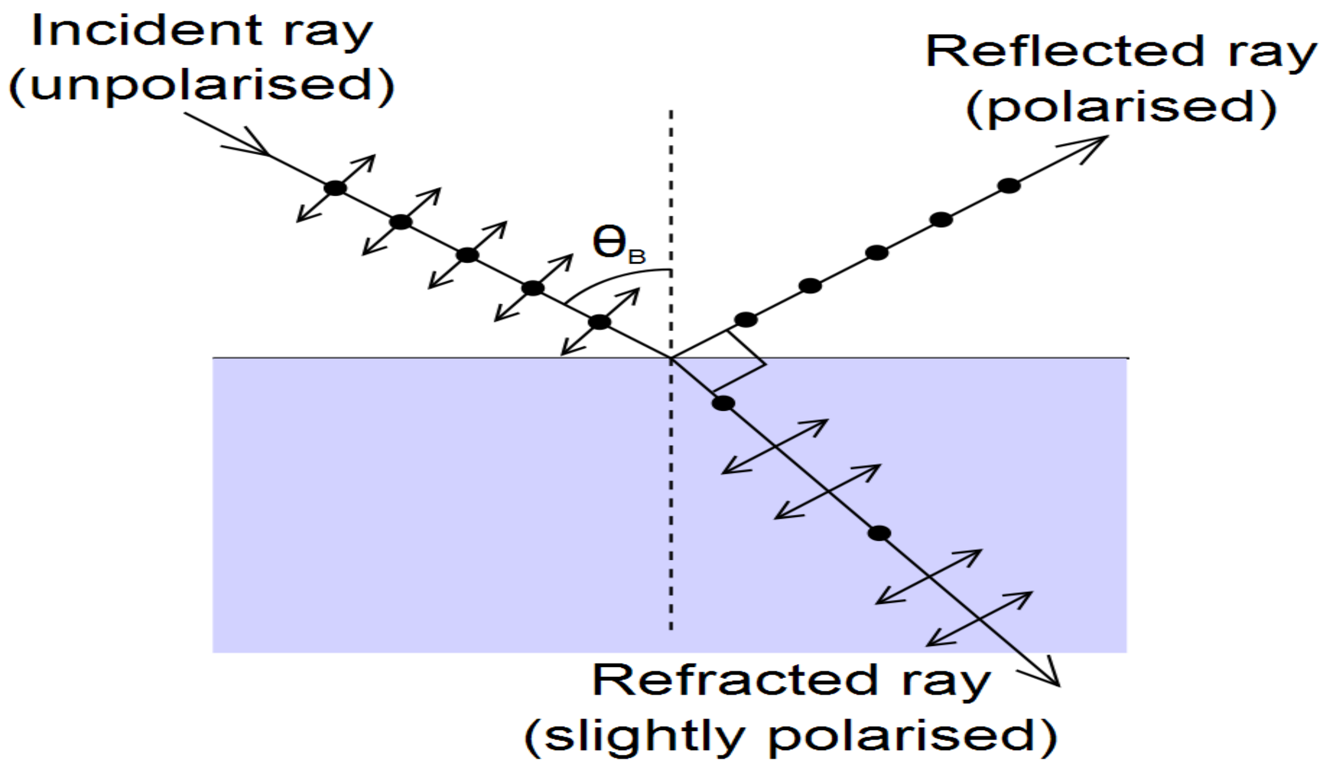


When unpolarized wave is incident obliquely at Brewster angle θ_B, only the component with perpendicular polarization (Horizontal polarization) will be reflected, while component with parallel polarization will not be reflected.

Brewster Angle

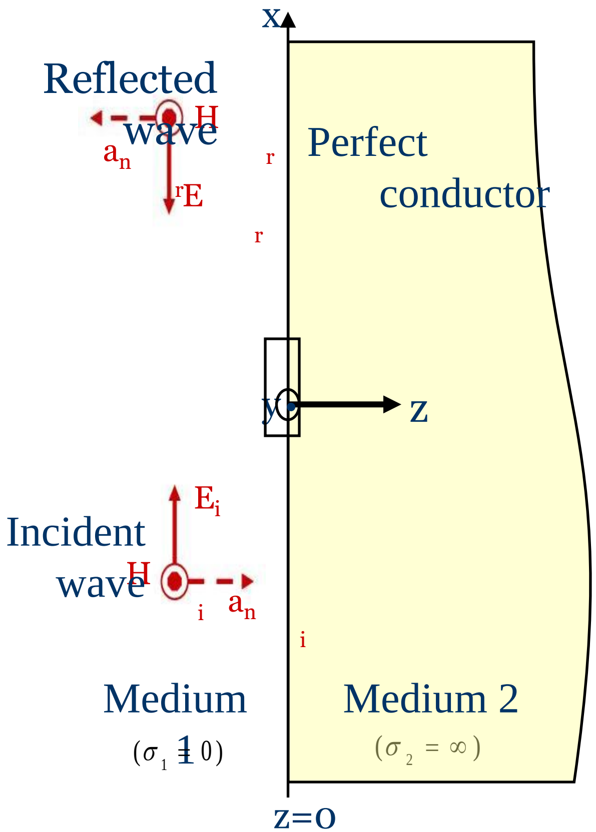
It is also called as Polarizing angle.

At Brewster angle, the angle between reflected ray and refracted ray is 90^0.



Normal Incidence - Conducting Boundary

- Consider the boundary is an interface with a perfect conductor $(\sigma_1 = \infty)$
- The incident wave travels in a lossless medium (Medium 1 $(\sigma_1 = 0)$)



- Incident wave (inside medium 1)

$$\vec{E}_i(z) = \hat{a}_x E_{i0} e^{-j\beta_1 z}$$

$$\vec{H}_i(z) = \hat{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z}$$

E_{i0}: the magnitude of $\vec{E}_i$ β_1 : phase constant

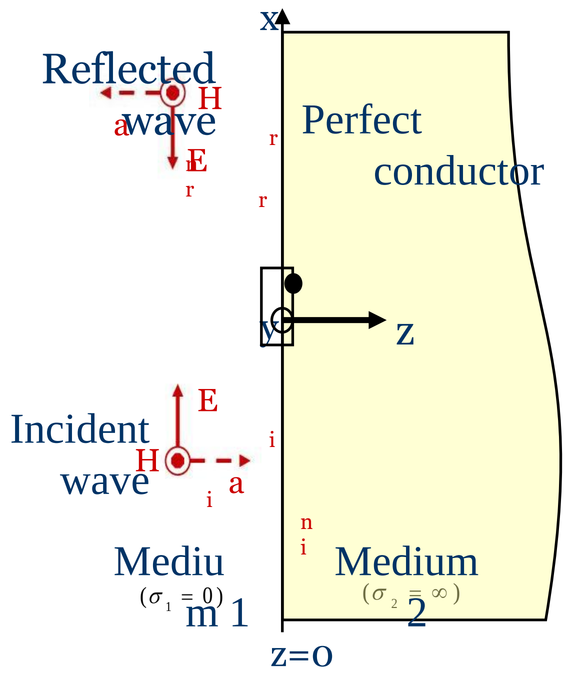
η_1 : intrinsic impedance of medium 1

Normal Incidence -Conducting Boundary contd..,

- The boundary is an interface with a perfect conductor(medium 2) both electric and magnetic fields vanish.

$$\vec{E}_2 = 0, \quad \vec{H}_2 = 0$$

- No wave is transmitted across the boundary into the z > 0 i.e.



Reflected wave (inside medium 1)

$$\vec{E}_r(z) = \hat{a}_x E_{r0} e^{+j\beta_1 z}$$

$$\vec{H}_r(z) = \frac{1}{\eta_1}\hat{a}_{nr} \times \vec{E}_r(z) = \frac{1}{\eta_1}(-\hat{a}_z) \times \vec{E}_r(z)$$

$$= -\hat{a}_y \frac{1}{\eta_1} E_{r0} e^{+j\beta_1 z}$$

Normal Incidence - Conducting Boundary contd..,

- Total wave in medium 1

$$\vec{E}_1(z) = \vec{E}_i(z) + \vec{E}_r(z) = \hat{a}_x(E_{i0}e^{-j\beta_1 z} + E_{r0}e^{+j\beta_1 z})$$

- Continuity of tangential component of the E-field at the boundary z = 0

$$\vec{E}_1(0) = \hat{a}_x(E_{i0} + E_{r0}) = E_2(0) = 0$$

$$\Rightarrow \quad E_{r0} = -E_{i0}$$

- Incident and reflected field are of equal amplitude, so all incident energy is reflected by a perfect conductor
- Negative sign indicates that the reflected field is shifted in phase by 180° relative to incident field at the boundary
- The magnetic field must be reflected without phase reversal, If both were reversed there would be no reversal of direction .

Normal Incidence - Conducting Boundary contd..,

- In medium 1, the total field can be written as (α=0)

$$\therefore\ \vec{E}_1(z) = \hat{a}_x E_{i0}(e^{-j\beta_1 z} - e^{+j\beta_1 z}) \quad = -\hat{a}_x j2E_{i0}\sin\beta_1 z$$

$$\therefore\ \vec{H}_1(z) = \vec{H}_i(z) + \vec{H}_r(z) = \hat{a}_y 2\frac{E_{i0}}{\eta_1}\cos\beta_1 z$$

- The space-time behavior of the total field in medium 1(multiply with $e^{j\omega t}$)

- After multiplying a time factor and take a real instantaneous part of the fields in medium 1 as a function of z and t can be written as

$$\vec{E}_1(z,t) = \mathrm{Re}[\vec{E}_1(z)e^{j\omega t}] = \hat{a}_x 2E_{i0}\sin\beta_1 z\sin\omega t$$

$$\vec{H}_1(z,t) = \mathrm{Re}[\vec{H}_1(z)e^{j\omega t}] = \hat{a}_y 2\frac{E_{i0}}{\eta_1}\cos\beta_1 z\cos\omega t$$

Normal Incidence - Conducting Boundary contd..,

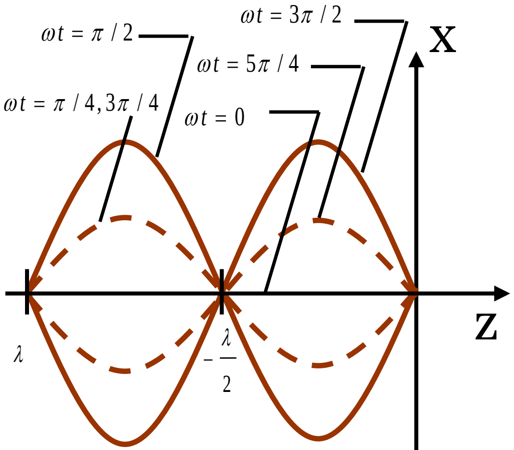
- From the total fields in medium 1, we conclude that there is a standing wave in medium 1, which does not progress
- The magnitude of field varies sinusoidal with distance from the reflecting plane
- Electric field has zero at the surface and multiple of half wave length from the surface and maxima for magnetic field

$$\left.\begin{array}{r}\text{Zeros of } \vec{E}_1(z,t) \\ \text{Maxima of } \vec{H}_1(z,t)\end{array}\right\} \text{occur at } \beta_1 z = -n\pi, \text{ or } z = -n\frac{\lambda}{2}, \quad n = 0,1,2,\ldots$$

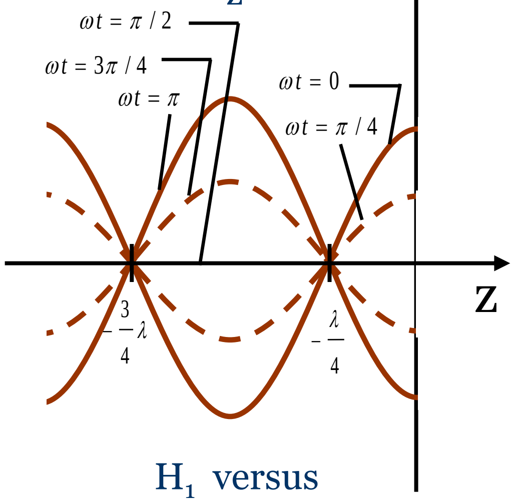
- It has a maximum value of twice the electric field strength of the incident wave at the distance from the odd multiple of the quarter wave length and zero for magnetic field

$$\left.\begin{array}{r}\text{Maxima of } \vec{E}_1(z,t) \\ \text{Zeros of } \vec{H}_1(z,t)\end{array}\right\} \text{occur at } \beta_1 z = -(2n+1)\frac{\pi}{2}, \text{ or } z = -(2n+1)\frac{\lambda}{4}, \quad n = 0,1,2,\ldots$$

Normal Incidence - Conducting Boundary contd..,



E$_1$ versus z



H$_1$ versus z

$$\vec{E}_1(z,t) = \mathrm{Re}[\vec{E}_1(z)e^{j\omega t}] = \hat{a}_x 2E_{i0} \sin\beta_1 z \sin\omega t$$

$$\vec{H}_1(z,t) = \mathrm{Re}[\vec{H}_1(z)e^{j\omega t}] = \hat{a}_y 2\frac{E_{i0}}{\eta_1} \cos\beta_1 z \cos\omega t$$

- The totally reflected wave combines with the incident wave to form a standing wave The total wave in medium 1 is not a traveling wave
- It stands and does not travel, it consists of two traveling waves $\mathbf{E_i}$ and $\mathbf{E_r}$ of equal amplitudes but in opposite directions.

Normal Incidence - Conducting Boundary contd..,

- The equations also shows that there is a 90° out of time phase between the electric and magnetic fields

$$\vec{E}_1(z,t) = \mathrm{Re}[\vec{E}_1(z)e^{j\omega t}] = \hat{a}_x 2E_{i0} \sin \beta_1 z \sin \omega t$$

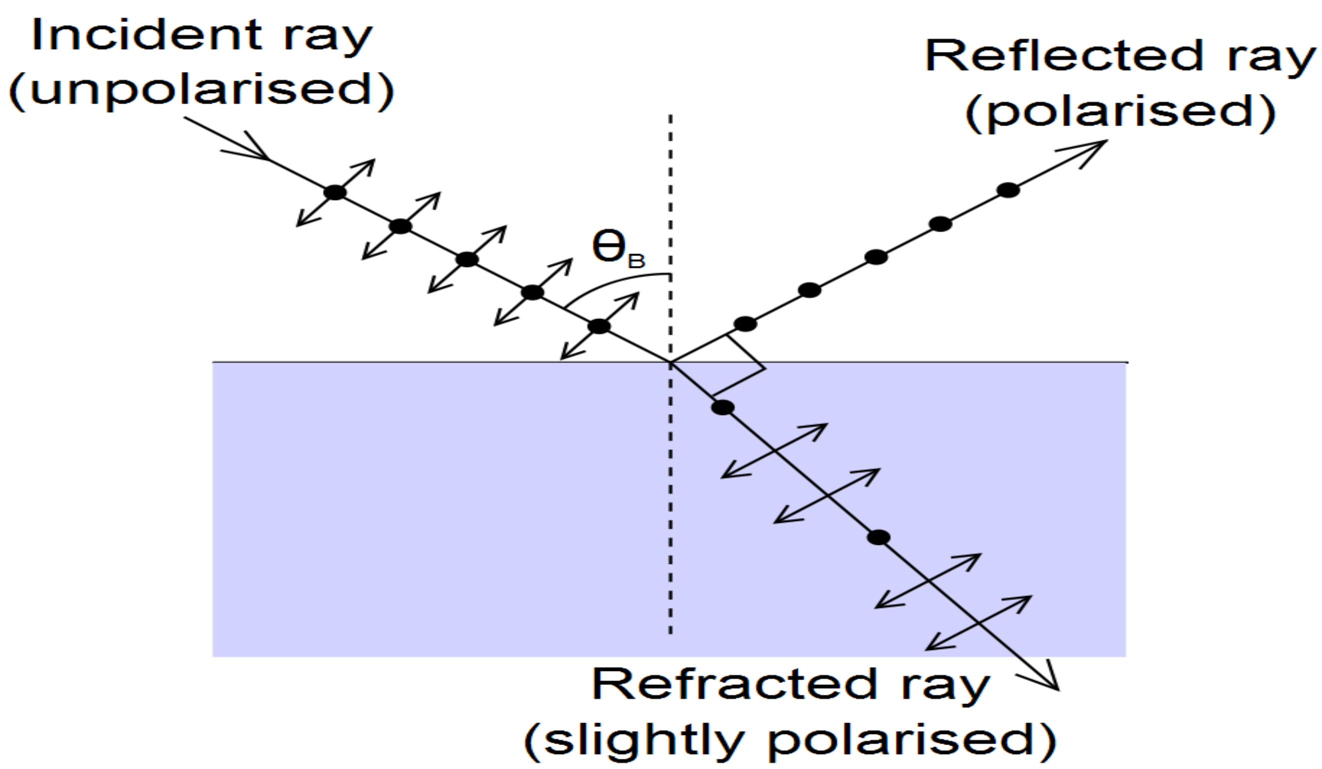
$$= \hat{a}_x 2E_{i0} \sin \beta_1 z \cos(\omega t - \frac{\pi}{2})$$

$$\vec{H}_1(z,t) = \mathrm{Re}[\vec{H}_1(z)e^{j\omega t}] = \hat{a}_y 2 \frac{E_{i0}}{\eta_1} \cos \beta_1 z \cos \omega t$$

Brewster Angle

It is also called as Polarizing angle.

At Brewster angle, the angle between reflected ray and refracted ray is 90^0.



<u>UNIT-IV</u>

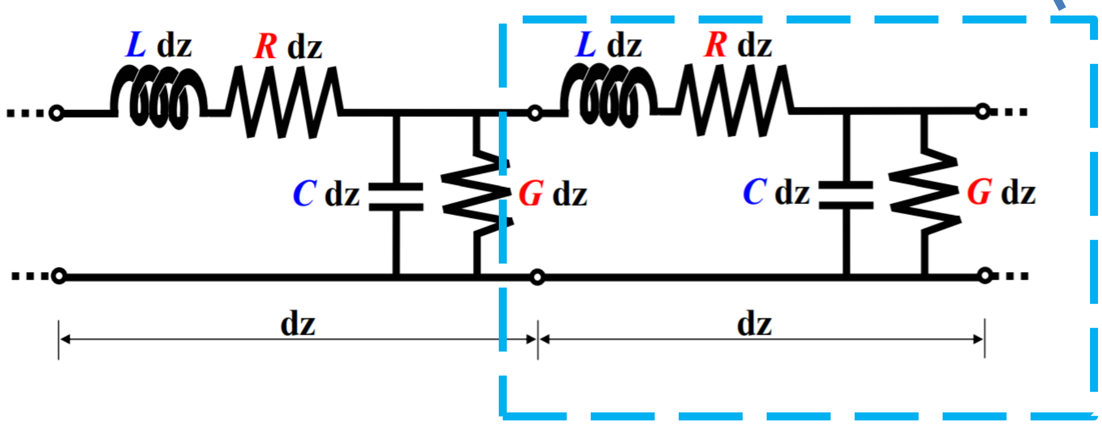
Transmission line Characteristics

Transmission line equivalent circuit

- If we are familiar with low frequency circuits and the circuit consists of lumped impedance elements (R,L,C), we treat all lines(wires) connecting the various circuit elements as perfect wires, with no voltage drop and no impedance associated to them.
- As long as the length of the wires is much smaller than the wavelength of the signal and at any given time, the measured voltage and current are the same for each location on the same wire.
- Let us try to explore what happens, when the signal propagates as a wave of voltage and current along the line at sufficiently high frequencies
- For sufficiently high frequencies, wavelength is comparable to the length of a conductor, so the positional dependence impedance properties (position dependent voltage and current) of wire can not be neglected, because it cannot change instantaneously at all locations.

Transmission line equivalent circuit

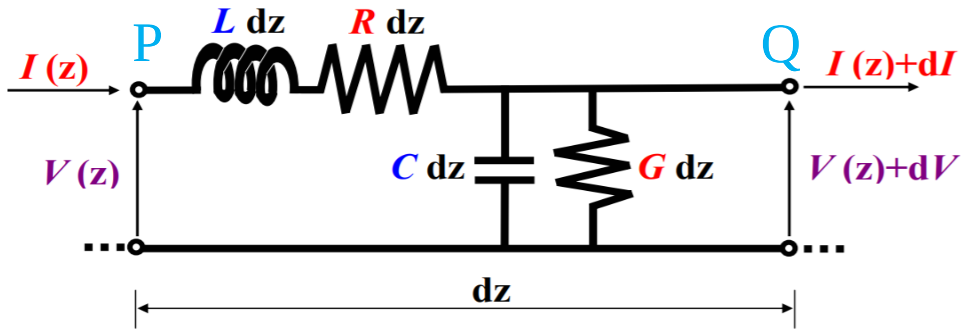
- So our first goal is to represent the uniform transmission line as a distributed circuit and determine the differential voltage and current behavior of an elementary cell of the distributed circuit.
- Once that is known, we can find a global differential equation that describes the entire transmission line by considering a cascaded network (subsections) of these equivalent models.
- So a uniform transmission line is represented as a distributed circuit shown below



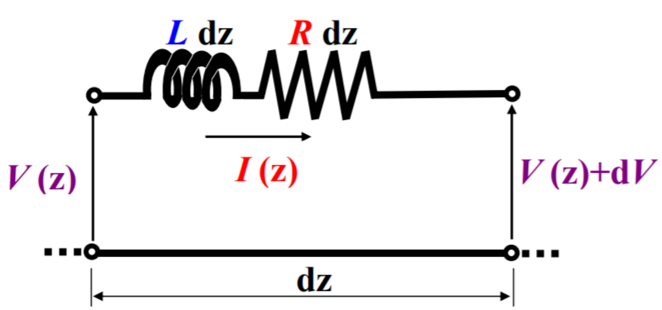
L→ series inductance
R→ series resistance
C→ shunt capacitance
G→ shunt conductance

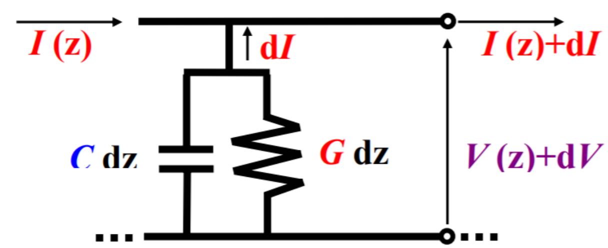
Transmission line equivalent circuit

- Let us assume a differential length dz of transmission line ,V(z),I(z) are voltage and current at point P and V(z)+dV, I(z)+dI are voltage and current at point Q



- The series impedance and shunt admittance determines the variation of the voltage and current from input to output of the cell shown below





Transmission line equations

- For a differential length dz of transmission line ,the series impedance and shunt admittance of the elemental length can be written as

$$(R + j\omega L)dz$$

$$(G + j\omega C)dz$$

- The corresponding circuit voltage and current equation is

$$V + dV - V = -I(R + j\omega L)dz$$

$$I + dI - I = -V(G + j\omega C)dz$$

- From which we obtain a first order differential equation

$$\frac{dV}{dz} = -I(R + j\omega L)$$

$$\frac{dI}{dz} = -V(G + j\omega C)$$

Transmission line equations

- We have a system of coupled first order differential equations that describe the behavior of voltage and current on the transmission line
- It can be easily obtain a set of uncoupled equations by differentiating with respect to the coordinate z

$$\frac{d^2V}{dz^2} = -\frac{dI}{dz}(R + j\omega L)$$

$$\frac{d^2I}{dz^2} = -\frac{dV}{dz}(G + j\omega C)$$

- Substitute $\frac{dV}{dz}, \frac{dI}{dz}$ in the above equations ,we obtain a second order differential equations

$$\frac{d^2V}{dz^2} = (G + j\omega C)(R + j\omega L)V$$

$$\frac{d^2I}{dz^2} = (G + j\omega C)(R + j\omega L)I$$

Transmission line equations

- Let the constant term can be represented as propagation constant, which is written as

$$\gamma = \alpha + j\beta = \sqrt{(G + j\omega C)(R + j\omega L)}$$

- The transmission line equations can be written as

$$\frac{d^2V}{dz^2} = \gamma^2 V$$

$$\frac{d^2I}{dz^2} = \gamma^2 I$$

- γ is the complex propagation constant, which is function of frequency

- α is the attenuation constant in nepers per unit length, β is the phase constant in radians per unit length

Transmission line equations

- The solution of the second order transmission line equation is

$$V = ae^{+\gamma z} + be^{-\gamma z}$$

$$I = ce^{+\gamma z} + de^{-\gamma z}$$

- Where, a,b,c, and d are the constants
- Above equations represent the standard solutions of the wave equations, which are similar to the solution of uniform plane wave equations.
- The terms $e^{+\gamma z}$ and $e^{-\gamma z}$ can be represented as backward and forward wave along z-direction.

Determination of constant terms A and B

Determination of the constant terms A and B

- Let the solutions of the transmission line wave equations can written as

$$V = ae^{+\gamma z} + be^{-\gamma z}$$

$$I = ce^{+\gamma z} + de^{-\gamma z}$$

- To determine the constants a,b,c, and d , the above equations can be written in terms of hyperbolic functions, where substitute

$$e^{\gamma z} = \cosh \gamma z + \cosh \gamma z$$

$$e^{-\gamma z} = \cosh \gamma z - \cosh \gamma z$$

- Substitute above equations in the solutions of transmission line equations V and I

$$V = a(\cosh \gamma z + \cosh \gamma z) + b(\cosh \gamma z - \cosh \gamma z)$$

$$I = c(\cosh \gamma z + \cosh \gamma z) + d(\cosh \gamma z - \cosh \gamma z)$$

Determination of the constant terms A and B

- The constants a+b, a-b, c+d, and c-d can be replaced by another constant terms A,B,C, and D respectively

$$V = (a+b)\cosh \gamma z + (a-b)\sinh \gamma z$$

$$I = (c+d)\cosh \gamma z + (c-d)\sinh \gamma z$$

- So, the above equations can be written as

$$V = A\cosh \gamma z + B\sinh \gamma z$$

$$I = C\cosh \gamma z + D\sinh \gamma z$$

- In order to reduce the four constant terms to two constant terms, we write the relation between V and I by considering the following basic differential equations

$$-\frac{dV}{dz} = I(R + j\omega L)$$

$$-\frac{dI}{dz} = V(G + j\omega C)$$

Determination of the constant terms A and B

- Substitute V in $\frac{dV}{dz}$

$$-\frac{d(A\cosh\gamma z + B\sinh\gamma z)}{dz} = I(R + j\omega L)$$

- Differentiating in terms of z, we obtain

$$-\gamma(A\sinh\gamma z + B\cosh\gamma z = I(R + j\omega L)$$

- So, the current I can be written in terms of constants A and B as

$$-\frac{\gamma}{(R + j\omega L)}(A\sinh\gamma z + B\cosh\gamma z = I$$

- Where γ is propagation constant, which is

$$\gamma = \sqrt{(G + j\omega C)(R + j\omega L)}$$

Determination of the constant terms A and B

- Substitute γ in current equation, we obtain

$$-\frac{\sqrt{(G+j\omega C)(R+j\omega L)}}{(R+j\omega L)}(A\sinh\gamma z + B\cosh\gamma z = I$$

- The above equation can be simplified and written as

$$V_s = A$$

$$-I_s Z_0 = B$$

$$-\frac{1}{Z_0}(A\sinh\gamma z + B\cosh\gamma z = I$$

- Where Z_0 is another constant and can be called as characteristic impedance along the line

$$Z_0 = \sqrt{\frac{(R+j\omega L)}{(G+j\omega C)}}$$

Determination of the constant terms A and B

- So, the voltage and current wave equation in terms of constants A and B can be re written as

$$V = A\cosh\gamma z + B\sinh\gamma z$$

$$I = -\frac{1}{Z_0}(A\sinh\gamma z + B\cosh\gamma z)$$

- Now the constants A and B can be obtained by applying initial conditions of the transmission line at Z=0
- Let Vs and Is be the source voltage and current respectively. At the source end, Z=0 the voltage V= Vs, current I= Is, then the above equations can be simplified as

$$V_s = A\cosh\gamma(0) + B\sinh\gamma(0) \qquad V_s = A$$

$$I_s = -\frac{1}{Z_0}(A\sinh\gamma(0) + B\cosh\gamma(0)) \qquad -I_s Z_0 = B$$

Determination of the constant terms A and B

- The constants A and B can be simplified as

$$V_s = A$$

$$-I_s Z_0 = B$$

- Substitute A and B values in basic voltage and current equations we obtain

$$V = V_s \cosh \gamma z - I_s Z_0 \sinh \gamma z$$

$$I = -\frac{V_s}{Z_0} \sinh \gamma z + I_s \cosh \gamma z$$

- These equations are called transmission line equations. The can represent voltage and current at any point z from the source voltage and current.

Determination of the constant terms A and B

- The constants A and B can be simplified as

$$V_s = A$$

$$-I_s Z_0 = B$$

- Substitute A and B values in basic voltage and current equations we obtain

$$V = V_s \cosh \gamma z - I_s Z_0 \sinh \gamma z$$

$$I = -\frac{V_s}{Z_0} \sinh \gamma z + I_s \cosh \gamma z$$

- These equations are called transmission line equations. The can represent voltage and current at any point z from the source voltage and current.

Input Impedance of Transmission Line from Load Impedance

Transmission Line with Load Impedance

- If the source voltage and source current is known to us, input impedance of a transmission line is derived from source end as

$$Z_{in} = Z_0 \frac{(Z_0 \tanh \gamma l + Z_R)}{(Z_0 + Z_R \tanh \gamma l)}$$

- Similar expression can be derived from terminating voltage or current is known
- This can be derived from a transmission line length *l* with known terminating voltage and current
- This can be start from basic transmission line equations with two unknown constants A and B

$$V = A \cosh \gamma z + B \sinh \gamma z$$

$$I = -\frac{1}{Z_0}(A \sinh \gamma z + B \cosh \gamma z)$$

Transmission Line with Load Impedance

- Now the terminating voltage and current at a distance z=l can be written as

$$V_R = A\cosh\gamma l + B\sinh\gamma l$$

$$I_R = -\frac{1}{Z_0}(A\sinh\gamma l + B\cosh\gamma l)$$

- To derive the constants A the above voltage and current equations can be multiplied with coshγl and sinhγl

$$V_R\cosh\gamma l = A\cosh^2\gamma l + B\sinh\gamma l\cosh\gamma l$$

$$I_R\sinh\gamma l = -\frac{1}{Z_0}(A\sinh^2\gamma l + B\sinh\gamma l\cosh\gamma l)$$

- Then we get the constants A as

$$V_R\cosh\gamma l + Z_0 I_R\sinh\gamma l = A$$

Transmission Line with Load Impedance

- Similarly, the constant B can be obtained by multiplying voltage equation with sinhγl and current equation with coshγl , and adding both the equations, we get B as

$$V_R \cosh \gamma l - Z_0 I_R \sinh \gamma l = B$$

- Then substitute A and B constants in basic equation. we obtain voltage and current at any point from the terminating point as

$$V = V_R \cosh \gamma(l - z) + \frac{V_R}{Z_0} \sinh \gamma(l - z)$$

$$I = \frac{V_R}{Z_0} \sinh \gamma(l - z) + I_R \cosh \gamma(l - z)$$

- Where, l-z is the distance from the terminating end

Transmission Line with Load Impedance

- Now from the above expressions, the input impedance z=0 can be written as

$$V_S = V_R \cosh \gamma(l-0) + I_R Z_0 \sinh \gamma(l-0)$$

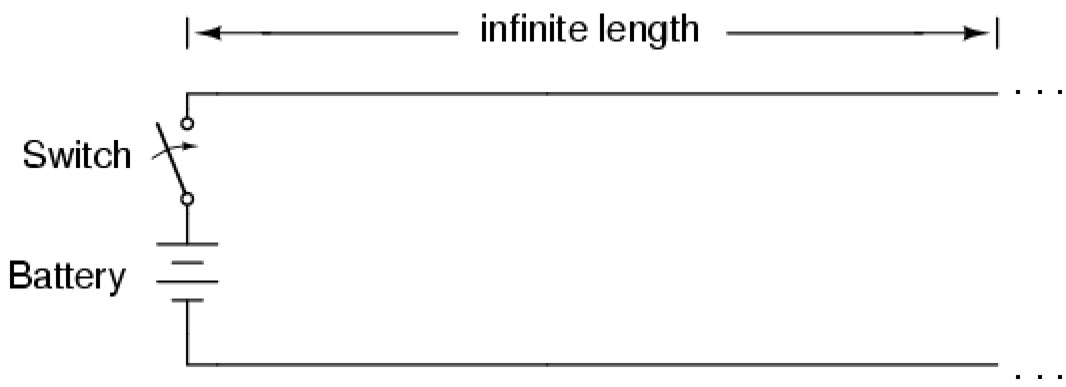
$$I_S = \frac{V_R}{Z_0} \sinh \gamma(l-0) + I_R \cosh \gamma(l-0)$$

$$\frac{V_S}{I_S} = \frac{V_R \cosh \gamma l + I_R Z_0 \sinh \gamma l}{\frac{V_R}{Z_0} \sinh \gamma l + I_R \cosh \gamma l}$$

$$Z_{in} = Z_0 \frac{(Z_0 \tanh \gamma l + Z_R)}{(Z_0 + Z_R \tanh \gamma l)}$$

Infinite Transmission Line

Infinite Transmission Line



>If a line of infinite length is considered then all the power fed into it will be absorbed. The reason is as we move away from the input terminals towards the load, the current and voltage will decrease along the line and become zero at an infinite distance, because the voltage drops across the inductor and current leaks through the capacitor.

>By considering this hypothetical line of infinite line an important terminal condition is formed.

Infinite Transmission Line

Let V_s be the sending end voltage and I_s be the sending end current and Z_s be the input impedance which is given by

$$Zs \ = Vs/Is$$

Current at any point distance x from sending end is given by

$$I = ce^{Px} + de^{-Px}$$

The value of c & d can now determined by considering an infinite line.

At the sending end x=0 and $I = I_s$.

$$I_s = c + d$$

At the receiving end, I=0 and x=∞.

$$0 = c * \infty \qquad \infty \neq 0$$

Infinite Transmission Line

Therefore $c = 0$

If $c = 0$ then $I_s = d$

Therefore

$$I = I_s e^{-Px} \qquad V = V_s e^{-Px}$$

The above equation gives current at any point of an infinite line

And

$$V = V_s e^{-Px}$$

Similarly the above equation gives voltage at any point of an infinite line

Infinite line terminated in its Z_0

When a finite length of line is joined with a similar kind of infinite line, their total input impedance is same as that of infinite line itself, because they together make one infinite line however the infinite line alone presents an impedance Z_o at its input PQ because the input impedance of an infinite line is Z_o.

It is therefore concluded that a finite line has an impedance Z_o when it is terminated in $Z_{o.}$

Or A finite line terminated by its Z_o behaves as an infinite line.

Let a finite length of ‘l’ is terminated by its characteristic impedance Z_o and is having voltage and current V_R and I_R at terminating end.

Infinite line terminated in its Z_0

Therefore

$$Zo = \frac{V_R}{I_R}$$

Putting $x = l$, $V = V_R$, $I = I_R$ in general equations, we have

$$I_R = I_S \cosh pl - \frac{V_S}{Z_O} \sinh pl$$

Dividing V_R by I_R we get Z_o

$$Z_o = \frac{V_R}{I_R} = \frac{V_S \cosh pl - I_S Z_O \sinh pl}{I_S \cosh pl - \frac{V_S}{Z_O} \sinh pl}$$

Infinite line terminated in its Z_0

Multiplying right hand side numerator and denominator by Z0, we get

$$Z_o = \frac{V_S coshpl - I_S Z_O sinhpl}{I_S coshpl - \frac{V_S}{Z_O} sinhpl} \frac{Z_o}{Z_o}$$

$$Z_o I_S coshpl - V_S sinhpl = V_S coshpl - I_S Z_O sinhpl$$

$$Z_O I_S (coshpl + sinhpl) = Z_o (coshpl + sinhpl)$$

Therefore $V_S = Z_O I_S$ and $Z_O = \frac{V_S}{I_S}$

But Vs/Is is equal to the input impedance.

Thus the input impedance of a finite line terminated in its characteristic impedance is the characteristic impedance of the line. Since the input impedance of an infinite line is the characteristic impedance of the line.

Characteristic Impedance, Phase Velocity

Characteristic impedance, Z_0 :

- The characteristic impedance, Z_0 can be defined as:

$$Z_0 = \frac{R + j\omega L}{\gamma} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

- characteristic impedance (Z_o) is the ratio of voltage to current in a forward travelling wave, assuming there is no backward wave
  - Z_o determines relationship between voltage and current waves
  - Z_o is a function of physical dimensions and εr
  - Z_o is usually a real impedance (e.g. 50 or 75 ohms)

Characteristic impedance, Z_0 :

- Voltage waveform can be expressed in time domain as:

$$v(z,t) = \left|V_0^+\right| \cos\left(\omega t - \beta z + \varphi^+\right) e^{-\alpha z} + \left|V_0^-\right| \cos\left(\omega t + \beta z + \varphi^-\right) e^{\alpha z}$$

- The factors V_0^+ and V_0^- represent the complex quantities. The $\Phi^{\pm}$ is the phase angle of $V_0^{\pm}$. The quantity βz is called the ***electrical length of line*** and is measured in radians.

Phase Velocity

- Phase Velocity For a sinusoidal wave, or a waveform comprised of many sinusoidal components that all propagate at the same velocity, the waveform will move at the phase velocity of the sinusoidal components We've seen already that the phase velocity is
- $v_p = \omega / k$

Group Velocity

- When the various frequency components of a waveform have different phase velocities

- The phase velocity of the waveform is an average of these velocities (the phase velocity of the carrier wave), but the waveform itself moves at a different speed than the underlying carrier wave called the group velocity

Group vs Phase velocity

- An analogy that may be useful for understanding the difference comes from velodrome cycling
- Riders race as a team and take turns as leader with the old leader peeling away and going to the back of the pack

- As riders make their way from the rear of the pack to the front they are moving faster than the group that they are in.

Group vs Phase velocity

- The phase velocity of a wave is

$V = v_p = \lambda T = f\lambda = \omega/k$

and comes from the change in the position of the wavefronts as a function of time

- The waveform moves at a rate that depends on the relative position of the component wavefronts as a function of time. This is the group velocity and is
- $v_g = d\omega /dk$

Condition for distortion less and minimum attenuation

Condition for Distortion Less Line

- A transmission line is said to be distortion less if the attenuation constant is frequency independent.

- The phase constant is linearly dependent of frequency.

- From the general equations of α&β, The distortion less line results, if the line parameters are such that,

$R/L=G/C$

- The attenuation constant and phase constant are

$\alpha= \sqrt{RG}$ and $\beta= \omega \sqrt{LC}$

Condition for Minimum Attenuation

- From the equation the attenuation of a line is expressed by attenuation constant DC as

$$\alpha = ½\sqrt{(RG + \omega^2 LC) + \sqrt{R^2 + \omega^2 L^2 (G^2 + \omega^2 C^2)}}$$

- It is observed that depends on the four primary constants in addition to the frequency. The Value of L for minimum attenuation.
- Let us assume the three line constants C,G and R including are constant and only L may be varied .
- Therefore, differentiating above value of , W.R.T L and equating it to zero, we get

Condition for Minimum Attenuation:

$$\frac{dx}{dL}\left[\frac{1}{2}\left\{\frac{2\omega^2 L(G^2+\omega^2 C^2)}{\sqrt{R^2+\omega^2 L^2(G^2+\omega^2 C^2)}}-\omega^2 C\right\}\right] = 0$$

$$\frac{\omega^2 L(G^2+\omega^2 C^2)}{\sqrt{R^2+\omega^2 L^2(G^2+\omega^2 C^2)}}-\omega^2 C = 0$$

then by solving above equation we get

R/L=G/C

This is the condition for minimum attenuation and distortion less line.

Condition for Distortion Less Line

CONCLUSIONS:

- The phase velocity is independent of frequency because, the phase constant of linearly depends upon frequency
- Both V_P and Z_0 remains the same for loss less line.
- A loss less line is also a distortion less line, but a distortion less line is not necessarily loss less.
- Loss less lines are desirable in power transmission telephone lines are to b distortion less.

UNIT-V

UHF Transmission Lines and Applications

UHF Transmission Lines

- UHF spectrum is used worldwide for land mobile radio systems for commercial, industrial, public safety, and military purposes.

- Many personal radio services use frequencies allocated in the UHF band, although exact frequencies in use differ significantly between countries.
- Major telecommunications providers have deployed voice and data cellular networks in UHF/VHF range. This allows mobile phones and mobile computing devices to be connected to the public switched telephone network and public internet.
- UHF radars are said to be effective at tracking stealth fighters, if not stealth bombers

UHF Transmission Lines

- Ultra High Frequency lines commonly abbreviated as U.H.F lines are one of the types of the transmission lines.

- Ultra high frequency lines have operational frequency range from 300 to 3000 MHz or wavelength from 100 cm to 10 cm.

- Under normal frequencies the transmission lines are used as wave guides for transferring power and information from one point to another.

UHF Transmission Lines

- At Ultra High Frequencies, the transmission lines can be used as circuit elements like capacitor or an inductor .

- It means they can be used in circuits like a capacitor or an inductor.

Applications:

- UHF Television Broad casting fulfilled the demand for additional over-the-air television channels in urban areas. Today, much of the bandwidth has been reallocated to land mobile, trunked radioand mobile telephone use. UHF channels are still used for digital television

UHF Transmission Lines

- UHF spectrum is used worldwide for land mobile radio systems for commercial, industrial, public safety, and military purposes.

- Many personal radio services use frequencies allocated in the UHF band, although exact frequencies in use differ significantly between countries.
- Major telecommunications providers have deployed voice and data cellular networks in UHF/VHF range. This allows mobile phones and mobile computing devices to be connected to the public switched telephone network and public internet.
- UHF radars are said to be effective at tracking stealth fighters, if not stealth bombers

Input impedance Relations; SC and OC lines

Input Impedance Relations

- Let the voltage and current transmission line equations at any point on the line from the source end can be written as

$$V = V_s \cosh \gamma z - I_s Z_0 \sinh \gamma z$$

$$I = -\frac{V_s}{Z_0} \sinh \gamma z + I_s \cosh \gamma z$$

- A transmission line, which is terminated with some load impedance Z_R at a distance '*l* 'from the load
- The voltage and current at the terminating end is V_R and I_R
- At z= *l*, the voltage and current can be written as

$$V_R = V_s \cosh \gamma l - I_s Z_0 \sinh \gamma l$$

$$I_R = -\frac{V_s}{Z_0} \sinh \gamma l + I_s \cosh \gamma l$$

Input Impedance Relations

- Now the load impedance from the terminating point can be written as

$$Z_R = \frac{V_R}{I_R} = \frac{V_s \cosh \gamma l - I_s Z_0 \sinh \gamma l}{-\frac{V_s}{Z_0} \sinh \gamma l + I_s \cosh \gamma l}$$

- By solving the above equation, we obtain

$$Z_s = Z_0 \frac{(Z_0 \sinh \gamma l + Z_R \cosh \gamma l)}{(Z_0 \cosh \gamma l + Z_R \sinh \gamma l)}$$

- Where Z_S is the source impedance, also called input impedance
- Above expression can be written in terms of hyperbolic tan functions as

$$Z_{in} = Z_s = Z_0 \frac{(Z_0 \tanh \gamma l + Z_R)}{(Z_0 + Z_R \tanh \gamma l)}$$

Input Impedance Relations

➢ Lossless Transmission line(α=0)

- For a loss less line the input impedance can be written as

$$Z_{in} = Z_s = Z_0 \frac{(Z_0 \tanh j\beta l + Z_R)}{(Z_0 + Z_R \tanh j\beta l)}$$

$$Z_{in} = Z_s = Z_0 \frac{(jZ_0 \tan \beta l + Z_R)}{(Z_0 + jZ_R \tan \beta l)}$$

- A special cases from the above general lossless line input impedance relations
  - When the line is terminated with characteristics impedance, $Z_R=Z_0$
  - When the line is terminated with open circuited, Z_R=infinity(∞)
  - When the line is terminated with short circuited, $Z_R=0$

Input Impedance Relations

➢ Matched Load $Z_R = Z_0$

- For a loss less line the input impedance can be written as

$$Z_{in} = Z_s = Z_0 \frac{(Z_0 \tanh j\beta l + Z_R)}{(Z_0 + Z_R \tanh j\beta l)}$$

$$Z_{in} = Z_0 \frac{(jZ_0 \tan \beta l + Z_0)}{(Z_0 + jZ_0 \tan \beta l)} = Z_0$$

- The line is terminated with characteristic impedance, the input impedance is equal to the characteristic impedance
- This condition is called matched load condition
- There is no reflections on the line

Input Impedance Relations

➢ Open Circuited, i.e. Z_R=infinity(∞)

- For a loss less line the input impedance can be written as

$$Z_{in} = Z_0 \frac{(jZ_0 \tan\beta l + Z_R)}{(Z_0 + jZ_R \tan\beta l)}$$

$$Z_{in} = Z_{OC} = Z_0 \frac{\left(j\frac{Z_0}{Z_R}\tan\beta l + 1\right)}{\left(\frac{Z_0}{Z_R} + j\tan\beta l\right)}$$

$$Z_{in} = Z_{OC} = -jZ_0 \cot\beta l$$

➢ Short Circuited, i.e. Z_R=0

$$Z_{in} = Z_0 \frac{(jZ_0 \tan\beta l + Z_R)}{(Z_0 + jZ_R \tan\beta l)}$$

$$Z_{in} = Z_{SC} = jZ_0 \tan\beta l$$

- If the line is terminated with open circuit or short circuit, the input impedance can be purely imaginary

Input Impedance Relations

- From the short circuit and open circuit impedance relations, the characteristic impedance can be written as

$$Z_{OC} Z_{SC} = (jZ_0 \tan \beta l) \frac{Z_0}{j \tan \beta l}$$

$$Z_{OC} Z_{SC} = Z_0^2$$

$$\sqrt{Z_{OC} Z_{SC}} = Z_0$$

- The characteristic impedance of the line can be measured from open and short circuit Transmission line

Impedance Matching, Single Stub Matching

Impedance Matching

- Impedance matching is one of the important aspects of high frequency circuit analysis.
- To avoid reflections and power loss from transmission line sections impedance matching techniques can be used
  - Quarter Wavelength Transformer,
  - Stub Matching
    - ✓ Single Stub Matching
    - ✓ Double Stub Matching
- The quarter wave transformer needs special line of characteristics impedance for every pair of resistances to be matched
- To add quarter wave transformer, cut the main line
- To avoid above difficulties, either open or short circuited transmission line attached at some position parallel to line

Impedance Matching

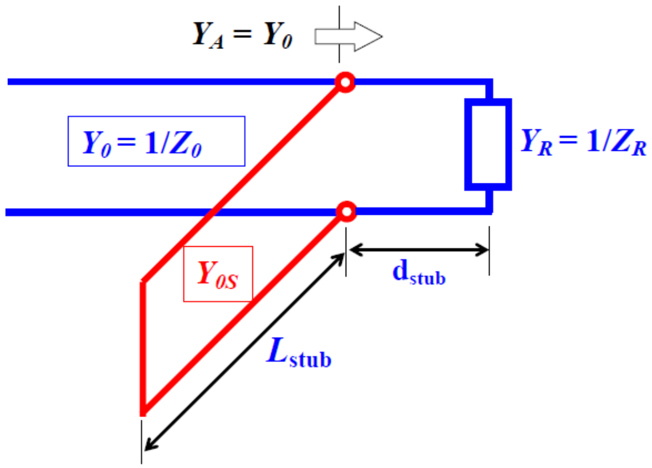
- Adding of either open or short line in parallel to main line is call stub
- The main advantage of this stub matching is
  - The length and characteristic impedance remains same
  - Since the stub is added in shunt, there is no need to cut the main line
  - The susceptance of the stub can be adjusted for perfect matching
- A short-circuited stub is less prone to leakage of electromagnetic radiation and is somewhat easier to realize.
- Open circuited stub may be more practical for certain types of transmission lines, for example microstrips where one would have to drill the insulating substrate to short circuit the two conductors of the line

Single Stub Matching

- Based on the number of parallel stubs connected to the line stub matching techniques can be classified into two types
  - Single-Stub Matching Technique
  - Double- Stub Matching Technique

➢ Single Stub Matching

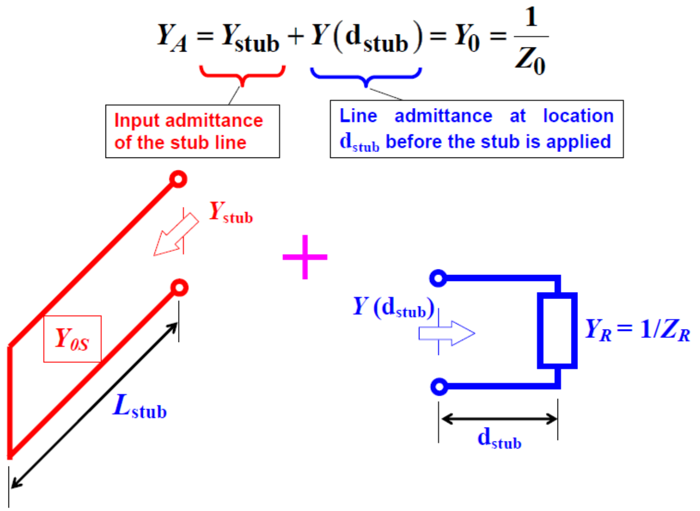
- A short-circuited section (stub) of a transmission line connected in parallel to the main transmission line is shown in Figure
- The stub is connected in parallel, so it is easy to deal with admittance analysis instead of impedance



Single Stub Matching

- There are two design parameters for single stub matching
  - The location of the stub with reference to the load can be represented as d_{stub}
  - The length of the stub line L_{stub}
- For proper impedance matching, the admittance at the location of the stub can be written as

$$Y_A = Y_{stub} + Y(d_{stub}) = Y_0 = \frac{1}{Z_0}$$



Single Stub Matching

➢ Location of the stub(d_{stub})

- To determine the location of the short-circuited stubs, the input impedance of a lossless transmission line and convert it to admittance

$$Y_{in} = Y_0 \frac{Y_R + jY_0 \tan \beta l}{Y_0 + jY_R \tan \beta l}$$

- The normalized admittance can be written as

$$y_{in} = \frac{Y_{in}}{Y_0} = \frac{y_R + j \tan \beta l}{1 + jy_R \tan \beta l}$$

- Separating the real and imaginary parts by rationalizing

$$y_{in} = \frac{y_R + y_R \tan^2 \beta l - jy_R^2 \tan \beta l + j \tan \beta l}{1 + y_R^2 \tan^2 \beta l}$$

$$= \frac{y_R \left(1 + \tan^2 \beta l\right)}{1 + y_R^2 \tan^2 \beta l} + \frac{j \left(1 - y_R^2\right) \tan \beta l}{1 + y_R^2 \tan^2 \beta l}$$

Single Stub Matching

- For no reflection, at a distance l=d_{stub} the real part of the admittance is unity

$$\mathrm{Re}[Y_{in}] = Y_0 \qquad or \qquad y_{in} = 1$$

- At $l=d_{stub}$

$$\frac{y_R\left(1+\tan^2\beta d_{stub}\right)}{1+y_R^2\tan^2\beta d_{stub}} = 1$$

- Simplifying the above expression, we obtain

$$y_R\left(1+\tan^2\beta d_{stub}\right) = 1 + y_R^2\tan^2\beta d_{stub}$$

$$\tan^2\beta d_{stub} = \frac{1-y_R}{\left(y_R - y_R^2\right)} = \frac{1}{y_R}$$

$$d_{stub} = \frac{1}{\beta}\tan^{-1}\left(\frac{1}{\sqrt{y_R}}\right)$$

$$d_{stub} = \frac{2\pi}{\lambda}\tan^{-1}\left(\sqrt{\frac{Y_0}{Y_R}}\right)$$

Single Stub Matching

- At this location the imaginary part of susceptance of b_s can be written as

$$\mathrm{Im}[y_{in}] = b_s = \frac{(1 - y_R^2)\tan\beta l}{1 + y_R^2 \tan^2\beta l} = \frac{(1 - y_R^2)\tan\beta d_{stub}}{1 + y_R^2 \tan^2\beta d_{stub}}$$

$$b_s = \frac{(1 - y_R^2)\tan\beta l}{1 + y_R^2 \tan^2\beta l} = \frac{(1 - y_R^2)\tan\beta d_{stub}}{1 + y_R^2 \tan^2\beta d_{stub}}$$

$$b_s = \frac{(1 - y_R^2)\dfrac{1}{\sqrt{y_R}}}{1 + y_R^2 \dfrac{1}{y_R}} = \frac{(1 - y_R^2)}{\sqrt{y_R}(1 + y_R)} = \frac{1 - y_R}{\sqrt{y_R}}$$

$$i.e. \qquad b_s = \frac{1 - \dfrac{Y_R}{Y_0}}{\sqrt{\dfrac{Y_R}{Y_0}}} = \frac{Y_0 - Y_R}{\sqrt{Y_R Y_0}}$$

- Therefore at length dstub the input admittance is $y_{in} = 1 + jb_s$

Single Stub Matching

➢ Length of the stub(L_{stub})

- To determine the length of the short-circuited stub consider the short circuited impedance and write in terms of admittance
- That is

$$Z_{sc} = jX_s = jZ_0 \tan \beta l$$

$$Y_{sc} = -jB_s = -jY_0 \cot \beta l$$

- The required normalized suspetance of the short circuited stub

$$b_s = \frac{B_s}{Y_0} = \cot \beta l$$

- Equating with y_{sc}

$$\cot \beta L_{stub} = \frac{Y_0 - Y_R}{\sqrt{Y_R Y_0}}$$

$$\tan \beta L_{stub} = \frac{\sqrt{Y_R Y_0}}{Y_0 - Y_R}$$

Single Stub Matching

- Converting into impedance

$$\tan \beta L_{stub} = \frac{\frac{1}{\sqrt{Z_R Z_0}}}{\frac{1}{Z_R} - \frac{1}{Z_0}}$$

$$= \frac{\sqrt{Z_R Z_0}}{Z_R - Z_0}$$

- Therefore, the length of short circuited stub is

$$L_{stub} = \frac{\lambda}{2\pi} \tan^{-1}\left(\frac{\sqrt{Z_R Z_0}}{Z_R - Z_0}\right) \quad \text{for} \quad Z_R > Z_0$$

$$L_{stub} = \frac{\lambda}{2\pi} \tan^{-1}\left(\frac{\sqrt{Z_R Z_0}}{Z_0 - Z_R}\right) \quad \text{for} \quad Z_0 > Z_R$$

- The drawback of this approach is that if the load is changed, the location of insertion may have to be moved

Smith Chart

Smith Chart

- It is a polar plot of the complex reflection coefficient.
- It is the transformation of complex impedance into reflection coefficient plane.

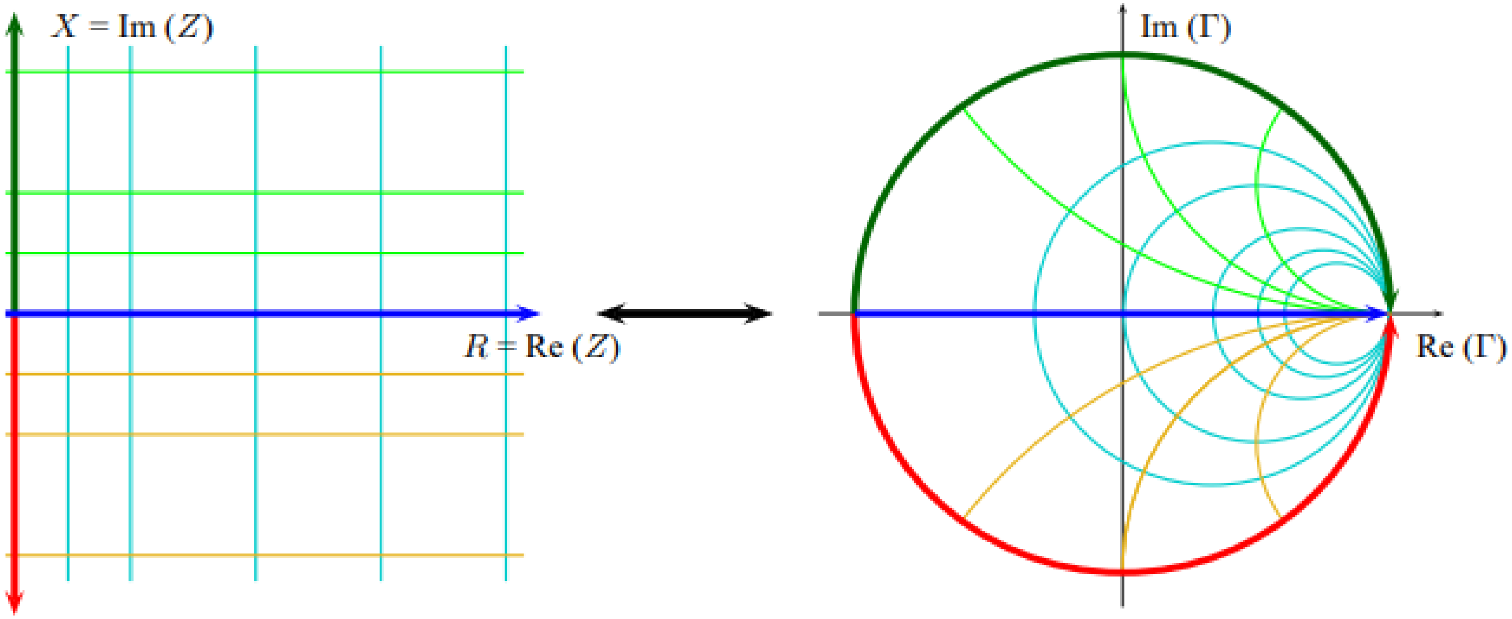
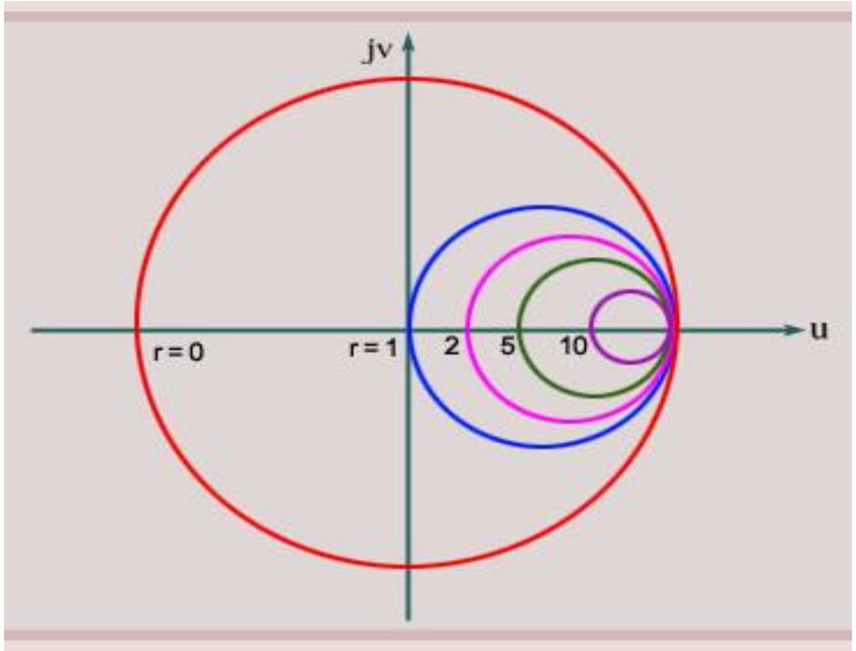


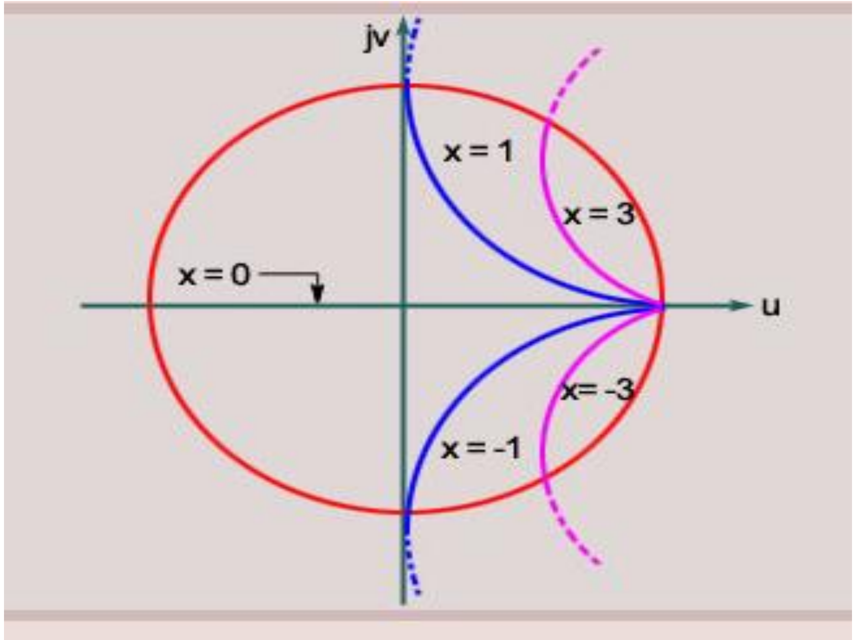
Figure 1: Transformation of Z into Γ

Smith Chart

- <u>Construction of smith chart</u>:-
- It consist of <u>r-circles</u> and <u>x-circles</u>



<u>Fig-2: r-circles</u>



<u>Fig-3: x-circles</u>

Smith Chart

- <u>Points to remember regarding smith chart:</u>

- The value of r is always positive, x can be positive (for inducatance impedance) and negative (for capacitance impedance)
- Apart from the r and x circles, we can draw the VSWR-circles or (S-circles)(ALWAYS NOT SHOWN ON THE CHART)

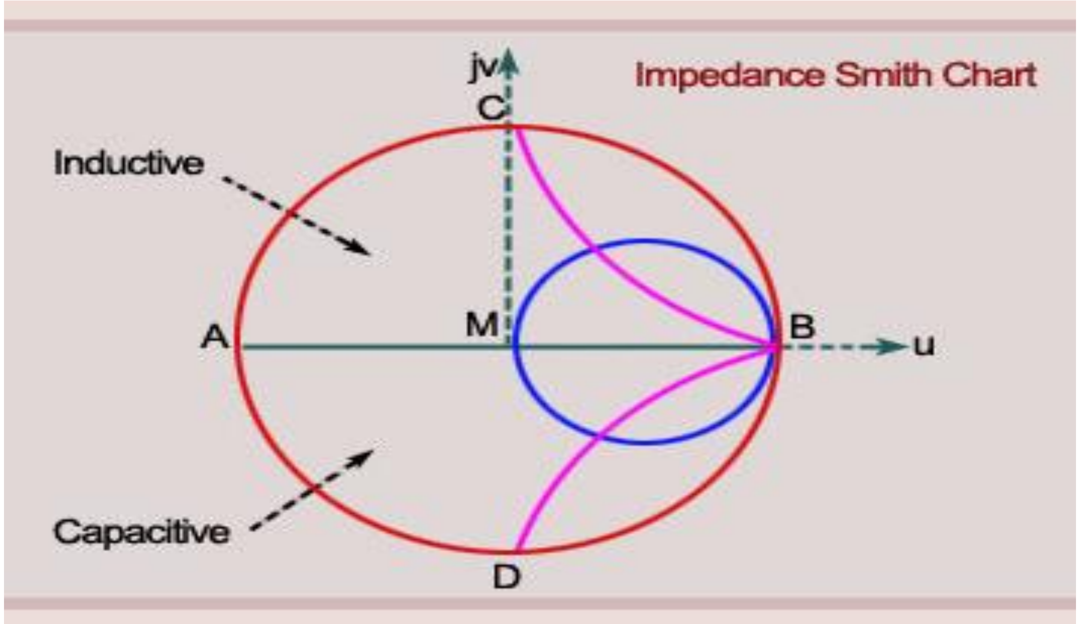


Figure 4: Some points on smith chart

Smith Chart

- At point Psc on the chart r = 0 and x = 0, it represents short circuit point. Similarly Poc on the chart r = infinity and x = infinity represents open circuit point.

- A complete revolution around the smith chart represents a distance of on the line(360 degress).

- Clockwise movement on the chart is regarded as moving toward the generator. Similarly, counter clockwise movement on the chart corresponds to moving towards load.

- There are 3 scales around periphery of the smith chart

  > Outermost scale used to determine distance on the from generator end in terms of wave length.

  > The next scale determines distance from the load end interms of WL

  > Outermost scale used to determine distance on the from generator end in terms of wave length.

  > The innermost scale is a protractor and is primarily used in determining angle of reflection coefficient

Smith Chart

- To the horizontal line upper part is inductive in nature and bottom part is capacitive in nature.

- As shown in figure (4):

  > The left most point A on the smith chart corresponds to and therefore represents ideal short-circuit load.

  > The right most point B on the Smith chart corresponds to , and therefore represents ideal open circuit load.

  >The upper most point C represents a pure inductive load of unity reactance and the lower most point D represents a pure capacitive load of unity reactance.

- Voltage Vmax occurs where impedance is maximum i.e point A and Vmin occurs where impedance is minimum i.e point B.