INSTITUTE OF AERONAUTICAL ENGINEERING
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Finite Element Methods

III B Tech II Semester

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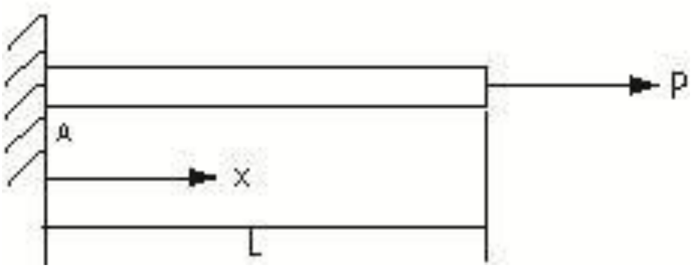
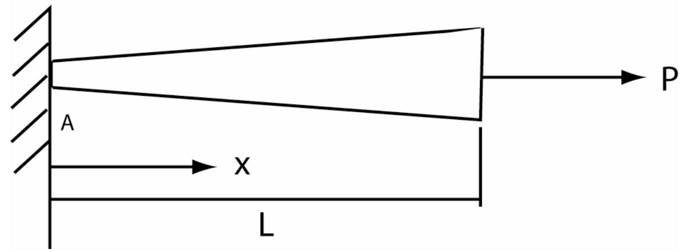
UNIT - 1

- **Introduction to FEM:**
  - Stiffness equations for a axial bar element in local co-ordinates using Potential Energy approach and Virtual energy principle
  - Finite element analysis of uniform, stepped and tapered bars subjected to mechanical and thermal loads
  - Assembly of Global stiffness matrix and load vector
  - Quadratic shape functions
  - properties of stiffness matrix

Axially Loaded Bar

Review:





Stress:

Strain:

Deformation:

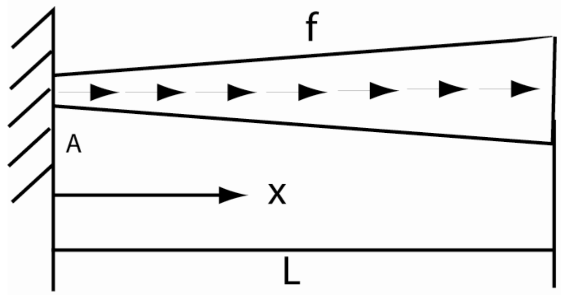
Stress:

Strain:

Deformation:

Axially Loaded Bar

Review:



Stress:

Strain:

Deformation:

Axially Loaded Bar – Governing Equations and Boundary Conditions

- ***Differential Equation***

$$\frac{d}{dx}\left[EA(x)\frac{du}{dx}\right]+f(x)=0 \qquad 0<x<L$$

- ***Boundary Condition Types***
  - ***prescribed displacement (essential BC)***

  - ***prescribed force/derivative of displacement (natural BC)***

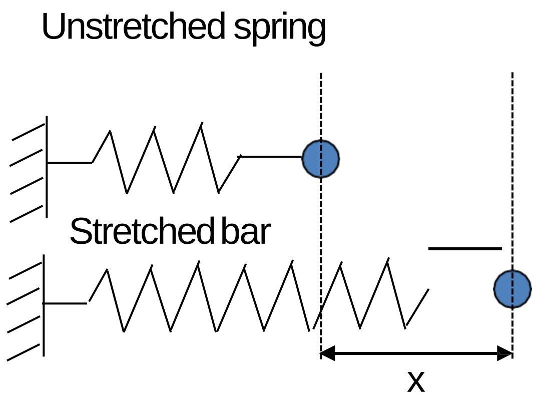
Axially Loaded Bar – Boundary Conditions

- *Examples*
  - *fixed end*
  - *simple support*
  - *free end*

Potential Energy

- **Elastic Potential Energy (PE)**

  - Spring case



$$\mathrm{PE} = 0$$

$$\mathrm{PE} = \frac{1}{2} k x^2$$

  - Axially loaded bar

undeformed: $\mathrm{PE} = 0$

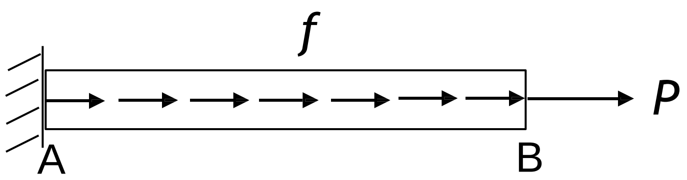
deformed: $\mathrm{PE} = \frac{1}{2} \int_0^L \sigma \varepsilon A dx$

  - Elastic body

$$\mathrm{PE} = \frac{1}{2} \int_V \boldsymbol{\sigma}^T \boldsymbol{\varepsilon} dv$$

Potential Energy

- **Work Potential (WE)**



$$\mathrm{WP} = -\int_0^L u \cdot f dx - P \cdot u_B$$

f: distributed force over a line
P: point force
u: displacement

- **Total Potential Energy**

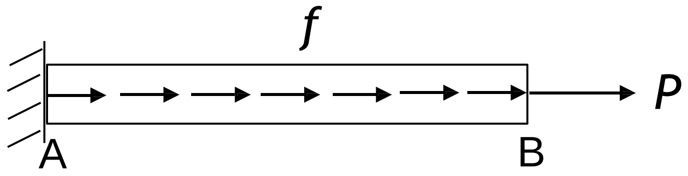
$$\Pi = \frac{1}{2}\int_0^L \sigma \varepsilon A dx - \int_0^L u \cdot f dx - P \cdot u_B$$

- **Principle of Minimum Potential Energy**

For conservative systems, of all the kinematically admissible displacement fields, those corresponding to equilibrium extremize the total potential energy. If the extremum condition is a minimum, the equilibrium state is stable.

Potential Energy + Rayleigh-Ritz Approach

Example:



Step 1: assume a displacement field $u = \sum_i a_i \phi_i(x) \quad i = 1 \text{ to } n$

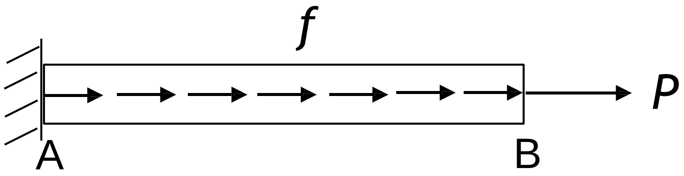
ϕ is shape function / basis function

n is the order of approximation

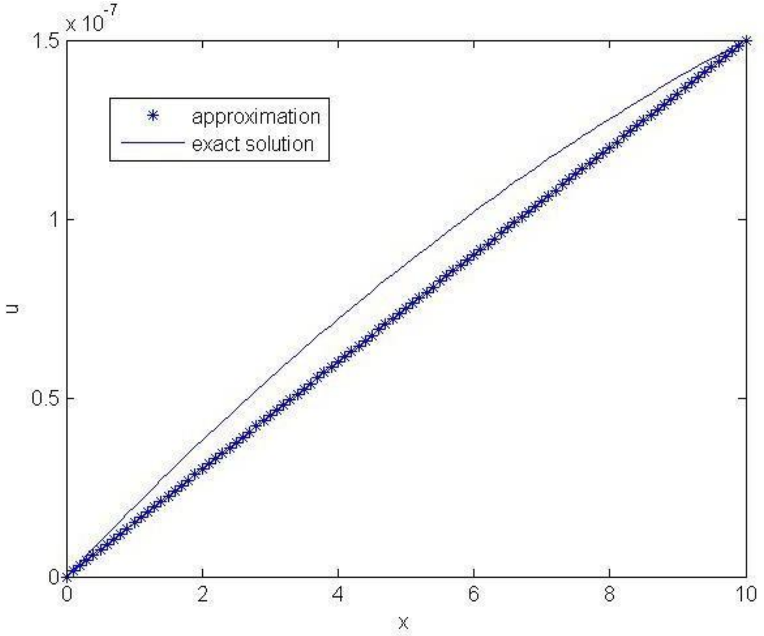
Step 2: calculate total potential energy

Potential Energy + Rayleigh-Ritz Approach

Example:

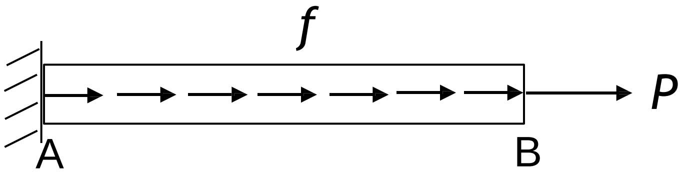


Step 3:select a_i so that the total potential energy is minimum



Galerkin's Method

Example:



$$\begin{cases} \dfrac{d}{dx}\left[EA(x)\dfrac{du}{dx} \right] + f(x) = 0 \\ u(x=0) = 0 \\ EA(x)\dfrac{du}{dx}\Big|_{x=L} = P \end{cases}$$

Seek an approximation $\tilde{u}$ so

$$\int_V w_i \left(\frac{d}{dx}\left[EA(x)\frac{d\tilde{u}}{dx} \right] + f(x) \right) dV = 0$$

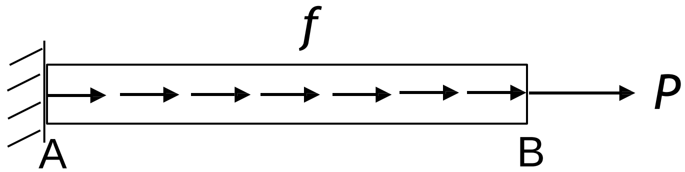
$$\tilde{u}(x=0) = 0$$

$$EA(x)\frac{d\tilde{u}}{dx}\Big|_{x=L} = P$$

In the Galerkin's method, the weight function is chosen to be the same as the shape function.

Galerkin's Method

Example:



$$\int_V w_i \left(\frac{d}{dx}\left[EA(x)\frac{d\tilde{u}}{dx} \right] + f(x) \right) dV = 0 \Longrightarrow -\int_0^L EA(x)\frac{d\tilde{u}}{dx}\frac{dw_i}{dx}dx + \int_0^L w_i f dx + w_i EA(x)\frac{d\tilde{u}}{dx}\Bigg|_0^L = 0$$

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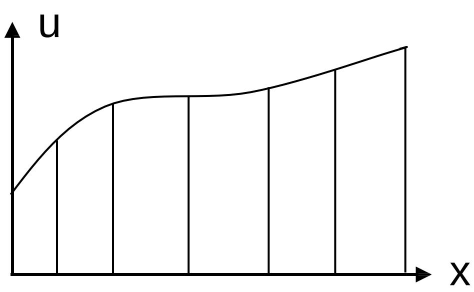
Finite Element Method – Piecewise Approximation



FEM Formulation of Axially Loaded Bar – Governing Equations

- ***Differential Equation***

$$\frac{d}{dx}\left[EA(x)\frac{du}{dx}\right]+f(x)=0 \qquad 0<x<L$$

- ***Weighted-Integral Formulation***

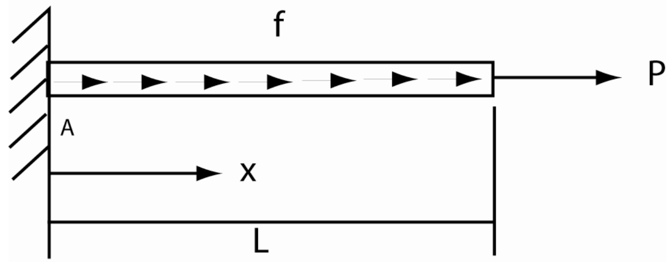
$$\int_0^L w\left(\frac{d}{dx}\left[EA(x)\frac{du}{dx}\right]+f(x)\right)dx=0$$

- ***Weak Form***

$$0=\int_0^L\left[\frac{dw}{dx}\left(EA(x)\frac{du}{dx}\right)-wf(x)\right]dx-\left.w\left(EA(x)\frac{du}{dx}\right)\right|_0^L$$

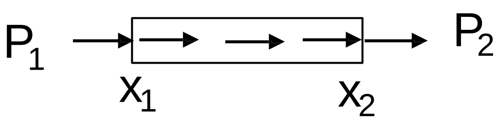
Approximation Methods – Finite Element Method

Example:



Step 1: Discretization

Step 2: Weak form of one element

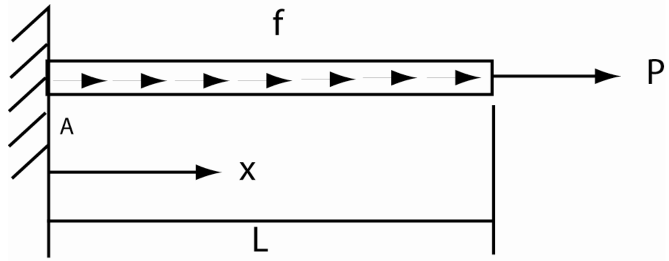


$$\int_{x_1}^{x_2}\left[\frac{dw}{dx}\left(EA(x)\frac{du}{dx}\right)-w(x)f(x)\right]dx-w(x)\left(EA(x)\frac{du}{dx}\right)\Bigg|_{x_1}^{x_2}=0$$

$$\Rightarrow\quad \int_{x_1}^{x_2}\left[\frac{dw}{dx}\left(EA(x)\frac{du}{dx}\right)-w(x)f(x)\right]dx-w(x_2)P_2-w(x_1)P_1=0$$

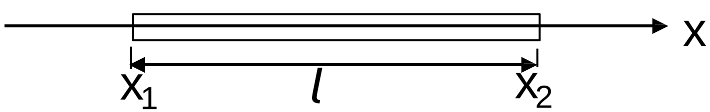
Approximation Methods – Finite Element Method

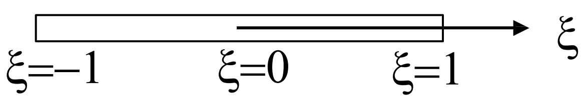
Example (*cont*):



Step 3: Choosing shape functions

- linear shape functions $$u = \phi_1 u_1 + \phi_2 u_2$$





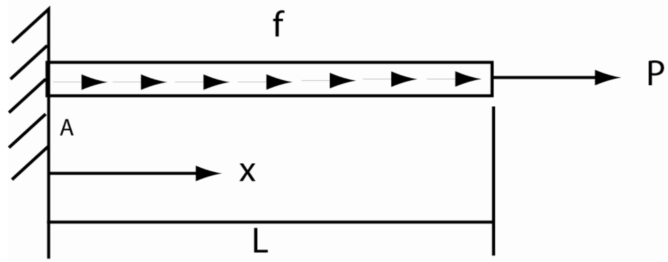
$$\phi_1 = \frac{x_2 - x}{l}; \quad \phi_2 = \frac{x - x_1}{l}$$

$$\phi_1 = \frac{1-\xi}{2}; \quad \phi_2 = \frac{1+\xi}{2}$$

$$\xi = \frac{2}{l}(x - x_1) - 1; \quad x = \frac{(\xi + 1)l}{2} + x_1$$

Approximation Methods – Finite Element Method

Example (*cont*):



Step 4: Forming element equation

E,A are constant

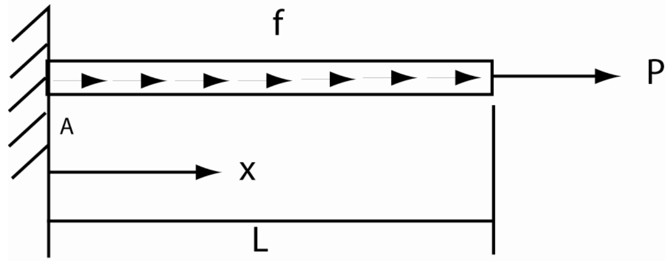
Let $w=\phi_1$, weak form becomes $\int_{x_1}^{x_2} -\frac{1}{l}\left(EA\cdot\frac{u_2-u_1}{l}\right)dx - \int_{x_1}^{x_2}\phi_1 f\,dx - \phi_1 P_2 - \phi_1 P_1 = 0 \longrightarrow \frac{EA}{l}u_1 - \frac{EA}{l}u_2 = \int_{x_1}^{x_2}\phi_1 f\,dx + P_1$

Let $w=\phi_2$, weak form becomes $\int_{x_1}^{x_2} \frac{1}{l}\left(EA\cdot\frac{u_2-u_1}{l}\right)dx - \int_{x_1}^{x_2}\phi_2 f\,dx - \phi_2 P_2 - \phi_2 P_1 = 0 \longrightarrow -\frac{EA}{l}u_1 + \frac{EA}{l}u_2 = \int_{x_1}^{x_2}\phi_2 f\,dx + P_2$

$$\longrightarrow \frac{EA}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \begin{Bmatrix}\int_{x_1}^{x_2}\phi_1 f dx\\ \int_{x_1}^{x_2}\phi_2 f dx\end{Bmatrix} + \begin{Bmatrix}P_1\\ P_2\end{Bmatrix} = \begin{Bmatrix}f_1\\ f_2\end{Bmatrix} + \begin{Bmatrix}P_1\\ P_2\end{Bmatrix}$$

Approximation Methods – Finite Element Method

Example (*cont*):



Step 5: Assembling to form system equation

Approach 1:

Element 1:

$$\frac{E^I A^I}{l^I}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_1^I \\ u_2^I \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} f_1^I \\ f_2^I \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} P_1^I \\ P_2^I \\ 0 \\ 0 \end{Bmatrix}$$

Element 2:

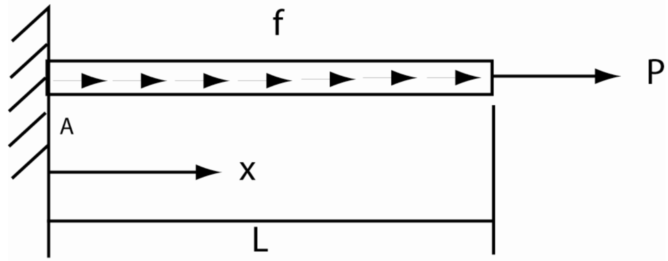
$$\frac{E^{II} A^{II}}{l^{II}}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ u_1^{II} \\ u_2^{II} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f_1^{II} \\ f_2^{II} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ P_1^{II} \\ P_2^{II} \\ 0 \end{Bmatrix}$$

Element 3:

$$\frac{E^{III} A^{III}}{l^{III}}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_1^{III} \\ u_2^{III} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ f_1^{III} \\ f_2^{III} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ P_1^{III} \\ P_2^{III} \end{Bmatrix}$$

Approximation Methods – Finite Element Method

Example (*cont*):



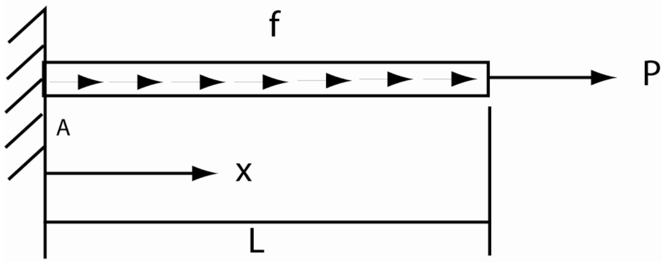
Step 5: Assembling to form system equation

Assembled System:

$$\begin{bmatrix} \frac{E^I A^I}{l^I} & -\frac{E^I A^I}{l^I} & 0 & 0 \\ -\frac{E^I A^I}{l^I} & \frac{E^I A^I}{l^I}+\frac{E^{II} A^{II}}{l^{II}} & -\frac{E^{II} A^{II}}{l^{II}} & 0 \\ 0 & -\frac{E^{II} A^{II}}{l^{II}} & \frac{E^{II} A^{II}}{l^{II}}+\frac{E^{III} A^{III}}{l^{III}} & -\frac{E^{III} A^{III}}{l^{III}} \\ 0 & 0 & -\frac{E^{III} A^{III}}{l^{III}} & \frac{E^{III} A^{III}}{l^{III}} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} + \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \begin{Bmatrix} f_1^I \\ f_2^I + f_1^{II} \\ f_2^{II} + f_1^{III} \\ f_2^{III} \end{Bmatrix} + \begin{Bmatrix} P_1^I \\ P_2^I + P_1^{II} \\ P_2^{II} + P_1^{III} \\ P_2^{III} \end{Bmatrix}$$

Approximation Methods – Finite Element Method

Example (*cont*):



Step 5: Assembling to form system equation

Approach 2: Element connectivity table

$$k_{ij}^{e} \rightarrow K_{IJ}$$

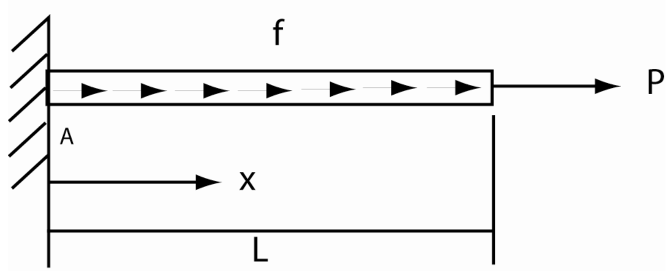
	Element 1	Element 2	Element 3
1	1	2	3
2	2	3	4

local node (i,j)

global node index (I,J)

Approximation Methods – Finite Element Method

Example (*cont*):



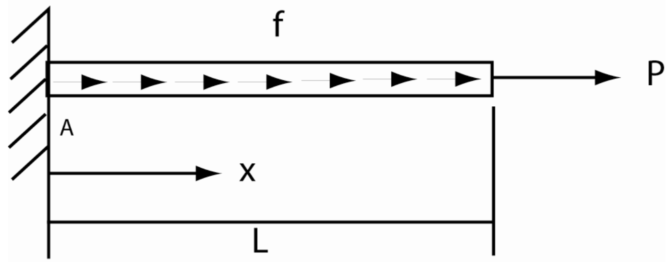
Step 6: Imposing boundary conditions and forming condense system

Condensed system:

$$\begin{pmatrix} \frac{E^I A^I}{l^I} + \frac{E^{II} A^{II}}{l^{II}} & -\frac{E^{II} A^{II}}{l^{II}} & 0 \\ -\frac{E^{II} A^{II}}{l^{II}} & \frac{E^{II} A^{II}}{l^{II}} + \frac{E^{III} A^{III}}{l^{III}} & -\frac{E_{III} A_{III}}{l^{III}} \\ 0 & -\frac{E^{III} A^{III}}{l^{III}} & \frac{E^{III} A^{III}}{l^{III}} \end{pmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_2 \\ f_3 \\ f_4 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ P \end{Bmatrix}$$

Approximation Methods – Finite Element Method

Example (*cont*):



Step 7: solution

Step 8: post calculation

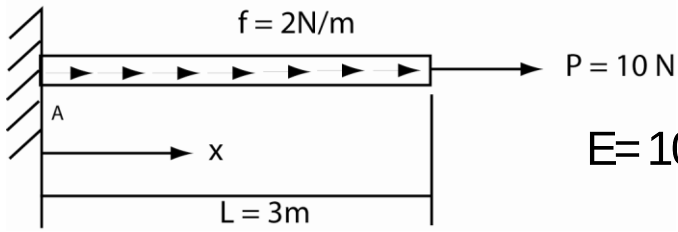
$$u = u_1\phi_1 + u_2\phi_2 \quad \Longrightarrow \quad \varepsilon = \frac{du}{dx} = u_1\frac{d\phi_1}{dx} + u_2\frac{d\phi_2}{dx} \quad \Longrightarrow \quad \sigma = E\varepsilon = Eu_1\frac{d\phi_1}{dx} + Eu_2\frac{d\phi_2}{dx}$$

Summary - Major Steps inFEM

- *Discretization*
- *Derivation of element equation*
  - *weak form*
  - *construct form of approximation solution over one element*
  - *derive finite element model*
- *Assembling – putting elements together*
- *Imposing boundary conditions*
- *Solving equations*
- *Postcomputation*

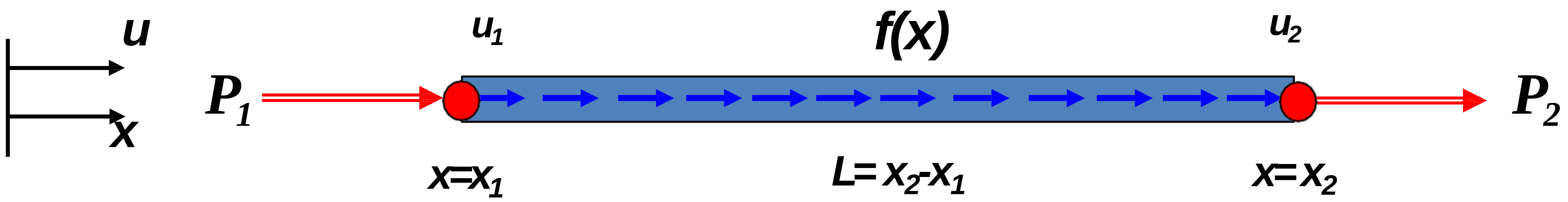
Exercises – Linear Element

Example 1:



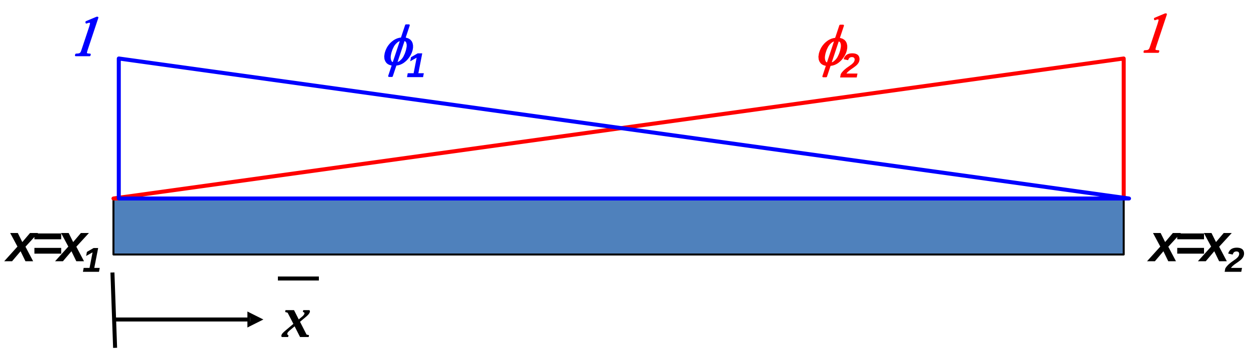
$E = 100$ GPa, $A = 1$ cm^2

Linear Formulation for Bar Element

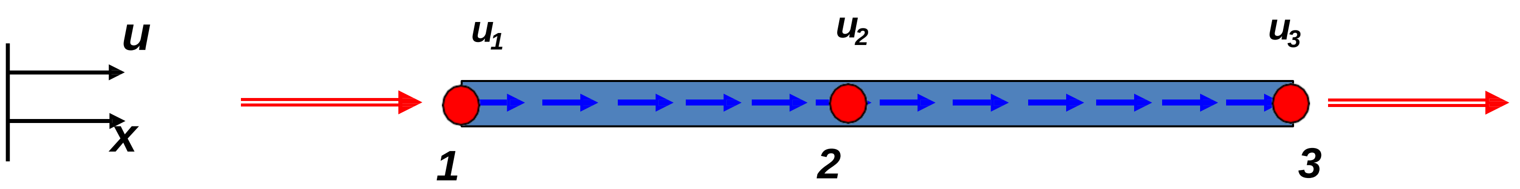


$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

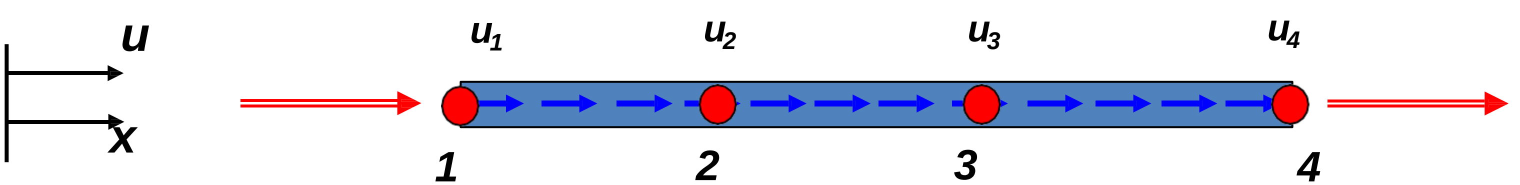
$$where \quad K_{ij} = \int_{x_1}^{x_2} EA\left(\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\right)dx = K_{ji}, \quad f_i = \int_{x_1}^{x_2} (\phi_i f)\,dx$$



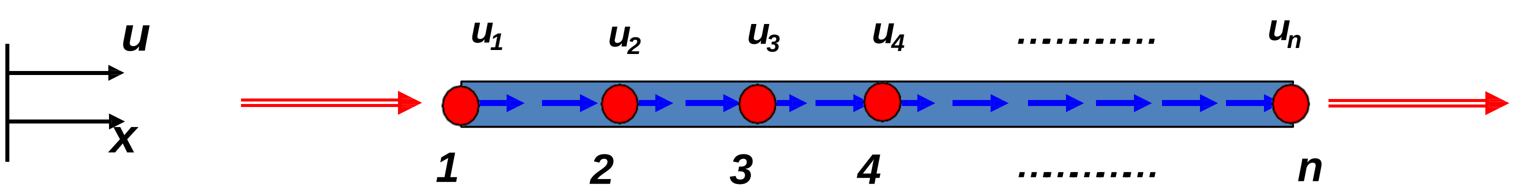
Higher Order Formulation for Bar Element



$$u(x) = u_1\phi_1(x) + u_2\phi_2(x) + u_3\phi_3(x)$$

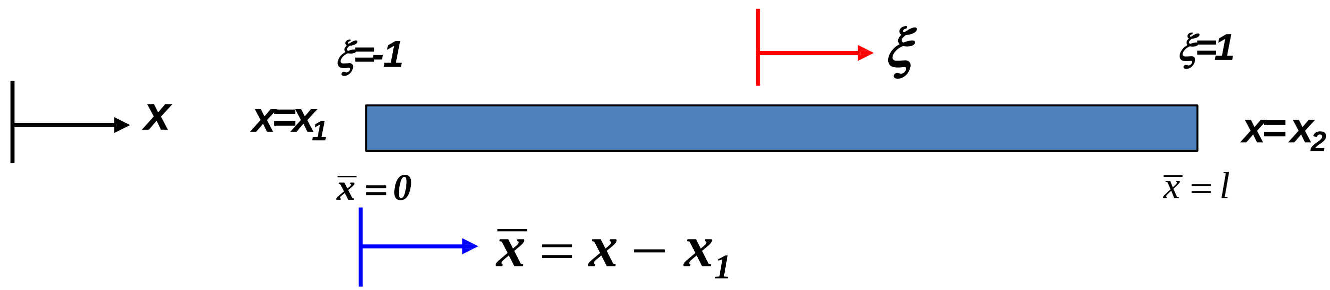


$$u(x) = u_1\phi_1(x) + u_2\phi_2(x) + u_3\phi_3(x) + u_4\phi_4(x)$$

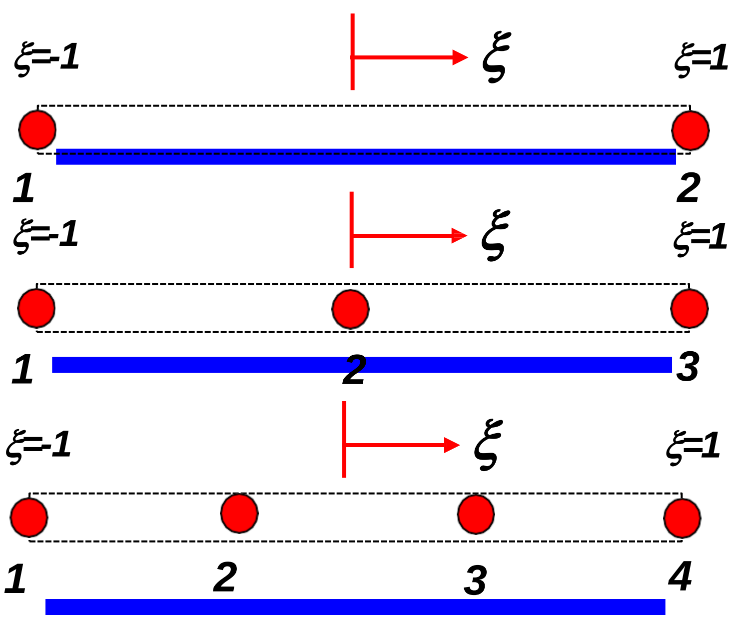


$$u(x) = u_1\phi_1(x) + u_2\phi_2(x) + u_3\phi_3(x) + u_4\phi_4(x) + \bullet\bullet\bullet\bullet\bullet\bullet + u_n\phi_n(x)$$

Natural Coordinates and Interpolation Functions



Natural (or Normal) Coordinate: $\xi = \dfrac{x - \dfrac{x_1 + x_2}{2}}{l/2}$



$$\phi_1 = -\frac{\xi - 1}{2}, \quad \phi_2 = \frac{\xi + 1}{2}$$

$$\phi_1 = \frac{\xi(\xi - 1)}{2}, \quad \phi_2 = -(\xi + 1)(\xi - 1), \quad \phi_3 = \frac{(\xi + 1)\xi}{2}$$

$$\phi_1 = -\frac{9}{16}\left(\xi + \frac{1}{3}\right)\left(\xi - \frac{1}{3}\right)(\xi - 1), \quad \phi_2 = \frac{27}{16}(\xi + 1)\left(\xi - \frac{1}{3}\right)(\xi - 1)$$

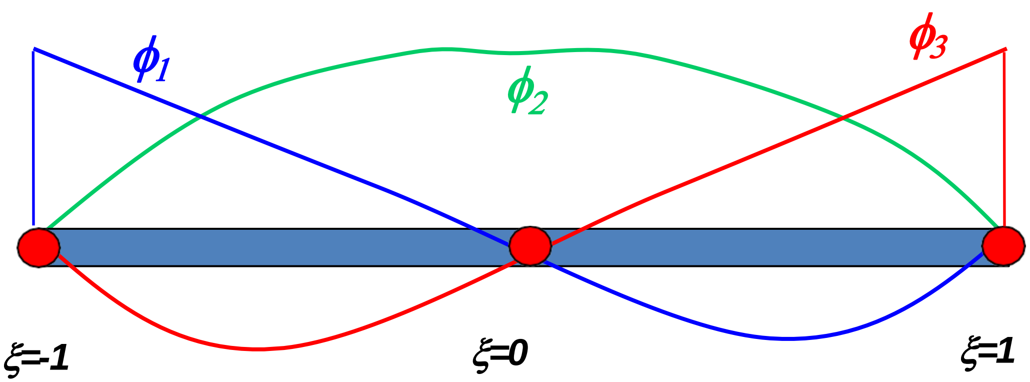
$$\phi_3 = -\frac{27}{16}(\xi + 1)\left(\xi + \frac{1}{3}\right)(\xi - 1), \quad \phi_4 = \frac{9}{16}(\xi + 1)\left(\xi + \frac{1}{3}\right)\left(\xi - \frac{1}{3}\right)$$

Quadratic Formulation for Bar Element

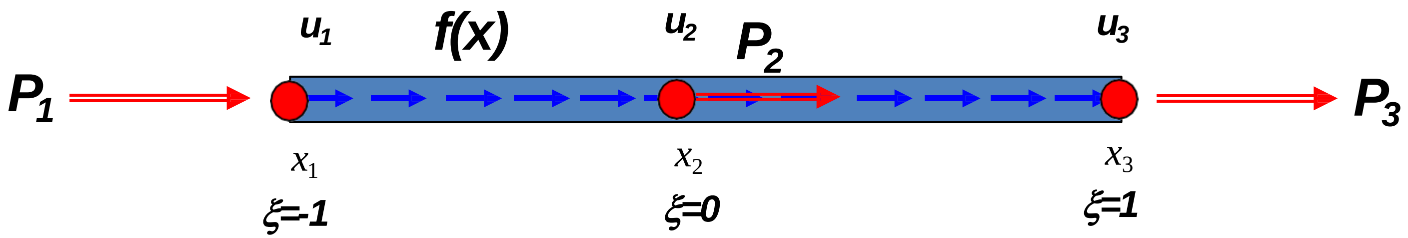
$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\text{where } K_{ij} = \int_{x_1}^{x_2} EA \left(\frac{d\phi_i}{dx} \frac{d\phi_j}{dx} \right) dx = \int_{-1}^{1} EA \left(\frac{d\phi_i}{d\xi} \frac{d\phi_j}{d\xi} \right) \frac{2}{l} \, d\xi = K_{ji}$$

$$\text{and } f_i = \int_{x_1}^{x_2} (\varphi_i f) dx = \int_{-1}^{1} (\varphi_i f) \frac{l}{2} \, d\xi, \quad i, j = 1, 2, 3$$



Quadratic Formulation for Bar Element



$$u(\xi)=u_1\phi_1(\xi)+u_2\phi_2(\xi)+u_3\phi_3(\xi)=u_1\frac{\xi(\xi-1)}{2}-u_2(\xi+1)(\xi-1)+u_3\frac{(\xi+1)\xi}{2}$$

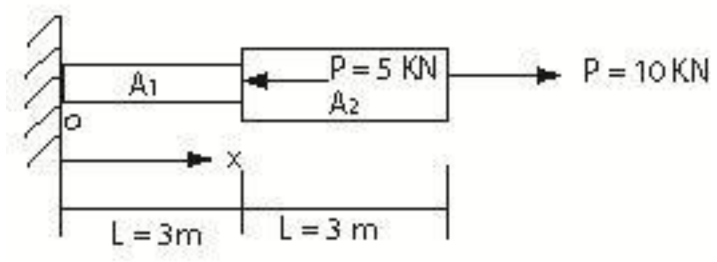
$$\phi_1=\frac{\xi(\xi-1)}{2},\quad \phi_2=-(\xi+1)(\xi-1),\quad \phi_3=\frac{(\xi+1)\xi}{2}$$

$$\xi=\frac{x-\frac{x_1+x_2}{2}}{l/2}\quad\Longrightarrow\quad \frac{l}{2}d\xi=dx\quad\Longrightarrow\quad \frac{d\xi}{dx}=\frac{2}{l}$$

$$\frac{d\phi_1}{dx}=\frac{2}{l}\frac{d\phi_1}{d\xi}=\frac{2\xi-1}{l},\qquad \frac{d\phi_2}{dx}=\frac{2}{l}\frac{d\phi_2}{d\xi}=-\frac{4\xi}{l},\qquad \frac{d\phi_3}{dx}=\frac{2}{l}\frac{d\phi_3}{d\xi}=\frac{2\xi+1}{l}$$

Exercises – Quadratic Element

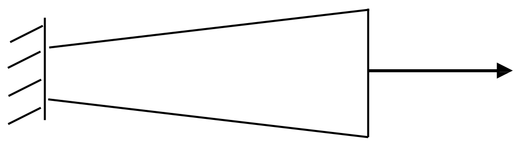
Example 2:



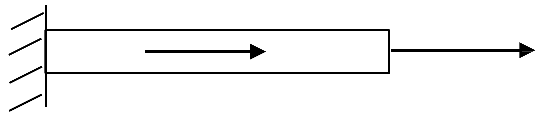
$E = 100$ GPa, $A_1 = 1$ cm^2; $A_1 = 2$ cm^2

Some Issues

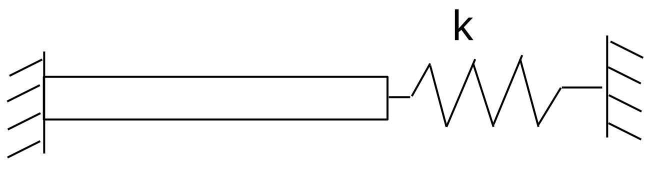
Non-constant cross section:



Interior load point:



Mixed boundary condition:

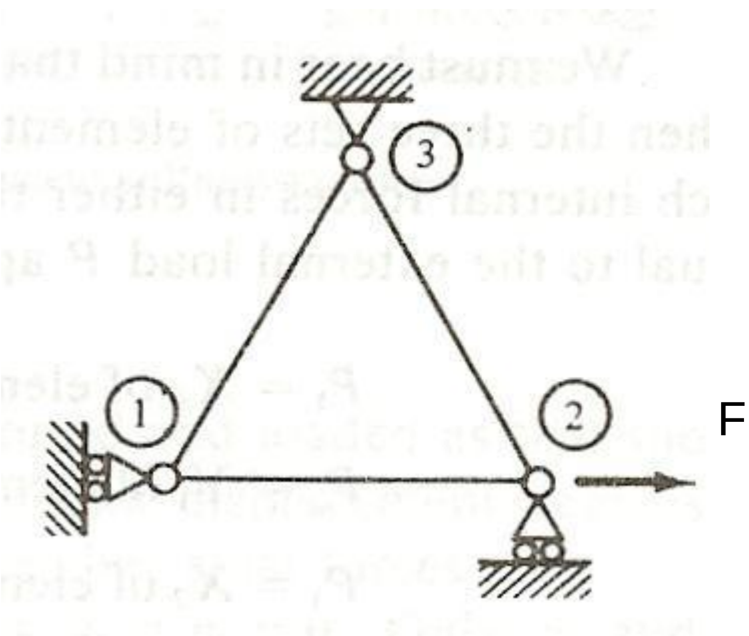


UNIT – 2

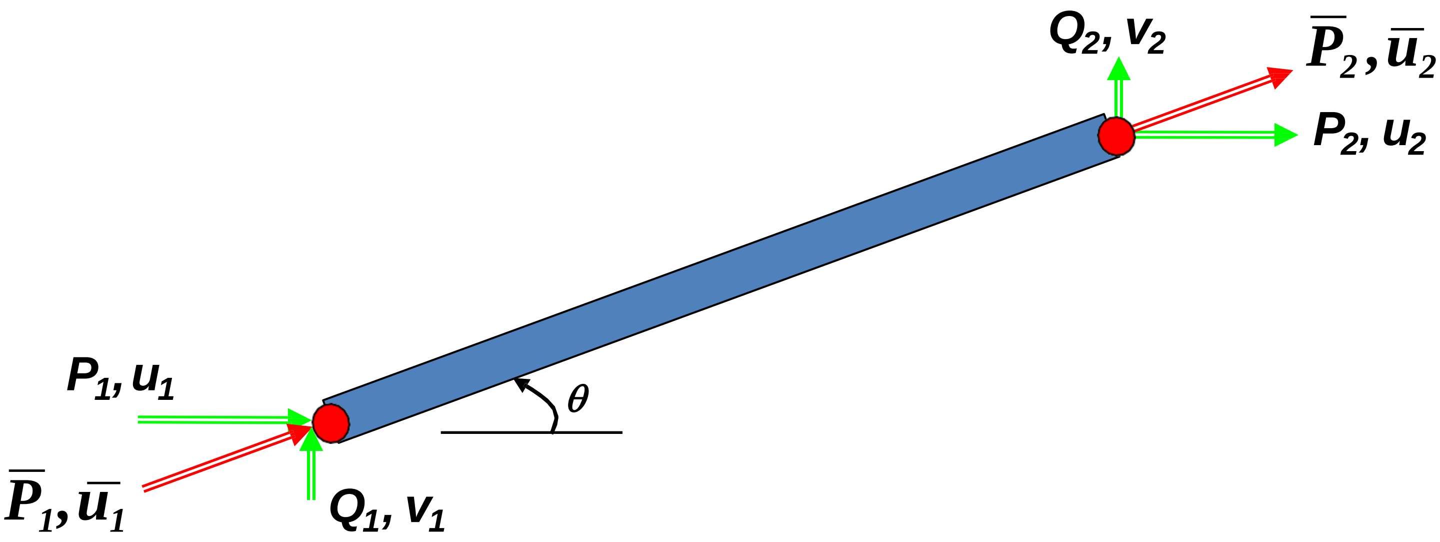
Finite Element Analysis of Trusses

Plane Truss Problems

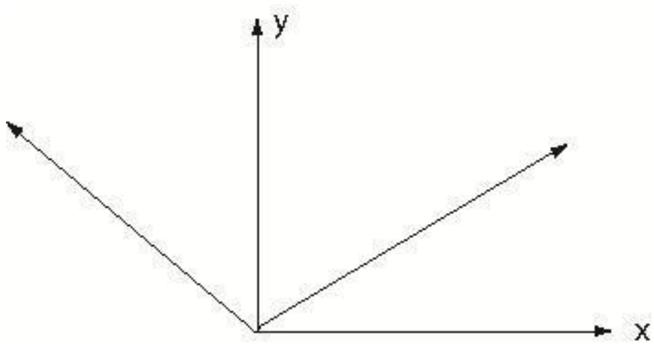
Example 1: Find forces inside each member. All members have the same length.



Arbitrarily Oriented 1-D Bar Element on 2-D Plane



Relationship Between Local Coordinates and Global Coordinates

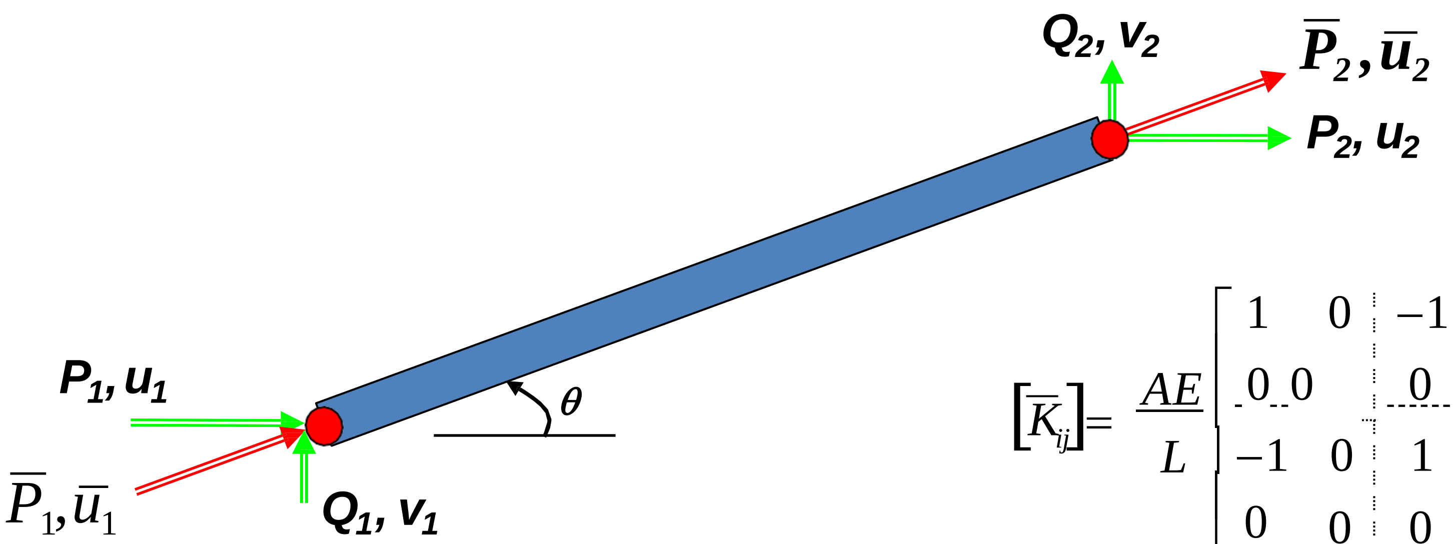


$$\begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 = 0 \\ \bar{u}_2 \\ \bar{v}_2 = 0 \end{Bmatrix} = \left[\begin{array}{cc|cc} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ \hline 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{array}\right] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

Relationship Between Local Coordinates and Global Coordinates

$$\begin{Bmatrix} \bar{P}_1 \\ 0 \\ \bar{P}_2 \\ 0 \end{Bmatrix} = \left[\begin{array}{cc:cc} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ \hdashline 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{array}\right] \begin{Bmatrix} P_1 \\ Q_1 \\ P_2 \\ Q_2 \end{Bmatrix}$$

Stiffness Matrix of 1-D Bar Element on 2-D Plane

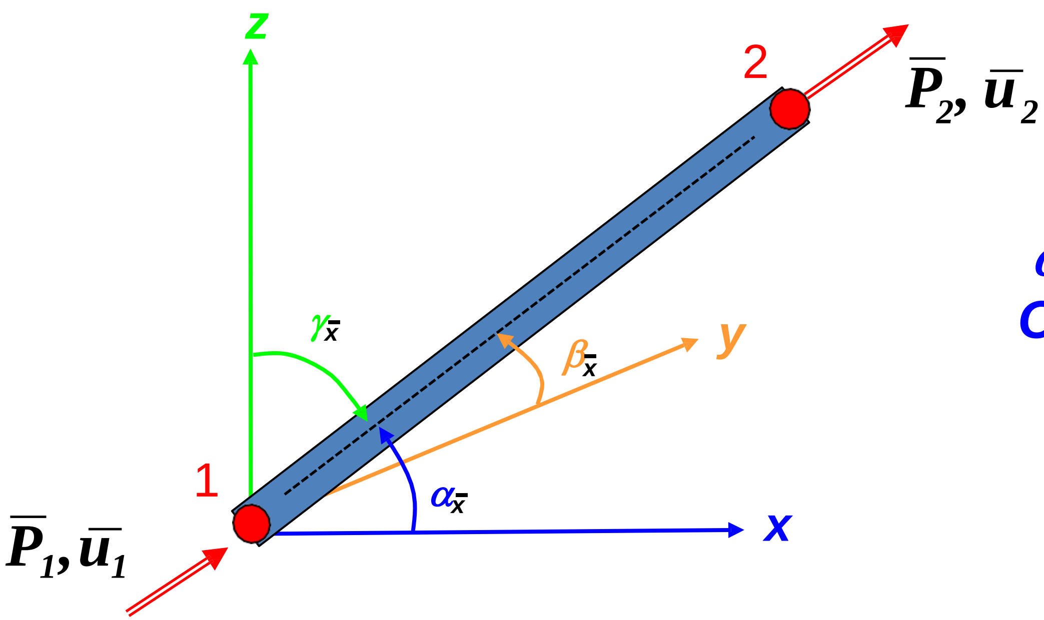


$$[\overline{K}_{ij}] = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{Bmatrix} P_1 \\ Q_1 \\ P_2 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} [\overline{K}_{ij}] \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$\begin{Bmatrix} P_1 \\ Q_1 \\ P_2 \\ Q_2 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta & -\cos^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\sin\theta\cos\theta & \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

Arbitrarily Oriented 1-D Bar Element in 3-D Space

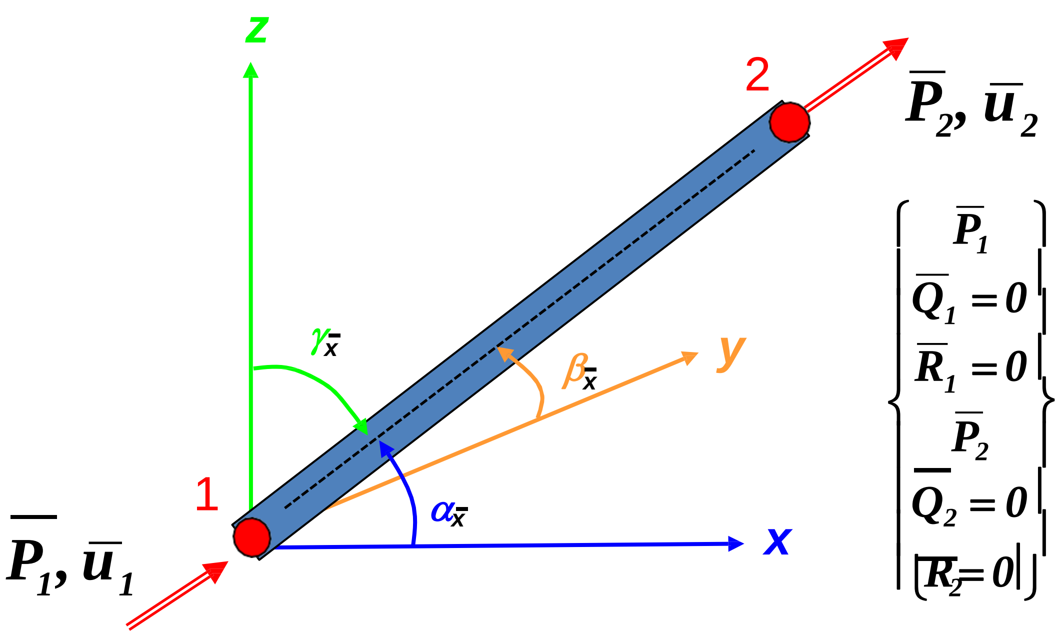


$\alpha_{\bar{x}}$, $\beta_{\bar{x}}$, $\gamma_{\bar{x}}$ are the Direction Cosines of the bar in the x-y-z coordinate system

$$\begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 = 0 \\ \bar{w}_1 = 0 \\ \bar{u}_2 \\ \bar{v}_2 = 0 \\ \bar{w}_2 = 0 \end{Bmatrix} = \begin{bmatrix} \alpha_{\bar{x}} & \beta_{\bar{x}} & \gamma_{\bar{x}} & 0 & 0 & 0 \\ \alpha_{\bar{y}} & \beta_{\bar{y}} & \gamma_{\bar{y}} & 0 & 0 & 0 \\ \alpha_{\bar{z}} & \beta_{\bar{z}} & \gamma_{\bar{z}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_{\bar{x}} & \beta_{\bar{x}} & \gamma_{\bar{x}} \\ 0 & 0 & 0 & \alpha_{\bar{y}} & \beta_{\bar{y}} & \gamma_{\bar{y}} \\ 0 & 0 & 0 & \alpha_{\bar{z}} & \beta_{\bar{z}} & \gamma_{\bar{z}} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix}$$

$$\begin{Bmatrix} \overline{P}_1 \\ \overline{Q}_1 = 0 \\ \overline{R}_1 = 0 \\ \overline{P}_2 \\ \overline{Q}_2 = 0 \\ \overline{R}_2 = 0 \end{Bmatrix} = \begin{bmatrix} \alpha_{\bar{x}} & \beta_{\bar{x}} & \gamma_{\bar{x}} & 0 & 0 & 0 \\ \alpha_{\bar{y}} & \beta_{\bar{y}} & \gamma_{\bar{y}} & 0 & 0 & 0 \\ \alpha_{\bar{z}} & \beta_{\bar{z}} & \gamma_{\bar{z}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_{\bar{x}} & \beta_{\bar{x}} & \gamma_{\bar{x}} \\ 0 & 0 & 0 & \alpha_{\bar{y}} & \beta_{\bar{y}} & \gamma_{\bar{y}} \\ 0 & 0 & 0 & \alpha_{\bar{z}} & \beta_{\bar{z}} & \gamma_{\bar{z}} \end{bmatrix} \begin{Bmatrix} P_1 \\ Q_1 \\ R_1 \\ P_2 \\ Q_2 \\ R_2 \end{Bmatrix}$$

Stiffness Matrix of 1-D Bar Element in 3-D Space



$$\begin{Bmatrix} \bar{P}_1 \\ \bar{Q}_1 = 0 \\ \bar{R}_1 = 0 \\ \bar{P}_2 \\ \bar{Q}_2 = 0 \\ \bar{R}_2 = 0 \end{Bmatrix} = \frac{AE}{L} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 = 0 \\ \bar{w}_1 = 0 \\ \bar{u}_2 \\ \bar{v}_2 = 0 \\ \bar{w}_2 = 0 \end{Bmatrix}$$

$$\begin{Bmatrix} P_1 \\ Q_1 \\ R_1 \\ P_2 \\ Q_2 \\ R_2 \end{Bmatrix} = \frac{AE}{L} \left[\begin{array}{ccc:ccc} \alpha_{\bar{x}}^2 & \alpha_{\bar{x}}\beta_{\bar{x}} & \alpha_{\bar{x}}\gamma_{\bar{x}} & -\alpha_{\bar{x}}^2 & -\alpha_{\bar{x}}\beta_{\bar{x}} & -\alpha_{\bar{x}}\gamma_{\bar{x}} \\ \alpha_{\bar{x}}\beta_{\bar{x}} & \beta_{\bar{x}}^2 & \beta_{\bar{x}}\gamma_{\bar{x}} & -\alpha_{\bar{x}}\beta_{\bar{x}} & -\beta_{\bar{x}}^2 & -\beta_{\bar{x}}\gamma_{\bar{x}} \\ \alpha_{\bar{x}}\gamma_{\bar{x}} & \beta_{\bar{x}}\gamma_{\bar{x}} & \gamma_{\bar{x}}^2 & -\alpha_{\bar{x}}\gamma_{\bar{x}} & -\beta_{\bar{x}}\gamma_{\bar{x}} & -\gamma_{\bar{x}}^2 \\ -\alpha_{\bar{x}}^2 & -\alpha_{\bar{x}}\beta_{\bar{x}} & -\alpha_{\bar{x}}\gamma_{\bar{x}} & \alpha_{\bar{x}}^2 & \alpha_{\bar{x}}\beta_{\bar{x}} & \alpha_{\bar{x}}\gamma_{\bar{x}} \\ -\alpha_{\bar{x}}\beta_{\bar{x}} & -\beta_{\bar{x}}^2 & -\beta_{\bar{x}}\gamma_{\bar{x}} & \alpha_{\bar{x}}\beta_{\bar{x}} & \beta_{\bar{x}}^2 & \beta_{\bar{x}}\gamma_{\bar{x}} \\ -\alpha_{\bar{x}}\gamma_{\bar{x}} & -\beta_{\bar{x}}\gamma_{\bar{x}} & -\gamma_{\bar{x}}^2 & \alpha_{\bar{x}}\gamma_{\bar{x}} & \beta_{\bar{x}}\gamma_{\bar{x}} & \gamma_{\bar{x}}^2 \end{array} \right] \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix}$$

Matrix Assembly of Multiple Bar Elements

Element (I)

$$\begin{Bmatrix} P_1 \\ Q_1 \\ P_2 \\ Q_2 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

Element (II)

$$\begin{Bmatrix} P_2 \\ Q_2 \\ P_3 \\ Q_3 \end{Bmatrix} = \frac{AE}{4L}\begin{bmatrix} 1 & -\sqrt{3} & -1 & \sqrt{3} \\ -\sqrt{3} & 3 & \sqrt{3} & -3 \\ -1 & \sqrt{3} & 1 & -\sqrt{3} \\ \sqrt{3} & -3 & -\sqrt{3} & 3 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

Element (III)

$$\begin{Bmatrix} P_1 \\ Q_1 \\ P_3 \\ Q_3 \end{Bmatrix} = \frac{AE}{4L}\begin{bmatrix} 1 & \sqrt{3} & -1 & -\sqrt{3} \\ \sqrt{3} & 3 & -\sqrt{3} & -3 \\ -1 & -\sqrt{3} & 1 & \sqrt{3} \\ -\sqrt{3} & -3 & \sqrt{3} & 3 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{Bmatrix}$$

Matrix Assembly of Multiple Bar Elements

Element (I)

$$\begin{Bmatrix} P_1 \\ Q_1 \\ P_2 \\ Q_2 \\ P_3 \\ Q_3 \end{Bmatrix} = \frac{AE}{4L} \begin{bmatrix} 4 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

Element (II)

$$\begin{Bmatrix} P_1 \\ Q_1 \\ P_2 \\ Q_2 \\ P_3 \\ Q_3 \end{Bmatrix} = \frac{AE}{4L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\sqrt{3} & -1 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 3 & \sqrt{3} & -3 \\ 0 & 0 & -1 & \sqrt{3} & 1 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & -3 & -\sqrt{3} & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

Element (III)

$$\begin{Bmatrix} P_1 \\ Q_1 \\ P_2 \\ Q_2 \\ P_3 \\ Q_3 \end{Bmatrix} = \frac{AE}{4L} \begin{bmatrix} 1 & \sqrt{3} & 0 & 0 & -1 & -\sqrt{3} \\ \sqrt{3} & 3 & 0 & 0 & -\sqrt{3} & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -\sqrt{3} & 0 & 0 & 1 & \sqrt{3} \\ -\sqrt{3} & -3 & 0 & 0 & \sqrt{3} & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

Matrix Assembly of Multiple Bar Elements

$$\begin{Bmatrix} R_1 \\ S_1 \\ R_2 \\ S_2 \\ R_3 \\ S_3 \end{Bmatrix} = \frac{AE}{4L} \left[\begin{array}{cc|cc|cc} 4+1 & 0+\sqrt{3} & -4 & 0 & -1 & -\sqrt{3} \\ 0+\sqrt{3} & 0+3 & 0 & 0 & -\sqrt{3} & -3 \\ \hline -4 & 0 & 4+1 & 0-\sqrt{3} & -1 & \sqrt{3} \\ 0 & 0 & 0-\sqrt{3} & 0+3 & \sqrt{3} & -3 \\ \hline -1 & -\sqrt{3} & -1 & \sqrt{3} & 1+1 & \sqrt{3}-\sqrt{3} \\ -\sqrt{3} & -3 & \sqrt{3} & -3 & \sqrt{3}-\sqrt{3} & 3+3 \end{array} \right] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

Apply known boundary conditions

$$\begin{Bmatrix} R_1 = ? \\ S_1 = 0 \\ R_2 = F \\ S_2 = ? \\ R_3 = ? \\ S_3 = ? \end{Bmatrix} = \frac{AE}{4L} \left[\begin{array}{cc|cc|cc} 5 & \sqrt{3} & -4 & 0 & -1 & -\sqrt{3} \\ \sqrt{3} & 3 & 0 & 0 & -\sqrt{3} & -3 \\ \hline -4 & 0 & 5 & -\sqrt{3} & -1 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 3 & \sqrt{3} & -3 \\ \hline -1 & -\sqrt{3} & -1 & \sqrt{3} & 2 & 0 \\ -\sqrt{3} & -3 & \sqrt{3} & -3 & 0 & 6 \end{array} \right] \begin{Bmatrix} u_1 = 0 \\ v_1 = ? \\ u_2 = ? \\ v_2 = 0 \\ u_3 = 0 \\ v_3 = 0 \end{Bmatrix}$$

Solution Procedures

$$\begin{Bmatrix} R_2 = F \\ S_1 = 0 \\ R_1 = ? \\ S_2 = ? \\ R_3 = ? \\ S_3 = ? \end{Bmatrix} = \frac{AE}{4L} \begin{bmatrix} -4 & 0 & 5 & -\sqrt{3} & -1 & \sqrt{3} \\ \sqrt{3} & 3 & 0 & 0 & -\sqrt{3} & -3 \\ 5 & \sqrt{3} & -4 & 0 & -1 & -\sqrt{3} \\ 0 & 0 & -\sqrt{3} & 3 & \sqrt{3} & -3 \\ -1 & -\sqrt{3} & 1 & -\sqrt{3} & 2 & 0 \\ -\sqrt{3} & -3 & \sqrt{3} & -3 & 0 & 6 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ v_1 = ? \\ u_2 = ? \\ v_2 = 0 \\ u_3 = 0 \\ v_3 = 0 \end{Bmatrix}$$

$\Longrightarrow$ $u_2 = 4FL/5AE,\ v_1 = 0$

$$\begin{Bmatrix} R_2 = F \\ S_1 = 0 \\ R_1 = ? \\ S_2 = ? \\ R_3 = ? \\ S_3 = ? \end{Bmatrix} = \frac{AE}{4L} \begin{bmatrix} -4 & 0 & 5 & -\sqrt{3} & -1 & \sqrt{3} \\ \sqrt{3} & 3 & 0 & 0 & -\sqrt{3} & -3 \\ 5 & \sqrt{3} & -4 & 0 & -1 & -\sqrt{3} \\ 0 & 0 & -\sqrt{3} & 3 & \sqrt{3} & -3 \\ -1 & -\sqrt{3} & 1 & -\sqrt{3} & 2 & 0 \\ -\sqrt{3} & -3 & \sqrt{3} & -3 & 0 & 6 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 = \dfrac{4FL}{5AE} \\ v_2 = 0 \\ u_3 = 0 \\ v_3 = 0 \end{Bmatrix}$$

Recovery of Axial Forces

Element (I)

$$\begin{Bmatrix} P_1 \\ Q_1 \\ P_2 \\ Q_2 \end{Bmatrix} = \frac{AE}{L}\left[\begin{array}{cc:cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hdashline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 = \dfrac{4FL}{5AE} \\ v_2 = 0 \end{Bmatrix} = F\begin{Bmatrix} -4/5 \\ 0 \\ 4/5 \\ 0 \end{Bmatrix}$$

Element (II)

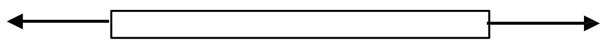
$$\begin{Bmatrix} P_2 \\ Q_2 \\ P_3 \\ Q_3 \end{Bmatrix} = \frac{AE}{4L}\left[\begin{array}{cc:cc} 1 & -\sqrt{3} & -1 & \sqrt{3} \\ -\sqrt{3} & 3 & \sqrt{3} & -3 \\ \hdashline -1 & \sqrt{3} & 1 & -\sqrt{3} \\ \sqrt{3} & -3 & -\sqrt{3} & 3 \end{array}\right]\begin{Bmatrix} u_2 = \dfrac{4FL}{5AE} \\ v_2 = 0 \\ u_3 = 0 \\ v_3 = 0 \end{Bmatrix} = F\begin{Bmatrix} 1/5 \\ -\sqrt{3}/5 \\ -1/5 \\ \sqrt{3}/5 \end{Bmatrix}$$

Element (III)

$$\begin{Bmatrix} P_1 \\ Q_1 \\ P_3 \\ Q_3 \end{Bmatrix} = \frac{AE}{4L}\left[\begin{array}{cc:cc} 1 & \sqrt{3} & -1 & -\sqrt{3} \\ \sqrt{3} & 3 & -\sqrt{3} & -3 \\ -1 & -\sqrt{3} & 1 & \sqrt{3} \\ -\sqrt{3} & -3 & \sqrt{3} & 3 \end{array}\right]\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_3 = 0 \\ v_3 = 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Stresses inside members

Element (I)

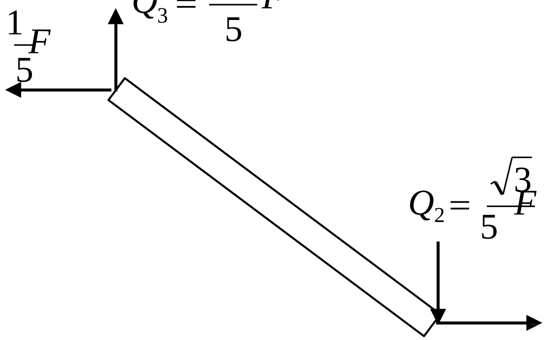


$P_1 = -\frac{4F}{5}$ $P_2 = -\frac{4F}{5}$

$$\sigma = \frac{4F}{5A}$$

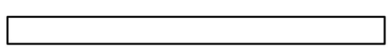
Element (II)

$P_3 = \frac{1}{5}F$ $Q_3 = \frac{\sqrt{3}}{5}F$



$Q_2 = \frac{\sqrt{3}}{5}F$ $P_2 = \frac{1}{5}F$

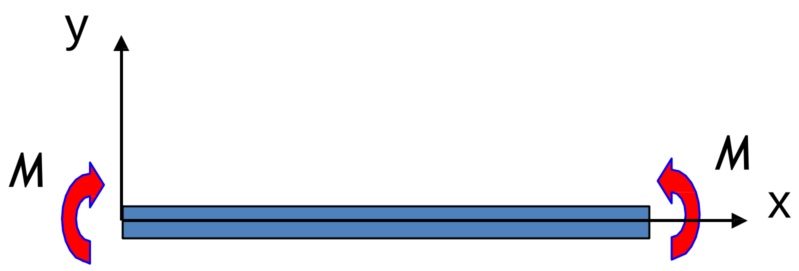
Element (III)



Finite Element Analysis of Beams

Bending Beam

Review



Pure bending problems:

Normal strain: $$\varepsilon_x = -\frac{y}{\rho}$$

Normal stress: $$\sigma_x = -\frac{Ey}{\rho}$$

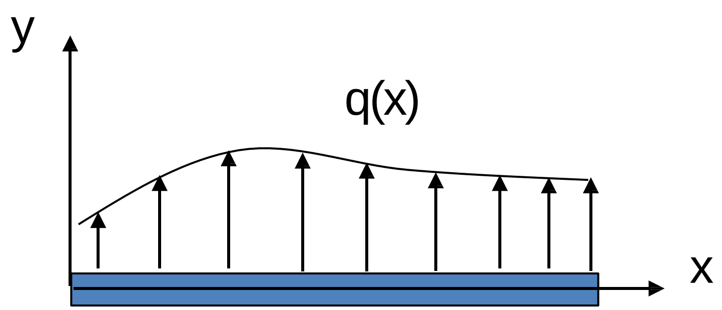
Normal stress with bending moment: $$\int -\sigma_x y dA = M$$

Moment-curvature relationship: $$\frac{1}{\rho} = \frac{M}{EI} \quad \Rightarrow \quad M = EI\frac{1}{\rho} \approx EI\frac{d^2y}{dx^2}$$

Flexure formula: $$\sigma_x = -\frac{My}{I} \qquad I = \int y^2 dA$$

Bending Beam

Review



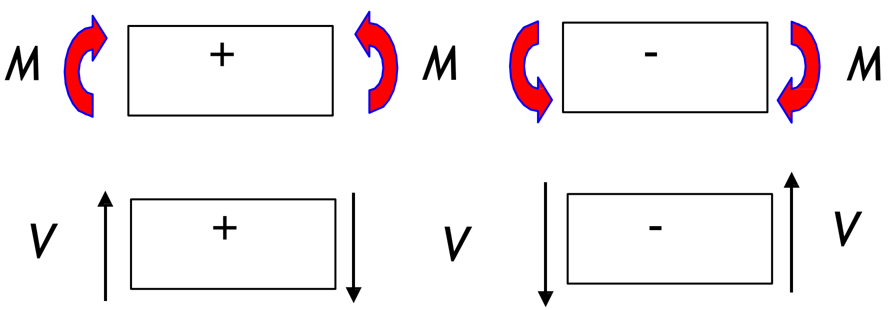
Relationship between shear force, bending moment and transverse load:

$$\frac{dV}{dx} = q \qquad \frac{dM}{dx} = V$$

Deflection:

$$EI\frac{d^4y}{dx^4} = q$$

Sign convention:



Governing Equation and Boundary Condition

- ***Governing Equation***

$$\frac{d^2}{dx^2}\left(EI \frac{d^2 v(x)}{dx^2} \right) - q(x) = 0, \qquad 0<x<L$$

- ***Boundary Conditions -----***

$$v = ?\ \&\ \frac{dv}{dx} = ?\ \&\ EI\frac{d^2v}{dx^2} = ?\ \&\ \frac{d}{dx}\left(EI\frac{d^2v}{dx^2} \right) = ?, \qquad \text{at}\ \ x = 0$$

$$v = ?\ \&\ \frac{dv}{dx} = ?\ \&\ EI\frac{d^2v}{dx^2} = ?\ \&\ \frac{d}{dx}\left(EI\frac{d^2v}{dx^2} \right) = ?, \qquad \text{at}\ \ x = L$$

Essential BCs – if ***v*** *or* $\frac{dv}{dx}$ *is specified at the boundary.*

Natural BCs – if $EI\frac{d^2v}{dx^2}$ *or* $\frac{d}{dx}\left(EI\frac{d^2v}{dx^2} \right)$ *is specified at the boundary.*

Weak Formulation for Beam Element

- ***Governing Equation***

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v(x)}{dx^2}\right)-q(x)=0, \qquad x_1 \le x \le x_2$$

- ***Weighted-Integral Formulation for one element***

$$0=\int_{x_1}^{x_2} w(x)\left[\frac{d^2}{dx^2}\left(EI\frac{d^2v(x)}{dx^2}\right)-q(x)\right]dx$$

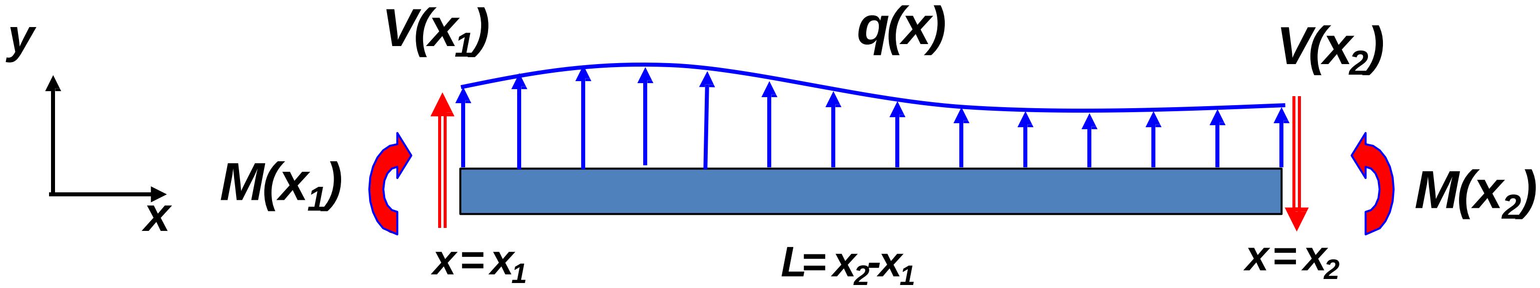
- ***Weak Form from Integration-by-Parts ----- (1st time)***

$$0=\int_{x_1}^{x_2}\left[-\frac{dw}{dx}\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right)-wq\right]dx+w\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right)\Bigg|_{x_1}^{x_2}$$

Weak Formulation

- ***Weak Form from Integration-by-Parts ----- (2nd time)***

$$0=\int_{x_1}^{x_2}\left[\frac{d^2w}{dx^2}\left(EI\frac{d^2v}{dx^2}\right)-wq\right]dx+w\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right)\Bigg|_{x_1}^{x_2}-\frac{dw}{dx}\left(EI\frac{d^2v}{dx^2}\right)\Bigg|_{x_1}^{x_2}$$

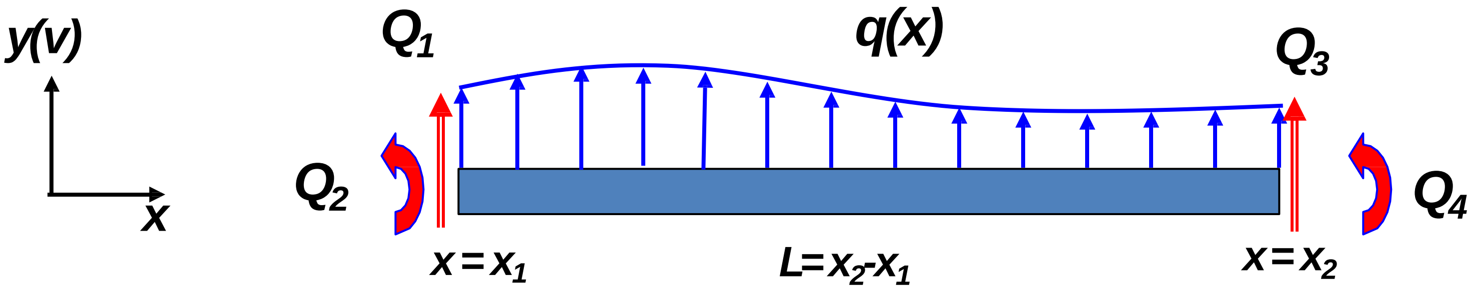


$$0=\int_{x_1}^{x_2}\left[\frac{d^2w}{dx^2}\left(EI\frac{d^2v}{dx^2}\right)-wq\right]dx+\left[wV-\frac{dw}{dx}M\right]\Bigg|_{x_1}^{x_2}$$

Weak Formulation

- ***Weak Form***

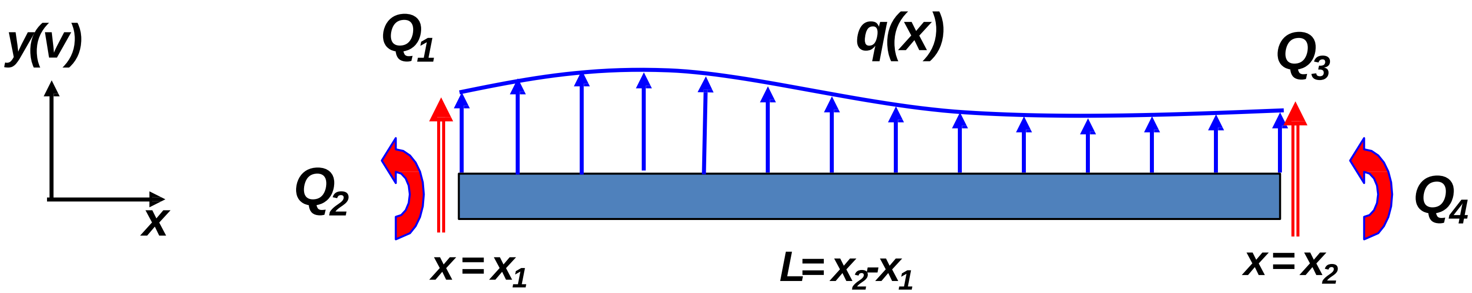
$$0=\int_{x_1}^{x_2}\left[\frac{d^2w}{dx^2}\left(EI\frac{d^2v}{dx^2}\right)-wq\right]dx+\left[wV-\frac{dw}{dx}M\right]\Bigg|_{x_1}^{x_2}$$



$$Q_1=V(x_1),\quad Q_2=-M(x_1),\quad Q_3=-V(x_2),\quad Q_4=M(x_2)$$

$$\int_{x_1}^{x_2}\left[\frac{d^2w}{dx^2}\left(EI\frac{d^2v}{dx^2}\right)-wq\right]dx=w(x_1)Q_1+w(x_2)Q_3+\frac{dw}{dx}\bigg|_1 Q_2+\frac{dw}{dx}\bigg|_2 Q_4$$

Ritz Method for Approximation



$$Let \quad v(x)=\sum_{j=1}^{n} u_j \phi_j(x) \text{ and } n=4$$

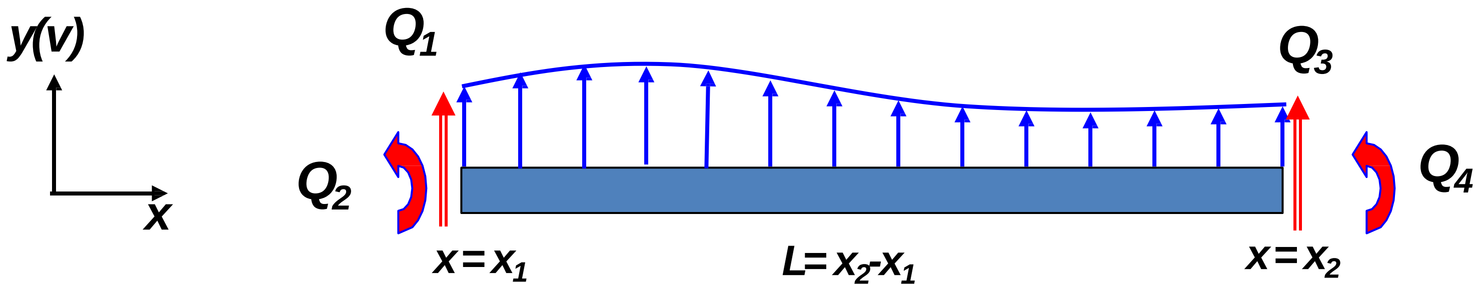
where $u_1 = v(x_1); \quad u_2 = \left.\frac{dv}{dx}\right|_{x=x_1}; \quad u_3 = v(x_2); \quad u_4 = \left.\frac{dv}{dx}\right|_{x=x_2};$

$$\int_{x_1}^{x_2}\left[\frac{d^2w}{dx^2}\left(EI\sum_{j=1}^{4} u_j \frac{d^2\phi_j}{dx^2}\right) - wq\right]dx = w(x_1)Q_1 + w(x_2)Q_3 + \left.\frac{dw}{dx}\right|_1 Q_2 + \left.\frac{dw}{dx}\right|_2 Q_4$$

Let w(x)= $\phi_i(x)$, $i=1, 2, 3, 4$

$$\int_{x_1}^{x_2}\left[\frac{d^2\phi_i}{dx^2}\left(EI\sum_{j=1}^{4} u_j \frac{d^2\phi_j}{dx^2}\right) - \phi_i q\right]dx = \phi_i(x_1)Q_1 + \phi_i(x_2)Q_3 + \left.\frac{d\phi_i}{dx}\right|_1 Q_2 + \left.\frac{d\phi_i}{dx}\right|_2 Q_4$$

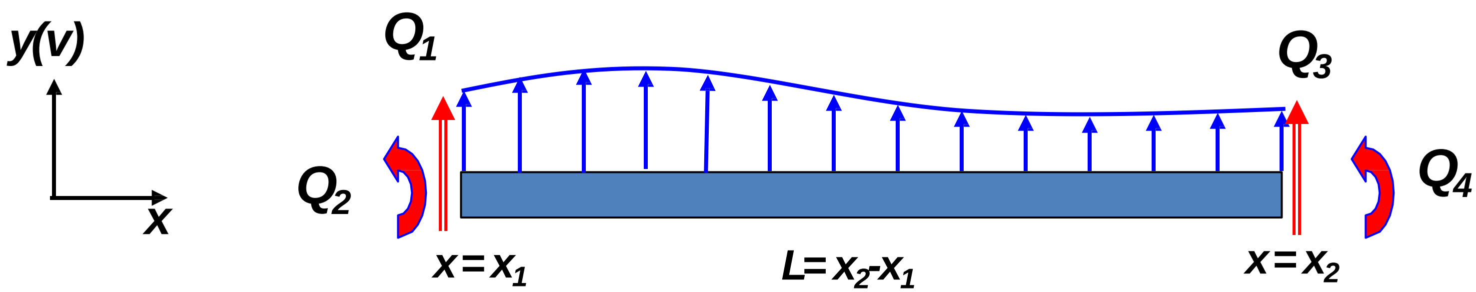
Ritz Method for Approximation



$$\left[\left(\phi_i\big|_{x_1}\right) Q_1 + \left(\frac{d\phi_i}{dx}\bigg|_{x_1}\right) Q_2 + \left(\phi_i\big|_{x_2}\right) Q_3 + \left(-\frac{d\phi_i}{dx}\bigg|_{x_2}\right) Q_4 \right] = \sum_{j=1}^{4} K_{ij} u_j - q_i$$

$$\text{where } K_{ij} = \int_{x_1}^{x_2} EI \left(\frac{d^2\phi_i}{dx^2} \frac{d^2\phi_j}{dx^2} \right) dx \text{ and } q_i = \int_{x_1}^{x_2} \phi_i q \, dx$$

Ritz Method for Approximation



$$\begin{bmatrix} \left(\phi_1\big|_{x_1}\right) & \left(\frac{d\phi_1}{dx}\Big|_{x_1}\right) & \left(\phi_1\big|_{x_2}\right) & \left(\frac{d\phi_1}{dx}\Big|_{x_2}\right) \\ \left(\phi_2\big|_{x_1}\right) & \left(\frac{d\phi_2}{dx}\Big|_{x_1}\right) & \left(\phi_2\big|_{x_2}\right) & \left(\frac{d\phi_2}{dx}\Big|_{x_2}\right) \\ \left(\phi_3\big|_{x_1}\right) & \left(\frac{d\phi_3}{dx}\Big|_{x_1}\right) & \left(\phi_3\big|_{x_2}\right) & \left(\frac{d\phi_3}{dx}\Big|_{x_2}\right) \\ \left(\phi_4\big|_{x_1}\right) & \left(\frac{d\phi_4}{dx}\Big|_{x_1}\right) & \left(\phi_4\big|_{x_2}\right) & \left(\frac{d\phi_4}{dx}\Big|_{x_2}\right) \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} - \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

$$\left(\textbf{where } K_{ij} = K_{ji}\right)$$

Selection of Shape Function

The best situation is -----

$$\begin{bmatrix} \left(\phi_1|_{x_1}\right) & \left(\left.\frac{d\phi_1}{dx}\right|_{x_1}\right) & \left(\phi_1|_{x_2}\right) & \left(\left.\frac{d\phi_1}{dx}\right|_{x_2}\right) \\ \left(\phi_2|_{x_1}\right) & \left(\left.\frac{d\phi_2}{dx}\right|_{x_1}\right) & \left(\phi_2|_{x_2}\right) & \left(\left.\frac{d\phi_2}{dx}\right|_{x_2}\right) \\ \left(\phi_3|_{x_1}\right) & \left(\left.\frac{d\phi_3}{dx}\right|_{x_1}\right) & \left(\phi_3|_{x_2}\right) & \left(\left.\frac{d\phi_3}{dx}\right|_{x_2}\right) \\ \left(\phi_4|_{x_1}\right) & \left(\left.\frac{d\phi_4}{dx}\right|_{x_1}\right) & \left(\phi_4|_{x_2}\right) & \left(\left.\frac{d\phi_4}{dx}\right|_{x_2}\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ Interpolation Properties

$$\Rightarrow \quad \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{12} & K_{22} & K_{23} & K_{24} \\ K_{13} & K_{23} & K_{33} & K_{34} \\ K_{14} & K_{24} & K_{34} & K_{44} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} - \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

Derivation of Shape Function for Beam Element – Local Coordinates

How to select ϕ_i???

$$v(\xi) = \tilde{u}_1\phi_1 + \tilde{u}_2\phi_2 + \tilde{u}_3\phi_3 + \tilde{u}_4\phi_4$$

and
$$\frac{dv(\xi)}{d\xi} = \tilde{u}_1\frac{d\phi_1}{d\xi} + \tilde{u}_2\frac{d\phi_2}{d\xi} + \tilde{u}_3\frac{d\phi_3}{d\xi} + \tilde{u}_4\frac{d\phi_4}{d\xi}$$

where
$$\tilde{u}_1 = v_1 \quad \tilde{u}_2 = \frac{dv_1}{d\xi} \quad \tilde{u}_3 = v_2 \quad \tilde{u}_4 = \frac{dv_2}{d\xi}$$

Let
$$\phi_i = a_i + b_i\xi + c_i\xi^2 + d_i\xi^3$$

Find coefficients to satisfy the interpolation properties.

Derivation of Shape Function for Beam Element

How to select ϕ_i???

e.g. Let $$\phi_1 = a_1 + b_1\xi + c_1\xi^2 + d_1\xi^3$$

$$\Longrightarrow \quad \phi_1 = \frac{1}{4}(1-\xi)^2(2+\xi)$$

Similarly

$$\phi_2 = \frac{1}{4}(1-\xi)^2(1+\xi)$$

$$\phi_3 = \frac{1}{4}(1+\xi)^2(2-\xi)$$

$$\phi_2 = \frac{1}{4}(1+\xi)^2(\xi-1)$$

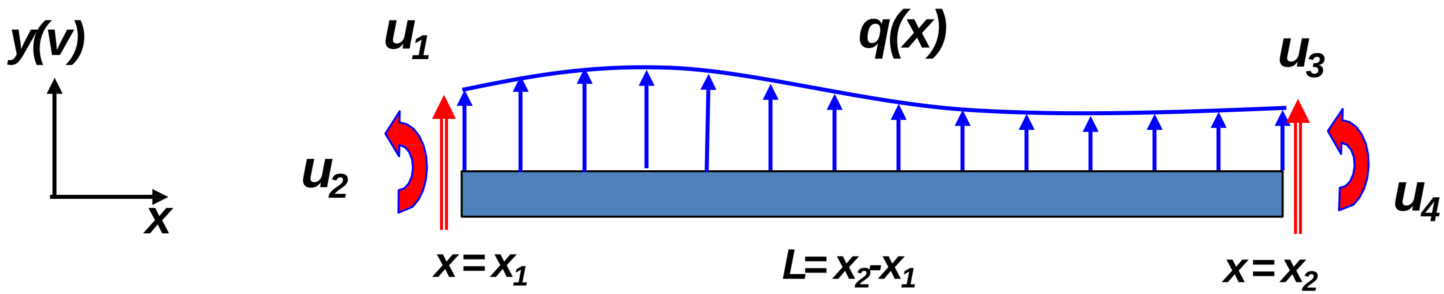
Derivation of Shape Function for Beam Element

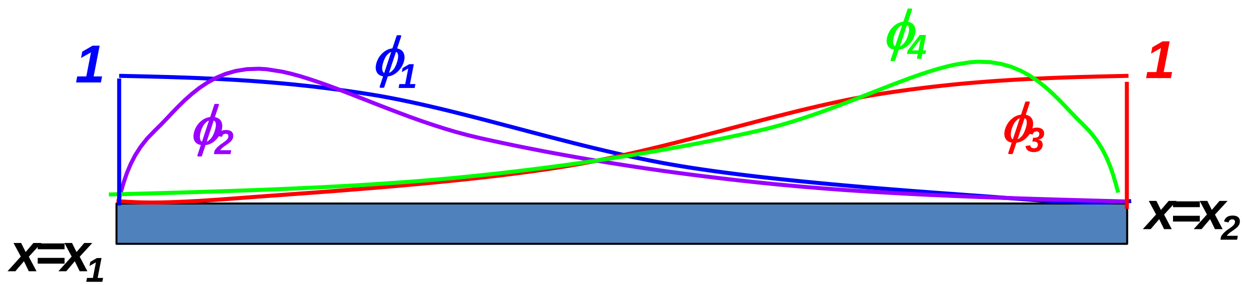
In the global coordinates:

$$v(x) = v_1\phi_1(x) + \frac{l}{2}\frac{dv_1}{dx}\phi_2(x) + v_2\phi_3(x) + \frac{l}{2}\frac{dv_2}{dx}\phi_4(x)$$

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} 1 - 3\left(\dfrac{x - x_1}{x_2 - x_1}\right)^2 + 2\left(\dfrac{x - x_1}{x_2 - x_1}\right)^3 \\ \dfrac{2}{l}(x - x_1)\left(1 - \dfrac{x - x_1}{x_2 - x_1}\right)^2 \\ 3\left(\dfrac{x - x_1}{x_2 - x_1}\right)^2 - 2\left(\dfrac{x - x_1}{x_2 - x_1}\right)^3 \\ \dfrac{2}{l}(x - x_1)\left[\left(\dfrac{x - x_1}{x_2 - x_1}\right)^2 - \dfrac{x - x_1}{x_2 - x_1}\right] \end{Bmatrix}$$

Element Equations of 4th Order 1-D Model

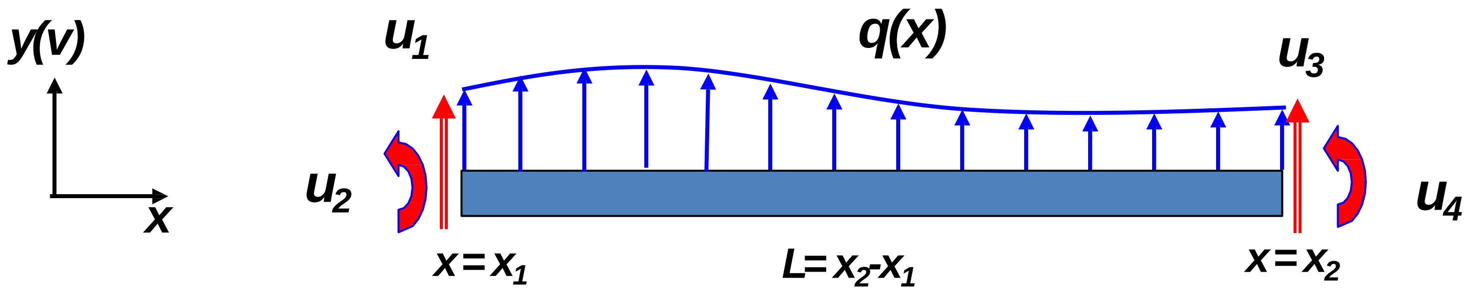




$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} + \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{12} & K_{22} & K_{23} & K_{24} \\ K_{13} & K_{23} & K_{33} & K_{34} \\ K_{14} & K_{24} & K_{34} & K_{44} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$where \quad K_{ij} = \int_{x_1}^{x_2} EI \left(\frac{d^2\phi_i}{dx^2} \frac{d^2\phi_j}{dx^2} \right) dx = K_{ji} \quad and \quad q_i = \int_{x_1}^{x_2} \phi_i q \, dx$$

Element Equations of 4th Order 1-D Model

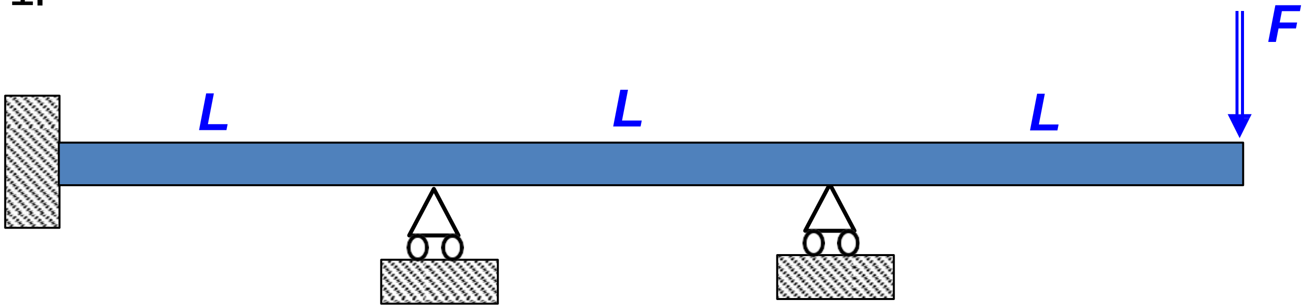


$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} + \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \frac{2EI}{L^3} \begin{bmatrix} 6 & 3L & -6 & 3L \\ 3L & 2L^2 & -3L & L^2 \\ -6 & -3L & 6 & -3L \\ 3L & L^2 & -3L & 2L^2 \end{bmatrix} \begin{Bmatrix} u_1 = v_1 \\ u_2 = \theta_1 \\ u_3 = v_2 \\ u_4 = \theta_2 \end{Bmatrix}$$

$$where \quad q_i = \int_{x_1}^{x_2} \phi_i q\, dx$$

Finite Element Analysis of 1-D Problems - Applications

Example 1.



Governing equation:

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2 v}{dx^2}\right) - q(x) = 0 \qquad 0 < x < L$$

Weak form for one element

$$\int_{x_1}^{x_2}\left(EI\,\frac{d^2 w}{dx^2}\frac{d^2 v}{dx^2} - wq\right)dx - w(x_1)Q_1 - \left.\frac{dw}{dx}\right|_{x_1} Q_2 - w(x_2)Q_3 - \left.\frac{dw}{dx}\right|_{x_2} Q_4 = 0$$

where $\quad Q_1 = V(x_1) \quad Q_2 = -M(x_1) \quad Q_3 = -V(x_2) \quad Q_4 = M(x_2)$

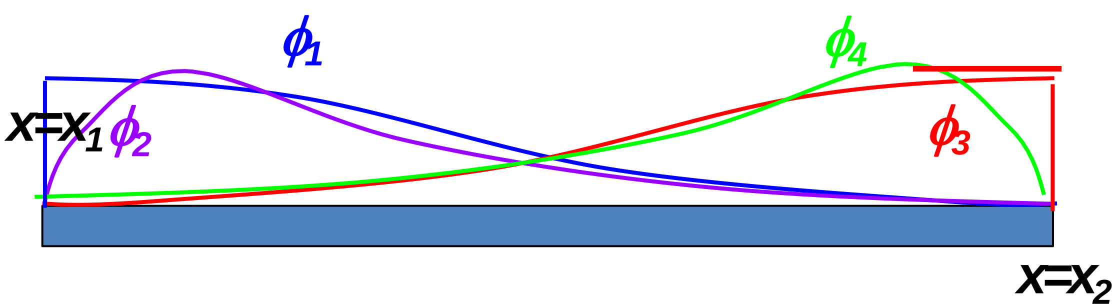
Finite Element Analysis of 1-D Problems

Example 1.

Approximation function:

$$v(x) = v_1\phi_1(x) + \frac{l}{2}\frac{dv_1}{dx}\phi_2(x) + v_2\phi_3(x) + \frac{l}{2}\frac{dv_2}{dx}\phi_4(x)$$

$$\begin{Bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{Bmatrix} = \begin{Bmatrix} 1-3\left(\frac{x-x_1}{x_2-x_1}\right)^2 + 2\left(\frac{x-x_1}{x_2-x_1}\right)^3 \\ \frac{2}{l}(x-x_1)\left(1-\frac{x-x_1}{x_2-x_1}\right)^2 \\ 3\left(\frac{x-x_1}{x_2-x_1}\right)^2 - 2\left(\frac{x-x_1}{x_2-x_1}\right)^3 \\ \frac{2}{l}(x-x_1)\left[\left(\frac{x-x_1}{x_2-x_1}\right)^2 - \frac{x-x_1}{x_2-x_1}\right] \end{Bmatrix}$$



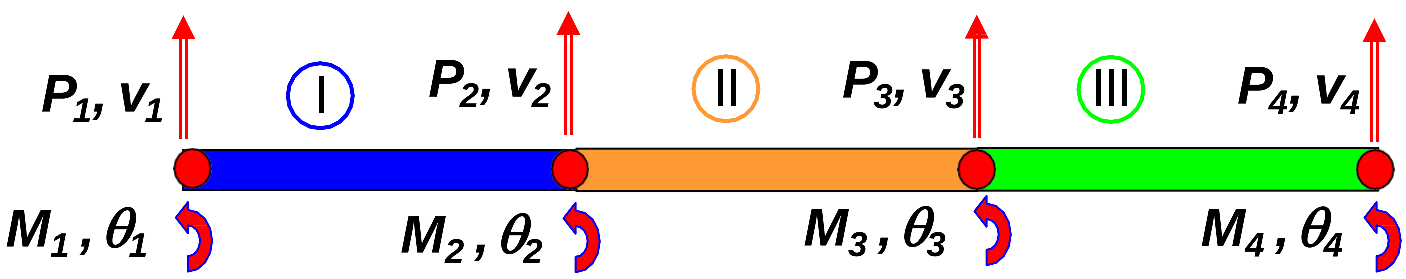
Finite Element Analysis of 1-D Problems

Example 1.

Finite element model:

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \frac{2EI}{L^3} \begin{bmatrix} 6 & 3L & -6 & 3L \\ 3L & 2L^2 & -3L & L^2 \\ -6 & -3L & 6 & -3L \\ 3L & L^2 & -3L & 2L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

Discretization:



Matrix Assembly of Multiple Beam Elements

Element I

$$\begin{Bmatrix} Q_1^I \\ Q_2^I \\ Q_3^I \\ Q_4^I \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \frac{2EI}{L^3} \left[\begin{array}{cccc:cccc} 6 & 3L & -6 & 3L & 0 & 0 & 0 & 0 \\ 3L & 2L^2 & -3L & L^2 & 0 & 0 & 0 & 0 \\ -6 & -3L & 6 & -3L & 0 & 0 & 0 & 0 \\ 3L & L^2 & -3L & 2L^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ v_4 \\ \theta_4 \end{Bmatrix}$$

Element II

$$\begin{Bmatrix} 0 \\ 0 \\ Q_1^{II} \\ Q_2^{II} \\ Q_3^{II} \\ Q_4^{II} \\ 0 \\ 0 \end{Bmatrix} = \frac{2EI}{L^3} \left[\begin{array}{cc:cccc:cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 3L & -6 & 3L & 0 & 0 \\ 0 & 0 & 3L & 2L^2 & -3L & L^2 & 0 & 0 \\ 0 & 0 & -6 & -3L & 6 & -3L & 0 & 0 \\ 0 & 0 & 3L & L^2 & -3L & 2L^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ v_4 \\ \theta_4 \end{Bmatrix}$$

Matrix Assembly of Multiple Beam Elements

Element III

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ Q_1^{III} \\ Q_2^{III} \\ Q_3^{III} \\ Q_4^{III} \end{Bmatrix} = \frac{2EI}{L^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 3L & -6 & 3L \\ 0 & 0 & 0 & 0 & 3L & 2L^2 & -3L & L^2 \\ 0 & 0 & 0 & 0 & -6 & -3L & 6 & -3L \\ 0 & 0 & 0 & 0 & 3L & L^2 & -3L & 2L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ v_4 \\ \theta_4 \end{Bmatrix}$$

$$\begin{Bmatrix} P_1 \\ M_1 \\ P_2 \\ M_2 \\ P_3 \\ M_3 \\ P_4 \\ M_4 \end{Bmatrix} = \frac{2EI}{L^3}\begin{bmatrix} 6 & 3L & -6 & 3L & 0 & 0 & 0 & 0 \\ 3L & 2L^2 & -3L & L^2 & 0 & 0 & 0 & 0 \\ -6 & -3L & 6+6 & -3L+3L & -6 & 3L & 0 & 0 \\ 3L & L^2 & -3L+3L & 2L^2+2L^2 & -3L & L^2 & 0 & 0 \\ 0 & 0 & -6 & -3L & 6+6 & -3L+3L & -6 & 3L \\ 0 & 0 & 3L & L^2 & -3L+3L & 2L^2+2L^2 & -3L & L^2 \\ 0 & 0 & 0 & 0 & -6 & -3L & 6 & -3L \\ 0 & 0 & 0 & 0 & 3L & L^2 & -3L & 2L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ v_4 \\ \theta_4 \end{Bmatrix}$$

Solution Procedures

Apply known boundary conditions

$$
\begin{Bmatrix} P_1 = ? \\ M_1 = ? \\ P_2 = ? \\ M_2 = 0 \\ P_3 = ? \\ M_3 = 0 \\ P_4 = -F \\ M_4 = 0 \end{Bmatrix}
= \frac{2EI}{L^3}
\begin{bmatrix}
6 & 3L & -6 & 3L & 0 & 0 & 0 & 0 \\
3L & 2L^2 & -3L & L^2 & 0 & 0 & 0 & 0 \\
-6 & -3L & 12 & 0 & -6 & 3L & 0 & 0 \\
3L & L^2 & 0 & 4L^2 & -3L & L^2 & 0 & 0 \\
0 & 0 & -6 & -3L & 12 & 0 & -6 & 3L \\
0 & 0 & 3L & L^2 & 0 & 4L^2 & -3L & L^2 \\
0 & 0 & 0 & 0 & -6 & -3L & 6 & -3L \\
0 & 0 & 0 & 0 & 3L & L^2 & -3L & 2L^2
\end{bmatrix}
\begin{Bmatrix} v_1 = 0 \\ \theta_1 = 0 \\ v_2 = 0 \\ \theta_2 = ? \\ v_3 = 0 \\ \theta_3 = ? \\ v_4 = ? \\ \theta_4 = ? \end{Bmatrix}
$$

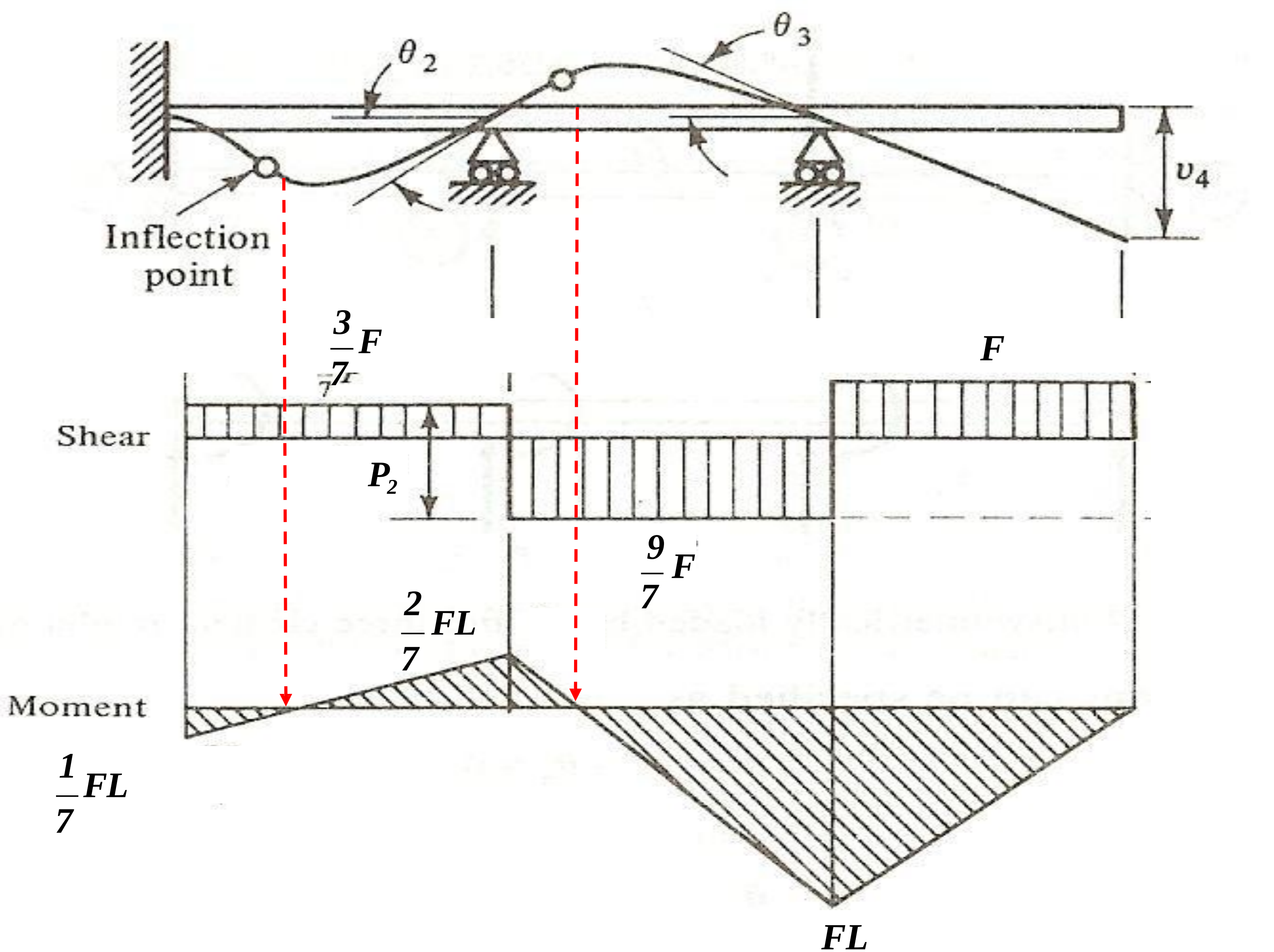
$$
\begin{Bmatrix} M_2 = 0 \\ M_3 = 0 \\ P_4 = -F \\ M_4 = 0 \\ P_1 = ? \\ M_1 = ? \\ P_2 = ? \\ P_3 = ? \end{Bmatrix}
= \frac{2EI}{L^3}
\begin{bmatrix}
3L & L^2 & 0 & 4L^2 & -3L & L^2 & 0 & 0 \\
0 & 0 & 3L & L^2 & 0 & 4L^2 & -3L & L^2 \\
0 & 0 & 0 & 0 & -6 & -3L & 6 & -3L \\
0 & 0 & 0 & 0 & 3L & L^2 & -3L & 2L^2 \\
\hline
6 & 3L & -6 & 3L & 0 & 0 & 0 & 0 \\
3L & 2L^2 & -3L & L^2 & 0 & 0 & 0 & 0 \\
-6 & -3L & 12 & 0 & -6 & 3L & 0 & 0 \\
0 & 0 & -6 & -3L & 12 & 0 & -6 & 3L
\end{bmatrix}
\begin{Bmatrix} v_1 = 0 \\ \theta_1 = 0 \\ v_2 = 0 \\ \theta_2 = ? \\ v_3 = 0 \\ \theta_3 = ? \\ v_4 = ? \\ \theta_4 = ? \end{Bmatrix}
$$

Solution Procedures

$$
\begin{Bmatrix} M_2 = 0 \\ M_3 = 0 \\ P_4 = -F \\ M_4 = 0 \\ P_1 = ? \\ M_1 = ? \\ P_2 = ? \\ P_3 = ? \end{Bmatrix}
= \frac{2EI}{L^3}
\left[\begin{array}{ccc|c|c|ccc}
3L & L^2 & 0 & -3L & 4L^2 & L^2 & 0 & 0 \\
0 & 0 & 3L & 0 & L^2 & 4L^2 & -3L & L^2 \\
0 & 0 & 0 & -6 & 0 & -3L & 6 & -3L \\
0 & 0 & 0 & 3L & 0 & L^2 & -3L & 2L^2 \\
\hline
6 & 3L & -6 & 0 & 3L & 0 & 0 & 0 \\
3L & 2L^2 & -3L & 0 & L^2 & 0 & 0 & 0 \\
-6 & -3L & 12 & -6 & 0 & 3L & 0 & 0 \\
0 & 0 & -6 & -12 & -3L & 0 & -6 & 3L
\end{array}\right]
\begin{Bmatrix} v_1 = 0 \\ \theta_1 = 0 \\ v_2 = 0 \\ v_3 = 0 \\ \theta_2 = ? \\ \theta_3 = ? \\ v_4 = ? \\ \theta_4 = ? \end{Bmatrix}
$$

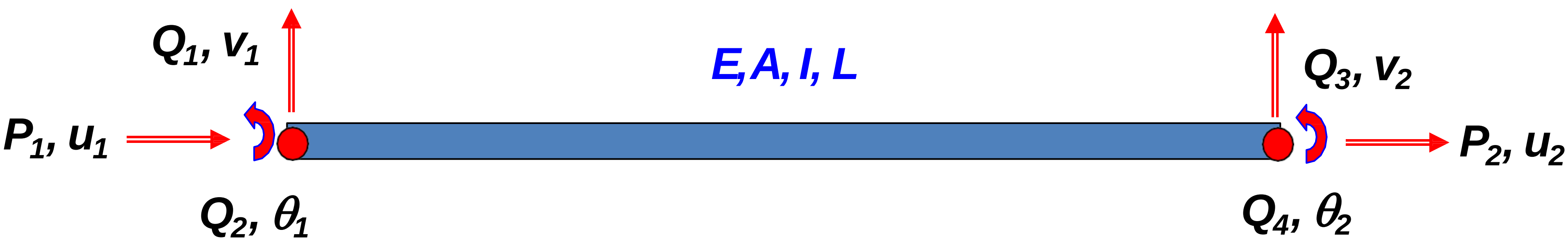
$$
\begin{Bmatrix} M_2 = 0 \\ M_3 = 0 \\ P_4 = -F \\ M_4 = 0 \end{Bmatrix}
= \frac{2EI}{L^3}
\begin{bmatrix}
4L^2 & L^2 & 0 & 0 \\
L^2 & 4L^2 & -3L & L^2 \\
0 & -3L & 6 & -3L \\
0 & L^2 & -3L & 2L^2
\end{bmatrix}
\begin{Bmatrix} \theta_2 = ? \\ \theta_3 = ? \\ v_4 = ? \\ \theta_4 = ? \end{Bmatrix}
\qquad
\begin{Bmatrix} P_1 = ? \\ M_1 = ? \\ P_2 = ? \\ P_3 = ? \end{Bmatrix}
= \frac{2EI}{L^3}
\begin{bmatrix}
3L & 0 & 0 & 0 \\
L^2 & 0 & 0 & 0 \\
0 & 3L & 0 & 0 \\
-3L & 0 & -6 & 3L
\end{bmatrix}
\begin{Bmatrix} \theta_2 \\ \theta_3 \\ v_4 \\ \theta_4 \end{Bmatrix}
$$

Shear Resultant & Bending Moment Diagram



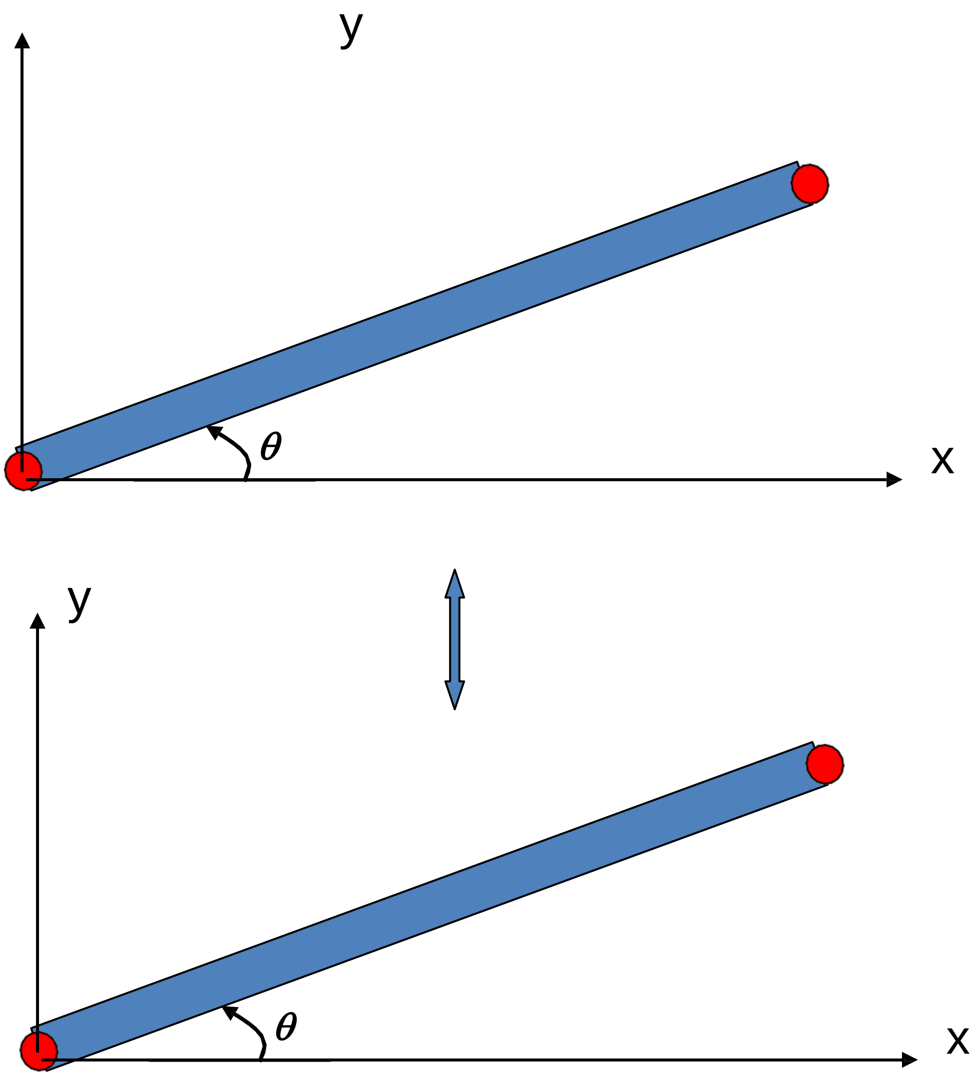
Plane Flame

Frame: combination of bar and beam



$$
\begin{Bmatrix} P_1 \\ Q_1 \\ Q_2 \\ P_2 \\ Q_3 \\ Q_4 \end{Bmatrix}
=
\begin{bmatrix}
\frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\
0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
-\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\
0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
\begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix}
$$

Finite Element Model of an Arbitrarily Oriented Frame



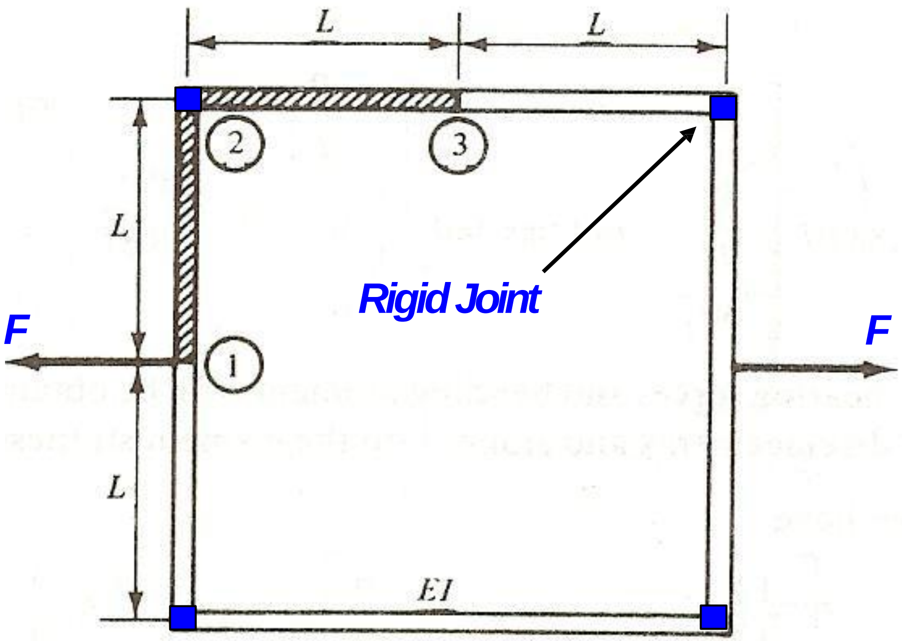
Finite Element Model of an Arbitrarily Oriented Frame

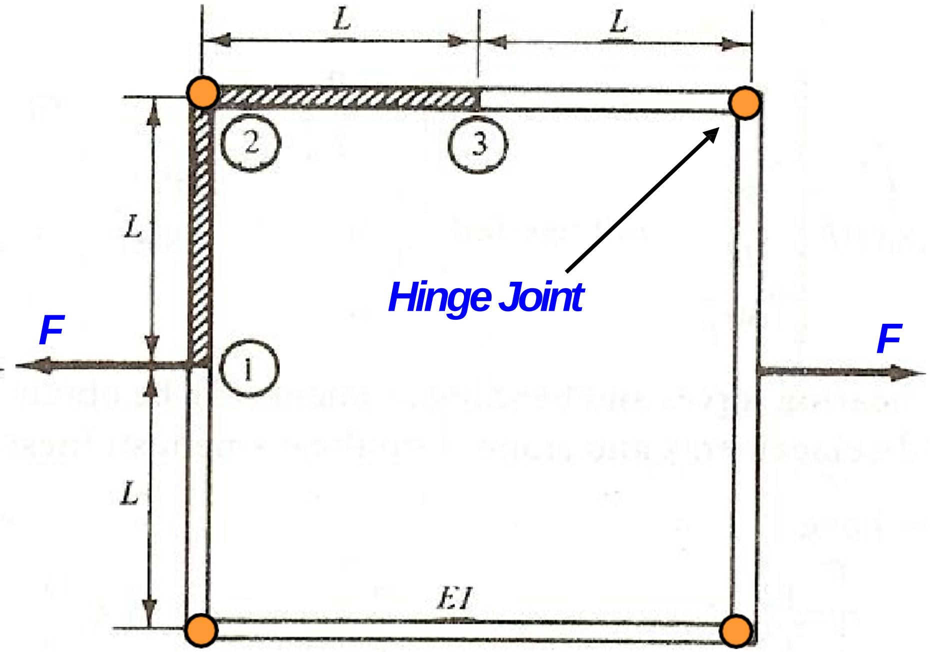
local

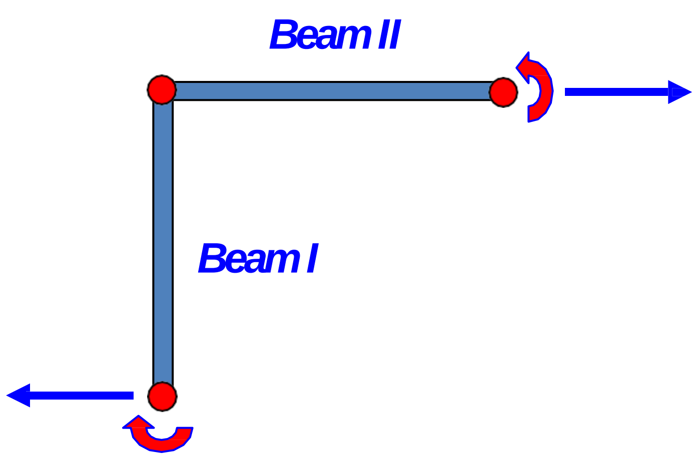
global

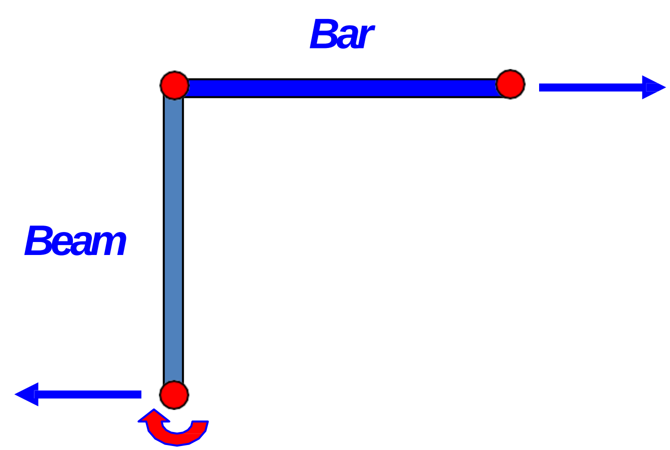
$$
\begin{Bmatrix} P_1 \\ Q_1 \\ Q_2 \\ P_2 \\ Q_3 \\ Q_4 \end{Bmatrix}
=
\left[\begin{array}{ccc:ccc}
\frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} & -\frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} \\
0 & \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 \\
-\frac{6EI}{L^2} & 0 & \frac{4EI}{L} & \frac{6EI}{L^2} & 0 & \frac{2EI}{L} \\
\hdashline
-\frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} & \frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} \\
0 & \frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 \\
-\frac{6EI}{L^2} & 0 & \frac{2EI}{L} & \frac{6EI}{L^2} & 0 & \frac{4EI}{L}
\end{array}\right]
\begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix}
$$

Plane Frame Analysis - Example

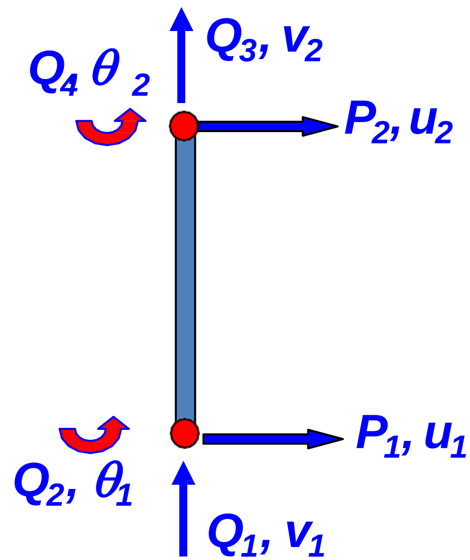






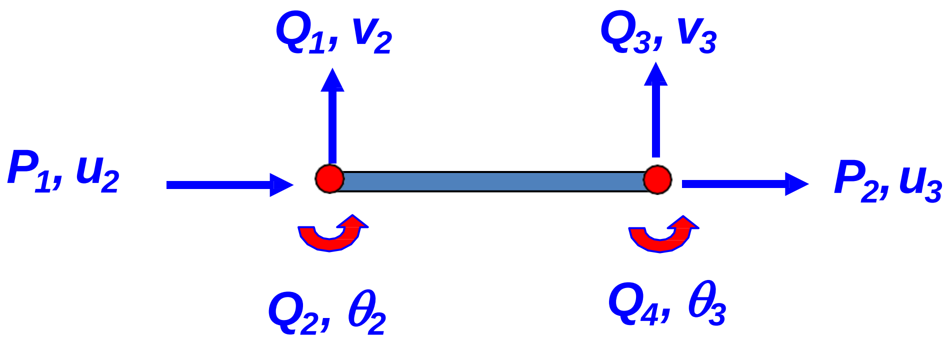


Plane Frame Analysis



$$
\begin{Bmatrix} P_1 \\ Q_1 \\ Q_2 \\ P_2 \\ Q_3 \\ Q_4 \end{Bmatrix}^I =
\left[\begin{array}{ccc|ccc}
\frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} & -\frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} \\
0 & \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 \\
-\frac{6EI}{L^2} & 0 & \frac{4EI}{L} & \frac{6EI}{L^2} & 0 & \frac{2EI}{L} \\
\hline
-\frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} & \frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} \\
0 & -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 \\
-\frac{6EI}{L^2} & 0 & \frac{2EI}{L} & \frac{6EI}{L^2} & 0 & \frac{4EI}{L}
\end{array}\right]^I
\begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix}
$$

Plane Frame Analysis



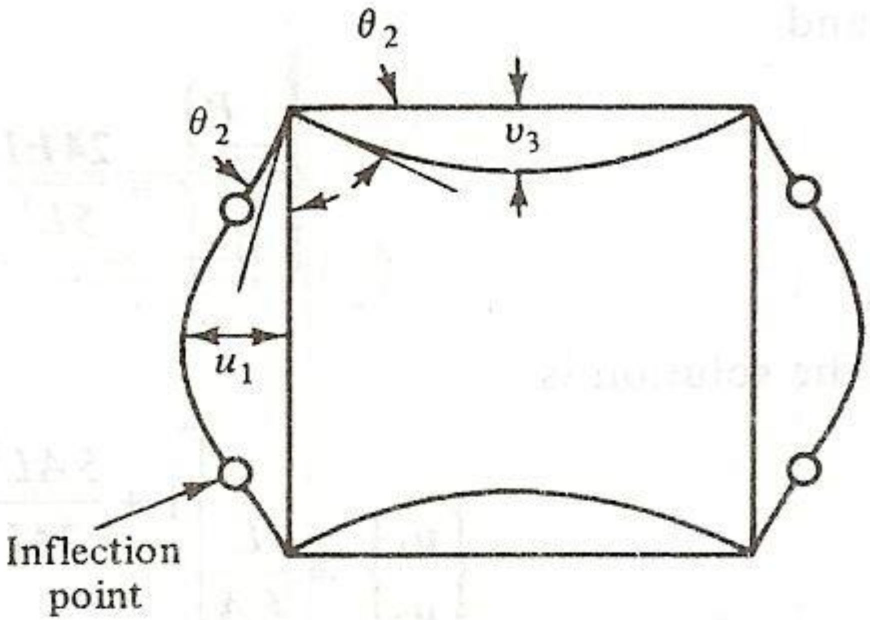
$$\begin{Bmatrix} P_1 \\ Q_2 \\ Q_2 \\ P_2 \\ Q_3 \\ Q_4 \end{Bmatrix}^{II} = \left[\begin{array}{ccc:ccc} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ \hdashline -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{array}\right] \begin{Bmatrix} u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_4 \end{Bmatrix}$$

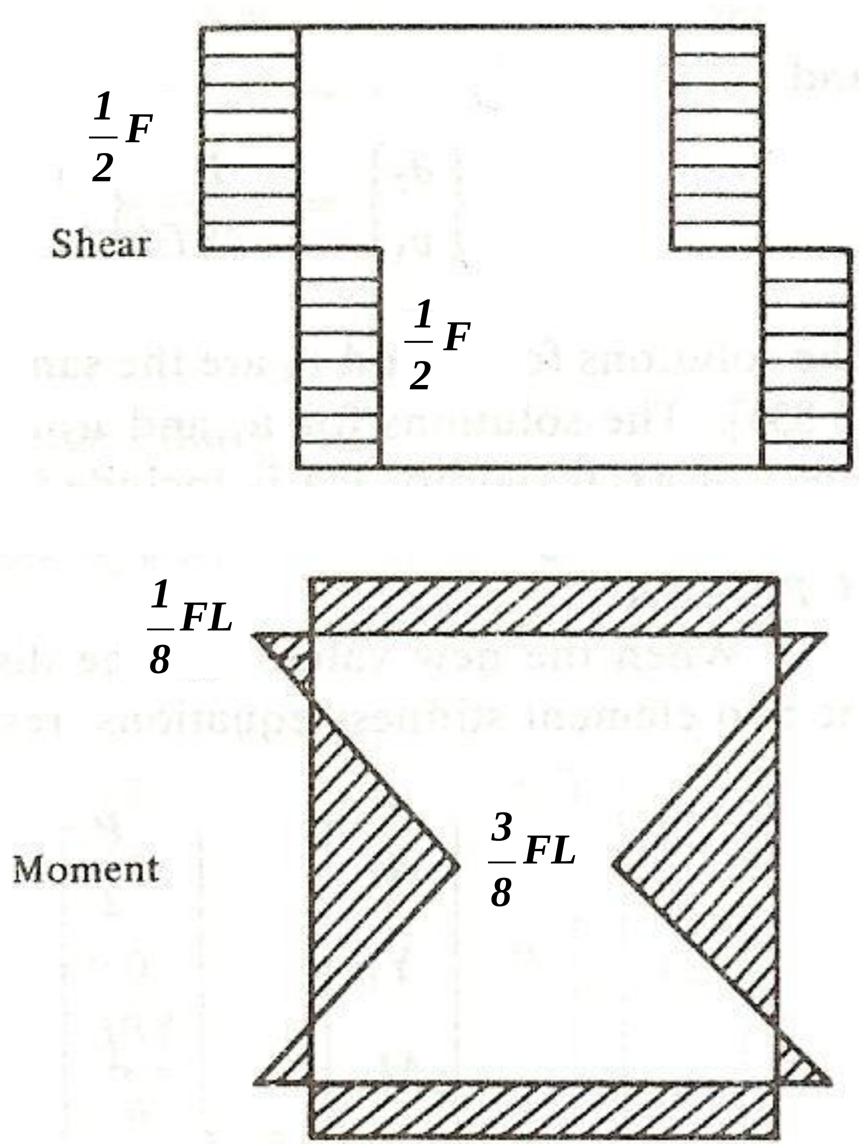
Plane Frame Analysis

$$
\begin{Bmatrix} P_1 \\ Q_1 \\ Q_2 \\ P_2 \\ Q_3 \\ Q_4 \end{Bmatrix}^{I}
=
\left[\begin{array}{ccc|ccc}
\frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} & -\frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} \\
0 & \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 \\
-\frac{6EI}{L^2} & 0 & \frac{4EI}{L} & \frac{6EI}{L^2} & 0 & \frac{2EI}{L} \\
\hline
-\frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} & \frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} \\
0 & -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 \\
-\frac{6EI}{L^2} & 0 & \frac{2EI}{L} & \frac{6EI}{L^2} & 0 & \frac{4EI}{L}
\end{array}\right]^{I}
\begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix}
$$

$$
\begin{Bmatrix} P_1 \\ Q_2 \\ Q_2 \\ P_2 \\ Q_3 \\ Q_4 \end{Bmatrix}^{II}
=
\left[\begin{array}{ccc|ccc}
\frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\
0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
\hline
-\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\
0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{array}\right]
\begin{Bmatrix} u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_4 \end{Bmatrix}
$$

Plane Frame Analysis



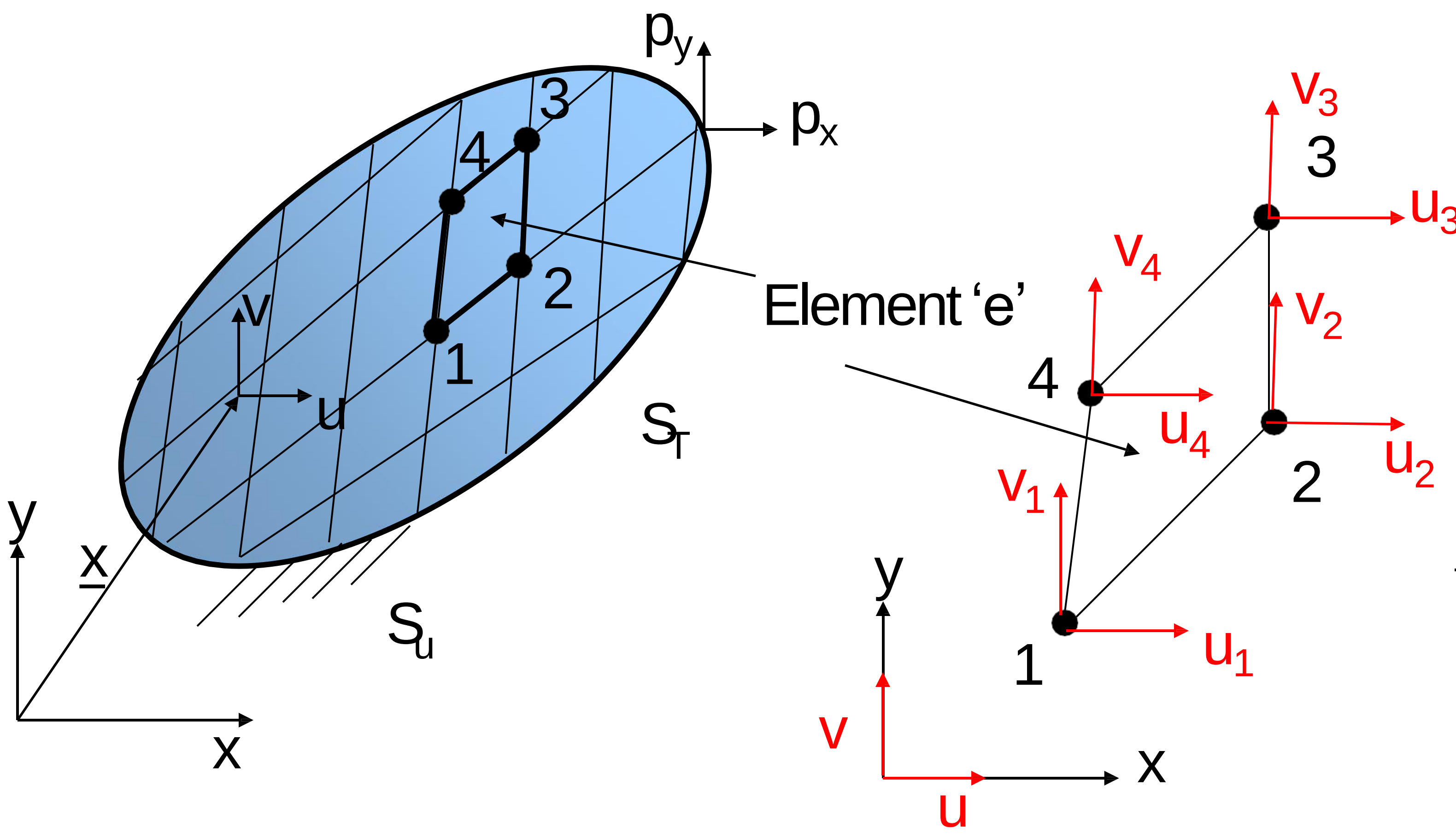


UNIT- 3

Finite element analysis constant strain triangle

Finite element formulation for 2D:

Step 1: Divide the body into **finite elements** connected to each other through special points ("**nodes**")



$$\underline{d} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$u(x,y) \approx N_1(x,y)u_1 + N_2(x,y)u_2 + N_3(x,y)u_3 + N_4(x,y)u_4$$

$$v(x,y) \approx N_1(x,y)v_1 + N_2(x,y)v_2 + N_3(x,y)v_3 + N_4(x,y)v_4$$

$$\underline{u} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \left[\begin{array}{cc|cc|cc|cc} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{array}\right] \left\{\begin{array}{c} u_1 \\ v_1 \\ \hline u_2 \\ v_2 \\ \hline u_3 \\ v_3 \\ \hline u_4 \\ v_4 \end{array}\right\}$$

$$\boxed{\underline{u} = \underline{N}\ \underline{d}}$$

TASK 2: APPROXIMATE THE STRAIN and STRESS WITHIN EACH ELEMENT

Approximation of the strain in element 'e'

$$\varepsilon_x = \frac{\partial u(x, y)}{\partial x} \approx \frac{\partial N_1(x, y)}{\partial x} u_1 + \frac{\partial N_2(x, y)}{\partial x} u_2 + \frac{\partial N_3(x, y)}{\partial x} u_3 + \frac{\partial N_4(x, y)}{\partial x} u_4$$

$$\varepsilon_y = \frac{\partial v(x, y)}{\partial y} \approx \frac{\partial N_1(x, y)}{\partial y} v_1 + \frac{\partial N_2(x, y)}{\partial y} v_2 + \frac{\partial N_3(x, y)}{\partial y} v_3 + \frac{\partial N_4(x, y)}{\partial y} v_4$$

$$\gamma_{xy} = \frac{\partial u(x, y)}{\partial y} + \frac{\partial v(x, y)}{\partial x} \approx \frac{\partial N_1(x, y)}{\partial y} u_1 + \frac{\partial N_1(x, y)}{\partial x} v_1 + \ldots\ldots$$

$$\underline{\varepsilon}=\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$$

$$=\underbrace{\begin{bmatrix}\frac{\partial N_1(x,y)}{\partial x} & 0 & \frac{\partial N_2(x,y)}{\partial x} & 0 & \frac{\partial N_3(x,y)}{\partial x} & 0 & \frac{\partial N_4(x,y)}{\partial x} & 0\\ 0 & \frac{\partial N_1(x,y)}{\partial y} & 0 & \frac{\partial N_2(x,y)}{\partial y} & 0 & \frac{\partial N_3(x,y)}{\partial y} & 0 & \frac{\partial N_4(x,y)}{\partial y}\\ \frac{\partial N_1(x,y)}{\partial y} & \frac{\partial N_1(x,y)}{\partial x} & \frac{\partial N_2(x,y)}{\partial y} & \frac{\partial N_2(x,y)}{\partial x} & \frac{\partial N_3(x,y)}{\partial y} & \frac{\partial N_3(x,y)}{\partial x} & \frac{\partial N_4(x,y)}{\partial y} & \frac{\partial N_4(x,y)}{\partial x}\end{bmatrix}}_{\underline{B}}\begin{Bmatrix}u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4\end{Bmatrix}$$

$$\boxed{\underline{\varepsilon}=\underline{B}\ \underline{d}}$$

Summary: For each element

Displacement approximation in terms of shape functions

$$\underline{u} = \underline{N}\ \underline{d}$$

Strain approximation in terms of strain-displacement matrix

$$\underline{\varepsilon} = \underline{B}\ \underline{d}$$

Stress approximation

$$\sigma = \underline{DB}\ \underline{d}$$

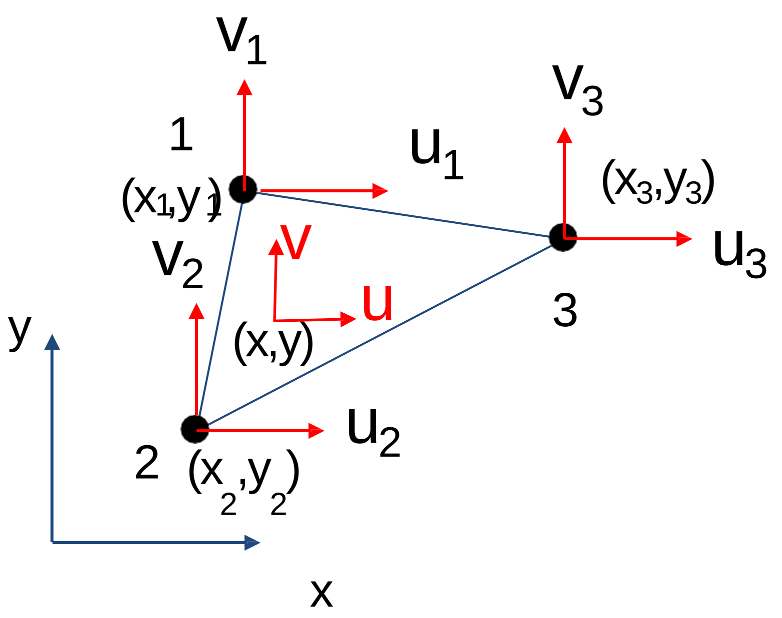
Element stiffness matrix

$$\underline{k} = \int_{V^e} \underline{B}^T \underline{D}\,\underline{B}\, dV$$

Element nodal load vector

$$\underline{f} = \underbrace{\int_{V^e} \underline{N}^T \underline{X}\, dV}_{\underline{f}_b} + \underbrace{\int_{S_T^e} \underline{N}^T \underline{T}_S\, dS}_{\underline{f}_S}$$

<u>Constant Strain Triangle (CST) :</u> Simplest 2D finite element



- 3 nodes per element
- 2 dofs per node (each node can move in x- and y-directions)
- Hence 6 dofs per element

The <u>displacement approximation</u> in terms of shape functions is

$$u(x,y) \approx N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v(x,y) \approx N_1 v_1 + N_2 v_2 + N_3 v_3$$

$$\underline{u} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \left[\begin{array}{cc:cc:cc} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{array}\right] \begin{Bmatrix} u_1 \\ v_1 \\ \hdashline u_2 \\ v_2 \\ \hdashline u_3 \\ v_3 \end{Bmatrix}$$

$$\boxed{\underline{u}_{2\times1} = \underline{N}_{2\times6}\, \underline{d}_{6\times1}}$$

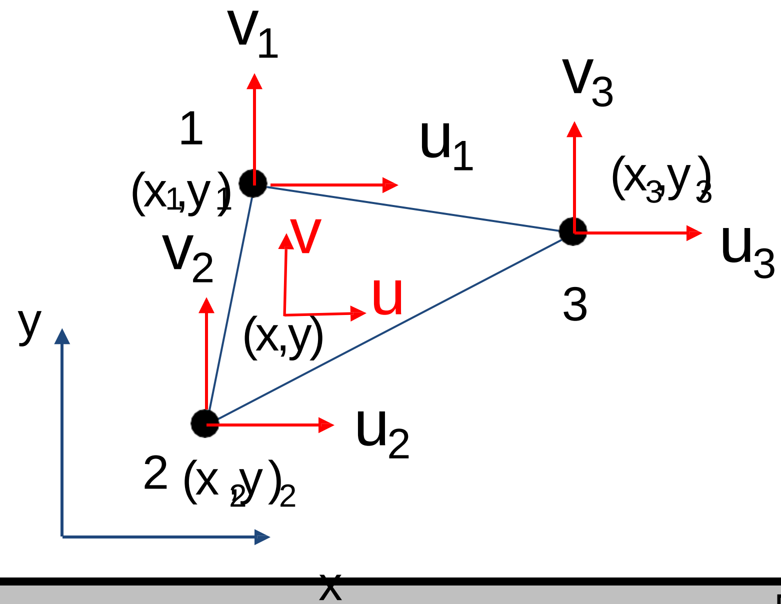
$$\underline{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

Formula for the shape functions are

$$N_1 = \frac{a_1 + b_1 x + c_1 y}{2A}$$

$$N_2 = \frac{a_2 + b_2 x + c_2 y}{2A}$$

$$N_3 = \frac{a_3 + b_3 x + c_3 y}{2A}$$



where

$$A = area\ of\ triangle = \frac{1}{2}\det\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

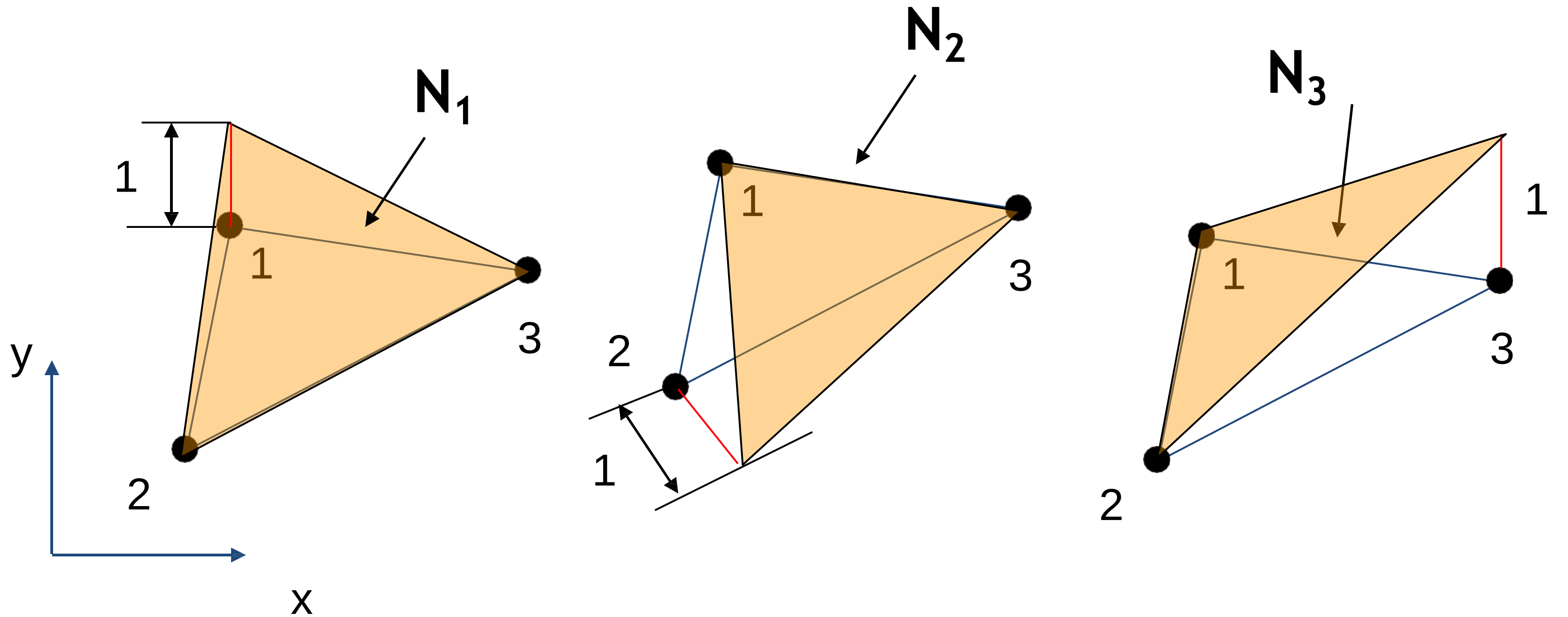
$$a_1 = x_2 y_3 - x_3 y_2 \qquad b_1 = y_2 - y_3 \qquad c_1 = x_3 - x_2$$

$$a_2 = x_3 y_1 - x_1 y_3 \qquad b_2 = y_3 - y_1 \qquad c_2 = x_1 - x_3$$

$$a_3 = x_1 y_2 - x_2 y_1 \qquad b_3 = y_1 - y_2 \qquad c_3 = x_2 - x_1$$

Properties of the shape functions:

1. The shape functions N_1, N_2 and N_3 are linear functions of x and y



$$N_i = \begin{cases} 1 & at\ node\ 'i' \\ 0 & at\ other\ nodes \end{cases}$$

2. At every point in the domain

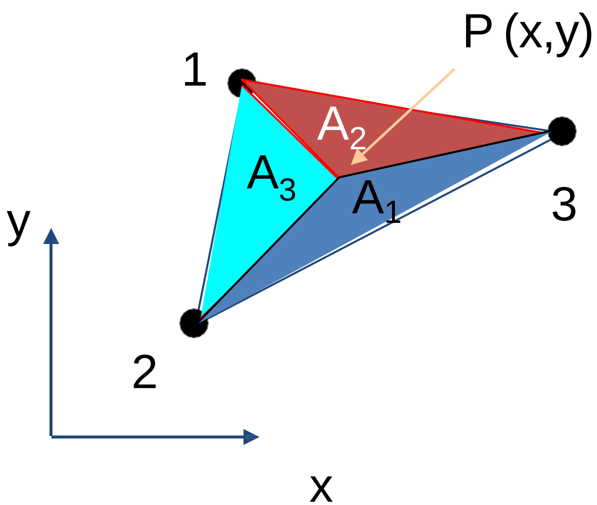
$$\sum_{i=1}^{3} N_i = 1$$

$$\sum_{i=1}^{3} N_i x_i = x$$

$$\sum_{i=1}^{3} N_i y_i = y$$

3. Geometric interpretation of the shape functions

At any point P(x,y) that the shape functions are evaluated,



$$N_1 = \frac{A_1}{A}$$

$$N_2 = \frac{A_2}{A}$$

$$N_3 = \frac{A_3}{A}$$

Approximation of the strains

$$\underline{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \approx \underline{B\,d}$$

$$\underline{B} = \begin{bmatrix} \dfrac{\partial N_1(x,y)}{\partial x} & 0 & \dfrac{\partial N_2(x,y)}{\partial x} & 0 & \dfrac{\partial N_3(x,y)}{\partial x} & 0 \\ 0 & \dfrac{\partial N_1(x,y)}{\partial y} & 0 & \dfrac{\partial N_2(x,y)}{\partial y} & 0 & \dfrac{\partial N_3(x,y)}{\partial y} \\ \dfrac{\partial N_1(x,y)}{\partial y} & \dfrac{\partial N_1(x,y)}{\partial x} & \dfrac{\partial N_2(x,y)}{\partial y} & \dfrac{\partial N_2(x,y)}{\partial x} & \dfrac{\partial N_3(x,y)}{\partial y} & \dfrac{\partial N_3(x,y)}{\partial x} \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

Inside each element, all components of strain are constant: hence the name **Constant Strain Triangle**

Element stresses (constant inside each element)

$$\sigma = \underline{DB}\ \underline{d}$$

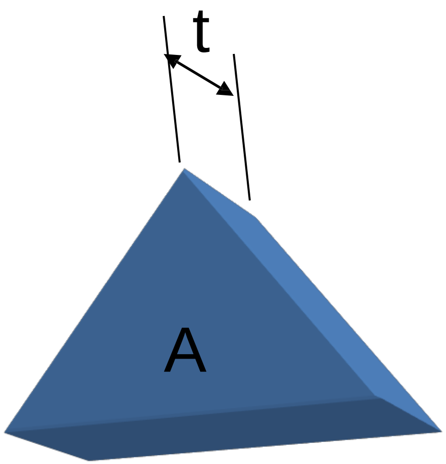
IMPORTANT NOTE:

1. The displacement field is continuous across element boundaries

2. The strains and stresses are NOT continuous across element boundaries

Element stiffness matrix

$$\underline{k} = \int_{V^e} \underline{B}^T \underline{D}\, \underline{B}\, dV$$



Since $\underline{B}$ is constant

$$\underline{k} = \underline{B}^T \underline{D}\, \underline{B} \int_{V^e} dV = \underline{B}^T \underline{D}\, \underline{B}\, A t$$

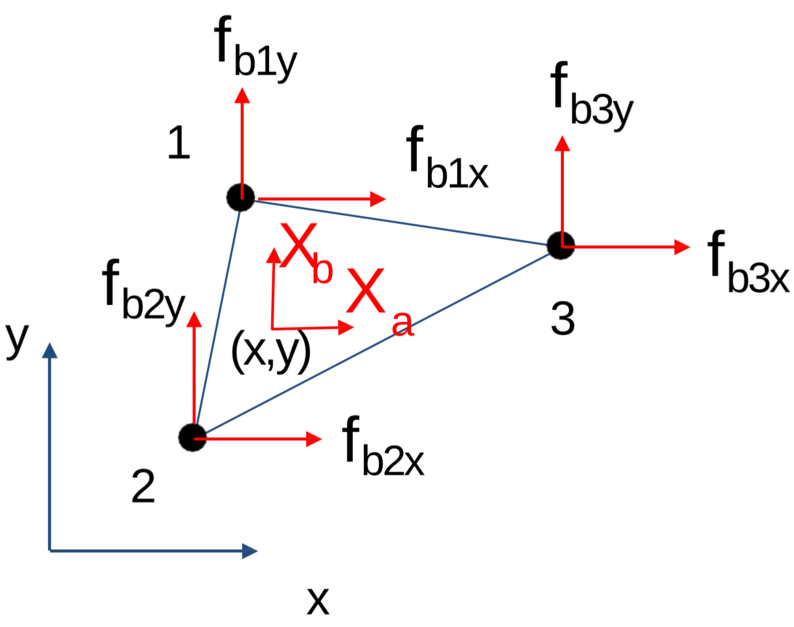
t=thickness of the element
A=surface area of the element

Element nodal load vector

$$\underline{f} = \underbrace{\int_{V^e} \underline{\mathrm{N}}^T \underline{X}\, dV}_{\underline{f}_b} + \underbrace{\int_{S_T^e} \underline{\mathrm{N}}^T \underline{T}_S\, dS}_{\underline{f}_S}$$

Element nodal load vector due to bodyforces

$$\underline{f}_b = \int_{V^e} \underline{\mathbf{N}}^T \underline{X}\, dV = t\int_{A^e} \underline{\mathbf{N}}^T \underline{X}\, dA$$



$$\underline{f}_b = \begin{Bmatrix} f_{b1x} \\ f_{b1y} \\ f_{b2x} \\ f_{b2y} \\ f_{b3x} \\ f_{b3y} \end{Bmatrix} = \begin{Bmatrix} t\int_{A^e} N_1 X_a \, dA \\ t\int_{A^e} N_1 X_b \, dA \\ t\int_{A^e} N_2 X_a \, dA \\ t\int_{A^e} N_2 X_b \, dA \\ t\int_{A^e} N_3 X_a \, dA \\ t\int_{A^e} N_3 X_b \, dA \end{Bmatrix}$$

EXAMPLE:

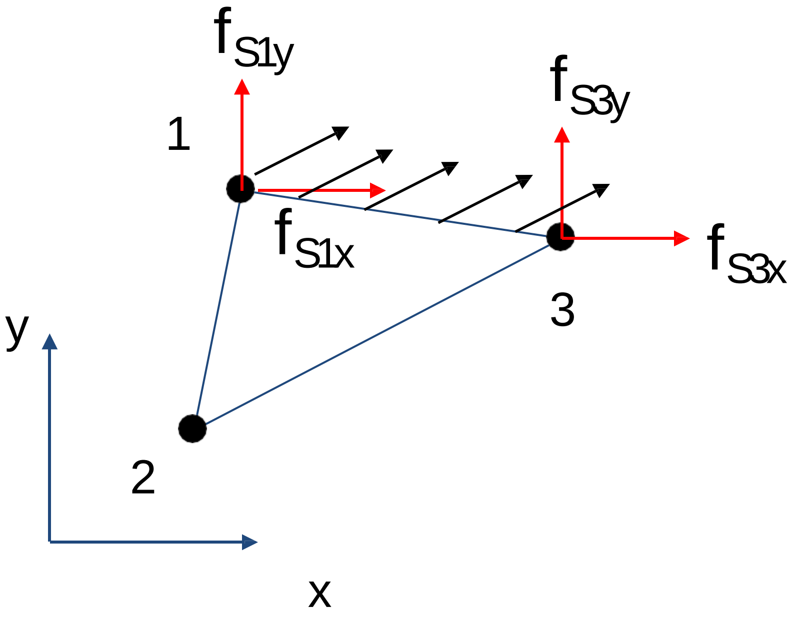
If X_a=1 and X_b=0

$$
\underline{f}_b = \begin{Bmatrix} f_{b1x} \\ f_{b1y} \\ f_{b2x} \\ f_{b2y} \\ f_{b3x} \\ f_{b3y} \end{Bmatrix} = \begin{Bmatrix} t\int_{A^e} N_1 X_a \ dA \\ t\int_{A^e} N_1 X_b \ dA \\ t\int_{A^e} N_2 X_a \ dA \\ t\int_{A^e} N_2 X_b \ dA \\ t\int_{A^e} N_3 X_a \ dA \\ t\int_{A^e} N_3 X_b \ dA \end{Bmatrix} = \begin{Bmatrix} t\int_{A^e} N_1 dA \\ 0 \\ t\int_{A^e} N_2 \ dA \\ 0 \\ t\int_{A^e} N_3 dA \\ 0 \end{Bmatrix} = \begin{Bmatrix} \frac{tA}{3} \\ 0 \\ \frac{tA}{3} \\ 0 \\ \frac{tA}{3} \\ 0 \end{Bmatrix}
$$

Element nodal load vector due to traction

$$\underline{f}_S = \int_{S_T^e} \underline{\mathbf{N}}^T \underline{T}_S \, dS$$

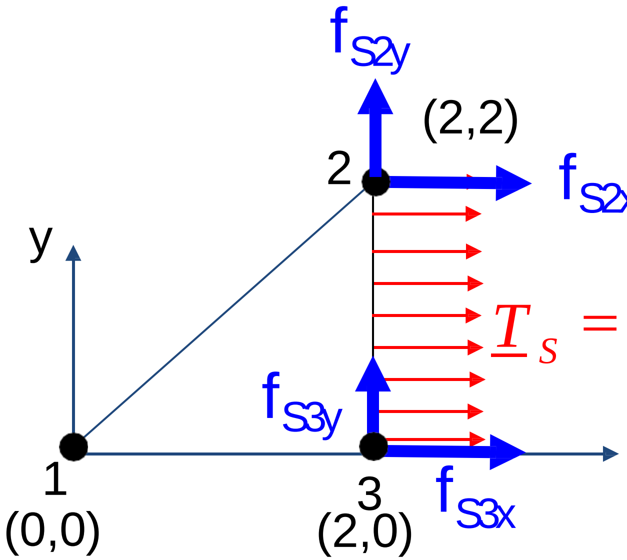
EXAMPLE:



$$\underline{f}_S = t \int_{l_{1-3}^e} \underline{\mathbf{N}}^T \Big|_{along\,1-3} \underline{T}_S \, dS$$

Element nodal load vector due to traction

EXAMPLE:

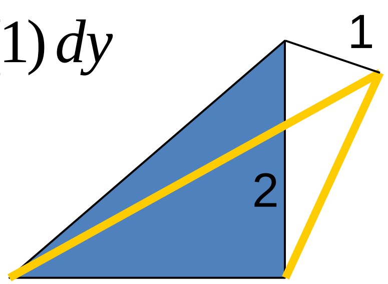


$$\underline{T}_S = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$\underline{f}_S = t\int_{l_{2-3}^e} \underline{\mathrm{N}}^T \Big|_{along\ 2-3} \underline{T}_S\, dS$$

$$f_{S_{2x}} = t\int_{l_{2-3}^e} N_2 \Big|_{along\ 2-3} (1)\, dy$$

$$= t\left(\frac{1}{2}\right) \times 2 \times 1 = t$$



Similarly, compute

$$f_{S_{2y}} = 0$$

$$f_{S_{3x}} = t$$

$$f_{S_{3y}} = 0$$

Recommendations for use of CST

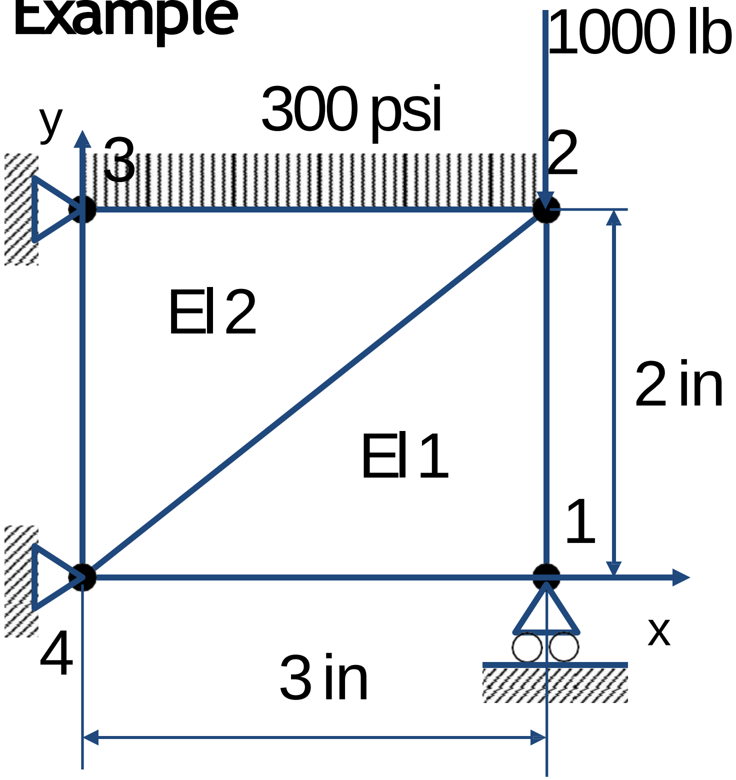
1. Use in areas where strain gradients are small

2. Use in mesh transition areas (fine mesh to coarse mesh)

3. Avoid CST in critical areas of structures (e.g., stress concentrations, edges of holes, corners)

4. In general CSTs are not recommended for general analysis purposes as a very large number of these elements are required for reasonable accuracy.

Example



Thickness (t) = 0.5 in
$E = 30 \times 10^6$ psi
$\nu = 0.25$

(a) Compute the unknown nodal displacements.
(b) Compute the stresses in the two elements.

Realize that this is a plane stress problem and therefore we need to use

$$\underline{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} 3.2 & 0.8 & 0 \\ 0.8 & 3.2 & 0 \\ 0 & 0 & 1.2 \end{bmatrix} \times 10^7 \ psi$$

Step 1: Node-element connectivity chart

ELEMENT	Node 1	Node 2	Node 3	Area (sqin)
1	1	2	4	3
2	3	4	2	3

Node	x	y
1	3	0
2	3	2
3	0	2
4	0	0

Nodal coordinates

Step 2: Compute strain-displacement matrices for the elements

Recall

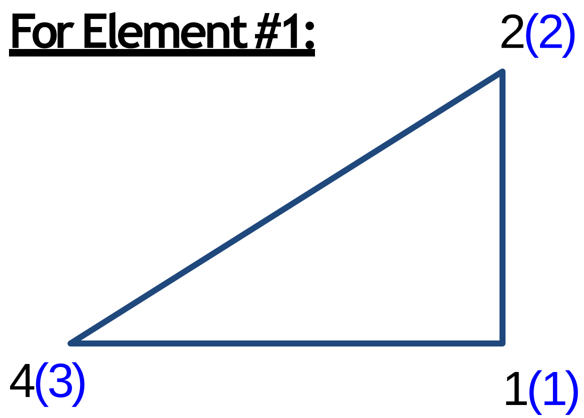
$$\underline{B}=\frac{1}{2A}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

with

$$b_1 = y_2 - y_3 \qquad b_2 = y_3 - y_1 \qquad b_3 = y_1 - y_2$$

$$c_1 = x_3 - x_2 \qquad c_2 = x_1 - x_3 \qquad c_3 = x_2 - x_1$$

For Element #1:



(local numbers within brackets)

$$y_1 = 0;\ y_2 = 2;\ y_3 = 0$$

$$x_1 = 3;\ x_2 = 3;\ x_3 = 0$$

Hence

$$b_1 = 2 \qquad b_2 = 0 \qquad b_3 = -2$$

$$c_1 = -3 \qquad c_2 = 3 \qquad c_3 = 0$$

Therefore

$$\underline{B}^{(1)}=\frac{1}{6}\begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & -3 & 0 & 3 & 0 & 0 \\ -3 & 2 & 3 & 0 & 0 & -2 \end{bmatrix}$$

For Element #2:

$$\underline{B}^{(2)}=\frac{1}{6}\begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 3 & -2 & -3 & 0 & 0 & 2 \end{bmatrix}$$

Step 3: Compute element stiffness matrices

$$\underline{k}^{(1)} = At\underline{B}^{(1)^T}\underline{D}\,\underline{B}^{(1)} = (3)(0.5)\underline{B}^{(1)^T}\underline{D}\,\underline{B}^{(1)}$$

$$= \begin{bmatrix} 0.9833 & -0.5 & -0.45 & 0.2 & -0.5333 & 0.3 \\ & 1.4 & 0.3 & -1.2 & 0.2 & -0.2 \\ & & 0.45 & 0 & 0 & -0.3 \\ & & & 1.2 & -0.2 & 0 \\ & & & & 0.5333 & 0 \\ & & & & & 0.2 \end{bmatrix} \times 10^7$$

$u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_4 \quad v_4$

$$\underline{k}^{(2)} = At\underline{B}^{(2)^T}\,\underline{D}\,\underline{B}^{(2)} = (3)(0.5)\underline{B}^{(2)^T}\,\underline{D}\,\underline{B}^{(2)}$$

$$= \begin{bmatrix} 0.9833 & -0.5 & -0.45 & 0.2 & -0.5333 & 0.3 \\ & 1.4 & 0.3 & -1.2 & 0.2 & -0.2 \\ & & 0.45 & 0 & 0 & -0.3 \\ & & & 1.2 & -0.2 & 0 \\ & & & & 0.5333 & 0 \\ & & & & & 0.2 \end{bmatrix} \times 10^7$$

$u_3 \quad v_3 \quad u_4 \quad v_4 \quad u_2 \quad v_2$

<u>Step 4</u>: Assemble the global stiffness matrix corresponding to the *<u>nonzero</u>* degrees of freedom

Notice that

$$u_3 = v_3 = u_4 = v_4 = v_1 = 0$$

Hence we need to calculate only a small (3x3) stiffness matrix

$$\underline{K} = \begin{bmatrix} 0.983 & -0.45 & 0.2 \\ -0.45 & 0.983 & 0 \\ 0.2 & 0 & 1.4 \end{bmatrix} \times 10^7 \quad \begin{matrix} u_1 \\ u_2 \\ v_2 \end{matrix}$$

$$\begin{matrix} u_1 & u_2 & v_2 \end{matrix}$$

Step 5: Compute consistent nodal loads

$$\underline{f} = \begin{Bmatrix} f_{1x} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ f_{2y} \end{Bmatrix}$$

$$f_{2y} = -1000 + f_{S_{2y}}$$

The consistent nodal load due to traction on the edge 3-2

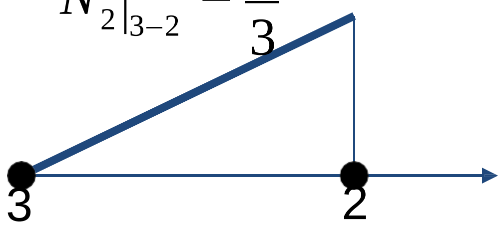
$$f_{S_{2y}} = \int_{x=0}^{3} N_3\big|_{3-2}(-300)t\,dx$$

$$= (-300)(0.5)\int_{x=0}^{3} N_3\big|_{3-2}\,dx$$

$$= -150\int_{x=0}^{3} \frac{x}{3}\,dx$$

$$= -50\left[\frac{x^2}{2}\right]_0^3 = -50\left(\frac{9}{2}\right) = -225\,lb$$

$$N_2\big|_{3-2} = \frac{x}{3}$$



Hence

$$f_{2y} = -1000 + f_{S_{2y}}$$
$$= -1225\ lb$$

<u>Step 6</u>: Solve the system equations to obtain the unknown nodal loads

$$\underline{K}\,\underline{d} = \underline{f}$$

$$10^7 \times \begin{bmatrix} 0.983 & -0.45 & 0.2 \\ -0.45 & 0.983 & 0 \\ 0.2 & 0 & 1.4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -1225 \end{Bmatrix}$$

Solve to get

$$\begin{Bmatrix} u_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0.2337 \times 10^{-4}\ in \\ 0.1069 \times 10^{-4}\ in \\ -0.9084 \times 10^{-4}\ in \end{Bmatrix}$$

Step 7: Compute the stresses in the elements

In Element #1

$$\underline{\sigma}^{(1)} = \underline{D}\,\underline{B}^{(1)}\underline{d}^{(1)}$$

With

$$\underline{d}^{(1)T} = \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_4 & v_4 \end{bmatrix}$$
$$= \begin{bmatrix} 0.2337\times10^{-4} & 0 & 0.1069\times10^{-4} & -0.9084\times10^{-4} & 0 & 0 \end{bmatrix}$$

Calculate

$$\underline{\sigma}^{(1)} = \begin{bmatrix} -114.1 \\ -1391.1 \\ -76.1 \end{bmatrix} psi$$

In Element #2

$$\underline{\sigma}^{(2)} = \underline{D}\underline{B}^{(2)}\underline{d}^{(2)}$$

With

$$\underline{d}^{(2)T} = \begin{bmatrix} u_3 & v_3 & u_4 & v_4 & u_2 & v_2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0.1069\times10^{-4} & -0.9084\times10^{-4} \end{bmatrix}$$

Calculate

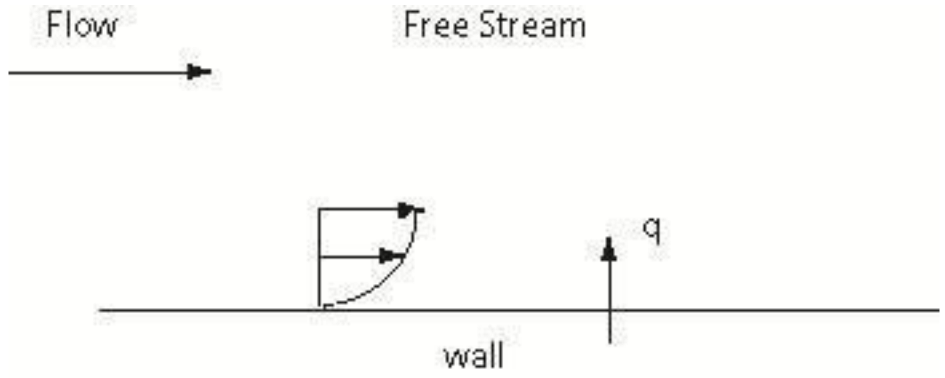
$$\sigma^{(2)} = \begin{bmatrix} 114.1 \\ 28.52 \\ -363.35 \end{bmatrix} psi$$

Notice that the stresses are constant in each element

UNIT - 4

Heat transfer analysis

Thermal Convection



Newton's Law of Cooling

$$q = h(T_s - T_\infty)$$

h: convective heat transfer coefficient $(W/m^2 \cdot C^o)$

Thermal Conduction in 1-D

Boundary conditions:

Dirichlet BC:

Natural BC:

Mixed BC:

Weak Formulation of 1-D Heat Conduction

(*Steady State Analysis*)

- ***Governing Equation of 1-D Heat Conduction -----***

$$-\frac{d}{dx}\left(\kappa(x)A(x)\frac{dT(x)}{dx}\right)-AQ(x)=0 \qquad 0<x<L$$

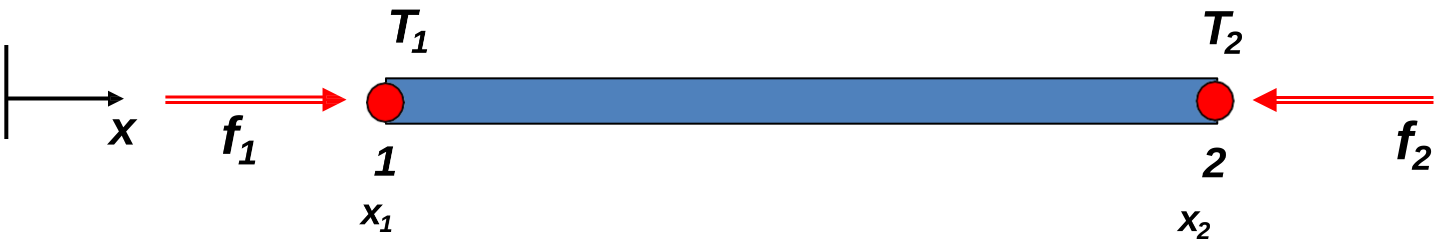
- ***Weighted Integral Formulation -----***

$$0=\int_0^L w(x)\left[-\frac{d}{dx}\left(\kappa(x)A(x)\frac{dT(x)}{dx}\right)-AQ(x)\right]dx$$

- ***Weak Form from Integration-by-Parts -----***

$$0=\int_0^L\left[\frac{dw}{dx}\left(\kappa A\frac{dT}{dx}\right)-wAQ\right]dx-w\left(\kappa A\frac{dT}{dx}\right)\Bigg|_0^L$$

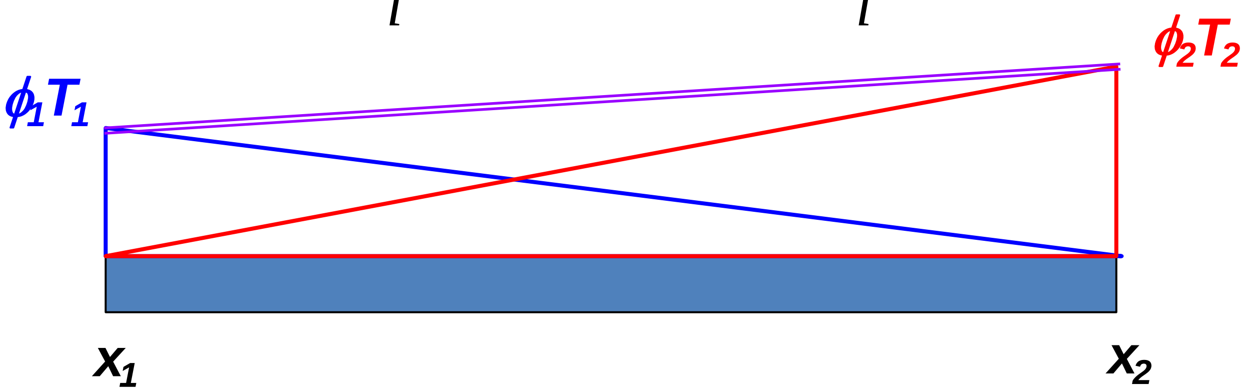
Formulation for 1-D Linear Element



$$f_1(x) = -\kappa A \frac{\partial T}{\partial x}\bigg|_1, \qquad f_2(x) = \kappa A \frac{\partial T}{\partial x}\bigg|_2$$

Let $$T(\mathrm{x}) = T_1\phi_1(\mathrm{x}) + T_2\phi_2(\mathrm{x})$$

$$\phi_1(x) = \frac{x_2 - x}{l}, \qquad \phi_2(x) = \frac{x - x_1}{l}$$



Formulation for 1-D Linear Element

Let $w(x) = \phi_i(x), \quad i = 1, 2$

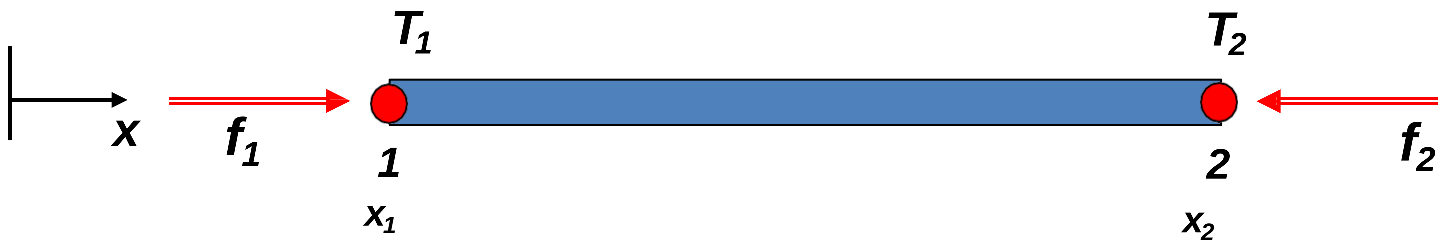
$$0 = \sum_{j=1}^{2} T_j \left[\int_{x_1}^{x_2} \kappa A \left(\frac{d\phi_i}{dx} \frac{d\phi_j}{dx} \right) dx \right] - \int_{x_1}^{x_2} \left(\phi_i A Q \right) dx - \left[\phi_i(x_2) f_2 + \phi_i(x_1) f_1 \right]$$

$$= \sum_{j=1}^{2} K_{ij} T_j - Q_i - \left[\phi_i(x_2) f_2 + \phi_i(x_1) f_1 \right]$$

$$\Longrightarrow \quad \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$where \quad K_{ij} = \int_{x_1}^{x_2} \kappa A \left(\frac{d\phi_i}{dx} \frac{d\phi_j}{dx} \right) dx, \quad \mathrm{Q}_i = \int_{x_1}^{x_2} \left(\phi_i A Q \right) dx, \quad \mathrm{f}_1 = -\kappa A \left. \frac{dT}{dx} \right|_{x_1}, \quad \mathrm{f}_2 = \kappa A \left. \frac{dT}{dx} \right|_{x_2}$$

Element Equations of 1-D Linear Element



$$\Rightarrow \quad \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \frac{\kappa A}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

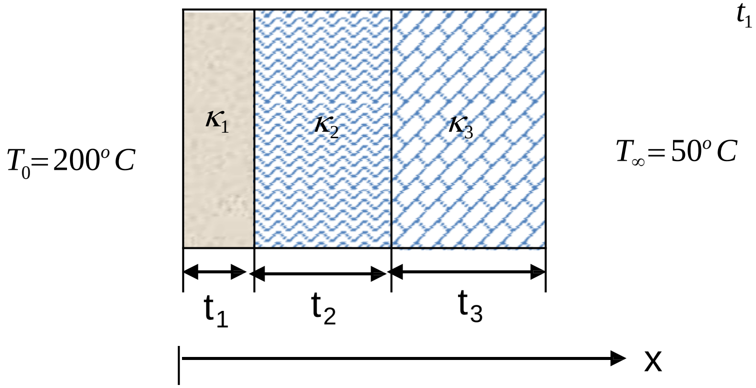
$$where \quad \mathrm{Q}_i = \int_{x_1}^{x_2} (\phi_i A Q) dx, \quad \mathrm{f}_1 = -\kappa A \left.\frac{dT}{dx}\right|_{x=x_1}, \quad \mathrm{f}_2 = \kappa A \left.\frac{dT}{dx}\right|_{x=x_2}$$

1-D Heat Conduction - Example

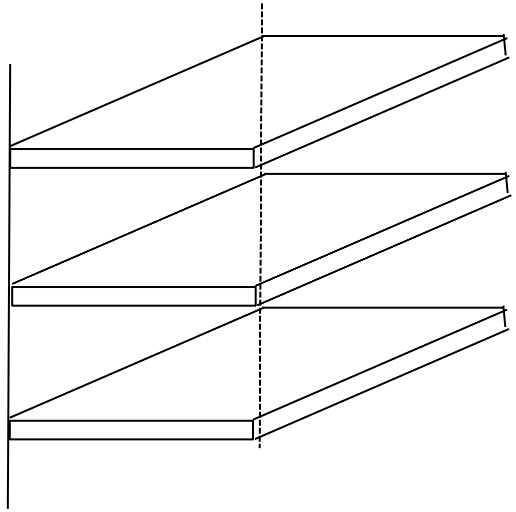
A composite wall consists of three materials, as shown in the figure below. The inside wall temperature is 200°C and the outside air temperature is 50°C with a convection coefficient of h = 10 W(m^2.K). Find the temperature along the composite wall.

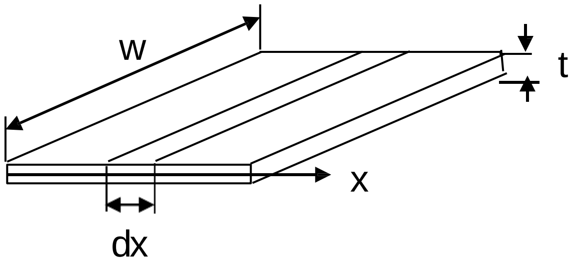
$$\kappa_1 = 70W/(m \cdot K), \quad \kappa_2 = 40W/(m \cdot K), \quad \kappa_3 = 20W/(m \cdot K)$$

$$t_1 = 2cm, \quad t_2 = 2.5cm, \quad t_3 = 4cm$$



Thermal Conduction and Convection-Fin





Objective: to enhance heat transfer

Governing equation for 1-D heat transfer in thin fin

$$\frac{d}{dx}\left(\kappa A_c \frac{dT}{dx}\right) + A_c Q = 0$$

$$Q_{loss} = \frac{2h(T-T_\infty)\cdot dx \cdot w + 2h(T-T_\infty)\cdot dx \cdot t}{A_c \cdot dx} = \frac{2h(T-T_\infty)\cdot(w+t)}{A_c}$$

$$\Rightarrow \quad \frac{d}{dx}\left(\kappa A_c \frac{dT}{dx}\right) - Ph\,(T - T_\infty) + A_c Q = 0$$

where $P = 2(w+t)$

Fin - Weak Formulation

(*Steady State Analysis*)

- ***Governing Equation of 1-D Heat Conduction -----***

$$-\frac{d}{dx}\left(\kappa(x)A(x)\frac{dT(x)}{dx}\right)+Ph\left(T-T_{\infty}\right)-AQ=0 \qquad 0<x<L$$

- ***Weighted Integral Formulation -----***

$$0=\int_0^L w(x)\left[-\frac{d}{dx}\left(\kappa(x)A(x)\frac{dT(x)}{dx}\right)+Ph(T-T_{\infty})-AQ(x)\right]dx$$

- ***Weak Form from Integration-by-Parts -----***

$$0=\int_0^L\left[\frac{dw}{dx}\left(\kappa A\frac{dT}{dx}\right)+wPh(T-T_{\infty})-wAQ\right]dx-w\left(\kappa A\frac{dT}{dx}\right)\Bigg|_0^L$$

Formulation for 1-D Linear Element

Let $w(x) = \phi_i(x)$, $i = 1, 2$

$$0 = \sum_{j=1}^{2} T_j \left[\int_{x_1}^{x_2} \left(\kappa A \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} + Ph\phi_i\phi_j \right) dx \right] - \int_{x_1}^{x_2} \phi_i \left(AQ + PhT_\infty \right) dx$$

$$- \left[\phi_i(x_2) f_2 + \phi_i(x_1) f_1 \right]$$

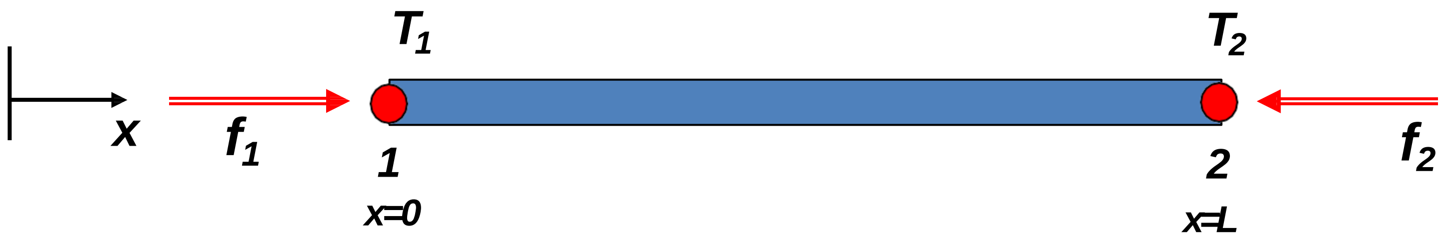
$$= \sum_{j=1}^{2} K_{ij} T_j - Q_i - \left[\phi_i(x_2) f_2 + \phi_i(x_1) f_1 \right]$$

$$\Longrightarrow \quad \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$\text{where} \quad K_{ij} = \int_{x_1}^{x_2} \left(\kappa A \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} + Ph\phi_i\phi_j \right) dx, \quad Q_i = \int_{x_1}^{x_2} \phi_i \left(AQ + PhT_\infty \right) dx,$$

$$f_1 = -\kappa A \left. \frac{dT}{dx} \right|_{x=x_1}, \quad f_2 = \kappa A \left. \frac{dT}{dx} \right|_{x=x_2}$$

Element Equations of 1-D Linear Element



$$\Rightarrow \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \left\{ \frac{\underline{\kappa A}}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{Phl}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right\} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$\text{where} \quad Q_i = \int_{x_1}^{x_2} \phi_i \left(AQ + PhT_\infty \right) dx, \quad f_1 = -\kappa A \frac{dT}{dx}\bigg|_{x=x_1}, \quad f_2 = \kappa A \frac{dT}{dx}\bigg|_{x=x_2}$$

Time-Dependent Problems

Time-Dependent Problems

In general, $u(x,t)$

Key question: How to choose approximate functions?

Two approaches:

$$u(x,t) = \sum u_j \phi_j(x,t)$$

$$u(x,t) = \sum u_j(t) \phi_j(x)$$

Model Problem I – Transient Heat Conduction

$$c\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) + f\left(x,t\right)$$

Weak form:

$$0 = \int_{x_1}^{x_2}\left(a\frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + cw\frac{\partial u}{\partial t} - wf\right)dx - Q_1 w(x_1) - Q_2 w(x_2)$$

$$Q_1 = -\left[a\frac{du}{dx}\right]_{x_1}; \qquad Q_2 = \left[a\frac{du}{dx}\right]_{x_2}$$

Transient Heat Conduction

let: $u(x,t) = \sum_{j=1}^{n} u_j(t)\phi_j(x)$ and $w = \phi_i(x)$

$$0 = \int_{x_1}^{x_2} \left(a \frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + cw\frac{\partial u}{\partial t} - wf \right) dx - Q_1 w(x_1) - Q_2 w(x_2)$$

$$[K]\{u\} + [M]\{\dot{u}\} = \{F\}$$

ODE!

$$K_{ij} = \int_{x_1}^{x_2} a \frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x} dx \qquad M_{ij} = \int_{x_1}^{x_2} c\phi_i\phi_j dx$$

$$F_i = \int_{x_1}^{x_2} \phi_i f dx + Q_i$$

Time Approximation – First Order ODE

$$a\frac{du}{dt} + bu = f(t) \qquad 0 < t < T \qquad u(0) = u_0$$

Forward difference approximation - explicit

$$u_{k+1} = u_k + \frac{\Delta t}{a}\left[f_k - bu_k\right]$$

Backward difference approximation - implicit

$$u_{k+1} = u_k + \frac{\Delta t}{a + b\Delta t}\left[f_k - bu_k\right]$$

Stability Requirment

$$\Delta t \le \Delta t_{cri} = \frac{2}{(1-2\alpha)\lambda_{\max}}$$

where $$([K]-\lambda[M])\{u\}=\{Q\}$$

Note: One must use the same discretization for solving the eigenvalue problem.

Transient Heat Conduction - Example

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad 0 < x < 1$$

$$u(0,t) = 0 \qquad \frac{\partial u}{\partial t}(1,t) = 0 \qquad t > 0$$

$$u(x,0) = 1.0$$

Transient Heat Conduction - Example

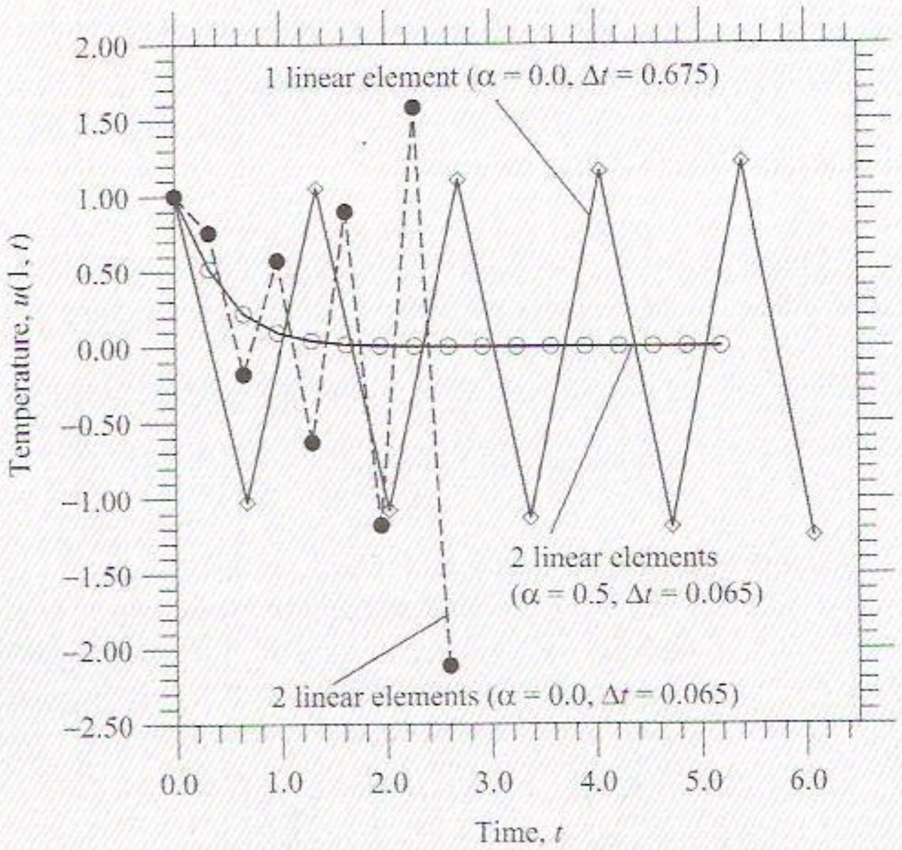


Figure 6.2.3 Stability of the forward difference ($\alpha = 0.0$) and Crank–Nicolson ($\alpha = 0.5$) schemes as applied to a parabolic equation.

Transient Heat Conduction - Example

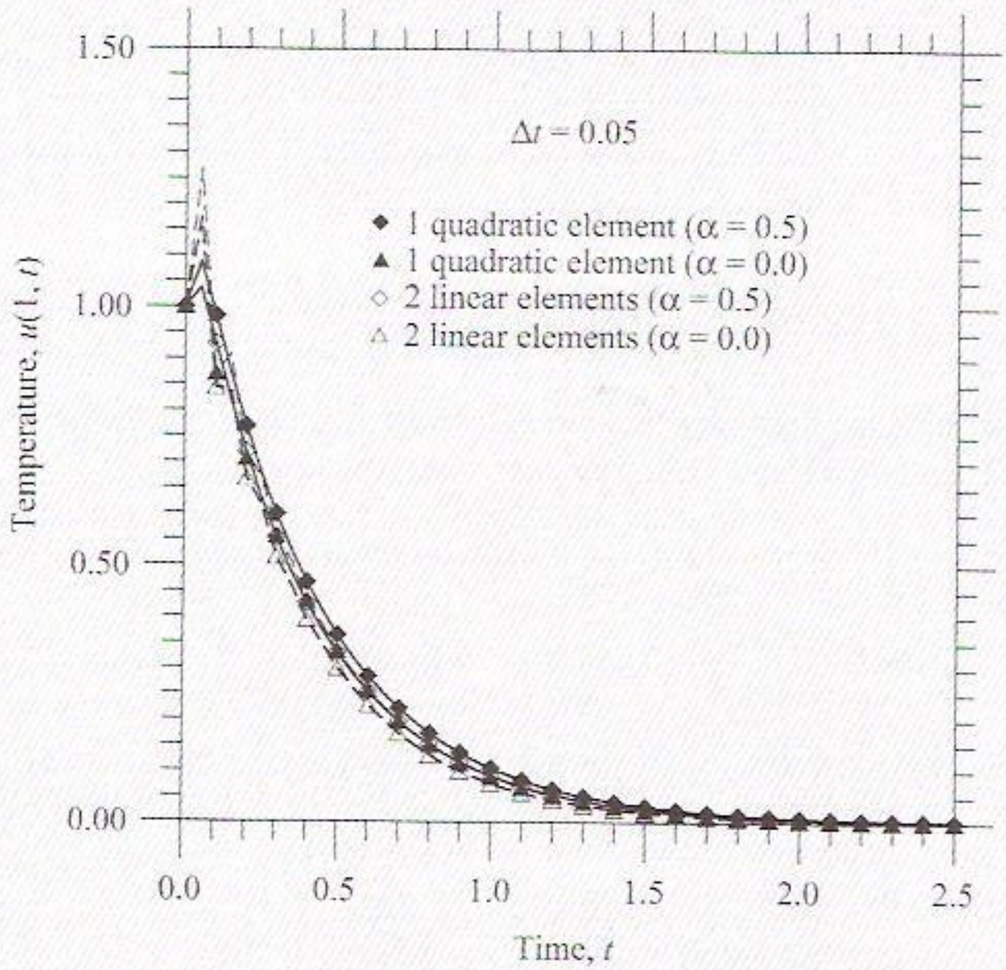
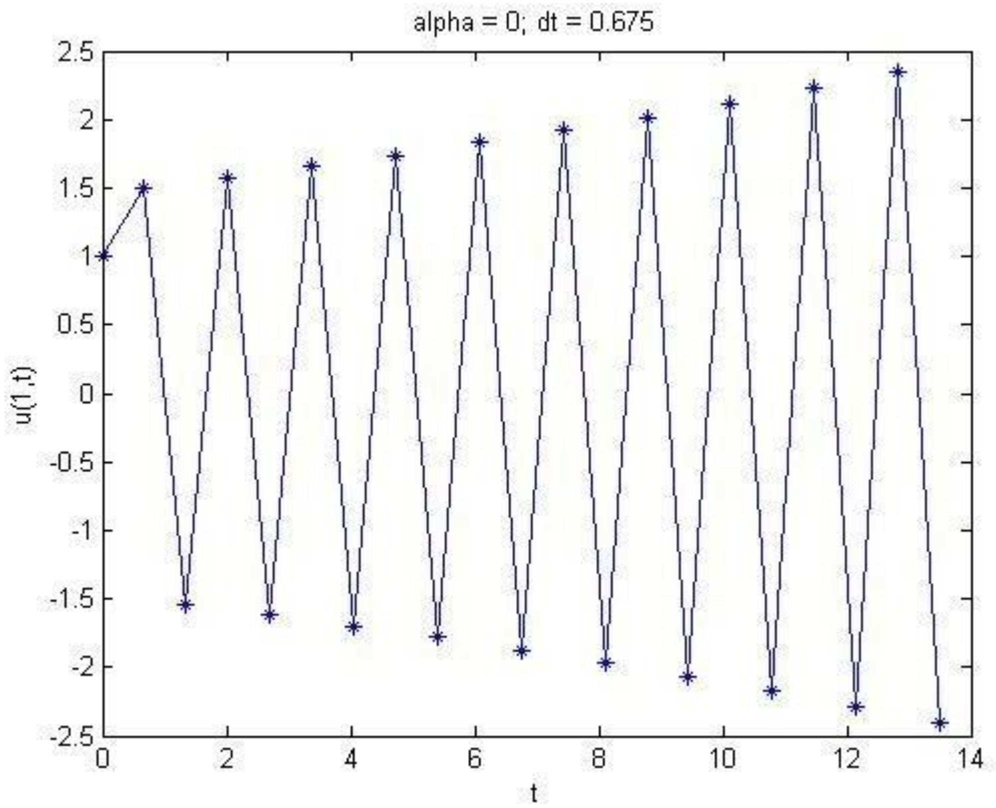
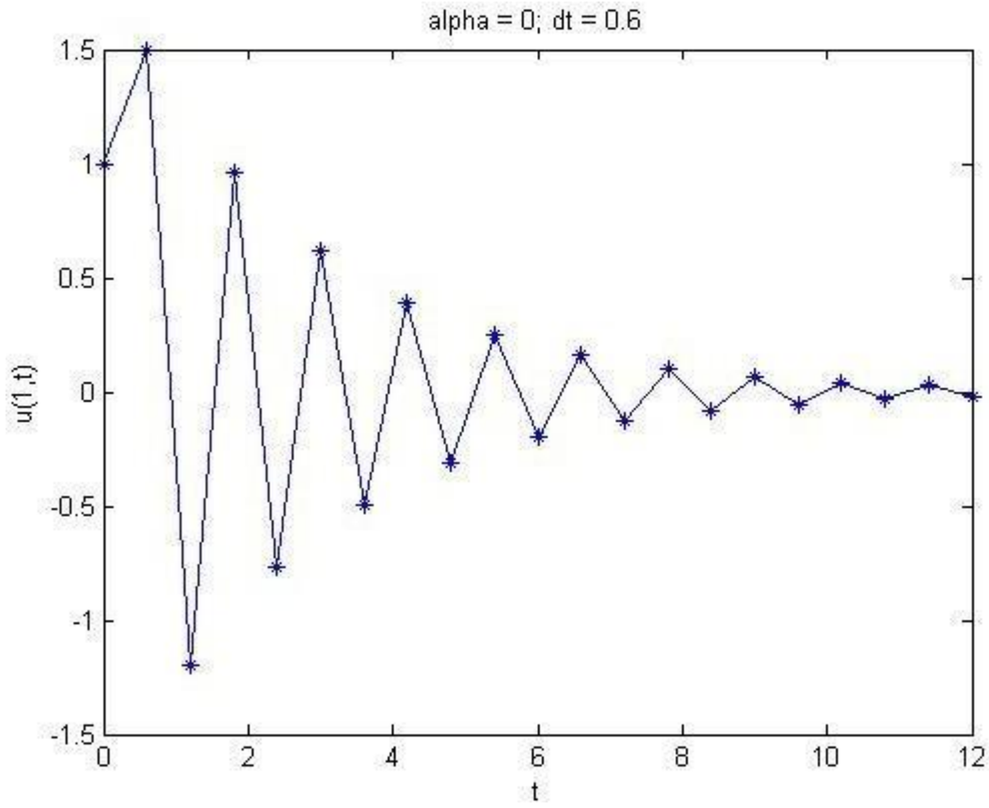


Figure 6.2.4 Transient solution of a parabolic equation according to linear and quadratic finite elements.

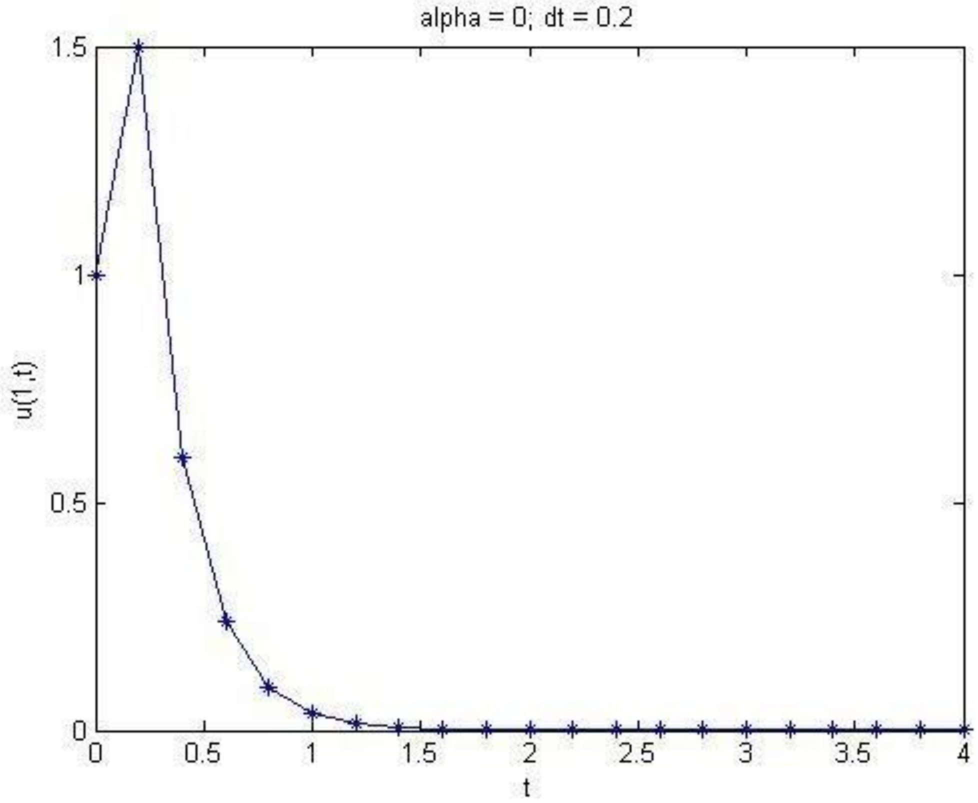
Transient Heat Conduction - Example



Transient Heat Conduction - Example



Transient Heat Conduction - Example



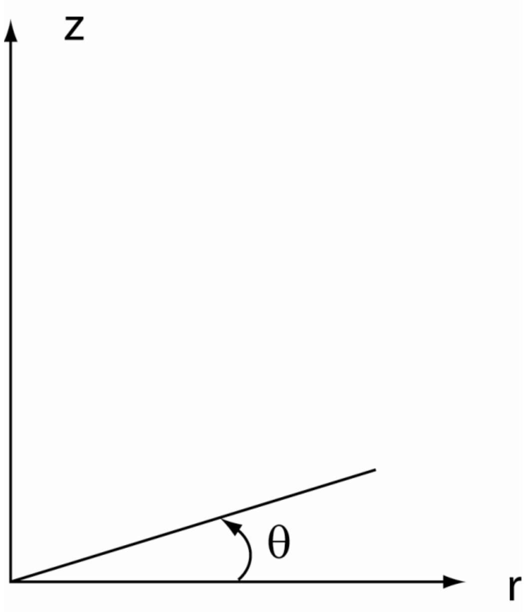
UNIT – 5

Dynamic Analysis

Axi-symmetric Analysis

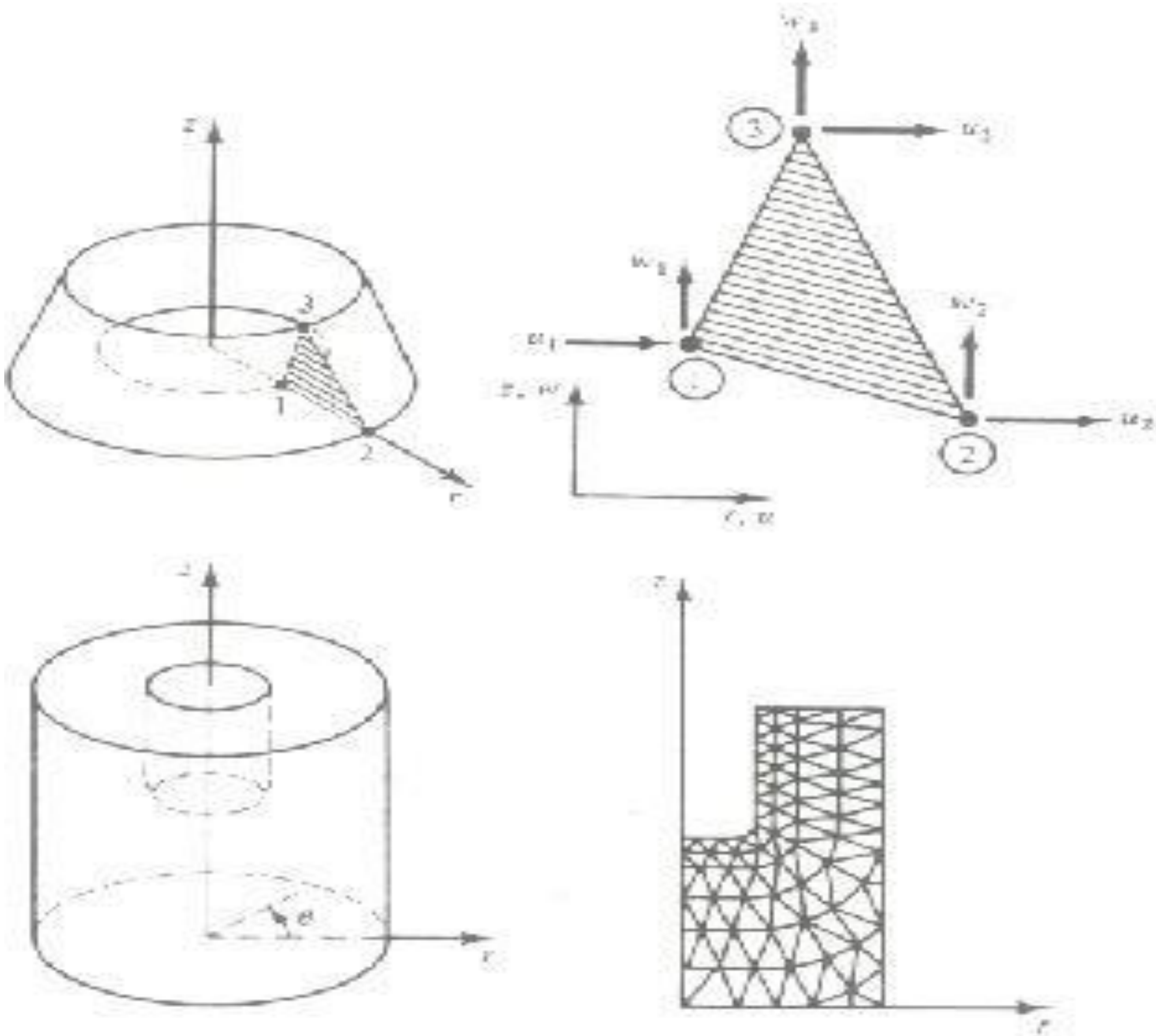
Cylindrical coordinates: $(r,\ \theta,\ z)$

$$x = r\cos\theta;\ \ y = r\sin\theta;\ \ z = z$$



- quantities depend on *r* and *z* only
- 3-D problem → 2-D problem

Axi-symmetric Analysis



Axi-symmetric Analysis – Single-Variable Problem

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r a_{11}\frac{\partial u(r,z)}{\partial r}\right) - \frac{\partial}{\partial z}\left(a_{22}\frac{\partial u(r,z)}{\partial z}\right) + a_{00}u - f(r,z) = 0$$

Weak form:

$$0 = \int_{\Omega_e}\left[\frac{\partial w}{\partial r}\left(a_{11}\frac{\partial u}{\partial r}\right) + \frac{\partial w}{\partial z}\left(a_{22}\frac{\partial u}{\partial z}\right) + a_{00}wu - wf(r,z)\right] r\,dr\,dz$$
$$-\oint_{\Gamma_e} w q_n\, ds$$

where $$q_n = a_{11}\frac{\partial u(r,z)}{\partial r}n_r + a_{22}\frac{\partial u(r,z)}{\partial z}n_z$$

Finite Element Model – Single-Variable Problem

$$u = \sum_j u_j \phi_j$$ where $$\phi_j(r,z) = \phi_j(x,y)$$

Ritz method: $$w = \phi_i$$

Weak form → $$\sum_{j=1}^{n} K_{ij}^e u_j^e = f_i^e + Q_i^e$$

where
$$K_{ij}^e = \int_{\Omega_e} \left(a_{11} \frac{\partial \phi_i}{\partial r} \frac{\partial \phi_j}{\partial r} + a_{22} \frac{\partial \phi_i}{\partial z} \frac{\partial \phi_j}{\partial z} + a_{00} \phi_i \phi_j \right) r dr dz$$

$$f_i^e = \int_{\Omega_e} \phi_i f r dr dz$$

$$Q_i^e = \int_{\Gamma_e} \phi_i q_n ds$$

Single-Variable Problem – Heat Transfer

Heat Transfer:

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T(r,z)}{\partial r}\right)-\frac{\partial}{\partial z}\left(k\frac{\partial T(r,z)}{\partial z}\right)-f(r,z)=0$$

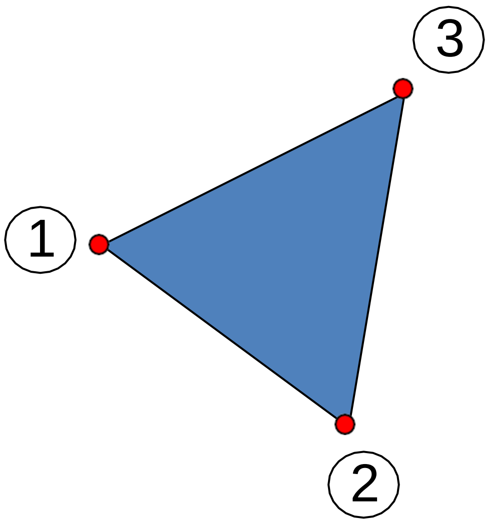
Weak form

$$0=\int_{\Omega_e}\left[\frac{\partial w}{\partial r}\left(k\frac{\partial T}{\partial r}\right)+\frac{\partial w}{\partial z}\left(k\frac{\partial T}{\partial z}\right)-wf(r,z)\right]rdrdz$$

$$-\oint_{\Gamma_e} wq_n\,ds$$

where $$q_n=k\frac{\partial T(r,z)}{\partial r}n_r+k\frac{\partial T(r,z)}{\partial z}n_z$$

3-Node Axi-symmetric Element



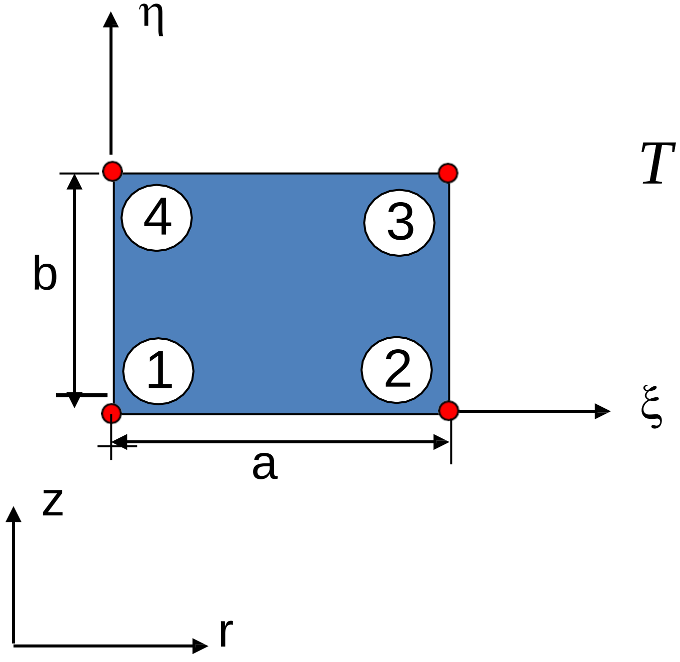
$$T(r,z) = T_1\phi_1 + T_2\phi_2 + T_3\phi_3$$

$$\phi_1 = \frac{\begin{pmatrix}1 & r & z\end{pmatrix}}{2A_e}\begin{Bmatrix} r_2 z_3 - r_3 z_2 \\ z_2 - z_3 \\ r_3 - r_2 \end{Bmatrix}$$

$$\phi_2 = \frac{\begin{pmatrix}1 & r & z\end{pmatrix}}{2A_e}\begin{Bmatrix} r_3 z_1 - r_1 z_3 \\ z_3 - z_1 \\ r_1 - r_3 \end{Bmatrix}$$

$$\phi_3 = \frac{\begin{pmatrix}1 & r & z\end{pmatrix}}{2A_e}\begin{Bmatrix} r_1 z_2 - r_2 z_1 \\ z_1 - z_2 \\ r_2 - r_1 \end{Bmatrix}$$

4-Node Axi-symmetric Element

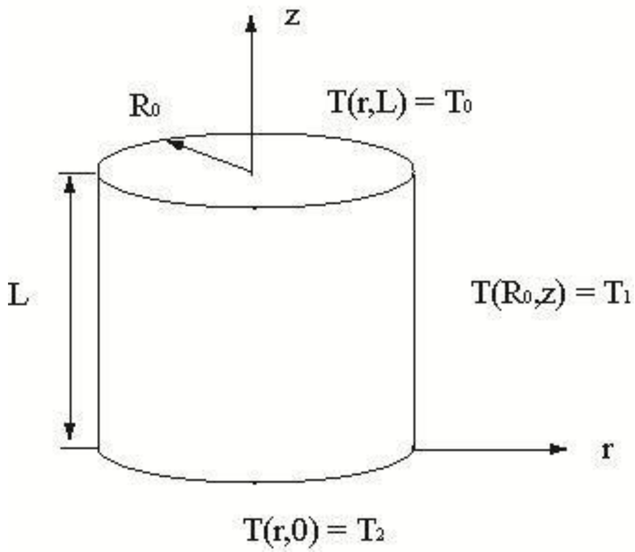


$$T(r,z) = T_1\phi_1 + T_2\phi_2 + T_3\phi_3 + T_4\phi_4$$

$$\phi_1 = \left(1 - \frac{\xi}{a}\right)\left(1 - \frac{\eta}{b}\right) \qquad \phi_2 = \frac{\xi}{a}\left(1 - \frac{\eta}{b}\right)$$

$$\phi_3 = \frac{\xi}{a}\frac{\eta}{b} \qquad \phi_4 = \left(1 - \frac{\xi}{a}\right)\frac{\eta}{b}$$

Single-Variable Problem – Example



Step 1: Discretization

Step 2: Element equation

$$K_{ij}^{e} = \int_{\Omega_e} \left(\kappa \frac{\partial \phi_i}{\partial r} \frac{\partial \phi_j}{\partial r} + \kappa \frac{\partial \phi_i}{\partial z} \frac{\partial \phi_j}{\partial z} \right) r dr dz$$

$$f_i^{e} = \int_{\Omega_e} \phi_i f r dr dz \qquad Q_i^{e} = \int_{\Gamma_e} \phi_i q_n ds$$

Review of CST Element

- Constant Strain Triangle (CST) - easiest and simplest finite element
  - Displacement field in terms of generalized coordinates

$$u = \beta_1 + \beta_2 x + \beta_3 y$$
$$v = \beta_4 + \beta_5 x + \beta_6 y \qquad (3.2\text{-}1)$$

  - Resulting strain field is

$$\varepsilon_x = \beta_2 \qquad \varepsilon_y = \beta_6 \qquad \gamma_{xy} = \beta_3 + \beta_5 \qquad (3.2\text{-}2)$$

  - Strains do not vary within the element. Hence, the name constant strain triangle (CST)
    - Other elements are not so lucky.
    - Can also be called linear triangle because displacement field is linear in x and y - sides remain straight.

Constant Strain Triangle

- The strain field from the shape functions looks like:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \qquad (3.2\text{-}3)$$

  - Where, x_i and y_i are nodal coordinates (i=1, 2, 3)
  - $x_{ij} = x_i - x_j$ and $y_{ij}=y_i - y_j$
  - 2A is twice the area of the triangle, $2A = x_{21}y_{31} - x_{31}y_{21}$
- Node numbering is arbitrary except that the sequence 123 must go clockwise around the element if A is to be positive.

Constant Strain Triangle

- Stiffness matrix for element $k = B^T E B t A$
- The CST gives good results in regions of the FE model where there is little strain gradient
  - Otherwise it does not work well.

Linear Strain Triangle

- Changes the shape functions and results in quadratic displacement distributions and linear strain distributions within the element.

$$u = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2 + \beta_5 xy + \beta_6 y^2$$
$$v = \beta_7 + \beta_8 x + \beta_9 y + \beta_{10} x^2 + \beta_{11} xy + \beta_{12} y^2 \qquad (3.3\text{-}1)$$

$$\varepsilon_x = \beta_2 + 2\beta_4 x + \beta_5 y$$
$$\varepsilon_y = \beta_9 + \beta_{11} x + 2\beta_{12} y \qquad (3.3\text{-}2)$$
$$\gamma_{xy} = (\beta_3 + \beta_8) + (\beta_5 + 2\beta_{10})x + (2\beta_6 + \beta_{11})y$$

Linear Strain Triangle

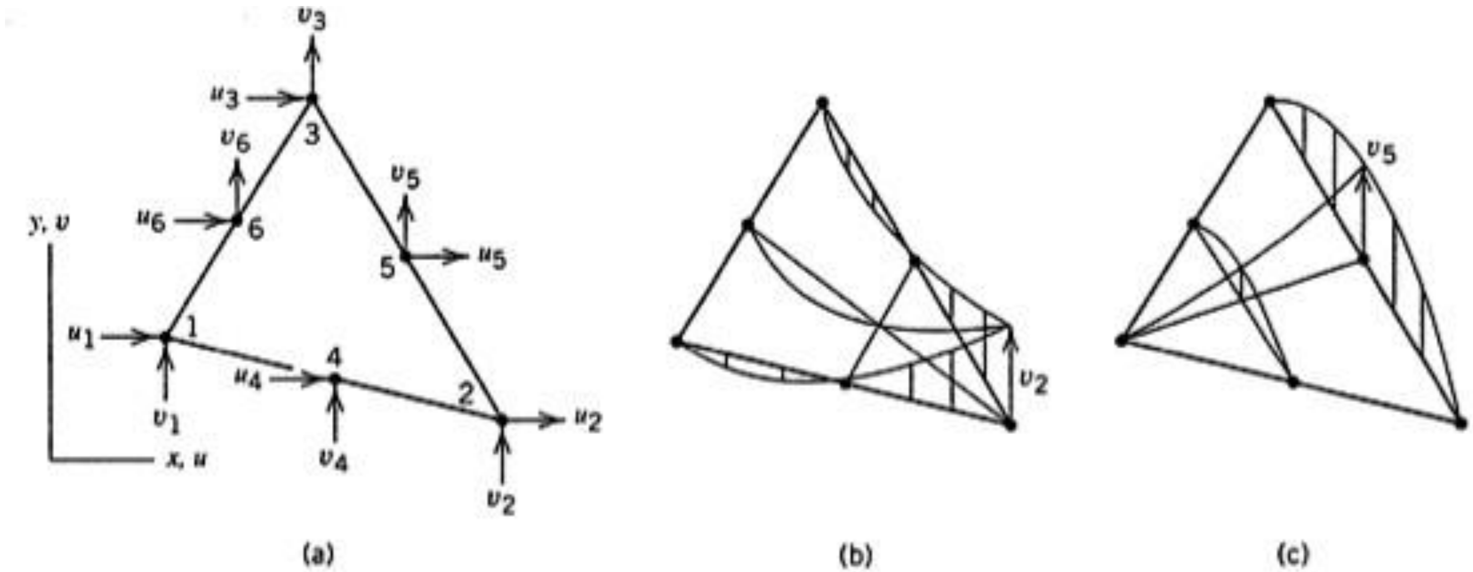
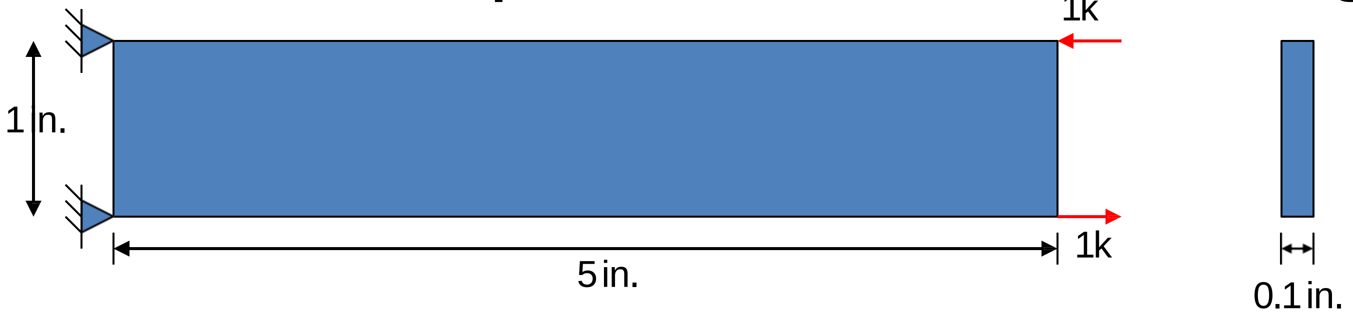


Fig. 3.3-1. (a) A linear strain triangle and its six nodal d.o.f. (b) Displacement mode associated with nodal d.o.f. v_2. (c) Displacement mode associated with nodal d.o.f. v_5. (For visualization only, imagine that displacement occurs normal to the plane of the element.) (*b* and *c* reprinted from [2.2] by permission of John Wiley & Sons, Inc.)

- Will this element work better for the problem?

Example Problem

- Consider the problem we were looking at:



$$I = 0.1 \times 1^3 / 12 = 0.008333\ in^4$$

$$\sigma = \frac{M \times c}{I} = \frac{1 \times 0.5}{0.008333} = 60\ ksi$$

$$\varepsilon = \frac{\sigma}{E} = 0.00207$$

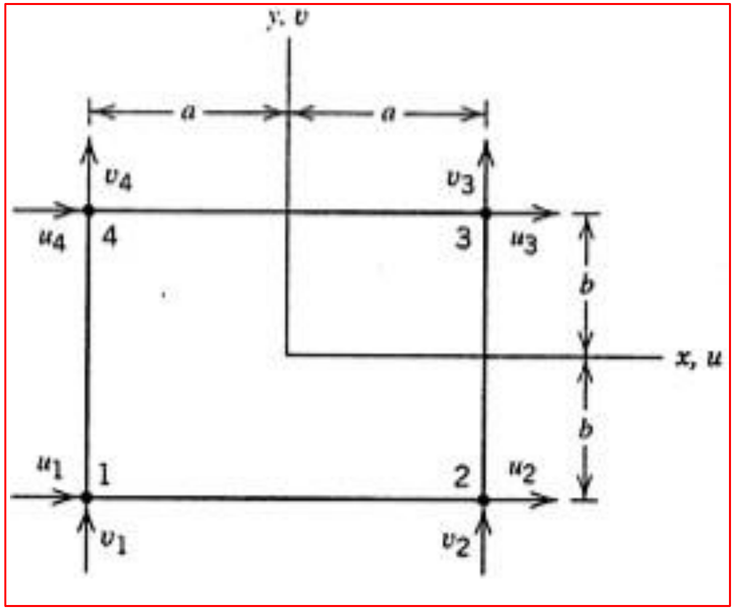
$$\delta = \frac{ML^2}{2EI} = \frac{25}{2 \times 29000 \times 0.008333} = 0.0517\ in.$$

Bilinear Quadratic

- The Q4 element is a quadrilateral element that has four nodes. In terms of generalized coordinates, its displacement field is:

$$u = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy$$
$$v = \beta_5 + \beta_6 x + \beta_7 y + \beta_8 xy \qquad (3.4\text{-}1)$$

$$\varepsilon_x = \beta_2 + \beta_4 y$$
$$\varepsilon_y = \beta_7 + \beta_8 x \qquad (3.4\text{-}2)$$
$$\gamma_{xy} = (\beta_3 + \beta_6) + \beta_4 x + \beta_8 y$$



Bilinear Quadratic

- Shape functions and strain-displacement matrix

$$N_1 = \frac{(a-x)(b-y)}{4ab} \qquad N_2 = \frac{(a+x)(b-y)}{4ab}$$
$$N_3 = \frac{(a+x)(b+y)}{4ab} \qquad N_4 = \frac{(a-x)(b+y)}{4ab} \qquad (3.4\text{-}3)$$

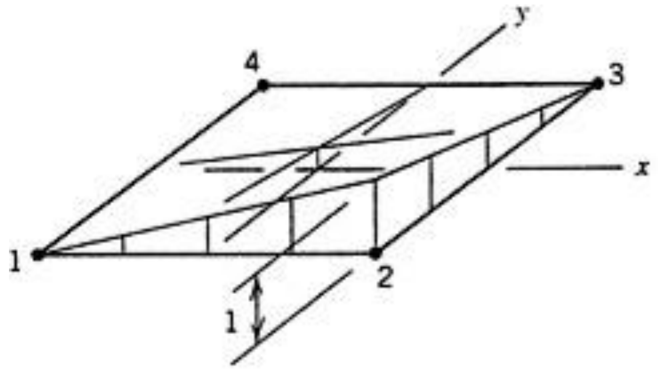
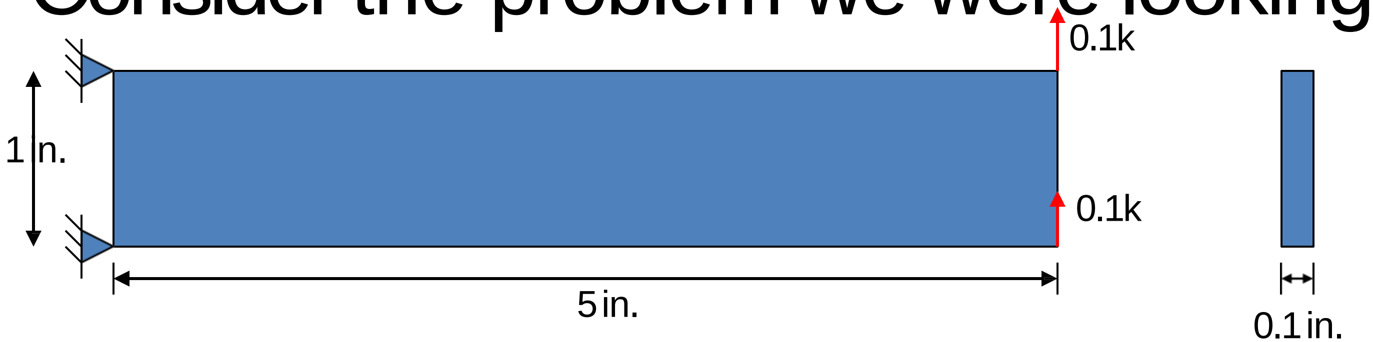


Fig. 3.4-3. Shape function N_2 of the bilinear quadrilateral. (For visualization only, imagine that displacement occurs normal to the *xy* plane.)

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{4ab} \begin{bmatrix} -(b-y) & 0 & (b-y) & 0 & \cdots \\ 0 & -(a-x) & 0 & -(a+x) & \cdots \\ -(a-x) & -(b-y) & -(a+x) & (b-y) & \cdots \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ v_4 \end{Bmatrix} \qquad (3.4\text{-}4)$$

Example Problem

- Consider the problem we were looking at:



$$I = 0.1 \times 1^3 / 12 = 0.008333 in^4$$

$$\sigma = \frac{M \times c}{I} = \frac{1 \times 0.5}{0.008333} = 60ksi$$

$$\varepsilon = \frac{\sigma}{E} = 0.00207$$

$$\delta = \frac{PL^3}{3EI} = \frac{0.2 \times 125}{3 \times 29000 \times 0.008333} = 0.0345 in.$$

Quadratic Quadrilateral Element

- The 8 noded quadratic quadrilateral element uses quadratic functions for the displacements

$$u = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2 + \beta_5 xy + \beta_6 y^2 + \beta_7 x^2 y + \beta_8 xy^2$$
$$v = \beta_9 + \beta_{10} x + \beta_{11} y + \beta_{12} x^2 + \beta_{13} xy + \beta_{14} y^2 + \beta_{15} x^2 y + \beta_{16} xy^2 \qquad (3.5\text{-}1)$$

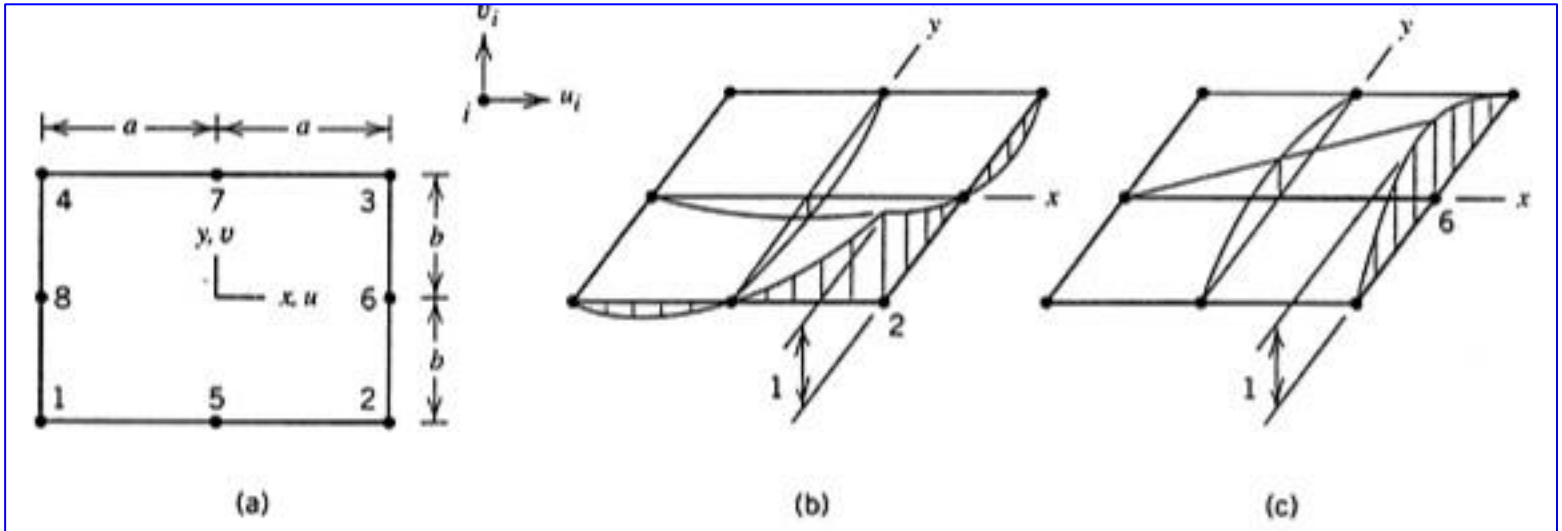


Fig. 3.5-1. (a) A quadratic quadrilateral. (b,c) Shape functions N_2 and N_6 . (For visualization only, imagine that displacement occurs normal to the xy plane.)

Quadratic Quadrilateral Element

- Shape function examples:

$$u = \sum N_i \, u_i \qquad v = \sum N_i \, v_i \tag{3.5-2}$$

where index i runs from 1 to 8, which explains the "8" in the name Q8. As examples, two of the eight shape functions are

$$\begin{aligned} N_2 &= \tfrac{1}{4}(1+\xi)(1-\eta) - \tfrac{1}{4}(1-\xi^2)(1-\eta) - \tfrac{1}{4}(1+\xi)(1-\eta^2) \\ N_6 &= \tfrac{1}{2}(1+\xi)(1-\eta^2) \end{aligned} \tag{3.5-3}$$

- Strain distribution within the element

$$\begin{aligned} \varepsilon_x &= \beta_2 + 2\beta_4 x + \beta_5 y + 2\beta_7 xy + \beta_8 y^2 \\ \varepsilon_y &= \beta_{11} + \beta_{13} x + 2\beta_{14} y + \beta_{15} x^2 + 2\beta_{16} xy \\ \gamma_{xy} &= (\beta_3 + \beta_{10}) + (\beta_5 + 2\beta_{12})x + (2\beta_6 + \beta_{13})y \\ &\quad + \beta_7 x^2 + 2(\beta_8 + \beta_{15})xy + \beta_{16} y^2 \end{aligned} \tag{3.5-4}$$

Quadratic Quadrilateral Element

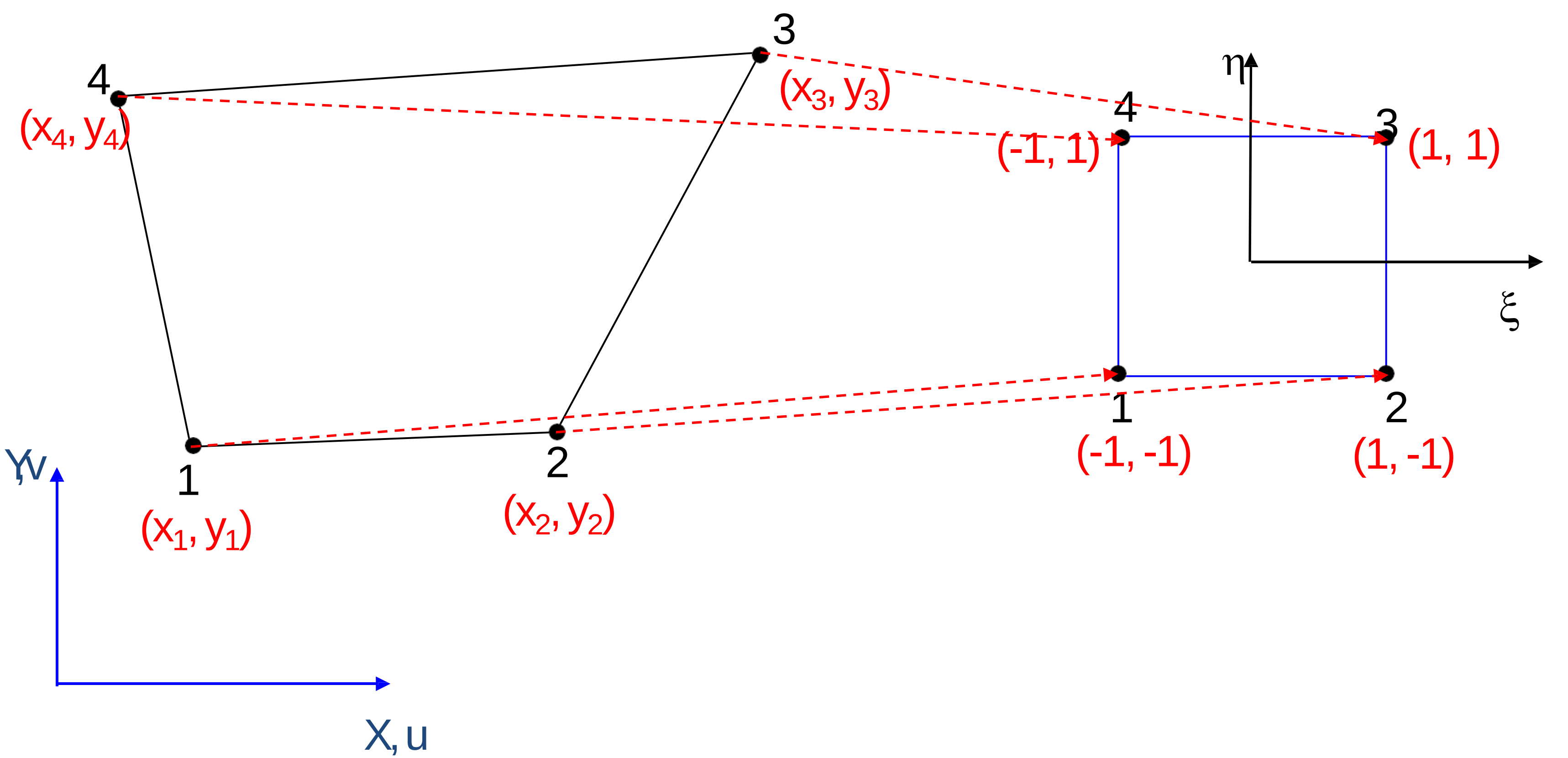
- Should we try to use this element to solve our problem?

- Or try fixing the Q4 element for our purposes.
  - Hmm…tough choice.

Isoparametric Elements and Solution

- Biggest breakthrough in the implementation of the finite element method is the development of an isoparametric element with capabilities to model structure (problem) geometries of any shape and size.
- The whole idea works on mapping.
  - The element in the real structure is mapped to an 'imaginary' element in an ideal coordinate system
  - The solution to the stress analysis problem is easy and known for the 'imaginary' element
  - These solutions are mapped back to the element in the real structure.
  - All the loads and boundary conditions are also mapped from the real to the 'imaginary' element in this approach

Isoparametric Element



Isoparametric element

- The mapping functions are quite simple:

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

Basically, the x and y coordinates of any point in the element are interpolations of the nodal (corner) coordinates.

From the Q4 element, the bilinear shape functions are borrowed to be used as the interpolation functions. They readily satisfy the boundary values too.

Isoparametric element

- Nodal shape functions for displacements

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix}$$

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

- The displacement strain relationships:

$$\varepsilon_x = \frac{\partial u}{\partial X} = \frac{\partial u}{\partial \xi} \bullet \frac{\partial \xi}{\partial X} + \frac{\partial u}{\partial \eta} \bullet \frac{\partial \eta}{\partial X}$$

$$\varepsilon_y = \frac{\partial v}{\partial Y} = \frac{\partial v}{\partial \xi} \bullet \frac{\partial \xi}{\partial Y} + \frac{\partial v}{\partial \eta} \bullet \frac{\partial \eta}{\partial Y}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial X} \\ \dfrac{\partial v}{\partial Y} \\ \dfrac{\partial u}{\partial Y} + \dfrac{\partial v}{\partial X} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial \xi}{\partial X} & \dfrac{\partial \eta}{\partial X} & 0 & 0 \\ 0 & 0 & \dfrac{\partial \xi}{\partial Y} & \dfrac{\partial \eta}{\partial Y} \\ \dfrac{\partial \xi}{\partial Y} & \dfrac{\partial \eta}{\partial Y} & \dfrac{\partial \xi}{\partial X} & \dfrac{\partial \eta}{\partial X} \end{bmatrix} \bullet \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix}$$

But, it is too difficult to obtain $\dfrac{\partial \xi}{\partial X}$ *and* $\dfrac{\partial \eta}{\partial X}$

Isoparametric Element

Hence we will do it another way

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial X} \bullet \frac{\partial X}{\partial \xi} + \frac{\partial u}{\partial Y} \bullet \frac{\partial Y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial X} \bullet \frac{\partial X}{\partial \eta} + \frac{\partial u}{\partial Y} \bullet \frac{\partial Y}{\partial \eta}$$

$$\begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial X}{\partial \xi} & \frac{\partial Y}{\partial \xi} \\ \frac{\partial X}{\partial \eta} & \frac{\partial Y}{\partial \eta} \end{bmatrix} \bullet \begin{Bmatrix} \frac{\partial u}{\partial X} \\ \frac{\partial u}{\partial Y} \end{Bmatrix}$$

It is easier to obtain $\frac{\partial X}{\partial \xi}$ *and* $\frac{\partial Y}{\partial \xi}$

$$J = \begin{bmatrix} \frac{\partial X}{\partial \xi} & \frac{\partial Y}{\partial \xi} \\ \frac{\partial X}{\partial \eta} & \frac{\partial Y}{\partial \eta} \end{bmatrix} = Jacobian$$

defines coordinate transformation

$$\frac{\partial X}{\partial \xi} = \sum \frac{\partial N_i}{\partial \xi} X_i \qquad \frac{\partial Y}{\partial \xi} = \sum \frac{\partial N_i}{\partial \xi} Y_i$$

$$\frac{\partial X}{\partial \eta} = \sum \frac{\partial N_i}{\partial \eta} X_i \qquad \frac{\partial Y}{\partial \eta} = \sum \frac{\partial N_i}{\partial \eta} Y_i$$

$$\therefore \begin{Bmatrix} \frac{\partial u}{\partial X} \\ \frac{\partial u}{\partial Y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix}$$

Gauss Quadrature

- The mapping approach requires us to be able to evaluate the integrations within the domain (-1…1) of the functions shown.
- Integration can be done analytically by using closed-form formulas from a table of integrals (Nah..)
  - Or numerical integration can be performed
- Gauss quadrature is the more common form of numerical integration - better suited for numerical analysis and finite element method.
- It evaluated the integral of a function as a sum of a finite number of terms

$$I = \int_{-1}^{1} \phi \, d\xi \qquad becomes \qquad I \approx \sum_{i=1}^{n} W_i \phi_i$$

Gauss Quadrature

- W_i is the 'weight' and ϕ_i is the value of $f(\xi=i)$

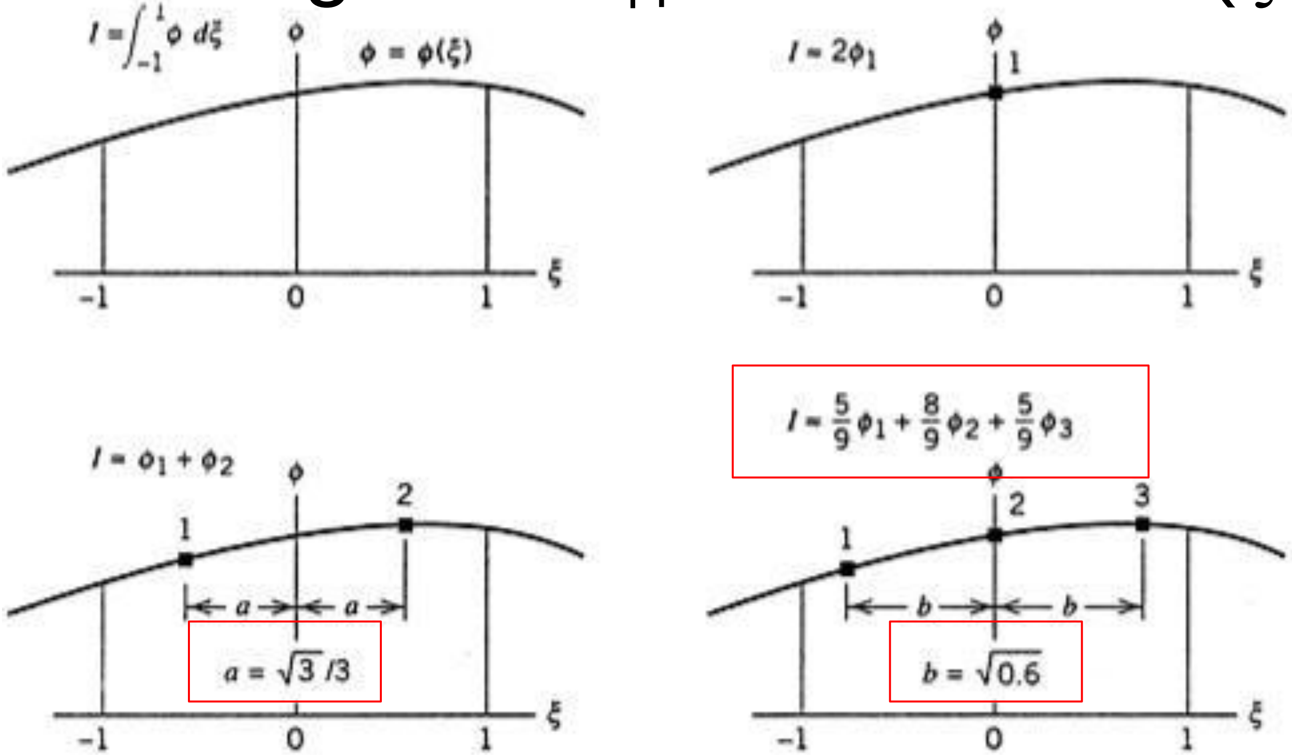
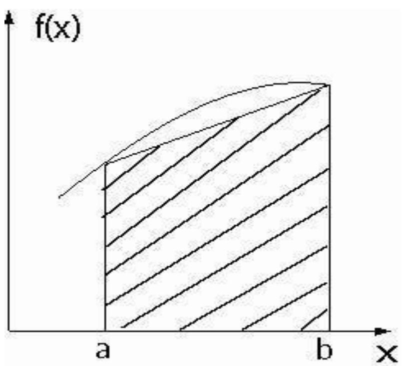


Fig. 4.5-1. Integration of a function $\phi = \phi(\xi)$ in one dimension by Gauss quadrature of orders 1, 2, and 3. Gauss points are numbered.

Numerical Integration

Calculate: $$I = \int_a^b f(x)dx$$

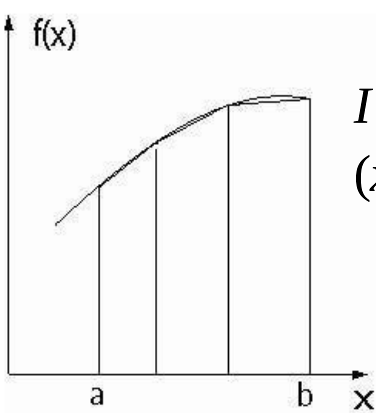
- ***Newton – Cotes integration***
  - ***Trapezoidal rule – 1st order Newton-Cotes integration***



$$f(x) \approx f_1(x) = f(a) + \frac{f(b) - f(a)}{b-a}(x-a)$$

$$I = \int_a^b f(x)dx \approx \int_a^b f_1(x)dx = (b-a)\,\frac{f(a)+f(b)}{2}$$

  - ***Trapezoidal rule – multiple application***



$$I = \int_a^b f(x)dx \approx \int_{x_0}^{x_n} f_n(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx$$

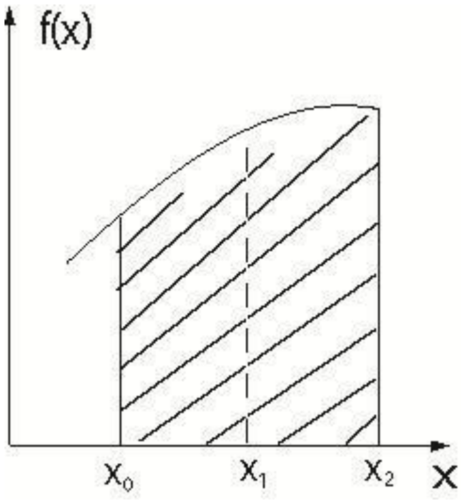
$$I = \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b) \right]$$

Numerical Integration

Calculate: $$I = \int_a^b f(x)dx$$

- ***Newton – Cotes integration***
  - ***Simpson 1/3 rule – 2nd order Newton-Cotes integration***

$$f(x) \approx f_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$



$$I = \int_a^b f(x)dx \approx \int_a^b f_2(x)dx = (x_2 - x_0)\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}$$

Numerical Integration

Calculate: $$I = \int_a^b f(x)\,dx$$

- ***Gaussian Quadrature***

Trapezoidal Rule:

$$I = (b-a)\frac{f(a)+f(b)}{2}$$

$$= \frac{(b-a)}{2} f(a) + \frac{(b-a)}{2} f(b)$$

Gaussian Quadrature:

$$I = c_0 f(x_0) + c_1 f(x_1)$$

Choose c_0, c_1, x_0, x_1 according to certain criteria

Numerical Integration

Calculate: $$I = \int_a^b f(x)dx$$

- **Gaussian Quadrature** $$I = \int_{-1}^{1} f(x)dx = c_0 f(x_0) + c_1 f(x_1) + \cdots + c_{n-1} f(x_{n-1})$$
  - **2pt Gaussian Quadrature**

$$I = \int_{-1}^{1} f(x)dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

  - **3pt Gaussian Quadrature**

$$I = \int_{-1}^{1} f(x)dx = 0.55 \cdot f(-0.77) + 0.89 \cdot f(0) + 0.55 \cdot f(0.77)$$

Let: $$\tilde{x} = -1 + \frac{2(x-a)}{b-a}$$

$$\int_a^b f(x)dx = \frac{1}{2}(b-a)\int_{-1}^{1} f\left[\frac{1}{2}(a+b) + \frac{1}{2}(b-a)\tilde{x}\right]d\tilde{x}$$