INSTITUTE OF AERONAUTICAL ENGINEERING

Dundigal, Hyderabad -500 043

Department of Civil Engineering

Strength of Material- I

Course Lecturer:

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(Professor)

Course Goal

- To introduce students to concepts of stresses and strain, shearing force and bending diagrams as well as deflection of different structural elements like beams.

- To develop theoretical and analytical skills relevant to the areas mentioned in above.

COURSE OUTLINE

UNIT	TITLE	CONTENTS
I	**Simple stresses-strains and strain energy**	Elasticity and plasticity – Types of stresses and strains – Hooke's law – stress – strain diagram for mild steel – Working stress – Factor of safety – Lateral strain, Poisson's ratio and volumetric strain – Elastic modulii and the relationship between them – Bars of varying section – composite bars – Temperature stresses. Elastic constants. Gradual, sudden, impact and shock loadings – simple applications
II	**Shear force and bending moment**	Definition of beam – Types of beams – Concept of shear force and bending moment – S.F and B.M diagrams for cantilever, simply supported and overhanging beams subjected to point loads, uniformly distributed load, uniformly varying loads and combination of these loads – Point of contraflexure – Relation between S.F., B.M and rate of loading at a section of a beam.

UNIT	TITLE	CONTENTS
III	**Flexural stresses and Shear stresses**	Theory of simple bending – Assumptions – Derivation of bending equation: M/I = f/y = E/R - Neutral axis – Determination of bending stresses – Section modulus of rectangular and circular sections (Solid and Hollow), I,T, Angle and Channel sections – Design of simple beam sections. Derivation of formula – Shear stress distribution across various beam sections like rectangular, circular, triangular, I, T angle sections.
IV	**Principal stresses and strains and Theories of failure**	Introduction – Stresses on an inclined section of a bar under axial loading – compound stresses – Normal and tangential stresses on an inclined plane for biaxial stresses – Two perpendicular normal stresses accompanied by a state of simple shear – Mohr's circle of stresses – Principal stresses and strains – Analytical and graphical solutions. Introduction – Various theories of failure - Maximum Principal Stress Theory, Maximum Principal Strain Theory, Strain Energy and Shear Strain Energy Theory (Von Mises Theory).

UNIT	TITLE	CONTENTS
V	**Deflection of beams and Conjugate beam method**	Bending into a circular arc – slope, deflection and radius of curvature – Differential equation for the elastic line of a beam – Double integration and Macaulay's methods – Determination of slope and deflection for cantilever and simply supported beams subjected to point loads, U.D.L, Uniformly varying load-Mohr's theorems – Moment area method – application to simple cases including overhanging beams. Introduction – Concept of conjugate beam method, Difference between a real beam and a conjugate beam, Deflections of determinate beams with constant and different moments of inertia.

TEXT BOOKS

- Strength of materials by R. K. Bansal, Laxmi Publications (P) ltd., New Delhi, India.
- Strength of Materials by R. S. Khurmi, S. Chand publication New Delhi, India
- Strength of materials by Dr. Sadhu Singh, Khanna Publications Ltd

Course Objectives

Upon successful completion of this course, students should be able to:

- To impart adequate knowledge to find stresses and strain in various structural parts used in buildings beams, bridges etc.
- Understand the difference between statically determinate and indeterminate problems.
- Analyze stresses in two dimensions and understand the concepts of principal stresses and the use of Mohr circles to solve two-dimensional stress problems.

Course objectives contd.

- Draw shear force and bending moment diagrams of simple beams and understand the relationships between loading intensity, shearing force and bending moment.
- Compute the bending stresses in beams with one or two materials.
- Calculate the deflection of beams using the direct integration and moment-area method.

Course objectives

- To understand the theory of failure phenomenon and to learn how to prevent the failure.
- To impart adequate knowledge to continue the design and research activity in structural analysis.
- To apply this knowledge in practical application.

Teaching Strategies

- The course will be taught via Lectures. Lectures will also involve the solution of tutorial questions. Tutorial questions are designed to complement and enhance both the lectures and the students appreciation of the subject.
- Course work assignments will be reviewed with the students.
- Daily assessment through questioning and class notes.

UNITS:

	British	Metric	S.I.
1. Force	Ib, kip, Ton	g, kg,	N, kN
	1 kip = 1000 Ib 1 ton = 2240 Ib	1 kg = 1000 g Ton = 1000 kg	1 kN = 1000 N 1 kg = 10 N
2. Long	in, ft	m, cm, mm	m, cm, mm
	1 f = 12 in	1 m = 100 cm 1 cm = 10 mm 1 m =1000 mm 1 in = 2.54 cm	1 m = 100 cm 1 cm = 10 mm 1 m =1000 mm 1 in = 2.54 cm
3. Stress	psi, ksi $\frac{p}{in^2}, \frac{kip}{in^2}$	Pa ($\frac{N}{mm^2}$), MPa, GPa	

$$MPa = 10^6\ Pa = 10^6\ N/mm^2 \times \frac{1}{1000^2 \frac{mm^2}{m^2}}$$

$$MPa = \frac{N}{mm^2}$$

$$GPa = 10^9\ Pa = 10^9\ N/mm^2 \times \frac{1}{1000^2 \frac{mm^2}{m^2}} = 10^3 \frac{N}{mm^2} \times \frac{1}{1000 \frac{N}{kN}}$$

$$GPa = kN/mm^2$$

UNIT –I

Simple stress and strain

Syllabus

- **SIMPLE STRESSES AND STRAINS :** Elasticity and plasticity – Types of stresses and strains – Hooke's law – stress – strain diagram for mild steel – Working stress – Factor of safety – Lateral strain, Poisson's ratio and volumetric strain – Elastic modulii and the relationship between them – Bars of varying section – composite bars – Temperature stresses. Elastic constants.
- **STRAIN ENERGY:** Resilience – Gradual, sudden, impact and shock loadings – simple applications.

Concept of elasticity and plasticity

- **Strength** of Material is its ability to withstand and applied load without failure.
- **Elasticity**: Property of material by which it return to its original shape and size after removing the applied load , is called elasticity. And material itself is said to elastic.
- Plasticity: Characteristics of material by which it undergoes inelastic strains (Permanent Deformation) beyond the elastic limit, known as **plasticity**. This property is useful for pressing and forging.

Direct or Normal Stress

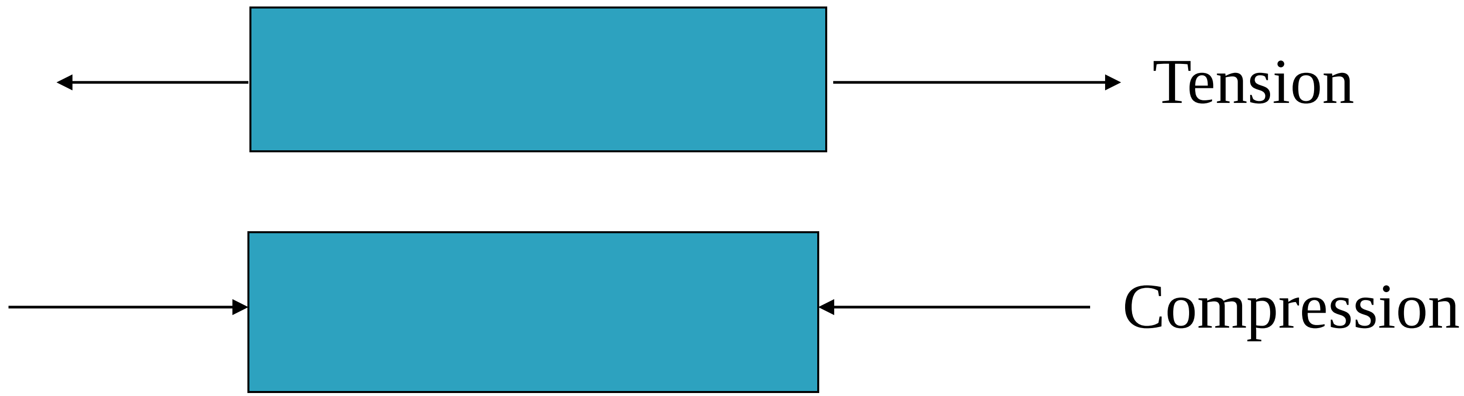
- When a force is transmitted through a body, the body tends to change its shape or deform. The body is said to be strained.

- Direct Stress = $\dfrac{\text{Applied Force (F)}}{\text{Cross Sectional Area (A)}}$

- **Units:** Usually N/m^2 (Pa), N/mm^2, MN/m^2, GN/m^2 or N/cm^2
- **Note:** $1\ N/mm^2 = 1\ MN/m^2 = 1\ MPa$

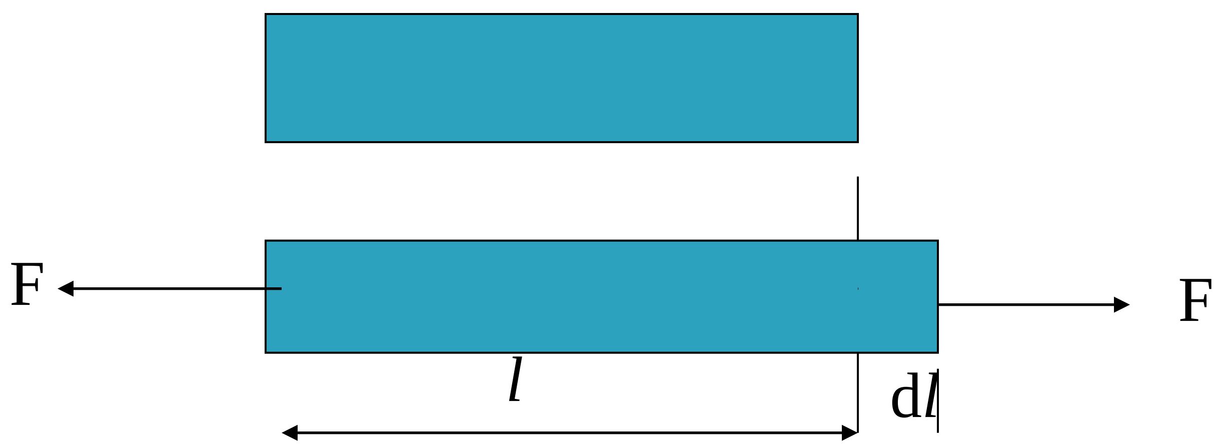
Direct Stress Contd.

- Direct stress may be tensile or compressive and result from forces acting perpendicular to the plane of the cross-section



Direct or Normal Strain

- When loads are applied to a body, some deformation will occur resulting to a change in dimension.
- Consider a bar, subjected to axial tensile loading force, F. If the bar extension is d*l* and its original length (before loading) is *l*, then tensile strain is:



Direct Strain (ε) = $\dfrac{\text{Change in Length}}{\text{Original Length}}$

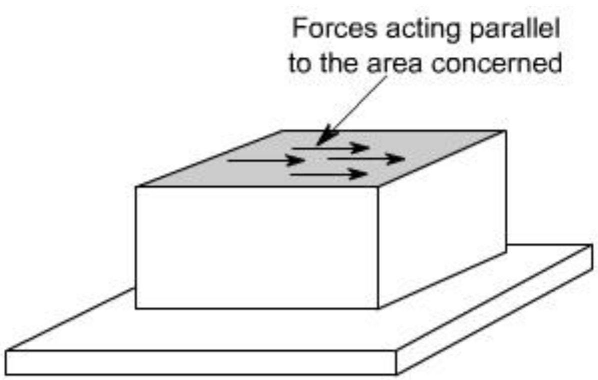
i.e. $\varepsilon = dl/l$

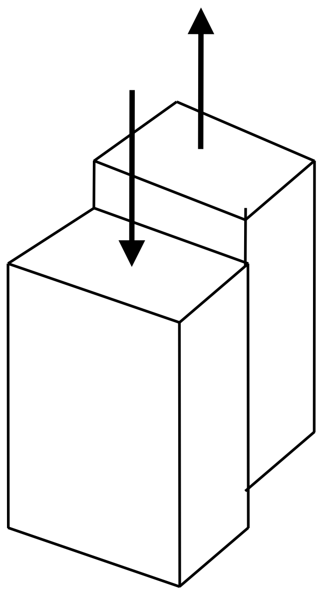
Direct or Normal Strain Contd.

- As strain is a ratio of lengths, it is dimensionless.
- Similarly, for compression by amount, dl: Compressive strain = - dl/L
- **Note:** Strain is positive for an increase in dimension and negative for a reduction in dimension.

Shear Stress and Shear Strain

- Shear stresses are produced by equal and opposite parallel forces not in line.
- The forces tend to make one part of the material slide over the other part.
- Shear stress is tangential to the area over which it acts.

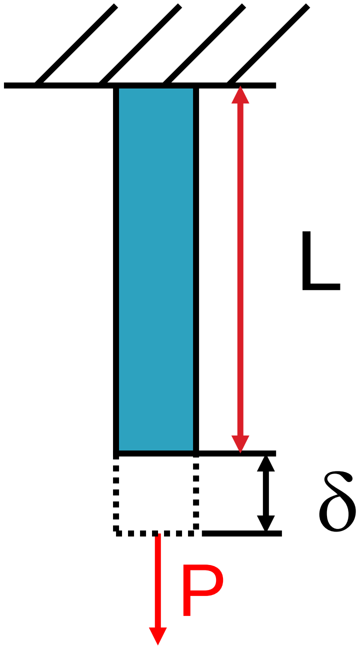




- Strain
- It is defined as deformation per unit length

- it is the ratio of change in length to original length
- Tensile strain (+ Ve) (ε) $= \dfrac{\text{increase in length}}{\text{Original length}} = \dfrac{\delta}{L}$

Compressive strain (- Ve) (ε) $= \dfrac{\text{decrease in length}}{\text{Original length}} = \dfrac{\delta}{L}$



Ultimate Strength

The **strength** of a material is a measure of the stress that it can take when in use. The **ultimate strength** is the measured stress at failure but this is not normally used for design because safety factors are required. The normal way to define a safety factor is :

$$safety\ \ factor = \frac{stress\, at\ \ failure}{stress\ \ when\, loaded} = \frac{Ultimate\ \ \ stress}{Permissible\ \ \ stress}$$

Strain

We must also define **strain**. In engineering this is not a measure of force but is a measure of the deformation produced by the influence of stress. For tensile and compressive loads:

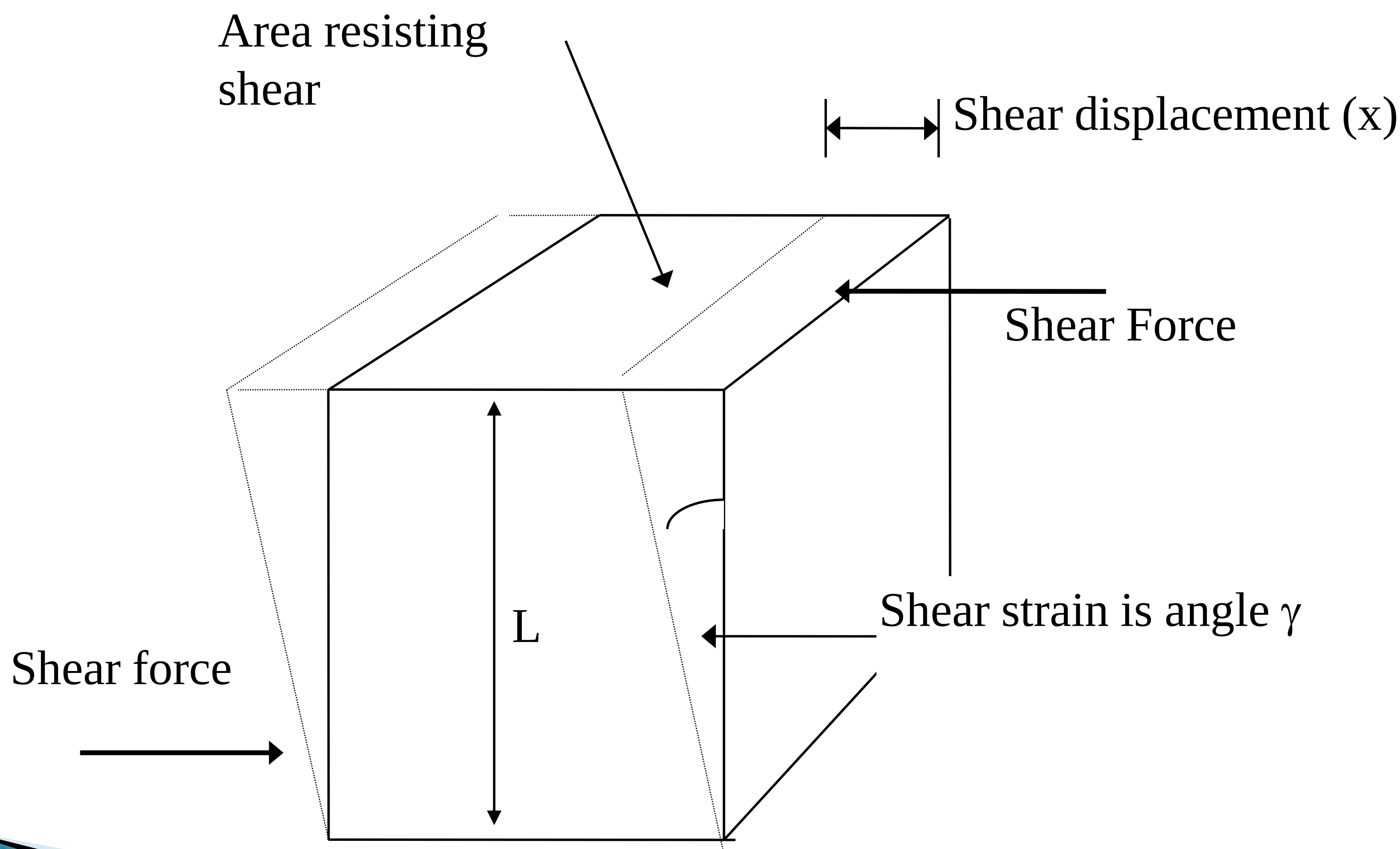
Strain is dimensionless, i.e. it is not measured in metres, kilograms etc.

$$strain \quad \varepsilon = \frac{increase\ in\ length\ \ x}{original\ length\ \ L}$$

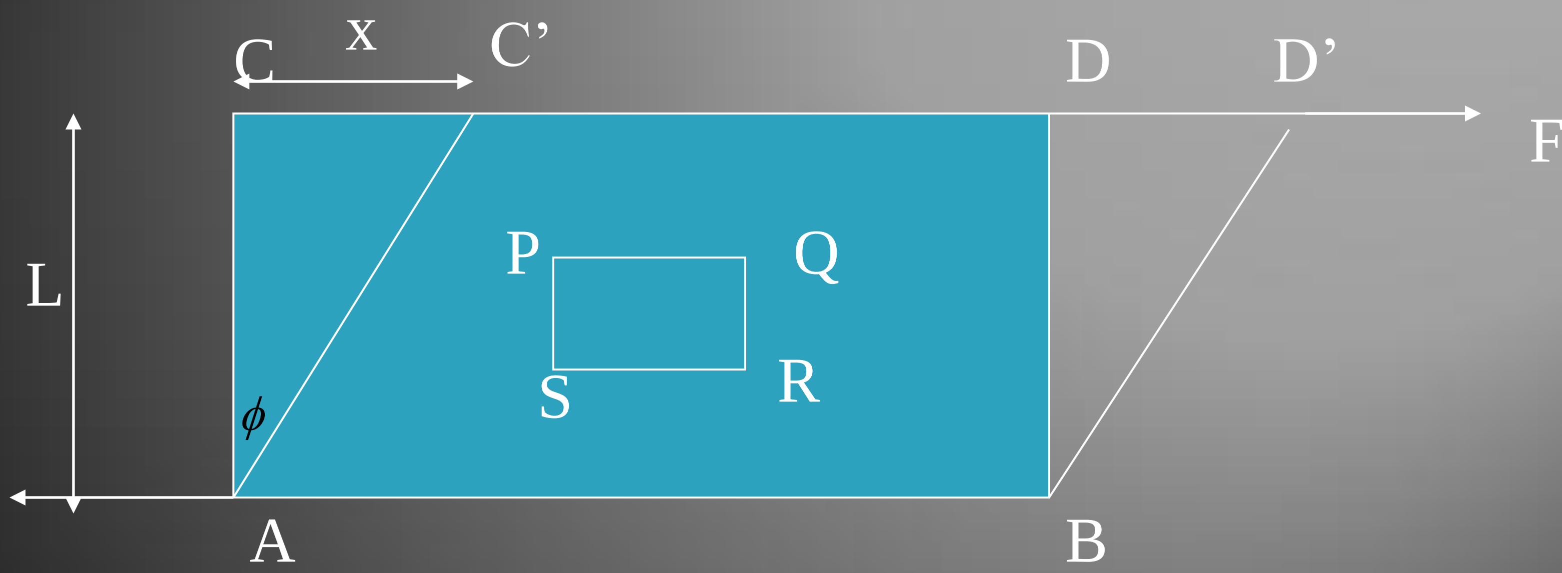
For shear loads the strain is defined as the angle γ This is measured in radians

$$shear\ strain \quad \gamma \approx \frac{shear\ displacement\ \ x}{width\ \ L}$$

Shear stress and strain



Shear Stress and Shear Strain Contd.



Shear strain is the distortion produced by shear stress on an element or rectangular block as above. The shear strain, γ (gamma) is given as:

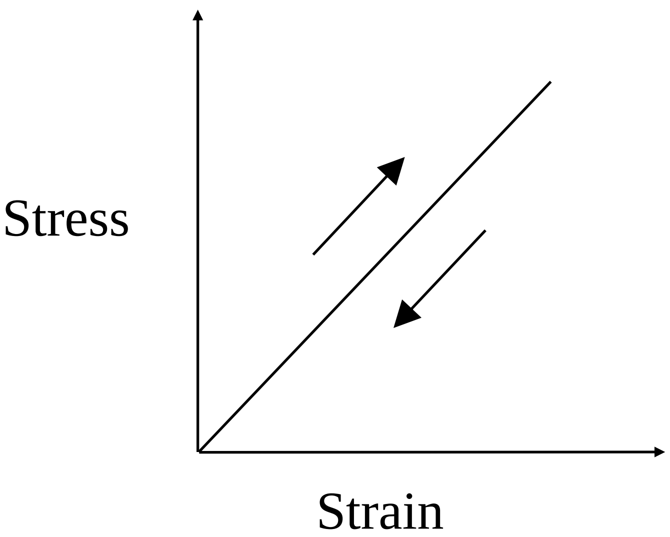
$$\gamma = x/L = \tan\phi$$

Shear Stress and Shear Strain Concluded

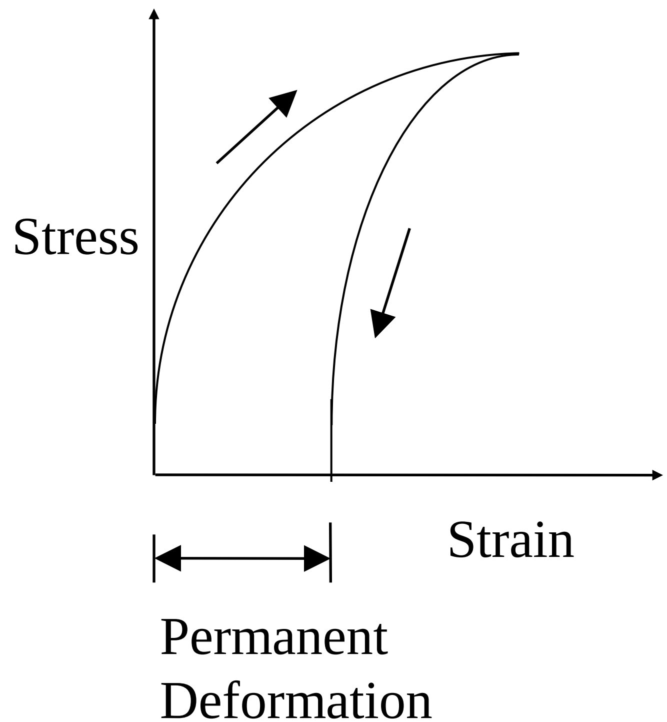
- For small , ϕ $\gamma = \phi$

- Shear strain then becomes the change in the right angle.
- It is dimensionless and is measured in radians.

Elastic and Plastic deformation



Elastic deformation



Plastic deformation

Modulus of Elasticity

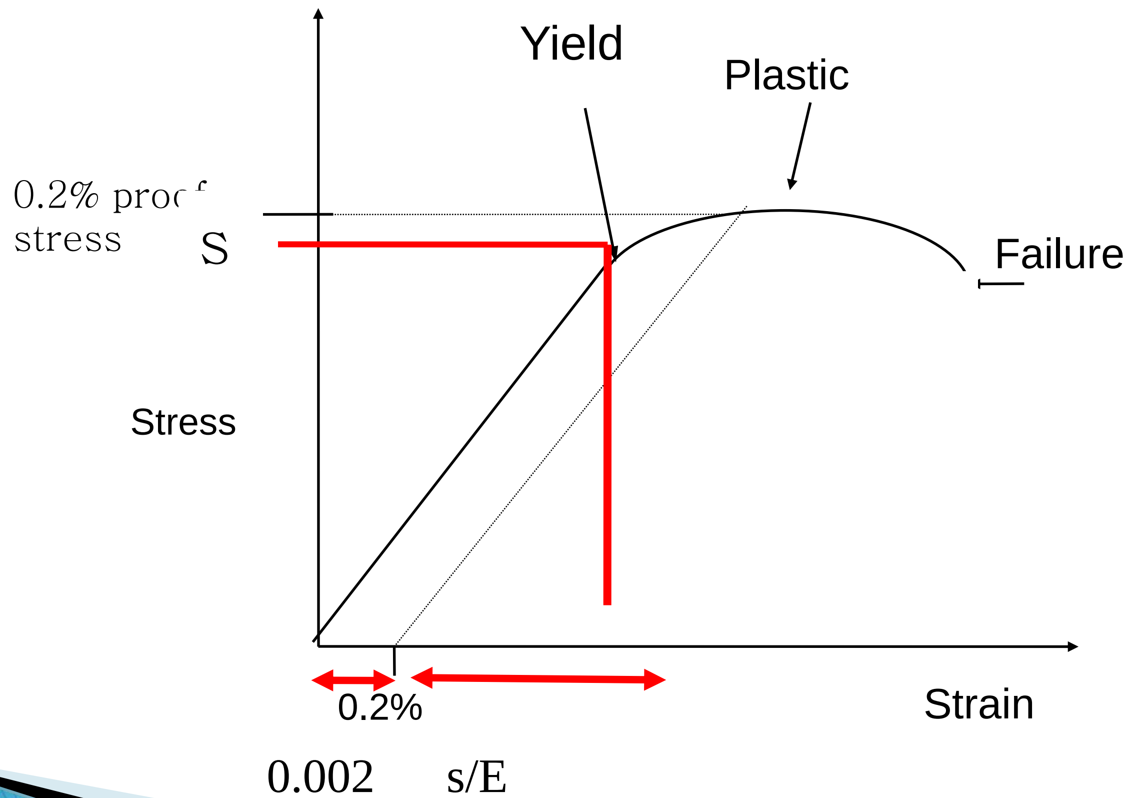
If the strain is "elastic" Hooke's law may be used to define

$$Youngs\ Modulus \quad E = \frac{Stress}{Strain} = \frac{W}{x} \times \frac{L}{A}$$

Young's modulus is also called the modulus of elasticity or stiffness and is a measure of how much strain occurs due to a given stress. Because strain is dimensionless Young's modulus has the units of stress or pressure

How to calculate deflection if the proof stress is applied and then partially removed.

If a sample is loaded up to the 0.2% proof stress and then unloaded to a stress s the strain x = 0.2% + s/E   where E is the Young's modulus



Volumetric Strain

- Hydrostatic stress refers to tensile or compressive stress in all dimensions within or external to a body.
- Hydrostatic stress results in change in volume of the material.
- Consider a cube with sides x, y, z. Let dx, dy, and dz represent increase in length in all directions.
- i.e. new volume = (x + dx) (y + dy) (z + dz)

Volumetric Strain Contd.

- Neglecting products of small quantities:
- New volume = x y z + z y dx + x z dy + x y dz
- Original volume = x y z
- ΔV = z y dx + x z dy + x y dz
- Volumetric strain, $\varepsilon_v = \dfrac{z\,y\,dx + x\,z\,dy + x\,y\,dz}{x\,y\,z}$
- ε_v = dx/x + dy/y + dz/z

$$\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Elasticity and Hooke’s Law

- All solid materials deform when they are stressed, and as stress is increased, deformation also increases.
- If a material returns to its original size and shape on removal of load causing deformation, it is said to be **elastic**.
- If the stress is steadily increased, a point is reached when, after the removal of load, not all the induced strain is removed.
- This is called the elastic limit.

Hooke's Law

- States that providing the limit of proportionality of a material is not exceeded, the stress is directly proportional to the strain produced.
- If a graph of stress and strain is plotted as load is gradually applied, the first portion of the graph will be a straight line.
- The slope of this line is the constant of proportionality called modulus of Elasticity, E or Young's Modulus.
- It is a measure of the stiffness of a material.

Hooke's Law

Modulus of Elasticity, E = $\dfrac{Direct\ stress}{Direct\ strain} = \dfrac{\sigma}{\varepsilon}$

Also: For Shear stress: Modulus of rigidity or shear modulus, G = $\dfrac{Shear\ stress}{Shear\ strain} = \dfrac{\tau}{\gamma}$

Also: Volumetric strain , is proportional to hydrostatic stress, within the elastic range i.e. :

$\sigma / \varepsilon_v = K$ called **bulk modulus**.

Stress-Strain Relations of Mild Steel

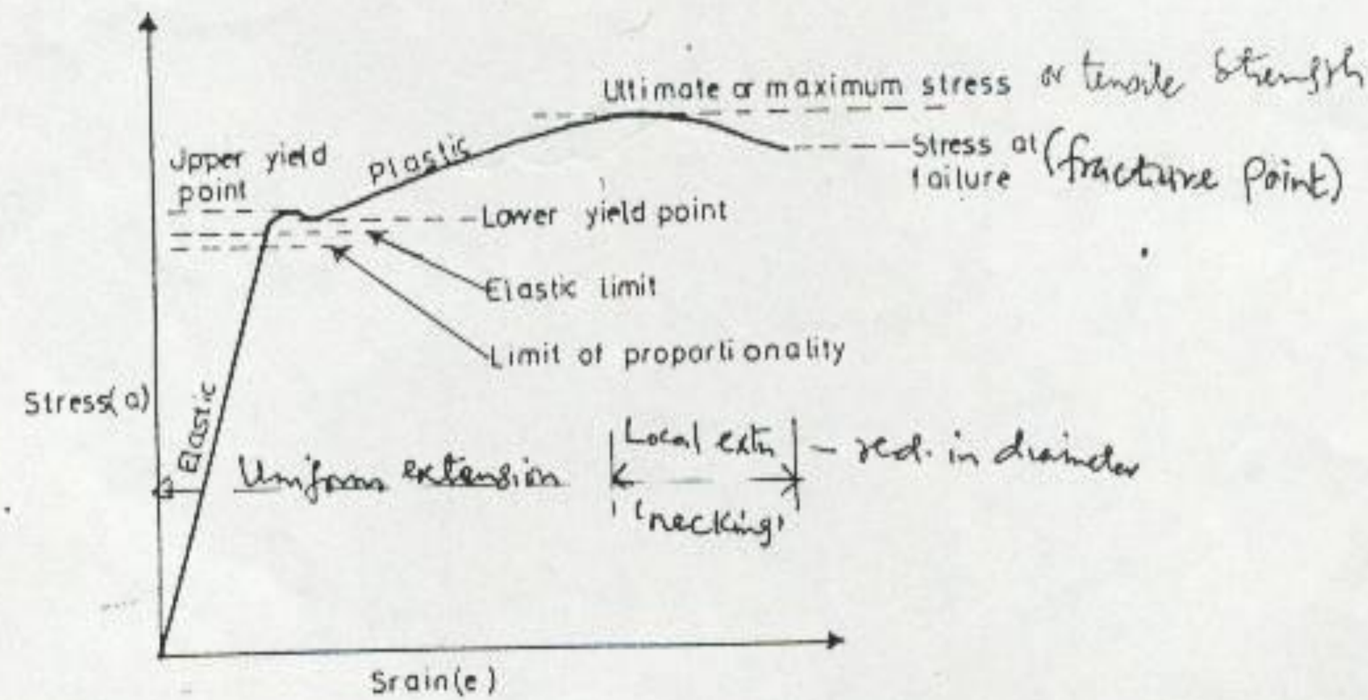


Fig: Behaviour of mild-steel rod under tension.

Equation For Extension

From the above equations:

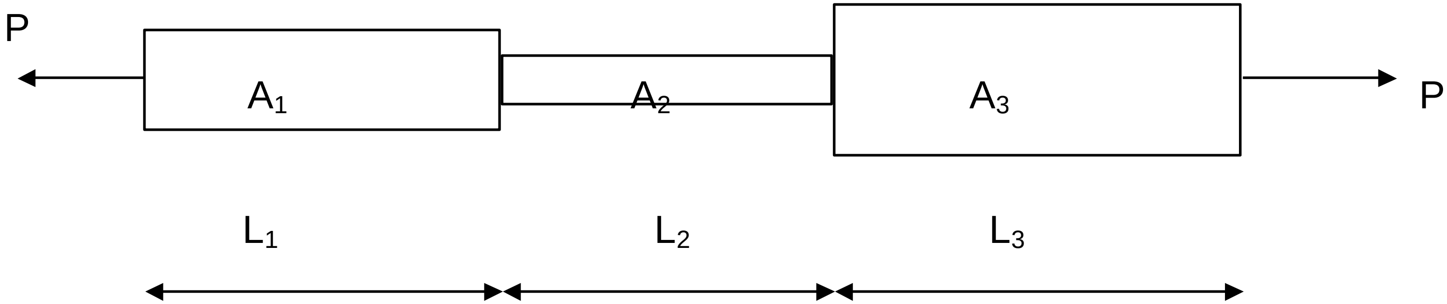
$$E = \frac{\sigma}{\varepsilon} = \frac{F / A}{dl / L} = \frac{F\ L}{A\ dl}$$

$$dl = \boxed{\frac{F\ L}{A\ E}}$$

This equation for extension is very important

Extension For Bar of Varying Cross Section

<u>For a bar of varying cross section:</u>



$$dl = \frac{F}{E}\left[\frac{L_1}{A_1} + \frac{L_2}{A_2} + \frac{L_3}{A_3}\right]$$

Factor of Safety

- The load which any member of a machine carries is called working load, and stress produced by this load is the working stress.
- Obviously, the working stress must be less than the yield stress, tensile strength or the ultimate stress.
- This working stress is also called the permissible stress or the allowable stress or the design stress.

Factor of Safety Contd.

- Some reasons for factor of safety include the inexactness or inaccuracies in the estimation of stresses and the non-uniformity of some materials.

$$\text{Factor of safety} = \frac{\textit{Ultimate or yield stress}}{\textit{Design or working stress}}$$

Note: Ultimate stress is used for materials e.g. concrete which do not have a well-defined yield point, or brittle materials which behave in a linear manner up to failure. Yield stress is used for other materials e.g. steel with well defined yield stress.

Results From a Tensile Test

(a) Modulus of Elasticity, $E = \dfrac{Stress\ up\ to\ \lim it\ of\ \ proportionality}{Strain}$

(b) Yield Stress or Proof Stress (See below)

(c) Percentage elongation = $\dfrac{Increase\ in\ gauge\ length}{Original\ gauge\ length}\ x\,100$

(d) Percentage reduction in area = $\dfrac{Original\ area - area\ at\ \ fracture}{Original\ area}\ x\,100$

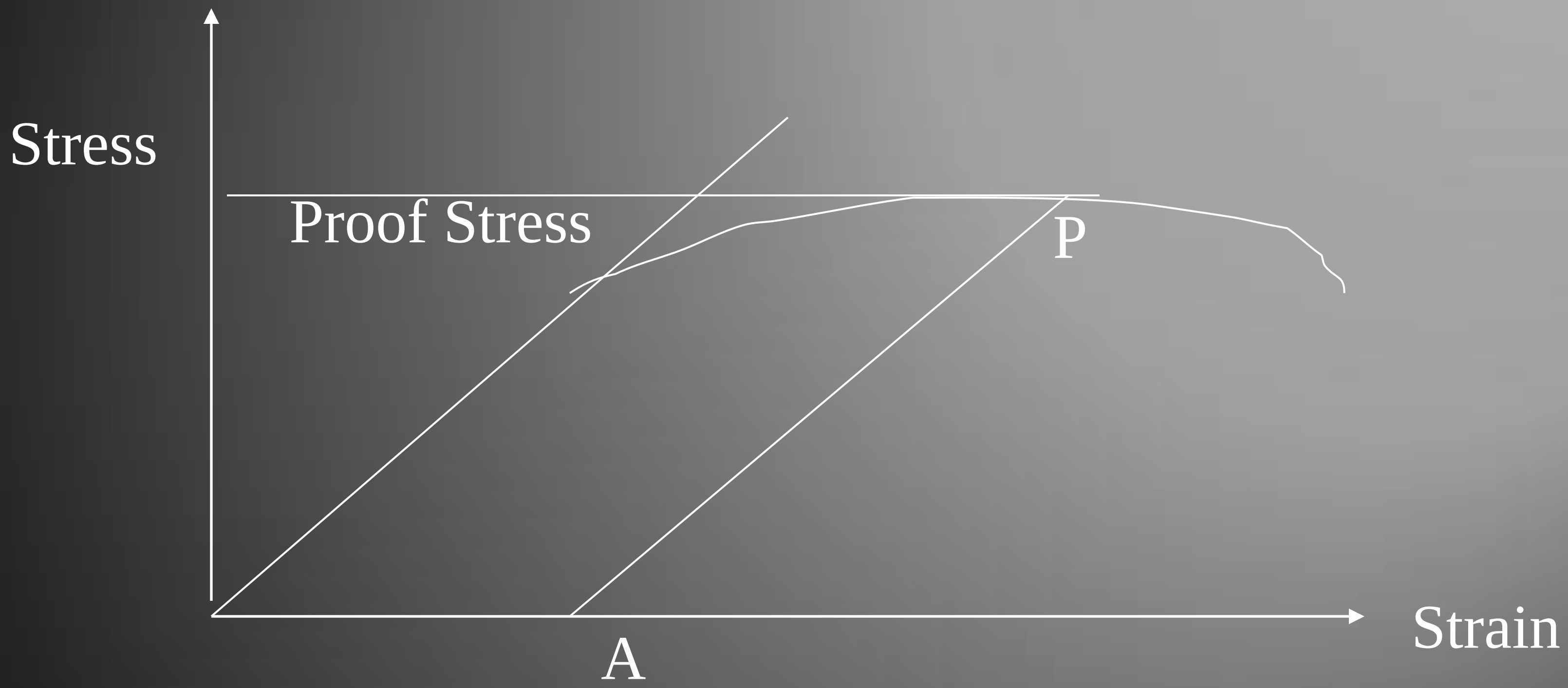
(e) Tensile Strength = $\dfrac{Maximum\ load}{Original\ cross\ \sec tional\ area}$

The percentage of elongation and percentage reduction in area give an indication of the ductility of the material i.e. its ability to withstand strain without fracture occurring.

Proof Stress

- High carbon steels, cast iron and most of the non-ferrous alloys do not exhibit a well defined yield as is the case with mild steel.
- For these materials, a limiting stress called proof stress is specified, corresponding to a non-proportional extension.
- The non-proportional extension is a specified percentage of the original length e.g. 0.05, 0.10, 0.20 or 0.50%.

Determination of Proof Stress



The proof stress is obtained by drawing AP parallel to the initial slope of the stress/strain graph, the distance, OA being the strain corresponding to the required non-proportional extension e.g. for 0.05% proof stress, the strain is 0.0005.

Thermal Strain

Most structural materials expand when heated,

in accordance to the law: $\varepsilon = \alpha\, T$

where ε is linear strain and

α is the coefficient of linear expansion;

T is the rise in temperature.

That is for a rod of Length, L;

if its temperature increased by t, the extension,

dl = α L T.

Thermal Strain Contd.

As in the case of lateral strains, thermal strains do not induce stresses unless they are constrained. The total strain in a body experiencing thermal stress may be divided into two components:

Strain due to stress, ε_σ and

That due to temperature, ε_T .

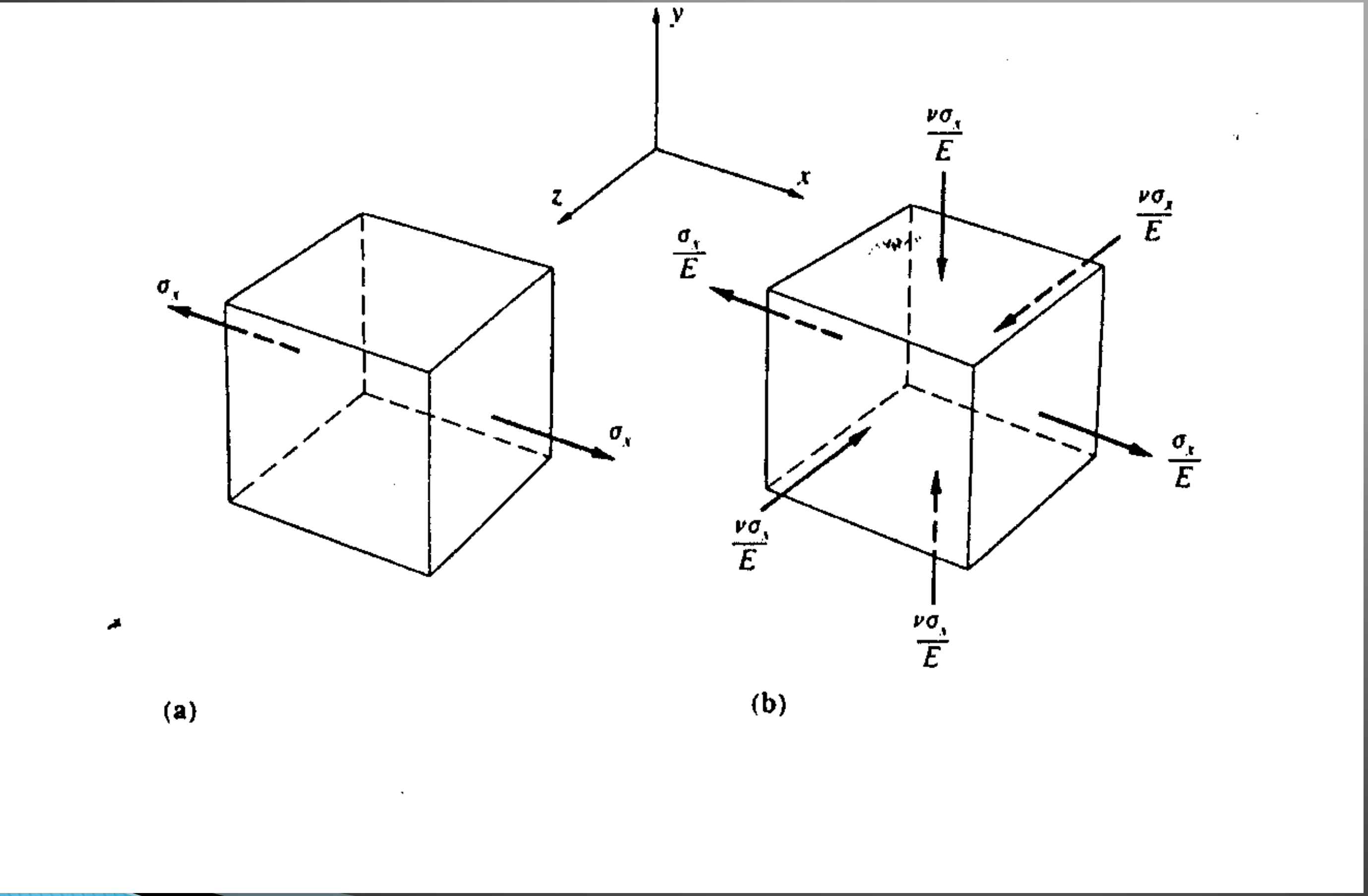
Thus: $\varepsilon = \varepsilon_\sigma + \varepsilon_T$

$$\boxed{\varepsilon = \frac{\sigma}{E} + \alpha T}$$

Principle of Superposition

- It states that the effects of several actions taking place simultaneously can be reproduced exactly by adding the effect of each action separately.
- The principle is general and has wide applications and holds true if:
- (i) The structure is elastic
- (ii) The stress-strain relationship is linear
- (iii) The deformations are small.

General Stress-Strain Relationships



Relationship between Elastic Modulus (E) and Bulk Modulus, K

It has been shown that : $\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$

$$\varepsilon_x = \frac{1}{E} \; \sigma_x - \upsilon\,(\sigma_y + \sigma_z)$$

$$For\ hydrostatic\ stress, \quad \sigma_x = \sigma_y = \sigma_z = \sigma$$

$$i.e. \quad \varepsilon_x = \frac{1}{E} \; \sigma - 2\,\sigma\,\upsilon \; = \frac{\sigma}{E} \; 1 - 2\,\upsilon$$

$$Similarly, \quad \varepsilon_y\ and\ \varepsilon_z\ are\ each \quad \frac{\sigma}{E} \; 1 - 2\,\upsilon$$

$$\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z \quad = Volumetric\ strain$$

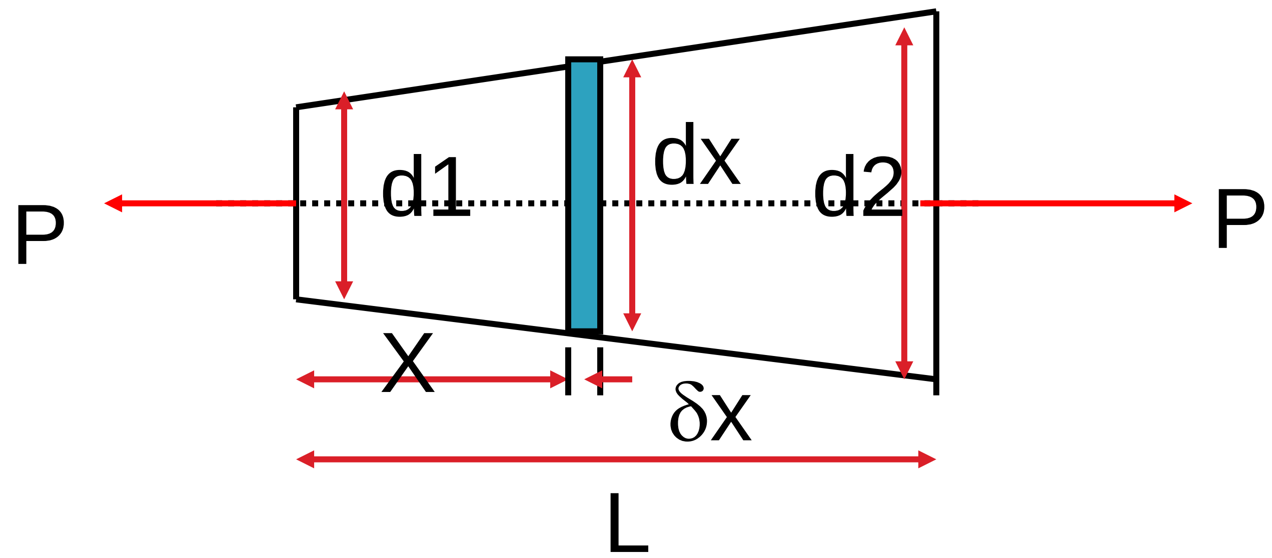
$$\varepsilon_v = \frac{3\,\sigma}{E} \; 1 - 2\,\upsilon$$

$$E = \frac{3\,\sigma}{\varepsilon_v} \; 1 - 2\,\upsilon$$

$$Bulk\ \ Modulus,\ K = \frac{Volumetric\ or\ hydrostatic\ stress}{Volumetric\ strain} = \frac{\sigma}{\varepsilon_v}$$

$$i.e. \quad E = 3\,K \;\; 1 - 2\,\upsilon \quad and \quad K = \frac{E}{3 \;\, 1 - 2\,\upsilon}$$

Extension of Bar of Tapering cross Section from diameter d1 to d2:-



Bar of Tapering Section:

$dx = d1 + [(d2 - d1) / L] * X$

$\delta\Delta = P\delta x / E[\pi /4\{d1 + [(d2 - d1) / L] * X\}^2]$

$$\Delta = 4P\int_0^L dx\,/[E\,\pi\{d1+kx\}^2]$$

$$= -\,[4P/\,\pi\,E]\ x\ \ 1/k\ \left[\{1/(d1+kx)\}\right]_0^L dx$$

$$=-\,[4PL/\,\pi\,E(d2-d1)]\ \{1/(d1+d2-d1) - 1/d1\}$$

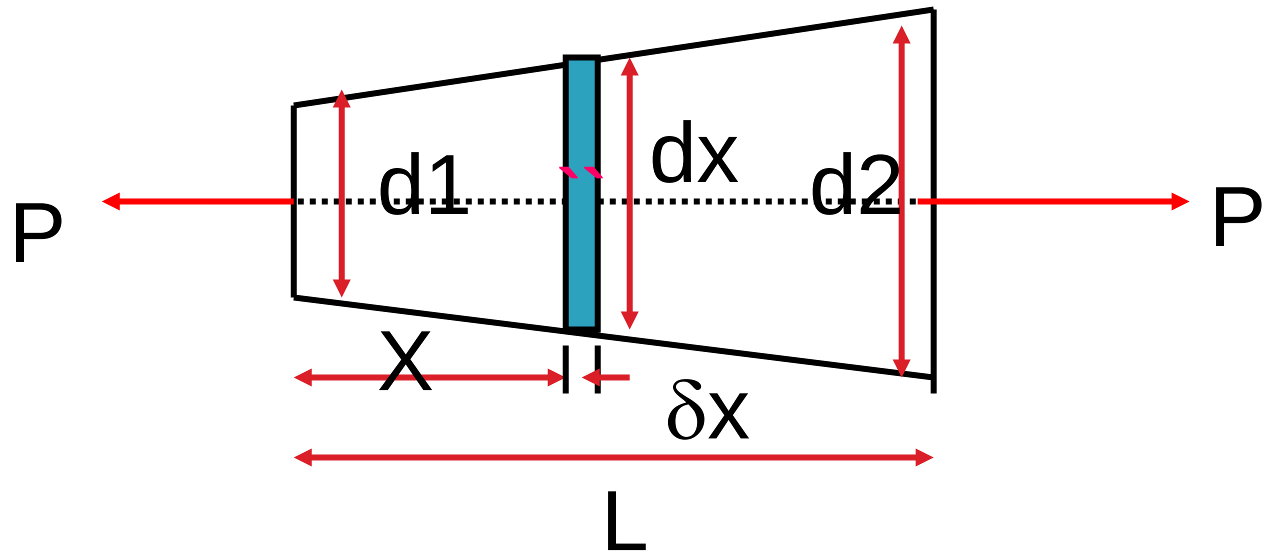
$$\Delta = 4PL/(\pi\,E\,d1\,d2)$$

Check :-

When d = d1=d2

Δ =PL/ [(π /4)* d^2E] = PL /AE (refer -24)

Q. Find extension of tapering circular bar under axial pull for the following data: d1 = 20mm, d2 = 40mm, L = 600mm, E = 200GPa. P = 40kN

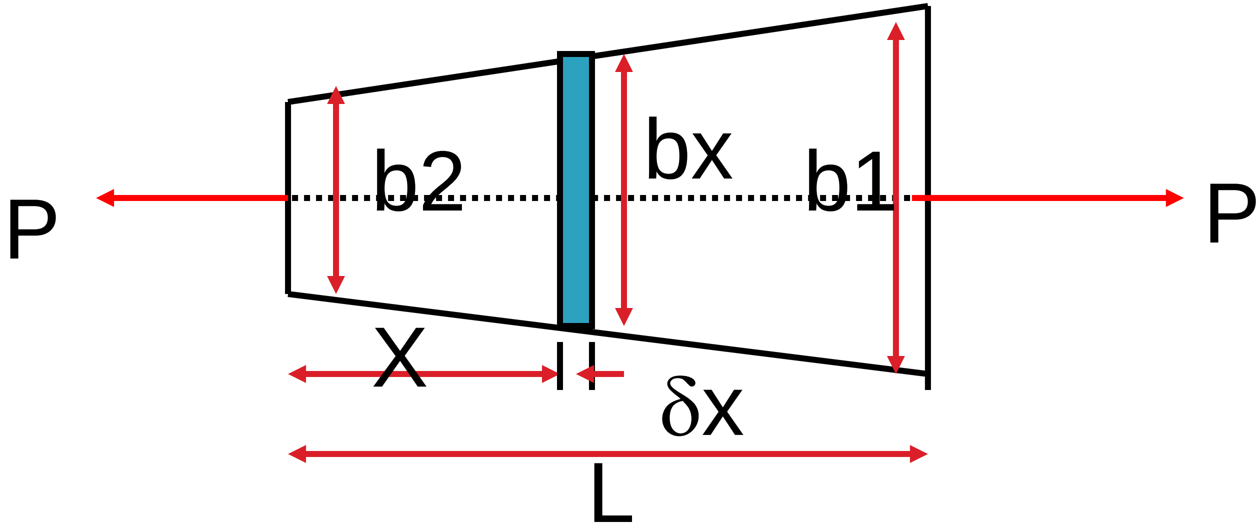


$\Delta L = 4PL/(\pi\ E\ d1\ d2)$

$= 4*40{,}000*600/(\pi*\ 200{,}000*20*40)$

$= 0.38$mm. Ans.

Extension of Tapering bar of uniform thickness t, width varies from b1 to b2:-



$$\mathbf{P/Et} \int \delta x \, / \, [\, (b1 + k^*X)],$$

Bar of Tapering Section:

bx = b1 + [(b2 - b1) / L] * X = b1 + k*x,

δΔ = Pδx / [Et(b1 + k*X)],   k = **(b2 - b1) / L**

$$\Delta L \int_0^L = \int_0^L \Delta L = \int_0^L P\delta x / [Et(b1 - k*X)],$$

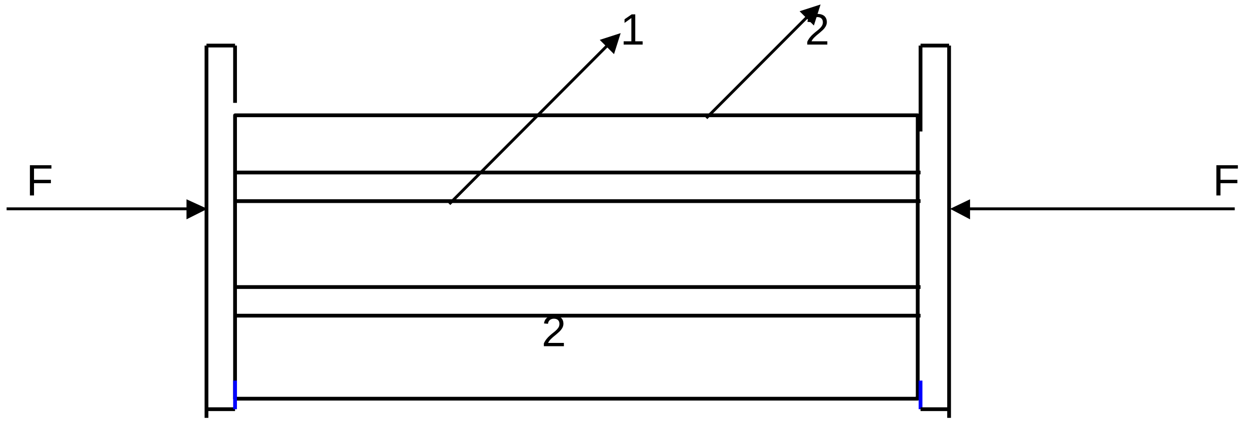
$$= P/Et \int \delta x / [\ (b1 - k*X)],$$

$$= - P/Etk * \log_e\ [\ (b1 - k*X)]_0^L,$$

$$= PL\log_e(b1/b2) / [Et(b1 – b2)]$$

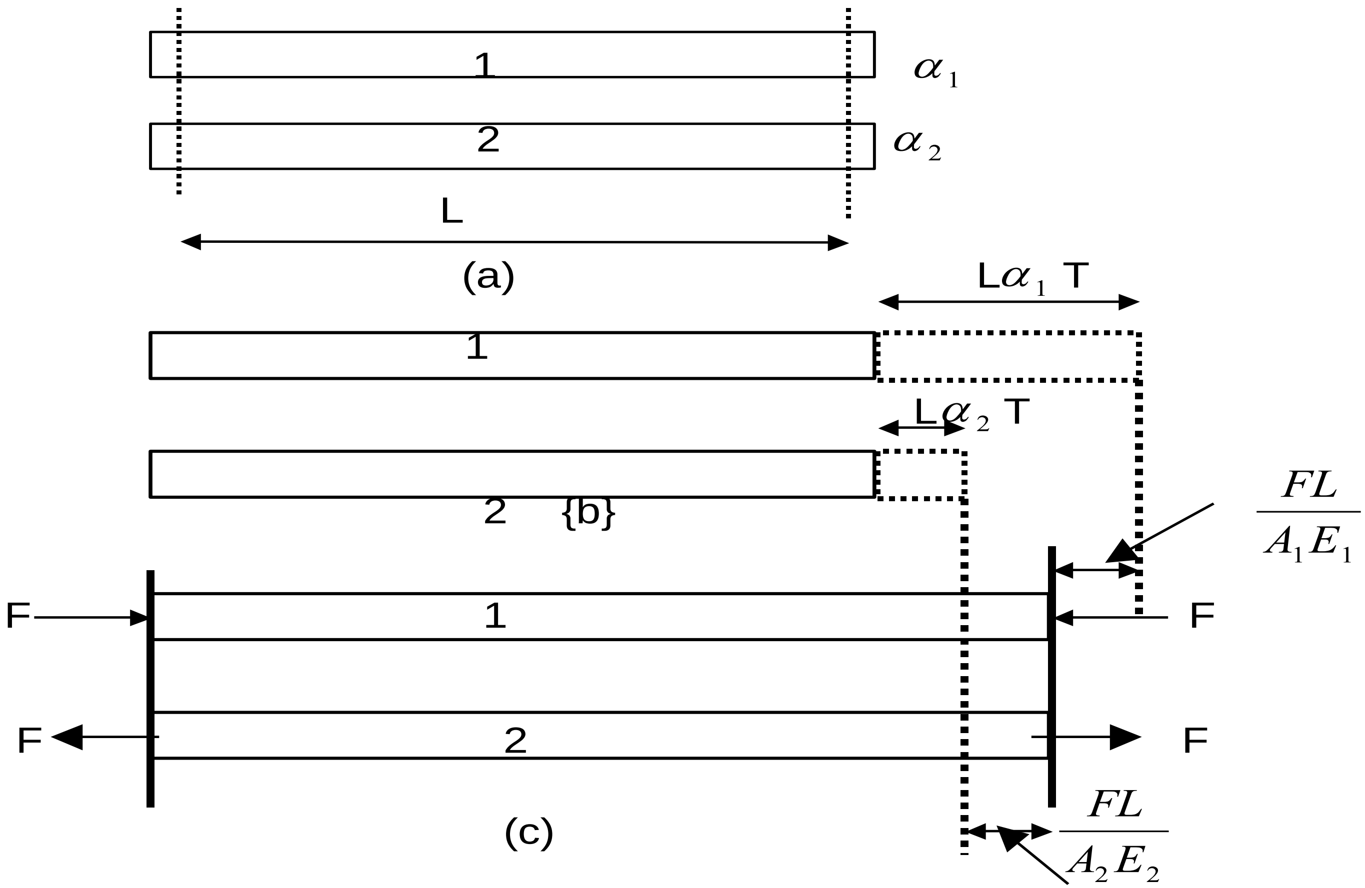
Compound Bars

A compound bar is one comprising two or more parallel elements, of different materials, which are fixed together at their end. The compound bar may be loaded in tension or compression.



Section through a typical compound bar consisting of a circular bar (1) surrounded by a tube (2)

Temperature stresses in compound bars

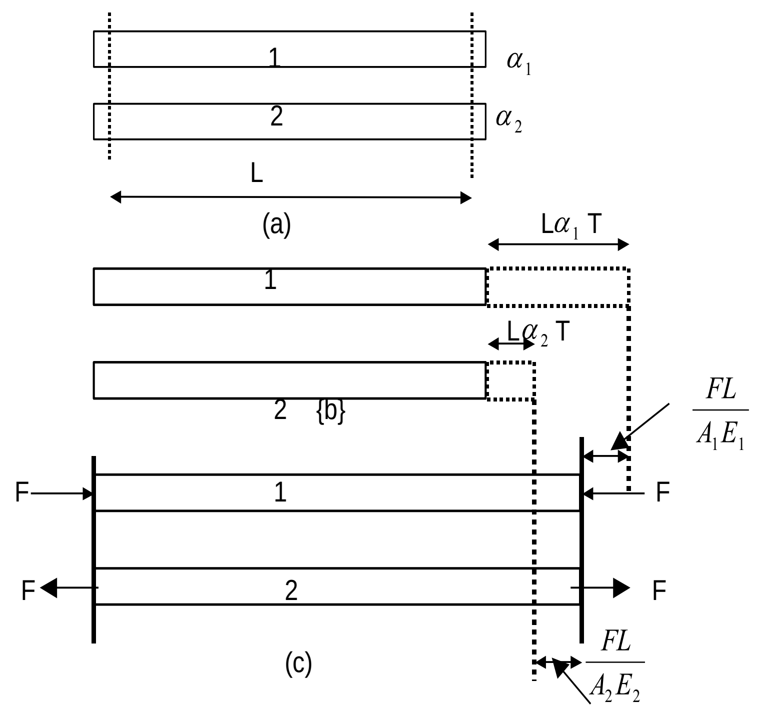


Temperature Stresses Contd.

Free expansions in bars (1) and (2) are $L\alpha_1 T$ *and* $L\alpha_2 T$ respectively.

Due to end fixing force, F: the decrease in length of bar (1) is

$\frac{FL}{A_1E_1}$ and the increase in length of (2) is $\frac{FL}{A_2E_2}$.



At Equilibrium:

$$L\alpha_1 T - \frac{FL}{A_1E_1} = L\alpha_2 T + \frac{FL}{A_2E_2}$$

$$i.e.\quad F\left[\frac{1}{A_1E_1} + \frac{1}{A_2E_2}\right] = T(\alpha_1 - \alpha_2)$$

$$i.e.\quad \sigma_1 A_1\left[\frac{A_2E_2 + A_1E_1}{E_1E_2A_1A_2}\right] = T(\alpha_1 - \alpha_2)$$

$$\boxed{\sigma_1 = \frac{T(\alpha_1 - \alpha_2)A_2E_1E_2}{A_1E_1 + A_2E_2}}$$

$$\boxed{\sigma_2 = \frac{T(\alpha_1 - \alpha_2)A_1E_1E_2}{A_1E_1 + A_2E_2}}$$

Note: As a result of force, F, bar (1) will be in compression while (2) will be in tension.

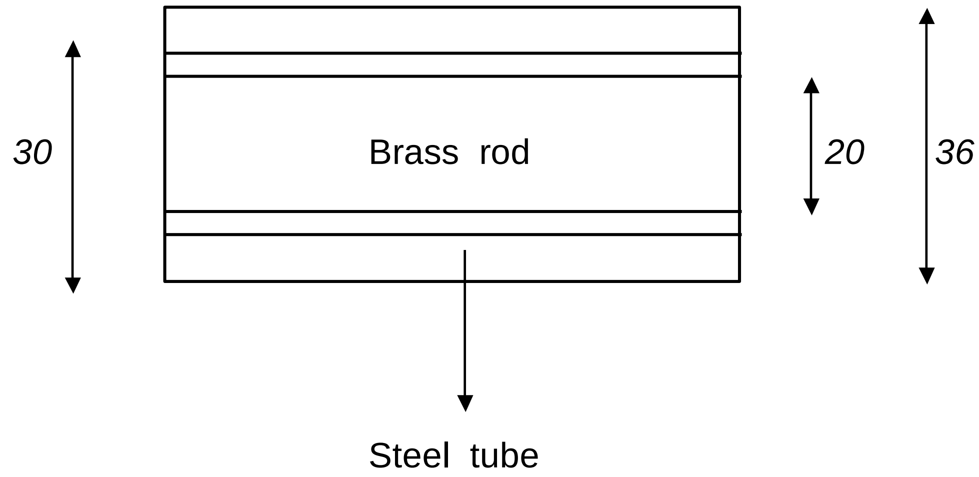
Example

- A steel tube having an external diameter of 36 mm and an internal diameter of 30 mm has a brass rod of 20 mm diameter inside it, the two materials being joined rigidly at their ends when the ambient temperature is 18 ^{0}C. Determine the stresses in the two materials: (a) when the temperature is raised to 68 ^{0}C (b) when a compressive load of 20 kN is applied at the increased temperature.

For brass: Modulus of elasticity = 80 GN/m^2; Coefficient of expansion = 17 x 10 -6 /^{0}C

For steel: Modulus of elasticity = 210 GN/m^2; Coefficient of expansion = 11 x 10 -6 /^{0}C

Solution



Area of brass rod (A_b) = $\dfrac{\pi \ x \ 20^2}{4} = 314.16 \ mm^2$

Area of steel tube (A_s) = $\dfrac{\pi \ x \ (36^2 - 30^2)}{4} = 311.02 \ mm^2$

$$A_s E_s = 311.02 \ x \ 10^{-6} m^2 \ x \ 210 \ x \ 10^9 N / m^2 = 0.653142 \ x \ 10^8 \ N$$

$$\frac{1}{A_s E_s} = 1.53106 \ x \ 10^{-8}$$

Solution Contd.

$$A_b E_b = 314.16\ x\ 10^{-6} m^2\ \ x\ 80\ x\ 10^{9}\ N/m^2 = 0.251327\ x\ 10^{8}\ N$$

$$\frac{1}{A_b E_b} = 3.9788736\ x\ 10^{-8}$$

$$T(\alpha_b - \alpha_s) = 50(17-11)\ x\ 10^{-6} = 3\ x\ 10^{-4}$$

With increase in temperature, brass will be in compression while steel will be in tension. This is because expands more than steel.

$$i.e.\quad F[\frac{1}{A_s E_s} + \frac{1}{A_b E_b}] = T(\alpha_b - \alpha_s)$$

i.e. F[1.53106 + 3.9788736] x 10 $^{-8}$ = 3 x 10 $^{-4}$

F = 5444.71 N

Solution Concluded

Stress in steel tube = $\dfrac{5444.71N}{311.02\,mm^2} = 17.51N/mm^2 = 17.51MN/m^2\,(Tension)$

Stress in brass rod = $\dfrac{5444.71N}{314.16\,mm^2} = 17.33N/mm^2 = 17.33MN/m^2\,(Compression)$

(b) Stresses due to compression force, F' of 20 kN

$$\sigma_s = \frac{F' E_s}{E_s A_s + E_b A_b} = \frac{20\ x\ 10^3\ N\ \ x\ 210\ x\ 10^9\ N/m^2}{0.653142 + 0.251327\ x\ 10^8} = 46.44\,MN/m^2\,(Compression)$$

$$\sigma_b = \frac{F' E_b}{E_s A_s + E_b A_b} = \frac{20\ x\ 10^3\ N\ \ x\ 80\ x\ 10^9\ N/m^2}{0.653142 + 0.251327\ x\ 10^8} = 17.69\,MN/m^2\,(Compression)$$

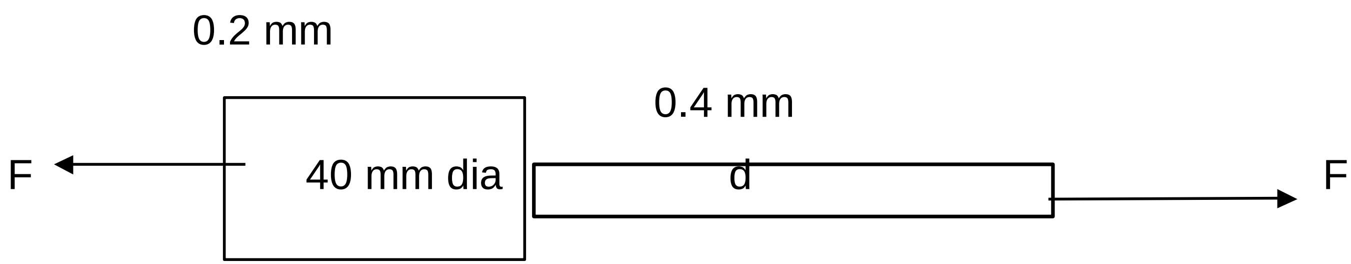
Resultant stress in steel tube = - 46.44 + 17.51 = 28.93 MN/m^2 (Compression)

Resultant stress in brass rod = -17.69 - 17.33 = 35.02 MN/m^2 (Compression)

Example

- A composite bar, 0.6 m long comprises a steel bar 0.2 m long and 40 mm diameter which is fixed at one end to a copper bar having a length of 0.4 m.
- Determine the necessary diameter of the copper bar in order that the extension of each material shall be the same when the composite bar is subjected to an axial load.
- What will be the stresses in the steel and copper when the bar is subjected to an axial tensile loading of 30 kN? (For steel, $E = 210\ GN/m^2$; for copper, $E = 110\ GN/m^2$)

Solution



Let the diameter of the copper bar be d mm

Specified condition: Extensions in the two bars are equal

$$dl_c = dl_s$$

$$dl = \varepsilon \, L = \frac{\sigma}{E} \, L = \frac{FL}{AE}$$

Thus: $$\frac{F_c L_c}{A_c E_c} = \frac{F_s L_s}{A_s E_s}$$

Solution Concluded

Also: Total force, F is transmitted by both copper and steel

i.e. F_c = Fs = F

$$i.e. \quad \frac{L_c}{A_c E_c} = \frac{L_s}{A_s E_s}$$

Substitute values given in problem:

$$\frac{0.4\,m}{\pi\,d^2 / 4\,m^2\;110\,x\,10^9\,N/m^2} = \frac{0.2\,m}{\pi/4\,x\,0.040^2\;x\,210\,x\,10^9\;N/m^2}$$

$$d^2 = \frac{2\,x\,210\,x\,0.040^2}{110}\,m^2; \quad d = 0.07816\,m = 78.16\,mm.$$

Thus for a loading of 30 kN

Stress in steel, $\sigma_s = \dfrac{30\,x\,10^3\,N}{\pi/4\,x\,0.040^2\;x\,10^{-6}} = 23.87\;MN/m^2$

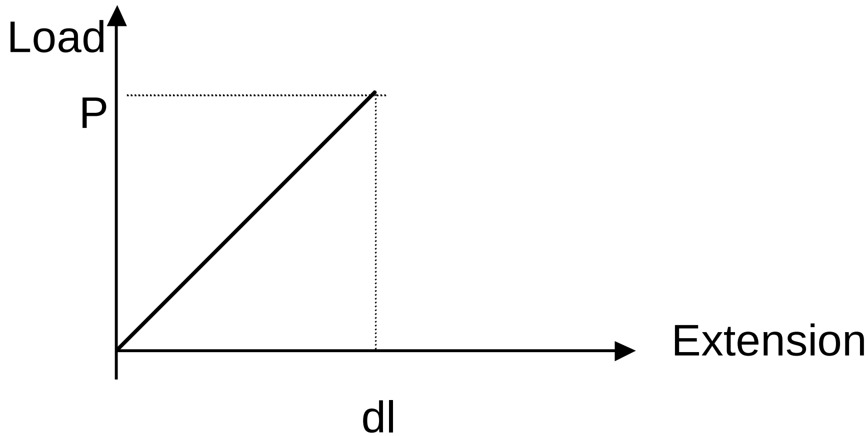
Stress in copper, $\sigma_c = \dfrac{30\,x\,10^3\,N}{\pi/4\,x\,0.07816^2\;x\,10^{-6}} = 9\;MN/m^2$

Elastic Strain Energy

- If a material is strained by a gradually applied load, then work is done on the material by the applied load.
- The work is stored in the material in the form of strain energy.
- If the strain is within the elastic range of the material, this energy is not retained by the material upon the removal of load.

Elastic Strain Energy Contd.

Figure below shows the load-extension graph of a uniform bar. The extension dl is associated with a gradually applied load, P which is within the elastic range. The shaded area represents the work done in increasing the load from zero to its value



Work done = strain energy of bar = shaded area

Elastic Strain Energy Concluded

$W = U = 1/2\ P\ dl$ (1)

Stress, $\sigma = P/A$ i.e $P = \sigma A$

Strain = Stress/E

i.e $dl/L = \sigma/E$, $dl = (\sigma L)/E$ L= original length

Substituting for P and dl in Eqn (1) gives:

$W = U = 1/2\ \sigma\ A\ .\ (\sigma\ L)/E = \sigma^2/2E \times A\ L$

A L is the volume of the bar.

i.e $\boxed{U = \sigma^2/2E \times \text{Volume}}$

The units of strain energy are same as those of work i.e. Joules. Strain energy per unit volume, $\sigma^2/2E$ is known as resilience. The greatest amount of energy that can stored in a material without permanent set occurring will be when σ is equal to the elastic limit stress.

UNIT 2

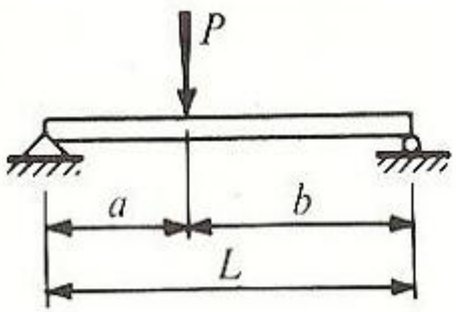
Shear Force and Bending Moment Diagram

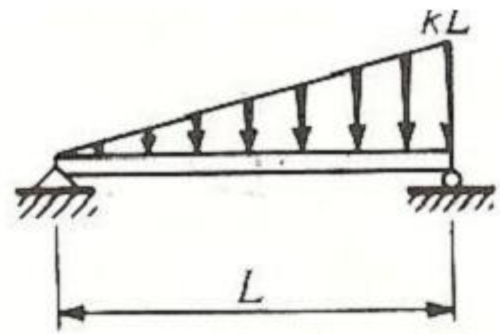
Syllabus

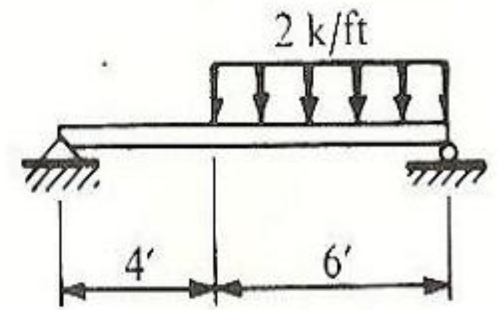
- **SHEAR FORCE AND BENDING MOMENT:** Definition of beam – Types of beams – Concept of shear force and bending moment – S.F and B.M diagrams for cantilever, simply supported and overhanging beams subjected to point loads, uniformly distributed load, uniformly varying loads and combination of these loads – Point of contraflexure – Relation between S.F., B.M and rate of loading at a section of a beam.

4-Classification of Beams:

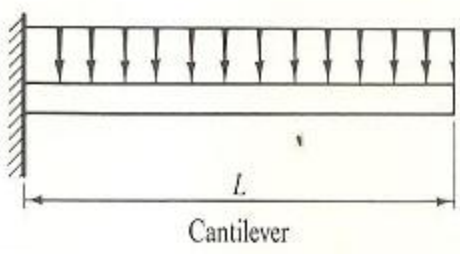
1) Simple Beam

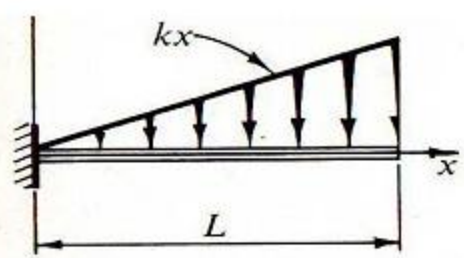


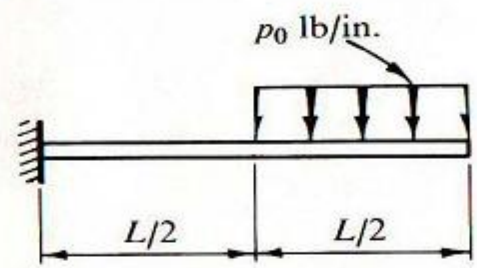




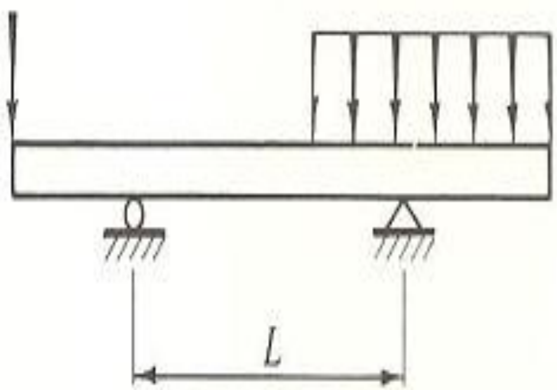
Cantilever Beam



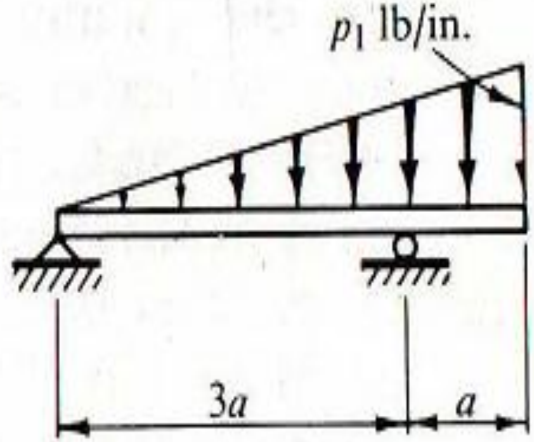




3) Simple Beam with Overhanging OR "Overhanging Beam"

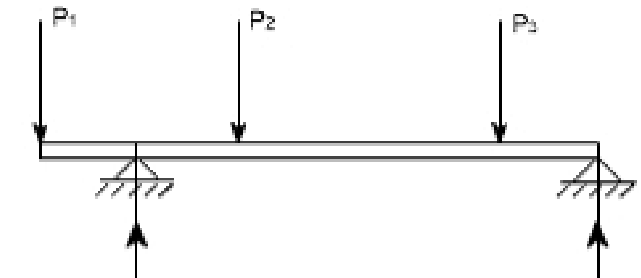


Overhanging beam

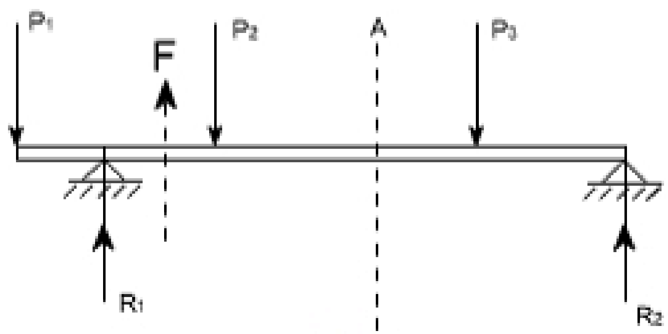


Concept of Shear Force and Bending moment in beams:

When the beam is loaded in some arbitrarily manner, the internal forces and moments are developed and the terms shear force and bending moments come into pictures which are helpful to analyze the beams further. Let us define these terms



(a)



(b)

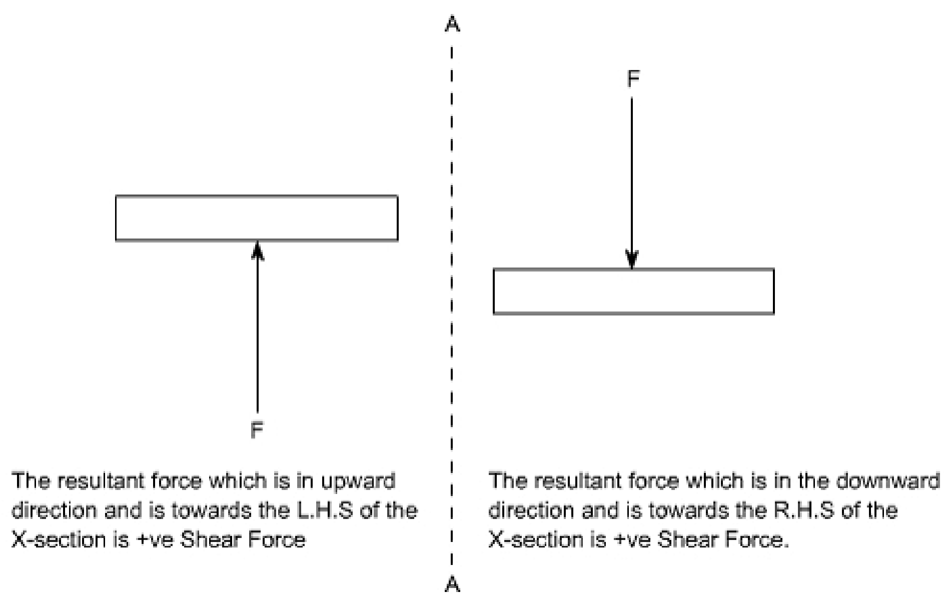
Now let us consider the beam as shown in fig 1(a) which is supporting the loads P_1, P_2, P_3 and is simply supported at two points creating the reactions R_1 and R_2 respectively. Now let us assume that the beam is to divided into or imagined to be cut into two portions at a section AA. Now let us assume that the resultant of loads and reactions to the left of AA is 'F' vertically upwards, and since the entire beam is to remain in equilibrium, thus the resultant of forces to the right of AA must also be F, acting downwards. This forces 'F' is as a shear force. The shearing force at any x-section of a beam represents the tendency for the portion of the beam to one side of the section to slide or shear laterally relative to the other portion.

Therefore, now we are in a position to define the shear force 'F' to as follows:

At any x-section of a beam, the shear force 'F' is the algebraic sum of all the lateral components of the forces acting on either side of the x-section.

Sign Convention for Shear Force:

The usual sign conventions to be followed for the shear forces have been illustrated in figures 2 and 3.



Positive Shear Force

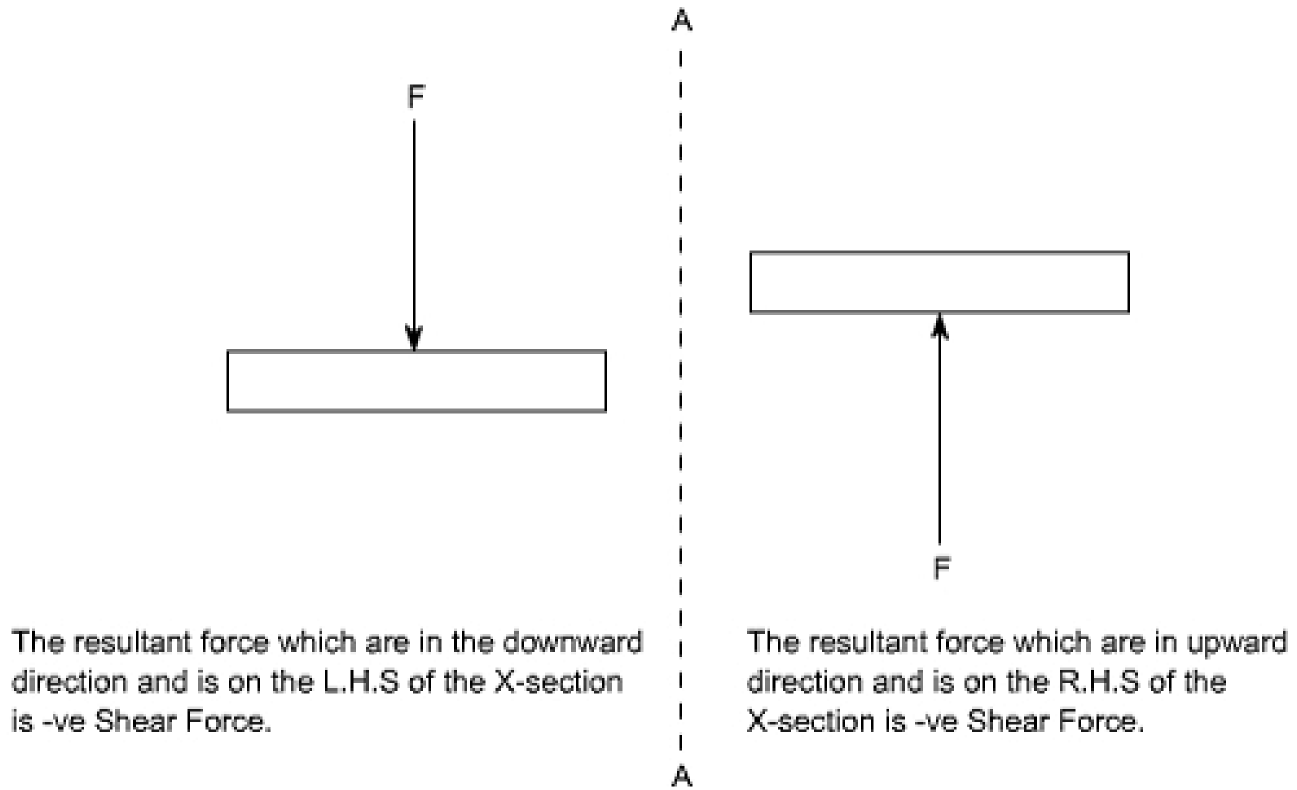
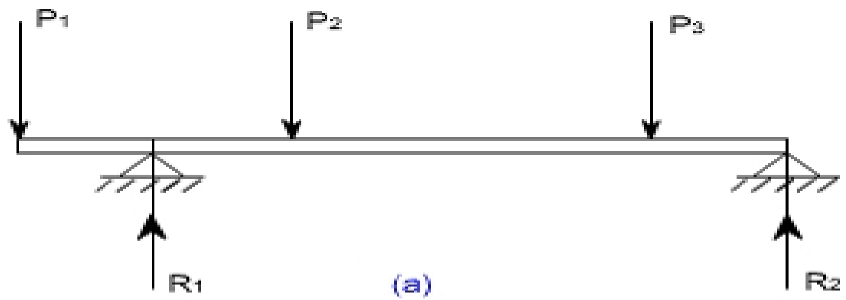
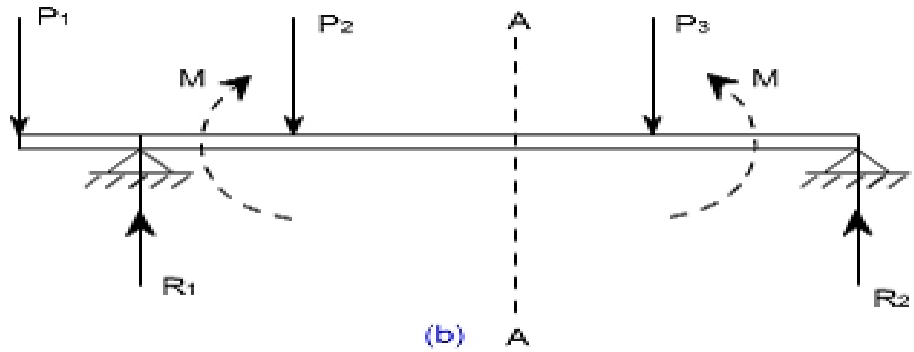


Fig 3: Negative Shear Force

BENDING MOMENT





Let us again consider the beam which is simply supported at the two prints, carrying loads P_1, P_2 and P_3 and having the reactions R_1 and R_2 at the supports Fig 4. Now, let us imagine that the beam is cut into two potions at the x-section AA. In a similar manner, as done for the case of shear force, if we say that the resultant moment about the section AA of all the loads and reactions to the left of the x-section at AA is M in C.W direction, then moment of forces to the right of x-section AA must be 'M' in C.C.W. Then 'M' is called as the Bending moment and is abbreviated as B.M. Now one can define the bending moment to be simply as <u>the algebraic sum of the moments about an x-section of all the forces acting on either side of the section</u>

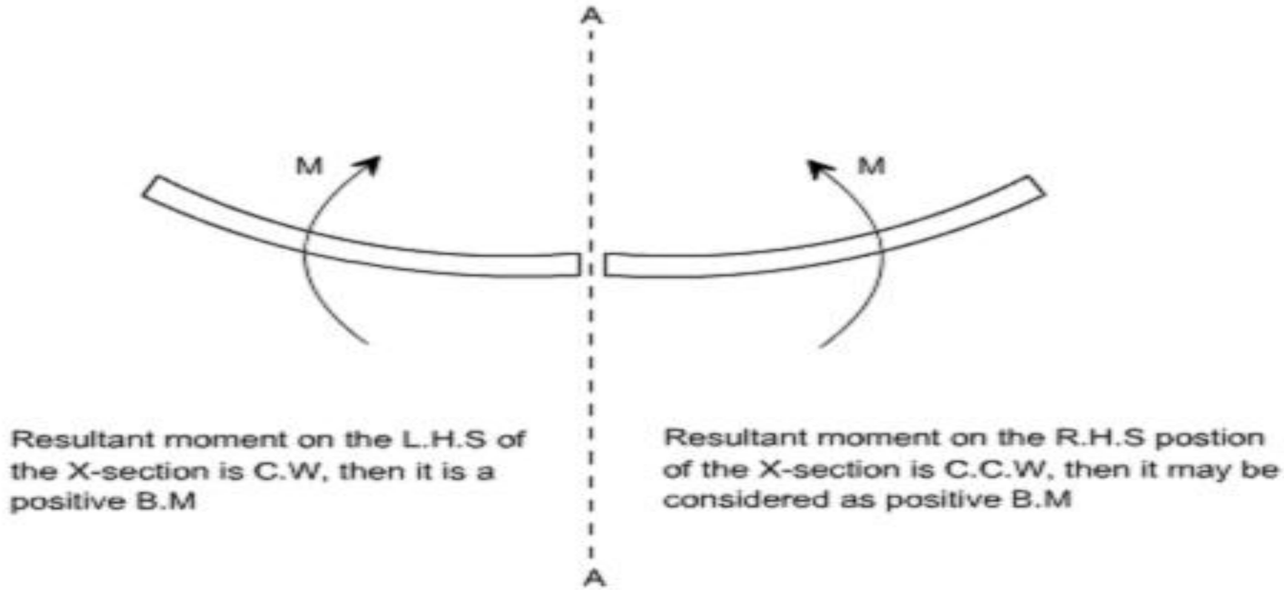
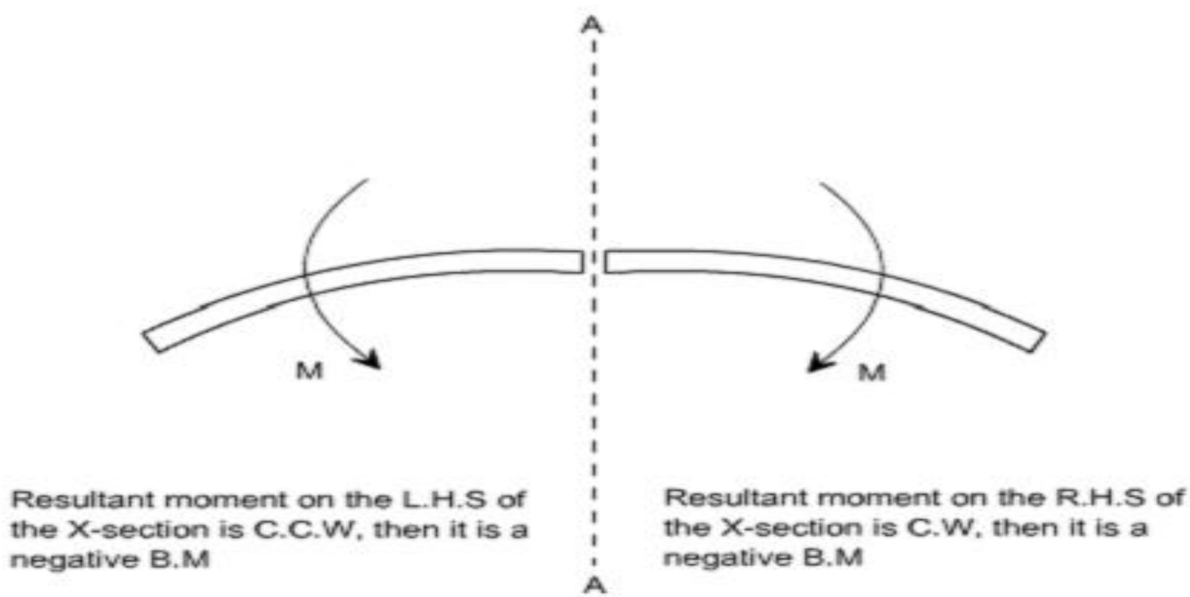
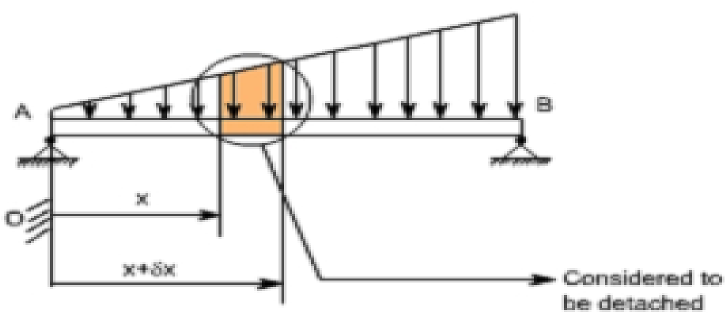


Fig 5: Positive Bending Moment

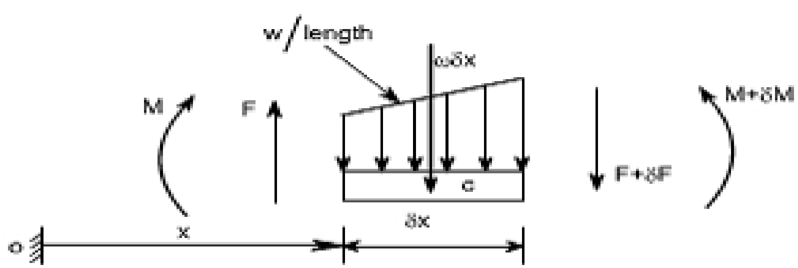


Basic Relationship Between The Rate of Loading, Shear Force and Bending Moment:

The construction of the shear force diagram and bending moment diagrams is greatly simplified if the relationship among load, shear force and bending moment is established. Let us consider a simply supported beam AB carrying a uniformly distributed load w/length. Let us imagine to cut a short slice of length dx cut out from this loaded beam at distance ‘x' from the origin ‘0'.



Let us detach this portion of the beam and draw its free body diagram.



The forces acting on the free body diagram of the detached portion of this loaded beam are the following

- The shearing force F and F+ dF at the section x and x + dx respectively.

•The bending moment at the sections x and x + dx be M and M + dM respectively.

- Force due to external loading, if ‘w' is the mean rate of loading per unit length then the total loading on this slice of length dx is w. dx, which is approximately acting through the centre ‘c'. If the loading is assumed to be uniformly distributed then it would pass exactly through the centre ‘c'.

This small element must be in equilibrium under the action of these forces and couples. Now let us take the moments at the point ‘c'. Such that

$$M + F.\frac{\delta x}{2} + (F + \delta F).\frac{\delta x}{2} = M + \delta M$$

$$\Rightarrow F.\frac{\delta x}{2} + (F + \delta F).\frac{\delta x}{2} = \delta M$$

$$\Rightarrow F.\frac{\delta x}{2} + F.\frac{\delta x}{2} + \delta F.\frac{\delta x}{2} = \delta M \text{ [Neglecting the product of } \delta F \text{ and } \delta x \text{ being small quantities]}$$

$$\Rightarrow F.\delta x = \delta M$$

$$\Rightarrow F = \frac{\delta M}{\delta x}$$

Under the limits $\delta x \rightarrow 0$

$$\boxed{F = \frac{dM}{dx}} \quad \text{............} \quad (1)$$

Resolving the forces vertically we get

$$w.\delta x + (F + \delta F) = F$$

$$\Rightarrow w = -\frac{\delta F}{\delta x}$$

Under the limits $\delta x \rightarrow 0$

$$\Rightarrow w = -\frac{dF}{dx} \text{ or } -\frac{d}{dx}\left(\frac{dM}{dx}\right)$$

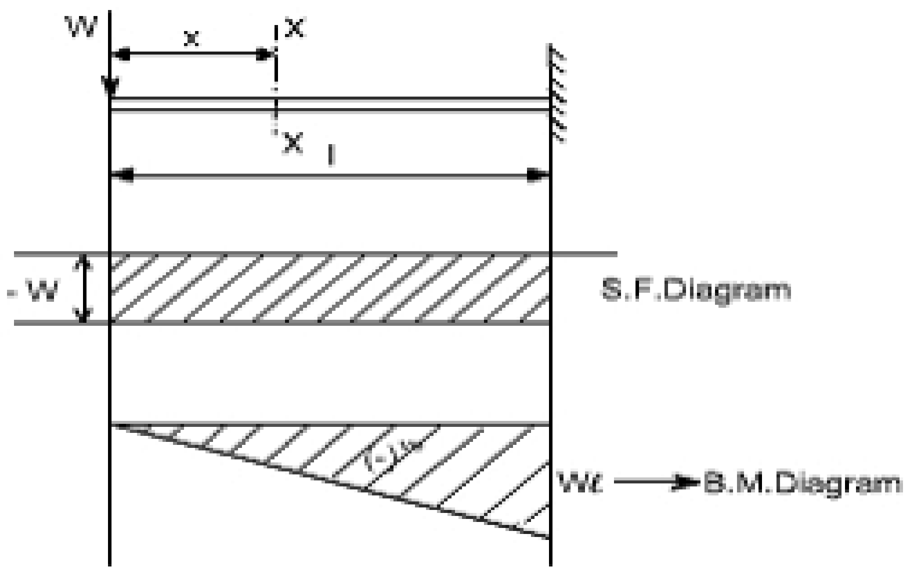
$$\boxed{w = -\frac{dF}{dx} = -\frac{d^2M}{dx^2}} \quad \text{............} (2)$$

A cantilever of length carries a concentrated load ‘W' at its free end.

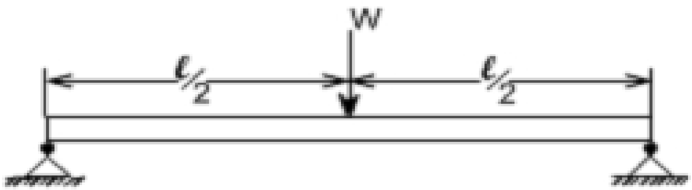
Draw shear force and bending moment.

Solution:

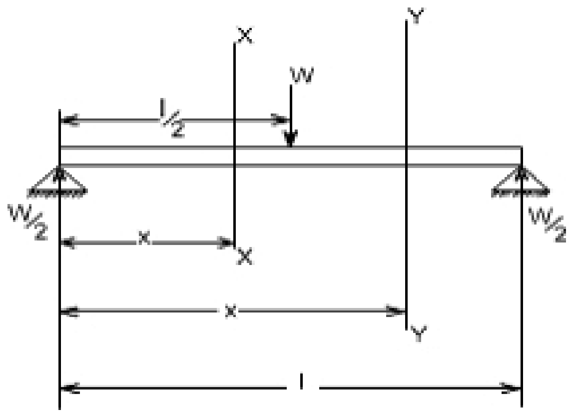
At a section a distance x from free end consider the forces to the left, then F = -W (for all values of x) -ve sign means the shear force to the left of the x-section are in downward direction and therefore negative. Taking moments about the section gives (obviously to the left of the section) M = -Wx (-ve sign means that the moment on the left hand side of the portion is in the anticlockwise direction and is therefore taken as –ve according to the sign convention) so that the maximum bending moment occurs at the fixed end i.e. M = -W l From equilibrium consideration, the fixing moment applied at the fixed end is Wl and the reaction is W. the shear force and bending moment are shown as,



Simply supported beam subjected to a central load (i.e. load acting at the mid-



By symmetry the reactions at the two supports would be W/2 and W/2. now consider any section X-X from the left end then, the beam is under the action of following forces.



.So the shear force at any X-section would be = W/2 [Which is constant upto x < l/2]

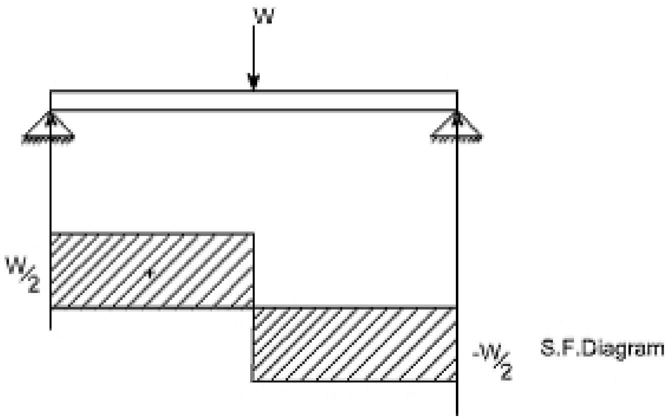
If we consider another section Y-Y which is beyond l/2 then

$S.F_{Y\text{-}Y} = \frac{W}{2} - W = \frac{-W}{2}$ for all values greater = l/2

Hence S.F diagram can be plotted as,

.For B.M diagram:

If we just take the moments to the left of the cross-section,



$$B.M_{x\text{-}x} = \frac{W}{2}x \text{ for } x \text{ lies between } 0 \text{ and } l/2$$

$$B.M_{\text{at } x = \frac{l}{2}} = \frac{W}{2}\,\frac{l}{2} \text{ i.e B.M at } x = 0$$

$$= \frac{Wl}{4}$$

$$B.M_{Y\text{-}Y} = \frac{W}{2}x - W\left(x - \frac{l}{2}\right)$$

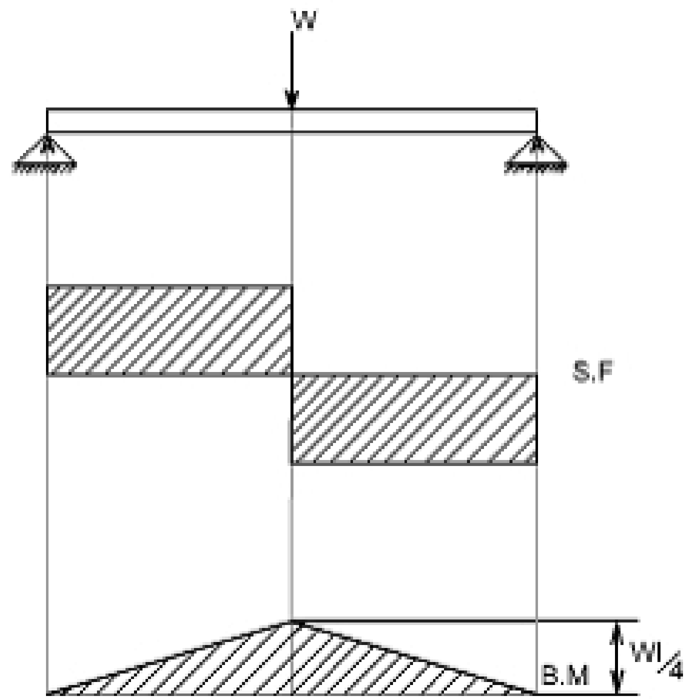
Again

$$= \frac{W}{2}x - Wx + \frac{Wl}{2}$$

$$= -\frac{W}{2}x + \frac{Wl}{2}$$

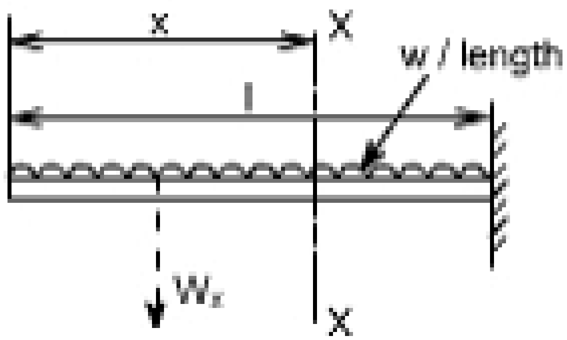
$$B.M_{\text{at } x = l} = -\frac{Wl}{2} + \frac{Wl}{2}$$

$$= 0$$



Which when plotted will give a straight relation i.e.

A cantilever beam subjected to U.d.L, draw S.F and B.M diagram.



Here the cantilever beam is subjected to a uniformly distributed load whose intensity is given w / length.

Consider any cross-section XX which is at a distance of x from the free end. If we just take the resultant of all the forces on the left of the X-section, then

$S.F_{xx}$ = -Wx for all values of 'x'. --------- (1)

$S.F_{xx} = 0$

$S.F_{xx \text{ at } x=l} = -Wl$

So if we just plot the equation No. (1), then it will give a straight line relation. Bending Moment at X-X is obtained by treating the load to the left of X-X as a concentrated load of the same value acting through the centre of gravity.

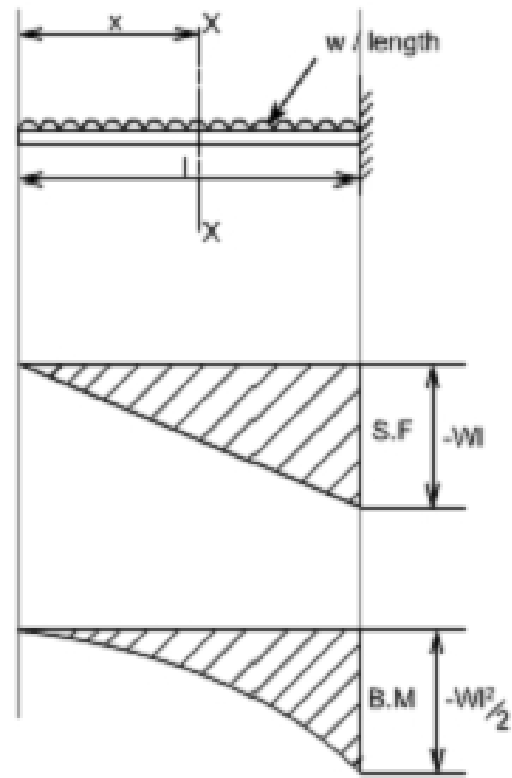
Therefore, the bending moment at any cross-section X-X is

$$B.M_{X\text{-}X} = - W x \frac{x}{2}$$
$$= - W\frac{x^2}{2}$$

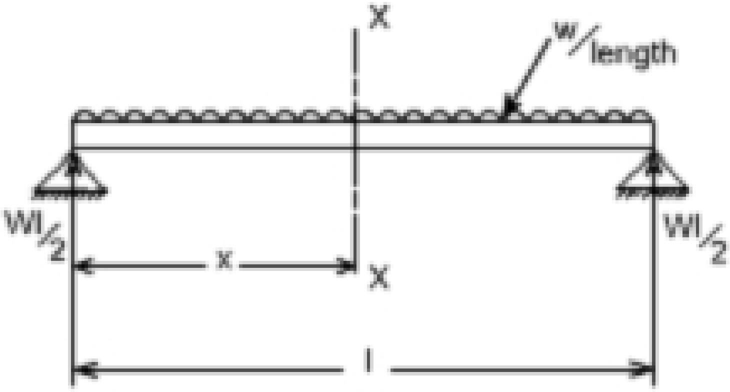
The above equation is a quadratic in x, when B.M is plotted against x this will produces a parabolic variation.

The extreme values of this would be at x = 0 and x = l

$$B.M_{at\ x=l} = -\frac{Wl^2}{2}$$
$$= \frac{Wl}{2} - Wx$$



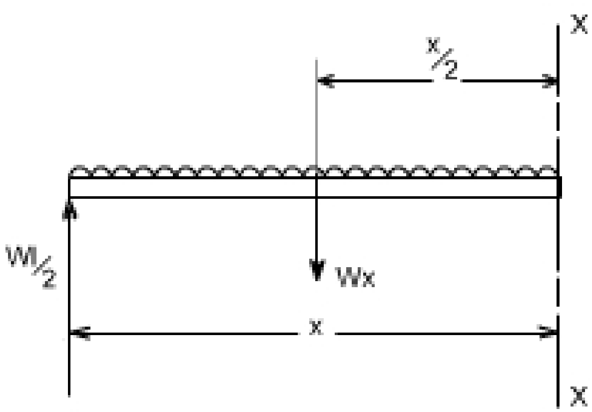
Simply supported beam subjected to a uniformly distributed load U.D.L



S.F at any X-section X-X is

$$= \frac{Wl}{2} - Wx$$
$$= W\left(\frac{l}{2} - x\right)$$

The bending moment at the section x is found by treating the distributed load as acting at its centre of gravity, which at a distance of x/2 from the section



$$B.M_{x-x} = \frac{Wl}{2}x - Wx.\frac{x}{2}$$

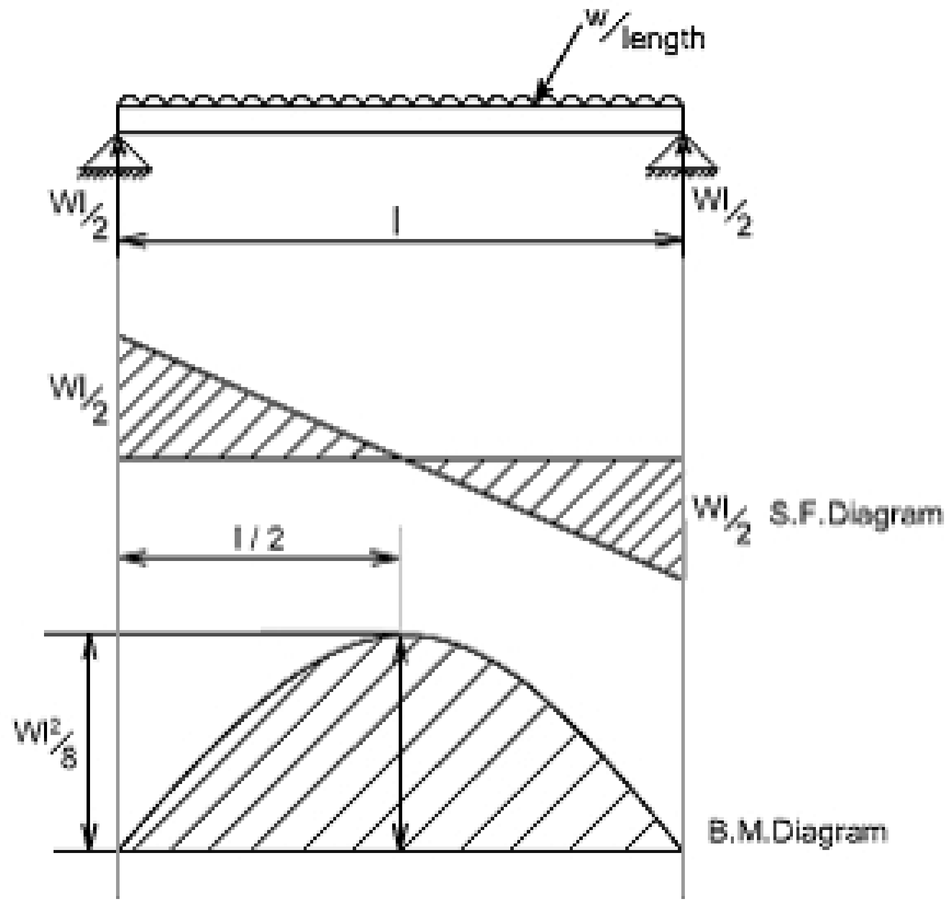
so the

$$= W.\frac{x}{2}(l - 2) \quad ....(2)$$

$B.M_{at\ x = 0} = 0$

$B.M_{at\ x = l} = 0$

$$B.M\Big|_{at\ x = l} = -\frac{Wl^2}{8}$$



An I - section girder, 200mm wide by 300 mm depth flange and web of thickness is 20 mm is used as simply supported beam for a span of 7 m. The girder carries a distributed
load of 5 KN /m and a concentrated load of 20 KN at mid-span.
Determine the
(i). The second moment of area of the cross-section of the girder
(ii). The maximum stress set up.
Solution:
The second moment of area of the cross-section can be determined as follows :
For sections with symmetry about the neutral axis, use can be made of standard I value for a rectangle about an axis through centroid i.e. (b.d3)/12. The section can thus be divided into convenient rectangles for each of which the neutral axis passes through the centroid. Example in the case enclosing the girder by a rectangle

$$I_{girder} = I_{rectangle} - I_{shaded\ portion}$$

$$= \left[\frac{200 \times 300^3}{12}\right] 10^{-12} - 2\left[\frac{90 \times 260^3}{12}\right] 10^{-12}$$

$$= (4.5 - 2.64)10^{-4}$$

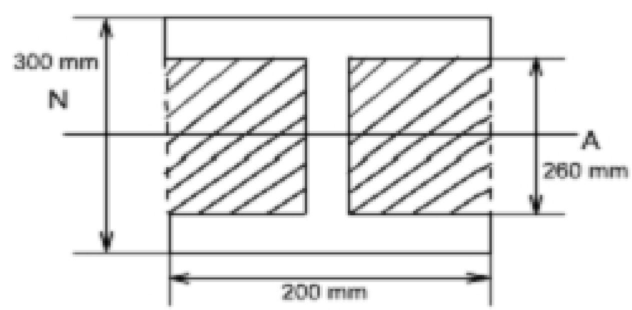
$$= 1.86 \times 10^{-4}\ m^4$$

The maximum stress may be found from the simple bending theory by equation

$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$$

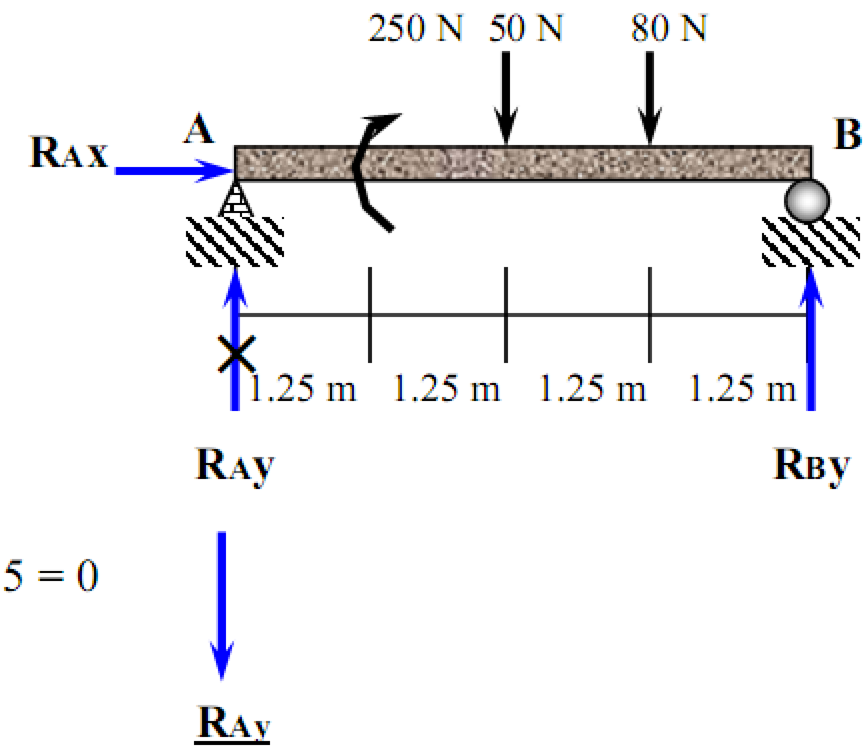
i.e.

$$\sigma_{max^m} = \frac{M_{max^m}}{I}\ y_{max^m}$$



Calculations of Beam Reactions

Ex3:



$\rightarrow \sum Fx = 0$ --- (1)

$R_{AX} = 0$

$+ \sum M@A = 0$ --- (2)

$250 + 80 \times 2.5 + 80 \times 3.75 - R_B \times 5 = 0$

$\therefore R_{By} = +135\ N \uparrow$

$\uparrow \sum Fy = 0$ --- (3)

$R_{Ay} = -5\ N \uparrow \Rightarrow R_{Ay} = 5\ N \downarrow$

UNIT -3

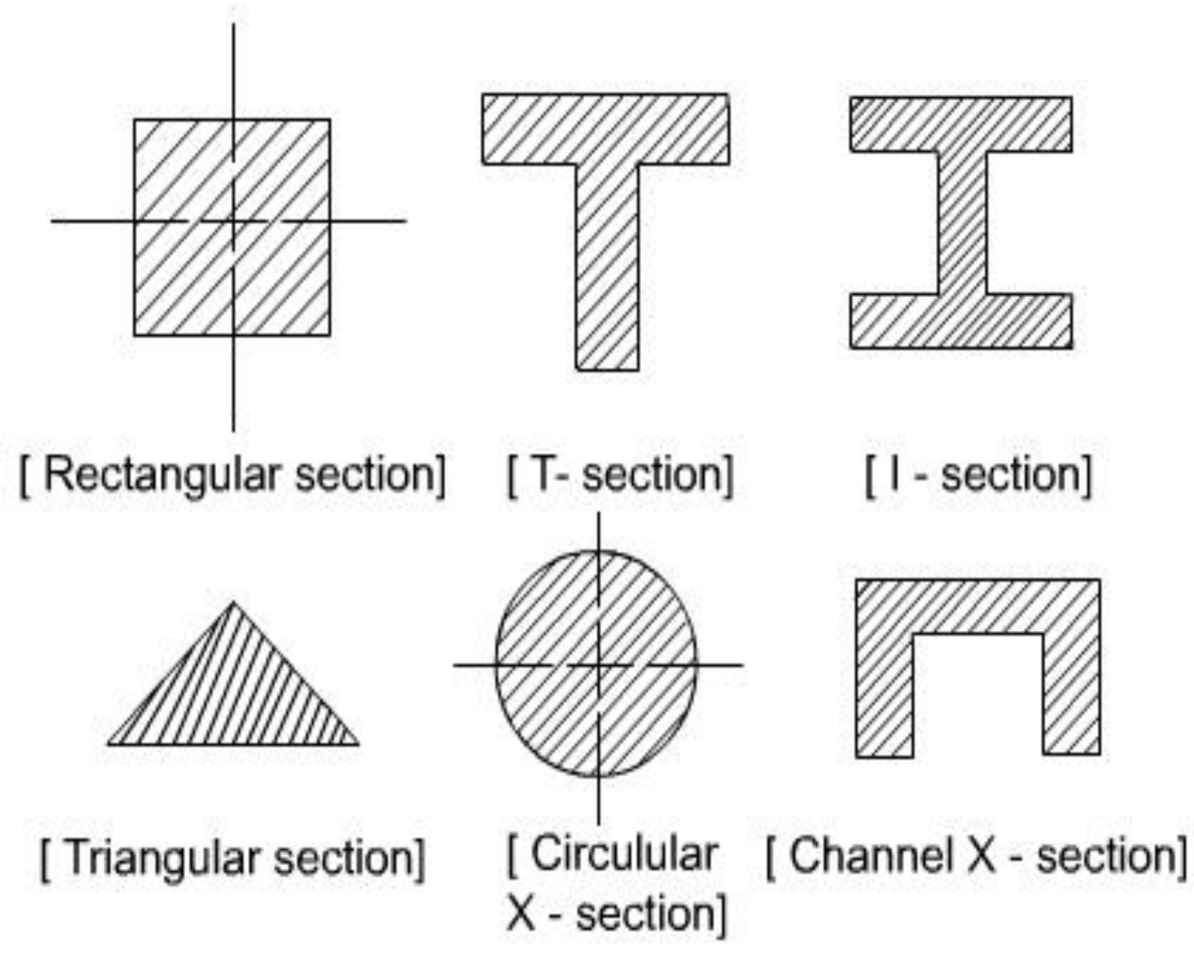
Flexural and shear stresses

Syllabus

- **FLEXURAL STRESSES:** Theory of simple bending – Assumptions – Derivation of bending equation: M/I = f/y = E/R - Neutral axis – Determination of bending stresses – Section modulus of rectangular and circular sections (Solid and Hollow), I,T, Angle and Channel sections – Design of simple beam sections.
- **SHEAR STRESSES**: Derivation of formula – Shear stress distribution across various beam sections like rectangular, circular, triangular, I, T angle sections.

- **Members Subjected to Flexural Loads**
- **Introduction:**
- In many engineering structures members are required to resist forces that are applied laterally or transversely to their axes. These type of members are termed as beam.
- There are various ways to define the beams such as
- **Definition I:** A beam is a laterally loaded member, whose cross-sectional dimensions are small as compared to its length.
- **Definition II:** A beam is nothing simply a bar which is subjected to forces or couples that lie in a plane containing the longitudinal axis of the bar. The forces are understood to act perpendicular to the longitudinal axis of the bar.
- **Definition III:** A bar working under bending is generally termed as a beam.
- **Materials for Beam:**
- The beams may be made from several usable engineering materials such commonly among them are as follows:
- Metal
- Wood
- Concrete
- Plastic

- **Geometric forms of Beams:**
- The Area of X-section of the beam may take several forms some of them have been shown below:



- **Loading restrictions:**
- **Concept of pure bending:**
- As we are aware of the fact internal reactions developed on any cross-section of a beam may consists of a resultant normal force, a resultant shear force and a resultant couple. In order to ensure that the bending effects alone are investigated, we shall put a constraint on the loading such that the resultant normal and the resultant shear forces are zero on any cross-section perpendicular to the longitudinal axis of the member,
- That means F = 0
- since   or M = constant.
- Thus, the zero shear force means that the bending moment is constant or the bending is same at every cross-section of the beam. Such a situation may be visualized or envisaged when the beam or some portion of the beam, as been loaded only by pure couples at its ends. It must be recalled that the couples are assumed to be loaded in the plane of symmetry.

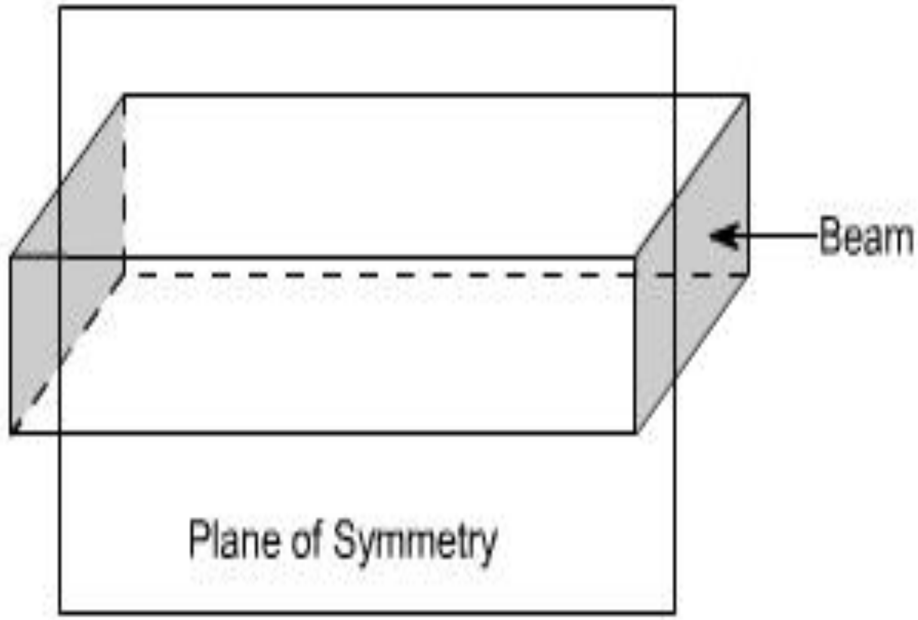


Fig (1)

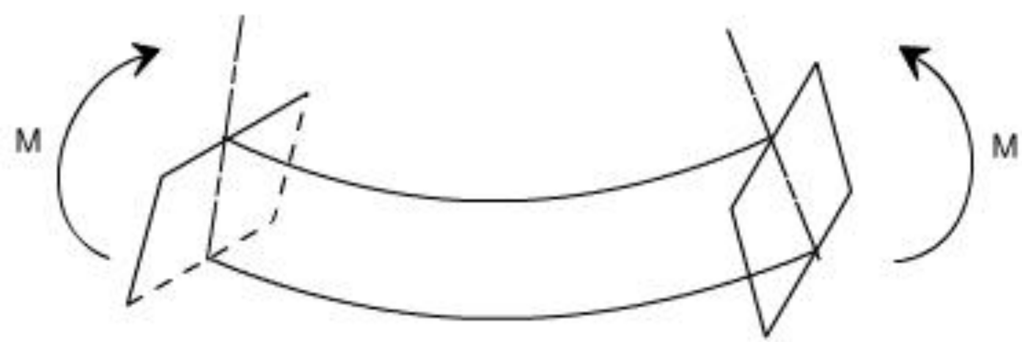


Fig (2)

- **Bending Stresses in Beams or Derivation of Elastic Flexural formula :**
- In order to compute the value of bending stresses developed in a loaded beam, let us consider the two cross-sections of a beam **HE** and **GF** , originally parallel as shown in fig 1(a).when the beam is to bend it is assumed that these sections remain parallel i.e. **H'E'** and **G'F'** , the final position of the sections, are still straight lines, they then subtend some angle
- Consider now fibre AB in the material, at a distance y from the N.A, when the beam bends this will stretch to A'B'

Therefore,

$$\text{strain in fibre AB} = \frac{\text{change in length}}{\text{orginal length}}$$

$$= \frac{A'B' - AB}{AB}$$

But AB = CD and CD = C'D'

refer to fig1(a) and fig1(b)

$$\therefore \text{strain} = \frac{A'B' - C'D'}{C'D'}$$

- Consider now fibre AB in the material, at a distance y from the N.A, when the beam bends this will stretch to A'B'
- Since CD and C'D' are on the neutral axis and it is assumed that the Stress on the neutral axis zero. Therefore, there won't be any strain on the neutral axis

$$= \frac{(R+y)\theta - R\theta}{R\theta} = \frac{R\theta + y\theta - R\theta}{R\theta} = \frac{y}{R}$$

However $\frac{\text{stress}}{\text{strain}} = E$ where E = Young's Modulus of elasticity

Therefore, equating the two strains as obtained from the two relations i.e,

$$\frac{\sigma}{E} = \frac{y}{R} \text{ or } \frac{\sigma}{y} = \frac{E}{R} \qquad \ldots\ldots\ldots\ldots(1)$$

$$\sigma = \frac{E}{R}\, y$$

if the shaded strip is of area 'dA'

then the force on the strip is

$$F = \sigma\, \delta A = \frac{E}{R}\, y\, \delta A$$

Moment about the neutral axis would be $= F.y = \frac{E}{R}\, y^2 \delta A$

The toatl moment for the whole cross-section is therefore equal to

$$M = \sum \frac{E}{R}\, y^2\, \delta A = \frac{E}{R} \sum y^2\, \delta A$$

- Now the term is the property of the material and is called as a second moment of area of the cross-section and is denoted by a symbol I.
- Therefore

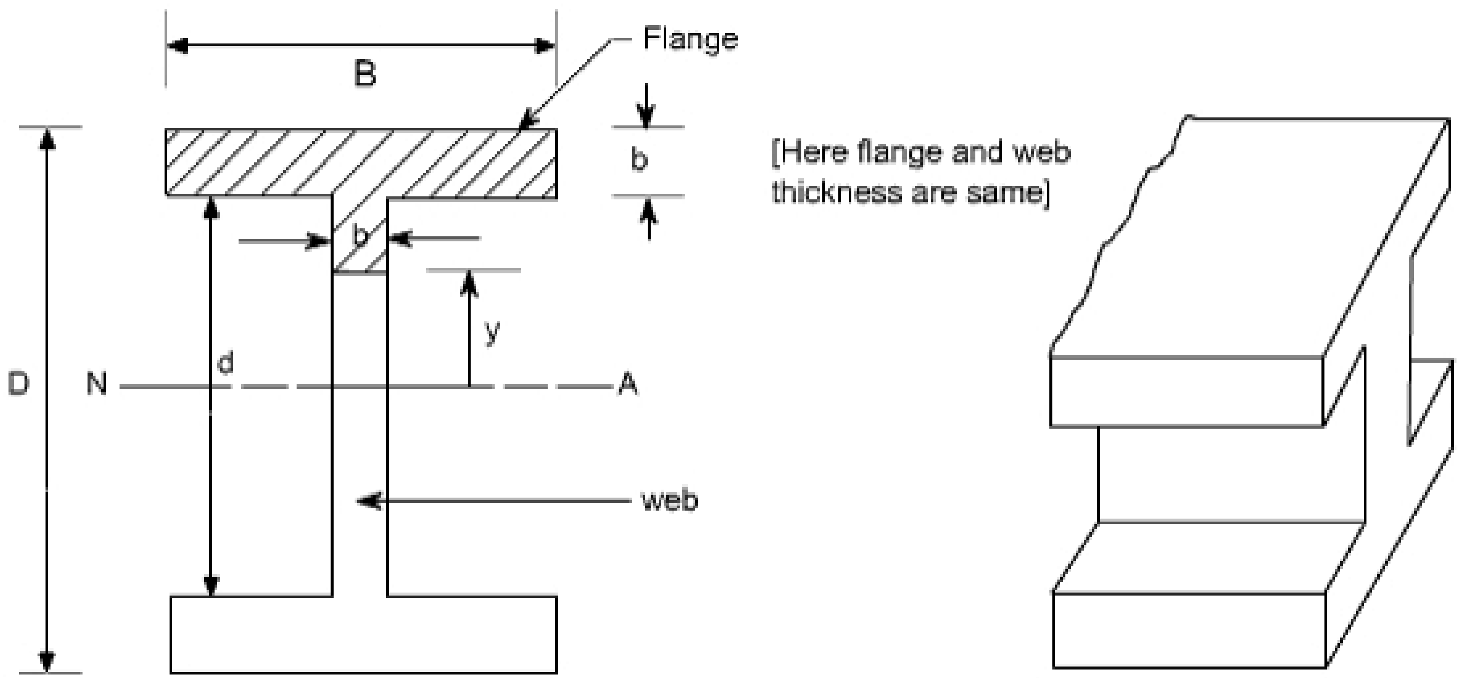
$$M = \frac{E}{R} I \qquad \ldots\ldots\ldots (2)$$

combining equation 1 and 2 we get

$$\boxed{\frac{\sigma}{y} = \frac{M}{T} = \frac{E}{R}}$$

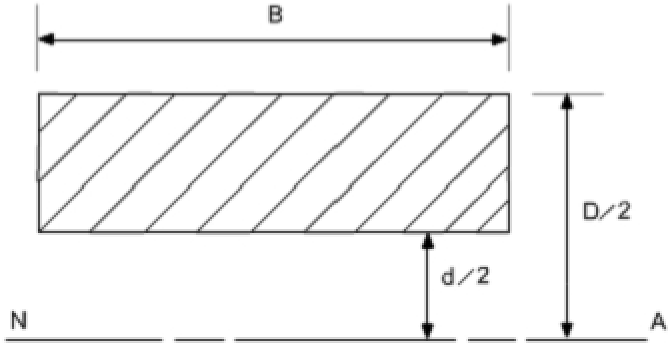
- **This equation is known as the Bending Theory Equation.** The above proof has involved the assumption of pure bending without any shear force being present. Therefore this termed as the pure bending equation. This equation gives distribution of stresses which are normal to cross-section i.e. in x-direction.

Consider an I - section of the dimension shown below.



The shear stress distribution for any arbitrary shape is given as $\tau = \frac{F A \bar{y}}{Z I}$

Let us evaluate the quantity $A\bar{y}$, the $A\bar{y}$ quantity for this case comprise the contribution due to flange area and web area



Flange area

Area of the flange $= B\left(\frac{D-d}{2}\right)$

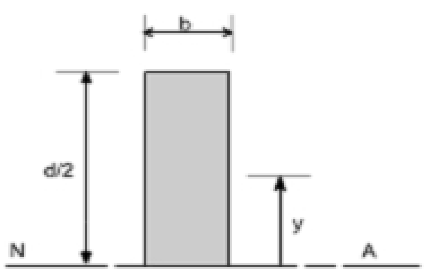
Distance of the centroid of the flange from the N.A

$$\bar{y} = \frac{1}{2}\left(\frac{D-d}{2}\right) + \frac{d}{2}$$

$$\bar{y} = \left(\frac{D+d}{4}\right)$$

Hence,

$$A\bar{y}\big|_{Flange} = B\left(\frac{D-d}{2}\right)\left(\frac{D-d}{4}\right)$$



Web Area

Area of the web

$$A = b\left(\frac{d}{2} - y\right)$$

Distance of the centroid from N.A

$$\bar{y} = \frac{1}{2}\left(\frac{d}{2} - y\right) + y$$

$$\bar{y} = \frac{1}{2}\left(\frac{d}{2} + y\right)$$

Therefore,

$$A\bar{y}\big|_{web} = b\left(\frac{d}{2} - y\right)\frac{1}{2}\left(\frac{d}{2} + y\right)$$

Hence,

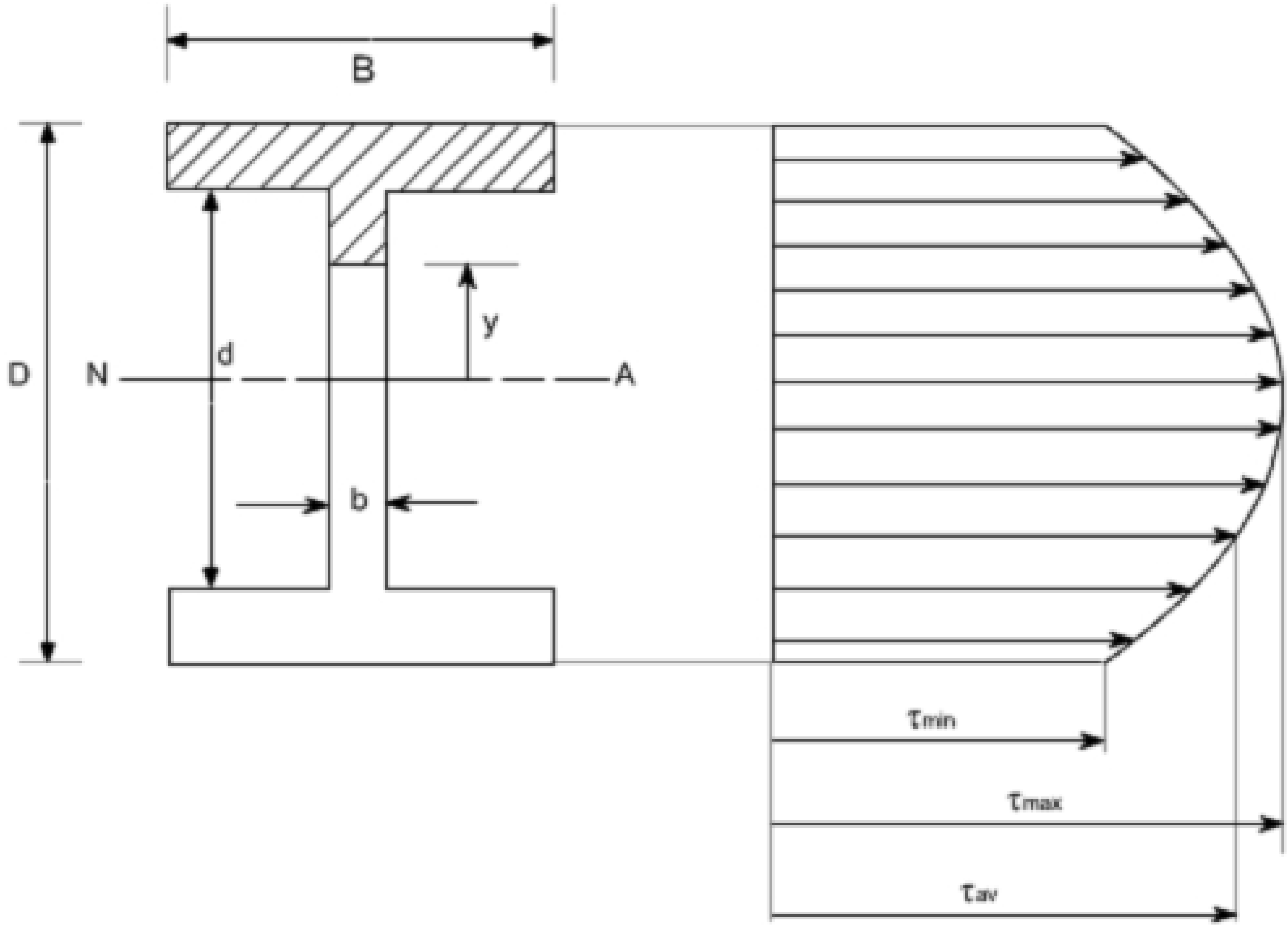
$$A\bar{y}\big|_{Total} = B\left(\frac{D-d}{2}\right)\left(\frac{D+d}{4}\right) + b\left(\frac{d}{2} - y\right)\left(\frac{d}{2} + y\right)\frac{1}{2}$$

Thus,

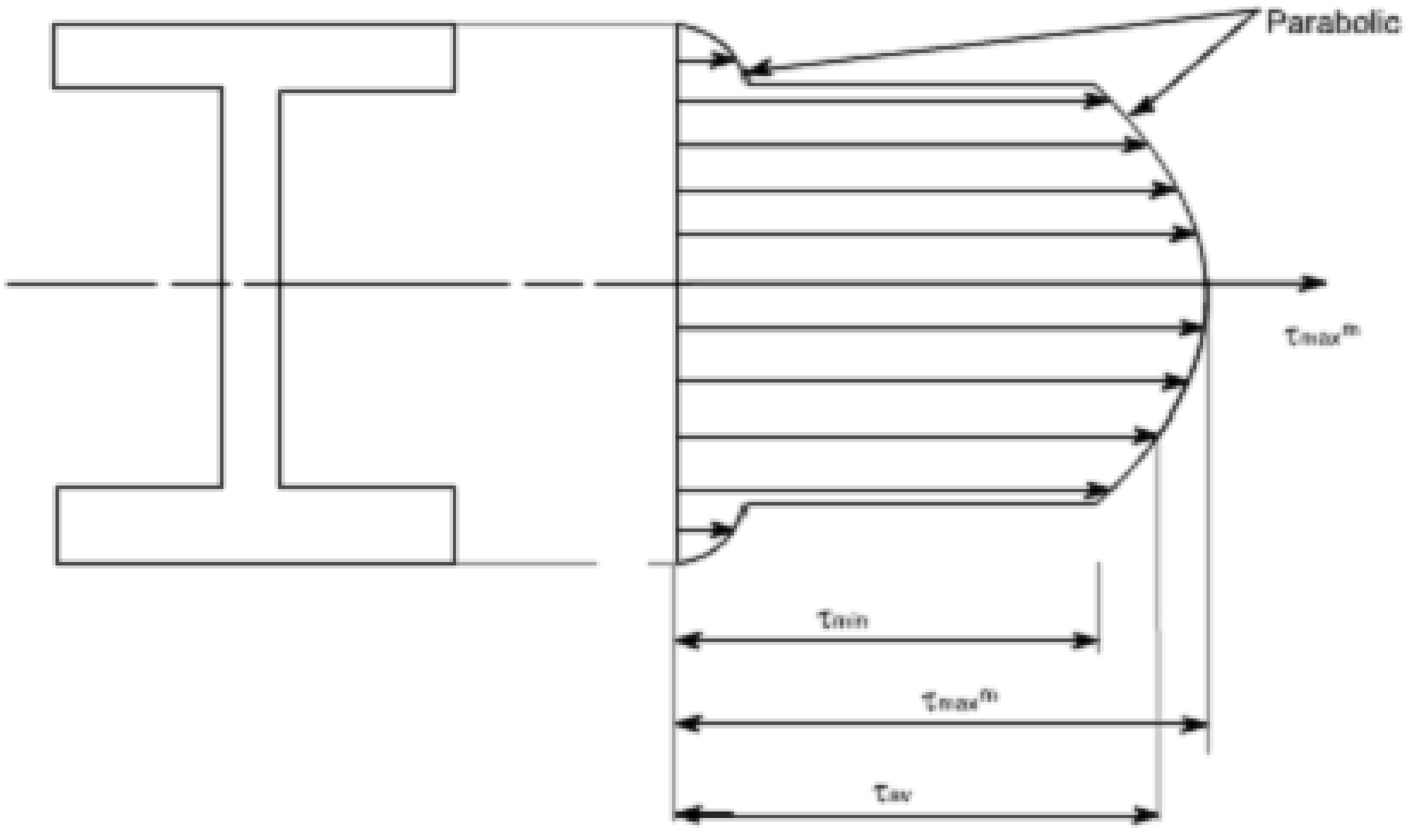
$$A\bar{y}\big|_{Total} = B\left(\frac{D^2 - d^2}{8}\right) + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

Therefore shear stress,

$$\tau = \frac{F}{bI}\left[\frac{B\left(D^2 - d^2\right)}{8} + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)\right]$$



$$\tau_{\max m} = \frac{F}{8bI}\left[B\left(D^2 - d^2\right) + bd^2\right]$$



This distribution is known as the "top – hat" distribution. Clearly the web bears the most of the shear stress and bending theory we can say that the flange will bear most of the bending stress.

UNIT -4
Principal stresses – strains and theory of failure

Syllabus

- **PRINCIPAL STRESSES AND STRAINS:** Introduction – Stresses on an inclined section of a bar under axial loading – compound stresses – Normal and tangential stresses on an inclined plane for biaxial stresses – Two perpendicular normal stresses accompanied by a state of simple shear – Mohr's circle of stresses – Principal stresses and strains – Analytical and graphical solutions.
- **THEORIES OF FAILURE:** Introduction – Various theories of failure - Maximum Principal Stress Theory, Maximum Principal Strain Theory, Strain Energy and Shear Strain Energy Theory (Von Mises Theory).

DERIVATION OF GENERAL EQUATIONS

Consider the complex stress system in Figure 4.1 acting on an element of material. The stresses σ_x and σ_y may be compressive or tensile and may be the result of direct forces or bending. The shear stresses may be as shown or completely reversed and occur as a result of either shear forces or torsion. Since the applied and complementary shear stresses are of equal value on the x and y planes, they are both given the symbol, τ_{xy}.

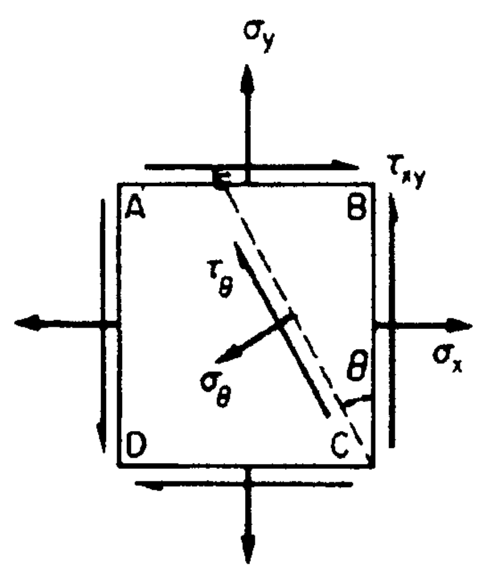


Fig 4.1 : Two-dimensional complex stress system.

The diagram thus represents a complete stress system for any condition of applied load in two dimensions. Consider the rectangular element of unit depth shown in Figure 4.1 . subjected to a system of two direct stresses and shear stresses.

For equilibrium of the portion EBC (Figure 4.1), redrawn in Figure 4.2:

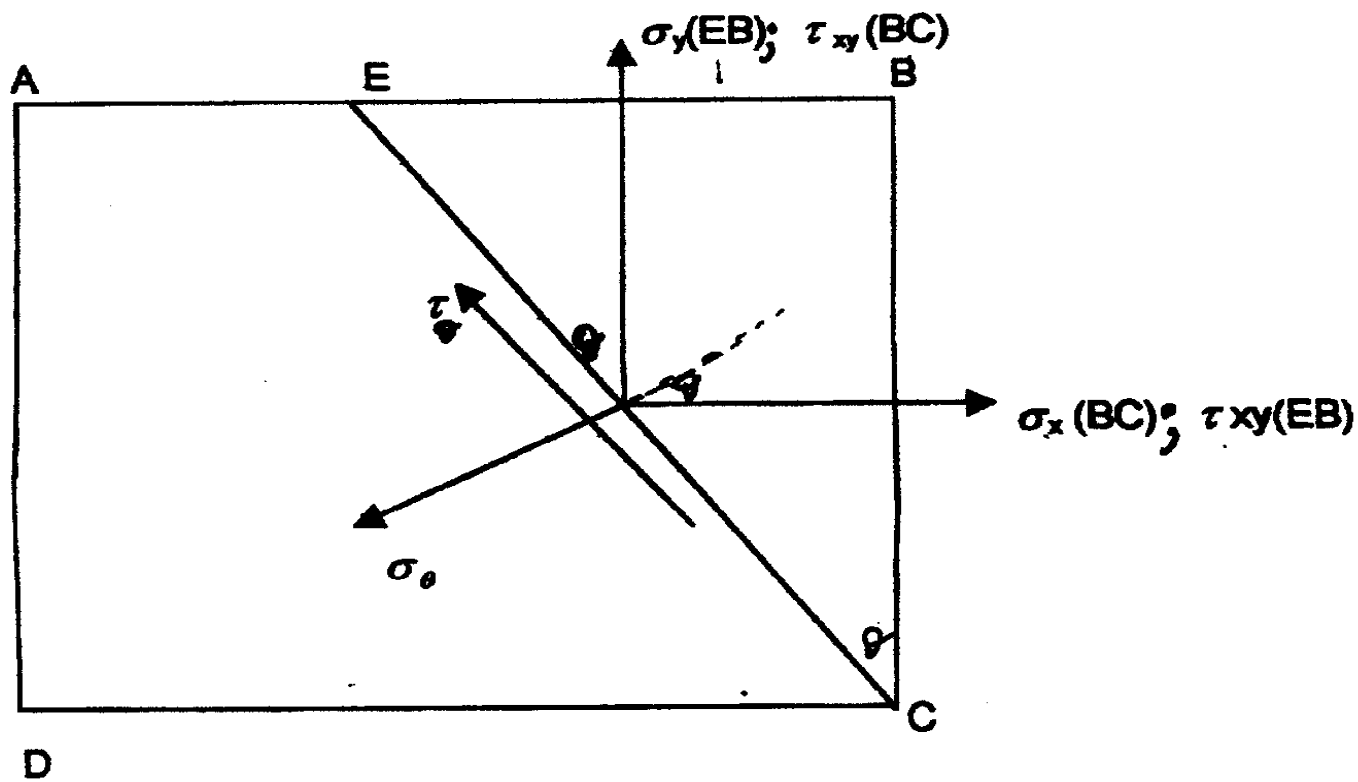
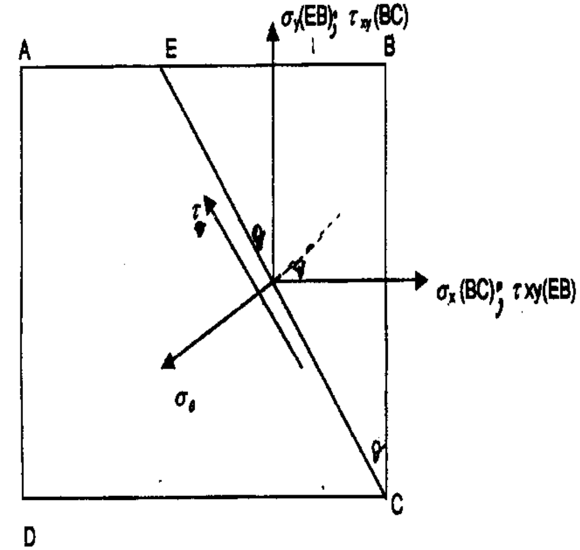


Figure 4.2: Forces Acting on Element EBC

Resolving perpendicular to EC:



$$\sigma_\theta \text{ x } 1 \text{ x } EC = \sigma_x \text{ x } BC \text{ x } 1 \text{ x } \cos\theta$$

$$+ \sigma_y \text{ x } EB \text{ x } 1 \text{ x } \sin\theta$$

$$+ \tau_{xy} \text{ x } 1 \text{ x } EB \text{ x } \cos\theta$$

$$+ \tau_{xy} \text{ x } 1 \text{ x } BC \text{ x } \sin\theta$$

Note that $EB = EC \sin\theta$ and $BC = EC \cos\theta$

$$\sigma_\theta \text{ x } EC = \sigma_x \text{ x } EC \cos^2\theta + \sigma_y \text{ x } EC \sin^2\theta$$

$$+ \tau_{xy} \text{ x } EC \text{ x } \sin\theta \cos\theta$$

$$+ \tau_{xy} \text{ x } EC \sin\theta \cos\theta$$

$$\sigma_\theta = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\,\tau_{xy} \sin\theta \cos\theta$$

Recall that : $\cos^2\theta = (1 + \cos 2\theta)/2$, $\sin^2\theta = (1 - \cos 2\theta)/2$ and

$$\sin 2\theta = 2 \sin\theta \cos\theta$$

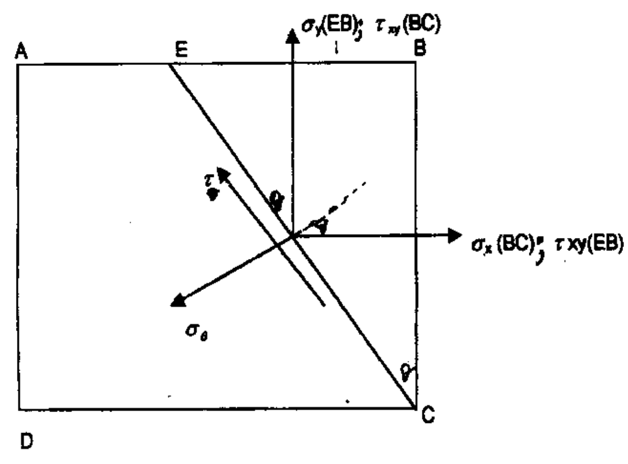
$$\sigma_\theta = \sigma_x/2\,(1 + \cos 2\theta) + \sigma_y/2\,(1 - \cos 2\theta) + \tau_{xy} \sin 2\theta$$

$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4.1)$$

Resolving parallel to EC:

$$\tau_\theta \times 1 \times EC = \sigma_x \times BC \times 1 \times \sin\theta + \sigma_y \times EB \times 1 \times \cos\theta$$

$$+\ \tau_{xy} \times 1 \times EB \times \sin\theta + \tau_{xy} \times 1 \times BC \times \cos\theta$$



Derivation of General Equation Concluded

τ_θ x EC = σ_x x EC $\sin\theta\cos\theta$ - σ_y x EC $\sin\theta$ $\cos\theta$ +

τ_{xy} x EC x $\sin^2\theta$ - τ_{xy} x EC $\cos^2\theta$

τ_θ = σ_x $\sin\theta$ $\cos\theta$ - σ_y $\sin\theta$ $\cos\theta$ + τ_{xy} $\sin^2\theta$ - τ_{xy} $\cos^2\theta$

Recall that $\sin 2\theta$ = $2\sin\theta\cos\theta$ and $\cos 2\theta$ = $\cos^2\theta - \sin^2\theta$

$$\tau_\theta = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4.2)$$

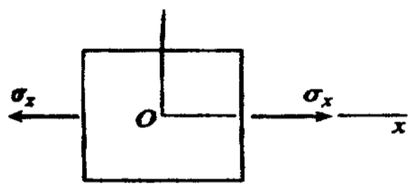
SPECIAL CASES OF PLANE STRESS

The general case of plane stress reduces to simpler states of stress under special conditions:

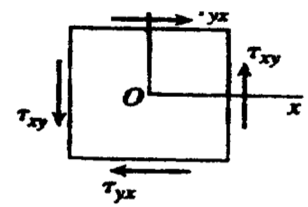
4.1.1 **Uniaxial Stress**: This is the situation where all the stresses acting on the xy element are zero except for the normal stress σ_x, then the element is in uniaxial stress. The corresponding transformation equations, obtained by setting σ_y and τ_{xy} equal to zero in the Equations 4.1 and 4.2 above:

$$\sigma_\theta = \frac{\sigma_x}{2}(1+\cos 2\theta), \qquad \tau_\theta = \frac{\sigma_x}{2}\sin 2\theta$$

Special Cases of Plane Stress Contd.



Element in uniaxial stress



Element in pure shear

4.2.2 **Pure Shear:** The transformation equations are obtained by substituting $\sigma_x = 0$ and $\sigma_y = 0$ into Equations 4.1 and 4.

$$\sigma_\theta = \tau_{xy} \sin 2\theta \qquad \tau_\theta = \tau_{xy} \cos 2\theta$$

4.2.3 **Biaxial Stress:** The xy element is subjected to normal stresses in both x and y directions but without any shear stresses. τ_{xy} is merely dropped from the general equations to obtain:

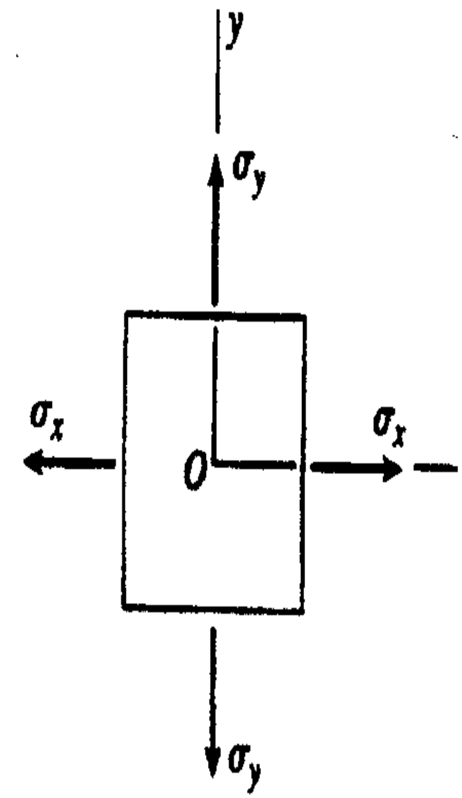
$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta \qquad \tau_\theta = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta$$

The maximum direct stress will equal σ_x or σ_y, whichever is the greater, when $\theta = 0$ or $90°$.

Maximum Shear Stress

The maximum shear stress in the plane of the applied stresses occurs when $\theta = 45°$, i.e.

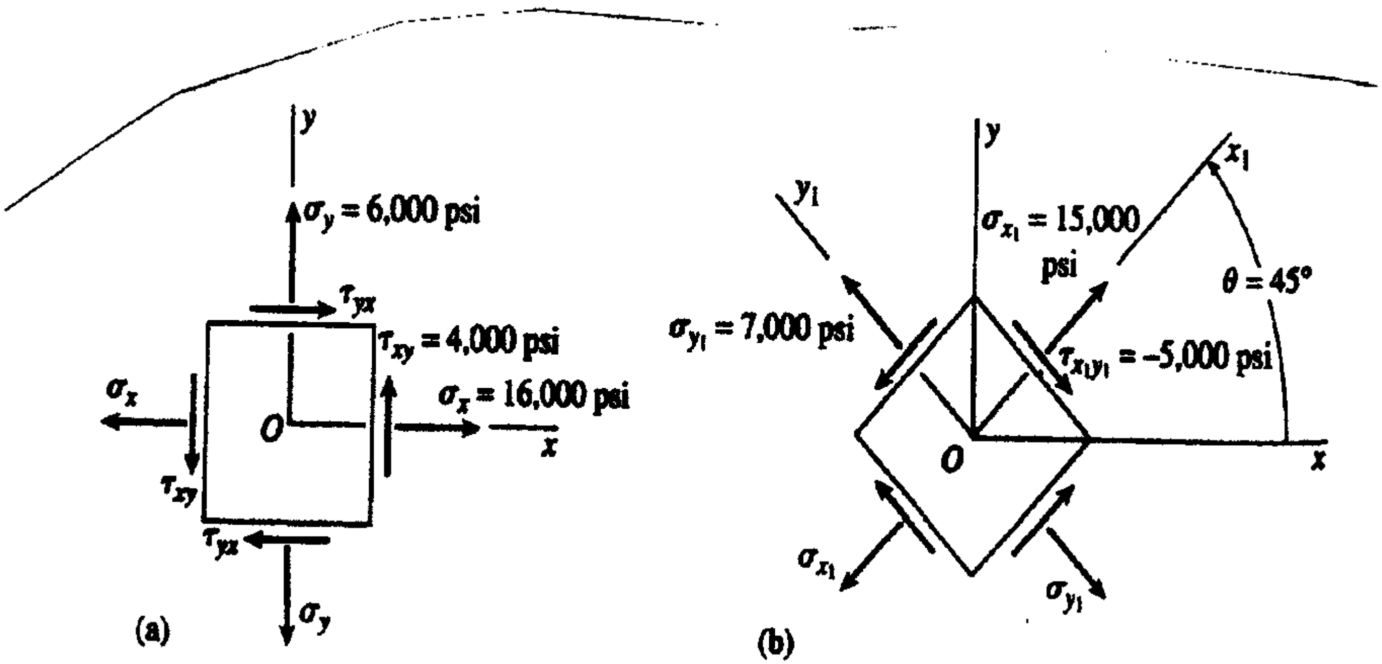
$$\tau_{max} = \frac{1}{2}(\sigma_x - \sigma_y) \qquad \ldots\ldots\ldots\ldots \quad (4.3)$$



Element in biaxial stress

Example

Example: An element in plane stress is subjected to stresses σ_x = 16,000 N/mm^2, σ_y =6000 N/mm^2, and τ_{xy} = 4000 N/mm^2. Determine the stresses acting on an element inclined at an angle of θ = 45°.



Solution

Solution: To obtain the stresses on an inclined element, use equations (4.1) and (4.2)

$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_\theta = \frac{16{,}000 + 6{,}000}{2} + \frac{16{,}000 - 6000}{2}\cos 90^\circ + 4000\sin 90 = 15{,}000 N/mm^2$$

$$\tau_\theta = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta = \frac{16000 - 6000}{2}\sin 90 + 4000\cos 90^\circ = 5000 N/mm^2$$

Principal Stresses and Maximum Shear Stresses

4.3 PRINCIPAL STRESSES AND MAXIMUM SHEAR STRESSES

The maximum and minimum stresses which occur on any plane in the material can now be determined as follows:

$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \quad \text{-- -- .} \quad (4.1)$$

For σ_θ to be a maximum or minimum, $d\sigma_\theta / d\theta = 0$

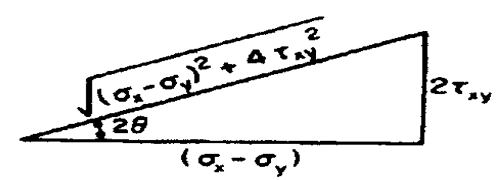
$$\frac{d\sigma_\theta}{d_\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0 \qquad \div \cos 2\theta$$

$$\tan 2\theta = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4.4)$$

From Figure below:

$$\sin 2\theta = \frac{2\tau_{xy}}{\sqrt{\left[(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}\right]}}$$

$$\cos 2\theta = \frac{(\sigma_x - \sigma_y)}{\sqrt{\left[(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}\right]}}$$



Principal Stresses and Maximum Shear Stresses Contd.

The solution of equation 4.4 yields two values of 2θ separated by 180^o, i.e. two values of θ separated by 90^o. Thus the two principal stresses occur on mutually perpendicular planes termed principal planes,

Substituting in equation 4.1:

$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}} + \tau_{xy} \frac{2\,\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} \pm \frac{(\sigma_x - \sigma_y)^2}{2\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}} + \frac{2\,\tau^2_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{2} \frac{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

Shear Stresses at Principal Planes are Zero

$$\sigma_1 \text{ or } \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{2}\sqrt{\sigma_x - \sigma_y)^2 + 4\,\tau^2{}_{xy}} \quad \text{........ (4.5)}$$

These are termed the principal stresses of the system. By substitution for θ from equation 4.4 , into the shear stress expression (equation 4.2):

$$\tau_\theta = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta \quad \text{.................... (4.2)}$$

$$\tau_\theta = \frac{\sigma_x - \sigma_y}{2}\;\frac{2\,\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}} - \tau_{xy}\;\frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}}$$

$$\tau_\theta = \frac{\tau_{xy}\;(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau^2{}_{xy}}} - \frac{\tau_{xy}\,(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}} = 0$$

Principal Planes and Stresses Contd.

Thus at principal planes, $\tau_{\theta} = 0$. Shear stresses do not occur at the principal planes.

The complex stress system of Figure 4.1 can now be reduced to the equivalent system of principal stresses shown in Figure 4.2 below.

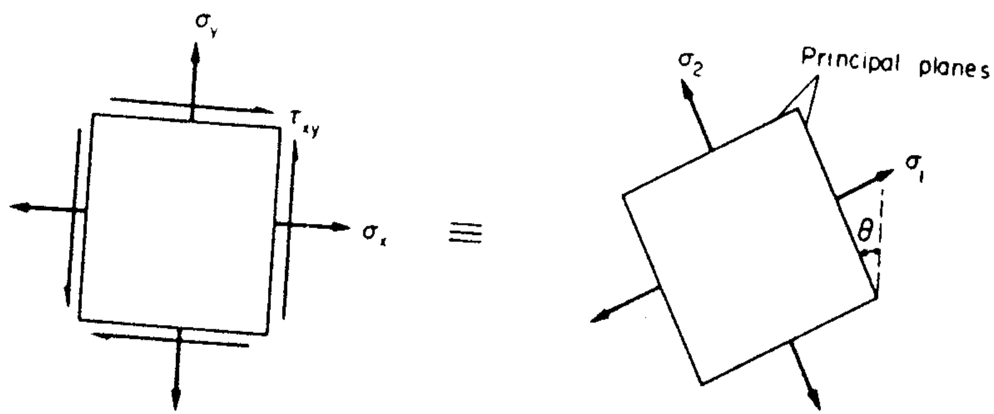


Figure 4.3: Principal planes and stresses

Equation For Maximum Shear Stress

From equation 4.3, the maximum shear stress present in the system is given by:

$$\tau_{\max} = \frac{1}{2}(\sigma_x - \sigma_y) = \frac{1}{2}\sqrt{\sigma_x - \sigma_y)^2 + 4\,\tau^2{}_{xy}}$$

and this occurs on planes at 45^o to the principal planes.

Note: This result could have been obtained using a similar procedure to that used for determining the principal stresses, i.e. by differentiating expression 4.2, equating to zero and substituting the resulting expression for θ

PRINCIPAL PLANE INCLINATION IN TERMS OF THE ASSOCIATED PRINCIPAL STRESS

It has been stated in the previous section that expression (4.4), namely

$$\tan 2\theta = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

yields two values of θ, i.e. the inclination of the two principal planes on which the principal stresses σ_1 or σ_2. It is uncertain, however, which stress acts on which plane unless eqn. (4.1) is used, substituting *one* value of θ obtained from eqn. (4.4) and observing which one of the two principal stresses is obtained. The following alternative solution is therefore to be preferred.

PRINCIPAL PLANE INCLINATION CONTD.

- Consider once again the equilibrium of a triangular block of material of unit depth (Fig. 4.3); this time *EC* is a principal plane on which a principal stress acts, and the shear stress is zero (from the property of principal planes).

PRINCIPAL PLANE INCLINATION CONTD.

Resolving forces horizontally,

$$(,\sigma_x x \; BC \, x \, 1) + (\tau_{xy} \; x \; EB \, x \, 1) = (\sigma_p \, x \, EC \, x \, l) \cos\theta$$

$$\sigma_x \, EC \cos\theta + \tau_{xy} \; x \; EC \sin\theta = \sigma_p \; x \; EC \cos\theta$$

$$\sigma_x + \tau_{xy} \tan\theta = \sigma_p$$

$$\tan\theta = \frac{\sigma_p - \sigma_x}{\tau_{xy}} \qquad \ldots \; (4.7)$$

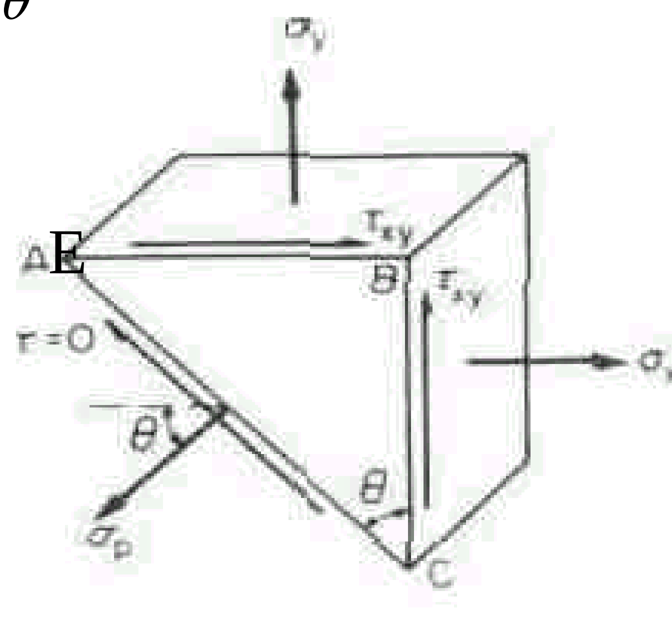


Fig. 13.8.

PRINCIPAL PLANE INCLINATION CONTD.

- Thus we have an equation for the inclination of the principal planes *in terms of the* principal stress. If, therefore, the principal stresses are determined and substituted in the above equation, each will give the corresponding angle of the plane on which it acts and there can then be no confusion.

PRINCIPAL PLANE INCLINATION CONTD.

- The above formula has been derived with two tensile direct stresses and a shear stress system, as shown in the figure; should any of these be reversed in action, then the appropriate minus sign must be inserted in the equation.

Graphical Solution Using the Mohr's Stress Circle

4.5. GRAPHICAL SOLUTION-MOHR'S STRESS CIRCLE

Consider the complex stress system of Figure below. As stated previously this represents a complete stress system for any condition of applied load in two dimensions. In order to find graphically the direct stress σ_p and shear stress τ_θ on any plane inclined at θ to the plane on which σ_x acts, proceed as follows:

(1) Label the block *ABCD.*

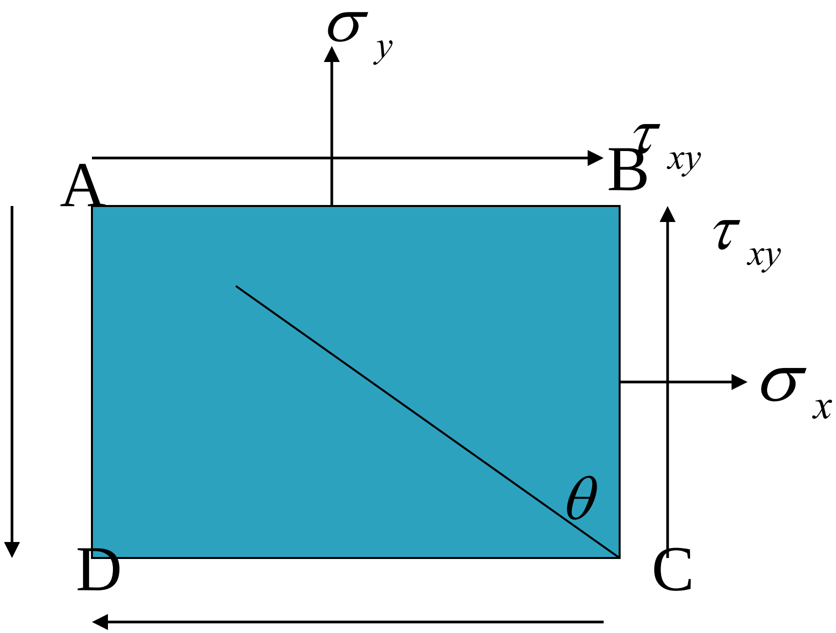
(2) Set up axes for direct stress (as abscissa) and shear stress (as ordinate)

(3) Plot the stresses acting on two *adjacent* faces, e.g. *AB* and *BC,* using the following sign conventions:

Mohr's Circle Contd.

- *Direct stresses:* tensile, positive; compressive, negative;
- *Shear stresses:* tending to turn block clockwise, positive; tending to turn block
  counterclockwise, negative.
- This gives two points on the graph which may then be labeled *AB* and *BC* respectively to denote stresses on these planes

Mohr’s Circle Contd.



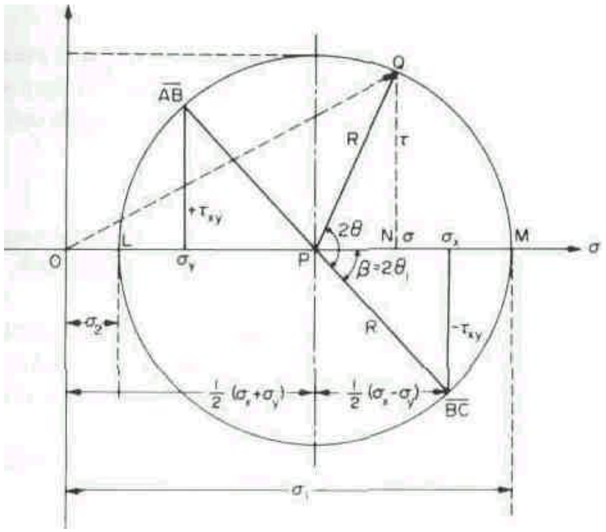


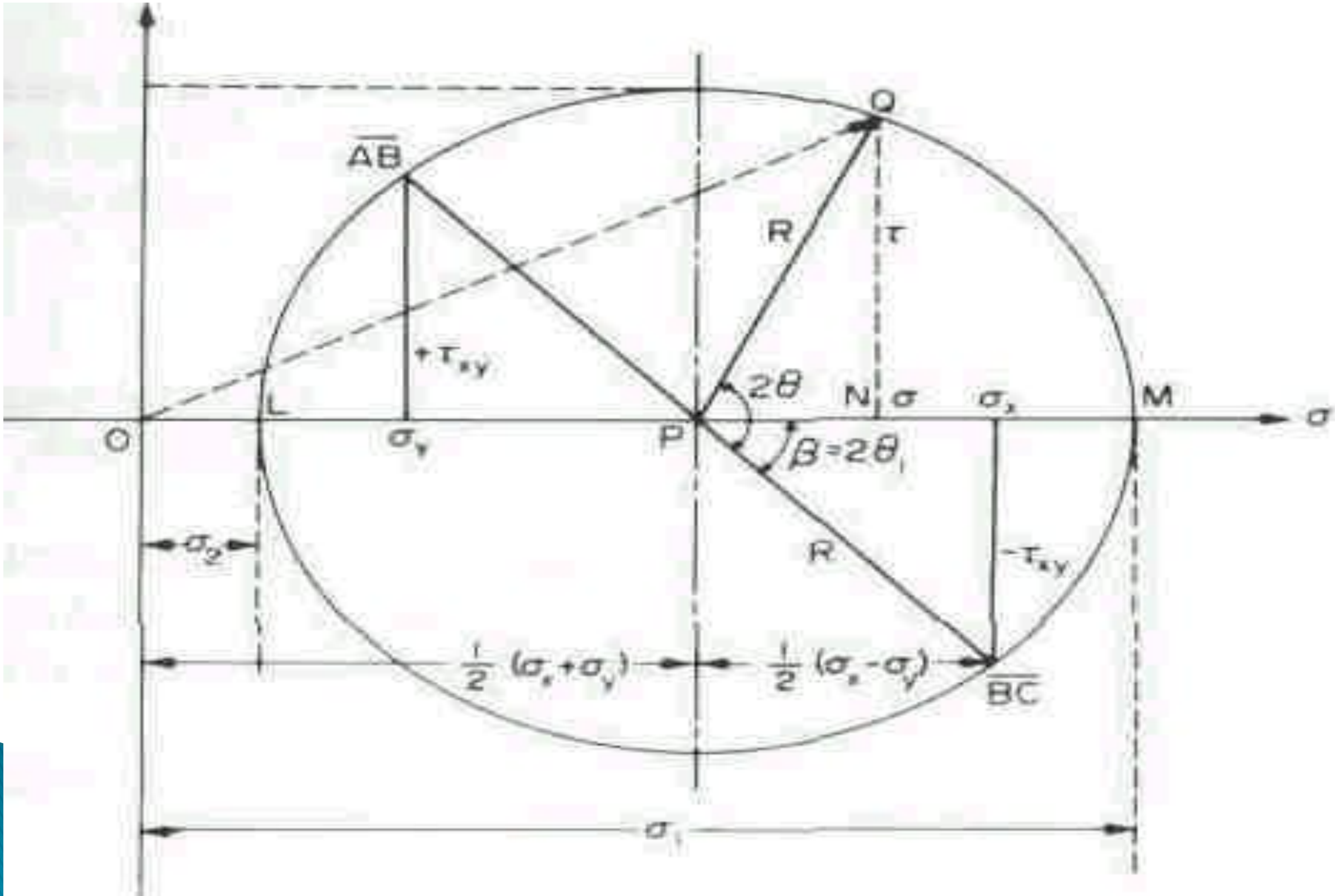
Fig. 4.5 Mohr's stress circle.

(4) Join *AB* and *BC.*

(5) The point *P* where this line cuts the *a* axis is then the centre of Mohr's circle, and the

line is the diameter; therefore the circle can now be drawn. Every point on the circumference of the circle then represents a state of stress on some **plane** through C.

Mohr's stress circle.



Proof

Consider any point Q on the circumference of the circle, such that PQ makes an angle 2θ with BC, and drop a perpendicular from Q to meet the a axis at N.

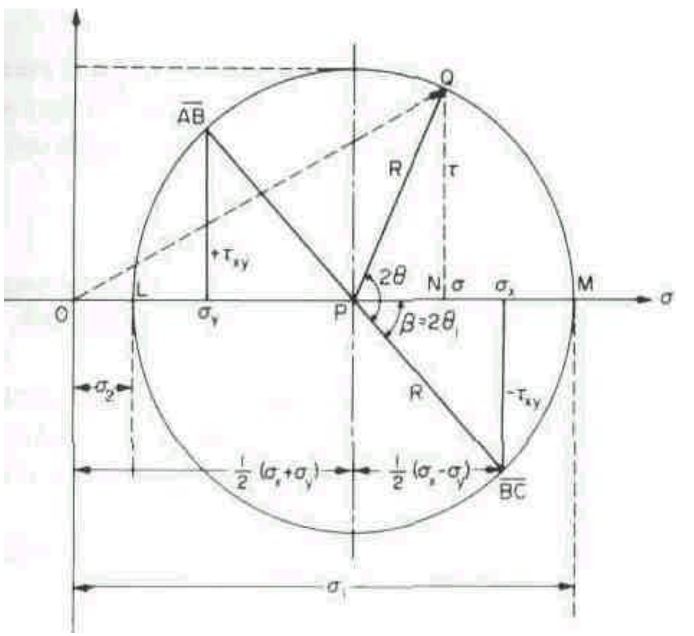
***Coordinates of** Q:*

$$ON = OP + PN = \frac{1}{2}(\sigma_x + \sigma_y) + R\cos(2\theta - \beta)$$

$$\frac{1}{2}(\sigma_x + \sigma_y) + R\cos 2\theta \cos\beta + R\sin 2\theta \sin\beta$$

$$R\cos\beta = \frac{1}{2}(\sigma_x - \sigma_y) \quad \text{and} \quad R\sin\beta = \tau_{xy}$$

$$ON = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$



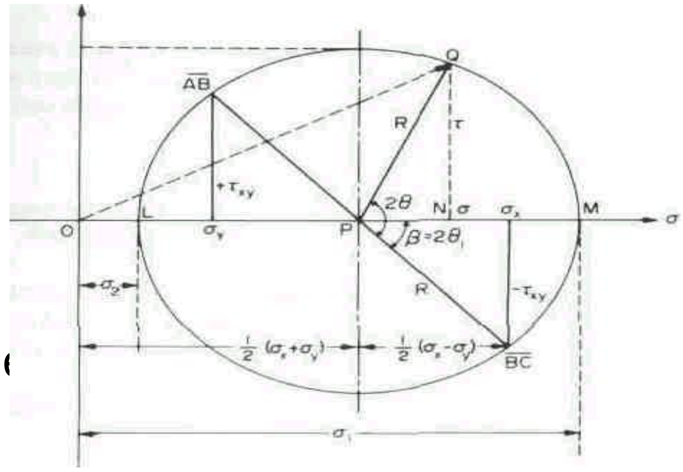
On inspection this is seen to be eqn. (4.1) for the direct stress σ_θ on the plane inclined at θ to *BC* in the figure for the two-dimensional complex system.

Similarly,

QN sin (2θ - β)

= *R* sin 2 θ ***cos*** β **-** ***R cos*** *2* θ ***sin*** β

$$= \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$



Again, on inspection this is seen to be eqn. (4.2) for the

inclined at θ to *BC.*

Note

Thus the coordinates of *Q* are the normal and shear stresses on a plane inclined at θ to BC in the original stress system.

N.B. - Single angle *BCPQ* is 2θ on Mohr's circle and not θ , it is evident that angles *are doubled on Mohr's circle.* This is the only difference, however, as they are measured in the same direction and from the same plane in both figures (in this case counterclockwise from

~BC).

Further Notes on Mohr's Circle

Further points to note are:

(1) The direct stress is a maximum when *Q* is at *M,* i.e. *OM* is the length representing the maximum principal stress σ_1 and $2\theta_1$ gives the angle of the plane θ_1 from *BC.* Similarly, *OL is* the other principal stress.

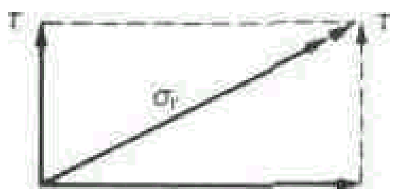
(2) The maximum shear stress is given by the highest point on the circle and is represented by the radius of the circle. This follows since shear stresses and complementary shear stresses have the same value; *therefore the centre of the circle will always lie on the* σ_1 *axis midway between* σ_x *and* σ_y *.*

(3) From the above point the direct stress on the plane of maximum shear must be midway between σ_x *and* σ_y .

Further Notes on Mohr Circle Contd.

(4) The shear stress on the principal planes is zero.

(5) Since the resultant of two stresses at 90° can be found from the parallelogram of vectors as the diagonal, as shown in Figure below, the resultant stress on the plane at θ to BC is given by OQ on Mohr's circle.



Resultant stress σ_r on any plane.

Preference of Mohr Circle

- The graphical method of solution of complex stress problems using Mohr's circle is a very powerful technique since all the information relating to any plane within the stressed element is contained in the single construction.
- It thus provides a convenient and rapid means of solution which is less prone to arithmetical errors and is highly recommended.

UNIT -5
Deflection of Beams and Conjugate Beam Method

Deflection of Beams

The deformation of a beam is usually expressed in terms of its deflection from its original unloaded position. The deflection is measured from the original neutral surface of the beam to the neutral surface of the deformed beam. The configuration assumed by the deformed neutral surface is known as the elastic curve of the beam.

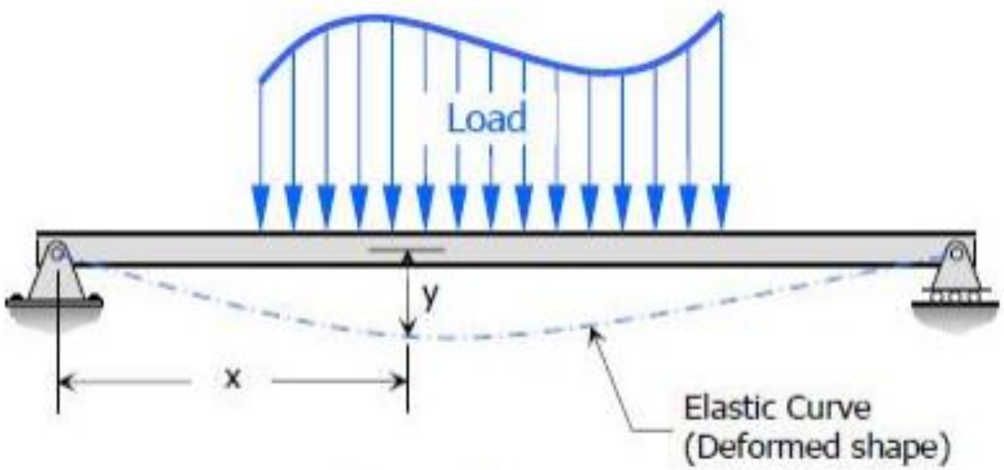


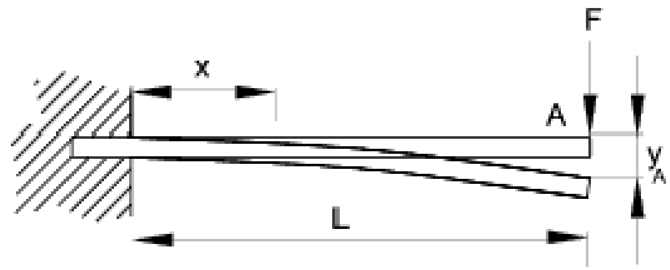
Figure: Elastic curve

METHODS OF DETERMINING DEFLECTION OF BEAMS

- Double integration method
- Moment area method
- Conjugate method
- Macaulay's method

Example - Cantilever beam

Consider a cantilever beam (uniform section) with a single concentrated load at the end. At the fixed end x = 0, dy = 0 , dy/dx = 0



From the equilibrium balance ..At the support there is a resisting moment -FL and a vertical upward force F.
At any point x along the beam there is a moment $F(x - L) = M_x = EI\, d^2y/dx^2$

$$EI\frac{d^2y}{dx^2} = -F\,(L-x) \quad \text{Integrating}$$

$$EI\frac{dy}{dx} = -F\left(Lx - \frac{x^2}{2}\right) + C_1 \text{ .....}(C_1 = 0 \text{ because } dy/dx = 0 \text{ at } x = 0)$$

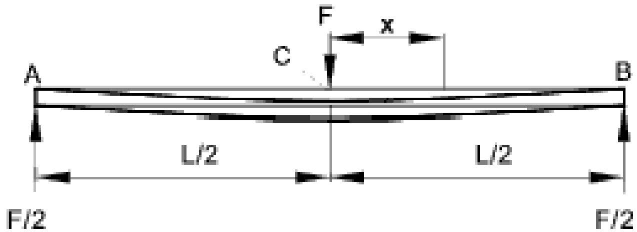
Integrating again

$$EI\,y = -F\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + C_2 \text{ .....}(C = 0 \text{ because } y = 0 \text{ at } x = 0)$$

$$\text{At end A} \quad \left(\frac{dy}{dx}\right)_A = -\frac{F}{EI}\left(L^2 - \frac{L^2}{2}\right) = -\frac{FL^2}{2EI} \quad \text{and} \quad y_A = -\frac{F}{EI}\left(\frac{L^3}{2} - \frac{L^3}{6}\right) = -\frac{FL^3}{3EI}$$

Example - Simply supported beam

Consider a simply supported uniform section beam with a single load F at the centre. The beam will be deflect symmetrically about the centre line with 0 slope (dy/dx) at the centre line. It is convenient to select the origin at the centre li



$$\frac{d^2y}{dx^2} = \frac{1}{EI}\left[\frac{F}{2}\left(\frac{L}{2}+x\right) - Fx\right] = \frac{F}{2EI}\left(\frac{L}{2}-x\right)$$ Integrating

$$\frac{dy}{dx} = \frac{F}{2EI}\left(\frac{Lx}{2}-\frac{x^2}{2}\right) + C_1 \ldots$$ ($C_1 = 0$ because $dy/dx = 0$ at $x = 0$)

Integrating again $y = \frac{F}{2EI}\left(\frac{Lx^2}{4}-\frac{x^3}{6}\right) + C_2$

$y = 0$ when $x = L/2$ therefore $\frac{F}{2EI}\left(\frac{L^3}{8}-\frac{L^3}{12}\right) + c_2 = 0$

and thus $c_2 = -\frac{FL^3}{48EI}$

At end B $\left(\frac{dy}{dx}\right)_b = -\frac{F}{2EI}\left(\frac{L^2}{4}-\frac{L^2}{8}\right) = \frac{FL^2}{16EI}$ and $y_B = \frac{F}{2EI}\left(\frac{L^3}{8}-\frac{L^3}{12}\right) - \frac{FL^3}{48EI} = 0$

At centre C $x = 0$ $\quad y_c = -\frac{Fl^3}{48EI}$ $\quad$ (slope $\frac{dy}{dx} = 0$ by symmetry)

Moment Area Method

This is a method of determining the change in slope or the deflection between two points on a beam. It is expressed as two theorems...

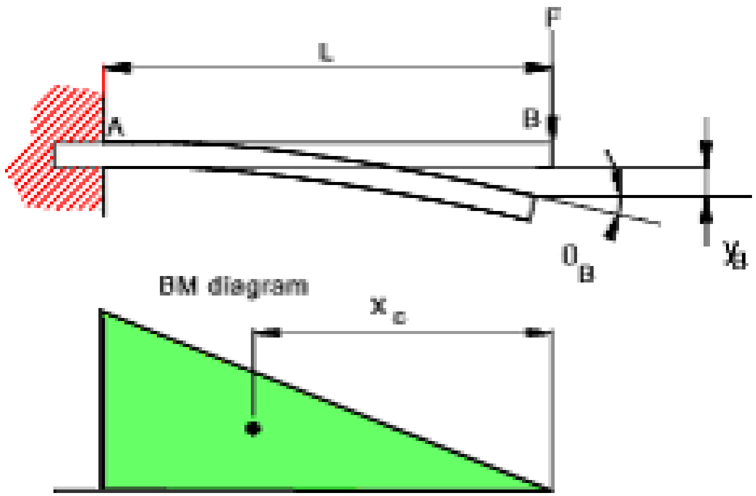
Theorem 1

If A and B are two points on a beam the change in angle (radians) between the tangent at A and the tangent at B is equal to the area of the bending moment diagram between the points divided by the relevant value of EI (the flexural rigidity constant).

Theorem 2

If A and B are two points on a beam the displacement of B relative to the tangent of the beam at A is equal to the moment of the area of the bending moment diagram between A and B about the ordinate through B divided by the relevant value of EI (the flexural rigidity constant).

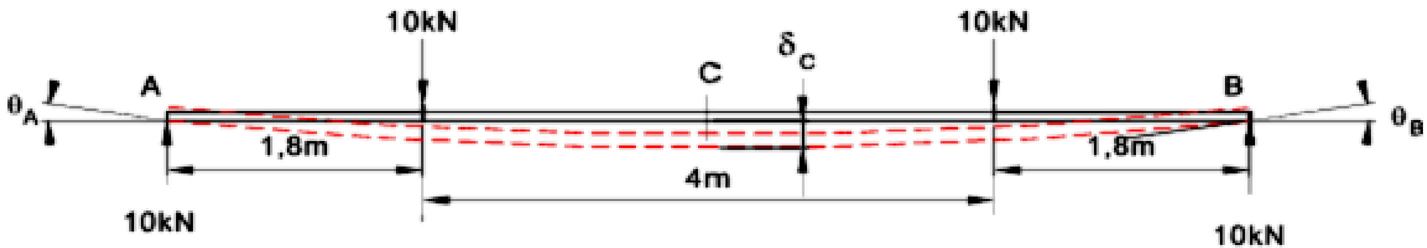
Examples .Two simple examples are provide below to illustrate these theorems
Example 1) Determine the deflection and slope of a cantilever as shown..

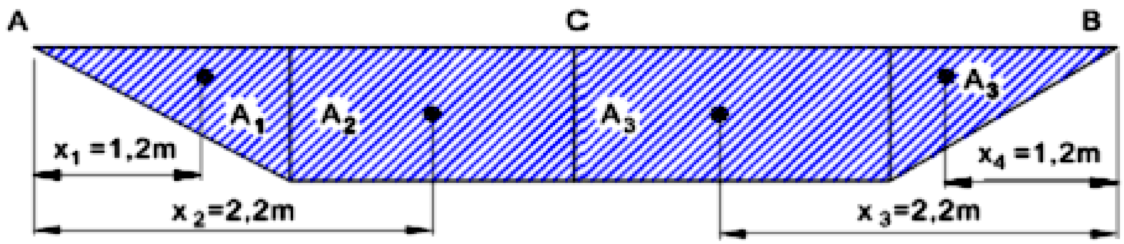


The bending moment at A = M_A = FL
The area of the bending moment diagram $A_M = F.L^2/2$
The distance to the centroid of the BM diagram from B= $x_c = 2L/3$
The deflection of B = $y_b = A_M.x_c/EI = F.L^3/3EI$
The slope at B relative to the tan at A = $\theta_b = A_M/EI = FL^2/2EI$

Example 2) Determine the central deflection and end slopes of the simply supported beam as shown..

$E = 210$ GPa ......$I = 834$ cm^4...... $EI = 1{,}7514.\ 10^{6}$Nm ^{2}





Bending Moment Diagram

$A_1 = 10.1{,}8.1{,}8/2 = 16{,}2$kNm

$A_2 = 10.1{,}8.2 = 36$kNm

$A_2 = 10.1{,}8.2 = 36$kNm

$A_1 = 10.1{,}8.1{,}8/2 = 16{,}2$kNm

$x_1 =$ Centroid of $A_1 = (2/3).1{,}8 = 1{,}2$

$x_2 =$ Centroid of $A_2 = 1{,}8 + 1 = 2{,}8$

$x_3 =$ Centroid of $A_3 = 1{,}8 + 1 = 2{,}8$

$x_4 =$ Centroid of $A_4 = (2/3).1{,}8 = 1{,}2$

The slope at A is given by the area of the moment diagram between A and C divided by EI.

$$\theta_A = (A_1 + A_2)/EI = (16,2+36).10^3 / (1,7514.\ 10^6)$$
$$= 0,029\text{rads} = 1,7 \text{ degrees}$$

The deflection at the centre (C) is equal to the deviation of the point A above a line that is tangent to C.
Moments must therefore be taken about the deviation line at A.

$$\delta_C = (A_M.x_M)/EI = (A_1 x_1 + A_2 x_2)/EI = 120,24.10^3/(1,7514.\ 10^6)$$
$$= 0,0686\text{m} = 68,6\text{mm}$$

Moment Area Method

This method is based on two theorems which are stated through an example. Consider a beam AB subjected to some arbitrary load as shown in Figure 1.

Let the flexural rigidity of the beam be EI. Due to the load, there would be bending moment and BMD would be as shown in Figure 2. The deflected shape of the beam which is the elastic curve is shown in Figure 3. Let C and D be two points arbitrarily chosen on the beam. On the elastic curve, tangents are drawn at deflected positions of C and D. The angles made by these tangents with respect to the horizontal are marked as and . These angles are nothing but slopes. The change is the angle between these two tangents is demoted as . This change in the angel is equal to the area of the diagram between the two points C and D. This is the area of the shaded portion in figure 2.

Hence $\theta_{CD} = \theta_C \sim \theta_D$ = Area of $\dfrac{M}{EI}$ diagram between C and D

$$\theta_{CD} = \frac{\text{Area BM}}{\text{EI}} \longrightarrow 1\,(a)$$

It is also expressed in the integration mode as

$$\theta_{CD} = \int_{CD} \frac{M}{EI} dx \longrightarrow 1\,(b)$$

Equation 1 is the first moment area theorem which is stated as follows:

Statement of theorem I:

The change in slope between any two points on the elastic curve for a member subjected to bending is equal to the area of $\dfrac{M}{EI}$ ***diagram between those two points.***

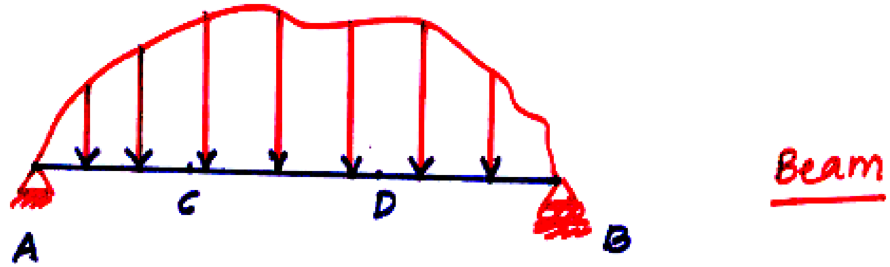


Fig. 1

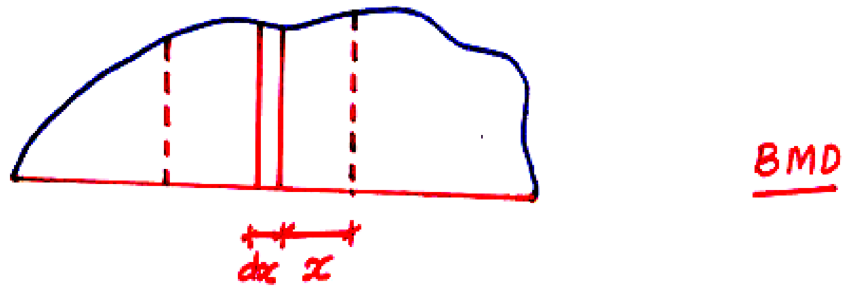


Fig. 2

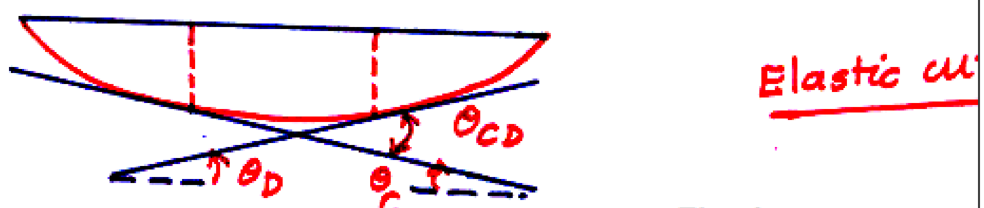


Fig. 3

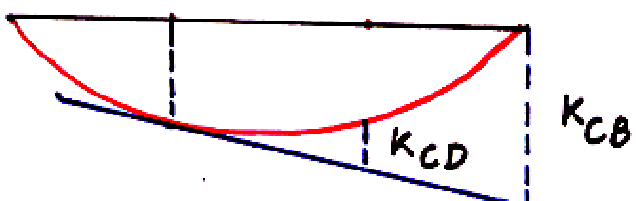
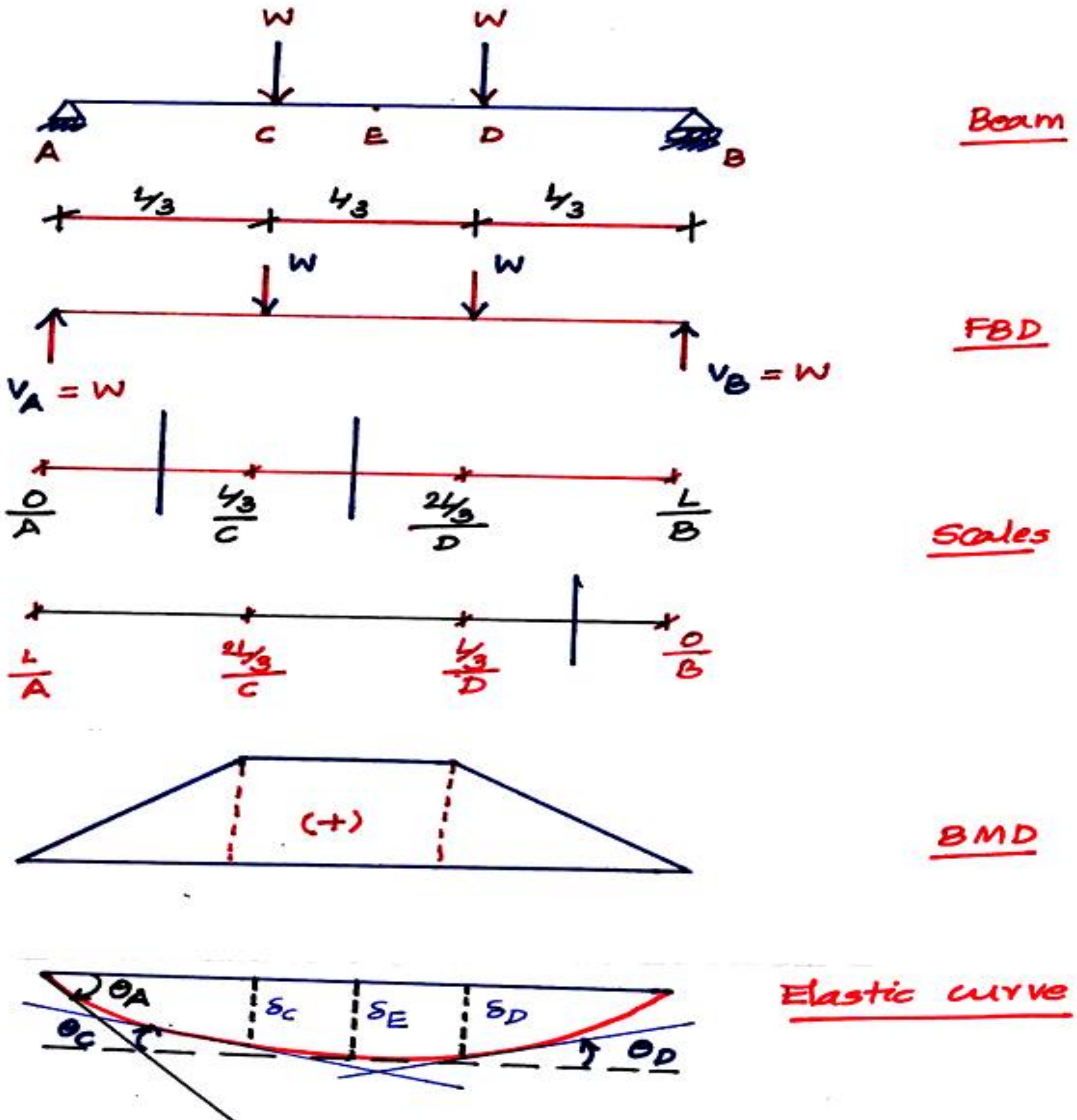


Fig. 4

Problem 1 : Compute deflections and slopes at C,D and E. Also compute slopes at A and B.



To Compute Reactions:

$$\overset{\rightarrow +}{\sum} fx = 0 \Rightarrow H_A = 0$$

$$\uparrow \sum fy = 0 \Rightarrow V_A + V_B - W - W = 0$$

$$V_A + V_B = 2W$$

$$+ \circlearrowright \sum M_B = 0 \Rightarrow LV_A - \frac{WL}{3} - W\frac{(2L)}{3} = 0$$

$$LV_A = \frac{WL}{3} + \frac{2WL}{3} = WL$$

$V_A = W$; $V_B = W$

Bending Moment Calculations:

Section (1) – (1) (LHP, 0 to L/3)

$+ \circlearrowright$ $M_{x-x} = Wx$

At x = 0; BM at A = 0

$x = L/3$; BM @ C = $WL/3$

Section (2) – (2) (LHP, $L/3$ to $2L/3$)

$+ \circlearrowright$ $M_{x-x} = Wx - W(x - L/3)$

At $x = L/3$, BM @ C = $W L/3 - W L/3 + W L/3$

$= W L/3$

$$\text{At } x = \frac{2L}{3}, \text{ BM @ D} = W\left(\frac{2L}{3}\right) - W\left(\frac{2L}{3} - \frac{L}{3}\right)$$

$$= \left(\frac{2WL}{3} - \frac{2WL}{3} + \frac{WL}{3}\right)$$

$$= \frac{WL}{3}$$

Section (3) – (3) RHP (0 to $L/3$)

$\circlearrowleft +$ $M_{x-x} = Wx$

At x = 0; BM @ B = 0

$$\text{At } x = L/3, \text{ BM @ D} = \frac{WL}{3}$$

This beam is symmetrical. Hence the BMD & elastic curve is also symmetrical. In such a case, maximum deflection occurs at mid span, marked as δ_E. Thus, the tangent drawn at E will be parallel to the beam line and θ_E is zero.

Also, $\delta_c = \delta_D$, $\theta_A = \theta_B$ and $\theta_C = \theta_D$

To compute θ_C

From first theorem,

$$\theta_{CE} = \frac{\text{Area of BMD between E\&}}{EI}$$

$$\theta_{C\sim}\theta_E = \frac{W\,{}^{L}\!/\!_{3}\left({}^{L}\!/\!_{6}\right)}{EI}$$

$$= \frac{WL^2}{18EI}$$

θ_E being zero, $\theta_C = \dfrac{WL^2}{18EI}$ ( )

To compute θ_A

From First theorem,

$$\theta_{AE} = \frac{\text{Area of BMD between A\&E}}{EI}$$

$$\theta_{A\sim}\theta_E = \frac{\frac{1}{2}\left(\frac{L}{3}\right)\frac{WL}{3} + \frac{WL}{3}\left(\frac{L}{6}\right)}{EI}$$

$$= \frac{\frac{WL^2}{18} + \frac{WL^2}{18}}{EI}$$

θ_E being zero, $\theta_A = \dfrac{WL^2}{9EI}$ ( )

$$\theta_B = \frac{WL^2}{9EI} \;(\;)$$

To compute δ_E

From 2nd theorem

$$K_{EA} = \frac{\left(\text{Area of BM } \overline{X}\right)_{EA}}{EI}$$

$$K_{EA} = \frac{\left(\frac{1}{2}\frac{L}{3}\frac{WL}{3}\right)\left(\frac{2}{3}\frac{L}{3}\right) + \left(\frac{WL}{3}\frac{L}{6}\right)\left(\frac{L}{3} + \frac{L}{12}\right)}{EI}$$

$$= \frac{\frac{WL^3}{81} + \frac{5WL^3}{216}}{EI}$$

$$= \frac{1}{EI}\left\{\frac{8WL^3 + 15WL^3}{648}\right\}$$

$$= \frac{23WL^3}{648EI}$$

From figure, K_{EA} is equal to δ_E.

$$\text{Therefore } \delta_E = \frac{23WL^3}{648EI} \quad (\downarrow)$$

To compute θ_C

From 2nd theorem

$$K_{EC} = \frac{\left(\text{Area of BMD } \overline{X}\right)_{CE}}{EI}$$

$$= \frac{\left(W\,L/3\right)\left(L/6\right)\left(L/12\right)}{EI}$$

$$= \frac{WL^3}{EI}\left(\frac{1}{216}\right)$$

$$= \frac{WL^3}{216EI}$$

$\delta_c = \delta_E - K_{EC}$

$$\therefore \delta_c = \frac{23WL^3}{648EI} - \frac{WL^3}{216EI}$$

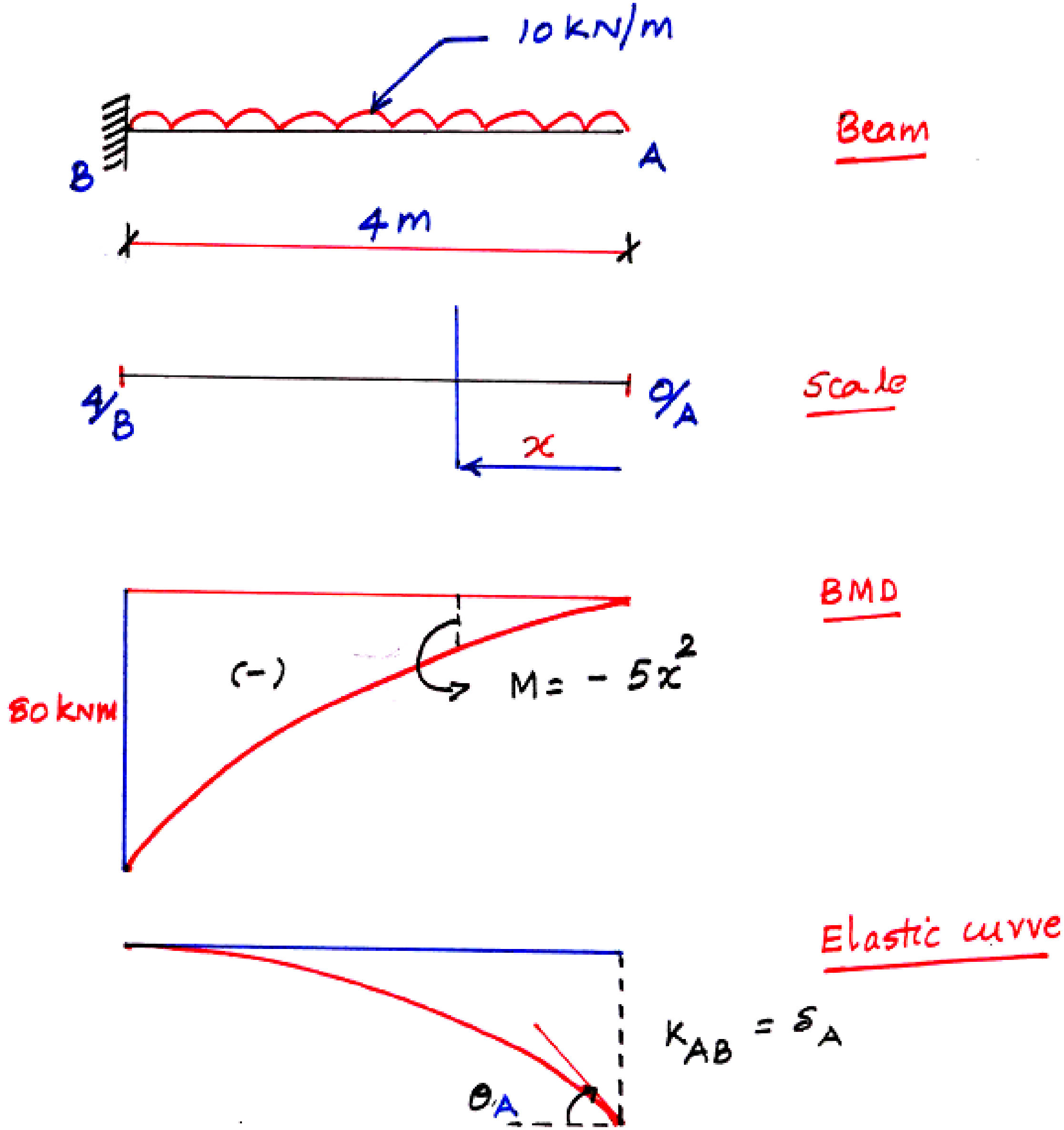
$$= \frac{23WL^3 - 3WL^3}{648EI}$$

$$= \frac{20WL^3}{648EI}$$

$$= \frac{5WL^3}{162EI} \quad (\downarrow)$$

$$= \delta_D = \delta_C = \frac{5WL^3}{162EI} \quad (\downarrow)$$

Problem 2. For the cantilever beam shows in figure, compute deflection and slope at the free end.



Consider a section x-x at a distance x from the free end. The FBD of RHP is taken into account.

$$(\text{RHP} \curvearrowleft +) \quad \text{BM @ X-X} = M_{X\text{-}X} = -10\,(x)\,(x/2) = -5x^2$$

At x = 0; BM @ B = 0

At x = 4m; BM @ A = -5(16) = -80 kNm

The BMD is sketched as shown in figure. Note that it is Hogging Bending Moment. The elastic curve is sketched as shown in figure.

To compute θ_B

For the cantilever beam, at the fixed support, there will be no rotation and hence in this case $\theta_A = 0$. This implies that the tangent drawn to the elastic curve at A will be the same as the beam line.

From I theorem,

$$\theta_{AB} = \theta_A \sim \theta_B = \int_0^4 \frac{M\,dx}{EI}$$

$$= \frac{1}{EI}\int_0^4 \left(-5X^2\right)dx$$

$$= \frac{-5}{EI}\left[x^3/3\right]_0^4$$

$$= \frac{-5}{3EI}(64) = \frac{-320}{3EI}$$

θ_A being zero,

$$\theta_B = \frac{320}{3EI}\ (\curvearrowright)$$

To compute δ_B

From II theorem

$$K_{AB} = \int_0^4 \frac{M\,x\,dx}{EI}$$

$$= \frac{1}{EI}\int_0^4 \left(-5X^2\right)x\,dx$$

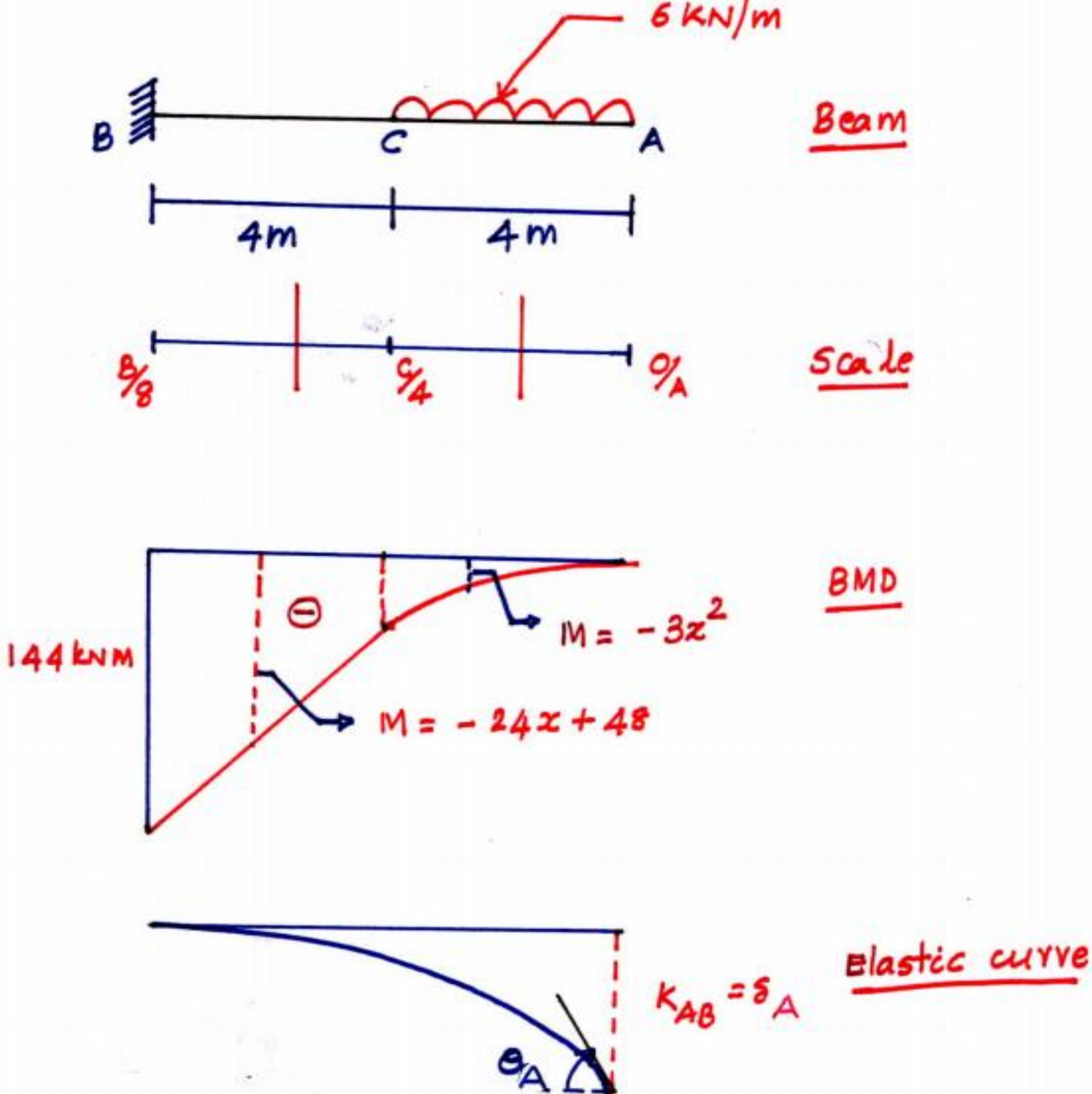
$$= \frac{-5}{EI}\left[x^4/4\right]_0^4 = \frac{-5}{4EI}(256)$$

$$= \frac{-320}{EI}$$

From the elastic curve,

$$K_{AB} = \delta_B = \frac{320}{EI}(\downarrow)$$

Problem 3: Find deflection and slope at the free end for the beam shown in figure by using moment area theorems. Take $EI = 40000\ KNm^{-2}$



Calculations of Bending Moment:

Region AC: Taking RHP $\curvearrowleft$ +

Moment at section = $-6x^2/2$

At x = 0, BM @ A = 0

x = 4m; BM @ C = -3(16) = - 48kNm

Region CB: (x = 4 to x = 8)

Taking RHP $\curvearrowleft$+, moment @ section = -24 (x-2)

= -24x+48;

At x = 4m; BM @ C = -24(4) + 48 = -48kNm;

x = 8 m BM @ B = -144 kNm;

To compute θ_B:

First moment area theorem is used. For the elastic curve shown in figure. We know that $\theta_A = 0$.

$$\theta_{AB} = \theta_A \sim \theta_B = \int \frac{M\,dx}{EI}$$

$$= \frac{1}{EI}\int_0^4 -3x^2 dx + \frac{1}{EI}\int_4^8 (-24x + 48)dx$$

$$\theta_A = \frac{-3}{EI}\left[x^3/3\right]_0^4 + \frac{1}{EI}\left[-24\,x^2/2 + 48x\right]_4^8$$

$$= \frac{-64}{EI} + \frac{1}{EI}\left[-12(64-16) + 48(8-4)\right]$$

= -0.0112 Radians

= 0.0112 Radians ($\curvearrowright$)

To compute δ_B

$$K_{AB} = \int \frac{Mxdx}{EI}$$

$$= \frac{1}{EI}\int_0^4 -3x^2 xdx + \frac{1}{EI}\int_4^8 (-24x + 48)xdx$$

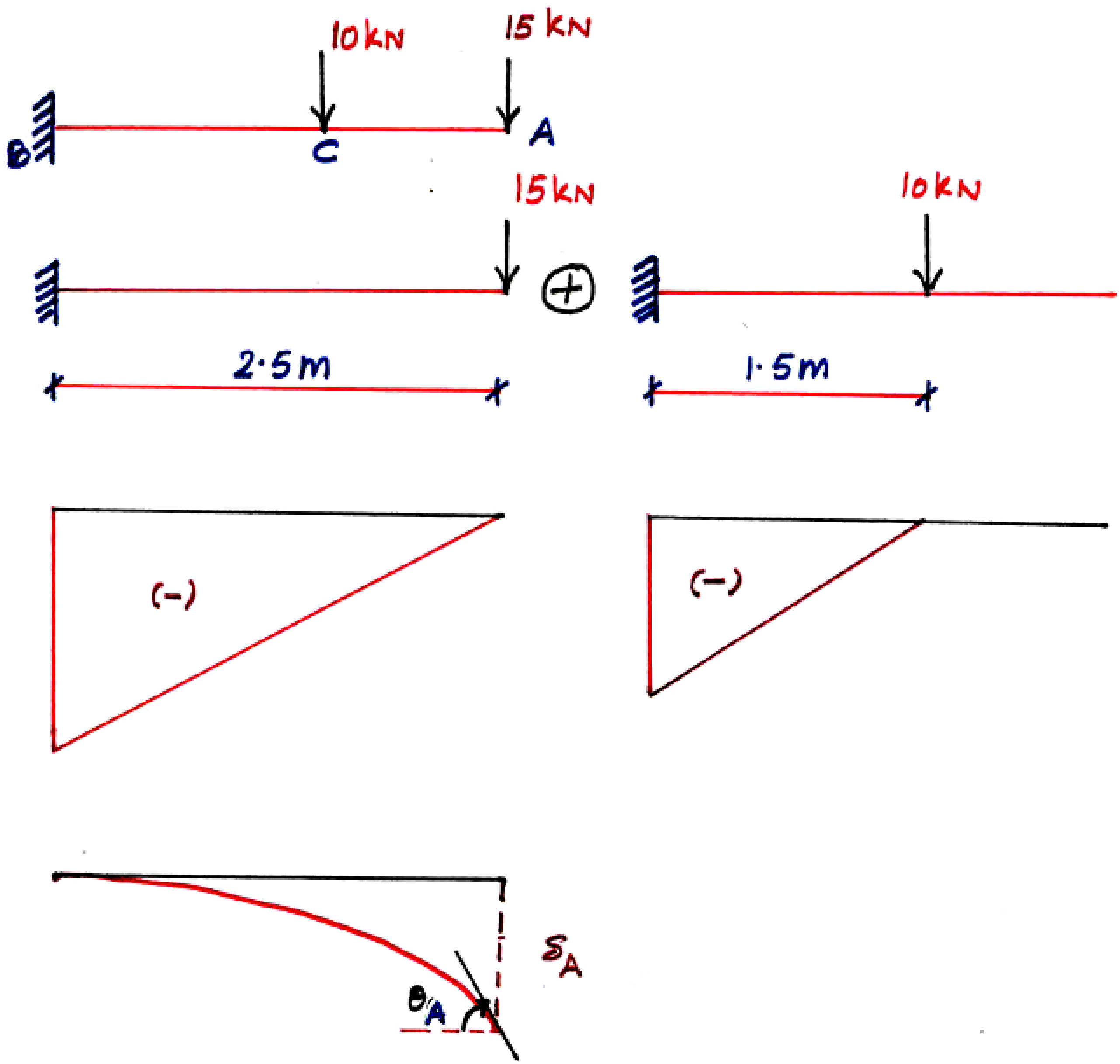
$$= \frac{-3}{EI}\left[x^4/4\right]_0^4 + \frac{1}{EI}\left[-24\left(x^3/3\right)_4^8 + 48\left(x^2/2\right)_4^8\right]$$

$$= \frac{-3}{4EI}[256] + \frac{1}{EI}\left[\frac{-24}{3}(512-64) + 24(64-16)\right]$$

$$= \frac{-192}{EI} + \frac{1}{EI}[-3584 + 1152]$$

$$= \frac{-2624}{EI} = -0.0656m = 0.0656m \downarrow$$

Problem 4: For the cantilever shown in figure, compute deflection and at the points where they are loaded.



To compute θ_B :

$$\theta_{BA} = \theta_B \sim \theta_A = \frac{1}{EI}\left[-\tfrac{1}{2}(2.5)(37.5) - \tfrac{1}{2}(1.5)(15)\right]$$

$$\theta_B = \frac{58.125}{EI}\ (\circlearrowright)$$

$$\theta_C = \frac{1}{EI}\left[-\tfrac{1}{2}(1.5)(37.5+15) - \tfrac{1}{2}(1.5)(15)\right]$$

$$= \frac{50.625}{EI}\ (\circlearrowright)$$

$$\delta_B = -\frac{\tfrac{1}{2}(2.5)(37.5)(\tfrac{2}{3})(2.5)}{EI} - \frac{1}{EI}(\tfrac{1}{2})(1.5)45(1)$$

$$= -\frac{100.625}{EI}$$

$$= -\frac{100.625}{EI}(\downarrow)$$

$$\delta_C = \frac{1}{EI}\int \tfrac{1}{2}(1.5)(37.5+15)0.857 + \tfrac{1}{2}(1.5)(45)(1)$$

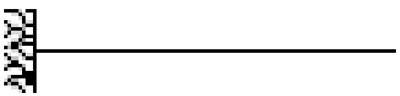
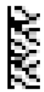
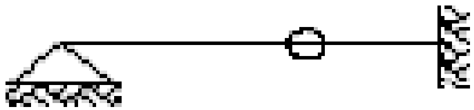
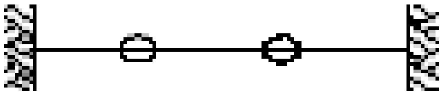
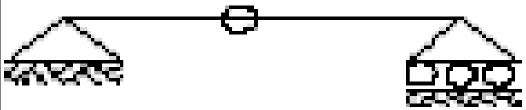
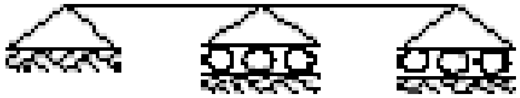
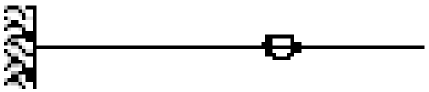
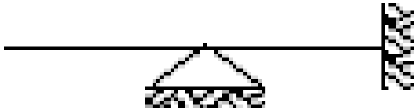
$$\delta_C = \frac{44.99}{EI}(\downarrow)$$

CONJUGATE BEAM METHOD

This is another elegant method for computing deflections and slopes in beams. The principle of the method lies in calculating BM and SF in an imaginary beam called as Conjugate Beam which is loaded with M/EI diagram obtained for real beam. Conjugate Beam is nothing but an imaginary beam which is of the same span as the real beam carrying M/EI diagram of real beam as the load. The SF and BM at any section in the conjugate beam will represent the rotation and deflection at that section in the real beam. Following are the concepts to be used while preparing the Conjugate beam.

- It is of the same span as the real beam.
- The support conditions of Conjugate beam are decided as follows:

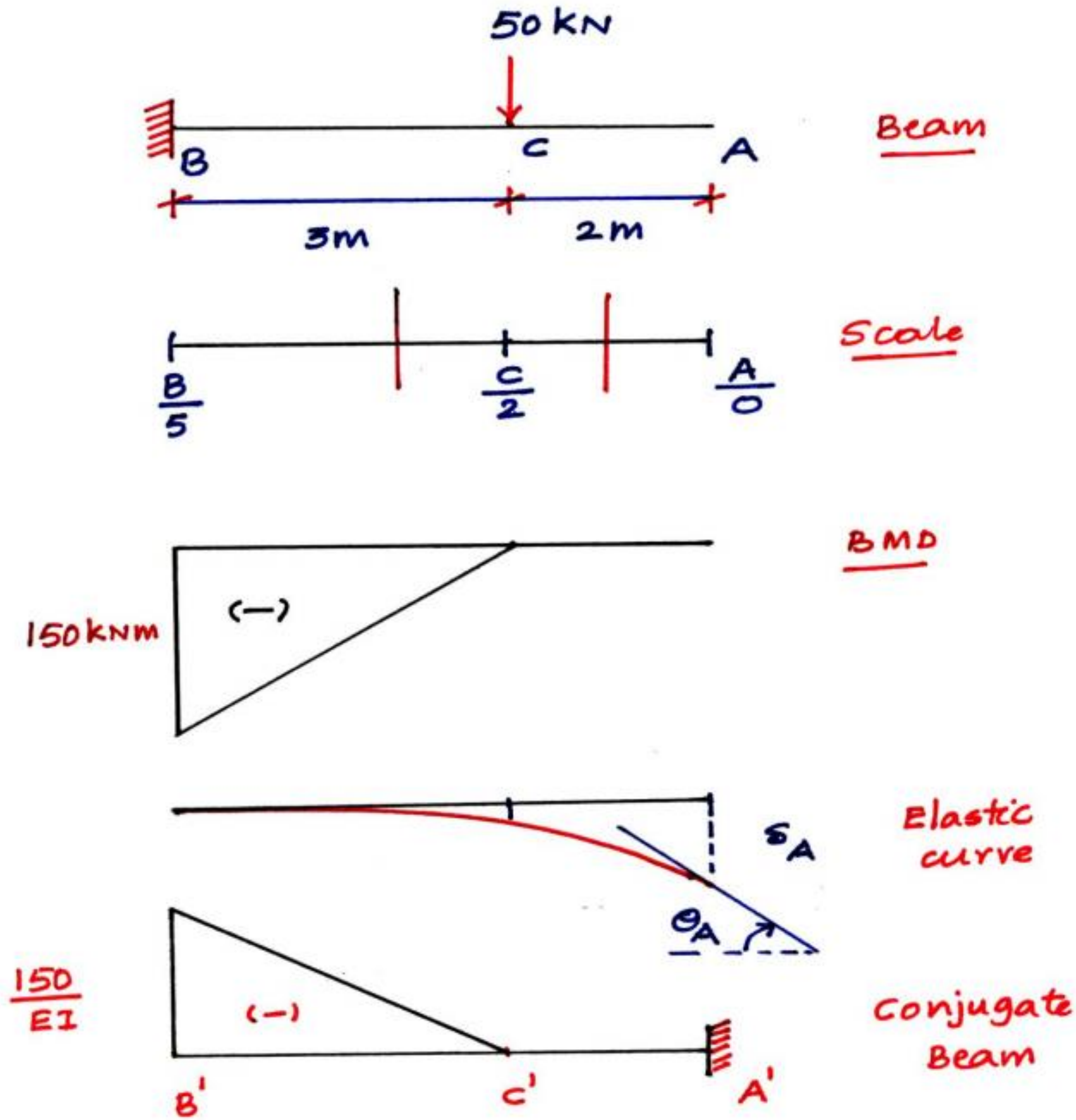
SL NO	SUUPORT IN REAL BEAM	SUUPORT IN CONJUGATE BEAM
1	ROLLER	ROLLER
2	HINGE	HINGE
3	FIXED	FREE
4	FREE	FIXED
5	INTERIOR SUPPORT	INTERNAL HINGE
6	INTERNAL HINGE	INTERIOR SUPPORT

SL NO	REAL BEAM	CONJUGATE BEAM
1		
2		
3		
4		
5		
6		
7		

Problem 1 : For the Cantilever beam shown in figure, compute deflection and rotation at

(i) the free end

(ii) under the load



Conjugate Beam:

By taking a section @ C′ and considering FBD of LHP,

$$\overset{\uparrow}{\underset{+}{}} SF = \sum f_x = \frac{-150}{EI}(3)(½) = \frac{-225}{EI}$$

BM @ C′= $\frac{-150}{EI}(3)(½)(2) = \frac{-450}{EI}$;

Similarly by taking a section at A' and considering FBD of LHP;

SF @ A' = $\frac{-225}{EI}$

BM @ A' = $\frac{-225}{EI}(2+2) = \frac{-900}{EI}$

SF @ a section in Conjugate Beam gives rotation at the same section in Real Beam

BM @ a section in Conjugate Beam gives deflection at the same section in Real Beam

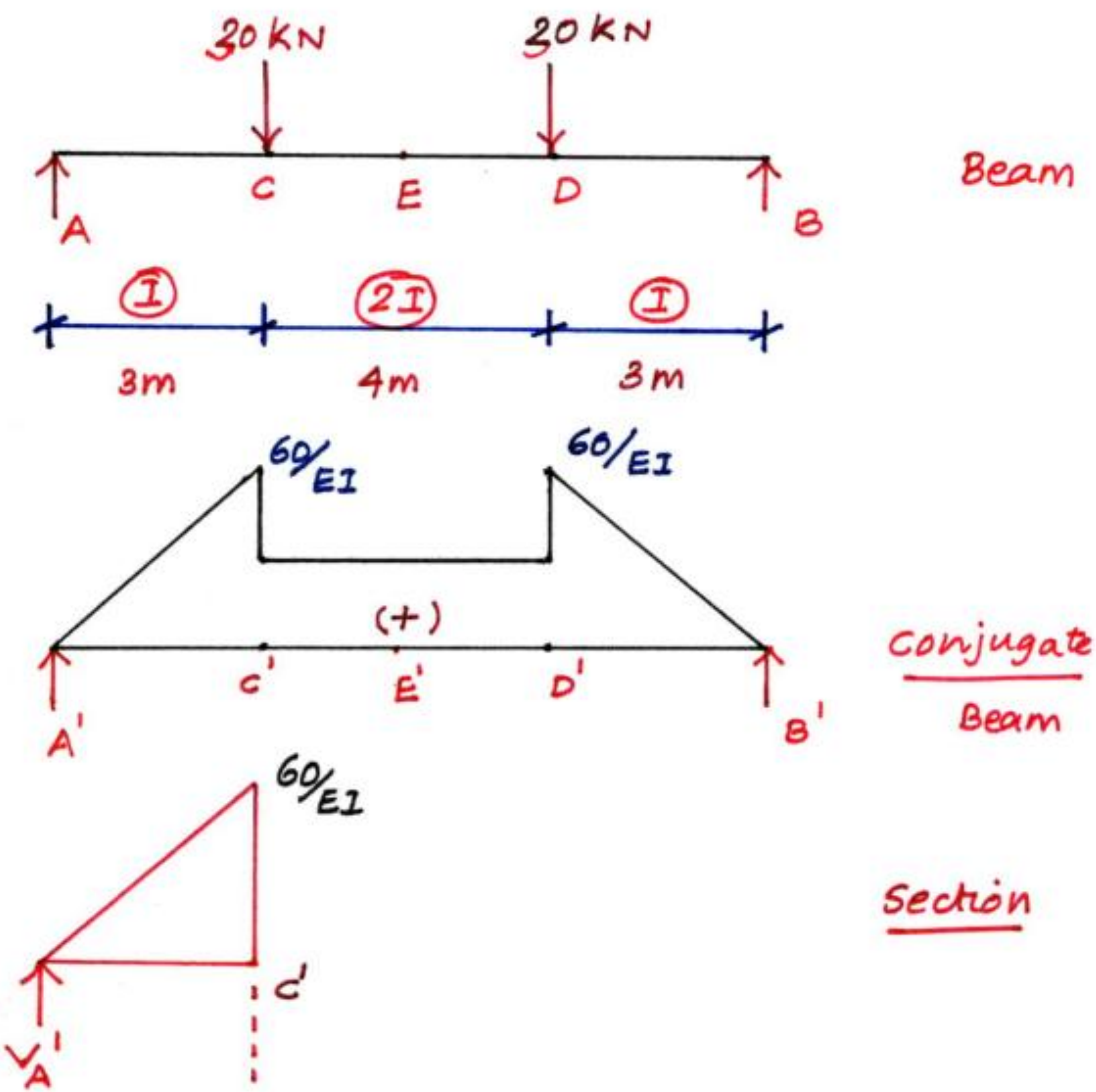
Therefore, Rotation @ C = $\frac{225}{EI}$ (↻)

Deflection @ C= $\frac{450}{EI}(\downarrow)$

Rotation @ A = $\frac{225}{EI}$ (↻)

Deflection @ A = $\frac{900}{EI}(\downarrow)$

Problem 2: For the beam shown in figure, compute deflections under the loaded points. Also compute the maximum deflection. Compute, also the slopes at supports.



For the conjugate beam:

$$V_A' = V_B' = \frac{1}{2}[\text{Total load on Conjugate Beam}]$$

$$= \frac{1}{2}\left[2\left(\frac{1}{2}\right)\left(\frac{60}{EI}\right)(3) + 4\left(\frac{30}{EI}\right)\right]$$

$$= \frac{1}{2}\left[\frac{180}{EI} + \frac{120}{EI}\right] = \frac{150}{EI}$$

To compute δ_C :

A section at C' is placed on conjugate beam. Then considering FBD of LHP;

$$\circlearrowleft + \text{ BM @ C'} = \frac{150}{EI}(3) - \frac{1}{2}(3)\left(\frac{60}{EI}\right)(1)$$

$$= \frac{450}{EI} - \frac{90}{EI} = \frac{360}{EI}$$

$$\therefore \delta_C = \frac{360}{EI}(\downarrow);$$

To compute δ_E:

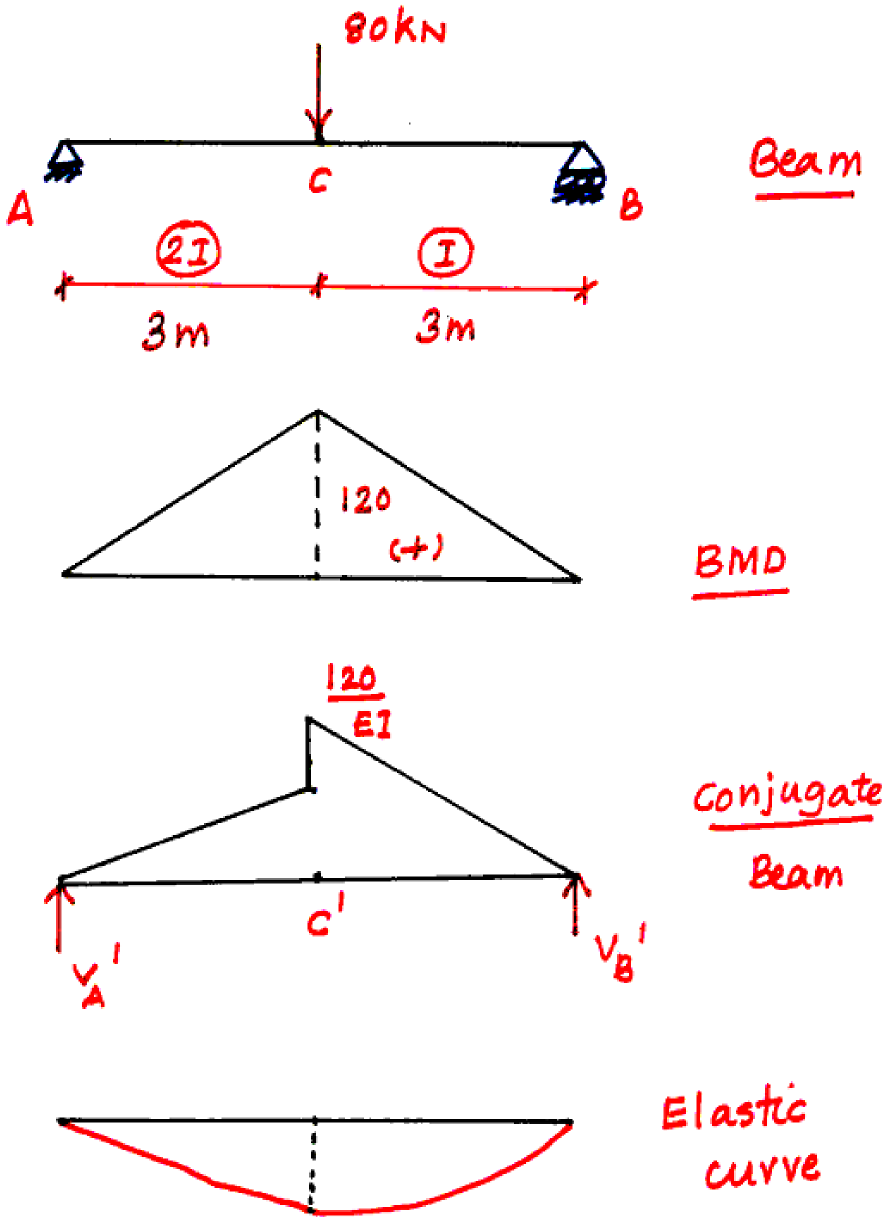
A section @ E' is placed on conjugate beam. Then considering FBD of LHP;

$$+\circlearrowright \text{ BM @ E'} = \frac{150}{EI}(5) - \frac{1}{2}(3)\left(\frac{60}{EI}\right)(3) - \frac{30}{EI}(2)(1)$$

$$\text{i.e } \delta_E = \frac{750}{EI} - \frac{270}{EI} - \frac{60}{EI} = \frac{420}{EI}(\downarrow)$$

$$\theta_A = \frac{150}{EI}\ (\circlearrowright) \quad \theta_B = \frac{150}{EI}\ (\circlearrowleft)$$

Problem 3: Compute deflection and slope at the loaded point for the beam shown in figure. Given E = 210 Gpa and I = 120 x 10^6mm^4. Also calculate slopes at A and B.



To Compute reactions in Conjugate Beam:

$$\sum fy = 0 \Rightarrow V_A' + V_B' - \left(\frac{1}{2}\right)\left(\frac{60}{EI}\right)(3) - \frac{1}{2}\left(\frac{120}{EI}\right)(3) = 0$$

$$V_A' + V_B' - \frac{90}{EI} - \frac{180}{EI} = 0;$$

$$V_A' + V_B' = \frac{270}{EI}$$

$$\sum m_{B'} = 0 + ↻ \; V_A'(6) - \left(\frac{1}{2}\right)\left(\frac{60}{EI}\right)(3)(4) - \frac{1}{2}\left(\frac{120}{EI}\right)(3)(2$$

$$6V_A' = \frac{360}{EI} + \frac{360}{EI} = \frac{720}{EI}$$

$$V_A' = \frac{120}{EI} \quad ; \quad V_B' = \frac{150}{EI}$$

SF and BM at C' is obtained by placing a section at C' in the conjugate beam.

$$SF @ C' = \frac{120}{EI} - \frac{1}{2}\left(\frac{60}{EI}\right)(3)$$

$$= \frac{30}{EI}$$

$$+ ↻ \; BM @ C' = \frac{120}{EI}(3) - \frac{1}{2}\left(\frac{60}{EI}\right)(3)(1)$$

$$= \frac{360}{EI} - \frac{90}{EI} = \frac{270}{EI}$$

Given $E = 210 \times 10^9 \ N/m^2$
$= 210 \times 10^6 \ kN/m^2$
$I = 120 \times 10^6 \ mm^4$
$= 120 \times 10^6 \ (10^{-3} \ m)^4$
$= 120 \times 10^6 \ (10^{-12})$
$= 120 \times 10^{-6} \ m^4;$
$EI = 210 \times 10^6 \ (120 \times 10^{-6}) = 25200 \ kNm^{-2}$

$$\text{Rotation @ C} = \frac{30}{25200} = 1.19 \times 10^{-3} \text{ Radians } (↻)$$

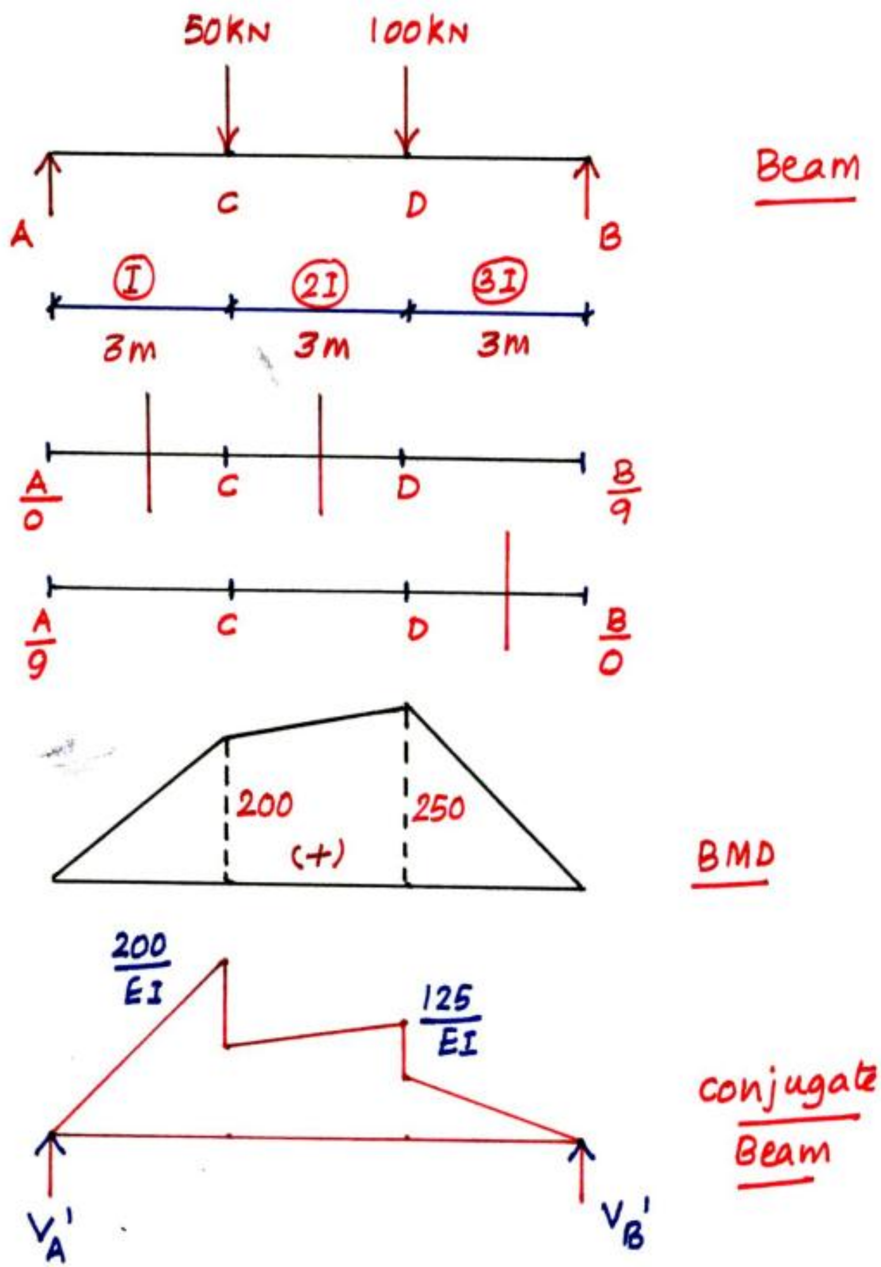
$$\text{Deflection @ C} = \frac{270}{25200} = 0.0107 \ m$$

$$= 10.71 \ mm \ (\downarrow)$$

$\theta_A = 4.76 \ X \ 10^{-3}$ Radians

$\theta_B = 5.95 \ X \ 10^{-3}$ Radians:

Problem 4: Compute slopes at supports and deflections under loaded points for the beam shown in figure.



To compute reactions and BM in real beam:

$$\uparrow_{+} \sum fy = 0 \Rightarrow V_A + V_B = 150$$

$$+\curvearrowright \sum M_B = 0 \quad 9V_A - 50(6) - 100(3) = 0$$

$$V_A = \frac{600}{9} = 66.67\text{kN} \quad V_B = 83.33\text{kN}$$

BM at (1) – (1) = 66.67 x
At x = 0; BM at A = 0, At x = 3m, BM at C = 200 kNm

BM at (2) – (2) = 66.67 x – 50 (x-3) = 16.67 x + 150
At x = 3m; BM at C = 200 kNm, At x = 6m, BM at D = 250 kNm

BM at (3) – (3) is computed by taking FBD of RHP. Then
BM at (3)-(3) = 83.33 x (x is measured from B)
At x = 0, BM at B = 0, At x = 3m, BM at D = 250 kNm

To compute reactions in conjugate beam:

$$\uparrow_{+} \sum fy = 0 \Rightarrow V_A^{'} + V_B^{'} = \frac{1}{2}(3)\left(\frac{200}{EI}\right) + 3\left(\frac{100}{EI}\right)$$

$$+\frac{1}{2}(3)\left(\frac{25}{EI}\right) + \frac{1}{2}(3)\left(\frac{83.33}{EI}\right)$$

$$= \frac{762.5}{EI}$$

$$+\curvearrowright \sum M_B^{'} = 0$$

$$\text{i.e } 9V_A^{'} - \left(\frac{1}{2}\right)(3)\left(\frac{200}{EI}\right)(7) - 3\left(\frac{100}{EI}\right)(4.5) - \left(\frac{1}{2}\right)(3)\left(\frac{25}{EI}\right)(4) - \left(\frac{1}{2}\right)(3)\left(\frac{83.33}{EI}\right)(2) = 0$$

$$9V_A^{'} = \frac{3850}{EI}$$

$$V_A^{'} = \frac{427.77}{EI}$$

$$V_B^{'} = \frac{334.73}{EI}$$

$$\therefore \ \theta_A = \frac{427.77}{EI}\ (\curvearrowright) \qquad \theta_B = \frac{334.73}{EI}\ (\curvearrowleft)$$

To Compute δ_C :

A Section at C' is chosen in the conjugate beam:

$$+\curvearrowright \ \text{BM at C}' = \frac{427.77}{EI}(3) - \left(\frac{1}{2}\right)(3)\left(\frac{200}{EI}\right)(1)$$

$$= \frac{983.31}{EI}$$

$$\therefore \ \delta_C = \frac{983.31}{EI}(\downarrow)$$

To compute δ_D:

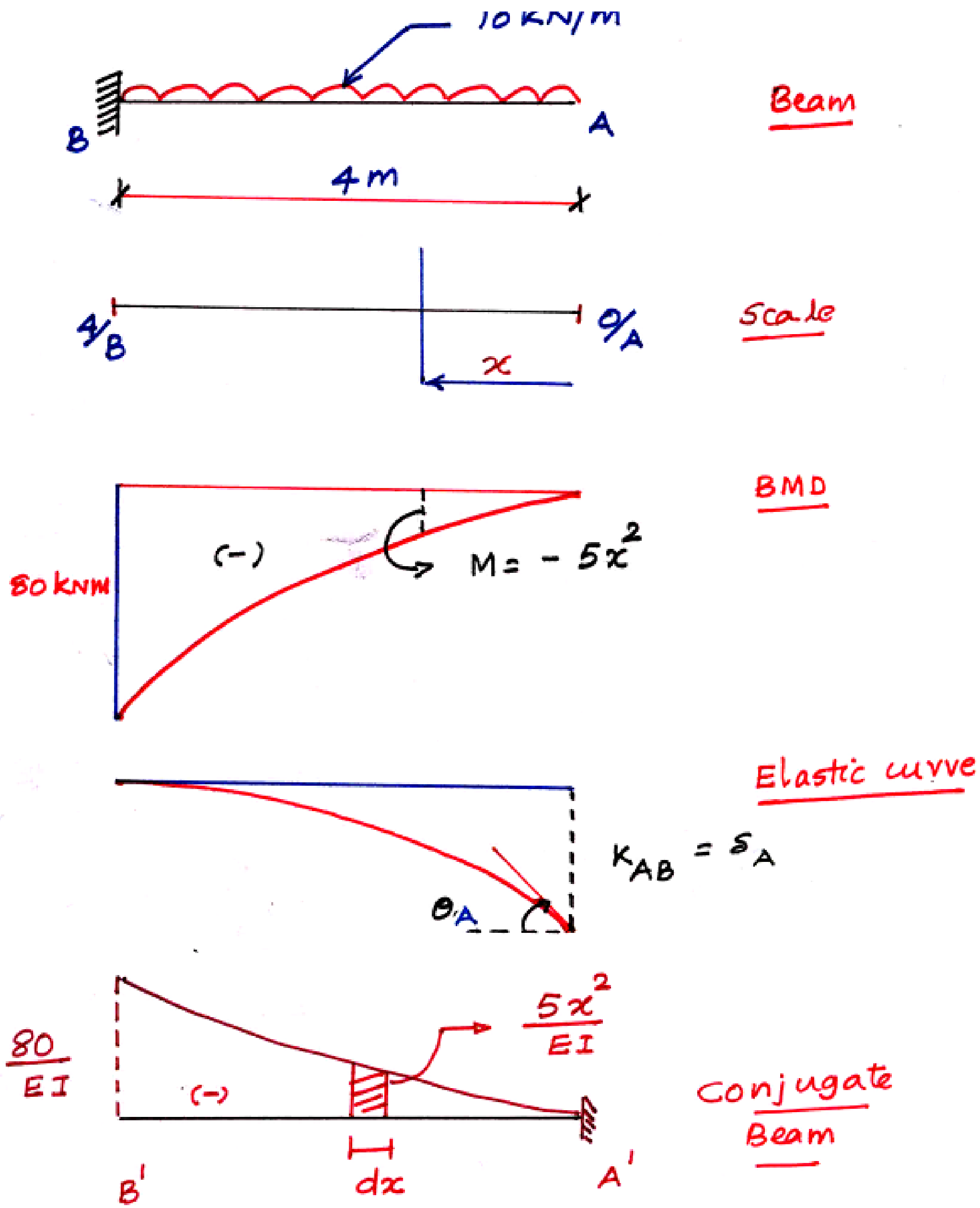
Section at D' is chosen and FBD of RHP is considered.

$$\curvearrowleft + \text{BM at D'} = \frac{334.73}{EI}(3) - \frac{1}{2}(3)\left(\frac{83.33}{EI}\right)(1)$$

$$= \frac{879.19}{EI}$$

$$\delta_D = \frac{879.19}{EI}(\downarrow)$$

Problem 5: Compute to the slope and deflection at the free end for the beam shown in figure.



The Bending moment for the real beam is as shown in the figure. The conjugate beam also is as shown.

Section at A' in the conjugate beam gives

$$\text{SF @ A'} = \int_0^4 \frac{-5x^2}{EI} dx$$

$$= \frac{-5}{EI}\left(x^3/3\right)_0^4 = \frac{-5}{3EI}(64)$$

$$= \frac{-320}{3EI}$$

$$\therefore \quad \theta_A = \frac{320}{3EI} \; (\circlearrowright)$$

$$\text{BM @ A'} = \frac{1}{EI}\int_0^4 -5x^2(x)dx$$

$$= \frac{-5}{EI}\left[\frac{x^4}{4}\right]_0^4 = \frac{-5}{4EI}[256]$$

$$\therefore \delta_A = \frac{320}{EI}(\downarrow)$$

Macaulay's Methods

If the loading conditions change along the span of beam, there is corresponding change in moment equation. This requires that a separate moment equation be written between each change of load point and that two integration be made for each such moment equation. Evaluation of the constants introduced by each integration can become very involved. Fortunately, these complications can be avoided by writing single moment equation in such a way that it becomes continuous for entire length of the beam in spite of the discontinuity of loading.

Note : In Macaulay's method some author's take the help of unit function approximation (i.e. Laplace transform) in order to illustrate this method, however both are essentially the same.

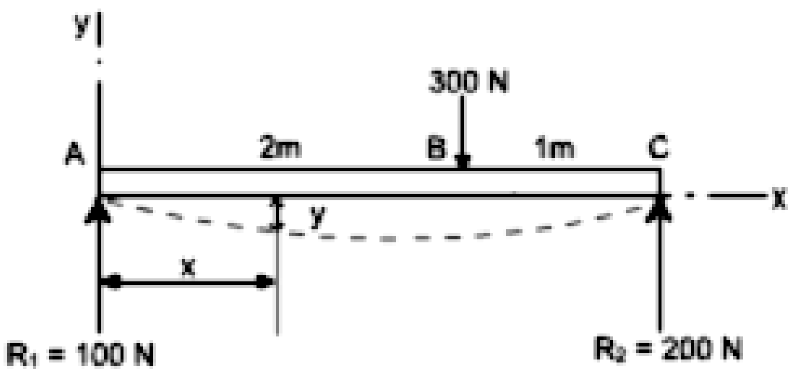
Procedure to solve the problems

(i). After writing down the moment equation which is valid for all values of ‘x' i.e. containing pointed brackets, integrate the moment equation like an ordinary equation.

(ii). While applying the B.C's keep in mind the necessary changes to be made regarding the pointed brackets.

illustrative Examples :

1. A concentrated load of 300 N is applied to the simply supported beam as shown in Fig. Determine the equations of the elastic curve between each change of load point and the maximum deflection in the beam.



Solution : writing the general moment equation for the last portion BC of the loaded beam,

$$EI\frac{d^2y}{dx^2} = M = \left(100x - 300\langle x-2\rangle\right)N.m \qquad ....(1)$$

Integrating twice the above equation to obtain slope and the deflection

$$EI\frac{dy}{dx} = \left(50x^2 - 150\langle x-2\rangle^2 + C_1\right)N.m^2 \qquad ....(2)$$

$$EIy = \left(\frac{50}{3}x^3 - 50\langle x-2\rangle^3 + C_1x + C_2\right)N.m^3 \qquad ....(3)$$

To evaluate the two constants of integration. Let us apply the following boundary conditions:

1. At point A where x = 0, the value of deflection y = 0. Substituting these values in Eq. (3) we find C2 = 0.keep in mind that < x -2 >3 is to be neglected for negative values.
2. At the other support where x = 3m, the value of deflection y is also zero. substituting these values in the deflection Eq. (3), we obtain

$$0 = \left(\frac{50}{3}3^3 - 50(3-2)^3 + 3.C_1\right) \text{or } C_1 = -133N.m^2$$

Having determined the constants of integration, let us make use of Eqs. (2) and (3) to rewrite the slope and deflection equations in the conventional form for the two portions.

segment AB ($0 \le x \le 2m$)

$$EI\frac{dy}{dx} = \left(50x^2 - 133\right)N.m^2 \qquad ....(4)$$

$$EIy = \left(\frac{50}{3}x^3 - 133x\right)N.m^3 \qquad ....(5)$$

segment BC ($2m \le x \le 3m$)

$$EI\frac{dy}{dx} = \left(50x^2 - 150(x-2)^2 - 133x\right)N.m^2 \qquad ....(6)$$

$$EIy = \left(\frac{50}{3}x^3 - 50(x-2)^3 - 133x\right)N.m^3 \qquad ....(7)$$

Continuing the solution, we assume that the maximum deflection will occur in the segment AB. Its location may be found by differentiating Eq. (5) with respect to x and setting the derivative to be equal to zero, or, what amounts to the same thing, setting the slope equation (4) equal to zero and solving for the point of zero slope.

50 x2– 133 = 0 or x = 1.63 m (It may be kept in mind that if the solution of the equation does not yield a value < 2 m then we have to try the other equations which are valid for segment BC)

Since this value of x is valid for segment AB, our assumption that the maximum deflection occurs in this region is correct. Hence, to determine the maximum deflection, we substitute x = 1.63 m in Eq (5), which yields

$$EIy\big|_{max} m = -145N.m^3 \qquad ....(8)$$

The negative value obtained indicates that the deflection y is downward from the x axis. quite usually only the magnitude of the deflection, without regard to sign, is desired; this is denoted by d, the use of y may be reserved to indicate a directed value of deflection.

if $E = 30$ Gpa and $I = 1.9 \times 10^6\ mm^4 = 1.9 \times 10^{-6}\ m^4$, Eq. (h) becomes

$$y|_{max} m = (30 \times 10^9)(1.9 \times 10^{-6})$$

Then $$= -2.54 mm$$