MATHEMATICS-III

CONTENTS

- Linear ODE with variable coefficients and series solutions
- Special functions
- Complex function – Differentiation and integration
- Power series expansions of complex functions and contour integration
- Conformal mapping

TEXT BOOKS:

- Advanced Engineering Mathematics by Kreyszig, John Wiley & Sons.
- Higher Engineering Mathematics by Dr. B.S. Grewal, Khanna Publishers.

REFERENCES:

1.Complex Variables Principles And Problem Sessions By A.K.Kapoor, World Scientific Publishers

2.Engineering Mathematics-3 By T.K.V.Iyengar andB.Krishna Gandhi Etc

3.A Text Book Of Engineering Mathematics By N P Bali, Manesh Goyal

4.Mathematics for Engineers and Scientists, Alan Jeffrey, 6th Edit. 2013, Chapman & Hall/CRC

5.Advanced Engineering Mathematics, Michael Greenberg, Second Edition. Person Education

6.Mathematics For Engineers By K.B.Datta And M.A S.Srinivas,Cengage Publications

Power Series Method

The **power series method** is the standard method for solving linear ODEs with ***variable*** coefficients.

It gives solutions in the form of power series.

These series can be used for computing values, graphing curves, proving formulas, and exploring properties of solutions. In this section we begin by explaining the idea of the power series method.

From calculus we remember that a **power series** (in powers of $x - x_0$) is an infinite series of the form

(1)

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots.$$

Here, x is a variable. a_0, a_1, a_2, … are constants, called the **coefficients** of the series. x_0 is a constant, called the **center** of the series. In particular, if $x_0 = 0$, we obtain a **power series in powers of *x***

(2)

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$$

We shall assume that all variables and constants are real.

Idea and Technique of the Power Series Method (continued)

Then we collect like powers of x and equate the sum of the coefficients of each occurring power of x to zero, starting with the constant terms, then taking the terms containing x, then the terms in x^2, and so on.
This gives equations from which we can determine the unknown coefficients of (3) successively.

Theory of the Power Series Method

The ***n*th partial sum** of (1) is

$$(6)\quad s_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n$$

where $n = 0, 1, \dots$.

If we omit the terms of s_n from (1), the remaining expression is

This expression is called the **remainder** *of* (1) *after the term* $a_n(x - x_0)^n$.

$$(7)\qquad R_n(x) = a_{n+1}(x - x_0)^{n+1} + a_{n+2}(x - x_0)^{n+2} + \cdots.$$

Theory of the Power Series Method (continued)

In this way we have now associated with (1) the sequence of the partial sums $s_0(x)$, $s_1(x)$, $s_2(x)$, …. If for some $x = x_1$ this sequence converges, say,

$$\lim_{n\to\infty} s_n(x_1) = s(x_1),$$

then the series (1) is called **convergent** *at* $x = x_1$, the number $s(x_1)$ is called the **value** or *sum* of (1) at x_1, and we write

$$s(x_1) = \sum_{m=0}^{\infty} a_m (x_1 - x_0)^m.$$

Then we have for every n,

(8)

$$s(x_1) = s_n(x_1) + R_n(x_1).$$

If that sequence diverges at $x = x_1$, the series (1) is called **divergent** at $x = x_1$.

Theory of the Power Series Method (continued)

Where does a power series converge? Now if we choose $x = x_0$ in (1), the series reduces to the single term a_0 because the other terms are zero. Hence the series converges at x_0.

In some cases this may be the only value of x for which (1) converges. If there are other values of x for which the series converges, these values form an interval, the **convergence interval**. This interval may be finite, as in Fig. 105, with midpoint x_0. Then the series (1) converges for all x in the interior of the interval, that is, for all x for which

$$|x - x_0| < R \tag{10}$$

and diverges for $|x - x_0| > R$. The interval may also be infinite, that is, the series may converge for all x.

Legendre’s Equation. Legendre Polynomials $P_n(x)$

Legendre's differential equation

(1) $$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0 \qquad (n \text{ constant})$$

is one of the most important ODEs in physics. It arises in numerous problems, particularly in boundary value problems for spheres.

The equation involves a **parameter** n, whose value depends on the physical or engineering problem.

So (1) is actually a whole family of ODEs. For $n = 1$
we solved it in Example 3 of Sec. 5.1 (look back at it).

Any solution of (1) is called a **Legendre function**.

The study of these and other "higher" functions not occurring in calculus is called the **theory of special functions**.

Dividing (1) by $1 - x^2$, we obtain the standard form needed in Theorem 1 of Sec. 5.1 and we see that the coefficients $-2x/(1 - x^2)$ and $n(n + 1)/(1 - x^2)$ of the new equation are analytic at $x = 0$, so that we may apply the power series method. Substituting

(2)

$$y(x) = \sum_{m=0}^{\infty} a_m x^m$$

and its derivatives into (1), and denoting the constant $n(n + 1)$ simply by k, we obtain

By writing the first expression as two separate series we have the equation

$$(1 - x^2) \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - 2x \sum_{m=1}^{\infty} m a_m x^{m-1} + k \sum_{m=0}^{\infty} a_m x^m = 0.$$

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1) a_m x^m - \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} k a_m x^m = 0.$$

Polynomial Solutions. Legendre Polynomials $P_n(x)$

The reduction of power series to polynomials is a great advantage because then we have solutions for all x, without convergence restrictions. For special functions arising as solutions of ODEs this happens quite frequently, leading to various important families of polynomials.

For Legendre's equation this happens when the parameter n is a nonnegative integer because then the right side

of (4) is zero for $s = n$, so that $a_{n+2} = 0$, $a_{n+4} = 0$, $a_{n+6} = 0$, ….

Hence if n is even, $y_1(x)$ reduces to a polynomial of degree n. If n is odd, the same is true for $y_2(x)$.

These polynomials, multiplied by some constants, are called **Legendre polynomials** and are denoted by $P_n(x)$.

Series Solutions Near a Regular Singular Point, Part I

- We now consider solving the general second order linear equation in the neighborhood of a regular singular point x_0. For convenience, will will take $x_0 = 0$.
- Recall that the point $x_0 = 0$ is a regular singular point of

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

iff

$$x\frac{Q(x)}{P(x)} = xp(x) \text{ and } x^2\frac{R(x)}{P(x)} = x^2q(x) \text{ are analytic at } x = 0$$

iff

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n \text{ and } x^2q(x) = \sum_{n=0}^{\infty} q_n x^n, \text{ convergent on } |x| < \rho$$

Series Solutions Near a Regular Singular Point, Part I

- We now consider solving the general second order linear equation in the neighborhood of a regular singular point x_0. For convenience, will will take $x_0 = 0$.
- Recall that the point $x_0 = 0$ is a regular singular point of

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

iff

$$x\frac{Q(x)}{P(x)} = xp(x) \text{ and } x^2\frac{R(x)}{P(x)} = x^2 q(x) \text{ are analytic at } x = 0$$

iff

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n \text{ and } x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n, \text{ convergent on } |x| < \rho$$

Transforming Differential Equation

- Our differential equation has the form

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

- Dividing by $P(x)$ and multiplying by x^2, we obtain

$$x^2 y'' + x[xp(x)]y' + [x^2 q(x)]y = 0$$

- Substituting in the power series representations of p and q,

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \;\; x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n,$$

we obtain

$$x^2 y'' + x\left(p_0 + p_1 x + p_2 x^2 + \cdots\right)y' + \left(q_0 + q_1 x + q_2 x^2 + \cdots\right)y = 0$$

Comparison with Euler Equations

- Our differential equation now has the form

$$x^2 y'' + x\left(p_0 + p_1 x + p_2 x^2 + \cdots\right)y' + \left(q_0 + q_1 x + q_2 x^2 + \cdots\right)y = 0$$

- Note that if

$$p_1 = p_2 = \cdots = q_1 = q_2 = \cdots = 0$$

then our differential equation reduces to the Euler Equation

$$x^2 y'' + p_0 x y' + q_0 y = 0$$

- In any case, our equation is similar to an Euler Equation but with power series coefficients.
- Thus our solution method: assume solutions have the form

$$y(x) = x^r\left(a_0 + a_1 x + a_2 x^2 + \cdots\right) = \sum_{n=0}^{\infty} a_n x^{r+n}, \text{ for } a_0 \neq 0,\ x > 0$$

Example 1: Regular Singular Point

- Consider the differential equation

$$2x^2 y'' - xy' + (1+x)y = 0$$

- This equation can be rewritten as

$$x^2 y'' - \frac{x}{2} y' + \frac{1+x}{2} y = 0$$

- Since the coefficients are polynomials, it follows that $x = 0$ is a regular singular point, since both limits below are finite:

$$\lim_{x \to 0} x\left(-\frac{x}{2x^2}\right) = -\frac{1}{2} < \infty \quad \text{and} \quad \lim_{x \to 0} x^2\left(\frac{1+x}{2x^2}\right) = \frac{1}{2} < \infty$$

$$2x^2y'' - xy' + (1+x)y = 0$$

Example 1: Euler Equation (2 of 13)

- Now $xp(x) = -1/2$ and $x^2q(x) = (1 + x)/2$, and thus for

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \quad x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n,$$

it follows that

$$p_0 = -1/2,\ q_0 = 1/2,\ q_1 = 1/2,\ p_1 = p_2 = \cdots = q_2 = q_3 = \cdots = 0$$

- Thus the corresponding Euler Equation is

$$x^2y'' + p_0 xy' + q_0 y = 0 \iff 2x^2y'' - xy' + y = 0$$

- As in Section 5.5, we obtain

$$x^r\left[2r(r-1) - r + 1\right] = 0 \iff (2r-1)(r-1) = 0 \iff r = 1,\ r = 1/2$$

- We will refer to this result later.

$$2x^2y'' - xy' + (1+x)y = 0$$

Example 1: Differential Equation

- For our differential equation, we assume a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}, \;\; y'(x) = \sum_{n=0}^{\infty} a_n (r+n) x^{r+n-1},$$

$$y''(x) = \sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n-2}$$

- By substitution, our differential equation becomes

$$\sum_{n=0}^{\infty} 2a_n (r+n)(r+n-1) x^{r+n} - \sum_{n=0}^{\infty} a_n (r+n) x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+1} = 0$$

or

$$\sum_{n=0}^{\infty} 2a_n (r+n)(r+n-1) x^{r+n} - \sum_{n=0}^{\infty} a_n (r+n) x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} + \sum_{n=1}^{\infty} a_{n-1} x^{r+n} = 0$$

Example 1: Combining Series (4 of 13)

- Our equation

$$\sum_{n=0}^{\infty} 2a_n(r+n)(r+n-1)x^{r+n} - \sum_{n=0}^{\infty} a_n(r+n)x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} + \sum_{n=1}^{\infty} a_{n-1} x^{r+n} = 0$$

can next be written as

$$a_0\left[2r(r-1)-r+1\right]x^r + \sum_{n=1}^{\infty}\left\{a_n\left[2(r+n)(r+n-1)-(r+n)+1\right]+a_{n-1}\right\}x^{r+n} = 0$$

- It follows that

$$a_0\left[2r(r-1)-r+1\right]=0$$

and

$$a_n\left[2(r+n)(r+n-1)-(r+n)+1\right]+a_{n-1}=0, \quad n=1,2,\ldots$$

Example 1: Indicial Equation (5 of 13)

- From the previous slide, we have

$$a_0\left[2r(r-1)-r+1\right]x^r+\sum_{n=1}^{\infty}\left\{a_n\left[2(r+n)(r+n-1)-(r+n)+1\right]+a_{n-1}\right\}x^{r+n}=0$$

- The equation

$$a_0\left[2r(r-1)-r+1\right]=0 \quad \overset{a_0\neq 0}{\Leftrightarrow} \quad 2r^2-3r+1=(2r-1)(r-1)=0$$

  is called the **indicial equation**, and was obtained earlier when we examined the corresponding Euler Equation.
- The roots $r_1 = 1$, $r_2 = ½$, of the indicial equation are called the **exponents of the singularity,** for regular singular point $x = 0$.
- The exponents of the singularity determine the qualitative behavior of solution in neighborhood of regular singular point.

Example 1: Recursion Relation

- Recall that

$$a_0\left[2r(r-1)-r+1\right]x^r+\sum_{n=1}^{\infty}\left\{a_n\left[2(r+n)(r+n-1)-(r+n)+1\right]+a_{n-1}\right\}x^{r+n}=0$$

- We now work with the coefficient on x^{r+n}:

$$a_n\left[2(r+n)(r+n-1)-(r+n)+1\right]+a_{n-1}=0$$

- It follows that

$$\begin{aligned}a_n&=-\frac{a_{n-1}}{2(r+n)(r+n-1)-(r+n)+1}\\&=-\frac{a_{n-1}}{2(r+n)^2-3(r+n)+1}\\&=-\frac{a_{n-1}}{\left[2(r+n)-1\right]\left[(r+n)-1\right]},\quad n\geq 1\end{aligned}$$

Example 1: First Root (7 of 13)

- We have

$$a_n = -\frac{a_{n-1}}{[2(r+n)-1][(r+n)-1]}, \quad \text{for } n \geq 1, \; r_1 = 1 \text{ and } r_1 = 1/2$$

- Starting with $r_1 = 1$, this recursion becomes

$$a_n = -\frac{a_{n-1}}{[2(1+n)-1][(1+n)-1]} = -\frac{a_{n-1}}{(2n+1)n}, \quad n \geq 1$$

- Thus

$$a_1 = -\frac{a_0}{3 \cdot 1} \qquad a_3 = -\frac{a_2}{7 \cdot 3} = -\frac{a_0}{(3 \cdot 5 \cdot 7)(1 \cdot 2 \cdot 3)}, \quad \text{etc}$$

$$a_2 = -\frac{a_1}{5 \cdot 2} = \frac{a_0}{(3 \cdot 5)(1 \cdot 2)} \qquad a_n = \frac{(-1)^n a_0}{(3 \cdot 5 \cdot 7 \cdots (2n+1))n!}, \quad n \geq 1$$

Example 1: First Solution (8 of 13)

- Thus we have an expression for the *n*-th term:

$$a_n = \frac{(-1)^n a_0}{\big(3\cdot 5\cdot 7\cdots(2n+1)\big)n!},\ \ n \geq 1$$

- Hence for $x > 0$, one solution to our differential equation is

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ &= a_0 x + \sum_{n=1}^{\infty} \frac{(-1)^n a_0 x^{n+1}}{\big(3\cdot 5\cdot 7\cdots(2n+1)\big)n!} \\ &= a_0 x\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\big(3\cdot 5\cdot 7\cdots(2n+1)\big)n!}\right] \end{aligned}$$

Example 1: Radius of Convergence for First Solution (9 of 13)

- Thus if we omit a_0, one solution of our differential equation is

$$y_1(x) = x\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\left(3\cdot 5\cdot 7\cdots(2n+1)\right)n!}\right], \quad x > 0$$

- To determine the radius of convergence, use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty}\left|\frac{\left(3\cdot 5\cdot 7\cdots(2n+1)\right)n!(-1)^{n+1}x^{n+1}}{\left(3\cdot 5\cdot 7\cdots(2n+1)(2n+3)\right)(n+1)!(-1)^n x^n}\right|$$

$$= \lim_{n\to\infty}\frac{|x|}{(2n+3)(n+1)} = 0 < 1$$

- Thus the radius of convergence is infinite, and hence the series converges for all *x*.

Example 1: Second Root (10 of 13)

- Recall that

$$a_n = -\frac{a_{n-1}}{[2(r+n)-1][(r+n)-1]}, \text{ for } n \geq 1, \; r_1 = 1 \text{ and } r_1 = 1/2$$

- When $r_1 = 1/2$, this recursion becomes

$$a_n = -\frac{a_{n-1}}{[2(1/2+n)-1][(1/2+n)-1]} = -\frac{a_{n-1}}{2n(n-1/2)} = -\frac{a_{n-1}}{n(2n-1)}, \; n \geq 1$$

- Thus

$$a_1 = -\frac{a_0}{1 \cdot 1} \qquad a_3 = -\frac{a_2}{3 \cdot 5} = -\frac{a_0}{(1 \cdot 2 \cdot 3)(1 \cdot 3 \cdot 5)}, \text{ etc}$$

$$a_2 = -\frac{a_1}{2 \cdot 3} = \frac{a_0}{(1 \cdot 2)(1 \cdot 3)} \qquad a_n = \frac{(-1)^n a_0}{((1 \cdot 3 \cdot 5) \cdots (2n-1))n!}, \; n \geq 1$$

Example 1: Second Solution (11 of 13)

- Thus we have an expression for the *n*-th term:

$$a_n = \frac{(-1)^n a_0}{\big(1\cdot 3\cdot 5\cdots(2n-1)\big)n!},\ \ n \geq 1$$

- Hence for $x > 0$, a second solution to our equation is

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ &= a_0 x^{1/2} + \sum_{n=1}^{\infty} \frac{(-1)^n a_0 x^{n+1/2}}{\big(1\cdot 3\cdot 5\cdots(2n-1)\big)n!} \\ &= a_0 x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\big(1\cdot 3\cdot 5\cdots(2n-1)\big)n!}\right] \end{aligned}$$

Example 1: Radius of Convergence for Second Solution (12 of 13)

- Thus if we omit a_0, the second solution is

$$y_2(x) = x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\big(1\cdot 3\cdot 5\cdots(2n-1)\big)n!}\right]$$

- To determine the radius of convergence for this series, we can use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty}\left|\frac{\big(1\cdot 3\cdot 5\cdots(2n-1)\big)n!(-1)^{n+1}x^{n+1}}{\big(1\cdot 3\cdot 5\cdots(2n-1)(2n+1)\big)(n+1)!(-1)^n x^n}\right|$$

$$= \lim_{n\to\infty}\frac{|x|}{(2n+1)n} = 0 < 1$$

- Thus the radius of convergence is infinite, and hence the series converges for all x.

Example 1: General Solution (13 of 13)

- The two solutions to our differential equation are

$$y_1(x) = x\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{\left(3\cdot 5\cdot 7\cdots(2n+1)\right)n!}\right]$$

$$y_2(x) = x^{1/2}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{\left(1\cdot 3\cdot 5\cdots(2n-1)\right)n!}\right]$$

- Since the leading terms of y_1 and y_2 are x and $x^{1/2}$, respectively, it follows that y_1 and y_2 are linearly independent, and hence form a fundamental set of solutions for differential equation.
- Therefore the general solution of the differential equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x), \quad x > 0,$$

where y_1 and y_2 are as given above.

Shifted Expansions & Discussion

- For the analysis given in this section, we focused on $x = 0$ as the regular singular point. In the more general case of a singular point at $x = x_0$, our series solution will have the form

$$y(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

- If the roots r_1, r_2 of the indicial equation are equal or differ by an integer, then the second solution y_2 normally has a more complicated structure. These cases are discussed in Section 5.7.
- If the roots of the indicial equation are complex, then there are always two solutions with the above form. These solutions are complex valued, but we can obtain real-valued solutions from the real and imaginary parts of the complex solutions.

Complex variables

$f(z) = u(x, y) + iv(x, y)$ for $z = x + iy$

$$f'(z) = \lim_{\Delta z \to 0}\left[\frac{f(z+\Delta z) - f(z)}{\Delta z}\right] \text{ exists}$$

Its value does not depend on the direction.

Ex : Show that the function $f(z) = x^2 - y^2 + i2xy$ **is differentiable for all values of** z.

for $\Delta z = \Delta x + i\Delta y$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \frac{(x+\Delta x)^2 - (y+\Delta y)^2 + 2i(x+\Delta x)(y+\Delta y) - x^2 + y^2 - 2ixy}{\Delta x + i\Delta y}$$

$$= 2x + i2y + \frac{(\Delta x)^2 - (\Delta y)^2 + 2i\Delta x\Delta y}{\Delta x + i\Delta y}$$

(1) choose $\Delta y = 0, \Delta x \to 0 \Rightarrow f'(z) = 2x + i2y$

(2) choose $\Delta x = 0, \Delta y \to 0 \Rightarrow f'(z) = 2x + i2y$

Complex variables

$f(z) = u(x,y) + iv(x,y)$ for $z = x + iy$

$$f'(z) = \lim_{\Delta z \to 0}\left[\frac{f(z+\Delta z) - f(z)}{\Delta z}\right] \text{ exists}$$

Its value does not depend on the direction.

Ex : Show that the function $f(z) = x^2 - y^2 + i2xy$ is differentiable for all values of z.

for $\Delta z = \Delta x + i\Delta y$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \frac{(x+\Delta x)^2 - (y+\Delta y)^2 + 2i(x+\Delta x)(y+\Delta y) - x^2 + y^2 - 2ixy}{\Delta x + i\Delta y}$$

$$= 2x + i2y + \frac{(\Delta x)^2 - (\Delta y)^2 + 2i\Delta x\Delta y}{\Delta x + i\Delta y}$$

(1) choose $\Delta y = 0, \Delta x \to 0 \Rightarrow f'(z) = 2x + i2y$

(2) choose $\Delta x = 0, \Delta y \to 0 \Rightarrow f'(z) = 2x + i2y$

Complex variables

$$f(z) = u(x, y) + iv(x, y) \text{ for } z = x + iy$$

$$f'(z) = \lim_{\Delta z \to 0}\left[\frac{f(z + \Delta z) - f(z)}{\Delta z}\right] \text{ exists}$$

Its value does not depend on the direction.

Ex : Show that the function $f(z) = x^2 - y^2 + i2xy$ **is differentiable for all values of** z.

for $\Delta z = \Delta x + i\Delta y$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \frac{(x + \Delta x)^2 - (y + \Delta y)^2 + 2i(x + \Delta x)(y + \Delta y) - x^2 + y^2 - 2ixy}{\Delta x + i\Delta y}$$

$$= 2x + i2y + \frac{(\Delta x)^2 - (\Delta y)^2 + 2i\Delta x\Delta y}{\Delta x + i\Delta y}$$

(1) choose $\Delta y = 0, \Delta x \to 0 \Rightarrow f'(z) = 2x + i2y$

(2) choose $\Delta x = 0, \Delta y \to 0 \Rightarrow f'(z) = 2x + i2y$

Complex variables

**** Another method:**

$$f(z) = (x+iy)^2 = z^2$$

$$f'(z) = \lim_{\Delta z \to 0}\left[\frac{(z+\Delta z)^2 - z^2}{\Delta z}\right] = \lim_{\Delta z \to 0}\left[\frac{(\Delta z)^2 + 2z\Delta z}{\Delta z}\right]$$

$$= \lim_{\Delta z \to 0} \Delta z + 2z = 2z$$

Ex : Show that the function $f(z) = 2y + ix$ is not differentiable anywhere in the complex plane.

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{2y + 2\Delta y + ix + i\Delta x - 2y - ix}{\Delta x + i\Delta y} = \frac{2\Delta y + i\Delta x}{\Delta x + i\Delta y}$$

if $\Delta z \to 0$ along a line thriugh z of slope $m \Rightarrow \Delta y = m\Delta x$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta x, \Delta y \to 0}\left[\frac{2\Delta y + i\Delta x}{\Delta x + i\Delta y}\right] = \frac{2m + i}{1 + im}$$

The limit depends on m (the direction), so $f(z)$ is nowhere differentiable.

Complex variables

Ex : Show that the function $f(z) = 1/(1-z)$ is analytic everywhere except at $z = 1$.

$$f'(z) = \lim_{\Delta z \to 0}\left[\frac{f(z+\Delta z) - f(z)}{\Delta z}\right] = \lim_{\Delta z \to 0}\left[\frac{1}{\Delta z}\left(\frac{1}{1-z-\Delta z} - \frac{1}{1-z}\right)\right]$$

$$= \lim_{\Delta z \to 0}\left[\frac{1}{(1-z-\Delta z)(1-z)}\right] = \frac{1}{(1-z)^2}$$

Provided $z \neq 1$, $f(z)$ is analytic everywhere such that $f'(z)$ is independent of the direction.

Cauchy-Riemann relation

A function f(z)=u(x,y)+iv(x,y) is differentiable and analytic, there must be particular connection between u(x,y) and v(x,y)

$$L = \lim_{\Delta z \to 0} [\frac{f(z+\Delta z) - f(z)}{\Delta z}]$$

$$f(z) = u(x,y) + iv(x,y) \qquad \Delta z = \Delta x + i\Delta y$$

$$f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)$$

$$\Rightarrow L = \lim_{\Delta x, \Delta y \to 0} [\frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)}{\Delta x + i\Delta y}]$$

$$\text{(1) if suppose } \Delta z \text{ is real} \Rightarrow \Delta y = 0$$

$$\Rightarrow L = \lim_{\Delta x \to 0} [\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}] = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\text{(2) if suppose } \Delta z \text{ is imaginary} \Rightarrow \Delta x = 0$$

$$\Rightarrow L = \lim_{\Delta y \to 0} [\frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i\frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}] = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \text{Cauchy - Riemann relations}}$$

Ex : In which domain of the complex plane is

$f(z) = |x| - i|y|$ an analytic function?

$u(x,y) = |x|,\ v(x,y) = -|y|$

(1) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial}{\partial x}|x| = \frac{\partial}{\partial y}[-|y|] \Rightarrow$ (a) $x > 0,\ y < 0$ the fouth quatrant

(b) $x < 0,\ y > 0$ the second quatrant

(2) $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow \frac{\partial}{\partial x}[-|y|] = -\frac{\partial}{\partial y}|x|$

$z = x + iy$ and complex conjugate of z is $z^* = x - iy$

$\Rightarrow x = (z + z^*)/2$ and $y = (z - z^*)/2i$

$$\Rightarrow \frac{\partial f}{\partial z^*} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z^*} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

If $f(z)$ is analytic, then the Cauchy - Riemann relations

are satisfied. $\Rightarrow \partial f / \partial z^* = 0$ implies an analytic fonction of z contains

the combination of $x + iy$, not $x - iy$

If Cauchy-Riemann relations are satisfied

$$(1)\ \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)=-\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\Rightarrow\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial^2 y^2}=0$$

$$(2)\ \text{the same result for function } v(x,y)\Rightarrow\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial^2 y^2}=0$$

$\Rightarrow u(x,y)$ **and** $v(x,y)$ **are solutions of Laplace's equation in two dimension.**

For two families of curves $u(x,y)=$ **conctant and** $v(x,y)=$ **constant, the normal vectors corresponding the two curves, respectively, are**

$$\vec{\nabla}u(x,y)=\frac{\partial u}{\partial x}\hat{i}+\frac{\partial u}{\partial y}\hat{j}\ \text{ and }\ \vec{\nabla}v(x,y)=\frac{\partial v}{\partial x}\hat{i}+\frac{\partial v}{\partial y}\hat{j}$$

$$\vec{\nabla}u\cdot\vec{\nabla}v=\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}=-\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}=0\quad\text{orthogonal}$$

Power series in a complex variable

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n r^n \exp(in\theta)$$

$$\text{if } \sum_{n=0}^{\infty} |a_n| r^n \text{ is convergent} \Rightarrow f(z) \text{ is absolutely convergent}$$

$$\text{Is } \sum_{n=0}^{\infty} |a_n| r^n \text{ convergent or not, can be justisfied by " Cauchy root test".}$$

$$\text{The radius of convergence } R \Rightarrow \frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} \Rightarrow (1)\ |z| < R \text{ absolutely convergent}$$

$$(2)\ |z| > R \text{ divergent}$$

$$(3)\ |z| = R \text{ undetermined}$$

$$(1)\ \sum_{n=0}^{\infty} \frac{z^n}{n!} \Rightarrow \lim_{n\to\infty} \left(\frac{1}{n!}\right)^{1/n} = 0 \Rightarrow R = \infty \text{ converges for all } z$$

$$(2)\ \sum_{n=0}^{\infty} n! z^n \Rightarrow \lim_{n\to\infty} (n!)^{1/n} = \infty \Rightarrow R = 0 \text{ converges only at } z = 0$$

Some elementary functions

$$\text{Define } \exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Ex : Show that $\exp z_1 \exp z_2 = \exp(z_1 + z_2)$

$$\exp(z_1 + z_2) = \sum_{n=0}^{\infty} \frac{(z_1 + z_2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}(C_0^n z_1^n + C_1^n z_1^{n-1} z_2 + C_2^n z_1^{n-2} z_2^2 + C_r^n z_1^{n-r} z_2^r + \ldots + C_n^n z_2^n)$$

set $n = r + s \Rightarrow$ the coeff. of $z_1^s z_2^r$ is $\dfrac{C_r^n}{n!} = \dfrac{1}{n!} \dfrac{n!}{(n-r)!r!} = \dfrac{1}{s!r!}$

$$\exp z_1 \exp z_2 = \sum_{s=0}^{\infty} \frac{z^s}{s!} \sum_{r=0}^{\infty} \frac{z^r}{r!} = \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \frac{1}{s!r!} z_1^s z_2^r$$

There are the same coeff. of $z_1^s z_2^r$ for the above two terms.

$$\boxed{\begin{array}{l}\text{Define the complex comonent of a real number } a>0 \\ a^z = \exp(z\ln a) = \sum_{n=0}^{\infty}\frac{z^n(\ln a)^n}{n!}\end{array}}$$

(1) if $a = e \Rightarrow e^z = \exp(z\ln e) = \exp z$ the same as real number

(2) if $a = e, z = iy \Rightarrow e^{iy} = \exp(iy) = 1 - \frac{y^2}{2!} - \frac{iy^3}{3!} + \ldots$

$$= 1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \ldots\ldots + i\left(y - \frac{y^3}{3!} + \ldots\right) = \cos y + i\sin y$$

(3) if $a = e, z = x + iy \Rightarrow e^{x+iy} = e^x e^{iy} = \exp(x)(\cos y + i\sin y)$

Set $\exp w = z$

Write $z = r\exp i\theta$ for r is real and $-\pi < \theta \le \pi$

$\Rightarrow z = r\exp[i(\theta + 2n\pi)] \Rightarrow w = Lnz = \ln r + i(\theta + 2n\pi)$

Lnz is a multivalued function of z.

Take its principal value by choosing $n = 0$

$\Rightarrow \ln z = \ln r + i\theta \quad -\pi < \theta \le \pi$

If $t \ne 0$ and z are both complex numbers, we define

$$t^z = \exp(zLnt)$$

Ex : Show that there are exactly n distinct nth roots of t.

$$t^{\frac{1}{n}} = \exp\left(\frac{1}{n} Lnt\right) \quad \text{and} \quad t = r\exp[i(\theta + 2k\pi)]$$

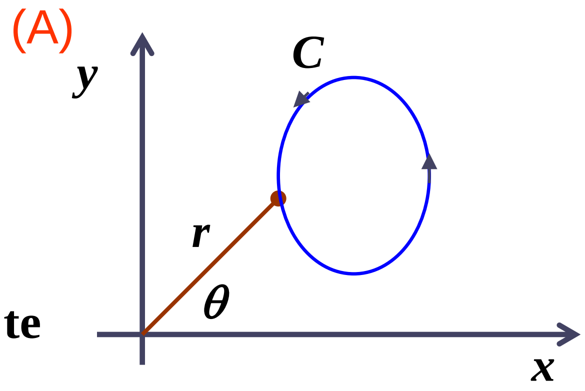
$$\Rightarrow \quad t^{\frac{1}{n}} = \exp\left[\frac{1}{n}\ln r + i\frac{(\theta + 2k\pi)}{n}\right] = r^{\frac{1}{n}}\exp\left[i\frac{(\theta + 2k\pi)}{n}\right]$$

Multivalued functions and branch cuts

A logarithmic function, a complex power and a complex root are all multivalued. Is the properties of analytic function still applied?

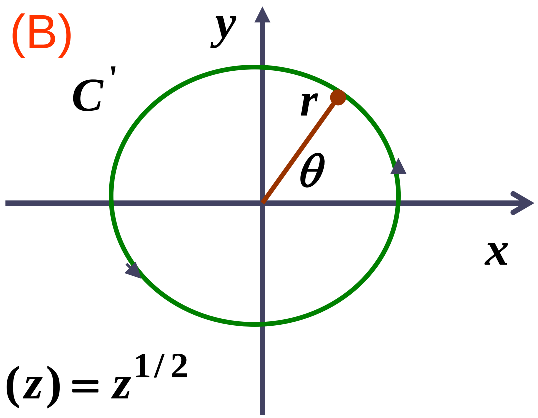
$$\text{Ex}: \; f(z) = z^{1/2} \text{ and } z = r\exp(i\theta)$$

(A) z traverse any closed contour C that dose not enclose the origin, θ return to its original value after one complete circuit.



(B) $\theta \to \theta + 2\pi$ enclose the origin

$$r^{1/2}\exp(i\theta/2) \to r^{1/2}\exp[i(\theta+2\pi)/2]$$
$$= -r^{1/2}\exp(i\theta/2)$$
$$\Rightarrow f(z) \to -f(z)$$



$z = 0$ is a branch point of the function $f(z) = z^{1/2}$

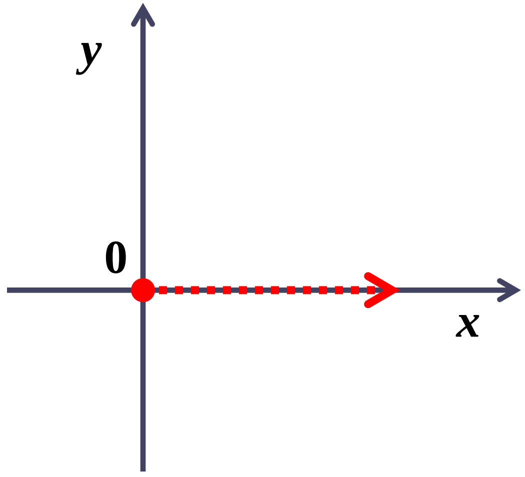
Branch point: z remains unchanged while z traverse a closed contour C about some point. But a function f(z) changes after one complete circuit.

Branch cut: It is a line (or curve) in the complex plane that we must cross , so the function remains single-valued.

$$\text{Ex}: \ f(z) = z^{1/2}$$

$$\text{restrict}\ \theta \Rightarrow 0 \le \theta < 2\pi$$

$$\Rightarrow f(z)\ \text{is single-valued}$$



Ex : Find the branch points of $f(z)=\sqrt{z^2+1}$, and hence sketch suitable arrangements of branch cuts.

$f(z)=\sqrt{z^2+1}=\sqrt{(z+i)(z-i)}$ expected branch points : $z=\pm i$

set $z-i=r_1\exp(i\theta_1)$ and $z+i=r_2\exp(i\theta_2)$

$$\Rightarrow f(z)=\sqrt{r_1r_2}\exp(i\theta_1/2)\exp(i\theta_2/2)$$
$$=\sqrt{r_1r_2}\exp[i(\theta_1+\theta_2)]$$

If contour C encloses

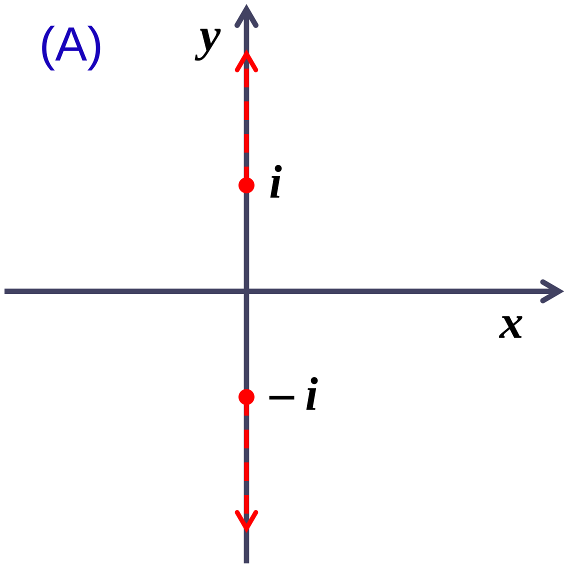
(1) neither branch point, then $\theta_1\to\theta_1,\theta_2\to\theta_2\Rightarrow f(z)\to f(z)$

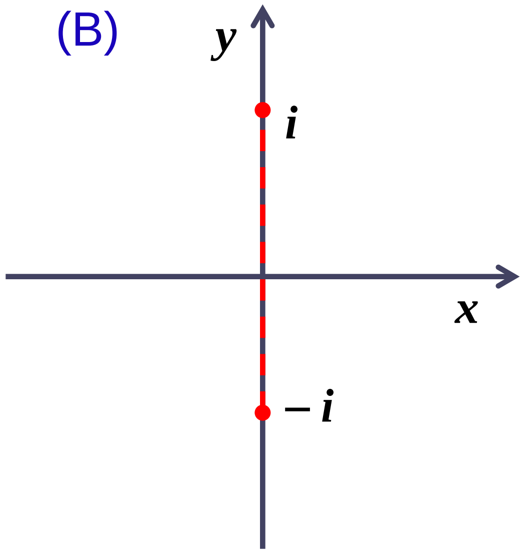
(2) $z=i$ but not $z=-i$, then $\theta_1\to\theta_1+2\pi,\theta_2\to\theta_2\Rightarrow f(z)\to -f(z)$

(3) $z=-i$ but not $z=i$, then $\theta_1\to\theta_1,\theta_2\to\theta_2+2\pi\Rightarrow f(z)\to -f(z)$

(4) both branch points, then $\theta_1\to\theta_1+2\pi,\theta_2\to\theta_2+2\pi\Rightarrow f(z)\to f(z)$

$f(z)$ changes value around loops containing either $z = i$ or $z = -i$. We choose branch cut as follows:





Singularities and zeros of complex function

Isolated singularity (pole): $f(z) = \dfrac{g(z)}{(z - z_0)^n}$

n is a positive integer, $g(z)$ is analytic at all points in some neighborhood containing $z = z_0$ and $g(z_0) \neq 0$, the $f(z)$ has a pole of order n at $z = z_0$.

** An alternate definition for that $f(z)$ has a pole of order n at $z = z_0$ is

$$\lim_{z \to z_0} [(z - z_0)^n f(z)] = a$$

$f(z)$ is analytic and a is a finite, non-zero complex number

(1) if $a = 0$, then $z = z_0$ is a pole of order less than n.

(2) if a is infinite, then $z = z_0$ is a pole of order greater than n.

(3) if $z = z_0$ is a pole of $f(z) \Rightarrow |f(z)| \to \infty$ as $z \to z_0$

(4) from any direction, if no finite n satisfies the limit $\Rightarrow$ essential singularity

Ex : Find the singularities of the function

(1) $f(z)=\dfrac{1}{1-z}-\dfrac{1}{1+z}$

$\Rightarrow f(z)=\dfrac{2z}{(1-z)(1+z)}$ poles of order 1 at $z=1$ and $z=-1$

(2) $f(z)=\tanh z$

$$=\frac{\sinh z}{\cosh z}=\frac{\exp z-\exp(-z)}{\exp z+\exp(-z)}$$

$f(z)$ has a singularity when $\exp z=-\exp(-z)$

$\Rightarrow \exp z=\exp[i(2n+1)\pi]=\exp(-z)$ n is any integer

$\Rightarrow 2z=i(2n+1)\pi \Rightarrow z=(n+\dfrac{1}{2})\pi i$

Using l'Hospital's rule

$$\lim_{z\to(n+1/2)\pi i}\left\{\frac{[z-(n+1/2)\pi i]\sinh z}{\cosh z}\right\}=\lim_{z\to(n+1/2)\pi i}\left\{\frac{[z-(n+1/2)\pi i]\cosh z+\sinh z}{\sinh z}\right\}=1$$

each singularity is a simple pole (n = 1)

Remove singularties:

Singularity makes the value of $f(z)$ undetermined, but $\lim_{z \to z_0} f(z)$ exists and independent of the direction from which z_0 is approached.

Ex : Show that $f(z) = \sin z / z$ is a removable singularity at $z = 0$

Sol : $\lim_{z \to 0} f(z) = 0/0$ undetermined

$$f(z) = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - ....\right) = 1 - \frac{z^3}{3!} + \frac{z^5}{5!} - ....$$

$\lim_{z \to 0} f(z) = 1$ is independent of the way $z \to 0$, so

$f(z)$ has a removable singularity at $z = 0$.

The behavior of $f(z)$ at infinity is given by that of $f(1/\xi)$ at $\xi = 0$, where $\xi = 1/z$

Ex : Find the behavior at infinity of (i) $f(z) = a + bz^{-2}$

(ii) $f(z) = z(1+z^2)$ and (iii) $f(z) = \exp z$

(i) $f(z) = a + bz^{-2} \Rightarrow$ set $z = 1/\xi \Rightarrow f(1/\xi) = a + b\xi^2$

is analytic at $\xi = 0 \Rightarrow f(z)$ is analytic at $z = \infty$

(ii) $f(z) = z(1-z^2) \Rightarrow f(1/\xi) = 1/\xi + 1/\xi^3$ has a pole of order 3 at $z = \infty$

(iii) $f(z) = \exp z \Rightarrow f(1/\xi) = \sum_{n=0}^{\infty} (n!)^{-1} \xi^{-n}$

$f(z)$ has an essential singularity at $z = \infty$

If $f(z_0) = 0$ and $f(z) = (z - z_0)^n g(z)$, if n is a positive integer, and $g(z_0) \neq 0$

(i) $z = z_0$ is called a zero of order n.

(ii) if $n = 1$, $z = z_0$ is called a simple zero.

(iii) $z = z_0$ is also a pole of order n of $1/f(z)$

Complex integral

A real continuous parameter t, for $\alpha \le t \le \beta$

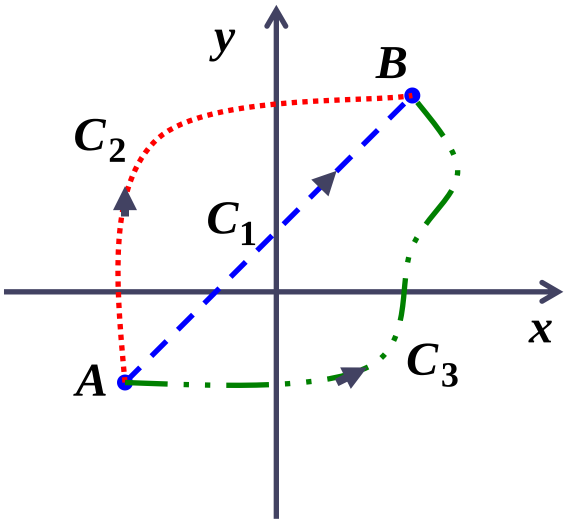
$x = x(t), y = y(t)$ and point A is $t = \alpha$,

point B is $t = \beta$

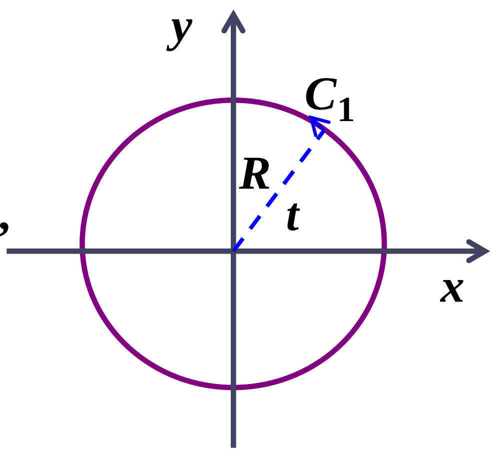
$$\Rightarrow \int_C f(z)dz = \int_C (u+iv)(dx+idy)$$

$$= \int_C udx - \int_C vdy + i\int_C udy + i\int_C vdx$$

$$= \int_\alpha^\beta u\frac{dx}{dt}dt - \int_\alpha^\beta v\frac{dy}{dt}dt + i\int_\alpha^\beta u\frac{dy}{dt}dt + i\int_\alpha^\beta v\frac{dx}{dt}dt$$



$$\text{Ex : Evaluate the complex integral of } f(z)=1/z, \text{ along the circle } |z|=R, \text{ starting and finishing at } z=R.$$



$$z(t)=R\cos t+iR\sin t,\ 0\le t\le 2\pi$$

$$\frac{dx}{dt}=-R\sin t,\ \frac{dy}{dt}=R\cos t,\ f(z)=\frac{1}{x+iy}=\frac{x-iy}{x^2+y^2}=u+iv,$$

$$u=\frac{x}{x^2+y^2}=\frac{\cos t}{R},\ v=\frac{-y}{x^2+y^2}=\frac{-\sin t}{R}$$

$$\int_{C_1}\frac{1}{z}dz=\int_0^{2\pi}\frac{\cos t}{R}(-R\sin t)dt-\int_0^{2\pi}\left(\frac{-\sin t}{R}\right)R\cos t\,dt$$

$$+i\int_0^{2\pi}\frac{\cos t}{R}R\cos t\,dt+i\int_0^{2\pi}\left(\frac{-\sin t}{R}\right)(-R\sin t)dt$$

$$=0+0+i\pi+i\pi=2\pi i$$

** The integral is also calculated by

$$\int_{C_1}\frac{dz}{z}=\int_0^{2\pi}\frac{-R\sin t+iR\cos t}{R\cos t+iR\sin t}dt=\int_0^{2\pi}i\,dt=2\pi i$$

The calculated result is independent of R.

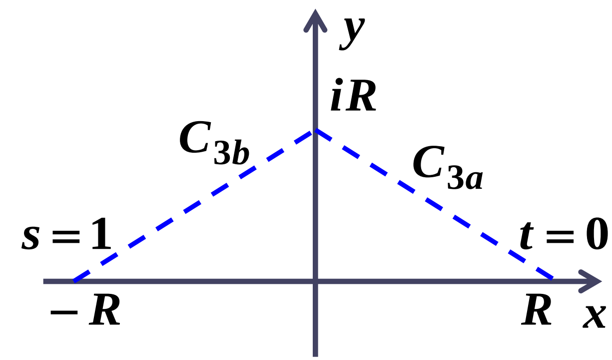
Ex : Evaluate the complex integral of $f(z) = 1/z$ along

(i) the contour C_2 consisting of the semicircle $|z| = R$ in the half-plane $y \geq 0$

(ii) the contour C_3 made up of two straight lines C_{3a} and C_{3b}

(i) This is just as in the previous example, but for

$$0 \leq t \leq \pi \Rightarrow \int_{C_2} dz/z = \pi i$$



(ii) $C_{3a} : z = (1-t)R + itR$ for $0 \leq t \leq 1$

$C_{3b} : -sR + i(1-s)R$ for $0 \leq s \leq 1$

$$\int_{C_3} \frac{dz}{z} = \int_0^1 \frac{-R+iR}{R+t(-R+iR)} dt + \int_0^1 \frac{-R-iR}{iR+s(-R-iR)} dt$$

$$\text{1st term} \Rightarrow \int_0^1 \frac{-1+i}{1-t+it} dt = \int_0^1 \frac{2t-1}{1-2t+2t^2} dt + i\int_0^1 \frac{1}{1-2t+2t^2} dt$$

$$= \frac{1}{2}[\ln(1-2t+2t^2)]\Big|_0^1 + \frac{i}{2}[2\tan^{-1}(\frac{t-1/2}{1/2})]\Big|_0^1$$

$$= 0 + \frac{i}{2}[\frac{\pi}{2} - (-\frac{\pi}{2})] = \frac{\pi i}{2}$$

$$\boxed{\int \frac{a}{a^2+x^2} dx = \tan^{-1}(\frac{x}{a}) + c}$$

$$\text{2nd term} \Rightarrow \int_0^1 \frac{1+i}{s+i(s-1)}ds = \int_0^1 \frac{(1+i)[s-i(s-1)]}{s^2+(s-1)^2}ds$$

$$= \int_0^1 \frac{2s-1}{2s^2-2s+1}ds + i\int_0^1 \frac{1}{2s^2-2s+1}ds$$

$$= \frac{1}{2}[\ln(2s^2-2s+1)]\Big|_0^1 + i\tan^{-1}\left(\frac{s-1/2}{1/2}\right)\Big|_0^1$$

$$= 0 + i\left[\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right] = \frac{\pi i}{2}$$

$$\Rightarrow \int_{C_3} \frac{dz}{z} = \pi i$$

The integral is independent of the different path.

$\text{Ex : Evaluate the complex integral of } f(z) = \text{Re}(z) \text{ along}$
$\text{the path } C_1, C_2 \text{ and } C_3 \text{ as shown in the previous examples.}$

$$\text{(i)} \quad C_1 : \int_0^{2\pi} R\cos t(-R\sin t + iR\cos t)dt = i\pi R^2$$

$$\text{(ii)} \quad C_2 : \int_0^{\pi} R\cos t(-R\sin t + iR\cos t)dt = \frac{i\pi}{2}R^2$$

$$\text{(iii)} C_3 = C_{3a} + C_{3b} :$$

$$\int_0^1 (1-t)R(-R+iR)dt + \int_0^1 (-sR)(-R-iR)ds$$

$$= R^2\int_0^1 (1-t)(-1+i)dt + R^2\int_0^1 s(1+i)ds$$

$$= \frac{1}{2}R^2(-1+i) + \frac{1}{2}R^2(1+i) = iR^2$$

The integral depends on the different path.

Cauchy theorem

If $f(z)$ is an analytic function, and $f'(z)$ is continuous at each point within and on a closed contour C

$$\Rightarrow \oint_C f(z)dz = 0$$

If $\dfrac{\partial p(x,y)}{\partial x}$ and $\dfrac{\partial q(x,y)}{\partial y}$ are continuous within and on a closed contour C, then by two - demensional divergence theorem $\Rightarrow \iint_R (\dfrac{\partial p}{\partial x} + \dfrac{\partial q}{\partial y})dxdy = \oint_C (pdy - qdx)$

$f(z) = u + iv$ and $dz = dx + idy$

$$I = \oint_C f(z)dz = \oint_C (udx - vdy) + i\oint_C (vdx + udy)$$

$$= \iint_R [\frac{\partial(-u)}{\partial y} + \frac{\partial(-v)}{\partial x}]dxdy + i\iint_R [\frac{\partial(-v)}{\partial y} + \frac{\partial u}{\partial x}]dxdy = 0$$

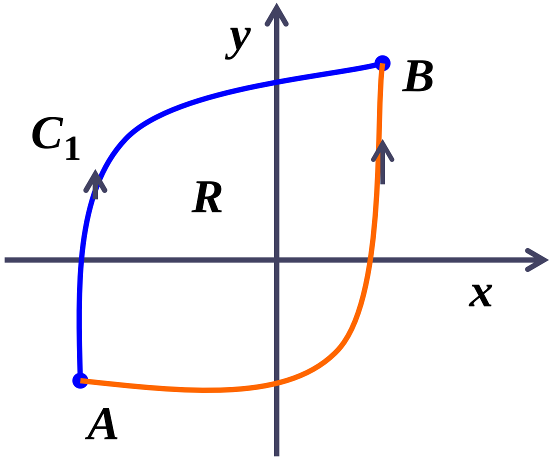
$f(z)$ is analytic and the Cauchy - Riemann relations apply.

Ex : Suppose twos points A and B in the complex plane are joined by two different paths C_1 and C_2. Show that if $f(z)$ is an analytic function on each path and in the region enclosed by the two paths then the integral of $f(z)$ is the same along C_1 and C_2.

$$\int_{C_1} f(z)dz - \int_{C_2} f(z)dz = \oint_{C_1 - C_2} f(z)dz = 0$$

path $C_1 - C_2$ forms a closed contour enclosing R

$$\Rightarrow \int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$



Ex : Consider two closed contour C and γ in the Argand diagram, γ being sufficiently small that it lies completely with C. Show that if the function $f(z)$ is analytic in the region between the two contours then $\oint_C f(z)dz = \oint_\gamma f(z)dz$

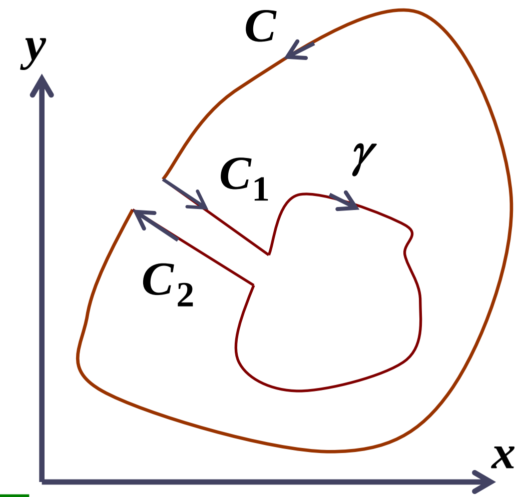
the area is bounded by Γ, and

$f(z)$ is analytic

$$\oint_\Gamma f(z)dz = 0$$

$$= \oint_C f(z)dz + \oint_\gamma f(z)dz + \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz$$

If take the direction of contour γ as that of contour $C \Rightarrow \oint_C f(z)dz = \oint_\gamma f(z)dz$



Morera's theorem:

if $f(z)$ is a continuous function of z in a closed domain R bounded by a curve C, for $\oint_C f(z)dz = 0 \Rightarrow f(z)$ is analytic.

Cauchy's integral formula

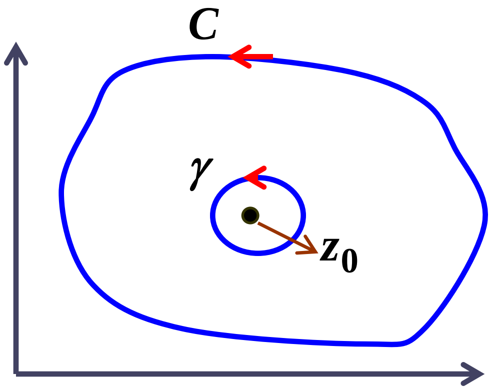
If $f(z)$ is analytic within and on a closed contour C and z_0 is a point within C then $f(z_0) = \dfrac{1}{2\pi i}\oint_C \dfrac{f(z)}{z - z_0}dz$

$$I = \oint_C \frac{f(z)}{z - z_0}dz = \oint_\gamma \frac{f(z)}{z - z_0}dz$$

for $z = z_0 + \rho\exp(i\theta)$, $dz = i\rho\exp(i\theta)d\theta$

$$I = \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{\rho e^{i\theta}} i\rho e^{i\theta} d\theta$$

$$= i\int_0^{2\pi} f(z_0 + \rho e^{i\theta})d\theta \overset{\rho\to 0}{=} 2\pi i f(z_0)$$



The integral form of the derivative of a complex function:

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz$$

$$f'(z_0) = \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}$$

$$= \lim_{h\to 0}[\frac{1}{2\pi i}\oint_C \frac{f(z)}{h}(\frac{1}{z-z_0-h}-\frac{1}{z-z_0})dz]$$

$$= \lim_{h\to 0}[\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0-h)(z-z_0)}dz]$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz$$

For nth derivative $f^{(n)}(z_0) = \dfrac{n!}{2\pi i}\oint_C \dfrac{f(z)}{(z-z_0)^{n+1}}dz$

Ex : Suppose that $f(z)$ is analytic inside and on a circle C of radius R centered on the point $z = z_0$. If $|f(z)| \le M$ on the circle, where M is some constant, show that $|f^{(n)}(z_0)| \le \dfrac{Mn!}{R^n}$.

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)dz}{(z-z_0)^{n+1}}\right| \le \frac{n!}{2\pi}\frac{M}{R^{n+1}}2\pi R = \frac{Mn!}{R^n}$$

Liouville's theorem: If $f(z)$ is analytic and bounded for all z then $f(z)$ is a constant.

Using Cauchy's inequality: $|f^{(n)}(z_0)| \le \dfrac{Mn!}{R^n}$

set $n = 1$ and let $R \to \infty \Rightarrow |f'(z_0)| = 0 \Rightarrow f'(z_0) = 0$

Since $f(z)$ is analytic for all z, we may take z_0 as any point in the z-plane. $f'(z) = 0$ for all $z \Rightarrow f(z) = \text{constant}$

Taylor and Laurent series

Taylor's theorem:

If $f(z)$ is analytic inside and on a circle C of radius R centered on the point $z = z_0$, and z is a point inside C, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

$f(z)$ is analytic inside and on C, so $f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{\xi - z} d\xi$ **where ξ lies on C**

expand $\frac{1}{\xi - z}$ **as a geometric series in** $\frac{z - z_0}{\xi - z_0} \Rightarrow \frac{1}{\xi - z} = \frac{1}{\xi - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{\xi - z_0}\right)^n$

$$\Rightarrow f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{\xi - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{\xi - z_0}\right)^n d\xi = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z - z_0)^n \oint_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z - z_0)^n \frac{2\pi i f^{(n)}(z_0)}{n!} = \sum_{n=0}^{\infty}(z - z_0)^n \frac{f^{(n)}(z_0)}{n!}$$

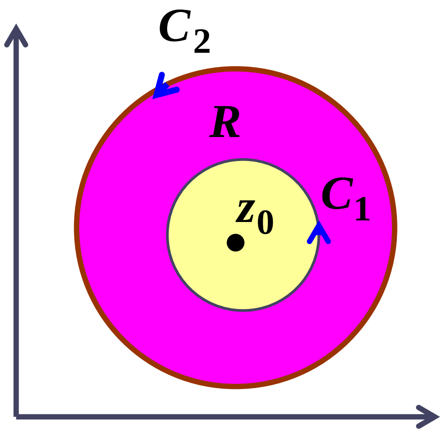
If $f(z)$ has a pole of order p at $z = z_0$ but is analytic at every other point inside and on C. Then $g(z) = (z - z_0)^p f(z)$ is analytic at $z = z_0$ and expanded as a Taylor series $g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$.

Thus, for all z inside C f(z) can be expanded *as a Laurent series*

$$f(z) = \frac{a_{-p}}{(z - z_0)^p} + \frac{a_{-p+1}}{(z - z_0)^{p-1}} + \ldots\ldots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2$$

$a_n = b_{n+p}$ and $b_n = \dfrac{g^{(n)}(z_0)}{n!} = \dfrac{1}{2\pi i} \oint \dfrac{g(z)}{(z - z_0)^{n+1}} dz$

$\Rightarrow a_n = \dfrac{1}{2\pi i} \oint \dfrac{g(z)}{(z - z_0)^{n+1+p}} dz = \dfrac{1}{2\pi i} \oint \dfrac{f(z)}{(z - z_0)^{n+1}} dz$



$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ is analytic in a region R between two circles C_1 and C_2 centered on $z = z_0$

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n$$

(1) If $f(z)$ is analytic at $z=z_0$, then all $a_n=0$ for $n<0$.

It may happen $a_n=0$ for $n\geq 0$, the first non-vanishing term is $a_m(z\text{-}z_0)^m$ with $m>0$, $f(z)$ is said to have a zero of order m at $z=z_0$.

(2) If $f(z)$ is not analytic at $z=z_0$

(i) possible to find $a_{-p}\neq 0$ but $a_{-p-k}=0$ for all $k>0$

$f(z)$ has a pole of order p at $z=z_0$, a_{-1} is called the residue of $f(z)$

(ii) impossible to find a lowest value of $-p\Rightarrow$ essential singularity

Ex : Find the Laurent series of $f(z)=\dfrac{1}{z(z-2)^3}$ about the singularities $z=0$ and $z=2$. Hence verify that $z=0$ is a pole of order 1 and $z=2$ is a pole of order 3, and find the residue of $f(z)$ at each pole.

(1) point $z=0$

$$f(z)=\frac{-1}{8z(1-z/2)^3}=\frac{-1}{8z}[1+(-3)(\frac{-z}{2})+\frac{(-3)(-4)}{2!}(\frac{-z}{2})^2+\frac{(-3)(-4)(-5)}{3!}(\frac{-z}{2})^3+...]$$

$$=-\frac{1}{8z}-\frac{3}{16}-\frac{3}{16}z-\frac{5z^2}{32}-... \qquad z=0 \text{ is a pole of order 1}$$

(2) point $z=2 \Rightarrow$ set $z-2=\xi \Rightarrow z(z-2)^3=(2+\xi)\xi^3=2\xi^3(1+\xi/2)$

$$f(z)=\frac{1}{2\xi^3(1+\xi/2)}=\frac{1}{2\xi^3}[1-(\frac{\xi}{2})+(\frac{\xi}{2})^2-(\frac{\xi}{2})^3+(\frac{\xi}{2})^4-...]$$

$$=\frac{1}{2\xi^3}-\frac{1}{4\xi^2}+\frac{1}{8\xi}-\frac{1}{16}+\frac{\xi}{32}-..=\frac{1}{2(z-2)^3}-\frac{1}{4(z-2)^2}+\frac{1}{8(z-2)}-\frac{1}{16}+\frac{z-2}{32}-$$

$z=2$ is a pole of order 3, the residue of $f(z)$ at $z=2$ is $1/8$.

How to obtain the residue ?

$$f(z)=\frac{a_{-m}}{(z-z_0)^m}+\ldots\ldots+\frac{a_{-1}}{(z-z_0)}+a_0+a_1(z-z_0)+a_2(z-z_0)^2+\ldots$$

$$\Rightarrow (z-z_0)^m f(z)=a_{-m}+a_{-m+1}(z-z_0)+\ldots\ldots+a_{-1}(z-z_0)^{m-1}+\ldots$$

$$\Rightarrow \frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]=(m-1)!a_{-1}+\sum_{n=1}^{\infty}b_n(z-z_0)^n$$

Take the limit $z\to z_0$

$$\boxed{R(z_0)=a_{-1}=\lim_{z\to z_0}\{\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\}\ \text{residue at } z=z_0}$$

(1) For a simple pole $m=1\Rightarrow R(z_0)=\lim_{z\to z_0}[(z-z_0)f(z)]$

(2) If $f(z)$ has a simple at $z=z_0$ and $f(z)=\dfrac{g(z)}{h(z)}$, $g(z)$ is analytic and non-zero at z_0 and $h(z_0)=0$

$$\Rightarrow R(z_0)=\lim_{z\to z_0}\frac{(z-z_0)g(z)}{h(z)}=g(z_0)\lim_{z\to z_0}\frac{(z-z_0)}{h(z)}=g(z_0)\lim_{z\to z_0}\frac{1}{h'(z)}=\frac{g(z_0)}{h'(z_0)}$$

Ex : Suppose that $f(z)$ has a pole of order m at the point $z = z_0$. By considering the Laurent series of $f(z)$ about z_0, deriving a general expression for the residue $R(z_0)$ of $f(z)$ at $z = z_0$. Hence evaluate the residue of the function $f(z) = \dfrac{\exp iz}{(z^2+1)^2}$ at the point $z = i$.

$$f(z) = \frac{\exp iz}{(z^2+1)^2} = \frac{\exp iz}{(z+i)^2(z-i)^2} \quad \text{poles of order 2 at } z = i \text{ and } z = -i$$

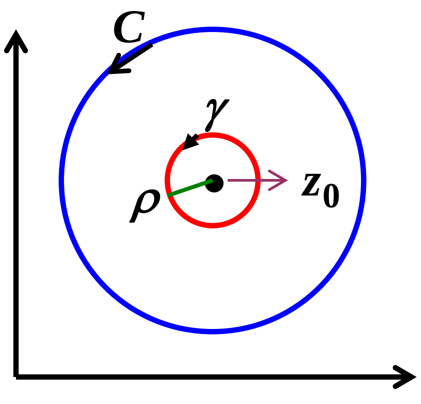
for pole at $z = i$:

$$\frac{d}{dz}[(z-i)^2 f(z)] = \frac{d}{dz}\left[\frac{\exp iz}{(z+i)^2}\right] = \frac{i}{(z+i)^2}\exp iz - \frac{2}{(z+i)^3}\exp iz$$

$$R(i) = \frac{1}{1!}\left[\frac{i}{(2i)^2}e^{-1} - \frac{2}{(2i)^3}e^{-1}\right] = \frac{-i}{2e}$$

20.14 Residue theorem

$f(z)$ has a pole of order m at $z = z_0$



$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n$$

$$I = \oint_C f(z)dz = \oint_\gamma f(z)dz$$

set $z = z_0 + \rho e^{i\theta} \Rightarrow dz = i\rho e^{i\theta} d\theta$

$$I = \sum_{n=-m}^{\infty} a_n \oint_C (z - z_0)^n dz = \sum_{n=-m}^{\infty} a_n \int_0^{2\pi} i\rho^{n+1} e^{i(n+1)\theta} d\theta$$

for $n \neq -1 \Rightarrow \int_0^{2\pi} i\rho^{n+1} e^{i(n+1)\theta} d\theta = \dfrac{i\rho^{n+1} e^{i(n+1)\theta}}{i(n+1)} \Big|_0^{2\pi} = 0$

for $n = 1 \Rightarrow \int_0^{2\pi} i d\theta = 2\pi i$

$$\boxed{I = \oint_C f(z)dz = 2\pi i a_{-1}}$$

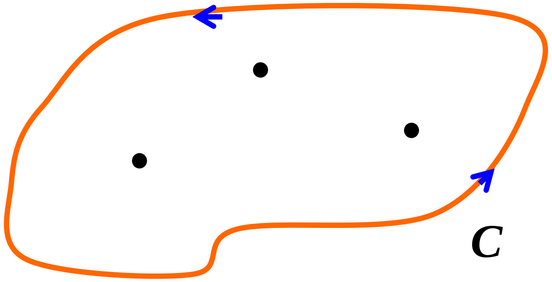
Residue theorem:

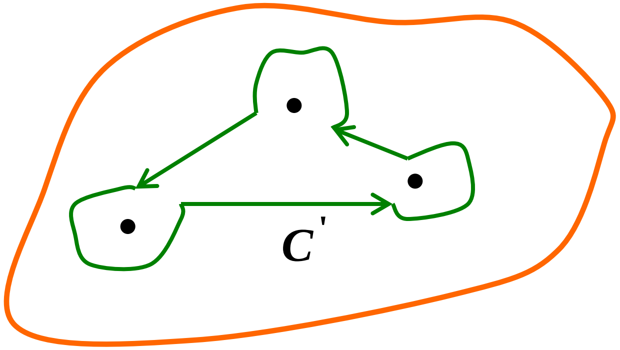
$f(z)$ is continuous within and on a closed contour C

and analytic, except for a finite number of poles within C

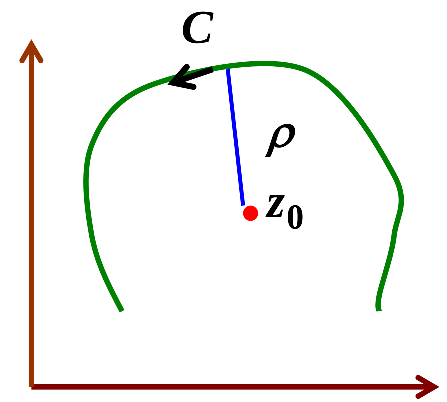
$$\oint_C f(z)dz = 2\pi i \sum_j R_j$$

$\sum_j R_j$ is the sum of the residues of $f(z)$ at its poles within C





The integral I of $f(z)$ along an open contour C



if $f(z)$ has a simple pole at $z = z_0$

$$\Rightarrow f(z) = \phi(z) + a_{-1}(z - z_0)^{-1}$$

$\phi(z)$ is analytic within some neighbour surrounding z_0

$|z - z_0| = \rho$ and $\theta_1 \le \arg(z - z_0) \le \theta_2$

ρ is chosen small enough that no singularity of $f(z)$ except $z = z_0$

$$I = \int_C f(z)dz = \int_C \phi(z)dz + a_{-1}\int_C (z - z_0)^{-1}dz$$

$$\lim_{\rho\to 0}\int_C \phi(z)dz = 0$$

$$I = \lim_{\rho\to 0}\int_C f(z)dz = \lim_{\rho\to 0}\left(a_{-1}\int_{\theta_1}^{\theta_2}\frac{1}{\rho e^{i\theta}} i\rho e^{i\theta}d\theta\right) = ia_{-1}(\theta_2 - \theta_1)$$

for a closed contour $\theta_2 = \theta_1 + 2\pi \Rightarrow I = 2\pi i a_{-1}$

20.16 Integrals of sinusoidal functions

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta \quad \text{set } z = \exp i\theta \text{ in unit circle}$$

$$\Rightarrow \cos\theta = \frac{1}{2}(z + \frac{1}{z}), \quad \sin\theta = \frac{1}{2i}(z - \frac{1}{z}), \; d\theta = -iz^{-1}dz$$

$$\boxed{\text{Ex : Evaluate } I = \int_0^{2\pi} \frac{\cos 2\theta}{a^2 + b^2 - 2ab\cos\theta} d\theta \quad \text{for } b > a > 0}$$

$$\cos n\theta = \frac{1}{2}(z^n + z^{-n}) \Rightarrow \cos 2\theta = \frac{1}{2}(z^2 + z^{-2})$$

$$\frac{\cos 2\theta}{a^2 + b^2 - 2ab\cos\theta} d\theta = \frac{\frac{1}{2}(z^2 + z^{-2})(-iz^{-1})dz}{a^2 + b^2 - 2ab \cdot \frac{1}{2}(z + z^{-1})} = \frac{-\frac{1}{2}(z^4 + 1)idz}{z^2(za^2 + zb^2 - abz^2 - ab)}$$

$$= \frac{i}{2ab} \frac{(z^4 + 1)dz}{z^2(z^2 - z(\frac{a}{b} - + \frac{b}{a}) + 1)} = \frac{i}{2ab} \frac{(z^4 + 1)}{z^2(z - \frac{a}{b})(z - \frac{b}{a})} dz$$

$$I = \frac{i}{2ab}\oint_C \frac{z^4+1}{z^2(z-\frac{a}{b})(z-\frac{b}{a})}dz \quad \text{double poles at } z=0 \text{ and } z=a/b \text{ within the unit circle}$$

$$\text{Residue: } R(z_0) = \lim_{z\to z_0}\{\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]$$

(1) pole at $z=0, m=2$

$$R(0) = \lim_{z\to 0}\{\frac{1}{1!}\frac{d}{dz}[z^2\frac{z^4+1}{z^2(z-a/b)(z-b/a)}]\}$$

$$= \lim_{z\to 0}\{\frac{4z^3}{(z-a/b)(z-b/a)} + \frac{(z^4+1)(-1)[2z-(a/b+b/a)]}{(z-a/b)^2(z-b/a)^2}\} = a/b+b/a$$

(2) pole at $z=a/b, m=1$

$$R(a/b) = \lim_{z\to a/b}[(z-a/b)\frac{z^4+1}{z^2(z-a/b)(z-b/a)}] = \frac{(a/b)^4+1}{(a/b)^2(a/b-b/a)} = \frac{-(a^4+b^4)}{ab(b^2-a^2)}$$

$$I = 2\pi i\times\frac{i}{2ab}[\frac{a^2+b^2}{ab} - \frac{a^4+b^4}{ab(b^2-a^2)}] = \frac{2\pi a^2}{b^2(b^2-a^2)}$$

Some infinite integrals

$$\int_{-\infty}^{\infty} f(x)dx$$

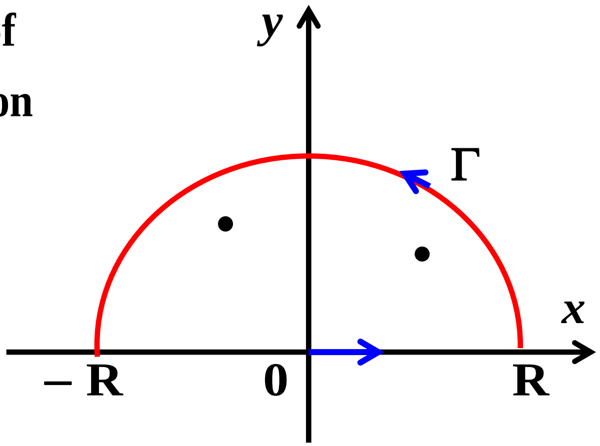
$f(z)$ has the following properties:

(1) $f(z)$ is analytic in the upper half-plane, $\operatorname{Im} z \geq 0$, except for a finite number of poles, none of which is on the real axis.

(2) on a semicircle Γ of radius R, R times the maximum of $|f|$ on Γ tends to zero as $R \to \infty$ (a sufficient condition is that $zf(z) \to 0$ as $|z| \to \infty$).

(3) $\int_{-\infty}^{0} f(x)dx$ and $\int_{0}^{\infty} f(x)dx$ both exist

$$\Rightarrow \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{j} R_j$$

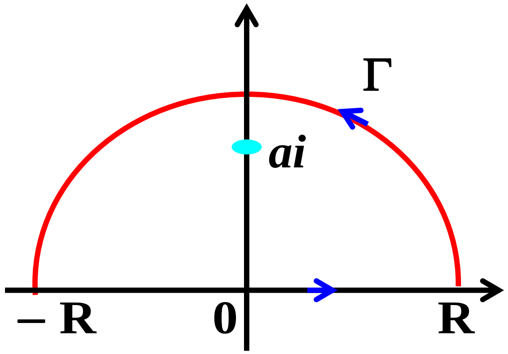


for $\left|\int_{\Gamma} f(z)dz\right| \leq 2\pi R \times$ (maximum of $|f|$ on Γ), the integral along Γ tends to zero as $R \to \infty$.

$$\boxed{\text{Ex : Evaluate } I = \int_0^{\infty} \frac{dx}{(x^2+a^2)^4} \quad a \text{ is real}}$$

$$\oint_C \frac{dz}{(z^2+a^2)^4} = \int_{-R}^{R} \frac{dx}{(x^2+a^2)^4} + \int_\Gamma \frac{dz}{(z^2+a^2)^4} \text{ as } R \to \infty$$

$$\Rightarrow \int_\Gamma \frac{dz}{(z^2+a^2)^4} \to 0 \Rightarrow \oint_C \frac{dz}{(z^2+a^2)^4} = \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^4}$$



$(z^2+a^2)^4 = 0 \Rightarrow$ poles of order 4 at $z = \pm ai$,

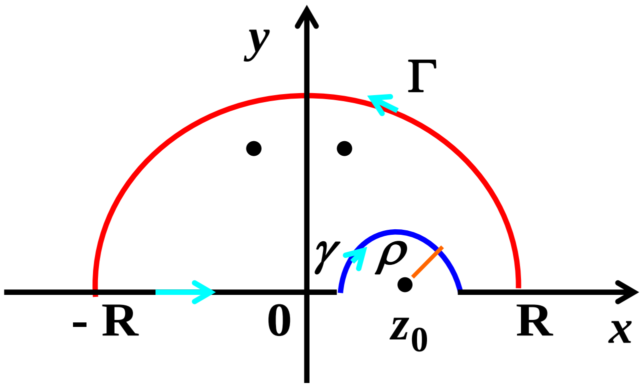
only $z = ai$ at the upper half - plane

$$\text{set } z = ai + \xi, \xi \to 0 \Rightarrow \frac{1}{(z^2+a^2)^4} = \frac{1}{(2ai\xi+\xi^2)^4} = \frac{1}{(2ai\xi)^4}\left(1-\frac{i\xi}{2a}\right)^{-4}$$

$$\text{the coefficient of } \xi^{-1} \text{ is } \frac{1}{(2a)^4}\frac{(-4)(-5)(-6)}{3!}\left(\frac{-i}{2a}\right)^3 = \frac{-5i}{32a^7}$$

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)^4} = 2\pi i\left(\frac{-5i}{32a^7}\right) = \frac{10\pi}{32a^7} \Rightarrow I = \frac{1}{2} \times \frac{10\pi}{32a^7} = \frac{5\pi}{32a^7}$$

For poles on the real axis:



Principal value of the integral, defined as $\rho \to 0$

$$P\int_{-R}^{R} f(x)dx = \int_{-R}^{z_0-\rho} f(x)dx + \int_{z_0+\rho}^{R} f(x)dx$$

for a closed contour C

$$\oint_C f(z)dz = \int_{-R}^{z_0-\rho} f(x)dx + \int_\gamma f(z)dz + \int_{z_0+\rho}^{R} f(x)dx + \int_\Gamma f(z)dz$$

$$= P\int_{-R}^{R} f(x)dx + \int_\gamma f(z)dz + \int_\Gamma f(z)dz$$

(1) for $\int_\gamma f(z)dz$ has a pole at $z = z_0 \Rightarrow \int_\gamma f(z)dz = -\pi i a_1$

(2) for $\int_\Gamma f(z)dz$ set $z = \mathrm{Re}^{i\theta}$ $dz = i\,\mathrm{Re}^{i\theta}\,d\theta$

$$\Rightarrow \int_\Gamma f(z)dz = \int_\Gamma f(\mathrm{Re}^{i\theta})i\,\mathrm{Re}^{i\theta}\,d\theta$$

If $f(z)$ vanishes faster than $1/R^2$ as $R \to \infty$, the integral is zero

Jordan's lemma

(1) $f(z)$ is analytic in the upper half - plane except for a finite number of poles in $\operatorname{Im} z > 0$

(2) the maximum of $|f(z)| \to 0$ as $|z| \to \infty$ in the upper half - plane

(3) $m > 0$, then

$$I_\Gamma = \int_\Gamma e^{imz} f(z)dz \to 0 \text{ as } R \to \infty, \Gamma \text{ is the semicircular contour}$$

for $0 \le \theta \le \pi/2, \quad 1 \ge \sin\theta/\theta \ge \pi/2$

$$|\exp(imz)| = |\exp(-mR\sin\theta)|$$

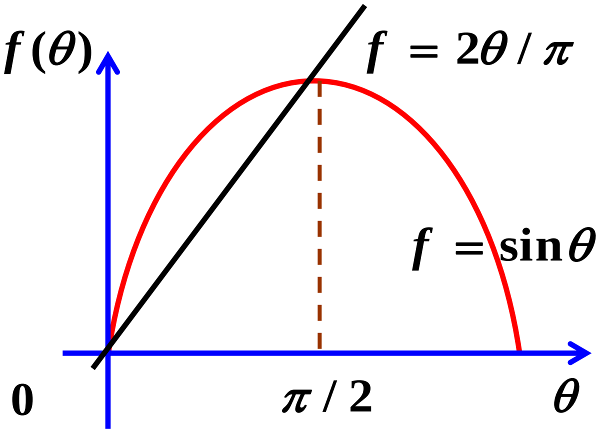
$$I_\Gamma \le \int_\Gamma |e^{imz} f(z)||dz| \le MR\int_0^\pi e^{-mR\sin\theta} d\theta$$

$$= 2MR\int_0^{\pi/2} e^{-mR\sin\theta} d\theta$$

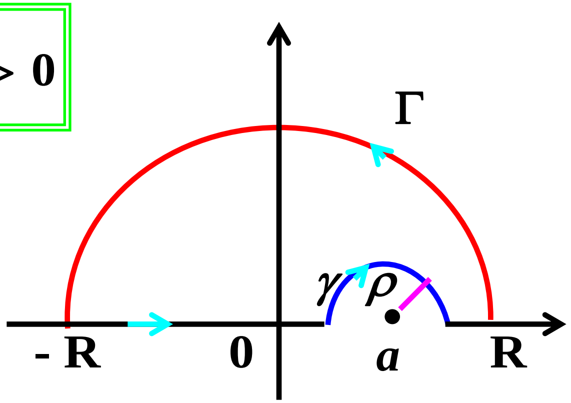
M is the maximum of $|f(z)|$ on $|z| = R$, $R \to \infty$ $M \to 0$

$$I_\Gamma < 2MR\int_0^{\pi/2} e^{-mR(2\theta/\pi)} d\theta = \frac{\pi M}{m}(1 - e^{-mR}) < \frac{\pi M}{m}$$

as $R \to \infty \Rightarrow M \to 0 \Rightarrow I_\Gamma \to 0$



$$\text{Ex : Find the principal value of } \int_{-\infty}^{\infty} \frac{\cos mx}{x-a}dx \quad a \text{ real}, m > 0$$



Consider the integral $I = \oint_C \frac{e^{imz}}{z-a} dz = 0$ no pole in the upper half - plane, and $|(z-a)^{-1}| \to 0$ as $|z| \to \infty$

$$I = \oint_C \frac{e^{imz}}{z-a}dz$$

$$= \int_{-R}^{a-\rho} \frac{e^{imx}}{x-a}dx + \oint_\gamma \frac{e^{imz}}{z-a}dz + \int_{a+\rho}^{R} \frac{e^{imx}}{x-a}dx + \int_\Gamma \frac{e^{imz}}{z-a} dz = 0$$

As $R \to \infty$ and $\rho \to 0 \Rightarrow \int_\Gamma \frac{e^{imz}}{z-a}dz \to 0$

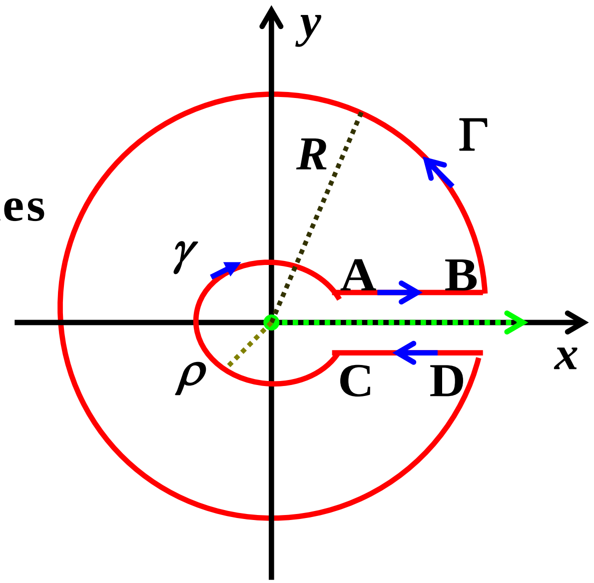
$$\Rightarrow P\int_{-\infty}^{\infty} \frac{e^{imx}}{x-a}dx - i\pi a_{-1} = 0 \text{ and } a_{-1} = e^{ima}$$

$$\Rightarrow P\int_{-\infty}^{\infty} \frac{\cos mx}{x-a}dx = -\pi \sin ma \text{ and } P\int_{-\infty}^{\infty} \frac{\sin mx}{x-a}dx = \pi \cos ma$$

Integral of multivalued functions

Multivalued functions such as $z^{1/2}$, Lnz

Single branch point is at the otigin. We let $R \to \infty$ and $\rho \to 0$. The integrand is multivalued, its values along two lines AB and CD joining $z = \rho$ to $z = R$ are not equal and opposite.



$$\boxed{\text{Ex} : I = \int_0^\infty \frac{dx}{(x+a)^3 x^{1/2}} \text{ for } a > 0}$$

(1) the integrand $f(z) = (z+a)^{-3} z^{-1/2}$, $|zf(z)| \to 0$ as $\rho \to 0$ and $R \to \infty$

the two circles make no contribution to the contour integral

(2) pole at $z = -a$, and $(-a)^{1/2} = a^{1/2} e^{i\pi/2} = ia^{1/2}$

$$R(-a) = \lim_{z \to -a} \frac{1}{(3-1)!} \frac{d^{3-1}}{dz^{3-1}} [(z+a)^3 \frac{1}{(z+a)^3 z^{1/2}}]$$

$$= \lim_{z \to -a} \frac{1}{2!} \frac{d^2}{dz^2} z^{-1/2} = \frac{-3i}{8a^{5/2}}$$

$$\int_{AB} dz + \int_{\Gamma} dz + \int_{DC} dz + \int_{\gamma} dz = 2\pi i\left(\frac{-3i}{8a^{5/2}}\right)$$

and $\int_{\Gamma} dz = 0$ and $\int_{\gamma} dz = 0$

along line AB $\Rightarrow z = xe^{i0}$, along line CD $\Rightarrow z = xe^{i2\pi}$

$$\int_{0,A\to B}^{\infty} \frac{dx}{(x+a)^3 x^{1/2}} + \int_{\infty,C\to D}^{0} \frac{dx}{(xe^{i2\pi}+a)^3 x^{1/2} e^{(1/2\times 2\pi i)}} = \frac{3\pi}{4a^{5/2}}$$

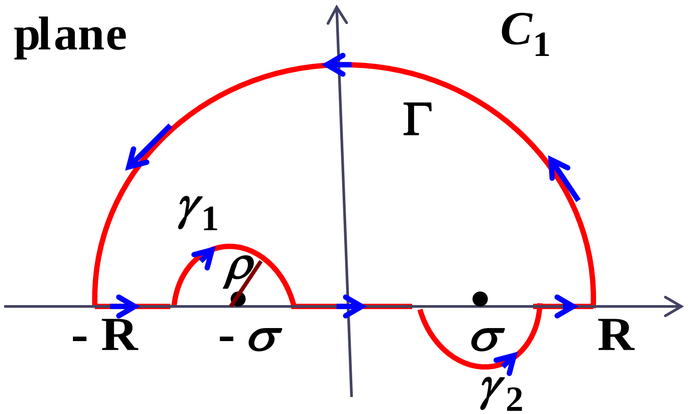
$$\Rightarrow \left(1 - \frac{1}{e^{i\pi}}\right)\int_0^{\infty} \frac{dx}{(x+a)^3 x^{1/2}} = \frac{3\pi}{4a^{5/2}}$$

$$\Rightarrow \int_0^{\infty} \frac{dx}{(x+a)^3 x^{1/2}} = \frac{3\pi}{8a^{5/2}}$$

$$\boxed{\text{Ex : Evaluate } I(\sigma) = \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 - \sigma^2} \mathrm{dx}}$$

$$\oint_C \frac{z \sin z}{z^2 - \sigma^2} dz = \frac{1}{2i} \oint_{C_1} \frac{ze^{iz}}{z^2 - \sigma^2} dz - \frac{1}{2i} \oint_{C_2} \frac{ze^{-iz}}{z^2 - \sigma^2} dz = I_1 + I_2$$

(1) for I_1, the contour is choosed on the upper half - plane due to the term e^{iz}, and only one pole at $z = \sigma$.



$$I_1 = \frac{1}{2i} \oint_{C_1} \frac{ze^{iz}}{z^2 - \sigma^2} dz = \frac{1}{2i} \int_{-R}^{-\sigma-\rho} \frac{xe^{ix}}{x^2 - \sigma^2} dx$$

$$+ \frac{1}{2i} \int_{-\sigma+\rho}^{\sigma-\rho} \frac{xe^{ix}}{x^2 - \sigma^2} dx + \frac{1}{2i} \int_{\sigma+\rho}^{\infty} \frac{xe^{ix}}{x^2 - \sigma^2} dx$$

$$+ \frac{1}{2i} \int_{\gamma_1} \frac{ze^{iz}}{z^2 - \sigma^2} dz + \frac{1}{2i} \int_{\gamma_2} \frac{ze^{iz}}{z^2 - \sigma^2} dz + \frac{1}{2i} \int_{\Gamma} \frac{ze^{iz}}{z^2 - \sigma^2} dz$$

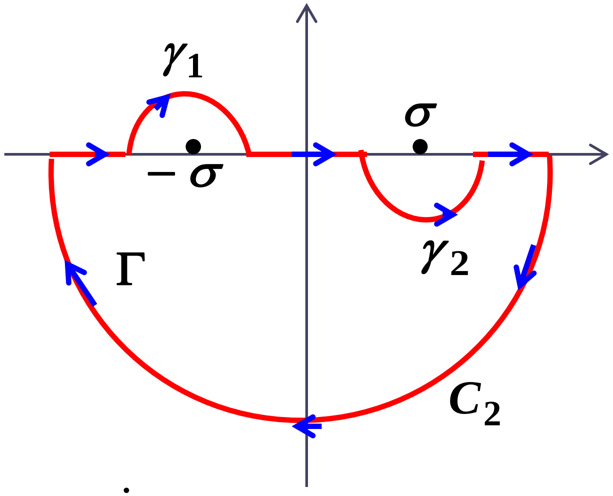
$$= \frac{1}{2i} 2\pi i \times \mathrm{Res}(z = \sigma) = \pi \frac{\sigma e^{i\sigma}}{2\sigma} = \frac{\pi}{2} e^{i\sigma}$$

As $\rho \to 0$ and $R \to \infty \Rightarrow \int_{\Gamma} dz \to 0$

$$\frac{1}{2i}\int_{\gamma_1} \frac{ze^{iz}}{(z+\sigma)(z-\sigma)}dz = \frac{1}{2i} \times (-\pi i)\mathrm{Res}(z=-\sigma) = \frac{-\pi}{4}e^{-i\sigma}$$

$$\frac{1}{2i}\int_{\gamma_2} \frac{ze^{iz}}{(z+\sigma)(z-\sigma)}dz = \frac{1}{2i} \times \pi i\mathrm{Res}(z=\sigma) = \frac{\pi}{4}e^{i\sigma}$$

$$I_1 = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2-\sigma^2}dx + \frac{\pi}{4}(e^{i\sigma} - e^{-i\sigma}) = \frac{\pi}{2}e^{i\sigma}$$



(2) for I_2, we choose the lower half - plane by the

term e^{-iz}, only one pole at $z=-\sigma$

$$I_2 = \frac{-1}{2i}\oint_{C_2} \frac{ze^{-iz}}{z^2-\sigma^2}dz = \frac{-1}{2i}\int_{-R}^{-\sigma-\rho} \frac{xe^{-ix}}{x^2-\sigma^2}dx$$

$$-\frac{1}{2i}\int_{-\sigma+\rho}^{\sigma-\rho} \frac{xe^{-ix}}{x^2-\sigma^2}dx - \frac{1}{2i}\int_{\sigma+\rho}^{\infty} \frac{xe^{-ix}}{x^2-\sigma^2}dx - \frac{1}{2i}\int_{\gamma_1} \frac{ze^{-iz}}{z^2-\sigma^2}dz$$

$$-\frac{1}{2i}\int_{\gamma_2} \frac{ze^{-iz}}{z^2-\sigma^2}dz - \frac{1}{2i}\int_{\Gamma} \frac{ze^{-iz}}{z^2-\sigma^2}dz = \left(\frac{-1}{2i}\right) \times (-2\pi i)\frac{(-\sigma)e^{i\sigma}}{-2\sigma} = \frac{\pi}{2}e^{i\sigma}$$

$$\text{As } \rho \to 0, R \to \infty \Rightarrow \int_\Gamma dz \to 0$$

$$\frac{-1}{2i}\int_{\gamma_1} \frac{ze^{-iz}}{(z+\sigma)(z-\sigma)}dz = (\frac{-1}{2i})(-\pi i)\frac{(-\sigma)e^{i\sigma}}{-2\sigma} = \frac{\pi}{4}e^{i\sigma}$$

$$\frac{-1}{2i}\int_{\gamma_2} \frac{ze^{-iz}}{(z+\sigma)(z-\sigma)}dz = (\frac{-1}{2i})(\pi i)\frac{\sigma e^{-i\sigma}}{2\sigma} = \frac{-\pi}{4}e^{-i\sigma}$$

$$I_2 = \frac{-1}{2i}\int_{-\infty}^{\infty} \frac{xe^{-ix}}{x^2-\sigma^2}dx + \frac{\pi}{4}(e^{i\sigma} - e^{-i\sigma}) = \frac{\pi}{2}e^{i\sigma}$$

$$\Rightarrow \frac{-1}{2i}\int_{-\infty}^{\infty} \frac{xe^{-ix}}{x^2-\sigma^2}dx = \frac{\pi}{2}e^{i\sigma} - \frac{1}{4}(e^{i\sigma} - e^{-i\sigma})$$

$$I(\sigma) = \int_{-\infty}^{\infty} \frac{x\sin x}{x^2-\sigma^2}dx = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2-\sigma^2}dx - \frac{1}{2i}\int_{-\infty}^{\infty} \frac{xe^{-ix}}{x^2-\sigma^2}dx$$

$$= \frac{\pi}{2}e^{i\sigma} - \frac{\pi}{4}(e^{i\sigma} - e^{-i\sigma}) + \frac{\pi}{2}e^{i\sigma} - \frac{\pi}{4}(e^{i\sigma} - e^{-i\sigma})$$

$$= \pi e^{i\sigma} - \frac{\pi}{2}e^{i\sigma} + \frac{\pi}{2}e^{-i\sigma} = \frac{\pi}{2}(e^{i\sigma} + e^{-i\sigma}) = \pi\cos\sigma$$