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MATHEMATICAL TRANSFORM TECHNIQUES(MTT)
B.TECH IVSEM(CIVIL)

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CONTENTS

TEXT BOOKS

- Advanced Engineering Mathematics by Kreyszig, John Wiley & Sons.
- Higher Engineering Mathematics by Dr. B.S. Grewal, Khanna Publishers

REFERENCE BOOKS

- S. S. Sastry, "Introduction methods of numerical analysis", Prentice-Hall of India Private Limited, 5th Edition, 2005
- G. Shanker Rao, "Mathematical Methods", I. K. International Publications, 1st Edition, 2011.

UNIT-I
Fourier Series

INTRODUCTION

- Suppose that a given function f(x) defined in ($-\pi,\pi$) or ($0, 2\pi$) or in any other interval can be expressed as a trigonometric series as

$$f(x)=a_0/2 + (a_1\cos x + a_2\cos 2x + a_3\cos 3x + \ldots + a_n\cos nx) + (b_1\sin x + b_2\sin 2x + \ldots\ldots b_n\sin nx) + \ldots\ldots$$

$$f(x) = a_0/2 + \sum(a_n\cos nx + b_n\sin nx)$$

- Where a and b are constants with in a desired range of values of the variable such series is known as the fourier series for f(x) and the constants a_0, a_n, b_n are called fourier coefficients of f(x)

- It has period 2π and hence any function represented by a series of the above form will also be periodic with period 2π

POINTS OF DISCONTINUITY

- In deriving the Euler's formulae for a0,an,bn it was assumed that f(x) is continuous. Instead a function may have a finite number of discontinuities. Even then such a function is expressable as a fourier series.

DISCONTINUITY FUNCTION

- For instance, let the function f(x) be defined by
- $f(x) = \varnothing(x),\ c < x < x_0$
- $= \Psi(x),\ x_o < x < c+2\pi$
- where x_0 is thepoint of discontinuity in $(c, c+2\pi)$.

DISCONTINUITY FUNCTION

In such cases also we obtain the fourier series for f(x) in the usual way. The values of a_0,$a_{n,}$$b_n$ are given by

$a_0 = 1/\pi\ [\ \int ø(x)\ dx + \int \Psi(x)\ dx\ ]$

$a_n = 1/\pi\ [\ \int ø(x)\cos nx\ dx + \int \Psi(x)\cos nx\ dx$

$b_n = 1/\pi\ [\ \int ø(x)\sin nx\ dx + \int \Psi(x)\sin nx\ dx]$

EULER'S FORMULAE

- The fourier series for the function f(x) in the interval $C \leq x \leq C+2\pi$ is given by

$$f(x) = a_0/2 + \sum(a_n \cos nx + b_n \sin nx)$$

where $a_0 = 1/\pi \int f(x)\, dx$

$$a_n = 1/\pi \int f(x) \cos nx\, dx$$

$$b_n = 1/\pi \int f(x) \sin nx\, dx$$

These values of a_o, a_n, b_n are known as Euler's formulae

EVEN AND ODD FUNCTIONS

- A function f(x) is said to be even if f(-x)=f(x) and odd if f(-x) = - f(x).
- If a function f(x) is even in (-π, π), its fourier series expansion contains only cosine terms, and their coefficients are a_o
- and a_n.
- $f(x)= a_0/2 + \sum a_n \cos nx$
- where $a_o= 2/ \pi \int f(x)\, dx$
- $a_n= 2/ \pi \int f(x) \cos nx\, dx$

ODD FUNCTION

- When f(x) is an odd function in (-π, π) its fourier expansion contains only sine terms.
- And their coefficient is bn
- $f(x) = \sum b_n \sin nx$
- where $b_n = 2/\pi \int f(x) \sin nx \, dx$

HALF RANGE FOURIER SERIES

- THE SINE SERIES: If it be required to express f(x) as a sine series in (0,π), we define an odd function f(x) in (-π, π) ,identical with f(x) in (0,π).
- Hence the half range sine series (0,π) is given by
- $f(x) = \sum bn \sin nx$
- Where $bn = 2/\pi \int f(x) \sin nx \, dx$

HALF RANGE SERIES

- The cosine series: If it be required to express f(x) as a cosine series, we define an even function f(x) in (- (-π, π) , identical with f(x) in (0, π) , i.e we extend the function reflecting it with respect to the y-axis, so that f(-x)=f(x).

HALF RANGE COSINE SERIES

- Hence the half range series in (o,π) is given by
- $f(x) = a0/2 + \sum an \cos nx$
- where $a0= 2/ \pi\int f(x)dx$
- $an= 2/ \pi\int f(x)\cos nx dx$

CHANGE OF INTERVAL

- So far we have expanded a given function in a Fourier series over the interval $(-\pi,\pi)$and $(0,2\pi)$ of length 2π. In most engineering problems the period of the function to be expanded is not 2π but some other quantity say $2l$. In order to apply earlier discussions to functions of period $2l$, this interval must be converted to the length 2π.

PERIODIC FUNCTION

- Let f(x) be a periodic function with period 2l defined in the interval $c<x<c+2l$. We must introduce a new variable z such that the period becomes 2π.

CHANGE OF INTERVAL

- The fourier expansion in the change of interval is given by
- $f(x) = a0/2+\sum an\cos n\pi x/l + \sum bn \sin n\pi x/l$
- Where $a0 = 1/l \int f(x)dx$
- $an = 1/l \int f(x)\cos n\pi x/l \, dx$
- $bn = 1/l \int f(x)\sin n\pi x/l \, dx$

EVEN AND ODD FUNCTION

- Fourier cosine series : Let f(x) be even function in (-l,l) then
- $f(x) = a0/2 + \sum ancos\ n\pi x/l$
- where $a0 = 2/l \int f(x)dx$
- $an = 2/l \int f(x)\cos n\pi x/l\ dx$

FOURIER SINE SERIES

- Fourier sine series : Let f(x) be an odd function in (-l,l) then
- $f(x) = \sum bn \sin n\pi x/l$
- where $bn = 2/l \int f(x)\sin n\pi x/l \, dx$
- Once ,again here we remarks that the even nature or odd nature of the function is to be considered only when we deal with the interval (-l,l).

HALF-RANGE EXPANSION

- Cosine series: If it is required to expand f(x) in the interval (0,l) then we extend the function reflecting in the y-axis, so that f(-x)=f(x).We can define a new function g(x) such that $f(x)= a_0/2 + \sum a_n \cos n\pi x/l$
- where $a_0= 2/l \int f(x)dx$
- $a_n= 2/l \int f(x)\cos n\pi x/l \, dx$

HALF RANGE SINE SERIES

- Sine series : If it be required to expand f(x)as a sine series in (0,l), we extend the function reflecting it in the origin so that f(-x) = f(x).we can define the fourier series in (-l,l) then,
- $f(x) = a0/2 + \sum bn \sin n\pi x/l$
- where $bn = 2/l \int f(x) \sin n\pi x/l \, dx$

UNIT-II

Fourier Transform

FOURIER INTEGRAL TRANSFORMS

- INTRODUCTION:A transformation is a mathematical device which converts or changes one function into another function. For example, differentiation and integration are transformations.
- In this we discuss the application of finite and infinite fourier integral transforms which are mathematical devices from which we obtain the solutions of boundary value.

- We obtain the solutions of boundary value problems related toengineering. For example conduction of heat, free and forced vibrations of a membrane, transverse vibrations of a string, transverse oscillations of an elastic beam etc.

- DEFINITION: The integral transforms of a function f(t) is defined by
- $F(p)=I[f(t) = \int f(t)\ k(p,t\ )dt$
- Where k(p,t) is called the kernel of the integral transform and is a function of p and t.

FOURIER COSINE AND SINE INTEGRAL

- When f(t) is an odd function cospt,f(t) is an odd function and sinpt f(t) is an even function. So the first integral in the right side becomes zero. Therefore we get
- f(x) = 2/π∫sinpx

FOURIER COSINE AND SINE INTEGRAL

- When f(t) is an odd function cospt,f(t) is an odd function and sinpt f(t) is an even function. So the first integral in the right side becomes zero. Therefore we get
- $f(x) = 2/\pi\int \sin px \int f(t) \sin pt \, dt \, dp$
- which is known as FOURIER SINE INTEGRAL.

- When f(t) is an even function, the second integral in the right side becomes zero.Therefore we get
- $f(x) = 2/\pi\int \cos px \int f(t) \cos pt \, dt \, dp$
- which is known as FOURIER COSINEINTEGRAL.

FOURIER INTEGRAL IN COMPLEX FORM

- Since cos p(t-x) is an even functionof p, we have
- $f(x) = 1/2\pi\iint e^{ip(t-x)} f(t)\, dt\, dp$
- which is the required complex form.

INFINITE FOURIER TRANSFORM

- The fourier transform of a function f(x) is given by
- $F\{f(x)\} = F(p) = \int f(x)\, e^{ipx}\, dx$
- The inverse fourier transform of F(p) is given by
- $f(x) = 1/2\pi \int F(p)\, e^{-ipx}\, dp$

FOURIER SINE TRANSFORM

- The finite Fourier sine transform of f(x) when 0<x<l, is definedas
- Fs{f(x) = Fs (n) sin (nπx)/l dx where n is an integer and the function f(x) is given by
- f(x) = 2/l∑ Fs (n) sin (nπx)/l is called the Inverse finite Fourier sine transform Fs(n)

FOURIER COSINE TRANSFORM

- We have $f(x) = 2/\pi \int \cos px \int f(t) \cos pt \, dt \, dp$
- Which is the fourier cosine integral .Now
- $Fc(p) = \int f(x) \cos px \, dx$ then
- $f(x)$ becomes $f(x) = 2/\pi \int Fc(p) \cos px \, dp$ which is the fourier cosine transform.

PROPERTIES

- Linear property of Fourier transform
- Change of Scale property
- Shifting property
- Modulation property

UNIT-III

Laplace Transform

DEFINITION

- Let f(t) be a function defined for all positive values of t. Then the Laplace transform of f(t), denoted by L{f(t)} or f(s) is defined by $L\{f(t)\}=f(s)=\int e^{-st} f(t)\, dt$
- *Example* 1:$L\{1\}=1/s$
- *Example* 2:$L\{e^{at}\}=1/(s-a)$
- *Example* 3:$L\{Sinat\}=a/(s^2+a^2)$

FIRST SHIFTING THEOREM

- If $L\{f(t)\}=f(s)$, then $L\{e^{at} f(t)\}=f(s-a)$, $s-a>0$ is known as a first shifting theorem.
- *Example* 1: By first shifting theorem the value of $L\{e^{at}Sinbt\}$ is $b/[(s-a)^2+b^2]$
- *Example* 2: $L\{e^{at}t^n\}=n!/(s-a)^{n+1}$
- *Example* 3: $L\{e^{at}Sinhbt\}=b/[(s-a)^2-b^2]$
- *Example* 4: $L\{e^{-at}Sinbt\}=b/[(s+a)^2+b^2]$

UNIT STEP FUNCTION(HEAVISIDES UNIT FUNCTION)

- The unit step function is defined as H(t-a) or u(t-a)=0, if t<a and 1 otherwise.
- $L\{u(t-a)\}=e^{-as} f(s)$
- *Example* 1: The laplace transform of $(t-2)^3u(t-2)$ is $6e^{-2s}/s^4$
- *Example* 2: The laplace transform of $e^{-3t}u(t-2)$ is $e^{-(s+3)}/(s+3)$

CHANGE OF SCALE PROPERTY

- If $L\{f(t)\}=f(s)$, then $L\{f(at)\}=1/a\ f(s/a)$ is known as a change of scale property.
- *Example* 1:By change of scale property the value of $L\{\sin^2 at\}$ is $2a^2/[s(s^2+4a^2]$
- *Example* 2:If $L\{f(t)\}=1/s\ e^{-1/s}$ then by change of scale property the value of $L\{e^{-t}f(3t)\}$ is $e^{-3/(s+1)}/(s+1)$

LAPLACE TRANSFORM OF INTEGRAL

- If $L\{f(t)\}=f(s)$ then $L\{\int f(u)du\}=1/s\ f(s)$ is known as laplace transform of integral.
- *Example* 1:By the integral formula,

  $L\{\int e^{-t}\cos t\ dt\}=(s+1)/[s(s^2+2s+2)]$
- *Example* 2:By the integral formula,

  $L\{\int\int \cosh at\ dt\ dt\}=1/[s(s^2-a^2)]$

LAPLACE TRANSFORM OF $t^n f(t)$

- If f(t) is sectionally continuous and of exponential order and if L{f(t)}=f(s) then L{t.f(t)}=-f(s)
- In general $L\{t^n.f(t)\}= (-1)^n d^n/ds^n f(s)$
- *Example* 1: By the above formula the value of

  $L\{t \cos at\}$ is $(s^2-a^2)/(s^2+a^2)^2$
- *Example* 2: By the above formula the value of

  $L\{t e^{-t} \cosh t\}$ is $(s^2+2s+2)/(s^2+2s)^2$

LAPLACE TRANSFORM OF f(t)/t

- If $L\{f(t)\}=f(s)$, then $L\{f(t)/t\}=\int f(s)ds$, provided the integral exists.
- *Example* 1: By the above formula, the value of $L\{\sin t/t\}$ is $\cot^{-1}s$
- *Example* 2: By the above formula, the value of $L\{(e^{-at}-e^{-bt})/t\}=\log(s+b)/(s+a)$

LAPLACE TRANSFORM OF PERIODIC FUNCTION

- PERIODIC FUNCTION: A function f(t) is said to be periodic, if and only if f(t+T)=f(t) for some value of T and for every value of t. The smallest positive value of T for which this equation is true for every value of t is called the period of the function.
- If f(t) is a periodic function then
- $L\{f(t)\}=1/(1-e^{-sT})\int e^{-st} f(t)\, dt$

INVERSE LAPLACE TRANSFORM

- So far we have considered laplace transforms of some functions f(t). Let us now consider the converse namely, given f(s), f(t) is to be determined. If f(s) is the laplace transform of f(t) then f(t) is called the inverse laplace transform of f(s) and is denoted by $f(t)=L^{-1}\{f(s)\}$

CONVOLUTION THEOREM

- Let f(t) and g(t) be two functions defined for positive numbers t. We define
- $f(t)*g(t)=\int f(u)g(t-u)\,du$
- Assuming that the integral on the right hand side exists.f(t)*g(t) is called the convolution product of f(t) and g(t).
- *Example*: By convolution theorem the value of $L^{-1}\{1/[(s-1)(s+2)]\}$ is $(e^{t}-e^{-2t})/3$

APPLICATION TO DIFFERENTIAL EQUATION

- Ordinary linear differential equations with constant coefficients can be easily solved by the laplace tranform method, without the necessity of first finding the general solution and then evaluating the arbitrary constants. This method, in general, shorter than our earlier methods and is especially suitable to obtain the solution of linear non-homogeneous ordinary differential equations with constant coefficients.

SOLUTION OF A DIFFERENTIAL EQUATION BY LAPLACE TRANSFORM

- Step 1:Take the laplace transform of both sides of the given differential equation.
- Step 2:Use the formula
- $L\{y'(t)\}=sy(s)-y(0)$
- Step 3:Replace y(0),y'(0) etc., with the given initial conditions
- Step 4:Transpose the terms with minus signs to the right
- Step 5:Divide by the coefficient of y, getting y as a known function of s.
- Step 6:Resolve this function of s into partial fractios.
- Step 7:Take the inverse laplace transform of y obtained in step 5. This gives the required solution.

UNIT-IV

Z- Transform

- The z-transform is the most general concept for the transformation of discrete-time series.
- The Laplace transform is the more general concept for the transformation of continuous time processes.
- For example, the Laplace transform allows you to transform a differential equation, and its corresponding initial and boundary value problems, into a space in which the equation can be solved by ordinary algebra.
- The switching of spaces to transform calculus problems into algebraic operations on transforms is called operational calculus. The Laplace and z transforms are the most important methods for this purpose.

The Laplace transform of a function *f*(*t*):

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

The one-sided z-transform of a function *x*(*n*):

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

The two-sided z-transform of a function *x*(*n*):

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Note that expressing the complex variable *z* in polar form reveals the relationship to the Fourier transform:

$$X(re^{i\omega}) = \sum_{n=-\infty}^{\infty} x(n)(re^{i\omega})^{-n}, \text{ or}$$

$$X(re^{i\omega}) = \sum_{n=-\infty}^{\infty} x(n)r^{-n}e^{-i\omega n}, \text{ and} \quad \text{if} \quad r = 1,$$

$$X(e^{i\omega}) = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n}$$

which is the Fourier transform of $x(n)$.

The z-transform of $x(n)$ can be viewed as the Fourier transform of $x(n)$ multiplied by an exponential sequence r^{-n}, and the z-transform may converge even when the Fourier transform does not.

By redefining convergence, it is possible that the Fourier transform may converge when the z-transform does not.

For the Fourier transform to converge, the sequence must have finite energy, or:

$$\sum_{n=-\infty}^{\infty} \left| x(n) r^{-n} \right| < \infty$$

The power series for the z-transform is called a Laurent series:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

The Laurent series, and therefore the z-transform, represents an analytic function at every point inside the region of convergence, and therefore the z-transform and all its derivatives must be continuous functions of *z* inside the region of convergence.

In general, the Laurent series will converge in an annular region of the z-plane.

First we introduce the Dirac delta function (or unit sample function):

$$\delta(n) = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases} \quad \textbf{or} \quad \delta(t) = \begin{cases} 0, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

This allows an arbitrary sequence $x(n)$ or continuous-time function $f(t)$ to be expressed as:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

$$f(t) = \int_{-\infty}^{\infty} f(x)\delta(x-t)dt$$

These are referred to as discrete-time or continuous-time convolution, and are denoted by:

$$x(n) = x(n) * \delta(n)$$

$$f(t) = f(t) * \delta(t)$$

We also introduce the unit step function:

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \quad \text{or} \quad u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Note also:

$$u(n) = \sum_{k=-\infty}^{\infty} \delta(k)$$

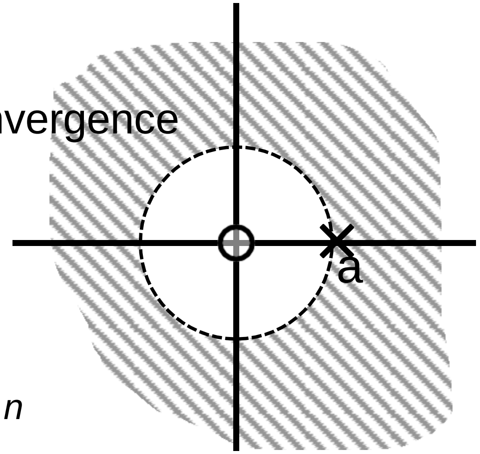
When $X(z)$ is a rational function, i.e., a ration of polynomials in z, then:

1. **The roots of the numerator polynomial are referred to as the zeros of $X(z)$, and**
2. **The roots of the denominator polynomial are referred to as the poles of $X(z)$.**

Note that no poles of $X(z)$ can occur within the region of convergence since the z-transform does not converge at a pole.

Furthermore, the region of convergence is bounded by poles.

Example



$$x(n) = a^{n} u(n)$$

The z-transform is given by:

$$X(z) = \sum_{n=-\infty}^{\infty} a^{n} u(n) z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^{n}$$

Which converges to:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a} \quad \mathit{for} \quad |z| > |a|$$

Clearly, *X*(*z*) has a zero at *z* = 0 and a pole at *z* = *a*.

Convergence of Finite Sequences

Suppose that only a finite number of sequence values are nonzero, so that:

$$X(z) = \sum_{n=n_1}^{n_2} x(n) z^{-n}$$

Where n_1 and n_2 are finite integers. Convergence requires

$$|x(n)| < \infty \quad \text{for} \quad n_1 \le n \le n_2 \ .$$

So that finite-length sequences have a region of convergence that is at least $0 < |z| < \infty$, and may include either $z = 0$ or $z = \infty$.

Inverse z-Transform

The inverse z-transform can be derived by using Cauchy's integral theorem. Start with the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Multiply both sides by z^{k-1} and integrate with a contour integral for which the contour of integration encloses the origin and lies entirely within the region of convergence of $X(z)$:

$$\frac{1}{2\pi i}\oint_C X(z) z^{k-1} dz = \frac{1}{2\pi i}\oint_C \sum_{n=-\infty}^{\infty} x(n) z^{-n+k-1} dz$$

$$= \sum_{n=-\infty}^{\infty} x(n) \frac{1}{2\pi i}\oint_C z^{-n+k-1} dz$$

$$\frac{1}{2\pi i}\oint_C X(z) z^{k-1} dz = x(n) \text{ is the inverse z-transform.}$$

Properties

- z-transforms are linear:

$$\mathcal{Z}[ax(n)+by(n)] = aX(z)+bY(z)$$

- The transform of a shifted sequence:

$$\mathcal{Z}[x(n+n_0)] = z^{n_0}X(z)$$

- Multiplication: $\mathcal{Z}\left[a^n x(n)\right] = Z(a^{-1}z)$

But multiplication will affect the region of convergence and all the pole-zero locations will be scaled by a factor of a.

Convolution of Sequences

$$w(n) = \sum_{k=-\infty}^{\infty} x(k)\,y(n-k)$$

Then

$$W(z) = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k)y(n-k) \right] z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} y(n-k)\,z^{-n}$$

let $m = n - k$

$$W(z) = \sum_{k=-\infty}^{\infty} x(k) \left[\sum_{m=-\infty}^{\infty} y(m)z^{-m} \right] z^{-k}$$

$W(z) = X(z)Y(z)$ for values of z inside the regions of convergence of both.

UNIT-V

Partial Differential Equations

INTRODUCTION

- Equations which contain one or more partial derivatives are called Partial Differential Equations. They must therefore involve atleast two independent variables and one dependent variable. When ever we consider the case of two independent variables we shall usually take them to be x and y and take z to be the dependent variable.The partial differential coefficients

FORMATION 0F P.D.E

- Unlike in the case of ordinary differential equations which arise from the elimination of arbitrary constants the partial differential equations can be formed either by the elimination of arbitrary constants or by the elimination of arbitrary functions from a relation involving three or more variables.

ELIMINATION OF ARBITRARY CONSTANTS

- Consider z to be a function of two independent variables x and y defined by
- f (x,y,z,a,b) = 0……….(1) in which a and b are constants. Differentiating (1) partially with respect to x and y, we obtains two differential equations,let it be equation 2 &3. By means of the 3 equations two constants a and b can be eliminated.This results in a partial differential equation of order one in the form F(x,y,z,p,q)=0.

ELIMINATION OF ARBITRARY FUNCTIONS

- Let u= u(x,y,z) and v=v(x,y,z) be independent functions of the variables x,y,z and let ø(u,v)=0………….(1) be an arbitrary relation between them.We shall obtain a partial differential equation by eliminating the functions u and v. Regarding z as the dependent variable and differentiating (1) partially with respect to x and y, we get

LINEAR P.D.E

- Equation takes the form

  $Pp + Qq = R$

- a partial differential equation in p and q and free of the arbitrary function $\phi(u,v)=0$ a partial differential equation which is linear. If the given relation between x,y,z contains two arbitrary functions then leaving a few exceptional cases the partial differential equations of higher order than the second will be formed.

SOLUTIONS OF P.D.E

- Through the earlier discussion we can understand that a partial differential equation can be formed by eliminating arbitrary constants or arbitrary functions from an equation involving them and three or more variables.
- Consider a partial differential equation of the form $F(x,y,z,p,q)=0 \ldots\ldots(1)$

LINEAR P.D.E

- If this is linear in p and q it is called a linear partial differential equation of first order, if it is non linear in p,q then it is called a non-linear partial differential equation of first order.
- A relation of the type F (x,y,z,a,b)=0…..(2) from which by eliminating a and b we can get the equation (1) is called complete integral or complete solution of P.D.E

PARTICULAR INTEGRAL

- A solution of (1) obtained by giving particular values to a and b in the complete integral (2) is called particular integral.
- If in the complete integral of the form (2) we take

f=(a,b).

COMPLETE INTEGRAL

- A solution of (1) obtained by giving particular values to a and b in the complete integral (2) is called particular integral.
- If in the complete integral of the form (2) we take

f= aø() where a is arbitrary and obtain the envelope of the family of surfaces f(x,y,z,ø(a0) =0

GENERAL INTEGRAL

- Then we get a solution containing an arbitrary function. This is called the general solution of (1) corresponding to the complete integral (2)
- If in this we use a definite function ø(a), we obtain a particular case of the general integral.

SINGULAR INTEGRAL

- If the envelope of the two parameter fmily of surfaces (2) exists, it will also be a solution of (1). It is called a singular integral of the equation (1).
- The singular integral differs from the particular integral. It cannot be obtained that way. A more elaborate discussion of these ideas is beyond the scope.

LINEAR P.D.E OF THE FIRST ORDER

- A differential equation involving partial derivatives p and q only and no higher order derivatives is called a first order equation. If p and q occur in the first degree, it is called a linear partial differential equation of first order, other wise it is called non- linear partial differential equation.

LAGRANGE'S LINEAR EQUATION

- A linear partial differential equation of order one, involving a dependent variable and two independent variables x and y, of the form
- $Pp + Qq = R$
- Where P,Q,R are functions of x,y,z is called Lagrange's linear equation.

PROCEDURE

- Working rule to solve Pp+Qq=R
- First step: write down the subsidary equations $dx/P = dy/Q = dz/R$
- Second step: Find any two independent solutions of the subsidary equations. Let the two solutions be u=a and v=b where a and b are constants.

METHODS OF SOLVING LANGRANGE'S LINEAR EQUATION

- Third step: Now the general solution of Pp+Qq=R is given by f(u,v) = 0 or u=f(v)
- T o solve dx/P = dy/Q = dz/R
- We have two methods
- (i) Method of grouping
- (ii) Method of multipliers

METHOD OF GROUPING

- In some problems it is possible that two of the equations $dx/P= dy/Q= dz/R$ are directly solvable to get solutions $u(x,y)=\text{constant}$ or $v(y,z)= \text{constant}$ or $w(z,x) = \text{constant}$. These give the complete solution.

METHOD OF GROUPING

- Sometimes one of them say $dx/P = dy/Q$ may give rise to solution $u(x,y) = c_1$. From this we may express y, as a function of x. Using this $dy/Q = dz/R$ and integrating we may get $v(y,z) = c_2$. These two relations $u=c_1$, $v=c_2$ give rise to the complete solution.

METHOD OF MULTIPLIERS

- If $a_1/b_1 = a_2/b_2 = a_3/b_3 = \ldots\ldots\ldots = a_n/b_n$ then each ratio is equal to
- $l_1a_1+l_2a_2+l_3a_3+\ldots\ldots+l_na_n$
- $l_1b_1+l_2b_2+l_3b_3+\ldots\ldots+l_nb_n$
- consider $dx/P = dy/Q = dz/R$
- If possible identify multipliers l,m,n not necessarily so that each ratio is equal to

METHOD OF MULTIPLIERS

- If, l,m,n are so chosen that lP+mQ+nR=0 then ldx + mdy + ndz =0, Integrating this we get u(x,y,z) = c_1 similarlyor otherwise get another solution v(x,y,z) = c_2 independent of the earlier one. We now have the complete solution constituted by u=c_1, v= c_2.

NON-LINEAR P.D.E OF FIRST ORDER

- A partial differential equation which involves first order partial derivatives p and q with degree higher than one and the products of p and q is called a non- linear partial differential equations.

DEFINITIONS

- Complete integral: A solution in which the number of arbitrary constants is equal to the number of arbitrary constants is equal to the number of independent variables is called complete integral or complete solution of the given equation

PARTICULAR INTEGRAL

- Particular integral : A solution obtained by giving particular values to the arbitrary constants in the complete integral is called a particular integral.
- Singular integral: Let $f(x,y,z,p,q) = 0$ be a partial differential equation whose complete integral is $\phi(x,y,z,p,q) = 0$

STANDARD FORM I

- Equations of the form $f(p,q)=0$ i.e equations containing p and q only.
- Let the required solution be $z= ax+by+c$
- Where $p=a$, $q=b$.substituting these values in $f(p,q)=0$ we get $f(a,b)=0$
- From this, we can obtainb in terms of a .Let $b=ø(a)$. Then the required solution is
- $Z=ax + ø(a)y+c$.

STANDARD FORM II

- Equations of the form $f(z,p,q)=0$ i.e not containing x and y.
- Let $u=x+ay$ and substitute p and q in the given equation.
- Solve the resulting ordinary differential equation in z and u.
- Substitute $x+ay$ for u.

STANDARD FORM III

- Equations of the form f(x,p)=f(y,q) i.e equations not involving z and the terms containing x and p can be separated from those containing y and q.We assume each side equal to an arbitrary constant a, solve for p and q from the resulting equations
- Solving for p and q, we obtain p= f(x,p) and q= f(y,q) since is a function of x and y we have pdx + q dy integrating which gives the required solution.

STANDARD FORM IV

- CLAUIRT'S FORM : Equations of the form $z=px+qy+f(p,q)$. An equation analogous to clairaut's ordinary differential equation $y=px+f(p)$. The complete solution of the equation $z=px+qy+f(p,q)$. Is
- $z=ax+by+f(a,b)$. Let the required solution be $z=ax+by+c$

METHOD OF SEPARATION OF VARIABLES

- When we have a partial differential equation involving two independent variables say x and y, we seek a solution in the form X(x), Y(Y) and write down various types of solutions.

Heat equation

- Consider heat equation

$$u_t = c\, u_{xx}, \qquad 0 \le x \le 1, \qquad t \ge 0$$

with initial and boundary conditions

$$u(0, x) = f(x), \qquad u(t, 0) = \alpha, \qquad u(t, 1) = \beta$$

- Define spatial mesh points $x_i = i\Delta x$, $i = 0, 1, \ldots, n+1$, where $\Delta x = 1/(n+1)$, and temporal mesh points $t_k = k\Delta t$, for suitably chosen Δt
- Let u_i^k denote approximate solution at (t_k, x_i)

wave equation

- With mesh points defined as before, using centered difference formulas for both u_{tt} and u_{xx} gives finite difference scheme

$$\frac{u_i^{k+1} - 2u_i^k + u_i^{k-1}}{(\Delta t)^2} = c\,\frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{(\Delta x)^2}, \quad \text{or}$$

$$u_i^{k+1} = 2u_i^k - u_i^{k-1} + c\left(\frac{\Delta t}{\Delta x}\right)^2 \left(u_{i+1}^k - 2u_i^k + u_{i-1}^k\right), \; i = 1, \ldots, n$$