Institute of Aeronautical Engineering

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LINEAR ALGEBRA
AND
ORDINARY DIFFERENTIAL EQUATIONS

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CONTENTS

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UNIT-I

Theory of Matrices

Matrix: A system of mn numbers(real or complex) arranged in the form of an ordered set of m rows,each row consisting of an ordered set of n numbers between [ ] or ( ) or || || is called a matrix of order mxn

- If A=[aij]mxn and m=n,then A is called <u>Square matrix</u>
- A matrix which is not a square matrix is called a <u>rectangular matrix</u>
- A matrix of order 1xm is called a <u>row matrix</u>
- A matrix of order nx1 is called a <u>column matrix</u>

- If A=[aij]nxn such that aij=1 for i=j and aij=0 for i≠j,then A is called unit matrix and it is denoted by I_n

- The matrix obtained from any given matrix A,by interchanging rows and columns is called the transpose of A. It is denoted by A^T

- A square matrix all of whose elements below the leading diagonal are zero is called upper triangular matrix

- A square matrix all of whose elements above the leading diagonal are zero is called lower triangular matrix

- A square matrix $A=[a_{ij}]$ is said to be symmetric if $a_{ij}=a_{ji}$ for every i and j

  Thus,A is symmetric matrix $\leftrightarrow A=A^T$

- A square matrix $A=[a_{ij}]$ is said to be Skew-symmetric if

  $a_{ij}=-a_{ji}$ for every i and j

  Thus,A is skew-symmetric $\leftrightarrow A= -A^T$

- Every square matrix can be expressed as the sum of symmetric and skew-symmetric matrices in uniquely

- Trace of a square matrix A is defined as sum of the diagonal elements and it is denoted by tr(A)

If A and B square matrices of order n and k is any scalar then

- tr(kA) = k.tr(A)
- tr(A+B) = tr(A) + tr(B)
- tr(AB) = tr(A).tr(B)

- If A is a square matrix such that $A^2=A$ is called <u>Idempotent</u>

- If A is a square matrix such that $A^m=O$ is called <u>Nilpotent</u> . If m is the least positive integer such that $A^m=O$ then A is called nilpotent of index

- If A is a square matrix such that $A^2=I$ is called <u>Involutory</u>

Minors and Cofactors of a square matrix:

- Let $A=[a_{ij}]_{nxn}$ be a square matrix. When from A the elements of ith row and jth column are deleted the determinant of (n-1) rowed matrix M_{ij} is called the Minor of a_{ij} and is denoted by $|M_{ij}|$

- The signed minor of $(-1)^{i+j}|M_{ij}|$ is called the Cofactor of a_{ij} and is denoted by A_{ij}

- Determinant of a square matrix can be defined as the sum of products of the elements of any row or column with their corresponding co-factors is equal to the value of the determinant.

  $|A|=a_{11}A_{11}+A_{12}a_{12}+a_{13}A_{13}$

Properties of deteminants

- If A is a square matrix of order n then $|kA|=k^n|A|$, where k is a scalar

- If A is a square matrix of order n, then $|A|=|A^T|$

- If A and B are two square matrices of the same order, then $|AB|=|A||B|$

Inverse of a matrix

- Let A be any square matrix, then a matrix B, if it exits such that $AB=BA=I$, then B is called inverse of A and is denoted by A^{-1}.
- Every invertible matrix possesses a unique inverse.
- The necessary and sufficient conditions for a square matrix to possess inverse is that $|A|\neq 0$
- A square matrix A is singular if $|A|=0$.
- A square matrix A is non singular if $|A|\neq 0$
- If A,B are invertable matrices of the same order, then

  i) $(AB)^{-1}=B^{-1}A^{-1}$ ii) $(A^{T})^{-1}=(A^{-1})^{T}$

Rank of a matrix

- Let A be an mxn matrix. If A is a null matrix, we define its rank to be zero.
- If A is a non null matrix , we say that r is the rank of A if

  i) every (r+1)th order minor of A is 0 and

  ii) there exists at least one rth order minor of A which is not zero

  iii) Rank of A is denoted by $\rho(A)$.

  iv) This definition is useful to understand what rank is.

Properties of rank

- Rank of a matrix is unique
- Every matrix will have a rank
- If A is a matrix of order mxn, rank of $A = \rho(A) \leq \min\{m,n\}$
- If $\rho(A)=r$ then every minor of A of order r+1,or more is zero
- Rank of the identity matrix I_n is n
- If A is a matrix of order n and A is non-singular then $\rho(A)=n$

Echelon form of a matrix

- A matrix is said to be in Echelon form if it has the following properties
- Zero rows,if any, are below any non zero row
- The number of zeros before the first non-zero element in a row is less than the number of such zeros in next row

Reduction to Normal form

- Every mxn matrix of rank r can be reduced to the form I_r,[I_r,O] by a finite chain of elementary row or column operations ,where I_r is the r-rowed unit matrix.This form is called Normal form or first Canonical form of a matrix

- The rank of mxn matrix A is r if and only if it can be reduced to the normal form by a finite chain of elementary row and column operations

- If A is an mxn matrix of rank r, there exists non-singular matrices P and Q such that PAQ=[I_r O]

The inverse of a matrix by elementary Transformations: **(Gauss – Jordan method)**

- suppose A is a non-singular matrix of order ‘n’ then we write $A = I_n A$
- Now we apply elementary row-operations only to the matrix A and the pre-factor I_n of the R.H.S
- We will do this till we get $I_n = BA$ then obviously B is the inverse of A.

LU Decomposition Method

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

This is in the form AX=B Where

Let A=LU where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Hence LUX=B

(a)LY=B (b)UX=Y

Solve for Y from (a) then Solve for X from (b)

LY=B Can be solved by forward substitution and UX=Y can be solved by backward substitution

UNIT-II

LINEAR TRANSFORMATIONS

Characteristic matrix: Let A be a square matrix of order n then the matrix (A-λI) is called Charecteristic matrix of A.where I is the unit matrix of order n and λ is any scalar

Ex: Let A= $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ then

A- λI= $\begin{pmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{pmatrix}$ is the characteristic matrix of A

Characteristic polynomial: Let A be square matrix of order n then | A- λI| is called characteristic polynomial of A

Ex: Let A= $\begin{pmatrix} 1 & -2 \\ 4 & 1 \end{pmatrix}$

then | A- λI| = λ^2-2λ+9 is the characteristic polynomial of A

Characteristic equation: Let A be square matrix of order n then

$|A-\lambda I|=0$ is called characteristic equation of A

Ex: Let $A=\begin{pmatrix} 1 & -2 \\ 4 & 1 \end{pmatrix}$

then $|A-\lambda I| = \lambda^2-2\lambda+9=0$ is the characteristic equation of A

Eigen values: The roots of the characteristic equation $|A-\lambda I|=0$ are called the eigen values

Ex: Let $A=\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ then $|A-\lambda I|=\lambda^2-6\lambda+5=0$

Therefore $\lambda=1,5$ are the eigen values of the matrix A

- Eigen vector: If λ is an eigen value of the square matrix A. If there exists a non- zero vector X such that $AX=\lambda X$ is said to be eigen vector corresponding to eigen value λ of a square matrix A

- Eigen vector must be a non-zero vector

- If λ is an eigen value of matrix A if and only if there exists a non-zero vector X such that $AX=\lambda X$

- I f X is an eigen vector of a matrix A corresponding to the eigen value λ, then kX is also an eigen vector of A corresponding to the same eigen vector λ. K is a non zero scalar.

Properties of eigen values and eigen vectors

- The matrices A and A^T have the same eigen values.

- If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A then
  $1/\lambda_1, 1/\lambda_2 \ldots 1/\lambda_n$ are the eigen values of A^{-1}.

- If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A then $\lambda_1^k, \lambda_2^k, \ldots \lambda_n^k$ are the eigen values of A^k.

- If λ is the eigen value of a non singular matrix A, then $|A|/\lambda$ is the eigen value of A.

- The sum of the eigen values of a matrix is the trace of the matrix

- If λ is the eigen value of A then the eigen values of $B= a_oA_2+ a_1A+a_2I$ is $a_o\lambda_2+a_1\lambda+a_2$.
  Similar Matrices : Two matrices A&B are said to be similar if their exists an invertable matrix P such that $B=P^{-1}AP$.

- Eigen values of two similar matrices are same

- If A & B are square matrices and if A is invertable then the matrices $A^{-1}B$ & BA^{-1} have the same eigen values

Determination of A^{-1} using Cayley-Hamilton theorem.

A satisfies it characteristic equation

$$=(-1)^n[A^n+a_1 A^{n-1}+a_2A^{n-2}+\ldots.+a_nI]=0$$

$$= [A^n+a_1 A^{n-1}+a_2A^{n-2}+\ldots.+a_nI]=0$$

$$=A^{-1} [A^n+a_1 A^{n-1}+a_2A^{n-2}+\ldots.+a_nI]= 0$$

If A is nonsingular, then we have

$$a_nA^{-1}=-[A^n+a_1 A^{n-1}+a_2A^{n-2}+\ldots.+a_nI]$$

$$A^{-1}=-1/a_n[A^n+a_1 A^{n-1}+a_2A^{n-2}+\ldots.+a_nI]$$

Real matrices:

- Symmetric matrix
- Skew-symmetric matrix
- Orthogonal matrix

Complex matrices:

- Hermitian matrix
- Skew-hermitian matrix
- Unitary matrix

Complex matrices: If the elements of a matrix are complex numbers, then the matrix is called a complex matrix.

$$\begin{bmatrix} 1+i & i \\ -2 & -2+i \end{bmatrix}$$ is a complex matrix

Conjugate matrix: If A=[aij]mxn is a complex matrix then conjugate of A is A=[aij]mxn

$$\begin{bmatrix} 1+i & 2i \\ 0 & 3+6i \end{bmatrix}$$ then A= $$\begin{bmatrix} 1-i & -2i \\ 0 & 3-6i \end{bmatrix}$$

Conjugate transpose: conjugate transpose of a matrix A is $(\overline{A})^{T} = A^{\theta}$

$$A = \begin{bmatrix} 2+i & 3i & 2i \\ -i & 6+2i & 9 \end{bmatrix}, \qquad \overline{A} = \begin{bmatrix} 2-i & -3i & -2i \\ i & 6-2i & 9 \end{bmatrix}$$

Then $A^{\theta} = \begin{bmatrix} 2-i & -i \\ -3i & 6-2i \\ -2i & 9 \end{bmatrix}$

Note: 1. $(A^{\theta})^{\theta} = A$

2. $(kA)^{\theta} = \overline{k} A^{\theta}$, k is a complex number

3. $(A+B)^{\theta} = A^{\theta} + B^{\theta}$

Hermitian matrix: A square matrix $A=[a_{ij}]$ is said to be hermitian if $a_{ij} = \overline{aij}$

aji- for all i and j. The diagonal elements aii= aii-, a is real.Thus every diagonal element of a Hermitian matrix must be real.

- Skew-Hermitian matrix : A square matrix $A=(a_{ij})$ is said to be skew-hermitian if

$a_{ij}=-\bar{a}_{ji}$ for all i and j. The diagonal elements must be either purely imaginary or must be zero.

Note:

1. The diagonal elements of a Hermitian matrix are real
2. The diagonal elements of a Skew-hermitian matrix are eigther zero or purely imaginary
3. If A is Hermitian(skew-hermitian) then iA is Skew-hermitian(hemitian).
4. For any complex square matrix A , AA^{θ} is Hermitian
5. If A is Hermitian matrix and its eigen values are real

Unitary matrix: A complex square matrix $A=[a_{ij}]$ is said to be unitary if $AA^{\theta} = A^{\theta}A = I$

$$A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \qquad \overline{A} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad A^{\theta} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

$$AA^{\theta} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is a unitary matrix

Note: 1. The determinant of an unitary matrix has unit modulus.

2. The eigen values of a unitary matrix are of unit modulus.

Diagonalization of a matrix: If a square matrix A of order n has n eigen vectors X_1, X_2 ………...X_n

Corresponding to n eigen values λ_1, λ_2… λ_n respectively then a matrix P can be found such that

$P^{-1}AP$ is a diagonal matrix

i.e., $P^{-1}AP=D$.

Modal and spectral matrices: The matrix P in the above result which diagonalize in the square matrix A is called the modal matrix of A and the resulting diagonal matrix D is known as spectral matrix.

Calculation of power of matrix: We can obtain the powers of a matrix by using diagonalization.

Let A be the square matrix. Then a non singular matrix P can be found such that

$D=P^{-1}AP$

$D^2=P^{-1}A^2P \implies A^2=PD^2P^{-1}$

$D^3=P^{-1}A^3P \implies A^3=PD^3P^{-1}$

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$D^n=P^{-1}A^nP \Rightarrow A^n=PD^nP^{-1}$

UNIT-III

DIFFERENTIAL EQUATIONS OF FIRST ORDER AND THEIR APPLICATIONS

INTRODUCTION

- An equation involving a dependent variable and its derivatives with respect to one or more independent variables is called a Differential Equation.
- *Example* 1: $y'' + 2y = 0$
- *Example* 2: $y_2 - 2y_1 + y = 23$
- *Example* 3: $d^2y/dx^2 + dy/dx - y = 1$

TYPES OF A DIFFERENTIAL EQUATION

- ORDINARY DIFFERENTIAL EQUATION: A differential equation is said to be ordinary, if the derivatives in the equation are ordinary derivatives.
- *Example*: $d^2y/dx^2 - dy/dx + y = 1$
- PARTIAL DIFFERENTIAL EQUATION: A differential equation is said to be partial if the derivatives in the equation have reference to two or more independent variables.
- *Example*: $\partial^4 y/\partial x^4 + \partial y/\partial x + y = 1$

DEFINITIONS

- ORDER OF A DIFFERENTIAL EQUATION: A differential equation is said to be of order n, if the n^{th} derivative is the highest derivative in that equation.
- *Example*: Order of $d^2y/dx^2+dy/dx+y=2$ is 2
- DEGREE OF A DIFFERENTIAL EQUATION: If the given differential equation is a polynomial in $y^{(n)}$, then the highest degree of $y^{(n)}$ is defined as the degree of the differential equation.
- *Example*: Degree of $(dy/dx)^4+y=0$ is 4

SOLUTION OF A DIFFERENTIAL EQUATION

- SOLUTION: Any relation connecting the variables of an equation and not involving their derivatives, which satisfies the given differential equation is called a solution.
- GENERAL SOLUTION: A solution of a differential equation in which the number of arbitrary constant is equal to the order of the equation is called a general or complete solution or complete primitive of the equation.
- *Example*: $y = Ax + B$
- PARTICULAR SOLUTION: The solution obtained by giving particular values to the arbitrary constants of the general solution, is called a particular solution of the equation.
- *Example*: $y = 3x + 5$

EXACT DIFFERENTIAL EQUATION

- Let $M(x,y)dx + N(x,y)dy = 0$ be a first order and first degree differential equation where M and N are real valued functions for some x, y. Then the equation $Mdx + Ndy = 0$ is said to be an exact differential equation if $\partial M/\partial y=\partial N/\partial x$
- *Example*: $(2y \sin x+\cos y)dx=(x \sin y+2\cos x+\tan y)dy$

Working rule to solve an exact equation

STEP 1: Check the condition for exactness,

if exact proceed to step 2.

STEP 2: After checking that the equation is

exact, solution can be obtained as

$$\int M\,dx+\int(\text{terms not containing } x)\,dy=c$$

INTEGRATING FACTOR

- Let Mdx + Ndy = 0 be not an exact differential equation. Then Mdx + Ndy = 0 can be made exact by multiplying it with a suitable function *u* is called an integrating factor.
- *Example* 1:ydx-xdy=0 is not an exact equation. Here $1/x^2$ is an integrating factor
- *Example* 2:$y(x^2y^2+2)dx+x(2-2x^2y^2)dy=0$ is not an exact equation. Here $1/(3x^3y^3)$ is an integrating factor

METHODS TO FIND INTEGRATING FACTORS

- METHOD 1: With some experience integrating factors can be found by inspection. That is, we have to use some known differential formulae.
- *Example* 1:$d(xy)=xdy+ydx$
- *Example* 2:$d(x/y)=(ydx-xdy)/y^2$
- *Example* 3:$d[\log(x^2+y^2)]=2(xdx+ydy)/(x^2+y^2)$

METHODS TO FIND INTEGRATING FACTORS

- METHOD 2: If $Mdx + Ndy = 0$ is a non-exact but homogeneous differential equation and $Mx + Ny \neq 0$ then $1/(Mx + Ny)$ is an integrating factor of $Mdx + Ndy = 0$.
- *Example* 1:$x^2ydx-(x^3+y^3)dy=0$ is a non-exact homogeneous equation. Here I.F.$=-1/y^4$
- *Example* 2:$y^2dx+(x^2-xy-y^2)dy=0$ is a non-exact homogeneous equation. Here I.F.$=1/(x^2y-y^3)$

METHODS TO FIND INTEGRATING FACTORS

- METHOD 3: If the equation $Mdx + Ndy = 0$ is of the form $y.f(xy)\, dx + x.g(xy)\, dy = 0$ and $Mx – Ny \neq 0$ then $1/(Mx – Ny)$ is an integrating factor of $Mdx + Ndy = 0$.
- *Example* 1:$y(x^2y^2+2)dx+x(2-2x^2y^2)dy=0$ is non-exact and in the above form. Here I.F$=1/(3x^3y^3)$
- *Example* 2:$(xy\sin xy+\cos xy)ydx+(xy\sin xy-\cos xy)xdy=0$ is non-exact and in the above form. Here I.F$=1/(2xy\cos xy)$

METHODS TO FIND INTEGRATING FACTORS

- METHOD 4: If there exists a continuous single variable function $f(x)$ such that $\partial M/\partial y - \partial N/\partial x = Nf(x)$ then $e^{\int f(x)dx}$ is an integrating factor of $Mdx + Ndy = 0$
- *Example* 1: $2xydy-(x^2+y^2+1)dx=0$ is non-exact and
  $\partial M/\partial y - \partial N/\partial x = N(-2/x)$. Here I.F$=1/x^2$
- *Example* 2: $(3xy-2ay^2)dx+(x^2-2axy)=0$ is non-exact and $\partial M/\partial y - \partial N/\partial x = N(1/x)$. Here I.F$=x$

METHODS TO FIND INTEGRATING FACTORS

- METHOD 5: If there exists a continuous single variable function $f(y)$ such that
  $\partial N/\partial x - \partial M/\partial y = Mg(y)$ then $e^{\int g(y)dy}$ is an integrating factor of $Mdx + Ndy = 0$
- *Example* 1:$(xy^3+y)dx+2(x^2y^2+x+y^4)dy=0$ is a non-exact equation and $\partial N/\partial x - \partial M/\partial y=M(1/y)$. Here I.F=y
- *Example* 2:$(y^4+2y)dx+(xy^3+2y^4-4x)dy=0$ is a non-exact equation and $\partial N/\partial x - \partial M/\partial y=M(-3/y)$.Here I.F=$1/y^3$

LEIBNITZ LINEAR EQUATION

- An equation of the form $y' + Py = Q$ is called a linear differential equation.
- Integrating Factor(I.F.)=$e^{\int p\,dx}$
- Solution is $y(I.F) = \int Q(I.F)dx + C$
- *Example* 1:$x\,dy/dx+y=\log x$. Here I.F=x and solution is $xy=x(\log x-1)+C$
- *Example* 2:$dy/dx+2xy=e^{-x}$.Here I.F=e^x and solution is $ye^x=x+C$

BERNOULLI'S LINEAR EQUATION

- An equation of the form $y' + Py = Qy^n$ is called a Bernoulli's linear differential equation. This differential equation can be solved by reducing it to the Leibnitz linear differential equation. For this dividing above equation by y^n
- *Example* 1: $xdy/dx+y=x^2y^6$.Here I.F=$1/x^5$ and solution is $1/(xy)^5=5x^3/2+Cx^5$
- *Example* 2: $dy/dx+y/x=y^2x\sin x$. Here I.F=$1/x$ and solution is $1/xy=\cos x+C$

ORTHOGONAL TRAJECTORIES

- If two families of curves are such that each member of family cuts each member of the other family at right angles, then the members of one family are known as the orthogonal trajectories of the other family.
- *Example* 1: The orthogonal trajectory of the family of parabolas through origin and foci on y-axis is $x^2/2c+y^2/c=1$
- *Example* 2: The orthogonal trajectory of rectangular hyperbolas is $xy=c^2$

PROCEDURE TO FIND ORTHOGONAL TRAJECTORIES

Suppose $f(x, y, c) = 0$ is the given family of curves, where c is the constant.

STEP 1: Form the differential equation by eliminating the arbitrary constant.

STEP 2: Replace y' by $-1/y'$ in the above equation.

STEP 3: Solve the above differential equation.

NEWTON'S LAW OF COOLING

- The rate at which the temperature of a hot body decreases is proportional to the difference between the temperature of the body and the temperature of the surrounding air.

$$\theta' \propto (\theta - \theta_0)$$

- *Example:* If a body is originally at 80^{o}C and cools down to 60^{o}C in 20 min.If the temperature of the air is at 40^{o}C then the temperture of the body after 40 min is 50^{o}C

LAW OF NATURAL GROWTH

- When a natural substance increases in Magnitude as a result of some action which affects all parts equally, the rate of increase depends on the amount of the substance present.

$$N' = k\,N$$

- *Example:* If the number N of bacteria in a culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in 1 hour. Then the value of N after one and half hour is 605

LAW OF NATURAL DECAY

- The rate of decrease or decay of any substance is proportion to N the number present at time.

$$N' = -k\,N$$

- *Example*: A radioactive substance disintegrates at a rate proportional to its mass. When mass is 10gms, the rate of disintegration is 0.051gms per day. The mass is reduced to 10 to 5gms in 136 days.

UNIT-IV

HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS

INTRODUCTION

- An equation of the form

  $D^n y + k_1 D^{n-1} y + ... + k_n y = X$

  Where $k_1,.....,k_n$ are real constants and X is a continuous function of *x* is called an ordinary linear equation of order *n* with constant coefficients.

  Its complete solution is

  $y = C.F + P.I$

  where C.F is a Complementary Function and

  P.I is a Particular Integral.

- *Example*: $d^2y/dx^2+3dy/dx+4y=\sin x$

COMPLEMENTARY FUNCTION

- If roots are real and distinct then
  C.F = $c_1 e^{m1x} + \ldots + c_k e^{mkx}$
- *Example* 1: Roots of an auxiliary equation are 1,2,3 then C.F = $c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$
- *Example* 2: For a differential equation
  (D-1)(D+1)y=0, roots are -1 and 1. Hence
  C.F = $c_1 e^{-x} + c_2 e^x$

COMPLEMENTARY FUNCTION

- If roots are real and equal then

  $$C.F = (c_1 + c_2x + ... + c_kx^k)e^{mx}$$

- *Example* 1: The roots of a differential equation $(D-1)^3y=0$ are 1,1,1. Hence $C.F.= (c_1+c_2x+c_3x^2)e^x$
- *Example* 2: The roots of a differential equation $(D+1)^2y=0$ are -1,-1. Hence $C.F=(c_1+c_2x)e^{-x}$

COMPLEMENTARY FUNCTION

- If two roots are real and equal and rest are real and different then $C.F=(c_1+c_2x)e^{m1x}+c_3e^{m3x}+\ldots.$
- Example : The roots of a differential equation $(D-2)^2(D+1)y=0$ are 2,2,-1. Hence $C.F.=(c_1+c_2x)e^{2x}+c_3e^{-X}$

COMPLEMENTARY FUNCTION

- If roots of Auxiliary equation are complex say p+iq and p-iq then C.F=$e^{px}(c_1 \cos qx + c_2 \sin qx)$
- Example: The roots of a differential equation $(D^2+1)y=0$ are 0+i(1) and 0-i(1). Hence C.F=$e^{0x}(c_1\cos x+c_2\sin x)$
- $=(c_1\cos x+c_2\sin x)$

COMPLEMENTARY FUNCTION

- A pair of conjugate complex roots say p+iq and p-iq are repeated twice then $C.F=e^{px}((c_1+c_2x)\cos qx+(c_3+c_4x)\sin qx)$
- *Example*: The roots of a differential equation $(D^2-D+1)^2y=0$ are ½+i(1.7/2) and ½-i(1.7/2) repeated twice. Hence $C.F=e^{1/2x}(c_1+c_2x)\cos(1.7/2)x+(c_3+c_4x)\sin(1.7/2)x$

PARTICULAR INTEGRAL

- When $X = \text{Sinax}$ or Cosax or $\text{Sin}(ax+b)$ or $\text{Cos}(ax+b)$ then put $D^2 = -a^2$ in Particular Integral.
- *Example* 1: $D^2y-3Dy+2y=\text{Cos}3x$. Here P.I$=(9\text{Sin}3x+7\text{Cos}3x)/130$
- *Example* 2: $(D^2+D+1)y=\text{Sin}2x$. Here P.I$= -1/13(2\text{Cos}2x+3\text{Sin}2x)$

PARTICULAR INTEGRAL

- When X = x^k or in the form of polynomial then convert f(D) into the form of binomial expansion from which we can obtain Particular Integral.
- *Example* 1: $(D^2+D+1)y=x^3$.Here P.I=x^3-3x^2+6
- *Example* 2: $(D^2+D)y=x^2+2x+4$. Here P.I=$x^3/3+4x$

PARTICULAR INTEGRAL

- When $X = e^{ax}$ put $D = a$ in Particular Integral. If $f(a) \neq 0$ then P.I. will be calculated directly. If $f(a) = 0$ then multiply P.I. by x and differentiate denominator. Again put $D = a$.Repeat the same process.
- *Example* 1:$y''+5y'+6y=e^x$. Here $P.I=e^x/12$
- *Example* 2:$4D^2y+4Dy-3y=e^{2x}$.Here $P.I=e^{2x}/21$

PARTICULAR INTEGRAL

- When $X = e^{ax}v$ then put $D = D+a$ and take out e^{ax} to the left of f(D). Now using previous methods we can obtain Particular Integral.
- *Example* 1:$(D^4-1)y=e^x$ Cosx. Here P.I=-e^xCosx/5
- *Example* 2: $(D^2-3D+2)y=xe^{3x}+\text{Sin}2x$. Here P.I=$e^{3x}/2(x-3/2)+1/20(3\text{Cos}2x-\text{Sin}2x)$

PARTICULAR INTEGRAL

- When X = x.v then

  $P.I = [\{x - f'(D)/f(D)\}/f(D)]v$

- *Example* 1: $(D^2+2D+1)y=x\,Cosx$. Here $P.I=x/2Sinx+1/2(Cosx-Sinx)$
- *Example* 2: $(D^2+3D+2)y=x\,e^x\,Sinx$. Here $P.I=e^x[x/10(Sinx-Cosx)-1/25Sinx+Cosx/10]$

PARTICULAR INTEGRAL

- When *X* is any other function then Particular Integral can be obtained by resolving 1/*f(D)* into partial fractions.
- *Example* 1: $(D^2+a^2)y=\text{Secax}$. Here P.I=x/a Sinax+Cosax log(Cosax)/a^2

CAUCHY'S LINEAR EQUATION

- Its general form is

  $x^nD^ny + \ldots. +y = X$

  then to solve this equation put $x = e^z$ and convert into ordinary form.
- *Example* 1: $x^2D^2y+xDy+y=1$
- *Example* 2: $x^3D^3y+3x^2D^2y+2xDy+6y=x^2$

LEGENDRE'S LINEAR EQUATION

- Its general form is

  $(ax + b)^n D^n y + \ldots\ldots + y = X$

  then to solve this equation put $ax + b = e^z$ and convert into ordinary form.
- *Example* 1: $(x+1)^2D^2y-3(x+1)Dy+4y=x^2+x+1$
- *Example* 2: $(2x-1)^3D^3y+(2x-1)Dy-2y=x$

METHOD OF VARIATION OF PARAMETERS

- Its general form is
  $D^2y + P\,Dy + Q = R$
  where P, Q, R are real valued functions of *x*.
  Let C.F $= C_1u + C_2v$
  P.I $= Au + Bv$
- *Example* 1: $(D^2+1)y=\text{Cosec}x$. Here $A=-x$, $B=\log(\text{Sin}x)$
- *Example* 2: $(D^2+1)y=\text{Cos}x$. Here $A=\text{Cos}2x/4$, $B=(x+\text{Sin}2x)/2$

Lecture-15
ELECTRIC CIRCUITS

- Consider the linear D.E that govern the flow of electricity in an electric circuit containing a resistance R, an inductance L and a capacitance C. If $I = \frac{dq}{dt}$ is the current in the circuit at any time t, then

By Kirchoff's law " In an electric circuit the sum of voltage drop is equal to the supplied voltage" i.e.,

$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = E$ is the second order linear D.E

- *Example*: The current I in the LCR circuit assuming zero initial current and charge for R=800 ohms, L=20 Henrys, C=0.01 Farads and E=100 volts is $5e^{-2t}\sin t$

Lecture-16

SIMPLE HARMONIC MOTION

- When the acceleration of a particle is proportional to its displacement from a fixed point and is always directed towards it is called Simple harmonic motion.

A’ O P A

$\longleftarrow x \longrightarrow$

$\longleftarrow a \longrightarrow$

Let P be the position of the particle at any time t, where OP=X. Since the acceleration is in the direction opposite to that in which x increases, the equation of motion of particle is $\frac{d^2x}{dt^2} = -\mu^2 x$

- Solution of $\frac{d^2x}{dt^2}+\mu^2x=0$ is $x=c_1\cos\mu t+c_2\sin\mu t$
  If the particle moves from rest A, where OA=a then at $t=0$, $x=a$ and velocity, $v=dx/dt=0$ then $x=a\cos\mu t$
- *Example 1* :A particle is executing SHM with amplitude 20cm and time 4 seconds. The time required by the particle in passing between points which are at distances 15cm and 5cm from the centre of the force and are on the same side of it is 0.38 seconds
- *Example 2*: At the end of 3 successive seconds, the distance of a point moving with SHM from its mean position are x_1,x_2,x_3 resp. The time of complete oscillation is $2\pi/\cos^{-1}\left(\frac{x_1+x_3}{x_2}\right)$

UNIT-V

FUNCTIONS OF SINGLE AND SEVERAL VARIABLES

INTRODUCTION

- Here we study about Mean value theorems.
- Continuous function: If limit of f(x) as x tends c is f(c) then the function f(x) is known as continuous function. Otherwise the function is known as discontinuous function.
- Example: If f(x) is a polynomial function then it is continuous.

ROLLE'S MEAN VALUE THEOREM

- Let f(x) be a function such that

1) it is continuous in closed interval [a, b];

2) it is differentiable in open interval (a, b) and

3) $f(a)=f(b)$

Then there exists at least one point c in open interval (a, b) such that $f'(c)=0$

- *Example*: $f(x)=(x+2)^3(x-3)^4$ in [-2,3]. Here $c=-2$ or 3 or 1/7

LAGRANGE'S MEAN VALUE THEOREM

- Let f(x) be a function such that

1) it is continuous in closed interval [a, b] and

2) it is differentiable in open interval (a, b)

Then there exists at least one point c in open interval (a, b) such that $f'(c)=[f(b)-f(a)]/[b-a]$

- *Example*: $f(x)=x^3-x^2-5x+3$ in [0,4]. Here $c=1+\sqrt{37}/3$

CAUCHY'S MEAN VALUE THEOREM

- If f:[a, b] →R, g:[a, b] →R are such that 1)f,g are continuous on [a, b] 2)f,g are differentiable on (a, b) and 3)g '(x)≠0 for all x∈(a, b) then there exists c∈(a, b) such that [f(b)-f(a)]/[g(b)-g(a)]=f ′(c)/g′(c)
- *Example*: f(x)=√x, g(x)=1/√x in [a,b]. Here c=√ab

FUNCTIONS OF SEVERAL VARIABLES

- We have already studied the notion of limit, continuity and differentiation in relation of functions of a single variable. In this chapter we introduce the notion of a function of several variables i.e., function of two or more variables.
- *Example* 1: Area A= ab
- *Example* 2: Volume V= abh

DEFINITIONS

- Neighbourhood of a point(a,b): A set of points lying within a circle of radius r centered at (ab) is called a neighbourhood of (a,b) surrounded by the circular region.
- Limit of a function: A function f(x,y) is said to tend to the limit l as (x,y) tends to (a,b) if corresponding to any given positive number p there exists a positive number q such that f(x,y)-l<p foa all points (x,y) whenever x-a≤q and y-b≤q

JACOBIAN

- Let $u=u(x,y)$, $v=v(x,y)$. Then these two simultaneous relations constitute a transformation from (x,y) to (u,v). Jacobian of u,v w.r.t x,y is denoted by $J[u,v]/[x,y]$ or $\partial(u,v)/\partial(x,y)$
- *Example*: $x=r\cos\theta, y=r\sin\theta$ then $\partial(x,y)/\partial(r,\theta)$ is r and $\partial(r,\theta)/\partial(x,y)=1/r$

MAXIMUM AND MINIMUM OF FUNCTIONS OF TWO VARIABLES

- Let $f(x,y)$ be a function of two variables x and y. At $x=a$, $y=b$, $f(x,y)$ is said to have maximum or minimum value, if $f(a,b)>f(a+h,b+k)$ or $f(a,b)<f(a+h,b+k)$ respectively where h and k are small values.
- *Example*: The maximum value of $f(x,y)=x^3+3xy^2-3y^2+4$ is 36 and minimum value is -36

EXTREME VALUE

- $f(a,b)$ is said to be an extreme value of f if it is a maximum or minimum value.
- *Example* 1: The extreme values of $u=x^2y^2-5x^2-8xy-5y^2$ are -8 and -80
- *Example* 2: The extreme value of $x^2+y^2+6x+12$ is 3

LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

- Suppose it is required to find the extremum for the function $f(x,y,z)=0$ subject to the condition $\phi(x,y,z)=0$

1)Form Lagrangean function $F=f+\lambda\phi$

2)Obtain $F_x=0, F_y=0, F_z=0$

3)Solve the above 3 equations along with condition.

- *Example*: The minimum value of $x^2+y^2+z^2$ with $xyz=a^3$ is $3a^2$