STRUCTURAL ANALYSIS

Prepared By

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MOMENT DISTRIBUTION METHOD
UNIT 1
MOMENT DISTRIBUTION METHOD - AN OVERVIEW

• MOMENT DISTRIBUTION METHOD - AN OVERVIEW
• INTRODUCTION
• STATEMENT OF BASIC PRINCIPLES
• SOME BASIC DEFINITIONS
• SOLUTION OF PROBLEMS
• MOMENT DISTRIBUTION METHOD FOR STRUCTURES HAVING NONPRISMATIC MEMBERS
Introduction

(Method developed by Prof. Hardy Cross in 1932)
The method solves for the joint moments in continuous beams and rigid frames by successive approximation.

Statement of Basic Principles

Consider the continuous beam ABCD, subjected to the given loads, as shown in Figure below. Assume that only rotation of joints occur at B, C and D, and that no support displacements occur at B, C and D. Due to the applied loads in spans AB, BC and CD, rotations occur at B, C and D.
In order to solve the problem in a successively approximating manner, it can be visualized to be made up of a continued two-stage problems viz., that of locking and releasing the joints in a continuous sequence.

**Step I**

The joints B, C and D are locked in position before any load is applied on the beam ABCD; then given loads are applied on the beam. Since the joints of beam ABCD are locked in position, beams AB, BC and CD acts as individual and separate fixed beams, subjected to the applied loads; these loads develop fixed end moments.
In beam AB
Fixed end moment at A = -\(wl^2/12\) = - (15)(8)(8)/12 = - 80 kN.m
Fixed end moment at B = +\(wl^2/12\) = +(15)(8)(8)/12 = + 80 kN.m

In beam BC
Fixed end moment at B = - (\(P_{ab}^2\))/l^2 = - (150)(3)(3)^2/6^2
= -112.5 kN.m
Fixed end moment at C = + (\(P_{ab}^2\))/l^2 = + (150)(3)(3)^2/6^2
= + 112.5 kN.m

In beam AB
Fixed end moment at C = -\(wl^2/12\) = - (10)(8)(8)/12 = - 53.33 kN.m
Fixed end moment at D = +\(wl^2/12\) = +(10)(8)(8)/12 = + 53.33 kN.m
Step II

Since the joints B, C and D were fixed artificially (to compute the the fixed-end moments), now the joints B, C and D are released and allowed to rotate. Due to the joint release, the joints rotate maintaining the continuous nature of the beam. Due to the joint release, the fixed end moments on either side of joints B, C and D act in the opposite direction now, and cause a net unbalanced moment to occur at the joint.
**Step III**

These unbalanced moments act at the joints and modify the joint moments at B, C and D, according to their relative stiffnesses at the respective joints. The joint moments are distributed to either side of the joint B, C or D, according to their relative stiffnesses. These distributed moments also modify the moments at the opposite side of the beam span, viz., at joint A in span AB, at joints B and C in span BC and at joints C and D in span CD. This modification is dependent on the carry-over factor (which is equal to 0.5 in this case); when this carry over is made, the joints on opposite side are assumed to be fixed.

**Step IV**

The carry-over moment becomes the unbalanced moment at the joints to which they are carried over. Steps 3 and 4 are repeated till the carry-over or distributed moment becomes small.

**Step V**

Sum up all the moments at each of the joint to obtain the joint moments.
7.3 SOME BASIC DEFINITIONS

In order to understand the five steps mentioned in section 7.3, some words need to be defined and relevant derivations made.

7.3.1 Stiffness and Carry-over Factors

Stiffness = Resistance offered by member to a unit displacement or rotation at a point, for given support constraint conditions.

A clockwise moment $M_A$ is applied at A to produce a +ve bending in beam AB. Find $\theta_A$ and $M_B$. 

E, I – Member properties
Using method of consistent deformations

\[ \Delta_A = + \frac{M_A L^2}{2 EI} \]

Applying the principle of consistent deformation,

\[ \Delta_A + R_A f_{AA} = 0 \quad \Rightarrow \quad R_A = - \frac{3M_A}{2L} \downarrow \]

\[ \theta_A = \frac{M_A L}{EI} + \frac{R_A L^2}{2 EI} = \frac{M_A L}{4 EI} \]

\[ \therefore \quad M_A = \frac{4 EI}{L} \theta_A ; \quad \text{hence} \quad k_\theta = \frac{M_A}{\theta_A} = \frac{4 EI}{L} \]

Stiffness factor = \( k_\theta = \frac{4 EI}{L} \)
Considering moment $M_B$,

$$M_B + M_A + R_A L = 0$$

$$\therefore M_B = M_A/2 = \frac{1}{2}M_A$$

**Carry - over Factor = 1/2**

**Distribution Factor**

Distribution factor is the ratio according to which an externally applied unbalanced moment $M$ at a joint is apportioned to the various members mating at the joint

At joint $B$

$$M - M_{BA} - M_{BC} - M_{BD} = 0$$
\[ M = M_{BA} + M_{BC} + M_{BD} \]

\[
= \left[ \left( \frac{4E_1I_1}{L_1} \right) + \left( \frac{4E_2I_2}{L_2} \right) + \left( \frac{4E_3I_3}{L_3} \right) \right] \theta_B
\]

\[
= \left( K_{BA} + K_{BC} + K_{BD} \right) \theta_B
\]

\[
\therefore \quad \theta_B = \frac{M}{K_{BA} + K_{BC} + K_{BD}} = \frac{M}{\sum K}
\]

\[
M_{BA} = K_{BA} \theta_B = \left( \frac{K_{BA}}{\sum K} \right) M = (D.F)_{BA} M
\]

Similarly

\[
M_{BC} = \left( \frac{K_{BC}}{\sum K} \right) M = (D.F)_{BC} M
\]

\[
M_{BD} = \left( \frac{K_{BD}}{\sum K} \right) M = (D.F)_{BD} M
\]
Modified Stiffness Factor

The stiffness factor changes when the far end of the beam is simply-supported.

As per earlier equations for deformation, given in Mechanics of Solids text-books.

\[
\theta_A = \frac{M_A L}{3EI}
\]

\[
K_{AB} = \frac{M_A}{\theta_A} = \frac{3EI}{L} \left( \frac{3}{4} \right) \left( \frac{4EI}{L} \right) = \frac{3}{4} (K_{AB})_{fixed}
\]
7.4 SOLUTION OF PROBLEMS -

7.4.1 Solve the previously given problem by the moment distribution method

7.4.1.1: Fixed end moments

\[ M_{AB} = -M_{BA} = -\frac{wl^2}{12} = -\frac{(15)(8)^2}{12} = -80 \text{ kN.m} \]

\[ M_{BC} = -M_{CB} = -\frac{wl}{8} = -\frac{(150)(6)}{8} = -112.5 \text{ kN.m} \]

\[ M_{CD} = -M_{DC} = -\frac{wl^2}{12} = -\frac{(10)(8)^2}{12} = -53.333 \text{ kN.m} \]

7.4.1.2 Stiffness Factors (Unmodified Stiffness)

\[ K_{AB} = K_{BA} = \frac{4EI}{L} = \frac{(4)(EI)}{8} = 0.5EI \]

\[ K_{BC} = K_{CB} = \frac{4EI}{L} = \frac{(4)(EI)}{6} = 0.667EI \]

\[ K_{CD} = \begin{bmatrix} 4EI \\ 8 \end{bmatrix} = \frac{4}{8}EI = 0.5EI \]

\[ K_{DC} = \frac{4EI}{8} = 0.5EI \]
Distribution Factors

\[
\begin{align*}
DF_{AB} &= \frac{K_{BA}}{K_{BA} + K_{wall}} = \frac{0.5 \cdot EI}{0.5 + \infty (\text{wall stiffness})} = 0.0 \\
DF_{BA} &= \frac{K_{BA}}{K_{BA} + K_{BC}} = \frac{0.5 \cdot EI}{0.5 \cdot EI + 0.667 \cdot EI} = 0.4284 \\
DF_{BC} &= \frac{K_{BC}}{K_{BA} + K_{BC}} = \frac{0.667 \cdot EI}{0.667 \cdot EI + 0.500 \cdot EI} = 0.5716 \\
DF_{CB} &= \frac{K_{CB}}{K_{CB} + K_{CD}} = \frac{0.667 \cdot EI}{0.667 \cdot EI + 0.500 \cdot EI} = 0.5716 \\
DF_{CD} &= \frac{K_{CD}}{K_{CB} + K_{CD}} = \frac{0.500 \cdot EI}{0.667 \cdot EI + 0.500 \cdot EI} = 0.4284 \\
DF_{DC} &= \frac{K_{DC}}{K_{DC}} = 1.00
\end{align*}
\]
### Moment Distribution Table

<table>
<thead>
<tr>
<th>Joint</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Member</td>
<td>AB</td>
<td>BA</td>
<td>BC</td>
<td>CB</td>
<td>CD</td>
<td>DC</td>
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<td>Distribution Factors</td>
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<td>0.5716</td>
<td>0.5716</td>
<td>0.4284</td>
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<td>Cycle 1</td>
<td>Computed end moments</td>
<td>-80</td>
<td>80</td>
<td>-112.5</td>
<td>112.5</td>
<td>-53.33</td>
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<tr>
<td>Distribution</td>
<td>13.923</td>
<td>18.577</td>
<td>-33.82</td>
<td>-25.35</td>
<td>-53.33</td>
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<tr>
<td>Distribution</td>
<td>7.244</td>
<td>9.662</td>
<td>9.935</td>
<td>7.446</td>
<td>12.35</td>
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<td>Carry-over moments</td>
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<td>4.968</td>
<td>4.831</td>
<td>6.175</td>
<td>3.723</td>
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<td>Distribution</td>
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<td>-6.129</td>
<td>-4.715</td>
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</tr>
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<td>Carry-over moments</td>
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<td>1.179</td>
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<td>-1.187</td>
<td>-0.891</td>
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<tr>
<td>Summed up moments</td>
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<td>99.985</td>
<td>-99.99</td>
<td>96.613</td>
<td>-96.61</td>
<td>0</td>
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</tbody>
</table>
**Computation of Shear Forces**

**Diagram:**
- **Load Distribution:**
  - 15 kN/m distributed along the beam.
  - 150 kN applied at point B.
  - 10 kN/m applied at point C.

**Table:**

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Simply-supported</strong></td>
<td>60</td>
<td>60</td>
<td>75</td>
<td>75</td>
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<tr>
<td><strong>reaction</strong></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>End reaction</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>due to left hand FEM</td>
<td>8.726</td>
<td>-8.726</td>
<td>16.665</td>
<td>-16.67</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>12.079</td>
<td>-12.08</td>
</tr>
<tr>
<td><strong>End reaction</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>due to right hand FEM</td>
<td>-12.5</td>
<td>12.498</td>
<td>-16.1</td>
<td>16.102</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td><strong>Summed-up moments</strong></td>
<td>56.228</td>
<td>63.772</td>
<td>75.563</td>
<td>74.437</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>53.077</td>
<td>27.923</td>
</tr>
</tbody>
</table>
Shear Force and Bending Moment Diagrams

S. F. D.

Max = +35.59 kN.m

M_{max} = +38.985 kN.m

B. M. D.
Simply-supported bending moments at center of span

\[ M_{\text{center in } AB} = (15)(8)^2/8 = +120 \text{ kN.m} \]

\[ M_{\text{center in } BC} = (150)(6)/4 = +225 \text{ kN.m} \]

\[ M_{\text{center in } AB} = (10)(8)^2/8 = +80 \text{ kN.m} \]
The section will discuss moment distribution method to analyze beams and frames composed of nonprismatic members. First the procedure to obtain the necessary carry-over factors, stiffness factors and fixed-end moments will be outlined. Then the use of values given in design tables will be illustrated. Finally the analysis of statically indeterminate structures using the moment distribution method will be outlined.
Stiffness and Carry-over Factors

Use moment-area method to find the stiffness and carry-over factors of the non-prismatic beam.

\[ P_A = (K_A)_{AB} \Delta_A \]

\[ M_A = (K_\theta)_{AB} \theta_A \]

\[ M_B = C_{AB} M_A \]

\( C_{AB} \) = Carry-over factor of moment \( M_A \) from A to B
Use of Betti-Maxwell’s reciprocal theorem requires that the work done by loads in case (a) acting through displacements in case (b) is equal to work done by loads in case (b) acting through displacements in case (a)

\[(M_A)(0) + (M_B)(1) = (M'_A)(1.0) + (M'_B)(0.0)\]

\[C_{AB} K_A = C_{BA} K_B\]
Tabulated Design Tables

Graphs and tables have been made available to determine fixed-end moments, stiffness factors and carry-over factors for common structural shapes used in design. One such source is the Handbook of Frame constants published by the Portland Cement Association, Chicago, Illinois, U. S. A. A portion of these tables, is listed here as Table 1 and 2

Nomenclature of the Tables

\[ a_A, a_b = \text{ratio of length of haunch (at end A and B to the length of span)} \]

\[ b = \text{ratio of the distance (from the concentrated load to end A) to the length of span} \]

\[ h_A, h_B = \text{depth of member at ends A and B, respectively} \]

\[ h_C = \text{depth of member at minimum section} \]
\( I_c = \text{moment of inertia of section at minimum section} = \frac{1}{12}B(h_c)^3, \)
with \( B \) as width of beam

\( k_{AB}, k_{BC} = \text{stiffness factor for rotation at end A and B, respectively} \)

\( L = \text{Length of member} \)

\( M_{AB}, M_{BA} = \text{Fixed-end moments at end A and B, respectively; specified in tables for uniform load w or concentrated force P} \)

\[ r_A = \frac{h_A - h_c}{h_c}, \quad r_B = \frac{h_B - h_c}{h_c} \]

Also \( K_A = \frac{k_{AB} EI_c}{L}, \quad K_B = \frac{k_{BA} EI_c}{L} \)


<table>
<thead>
<tr>
<th>Right Haunch</th>
<th>Carry-over Factors</th>
<th>Stiffness Factors</th>
<th>Unif. Load FEM Coef. × ωL²</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>rₜ₝</td>
<td>Cₜ₝</td>
<td>Cₜ₝</td>
<td>Mₜ₝</td>
<td>Mₜ₝</td>
<td>Mₜ₝</td>
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<td>Mₜ₝</td>
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<tr>
<td>0.2</td>
<td>0.4</td>
<td>0.543</td>
<td>0.766</td>
<td>9.19</td>
<td>6.52</td>
<td>0.1194</td>
<td>0.0791</td>
<td>0.0935</td>
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<tr>
<td>0.6</td>
<td>0.622</td>
<td>0.748</td>
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<td>0.0931</td>
<td>0.0042</td>
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<tr>
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<td>0.740</td>
<td>10.52</td>
<td>9.38</td>
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<td>0.0927</td>
<td>0.0047</td>
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<tr>
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<td>0.734</td>
<td>10.83</td>
<td>10.02</td>
<td>0.0924</td>
<td>0.0550</td>
<td>0.0622</td>
<td>0.0565</td>
</tr>
<tr>
<td></td>
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<td>0.579</td>
<td>0.714</td>
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<td>0.0934</td>
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<td>0.629</td>
<td>0.726</td>
<td>9.98</td>
<td>8.64</td>
<td>0.1120</td>
<td>0.0902</td>
<td>0.0931</td>
<td>0.0042</td>
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<tr>
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<td>0.705</td>
<td>10.85</td>
<td>10.85</td>
<td>0.1034</td>
<td>0.1034</td>
<td>0.0924</td>
<td>0.0552</td>
</tr>
<tr>
<td>2.0</td>
<td>0.771</td>
<td>0.689</td>
<td>11.70</td>
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<td>0.9566</td>
<td>0.1157</td>
<td>0.0917</td>
<td>0.0062</td>
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<tr>
<td></td>
<td>0.4</td>
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<td>14.85</td>
<td>0.0901</td>
<td>0.1246</td>
<td>0.0913</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>aₜ = 0.3</th>
<th>aₗ = variable</th>
<th>rₜ = 1.0</th>
<th>rₗ = variable</th>
</tr>
</thead>
</table>

Note: All carry-over factors are negative and all stiffness factors are positive.
### Table 12-2  Parabolic Haunches—Constant Width

![Diagram of Parabolic Haunches](image.png)

Note: All carry-over factors are negative and all stiffness factors are positive.

<table>
<thead>
<tr>
<th>Right Haunch</th>
<th>Carry-over Factors</th>
<th>Stiffness Factors</th>
<th>Uniform Load FEM Coef. $\times wL^2$</th>
<th>Concentrated Load FEM—Coef. $\times PL$</th>
<th>Haunch Load at FEM Coef. $\times wAL^2$</th>
<th>FEM Coef. $\times wBL^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_B$, $r_B$</td>
<td>$C_{AB}$, $C_{BA}$</td>
<td>$k_{AB}$, $k_{BA}$</td>
<td>$M_{AB}$, $M_{BA}$</td>
<td>$a_A = 0.2$, $a_B = \text{variable}$</td>
<td>$r_A = 1.0$, $r_B = \text{variable}$</td>
<td>$a_A = 0.5$, $a_B = \text{variable}$</td>
</tr>
<tr>
<td>0.4</td>
<td>0.558, 0.627</td>
<td>0.0202, 0.0841</td>
<td>0.0938, 0.0333</td>
<td>0.1022, 0.0502</td>
<td>0.1572, 0.1261</td>
<td>0.0715, 0.1618</td>
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<tr>
<td>0.5</td>
<td>0.582, 0.624</td>
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<td>0.1872, 0.0535</td>
<td>0.1527, 0.1339</td>
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<td>0.0926, 0.0040</td>
<td>0.2079, 0.0448</td>
<td>0.1890, 0.1245</td>
<td>0.0809, 0.1740</td>
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<td>0.571, 0.786</td>
<td>0.1093, 0.0922</td>
<td>0.0922, 0.0043</td>
<td>0.2055, 0.0485</td>
<td>0.1818, 0.1344</td>
<td>0.0719, 0.1862</td>
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<td>0.1063, 0.0961</td>
<td>0.0922, 0.0046</td>
<td>0.2041, 0.0506</td>
<td>0.1764, 0.1417</td>
<td>0.0661, 0.1948</td>
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<td>0.4</td>
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<td>0.1170, 0.0811</td>
<td>0.0926, 0.0040</td>
<td>0.2087, 0.0442</td>
<td>0.1924, 0.1205</td>
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<td>0.5</td>
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<td>0.0922, 0.0046</td>
<td>0.2045, 0.0506</td>
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<td>0.1025, 0.1025</td>
<td>0.0915, 0.0057</td>
<td>0.1970, 0.0626</td>
<td>0.1639, 0.1639</td>
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<td>0.0908, 0.0070</td>
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<td>0.1456, 0.1939</td>
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<td>0.0901, 0.0082</td>
<td>0.1825, 0.0877</td>
<td>0.1307, 0.2193</td>
<td>0.0376, 0.2348</td>
</tr>
</tbody>
</table>
UNIT 1
PART 2
Kani’s Method
Analysis by Kani’s Method:

- Framed structures are rarely symmetric and subjected to side sway, hence Kani’s method is best and much simpler than other methods.

PROCEDURE:

1. Rotation stiffness at each end of all members of a structure is determined depending upon the end conditions.
   - a. Both ends fixed $K_{ij} = K_{ji} = EI/L$
   - b. Near end fixed, far end simply supported $K_{ij} = \frac{3}{4} EI/L$; $K_{ji} = 0$
2. Rotational factors are computed for all the members at each joint it is given by $U_{ij} = -0.5 \left( \frac{K_{ij}}{K_{ji}} \right)$ \{THE SUM OF ROTATIONAL FACTORS AT A JOINT IS -0.5\} (Fixed end moments including transitional moments, moment releases and carry over moments are computed for members and entered. The sum of the FEM at a joint is entered in the central square drawn at the joint).
3. Iterations can be commenced at any joint however the iterations commence from the left end of the structure generally given by the equation $M_{ij} = U_{ij} [ (M_{fi} + M_{ji}) + ? M_{ji})]$
4. Initially the rotational components $M_{ji}$ (sum of the rotational moments at the far ends of the joint) can be assumed to be zero. Further iterations take into account the rotational moments of the previous joints. 5. Rotational moments are computed at each joint successively till all the joints are processed. This process completes one cycle of iteration.
• 6. Steps 4 and 5 are repeated till the difference in the values of rotation moments from successive cycles is neglected.

• 7. Final moments in the members at each joint are computed from the rotational members of the final iterations step. \( M_{ij} = (M_{fij} + M_{??ij}) + 2 M_{ij} + M_{jii} \)
• The lateral translation of joints (side sway) is taken into consideration by including column shear in the iterative procedure.

• 8. Displacement factors are calculated for each storey given by $U_{ij} = -1.5 \left( \frac{K_{ij}}{?K_{ij}} \right)$
• Application Of Analysis Methods For The Portal Frame
• Application of Rotation contribution Method (Kani’s Method) for the analysis of portal frame:
• Fixed end moments
• FEMAB = 0
• FEMBA = 0
• FEMBC = -120 kNm
• FEMCB = 120 kNm
• FEMCD = 0
• FEMDC = 0
• Stiffness and rotation factor (R.F.)
• Table 1.
• Stiffness and Rotation Factors – Kani’s Method
Stiffness and rotation factor (R.F.)

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>$K$</th>
<th>$\Sigma K$</th>
<th>$RF$</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>BA</td>
<td>0.333</td>
<td>0.666 I</td>
<td>-0.25</td>
</tr>
<tr>
<td></td>
<td>BC</td>
<td>0.333</td>
<td>0.666 I</td>
<td>-0.25</td>
</tr>
<tr>
<td>C</td>
<td>CB</td>
<td>0.333</td>
<td>0.583 I</td>
<td>-0.286</td>
</tr>
<tr>
<td></td>
<td>CD</td>
<td>0.25</td>
<td>0.583 I</td>
<td>-0.214</td>
</tr>
</tbody>
</table>
3. Displacement factors (δ)

Table 2. Calculation of Displacement factors (δ)

\[ \Sigma UCD = (-1.2) + (-0.3) = -1.5 \]

Checked.

Hence OK

Storey Moment (SM) Storey moment = 0 (since lack of nodal loads and lack of loadings on columns, SM=0) Iterations by Kani’s Method Figure 2. Calculations of rotation contributions in tabular form using Kani’s Method
• Final End Moments For columns:
  => F.E.M + 2 (near end contribution) + far end contribution of that particular column + L.D.C. of that column
• For beams: => F.E.M + 2 (near end contribution) + far end contribution of that particular beam or slab.
• MAB = 10.89 kNm
• MBA = 58.64 kNm
• MBC = -58.63 kNm
• MCB = 99.49 kNm
• MCD = -69.51 kNm
• MDC = 0 kNm
• MCE = -30 kNm
UNIT 2
SLOPE DEFLECTION METHOD

• Introduction
• Assumptions
• Sign conventions
• Derivation of slope deflection method
• Example
Introduction

• The methods of three moment equation, and consistent deformation method represent the FORCE METHOD of structural analysis. The slope deflection method use displacements as unknowns, hence this method is the displacement method.

• In this method, if the slopes at the ends and the relative displacement of the ends are known, the end moment can be found in terms of slopes, deflection, stiffness and length of the members.
ASSUMPTIONS IN THE SLOPE DEFLECTION METHOD

• This method is based on the following simplified assumptions.

• 1- All the joints of the frame are rigid, i.e., the angle between the members at the joints do not change, when the members of frame are loaded.

• 2- Distortion, due to axial and shear stresses, being very small, are neglected.
Sign Conventions:-

• Joint rotation & Fixed end moments are considered positive when occurring in a clockwise direction.
Derivation of slope deflection equation:
\[ M_{a1} = \frac{4EI}{L} \theta_A \]

\[ M_{b1} = \frac{2EI}{L} \theta_A \]

\[ M_{a2} = \frac{2EI}{L} \theta_B \]

\[ M_{b2} = \frac{4EI}{L} \theta_B \]
• Required $M_{ab}$ & $M_{ba}$ in term of
• $\theta_A$, $\theta_B$ at joint
• rotation of member (R)
• loads acting on member
• First assume ,
• Get $M_{f_{ab}}$ & $M_{f_{ba}}$ due to acting loads. These fixed end moment must be corrected to allow for the end rotations $\theta_A, \theta_B$ and the member rotation $R$.
• The effect of these rotations will be found separately.
\[ M_{a1} = \frac{4EI}{L} \cdot \theta_A \]

\[ M_{b1} = \frac{2EI}{L} \cdot \theta_A \]

\[ M_{b2} = \frac{4EI}{L} \cdot \theta_B \]

\[ M_{a2} = \frac{2EI}{L} \cdot \theta_B \]

\[ M_{b3} = M_{a3} = \frac{-6EI}{L^2} \cdot \Delta \]

\[ = \frac{-6EI}{L} \cdot R \]
by Superposition;

\[ M_{ab} = M_{f_{ab}} + M_{a1} + M_{a2} + M_{a3} \]

\[ M_{f_{ab}} + \frac{4EI}{L} \theta_A + \frac{2EI}{L} \theta_B + \frac{-6EI}{L} R \]

\[ M_{ab} = M_{f_{ab}} + \frac{2EI}{L} (2\theta_A + \theta_B - 3R) \]

\[ \frac{\Delta}{L} = R \]
Example

- Calculate the support moments in the continuous beam having constant flexural rigidity $EI$ throughout, due to vertical settlement of the support $B$ by $5\text{ mm}$. Assume $E = 200\text{ GPa}$ and $I = 4\times10^{-4}\text{ M}^4$. Also plot the quantitative elastic curve.
In the continuous beam, two rotations $B\theta$ and $C\theta$ need to be evaluated. Hence, beam is kinematically indeterminate to second degree. As there is no external load on the beam, in the restrained beam, the fixed end moments are zero.

\[
\begin{align*}
M'_{AA} &= 0 \\
M'_{BB} &= 0 \\
M'_{BC} &= 0 \\
M'_{CC} &= 0
\end{align*}
\]
• For each span, two slope-deflection equations need to be written. In span $AB$, $B$ is below $A$. Hence, the chord $AB$ rotates in clockwise direction. Thus, $\psi_{AB}$ is taken as negative.

Writing slope-deflection equation for span $AB$,

$$ M_{AB} = \frac{2EI}{L} \left(2\theta_A + \theta_B - 3\psi_{AB} \right) $$

For span $AB$, $\theta_A = 0$. Hence,

$$ M_{AB} = \frac{2EI}{5} \left(\theta_B + 3 \times 10^{-3} \right) $$

$$ M_{AB} = 0.4EI\theta_B + 0.0012EI \quad (2) $$

Similarly, for beam-end moment at end $B$, in span $AB$

$$ M_{BA} = 0.4EI \left(2\theta_B + 3 \times 10^{-3} \right) $$

$$ M_{BA} = 0.8EI\theta_B + 0.0012EI \quad (3) $$
In span $BC$, the support $C$ is above support $B$, Hence the chord joining $B'C$ rotates in anticlockwise direction.

$$\psi_{BC} = \psi_{CB} = 1 \times 10^{-3}$$  \hspace{1cm} (4)

Writing slope-deflection equations for span $BC$,

$$M_{BC} = 0.8EI\theta_B + 0.4EI\theta_C - 1.2 \times 10^{-3} EI$$

$$M_{CB} = 0.8EI\theta_C + 0.4EI\theta_B - 1.2 \times 10^{-3} EI$$  \hspace{1cm} (5)

Now, consider the joint equilibrium of support $B$
\[ M_{BA} + M_{BC} = 0 \] (6)

Substituting the values of \( M_{BA} \) and \( M_{BC} \) in equation (6),

\[ 0.8EI\theta_B + 1.2 \times 10^{-3} EI + 0.8EI\theta_B + 0.4EI\theta_C - 1.2 \times 10^{-3} EI = 0 \]

Simplifying,

\[ 1.6\theta_B + 0.4\theta_C = 1.2 \times 10^{-3} \] (7)

Also, the support \( C \) is simply supported and hence, \( M_{CB} = 0 \)

\[ M_{CB} = 0 = 0.8\theta_C + 0.4\theta_B - 1.2 \times 10^{-3} EI \]

\[ 0.8\theta_C + 0.4\theta_B = 1.2 \times 10^{-3} \] (8)

We have two unknowns \( \theta_B \) and \( \theta_C \) and there are two equations in \( \theta_B \) and \( \theta_C \). Solving equations (7) and (8)

\[ \theta_B = -0.4286 \times 10^{-3} \text{ radians} \]

\[ \theta_C = 1.7143 \times 10^{-3} \text{ radians} \] (9)

Substituting the values of \( \theta_B \), \( \theta_C \) and \( EI \) in slope-deflection equations,
Substituting the values of $\theta_B, \theta_C$ and $EI$ in slope-deflection equations,

\[ M_{AB} = 82.285 \text{kN.m} \]
\[ M_{BA} = 68.570 \text{kN.m} \]
\[ M_{BC} = -68.573 \text{kN.m} \]
\[ M_{CB} = 0 \text{kN.m} \quad (10) \]

Reactions are obtained from equations of static equilibrium.
In beam $AB$,

\[ \sum M_B = 0, \quad R_A = 30.171 \text{kN}(\uparrow) \]

\[ R_{BL} = -30.171 \text{kN}(\downarrow) \]

\[ R_{BR} = -13.714 \text{kN}(\downarrow) \]

\[ R_C = 13.714 \text{kN}(\uparrow) \]
• The shear force and bending moment diagram and elastic curve is shown in fig.
Elastic curve
UNIT 2

Two-Hinged Arch
Analysis of two-hinged arch

• A typical two-hinged arch is shown in Fig. 33.

• 1a. In the case of two-hinged arch, we have four unknown reactions, but there are only three equations of equilibrium available. Hence, the degree of statical indeterminacy is one for two-hinged arch.
Fig. 33.1a Two-hinged arch.
• ARCH
• It transfers the load to end support by axial compression & partly by bending & shear action
• Due to equal distribution of stress, the section is fully utilized.
• In arches, its bending moment is low compare to beam.
• Bending Moment=$W^*\times H^*y$
• CLASSIFICATION OF ARCHES Based on shape

• 1. Steel arches
• 2. R.C.C arches
• 3. Masonry arches

• Based on structural behavior
• [1]. Two hinged arches
• [2]. Fixed arches
• [3]. Three hinged arches
• ANALYSIS OF TWO HINGED ARCHES

• A two hinged arch is statically indeterminate to single degree, since there are four reaction components to be determined while the number of equations available from static equilibrium is only three. Considering H to be the redundant reaction, it can be found out by only by the use of Castigliano’s theorem of least work.
Thus, assuming the horizontal span remaining unchanged, we have,
\[ \partial U \partial H = 0, \]
Where
\[ U \] is the total strain energy stored in the arch. Here also, the strain energy stored due to thrust and shear will be
considered negligible in comparison to that due to bending.

\[
U = \int \frac{M^2 ds}{2EI}
\]

\[
\frac{\partial U}{\partial H} = \int \frac{2M}{2EI} \frac{\partial M}{\partial H} \cdot ds = \int \frac{M}{EI} \frac{\partial M}{\partial H} \cdot ds
\]

Now, \( M = \mu - Hy \); \( \frac{\partial M}{\partial H} = -y \)

\[
\frac{\partial U}{\partial H} = 0 = \int \frac{(\mu - Hy)(-y)}{EI} ds
\]

\[
H \int \frac{y^2 ds}{EI} = \int \frac{\mu y ds}{EI}
\]

\[
H = \frac{\int \mu y ds}{\int y^2 ds}
\]

Taking \( dx = ds \cos \theta \) so, \( ds = dx \sec \theta \)

From that we get,

\[
H = \frac{\int \mu y dx}{\int y^2 dx}
\]
• A parabolic Arch hinged at the ends has a span 30 m and rise 5m. A concentrated load of 12 kN acts at 10m from the left hinge. The second moment of area varies as the secant of the slope of the rib axis. Calculate the horizontal thrust and the reactions at the hinges. Also calculate the maximum bending moment anywhere on the arch.
$\Sigma M@A = 0,$
$V_b*30 = 12*10$
$V_b = 4$ KN
Now, $V_a + V_b = 12$ kN
So $V_a = 8$ kN
Equation of parabola,
$Y = \frac{4hx(L-x)}{L^2}$
$Y = \frac{4*5*x(30-x)}{30^2}$
$Y = \frac{x(30-x)}{45}$
Now, horizontal thrust can be found out from the equation
$H = \frac{\int \mu.y dx}{\int y^2 dx}$
$\mu = 8x$, for AC portion
$\mu = 8x - 12(x-10)$, for CB portion
$\mu = (120 - 4x)$
\[
\int_{0}^{30} \mu y \, dx = \int_{0}^{10} 8xy \, dx + \int_{10}^{30} (120 - 4x) y \, dx
\]

\[
= \int_{0}^{10} \frac{8x^2(30-x)}{45} + \int_{10}^{30} \frac{4x(30-x)^2}{45}
\]

\[
\int_{0}^{30} \mu y \, dx = \frac{44000}{9}
\]

\[
\int_{0}^{30} y^2 \, dx = \int_{0}^{30} \frac{x^2(30-x)^2}{45^2} \, dx
\]

\[
= \frac{1}{45^2} \int_{0}^{30} (900x^2 + x^4 - 60x^2x) \, dx
\]

\[
\int_{0}^{30} y^2 \, dx = 400
\]

So, \( H = \frac{44000}{9 \times 400} \)

\( H = 12.22 \) kN

Now, Resultant reaction \( R_a = \sqrt{8^2 + 12.22^2} = 14.61 \) kN

\[ \tan \theta_a = \frac{8}{12.22} = 0.655 \]

\( \theta_a = 33.21^\circ \)

\( R_b = \sqrt{4^2 + 12.22^2} = 12.85 \) kN

\( \theta_b = \tan^{-1} \frac{4}{12.22} \)

Maximum BM will occur in AC, just below the load

Rise of arch at that point

\[
Y = \frac{x(30-x)}{45} = \frac{10(30-10)}{45} = \frac{40}{9} m
\]

\( M_{\text{max}} = 8 \times 10 - 12.22 \times \frac{40}{9} = 25.49 \) kNm
UNIT 3
APPROXIMATE METHODS OF ANALYSIS
Introduction

- Using approximate methods to analyse statically indeterminate trusses and frames
- The methods are based on the way the structure deforms under the load
- Trusses
- Portal frames with trusses
- Vertical loads on building frames
- Lateral loads on building frames
  - Portal method
  - Cantilever method
Approximate Analysis

- Statically determinate structure – the force equilibrium equation is sufficient to find the support reactions
- **Approximate analysis** – to develop a simple model of the structure which is *statically determinate* to solve a statically indeterminate problem
- The method is based on the way the structure deforms under loads
- Their accuracy in most cases compares favourably with more exact methods of analysis (the statically indeterminate analysis)
Determinacy - truss

\[ b + r = 2j \]  \hspace{1cm} \text{Statically determinate}

\[ b + r > 2j \]  \hspace{1cm} \text{Statically indeterminate}

- \( b \) – total number of bars
- \( r \) – total number of external support reactions
- \( j \) – total number of joints
<table>
<thead>
<tr>
<th>real structure</th>
<th>approximation</th>
</tr>
</thead>
</table>

![Truss Diagram]

**Method 1:** Design long, slender diagonals - compressive diagonals are assumed to be a zero force member and all panel shear is resisted by tensile diagonal only.

**Method 2:** Design diagonals to support both tensile and compressive forces - each diagonal is assumed to carry half the panel shear.

\[ b=16, \ r=3, \ j=8 \]
\[ b+r = 19 > 2j=16 \]

The truss is statically indeterminate to the third degree.

Three assumptions regarding the bar forces will be required.
Example 1 - trusses

Determine (approximately) the forces in the members. The diagonals are to be designed to support both tensile and compressive forces.

\[ F_{FB} = F_{AE} = F \]

\[ F_{DB} = F_{EC} = F \]
Portal frames – lateral loads

- Portal frames are frequently used over the entrance of a bridge.
- Portals can be pin supported, fixed supported or supported by partial fixity.
## Portal frames – lateral loads

<table>
<thead>
<tr>
<th>real structure</th>
<th>approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Fixed-supported</strong></td>
<td>A point of inflection is located approximately at the girder’s midpoint</td>
</tr>
<tr>
<td>Three assumption must be made</td>
<td></td>
</tr>
<tr>
<td><strong>Pin-supported</strong></td>
<td>Points of inflection are located approximately at the midpoints of all three members</td>
</tr>
<tr>
<td>Three assumption must be made</td>
<td></td>
</tr>
<tr>
<td><strong>Pin-Jointed</strong></td>
<td>Points of inflection for columns are located approximately at h/3 and the centre of the girder</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Pin-Supported Portal Frames

- A point of inflection – where the moment changes from positive bending to negative bending.
- Bending moment is zero at this point.

The horizontal reactions (shear) at the base of each column are equal.
Fixed-Supported Portal Frames

- A point of inflection – where the moment changes from positive bending to negative bending.
- Bending moment is zero at this point.

The horizontal reactions (shear) at the base of each column are equal.
Frames with trusses

- When a portal is used to span large distance, a truss may be used in place of the horizontal girder
- The suspended truss is assumed to be pin connected at its points of attachment to the columns
- Use the same assumptions as those used for simple portal frames
<table>
<thead>
<tr>
<th>Real Structure</th>
<th>Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>![Real Structure Image]</td>
<td>![Approximation Image]</td>
</tr>
<tr>
<td>pin supported columns</td>
<td>the horizontal reactions (shear) are equal</td>
</tr>
<tr>
<td>pin connection truss-column</td>
<td></td>
</tr>
<tr>
<td>![Real Structure Image]</td>
<td>![Approximation Image]</td>
</tr>
<tr>
<td>fixed supported columns</td>
<td>horizontal reactions (shear) are equal</td>
</tr>
<tr>
<td>pin connection truss-column</td>
<td>there is a zero moment (hinge) on each column</td>
</tr>
</tbody>
</table>

Table showing frames with trusses, comparing real structures with approximations.
Example 2 – Frame with trusses

Determine by approximate methods the forces acting in the members of the Warren portal.
Example 2 (contd)
Building frames – vertical loads

- Building frames often consist of girders that are rigidly connected to columns.
- The girder is statically indeterminate to the third degree – require 3 assumptions.

If the columns are extremely stiff:
- Average point between the two extremes = \(0.21L + 0\)/2 \(\approx 0.1\).

If the columns are extremely flexible:
- Point of zero moment.
Building frames – vertical loads

<table>
<thead>
<tr>
<th>real structure</th>
<th>approximation</th>
</tr>
</thead>
</table>

1. There is zero moment (hinge) in the girder 0.1L from the left support
2. There is zero moment (hinge) in the girder 0.1L from the right support
3. The girder does not support an axial force.
Example 3 – Vertical loads

Determine (approximately) the moment at the joints E and C caused by members EF and CD.
Building frames – lateral loads: Portal method

- A building bent deflects in the same way as a portal frame
- The assumptions would be the same as those used for portal frames
- The *interior columns* would represent the effect of *two portal columns*
Building frames – lateral loads: Portal method

<table>
<thead>
<tr>
<th>real structure</th>
<th>approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="real structure diagram" /></td>
<td><img src="image2.png" alt="approximation diagram" /></td>
</tr>
</tbody>
</table>

1. A hinge is placed at the centre of each girder, since this is assumed to be a point of zero moment.
2. A hinge is placed at the centre of each column, this to be a point of zero moment.
3. At the given floor level the shear at the interior column column hinges is twice that at the exterior column hinges.

The method is most suitable for buildings having low elevation and uniform framing.
Example 4 – Portal method

Determine (approximately) the reactions at the base of the columns of the frame.
Building frames – lateral loads: Cantilever method

- The method is based on the same action as a long cantilevered beam subjected to a transverse load.
- It is reasonable to assume the axial stress has a linear variation from the centroid of the column areas.
Building frames – lateral loads: Cantilever method

<table>
<thead>
<tr>
<th>real structure</th>
<th>approximation</th>
</tr>
</thead>
</table>

- The method is most suitable if the frame is tall and slender, or has columns with different cross-sectional areas.

1. zero moment (hinge) at the centre of each girder
2. zero moment (hinge) at the centre of each column
3. The axial stress in a column is proportional to its distance from the centroid of the cross-sectional areas of the columns at a given floor level.

\[
\bar{x} = \frac{\sum x_i A_i}{\sum A_i}
\]
Example 5 – Cantilever method

Show how to determine (approximately) the reactions at the base of the columns of the frame.
Example 5 – Cantilever method
UNIT 4

• MATRIX METHODS OF ANALYSIS
Methods to Solve Indeterminate Problem

- Small degree of statical indeterminacy
  - Force method
  - Displacement methods
  - Displacement method in matrix formulation
- Large degree of statical indeterminacy
  - Numerical methods
ADVANTAGES AND DISADVANTAGES OF MATRICES METHODS

Advantages:
• very formalized and computer-friendly;
• versatile, suitable for large problems;
• applicable for both statically determinate and indeterminate problems.

Disadvantages:
• bulky calculations (not for hand calculations);
• structural members should have some certain number of unknown nodal forces and nodal displacements; for complex members such as beams and arbitrary solids this requires some discretization, so no analytical solution is possible.
FLOWCHART OF MATRIX METHOD

Press Esc to exit full screen

Classification of members

- Stiffness matrices for members
- Stiffness matrices are composed according to member models

Transformed stiffness matrices

- Stiffness matrices are transformed from local to global coordinates

Final equation
\[ F = K \cdot Z \]

- Unknown displacements and reaction forces are calculated

Stress-strain state of structure

- Stiffness matrices of separate members are assembled into a single stiffness matrix \( K \)
Stiffness matrix \( K \) gives the relation between the vector of nodal forces \( F \) and nodal displacements \( Z \).

\[
F = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{pmatrix} \quad F = K \cdot Z \\
Z = \begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{pmatrix}
\]
Stiffness relation for a rod:

\[ F_i = -\frac{EA}{L} \cdot (x_j - x_i) \]

\[ F_j = \frac{EA}{L} \cdot (x_j - x_i) \]

\[ F = K \cdot Z \]

Stiffness matrix:

\[ K = \begin{pmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{pmatrix} \]
To assemble stiffness matrices of separate members into a single matrix for the whole structure, we simply add terms for corresponding displacements. Physically, this procedure represents the usage of compatibility and equilibrium equations.
Let’s consider a system of two rods:

\[
\begin{pmatrix}
F_i \\
F_j \\
F_k
\end{pmatrix} = \frac{EA}{L} \cdot \begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix} \cdot \begin{pmatrix}
x_i \\
x_j \\
x_k
\end{pmatrix}
\]

\[
\begin{pmatrix}
F_i \\
F_j \\
F_k
\end{pmatrix} = \frac{EA}{L} \cdot \begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix} \cdot \begin{pmatrix}
x_i \\
x_j \\
x_k
\end{pmatrix}
\]
Given

Boundary conditions:

Equilibrium conditions:

Matrix equation:

Solution of system:

Forces:

\[
\begin{bmatrix}
-10 \\
0 \\
10
\end{bmatrix}
\]

Displacements:

\[
\begin{bmatrix}
0 \\
5 \cdot 10^{-6} \\
10
\end{bmatrix}
\]
Given

Boundary conditions:

Matrix equation:

Solution of system:

Forces:

\[
\text{Res}_0 = \begin{pmatrix}
-5 \\
10 \\
-5
\end{pmatrix}
\]

Displacements:

\[
\text{Res}_1 = \begin{pmatrix}
0 \\
2.5 \cdot 10^{-6} \\
0
\end{pmatrix}
\]

\[
F_j = 10 \quad Z_i = 0
\]

\[
F = \frac{E \cdot A}{L} \cdot K \cdot Z
\]

\[
\text{Res} := \text{Find}(F, Z)
\]
Transformation matrix is used to transform node displacements and forces from local to global coordinate system (CS) and vice versa:

\[ \bar{F} = T \cdot F \]
\[ \bar{Z} = T \cdot Z \]

Transformation matrix is always orthogonal, inverse matrix is equal to transposed matrix:

\[ T^{-1} = T^T \]

The transformation from local CS to global CS:

\[ F = T^T \cdot \bar{F} \]
\[ Z = T^T \cdot \bar{Z} \]
For simplest member (rod) we get:

\[
Z = \begin{pmatrix} 
  x_i \\
  y_i \\
  x_j \\
  y_j 
\end{pmatrix} \quad \bar{Z} = \begin{pmatrix} 
  \bar{x}_i \\
  \bar{y}_i \\
  \bar{x}_j \\
  \bar{y}_j 
\end{pmatrix}
\]

\[
T = \begin{pmatrix} 
  \cos(\theta) & \sin(\theta) & 0 \\
  -\sin(\theta) & \cos(\theta) & 0 \\
  0 & 0 & \cos(\theta) \\
  0 & 0 & -\sin(\theta) 
\end{pmatrix}
\]

\[
\bar{Z} = T \cdot Z
\]
To transform the stiffness matrix from local CS to global CS, the following formula is used:

\[ K = T^T \cdot \bar{K} \cdot T \]
The truss has three members, thus 6 degrees of freedom. The stiffness matrix will be 6x6.
EXAMPLE FOR A TRUSS

\[ \text{ORIGIN} := 1 \]

Number of DOFs \( d := 6 \quad u := 1 \ldots d \)

Number of members \( n := 3 \quad i := 1 \ldots n \)

Geometrical and physical properties: \( EA \)

Stiffness matrix in local CS:

\[
K_{\text{LCS}_i} := \frac{EA}{L_i} \cdot \begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
EXAMPLE FOR A TRUSS

Transformation matrix:

\[
T(\theta) := \begin{pmatrix}
\cos(\theta) & \sin(\theta) & 0 & 0 \\
-\sin(\theta) & \cos(\theta) & 0 & 0 \\
0 & 0 & \cos(\theta) & \sin(\theta) \\
0 & 0 & -\sin(\theta) & \cos(\theta)
\end{pmatrix}
\]

Angles and stiffness matrices for members:

\[
\theta := \begin{pmatrix}
0 \\
\frac{5\cdot\pi}{4} \\
\frac{3\cdot\pi}{2}
\end{pmatrix}
\]

\[
K_{GCS_i} := T(\theta_i)^T \cdot K_{LC}
\]
EXAMPLE FOR A TRUSS

Results for stiffness matrices:

\[
\begin{align*}
K_{GCS_1} &= \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & K_{GCS_3} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{pmatrix} \\
K_{GCS_2} &= \begin{pmatrix} 0.354 & 0.354 & -0.354 & -0.354 \\ 0.354 & 0.354 & -0.354 & -0.354 \\ -0.354 & -0.354 & 0.354 & 0.354 \\ -0.354 & -0.354 & 0.354 & 0.354 \end{pmatrix}
\end{align*}
\]
Indexes of nodes for members:

\[
\text{Ind} := \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 3 \end{pmatrix}
\]
EXAMPLE FOR A TRUSS

Assembly of stiffness matrix:

\[
K := \begin{cases} 
\text{for } j \in 1..n \\
\quad \text{for } k \in 1..n \\
\quad \quad K_{j,k} \leftarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
\quad \text{for } i \in 1..3 \\
\quad \quad K_{Ind_i,1,Ind_i,1} \leftarrow K_{Ind_i,1,Ind_i,1} + \text{submatrix}(K_{Ind_i,1,Ind_i,1}) \\
\quad \quad K_{Ind_i,1,Ind_i,2} \leftarrow K_{Ind_i,1,Ind_i,2} + \text{submatrix}(K_{Ind_i,1,Ind_i,2}) \\
\quad \quad K_{Ind_i,2,Ind_i,1} \leftarrow K_{Ind_i,2,Ind_i,1} + \text{submatrix}(K_{Ind_i,2,Ind_i,1}) \\
\quad \quad K_{Ind_i,2,Ind_i,2} \leftarrow K_{Ind_i,2,Ind_i,2} + \text{submatrix}(K_{Ind_i,2,Ind_i,2}) \\
\end{cases}
\]
EXAMPLE FOR A TRUSS

Result after previous step:

\[
K = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}\begin{pmatrix}
-1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}\begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
\begin{pmatrix}
1.354 & 0.354 & \cdots & 0.354 \\
0.354 & 0.354 & \cdots & 0.354 \\
\vdots & \vdots & \ddots & \vdots \\
-0.354 & -0.354 & \cdots & 0.354
\end{pmatrix}\begin{pmatrix}
-0.354 \\
-0.354 \\
\vdots \\
0.354
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
\]
EXAMPLE FOR A TRUSS

\[
K := \begin{cases}
\text{for } j \in 1..d \\
\quad \text{for } k \in 1..d \\
\quad \quad M \leftarrow K \ \text{ceil} \left( \frac{j}{2} \right), \text{ceil} \left( \frac{k}{2} \right) \\
\quad \quad KN_{j,k} \leftarrow M_{2-\text{mod}(j,2), 2-\text{mod}(k,2)}
\end{cases}
\]

\[
K = \begin{pmatrix}
1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 1.354 & 0.354 & -0.354 & -0.354 & 0 \\
0 & 0 & 0.354 & 0.354 & -0.354 & -0.354 & 0 \\
0 & 0 & -0.354 & -0.354 & 0.354 & 0.354 & 0 \\
0 & -1 & -0.354 & -0.354 & 0.354 & 1.354 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
THREE BASIC EQUATIONS
How are they implemented in matrix methods?

- Equilibrium equations
- Constitutive equations
  - Taken into account when global stiffness matrix is assembled from member matrices
- Compatibility equations
- Through member stiffness matrices
  - Taken into account when global stiffness matrix is assembled from member matrices
UNIT 5

INFLUENCE LINES FOR STATICALLY INDETERMINATE STRUCTURES
3. INFLUENCE LINES FOR STATICALLY DETERMINATE STRUCTURES - AN OVERVIEW

- Introduction - What is an influence line?
- Influence lines for beams
- Qualitative influence lines - Muller-Breslau Principle
- Influence lines for floor girders
- Influence lines for trusses
- Live loads for bridges
- Maximum influence at a point due to a series of concentrated loads
- Absolute maximum shear and moment
INTRODUCTION TO INFLUENCE LINES

- **Influence lines describe the variation of an analysis variable** (reaction, shear force, bending moment, twisting moment, deflection, etc.) at a point (say at C in Figure 6.1)

  ![Influence Lines Diagram](image)

  - Why do we need the influence lines? For instance, when loads pass over a structure, say a bridge, one needs to know when the maximum values of shear/reaction/bending-moment will occur at a point so that the section may be designed

- **Notations:**
  - **Normal Forces** - +ve forces cause +ve displacements in +ve directions
  - **Shear Forces** - +ve shear forces cause clockwise rotation & - ve shear force causes anti-clockwise rotation
  - **Bending Moments**: +ve bending moments cause “cup holding water” deformed shape
INFLUENCE LINES FOR BEAMS

• Procedure:
  (1) **Allow a unit load** (either 1b, 1N, 1kip, or 1 tonne) **to move over beam from left to right**
  (2) **Find the values** of shear force or bending moment, **at the point under consideration**, as the unit load moves over the beam from left to right
  (3) **Plot the values** of the shear force or bending moment, **over the length of the beam, computed for the point under consideration**
3.3 MOVING CONCENTRATED LOAD

3.3.1 Variation of Reactions $R_A$ and $R_B$ as functions of load position

$$\sum M_A = 0$$

$$(R_B)(10) - (1)(x) = 0$$

$R_B = x/10$

$R_A = 1 - R_B$

$= 1 - x/10$

---

$R_A = 1 - x/10$

$R_B = x/10$

---

---

---

---

120
$R_A$ occurs only at A; $R_B$ occurs only at B

Influence line for $R_A$

Influence line for $R_B$
3.3.2 Variation of Shear Force at C as a function of load position

$0 < x < 3$ ft (unit load to the left of C)

Shear force at C is $-ve$, $V_C = -x/10$
3 < x < 10 ft (unit load to the right of C)

Shear force at C is +ve = 1-x/10

Influence line for shear at C
3.3.3 Variation of Bending Moment at C as a function of load position

\[0 < x < 3.0 \text{ ft (Unit load to the left of C)}\]

Bending moment is +ve at C
3 < x < 10 ft (Unit load to the right of C)

Moment at C is +ve

Influence line for bending

Moment at C

(1-7/10)(3) = 2.1 kip-ft
3.4 QUALITATIVE INFLUENCED LINES - MULLER-BRESLAU’S PRINCIPLE

- The principle **gives only a procedure** to determine of the influence line of a parameter for a determinate or an indeterminate structure.
- But **using the basic understanding of the influence lines**, the **magnitudes** of the influence lines also **can be computed**.
- In order to **draw the shape of the influence lines properly**, the **capacity of the beam to resist the parameter investigated** (reaction, bending moment, shear force, etc.), **at that point, must be removed**.
- The principle states that: **The influence line for a parameter** (say, reaction, shear or bending moment), at a point, **is to the same scale as the deflected shape of the beam**, when the beam is acted upon by that parameter.
  - The **capacity of the beam to resist that parameter**, at that point, **must be removed**.
  - **Then allow the beam to deflect under that parameter**.
  - **Positive directions of the forces are the same** as before.
PROBLEMS - Influence Line for a Determinate Beam by Muller-Breslau’s Method

Influence line for Reaction at A

Fig. 6-12
Influence Lines for a Determinate Beam by Muller-Breslau’s Method

Influence Line for Shear at C

Influence Line for Bending Moment at C

Fig. 6–13

Fig. 6–14
Influence Lines for an Indeterminate Beam by Muller-Breslau’s Method

Influence Line for Bending Moment at E

Influence Line for Shear at E

Fig. 9–24

Influence Line for Bending Moment at E

Fig. 9–25
INFLUENCE LINE FOR FLOOR GIRDERS

Floor systems are constructed as shown in figure below,

![Diagram of floor system]

Fig. 8.14
INFLUENCE LINES FOR FLOOR GIRDERS (Cont’d)

Fig. 8.10
3.6 INFLUENCE LINES FOR FLOOR GIRDERS (Cont’d)

3.6.1 Force Equilibrium Method:

Draw the Influence Lines for: (a) Shear in panel CD of the girder; and (b) the moment at E.

5 spaces @ 10’ each = 50 ft
Place load over region A´B´ (0 < x < 10 ft)

Find the shear over panel CD

\[ V_{CD} = - \frac{x}{50} \]

At \( x = 0 \), \( V_{CD} = 0 \)

At \( x = 10 \), \( V_{CD} = -0.2 \)

Shear is -ve

\[ R_F = \frac{x}{50} \]

Find moment at \( E = +\left(\frac{x}{50}\right)(10) = +\frac{x}{5} \)

At \( x = 0 \), \( M_E = 0 \)

At \( x = 10 \), \( M_E = +2.0 \)

+ve moment
Continuation of the Problem

I. L. for $V_{CD}$

I. L. for $M_E$
Problem Continued - Place load over region B´C´ (10 ft < x < 20ft)

\[ V_{CD} = -\frac{x}{50} \text{ kip} \]

At \( x = 10 \) ft
\[ V_{CD} = -0.2 \]

At \( x = 20 \) ft
\[ V_{CD} = -0.4 \]

\[ M_E = \left(\frac{x}{50}\right)(10) = \frac{x}{5} \text{ kip.ft} \]

At \( x = 10 \) ft, \( M_E = +2.0 \) kip.ft

At \( x = 20 \) ft, \( M_E = +4.0 \) kip.ft

Shear is -ve

Moment is +ve
I. L. for $V_{CD}$

I. L. for $M_E$
Place load over region C’D’ (20 ft < x < 30 ft)

When the load is at C’ (x = 20 ft)

\[ V_{CD} = -0.4 \text{ kip} \]

When the load is at D’ (x = 30 ft)

\[ V_{CD} = + \frac{20}{50} = +0.4 \text{ kip} \]
\[ M_E = + \left( \frac{x}{50} \right) (10) = + \frac{x}{5} \]

\[ R_F = \frac{x}{50} \]

\[ \begin{align*}
A' & \rightarrow B' \rightarrow C' \rightarrow D' \\
\text{Load } P & \rightarrow E
\end{align*} \]

**I. L. for \( V_{CD} \)**

**I. L. for \( M_E \)**

\( +ve \) moment

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.2</th>
<th>-ve</th>
<th>0.4</th>
<th>+ve</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>2.0</td>
<td>4.0</td>
<td>6.0</td>
<td></td>
</tr>
</tbody>
</table>
Place load over region D′E′ (30 ft < x < 40 ft)

\[ V_{CD} = + (1-x/50) \text{ kip} \]

\[ R_A = (1-x/50) \]

Shear is +ve

\[ M_E = +(x/50)(10) = + x/5 \text{ kip.ft} \]

Moment is +ve

\[ R_F = x/50 \]

At x = 30 ft, \( M_E = +6.0 \)

At x = 40 ft, \( M_E = +8.0 \)
Problem continued

I. L. for $V_{CD}$

I. L. for $M_E$
Place load over region E’F’ (40 ft < x < 50 ft)

\[ V_{CD} = +1-x/50 \]

At \( x = 40 \) ft, \( V_{CD} = +0.2 \)

At \( x = 50 \) ft, \( V_{CD} = 0.0 \)

\[ R_A = 1-x/50 \]

Shear is +ve

\[ M_E = + (1-x/50)(40) = (50-x)*40/50 = +(4/5)(50-x) \]

At \( x = 40 \) ft, \( M_E = +8.0 \text{ kip.ft} \)

At \( x = 50 \) ft, \( M_E = 0.0 \)
INFLUENCE LINES FOR TRUSSES

Draw the influence lines for: (a) Force in Member GF; and 
(b) Force in member FC of the truss shown below in Figure below
Problem 3.7 continued -
3.7.1 Place unit load over AB

(i) To compute GF, cut section (1) - (1)

Taking moment about B to its right,
\[(R_D)(40) - (F_{GF})(10\sqrt{3}) = 0\]
\[F_{GF} = (x/60)(40)(1/10\sqrt{3}) = x/(15\sqrt{3}) (-ve)\]
PROBLEM CONTINUED -

(ii) To compute $F_{FC}$, cut section (2) - (2)

Resolving vertically over the right hand section

$$F_{FC} \cos 30^0 - R_D = 0$$

$$F_{FC} = \frac{R_D}{\cos 30} = \frac{x/60}{2/\sqrt{3}} = \frac{x}{30 \sqrt{3}} \text{ (ve)}$$
At $x = 0$, $F_{FC} = 0.0$
At $x = 20$ ft, $F_{FC} = -0.385$

I. L. for $F_{GF}$

I. L. for $F_{FC}$
The problem continues as follows:

**PROBLEM Continued -**

Place unit load over BC (20 ft < x < 40 ft)

[Section (1) - (1) is valid for 20 < x < 40 ft]

**i) To compute \( F_{GF} \) use section (1) -(1)**

Taking moment about B, to its left,

\[
(R_A)(20) - (F_{GF})(10\sqrt{3}) = 0
\]

\[
F_{GF} = \frac{(20R_A)}{(10\sqrt{3})} = \frac{(1-x/60)(2/\sqrt{3})}{1}
\]

At \( x = 20 \) ft, \( F_{FG} = 0.77 \) (-ve)

At \( x = 40 \) ft, \( F_{FG} = 0.385 \) (-ve)
PROBLEM Continued -
(ii) To compute $F_{FC}$, use section (2) - (2)

Section (2) - (2) is valid for $20 < x < 40$ ft

Resolving force vertically, over the right hand section,

$$F_{FC} \cos 30 - \left( \frac{x}{60} \right) + \left( \frac{x-20}{20} \right) = 0$$

$$F_{FC} \cos 30 = \frac{x}{60} - \frac{x}{20} + 1 = \frac{1-2x}{60} \text{ (-ve)}$$

$$F_{FC} = \left( \frac{60 - 2x}{60} \right)(2/\sqrt{3}) \text{ -ve}$$
At $x = 20$ ft, $F_{FC} = (20/60)(2/ \sqrt{3}) = 0.385$ (-ve)

At $x = 40$ ft, $F_{FC} = ((60-80)/60)(2/ \sqrt{3}) = 0.385$ (+ve)

I. L. for $F_{GF}$

I. L. for $F_{FC}$
PROBLEM Continued -
3.7.3 Place unit load over CD (40 ft < x < 60 ft)

(i) To compute $F_{GF}$, use section (1) - (1)

Take moment about B, to its left,

$$(F_{FG})(10\sqrt{3}) - (R_A)(20) = 0$$

$$F_{FG} = (1-x/60)(20/10\sqrt{3}) = (1-x/60)(2/\sqrt{3}) \text{-ve}$$

At $x = 40$ ft, $F_{FG} = 0.385$ kip (-ve)
At $x = 60$ ft, $F_{FG} = 0.0$
**PROBLEM Continued**

(ii) To compute $F_{FG}$, use section (2) - (2)

Resolving forces vertically, to the left of C,

\[(R_A) - F_{FC} \cos 30 = 0\]

\[F_{FC} = R_A / \cos 30 = (1-x/10) (2/\sqrt{3}) +ve\]
At $x = 40$ ft, $F_{FC} = 0.385$ (+ve)
At $x = 60$ ft, $F_{FC} = 0.0$
MAXIMUM SHEAR FORCE AND BENDING MOMENT UNDER A SERIES OF CONCENTRATED LOADS

Taking moment about A,

\[ R_E \times L = P_R \times [L/2 - (\bar{x} - x)] \]

\[ R_E = \frac{P_R}{L} (L/2 - \bar{x} + x) \]
Taking moment about E,

\[ R_A \times L = P_R \times [L / 2 + (\bar{x} - x)] \]

\[ R_A = \frac{P_R}{L} (L / 2 + \bar{x} - x) \]

\[ M_D = R_A \times (L / 2 + x) - P_1(a_1 + a_2) - P_2 \times a_2 \]

\[ = \frac{P_R}{L} (L / 2 + \bar{x} - x)(L / 2 + x) - P_1(a_1 + a_2) - P_2 \times (a_2) \]

\[ \frac{dM_D}{dx} = 0 \]

\[ 0 = \frac{P_R}{L} (L / 2 + \bar{x} - x) + \frac{P_R}{L} (L / 2 + x)(-1) \]

\[ = \frac{P_R}{L} [(L / 2) + \bar{x} - x - (L / 2) - x] \]

\[ i.e., \quad \bar{x} - 2x = 0 \]

\[ \bar{x} = 2x \]

\[ x = \frac{\bar{x}}{2} \]

The centerline must divide the distance between the resultant of all the loads in the moving series of loads and the load considered under which maximum bending moment occurs.