INSTITUTE OF AERONAUTICAL ENGINEERING
(Autonomous)
Dundigal, Hyderabad - 500 043

OPERATIONS RESEARCH

IV B.TECH-I SEM

PREPARED BY:

A. SOMAIAH, ASST. PROFESSOR
T. VANAJA, ASST. PROFESSOR
DEPT. OF MECHANICAL ENGINEERING

Syllabus

UNIT-I

Development – Definition– Characteristics and Phases – Types of models - Operations Research models – applications.

Allocation: Linear Programming Problem Formulation – Graphical solution – Simplex method – Artificial variables techniques: Two–phase method, Big-M method.

UNIT-II

Transportation Problem - Formulation – Optimal solution, unbalanced transportation problem – Degeneracy.

Assignment problem – Formulation – Optimal solution – Variants of Assignment Problem- Traveling Salesman problem.

UNIT-III

Sequencing - Introduction – Flow –Shop sequencing – n jobs through two machines – n jobs through three machines – Job shop sequencing – two jobs through ‘m’ machines.

Replacement: Introduction – Replacement of items that deteriorate with time – when money value is not counted and counted – Replacement of items that fail completely, Group Replacement.

UNIT-IV

Theory Of Games: Introduction – Terminology - Solution of games with saddle points and without saddle points – 2×2 games – dominance principle – m X 2 & 2 X n games - Graphical method.

Inventory: Introduction – Single item – Deterministic models – Purchase inventory models with one price break and multiple price breaks – Stochastic models – demand may be discrete variable or continuous variable – Single period model and no setup cost.

UNIT-V

Waiting Lines: Introduction – Terminology - Single Channel – Poisson arrivals and exponential service times – with infinite population and finite population models– Multichannel – Poisson arrivals and exponential service times with infinite population.

Dynamic Programming: Introduction –Terminology - Bellman's Principle of optimality –Applications of dynamic programming – shortest path problem – linear programming problem.

Simulation: Introduction, Definition, types of simulation models, steps involved in the simulation process - Advantages and Disadvantages – Application of Simulation to queuing and inventory.

Unit – I: Linear Programming Problem

- As its name implies, operations research involves "research on operations." Thus, operations research is applied to problems that concern how to conduct and coordinate the operations (i.e., the activities) within an organization .
- The nature of the organization is essentially immaterial, in fact, OR has been applied extensively in such diverse areas as manufacturing, transportation, construction, telecommunications, financial planning, health care, the military, and public services, to name just a few .

Operation Research

- The process begins by carefully observing and formulating the problem, including gathering all relevant data. Then construct a scientific model to represent the real problem ,while explaining its objectives with the system constraints.

- It attempts to resolve the conflicts of interest among the components of the organization in a way that is best for the organization as a whole.

HISTORY OF OPERATIONS RESEARCH

Operations Research came into existence during World War II, when the British and American military management called upon a group of scientists with diverse educational background namely, Physics, Biology, Statistics, Mathematics, Psychology, etc., to develop and apply a scientific approach to deal with strategic and tactical problems of various military operations.

HISTORY OF OPERATIONS RESEARCH

The objective was to allocate scarce resources in an effective manner to various military operations and to the activities within each operation. The name Operations Research (OR) came directly from the context in which it was used and developed, viz., research on military operations

During the 50s, Operations Research achieved recognition as a subject for study in the universities. Since then the subject has gained increasing importance for the students of Management, Public Administration, Behavioral Sciences, Engineering, Mathematics, Economics and Commerce, etc. Today, Operations Research is also widely used in regional planning, transportation, public health, communication etc., besides military and industrial operations.

In India, Operations Research came into existence in 1949 with the opening of an Operations Research Unit at the Regional Research Laboratory at Hyderabad and also in the Defence Science Laboratory at Delhi which devoted itself to the problems of stores, purchase and planning. For national planning and survey, an Operations Research Unit was established in 1953 at the India Statistical Institute, Calcutta. In 1957, Operations Research Society of India was formed. Almost all the universities and institutions in India are offering the input of Operations Research in their curriculum .

Definition

- **Operations Research (OR)**

  It is a scientific approach to determine the optimum (best) solution to a decision problem under the restriction of limited resources, using the mathematical techniques to model, analyze, and solve the problem

Phases of OR

- Definition of the problem
- Model Construction
- Solution of the model
- Model validity
- Implementation of the solution

Basic components of the model

1. Decision Variables
2. Objective Function
3. Constraints

Example 1:

- A company manufactures two products A&B. with 4 & 3 units of price. A&B take 3&2 minutes respectively to be machined. The total time available at machining department is 800 hours (100 days or 20 weeks). A market research showed that at least 10000 units of A and not more than 6000 units of B are needed. It is required to determine the number of units of A&B to be produced to maximize profit.

- Decision variables

  X1 = number of units produced of A.

  X2 = number of units produced of B.

- Objective Function

  Maximize $Z = 4X_1 + 3X_2$

- Constraints

$$
\begin{aligned}
3X_1 + 2X_2 &\le 800 \times 60 \\
X_1 &\ge 10000 \\
X_2 &\le 6000 \\
X_1,\ X_2 &\ge 0
\end{aligned}
$$

Example 2: Feed mix problem

- A farmer is interested in feeding his cattle at minimum cost. Two feeds are used A&B. Each cow must get at least 400 grams/day of protein, at least 800 grams/day of carbohydrates, and not more than 100 grams/day of fat. Given that A contains 10% protein, 80% carbohydrates and 10% fat while B contains 40% protein, 60% carbohydrates and no fat. A costs Rs 20/kg, and B costs Rs 50 /kg. Formulate the problem to determine the optimum amount of each feed to minimize cost.

- Decision variables

  X1 = weight of feed A kg/day/animal

  X2 = weight of feed B kg/day/animal

- Objective Function

  Minimize $Z = 20X_1 + 50X_2$

- Constraints

Cost function

Protein	$0.1\,X_1 + 0.4\,X_2$	$\geq$	0.4
Carbohydrates	$0.8\,X_1 + 0.6\,X_2$	$\geq$	0.8
Fats	$0.1\,X_1$	$\leq$	0.1
	$X_1, \quad X_2$	$\geq$	0

Example 3: Blending Problem

- An iron ore from 4 mines will be blended. The analysis has shown that, in order to obtain suitable tensile properties, minimum requirements must be met for 3 basic elements A, B, and C. Each of the 4 mines contains different amounts of the 3 elements (see the table). Formulate to find the least cost (Minimize) blend for one ton of iron ore.

Problem Formulation

- **Decision variables**

  X1= Fraction of ton to be selected from mine number 1

  X2= Fraction of ton to be selected from mine number 2

  X3= Fraction of ton to be selected from mine number 3

  X4= Fraction of ton to be selected from mine number 4

- **Objective Function**

  Minimize Z= 800 X1 + 400 X2 + 600 X3 + 500 X4

- **Constraints**

$10 X_1 + 3X_2 + 8X_3 + 2X_4 \quad 5$

$90 X_1 + 150 X_2 + 75 X_3 + 175 X_4 \quad 10 \geq$

$45 X_1 + 25 X_2 + 20 X_3 + 37 X_4 \quad 30 \geq$

$X_1 + X_2 + X_3 + X_4 \quad 1 \quad \geq$

$X_1, X_2, X_3, X_4 \quad \underline{0}$

$\geq$

Example 4: Inspection Problem

- A company has 2 grades of inspectors 1&2. It is required that at least 1800 pieces be inspected per 8 hour day. Grade 1 inspectors can check pieces at the rate of 25 per hour with an accuracy of 98%. Grade 2 inspectors can check at the rate of 15 pieces per hour with an accuracy of 95%. Grade 1 costs 4 L.E/hour, grade 2 costs 3 L.E/hour. Each time an error is made by an inspector costs the company 2 L.E. There are 8 grade 1 and 10 grade 2 inspectors available. The company wants to determine the optimal assignment of inspectors which will minimize the total cost of inspection/day.

Problem Formulation

- **Decision variables**

  X1= Number of grade 1 inspectors/day.

  X2= Number of grade 2 inspectors/day.

- **Objective Function**

  Cost of inspection = Cost of error + Inspector salary/day

  Cost of grade 1/hour = 4 + (2 X 25 X 0.02) = 5 L.E

  Cost of grade 2/hour = 3 + (2 X 15 X 0.05) = 4.5 L.E

  Minimize Z= 8 (5 X1 + 4.5 X2) = 40 X1 + 36 X2

- **Constraints**

$$X_1 \leq 8$$
$$X_2 \leq 10$$
$$8(25)\, X_1 + 8(15) X_2 \geq 1800$$
$$200\, X_1 + 120\, X_2 \geq 1800$$
$$X_1,\ X_2 \geq 0$$

Example 5: Trim-loss Problem.

- A company produces paper rolls with a standard width of 20 feet. Each special customer orders with different widths are produced by slitting the standard rolls. Typical orders are summarized in the following tables.

Order	Desired Width	Desired Number of Rolls
1	5	150
2	7	200
3	9	300

Possible knife settings

Required Width	Knife settings						Minimum Number of rolls
	X1	X2	X3	X4	X5	X6	
5	0	2	2	4	1	0	150
7	1	1	0	0	2	0	200
9	1	0	1	0	0	2	300
Trim loss/roll	4	3	1	0	1	2	

- Formulate to minimize the trim loss and the number of rolls needed to satisfy the order.

Problem Formulation

Decision variables

X_j = Number of standard rolls to be cut according to setting j $j = 1, 2, 3, 4, 5, 6$

- Number of 5 feet rolls produced $= 2X_2 + 2X_3 + 4X_4 + X_5$
- Number of 7 feet rolls produced $= X_1 + X_2 + 2X_5$
- Number of 9 feet rolls produced $= X_1 + X_3 + 2X_6$
- Let Y_1, Y_2, Y_3 be the number of surplus rolls of the 5, 7, 9 feet rolls thus
  - $Y_1 = 2X_2 + 2X_3 + 4X_4 + X_5 - 150$
  - $Y_2 = X_1 + X_2 + 2X_5 - 200$
  - $Y_3 = X_1 + X_3 + 2X_6 - 300$
- The total trim losses $= L(4X_1 + 3X_2 + X_3 + X_5 + 2X_6 + 5Y_1 + 7Y_2 + 9Y_3)$ *Where L is the length of the standard roll.

Objective Function

$$\text{Minimize } Z = L(4X_1 + 3X_2 + X_3 + X_5 + 2X_6 + 5Y_1 + 7Y_2 + 9Y_3)$$

Constraints

$$2X_2 + 2X_3 + 4X_4 + X_5 - Y_1 = 150$$

$$X_1 + X_2 + 2X_5 - Y_2 = 200$$

$$X_1 + X_3 + 2X_6 - Y_3 = 300$$

$$X1, X2, X3, X4, X5, X6 \geq 0$$

$$Y1, Y2, Y3 \geq 0$$

General form of a LP problem with m constraints and n decision variables is:

$$\text{Maximize } Z = C_1X_1 + C_2X_2 + \ldots + C_nX_n$$

- **Constraints**

$$A_{11}X_1 + A_{12}X_2 + \ldots\ldots\ldots\ldots\ldots + A_{1n}X_n \leq B_1$$
$$A_{21}X_1 + A_{22}X_2 + \ldots\ldots\ldots\ldots\ldots + A_{2n}X_n \leq B_2$$

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$$A_{m1}X_1 + A_{m2}X_2 + \ldots\ldots\ldots\ldots\ldots + A_{mn}X_n \leq B_m$$
$$X_1, X_2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, X_n \geq 0$$

OR

Maximize $z = \sum_{j=1}^{n} c_j x_j$

- **Constraints**

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad , i = 1,2,\ldots, m$$

$$x_j \geq 0 \quad , j = 1,2,\ldots, n$$

Terminology of solutions for a LP model:

- **A Solution**

  Any specifications of values of $X_1, X_2, \ldots, X_n$ is called a solution.

- **A Feasible Solution**

  Is a solution for which all the constraints are satisfied.

- **An Optimal Solution**

  Is a feasible solution that has the most favorable value of the objective function (largest of maximize or smallest for minimize)

Graphical Solution

- **Construction of the LP model**
  - *Example 1: The Reddy Mikks Company*

    Reddy Mikks produces both interior and exterior paints from two raw materials, M1&M2. The following table provides the basic data of the problem.

	Tons of raw material per ton of		Maximum daily availability (tons)
	Exterior paint	Interior paint	
Raw Material, M1	6	4	24
Raw Material, M2	1	2	6
Profit per ton ($1000)	5	4	

- A market survey indicates that the daily demand for interior paint cannot exceed that of exterior paint by more than 1 ton. Also, the maximum daily demand of interior paint is 2 ton.
- Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit

Problem Formulation

- **Decision variables**

  X_1= Tons produced daily of exterior paint.

  X_2= Tons produced daily of interior paint.

- **Objective Function**

  Maximize $Z= 5\ X_1 + 4\ X_2$

- **Constraints**

$$6\ X_1+4\ X_2 \leq 24$$
$$X_1+2\ X_2 \leq 6$$
$$-X_1+\ X_2 \leq 1$$
$$X_2 \leq 2$$
$$X_1,\quad X_2 \geq 0$$

- Any solution that satisfies all the constraints of the model is a *feasible solution*. For example, X1=3 tons and X2=1 ton is a feasible solution. We have an infinite number of feasible solutions, but we are interested in the optimum feasible solution that yields the maximum total profit.

Graphical Solution

- The graphical solution is valid only for two-variable problem .
- The graphical solution includes two basic steps:
    1. The determination of the solution space that defines the feasible solutions that satisfy all the constraints.
    2. The determination of the optimum solution from among all the points in the feasible solution space.

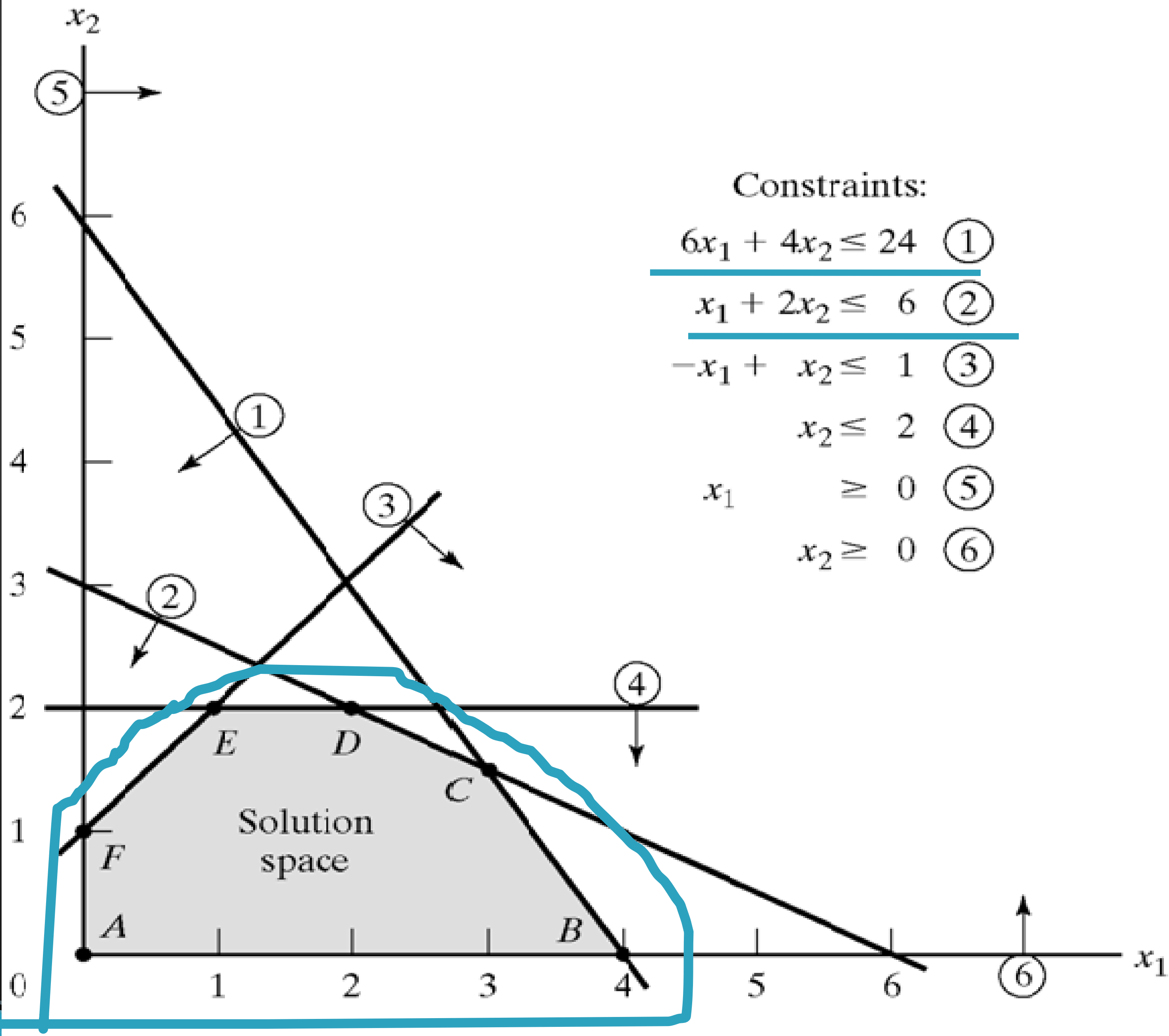


Figure 2.1

Feasible space of the Reddy Mikks model.

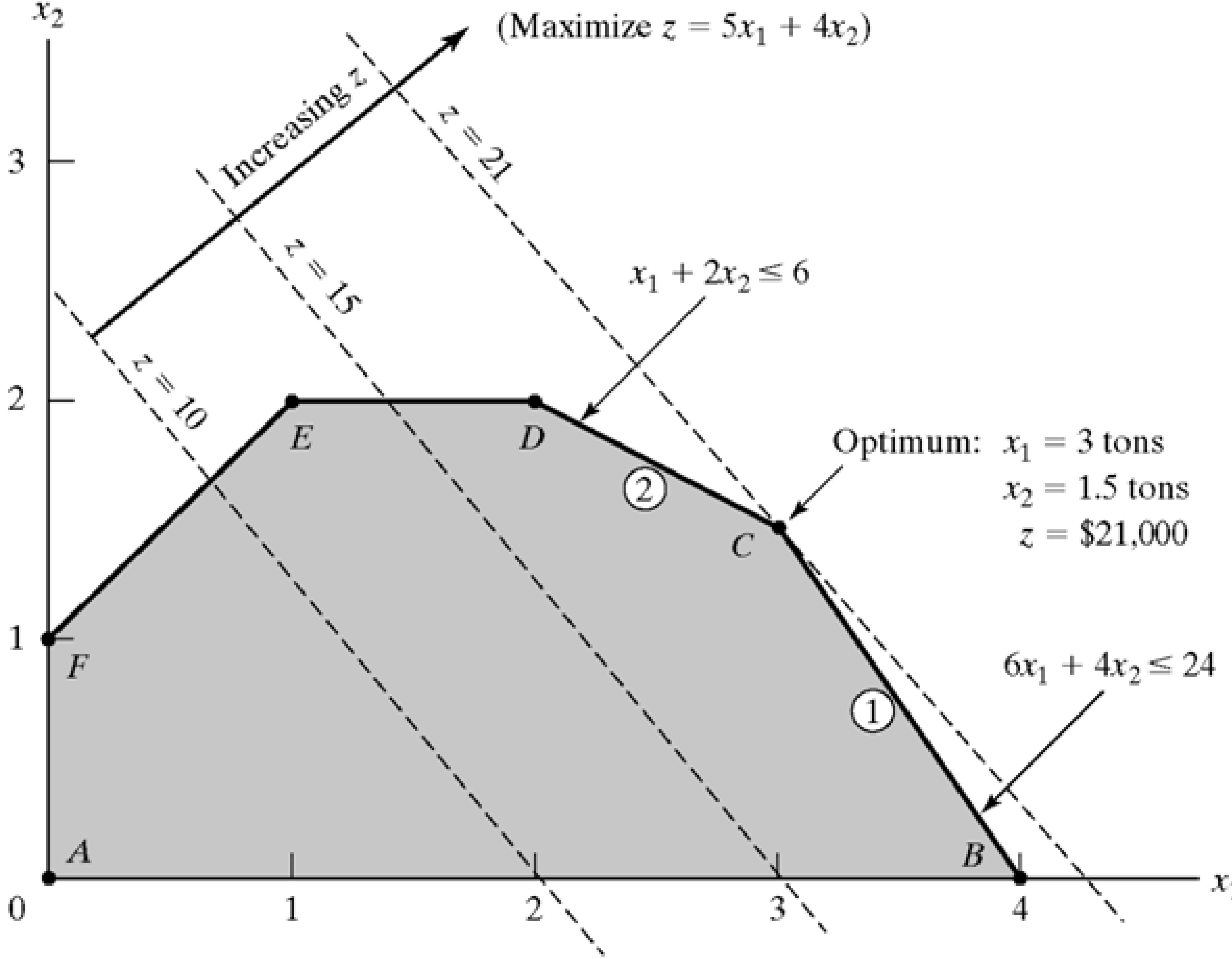


Figure 2.2
Optimum solution of the Reddy Mikks model.

- ABCDEF consists of an infinite number of points; we need a systematic procedure that identifies the optimum solutions. The optimum solution is associated with a corner point of the solution space.
- To determine the direction in which the profit function increases we assign arbitrary increasing values of 10 and 15

$$5\ X_1 + 4\ X_2 = 10$$

And $$5\ X_1 + 4\ X_2 = 15$$

- The optimum solution is mixture of 3 tons of exterior and 1.5 tons of interior paints will yield a daily profit of 21000$.

Unit – II: Transportation Problem

The Transportation Problem is a classic Operations Research problem where the objective is to determine the schedule for transporting goods from source to destination in a way that minimizes the shipping cost while satisfying supply and demand constraints.

Transportation Problem

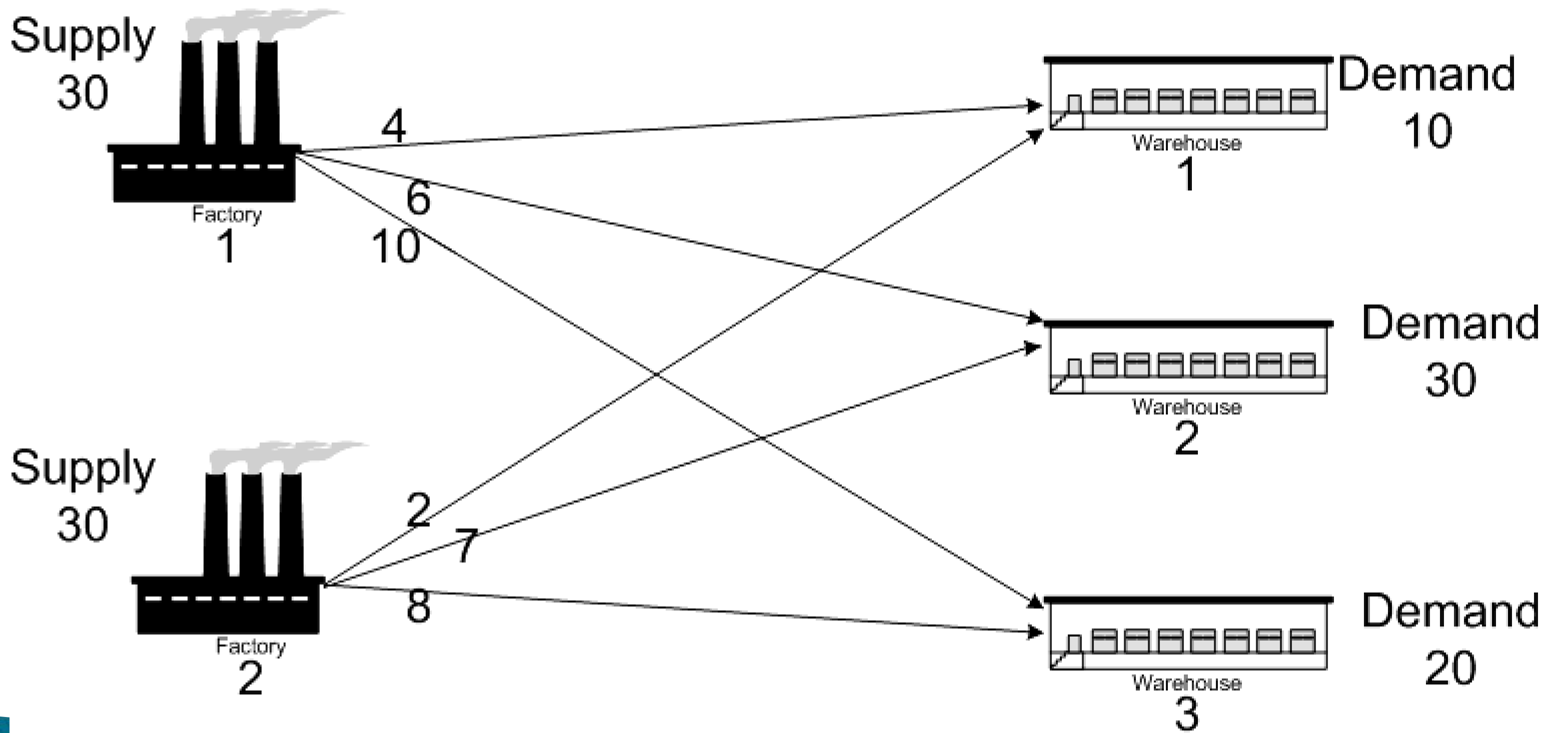
A typical Transportation Problem has the following elements:

1. Source(s)

2. Destination(s)

3. Weighted edge(s) showing cost of transportation

Transportation Problem



The Assignment Problem

- In many business situations, management needs to assign - personnel to jobs, - jobs to machines, - machines to job locations, or - salespersons to territories.
- Consider the situation of assigning *n* jobs to *n* machines.
- When a job i (=1,2,....,n) is assigned to machine j (=1,2, ...n) that incurs a cost Cij.
- The objective is to assign the jobs to machines at the least possible total cost.

The Assignment Problem

- This situation is a special case of the Transportation Model And it is known as the *assignment problem*.
- Here, jobs represent "sources" and machines represent "destinations."
- The supply available at each source is 1 unit And demand at each destination is 1 unit.

The Assignment Problem

$$\text{Min} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}$$

(Sum of assignments from a source should be exactly equal to 1):

$$\sum_{j=1}^{n} X_{ij} = 1 \qquad \text{For } i=1,2,\ldots,n$$

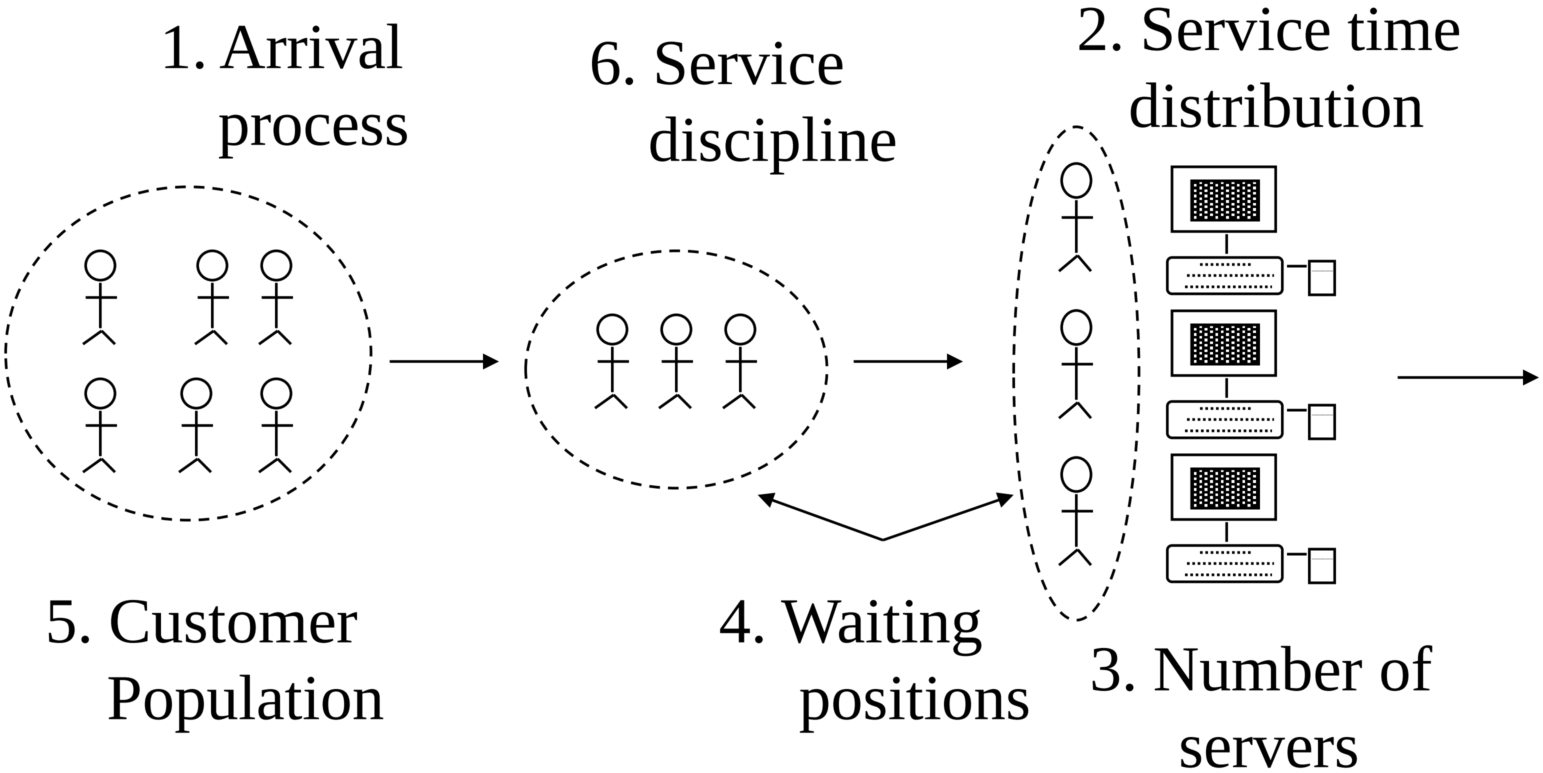
(Sum of assignments to a destination should be equal to the demanded quantity by that destination):

$$\sum_{i=1}^{n} X_{ij} = 1 \qquad \text{For } j=1,2,\ldots,n$$

(Quantities to be assigned can be either 0 or 1):

$$X_{ij} = 0 \text{ or } 1 \qquad \text{For all } i \text{ and } j.$$

Unit – III: Queuing Theory



Example: students at a typical computer terminal room with a number of terminals. If all terminals are busy, the arriving students wait in a queue.

Kendall Notation *A/S/m/B/K/SD*

- *A*: Arrival process
- *S*: Service time distribution
- *m*: Number of servers
- *B*: Number of buffers (system capacity)
- *K*: Population size, and
- *SD*: Service discipline

Arrival Process

- Arrival times: $t_1, t_2, \ldots, t_j$
- Interarrival times: $\tau_j = t_j - t_{j-1}$
- τ_j form a sequence of Independent and Identically Distributed (IID) random variables
- The most common arrival process: Poisson arrivals
  - Inter-arrival times are exponential + IID $\Rightarrow$ Poisson arrivals
- Notation:
  - M = Memoryless = Poisson
  - E = Erlang
  - H = Hyper-exponential
  - G = General $\Rightarrow$ Results valid for all distributions

Service Time Distribution

- Time each student spends at the terminal
- Service times are IID
- Distribution: M, E, H, or G
- Device = Service center = Queue
- Buffer = Waiting positions

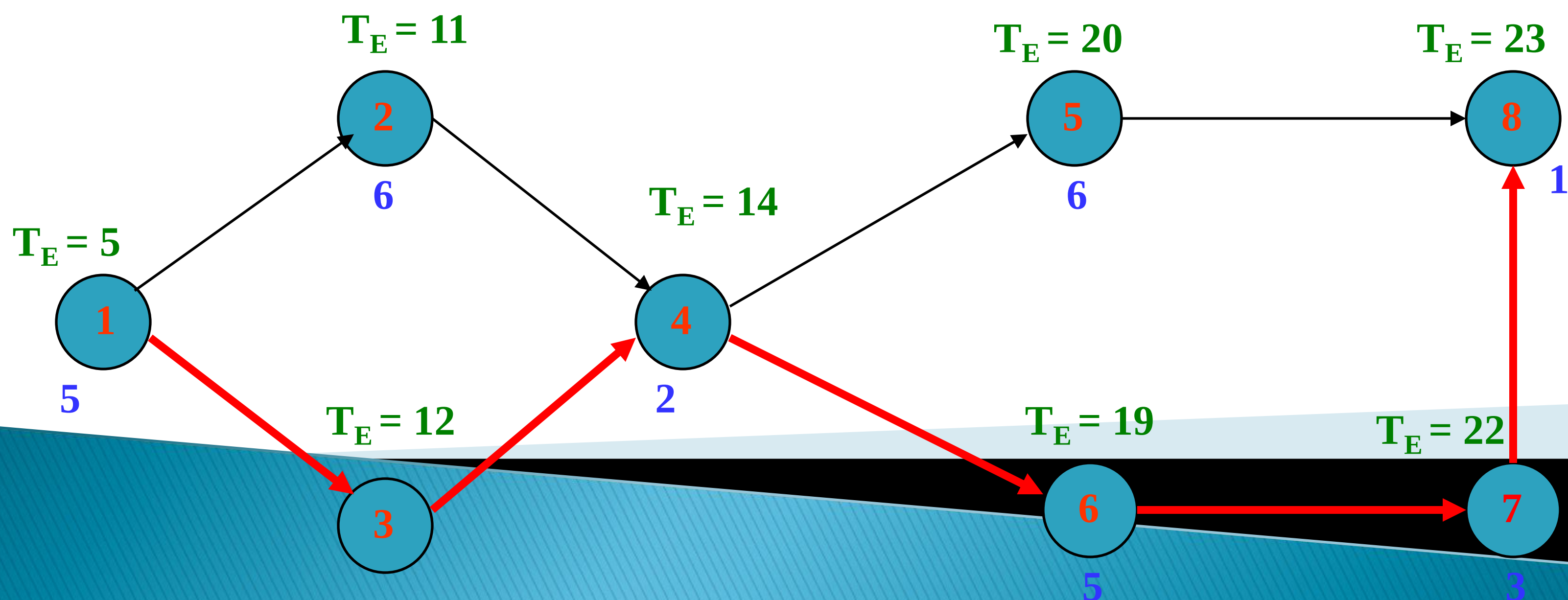
Service Disciplines

- First-Come-First-Served (FCFS)
- Last-Come-First-Served (LCFS)
- Last-Come-First-Served with Preempt and Resume (LCFS-PR)
- Round-Robin (RR) with a fixed quantum.
- Small Quantum $\Rightarrow$ Processor Sharing (PS)
- Infinite Server: (IS) = fixed delay
- Shortest Processing Time first (SPT)
- Shortest Remaining Processing Time first (SRPT)
- Shortest Expected Processing Time first (SEPT)
- Shortest Expected Remaining Processing Time first (SERPT).
- Biggest-In-First-Served (BIFS)
- Loudest-Voice-First-Served (LVFS)

Common Distributions

- M: Exponential
- E_k: Erlang with parameter k
- H_k: Hyper-exponential with parameter k
- D: Deterministic $\Rightarrow$ constant
- G: General $\Rightarrow$ All
- Memoryless:
  - Expected time to the next arrival is always $1/\lambda$ regardless of the time since the last arrival
  - Remembering the past history does not help

PERT/CPM Chart



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PERT/CPM Chart

Task. A project has been defined to contain the following list of activities along with their required times for completion:

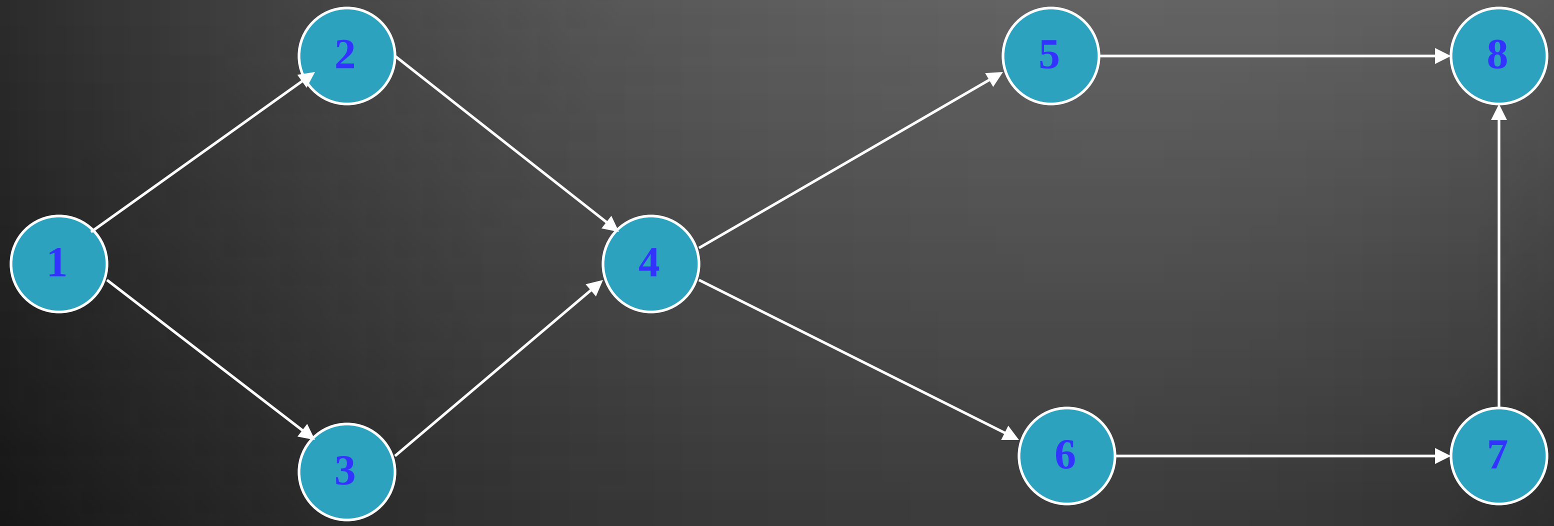
Activity No	Activity	Expected completion time	Dependency
1.	Requirements collection	5	-
2.	Screen design	6	1
3.	Report design	7	1
4.	Database design	2	2,3
5.	User documentation	6	4
6.	Programming	5	4
7.	Testing	3	6
8.	Installation	1	5,7

a. Draw a PERT chart for the activities.
b. Calculate the earliest expected completion time.
c. Show the critical path.

PERT/CPM Chart (cont'd)

a. Draw a PERT chart for the activities.

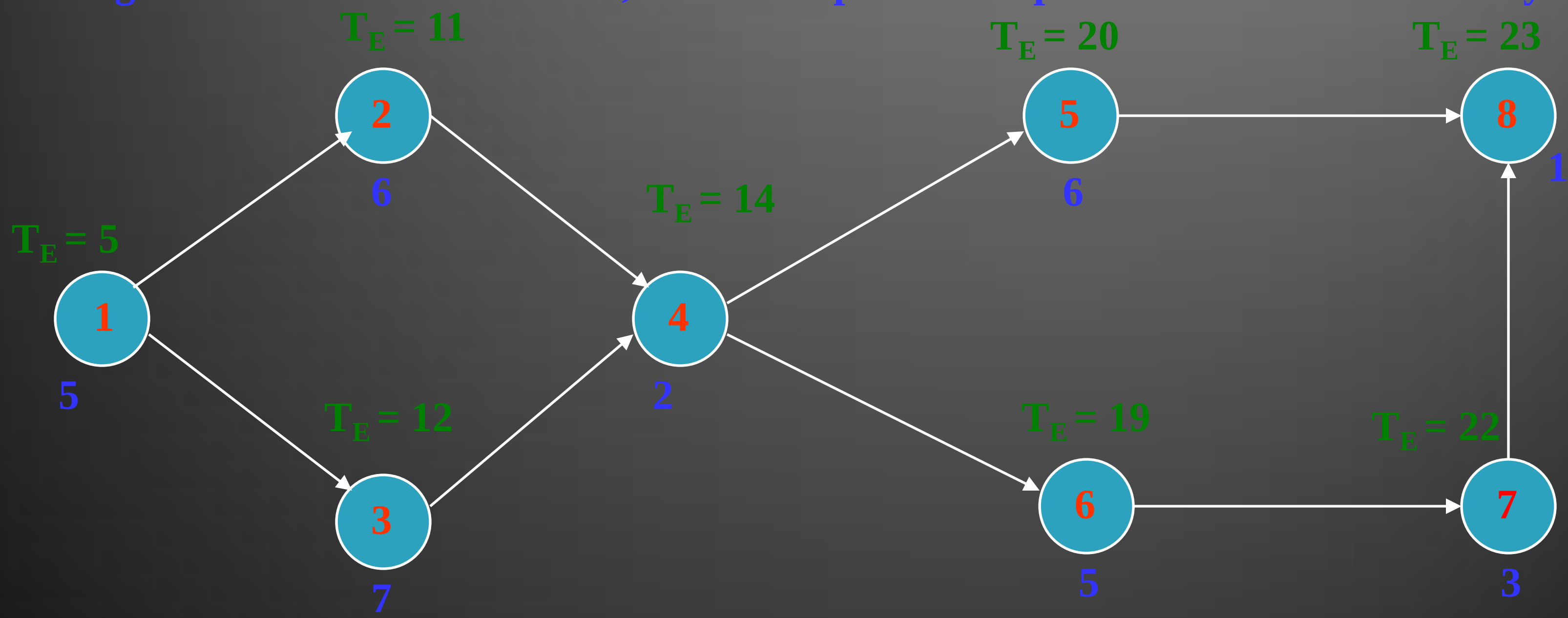
Using information from the table, show the sequence of activities.



PERT/CPM Chart (cont'd)

b. Calculate the earliest expected completion time.

1. Using information from the table, indicate expected completion time for each activity.



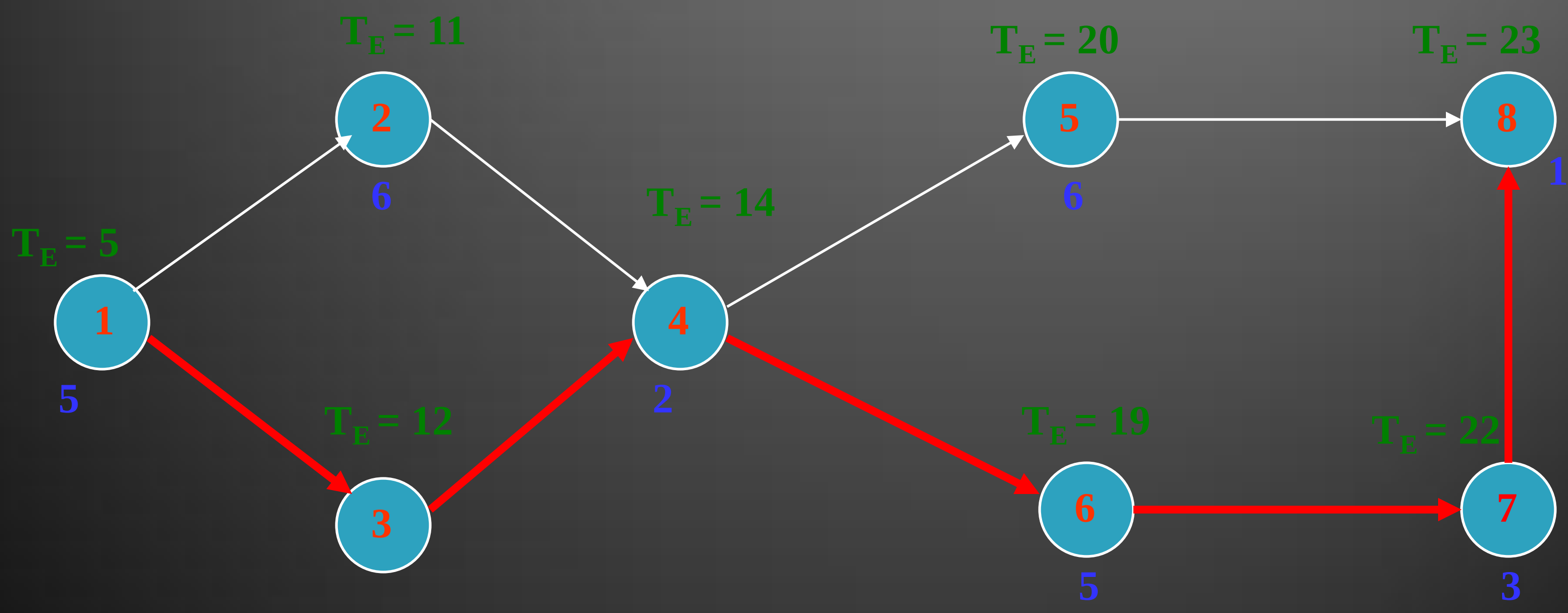
2. Calculate earliest expected completion time for each activity (T_E) and the entire project.

Hint: the earliest expected completion time for a given activity is determined by summing the expected completion time of this activity and the earliest expected completion time of the immediate predecessor.

Rule: if two or more activities precede an activity, the one with the largest T_E is used in calculation (e.g., for activity 4, we will use T_E of activity 3 but not 2 since 12 > 11).

PERT/CPM Chart (the end)

c. Show the critical path.



The critical path represents the shortest time, in which a project can be completed. Any activity on the critical path that is delayed in completion delays the entire project. Activities not on the critical path contain slack time and allow the project manager some flexibility in scheduling.

Scope of Operations Research

O.R is useful in the following various important fields.

1. <u>In Agriculture</u>:

   (i) Optimum allocation of land to various crops in accordance with the climatic conditions, and
   (ii) Optimum allocation of water from various resources like canal for irrigation purposes.

2. <u>In Finance</u>:

   (i) To maximize the per capita income with minimum resources
   (i) To find the profit plan for the country
   (ii) To determine the best replacement policies, etc.

Continued…

Scope of Operations Research

3. <u>In Industry</u>:

(i) *O.R is useful for optimum allocations of limited resources such as men materials, machines, money, time, etc. to arrive at the optimum decision.*

4. <u>In Marketing</u>:

With the help of O.R Techniques a marketing Administrator (manager) can decide where to distribute the products for sale so that the total cost of transportation etc. is minimum.

Continued…

Scope of Operations Research

Continuation…

(ii) The minimum per unit sale price
(iii) The size of the stock to meet the future demand
(iv) How to select the best advertising media with respect to time, cost etc.
(v) How when and what to purchase at the min. possible cost?

5. <u>In Personnel Management:</u>

(i) To appoint the most suitable persons on min. salary
(i) To determine the best age of retirement for the employees
(ii) To find out the number of persons to be appointed on full time basis when the work load is seasonal.

Continued…

Scope of Operations Research

Continuation…

6. In Production Management:

(i) To find out the number and size of the items to be produced

(ii) In scheduling and sequencing the production run by proper allocation of machines

(iii) In calculating the optimum product mix, and

(iv) To select, locate and design the sites for the production plants

7. <u>In L.I.C.:</u>

***(i) What** should be the premium rates for various modes of policies*

(ii) How best the profits could be distributed in the cases of with profit policies etc.

THE LINEAR PROGRAMMING PROBLEM

INTRODUCTION

A linear programming problem is a problem of minimizing or maximizing a linear function in the presence of linear constraints of the inequality and/or the equality type.

Consider the following linear programming problem.

$$
\begin{array}{llllllll}
\text{Minimize} & c_1x_1 + & c_2x_2 + & \cdots & + & c_nx_n & & \\
\text{Subject to} & a_{11}x_1 + & a_{12}x_2 + & \cdots & + & a_{1n}x_n & \geqslant & b_1 \\
 & a_{21}x_1 + & a_{22}x_2 + & \cdots & + & a_{2n}x_n & \geqslant & b_2 \\
 & \vdots & \vdots & & & \vdots & & \vdots \\
 & a_{m1}x_1 + & a_{m2}x_2 + & \cdots & + & a_{mn}x_n & \geqslant & b_m \\
 & x_1, & x_2, & \ldots, & & x_n & \geqslant & 0
\end{array}
$$

Here $c_1x_1 + c_2x_2 +, \ldots, + c_nx_n$ is the *objective function* (or criterion function) to be minimized and will be denoted by z. The coefficients $c_1, c_2, \ldots, c_n$ are the (known) *cost coefficients* and $x_1, x_2, \ldots, x_n$ are the *decision variables* (variables, or activity levels) to be determined. The inequality $\sum_{j=1}^{n} a_{ij}x_j \geqslant b_i$ denotes the ith *constraint* (or restriction). The coefficients a_{ij} for $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$ are called the *technological coefficients*. These technological coefficients form the *constraint matrix* **A** given below.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The column vector whose ith component is b_i, which is referred to as the *right-hand-side vector*, represents the minimal requirements to be satisfied. The constraints $x_1, x_2, \ldots, x_n \geqslant 0$ are the *nonnegativity constraints*. A set of variables $x_1, \ldots, x_n$ satisfying all the constraints is called a *feasible point* or a *feasible vector*. The set of all such points constitutes the *feasible region* or the *feasible space*.

Using the foregoing terminology, the linear programming problem can be

Linear Programming in Matrix Notation

A linear programming problem can be stated in a more convenient form using matrix notation. To illustrate, consider the following problem.

$$\text{Minimize} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{Subject to} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i \quad i = 1, 2, \ldots, m$$

$$x_j \geqslant 0 \quad j = 1, 2, \ldots, n$$

Denote the row vector $(c_1, c_2, \ldots, c_n)$ by $\mathbf{c}$, and consider the following column vectors $\mathbf{x}$ and $\mathbf{b}$, and the $m \times n$ matrix $\mathbf{A}$.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Then the above problem can be written as follows.

$$\begin{aligned} \text{Minimize} \quad & \mathbf{cx} \\ \text{Subject to} \quad & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0} \end{aligned}$$

The problem can also be conveniently represented via the columns of $\mathbf{A}$. Denoting $\mathbf{A}$ by $[\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$ where $\mathbf{a}_j$ is the jth column of $\mathbf{A}$, the problem can be formulated as follows.

Table 1.1 Standard and Canonical Forms

	MINIMIZATION PROBLEM	MAXIMIZATION PROBLEM
Standard Form	Minimize $\sum_{j=1}^{n} c_j x_j$ Subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i \quad i = 1, \ldots, m$ $x_j \geqslant 0 \quad j = 1, \ldots, n$	Maximize $\sum_{j=1}^{n} c_j x_j$ Subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i \quad i = 1, \ldots, m$ $x_j \geqslant 0 \quad j = 1, \ldots, n$
Canonical Form	Minimize $\sum_{j=1}^{n} c_j x_j$ Subject to $\sum_{j=1}^{n} a_{ij} x_j \geqslant b_i \quad i = 1, \ldots, m$ $x_j \geqslant 0 \quad j = 1, \ldots, n$	Maximize $\sum_{j=1}^{n} c_j x_j$ Subject to $\sum_{j=1}^{n} a_{ij} x_j \leqslant b_i \quad i = 1, \ldots, m$ $x_j \geqslant 0 \quad j = 1, \ldots, n$

Maximize $Z = 6X_1 + 8X_2$

Subject to $30X_1 + 20X_2 \leq 300$

$5X_1 + 10X_2 \leq 110$

And $X_1, X_2 \geq 0$

Method :

Step 1 : Convert the above inequality constraint into equality constraint by adding slack variables S1 and S2

The constraint equations are now

$30X1 + 20X2 + S1 = 300$, $S1 \geq 0$

$5X1 + 10X2 \quad + S2 = 110$, $S2 \geq 0$

The LP problem in standard is now

$Z = 6X1 + 8X2 + 0 \times S1 + 0 \times S2$

$30X1 + 20X2 + S1 = 300$

$5X1 + 10X2 \quad + S2 = 110$

And $X1, X2, S1, S2 \geq 0$

Variables with non-zero values are called basic variables.

Variables with zero values are called non-basic variables.

If there is no redundant constraint equation in the problem , there will be as many basic variables as many constraints, provided a basic feasible solution exists.

Step 2 : Form a table

Table I

Basic	Z	X1	X2	S1	S2	Solution	Ratio
Z	1	-6	-8	0	0	0	
S1	0	30	20	1	0	300	(300/20=15)
S2	0	5	10	0	1	110	(110/10=11)

Start with the current solution at the origin X1=0,X2=0

And therefore Z = 0. S1,S2 are the basic variables and X1,X2 are the non-basic variables.

- Z is 0 and it is not maximum . It has scope for improvement
- Z is found to be most sensitive to X2 since its coefficient is -8 ,so this is chosen as the pivot column .X2 **enter**s into the basic variable column. This becomes the pivot column.
- Search for leaving variable in the first column

by choosing the row which has the least value in the ratio column. It is S2 which leaves the basic variable

The modified table is now obtained by

1.New pivot row =current pivot row/pivot element

2. All other new row (including Z row) =

current row- its pivot column coefficient*new pivot-row

Table II

Basic	Z	X1	X2	S1	S2	Solution	Ratio
Z	1	-2	0	0	8/10	88	
S1	0	20	0	1	-2	80	(80/20 [illegible])
X2	8	5/10	1	0	1/10	11	(11/.5 =22)

Unit – IV: Game Theory

Z row has -2 in X1 column,hence there is scope for improvement in Z.

X1 now enters the basic variable and S1 row with least ratio of 4 will leave the basic variable .

Table III

Basic	Z	X1	X2	S1	S2	Solution	Ratio
Z	1	0	0	1/10	6/10	96	
X1	6	1	0	1/20	-1/10	4	
X2	8	0	1	1/20	3/20	9	

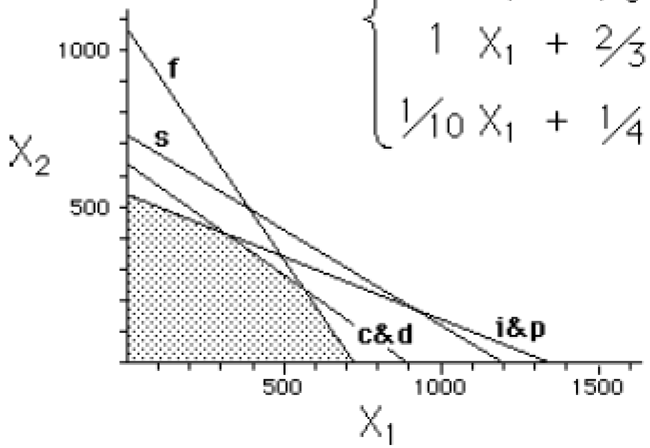
No negative coeff in Z-row for basic variable .0ptimal solution X1=4,X2=9 and Z=96

LP: Maximize $10 X_1 + 9 X_2$

$$\text{subject to} \begin{cases} \frac{7}{10} X_1 + 1 X_2 \le 630 \\ \frac{1}{2} X_1 + \frac{5}{6} X_2 \le 600 \\ 1 X_1 + \frac{2}{3} X_2 \le 708 \\ \frac{1}{10} X_1 + \frac{1}{4} X_2 \le 135 \\ X_1 \ge 0 \quad X_2 \ge 0 \end{cases}$$

a product-mix LP model

$$
\begin{array}{lr}
 & \textit{DEPT.} \\
\left\{ \begin{array}{l}
\frac{7}{10} X_1 + 1\, X_2 \le 630 \\
\frac{1}{2} X_1 + \frac{5}{6} X_2 \le 600 \\
1\, X_1 + \frac{2}{3} X_2 \le 708 \\
\frac{1}{10} X_1 + \frac{1}{4} X_2 \le 135
\end{array} \right. &
\begin{array}{l}
\textit{cut\&dye} \\
\textit{sewing} \\
\textit{finishing} \\
\textit{inspect\& pack}
\end{array}
\end{array}
$$



The shaded region satisfies all four inequality constraints!

Converting to "standard" LP model

Define "slack" variables

S_1 = unused hours in Cut-&-Dye Dept.

S_2 = unused hours in Sewing Dept.

S_3 = unused hours in Finishing Dept.

S_4 = unused hours in Inspect-&-Pack Dept.

and

Z = profit

By the introduction of the "slack" variables, the inequalities (with the exception of the non-negativity restrictions) become equations:

$$\text{Maximize} \quad 10X_1 + 9X_2 = Z$$

$$\text{subject to} \begin{cases} \tfrac{7}{10}X_1 + 1X_2 + S_1 = 630 \\ \tfrac{1}{2}X_1 + \tfrac{5}{6}X_2 + S_2 = 600 \\ 1X_1 + \tfrac{2}{3}X_2 + S_3 = 708 \\ \tfrac{1}{10}X_1 + \tfrac{1}{4}X_2 + S_4 = 135 \\ X_1 \geq 0 \quad X_2 \geq 0 \end{cases}$$

Simplex Method

Tableau

$-Z$	X_1	X_2	S_1	S_2	S_3	S_4		rhs
1	10	9	0	0	0	0	=	0
0	$7/10$	1	1	0	0	0	=	630
0	$1/2$	$5/6$	0	1	0	0	=	600
0	1	$2/3$	0	0	1	0	=	708
0	$1/10$	$1/4$	0	0	0	1	=	135

Notice that the system of equations represented by the tableau has essentially been "solved" for the variables Z, S_1, S_2, S_3, and S_4 in terms of the variables X_1 and X_2:

"basic" variables

$$\begin{cases} Z = 0 + 10X_1 + 9X_2 \\ S_1 = 630 - \frac{7}{10}X_1 - 1X_2 \\ S_2 = 600 - \frac{1}{2}X_1 - \frac{5}{6}X_2 \\ S_3 = 708 - 1X_1 - \frac{2}{3}X_2 \\ S_4 = 135 - \frac{1}{10}X_1 - \frac{1}{4}X_2 \end{cases}$$

X_1 and X_2 are parameters, or "nonbasic" variables

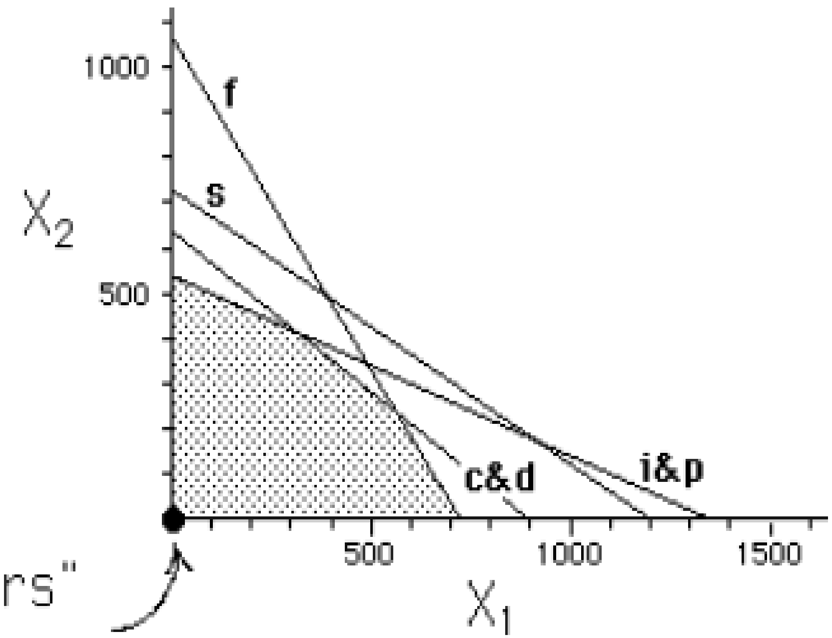
"complete" solution

$$
\begin{cases}
Z = 0 + 10X_1 + 9X_2 \\
S_1 = 630 - \tfrac{7}{10}X_1 - 1X_2 \\
S_2 = 600 - \tfrac{1}{2}X_1 - \tfrac{5}{6}X_2 \\
S_3 = 708 - 1X_1 - \tfrac{2}{3}X_2 \\
S_4 = 135 - \tfrac{1}{10}X_1 - \tfrac{1}{4}X_2
\end{cases}
$$

If we let the "nonbasic" variables X_1 & X_2 be zero, then we obtain a "**basic**" solution:

basic solution

$$
\begin{cases}
Z = 0 \ \$ \\
S_1 = 630 \ \textit{hrs.} \\
S_2 = 600 \ \textit{hrs.} \\
S_3 = 708 \ \textit{hrs.} \\
S_4 = 135 \ \textit{hrs.}
\end{cases}
$$



This basic solution
is one of the "corners"
of the feasible region,
which is a polyhedron.

Unit – V: Dynamic Programming

Looking at the PROFIT equation,

$$Z = 0 + 10X_1 + 9X_2$$

we see that this basic solution is not optimal, since an increase in ***either*** X_1 ***or*** X_2 results in an ***increase*** in the profit Z.

X1 contributes maximum in profit. This is selected for basic variable .

increase in X_1 results in a \$10 increase in Z (profit).

As X_1 is increased, the values of the basic variables S_1, S_2, S_3, and S_4 are also altered.

$$\begin{cases} S_1 = 630 - \frac{7}{10} X_1 - \ldots \\ S_2 = 600 - \frac{1}{2} X_1 - \ldots \\ S_3 = 708 - 1\ X_1 - \ldots \\ S_4 = 135 - \frac{1}{10} X_1 - \ldots \end{cases}$$

An **in**crease of 1 unit of X_1 $\Longrightarrow$

- $\frac{7}{10}$ unit **de**crease in S_1
- $\frac{1}{2}$ unit **de**crease in S_2
- 1 unit **de**crease in S_3
- $\frac{1}{10}$ unit **de**crease in S_4

"substitution rates"

An **in**crease of 1 unit of X_1 $\Longrightarrow$

$$\left\{ \begin{array}{ll} \tfrac{7}{10} & \text{unit } \textbf{de}\text{crease in } S_1 \\ \tfrac{1}{2} & \text{unit } \textbf{de}\text{crease in } S_2 \\ 1 & \text{unit } \textbf{de}\text{crease in } S_3 \\ \tfrac{1}{10} & \text{unit } \textbf{de}\text{crease in } S_4 \end{array} \right.$$

How much may X_1 be increased?

A further increase in X_1 is "blocked" when one of the (currently) basic variables reaches its lower bound (zero). To continue increasing X_1 would cause a violation in the nonnegativity of the basic variable.

$$\begin{cases} S_1 = 630 - \frac{7}{10} X_1 \geq 0 \\ S_2 = 600 - \frac{1}{2} X_1 \geq 0 \\ S_3 = 708 - 1\, X_1 \geq 0 \\ S_4 = 135 - \frac{1}{10} X_1 \geq 0 \end{cases} \Longrightarrow \begin{cases} \frac{7}{10} X_1 \leq 630 \\ \frac{1}{2} X_1 \leq 600 \\ 1\, X_1 \leq 708 \\ \frac{1}{10} X_1 \leq 135 \end{cases}$$

$$\Longrightarrow \begin{cases} X_1 \leq \dfrac{630}{7/10} \\ X_1 \leq \dfrac{600}{1/2} \\ X_1 \leq \dfrac{708}{1} \\ X_1 \leq \dfrac{135}{1/10} \end{cases} \Longrightarrow \begin{cases} X_1 \leq 900 \\ X_1 \leq 1200 \\ X_1 \leq 708 \\ X_1 \leq 1350 \end{cases}$$

least upper bound

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As we increase X_1 from zero, the first "block" occurs at **min**{900, 1200, 708, 1350} = 708, where S_3 becomes zero.

$$\begin{cases} X_1 \le 900 \\ X_1 \le 1200 \\ X_1 \le 708 \\ X_1 \le 1350 \end{cases}$$

least upper bound

We now wish to "re-solve" the system of equations so that X_1 is a basic variable and S_3 is nonbasic (and therefore zero).

Current tableau

$-Z$	X_1	X_2	S_1	S_2	S_3	S_4	rhs
1	10	9	0	0	0	0	0
0	$7/10$	1	1	0	0	0	630
0	$1/2$	$5/6$	0	1	0	0	600
0	1	$2/3$	0	0	1	0	708
0	$1/10$	$1/4$	0	0	0	1	135

"Pivot" on the element in the column of the new basic variable and the blocking row.

PIVOT

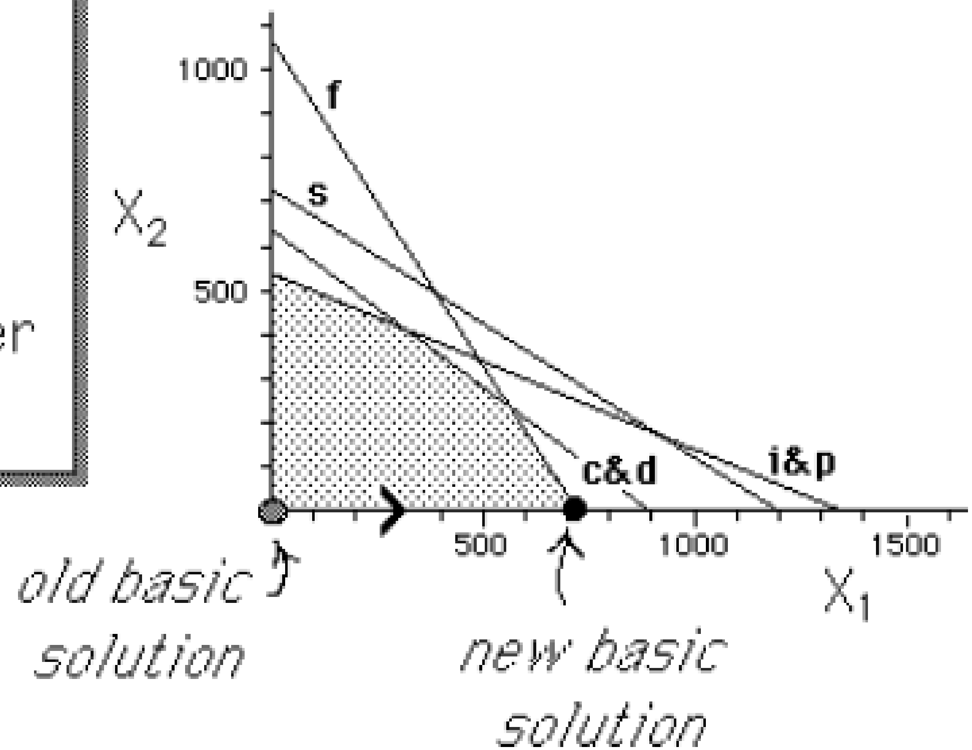
Subtract $10 \times$ ROW4 from ROW1
Subtract $(7/10)$ROW4 from ROW2
Subtract $(1/2)$ROW4 from ROW3
Subtract $(1/10)$ROW4 from ROW5

New tableau resulting from the pivot

$-Z$	X_1	X_2	S_1	S_2	S_3	S_4	rhs
1	0	$7/3$	0	0	-10	0	-7080
0	0	$8/15$	1	0	$-7/10$	0	134.4
0	0	$1/2$	0	1	$-1/2$	0	246
0	1	$2/3$	0	0	1	0	708
0	0	$11/60$	0	0	$-1/10$	1	64.2

Basic Variables

A pivot corresponds to a move along an edge from one corner to another!



"complete" solution

$$\begin{cases} Z = 7080 + \frac{7}{3} X_2 - 10 S_3 \\ S_1 = 134.4 - \frac{8}{15} X_2 + \frac{7}{10} S_3 \\ S_2 = 246 - \frac{1}{2} X_2 + \frac{1}{2} S_3 \\ X_1 = 708 - \frac{2}{3} X_2 - 1 S_3 \\ S_4 = 64.2 - \frac{11}{60} X_2 + \frac{1}{10} S_3 \end{cases}$$

The ***basic*** solution corresponding to this choice of basic variables is ***different***, however:

$X_2 = 0$ and $S_3 = 0$ yield

$Z = 7080$ \$

and

$$\begin{cases} S_1 = 134.4 \text{ } \textit{hrs.} \\ S_2 = 246 \text{ } \textit{hrs.} \\ X_1 = 708 \text{ } \textit{bags} \\ S_4 = 64.2 \text{ } \textit{hrs.} \end{cases}$$

Note that the current basic solution is *still* not optimal, however, since increasing X_2 will further increase the profit:

$$Z = 7080 + \tfrac{7}{3} X_2 - 10\ S_3$$

"relative profit"

The coefficient of a variable in the equation for the profit, Z, is called the "relative profit"

The variable X_2 is the *only* nonbasic variable with a positive relative profit, so we will select it to be increased.

$$\begin{cases} S_1 = 134.4 - \frac{8}{15} X_2 + \ldots \\ S_2 = 246 - \frac{1}{2} X_2 + \ldots \\ X_1 = 708 - \frac{2}{3} X_2 - \ldots \\ S_4 = 64.2 - \frac{11}{60} X_2 + \ldots \end{cases}$$

substitution rates

$$\begin{bmatrix} \frac{8}{15} \\ \frac{1}{2} \\ \frac{2}{3} \\ \frac{11}{60} \end{bmatrix}$$

As before, we will increase the nonbasic variable until one of the basic variables reaches its lower bound (zero), which "blocks" any further increase in X_2.

Nonnegativity of the basic variables provides bounds on X_2:

$$\begin{cases} S_1 = 134.4 - \frac{8}{15} X_2 \geq 0 \\ S_2 = 246 - \frac{1}{2} X_2 \geq 0 \\ X_1 = 708 - \frac{2}{3} X_2 \geq 0 \\ S_4 = 64.2 - \frac{11}{60} X_2 \geq 0 \end{cases} \Longrightarrow \begin{cases} X_2 \leq \dfrac{134.4}{8/15} = 252 \\ X_2 \leq \dfrac{246}{1/2} = 492 \\ X_2 \leq \dfrac{708}{2/3} = 1062 \\ X_2 \leq \dfrac{64.2}{11/60} = 350.18 \end{cases}$$

As soon as X_2 reaches the smallest of these bounds (in this case 252), any further increase is blocked, since it would force a basic variable (in this case S_1) to become negative!

Minimum Ratio Test

The increase of a nonbasic variable is blocked when it reaches the minimum of the ratios of right-hand-sides to ***positive*** substitution rates in the constraint rows.

The variable which is basic in the row with the minimum ratio will be replaced by the increased variable.

$-Z$	X_1	X_2	S_1	S_2	S_3	S_4	rhs
1	0	$7/3$	0	0	-10	0	-7080
0	0	$8/15$	1	0	$-7/10$	0	134.4
0	0	$1/2$	0	1	$-1/2$	0	246
0	1	$2/3$	0	0	1	0	708
0	0	$11/60$	0	0	$-1/10$	1	64.2

The pivot row is the row with the minimum ratio of rhs to (positive) substitution rate!

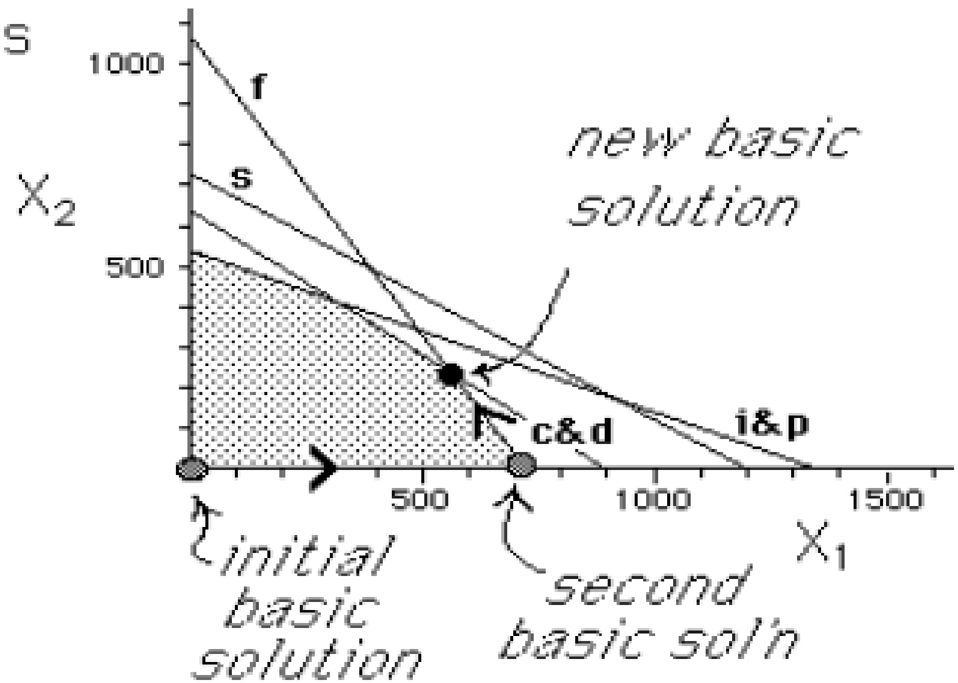
pivot column

Result of the pivot

$-Z$	X_1	X_2	S_1	S_2	S_3	S_4	rhs
1	0	0	$-35/8$	0	$-111/16$	0	-7668
0	0	1	$15/8$	0	$-21/16$	0	252
0	0	0	$-15/16$	1	$5/32$	0	120
0	1	0	$-10/8$	0	$15/8$	0	540
0	0	0	$-11/32$	0	$9/64$	1	18

A pivot corresponds to a move along an edge from one corner to an adjacent corner:

At this new basic solution, the nonbasic variables S_1 & S_3 are zero, i.e., the first and third constraints are "tight"



$$\boxed{\text{"complete" solution}}\quad \begin{cases} Z = 7668 - \frac{35}{8} S_1 - \frac{111}{16} S_3 \\ X_2 = 252 - \frac{15}{8} S_1 + \frac{21}{16} S_3 \\ S_2 = 120 + \frac{15}{16} S_1 - \frac{5}{32} S_3 \\ X_1 = 540 + \frac{10}{8} S_1 - \frac{15}{8} S_3 \\ S_4 = 18 + \frac{11}{32} S_1 - \frac{9}{64} S_3 \end{cases}$$

The basic solution corresponding to this choice of basis is to produce 540 STANDARD golf bags and 252 DELUXE golf bags, with 120 and 18 hours unused in the sewing and the inspect&pack depts., respectively.

Looking at the equation for PROFIT, we see that the "relative profits" of the nonbasic variables are both negative:

$$Z = 7668 - {}^{35}\!/_{8}\, S_1 - {}^{111}\!/_{16}\, S_3$$

This means that ***any*** positive values assigned to the variables S_1 and S_3 will result in a profit of ***less*** than $7668.

Therefore, the current basic solution ***must be optimal!***

Thank You