PROBABILITY AND STATISTICS

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CONTENTS

- Random Variables
- Probability Distributions
- Multiple Random Variables
- Sampling Distributions
- Large Sample Tests
- Small Sample Tests
- Queuing Theory
- Stochastic Process

TEXT BOOKS

- Higher Engineering Mathematics by Dr.B.S.Grewal,Khanna publishers
- Probability and Statistics for Engineering and Scientists by Sheldon M Ross,Academic press
- Operation Research by S.D.Sarma

REFERENCES

- Mathematics for Engineering by K.B.Datta and M.A.S.Srinivas, Cengage Publications
- Probability and Statistics by T.K.V.Iyengar & B.Krishna Gandhi Et
- Fundamentals of Mathematical Statistics by S C Gupta and V.K.Kapoor
- Probability and Statistics for Engineers and Scientists by Jay I Devore

UNIT-I

Single Random Variables and Probability Distributions

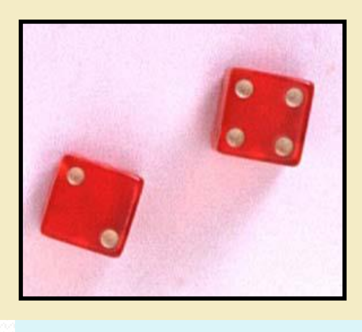
Basic Concepts

- An **experiment** is the process by which an observation (or measurement) is obtained.
- **Experiment: Record an age**
- **Experiment: Toss a die**
- **Experiment: Record an opinion (yes, no)**
- **Experiment: Toss two coins**

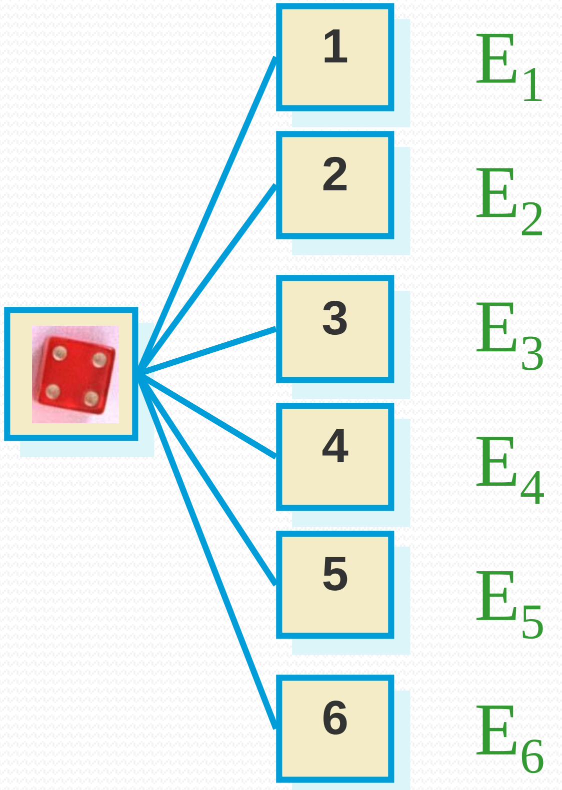
- A **simple event** is the outcome that is observed on a single repetition of the experiment.
  - The basic element to which probability is applied.
  - One and only one simple event can occur when the experiment is performed.
- A **simple event** is denoted by E with a subscript.

- Each simple event will be assigned a probability, measuring "how often" it occurs.
- The set of all simple events of an experiment is called the **sample space, S.**

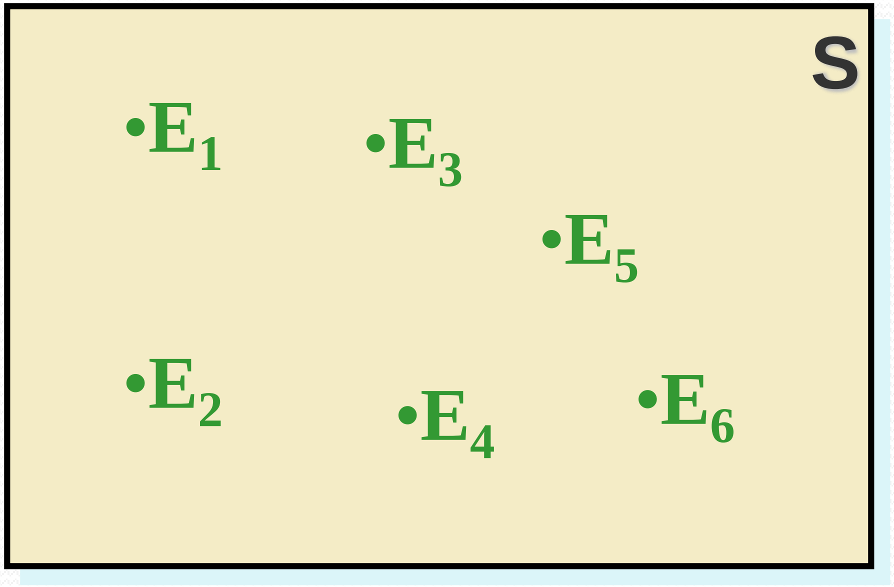
Example



- **The die toss:**
- Simple events: Sample space:



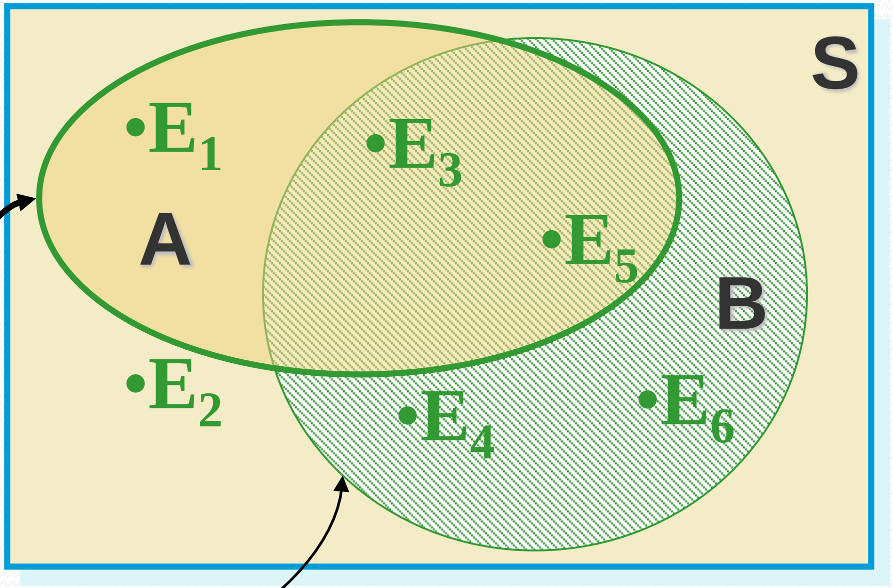
$$S = \{E_1, E_2, E_3, E_4, E_5, E_6\}$$



- An **event** is a collection of one or more **simple events.**

•The die toss:

- –A: an odd number
- –B: a number > 2



$A = \{E_1, E_3, E_5\}$

$B = \{E_3, E_4, E_5, E_6\}$

- Two events are **mutually exclusive** if, when one event occurs, the other cannot, and vice versa.

- **Experiment: Toss a die**
  - A: observe an odd number
  - B: observe a number greater than 2
  - C: observe a 6
  - D: observe a 3

Not Mutually Exclusive

Mutually Exclusive

B and C?
B and D?

- The probability of an event A measures "how often" we think A will occur. We write **P(A)**.
- Suppose that an experiment is performed n times. The relative frequency for an event A is

$$\frac{\text{Number of times A occurs}}{n} = \frac{f}{n}$$

- If we let n get infinitely large,

$$P(A) = \lim_{n \to \infty} \frac{f}{n}$$

- P(A) must be between 0 and 1.
  - If event A can never occur, P(A) = 0. If event A always occurs when the experiment is performed, P(A) =1.
- The sum of the probabilities for all simple events in S equals 1.

•The **probability of an event A** is found by adding the probabilities of all the simple events contained in A.

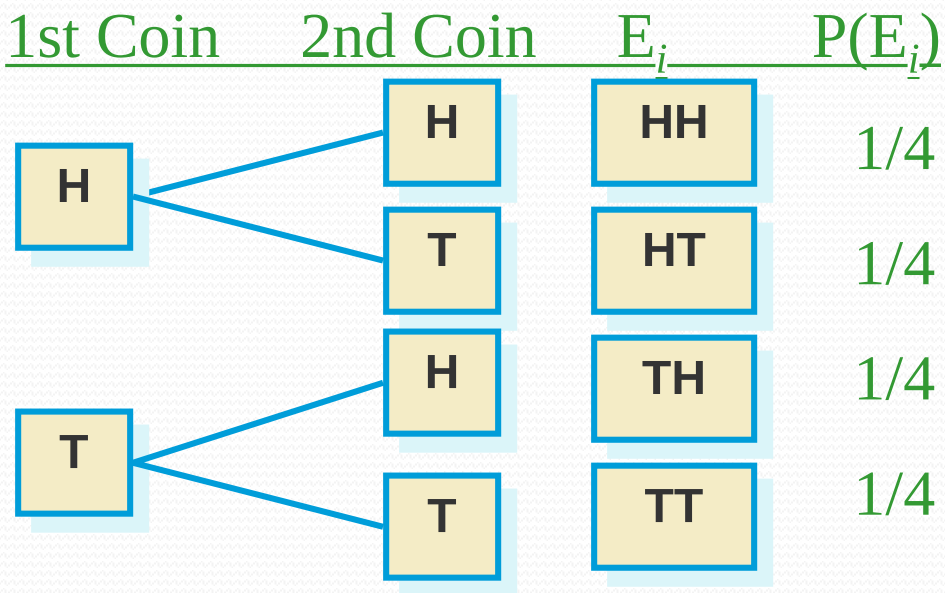
Finding Probabilities

- Probabilities can be found using
  - Estimates from empirical studies
  - Common sense estimates based on equally likely events.

- **Examples:**
  - –Toss a fair coin. P(Head) = 1/2
  - –10% of the U.S. population has red hair. Select a person at random. P(Red hair) = .10

Example

- Toss a fair coin twice. What is the probability of observing at least one head?



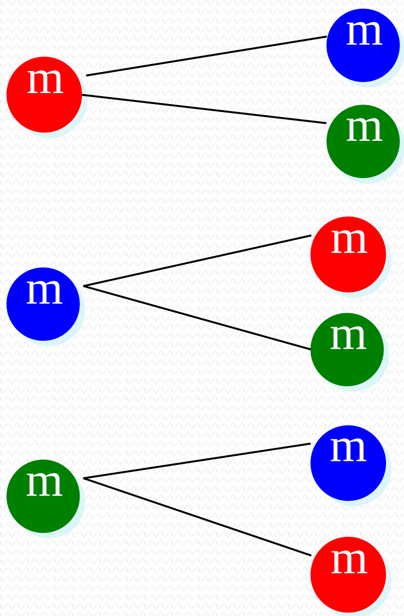
P(at least 1 head)

$= P(E_1) + P(E_2) + P(E_3)$

$= 1/4 + 1/4 + 1/4 = 3/4$

Example

- A bowl contains three M&Ms®, one red, one blue and one green. A child selects two M&Ms at random. What is the probability that at least one is red?



1st M&M	2nd M&M	E_i	$P(E_i)$
		RB	1/6
		RG	1/6
		BR	1/6
		BG	1/6
		GB	1/6
		GR	1/6

P(at least 1 red)

= P(RB) + P(BR)+ P(RG) + P(GR)

= 4/6 = 2/3

Counting Rules

- If the simple events in an experiment are **equally likely**, you can calculate

$$P(A) = \frac{n_A}{N} = \frac{\text{number of simple events in A}}{\text{total number of simple events}}$$

- You can use **counting rules** to find n_A and N.

The *mn* Rule

- If an experiment is performed in two stages, with ***m*** ways to accomplish the first stage and ***n*** ways to accomplish the second stage, then there are ***mn*** ways to accomplish the experiment.
- This rule is easily extended to ***k*** stages, with the number of ways equal to

$$n_1 n_2 n_3 \ldots n_k$$

Example: Toss two coins. The total number of simple events is:

$$2 \times 2 = 4$$

Examples

Example: Toss three coins. The total number of simple events is:

$2 \times 2 \times 2 = 8$

Example: Toss two dice. The total number of simple events is:

$6 \times 6 = 36$

Example: Two M&Ms are drawn from a dish containing two red and two blue candies. The total number of simple events is:

$4 \times 3 = 12$

Permutations

- The number of ways you can arrange ***n*** distinct objects, taking them ***r*** at a time is

$$P_r^n = \frac{n!}{(n-r)!}$$

where $n! = n(n-1)(n-2)...(2)(1)$ and $0! \equiv 1.$

Example: How many 3-digit lock combinations can we make from the numbers 1, 2, 3, and 4?

The order of the choice is important!

$$P_3^4 = \frac{4!}{1!} = 4(3)(2) = 24$$

Combinations

- The number of distinct combinations of ***n*** distinct objects that can be formed, taking them ***r*** at a time is

$$C_r^n = \frac{n!}{r!(n-r)!}$$

Example: Three members of a 5-person committee must be chosen to form a subcommittee. How many different subcommittees could be formed?

The order of the choice is not important!

$$C_3^5 = \frac{5!}{3!(5-3)!} = \frac{5(4)(3)(2)1}{3(2)(1)(2)1} = \frac{5(4)}{(2)1} = 10$$

Example

- A box contains six M&Ms®, four red
- and two green. A child selects two M&Ms at random. What is the probability that exactly one is red?

The order of the choice is not important!

$$C_2^6 = \frac{6!}{2!4!} = \frac{6(5)}{2(1)} = 15$$

ways to choose 2 M & Ms.

$$C_1^2 = \frac{2!}{1!1!} = 2$$

ways to choose 1 green M & M.

$$C_1^4 = \frac{4!}{1!3!} = 4$$

ways to choose 1 red M & M.

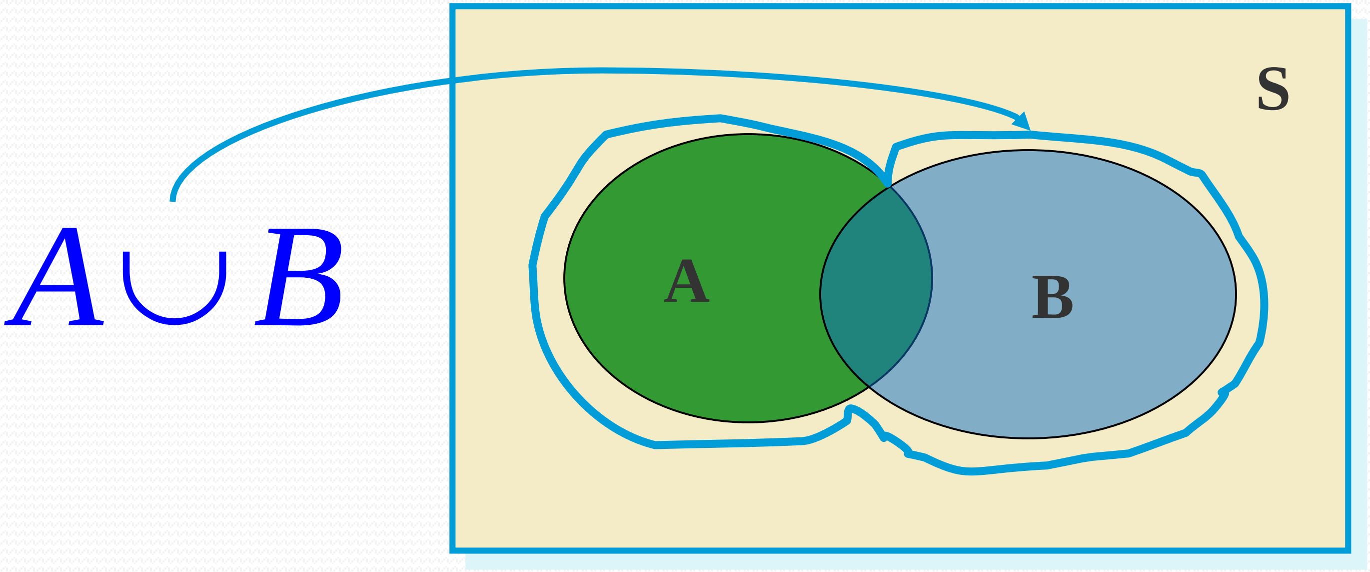
$4 \times 2 = 8$ ways to choose 1 red and 1 green M&M.

P(exactly one red) = 8/15

Event Relations

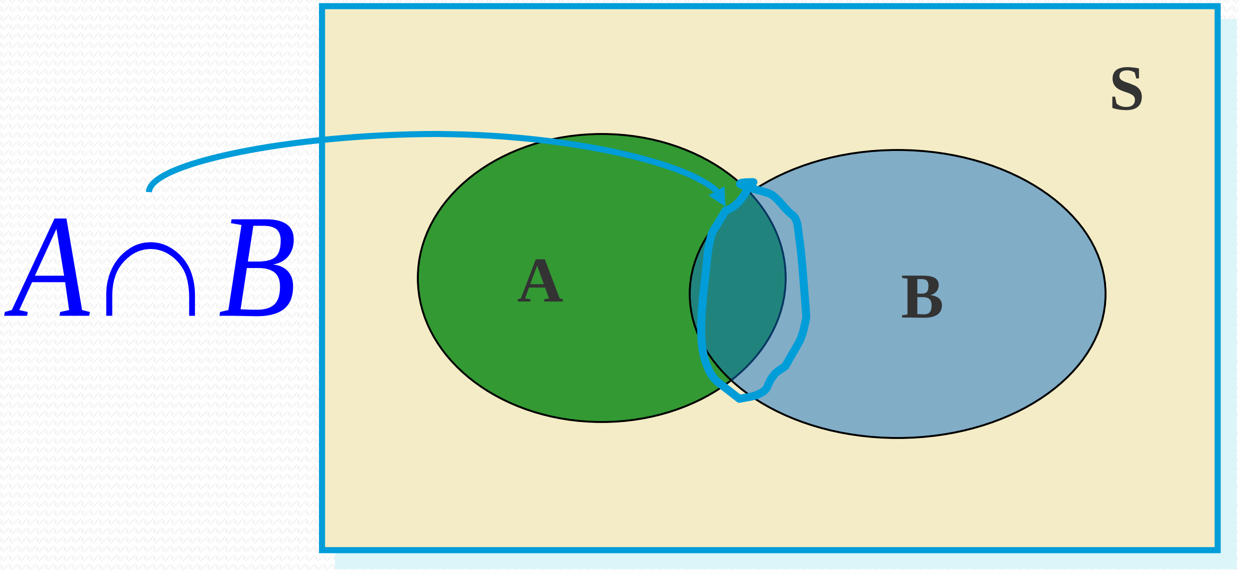
- The **union** of two events, A and B, is the event that either A **or** B **or both** occur when the experiment is performed. We write

$$A \cup B$$



Event Relations

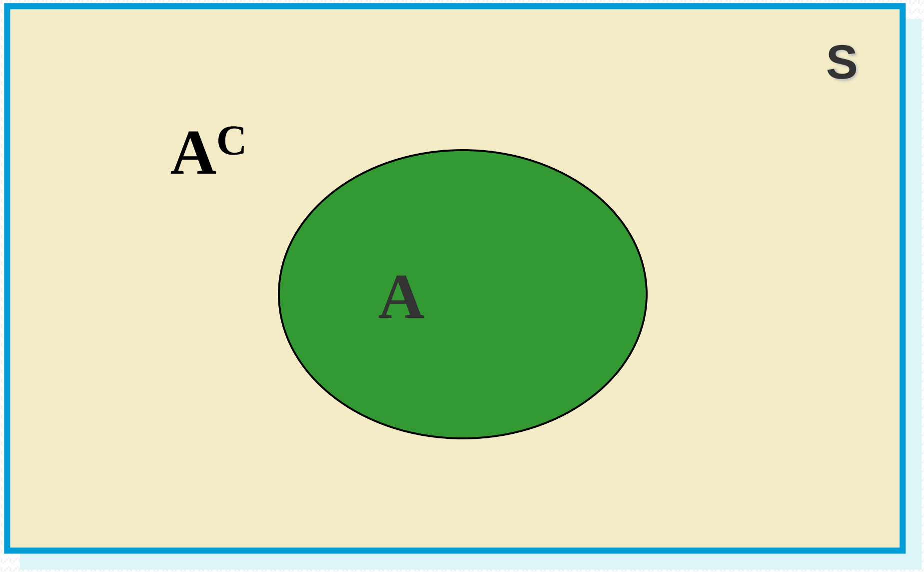
- The **intersection** of two events, **A** and **B**, is the event that both A **and** B occur when the experiment is performed. We write **A ∩ B.**



- If two events A and B are **mutually exclusive,** then P(**A ∩ B) = 0.**

Event Relations

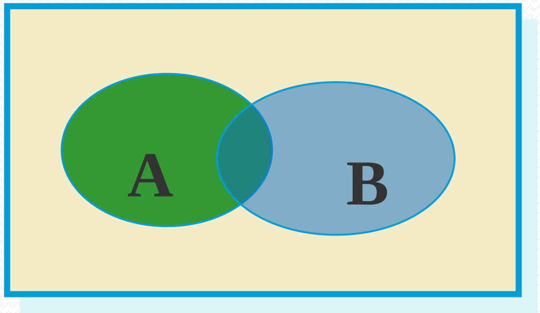
- The **complement** of an event **A** consists of all outcomes of the experiment that do not result in event A. We write A^C.



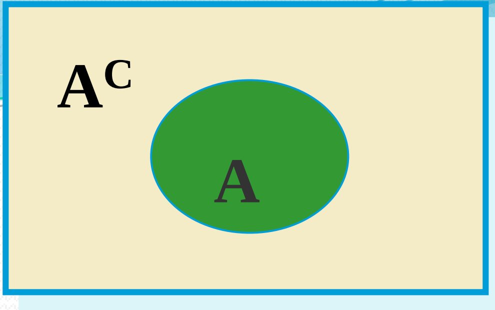
Calculating Probabilities for Unions and Complements

- There are special rules that will allow you to calculate probabilities for composite events.
- **The Additive Rule for Unions:**
- For any two events, **A** and **B**, the probability of their union, **P(A ∪ B)**, is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Calculating Probabilities for Complements



- We know that for any event **A**:
  - $P(A \cap A^C) = 0$
- Since either A or A^C must occur,

  $P(A \cup A^C) = 1$
- so that $P(A \cup A^C) = P(A) + P(A^C) = 1$

$$P(A^C) = 1 - P(A)$$

Calculating Probabilities for Intersections

- In the previous example, we found $P(A \cap B)$ directly from the table. Sometimes this is impractical or impossible. The rule for calculating $P(A \cap B)$ depends on the idea of **independent and dependent events.**

Two events, **A** and **B**, are said to be **independent** if and only if the probability that event **A** occurs does not change, depending on whether or not event **B** has occurred.

Conditional Probabilities

- The probability that A occurs, given that event B has occurred is called the **conditional probability** of A given B and is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \text{ if } P(B) \neq 0$$

"given"

Defining Independence

- We can redefine independence in terms of conditional probabilities:

Two events A and B are **independent** if and only if

$$P(A|B) = P(A) \quad \text{or} \quad P(B|A) = P(B)$$

Otherwise, they are **dependent**.

- Once you've decided whether or not two events are independent, you can use the following rule to calculate their intersection.

The Multiplicative Rule for Intersections

- For any two events, **A** and **B**, the probability that both **A** and **B** occur is

P(A ∩ B) = P(A) P(B given that A occurred) = P(A)P(B|A)

$$P(A \cap B) = P(A)\,P(B \mid A)$$

- If the events **A** and **B** are independent, then the probability that both **A** and **B** occur is

P(A ∩ B) = P(A) P(B)

$$P(A \cap B) = P(A)\,P(B)$$

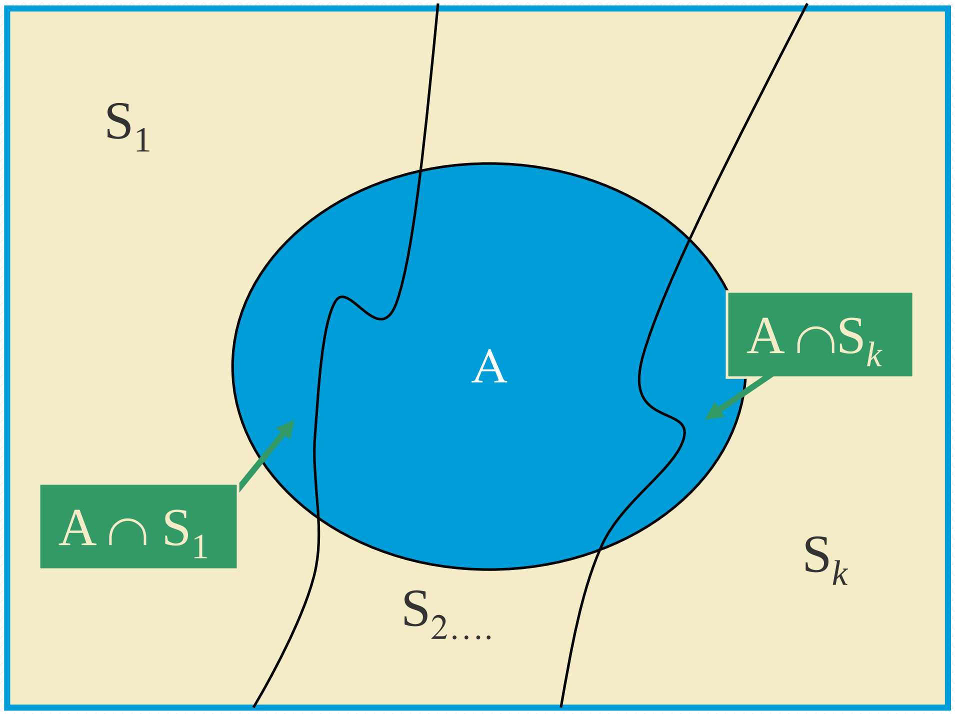
The Law of Total Probability

- Let S_1 , S_2 , S_3 ,…, S_k be mutually exclusive and exhaustive events (that is, one and only one must happen). Then the probability of another event A can be written as

$$P(A) = P(A \cap S_1) + P(A \cap S_2) + \ldots + P(A \cap S_k)$$

$$= P(S_1)P(A|S_1) + P(S_2)P(A|S_2) + \ldots + P(S_k)P(A|S_k)$$

The Law of Total Probability



$$P(A) = P(A \cap S_1) + P(A \cap S_2) + \ldots + P(A \cap S_k)$$

$$= P(S_1)P(A|S_1) + P(S_2)P(A|S_2) + \ldots + P(S_k)P(A|S_k)$$

Bayes' Rule

- Let S_1 , S_2 , S_3 ,...., S_k be mutually exclusive and exhaustive events with prior probabilities $P(S_1)$, $P(S_2)$,...,$P(S_k)$. If an event A occurs, the posterior probability of S_i, given that A occurred is

$$P(S_i \mid A) = \frac{P(S_i)P(A \mid S_i)}{\sum P(S_i)P(A \mid S_i)} \text{ for } i = 1, 2, \ldots k$$

Random Variables

- A quantitative variable *x* is a **random variable** if the value that it assumes, corresponding to the outcome of an experiment is a chance or random event.
- Random variables can be **discrete** or **continuous.**

- **Examples:**
  - ✓*x* = SAT score for a randomly selected student
  - ✓*x* = number of people in a room at a randomly selected time of day
  - ✓*x* = number on the upper face of a randomly tossed die

Probability Distributions for Discrete Random Variables

- The **probability distribution for a discrete random variable *x*** resembles the relative frequency distributions we constructed in Chapter 1. It is a graph, table or formula that gives the possible values of *x* and the probability *p(x)* associated with each value.

We must have

$$0 \le p(x) \le 1 \text{ and } \sum p(x) = 1$$

Probability Distributions

- Probability distributions can be used to describe the population, just as we described samples in Chapter 1.
  - **Shape:** Symmetric, skewed, mound-shaped…
  - **Outliers:** unusual or unlikely measurements
  - **Center and spread:** mean and standard deviation. A population mean is called μ and a population standard deviation is called σ.

The Mean and Standard Deviation

- Let *x* be a discrete random variable with probability distribution *p(x)*. Then the mean, variance and standard deviation of *x* are given as

$$\text{Mean}: \mu = \sum x p(x)$$

$$\text{Variance}: \sigma^2 = \sum (x-\mu)^2 p(x)$$

$$\text{Standard deviation}: \sigma = \sqrt{\sigma^2}$$

Example

- Toss a fair coin 3 times and record x the number of heads.

x	$p(x)$	$xp(x)$	$(x-\mu)^2 p(x)$
0	1/8	0	$(-1.5)^2(1/8)$
1	3/8	3/8	$(-0.5)^2(3/8)$
2	3/8	6/8	$(0.5)^2(3/8)$
3	1/8	3/8	$(1.5)^2(1/8)$

$$\mu = \sum xp(x) = \frac{12}{8} = 1.5$$

$$\sigma^2 = \sum (x-\mu)^2 p(x)$$

$$\sigma^2 = .28125 + .09375 + .09375 + .28125 = .75$$

$$\sigma = \sqrt{.75} = .688$$

Introduction

- Discrete random variables take on only a finite or countably number of values.
- Three discrete probability distributions serve as models for a large number of practical applications:

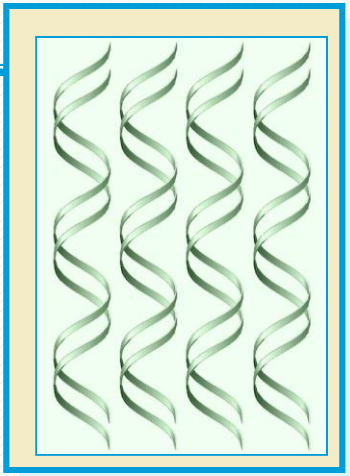
✓The **binomial** random variable

✓The **Poisson** random variable

✓The **hypergeometric** random variable

The Binomial Random Variable

- Many situations in real life resemble the coin toss, but the coin is not necessarily fair, so that $P(H) \neq 1/2$.



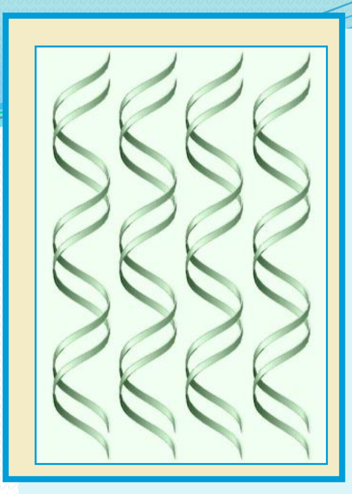
- Example: A geneticist samples 10 people and counts the number who have a gene linked to Alzheimer's disease.

- Coin: Person
- Head: Has gene
- Tail: Doesn't have gene
- Number of tosses: $n = 10$
- P(H): P(has gene) = proportion in the population who have the gene.

The Binomial Experiment

1. The experiment consists of ***n* identical trials.**
2. Each trial results in **one of two outcomes**, success (S) or failure (F).
3. The probability of success on a single trial is p and **remains constant** from trial to trial. The probability of failure is $q = 1 - p$.
4. The trials are **independent**.
5. We are interested in ***x*, the number of successes in *n* trials.**

Binomial or Not?



- Very few real life applications satisfy these requirements exactly.

- Select two people from the U.S. population, and suppose that 15% of the population has the Alzheimer's gene.
  - For the first person, p = P(gene) = .15
  - For the second person, $p \approx$ P(gene) = .15, even though one person has been removed from the population.

The Binomial Probability Distribution

- For a binomial experiment with n trials and probability p of success on a given trial, the probability of k successes in n trials is

$$P(x=k)=C_k^n p^k q^{n-k} = \frac{n!}{k!(n-k)!} p^k q^{n-k} \text{ for } k=0,1,2,\ldots n.$$

$$\text{Recall} \quad C_k^n = \frac{n!}{k!(n-k)!}$$

$$\text{with } n! = n(n-1)(n-2)\ldots(2)1 \text{ and } 0! \equiv 1.$$

The Mean and Standard Deviation

- For a binomial experiment with *n* trials and probability *p* of success on a given trial, the measures of center and spread are:

$$\text{Mean: } \mu = np$$

$$\text{Variance: } \sigma^2 = npq$$

$$\text{Standard deviation: } \sigma = \sqrt{npq}$$

Cumulative Probability Tables

You can use the cumulative probability tables to find probabilities for selected binomial distributions.

✓Find the table for the correct value of *n*.

✓Find the column for the correct value of *p*.

✓The row marked "*k*" gives the cumulative probability, $P(x \leq k) = P(x = 0) + \ldots + P(x = k)$

The Poisson Random Variable

- The Poisson random variable *x* is a model for data that represent the number of occurrences of a specified event in a given unit of time or space.

- Examples:
  - The number of calls received by a switchboard during a given period of time.
  - The number of machine breakdowns in a day
  - The number of traffic accidents at a given intersection during a given time period.

The Poisson Probability Distribution

- ***x*** is the number of events that occur in a period of time or space during which an average of **μ** such events can be expected to occur. The probability of *k* occurrences of this event is

$$P(x=k)=\frac{\mu^k e^{-\mu}}{k!}$$

For values of *k* = 0, 1, 2, … The mean and standard deviation of the Poisson random variable are

Mean: μ

Standard deviation:

$$\sigma=\sqrt{\mu}$$

Cumulative Probability Tables

You can use the cumulative probability tables to find probabilities for selected Poisson distributions.

✓Find the column for the correct value of μ.

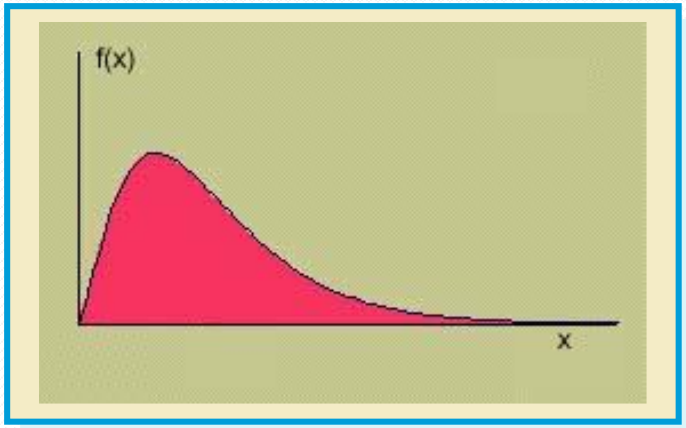
✓The row marked "k" gives the cumulative probability, $P(x \leq k) = P(x = 0) + \ldots + P(x = k)$

Continuous Random Variables

- Continuous random variables can assume the infinitely many values corresponding to points on a line interval.
- **Examples:**
  - Heights, weights
  - length of life of a particular product
  - experimental laboratory error

Continuous Random Variables

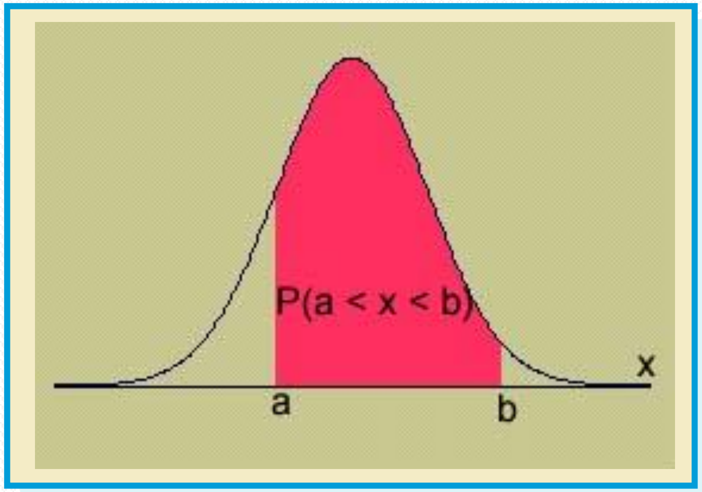
- A **smooth curve** describes the probability distribution of a continuous random variable.



•The depth or density of the probability, which varies with *x*, may be described by a mathematical formula *f* (*x*), called the probability distribution or probability density function for the random variable *x*.

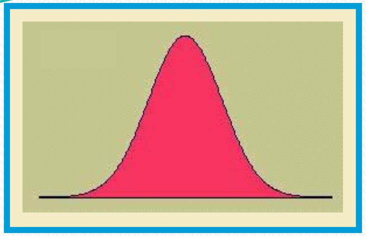
Properties of Continuous Probability Distributions

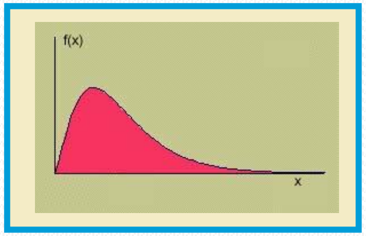
- The area under the curve is equal to **1.**
- P(a ≤ x ≤ b) = **area under the curve** between a and b.



•There is no probability attached to any single value of x. That is, **P(x = a) = 0.**

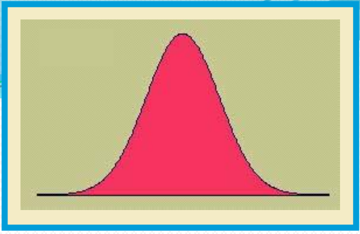
Continuous Probability Distributions





- There are many different types of continuous random variables
- We try to pick a model that
  - Fits the data well
  - Allows us to make the best possible inferences using the data.
- One important continuous random variable is the **normal random variable**.

The Normal Distribution



- The formula that generates the normal probability distribution is:

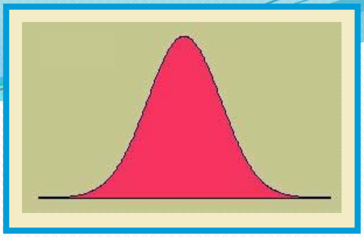
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{for } -\infty < x < \infty$$

$$e = 2.7183 \qquad \pi = 3.1416$$

μ and σ are the population mean and standard deviation.

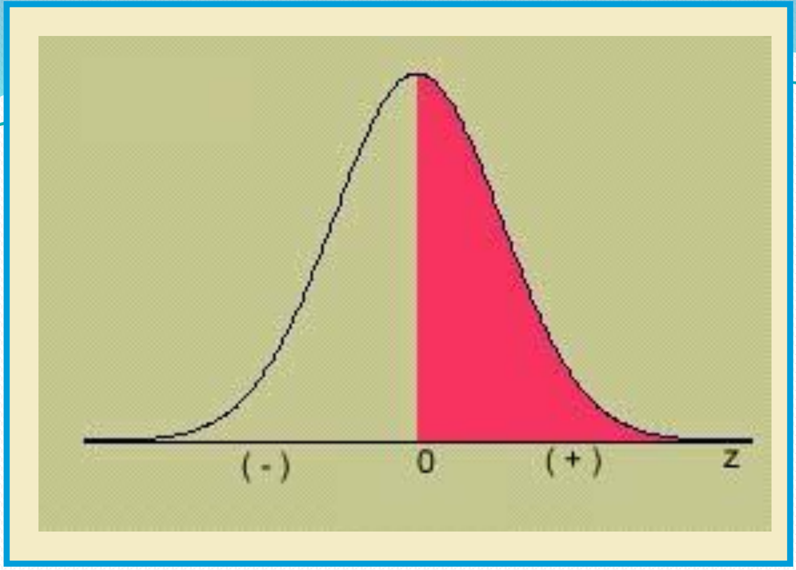
- The shape and location of the normal curve changes as the mean and standard deviation change.

The Standard Normal Distribution



- To find P(a < x < b), we need to find the area under the appropriate normal curve.
- To simplify the tabulation of these areas, we **standardize** each value of x by expressing it as a z-score, the number of standard deviations σ it lies from the mean μ.

$$z = \frac{x - \mu}{\sigma}$$



The Standard Normal (*z*) Distribution

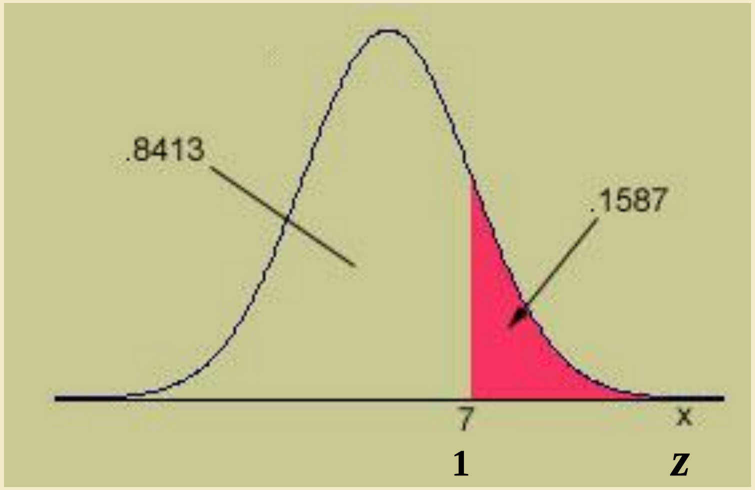
- Mean = 0; Standard deviation = 1
- When $x = \mu$, $z = 0$
- Symmetric about $z = 0$
- Values of z to the left of center are negative
- Values of z to the right of center are positive
- Total area under the curve is 1.

Finding Probabilities for the General Normal Random Variable

✓To find an area for a normal random variable *x* with mean μ and standard deviation σ, *standardize or rescale* the interval in terms of *z.*

✓Find the appropriate area using Table 3.

Example: *x* has a normal distribution with μ = 5 and σ = 2. Find P(*x* > 7).

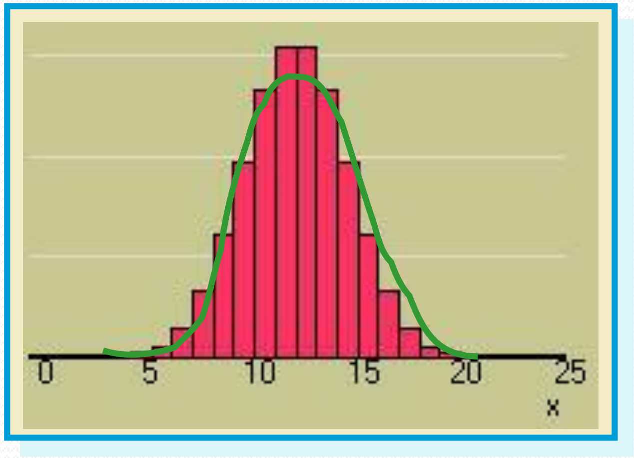


$$P(x > 7) = P(z > \frac{7-5}{2})$$

$$= P(z > 1) = 1 - .8413 = .1587$$

The Normal Approximation to the Binomial

- We can calculate binomial probabilities using
  - The binomial formula
  - The cumulative binomial tables
  - Java applets
- When n is large, and p is not too close to zero or one, areas under the normal curve with mean np and variance npq can be used to approximate binomial probabilities.



Approximating the Binomial

✓Make sure to include the entire rectangle for the values of *x* in the interval of interest. This is called the continuity correction.

✓Standardize the values of *x* using

$$z = \frac{x - np}{\sqrt{npq}}$$

✓Make sure that ***np* and *nq* are both greater than 5** to avoid inaccurate approximations!

UNIT-II

Multiple Random Variables, Correlation &Regression

Jointly Distributed Random Variables

- Joint Probability Distributions
  - Discrete $P(X = x_i, Y = y_j) = p_{ij} \geq 0$

    satisfying $\sum_i \sum_j p_{ij} = 1$

  - Continuous $f(x, y) \geq 0$ satisfying $\iint_{\text{state space}} f(x, y)\,dx\,dx = 1$

Jointly Distributed Random Variables

- Joint Cumulative Distribution Function
  - Discrete $F(x,y) = P(X \le x_i, Y \le y_j)$
  - Continuous $F(x,y) = \sum_{i:x_i \le x} \sum_{j:y_j \le y} p_{ij}$

$$F(x,y) = \int_{w=-\infty}^{x} \int_{z=-\infty}^{y} f(w,z)\,dz\,dw$$

Jointly Distributed Random Variables

- Example 19 : Air Conditioner Maintenance
  - A company that services air conditioner units in residences and office blocks is interested in how to schedule its technicians in the most efficient manner
  - The random variable X, taking the values 1,2,3 and 4, is the service time in hours
  - The random variable Y, taking the values 1,2 and 3, is the number of air conditioner units

Jointly Distributed Random Variables

Y= number of units	X=service time			
	1	2	3	4
1	0.12	0.08	0.07	0.05
2	0.08	0.15	0.21	0.13
3	0.01	0.01	0.02	0.07

- Joint p.m.f

$$\sum_i \sum_j p_{ij} = 0.12 + 0.18$$

$$+ \cdots + 0.07 = 1.00$$

- Joint cumulative distribution function

$$F(2,2) = p_{11} + p_{12} + p_{21} + p_{22}$$

$$= 0.12 + 0.18 + 0.08 + 0.15$$

$$= 0.43$$

Marginal Probability Distributions

- Marginal probability distribution
  - Obtained by summing or integrating the joint probability distribution over the values of the other random variable $P(X = i) = p_{i+} = \sum_j p_{ij}$
  - Discrete

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy$$

  - Continuous

Marginal Probability Distributions

- Example 19
  - Marginal p.m.f of X

$$P(X=1)=\sum_{j=1}^{3} p_{1j} = 0.12+0.08+0.01=0.21$$

  - Marginal p.m.f of Y

$$P(Y=1)=\sum_{i=1}^{4} p_{i1} = 0.12+0.08+0.07+0.05=0.32$$

- Example 20: (a jointly continuous case)
- Joint pdf: $f(x, y)$
- Marginal pdf's of X and Y:

$$f_X(x) = \int f(x, y)dy$$

$$f_Y(y) = \int f(x, y)dx$$

Conditional Probability Distributions

- Conditional probability distributions
  - The probabilistic properties of the random variable X under the knowledge provided by the value of Y
  - Discrete

$$p_{i|j} = P(X = i \mid Y = j) = \frac{P(X = i, Y = j)}{P(Y = j)} = \frac{p_{ij}}{p_{+j}}$$

  - Continuous

$$f_{X|Y=y}(x) = \frac{f(x, y)}{f_Y(y)}$$

  - The conditional probability distribution is a **probability distribution.**

Conditional Probability Distributions

- Example 19
  - Marginal probability distribution of Y

$$P(Y=3)=p_{+3}=0.01+0.01+0.02+0.07=0.11$$

  - Conditional distribution of X

$$p_{1|Y=3}=P(X=1\,|\,Y=3)=\frac{p_{13}}{p_{+3}}=\frac{0.01}{0.11}=0.091$$

Independence and Covariance

- Two random variables **X** and **Y** are said to be independent if
  - Discrete

$$p_{ij} = p_{i+} p_{+j} \quad \text{for all values } i \text{ of } X \text{ and } j \text{ of } Y$$

  - Continuous

$$f(x, y) = f_X(x) f_Y(y) \quad \textit{for all } x \textit{ and } y$$

  - How is this independency different from the independence among events?

Independence and Covariance

- Covariance

$$\mathrm{Cov}(X,Y) = E((X - E(X))(Y - E(Y)))$$
$$= E(XY) - E(X)E(Y)$$

$$\mathrm{Cov}(X,Y) = E((X - E(X))(Y - E(Y)))$$
$$= E(XY - XE(Y) - E(X)Y + E(X)E(Y))$$
$$= E(XY) - E(X)E(Y) - E(X)E(Y) + E(X)E(Y)$$
$$= E(XY) - E(X)E(Y)$$

  - May take any positive or negative numbers.
  - Independent random variables have a covariance of zero
  - What if the covariance is zero?

Independence and Covariance

- Example 19 (Air conditioner maintenance)

$$E(X) = 2.59, \qquad E(Y) = 1.79$$

$$E(XY) = \sum_{i=1}^{4}\sum_{j=1}^{3} ijp_{ij}$$

$$= (1\times1\times0.12) + (1\times2\times0.08)$$

$$+\cdots+ (4\times3\times0.07) = 4.86$$

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= 4.86 - (2.59\times1.79) = 0.224$$

Independence and Covariance

- Correlation:

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$

  - Values between -1 and 1, and independent random variables have a correlation of zero

Independence and Covariance

- Example 19: (Air conditioner maintenance)

$$\text{Var}(X) = 1.162, \quad \text{Var}(Y) = 0.384$$

$$\text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

$$= \frac{0.224}{\sqrt{1.162 \times 0.384}} = 0.34$$

- What if random variable X and Y have linear relationship, that is,

$$Y = aX + b \quad a \neq 0$$

where

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

$$= E[X(aX+b)] - E[X]E[aX+b]$$

$$= aE[X^2] + bE[X] - aE^2[X] - bE[X]$$

$$= a(E[X^2] - E^2[X]) = aVar(X)$$

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{aVar(X)}{\sqrt{Var(X)a^2Var(X)}}$$

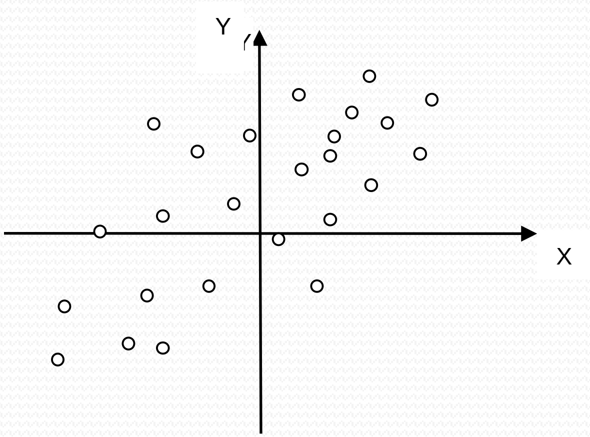
That is, Cov(X,Y)=1 if a>0; -1 if a<0.

The relationship between *x* and *y*

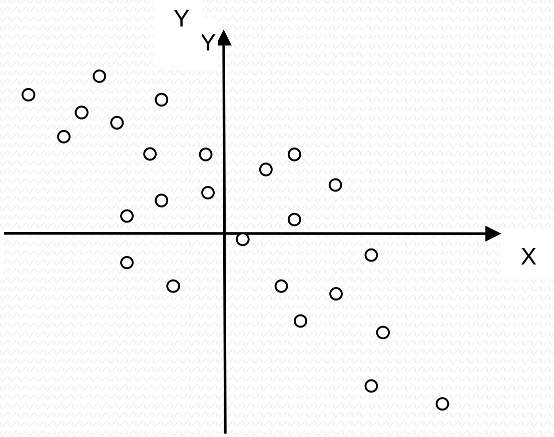
- Correlation: is there a relationship between 2 variables?
- Regression: how well a certain independent variable predict dependent variable?

- CORRELATION $\neq$ CAUSATION
  - In order to infer causality: manipulate independent variable and observe effect on dependent variable

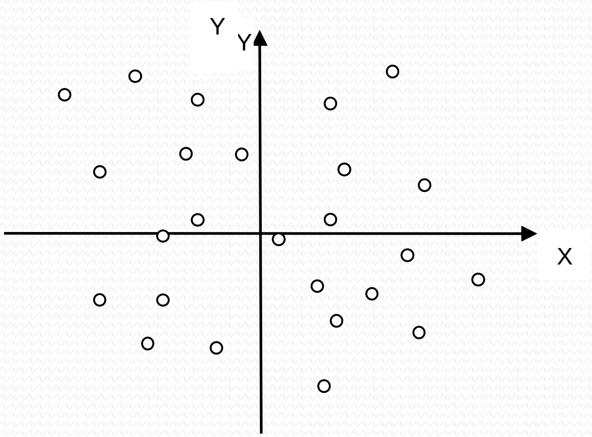
Scattergrams



Positive correlation



Negative correlation



No correlation

Variance vs Covariance

- *First, a note on your sample:*
  - *If you're wishing to assume that your sample is representative of the general population (RANDOM EFFECTS MODEL), use the degrees of freedom (n – 1) in your calculations of variance or covariance.*
  - *But if you're simply wanting to assess your current sample (FIXED EFFECTS MODEL), substitute n for the degrees of freedom.*

Variance vs Covariance

- Do two variables change together?

Variance:

- Gives information on variability of a single variable.

$$S_x^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$$

Covariance:

- Gives information on the degree to which two variables vary together.
- Note how similar the covariance is to variance: the equation simply multiplies x's error scores by y's error scores as opposed to squaring x's error scores.

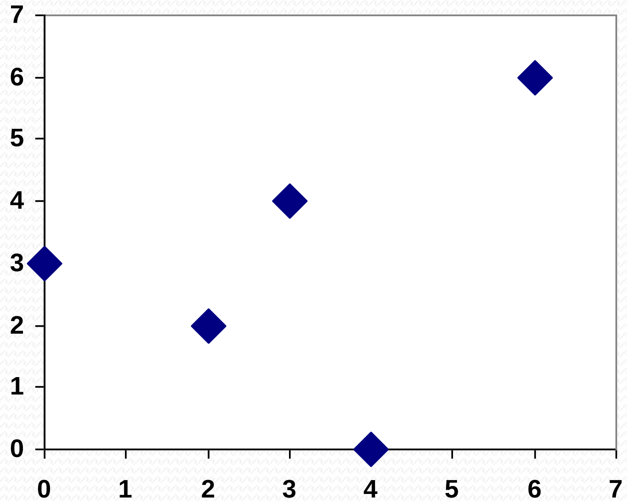
$$\text{cov}(x, y) = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{n-1}$$

Covariance

$$\operatorname{cov}(x, y) = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{n-1}$$

- When X↑ and Y↑ : cov (x,y) = pos.
- When X↓ and Y↑ : cov (x,y) = neg.
- When no constant relationship: cov (x,y) = 0

Example Covariance



x	y	$x_i - \bar{x}$	$y_i - \bar{y}$	$(x_i - \bar{x})(y_i - \bar{y})$
0	3	-3	0	0
2	2	-1	-1	1
3	4	0	1	0
4	0	1	-3	-3
6	6	3	3	9
$\bar{x} = 3$	$\bar{y} = 3$			$\sum = 7$

$$\operatorname{cov}(x, y) = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}))}{n-1} = \frac{7}{4} = 1.75$$

What does this number tell us?

Problem with Covariance:

- The value obtained by covariance is dependent on the size of the data's standard deviations: if large, the value will be greater than if small… *even if the relationship between x and y is exactly the same in the large versus small standard deviation datasets.*

Example of how covariance value relies on variance

	High variance data				Low variance data		
Subject	**x**	**y**	**x error * y error**		**x**	**y**	**X error * y error**
1	101	100	2500		54	53	9
2	81	80	900		53	52	4
3	61	60	100		52	51	1
4	51	50	0		51	50	0
5	41	40	100		50	49	1
6	21	20	900		49	48	4
7	1	0	2500		48	47	9
Mean	51	50			51	50	
Sum of x error * y error :			7000		**Sum of x error * y error :**		28
Covariance:			**1166.67**		**Covariance:**		**4.67**

Solution: Pearson's r

- **Covariance does not really tell us anything**
  - ***Solution: standardise this measure***
- **Pearson's R: standardises the covariance value.**
- **Divides the covariance by the multiplied standard deviations of X and Y:**

$$r_{xy} = \frac{\text{cov}(x, y)}{s_x s_y}$$

Pearson's R continued

$$\text{cov}(x,y) = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{n-1}$$

→

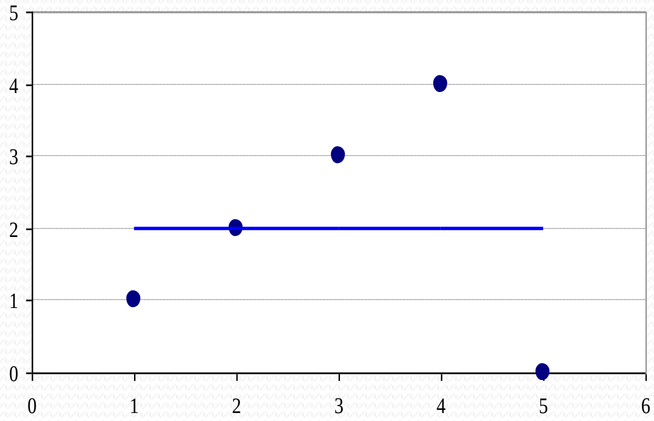
$$r_{xy} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{(n-1)s_x s_y}$$

↓

$$r_{xy} = \frac{\sum_{i=1}^{n} Z_{x_i} * Z_{y_i}}{n-1}$$

Limitations of r

- When r = 1 or r = -1:
  - We can predict y from x with certainty
  - all data points are on a straight line: y = ax + b
- r is actually $\hat{r}$
  - r = true r of whole population
  - $\hat{r}$ = estimate of r based on data
- r is very sensitive to extreme values:



Regression

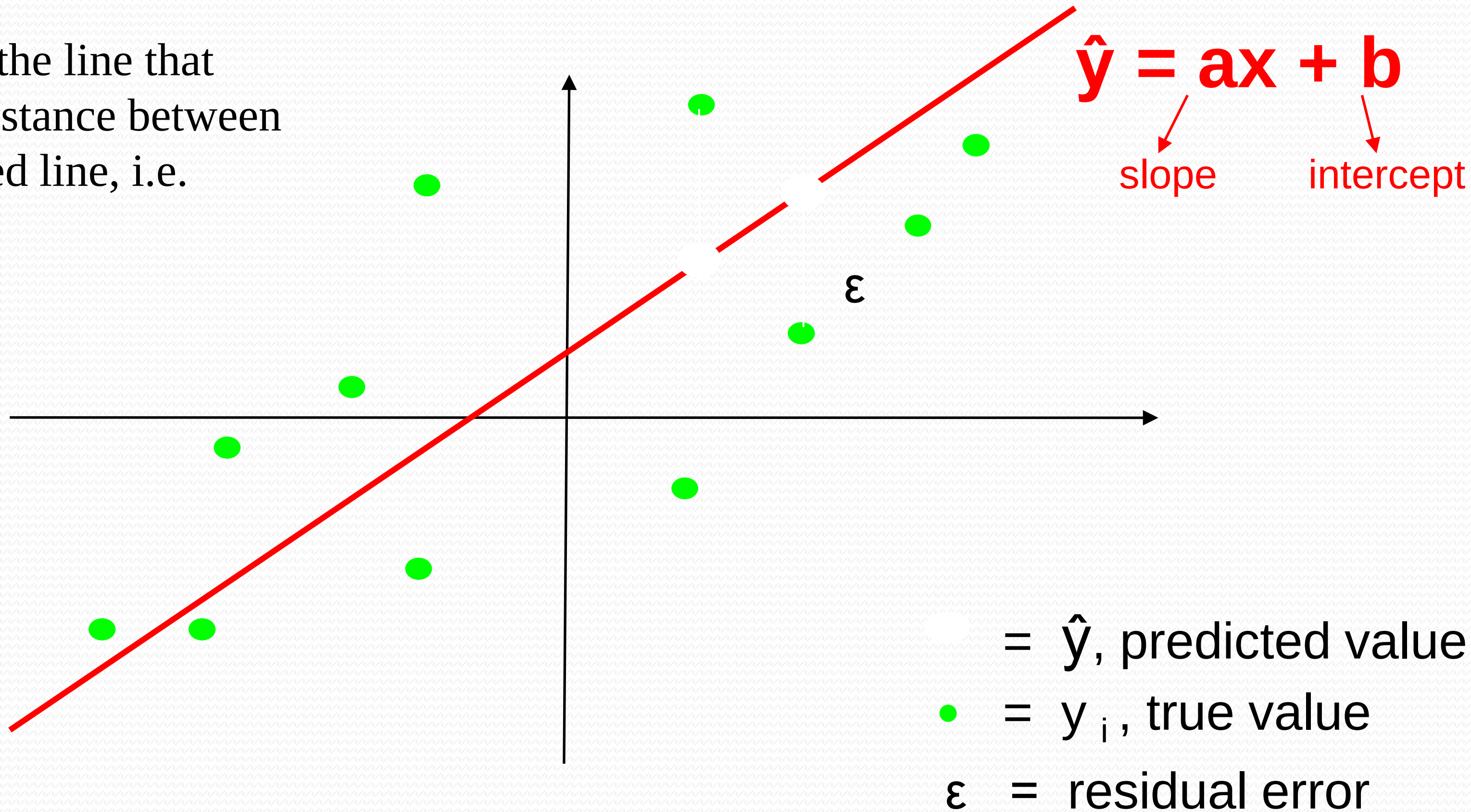
- Correlation tells you if there is an association between x and y but it doesn’t describe the relationship or allow you to predict one variable from the other.

- To do this we need REGRESSION!

Best-fit Line

- Aim of linear regression is to fit a straight line, $\hat{y} = ax + b$, to data that gives best prediction of y for any value of x

- This will be the line that minimises distance between data and fitted line, i.e. the residuals



Least Squares Regression

- To find the best line we must minimise the sum of the squares of the residuals (the vertical distances from the data points to our line)

Model line: $\hat{y} = ax + b$ a = slope, b = intercept

Residual (ε) = $y - \hat{y}$

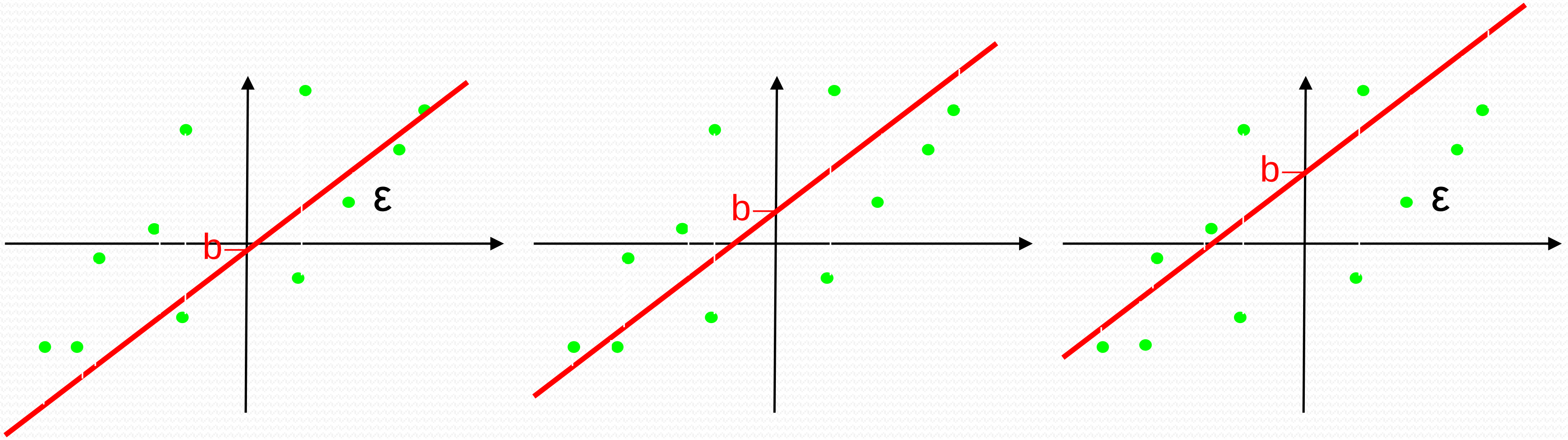
Sum of squares of residuals = $\Sigma (y - \hat{y})^2$

- we must find values of a and b that minimise

$$\Sigma (y - \hat{y})^2$$

Finding b

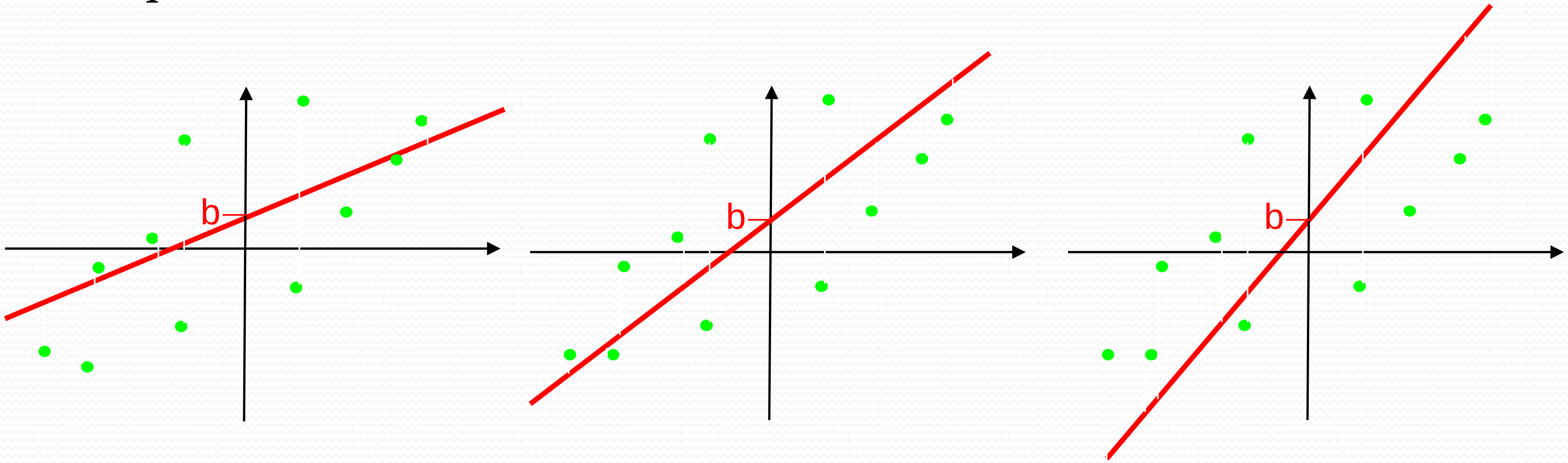
- First we find the value of b that gives the min sum of squares



- Trying different values of b is equivalent to shifting the line up and down the scatter plot

Finding a

- Now we find the value of a that gives the min sum of squares



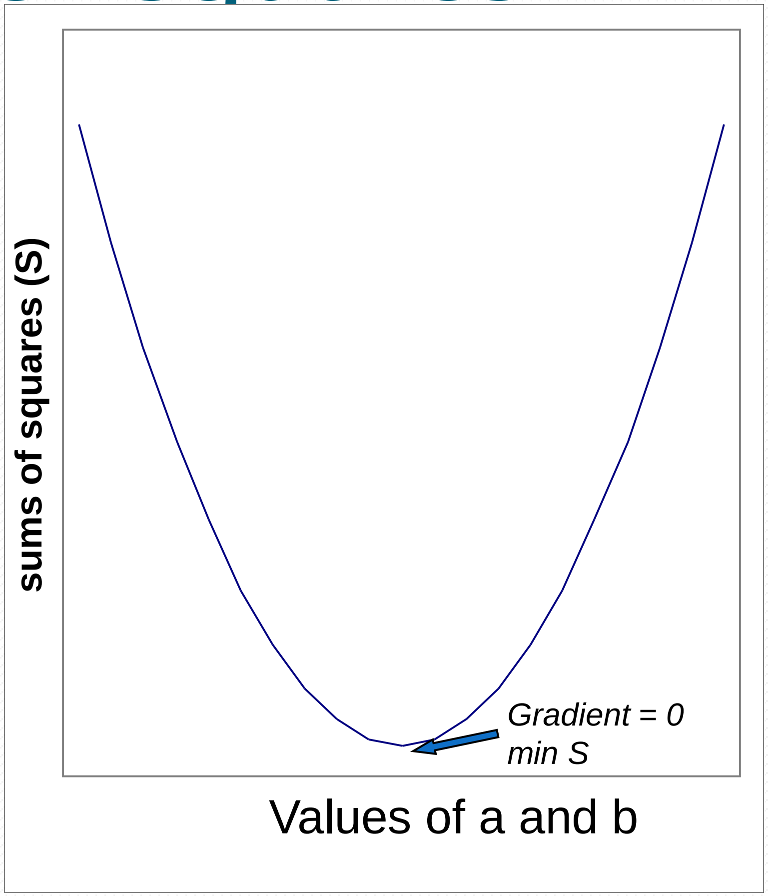
- Trying out different values of a is equivalent to changing the slope of the line, while b stays constant

Minimising sums of squares

- Need to minimise $\Sigma(y - \hat{y})^2$
- $\hat{y} = ax + b$
- so need to minimise:

$$\Sigma(y - ax - b)^2$$

- If we plot the sums of squares for all different values of a and b we get a parabola, because it is a squared term
- So the min sum of squares is at the bottom of the curve, where the gradient is zero.



The maths bit

- The min sum of squares is at the bottom of the curve where the gradient = 0

- So we can find a and b that give min sum of squares by taking partial derivatives of $\Sigma(y - ax - b)^2$ with respect to a and b separately

- Then we solve these for 0 to give us the values of a and b that give the min sum of squares

The solution

- Doing this gives the following equations for a and b:

$$a = \frac{r\, s_y}{s_x}$$

r = correlation coefficient of x and y
s_y = standard deviation of y
s_x = standard deviation of x

- From you can see that:
  - A low correlation coefficient gives a flatter slope (small value of a)
  - Large spread of y, i.e. high standard deviation, results in a steeper slope (high value of a)
  - Large spread of x, i.e. high standard deviation, results in a flatter slope (high value of a)

The solution cont.

- Our model equation is $\hat{y} = ax + b$
- This line must pass through the mean so:

$$\bar{y} = a\bar{x} + b \quad \Rightarrow \quad b = \bar{y} - a\bar{x}$$

- We can put our equation for a into this giving:

$$\mathbf{b = \bar{y} - \frac{r\, s_y}{s_x}\, \bar{x}}$$

r = correlation coefficient of x and y
s_y = standard deviation of y
s_x = standard deviation of x

- The smaller the correlation, the closer the intercept is to the mean of y

Back to the model

$$\hat{y} = ax + b = \underbrace{\frac{r\,s_y}{s_x}}_{a} x + \underbrace{\bar{y} - \underbrace{\frac{r\,s_y}{s_x}}_{a}\bar{x}}_{b}$$

Rearranges to: $\hat{y} = \underbrace{\frac{r\,s_y}{s_x}}_{a}(x - \bar{x}) + \bar{y}$

- If the correlation is zero, we will simply predict the mean of y for every value of x, and our regression line is just a flat straight line crossing the x-axis at y
- But this isn't very useful.
- We can calculate the regression line for any data, but the important question is how well does this line fit the data, or how good is it at predicting y from x

How good is our model?

- Total variance of y:

$$s_y^2 = \frac{\sum(y - \bar{y})^2}{n - 1} = \frac{SS_y}{df_y}$$

- Variance of predicted y values ($\hat{y}$):

$$s_{\hat{y}}^2 = \frac{\sum(\hat{y} - \bar{y})^2}{n - 1} = \frac{SS_{pred}}{df_{\hat{y}}}$$

This is the variance explained by our regression model

- Error variance:

$$s_{error}^2 = \frac{\sum(y - \hat{y})^2}{n - 2} = \frac{SS_{er}}{df_{er}}$$

This is the variance of the error between our predicted y values and the actual y values, and thus is the variance in y that is NOT explained by the regression model

How good is our model cont.

- Total variance = predicted variance + error variance

$$s_y^2 = s_{\hat{y}}^2 + s_{er}^2$$

- Conveniently, via some complicated rearranging

$$s_{\hat{y}}^2 = r^2 s_y^2$$

$$\Downarrow$$

$$r^2 = s_{\hat{y}}^2 / s_y^2$$

- so r^2 is the proportion of the variance in y that is explained by our regression model

How good is our model cont.

- Insert $r^2 s_y^2$ into $s_y^2 = s_{\hat{y}}^2 + s_{er}^2$ and rearrange to get:

$$s_{er}^2 = s_y^2 - r^2 s_y^2$$
$$= s_y^2 (1 - r^2)$$

- From this we can see that the greater the correlation the smaller the error variance, so the better our prediction

Is the model significant?

- i.e. do we get a significantly better prediction of y from our regression equation than by just predicting the mean?

- F-statistic:

complicated rearranging

$$F_{(df_{\hat{y}}, df_{er})} = \frac{s_{\hat{y}}^2}{s_{er}^2} = \ldots\ldots = \frac{r^2 (n - 2)^2}{1 - r^2}$$

- And it follows that:

(because F = t[2])

$$t_{(n-2)} = \frac{r\,(n - 2)}{\sqrt{1 - r^2}}$$

So all we need to know are r and n

General Linear Model

- Linear regression is actually a form of the General Linear Model where the parameters are a, the slope of the line, and b, the intercept.

$$y = ax + b + \varepsilon$$

- A General Linear Model is just any model that describes the data in terms of a straight line

Multiple regression

- Multiple regression is used to determine the effect of a number of independent variables, x_1, x_2, x_3 etc, on a single dependent variable, y
- The different x variables are combined in a linear way and each has its own regression coefficient:

$$y = a_1x_1 + a_2x_2 + \ldots\ldots + a_nx_n + b + \varepsilon$$

- The a parameters reflect the independent contribution of each independent variable, x, to the value of the dependent variable, y.
- i.e. the amount of variance in y that is accounted for by each x variable after all the other x variables have been accounted for

SPM

- Linear regression is a GLM that models the effect of one independent variable, x, on ONE dependent variable, y
- Multiple Regression models the effect of several independent variables, x_1, x_2 etc, on ONE dependent variable, y
- Both are types of General Linear Model
- GLM can also allow you to analyse the effects of several independent x variables on several dependent variables, y_1, y_2, y_3 etc, in a linear combination
- This is what SPM does and all will be explained next week!

UNIT-III

Sampling Distribution, Large Samples & Small Samples

Introduction

- Parameters are numerical descriptive measures for populations.
  - For the normal distribution, the location and shape are described by μ and σ.
  - For a binomial distribution consisting of n trials, the location and shape are determined by $\boldsymbol{p}$.
- Often the values of parameters that specify the exact form of a distribution are unknown.
- You must rely on the sample to learn about these parameters.

Sampling

Examples:

- A pollster is sure that the responses to his "agree/disagree" question will follow a binomial distribution, but p, the proportion of those who "agree" in the population, is unknown.
- An agronomist believes that the yield per acre of a variety of wheat is approximately normally distributed, but the mean μ and the standard deviation σ of the yields are unknown.

✓ If you want the sample to provide reliable information about the population, you must select your sample in a certain way!

Simple Random Sampling

- The **sampling plan** or **experimental design** determines the amount of information you can extract, and often allows you to measure the **reliability of your inference**.
- **Simple random sampling** is a method of sampling that allows each possible sample of size *n* an equal probability of being selected.

Types of Samples

- Sampling can occur in two types of practical situations:

1. Observational studies: The data existed before you decided to study it. Watch out for
   - ✓ Nonresponse: Are the responses biased because only opinionated people responded?
   - ✓ Undercoverage: Are certain segments of the population systematically excluded?
   - ✓ Wording bias: The question may be too complicated or poorly worded.

Types of Samples

- Sampling can occur in two types of practical situations:

2. Experimentation: The data are generated by imposing an experimental condition or treatment on the experimental units.

   ✓ Hypothetical populations can make random sampling difficult if not impossible.

   ✓ Samples must sometimes be chosen so that the experimenter believes they are representative of the whole population.

   ✓ Samples must behave like random samples!

Other Sampling Plans

- There are several other sampling plans that still involve randomization:

1. Stratified random sample: Divide the population into subpopulations or strata and select a simple random sample from each strata.
2. Cluster sample: Divide the population into subgroups called clusters; select a simple random sample of clusters and take a census of every element in the cluster.
3. 1-in-k systematic sample: Randomly select one of the first k elements in an ordered population, and then select every k-th element thereafter.

Non-Random Sampling Plans

- There are several other sampling plans that do not involve randomization. They should NOT be used for statistical inference!

1. Convenience sample: A sample that can be taken easily without random selection.
   - People walking by on the street
2. Judgment sample: The sampler decides who will and won't be included in the sample.
3. Quota sample: The makeup of the sample must reflect the makeup of the population on some selected characteristic.
   - Race, ethnic origin, gender, etc.

Sampling Distributions

- Numerical descriptive measures calculated from the sample are called statistics.
- Statistics vary from sample to sample and hence are random variables.
- The probability distributions for statistics are called sampling distributions.
- In repeated sampling, they tell us what values of the statistics can occur and how often each value occurs.

Sampling Distributions

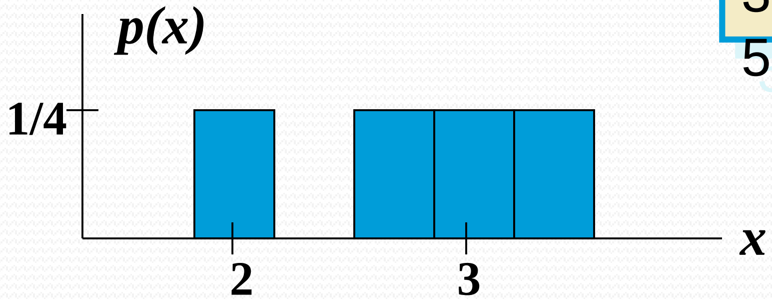
Definition: The sampling distribution of a statistic is the probability distribution for the possible values of the statistic that results when random samples of size *n* are repeatedly drawn from the population

Population: 3, 5, 2, 1

Draw samples of size *n* = 3 without replacement

Possible samples	$\bar{x}$
3, 5, 2	$10/3 = 3.33$
3, 5, 1	$9/3 = 3$
3, 2, 1	$6/3 = 2$
5, 2, 1	$8/3 = 2.67$

Each value of x-bar is equally likely, with probability 1/4



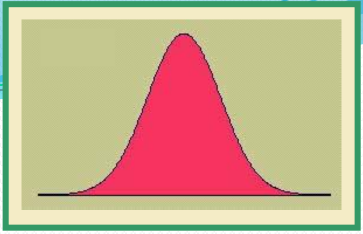
Sampling Distributions

Sampling distributions for statistics can be

- ✓Approximated with simulation techniques
- ✓Derived using mathematical theorems
- ✓The Central Limit Theorem is one such theorem.

Central Limit Theorem: If random samples of *n* observations are drawn from a nonnormal population with finite μ and standard deviation σ , then, when *n* is large, the sampling distribution of the sample mean $\bar{x}$ is approximately normally distributed, with mean μ and standard deviation $\sigma/\sqrt{n}$. The approximation becomes more accurate as *n* becomes large.

Why is this Important?



✓The Central Limit Theorem also implies that the sum of *n* measurements is approximately normal with mean $n\mu$ and standard deviation $\sigma\sqrt{n}$.

✓Many statistics that are used for statistical inference are sums or averages of sample measurements.

✓When *n* is large, these statistics will have approximately normal distributions.

✓This will allow us to describe their behavior and evaluate the reliability of our inferences.

How Large is Large?

If the sample is normal, then the sampling distribution of $\bar{x}$ will also be normal, no matter what the sample size.

When the sample population is approximately symmetric, the distribution becomes approximately normal for relatively small values of *n.*

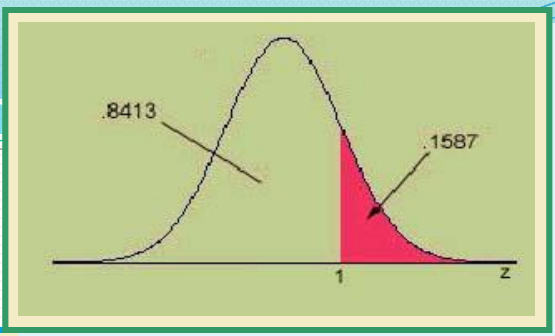
When the sample population is skewed, the sample size must be at least 30 before the sampling distribution of $\bar{x}$ becomes approximately normal.

The Sampling Distribution of the Sample Mean

- ✓A random sample of size *n* is selected from a population with mean μ and standard deviation σ.
- ✓The sampling distribution of the sample mean $\bar{x}$ will have mean μ and standard deviation $\sigma/\sqrt{n}$.
- ✓If the original population is normal, the sampling distribution will be normal for any sample size.
- ✓If the original population is nonnormal, the sampling distribution will be normal when *n* is large.

The standard deviation of *x*-bar is sometimes called the STANDARD ERROR (SE).

Finding Probabilities for the Sample Mean



✓If the sampling distribution of $\bar{x}$ is normal or approximately normal, *standardize or rescale* the interval of interest in terms of

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

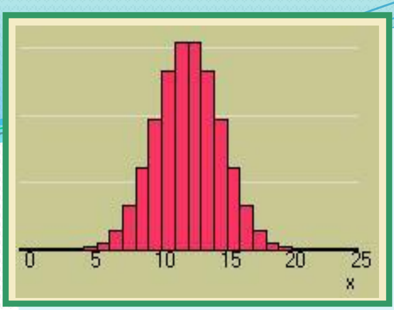
✓Find the appropriate area using Table 3.

Example: A random sample of size *n* = 16 from a normal distribution with μ = 10 and σ = 8.

$$P(\bar{x} > 12) = P(z > \frac{12 - 10}{8/\sqrt{16}})$$

$$= P(z > 1) = 1 - .8413 = .1587$$

The Sampling Distribution of the Sample Proportion

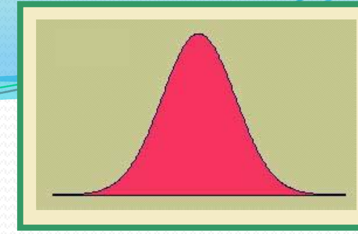


✓The Central Limit Theorem can be used to conclude that the binomial random variable *x* is approximately normal when *n* is large, with mean *np* and standard deviation .

✓The sample proportion, $\hat{p} = \frac{x}{n}$ is simply a *rescaling* of the binomial random variable *x*, dividing it by *n*.

✓From the Central Limit Theorem, the sampling distribution of $\hat{p}$ will also be approximately normal, with a *rescaled* mean and standard deviation.

The Sampling Distribution of the Sample Proportion

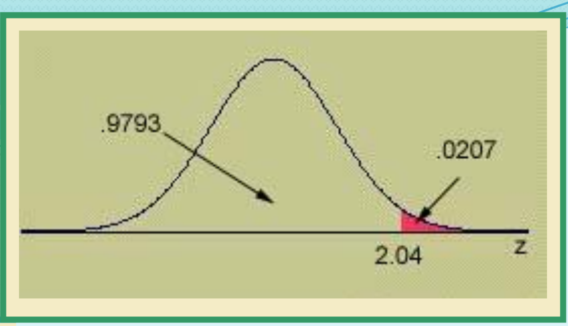


- ✓A random sample of size *n* is selected from a binomial population with parameter *p*.
- ✓The sampling distribution of the sample proportion,

$$\hat{p} = \frac{x}{n}$$

- ✓will have mean *p* and standard deviation $\sqrt{\frac{pq}{n}}$
- ✓If *n* is large, and *p* is not too close to zero or one, the sampling distribution of $\hat{p}$ will be approximately normal.

Finding Probabilities for the Sample Proportion



- ✓If the sampling distribution of $\hat{p}$ is normal or approximately normal, *standardize or rescale* the interval of interest in terms of $z = \dfrac{\hat{p} - p}{\sqrt{\dfrac{pq}{n}}}$
- ✓Find the appropriate area using Table 3.

Example: A random sample of size *n* = 100 from a binomial population with *p* = .4.

$$P(\hat{p} > .5) = P(z > \frac{.5 - .4}{\sqrt{\dfrac{.4(.6)}{100}}})$$

$$= P(z > 2.04) = 1 - .9793 = .0207$$

Types of Inference

- **Estimation:**
  - Estimating or predicting the value of the parameter
  - "What is (are) the most likely values of μ or p?"
- **Hypothesis Testing:**
  - Deciding about the value of a parameter based on some preconceived idea.
  - "Did the sample come from a population with $\mu = 5$ or $p = .2$?"

Types of Inference

- **Examples:**
  - A consumer wants to estimate the average price of similar homes in her city before putting her home on the market.

Estimation: Estimate μ, the average home price.

–A manufacturer wants to know if a new type of steel is more resistant to high temperatures than an old type was.

Hypothesis test: Is the new average resistance, μ_N equal to the old average resistance, μ_O?

Types of Inference

- Whether you are estimating parameters or testing hypotheses, statistical methods are important because they provide:
  - **Methods for making the inference**
  - **A numerical measure of the goodness or reliability of the inference**

Definitions

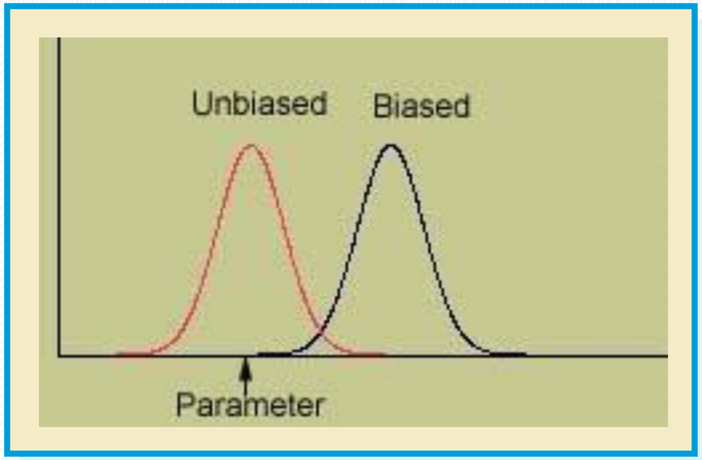
- An **estimator** is a rule, usually a formula, that tells you how to calculate the estimate based on the sample.
  - **Point estimation:** A single number is calculated to estimate the parameter.
  - **Interval estimation:** Two numbers are calculated to create an interval within which the parameter is expected to lie.

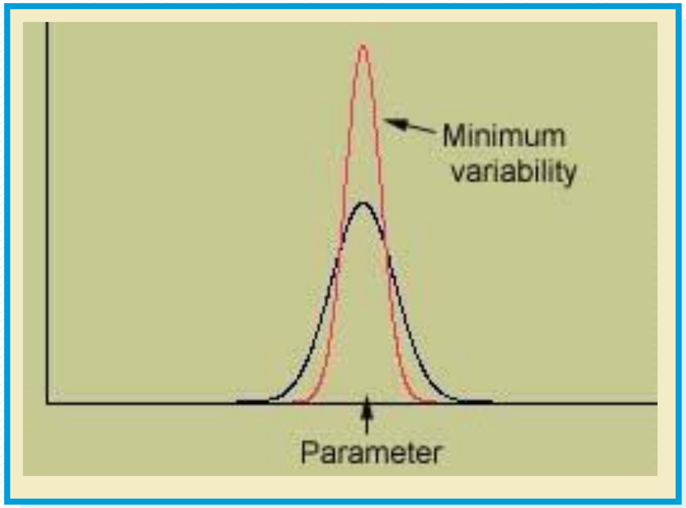
Properties of Point Estimators

- Since an estimator is calculated from sample values, it varies from sample to sample according to its **sampling distribution.**
- An **estimator** is **unbiased** if the mean of its sampling distribution equals the parameter of interest.
  - It does not systematically overestimate or underestimate the target parameter.

Properties of Point Estimators

- Of all the **unbiased** estimators, we prefer the estimator whose sampling distribution has the **smallest spread** or **variability**.





Measuring the Goodness of an Estimator

- The distance between an estimate and the true value of the parameter is the **error of estimation.**

The distance between the bullet and the bull's-eye.

- In this chapter, the sample sizes are large, so that our *unbiased* estimators will have normal dist

Because of the Central Limit Theorem.

Estimating Means and Proportions

- For a quantitative population,

Point estimator of population mean $\mu : \bar{x}$

Margin of error $(n \geq 30) : \pm 1.96 \frac{s}{\sqrt{n}}$

- For a binomial population,

Point estimator of population proportion $p : \hat{p} = x/n$

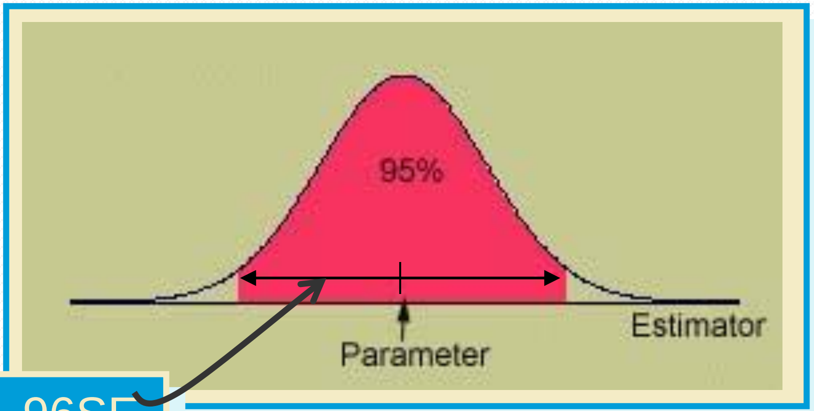
Margin of error $(n \geq 30) : \pm 1.96 \sqrt{\frac{\hat{p}\hat{q}}{n}}$

Interval Estimation

- Create an interval (a, b) so that you are fairly sure that the parameter lies between these two values.
- "Fairly sure" is means "with high probability", measured using the confidence coefficient,

Usually, $1-\alpha = .90, .95, .98, .99$

- Suppose $1-\alpha = .95$ and that the estimator has a normal distrib



Parameter ± 1.96SE

Confidence Intervals for Means and Proportions

- For a quantitative population,

Confidence interval for a population mean μ :

$$\bar{x} \pm z_{\alpha/2} \frac{s}{\sqrt{n}}$$

- For a binomial population,

Confidence interval for a population proportion p :

$$\hat{p} \pm z_{\alpha/2} \sqrt{\frac{\hat{p}\hat{q}}{n}}$$

Estimating the Difference between Two Means

- Sometimes we are interested in comparing the means of two populations.
  - The average growth of plants fed using two different nutrients.
  - The average scores for students taught with two different teaching methods.
- To make this comparison,

A random sample of size n_1 drawn from population 1 with mean μ_1 and variance σ_1^2.

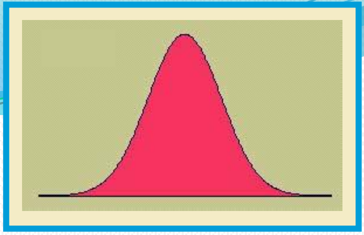
A random sample of size n_2 drawn from population 2 with mean μ_2 and variance σ_2^2.

Estimating the Difference between Two Means

- We compare the two averages by making inferences about μ_1-μ_2, the difference in the two population averages.
  - If the two population averages are the same, then μ_1-$\mu_2 = 0$.
  - The best estimate of μ_1-μ_2 is the difference in the two sample means,

$$\bar{x}_1 - \bar{x}_2$$

The Sampling Distribution of $\bar{x}_1 - \bar{x}_2$



1. The mean of $\bar{x}_1 - \bar{x}_2$ is $\mu_1 - \mu_2$, the difference in the population means.

2. The standard deviation of $\bar{x}_1 - \bar{x}_2$ is $SE = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$.

3. If the sample sizes are large, the sampling distribution of $\bar{x}_1 - \bar{x}_2$ is approximately normal, and SE can be estimated as $SE = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$.

Estimating μ_1-μ_2

- For large samples, point estimates and their margin of error as well as confidence intervals are based on the standard normal (z) distribution.

Point estimate for $\mu_1 - \mu_2 : \bar{x}_1 - \bar{x}_2$

Margin of Error : $\pm 1.96\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$

Confidence interval for $\mu_1 - \mu_2$:

$$(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Estimating the Difference between Two Proportions

- Sometimes we are interested in comparing the proportion of "successes" in two binomial populations.
  - The germination rates of untreated seeds and seeds treated with a fungicide.
  - The proportion of male and female voters who favor a particular candidate for governor.

A random sample of size n_1 drawn from binomial population 1 with parameter p_1.

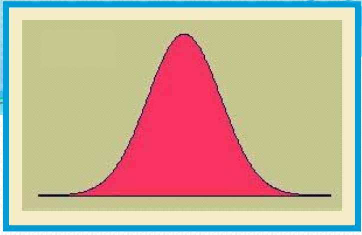
A random sample of size n_2 drawn from binomial population 2 with parameter p_2.

Estimating the Difference between Two Means

- We compare the two proportions by making inferences about p_1-p_2, the difference in the two population proportions.
  - If the two population proportions are the same, then p_1-$p_2 = 0$.
  - The best estimate of p_1-p_2 is the difference in the two sample proportions,

$$\hat{p}_1 - \hat{p}_2 = \frac{x_1}{n_1} - \frac{x_2}{n_2}$$

The Sampling Distribution of $\hat{p}_1 - \hat{p}_2$

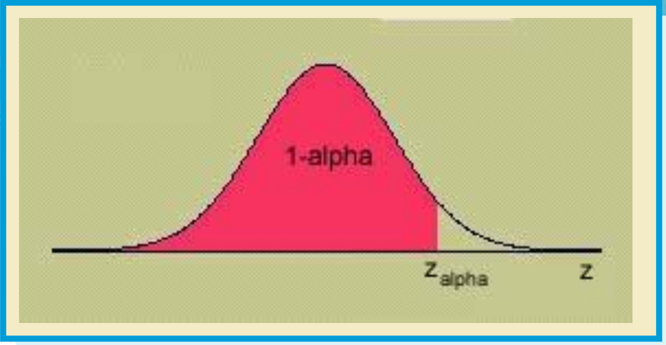


1. The mean of $\hat{p}_1 - \hat{p}_2$ is $p_1 - p_2$, the difference in the population proportions.

2. The standard deviation of $\hat{p}_1 - \hat{p}_2$ is $\text{SE} = \sqrt{\dfrac{p_1 q_1}{n_1} + \dfrac{p_2 q_2}{n_2}}$.

3. If the sample sizes are large, the sampling distribution of $\hat{p}_1 - \hat{p}_2$ is approximately normal, and SE can be estimated as $\text{SE} = \sqrt{\dfrac{\hat{p}_1 \hat{q}_1}{n_1} + \dfrac{\hat{p}_2 \hat{q}_2}{n_2}}$.

One Sided Confidence Bounds



- Confidence intervals are by their nature **two-sided** since they produce upper and lower bounds for the parameter.
- **One-sided bounds** can be constructed simply by using a value of z that puts α rather than α/2 in the tail of the *z* distribution.

$$\text{LCB}: \text{Estimator} - z_{\alpha} \times (\text{Std Error of Estimator})$$

$$\text{UCB}: \text{Estimator} + z_{\alpha} \times (\text{Std Error of Estimator})$$

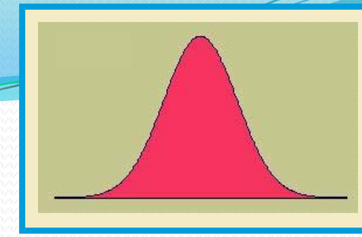
Parameter	Point Estimator	Margin of Error
μ	$\bar{x}$	$\pm 1.96\left(\frac{s}{\sqrt{n}}\right)$
p	$\hat{p} = \frac{x}{n}$	$\pm 1.96\sqrt{\frac{\hat{p}\hat{q}}{n}}$
$\mu_1 - \mu_2$	$\bar{x}_1 - \bar{x}_2$	$\pm 1.96\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$
$p_1 - p_2$	$(\hat{p}_1 - \hat{p}_2) = \left(\frac{x_1}{n_1} - \frac{x_2}{n_2}\right)$	$\pm 1.96\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}}$

IV. Large-Sample Interval Estimators

To estimate one of four population parameters when the sample sizes are large, use the following interval estimators.

Parameter	$(1-\alpha)100\%$ Confidence Interval
μ	$\bar{x} \pm z_{\alpha/2}\left(\frac{s}{\sqrt{n}}\right)$
p	$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}}$
$\mu_1 - \mu_2$	$(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$
$p_1 - p_2$	$(\hat{p}_1 - \hat{p}_2) \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}}$

The Sampling Distribution of the Sample Mean



- When we take a sample from a normal population, the sample mean $\bar{x}$ has a normal distribution for any sample size *n*, and

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

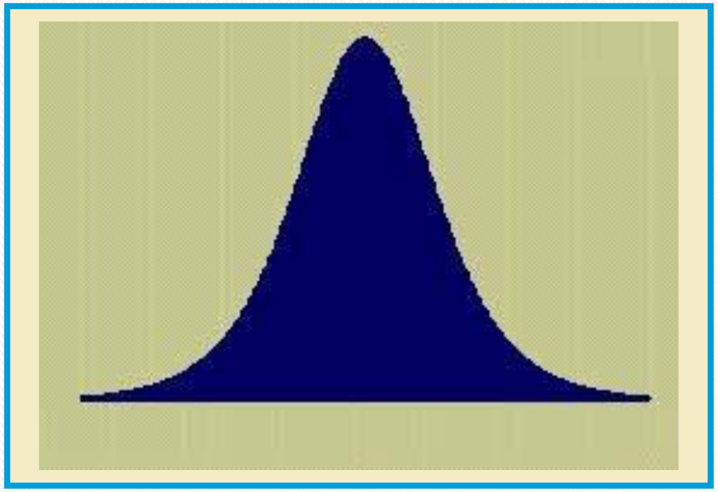
$$\frac{\bar{x} - \mu}{s / \sqrt{n}} \text{ is not normal!}$$

- has a standard normal distribution.
- But if σ is unknown, and we must use *s* to estimate it, the resulting statistic **is not normal**.

Student's t Distribution

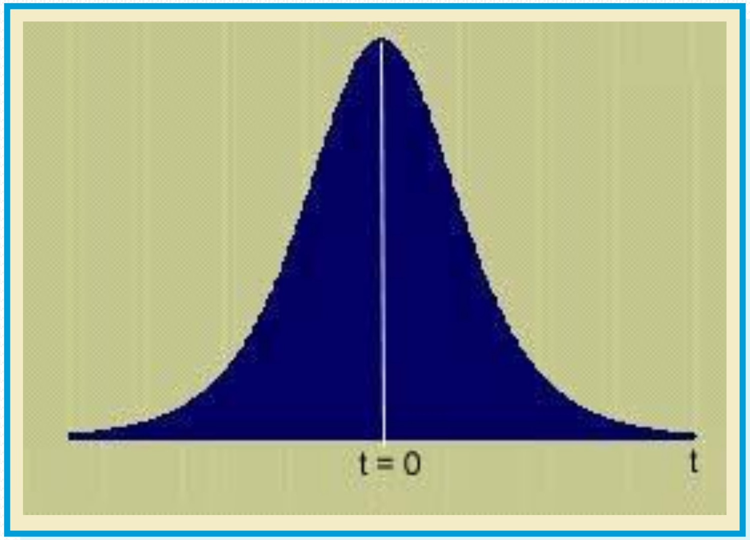
- Fortunately, this statistic does have a sampling distribution that is well known to statisticians, called the **Student's t distribution**, with ***n*-1 degrees of freedom.**

$$t = \frac{\bar{x} - \mu}{s / \sqrt{n}}$$



•We can use this distribution to create estimation testing procedures for the population mean μ.

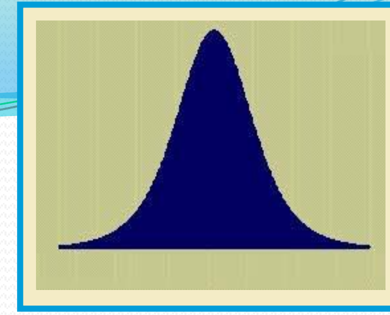
Properties of Student's *t*



- Mound-shaped and symmetric about 0.
- More variable than *z*, with "heavier tails"

- Shape depends on the sample size *n* or the **degrees of freedom, *n*-1.**
- As *n* increases the shapes of the *t* and *z* distributions become almost identical.

Small Sample Inference for a Population Mean μ



- The basic procedures are the same as those used for large samples. For a test of hypothesis:

Test $H_0: \mu = \mu_0$ versus H_a : one or two tailed

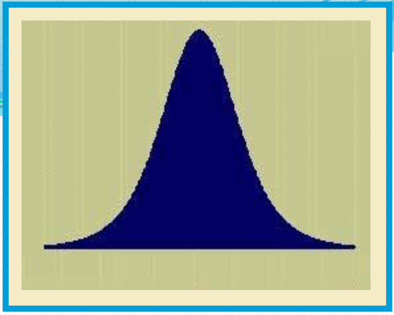
using the test statistic

$$t = \frac{\bar{x} - \mu_0}{s / \sqrt{n}}$$

using p-values or a rejection region based on

a t-distributi on with $df = n - 1$.

Small Sample Inference for a Population Mean μ



- For a 100(1–α)% confidence interval for the population mean μ:

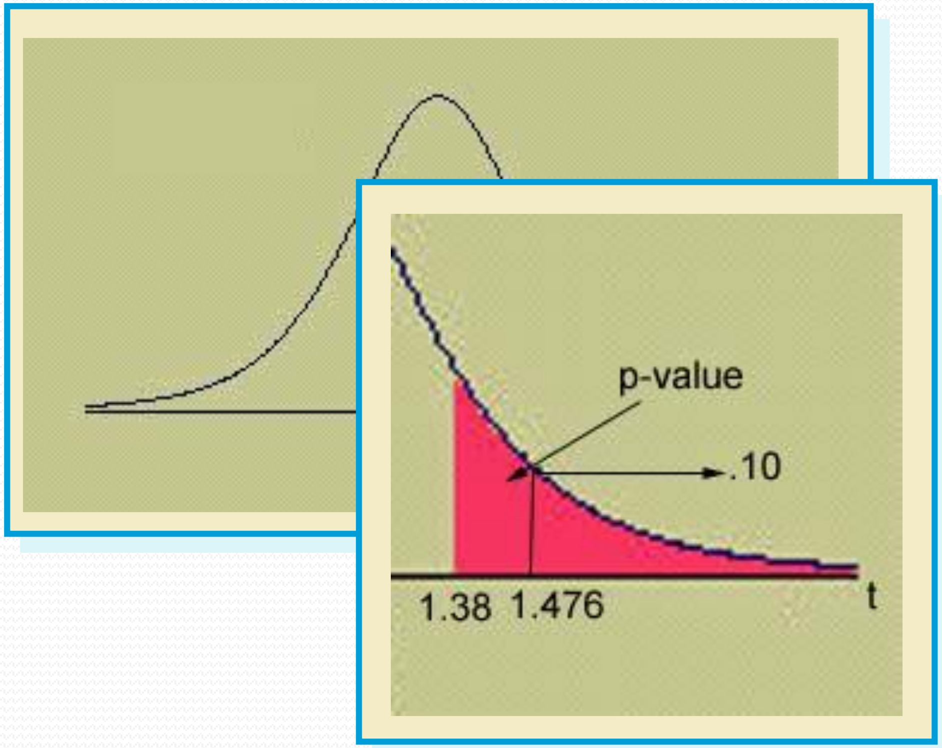
$$\bar{x} \pm t_{\alpha/2} \frac{s}{\sqrt{n}}$$

where $t_{\alpha/2}$ is the value of t that cuts off area $\alpha/2$ in the tail of a t-distribution with $df = n-1$.

Approximating the *p*-value

- You can only approximate the *p*-value for the test using Table 4.

df	$t_{.100}$	$t_{.050}$
1	3.078	6.314
2	1.886	2.920
3	1.638	2.353
4	1.533	2.132
5	1.476	2.015



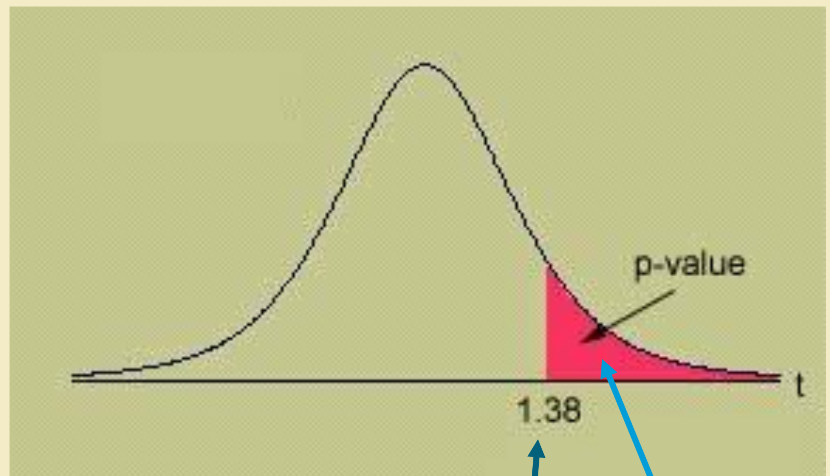
Since the observed value of $t = 1.38$ is smaller than $t_{.10} = 1.476$,

p-value $> .10$.

The exact *p*-value

- You can get the exact *p*-value using some calculators or a computer.

p-value = .113 which is greater than .10 as we approximated using Table 4.

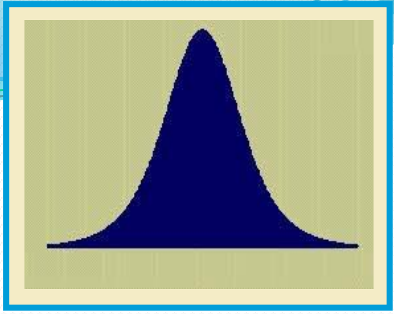


One-Sample T: Times

Test of mu = 15 vs > 15

Variable	N	Mean	StDev	SE Mean	95% Lower Bound	T	P
Times	6	19.1667	7.3869	3.0157	13.0899	1.38	0.113

Testing the Difference between Two Means



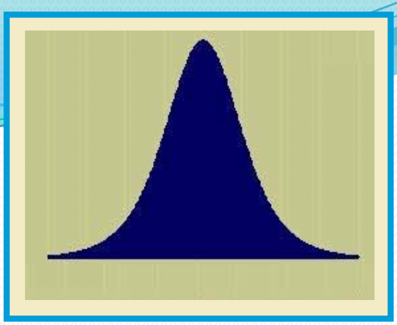
As in Chapter 9, independent random samples of size n_1 and n_2 are drawn from populations 1 and 2 with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2.

Since the sample sizes are small, the two populations must be normal.

- To test:
  - **H_0: $\mu_1 - \mu_2 = D_0$ versus H_a: one of three**

where D_0 is some hypothesized difference, usually 0.

Testing the Difference between Two Means

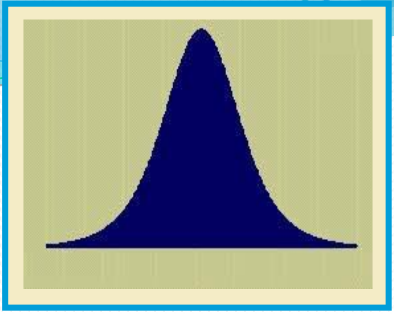


- The test statistic used in Chapter 9

$$z \approx \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

- does not have either a *z* or a *t* distribution, and cannot be used for small-sample inference.
- We need to make one more assumption, that **the population variances, although unknown, are equal.**

Testing the Difference between Two Means



- Instead of estimating each population variance separately, we estimate the common variance with

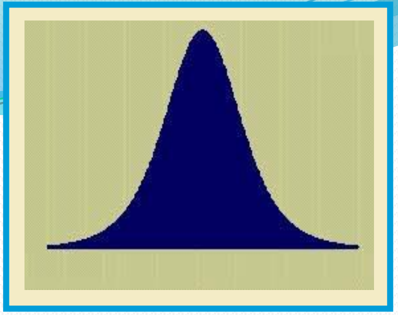
$$s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

- And the resulting test statistic,

$$t = \frac{\bar{x}_1 - \bar{x}_2 - D_0}{\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

has a *t* distribution with n_1+n_2-2 degrees of freedom.

Estimating the Difference between Two Means



- You can also create a 100(1-α)% confidence interval for $\mu_1 - \mu_2$.

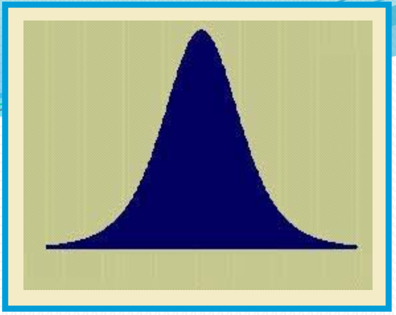
$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2} \sqrt{s^2 \left(\frac{1}{n_1} + \frac{1}{n_2} \right)}$$

$$\text{with } s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

Remember the three assumptions:

1. Original populations normal
2. Samples random and independent
3. Equal population variances.

Testing the Difference between Two Means



- How can you tell if the equal variance assumption is reasonable?

Rule of Thumb :

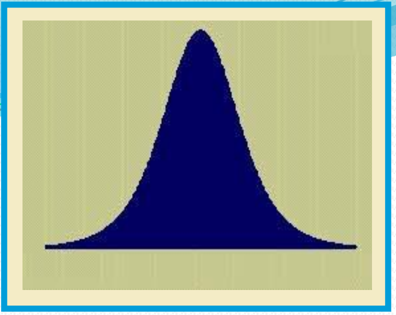
If the ratio, $\dfrac{\text{larger } s^2}{\text{smaller } s^2} \leq 3,$

the equal variance assumption is reasonable.

If the ratio, $\dfrac{\text{larger } s^2}{\text{smaller } s^2} > 3,$

use an alternative test statistic.

Testing the Difference between Two Means



- If the population variances cannot be assumed equal, the test statistic

$$t \approx \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

$$df \approx \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{(s_1^2 / n_1)^2}{n_1 - 1} + \frac{(s_2^2 / n_2)^2}{n_2 - 1}}$$

- has an approximate *t* distribution with degrees of freedom given above. This is most easily done by computer.

The Paired-Difference Test

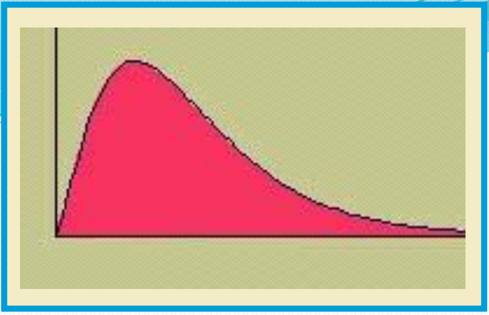
To test $H_0 : \mu_1 - \mu_2 = 0$ we test $H_0 : \mu_d = 0$ using the test statistic

$$t = \frac{\bar{d} - 0}{s_d / \sqrt{n}}$$

where n = number of pairs, $\bar{d}$ and s_d are the mean and standard deviation of the differences, d_i.

Use the p-value or a rejection region based on a t-distribution with $df = n - 1$.

Inference Concerning a Population Variance

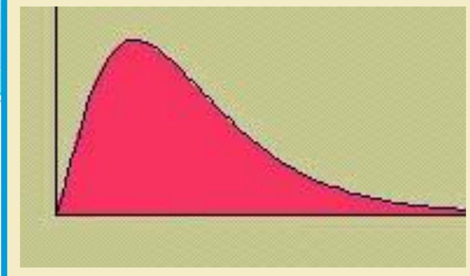


- Sometimes the primary parameter of interest is not the population mean μ but rather the population variance σ^2. We choose a random sample of size *n* from a normal distribution.
- The sample variance s^2 can be used in its standardized form:

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$$

- which has a Chi-Square distribution with *n* - 1 degrees of freedom.

Inference Concerning a Population Variance



To test $H_0 : \sigma^2 = \sigma_0^2$ versus H_a : one or two tailed

we use the test statistic

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$$ with a rejection region based on

a chi-square distribution with $df = n-1$.

Confidence interval :

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{(1-\alpha/2)}}$$

Inference Concerning Two Population Variances

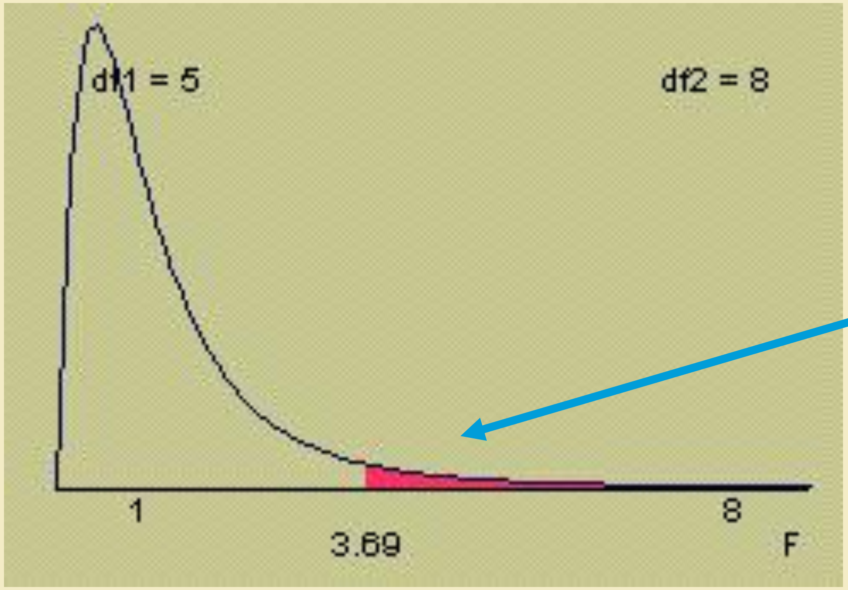
- We can make inferences about the ratio of two population variances in the form a ratio. We choose two independent random samples of size n_1 and n_2 from normal distributions.
- If the two population variances are equal, the statistic

$$F = \frac{s_1^2}{s_2^2}$$

- has an F distribution with $df_1 = n_1 - 1$ and $df_2 = n_2 - 1$ degrees of freedom.

Inference Concerning Two Population Variances

- Table 6 gives only upper critical values of the F statistic for a given pair of df_1 and df_2.



For example, the value of F that cuts off .05 in the upper tail of the distribution with df_1 = 5 and df_2 = 8 is F =3.69.

Inference Concerning Two Population Variances

To test $\text{H}_0 : \sigma_1^2 = \sigma_2^2$ versus H_a : one or two tailed
we use the test statistic

$$F = \frac{s_1^2}{s_2^2}$$ where s_1^2 is the larger of the two sample variances.

with a rejection region based on an F distribution with $df_1 = n_1 - 1$ and $df_2 = n_2 - 1$.

Confidence interval :

$$\frac{s_1^2}{s_2^2}\frac{1}{F_{df_1,df_2}} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{s_1^2}{s_2^2}F_{df_2,df_1}$$

Parameter	Test Statistic	Degrees of Freedom
μ	$t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$	$n - 1$
$\mu_1 - \mu_2$ (equal variances)	$t = \dfrac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{s^2\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$	$n_1 + n_2 - 2$
$\mu_1 - \mu_2$ (unequal variances)	$t \approx \dfrac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$	Satterthwaite's approximation
$\mu_1 - \mu_2$ (paired samples)	$t = \dfrac{\bar{d} - \mu_d}{s_d/\sqrt{n}}$	$n - 1$
σ^2	$\chi^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$	$n - 1$
σ_1^2/σ_2^2	$F = s_1^2/s_2^2$	$n_1 - 1$ and $n_2 - 1$

UNIT-IV

Queuing Theory

Queueing Theory

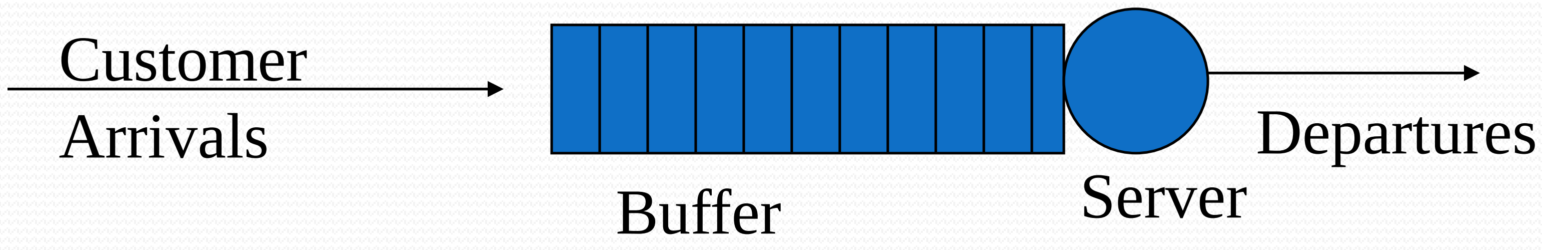
- Plan:
  - Introduce basics of Queueing Theory
  - Define notation and terminology used
  - Discuss properties of queuing models
  - Show examples of queueing analysis:
    - M/M/1 queue
    - Variations on the M/G/1 queue
    - Open queueing network models
    - Closed queueing network models

Queueing Theory Basics

- Queueing theory provides a very general framework for modeling systems in which customers must line up (queue) for service (use of resource)
  - Banks (tellers)
  - Restaurants (tables and seats)
  - Computer systems (CPU, disk I/O)
  - Networks (Web server, router, WLAN)

Queue-based Models

- Queueing model represents:
  - Arrival of jobs (customers) into system
  - Service time requirements of jobs
  - Waiting of jobs for service
  - Departures of jobs from the system
- Typical diagram:



Why Queue-based Models?

- In many cases, the use of a queuing model provides a quantitative way to assess system performance
  - Throughput (e.g., job completions per second)
  - Response time (e.g., Web page download time)
  - Expected waiting time for service
  - Number of buffers required to control loss
- Reveals key system insights (properties)
- Often with efficient, closed-form calculation

Caveats and Assumptions

- In many cases, using a queuing model has the following implicit underlying assumptions:
  - Poisson arrival process
  - Exponential service time distribution
  - Single server
  - Infinite capacity queue
  - First-Come-First-Serve (FCFS) discipline (also known as FIFO: First-In-First-Out)
- Note: important role of memoryless property!

Advanced Queueing Models

- There is TONS of published work on variations of the basic model:
  - Correlated arrival processes
  - General (G) service time distributions
  - Multiple servers
  - Finite capacity systems
  - Other scheduling disciplines (non-FIFO)
- We will start with the basics!

Queue Notation

- Queues are concisely described using the Kendall notation, which specifies:
  - Arrival process for jobs {M, D, G, …}
  - Service time distribution {M, D, G, …}
  - Number of servers {1, n}
  - Storage capacity (buffers) {B, infinite}
  - Service discipline {FIFO, PS, SRPT, …}
- Examples: M/M/1, M/G/1, M/M/c/c

The M/M/1 Queue

- Assumes Poisson arrival process, exponential service times, single server, FCFS service discipline, infinite capacity for storage, with no loss
- Notation: M/M/1
  - Markovian arrival process (Poisson)
  - Markovian service times (exponential)
  - Single server (FCFS, infinite capacity)

The M/M/1 Queue (cont'd)

- Arrival rate: λ (e.g., customers/sec)
  - Inter-arrival times are exponentially distributed (and independent) with mean $1 / \lambda$
- Service rate: μ (e.g., customers/sec)
  - Service times are exponentially distributed (and independent) with mean $1 / \mu$
- System load: $\rho = \lambda / \mu$

  $0 \leq \rho \leq 1$ (also known as utilization factor)
- Stability criterion: $\rho < 1$ (single server systems)

Queue Performance Metrics

- N: Avg number of customers in system as a whole, including any in service
- Q: Avg number of customers in the queue (only), excluding any in service
- W: Avg waiting time in queue (only)
- T: Avg time spent in system as a whole, including wait time plus service time
- Note: Little's Law: $N = \lambda T$

M/M/1 Queue Results

- Average number of customers in the system: $N = \rho / (1 - \rho)$
- Variance: $Var(N) = \rho / (1 - \rho)^2$

- Waiting time: $W = \rho / (\mu (1 - \rho))$
- Time in system: $T = 1 / (\mu (1 - \rho))$

The M/D/1 Queue

- Assumes Poisson arrival process, deterministic (constant) service times, single server, FCFS service discipline, infinite capacity for storage, no loss
- Notation: M/D/1
  - Markovian arrival process (Poisson)
  - Deterministic service times (constant)
  - Single server (FCFS, infinite capacity)

M/D/1 Queue Results

- Average number of customers: $Q = \rho/(1 - \rho) - \rho^2 / (2 (1 - \rho))$
- Waiting time: $W = x \rho / (2 (1 - \rho))$ where x is the mean service time
- Note that lower variance in service time means less queueing occurs ☺

UNIT-V

Stochastic Processes

Indexed collection of random variables

$\{X_t\}_{t \in T}$, for each $t \in T$, X_t is a random variable

T = Index Set

State Space = range (possible values) of all X_t

- Stationary Process: Joint Distribution of the X's dependent only on their relative positions. (not affected by time shift)
  $(X_{t1}, \ldots, X_{tn})$ has the same distribution as
  $(X_{t1+h}, X_{t2+h} \ldots, X_{tn+h})$

  e.g.) (X_8, X_{11}) has same distribution as (X_{20}, X_{23})

- Markov Process: P_r of any future event given present does not depend on past:

$t_0 < t_1 < \ldots < t_{n-1} < t_n < t$

$P(a \leq X_t \leq b \mid X_{tn} = x_{tn}, \ldots\ldots\ldots, X_{to} = x_{to})$

| ← future → | | present | | ← past → |

$P\,(a \leq X_t \leq b \mid X_{tn} = x_{tn})$

Another way of writing this:

$P\{X_{t+1} = j \mid X_0 = k_0, X_1 = k_1, \ldots, X_t = i\} =$

$P\{X_{t+1} = j \mid X_t = i\}$ for t=0,1,.. And

every sequence i, j, k_0, k_1,… k_{t-1},

- Markov Chains:

State Space {0, 1, …}

Discrete Time

{T = (0, 1, 2, …)}

Continuous Time

{T = [0, ∞)}

- Finite number of states
- The markovian property
- Stationary transition probabilities
- A set of initial probabilities $P\{X_0 = i\}$ for i

- Note:

$$P_{ij} = P(X_{t+1} = j \mid X_t = i)$$
$$= P(X_1 = j \mid X_0 = i)$$

Only depends on going <u>ONE</u> step

Stage (t) Stage (t + 1)

State i $\xrightarrow{\ P_{ij}\ }$ State j (with prob. P_{ij})

These are conditional probabilities!

Note that given $X_t = i$, must enter some state at stage t + 1

0		P_{i0}
1		P_{i1}
2	with	P_{i2}
......	prob.	.....
j		P_{ij}
......		.....
m		P_{im}

$$\sum_{j=0}^{m} P_{ij} = 1$$

Convenient to give transition probabilities in matrix form

$$P = P_{(m+1)\times (m+1)} = P_{ij}$$

go to i^{th} state

$$= \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ m \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & \cdots & j & \cdots & m \end{array} \\ \begin{bmatrix} P_{00} & & & & P_{0j} & & \\ P_{10} & & & & P_{1j} & & \\ P_{20} & & & & P_{2j} & & \\ & & & & & & \\ P_{i0} & & & & P_{ij} & & \\ & & & & & & \\ P_{m0} & & & & P_{mj} & & P_{mm} \end{bmatrix} \end{array}$$

Rows are given in this stage

Rows sum to 1

Example:

t = day index 0, 1, 2, ...

$X_t = 0$ high defective rate on t^{th} day

$\quad = 1$ low defective rate on t^{th} day

two states ===> $n = 1$ (0, 1)

$P_{00} = P(X_{t+1} = 0 \mid X_t = 0) = 1/4$ 0   0

$P_{01} = P(X_{t+1} = 1 \mid X_t = 0) = 3/4$ 0   1 $\rightarrow$

$P_{10} = P(X_{t+1} = 0 \mid X_t = 1) = 1/2$ 1   0 $\rightarrow$

$P_{11} = P(X_{t+1} = 1 \mid X_t = 1) = 1/2$ 1   1 $\rightarrow$

$\therefore \quad P =$ $\rightarrow$

$$\begin{bmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{bmatrix}$$

Note:

Row sum to 1

$P_{00} = P(X_1 = 0 \mid X_0 = 0) = 1/4$

$= P(X_{36} = 0 \mid X_{35} = 0)$

Also

$= P(X_2 = 0 \mid X_1 = 0, X_0 = 1)$

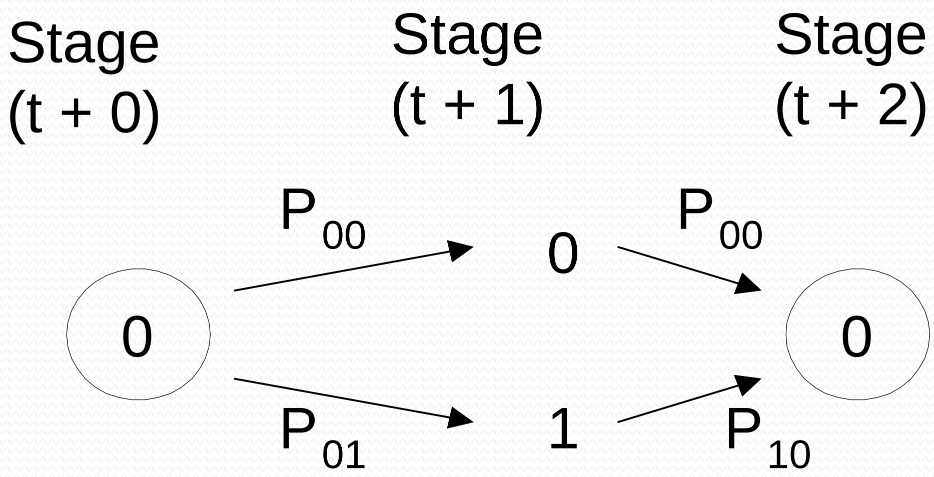
$= P(X_2 = 0 \mid X_1 = 0) = P_{00}$

What is $P(X_2 = 0 \mid X_0 = 0)$

This is a two-step trans.

stage 0 → stage 2

or t → t + 2



$P(X_2 = 0, X_1 = 0 \mid X_0 = 0) = P_{00} P_{00}$

$P(X_2 = 0 \mid X_0 = 0) = P_{00}^{(2)}$

$= P_{00} P_{00} + P_{01} P_{10}$

$= 1/4 * 1/4 + 3/4 * 1/2 = 7/16$ or 0.4575

Homogeneous, Irreducible, Aperiodic

$\Rightarrow$ Limiting State Probabilities:

$$P_j = \lim_{k \to \infty} P_j(k), \quad (j=0, 1, 2\ldots)$$

Exist and are Independent of the $P_j(0)$'s

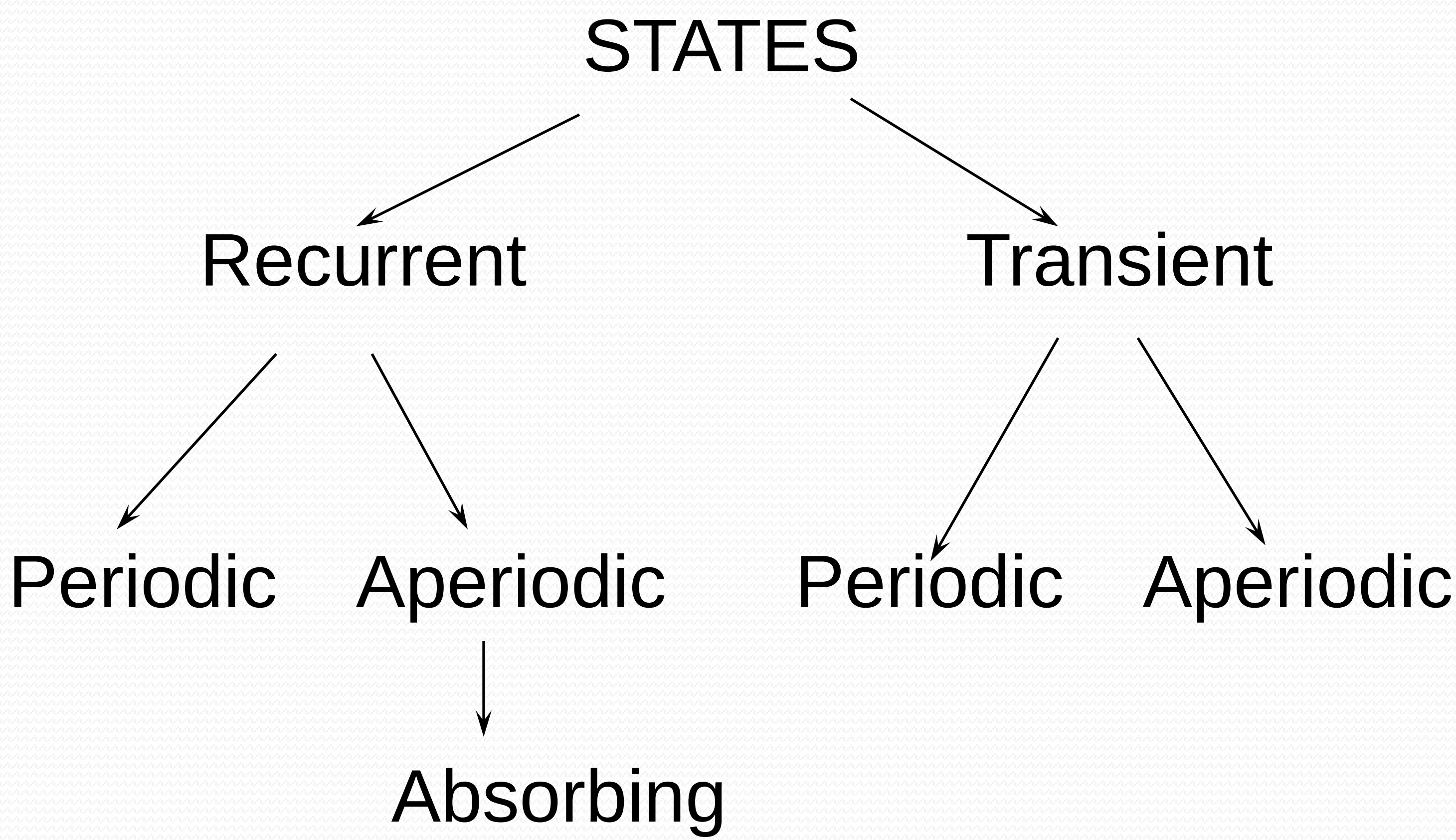
If all states of the chain are recurrent and their mean recurrence time is finite,

$\Rightarrow P_j$'s are a stationary probability distribution and can be determined by solving the equations

$P_j = \sum_i P_i P_{ij}$, (j=0,1,2..) and $\sum_i P_i = 1$

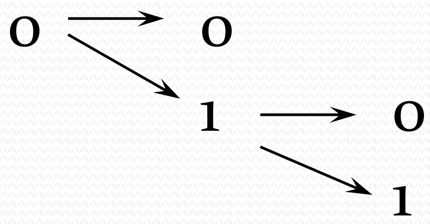
Solution ==> Equilibrium State Probabilities

Note: State Classification:



Example II

Example II: $P = \begin{bmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{bmatrix}$

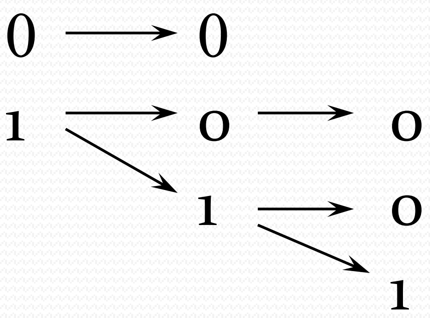


- Communicating Class {0, 1}
- Aperiodic chain
- Irreducible
- Positive Recurrent

Example III

Example III:

$$P = \begin{bmatrix} 1 & 0 \\ 1/4 & 3/4 \end{bmatrix}$$



- Absorbing State {0}
- Transient State {1}
- Aperiodic chain
- Communicating Classes {0} {1}

Exercise

Exercise: Classify States.

$$P = \begin{bmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.25 & 0 & 0.75 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0.2 & 0 & 0.8 & 0 \end{bmatrix}$$

Major Results

- Result I:

  j is transient

  $P(X_n = j \mid X_o = i) = P_{ij}^{(n)} \xrightarrow{\text{as } n \to \infty} 0$

- Result II:

  If chain is irreducible:

$$\frac{1}{n}\sum_{k=1}^{n} P_{ij}^{(k)} \to \Pi_j \quad \text{as } n \to \infty$$