MECHANICS OF SOLIDS
(AME004)

B.TECH - SEMESTER-III

PREPARED

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SIMPLE STRESSES AND STRAINS

Elasticity and plasticity – Types of stresses & strains–Hooke’s law– stress – strain diagram for mild steel – Working stress – Factor of safety – Lateral strain, Poisson’s ratio & volumetric strain – Elastic moduli & the relationship between them – Bars of varying section – composite bars – Temperature stresses. Strain energy – Resilience – Gradual, sudden, impact and shock loadings.

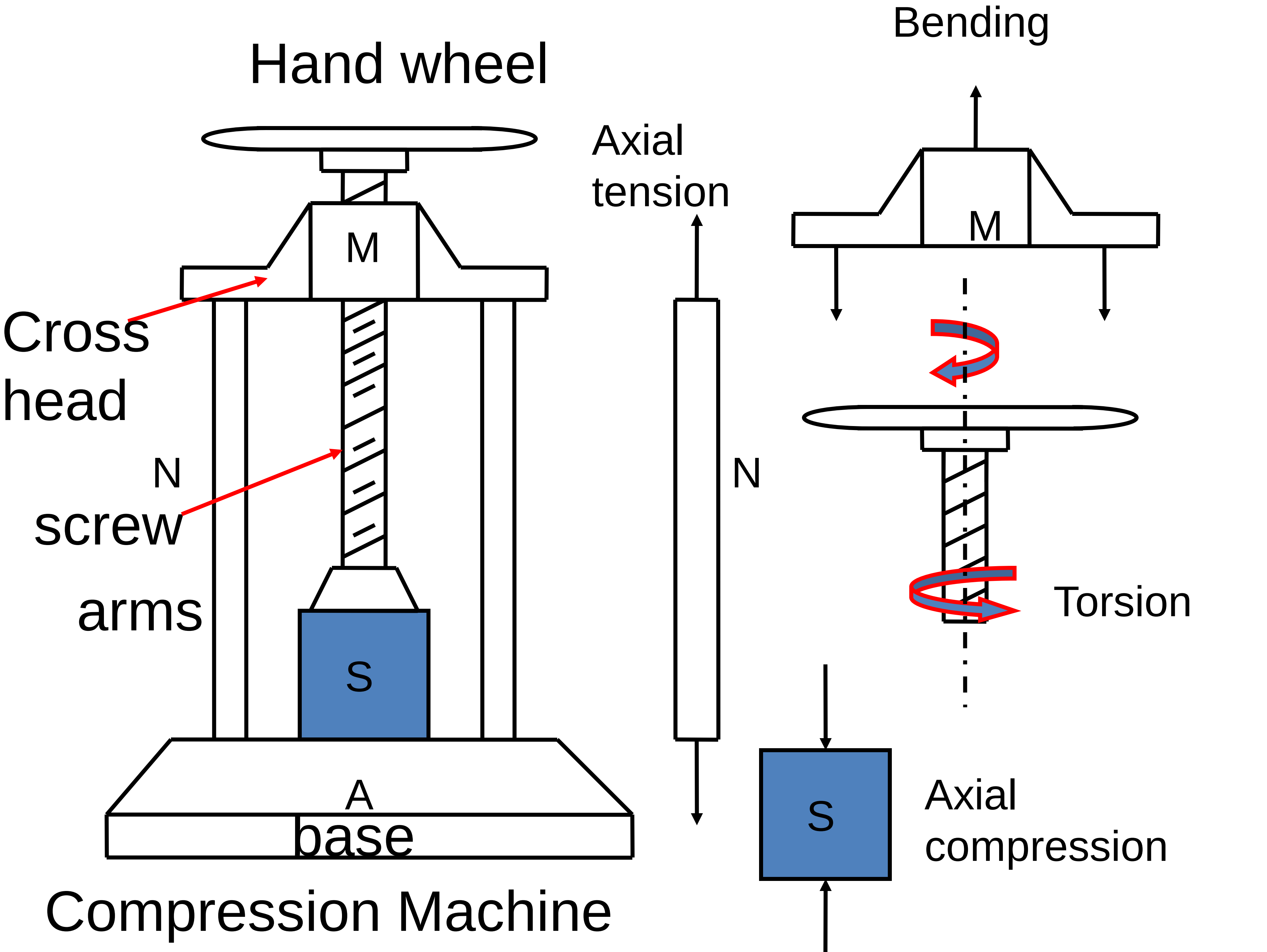
Mechanical Properties of Materials

Ductility, Brittleness, Toughness, Malleability, Behaviour of Ferrous & Non-Ferrous metals in Tension & Compression, Shear & Bending tests, Standard Test Pieces, Influence of Various Parameters on Test Results, True & Nominal Stress, Modes of Failure, Characteristic Stress-Strain Curves, Izod, Charpy & Tension Impact Tests,

Fatigue, Creep, Corelation between Different Mechanical Properties, Effect of Temperature, Testing Machines & Special Features, Different Types of Extensometers & Compressemeters, Measurement of Strain by Electrical Resistance Strain Gauges.

AIM OF MECHANICS OF SOLIDS:

Predicting how geometric and physical properties of structure will influence its behaviour under service conditions.

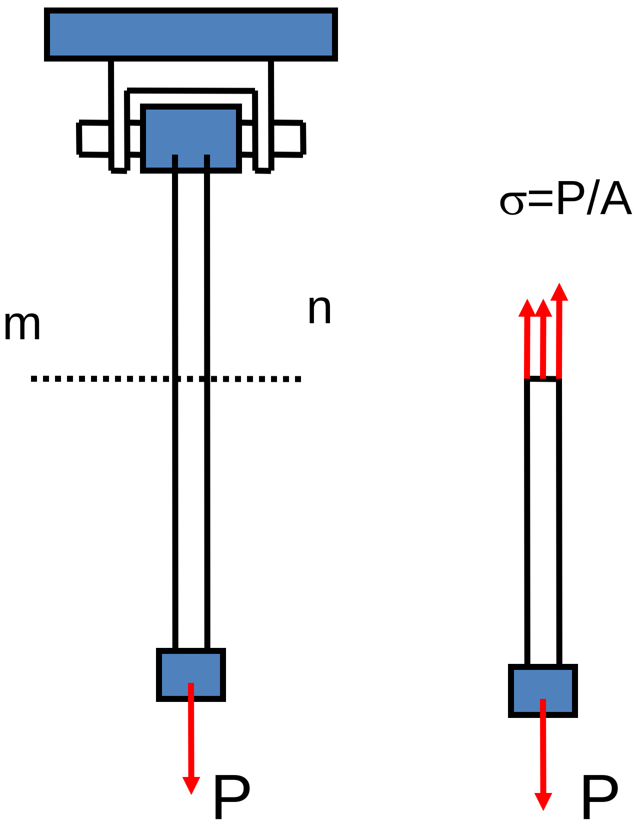

Hand wheel
Axial
tension
Bending
M
M
Cross
head
N
N
screw
arms
S
Torsion
A
base
S
Axial
compression
Compression Machine

- Stresses can occur isolated or in combination.
- Is structure strong enough to withstand loads applied to it ?
- Is it stiff enough to avoid excessive deformations and deflections?
- Engineering Mechanics---->   Statics----->

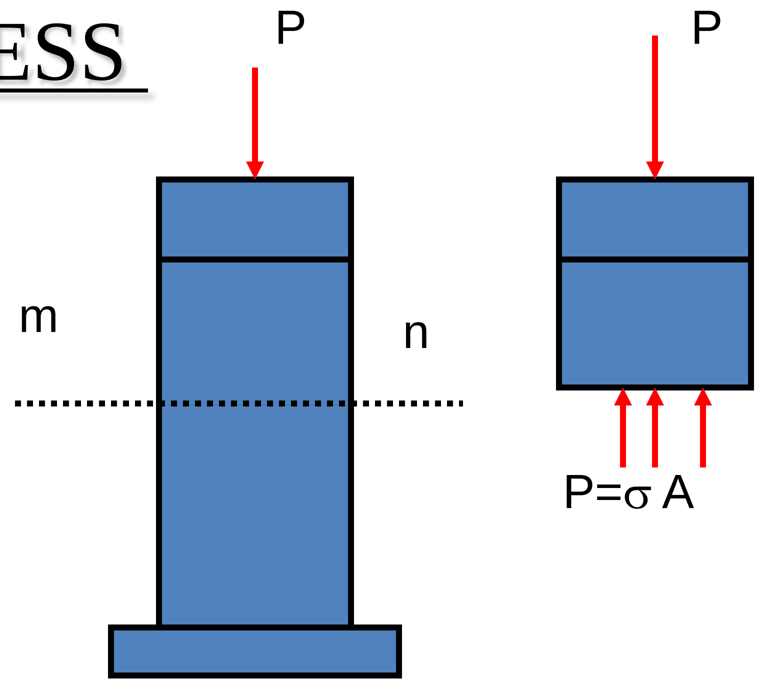
  deals with rigid bodies
- All materials are deformable and mechanics of solids takes this into account.

- Strength and stiffness of structures is function of size and shape, certain physical properties of material.
- Properties of Material:-
- Elasticity
- Plasticity
- Ductility
- Malleability
- Brittleness
- Toughness
- Hardness

INTERNAL FORCE:- STRESS



- Axial tension
- Stretches the bars & tends to pull it apart
- Rupture



- Axial Compression
- Shortens the bar
- Crushing
- Buckling

- Resistance offered by the material per unit cross- sectional area is called STRESS.

σ = P/A

Unit of Stress:

Pascal = 1 N/m^2

kN/m^2 , MN/m^2 , GN/m^2

1 MPa = 1 N/mm^2

Permissible stress or allowable stress or working stress = yield stress or ultimate stress /factor of safety.

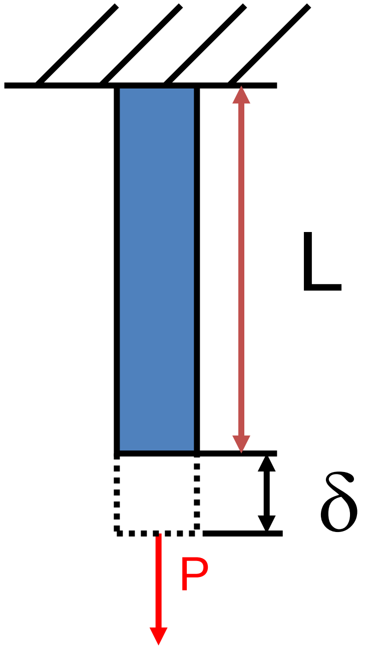
• Strain

•It is defined as deformation per unit length

• it is the ratio of change in length to original length

•Tensile strain (+ Ve) (ε) $= \dfrac{\text{increase in length}}{\text{Original length}} = \dfrac{\delta}{L}$

Compressive strain (- Ve) (ε) $= \dfrac{\text{decrease in length}}{\text{Original length}} = \dfrac{\delta}{L}$



•Strain is dimensionless quantity.

Example : 1

A short hollow, cast iron cylinder with wall thickness of 10 mm is to carry compressive load of 100 kN. Compute the required outside diameter `D' , if the working stress in compression is 80 N/mm^2. (D = 49.8 mm).

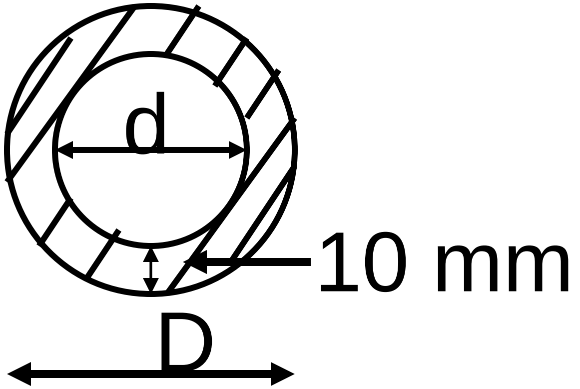
<u>Solution:</u> $\sigma = 80 N/mm^2$;

P= 100 kN = $100*10^3$ N

A =$(\pi/4) * \{D^2 - (D-20)^2\}$

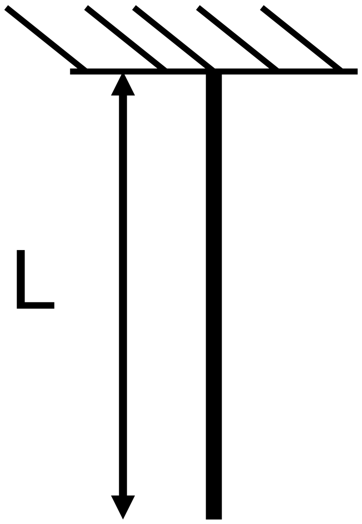
as $\sigma = P/A$

substituting in above eq. and solving. D = 49.8 mm



Example: 2

A Steel wire hangs vertically under its weight. What is the greatest length it can have if the allowable tensile stress σ_t =200 MPa? Density of steel γ=80 kN/m^3.(ans:-2500 m)



Solution:

σ_t =200 MPa= 200*10^3 kN/m^2 ;

γ=80 kN/m^3.

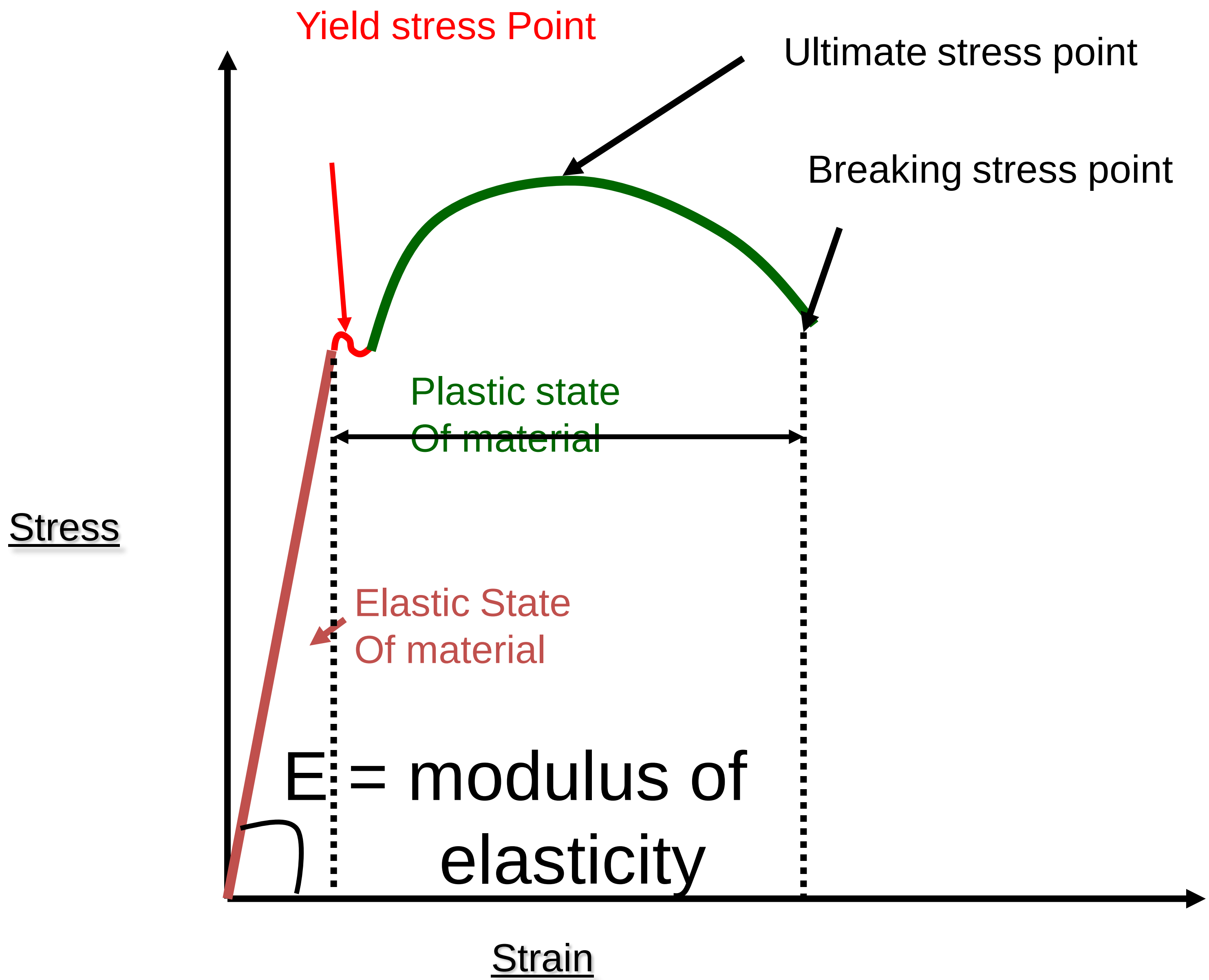
Wt. of wire $P=(\pi/4)*D^2*L*\gamma$

c/s area of wire $A=(\pi/4)*D^2$

$\sigma_t = P/A$

solving above eq. L =2500m

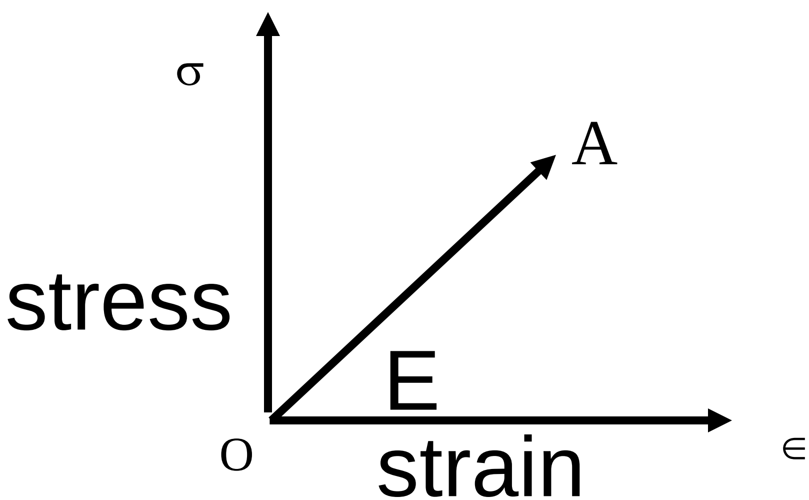
Stress- Strain Curve for Mild Steel (Ductile Material)



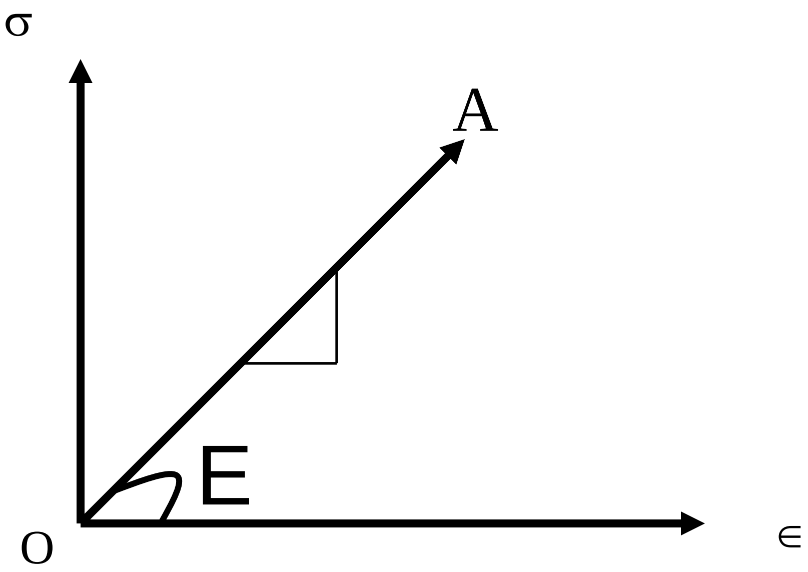
Modulus of Elasticity:

$$\sigma = E \in$$

- Stress required to produce a strain of unity.
- i.e. the stress under which the bar would be stretched to twice its original length . If the material remains elastic throughout , such excessive strain.
- Represents slope of stress-strain line OA.



Value of E is same in Tension & Compression.



- Hooke's Law:-

Up to elastic limit, Stress is proportional to strain

$\sigma \propto \in$

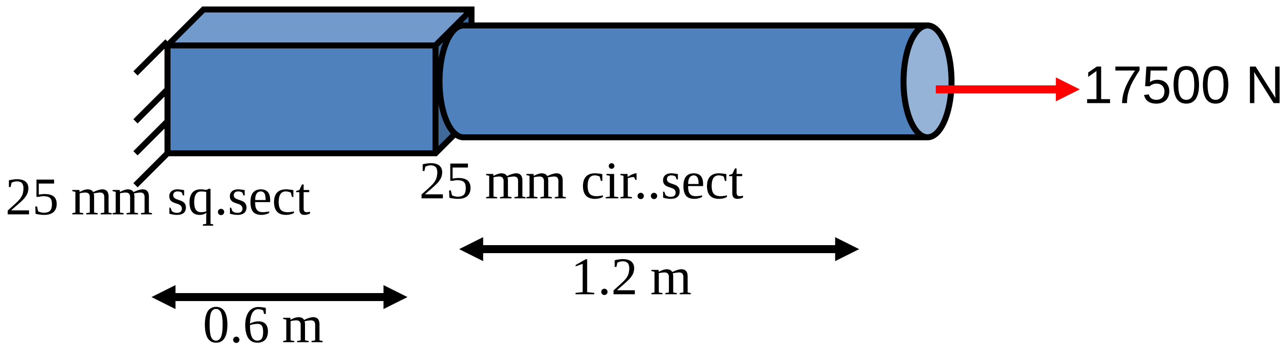
$\sigma = E \in$; where E=Young's modulus

$\sigma = P/A$ and $\in = \delta / L$

$P/A = E (\delta / L)$

$\delta = PL / AE$

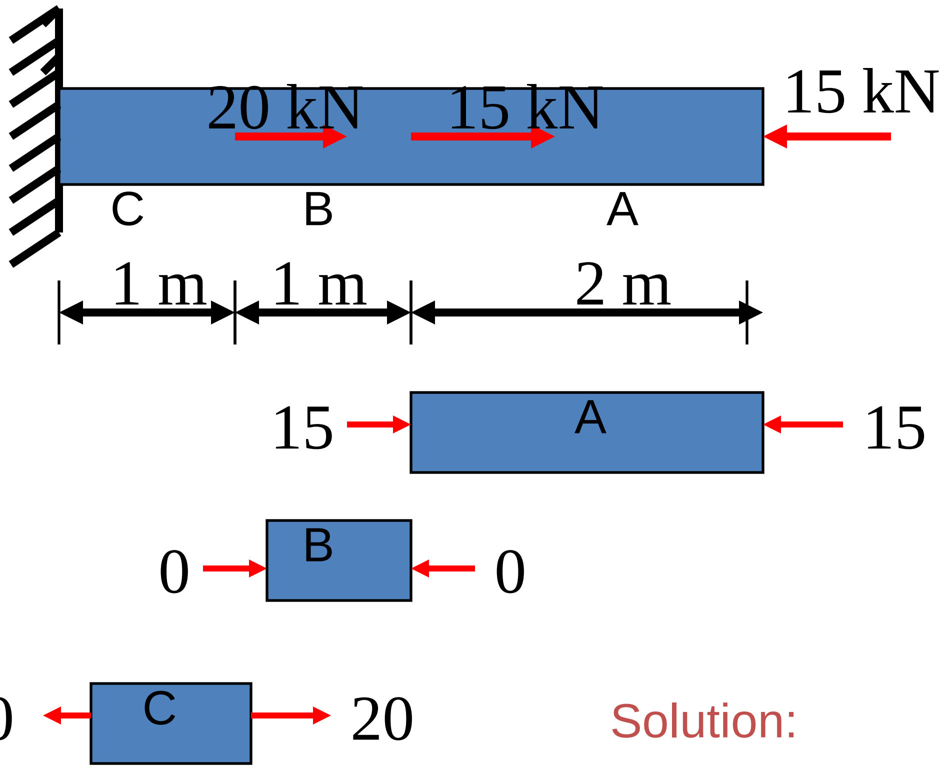
Example:4 An aluminium bar 1.8 meters long has a 25 mm square c/s over 0.6 meters of its length and 25 mm circular c/s over other 1.2 meters . How much will the bar elongate under a tensile load P=17500 N, if E = 75000 Mpa.



Solution :- $\delta = \sum PL/AE$

=17500*600 / (252*75000) + 17500*1200/(0.785*252*75000) =0.794 mm

Example: 5 A prismatic steel bar having cross sectional area of A=300 mm^2 is subjected to axial load as shown in figure . Find the net increase δ in the length of the bar. Assume E = 2 x 10^5 MPa.(Ans δ = -0.17mm)

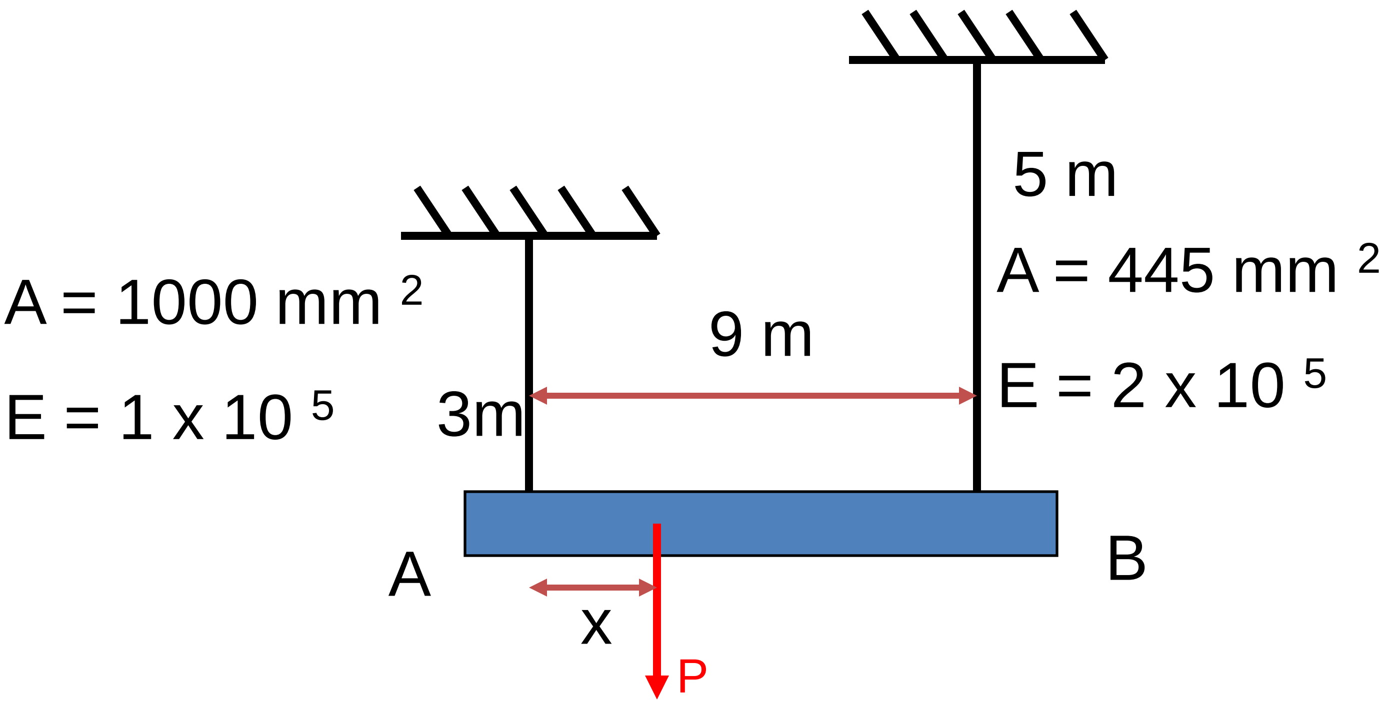


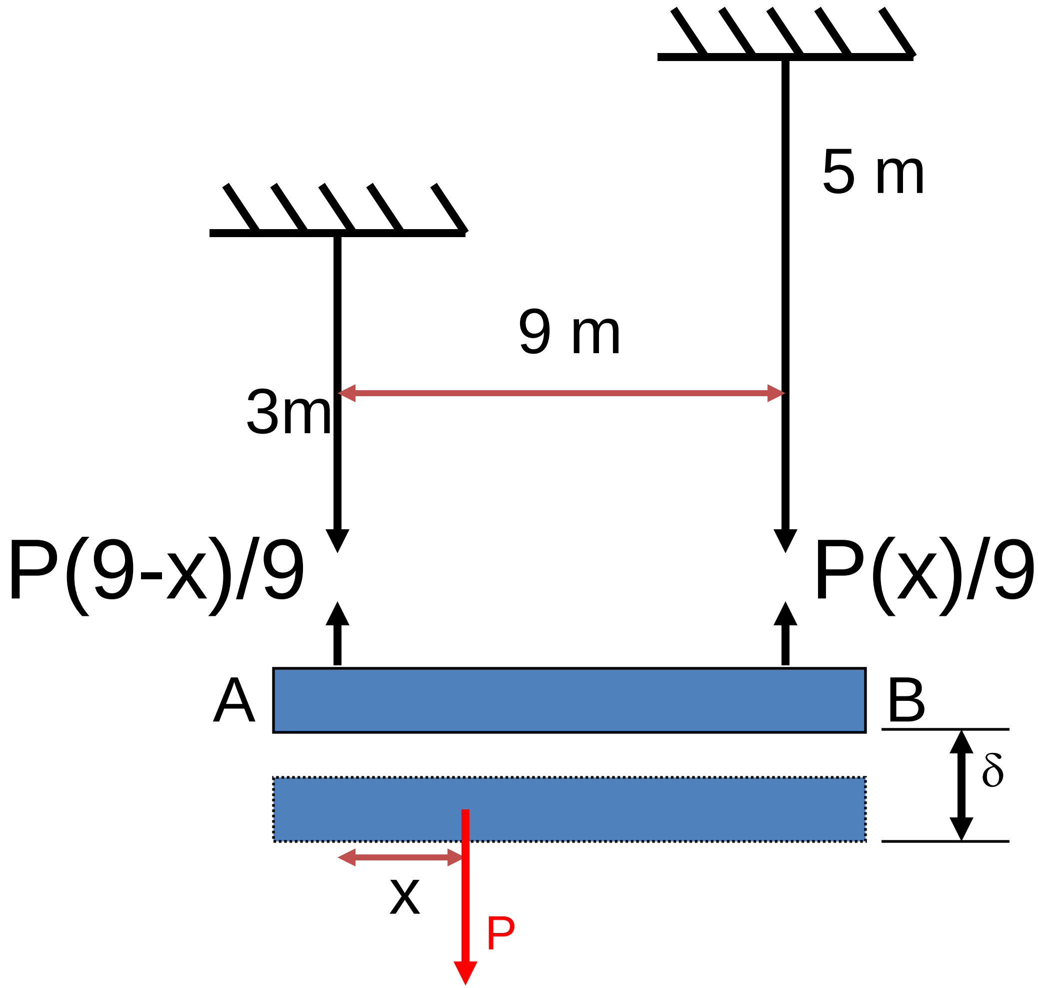
Solution:

$\delta = 20000*1000/(300*2x10^5)-15000*2000/(300*2 x10^5)$

$= 0.33 - 0.5 = -0.17$ mm (i.e.contraction)

Example: 6 A rigid bar AB, 9 m long, is supported by two vertical rods at its end and in a horizontal position under a load P as shown in figure. Find the position of the load P so that the bar AB remains horizontal.





5 m
9 m
3m
P(9-x)/9
P(x)/9
A
B
δ
x
P

For the bar to be in horizontal position, Displacements at A & B should be same,

$$\delta_A = \delta_B$$

$$(PL/AE)_A = (PL/AE)_B$$

$$\frac{\{P(9-x)/9\}*3}{(0.001*1*10^5)} = \frac{\{P(x)/9\}*5}{0.000445*2*10^5}$$

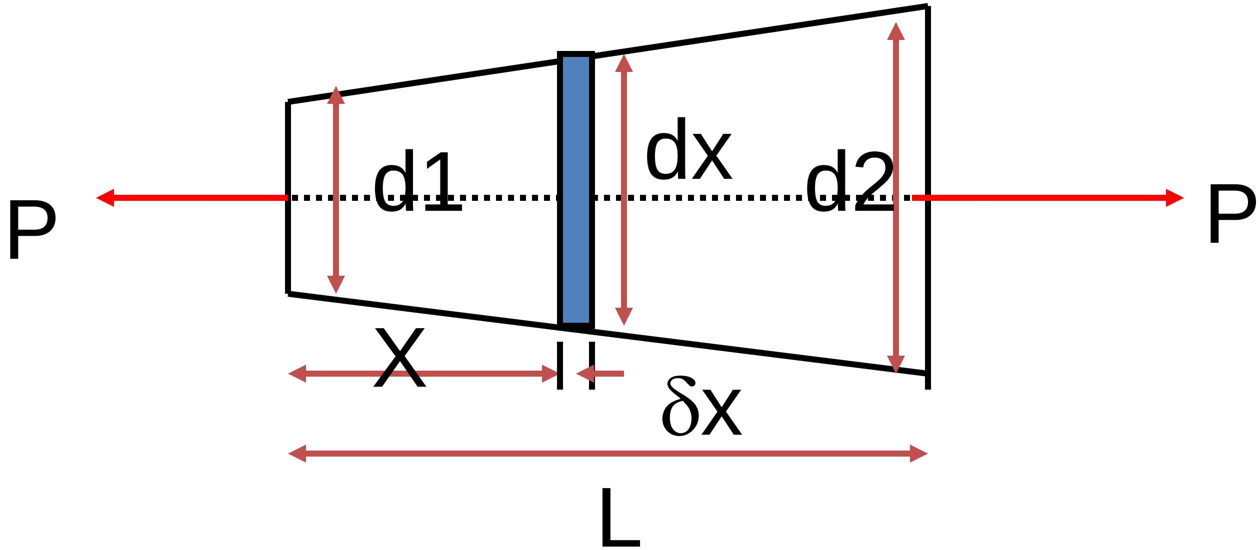
$$(9 - x)*3 = x*5*1.1236$$

$$27-3x=5.618\,x$$

$$8.618\,x=27$$

$$x = 3.13\text{ m}$$

Extension of Bar of Tapering cross Section from diameter d1 to d2:-



Bar of Tapering Section:

dx = d1 + [(d2 - d1) / L] * X

$\delta\Delta = P\delta x / E[\pi /4\{d1 + [(d2 - d1) / L] * X\}^2]$

$$\Delta = \int_0^L 4\,P\,dx / [E\,\pi\{d1+kx\}^2]$$

$$= -\,[4P/\,\pi\,E]\ x\ \ 1/k\ \ [\,\{1\,/(d1+kx)\}]_0^L\ dx$$

$$=-\,[4PL/\,\pi\,E(d2-d1)]\ \{1/(d1+d2\,-d1)\ -\ 1/d1\}$$

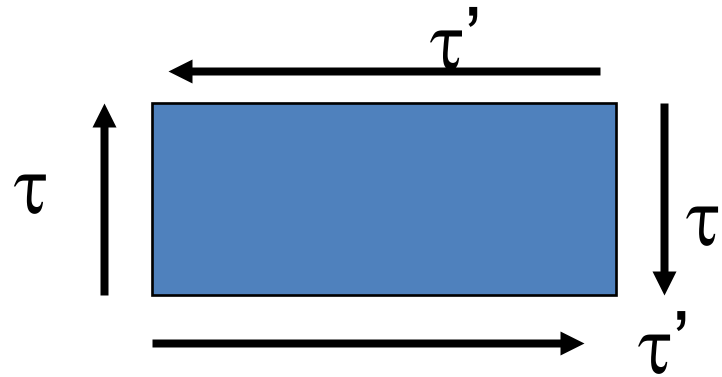
$$\Delta = 4PL/(\pi\,E\ d1\ d2)$$

Check :-

When d = d1=d2

$$\Delta = PL/\,[(\pi\,/4)^*\ d^2 E\,] = PL\,/AE \quad \text{(refer -24)}$$

COMPLEMENTRY STRESSES:“A stress in a given direction cannot exist without a balancing shear stress of equal intensity in a direction at right angles to it.”



Moment of given couple=Force *Lever arm

= (τ.AB)*AD

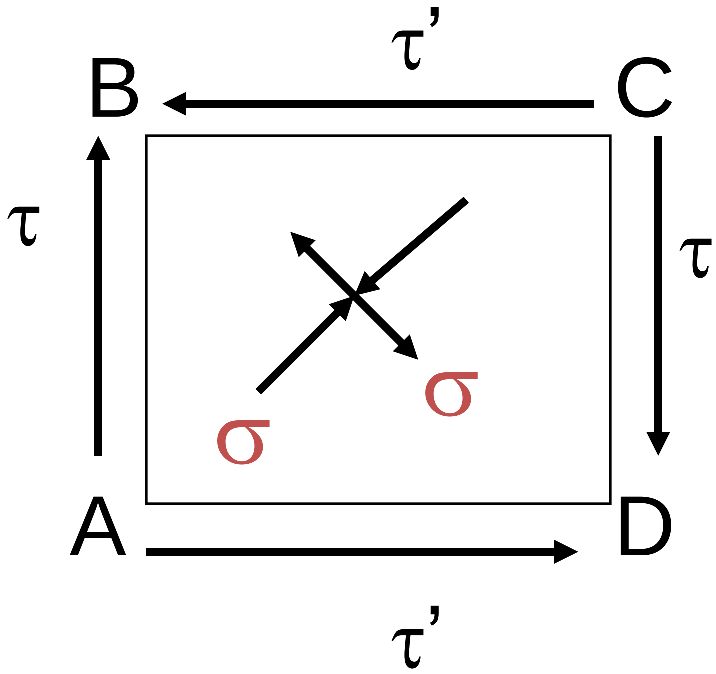
Moment of balancing couple= (τ’.AD)*AB

so (τ.AB)*AD=(τ’.AD)*AB => τ= τ’

Where τ=shear stress & τ’=Complementary shear stress

State of simple shear:

Here no other stress is acting - only simple shear.



Let side of square = b

length of diagonal AC =√2 .b

consider unit thickness perpendicular to block.

Equilibrium of piece ABC

the resolved sum of τ perpendicular to the diagonal = $2*(\tau*b*1)\cos 45^0 = \sqrt{2}\ \tau.b$

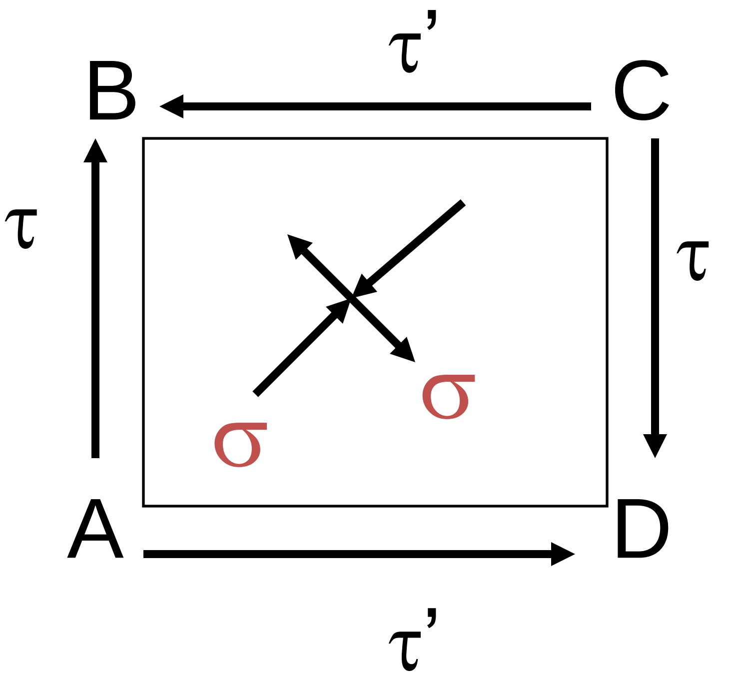
if σ is the tensile stress so produced on the diagonal

$\sigma(AC*1)=\sqrt{2}\ \tau.b$

$\sigma(\sqrt{2}\ .b)=\sqrt{2}\ \tau.b$

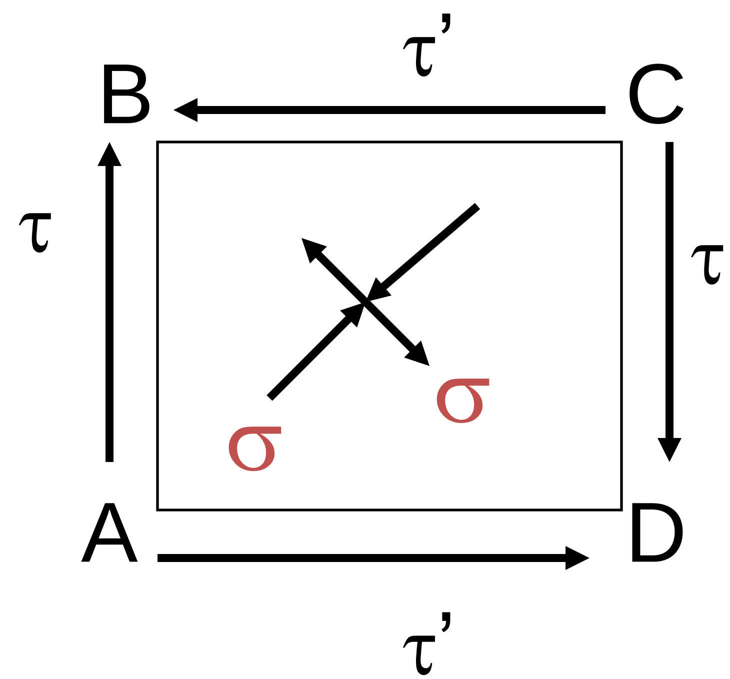
so

$\sigma= \tau$

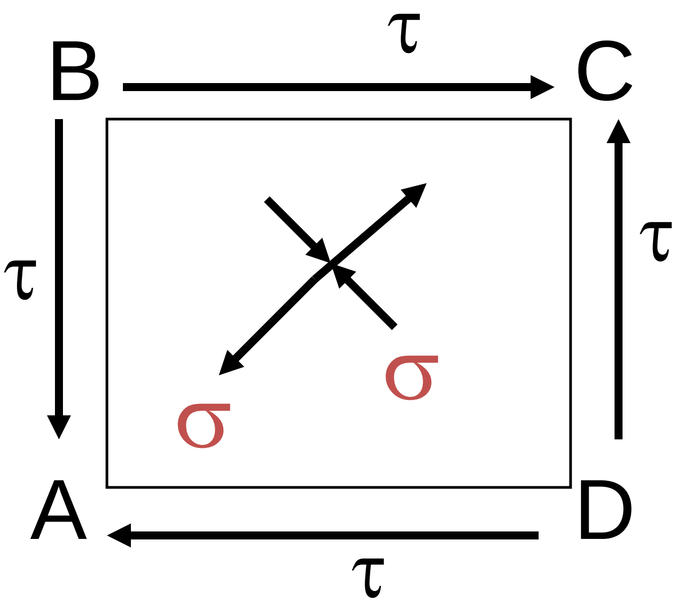


Similarly the intensity of compressive stress on plane BD is numerically equal to τ.

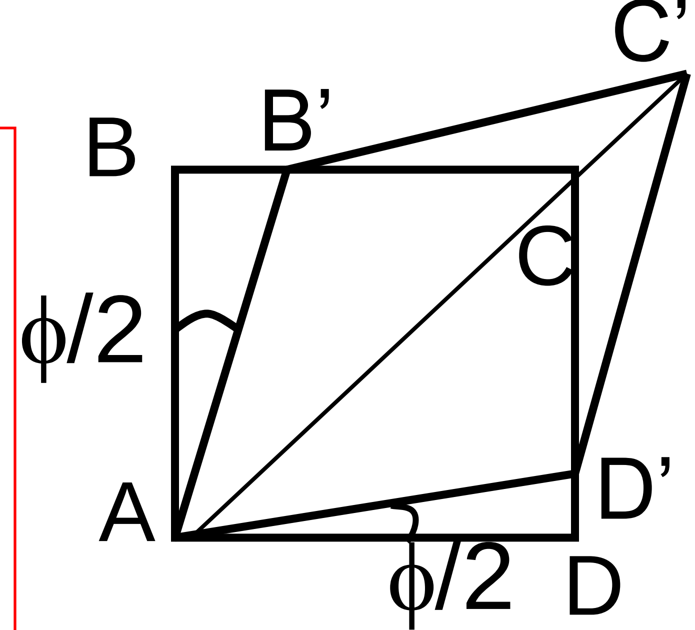
"Hence a state of simple shear produces pure tensile and compressive stresses across planes inclined at 45^0 to those of pure shear, and intensities of these direct stresses are each equal to pure shear stress."



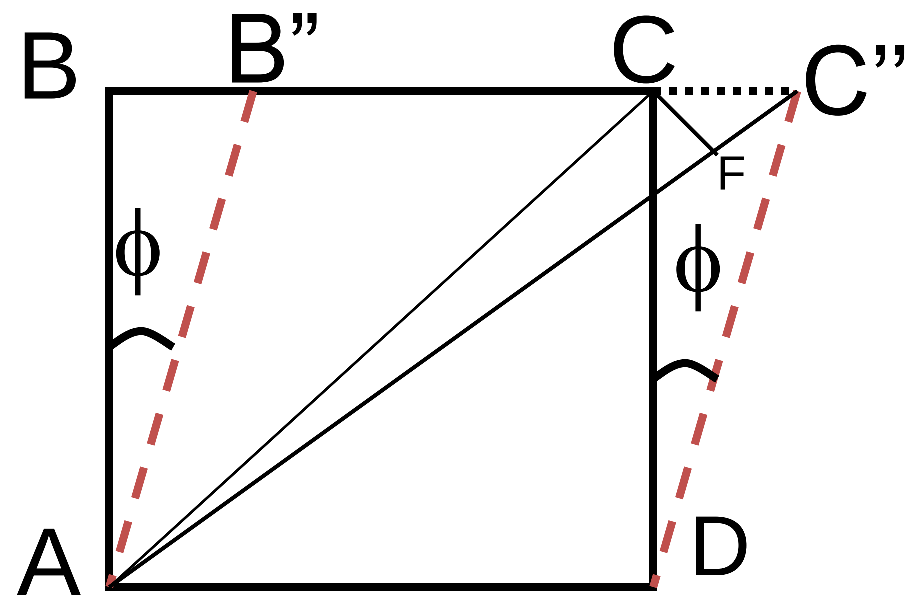
SHEAR STRAIN:



Total change in corner angles +/- ϕ



State of simple Shear on Block



Distortion with side AD fixed

Since

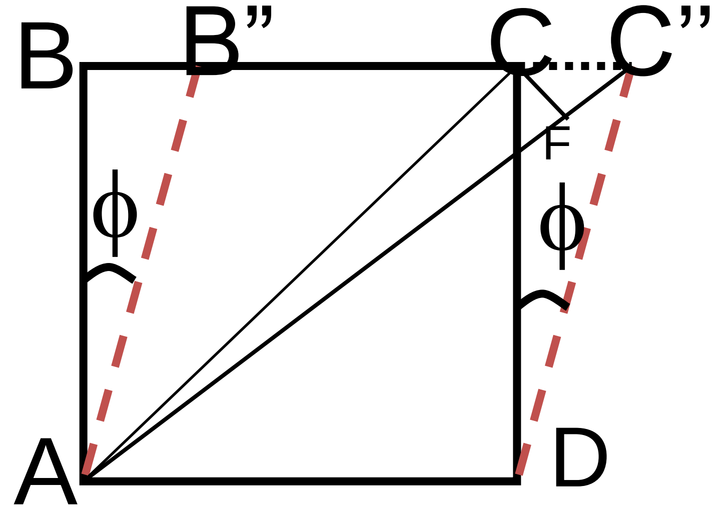
ϕ is extremely small,

we can assume

BB” = arc with A as centre ,

AB as radius.

So, ϕ=BB”/AB=CC”/CD



Elongation of diagonal AC can be nearly taken as FC”.

Linear strain of diagonal = FC”/AC

= CC”cos 45/CDsec45

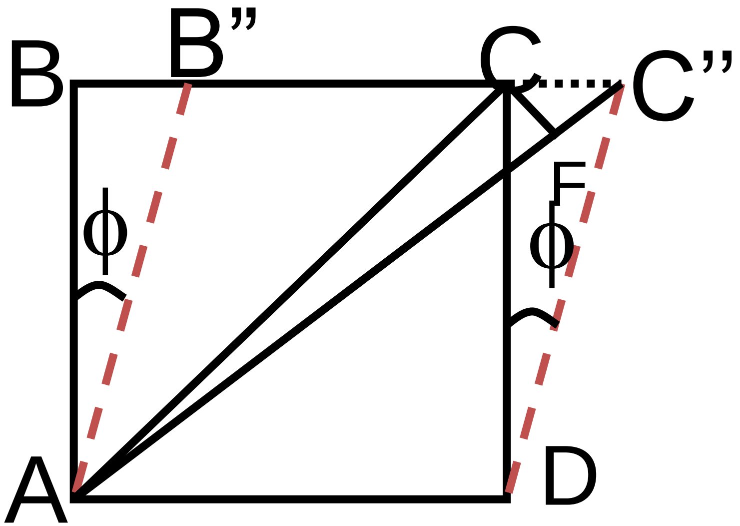
$\varepsilon = CC''/2CD = (1/2)\,\phi$

but $\phi=\tau/N$ (we know $N=\tau/\phi$)

so

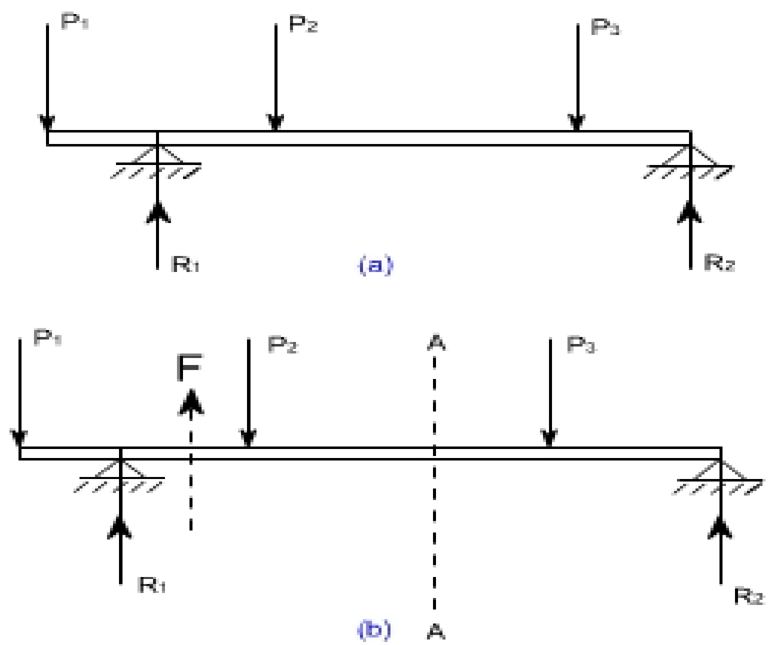
$$\varepsilon = \tau/2N \quad \text{------(8)}$$

Linear strain ‘ε’is half the shear strain ‘ϕ’.

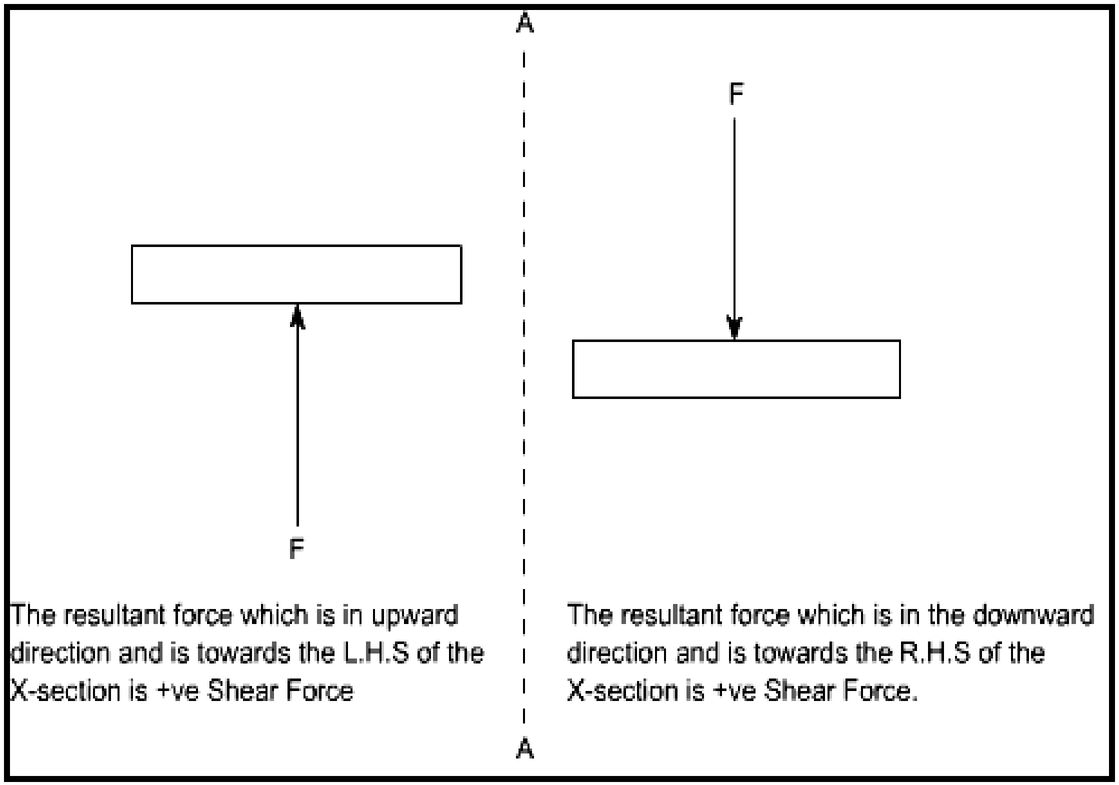


Concept of Shear Force and Bending moment in beams:

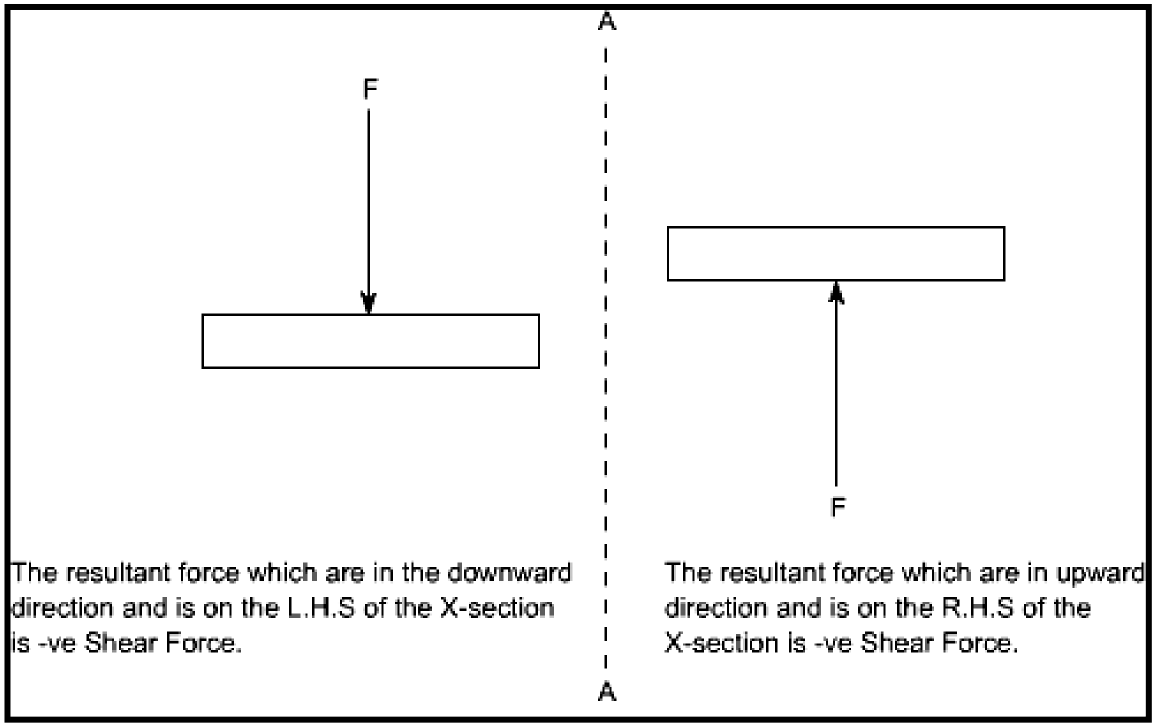
When the beam is loaded in some arbitrarily manner, the internal forces and moments are developed and the terms shear force and bending moments come into pictures which are helpful to analyze the beams further. Let us define these terms



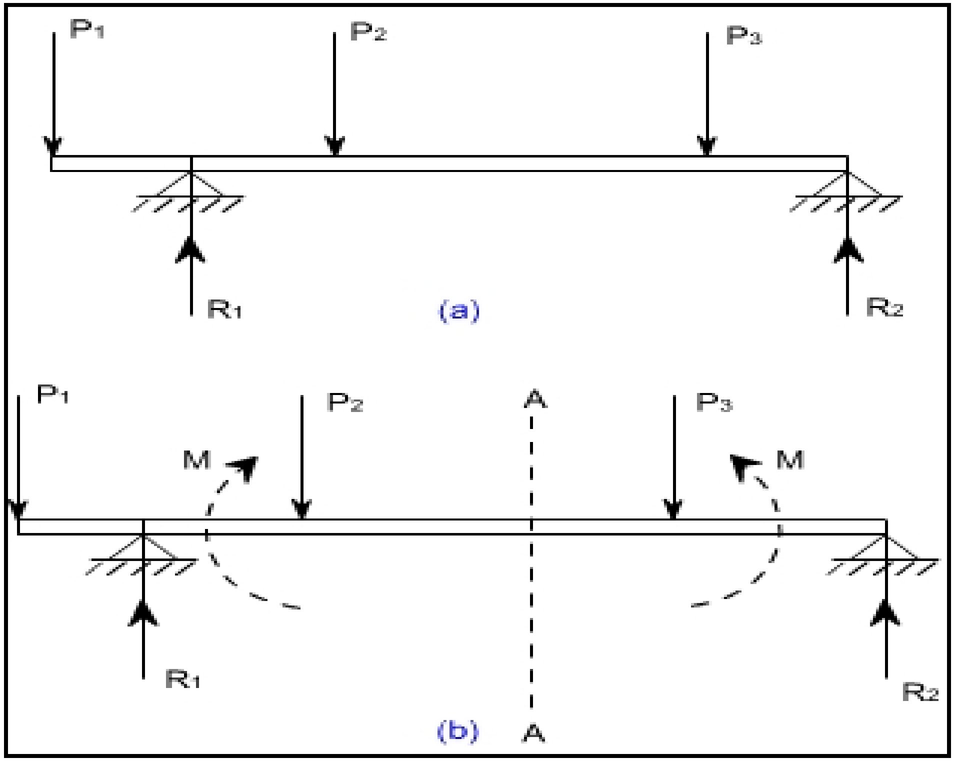
Sign Convention for Shear Force:



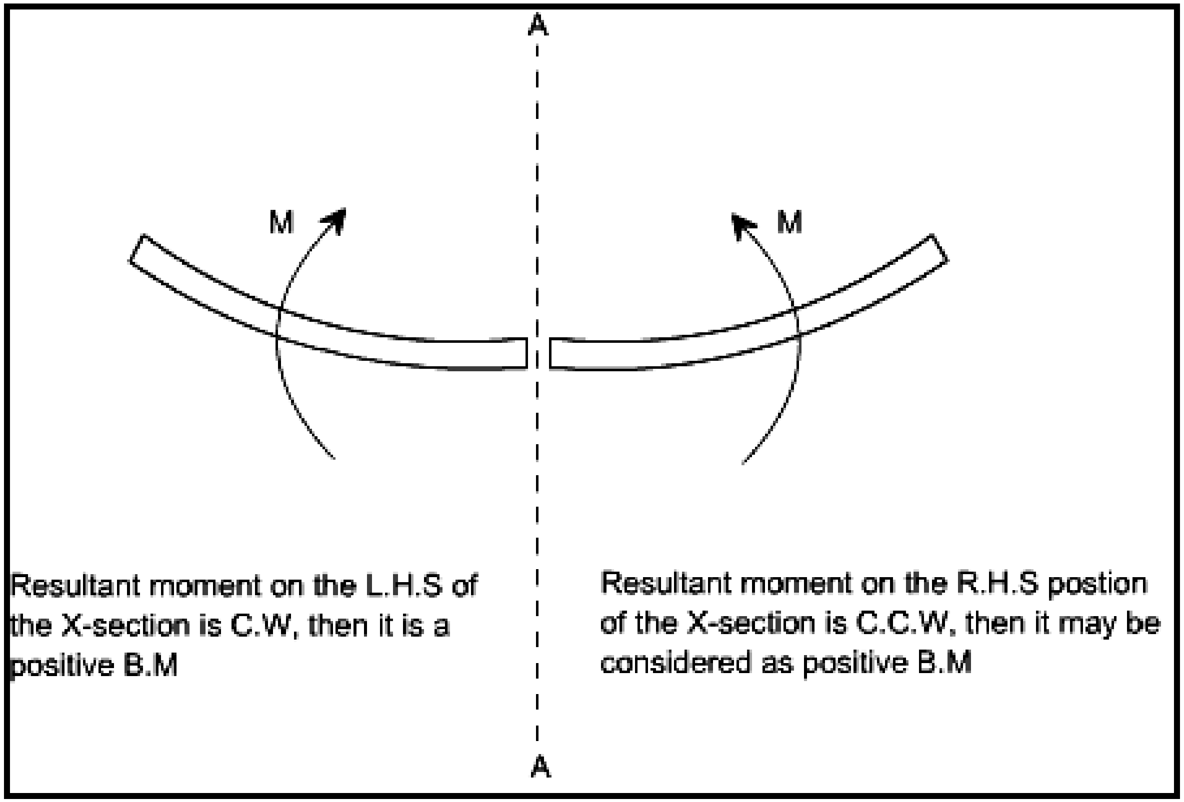
Sign Convention for Shear Force



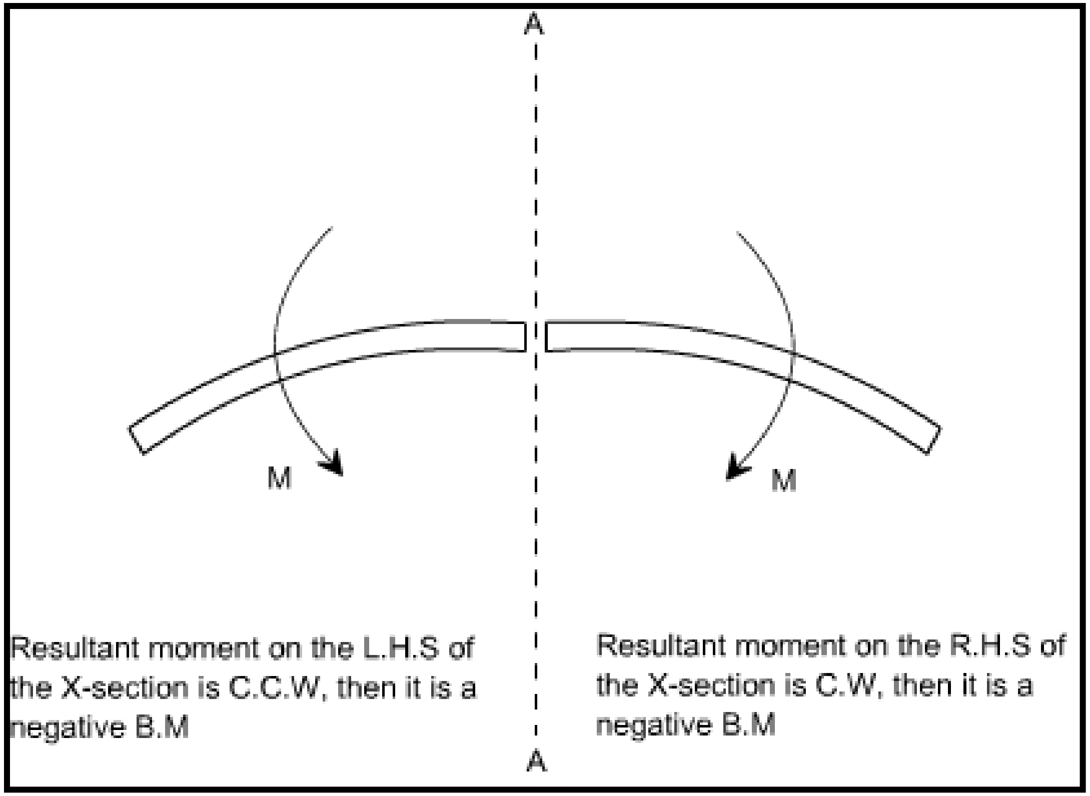
Bending Moment



Positive bending moment



Negative bending moment

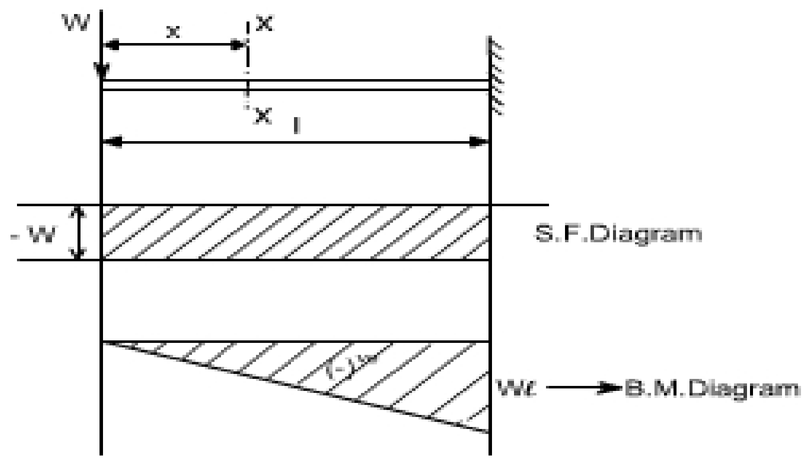


- **A cantilever of length carries a concentrated load ‘W' at its free end.**

Draw shear force and bending moment.

- At a section a distance x from free end consider the forces to the left, then F = -W (for all values of x) -ve sign means the shear force to the left of the x-section are in downward direction and therefore negative
- Taking moments about the section gives (obviously to the left of the section)

- M = -Wx (-ve sign means that the moment on the left hand side of the portion is in the anticlockwise direction and is therefore taken as –ve according to the sign convention)
- so that the maximum bending moment occurs at the fixed end i.e. M = -W l



Bending stresses

Loading restrictions:

- As we are aware of the fact internal reactions developed on any cross-section of a beam may consists of a resultant normal force, a resultant shear force and a resultant couple. In order to ensure that the bending effects alone are investigated, we shall put a constraint on the loading such that the resultant normal and the resultant shear forces are zero on any cross-section perpendicular to the longitudinal axis of the member,
- That means F = 0

- since or M = constant.
- Thus, the zero shear force means that the bending moment is constant or the bending is same at every cross-section of the beam. Such a situation may be visualized or envisaged when the beam or some portion of the beam, as been loaded only by pure couples at its ends. It must be recalled that the couples are assumed to be loaded in the plane of symmetry is shown in Fig (1).

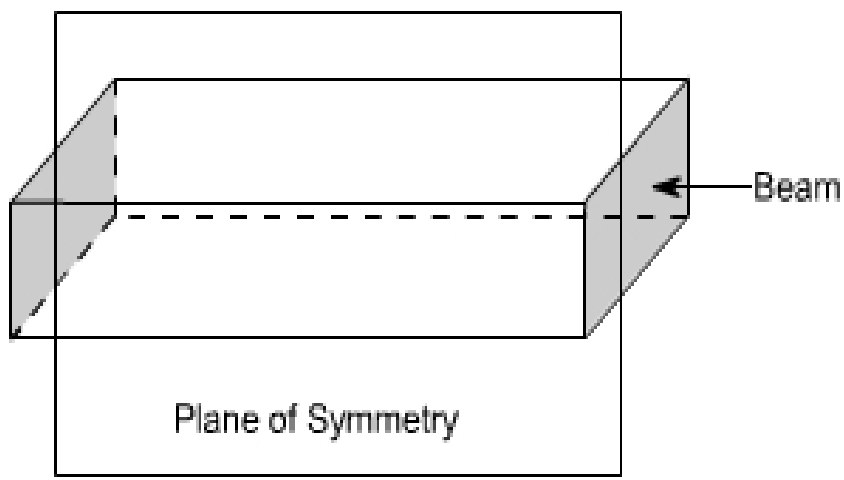
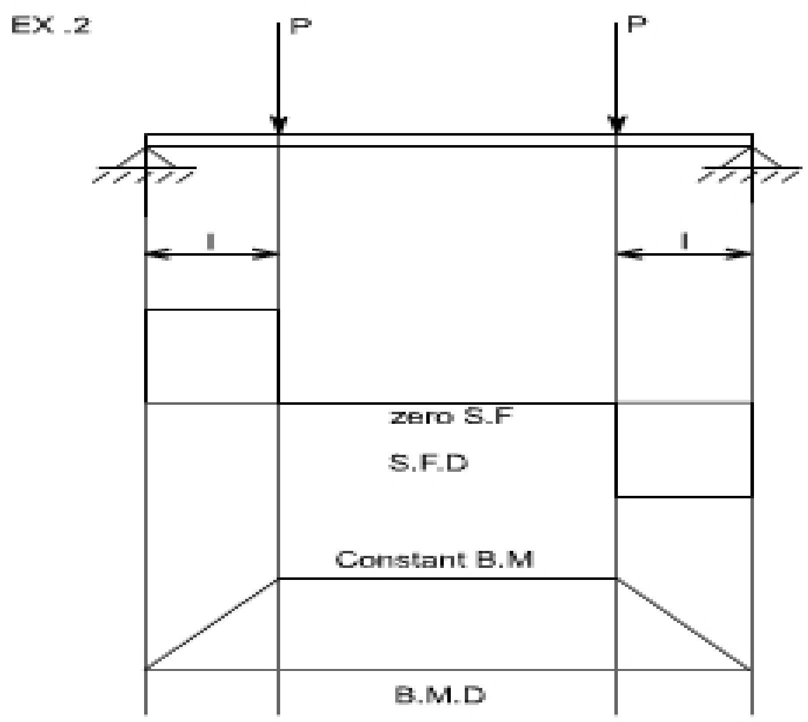
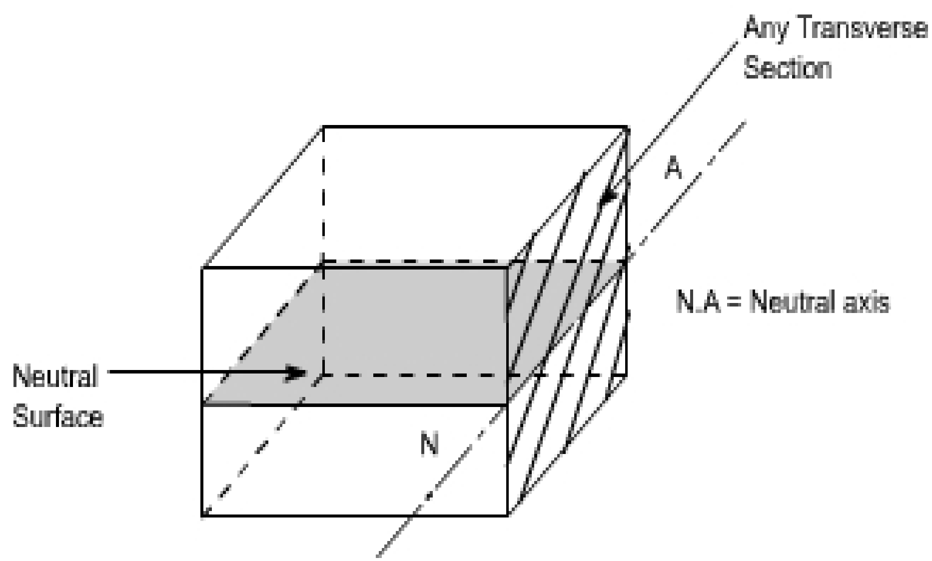


Fig (1)

- When a member is loaded in such a fashion it is said to be in **<u>pure bending.</u>** The examples of pure bending have been indicated in EX 1and EX 2 as shown below :



- When a beam is subjected to pure bending are loaded by the couples at the ends, certain cross-section gets deformed and we shall have to make out the conclusion that,
- Plane sections originally perpendicular to longitudinal axis of the beam remain plane and perpendicular to the longitudinal axis even after bending , i.e. the cross- section A'E', B'F' (refer Fig 1(a)) do not get warped or curved.
- In the deformed section, the planes of this cross-section have a common intersection i.e. any time originally parallel to the longitudinal axis of the beam becomes an arc of circle.



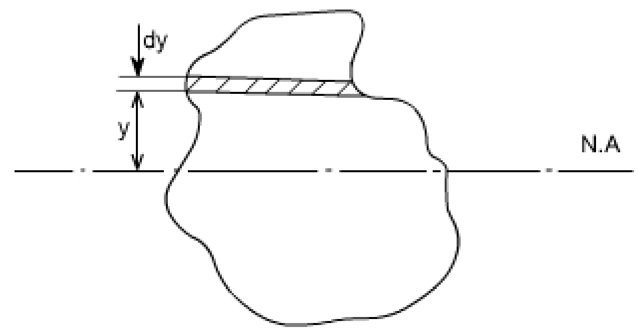
- We know that when a beam is under bending the fibres at the top will be lengthened while at the bottom will be shortened provided the bending moment M acts at the ends. In between these there are some fibres which remain unchanged in length that is they are not strained, that is they do not carry any stress. The plane containing such fibres is called neutral surface.
- The line of intersection between the neutral surface and the transverse exploratory section is called the neutral axisNeutral axis **(N A)** .

$$= \frac{(R+y)\theta - R\theta}{R\theta} = \frac{R\theta + y\theta - R\theta}{R\theta} = \frac{y}{R}$$

However $\dfrac{\text{stress}}{\text{strain}} = E$ where E = Young's Modulus of elasticity

Therefore, equating the two strains as obtained from the two relations i.e,

$$\frac{\sigma}{E} = \frac{y}{R} \text{ or } \frac{\sigma}{y} = \frac{E}{R} \qquad \ldots\ldots\ldots\ldots(1)$$



$$\sigma = \frac{E}{R} y$$

if the shaded strip is of area'dA'

then the force on the strip is

$$F = \sigma \, \delta A = \frac{E}{R} y \, \delta A$$

Moment about the neutral axis would be $= F.y = \frac{E}{R} y^2 \delta A$

The toatl moment for the whole

cross-section is therefore equal to

$$M = \sum \frac{E}{R} y^2 \delta A = \frac{E}{R} \sum y^2 \delta A$$

$$M = \frac{E}{R} I \qquad \ldots\ldots\ldots(2)$$

combining equation 1 and 2 we get

$$\boxed{\frac{\sigma}{y} = \frac{M}{T} = \frac{E}{R}}$$

This equation is known as the Bending Theory Equation.
The above proof has involved the assumption of pure bending without any shear force being present.

- J = polar moment of inertia $= \int r^2 dA$

$$= \int (x^2 + y^2) dA$$

$$= \int x^2 dA + \int y^2 dA$$

$$= I_x + I_y$$

$$\text{or } J = I_x + I_y \qquad \text{..........} (1)$$

The relation (1) is known as the **perpendicular axis theorem and may be stated as follows:**
The sum of the Moment of Inertia about any two axes in the plane is equal to the moment of inertia about an axis perpendicular to the plane, the three axes being concurrent, i.e, the three axes exist together.

CIRCULAR SECTION

$$J = \int r^2 dA$$

Taking the limits of intergration from 0 to d/2

$$J = \int_0^{\frac{d}{2}} r^2 2\pi r \delta r$$

$$= 2\pi \int_0^{\frac{d}{2}} r^3 \delta r$$

$$J = 2\pi \left[\frac{r^4}{4}\right]_0^{\frac{d}{2}} = \frac{\pi d^4}{32}$$

however, by perpendicular axis theorem

$$J = I_x + I_y$$

But for the circular cross-section, the I_x and I_y are both equal being moment of inertia about a diameter

$$I_{dia} = \frac{1}{2} J$$

$$I_{dia} = \frac{\pi d^4}{64}$$

for a hollow circular section of diameter D and d,

the values of J and I are defined as

$$J = \frac{\pi\left(D^4 - d^4\right)}{32}$$

$$I = \frac{\pi\left(D^4 - d^4\right)}{64}$$

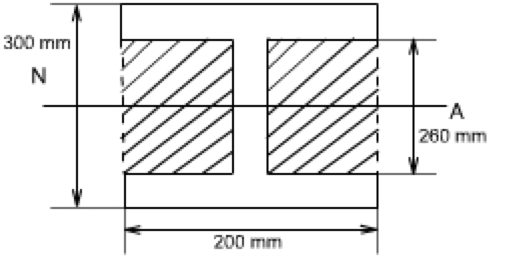
An I - section girder, 200mm wide by 300 mm depth flange and web of thickness is 20 mm is used as simply supported beam for a span of 7 m. The girder carries a distributed load of 5 KN /m and a concentrated load of 20 KN at mid-span. Determine the The second moment of area of the cross-section of the girder.

$$I_{girder} = I_{rectangle} - I_{shaded\ portion}$$

$$= \left[\frac{200 \times 300^3}{12}\right] 10^{-12} - 2\left[\frac{90 \times 260^3}{12}\right] 10^{-12}$$

$$= (4.5 - 2.64)10^{-4}$$

$$= 1.86 \times 10^{-4}\ m^4$$



The maximum stress may be found from the simple bending theory by equation

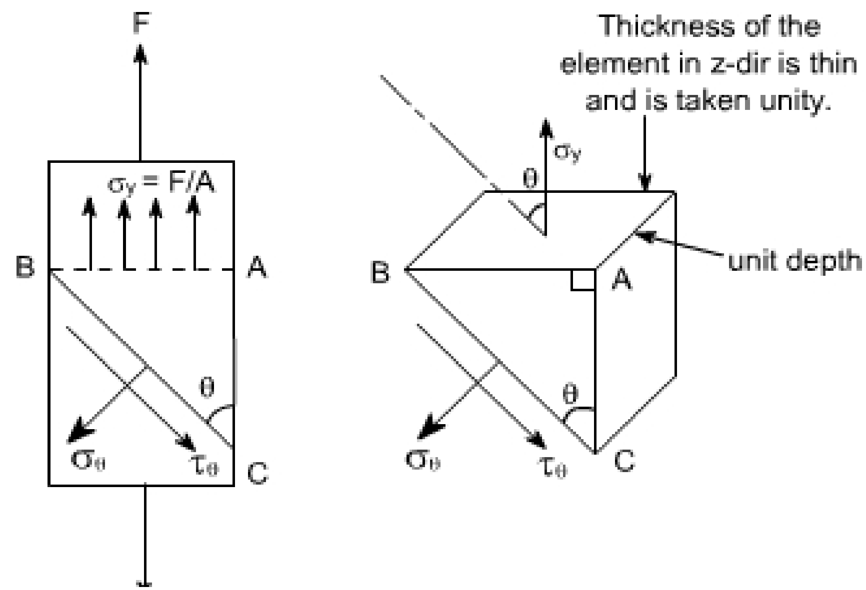
$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$$

i.e.

$$\sigma_{max^m} = \frac{M_{max^m}}{I} y_{max^m}$$

Two Dimensional State of Stress and Strain

Stresses on oblique plane: Till now we have dealt with either pure normal direct stress or pure shear stress. In many instances, however both direct and shear stresses acts and the resultant stress across any section will be neither normal nor tangential to the plane. A plane stse of stress is a 2 dimensional state of stress in a sense that the stress components in one direction are all zero



σ_θ .BC.1 = $\sigma_y \sin\theta$. BC $\sin\theta$. 1 or $\sigma_\theta = \sigma_y \sin^2 2\theta$ (1)

Now resolving the forces parallel to BC

τ_θ .BC.1 = $\sigma_y \cos\theta$. AB sin. 1

again AB = BC cos θ

σ_θ .BC.1 = $\sigma_y \cos\theta$. BC sin θ.1 or $\sigma_\theta = \sigma_y \sin\theta \cos\theta$

$$\tau_\theta = \frac{1}{2}.\sigma_y \sin 2\theta \qquad (2)$$

If θ = 90^0 the BC will be parallel to AB and τ_θ= 0, i.e. there will be only direct stress or normal stress.

Material subjected to pure shear

Assuming unit depth and resolving normal to PC or in the direction of σ_θ

σ_θ .PC.1 = τ_{xy} .PB.cos θ .1+ τ_{xy} .BC.sin θ .1

= τ_{xy} .PB.cos θ + τ_{xy} .BC.sin θ

Now writing PB and BC in terms of PC so that it cancels out from the two sides

PB/PC = sin θ BC/PC = cos θ

σ_θ .PC.1 = τ_{xy} .cos θ sin θ PC+ τ_{xy} .cos θ .sin θ .PC

$\sigma_\theta = 2\tau_{xy} \sin\theta \cos\theta$

Or, $\sigma_\theta = 2\tau_{xy} \sin 2\theta$ (1)

Now resolving forces parallel to PC or in the direction of σ_θ .then τ_{xy} PC.1

= τ_{xy} . PB sin θ - τ_{xy} BC cos θ

-ve sign has been put because this component is in the same direction as that of τ_θ .

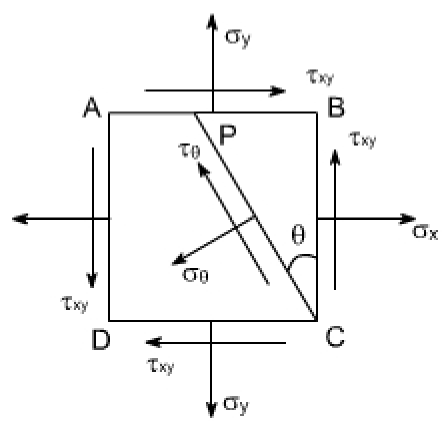
again converting the various quantities in terms of PC we have

τ_{xy} PC. 1 = τ_{xy} . PB.sin^2 θ τ_{xy} - τ_{xy} PCcos^2 θ

= - τ_{xy} [cos$^2\theta$ - sin$^2\theta$]

= - τ_{xy} cos2 θ (2)

- **Material subjected to combined direct and shear stresses:**
- Now consider a complex stress system shown below, acting on an element of material.
- The stresses $_x$ and $_y$ may be compressive or tensile and may be the result of direct forces or as a result of bending.The shear stresses may be as shown or



$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

For σ_θ to be a maximum or minimum $\frac{d\sigma_\theta}{d\theta} = 0$

Now

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\frac{d\sigma_\theta}{d\theta} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta.2 + \tau_{xy}\cos 2\theta.2$$

$$= 0$$

i.e. $-(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta.2 = 0$

$\tau_{xy}\cos 2\theta.2 = (\sigma_x - \sigma_y)\sin 2\theta$

Thus, $\boxed{\tan 2\theta = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}}$

Graphical Solution Using the Mohr's Stress Circle

4.5. GRAPHICAL SOLUTION -MOHR'S STRESS CIRCLE

Consider the complex stress system of Figure below. As stated previously this represents a complete stress system for any condition of applied load in two dimensions. In order to find graphically the direct stress σ_p and shear stress τ_θ on any plane inclined at θ to the plane on which σ_x acts, proceed as follows:

(1) Label the block *ABCD.*

(2) Set up axes for direct stress (as abscissa) and shear stress (as ordinate)

(3) Plot the stresses acting on two *adjacent* faces, e.g. *AB* and *BC*, using the following sign conventions:

- *Direct stresses:* tensile, positive; compressive, negative;
- *Shear stresses:* tending to turn block clockwise, positive; tending to turn block counterclockwise, negative.
- This gives two points on the graph which may then be labeled *AB* and *BC* respectively to denote stresses on these planes

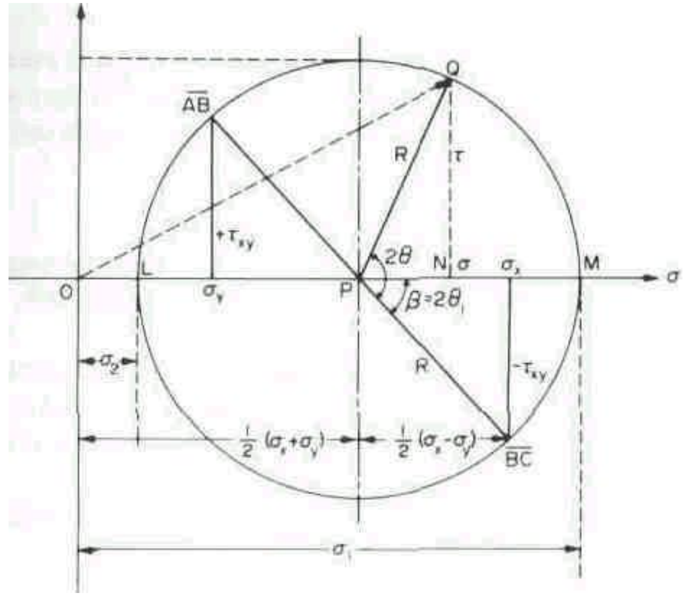
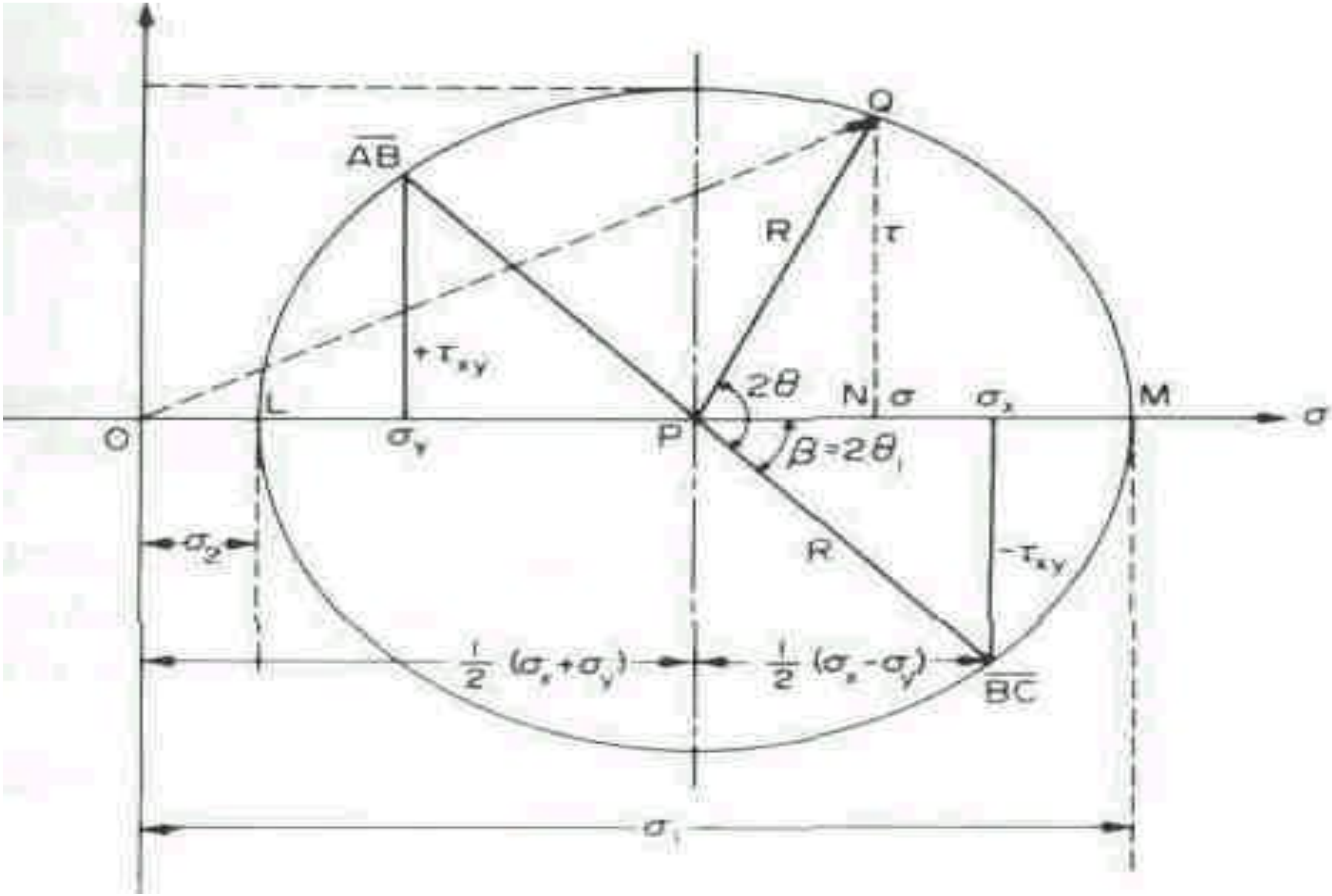


Fig. 4.5 Mohr's stress circle.

(4) Join *AB* and *BC.*

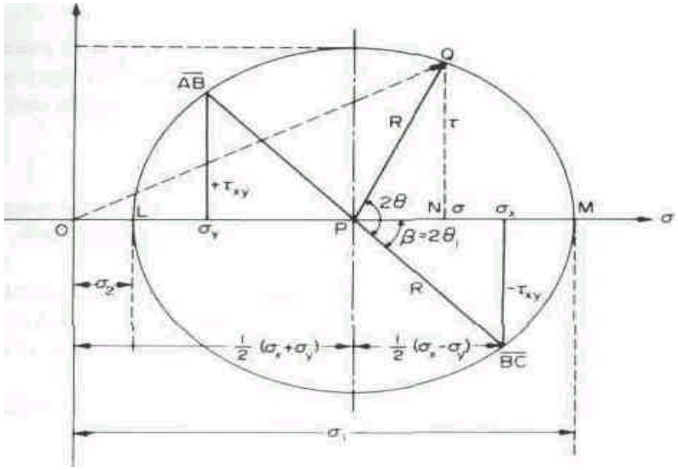
(5) The point *P* where this line cuts the *a* axis is then the centre of Mohr's circle, and the

line is the diameter; therefore the circle can now be drawn. Every point on the circumference of the circle then represents a state of stress on some **plane** through C.


Q
$\overline{AB}$
R
τ
$+\tau_{xy}$
2θ
N
σ
σ_x
M
σ
O
L
σ_y
P
$\beta = 2\theta_1$
σ_2
R
$-\tau_{xy}$
$\frac{1}{2}(\sigma_x + \sigma_y)$
$\frac{1}{2}(\sigma_x - \sigma_y)$
$\overline{BC}$
σ_1

On inspection this is seen to be eqn. (4.1) for the direct stress σ_θ on the plane inclined at θ to BC in the figure for the two -dimensional complex system.

Similarly,



QN $\sin(2\theta - \beta)$

$$= \boldsymbol{R \sin 2\theta \cos\beta - R\cos 2\theta \sin\beta}$$

$$= \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

Again, on inspection this is seen to be eqn. (4.2) for the shear stress τ_θ on the plane inclined at θ to BC.

- The graphical method of solution of complex stress problems using Mohr's circle is a very powerful technique since all the information relating to any plane within the stressed element is contained in the single construction.
- It thus provides a convenient and rapid means of solution which is less prone to arithmetical errors and is highly recommended.

THIN SHELLS

	British	Metric	S.I.
1. Force	Ib, kip, Ton	g, kg,	N, kN
	1 kip = 1000 Ib 1 ton = 2240 Ib	1 kg = 1000 g Ton = 1000 kg	1 kN = 1000 N 1 kg = 10 N
2. Long	in, ft	m, cm, mm	m, cm, mm
	1 f = 12 in	1 m = 100 cm 1 cm = 10 mm 1 m =1000 mm 1 in = 2.54 cm	1 m = 100 cm 1 cm = 10 mm 1 m =1000 mm 1 in = 2.54 cm
3. Stress	psi, ksi $\frac{\text{p}}{\text{in}^2}, \frac{\text{kip}}{\text{in}^2}$	Pa ($\frac{N}{mm^2}$), MPa, GPa	

$$\text{MPa} = 10^6 \text{ Pa} = 10^6 \text{ N/mm}^2 \times \frac{1}{1000^2 \frac{mm^2}{m^2}}$$

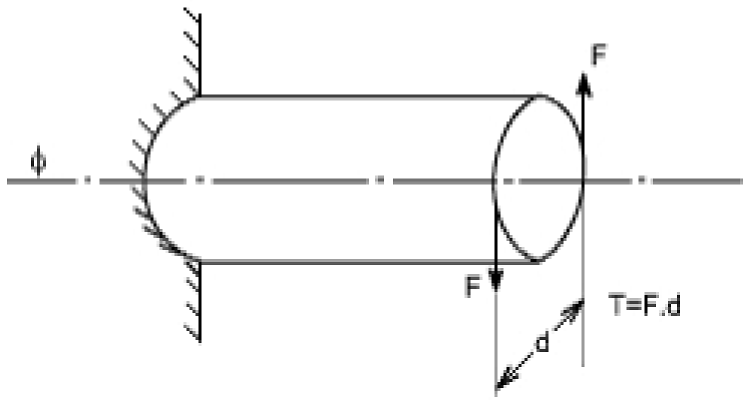
$$\text{MPa} = \frac{N}{mm^2}$$

$$\text{GPa} = 10^9 \text{ Pa} = 10^9 \text{ N/mm}^2 \times \frac{1}{1000^2 \frac{mm^2}{m^2}} = 10^3 \frac{N}{mm^2} \times \frac{1}{1000 \frac{N}{kN}}$$

$$\text{GPa} = \text{kN/mm}^2$$

Torsion of circular shafts

Definition of Torsion: Consider a shaft rigidly clamped at one end and twisted at the other end by a torque T = F.d applied in a plane perpendicular to the axis of the bar such a shaft is said to be in torsion.



Effects of Torsion: The effects of a torsional load applied to a bar are

(i) To impart an angular displacement of one end cross – section with respect to the other end.

(ii) To setup shear stresses on any cross section of the bar perpendicular to its axis.

Assumption:

(i) The materiel is homogenous i.e of uniform elastic properties exists throughout the material.

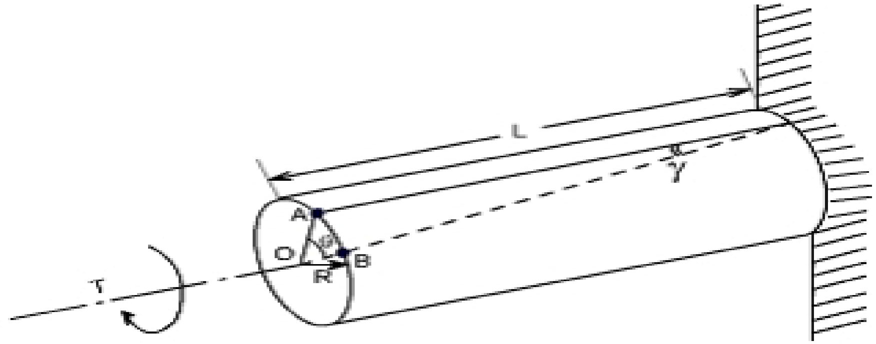
(ii) The material is elastic, follows Hook's law, with shear stress proportional to shear strain.

(iii) The stress does not exceed the elastic limit.

(iv) The circular section remains circular

(v) Cross section remain plane.

(vi) Cross section rotate as if rigid i.e. every diameter rotates through the same angle.



Consider now the solid circular shaft of radius R subjected to a torque T at one end, the other end being fixed Under the action of this torque a radial line at the free end of the shaft twists through an angle q , point A moves to B, and AB subtends an angle ‘ g ' at the fixed end. This is then the angle of distortion of the shaft i.e the shear strain.

Since angle in radius = arc / Radius

arc AB = Rq

= L g [since L and g also constitute the arc AB]

Thus, g = Rq / L (1)

From the definition of Modulus of rigidity or Modulus of elasticity in shear

$$G = \frac{\text{shear stress}(\tau)}{\text{shear strain}(\gamma)}$$

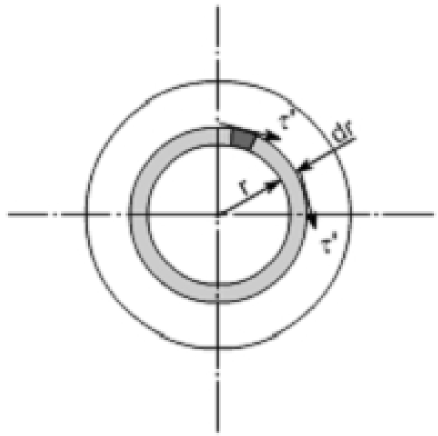
where γ is the shear stress set up at radius R.

Then $\frac{\tau}{G} = \gamma$

Equating the equations (1) and (2) we get $\frac{R\theta}{L} = \frac{\tau}{G}$

$\frac{\tau}{R} = \frac{G\theta}{L}\left(= \frac{\tau'}{r}\right)$ where τ' is the shear stress at any radius r.

Stresses: Let us consider a small strip of radius r and thickness dr which is subjected to shear stress t'.



The force set up on each element

= stress x area

= t' x 2p r dr (approximately)

This force will produce a moment or torque about the center axis of the shaft.

= t' . 2 p r dr . r

= 2 p t' . r^2. dr

The total torque T on the section, will be the sum of all the contributions. $T = \int_0^R 2\pi\tau' r^2 \, dr$

Since t' is a function of r, because it varies with radius so writing down t' in terms of r from the equation (1).

i.e $\tau' = \frac{G\theta.r}{L}$

we get $T = \int_0^R 2\pi \frac{G\theta}{L}.r^3 dr$

$T = \frac{2\pi G\theta}{L} \int_0^R r^3 dr$

$= \frac{2\pi G\theta}{L}.\left[\frac{R^4}{4}\right]_0^R$

$= \frac{G\theta}{L}.\frac{2\pi R^4}{4}$

$= \frac{G\theta}{L}.\frac{\pi R^4}{2}$

$= \frac{G\theta}{L}.\left[\frac{\pi d^4}{32}\right]$ now substituting $R = d/2$

$= \frac{G\theta}{L}.J$

since $\frac{\pi d^4}{32} = J$ the polar moment of inertia

or $\frac{T}{J} = \frac{G\theta}{L}$ ......(2)

if we combine the equation no.(1) and (2) we get $\boxed{\frac{T}{J} = \frac{\tau'}{r} = \frac{G.\theta}{L}}$

Where

T = applied external Torque, which is constant over Length L;

J = Polar moment of Inertia

$= \frac{\pi d^4}{32}$ for solid shaft

$= \frac{\pi(D^4 - d^4)}{32}$ for a hollow shaft. [D = Outside diameter ; d = inside diameter]

G = Modules of rigidity (or Modulus of elasticity in shear)

q = It is the angle of twist in radians on a length L.

Tensional Stiffness: The tensional stiffness k is defined as the torque per radius twist

Distribution of shear stresses in circular Shafts subjected to torsion :

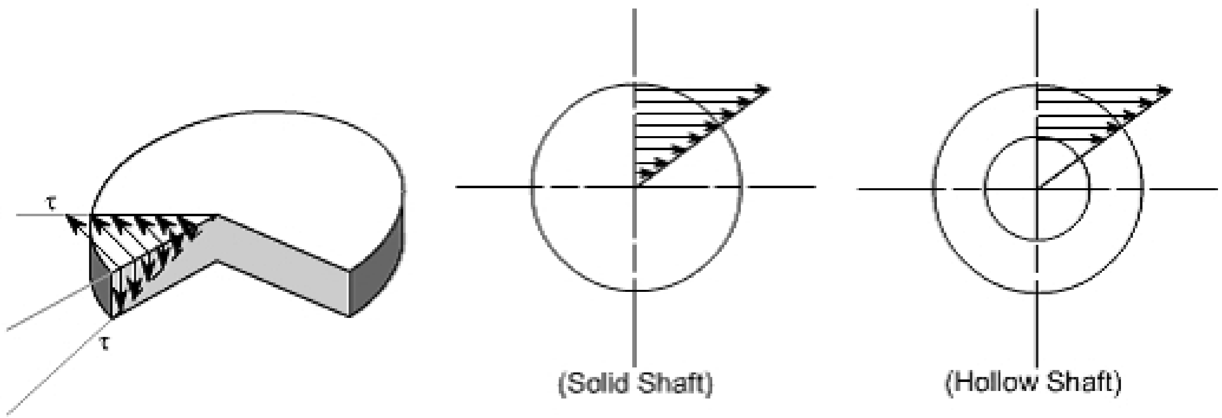
The simple torsion equation is written as

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$$

or

$$\tau = \frac{G\theta.r}{L}$$

This states that the shearing stress varies directly as the distance 'r' from the axis of the shaft and the following is the stress distribution in the plane of cross section and also the complementary shearing stresses in an axial plane.



Hence the maximum strear stress occurs on the outer surface of the shaft where r = R

The value of maximum shearing stress in the solid circular shaft can be determined as

$$\frac{\tau}{r} = \frac{T}{J}$$

$$\tau_{max}\Big|_{r=d/2} = \frac{T.R}{J} = \frac{T}{\frac{\pi d^4}{32}} . d/2$$

where d=diameter of solid shaft

$$\text{or} \quad \tau_{max^m} = \frac{16T}{\pi d^3}$$

From the above relation, following conclusion can be drawn

(i) t_{max}^{m} µ T

(ii) t_{max}^{m} µ $1/d^{3}$

Power Transmitted by a shaft:

In practical application, the diameter of the shaft must sometimes be calculated from the power which it is required to transmit.

Given the power required to be transmitted, speed in rpm ‘N' Torque T, the formula connecting

These quantities can be derived as follows

$$P = T.\omega$$

$$= \frac{T.2\pi N}{60} \text{watts}$$

$$= \frac{2\pi NT}{60 x 10^3} (\text{kw})$$

Torsional stiffness: The torsional stiffness k is defined as the torque per radian twist .

TORSION OF HOLLOW SHAFTS:

From the torsion of solid shafts of circular x – section , it is seen that only the material at the outer surface of the shaft can be stressed to the limit assigned as an allowable working stresses. All of the material within the shaft will work at a lower stress and is not being used to full capacity. Thus, in these cases where the weight reduction is important, it is advantageous to use hollow shafts. In discussing the torsion of hollow shafts the same assumptions will be made as in the case of a solid shaft. The general torsion equation as we have applied in the case of torsion of solid shaft will hold good

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$$

For the hollow shaft

$$J = \frac{\pi(D_0{}^4 - d_i{}^4)}{32} \qquad \text{where } D_0 = \text{Outside diameter}$$

$$d_i = \text{Inside diameter}$$

$$\text{Let } d_i = \frac{1}{2}.D_0$$

$$\tau_{max^m}\Big|_{solid} = \frac{16T}{\pi D_0{}^3} \qquad (1)$$

$$\tau_{max^m}\Big|_{hollow} = \frac{T.D_0/2}{\frac{\pi}{32}(D_0{}^4 - d_i{}^4)}$$

$$= \frac{16T.D_0}{\pi D_0{}^4\left[1-(d_i/D_0)^4\right]}$$

$$= \frac{16T}{\pi D_0{}^3\left[1-(1/2)^4\right]} = 1.066.\frac{16T}{\pi D_0{}^3} \qquad (2)$$

Members Subjected to Axisymmetric Loads

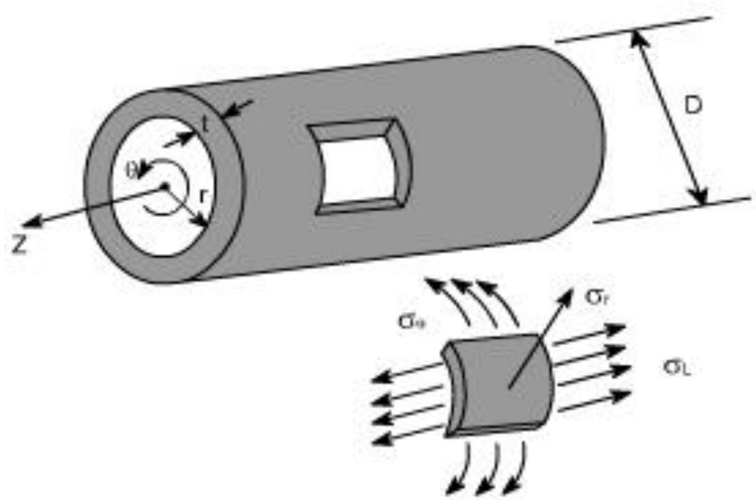
Pressurized thin walled cylinder:

Preamble : Pressure vessels are exceedingly important in industry. Normally two types of pressure vessel are used in common practice such as cylindrical pressure vessel and spherical pressure vessel.

In the analysis of this walled cylinders subjected to internal pressures it is assumed that the radial plans remains radial and the wall thickness dose not change due to internal pressure. Although the internal pressure acting on the wall causes a local compressive stresses (equal to pressure) but its value is negligibly small as compared to other stresses & hence the sate of stress of an element of a thin walled pressure is considered a biaxial one.

Further in the analysis of them walled cylinders, the weight of the fluid is considered negligible.

Let us consider a long cylinder of circular cross - section with an internal radius of R_2 and a constant wall thickness ‘t' as showing fig.



This cylinder is subjected to a difference of hydrostatic pressure of ‘p' between its inner and outer surfaces.

In many cases, ‘p' between gage pressure within the cylinder, taking outside pressure to be ambient.

By thin walled cylinder we mean that the thickness‘t' is very much smaller than the radius R_i and we may quantify this by stating than the ratio t / R_i of thickness of radius should be less than 0.1.

Type of failure:

Such a component fails in since when subjected to an excessively high internal pressure. While it might fail by bursting along a path following the circumference of the cylinder. Under normal circumstance it fails by circumstances it fails by bursting along a path parallel to the axis. This suggests that the hoop stress is significantly higher than the axial stress. In order to derive the expressions for various stresses we make following

Applications :

Liquid storage tanks and containers, water pipes, boilers, submarine hulls, and certain air plane components are common examples of thin walled cylinders and spheres, roof domes.

ANALYSIS : In order to analyse the thin walled cylinders, let us make the following assumptions :

- There are no shear stresses acting in the wall.
- The longitudinal and hoop stresses do not vary through the wall.

- Radial stresses $\square_r$ which acts normal to the curved plane of the isolated element are neglibly small as compared to other two stresses especially when $\left[\frac{t}{R_i} < \frac{1}{20}\right]$

Thin Cylinders Subjected to Internal Pressure:

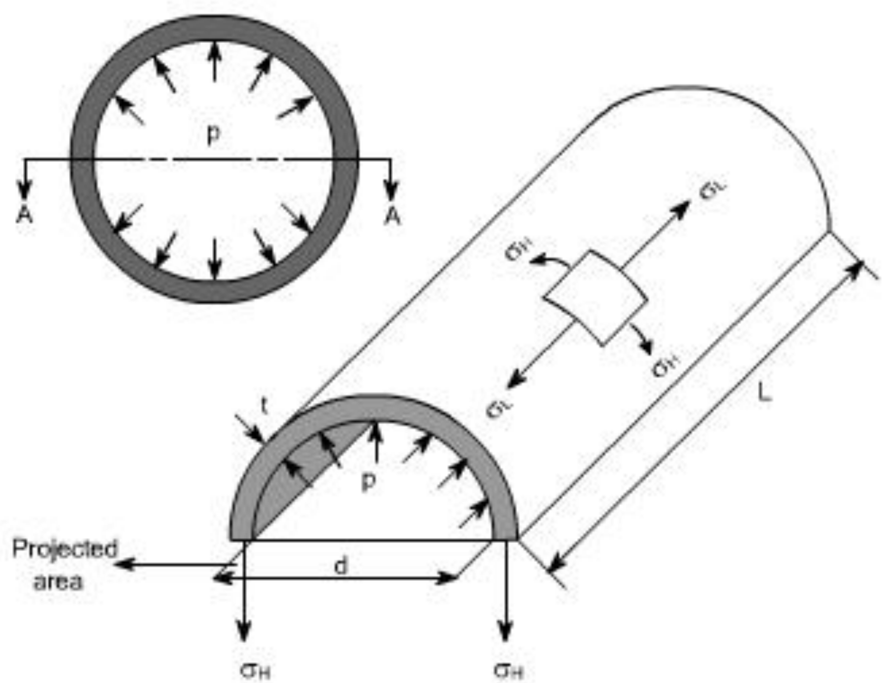
When a thin – walled cylinder is subjected to internal pressure, three mutually perpendicular principal stresses will be set up in the cylinder materials, namely

- Circumferential or hoop stress
- The radial stress
- Longitudinal stress

now let us define these stresses and determine the expressions for them

Hoop or circumferential stress:

This is the stress which is set up in resisting the bursting effect of the applied pressure and can be most conveniently treated by considering the equilibrium of the cylinder.



In the figure we have shown a one half of the cylinder. This cylinder is subjected to an internal
pressure p.

i.e. p = internal pressure

d = inside diametre

L = Length of the cylinder

t = thickness of the wall

Total force on one half of the cylinder owing to the internal pressure 'p'

= p x Projected Area

= p x d x L

= **p .d. L** ------- (1)

Volumetric Strain or Change in the Internal Volume:

When the thin cylinder is subjected to the internal pressure as we have already calculated that there is a change in the cylinder dimensions i.e, longitudinal strain and hoop strains come into picture. As a result of which there will be change in capacity of the cylinder or there is a change in the volume of the cylinder hence it becomes imperative to determine the change in volume or the volumetric strain. The capacity of a cylinder is defined as

V = Area X Length

= □$d^2/4$ x L

Let there be a change in dimensions occurs, when the thin cylinder is subjected to an internal pressure.

(i) The diameter **d** changes to □ **d +** □ **d**

(ii) The length **L** changes to □ **L +** □ **L**

Therefore, the change in volume = Final volume □ Original volume

$$= \frac{\pi}{4}[d + \delta d]^2.(L + \delta L) - \frac{\pi}{4}d^2.L$$

$$\text{Volumetric strain} = \frac{\text{Changein volume}}{\text{Original volume}} = \frac{\frac{\pi}{4}[d + \delta d]^2.(L + \delta L) - \frac{\pi}{4}d^2.L}{\frac{\pi}{4}d^2.L}$$

$$\in_v = \frac{\left\{[d + \delta d]^2.(L + \delta L) - d^2.L\right\}}{d^2.L} = \frac{\left\{(d^2 + \delta d^2 + 2d.\delta d).(L + \delta L) - d^2.L\right\}}{d^2.L}$$

simplifying and neglecting the products and squares of small quantities,i.e. δd & δL hence

$$= \frac{2d.\delta d.L + \delta L.d^2}{d^2 L} = \frac{\delta L}{L} + 2.\frac{\delta d}{d}$$

By definition $\frac{\delta L}{L}$ = Longitudnal strain

$\frac{\delta d}{d}$ = hoop strain,Thus

Volumetric strain = longitudnal strain + 2 x hoop strain

on substituting the value of longitudnal and hoop strains we get

$$\in_1 = \frac{pd}{4tE}[1 - 2\nu] \quad \& \quad \in_2 = \frac{pd}{4tE}[1 - 2\nu]$$

$$\text{or Volumetric} = \in_1 + 2\in_2 = \frac{pd}{4tE}[1 - 2\nu] + 2.\left(\frac{pd}{4tE}[1 - 2\nu]\right)$$

$$= \frac{pd}{4tE}\{1 - 2\nu + 4 - 2\nu\} = \frac{pd}{4tE}[5 - 4\nu]$$

$$\text{Volumetric Strain} = \frac{pd}{4tE}[5 - 4\nu] \qquad \text{or} \quad \boxed{\in_v = \frac{pd}{4tE}[5 - 4\nu]}$$

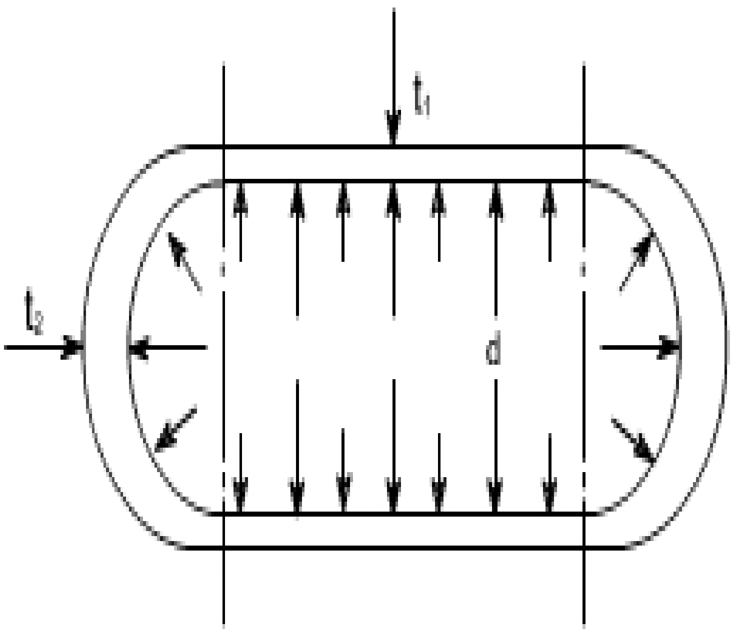
Therefore to find but the increase in capacity or volume, multiply the volumetric strain by original volume. Hence Change in Capacity / Volume

$$\boxed{\text{Increase in volume} = \frac{pd}{4tE}\left[5 - 4\nu\right]V}$$

Cylindrical Vessel with Hemispherical Ends:

Let us now consider the vessel with hemispherical ends. The wall thickness of the cylindrical and hemispherical portion is different. While the internal diameter of both the portions is assumed to be equal

Let the cylindrical vassal is subjected to an internal pressure p.



For the Cylindrical Portion

hoop or circumferential stress $= \sigma_{HC}$ 'c' here synifies the cylindrical portion.

$$= \frac{pd}{2t_1}$$

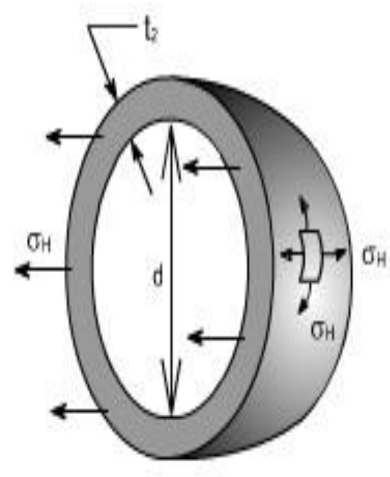
longitudnal stress $= \sigma_{LC}$

$$= \frac{pd}{4t_1}$$

hoop or circumferential strain $\in_2 = \frac{\sigma_{HC}}{E} - \nu\frac{\sigma_{LC}}{E} = \frac{pd}{4t_1E}[2-\nu]$

or $\boxed{\in_2 = \frac{pd}{4t_1E}[2-\nu]}$

For The Hemispherical Ends:



Because of the symmetry of the sphere the stresses set up owing to internal pressure will be two mutually perpendicular hoops or circumferential stresses of equal values. Again the radial stresses are neglected in comparison to the hoop stresses as with this cylinder having thickness to diametre less than1:20.

Consider the equilibrium of the half – sphere

Force on half-sphere owing to internal pressure = pressure x projected Area

$$= p. < d^2/4$$

Resisting force $= \sigma_H \,.\, \pi.d.t_2$

$$\therefore \qquad p.\frac{\pi.d^2}{4} = \sigma_H \,.\, \pi d.t_2$$

$$\Rightarrow \sigma_H \text{ (for sphere)} = \frac{pd}{4t_2}$$

similarly the hoop strain $= \frac{1}{E}\left[\sigma_H - \nu.\sigma_H\right] = \frac{\sigma_H}{E}[1-\nu] = \frac{pd}{4t_2E}[1-\nu]$ or $\boxed{\epsilon_{2s} = \frac{pd}{4t_2E}[1-\nu]}$

Fig – shown the (by way of dotted lines) the tendency, for the cylindrical portion and the spherical ends to expand by a different amount under the action of internal pressure. So owing to difference in stress, the two portions (i.e. cylindrical and spherical ends) expand by a different amount. This incompatibly of deformations causes a local bending and sheering stresses in the neighborhood of the joint. Since there must be physical continuity between the ends and the cylindrical portion, for this reason, properly curved ends must be used for pressure vessels.

Thus equating the two strains in order that there shall be no distortion of the junction

$$\frac{pd}{4t_1E}\left[2-\nu\right]=\frac{pd}{4t_2E}\left[1-\nu\right] \quad \text{or} \quad \frac{t_2}{t_1}=\frac{1-\nu}{2-\nu}$$

But for general steel works ν = 0.3, therefore, the thickness ratios becomes

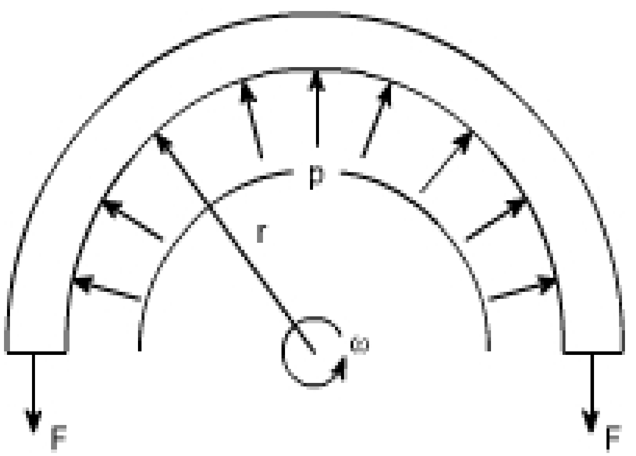
$$\mathbf{t_2 / t_1 = 0.7/1.7}$$

i.e. the thickness of the cylinder walls must be approximately 2.4 times that of the hemispheroid ends for no distortion of the junction to occur.

Thin rotating ring or cylinder

Consider a thin ring or cylinder as shown in Fig below subjected to a radial internal pressure p caused by the centrifugal effect of its own mass when rotating. The centrifugal effect on a unit length of the circumference is

$$p = m\,\omega^2\,r$$



Thin ring rotating with constant angular velocity

Here the radial pressure ‘p' is acting per unit length and is caused by the centrifugal effect if its own mass when rotating.

Thus considering the equilibrium of half the ring shown in the figure, 2F = p x 2r (assuming unit length), as 2r is the projected area

F = pr

Where F is the hoop tension set up owing to rotation.

The cylinder wall is assumed to be so thin that the centrifugal effect can be assumed constant across the wall thickness.

F = mass x acceleration = **m**
ω^2 r x r

This tension is transmitted through the complete circumference and therefore is resisted by the complete cross – sectional area hoop stress = F/A = $\mathbf{m\,\omega^2\,r^2\,/\,A}$

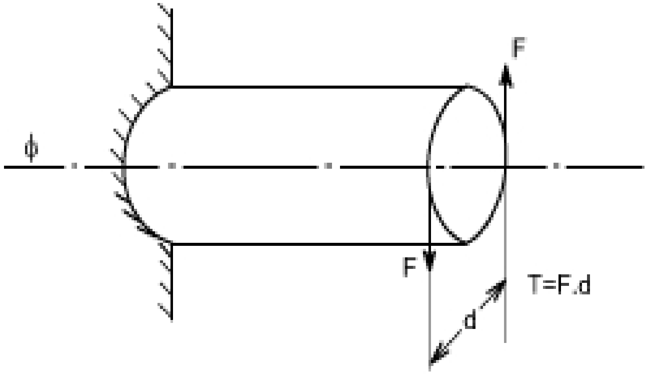
Where A is the cross – sectional area of the ring.

Now with unit length assumed m/A is the mass of the material per unit volume, i.e. the density < .

hoop stress □ H = □ $\omega^2\,\mathbf{r^2}$

Torsion of circular shafts

Definition of Torsion: Consider a shaft rigidly clamped at one end and twisted at the other end by a torque T =F.d applied in a plane perpendicular to the axis of the bar such a shaft is said to be in torsion.



Effects of Torsion: The effects of a torsional load applied to a bar are

(i) To impart an angular displacement of one end cross – section with respect to the other end.

(ii) To setup shear stresses on any cross section of the bar perpendicular to its axis.

Assumption:

(i)The materiel is homogenous i.e of uniform elastic properties exists throughout the material.

(ii)The material is elastic, follows Hook's law, with shear stress proportional to shear strain.

(iii)The stress does not exceed the elastic limit.

(iv)The circular section remains circular

(v)Cross section remain plane.

(vi)Cross section rotate as if rigid i.e. every diameter rotates through the same angle.

Consider now the solid circular shaft of radius R subjected to a torque T at one end, the other end being fixed Under the action of this torque a radial line at the free end of the shaft twists through an angle □ , point A moves to B, and AB subtends an angle ‘ □ ' at the fixed end. This is then the angle of distortion of the shaft i.e the shear strain.

Since angle in radius = arc / Radius arc AB = R

From the definition of Modulus of rigidity or Modulus of elasticity in shear

$$G = \frac{\text{shear stress}(\tau)}{\text{shear strain}(\gamma)}$$

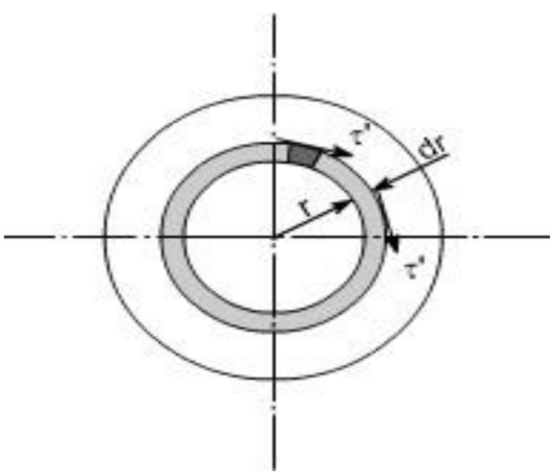
where γ is the shear stress set up at radius R.

Then $\frac{\tau}{G} = \gamma$

Equating the equations (1) and (2) we get $\frac{R\theta}{L} = \frac{\tau}{G}$

$\frac{\tau}{R} = \frac{G\theta}{L}\left(= \frac{\tau'}{r}\right)$ where τ' is the shear stress at any radius r.

Stresses: Let us consider a small strip of radius r and thickness dr which is subjected to shear stress'.



The force set up on each element= stress x area

The total torque T on the section, will be the sum of all the contributions.

Since □' is a function of r, because it varies with radius so writing down□□' in terms of r from the equation (1).

$$\text{i.e } \tau' = \frac{G\theta . r}{L}$$

$$\text{we get } T = \int_0^R 2\pi \frac{G\theta}{L} . r^3 dr$$

$$T = \frac{2\pi G\theta}{L} \int_0^R r^3 dr$$

$$= \frac{2\pi G\theta}{L} . \left[\frac{R^4}{4}\right]_0^R$$

$$= \frac{G\theta}{L} . \frac{2\pi R^4}{4}$$

$$= \frac{G\theta}{L} . \frac{\pi R^4}{2}$$

$$= \frac{G\theta}{L} . \left[\frac{\pi d^4}{32}\right] \text{ now substituting } R = d/2$$

$$= \frac{G\theta}{L} . J$$

$$\text{since } \frac{\pi d^4}{32} = J \text{ the polar moment of inertia}$$

$$\text{or } \frac{T}{J} = \frac{G\theta}{L} \qquad \ldots\ldots(2)$$

if we combine the equation no.(1) and (2) we get $\boxed{\dfrac{T}{J} = \dfrac{\tau'}{r} = \dfrac{G.\theta}{L}}$

$$= \frac{\pi d^4}{32} \text{ for solid shaft}$$

$$= \frac{\pi(D^4 - d^4)}{32} \text{ for a hollow shaft.}$$

Tensional Stiffness: The tensional stiffness k is defined as the torque per radius twist i.e, k = T /□= GJ / L

Power Transmitted by a shaft : If T is the applied Torque and □ is the angular velocity of the shaft, then the power transmitted by the shaft is

$$P = T.\omega = \frac{2\pi NT}{60} = \frac{2\pi NT}{60.10^3} \text{kw}$$

$$\text{where } N = \text{rpm}$$

TORSION OF HOLLOW SHAFTS:

From the torsion of solid shafts of circular x – section , it is seen that only the material at the outer surface of the shaft can be stressed to the limit assigned as an allowable working stresses. All of the material within the shaft will work at a lower stress and is not being used to full capacity. Thus, in these cases where the weight reduction is important, it is advantageous to use hollow shafts. In discussing the torsion of hollow shafts the same assumptions will be made as in the case of a solid shaft.

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$$

For the hollow shaft

$$J = \frac{\pi(D_0^4 - d_i^4)}{32} \quad \text{where } D_0 = \text{Outside diameter}$$

$$d_i = \text{Inside diameter}$$

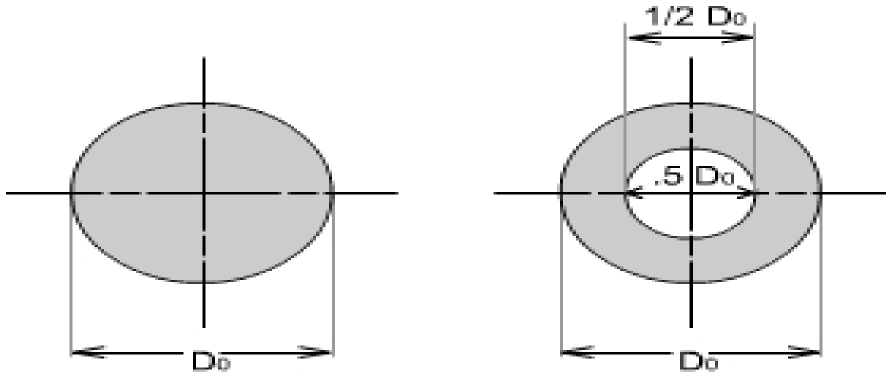
$$\text{Let } d_i = \frac{1}{2}.D_0$$

$$\tau_{max^m}\Big|_{solid} = \frac{16T}{\pi D_0^3} \qquad (1)$$

$$\tau_{max^m}\Big|_{hollow} = \frac{T.D_0/2}{\frac{\pi}{32}(D_0^4 - d_i^4)}$$

$$= \frac{16T.D_0}{\pi D_0^4\left[1-(d_i/D_0)^4\right]}$$

$$= \frac{16T}{\pi D_0^3\left[1-(1/2)^4\right]} = 1.066.\frac{16T}{\pi D_0^3} \qquad (2)$$



Weight of hollow shaft

$$= \left[\frac{\pi D_0{}^2}{4} - \frac{\pi (D_0/2)^2}{4}\right] l \times \rho$$

$$= \left[\frac{\pi D_0{}^2}{4} - \frac{\pi D_0{}^2}{16}\right] l \times \rho$$

$$= \frac{\pi D_0{}^2}{4}[1 - 1/4]\, l \times \rho$$

$$= 0.75 \frac{\pi D_0{}^2}{4} l \times \rho$$

Weight of solid shaft $= \dfrac{\pi D_0{}^2}{4} . l . \rho$

Reduction in weight $= (1 - 0.75) \dfrac{\pi D_0{}^2}{4} l \times \rho$

$$= 0.25 \frac{\pi D_0{}^2}{4} l \times \rho$$

Assignment Problem

A stepped solid circular shaft is built in at its ends and subjected to an externally applied torque. T_0 at the shoulder as shown in the figure. Determine the angle of rotation □$_0$ of the shoulder section where T_0 is applied ? along the entire length of the beam.