THEORY OF STRUCTURES

II B. Tech III Semester

(Autonomous IARE _R16)

BY

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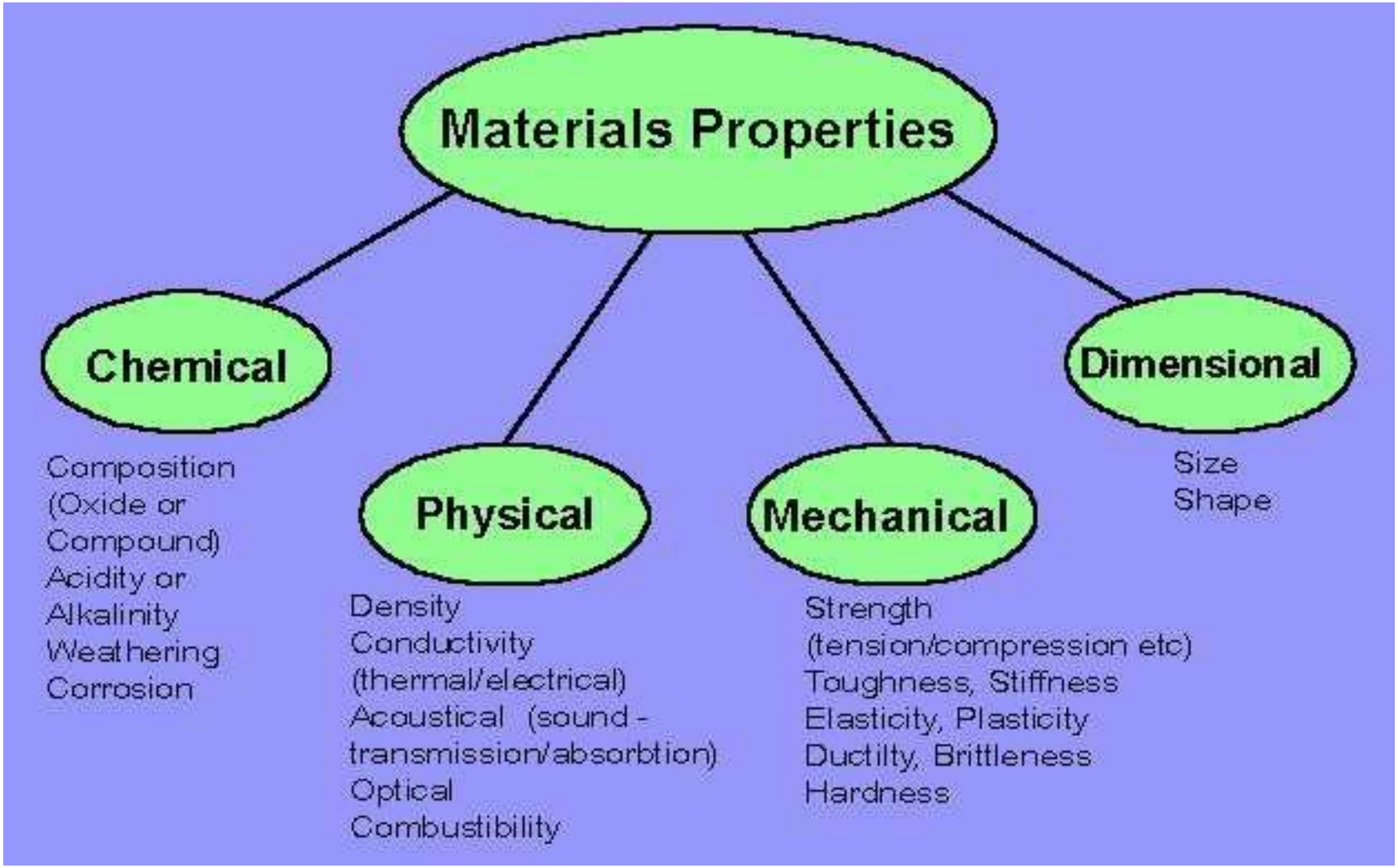
INTRODUCTION

“MECHANICAL PROPERTIES OF MATERIALS”

INTRODUCTION:

- The practical application of engineering materials in manufacturing engineering depends upon a thorough knowledge of their particular properties under a wide range of conditions.
- The term ” **property** ” is a qualitative or quantitative measure of response of materials to externally imposed conditions like forces and temperatures.
- However, the range of properties found in different classes of materials is very large.

Classification of material property:



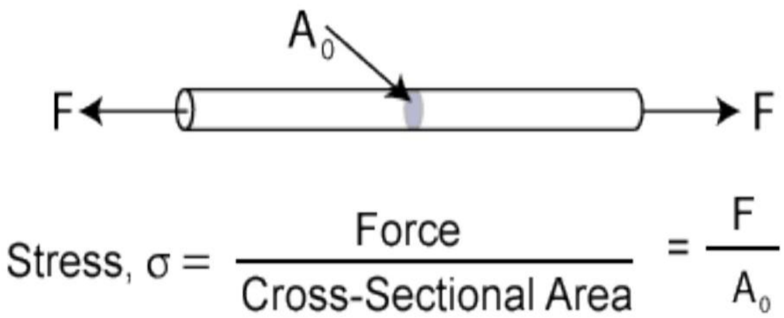
Mechanical properties:

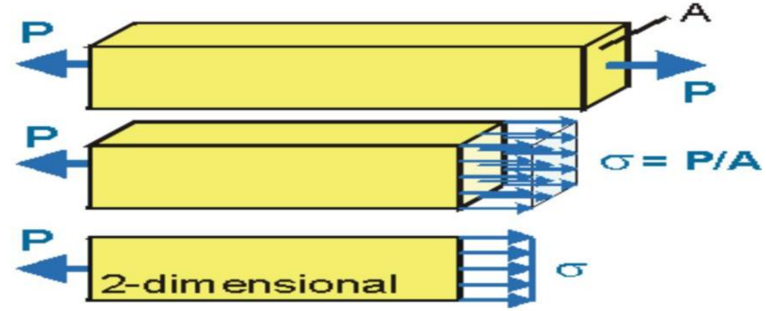
- The properties of material that determine its behavior under applied forces are known as mechanical properties.
- They are usually related to the elastic and plastic behavior of the material.
- These properties are expressed as functions of stress-strain, etc.
- A sound knowledge of mechanical properties of materials provides the basis for predicting behavior of materials under different load conditions and designing the components out of them.

Stress and Strain Introduction

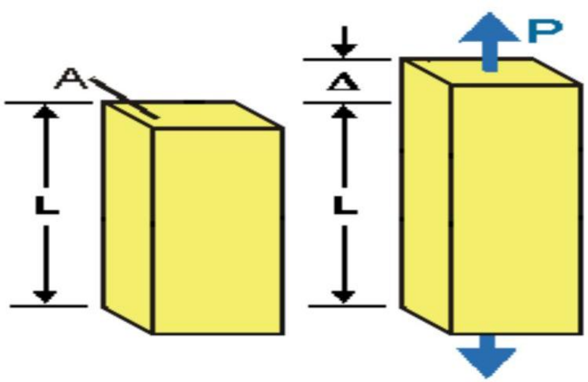
- Experience shows that any material subjected to a load may either deform, yield or break, depending upon the
    1. The Magnitude of load
    2. Nature of the material
    3. Cross sectional dime.
- STRESS : The sum total of all the elementary interatomic forces or internal resistances which the material is called upon to exert to counteract the applied load is called stress.
- Mathematically, the stress is expressed as force divided by cross-sectional area.

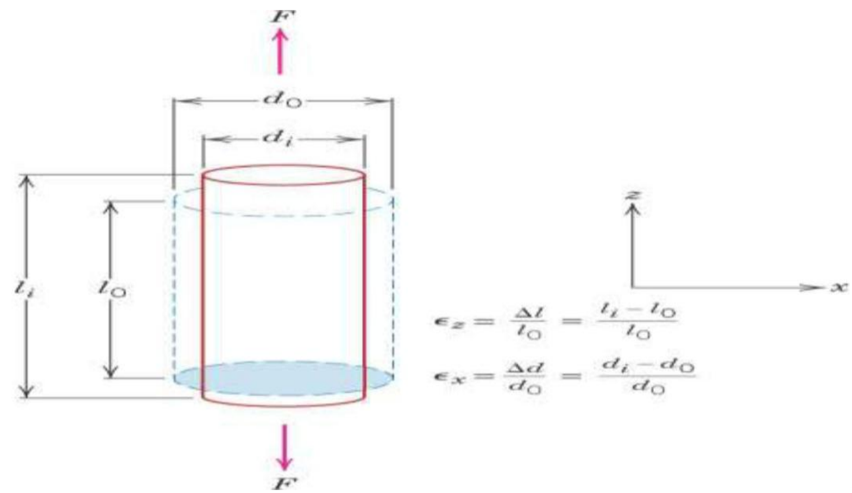
Stress and Strain





- **STRAIN :** Strain is the dimensional response given by material against mechanical loading/Deformation produced per unit length.
- Mathematically Strain is change in length divided by original length.



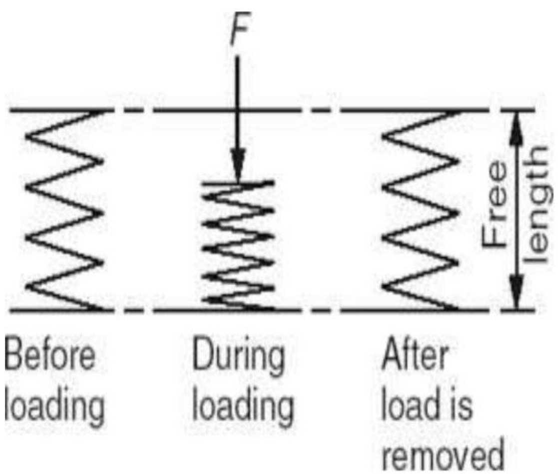


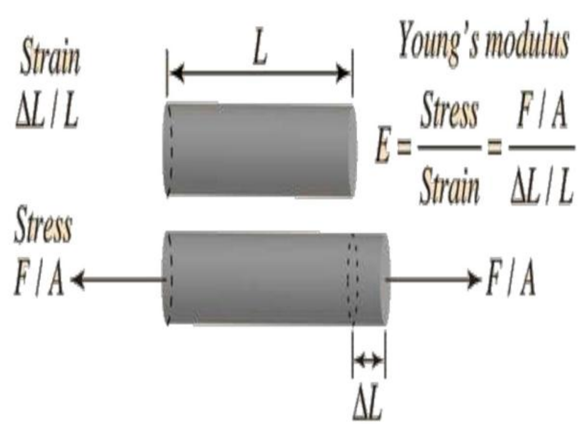
Strength:

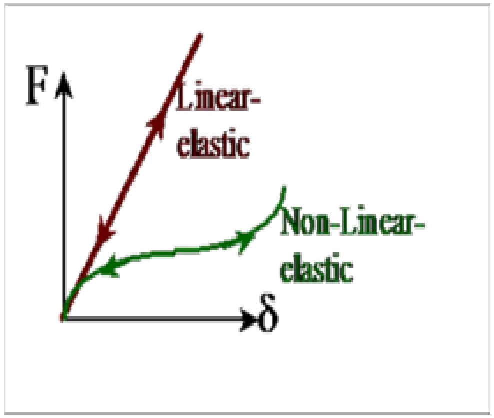
- The strength of a material is its capacity to withstand destruction under the action of external loads.
- It determines the ability of a material to withstand stress without failure.
- The maximum stress that any material will withstand before destruction is called ultimate strength.

Elasticity:

- The property of material by virtue of which deformation caused by applied load disappears upon removal of load.
- Elasticity of a material is the power of coming back to its original position after deformation when the stress or load is removed.(Elastic means reversible).

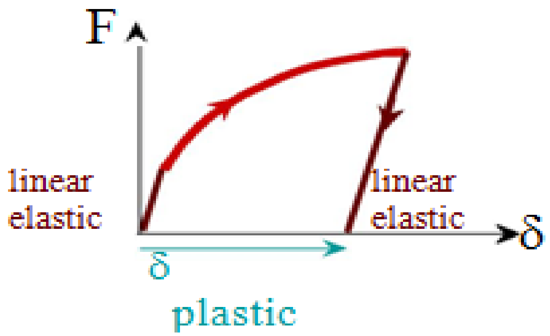




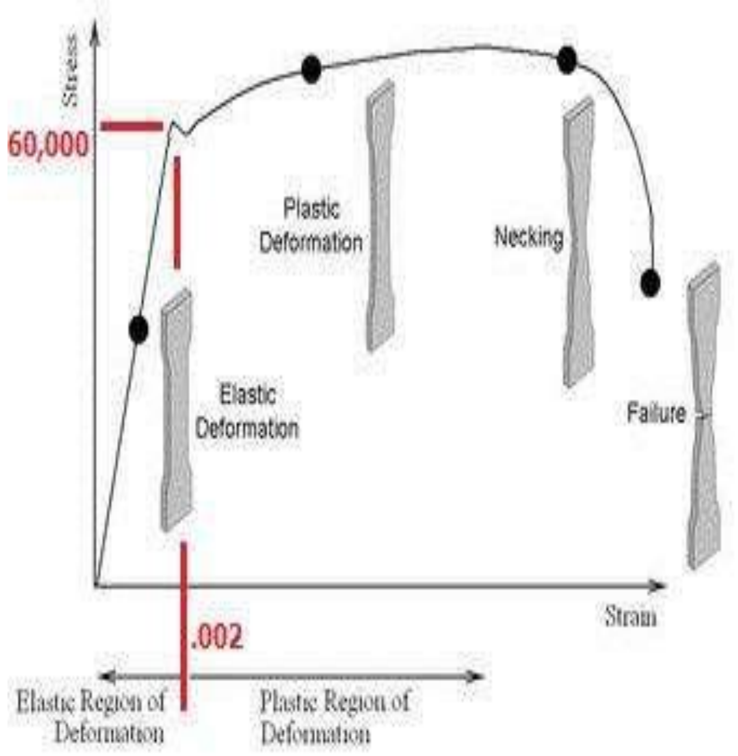


Plasticity:

- The plasticity of a material is its ability to undergo some degree of permanent deformation without rupture or failure. Plastic deformation will take only after the elastic limit is exceeded. It increases with increase in temperature.(Plastic means permanent).



- **STRESS STRAIN CURVE SHOWS ELASTICITY AND PLASTICITY FOR MATERIALS:**



Stiffness & Ductility:

STIFFNESS:

- The resistance of a material to elastic deformation or deflection is called stiffness or rigidity.
- A material which suffers slight deformation under load has a high degree of stiffness or rigidity.
- E.g. Steel beam is more stiffer or more rigid than aluminium beam.

DUCTILITY:

- It is the property of a material which enables it to draw out into thin wires.
- The percent elongation and the reduction in area in tension is often used as empirical measures of ductility.
- E.g., Mild steel is a ductile material.

Ductility:

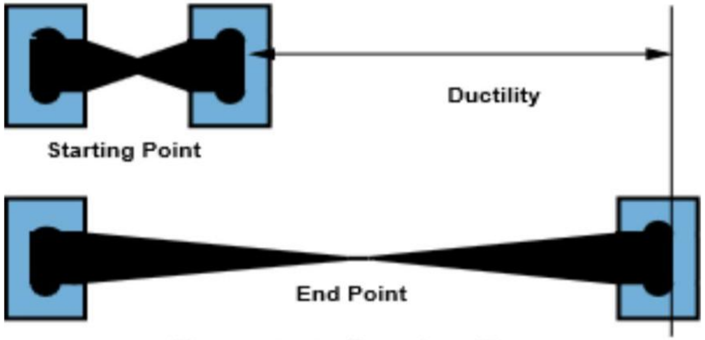
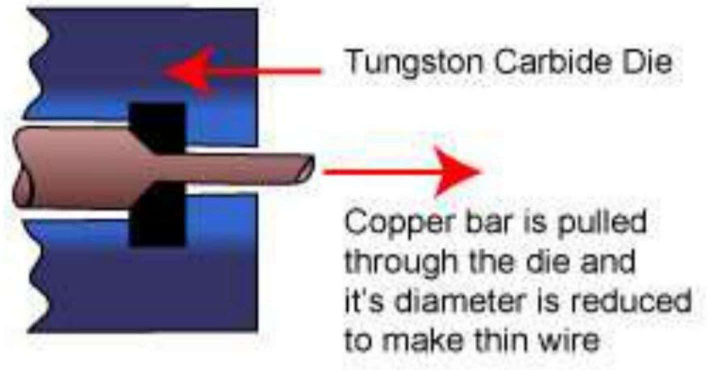
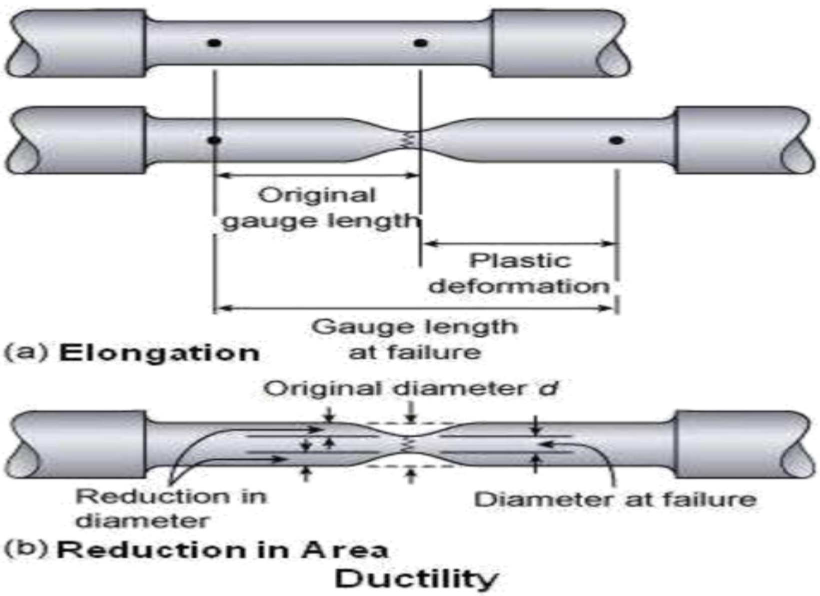
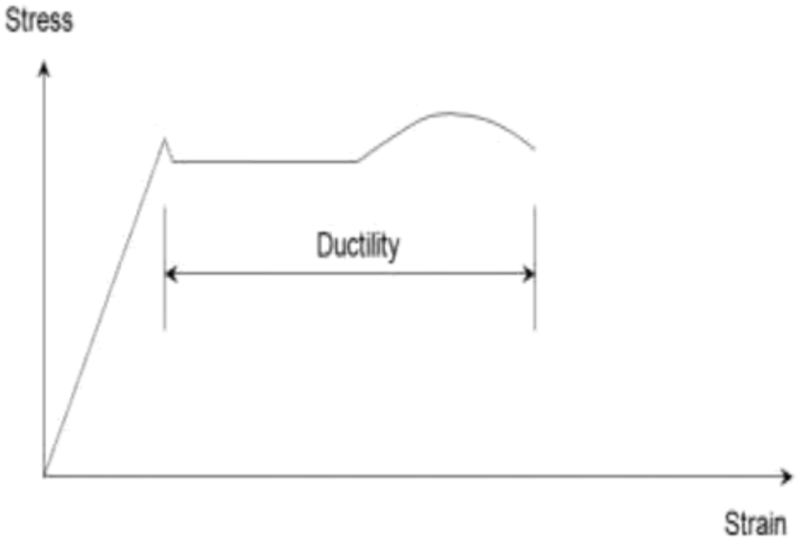


Figure 23:2: Ductility Test







Malleability:

- Malleability of a material is its ability to be flattened into thin sheets without cracking by hot or cold working.
- E.g. Lead can be readily rolled and hammered into thin sheets but can be drawn into wire.

Comparison of Ductility and Malleability:

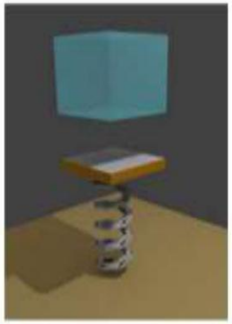
- Ductility and Malleability are frequently used interchangeably many times.
- Ductility is ***tensile quality, while*** malleability is ***compressive quality.***

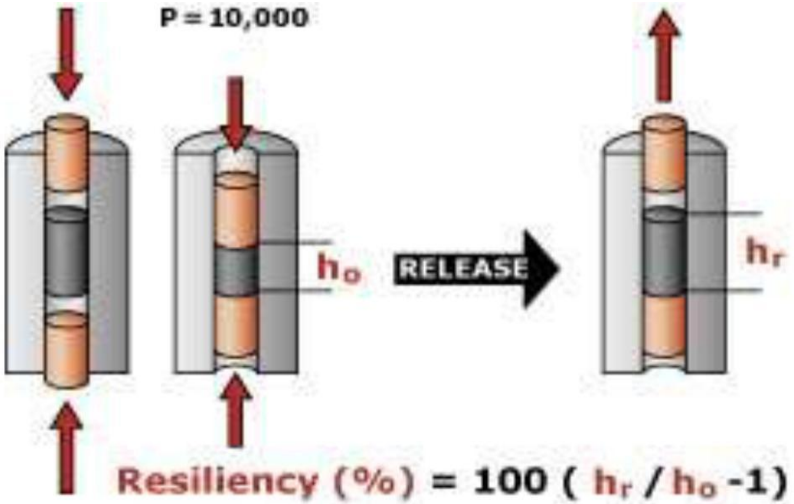
Resilience:

- It is the capacity of a material to absorb energy elastically.
- The maximum energy which can be stored in a body up to elastic limit is called the ***proof resilience, and*** the proof resilience per unit volume is called ***modulus of resilience***.
- The quantity gives capacity of the material to bear shocks and vibrations.

Resilience

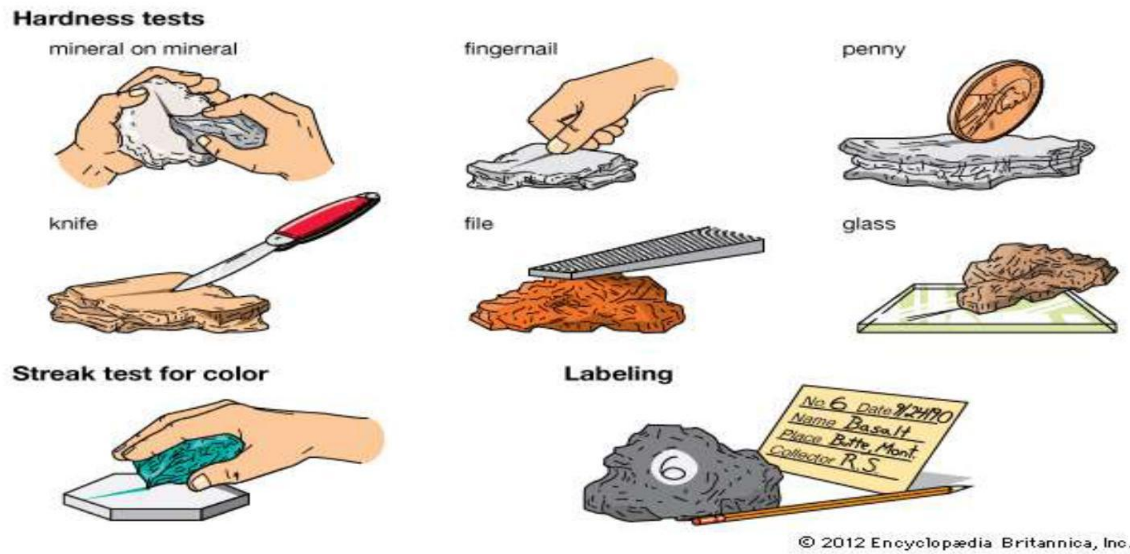
- It is the property of a material to absorb energy and to resist shock and impact loads. It is measured by the amount of energy absorbed per unit volume within elastic limit. This property is essential for spring materials.





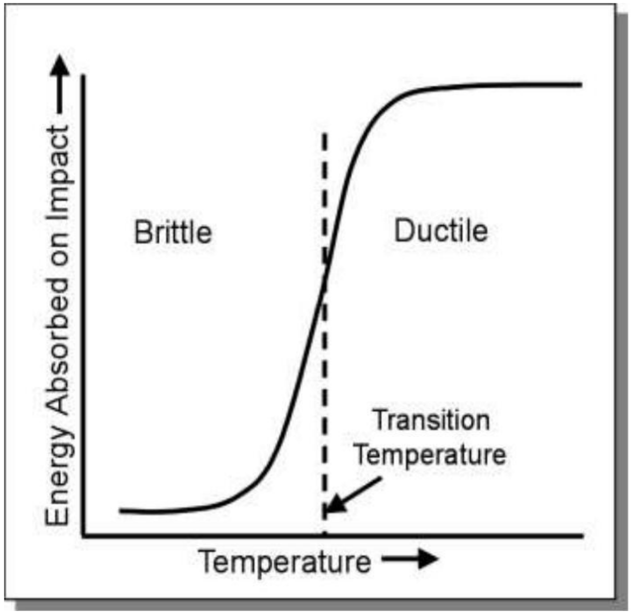
Hardness:

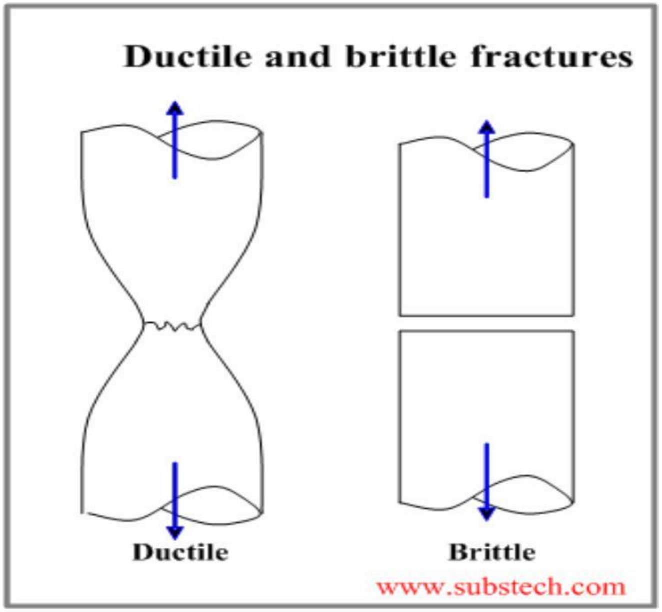
- Hardness is a fundamental property which is closely related to strength.
- Hardness is usually defined in terms material to resist to ***scratching,*** of the ability of a ***abrasion, cutting, indentation , or penetration.***
- Methods used for determining hardness: Brinel, Rockwell ,Vickers.



Brittleness:

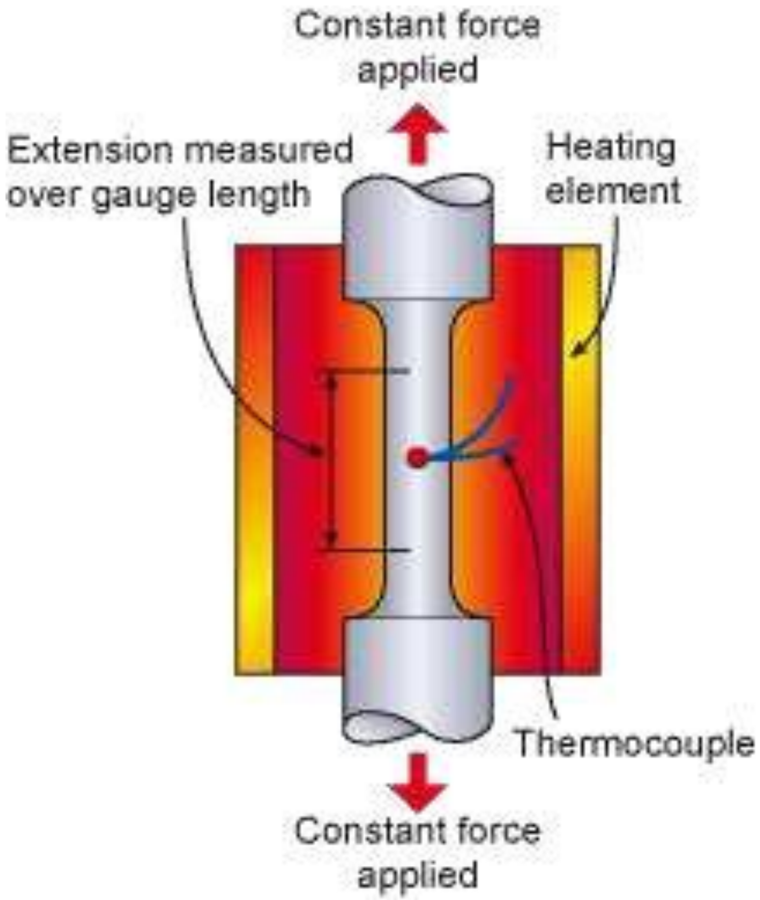
- It is the property of breaking without much permanent distortion.
- Non-Ductile material is considered to be brittle material.
- E.g., Glass, Cast iron, etc.





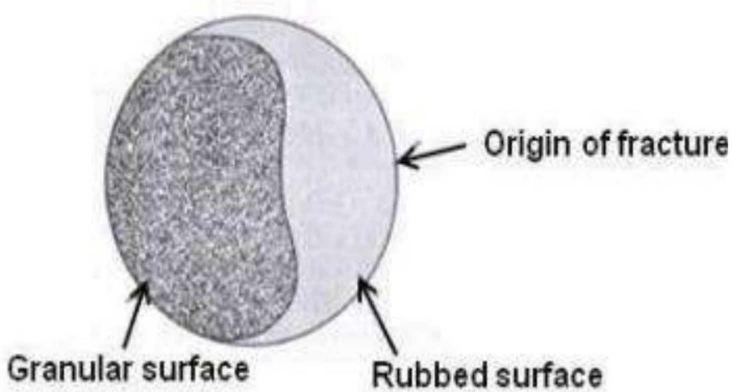
Creep:

- The slow and progressive deformation of a material with time at constant stress is called creep.
- Depending on temperature, stresses even below the elastic limit can cause some permanent deformation.
- It is most generally defined as time-dependent strain occurring under stress.



Fatigue:

- This phenomenon leads to fracture under repeated or fluctuating stress.
- Fatigue fractures are progressive beginning as minute cracks and grow under the action of fluctuating stress.
- Many components of high speed aero and turbine engines are of this type.



Surface of a Fatigue Fracture

Stress and Strain

1) Stress

1.1) Terminologies related to stress

1.2) Types of stress

2) Strain

2.1) Terminologies related to strain

2.2) Types of strain

3) Relation Between Stress and Strain

4) Stress and strain Diagram

Stress :

- Stresses are expressed as the ratio of the applied force divided by the resisting area
- Mathematically:

$$\sigma = \text{Force / Area}$$

- Units: N/m^2 or Pascal.
- 1kPa = 1000Pa, 1 MPa= 10^6 Pa
- **TERMINOLOGIES RELATED TO STRESS**
- **Stressor:**

  A stressor is anything that has the effect of causing stress.
- **Stress capacity:**

  While it is unclear precisely how much stress a person can carry, since each person has some stress in their lives, we say he/she has a capacity for stress. Similarly in case of Rocks, how much capacity they have to bear stress.
- **Stress-load:**

  Everyone, even children, must carry some amount of stress in their daily lives. When we think of stress as having an amount, or quantity, we refer to this as the person's stress-load. And here in case of rocks, we say that how much an already existing stress is applied on a rock.

Types of stress

- There are two types of stress

  1) Normal Stress

     1.1) Tensile stress

     1.2) Compressive stress

  2) Combine Stress

     2.1) Shear stress

     2.2) Tortional stress

- 1)**Normal Stress:**

  The resisting area is perpendicular to the applied force

  1.1) Tensile Stress:

  a. It is a stress induced in a body when it is subjected to two equal and opposite pulls (Tensile force) as a result of which there is tendency in increase in length.

  b. It acts normal to the area and pulls on the area.

- 1.2) **Compressive Stress:**

  a. Stress induced in a body, when subjected to two equal and opposite pushes as a result of which there is a tendency of decrease in length of the body.

  b. It acts normal to the area and it pushes on the area.

Types of stress Continuity

2) Combined Stress:

A condition of stress that cannot be represented by a single resultant stress.

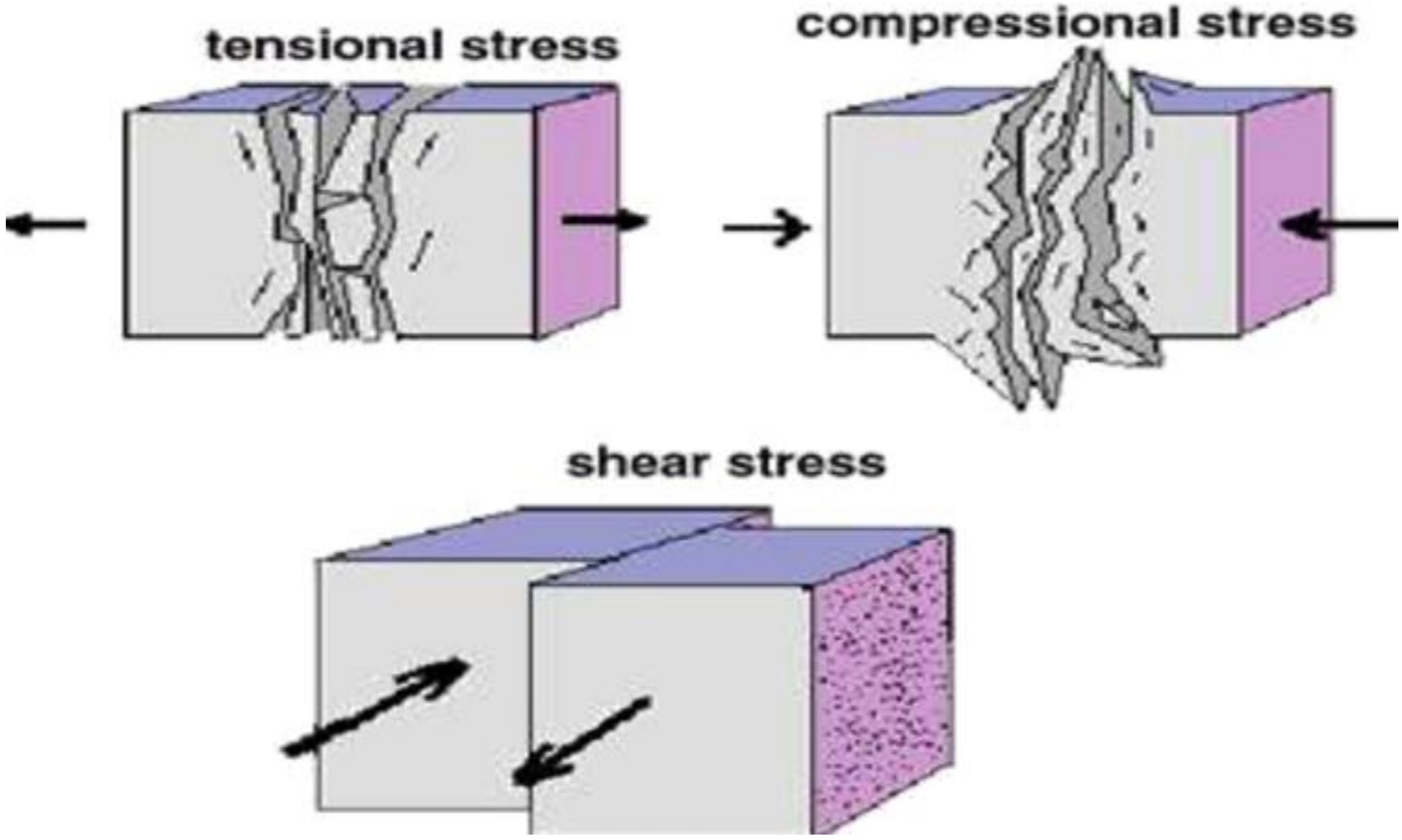
2.1) Shear stress:

- Forces parallel to the area resisting the force cause shearing stress.
- It differs to tensile and compressive stresses, which are caused by forces perpendicular to the area on which they act.
- Shearing stress is also known as tangential stress

2.2) Tortional stress:

- The stresses and deformations induced in a circular shaft by a twisting moment.

Types of stress diagrams



Strain:

- **STRAIN:**
- When a body is subjected to some external force, there is some change in the dimension of the body. The ratio of change in dimension of body to its original dimension is called as strain.
- Strain is a dimensionless quantity.

- TERMINOLOGIES RELATED TO STRAIN:

1. Longitudinal or Linear Strain
   - Strain that changes the length of a line without changing its direction.
   - Can be either compression or tensional.
2. Compression
   - Longitudinal strain that shortens an object.
3. Tension
   - Longitudinal strain that lengthens an object.

Strain: cont..,

- **Shear**
  - Strain that changes the angles of an object.
  - Shear causes lines to rotate.
- **Infinitesimal Strain**
  - Strain that is tiny, a few percent or less.
  - Allows a number of useful mathematical simplifications and approximations.
- **Finite Strain**
  - Strain larger than a few percent.
  - Requires a more complicated mathematical treatment than infinitesimal strain.
- **Homogeneous Strain**
  - Uniform strain.
  - Straight lines in the original object remain straight.
  - Parallel lines remain parallel.
  - Circles deform to ellipses.
  - Note that this definition rules out folding, since an originally straight layer has to remain straight.
- **Inhomogeneous Strain**
  - How real geology behaves.
  - Deformation varies from place to place.
  - Lines may bend and do not necessarily remain parallel.

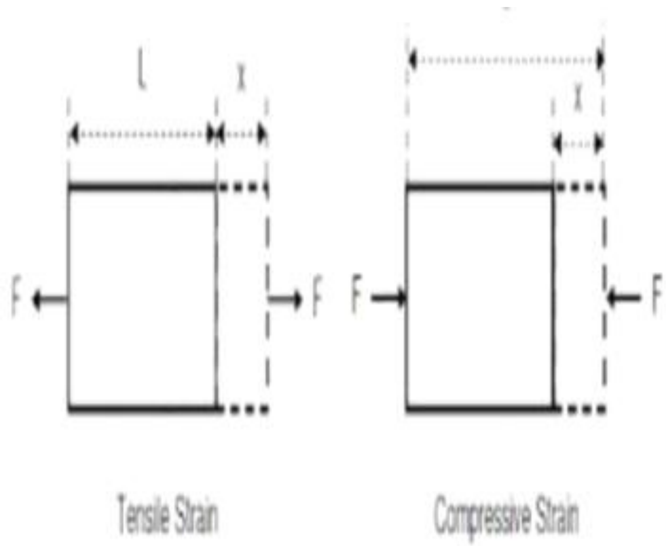
Types of strain

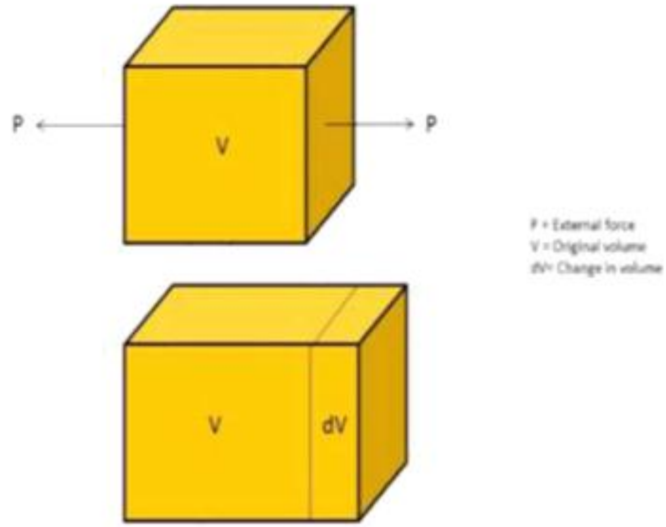
1. Tensile Strain
2. Compression Strain
3. Volumetric Strain
4. Shear Strain

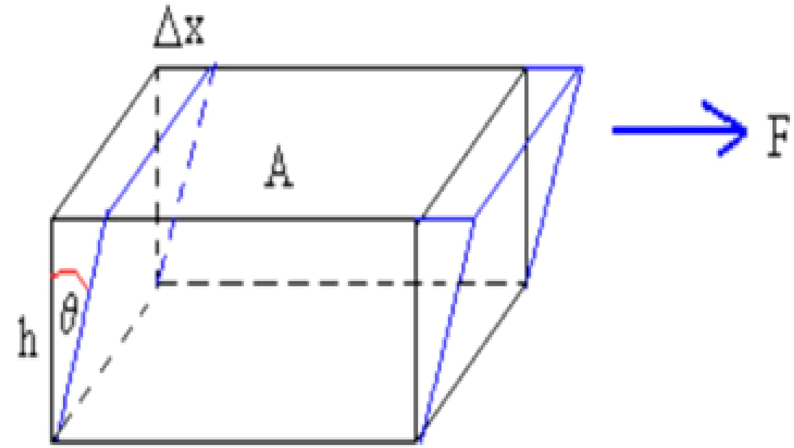
1. **Tensile Strain:**
   - Ratio of increase in length to the original length of the body when it is subjected to a pull force.
   - Tensile strain = Increase in length/ Original Length=dL/L
2. **Compressive Strain:**
   - Ratio of decrease in Length to the original length of body when it is subjected to push force.
   - Compressional Strain = Decrease in length/Original Length= dL/L
3. **Volumetric Strain:**
   - Ratio of change of volume to the original volume.
   - Volumetric Strain= dV/V
4. **Shear Strain**
   - Strain due to shear stresses.

- **<u>Sign convection for direct strain</u>**
  - Tensile strains are considered positive in case of producing increase in length.
  - Compressive strains are considered negative in case of producing decrease in length.

Types of Strain Diagrams







Shear and Moment Relationships

$$w = -\frac{dV}{dx}$$

Slope of the shear diagram = - Value of applied loading

$$V = \frac{dM}{dx}$$

Slope of the moment curve = Shear Force

Both equations not applicable at the point of loading because of discontinuity produced by the abrupt change in shear.

Beams – SFD and BMD

$w = -\frac{dV}{dx}$ Degree of *V* in *x* is one higher than that of *w*

$V = \frac{dM}{dx}$ Degree of *M* in *x* is one higher than that of *V*

→ Degree of *M* in *x* is two higher than that of *w*

Combining the two equations → $\boxed{\frac{d_2 M}{dx^2} = -w}$

→ M :: obtained by integrating this equation twice

→ **Method is usable** only if *w* **is a continuous function of** *x* (other cases not part of this course)

Beams – SFD and BMD

Shear and Moment Relationships

Expressing *V* in terms of *w* by integrating $w = -\frac{dV}{dx}$

$$\int_{V_0}^{V} dV = -\int_{x_0}^{x} w\,dx$$

OR $V = V_0$ + (**the negative of the area under the loading curve from x_0 to x**)

V_0 is the shear force at x_0 and V is the shear force at x

Expressing *M* in terms of *V* by integrating $V = \frac{dM}{dx}$

$$\int_{M_0}^{M} dM = \int_{x_0}^{x} V\,dx$$

OR $M = M_0$ + (**area under the shear diagram from x_0 to x**)

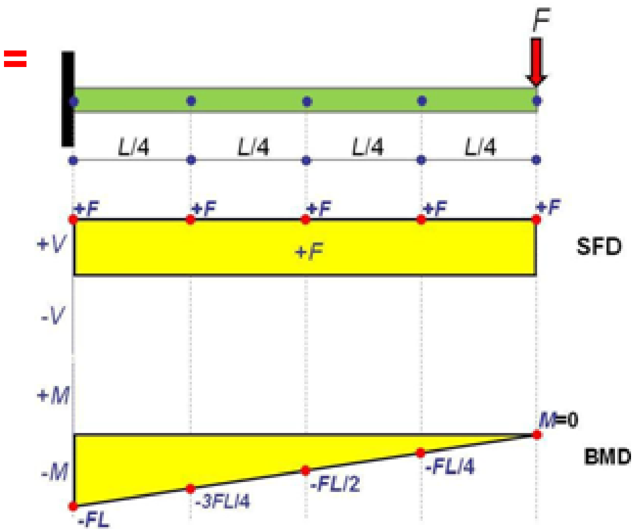
M_0 is the BM at x_0 and M is the BM at x

Beams – SFD and BMD

$V = V_0 +$ (negative of area under the loading curve from x_0 to x)

$M = M_0 +$ (area under the shear diagram from x_0 to x)

If there is no externally applied moment M_0 at x_0 = 0, total moment at any section equals the area under the shear diagram up to that section



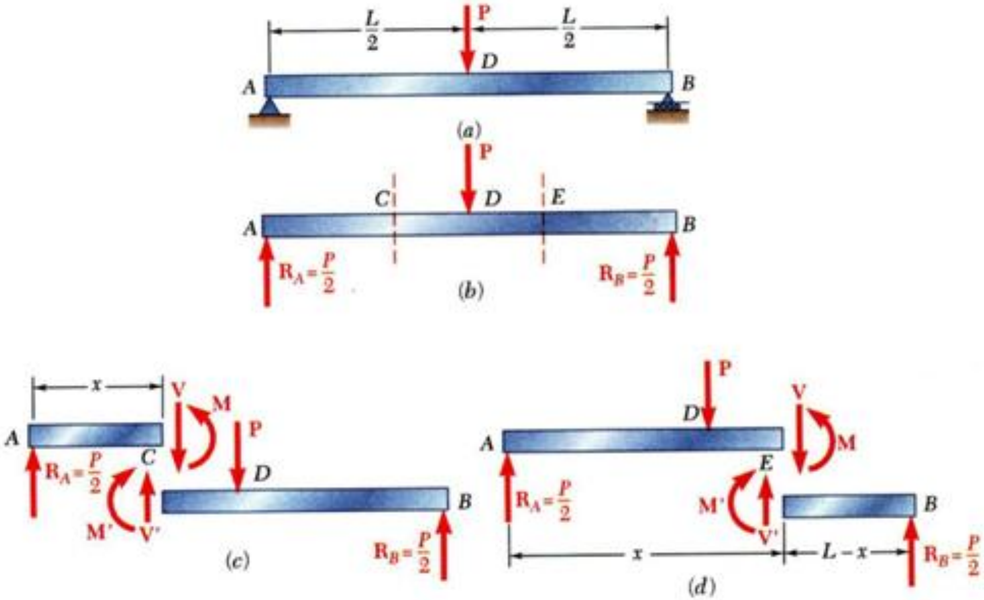
When V passes through zero and is a continuous function of x with $dV/dx \neq 0$ (i.e., nonzero loading)

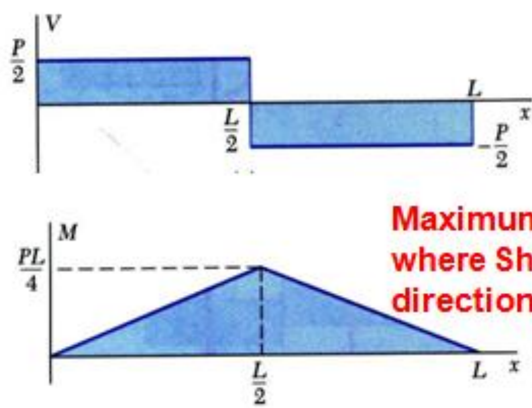
$$\frac{dM}{dx} = 0$$

→ BM will be a maximum or minimum at this point

Critical values of BM also occur when SF crosses the zero axis discontinuously (*e.g.*, Beams under concentrated loads)

Beams – SFD and BMD: Example (1)





Maximum BM occurs where Shear changes the direction

- Draw the SFD and BMD.
- Determine reactions at supports.
- Cut beam at *C* and consider member *AC*,

$$V = +P/2 \qquad M = +Px/2$$

- Cut beam at *E* and consider member *EB*,

$$V = -P/2 \qquad M = +P(L-x)/2$$

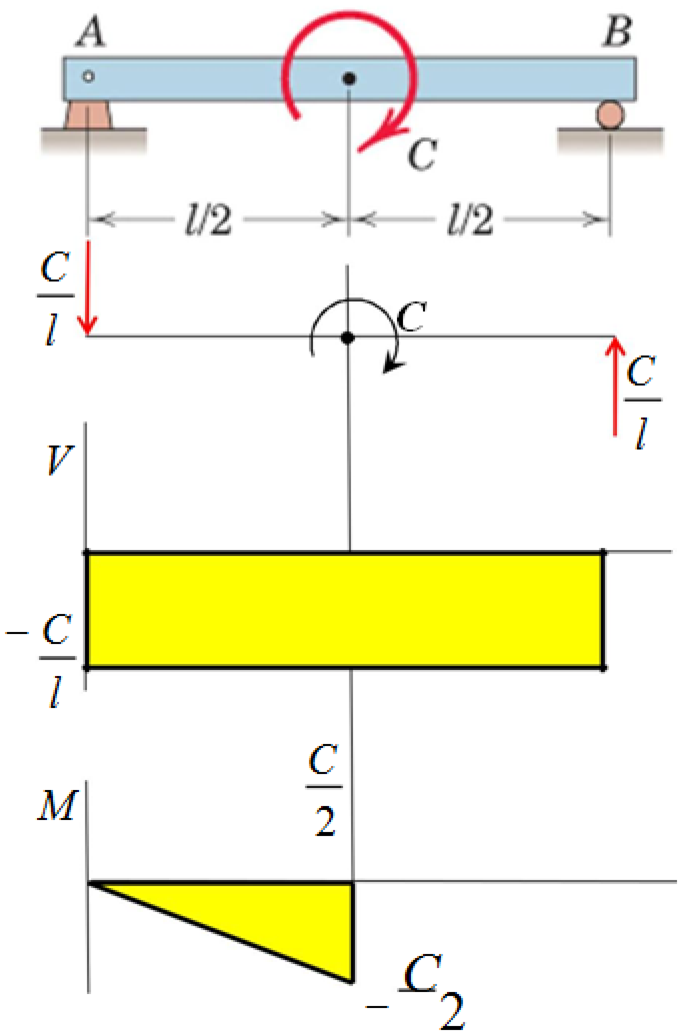
- For a beam subjected to <u>concentrated loads</u>, shear is constant between loading points and moment varies linearly

Beams – SFD and BMD: Example (2)

Draw the SFD and BMD for the beam acted upon by a clockwise couple at mid point

Solution: Draw FBD of the beam and Calculate the support reactions

Draw the SFD and the BMD starting From any one end



Beams – SFD and BMD: Example (3)

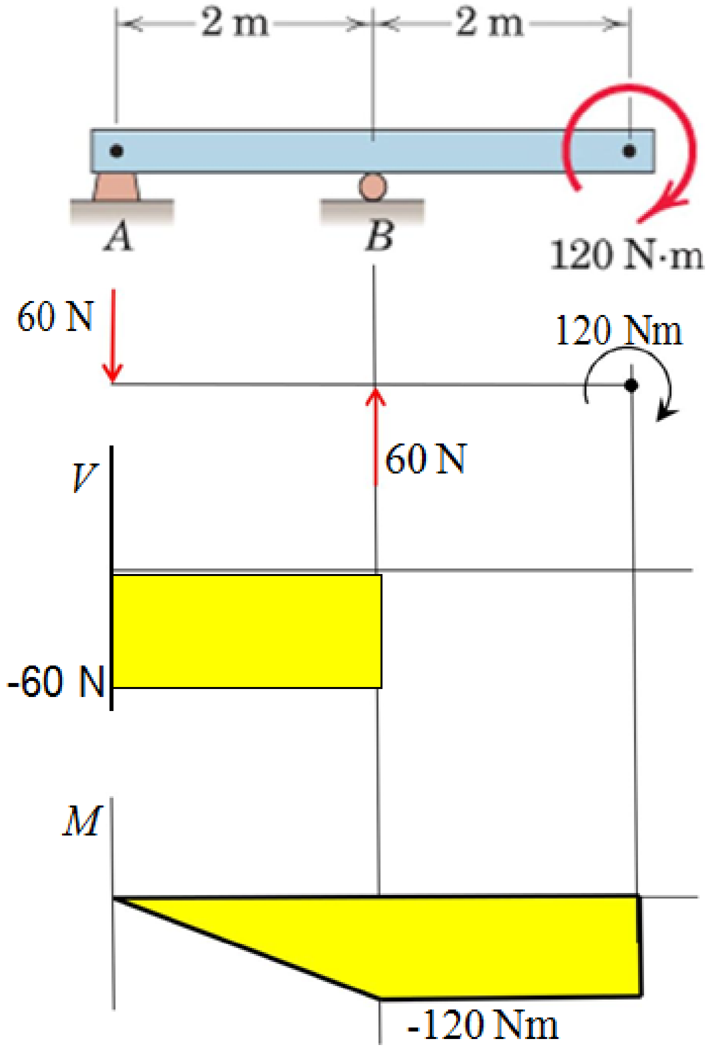
Draw the SFD and BMD for the beam

Solution: Draw FBD of the beam and Calculate the support reactions

$\sum M_A = 0 \rightarrow R_A = 60\ N\downarrow$

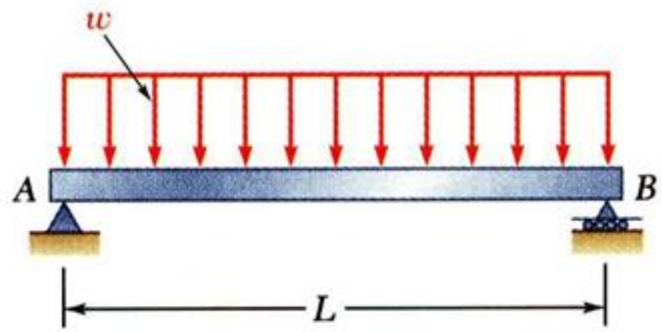
$\sum M_B = 0 \rightarrow R_B = 60\ N\uparrow$

Draw the SFD and the BMD starting from any one end



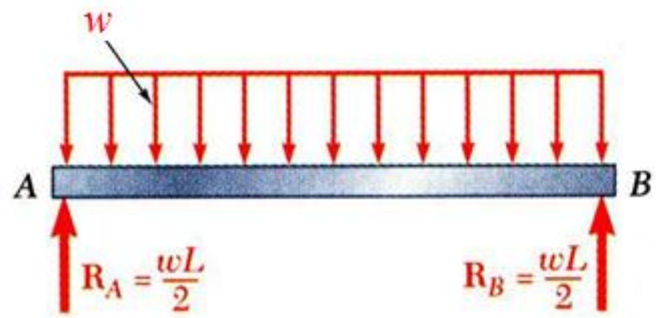
Beams – SFD and BMD: Example (4)

Draw the SFD and BMD for the beam



Solution:

Draw FBD of the entire beam and calculate support reactions using equilibrium equations

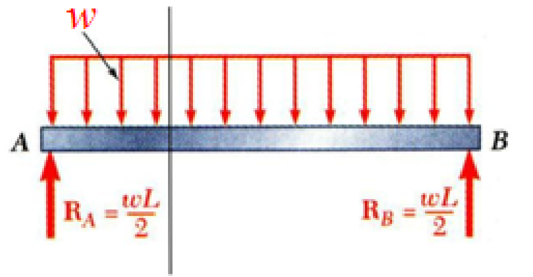


Reactions at supports: $R_A = R_B = \frac{wL}{2}$

Develop the relations between loading, shear force, and bending moment and plot the SFD and BMD

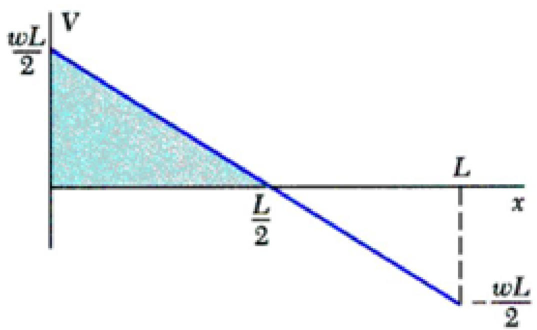
Beams – SFD and BMD: Example (4)

Shear Force at any section: $V = \frac{wL}{2} - wx = w\left(\frac{L}{2} - x\right)$



Alternatively, $V - V_A = -\int_0^x w\,dx = -wx$

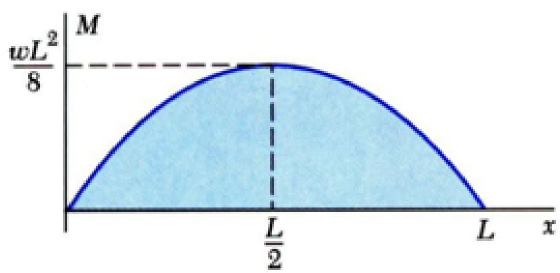
$$V = V_A - wx = \frac{wL}{2} - wx = w\left(\frac{L}{2} - x\right)$$



BM at any section: $M = \frac{wL}{2}x - wx\frac{x}{2} = \frac{w}{2}\left(Lx - x^2\right)$

Alternatively, $M - M_A = \int_0^x V dx$

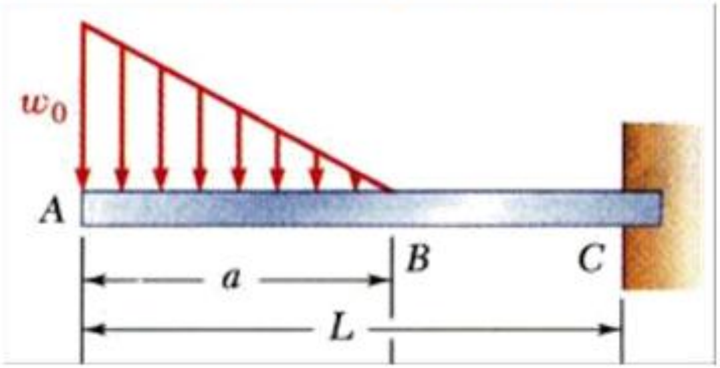
$$M = \int_0^x w\left(\frac{L}{2} - x\right) dx = \frac{w}{2}\left(Lx - x^2\right)$$



$$M_{\max} = \frac{wL^2}{8} \left(M \text{ at } \frac{dM}{dx} = V = 0 \right)$$

Beams – SFD and BMD: Example (5)

Draw the SFD and BMD for the Beam

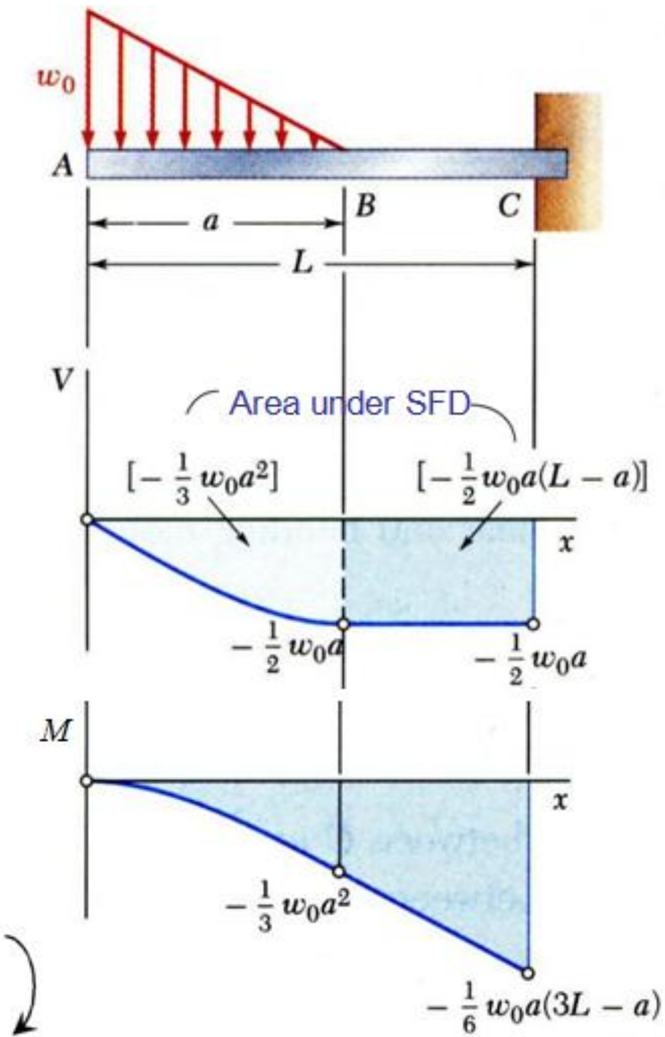


Solution:

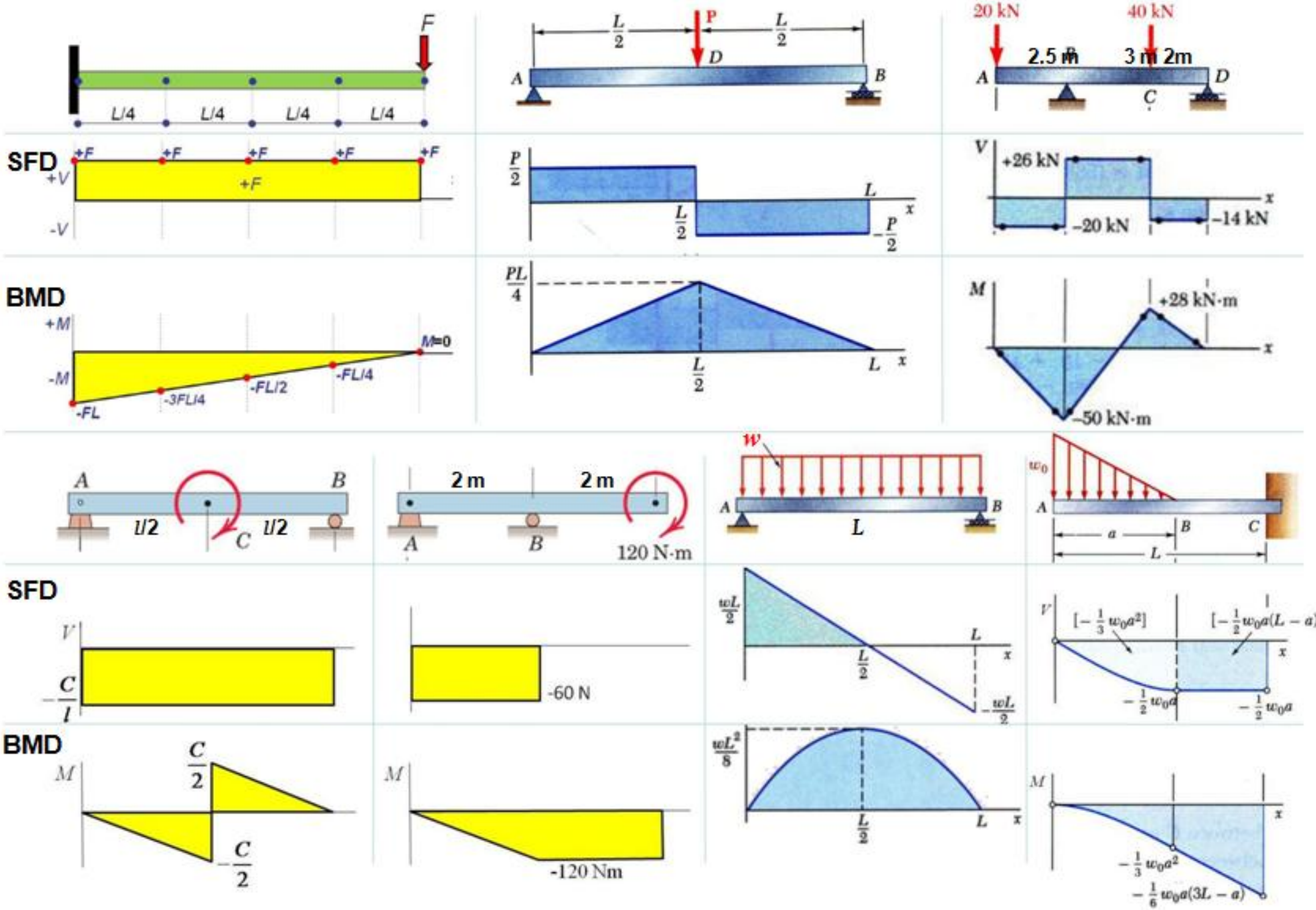
SFD and BMD can be plotted without determining support reactions since it is a cantilever beam.

However, values of SF and BM can be verified at the support if support reactions are known.

$$R_C = \frac{w_0 a}{2} \uparrow; \quad M_C = \frac{w_0 a}{2}\left(L - \frac{a}{3}\right) = \frac{w_0 a}{6}(3L - a)$$



Beams – SFD and BMD: Summary



3.1 *Torsion of Circular Shafts*

a. Simplifying assumptions

During the deformation, the cross sections are not distorted in any manner – they remain plane, and the radius r does not change. In addition, the length L of the shaft remains constant.

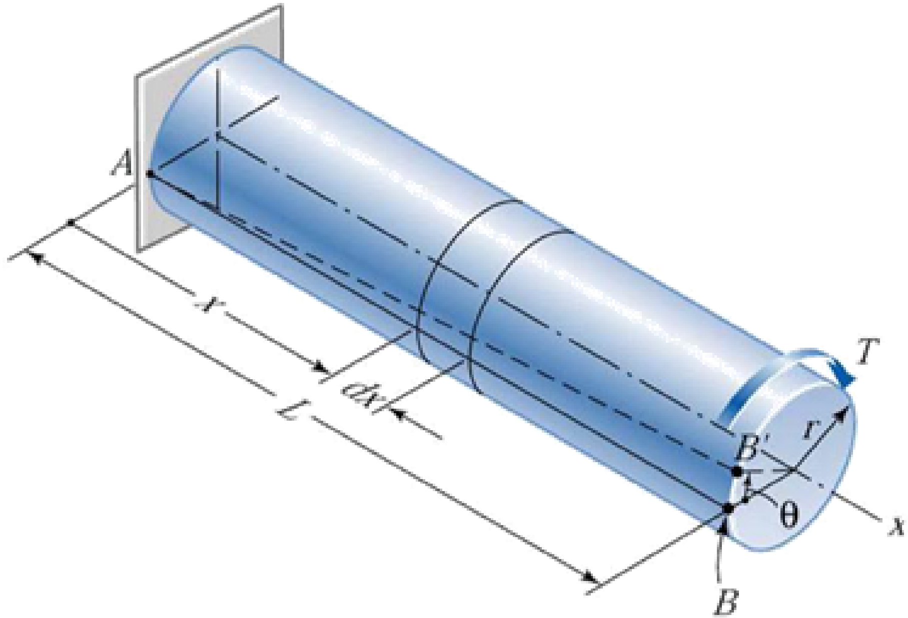


Figure 3.1 **Deformation of a circular shaft caused by the torque *T*. The initially straight line *AB* deforms into a helix.**

TORSION cont..,

Based on these observations, we make the following assumptions:

- Circular cross sections remain plane (do not warp) and perpendicular to the axis of the shaft.
- Cross sections do not deform (there is no strain in the plane of the cross section).
- The distances between cross sections do not change (the axial normal strain is zero).

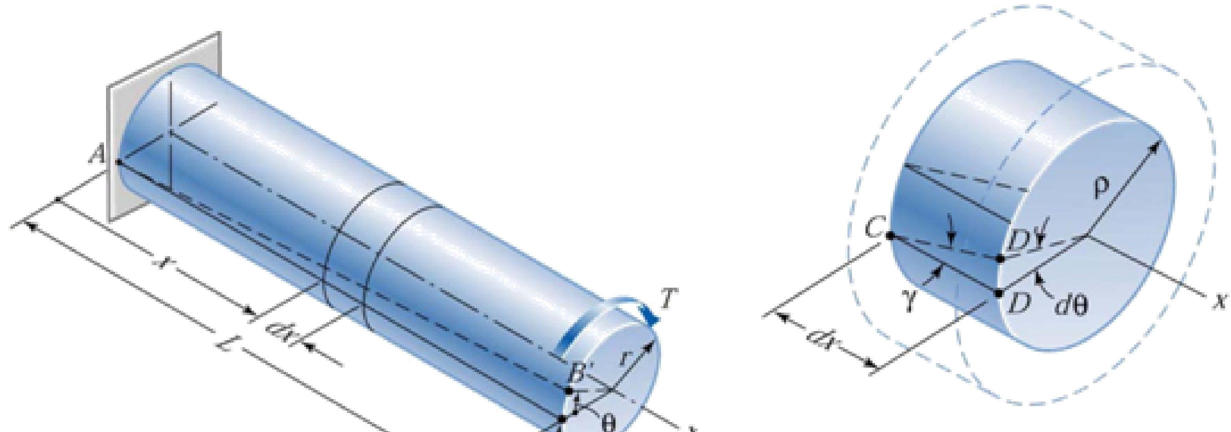
Each cross section rotates as a rigid entity about the axis of the shaft. Although this conclusion is based on the observed deformation of a cylindrical shaft carrying a **constant** internal torque, we assume that the result remains valid even if the diameter of the shaft or the internal torque **varies** along the length of the shaft.

Compatibility:

b. ***Compatibility***

Because the cross sections are separated by an infinitesimal distance, the difference in their rotations, denoted by the angle $d\theta$, is also infinitesimal.

As the cross sections undergo the relative rotation $d\theta$, *CD* deforms into the helix *CD*. By observing the distortion of the shaded element, we recognize that the helix angle γ is the *shear strain* of the element.



(a)

Compatibility: cont..,

From the geometry of Fig.3.2(a), we obtain DD′= $\rho\, d\theta = \gamma dx$, from which the shear strain γ is

$$\gamma = \frac{d\theta}{dx}\rho \qquad (3.1)$$

The quantity $d\theta/dx$ is the *angle of twist per unit length,* where θ is expressed in radians. The corresponding shear stress, illustrated in Fig. 3.2 (b), is determined from Hooke′s law:

$$\tau = G\gamma = G\frac{d\theta}{dx}\rho \qquad (3.2)$$

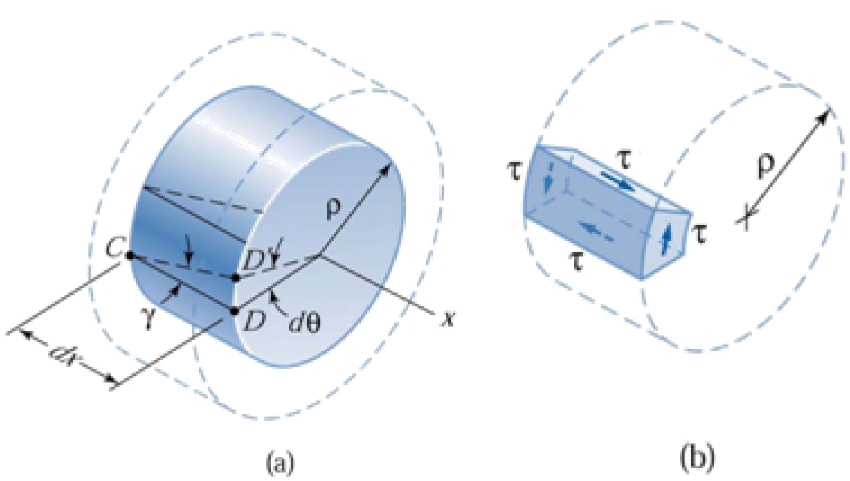


Figure 3.2 (a) Shear strain of a material element caused by twisting of the shaft; (b) the corresponding shear stress.

Compatibility: cont..,

the shear stress varies linearly with the radial distance ρ *from the axial of the shaft.* $\tau = G\gamma = G\frac{d\theta}{dx}\rho$

The variation of the shear stress acting on the cross section is illustrated in Fig. 3.3. The maximum shear stress, denoted by τ_{max}, occurs at the surface of the shaft.

Note that the above derivations assume neither a constant internal torque nor a constant cross section along the length of the shaft.

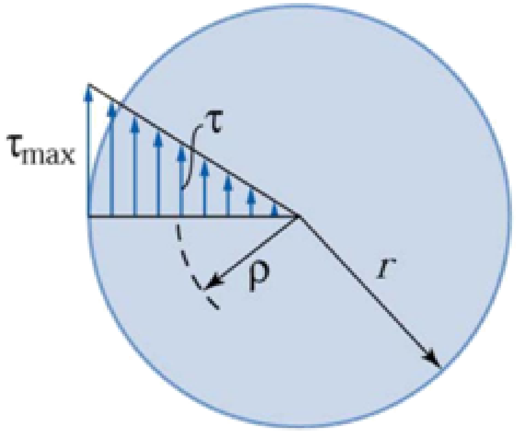


Figure 3.3 Distribution of shear stress along the radius of a circular shaft.

Equilibrium:

c. ***Equilibrium***

Fig. 3.4 shows a cross section of the shaft containing a differential element of area dA loaded at the radial distance ρ from the axis of the shaft.

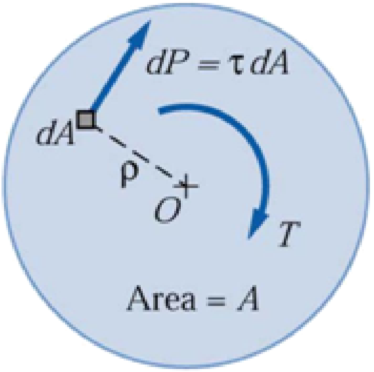


Figure 3.4

Calculating the Resultant of the shear stress acting on the cross section. Resultant is a couple equal to the internal torque T.

The shear force acting on this area is $dP = \tau dA = G\,(d\theta/dx)\,\rho\,dA$, directed perpendicular to the radius. Hence, the moment (torque) of dP about the center o is $\rho\, dP = G\,(d\theta/dx)\,\rho^2 dA$. Summing the contributions and equating the result to the internal torque yields $\int_A \rho dP = T$, or

$$G\frac{d\theta}{dx}\int_A \rho^2\, dA = T$$

Equilibrium: cont..,

Recognizing that is the polar moment of inertia of the cross-sectional area, we can write this equation as $G\,(d\theta/dx)\,J = T$, or

$$\boxed{\frac{d\theta}{dx} = \frac{T}{GJ}} \qquad (3.3)$$

The rotation of the cross section at the free end of the shaft, called **the angle of twist θ** , is obtained by integration:

$$\boxed{\theta = \int_o^L d\theta = \int_o^L \frac{T}{GJ}dx} \qquad (3.4a)$$

As in the case of a **prismatic bar** carrying a constant torque, then reduces **the torque-twist relationship**

$$\boxed{\theta = \frac{TL}{GJ}} \qquad (3.4b)$$

Note the similarity between Eqs. (3.4) and the corresponding formulas for axial deformation: $\delta = \int_o^L (P/EA)dx$ and $\delta = PL/(EA)$

Notes on the Computation of angle of Twist

- 1.In the U.S. Customary system, the consistent units are G [psi], T [$lb \cdot$ in] , and L [in.], and J [in.^{4}]; in the SI system, the consistent units are G [Pa], T [N · m], L [m], and J [m^4].
- 2.The unit of θ in Eqs. (3.4) is radians, regardless of which system of unit is used in the computation.
- 3.Represent torques as vectors using the right-hand rule, as illustrated in Fig. 3.5. The same sign convention applies to the angle of twist θ .

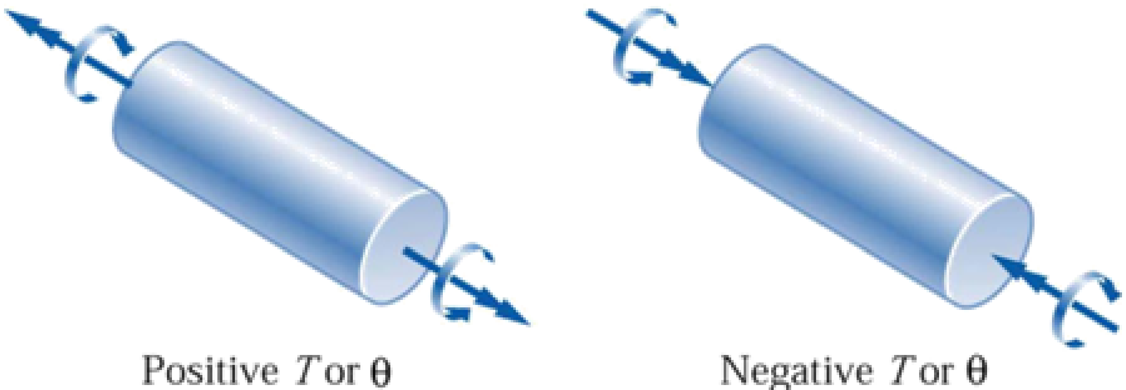


Figure 3.5 Sign Conventions for Torque T and angle of twist T.

formulas for axial deformation: $\delta = \int_o^L (P/EA)dx$ and δ = PL/(EA)

Torsion formulas

$G\,(d\theta/dx) = T/J$, which substitution into Eq. (3.2), $\tau = G\gamma = G\frac{d\theta}{dx}\rho$ gives the shear stress τ acting at the distance ρ from the center of the shaft, ***Torsion formulas*** :

$$\boxed{\tau = \frac{\rho T}{J}} \quad (3.5a)$$

The **maximum** shear stress τ_{max} is found by replacing ρ by the radius *r* of the shaft:

$$\boxed{\tau_{max} = \frac{Tr}{J}} \qquad (3.5b)$$

Because Hook´s law was used in the derivation of Eqs. (3.2)-(3.5), these formulas are valid if the shear stresses do not exceed the proportional limit of the material shear. Furthermore, these formulas are applicable only to **circular shafts**, either solid or hollow.

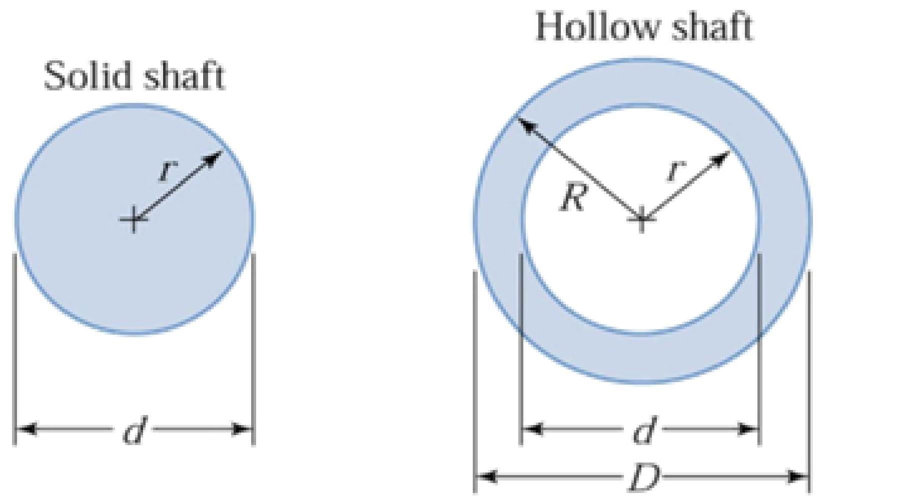
Torsion formulas cont..,

The expressions for the polar moments of circular areas are :

Solid shaft : $$\boxed{\tau_{max} = \frac{2T}{\pi r^3} = \frac{16T}{\pi d^3}} \qquad (3.5c)$$

Hollow shaft : $$\boxed{\tau_{max} = \frac{2TR}{\pi (R^4 - r^4)} = \frac{16TD}{\pi (D^4 - d^4)}} \qquad (3.5d)$$

Equations (3.5c) and (3.5d) are called the ***torsion formulas***.



$$J = \frac{\pi r^4}{2} = \frac{\pi d^4}{32} \qquad J = \frac{\pi}{2}(R^4 - r^4) = \frac{\pi}{32}(D^4 - d^4)$$

Figure 3.6 Polar moments of inertia of circular areas.

Power Transmission

Shafts are used to transmit power. The power ζ transmitted by a torque T rotating at the angular speed ω is given by $\zeta = T\omega$, where ω is measured in radians per unit time.

If the shaft is rotating with a frequency of f revolutions per unit time, then $\omega = 2\pi f$, which gives $\zeta = T(2\pi f)$. Therefore, the torque can be expressed as

$$\boxed{T = \frac{\zeta}{2\pi f}} \tag{3.6a}$$

In SI units, ζ in usually measured in watts (1.0 W=1.0 N · m/s) and f in hertz (1.0 Hz = 1.0 rev/s); Eq. (3.6a) then determines the torque T in N · m.

In U.S. Customary units with ζ in *lb* · in./s and f in hertz, Eq.(3.6a) calculates the torque T in *lb* · in.

Power Transmission cont..,

Because power in U.S. Customary units is often expressed in horsepower (1.0 hp = 550 lb · ft/s = 396×10^3 lb · in./min), a convenient form of Eq.(3.6a) is

$$T\,(lb \cdot in) = \frac{\zeta\,(hp)}{2\pi f\,(rev/\min)} \times \frac{396 \times 10^3\,(lb \cdot in./\min)}{1.0(hp)}$$

which simplifies to

$$T\,(lb \cdot in) = 63.0 \times 10^3\,\frac{\zeta\,(hp)}{f\,(rev/\min)} \qquad (3.6b)$$

Statically indeterminate problems

- Draw the required **free-body diagrams** and write the equations of **equilibrium**.
- Derive the **compatibility** equations from the restrictions imposed on the angles of twist.
- Use the **torque- twist relationships** in Eqs.(3.4) to express the angles of twist in the compatibility equations in terms of the torques.
- **Solve the equations** of equilibrium and compatibility for the torques.

Sample Problem.1

A solid steel shaft in a rolling mill transmits 20 kW of power at 2 Hz. Determine the smallest safe diameter of the shaft if the shear stress τ_w is not to exceed 40 MPa and the angle of twist θ is limited to 6° in a length of 3 m. Use G = 83 GPa.

Solution

Applying Eq. (3.6a) to determine the torque:

$$T = \frac{P}{2\pi f} = \frac{20 \times 10^3}{2\pi(2)} = 1591.5 N \cdot m$$

To satisfy the strength condition, we apply the torsion formula, Eq. (3.5c):

$$\tau_{max} = \frac{Tr}{J} \qquad \tau_{max} = \frac{16T}{\pi d^3} \qquad 4 \times 10^6 = \frac{16(1591.5)}{\pi d^3}$$

Which yields $d = 58.7 \times 10^{-3}$ m = 58.7 mm.

Sample Problem.1 cont...,

Apply the torque-twist relationship, Eq. (3.4b), to determine the diameter necessary to satisfy the requirement of rigidity (remembering to convert θ from degrees to radians):

$$\theta = \frac{TL}{GJ} \qquad 6\frac{\pi}{180} = \frac{1591.5(3)}{(83\times10^{9})(\pi d^{4}/32)}$$

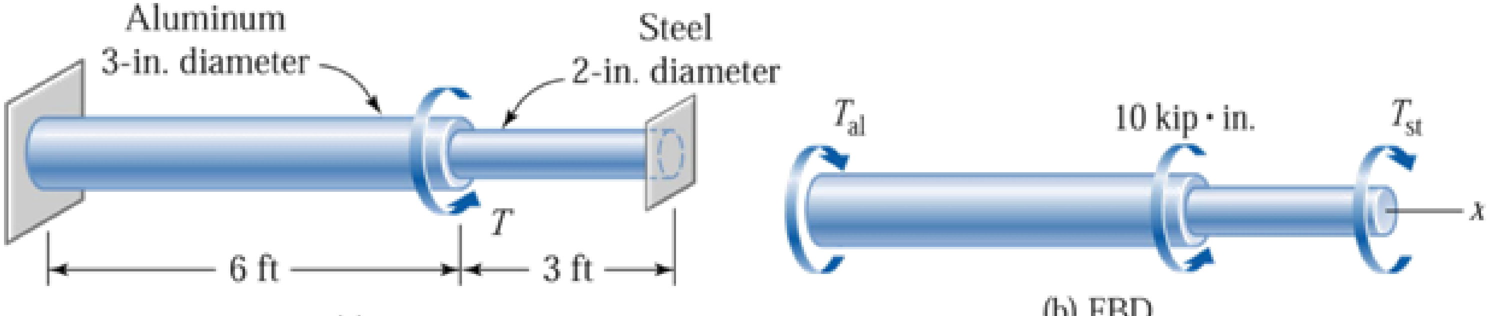
From which we obtain $d = 48.6\times10^{-3}$ m = 48.6 mm.

To satisfy both strength and rigidity requirements, we must choose the larger diameter-namely,

$d = 58.7$ mm. *Answer*

Sample problem .2

The shaft in Fig. (a) Consists of a 3-in. -diameter aluminum segment that is rigidly joined to a 2-in. -diameter steel segment. The ends of the shaft are attached to rigid supports, Calculate the maximum shear stress developed in each segment when the torque $T = 10$ kip in. is applied. Use $G = 4\times10^6$ psi for aluminum and $G = 12\times10^6$ psi for steel.



(a)

(b) FBD

Solution

Equilibrium $\Sigma M_x = 0,\quad (10 \times 10^3) - T_{st} - T_{al} = 0 \qquad (a)$

This problem is statically indeterminate.

Sample Problem.2 cont…,

Compatibility the two segments must have the same angle of twist; that is, $\theta_{st} = \theta_{al}$ From Eq. (3.4b), this condition between.

$$\left(\frac{TL}{GJ}\right)_{st} = \left(\frac{TL}{GJ}\right)_{al} \qquad \frac{T_{st}(3\times 12)}{(12\times 10^{6})\frac{\pi}{32}(2^{4})} = \frac{T_{al}(6\times 12)}{(4\times 10^{6})\frac{\pi}{32}(3^{4})}$$

from which

$$T_{st} = 1.1852\, T_{al} \qquad \text{(b)}$$

Solving Eqs. (a) and (b), we obtain

$$T_{al} = 4576\ lb\cdot\text{in.} \quad T_{st} = 5424\ lb\cdot\text{in.}$$

the maximum shear stresses are

$$(\tau_{max})_{al} = \frac{16T}{\pi d^{3}}_{al} = \frac{16(4576)}{\pi(3)^{3}} = 863\ psi \qquad \textit{Answer}$$

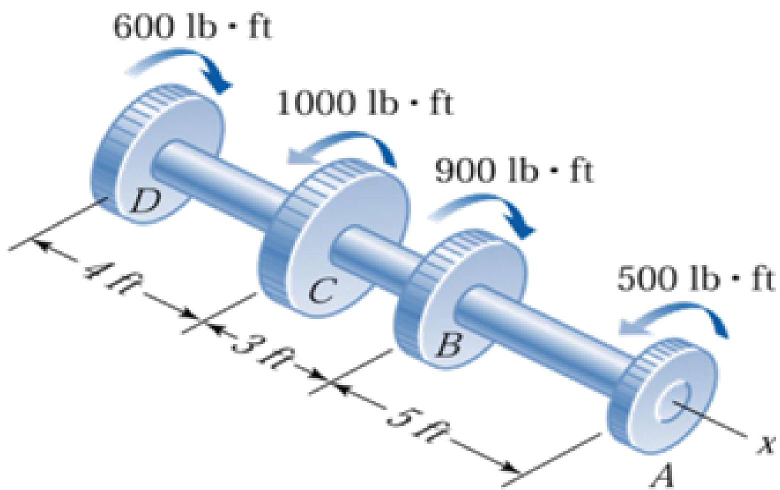
$$(\tau_{max})_{st} = \frac{16T}{\pi d^{3}}_{st} = \frac{16(5424)}{\pi(2)^{3}} = 3450\ psi \qquad \textit{Answer}$$

Sample problem.3

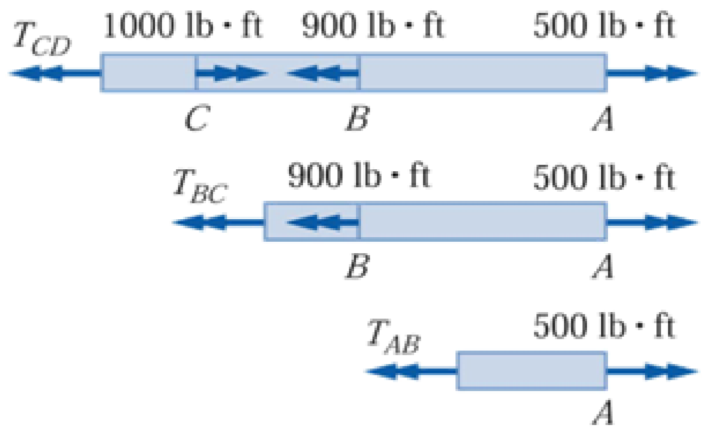
The four rigid gears, loaded as shown in Fig. (a), are attached to a 2-in.-diameter steel shaft. Compute the angle θ of rotation of gear A relative to gear D. Use $G = 12 \times 10^6$ psi for the shaft.

Solution

It is convenient to represent the torques as vectors (using the right-hand rule) on the FBDs in Fig. (b).



(a)



(b) FBDs

Sample Problem.3 cont...,

Solution

Assume that the internal torques T_{AB}, T_{BC}, and T_{CD} are positive according to the sign convention introduced earlier (positive torque vectors point away from the cross section). Applying the equilibrium condition $\sum M_x = 0$ to each FBD, we obtain

$$500 - 900 + 1000 - T_{CD} = 0$$

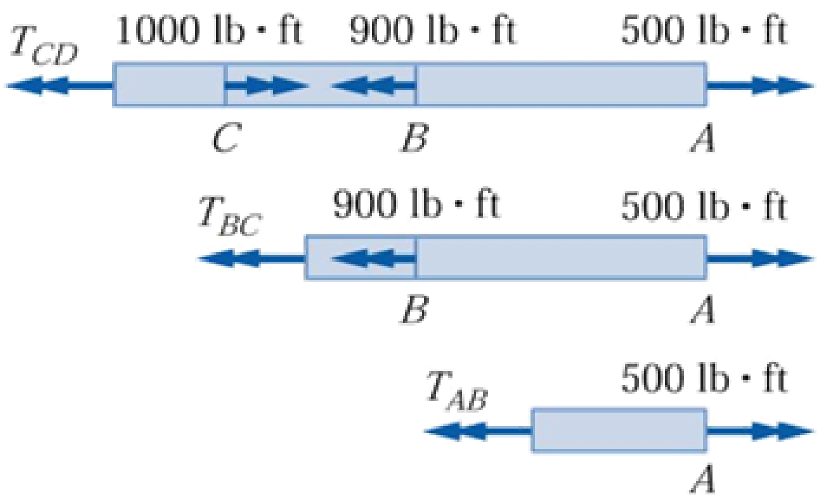
$$500 - 900 - T_{BC} = 0$$

$$500 - T_{AB} = 0$$

T_{AB} =500 $lb \cdot$ ft,

T_{BC} =-400 $lb \cdot$ ft

T_{CD} = 600 $lb \cdot$ ft



(b) FBDs

The minus sign indicates that the sense of T_{BC} is opposite to that shown on the FBD.A is gear D were fixed.

Sample Problem.3 cont…,

This rotation is obtained by summing the angles of twist of the three segments:

$$\theta_{A/D} = \theta_{A/B} + \theta_{B/C} + \theta_{C/D}$$

Using Eq. (3.4b), we obtain (converting the lengths to inches and torques to pound-inches)

$$\theta_{A/D} = \frac{T_{AB}L_{AB} + T_{BC}L_{BC} + T_{CD}L_{CD}}{GJ}$$

$$= \frac{(500\times12)(5\times12) - (400\times12)(3\times12) + (600\times12)(4\times12)}{[\pi\,(2)^4 / 32](12\times10)^6}$$

$$= 0.02827 \text{ rad} = 1.620^{\circ} \qquad \textit{Answer}$$

The positive result indicates that the rotation vector of A relative to D is in the positive x-direction: that is, θ_{AD} is directed counterclockwise when viewed from A toward D.

shown on the FBD.A is gear D were fixed.

Sample Problem.4

Figure (a) shows a steel shaft of length $L = 1.5$ m and diameter $d = 25$ mm that carries a distributed toque of intensity (torque per unit length) $t = t_B(x/L)$, where $t_B = 200$ N· m/m. Determine (1) the maximum shear stress in the shaft; and (2) The angle of twist. Use $G = 80$ GPa for steel.

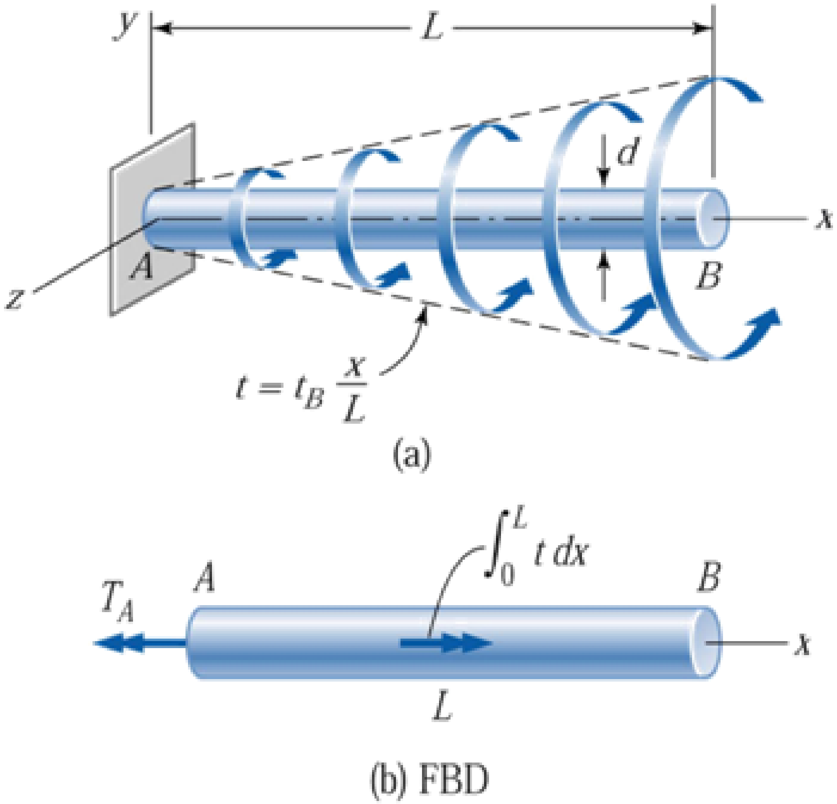
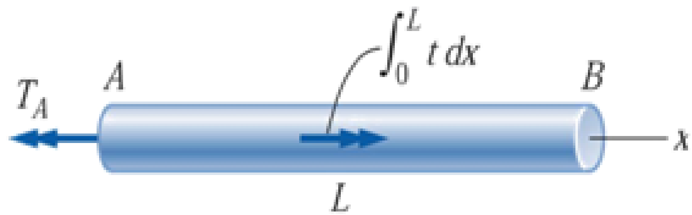


Figure (a) and (b) FBD

Solution



(b) FBD

Part 1

Figure (b) shows the FBD of the shaft. The total torque applied to the shaft is $\int_0^{t} t\,dx$. The maximum torque in the shaft is T_A, which occurs at the fixed support. From the FBD we get

$$\sum M_X = 0 \qquad \int_0^L t\,dx - T_A = 0$$

$$T_A = \int_0^L t\,dx = \int_0^L t_B \frac{x}{L}\,dx = \frac{t_B L}{2} = \frac{1}{2}(200)(1.5) = 150 N \cdot m$$

From Eq. (3.5c), the maximum stress in the shaft is

$$\tau_{max} = \frac{16 T_A}{\pi d^3} = \frac{16(150)}{\pi (0.025)^3} = 48.9 \times 10^6\ Pa = 48.9 MPa \quad Answer$$

Sample Problem.4 cont…,

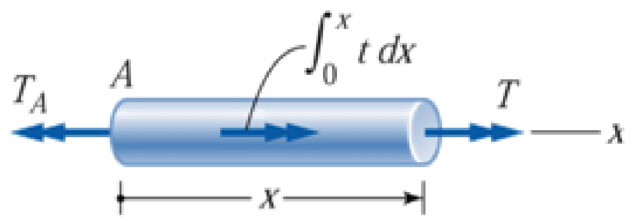
Part 2

The torque T acting on a cross section located at the distance x from the fixed end can be found from the FBD in Fig. (c):

$$\sum M_x = 0 \quad T + \int_0^x t\,dx - T_A = 0$$

$$T = T_A - \int_0^x t\,dx = \frac{t_B L}{2} - \int_0^x t_B \frac{x}{L}\,dx$$

$$= \frac{t_B}{2L}(L^2 - x^2)$$



(c) FBD

From Eq. (3.4a), the angle θ of twist of the shaft is

$$\theta = \int_0^L \frac{T}{GJ}\,dx = \frac{t_B}{2LGJ}\int_0^L (L^2 - x^2)\,dx = \frac{t_B L^2}{3GJ}$$

$$= \frac{200\,(1.5)^2}{3(80 \times 10^9)(\pi/32)(0.025)^4} = 0.0489\,rad = 2.8^\circ \quad \textit{Answer}$$

Torsion of Thin-Walled Tubes

Simple approximate formulas are available for thin-walled tubes. Such members are common in construction where light weight is of paramount importance.

The tube to be prismatic (constant cross section), but the wall thickness t is allowed to vary within the cross section. The surface that lies midway between the inner and outer boundaries of the tube is called the ***middle surface***.

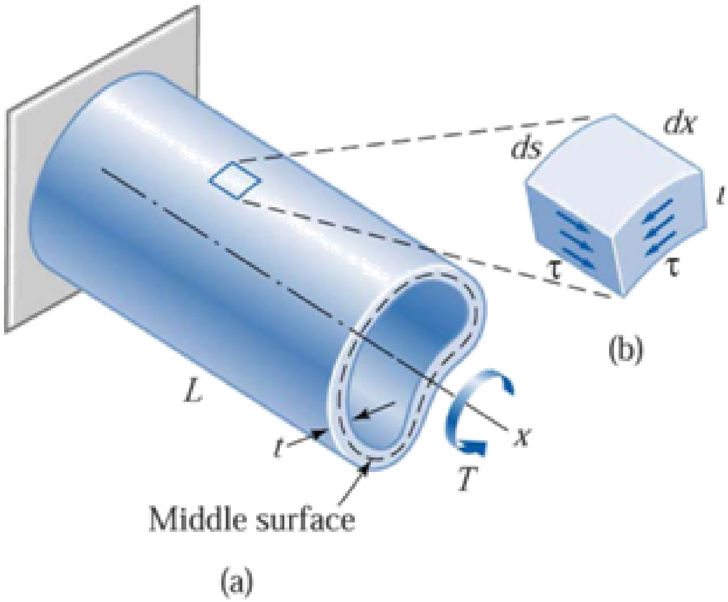


Figure 3.7 (a) Thin-walled tube in torsion; (b) shear stress in the wall of the tube.

Torsion of Thin-Walled Tubes cont..

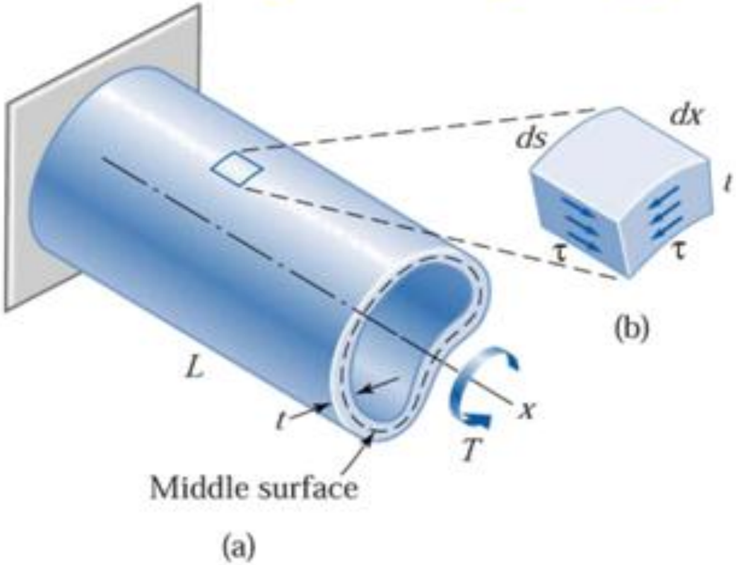
If thickness t is small compared to the overall dimensions of the cross section, the shear stress induced by torsion can be shown to be almost constant through the wall thickness of the tube and directed tangent to the middle surface, in Fig. (3.7b).

At this time, it is convenient to introduce the concept of ***shear flow q***, defined as the shear force per unit edge length of the middle surface.

the shear flow q is

$$q = \tau t \qquad (3.7)$$

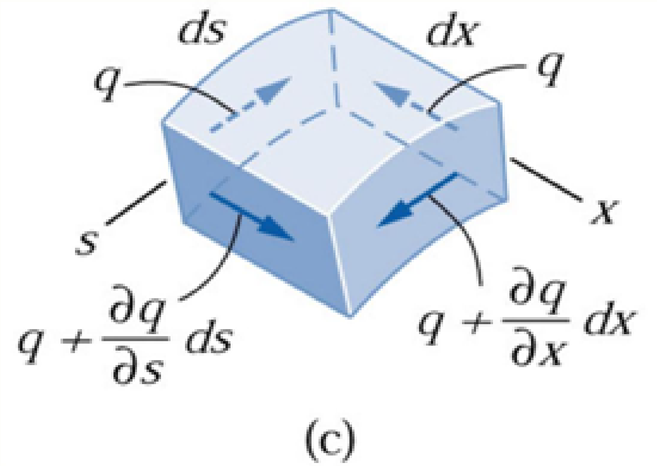
If the shear stress is not constant through the wall thickness, then τ in Eq. (3.7) should be viewed as the average shear stress.



Torsion of Thin-Walled Tubes cont..

The shear flow is constant throughout the tube. This result can be obtained by considering equilibrium of the element shown in Fig. 3.7(c).

In labeling the shear flows, we assume that q varies in the longitudinal (x) as well as the circumferential (s) directions. The force acting on each side of the element is equal to the shear flow multiplied by the edge length, resulting in the equilibrium equations.



(c) Shear flows on wall element.

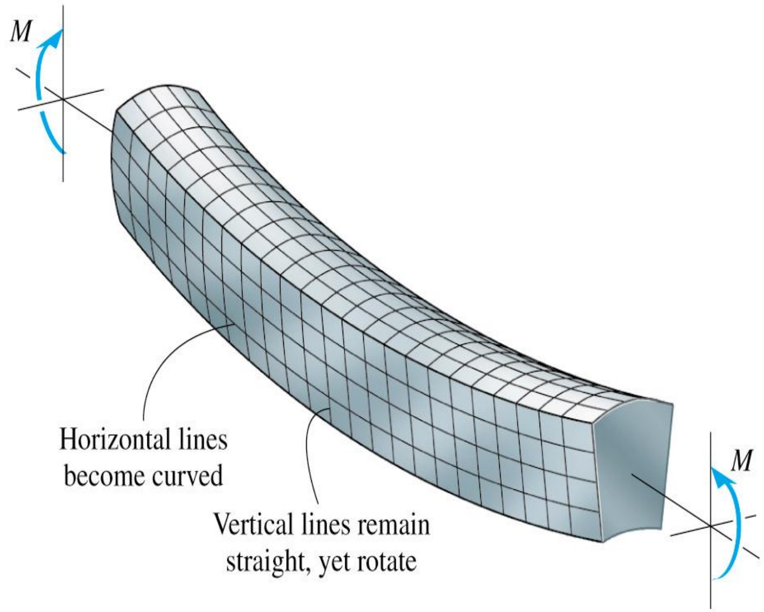
$$\sum F_X = 0 \quad q + \frac{\partial q}{\partial s} ds\, dx - q dx = 0$$

$$\sum F_S = 0 \quad q + \frac{\partial q}{\partial x} dx\, ds - q ds = 0$$

$\boxed{\partial q/ \partial x = \partial q/ \partial s} = 0$, thereby proving that the shear flow is constant throughout the tube.

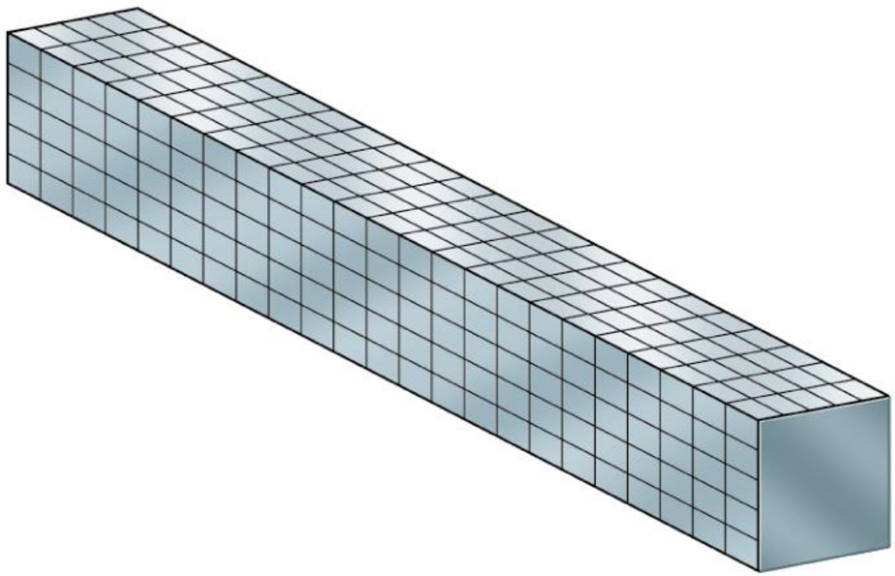
STRESSES IN BEAMS

- Bending Stresses in beams:
- Bending in beams:



After deformation

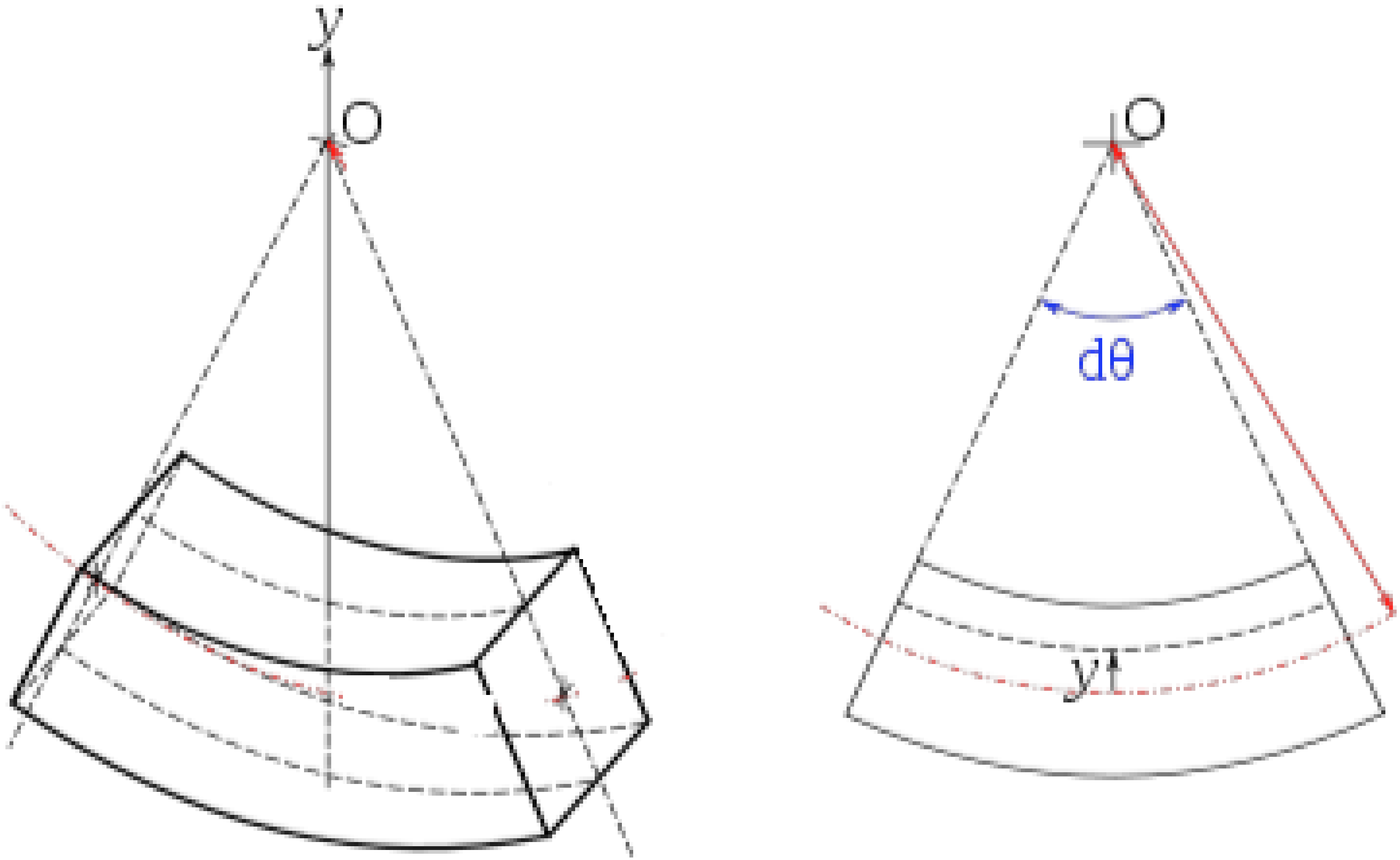
(b)



Before deformation

(a)

Stresses due to bending



SIMPLE BENDING OR PUREBENDING

- When some external force acts on a beam, the shear force and bending moments are set up at all the sections of the beam
- Due to shear force and bending moment, the beam undergoes deformation. The material of the beam offers resistance to deformation
- Stresses introduced by bending moment are known as bending stresses
- Bending stresses are indirect normal stresses

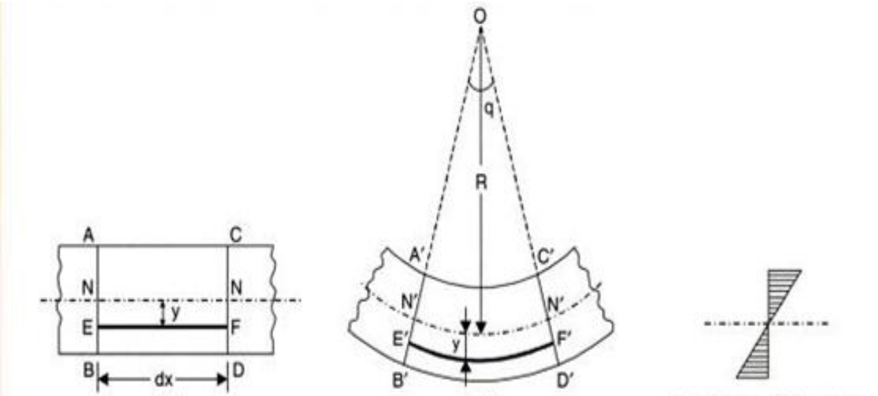
• ASSUMPTIONS FOR THE THEORY OF PURE BENDING:

- The material of the beam is isotropic and homogeneous. I.e. of same density and elastic properties throughout
- The beam is initially straight and unstressed and all the longitudinal filaments bend into a circular arc with a common centre of curvature
- The elastic limit is nowhere exceeded during the bending
- Young's modulus for the material is the same in tension and compression

THEORY OF SIMPLE BENDING

- The layer AC is shortened to A'C'. Hence it is subjected to compressive stress
- The layer BD is elongated to B'D'. Hence it is subjected to tensile stresses.
- Hence the amount of shortening decrease from the top layer towards bottom and the amount of elongation decreases from the bottom layer towards top
- Therefore there is a layer in between which neither elongates nor shortens. This layer is called neutral layer .

- <u>NEUTRAL AXIS:</u>
  - For a beam subjected to a pure bending moment, the stresses generated on the neutral layer is zero.
  - Neutral axis is the line of intersection of neutral layer with the transverse section
  - Consider the cross section of a beam subjected to pure bending. The stress at a distance y from the neutral axis is given by σ/y=E/R



(a) (b) (c) Stress Diagram

MOMENT OF RESISTANCE

- Due to the tensile and compressive stresses, forces are exerted on the layers of a beam subjected to simple bending
- These forces will have moment about the neutral axis. The total moment of these forces about the neutral axis is known as moment of resistance of that section
- We have seen that force on a layer of cross sectional area dA at a distance y from the neutral axis,

$$dF= (E \times y \times dA)/R$$

- Moment of force dF about the neutral axis= dF x y= (E x y x dA)/R x y= E/R x (y^2dA)
- Hence the total moment of force about the neutral axis= Bending moment applied= ∫ E/R x (y^2dA)= E/R x Ixx; Ixx is the moment of area about the neutral axis/centroidal axis.

  Hence $M=E/R \times I_{xx}$

  Or $M/I_{xx}=E/R$

  Hence $M/I_{xx}=E/R = \sigma b/y$; σb is also known as flexural stress (Fb)

  Hence $M/I_{xx}=E/R=F_b/y$
- The above equation is known as bending equation
- This can be remembered using the sentence “Elizabeth Rani May I Follow You”

CONDITION OF SIMPLE BENDING & FLEXURAL RIGIDITY

- Bending equation is applicable to a beam subjected to pure/simple bending. I.e. the bending moment acting on the beam is constant and the shear stress is zero
- However in practical applications, the bending moment varies from section to section and the shear force is not zero
- But in the section where bending moment is maximum, shear force (derivative of bending moment) is zero
- Hence the bending equation is valid for the section where bending moment is maximum

BENDING OF FLITCHED BEAMS

A beam made up of two or more different materials assumed to be rigidly connected together and behaving like a single piece is called a flitched beam or a composite beam.

- Consider a wooden beam reinforced by steel plates. Let

E1= Modulus of elasticity of steel plate

E2= Modulus of elasticity of wooden beam

M1= Moment of resistance of steel plate

M2= Moment of resistance of wooden beam

I1= MOI of steel plate about neutral axis

I2= MOI of wooden beam about neutral axis.

The bending stresses can be calculated using two conditions.

- Strain developed on a layer at a particular distance from the neutral axis is the same for both the materials
- Moment of resistance of composite beam is equal to the sum of individual moment of resistance of the members
  - Using condition-1:

    $\sigma 1/E1 = \sigma 2/E2$;

    $\sigma 1 = \sigma 2 \times (E1/E2)$ or $\sigma 1 = \sigma 2 \times m$; where m=

    E1/E2 is the modular ratio between steel and wood
  - Using condition-2:

    $M = M1 + M2$;

    $M1 = \sigma 1 \times I1/y$

    $M1 = \sigma 2 \times I2/y$
  - Hence $M = \sigma 1 \times I1/y + \sigma 2 \times I2/y$

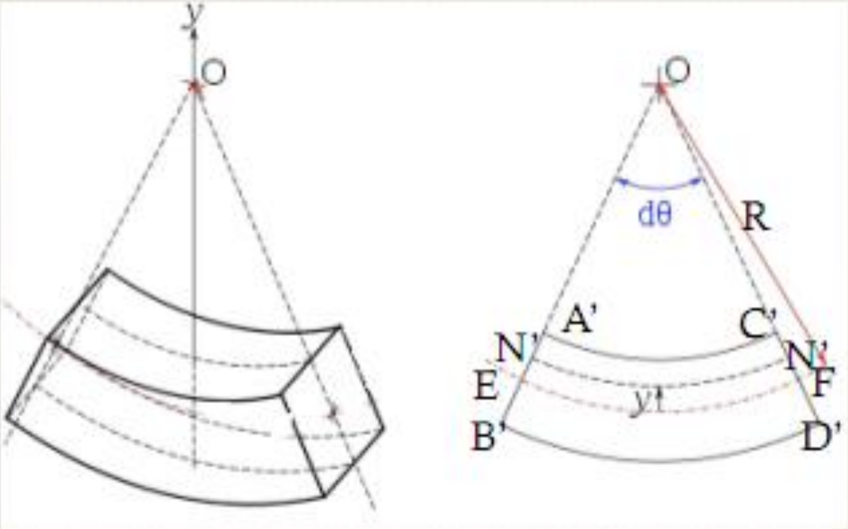
    $M = \sigma 2/y \times (I2 + I1 \times \sigma 1/\sigma 2)$

    $M = \sigma 2/y \times (I2 + I1 \times m)$

    $(I2 + I1 \times m) = I$ = equivalent moment of inertia of the cross section;

    Hence $M = \sigma 2/y \times I$

Stresses due to bending



Strain in layer EF $= \dfrac{y}{R}$

$$E = \frac{Stress\_in\_the\_layer\_EF}{Strain\_in\_the\_layer\_EF}$$

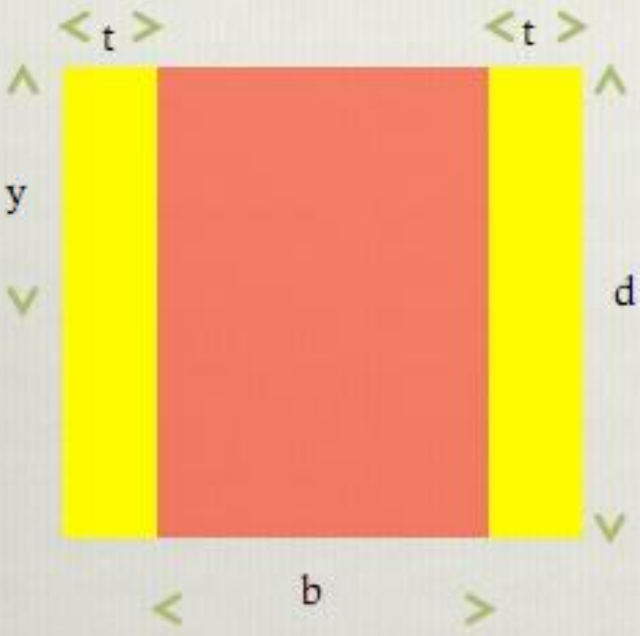
$$E = \frac{\sigma}{\left(\dfrac{y}{R}\right)}$$

$$\frac{\sigma}{y} = \frac{E}{R} \quad \Rightarrow \quad \sigma = \frac{E}{R}y$$

Flexure Formula

$$\frac{M}{I} = \frac{E}{R} = \frac{\sigma}{y}$$

Composite beams



$$\frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2}$$

$$\sigma_1 = \frac{E_1}{E_2}\sigma_2$$

$$= m\sigma_2$$

m=modular ratio

$$M = \frac{\sigma}{y} I$$

$$M = M_1 + M_2$$

$$= \frac{\sigma_1}{y} I_1 + \frac{\sigma_2}{y} I_2$$

$$= \frac{\sigma_2}{y}\left[m I_1 + I_2 \right]$$

WORKED EXAMPLE No.1

A beam has a rectangular cross section 80 mm wide and 120 mm deep. It is subjected to a bending moment of 15 kNm at a certain point along its length. It is made from metal with a modulus of elasticity of 180 GPa. Calculate the maximum stress on the section.

SOLUTION

B = 80 mm, D = 100 mm. It follows that the value of y that gives the maximum stress is 50 mm. Remember all quantities must be changed to metres in the final calculation.

$$I = \frac{BD^3}{12} = \frac{80 \times 100^3}{12} = 6.667 \times 10^6\,\text{mm}^4 = 6.667 \times 10^{-6}\,\text{m}^4$$

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\sigma = \frac{My}{I} = \frac{15 \times 10^3 \times 0.05}{6.667 \times 10^{-6}} = 112.5 \times 10^6\ \text{N/m}^2$$

WORKED EXAMPLE No.2

A beam has a hollow circular cross section 40 mm outer diameter and 30 mm inner diameter. It is made from metal with a modulus of elasticity of 205 GPa. The maximum tensile stress in the beam must not exceed 350 MPa.

Calculate the following.

(i) the maximum allowable bending moment.
(ii) the radius of curvature.

SOLUTION

D = 40 mm, d = 30 mm

$I = \pi(40^4 - 30^4)/64 = 85.9 \times 10^3 \text{ mm}^4$ or $85.9 \times 10^{-9} \text{ m}^4$.

The maximum value of y is D/2 so y = 20 mm or 0.02 m

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$M = \frac{\sigma I}{y} = \frac{350 \times 10^6 \times 85.9 \times 10^{-9}}{0.02} = 1503 \text{ Nm or } 1.503 \text{ M Nm}$$

$$\frac{\sigma}{y} = \frac{E}{R}$$

$$R = \frac{Ey}{\sigma} = \frac{205 \times 10^9 \times 0.02}{350 \times 10^6} = 11.71 \text{ m}$$

WORKED EXAMPLE No.3

Calculate the stress on the top and bottom of the section shown when the bending moment is 300 N m. Draw the stress distribution.

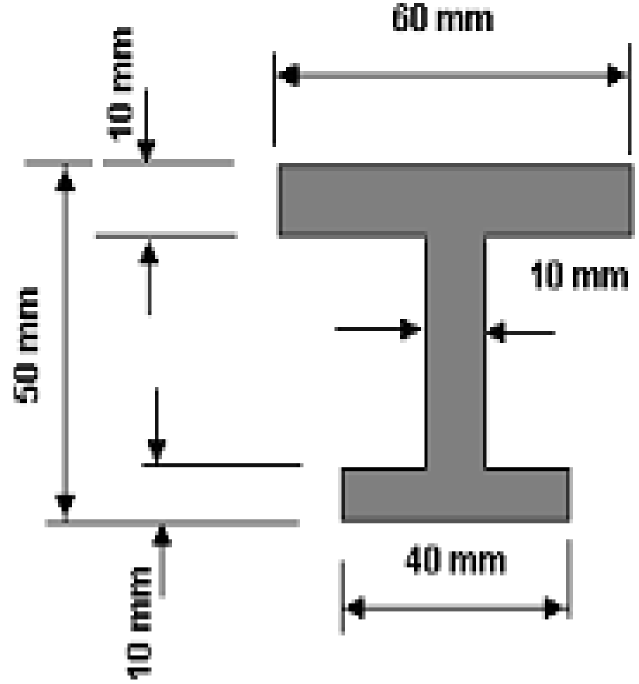


Figure 9

Now calculate the stress using the well known formula $\sigma_B = My/I$

Top edge y = distance from the centroid to the edge = 50 – 28.08 = 21.93 mm

$\sigma_B = 300 \times 0.02192/418.300 \times 10^{-9} = 15.72 \times 10^6$ Pa or 15.72 MPa (Tensile)

Bottom edge y = $\bar{y}$=28.07 mm

$\sigma_B = 300 \times 0.02808/418.300 \times 10^{-9} = 20.14$ MPa (Tensile)

The stress distribution looks like this.

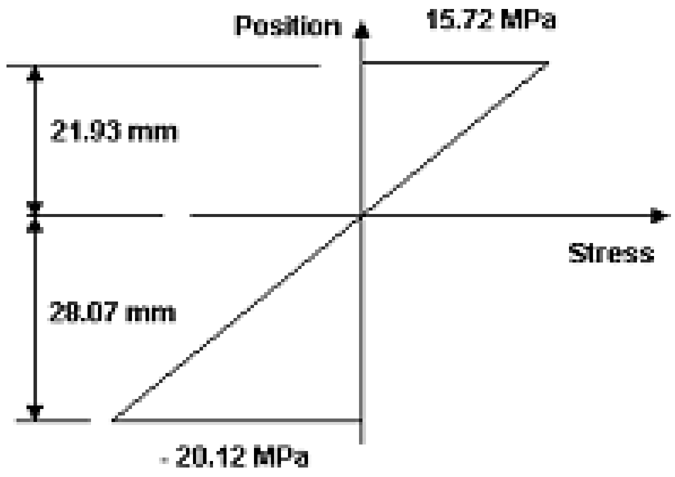


Figure 10

WORKED EXAMPLE No.4

The section solved in example 2 is subjected to a tensile force that adds a tensile stress of 10 MPa everywhere. Sketch the stress distribution and determine the new position of the neutral axis.

SOLUTION

The stress on the top edge will increase to 25.72 MPa and on the bottom edge it will decrease to -10.12 MPa. The new distribution will be as shown and the new position of the neutral axis may be calculated by ratios.

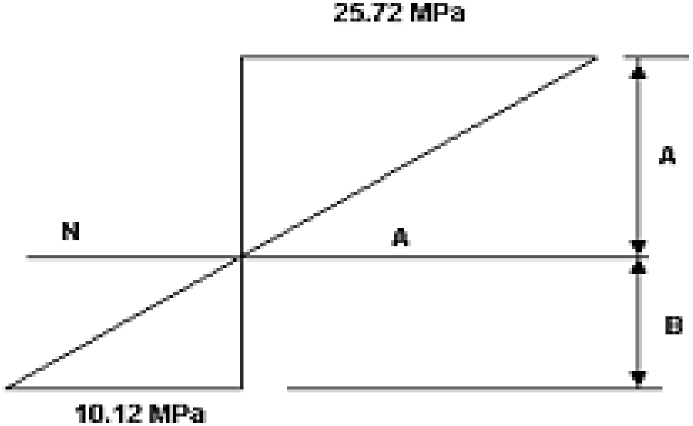


Figure 11

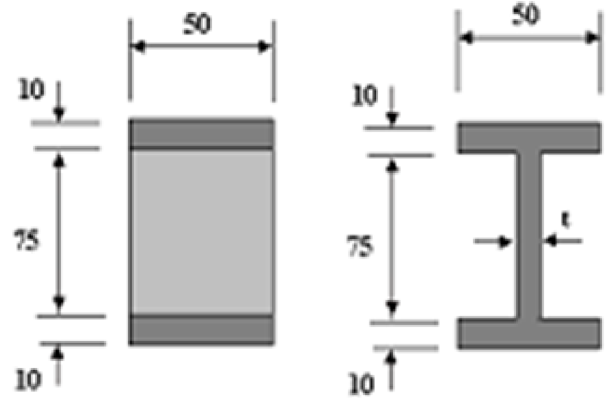
A + B = 50 mm so B = 50 - A
By similar triangles A/25.72 = B/10.12 A = (25.72/10.12)B = 2.54 B

B = 50 - 2.54 B 3.54 B = 50 B = 14.12 mm A= 50 - 14.12 = 35.88 mm

WORKED EXAMPLE No.5

A rectangular section timber beam is 50 mm wide and 75 mm deep. It is clad with steel plate 10 mm thick on the top and bottom. Calculate the maximum stress in the steel and the timber when a moment of 4 kNm is applied.
E for timber is 10 GPa and for steel 200 GPa

SOLUTION



The width of an equivalent steel web must be
$t = 50 \times E_t / E_s = 50 \times 10/200 = 2.5$ mm
Now calculate I_{gg} for the equivalent beam.
This is easy because it is symmetrical and involves finding I for the outer box and subtracting I for the missing parts.
$I_{gg} = 50 \times 95^3/12 - 47.5 \times 75^3/12$
$I_{gg} = 1.9025 \times 10^{-6}$ m^4

The stress at y = 37.5 mm $\sigma = My/I = 4000 \times 0.0375/1.9025 \times 10^{-6} = 78.845$ MPa
The stress in the timber at this level will be different because of the different E value.
$\sigma_t = \sigma_s E_t/E_s = 3.942$ MPa
The stress at y = 47.5 mm will be the stress at the edge of the steel.
$\sigma_s = My/I = 4000 \times 0.0475/1.9025 \times 10^{-6} = 99.87$ MPa

SOLUTION

First calculate the second moment of area using the tabular method that you should already know. Divide the shape into three sections A, B and C. First determine the position of the centroid from the bottom edge.

	Area	$\bar{y}$	$A\bar{y}$
A	600 mm^2	45 mm	27 000 mm^3
B	300 mm^2	25 mm	7500 mm^3
C	400 mm^2	5 mm	2000 mm^3
Totals	1300 mm^2		365000 mm^3

For the whole section the centroid position is $\bar{y}$ =365000/1300 = 28.07 mm

Now find the second moment of area about the base. Using the parallel axis theorem.

	$BD^3/12$	$A\bar{y}^2$	$I = BD^3/12 + A\bar{y}^2$
A	$60 \times 10^3/12 = 5000$ mm^4	$600 \times 45^2 = 1215000$	1220000 mm^4
B	$10 \times 30^3/12 = 22500$ mm^4	$300 \times 25^2 = 187500$	210000 mm^4
C	$40 \times 10^3/12 = 3333$ mm^4	$400 \times 5^2 = 10000$	13333 mm^4
			Total = 1443333 mm^4

The total second moment of area about the bottom is 1443333 mm4

Now move this to the centroid using the parallel axis theorem.

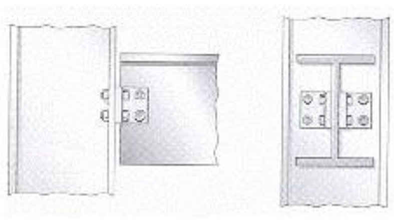
$$I = 1443333 - A\bar{y}^2 = 1443333 - 1300 \times 28.08^2 = 418300 \text{ mm}^4$$

Analysis of Statically Determinate Structures

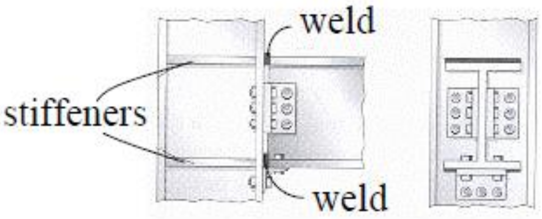
- Idealized Structure
- Principle of Superposition
- Equations of Equilibrium
- Determinacy and Stability
  - Beams
  - Frames
  - Gable Frames
- Application of the Equations of Equilibrium
- Analysis of Simple Diaphragm and Shear Wall Systems
  Problems

Classification of Structures

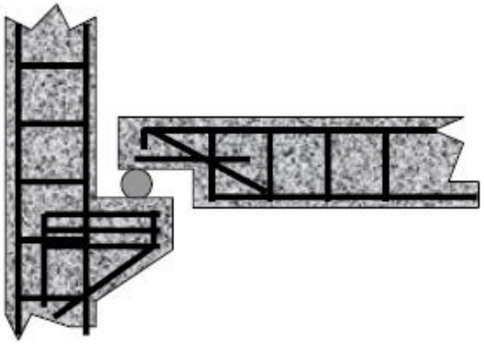
- **Support Connections**



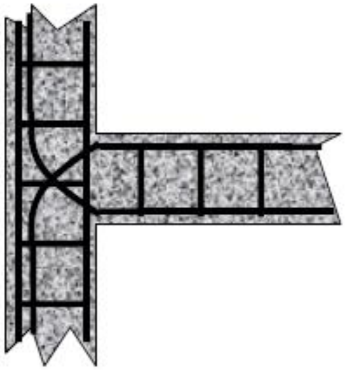
typical "pin-supported" connection (metal)



typical "fixed-supported" connection (metal)

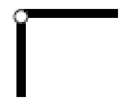


typical "roller-supported" connection (concrete)

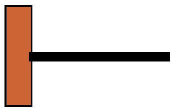


typical "fixed-supported" connection (concrete)

pin support

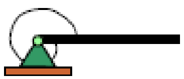


pin-connected joint

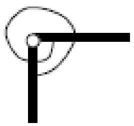


fixed support

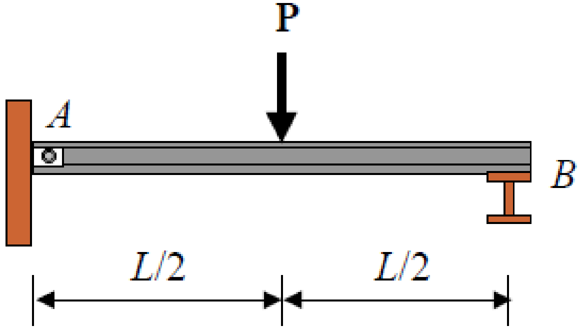
fixed-connected joint



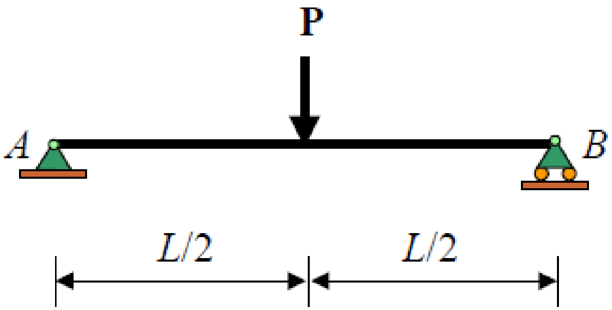
torsional spring support



torsional spring joint

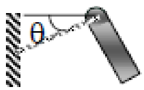
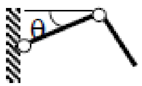
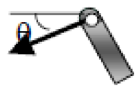
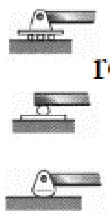
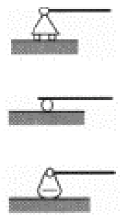
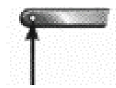
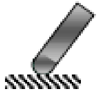
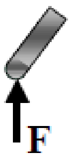
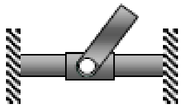
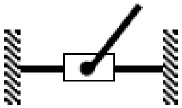
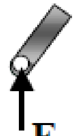


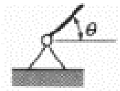
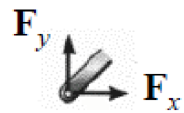
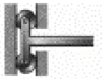
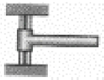
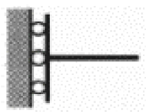
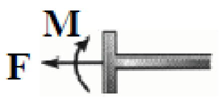
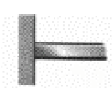
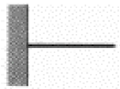
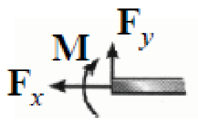
actual beam



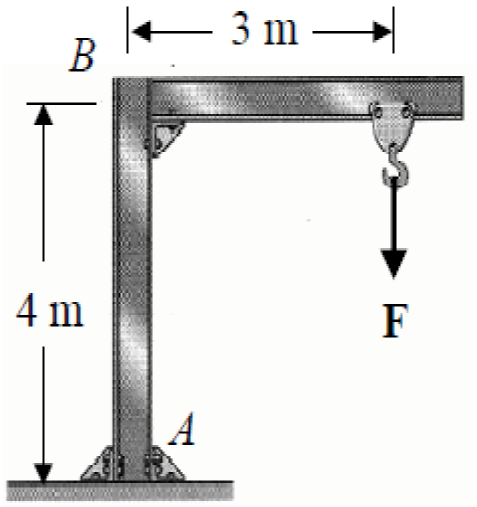
idealized beam

Supports for Coplanar Structures

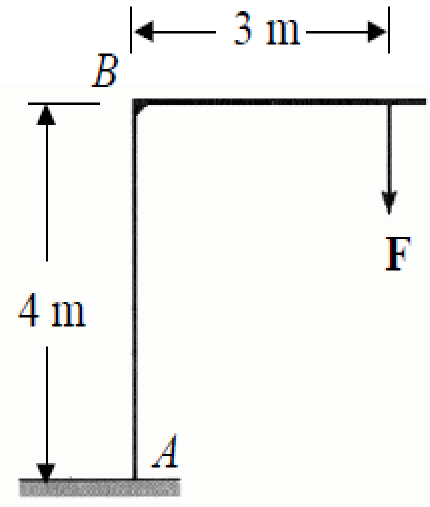
Type of Connection	Idealized Symbol	Reaction	Number of Unknowns
(1)  Light cable			One unknown. The reaction is a force that acts in the direction of the cable or link.
(2)  rollers rockers		 F	One unknown. The reaction is a force that acts perpendicular to the surface at the point of contact.
(3) 		 F	One unknown. The reaction is a force that acts perpendicular to the surface at the point of contact.
(4) 		 F	One unknown. The reaction is a force that acts perpendicular to the surface at the point of contact.

Type of Connection	Idealized Symbol	Reaction	Number of Unknowns
(5)  Smooth pin or hinge			Two unknowns. The reactions are two force components.
(6)  slider  fixed-connected collar			Two unknowns. The reactions are a force and moment.
(7)  fixed support			Three unknowns. The reactions are the moment and the two force components.

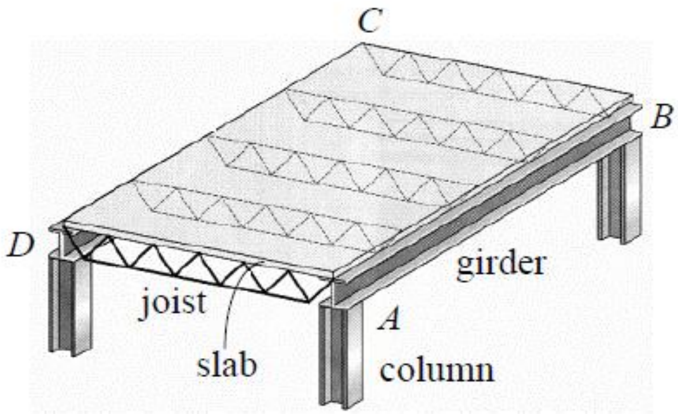
Idealized Structure

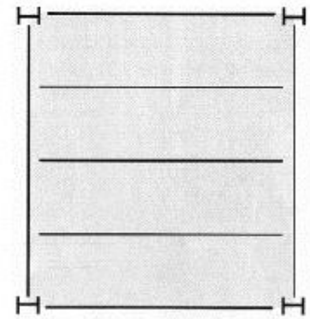


actual structure

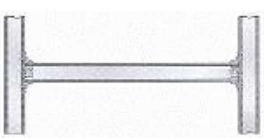


idealized structure





idealized framing plan

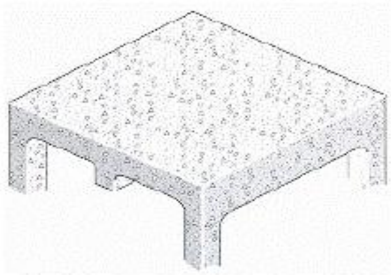


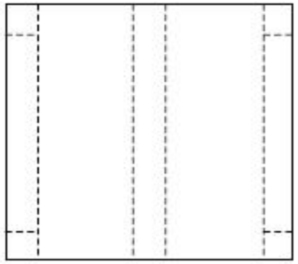
fixed-connected beam

idealize beam

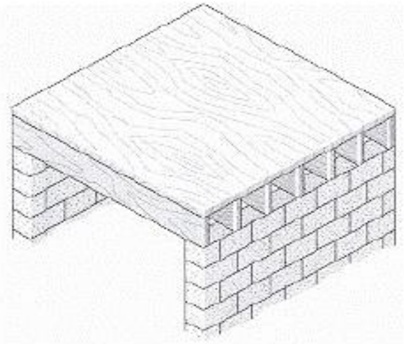
fixed-connected overhanging beam

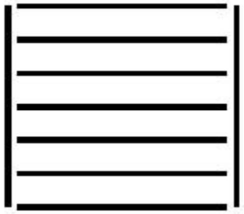
Idealized beam





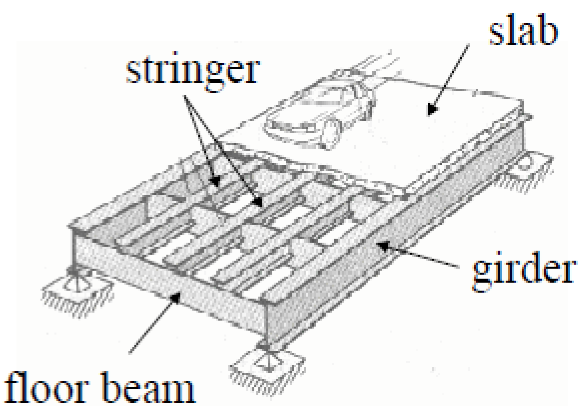
idealized framing plan

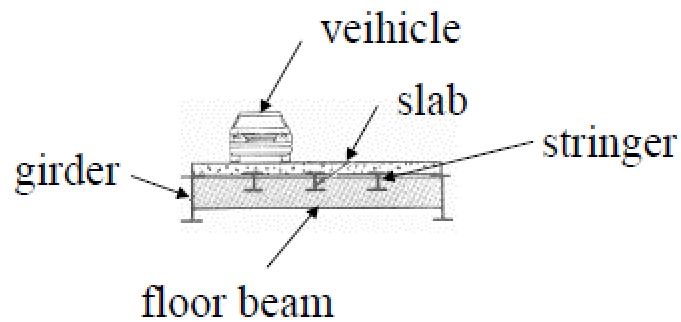


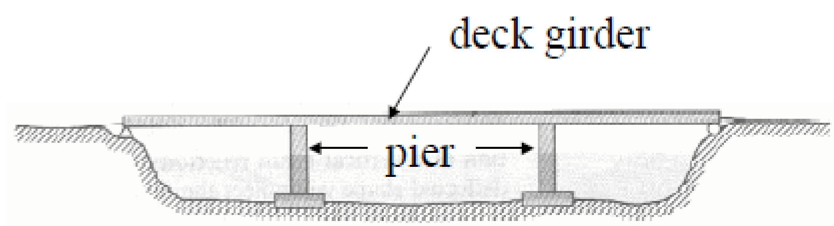


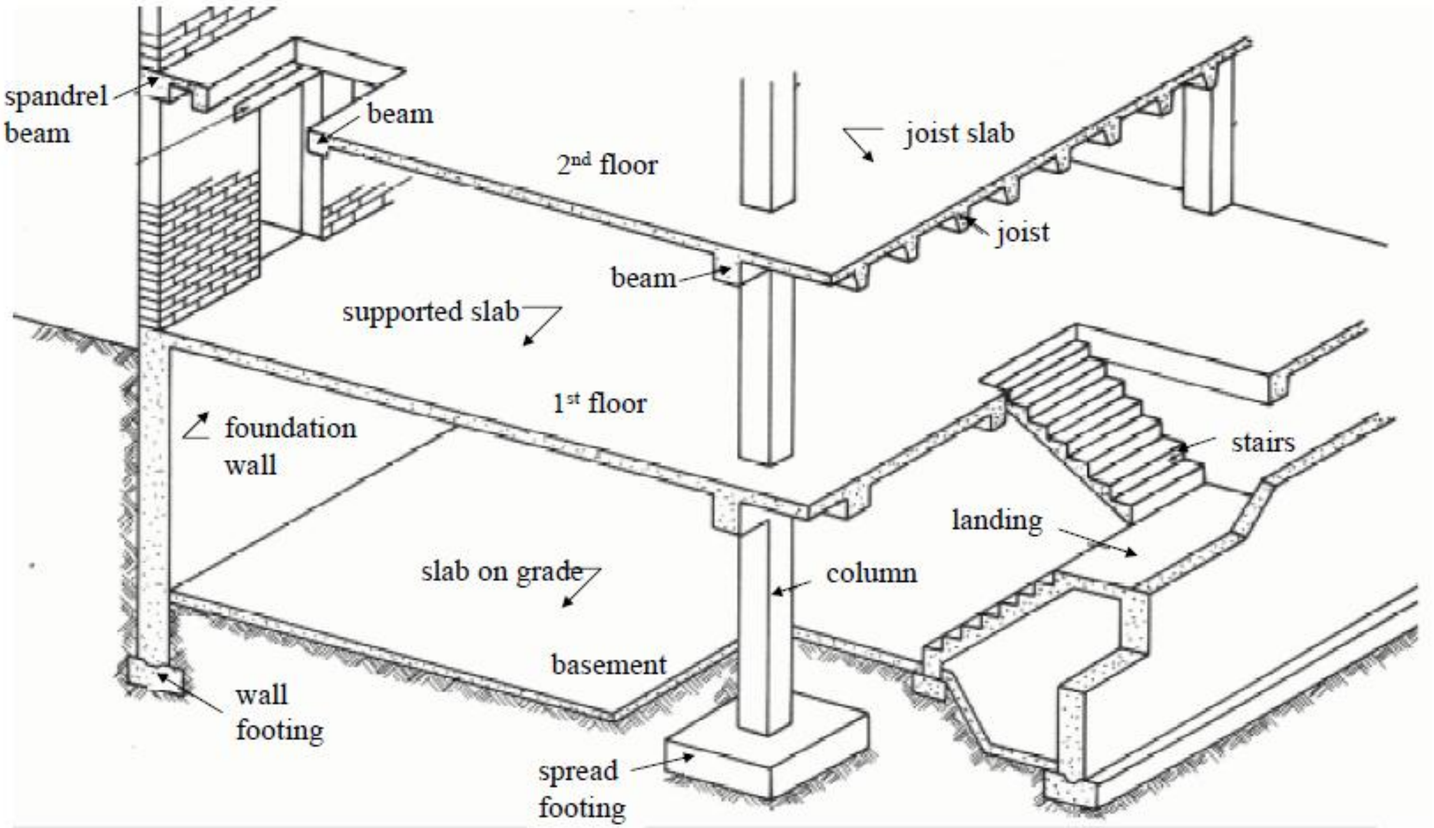
idealized framing plan

Tributary Loadings.

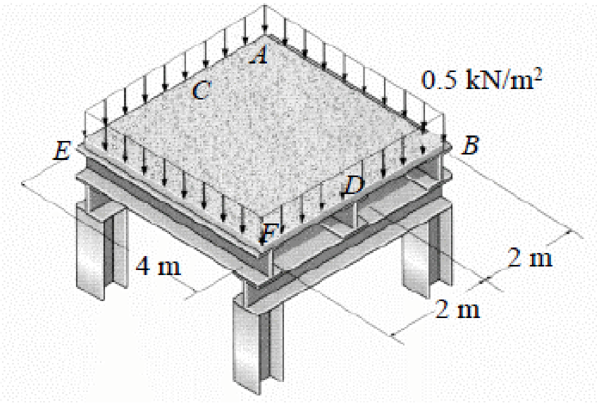


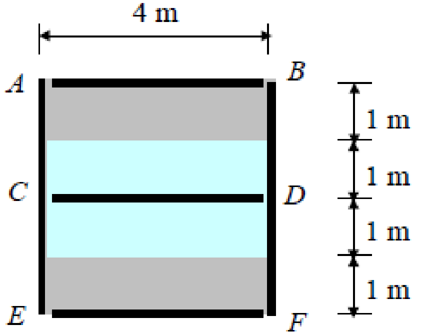




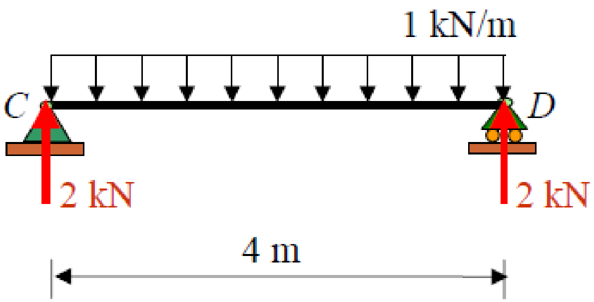

spandrel beam
beam
2nd floor
joist slab
joist
beam
supported slab
1st floor
foundation wall
stairs
landing
slab on grade
column
basement
wall footing
spread footing

One-Way System.

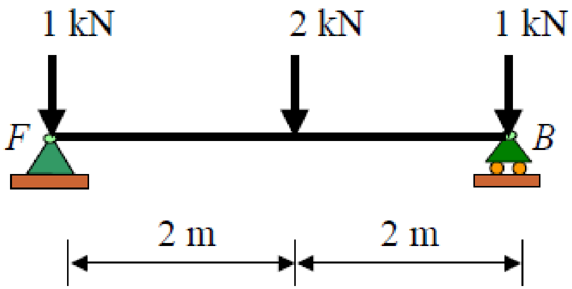




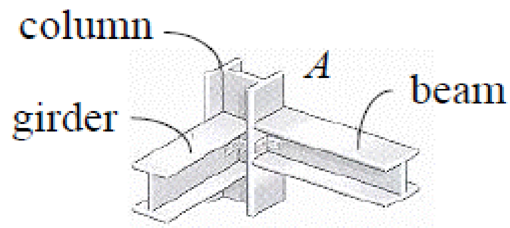
idealized framing plan

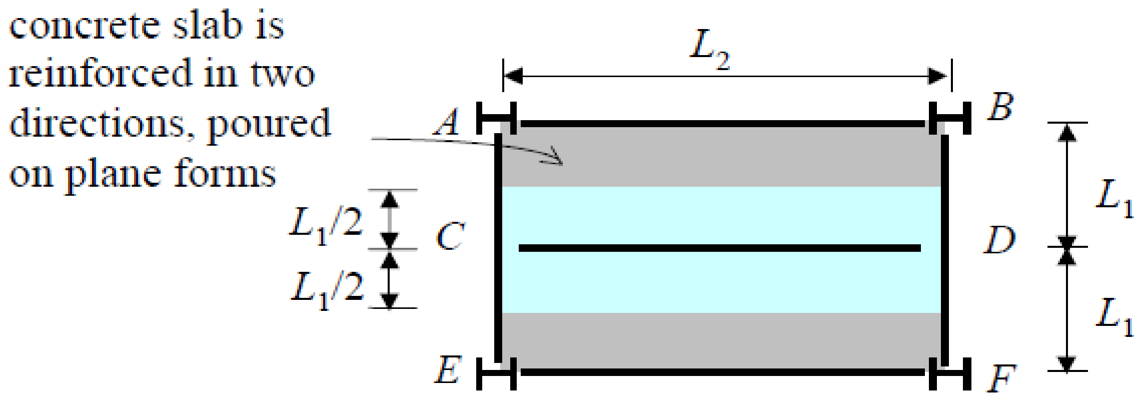


idealized beam



idealized girder

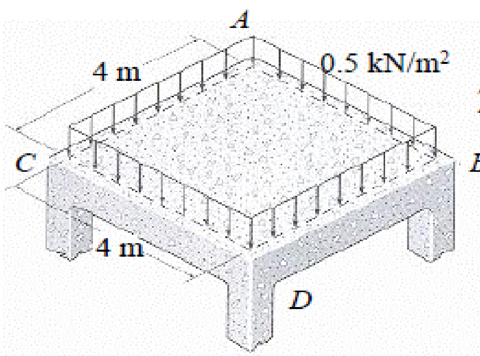


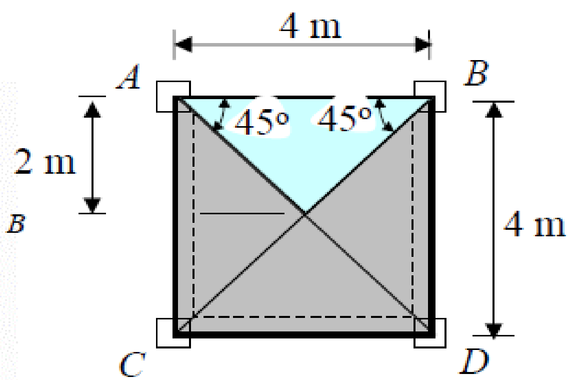


Idealized framing plan for one-way slab action requires $L_2 / L_1 \geq 2$

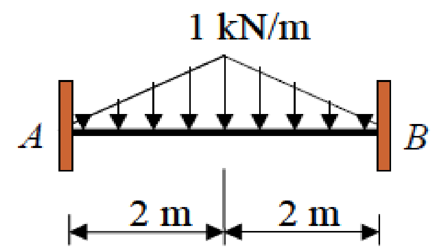
Two-Way System.

$L_2/L_1 = 1$



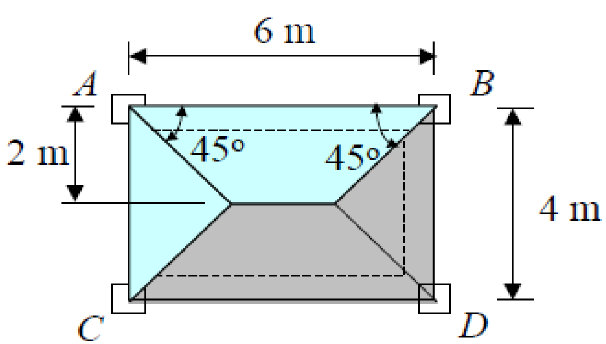


idealized framing plan

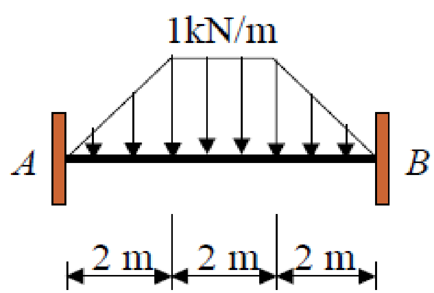


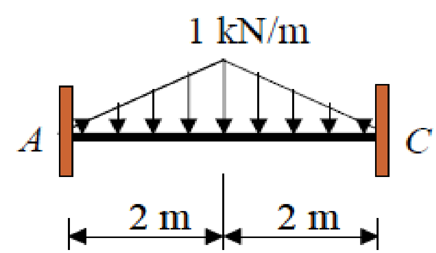
idealized beam, all

$L_2/L_1 = 1.0 < 2$



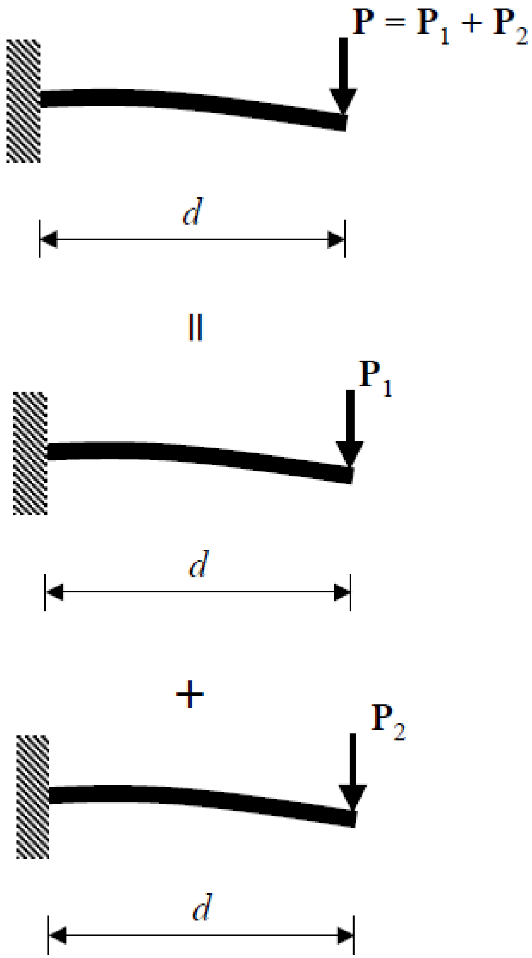
idealized framing plan





idealized beam

Principle of Superposition



Two requirements must be imposed for the principle of superposition to apply :

1. The material must behave in a linear-elastic manner, so that Hooke's law is valid, and therefore the load will be proportional to displacement.

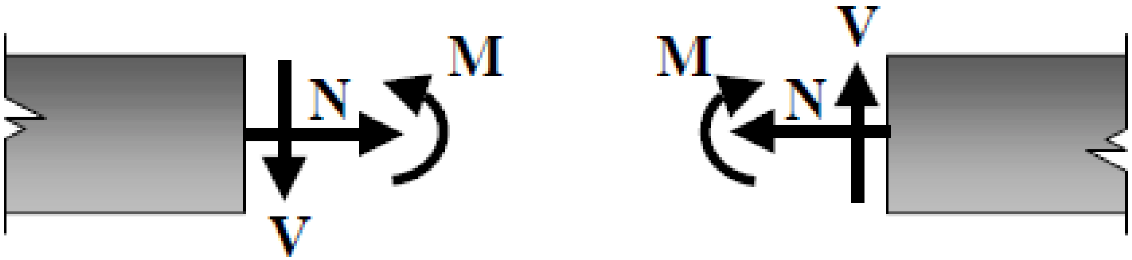
$$\sigma = P/A$$
$$\delta = PL/AE$$

2. The geometry of the structure must not undergo significant change when the loads are applied, i.e., small displacement theory applies. Large displacements will significantly change and orientation of the loads. An example would be a cantilevered thin rod subjected to a force at its end.

Equations of Equilibrium

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma F_z = 0$$

$$\Sigma M_x = 0 \qquad \Sigma M_y = 0 \qquad \Sigma M_z = 0$$



internal loadings

Determinacy and Stability

- **Determinacy**

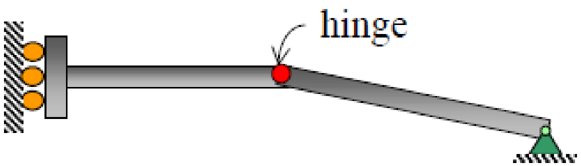
$r = 3n$, statically determinate

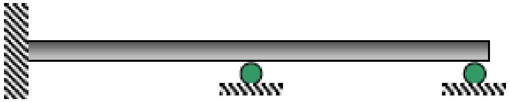
$r > 3n$, statically indeterminate

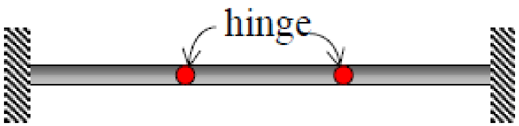
n = the total parts of structure members.
r = the total number of unknown reactive force and moment components

Example

Classify each of the beams shown below as statically determinate or statically indeterminate. If statically indeterminate, report the number of degrees of indeterminacy. The beams are subjected to external loadings that are assumed to be known and can act anywhere on the beams.

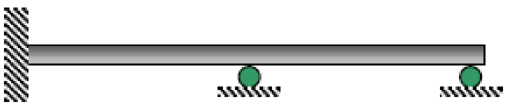


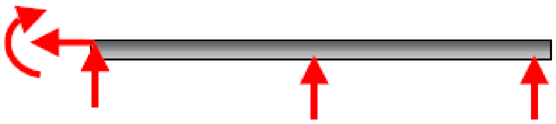




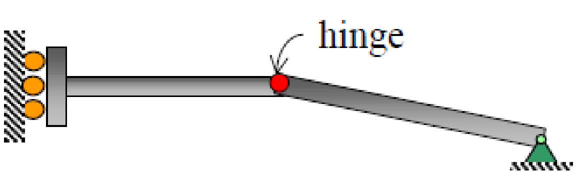
SOLUTION

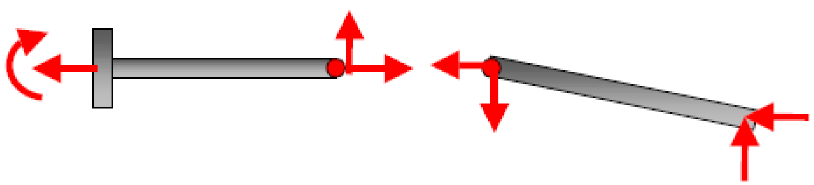
$r = 3$, $n = 1$, $3 = 3(1)$ Statically **determinate**



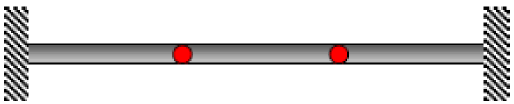


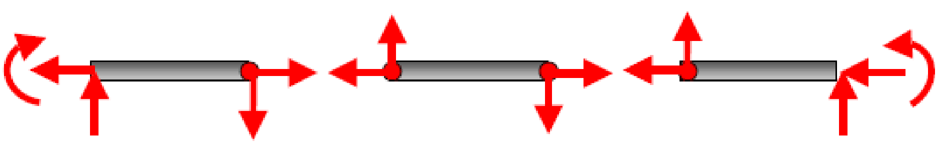
$r = 5$, $n = 1$, $5 - 3(1) = 2$ Statically **indeterminate** to the second degree





$r = 6$, $n = 2$, $6 = 3(2)$ Statically **determinate**

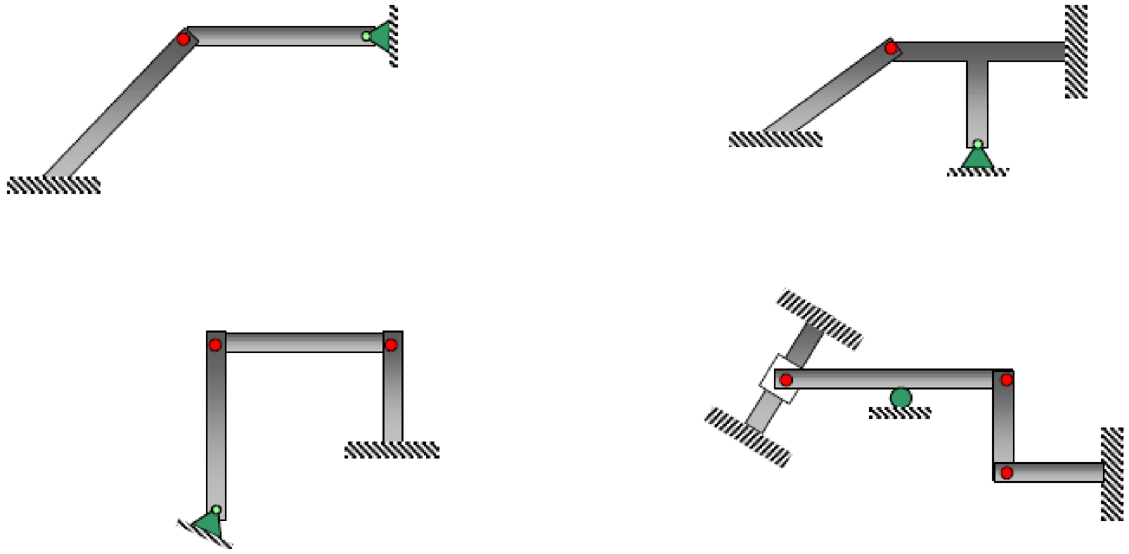




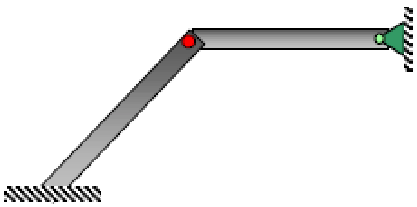
$r = 10$, $n = 3$, $10 - 3(3) = 1$ Statically **indeterminate** to the first degree

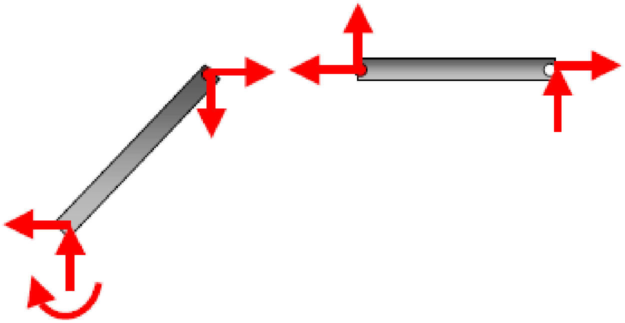
Example

Classify each of the pin-connected structures shown in figure below as statically determinate or statically indeterminate. If statically are subjected to arbitrary external loadings that are assumed to be known and can act anywhere on the structures.



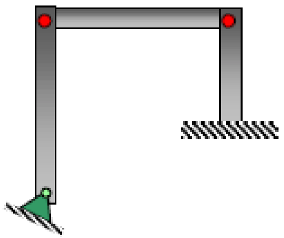
SOLUTION

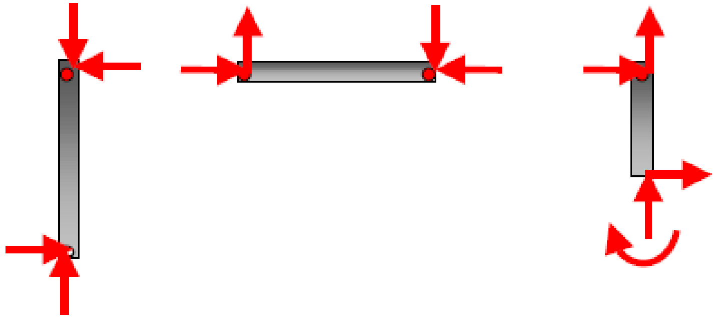




$r = 7$, $n = 2$, $7 - 3(2) = 1$

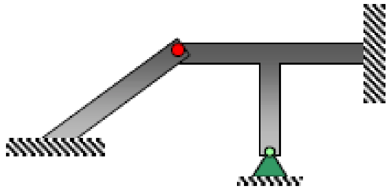
Statically **indeterminate** to the first degree

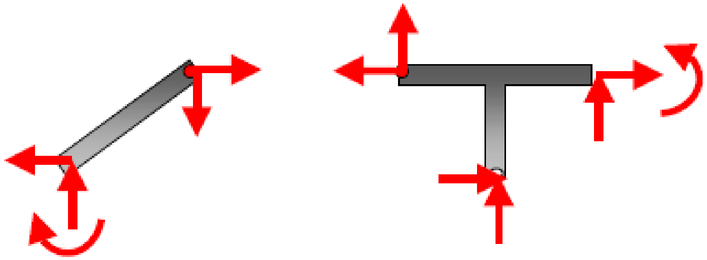




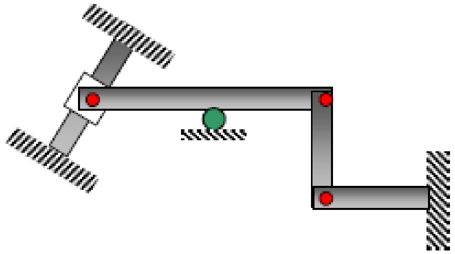
$r = 9$, $n = 3$, $9 = 3(3)$

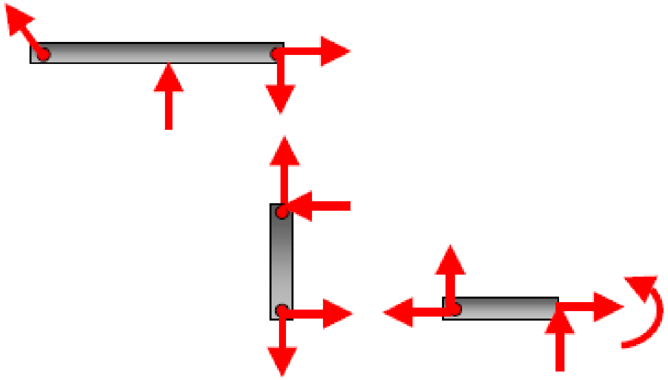
Statically **determinate**





$r = 10$, $n = 2$, $10 - 6 = 4$ Statically **indeterminate** to the fourth degree

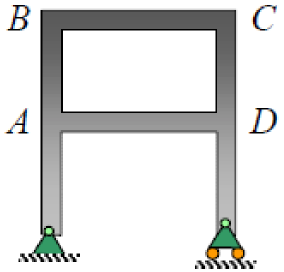


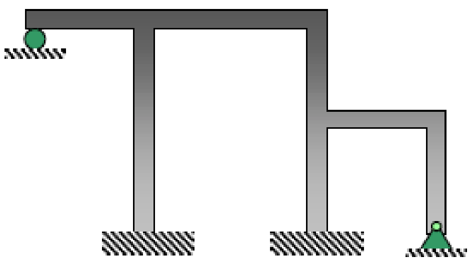


$r = 9$, $n = 3$, $9 = 3(3)$ Statically **determinate**

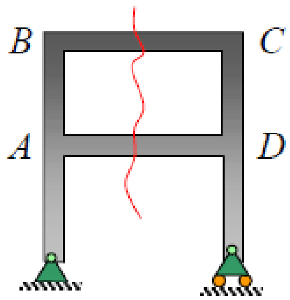
Example

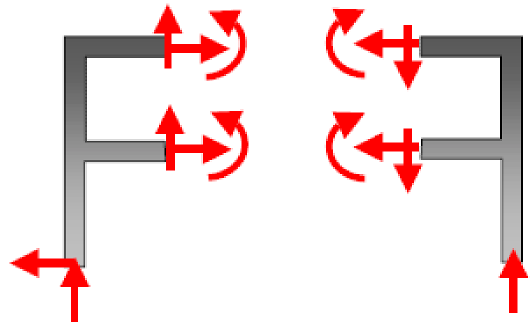
Classify each of the frames shown in figure below as statically determinate or statically indeterminate. If statically indeterminate, report the number of degrees of indeterminacy. The frames are subjected to external loadings that are assumed to be known and can act anywhere on the frames.





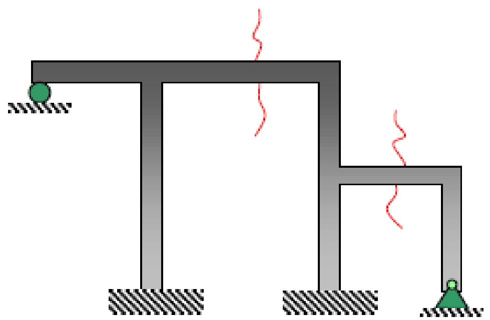
SOLUTION

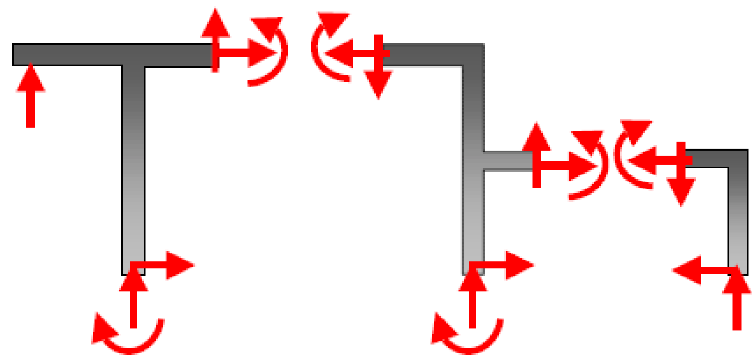




$r = 9$, $n = 2$, $9 - 6 = 3$

Statically **indeterminate** to the third degree





$r = 15$, $n = 3$, $15 - 9 = 6$

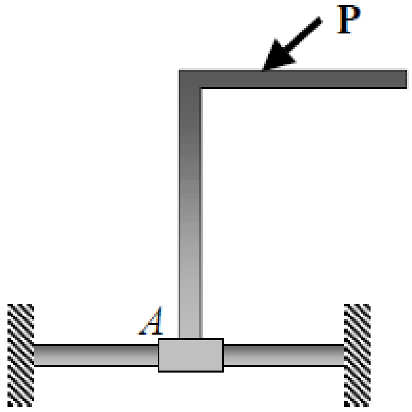
Statically **indeterminate** to the sixth degree

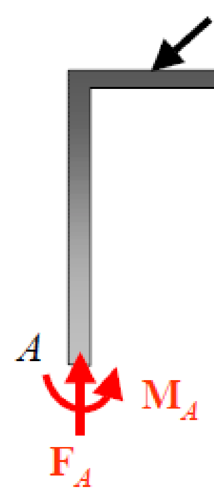
• **Stability**

$r < 3n$, unstable

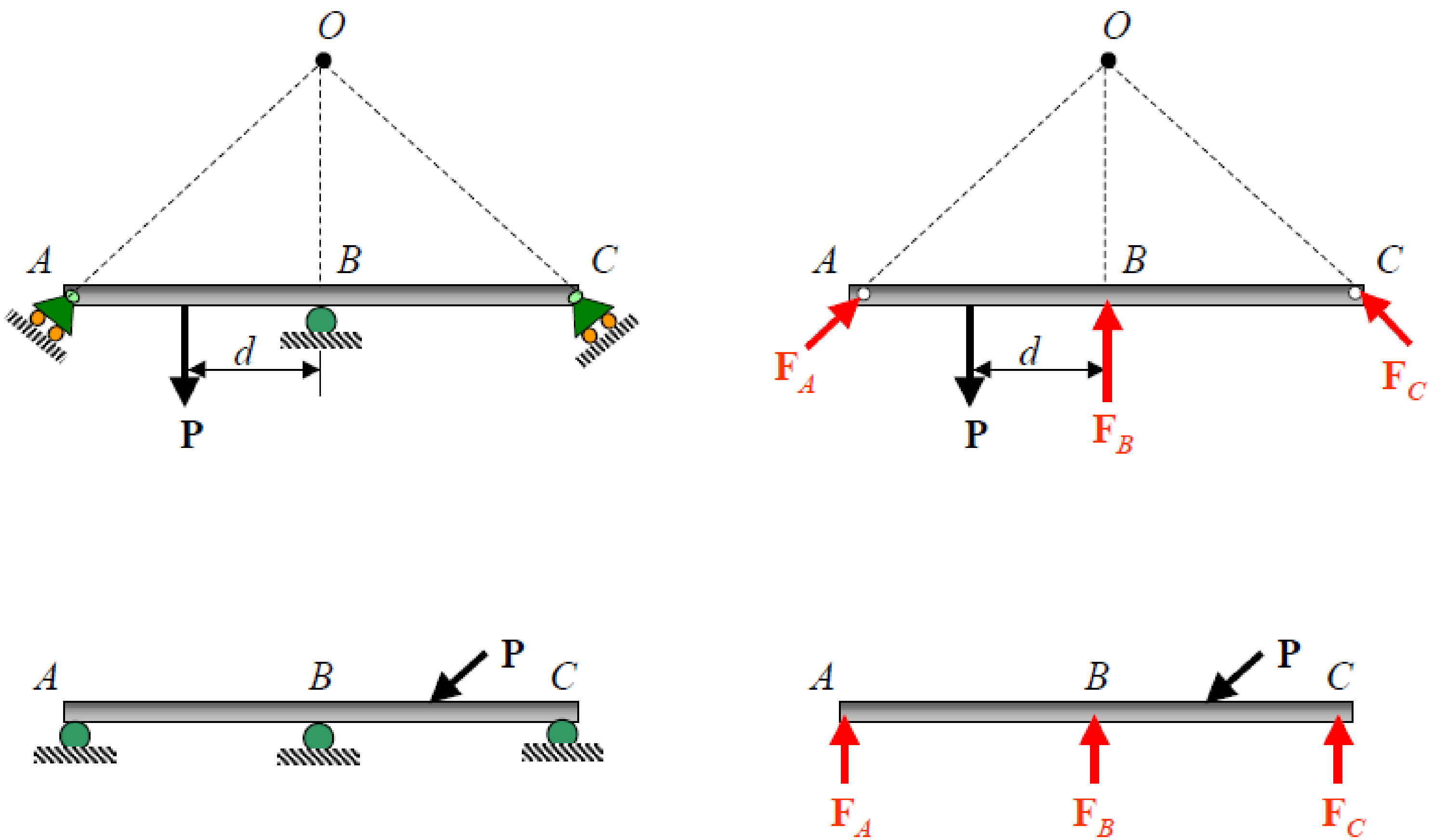
$r \geqslant 3n$, unstable if member reactions are concurrent or parallel or some of the components form a collapsible mechanism

<u>Partial Constrains</u>



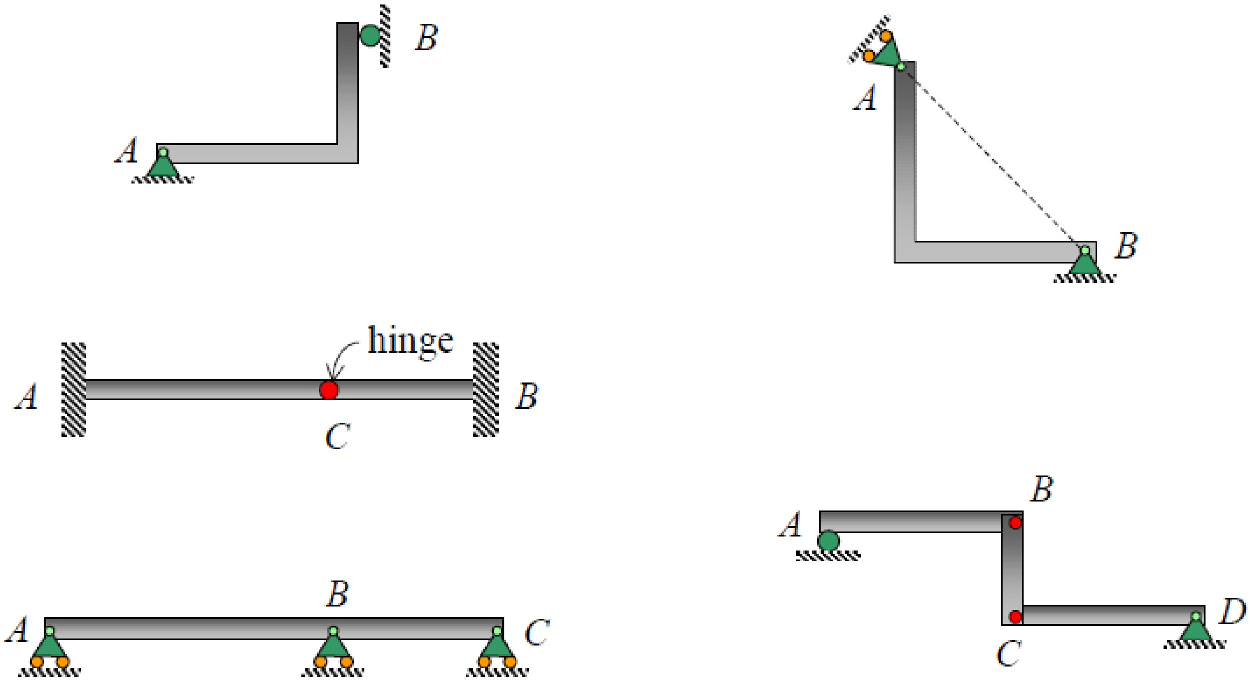


Improper Constraints

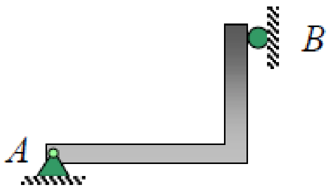


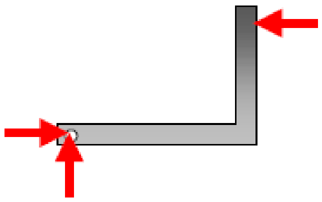
Example

Classify each of the structures in the figure below as stable or unstable. The structures are subjected to arbitrary external loads that are assumed to be known.

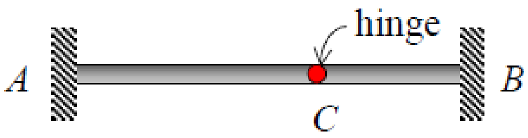


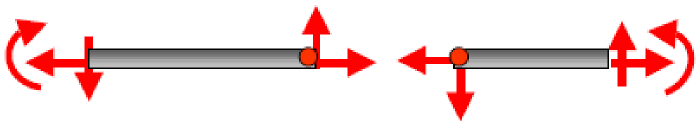
SOLUTION



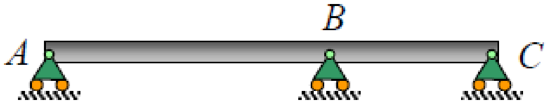


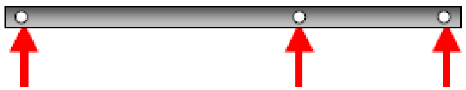
The member is ***stable*** since the reactions are non-concurrent and nonparallel. It is also statically **determinate**.



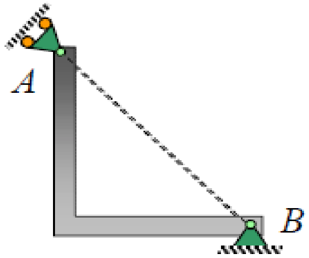


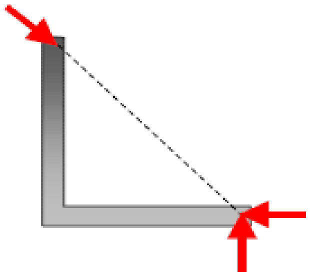
The compound beam is ***stable***. It is also **indeterminate** to the second degree.



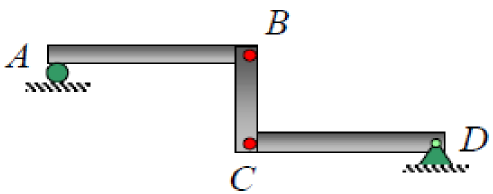


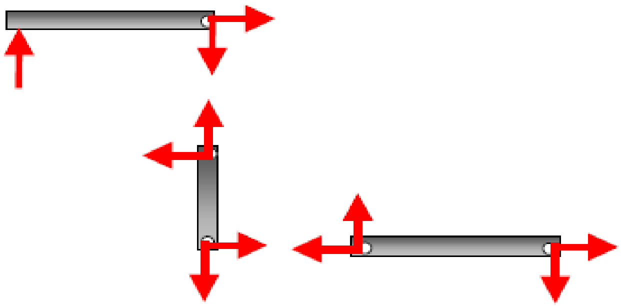
The compound beam is ***unstable*** since the three reactions are all parallel.





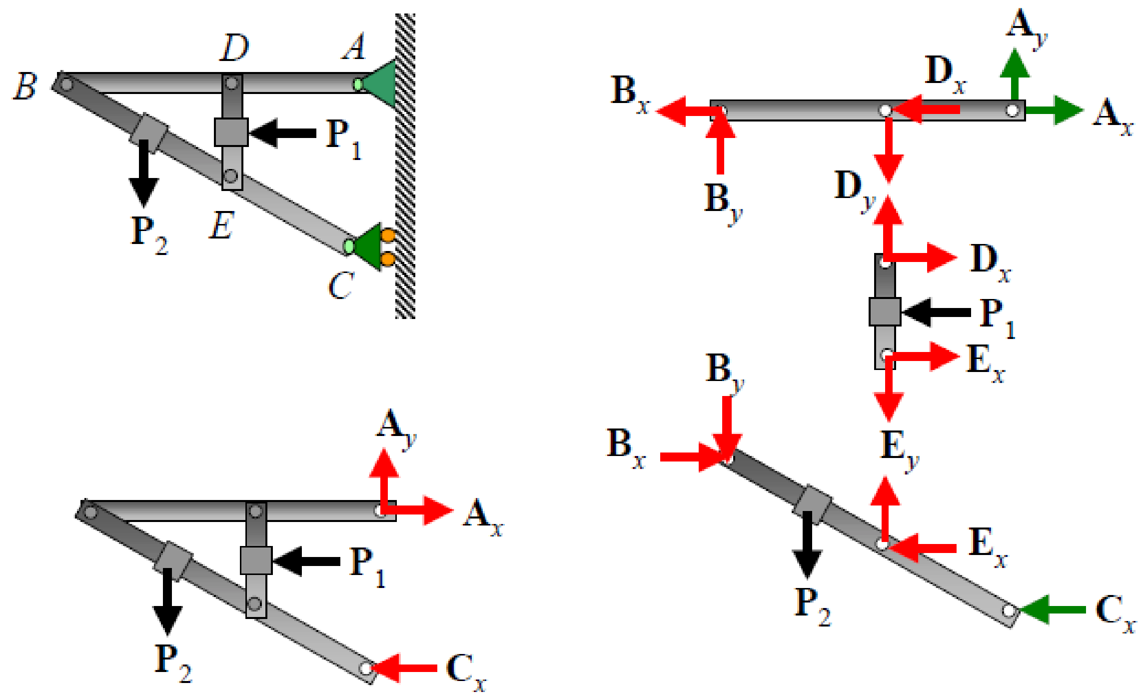
The member is ***unstable*** since the three reactions are concurrent at B.



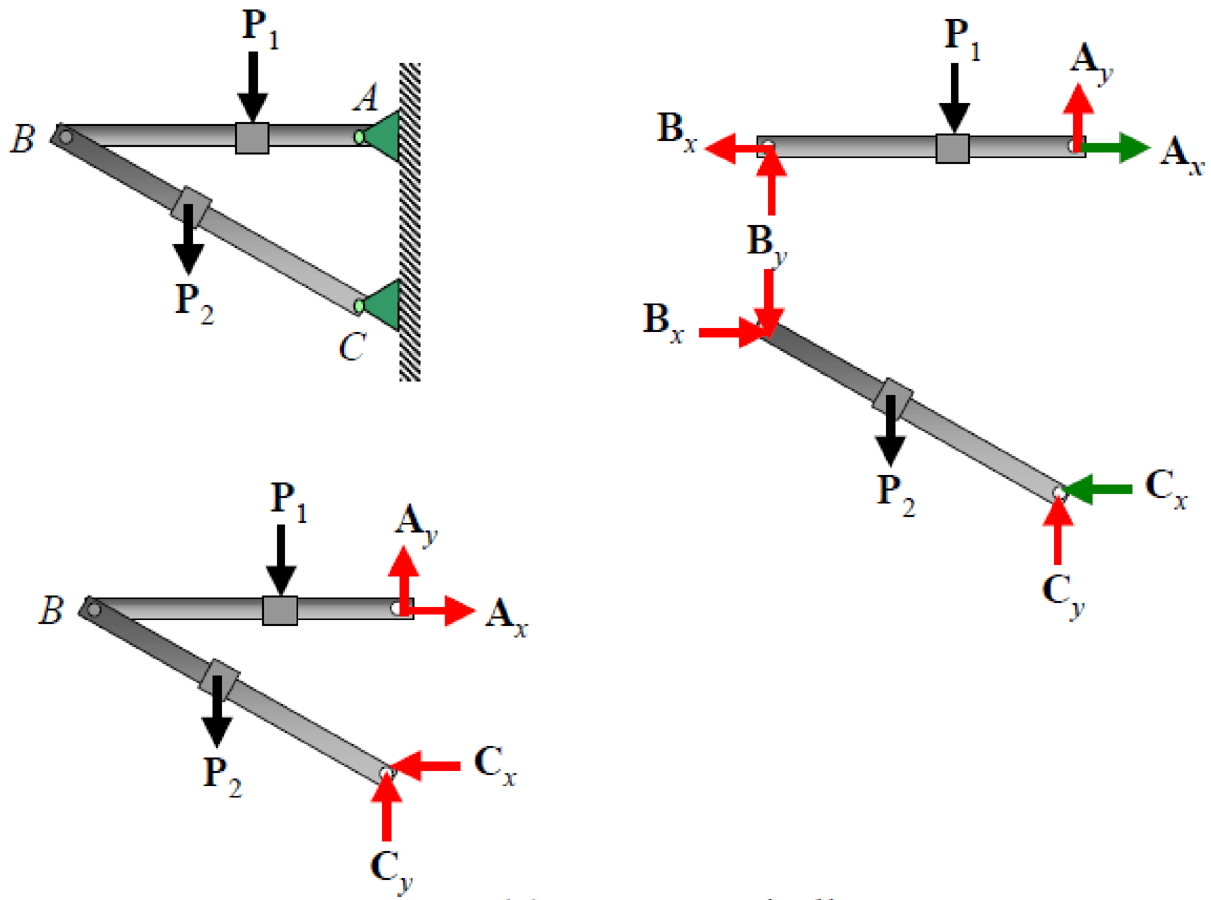


The structure is ***unstable*** since $r = 7$, $n = 3$, so that, $r < 3n$, $7 < 9$. Also, this can be seen by inspection, since AB can move horizontally without restraint.

Application of the Equations of Equilibrium



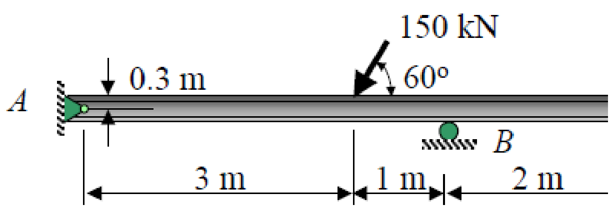
$r = 9$, $n = 3$, $9 = 3(3)$; statically **determinate**



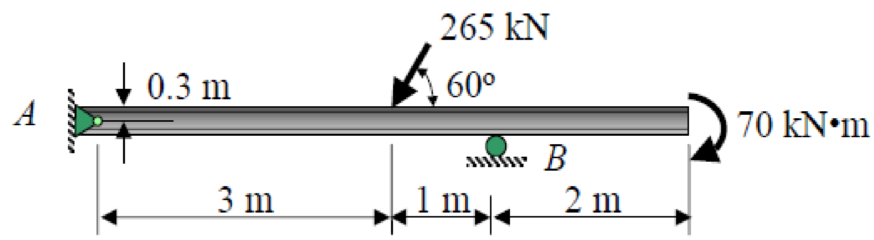
$r = 6$, $n = 2$, $6 = 3(2)$; statically **determinate**

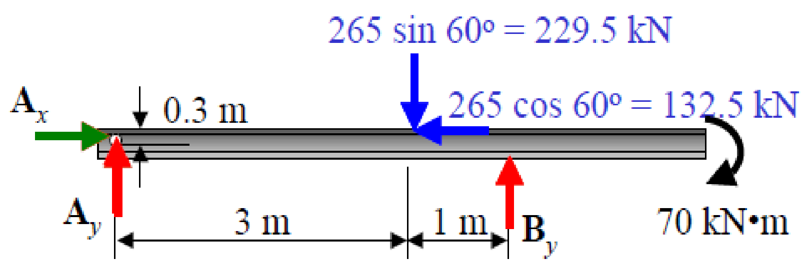
Example

Determine the reactions on the beam shown.



SOLUTION





$\xrightarrow{+} \Sigma F_x = 0$: $\quad A_x - 132.5 = 0$: $A_x = 132.5$ kN , →

$\circlearrowright + \ \Sigma M_A = 0$: $\quad B_y(4) - (229.5)(3) + (132.5)(0.3) - 70 = 0$

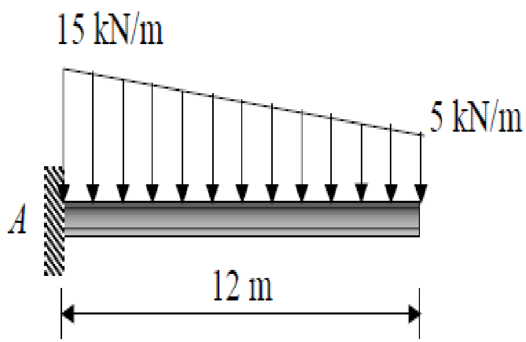
$B_y = 179.69$ kN, ↑

$+\uparrow \Sigma F_y = 0$: $\quad A_y - 229.5 + 179.69 = 0$

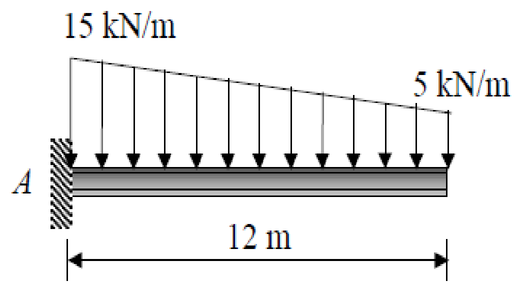
$A_y = 49.81$ kN , ↑

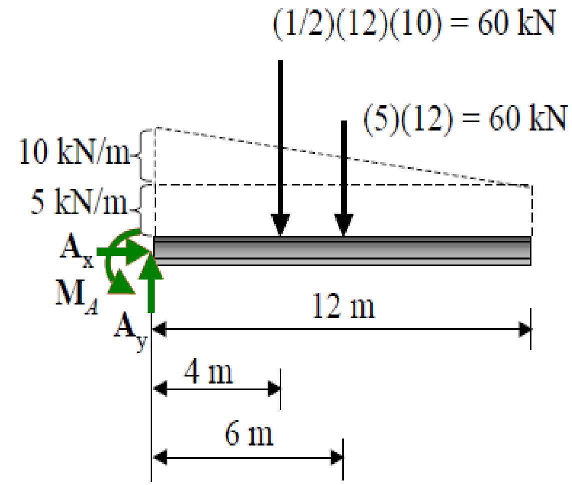
Example

Determine the reactions on the beam shown.



SOLUTION





$\xrightarrow{+}$ $\Sigma F_x = 0$: $\quad A_x = 0$

$+\uparrow$ $\Sigma F_y = 0$: $\quad A_y - 60 - 60 = 0$

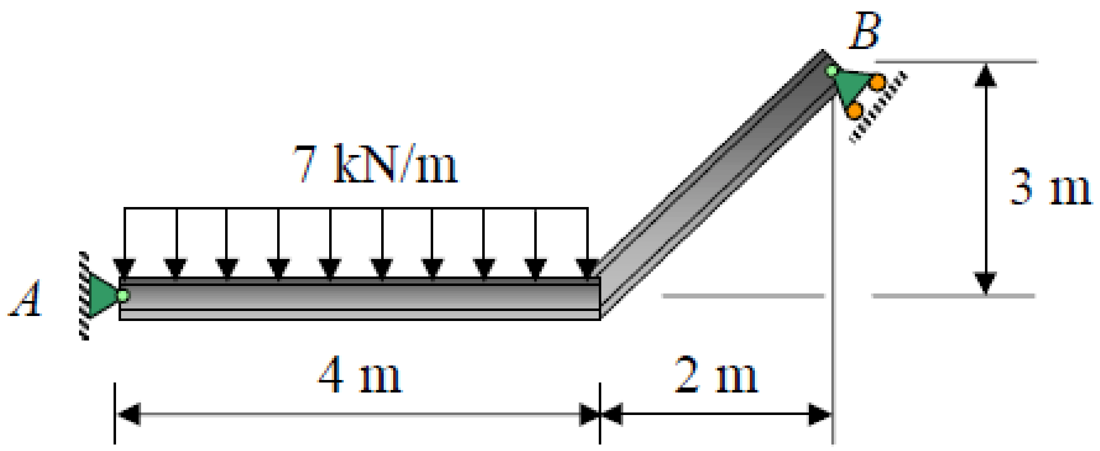
$A_y = 120$ kN , $\uparrow$

$+\circlearrowleft$ $\Sigma M_A = 0$: $\quad M_A - (60)(4) - (60)(6) = 0$

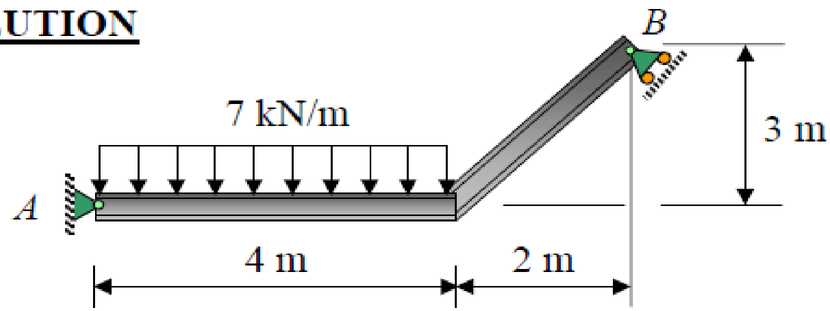
$M_A = 600$ kN•m

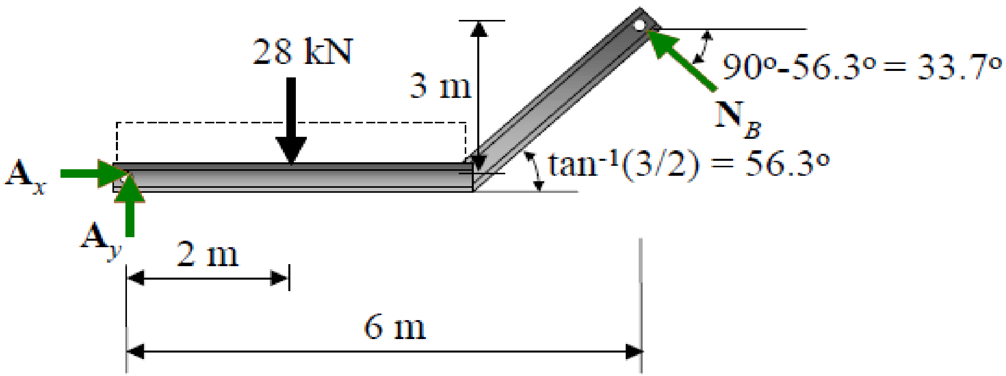
Example

Determine the reactions on the beam shown. Assume *A* is a pin and the support at *B* is a roller (smooth surface).



SOLUTION





$\circlearrowright^{+} \; \Sigma M_A = 0: \quad -28(2) + N_B \sin 33.7(6) + N_B \cos 33.7(3) = 0$

$N_B = 9.61$ kN

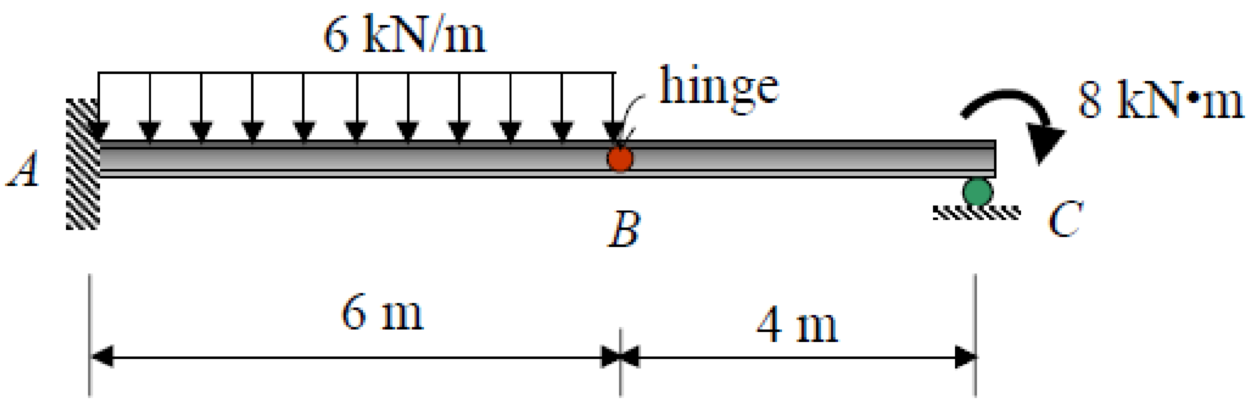
$\xrightarrow{+} \; \Sigma F_x = 0: \quad A_x - N_B \cos 33.7 = 0; \; A_x = 9.61 \cos 33.7 = 8$ kN , $\rightarrow$

$+\uparrow \; \Sigma F_y = 0: \quad A_y - 28 + 9.61 \cos 33.7 = 0$

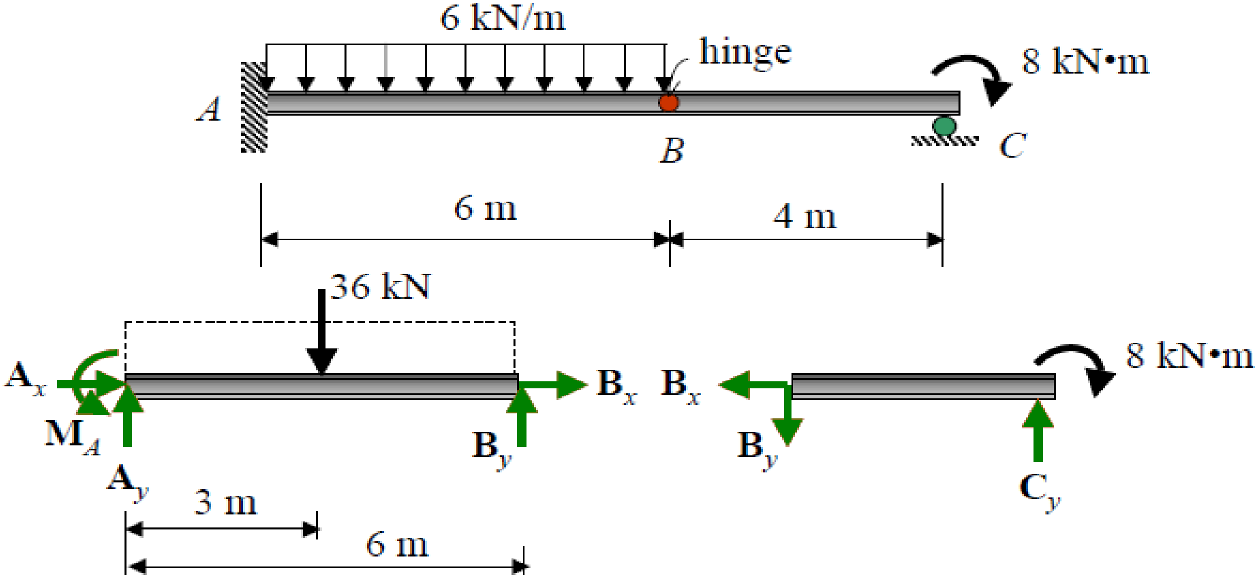
$A_y = 22.67$ kN , $\uparrow$

Example

The compound beam in figure below is fixed at *A*. Determine the reactions at *A*, *B*, and *C*. Assume that the connection at pin and *C* is a rooler.



SOLUTION



Member *BC*

$\circlearrowright + \; \Sigma M_B = 0: \quad C_y(4) - 8 = 0$

$C_y = 2 \text{ kN}, \uparrow$

$\xrightarrow{+} \; \Sigma F_x = 0: \quad B_x = 0$

$+\uparrow \; \Sigma F_y = 0: \quad C_y - B_y = 0;$

$B_y = C_y = 2 \text{ kN}, \uparrow$

Member *AB*

$\circlearrowright + \; \Sigma M_A = 0: \quad M_A - 36(3) + 2(6) = 0$

$M_A = 96 \text{ kN•m}$

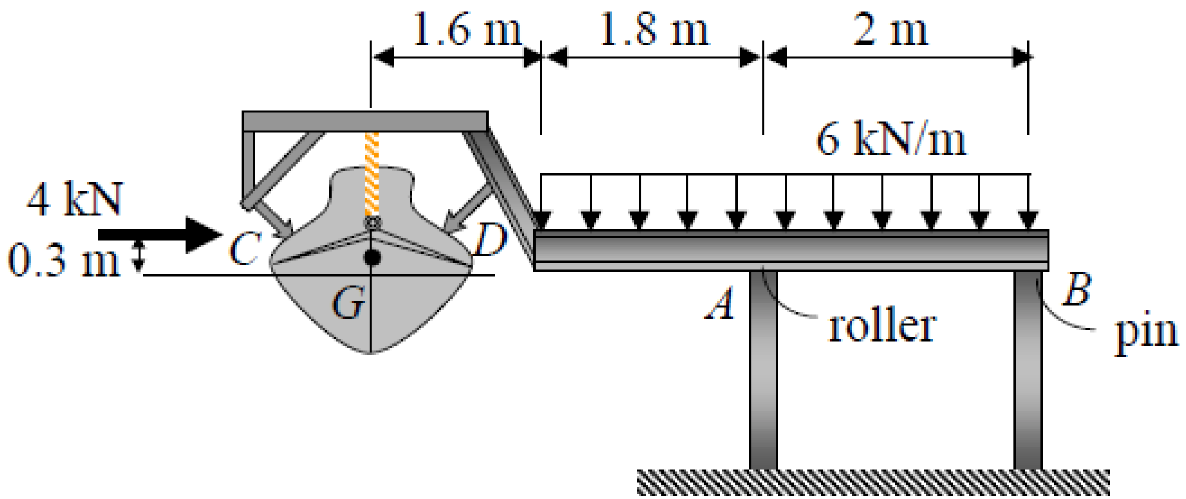
$\xrightarrow{+} \; \Sigma F_x = 0: \quad A_x - B = 0 \; ; \; A_x = B_x = 0$

$+\uparrow \; \Sigma F_y = 0: \quad A_y - 36 + 2 = 0$

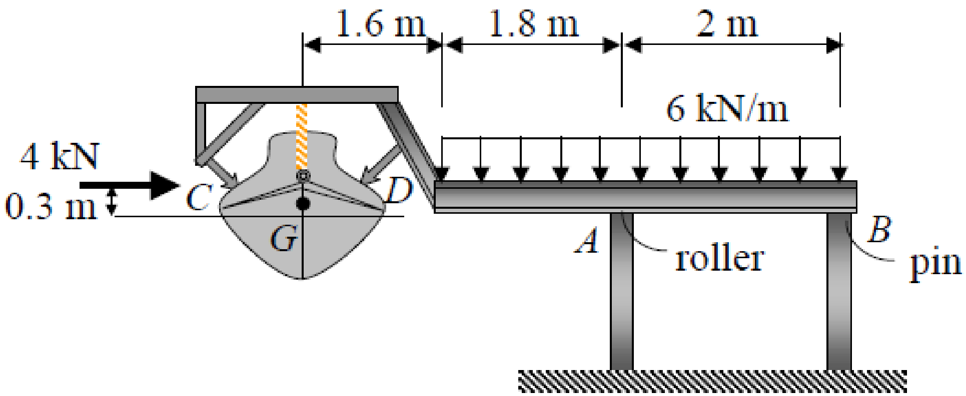
$A_y = 34 \text{ kN}, \uparrow$

Example

The side girder shown in the photo supports the boat and deck. An idealized model of this girder is shown in the figure below, where it can be assumed *A* is a roller and *B* is a pin. Using a local code the anticipated deck loading transmitted to the girder is 6 kN/m. Wind exerts a *resultant* horizontal force of 4 kN as shown, and the mass of the boat that is supported by the girder is 23 Mg. The boat's mass center is at *G*. Determine the reactions at the supports.

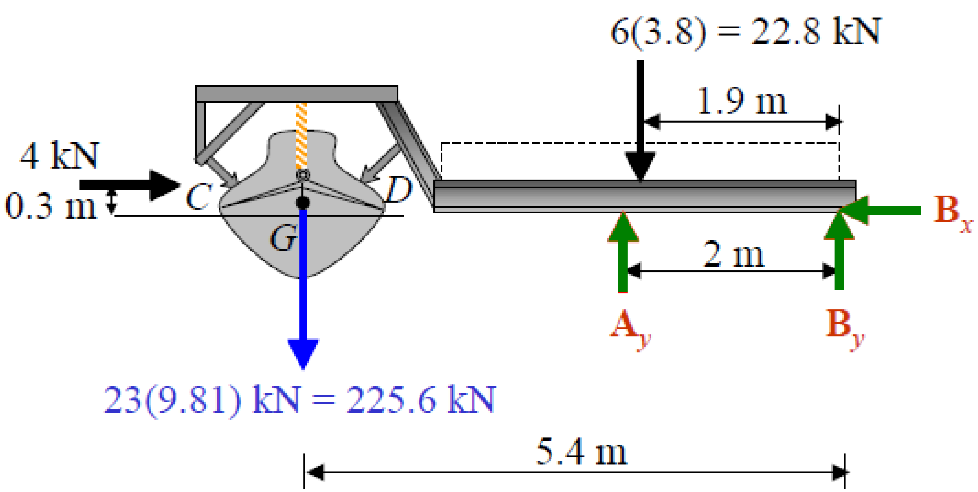


SOLUTION



$\xrightarrow{+} \Sigma F_x = 0:$

$$4 - B_x = 0$$
$$B_x = 4 \text{ kN}, \leftarrow$$



$\overset{+}{\circlearrowleft} \Sigma M_B = 0:$

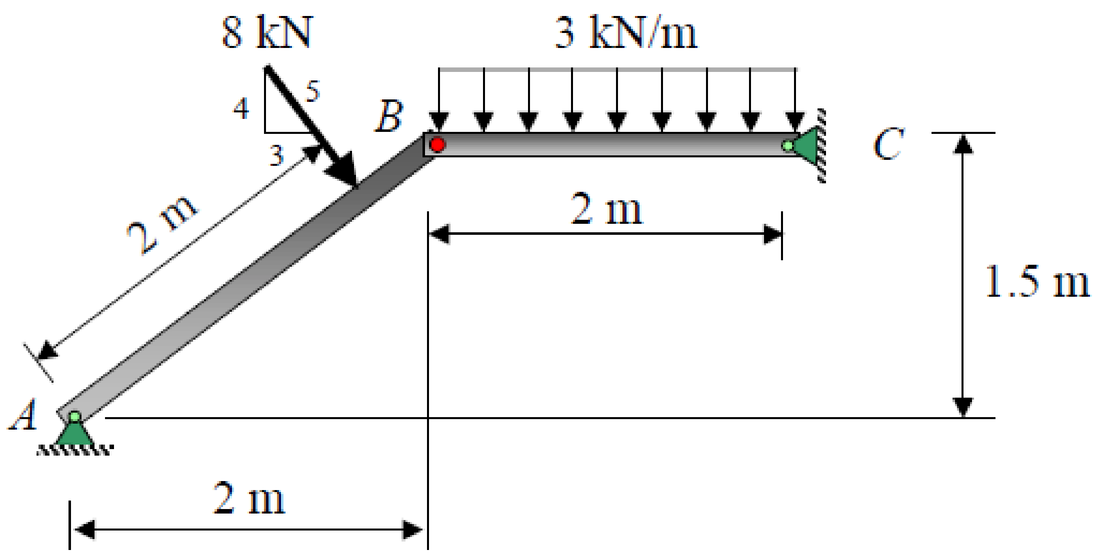
$$22.8(1.9) - A_y(2) + 225.6(5.4) - 4(0.3) = 0$$
$$A_y = 630.2 \text{ kN}, \uparrow$$

$+\uparrow \Sigma F_y = 0:$

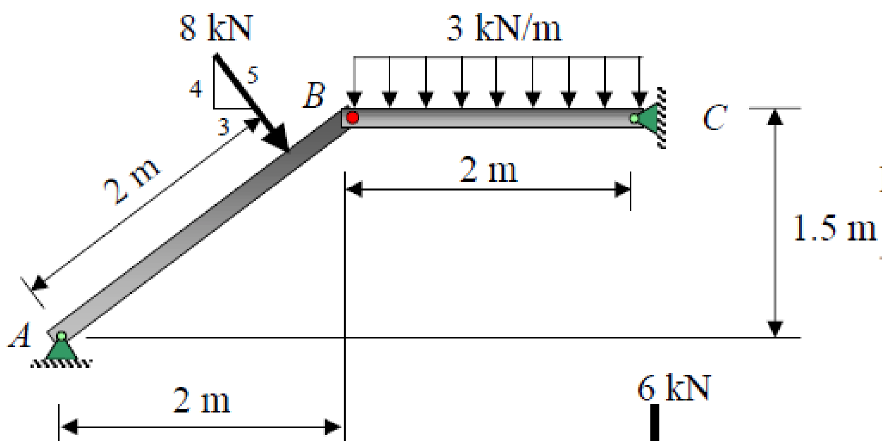
$$-225.6 + 630.2 - 22.8 + B_y = 0$$
$$B_y = 382 \text{ kN}, \uparrow$$

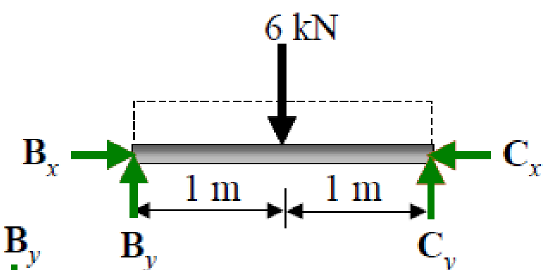
Example

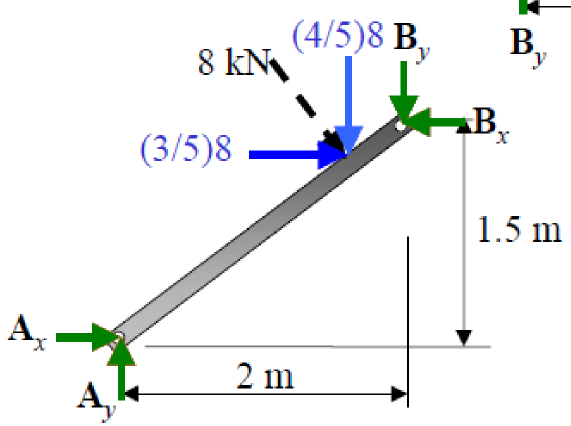
Determine the horizontal and vertical components of reaction at the pins *A*, *B*, and *C* of the two-member frame shown in the figure below.



SOLUTION







Member *BC*

$\circlearrowright + \ \Sigma M_C = 0$:

$-B_y(2) + 6(1) = 0$

$B_y = 3$ kN , ↑

Member *AB*

$\circlearrowright + \ \Sigma M_A = 0$:

$-8(2) - 3(2) + B_x(1.5) = 0$

$B_x = 14.7$ kN , ←

$\xrightarrow{+} \Sigma F_x = 0$:

$A_x + (3/5)8 - 14.7 = 0$

$A_x = 9.87$ kN , →

$+\uparrow \Sigma F_y = 0$:

$A_y - (4/5)8 - 3 = 0$

$A_y = 9.4$ kN , ↑

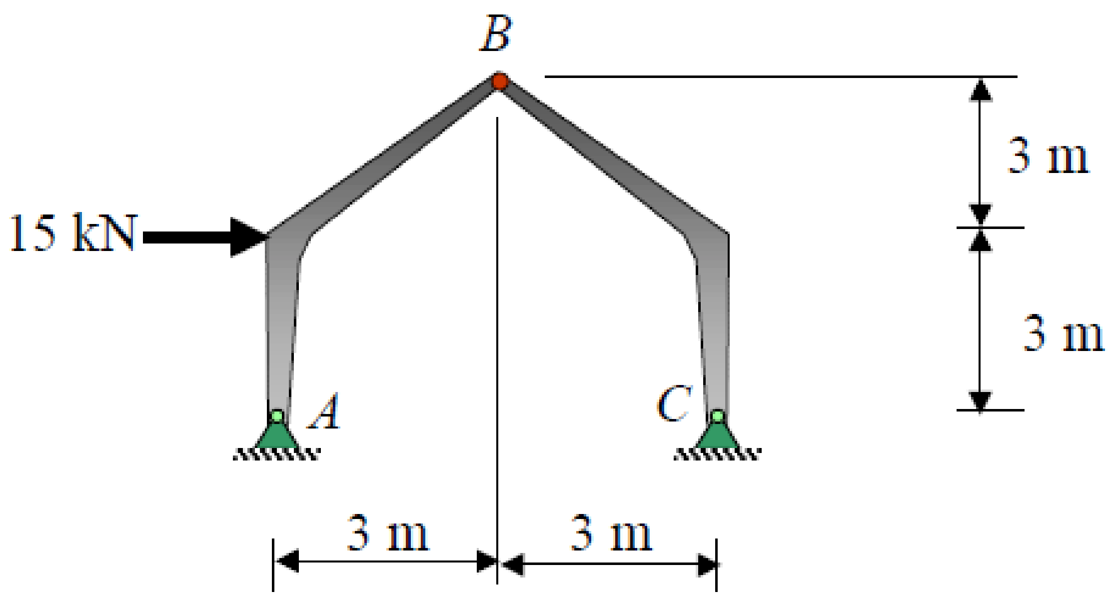
Member *BC*

$\xrightarrow{+} \Sigma F_x = 0$: $C_x - B_x = 0$; $C_x = B_x = 14.7$ kN , ←

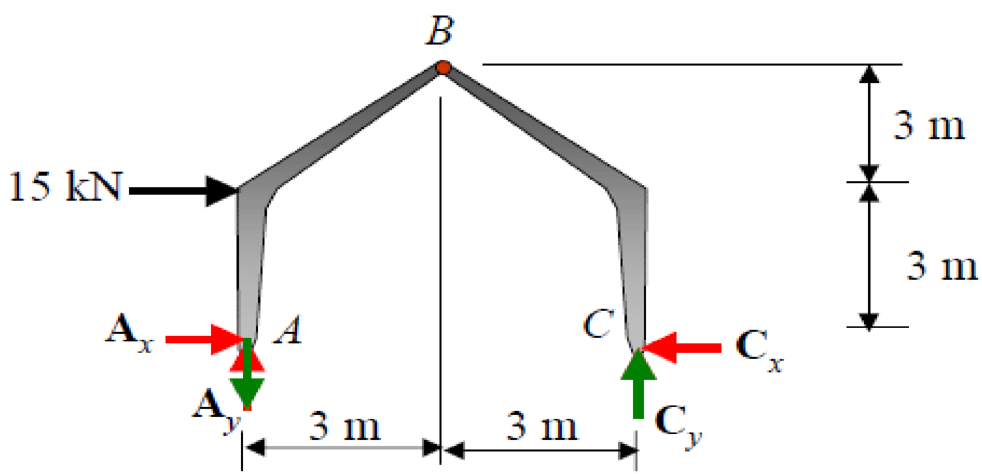
$+\uparrow \Sigma F_y = 0$: $3 - 6 + C_x = 0$; $C_y = 3$ kN , ↑

Example

From the figure below, determine the horizontal and vertical components of reaction at the pin connections *A*, *B*, and *C* of the supporting gable arch.



SOLUTION



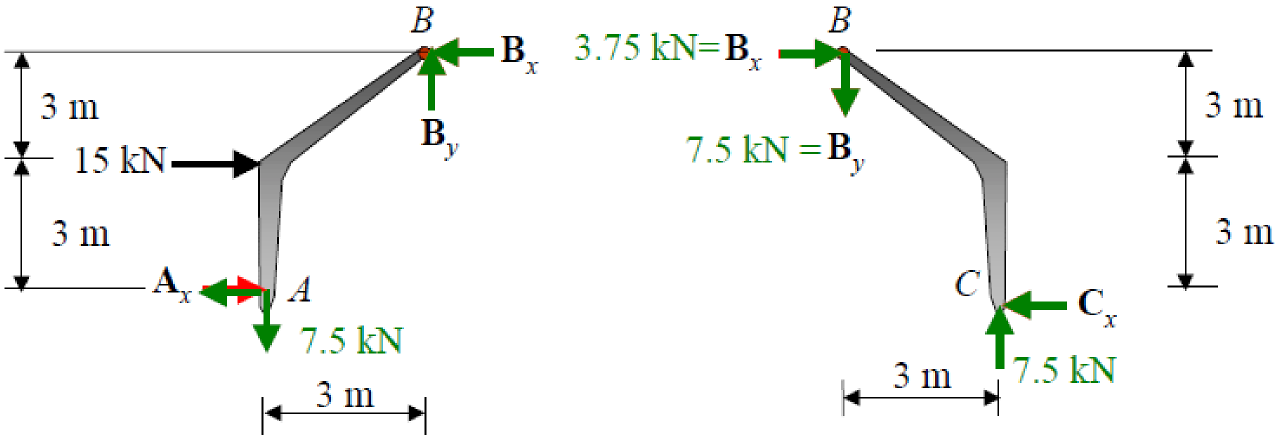
Entire Frame

$\circlearrowright + \; \Sigma M_A = 0: \qquad C_y(6) - 15(3) = 0$

$C_y = 7.5$ kN , ↑

$+\uparrow \; \Sigma F_y = 0: \qquad A_y + 7.5 = 0$

$A_y = -7.5$ kN , ↓



Member *AB*

$$\circlearrowright + \Sigma M_B = 0: \quad 15(3) + A_x(6) + 7.5(3) = 0$$

$$A_x = -11.25 \text{ kN}, \leftarrow$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad -11.25 + 15 - B_x = 0$$

$$B_x = 3.75 \text{ kN}, \leftarrow$$

$$+\uparrow \Sigma F_y = 0: \quad -7.5 + By = 0$$

$$B_y = 7.5 \text{ kN}$$

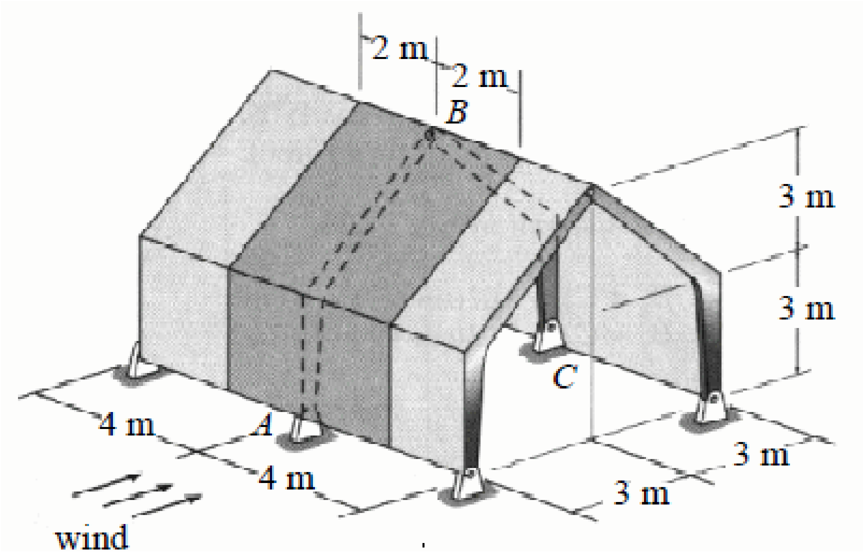
Member *BC*

$$\xrightarrow{+} \Sigma F_x = 0: \quad 3.75 - C_x = 0$$

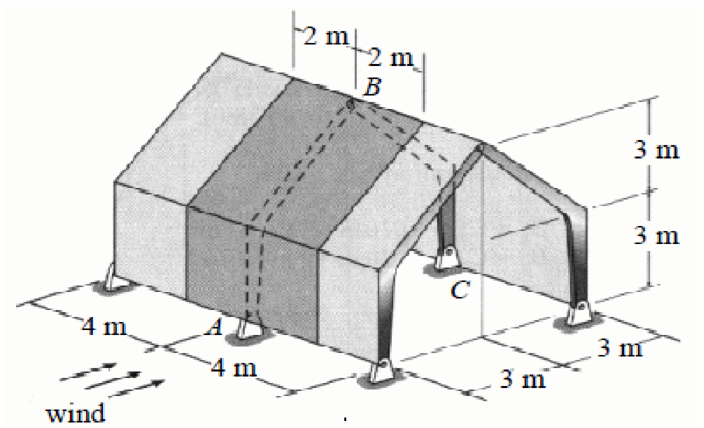
$$C_x = 3.75 \text{ kN}$$

Example

The side of the building in the figure below is subjected to a wind loading that creates a uniform *normal* pressure of 1.5 kPa on the windward side and a suction pressure of 0.5 kPa on the leeward side. Determine the horizontal and vertical components of reaction at the pin connections *A*, *B*, and *C* of the supporting gable arch.



SOLUTION

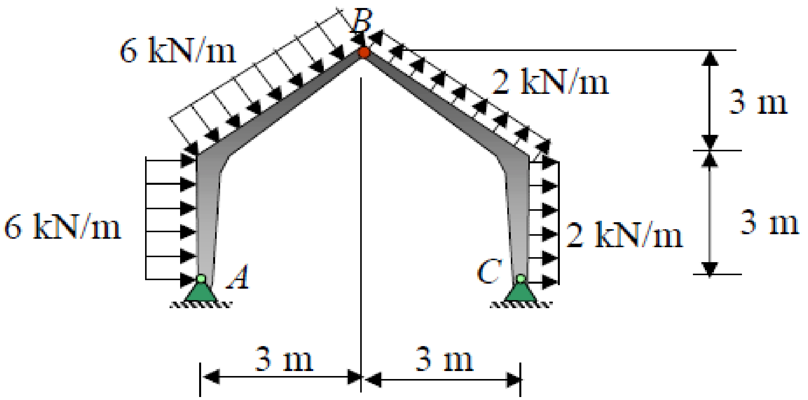


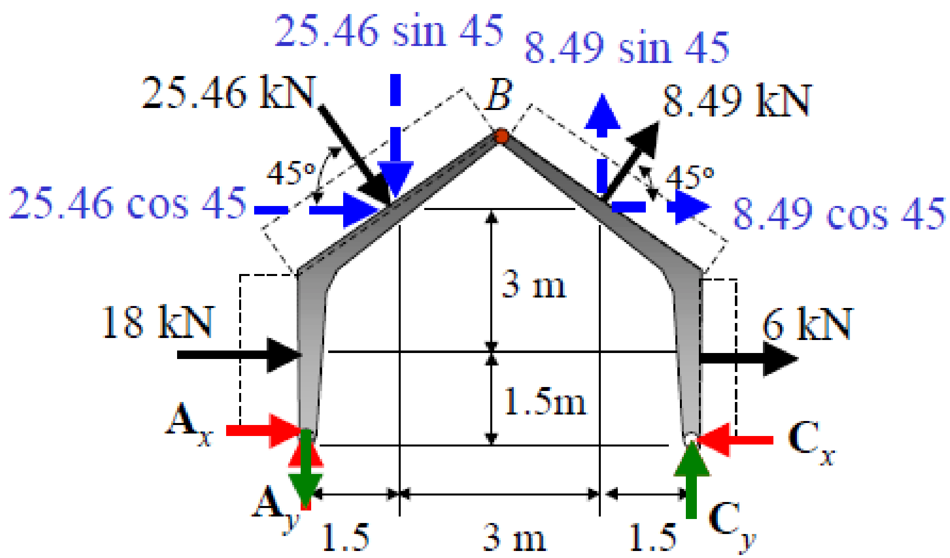
A uniform distributed load on the *windward* side is

$$(1.5\ \text{kN/m}^2)(4\ \text{m}) = 6\ \text{kN/m}$$

A uniform distributed load on the *leeward* side is

$$(0.5\ \text{kN/m}^2)(4\ \text{m}) = 2\ \text{kN/m}$$





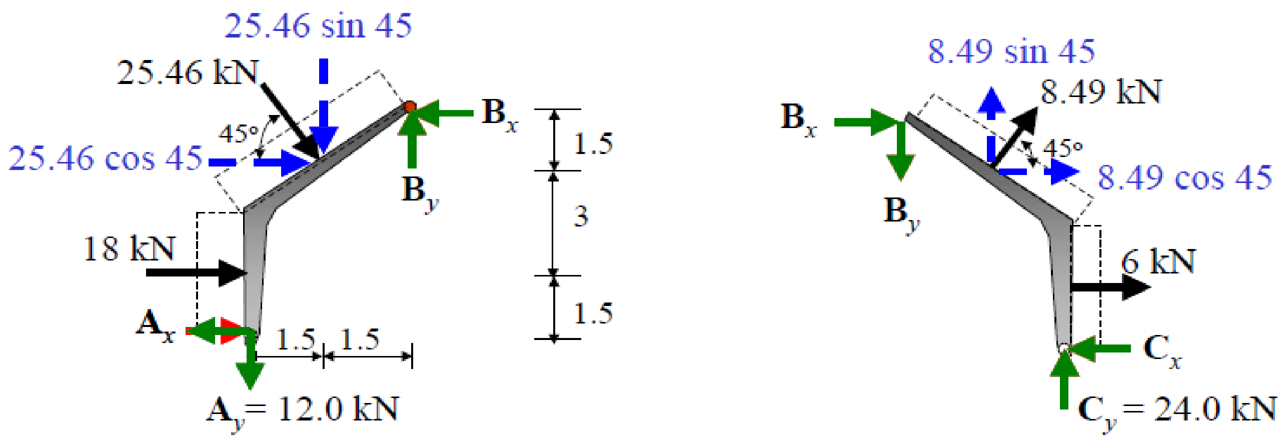
Entire Frame

$$+\circlearrowright \Sigma M_A = 0: \quad -(18+6)(1.5) - (25.46+8.49)\cos 45°(4.5) - (25.46 \sin 45°)(1.5) + (8.49 \sin 45°)(4.5) + C_y(6) = 0$$

$$C_y = 24.0 \text{ kN}, \uparrow$$

$$+\uparrow \Sigma F_y = 0: \quad A_y - 25.46 \sin 45° + 8.49 \sin 45° \; 3 + 24 = 0$$

$$A_y = -12.0 \text{ kN} \quad \downarrow$$



Member *AB*

$\overset{+}{\circlearrowright}\ \Sigma M_B = 0$: $(25.46 \sin 45°)(1.5) + (25.46 \cos 45°)(1.5) + (18)(4.5) + A_x(6) + 12(3) = 0$

$A_x = -28.5$ kN $\leftarrow$

$\overset{+}{\rightarrow}\ \Sigma F_x = 0$: $-28.5 + 18 + 25.46 \cos 45° - B_x = 0$

$B_x = 7.5$ kN , $\leftarrow$

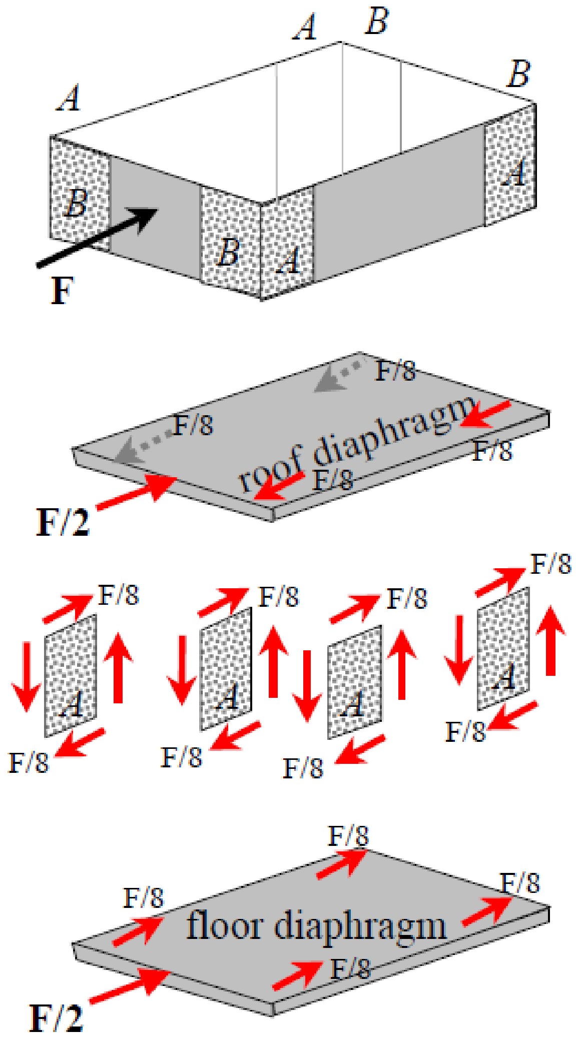
$+\uparrow\ \Sigma F_y = 0$: $-12 - 25.46 \sin 45° + B_y = 0$

$B_y = 30.0$ kN , $\uparrow$

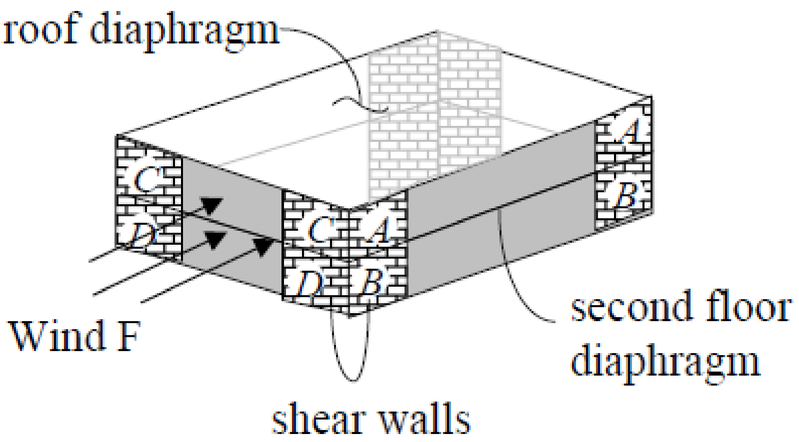
Member *CB*

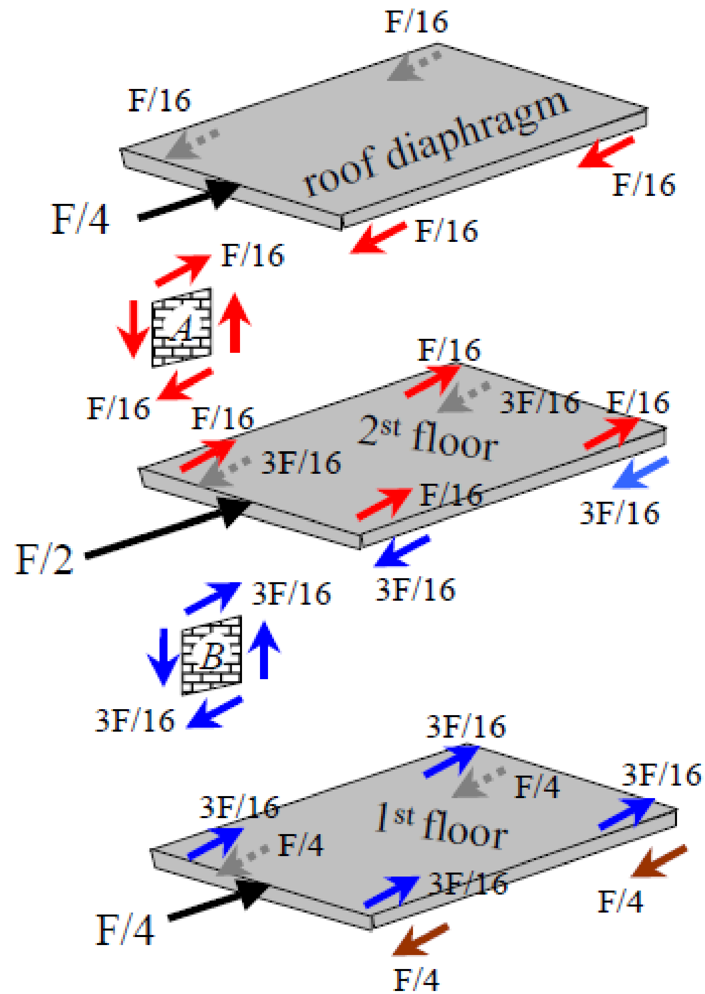
$\overset{+}{\rightarrow}\ \Sigma F_x = 0$: $7.5 + 8.49 \cos 45° + 6 - C_x = 0$

$C_x = 19.50$ kN , $\leftarrow$



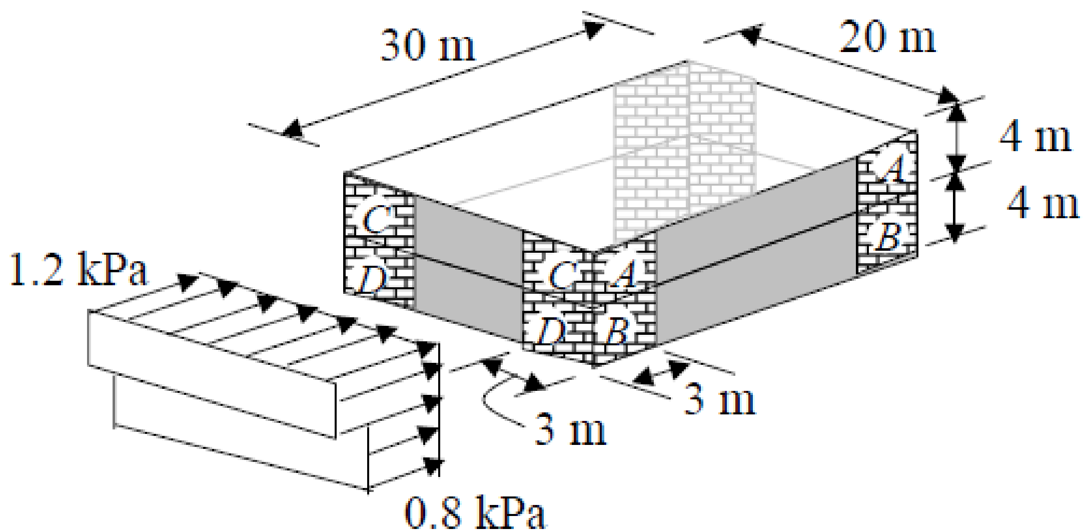
A
B
A
B
B
A
B
A
F
F/8
F/8
roof diaphragm
F/8
F/8
F/2
F/8
F/8
F/8
F/8
A
A
A
A
F/8
F/8
F/8
F/8
F/8
F/8
F/8
floor diaphragm
F/8
F/2


roof diaphragm
C
A
B
C
A
D
D
B
Wind F
second floor diaphragm
shear walls

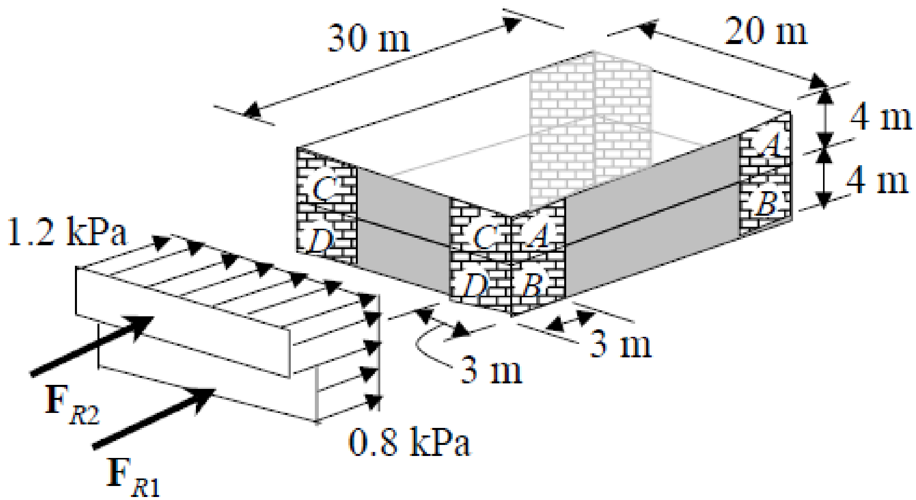

F/16
F/16
roof diaphragm
F/16
F/4
F/16
F/16
A
F/16
F/16
F/16
3F/16
F/16
3F/16
2st floor
F/16
F/2
3F/16
3F/16
3F/16
B
3F/16
3F/16
3F/16
F/4
3F/16
1st floor
F/4
3F/16
F/4
F/4
F/4

Example

Assume the wind loading acting on one side of a two-story building is as shown in the figure below. If shear walls are located at each of the corners as shown and flanked by columns, determine the shear in each panel located between the floors and the shear along the columns.



SOLUTION

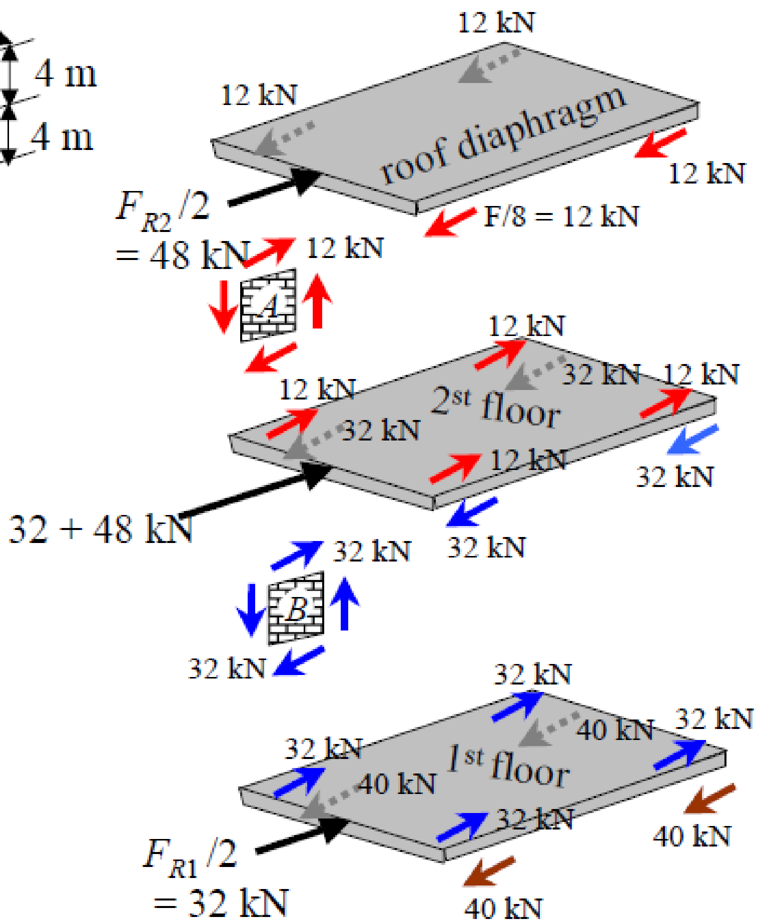


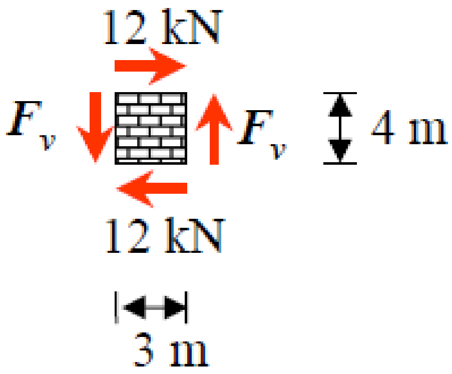
$$F_{R1} = 0.8(10^3)\ \mathrm{N/m^2}\,(20\ \mathrm{m})(4\ \mathrm{m}) = 64\ \mathrm{kN}$$

$$F_{R1}/2 = 32$$

$$F_{R2} = 1.2(10^3)\ \mathrm{N/m^2}\,(20\ \mathrm{m})(4\ \mathrm{m}) = 96\ \mathrm{kN}$$

$$F_{R2}/2 = 48$$

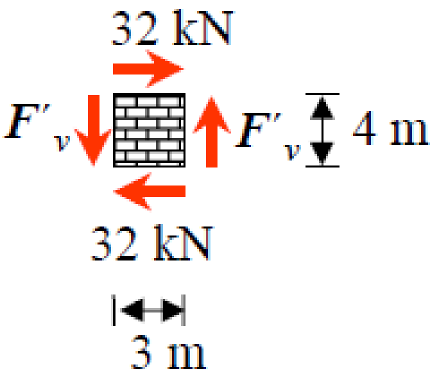




$\circlearrowright + \ \Sigma M = 0$:

$$F_v(3) - 12(4) = 0$$

$$F_v = 16 \text{ kN}$$



$\circlearrowright + \ \Sigma M = 0$:

$$F'_v(3) - 32(4) = 0$$

$$F'_v = 42.7 \text{ kN}$$

THEORY OF ELASTISITY

- INTRODUCTION:

    Elasticity

    Elasticity of Composites

    Viscoelasticity

    Elasticity of Crystals (Elastic Anisotropy)

- Let us start with some observations…

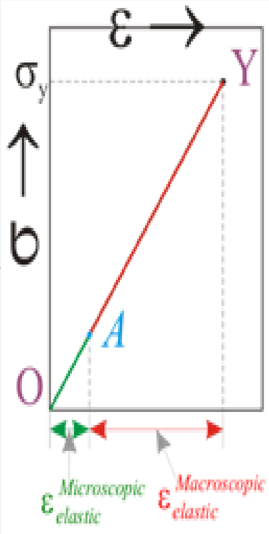
- ✓ When you pull a rubber band and release it, the band regains it original length.
- ✓ It is much more difficult (requires more load) to extend a metal wire as compared to a rubber ‘string’.
- ✓ It is difficult to extend a straight metal wire; however, if it is coiled in the form of a spring, one gets considerable extensions easily.
- ✓ A rubber string becomes brittle when dipped in liquid nitrogen and breaks, when one tries to extend the same.
- ✓ A diver gets lift-off using the elastic energy stored in the diving board. If the diving board is too compliant, the diver cannot get sufficient lift-off.
- ✓ A rim of metal is heated to expand the loop and then fitted around a wooden wheel (of say a bullock cart). On cooling of the rim it fits tightly around the wheel.

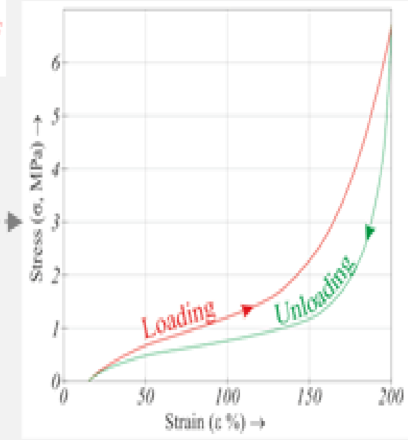
Elasticity

- Elastic deformation is reversible deformation- i.e. when load/forces/constraints are released the body returns to its original configuration (shape and size).
- Elastic deformation can be caused by tension/compression or shear forces.
- Usually in metals and ceramics elastic deformation is seen at low strains (less than ~10^{-3}). However, other materials can be stretched elastically to large strains (few 100%).
  So elastic deformation should not be assumed to be small deformation!
- The elastic behavior of metals and ceramics is usually linear.

Elasticity

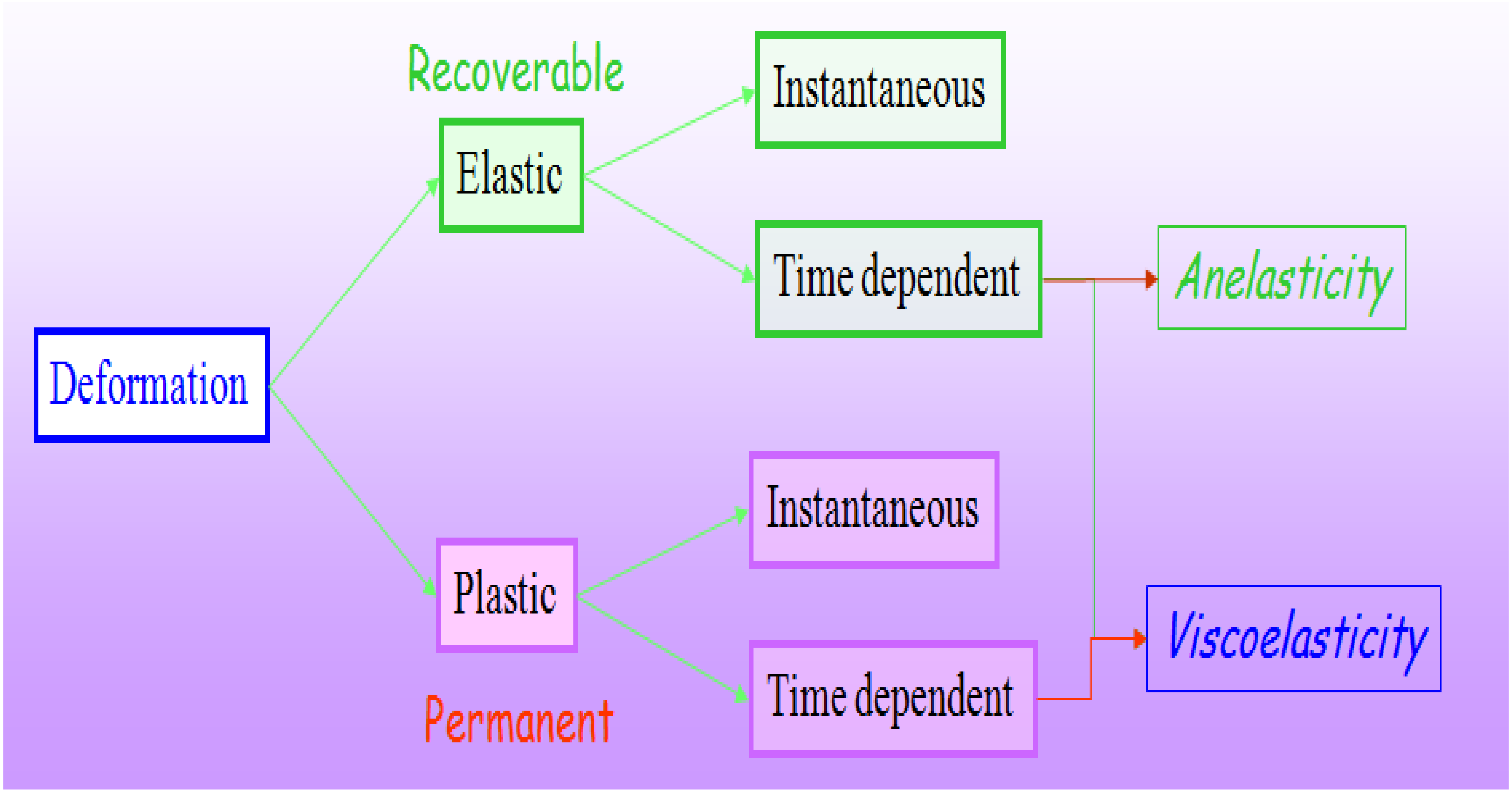
- Linear — *E.g. Al deformed at small strains*
- Non-linear — *E.g. deformation of an elastomer like rubber*





Time dependent aspects of elastic deformation

- In normal elastic behaviour (e.g. when 'small' load is applied to a metal wire, causing a strain within elastic limit) it is assumed that the strain is appears 'instantaneously'.
- Similarly when load beyond the elastic limit is applied (in an uniaxial tension test), it is assumed that the plastic strain develops instantaneously. It is further assumed that the load is applied 'quasi-statically' (i.e. very slowly).
- However, in some cases the elastic or plastic strain need not develop instantaneously. Elongation may take place at constant load with time. These effects are termed Anelasticity (for time dependent elastic deformation at constant load) and Viscoelasticity.
- Creep is one such phenomenon, where permanent deformation takes place at constant load (or stress). E.g. if sufficient weight is hung at the end of a lead wire at room temperature, it will elongate and finally fail.


Recoverable
Elastic
Instantaneous
Time dependent
Anelasticity
Deformation
Instantaneous
Plastic
Time dependent
Viscoelasticity
Permanent

Atomic model for elasticity

- At the atomic level elastic deformation takes place by the deformation of bonds (change in bond length or bond angle).
- Let us consider the stretching of bonds (leading to elastic deformation).
- Atoms in a solid feel an attractive force at larger atomic separations and feel a repulsive force (when electron clouds ‘overlap too much’) at shorter separations. (Of course, at very large separations there is no force felt).
- The energy and the force (which is a gradient of the energy field) display functional behaviour as in the equations below*. This implies under a state of compressive stress the atoms ‘want to’ go apart and under tensile stress they want to come closer.

Energy

$$F = -\frac{dU}{dr}$$

Force

$$U = -\frac{A}{r^n} + \frac{B}{r^m}$$

Repulsive

Attractive

A,B,m,n $\rightarrow$ constants

m > n

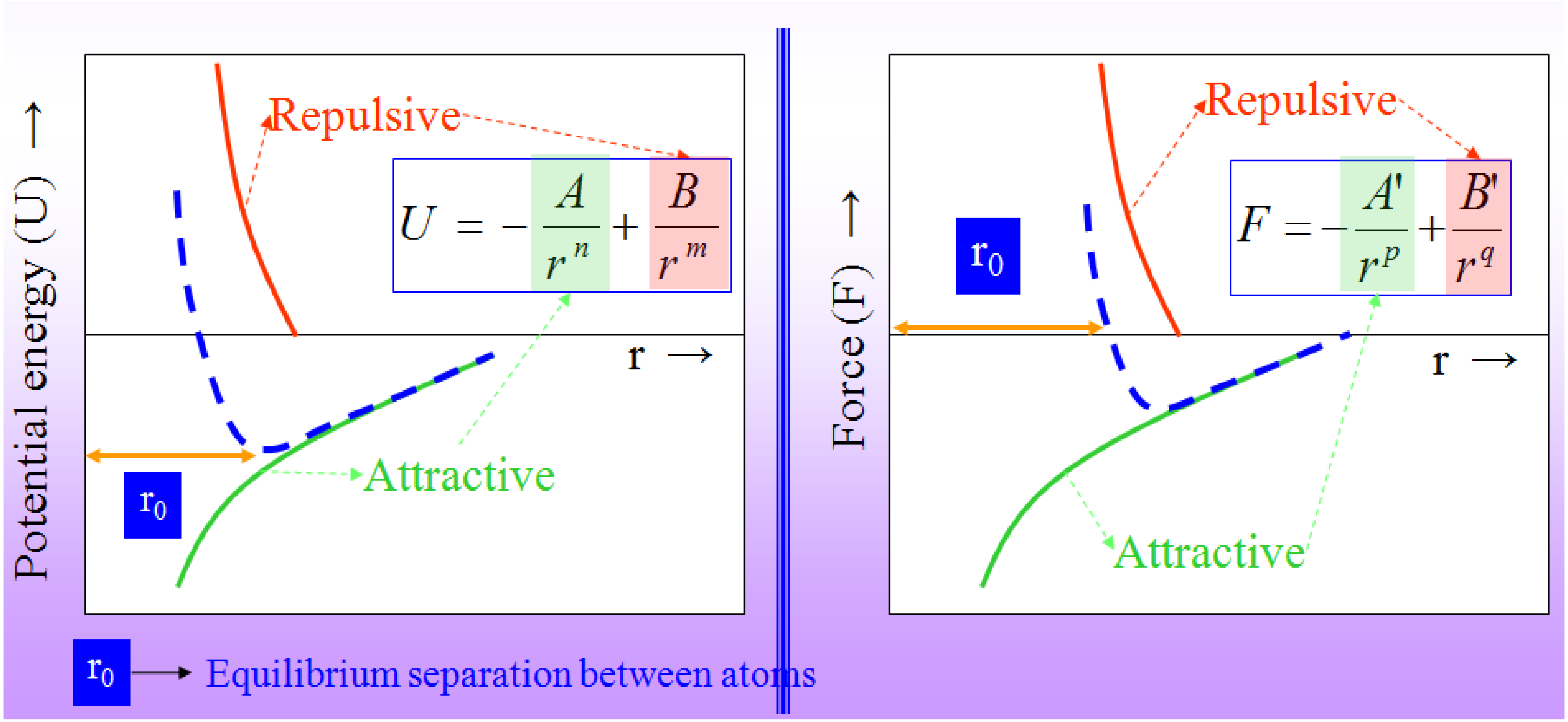
$$F = -\frac{nA}{r^{n+1}} + \frac{mB}{r^{m+1}}$$

$$F = -\frac{A'}{r^p} + \frac{B'}{r^q}$$

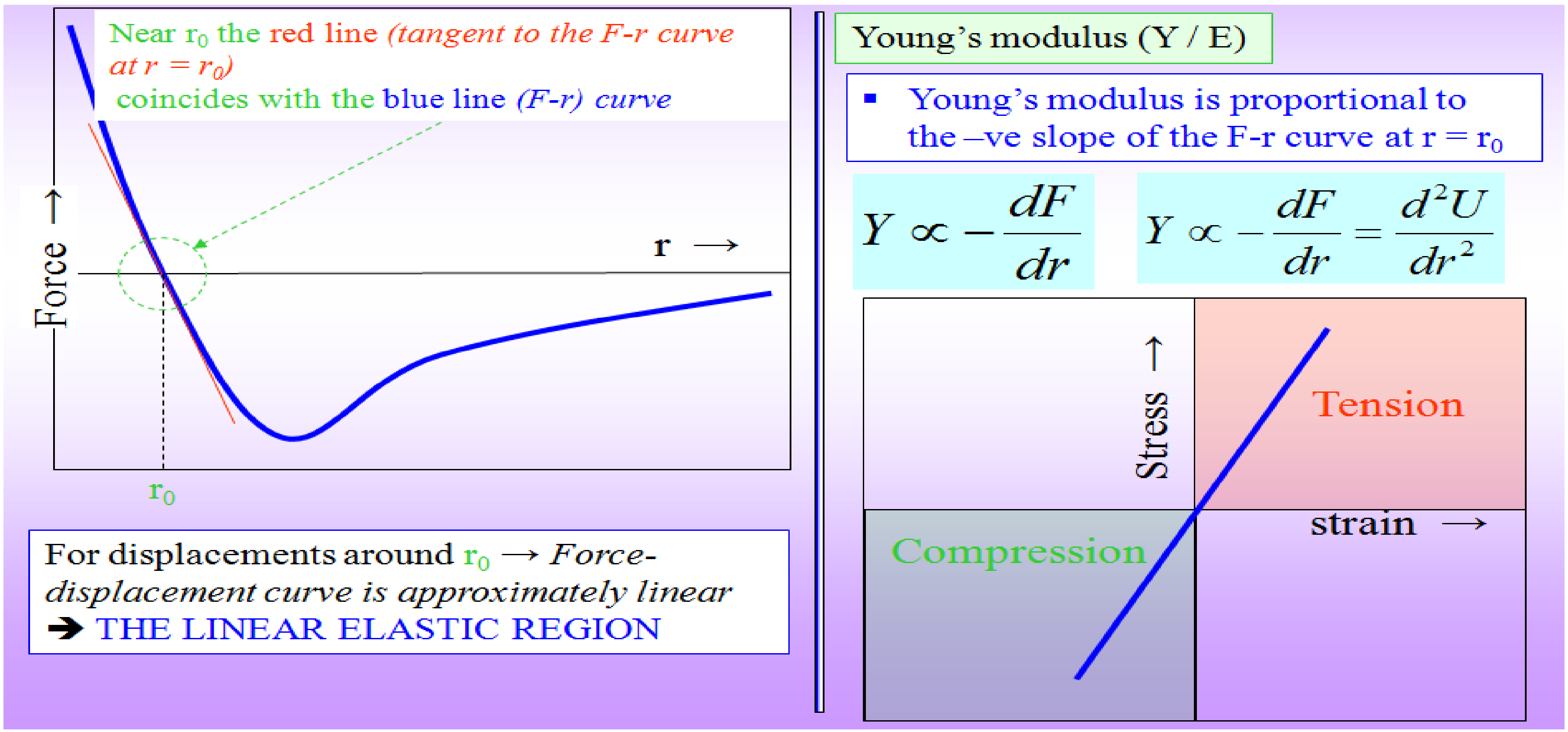
The plots of these functions is shown in the next slide

* *This is one simple form of interatomic potentials (also called Lennard-Jones potential, wherein m=12 and n=6).*

- The plot of the inter-atomic potential and force functions show that at the equilibrium inter-atomic separation (r_0) the potential energy of the system is a minimum and force experienced is zero.
- In reality the atoms are undergoing thermal vibration about this equilibrium position. The amplitude of vibration increases with increasing temperature. Due to the slight left-right asymmetry about the minimum in the U-r plot, increased thermal vibration leads to an expansion of the crystal.

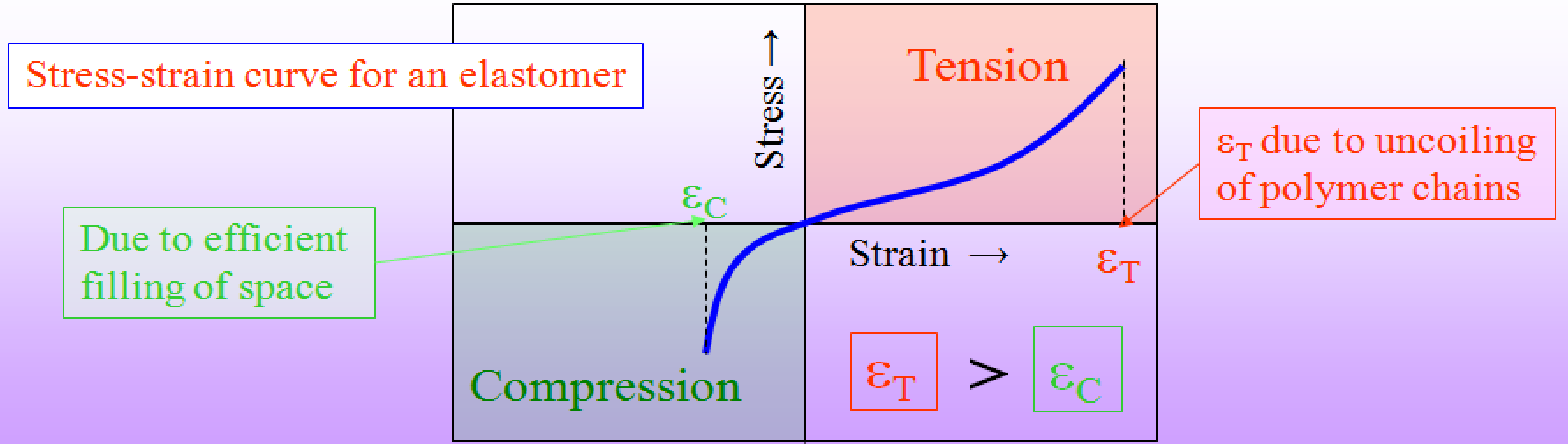


- Young's modulus is the slope of the Force-Interatomic spacing curve (F-r curve), at the equilibrium interatomic separation (r_0).
- In reality the Elastic modulus is 4th rank tensor (E_{ijkl}) and the curve below captures one aspect of it.



Stress-strain curves in the elastic region

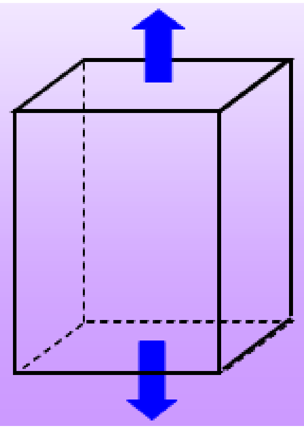
- In metals and ceramics the elastic strains (i.e. the strains beyond which plastic deformation sets in or fracture takes place*) are very small (~10^{-3}). As these strains are very small it does not matter if we use engineering stress-strain or true stress strain values (these concepts will be discussed later). The stress-strain plot is linear for such materials.
- Polymers (with special reference to elastomers) shown non-linear stress-strain behaviour in the elastic region. Rotation of the long chain molecules around a C-C bond can cause tensile elongation. The elastic strains can be large in elastomers (some can even extend a few hundred percent), but the modulus (slope of the stress-strain plot) is very small. Additionally, the behaviour of elastomers in compression is different from that in tension.



Brittle ceramics may show no plastic deformation and may fracture after elastic deformation.

Other elastic moduli

- We have noted that elastic modulus is a 4th rank tensor (with 81 components in general in 3D). In normal materials this is a symmetric tensor (i.e. it is sufficient to consider one set of the off-diagonal terms).
- In practice many of these components may be zero and additionally, many of them may have the same value (i.e. those surviving terms may not be independent). E.g. for a cubic crystal there are only 3 independent constants (in two index condensed notation these are E_{11}, E_{12}, E_{44}).
- For an isotropic material the number of independent constants is only 2. Typically these are chosen as E and ν (though in principle we could chose any other two as these moduli are interrelated for a isotropic material). More explained sooner.
- In a polycrystal (say made of grains of cubic crystal), due to average over all orientations, the material behaves like an isotropic material. More about this discussed elsewhere. Mathematically the isotropic material properties can be obtained from single crystal properties by Voigt or Reuss averaging.



In tensor notation

$$\boxed{\sigma = E.\varepsilon} \longrightarrow \boxed{\sigma_{ij} = E_{ijkl}\,\varepsilon_{kl}}$$

$$\boxed{\tau = G.\gamma}$$

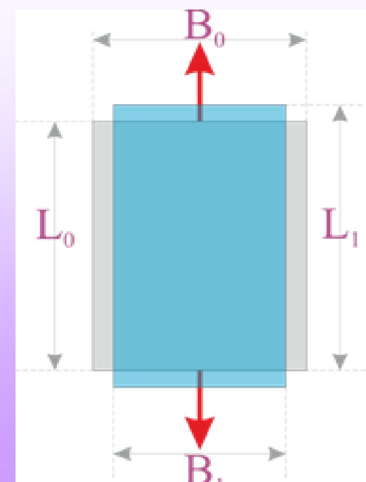
$$\boxed{\sigma_{hydrodynamic} = K.\varepsilon_{volumetric}}$$

- E → Young's modulus
- G → Shear modulus
- K → Bulk modulus
- ν → Poisson's ratio

- When a body is pulled (let us assume an isotropic body for now), it will elongate along the pulling direction and will contract along the orthogonal direction. The negative ratio of the transverse strain (ε_t) to the longitudinal strain (ε_t) is called the Poission's ratio (ν).
- In the equation below as $B_1 < B_0$, the term in square brackets '[]' in the numerator is –ve and hence, Poission's ratio is a positive quantity (for usual materials). I.e. usual materials shrink in the transverse direction, when they are pulled.
- The value of E, G and ν for some common materials are in the table (Table-E). Zero and even negative Possion's ratio have been reported in literature. The modulus of materials expected to be positive, i.e. the material resists deformation and stores energy in the deformed condition. However, structures and 'material-structures' can display negative stiffness (e.g. when a thin rod is pushed it will show negative stiffness during bulking– observed in displacement control mode).

$$\nu = -\frac{\varepsilon_t}{\varepsilon_l} = -\frac{[(B_1 - B_0)/B_0]}{[(L_1 - L_0)/L_0]}$$

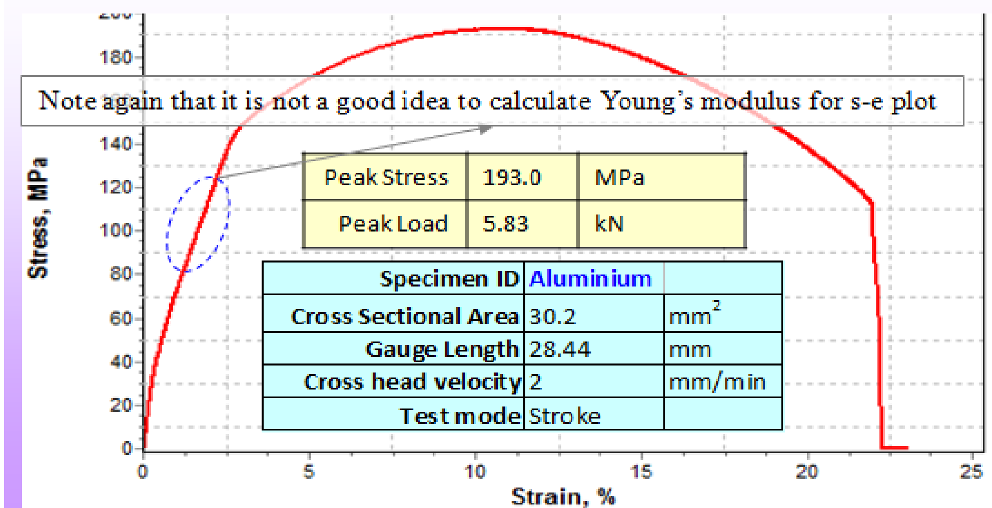
$$\boxed{\nu = -\frac{\varepsilon_t}{\varepsilon_l}} \quad \boxed{G = \frac{E}{2(1+\nu)}} \quad \boxed{K = \frac{E}{3(1-2\nu)}}$$



Initial configuration is represented by '0' in subscript

How to determine the elastic modulus?

- The Young's modulus (Y) of a isotropic material can be determined from the stress-strain diagram. But, this is not a very accurate method, as the machine compliance is in series with the specimen compliance. The slope of the stress-strain curve at any point is termed as the 'tangent modulus'. In the initial part of the s-e curve this measures the Young's modulus. Other kinds of mechanical tests can also be used for the measurement of 'Y'.
- A better method to measure 'Y' is using wave transmission (e.g. ultrasonic pulse echo transmission technique) in the material. This is best used for homogeneous, isotropic, non-dispersive material (wherein, the velocity of the wave does not change with frequency). Common polycrystalline metals, ceramics and inorganic glasses are best suited to this method. Soft plastics and rubber cannot be characterized by this method due to high dispersion, attenuation of sound wave and non-linear elastic properties. Porosity and other internal defects can affect the measurement.
- The essence of the ultrasound technique is to determine the longitudinal and shear wave velocity of sound in the material (symbols: v_l v_s). 'Y' and Poisson's ratio (ν) can be determined using the formulae as below.



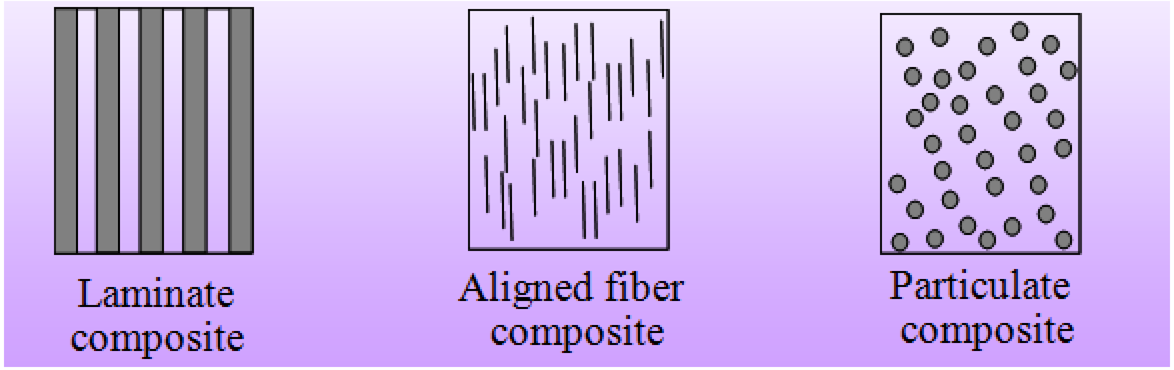
$$\nu = \frac{1 - 2\left(\dfrac{v_t}{v_l}\right)^2}{2 - 2\left(\dfrac{v_t}{v_l}\right)^2}$$

$$Y = \frac{v_l^2\, \rho (1+\nu)(1-2\nu)}{(1-\nu)}$$

- $\rho \rightarrow$ density

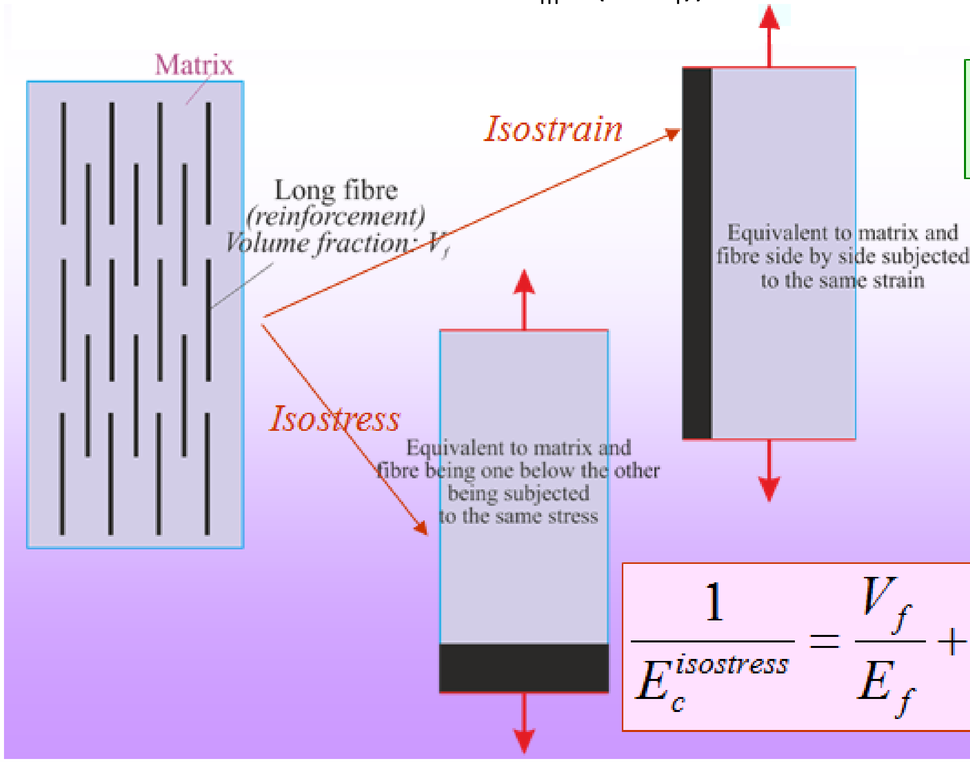
Increasing the modulus of a material

- The modulus of a metal can be increased by suitable alloying.
- E is a structure *(microstructure)* insensitive property and this implies that grain size, dislocation density, etc. do not play an important role in determining the elastic modulus of a material.
- One of the important strategies to increase the modulus of a material is by forming a hyrbrid/composite with an elastically 'harder' (stiffer) material. E.g. TiB_2 is added to Fe to increase the modulus of Fe.
- COMPOSITES
- A hard second phase *(termed as reinforcement)* can be added to a low E material to increase the modulus of the base material. The second phase can be in the form of *particles, fibres, laminates, etc.*
- Typically the second phase though harder is brittle and the ductility is provided by the matrix. If the reinforcement cracks the propagation of the crack is arrested by the matrix.



Modulus of a composite

- The modulus of the composite is between that of the matrix and the reinforcement.
- Let us consider two extreme cases.
  (A) Isostrain → the matrix and the reinforcement (say long fibres) are under identical strain.
  This is known as Voigt averaging.
  (B) Isostress → the matrix and fibre are under identical stress.
  This is known as Reuss averaging.
- Let the composite be loaded in uniaxial tension and the volume fraction of the fibre be V_f (automatically the volume fraction of the matrix is $V_m = (1 - V_f)$).



$$E_c^{isostrain} = E_f V_f + E_m V_m$$

Voigt averaging

$$\frac{1}{E_c^{isostress}} = \frac{V_f}{E_f} + \frac{V_m}{E_m}$$

Reuss averaging

- The modulus of a real composite will lie between these two extremes (usually closer to isostrain). The modulus of the composite will depend on the shape of the reinforcement and the nature of the interface (e.g. in a long aligned fibre composite having a perfectly bonded interface with no slippage will lead to isostrain conditions.
- Purely from a modulus perspective, a larger volume fraction will give a higher modulus; however, ductility and other considerations typically limit the volume fraction of reinforcement in a composite to about 30%.

Voigt averaging

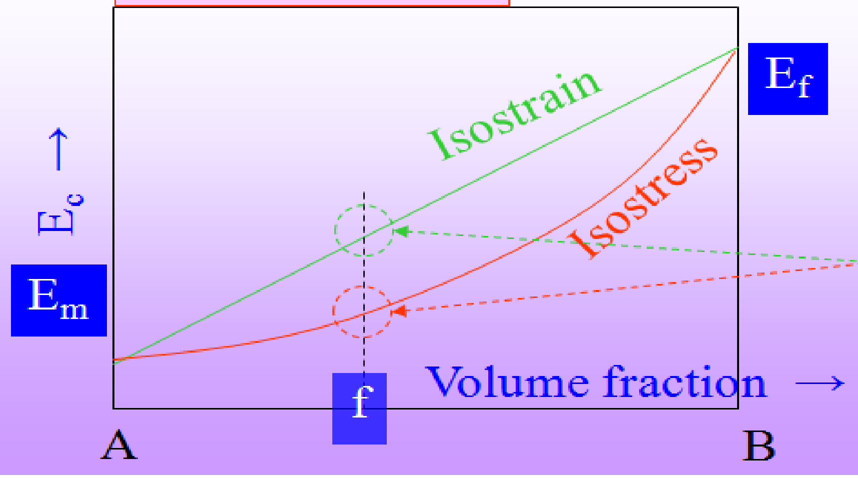
$$E_c = E_f V_f + E_m V_m$$

- Under iso-strain conditions [$\varepsilon_m = \varepsilon_f = \varepsilon_c$]
- I.e. ~ *resistances in series* configuration

$$\frac{1}{E_c} = \frac{V_f}{E_f} + \frac{V_m}{E_m}$$

- Under iso-stress conditions [$\sigma_m = \sigma_f = \sigma_c$]
- I.e. ~ *resistances in parallel* configuration
- Usually not found in practice

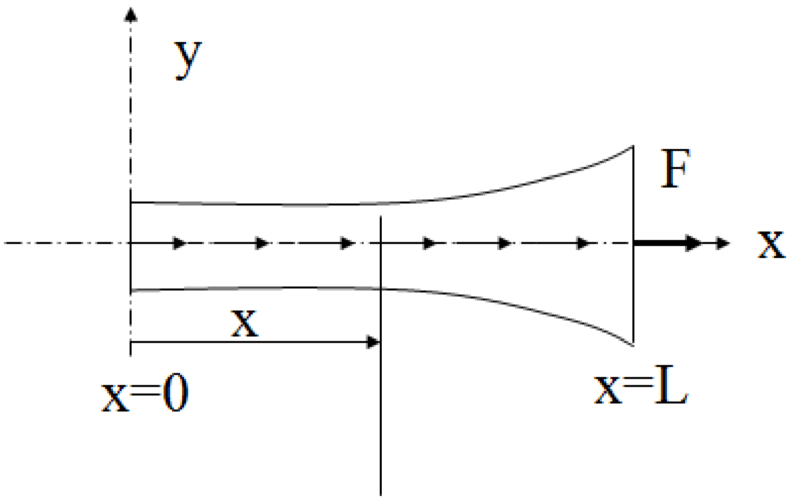
Reuss averaging



For a given fiber fraction f, the moduli of various conceivable composites lie between an upper bound given by isostrain condition and a lower bound given by isostress condition

1D Elasticity (axially loaded bar)

1D Elasticity (axially loaded bar)



A(x) = cross section at x
b(x) = body force distribution (force per unit length)
E(x) = Young's modulus
u(x) = displacement of the bar at x

1. Strong formulation: Equilibrium equation + boundary conditions

$\boxed{\text{Equilibrium equation}}$ $\quad \dfrac{d\sigma}{dx} + b = 0; \qquad 0 < x < L$

$\boxed{\text{Boundary conditions}}$ $\quad u = 0 \quad at \;\; x = 0$

$$EA\frac{du}{dx} = F \quad at \;\; x = L$$

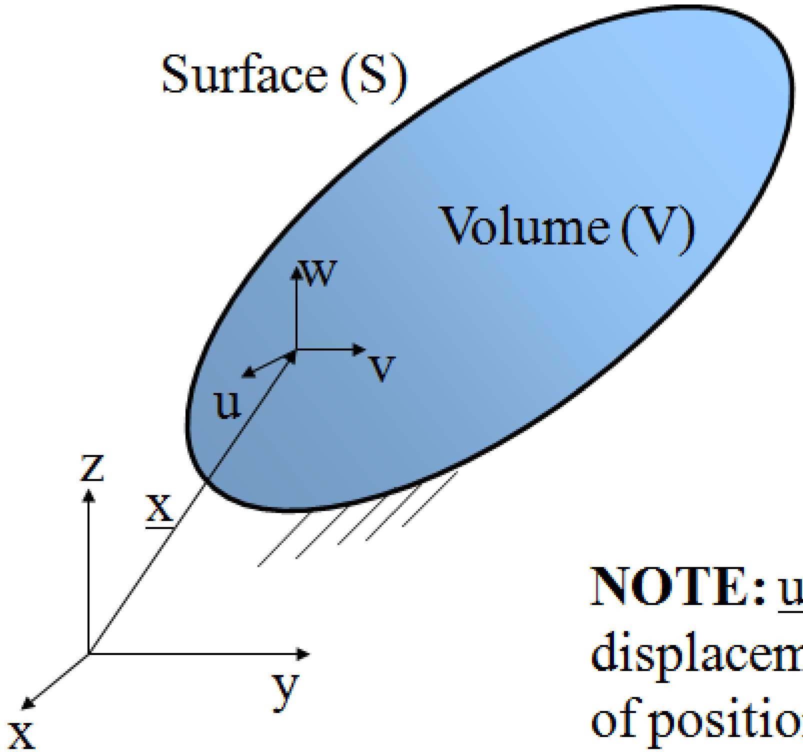
2. **Strain-displacement** relationship: $\varepsilon(x) = \dfrac{du}{dx}$

3. **Stress-strain** (constitutive) relation : $\sigma(x) = E\,\varepsilon(x)$

E: Elastic (Young's) modulus of bar

3D Elasticity

Problem definition



V: Volume of body

S: Total surface of the body

The deformation at point $\underline{x} = [x, y, z]^T$ is given by the 3 components of its displacement $\underline{u} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$

NOTE: $\underline{u} = \underline{u}(x,y,z)$, i.e., each displacement component is a function of position

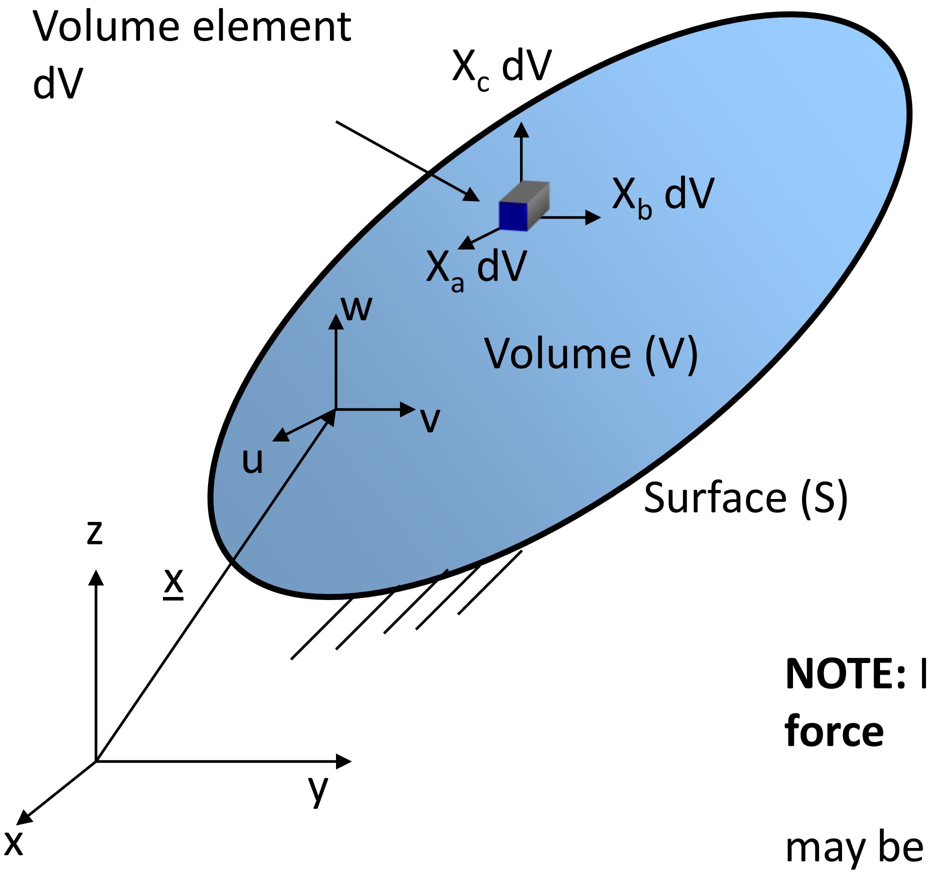
3D Elasticity:

EXTERNAL FORCES ACTING ON THE BODY

Two basic types of **external forces** act on a body

1. **Body force** (force per unit **volume**) e.g., weight, inertia, etc
2. **Surface traction** (force per unit **surface area**) e.g., friction

BODY FORCE



Body force: distributed <u>force per unit volume</u> (e.g., weight, inertia, etc)

$$\underline{X} = \begin{Bmatrix} X_a \\ X_b \\ X_c \end{Bmatrix}$$

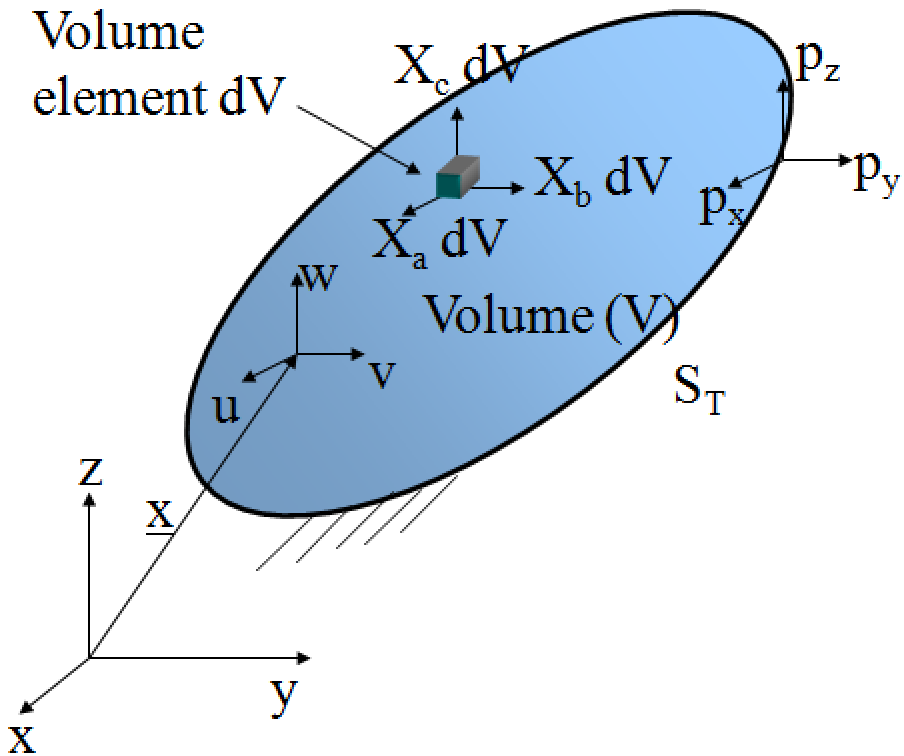
NOTE: If the body is accelerating, then the **inertia force**

$$\rho \underline{\ddot{u}} = \begin{Bmatrix} \rho \ddot{u} \\ \rho \ddot{v} \\ \rho \ddot{w} \end{Bmatrix}$$

may be considered as part of <u>X</u>

$$\underline{X} = \underline{\tilde{X}} - \rho \underline{\ddot{u}}$$

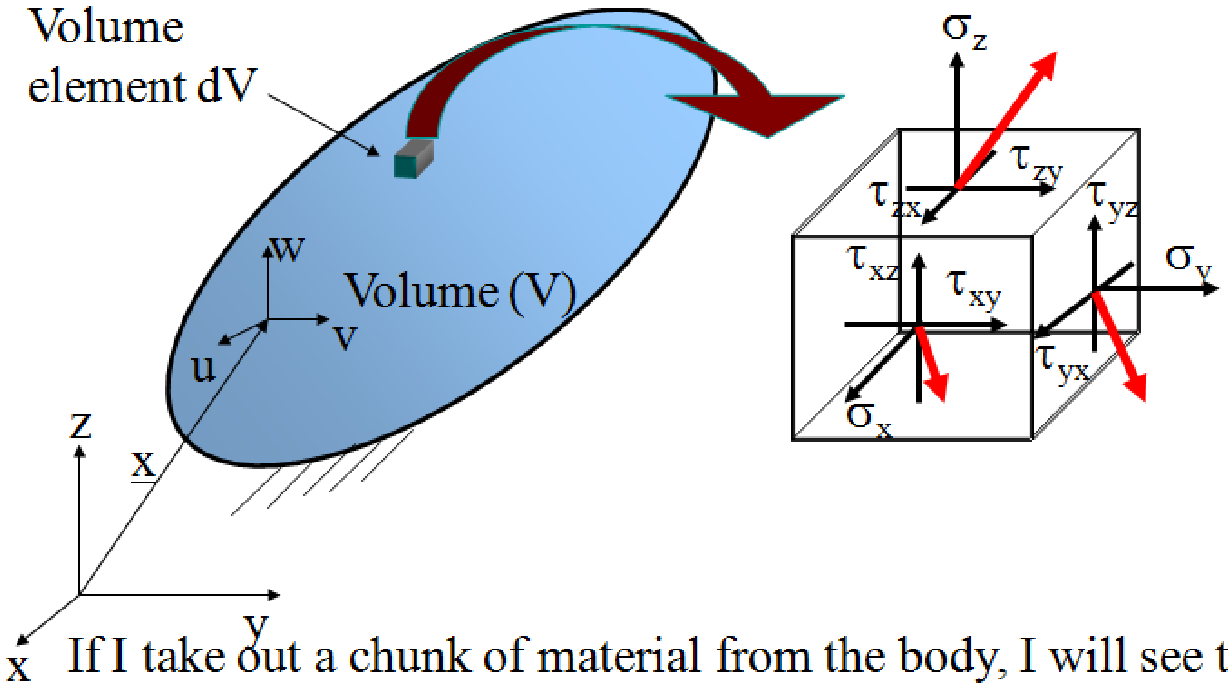
SURFACE TRACTION



Traction: Distributed force **per unit surface area**

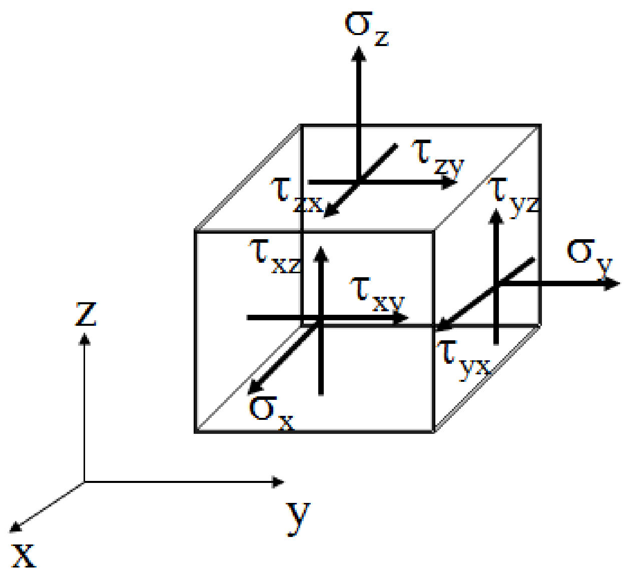
$$\underline{T}_S = \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}$$

3D Elasticity: INTERNAL FORCES



If I take out a chunk of material from the body, I will see that, due to the external forces applied to it, there are reaction forces (e.g., due to the loads applied to a truss structure, internal forces develop in each truss member). For the cube in the figure, the **internal reaction forces <u>per unit area</u>**(**red arrows**) , on each surface, may be decomposed into three orthogonal components.

3D Elasticity



σ_x, σ_y and σ_z are **normal stresses**.
The rest 6 are the **shear stresses**
Convention
τ_{xy} is the stress on the face perpendicular to the x-axis and points in the +ve y direction
Total of 9 stress components of which only 6 are independent since

$$\tau_{xy} = \tau_{yx}$$
$$\tau_{yz} = \tau_{zy}$$
$$\tau_{zx} = \tau_{xz}$$

The **stress vector** is therefore

$$\underline{\sigma} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

Strains: 6 independent **strain components**

$$\underline{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

Consider the equilibrium of a differential volume element to obtain the 3 **equilibrium equations** of elasticity

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X_a = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + X_b = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + X_c = 0$$

Compactly;

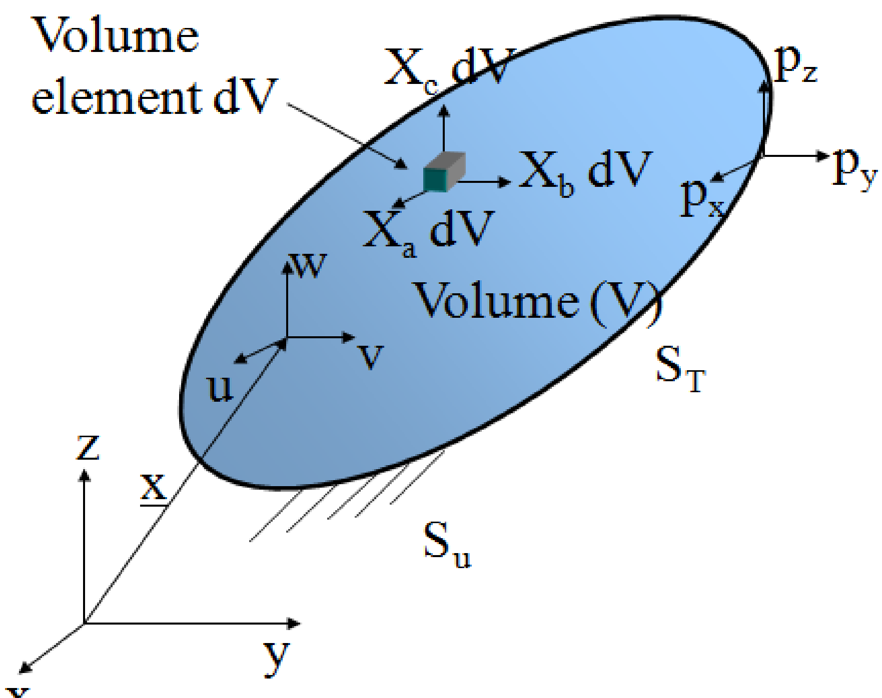
EQUILIBRIUM EQUATIONS

$$\underline{\partial}^{T}\underline{\sigma} + \underline{X} = \underline{0}$$

where

$$\underline{\partial} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix}$$

- 3D elasticity problem is completely defined once we understand the following three concepts
  - Strong formulation (governing differential equation + boundary conditions)
  - Strain-displacement relationship
  - Stress-strain relationship



<u>1. Strong formulation of the 3D elasticity problem</u>: "***Given the externally applied loads (on S_T and in V) and the specified displacements (on S_u) we want to solve for the resultant <u>displacements, strains and stresses</u> required to maintain equilibrium of the body.***"

Equilibrium equations $\underline{\partial}^T \underline{\sigma} + \underline{X} = \underline{0} \quad in \ V$ (1)

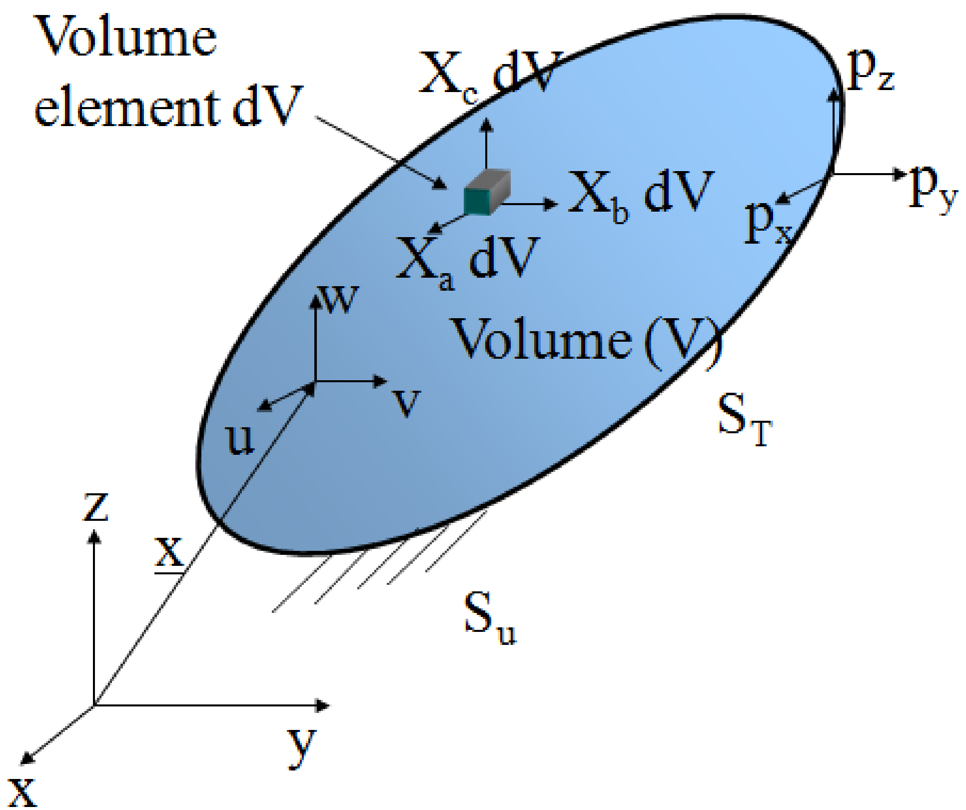
Boundary conditions

1. Displacement boundary conditions: Displacements are specified on portion S_u of the boundary

$$\underline{u} = \underline{u}^{specified} \quad on \quad S_u$$

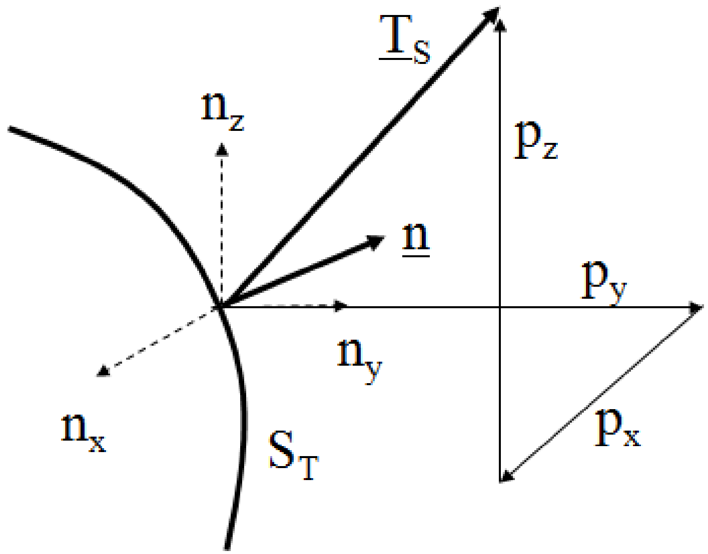
2. Traction (force) boundary conditions: **Tractions** are specified on portion S_T of the boundary

Now, how do I express this mathematically?



Traction: Distributed force **per unit area**

$$\underline{T}_S = \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}$$



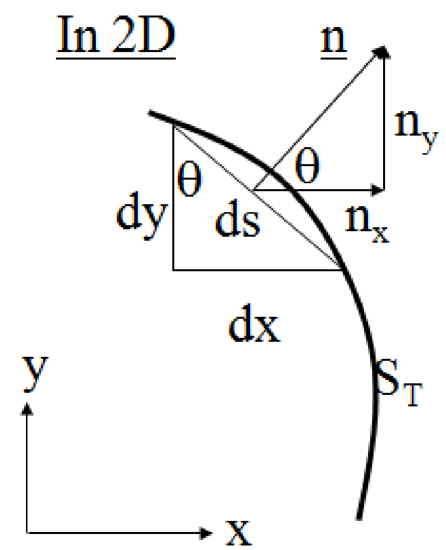
Traction: Distributed force **per unit area**

$$\underline{T}_S = \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}$$

If the unit outward normal to S_T : $\underline{n} = \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$

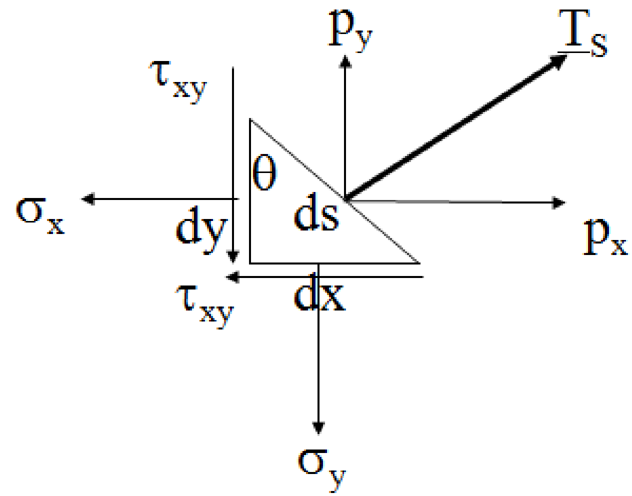
Then

$$\boxed{\begin{aligned} p_x &= \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z \\ p_y &= \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z \\ p_z &= \tau_{xz} n_x + \tau_{zy} n_y + \sigma_z n_z \end{aligned}}$$



$$\sin\theta = \frac{dx}{ds} = n_y$$

$$\cos\theta = \frac{dy}{ds} = n_x$$



Consider the equilibrium of the wedge in x-direction

$$p_x ds = \sigma_x dy + \tau_{xy} dx$$

$$\Rightarrow p_x = \sigma_x \frac{dy}{ds} + \tau_{xy} \frac{dx}{ds}$$

$$\Rightarrow p_x = \sigma_x n_x + \tau_{xy} n_y$$

Similarly

$$p_y = \tau_{xy} n_x + \sigma_y n_y$$

- 3D elasticity problem is completely defined once we understand the following three concepts
  - Strong formulation (governing differential equation + boundary conditions)
  - Strain-displacement relationship
  - Stress-strain relationship

2. Strain-displacement relationships:

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

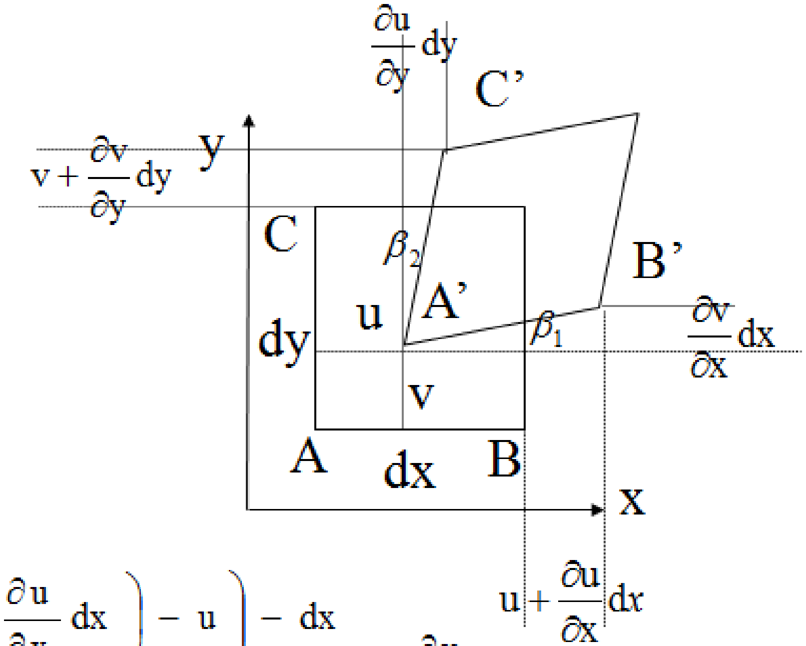
$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

Compactly; $\boxed{\underline{\varepsilon} = \underline{\partial}\ \underline{u}}$ (2)

$$\underline{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \qquad \underline{\partial} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \qquad \underline{\mathrm{u}} = \begin{Bmatrix} \mathrm{u} \\ \mathrm{v} \\ \mathrm{w} \end{Bmatrix}$$

In 2D



$$\varepsilon_x = \frac{A'B' - AB}{AB} = \frac{\left(dx + \left(u + \frac{\partial u}{\partial x}dx\right) - u\right) - dx}{dx} = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{A'C' - AC}{AC} = \frac{\left(dy + \left(v + \frac{\partial v}{\partial y}dy\right) - v\right) - dy}{dy} = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\pi}{2} - \text{angle}\ (C'A'B') = \beta_1 + \beta_2 \approx \tan\beta_1 + \tan\beta_2$$

$$\approx \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}$$

- 3D elasticity problem is completely defined once we understand the following three concepts
  - Strong formulation (governing differential equation + boundary conditions)
  - Strain-displacement relationship
  - Stress-strain relationship

3. Stress-Strain relationship:

Linear elastic material (Hooke's Law)

$$\boxed{\underline{\sigma} = \underline{D}\ \underline{\varepsilon}} \qquad (3)$$

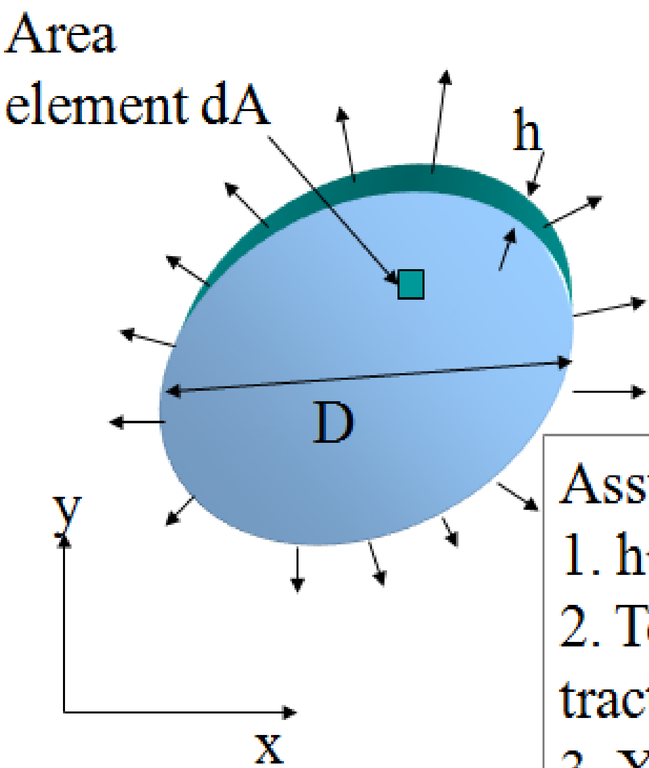
Linear elastic isotropic material

$$\underline{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

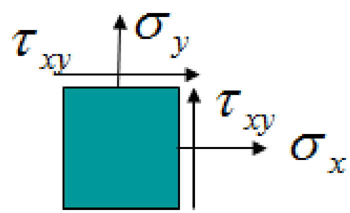
Special cases:

1. **1D elastic bar** (only 1 component of the stress (stress) is nonzero. All other stress (strain) components are zero)
Recall the (1) equilibrium, (2) strain-displacement and (3) stress-strain laws
2. **2D elastic problems:** 2 situations
PLANE STRESS
PLANE STRAIN
3. **3D elastic problem**: special case-**axisymmetric body with axisymmetric loading (we will skip this)**

<u>PLANE STRESS</u>: Only the in-plane <u>stress</u> components are nonzero



Nonzero <u>stress</u> components $\sigma_x, \sigma_y, \tau_{xy}$

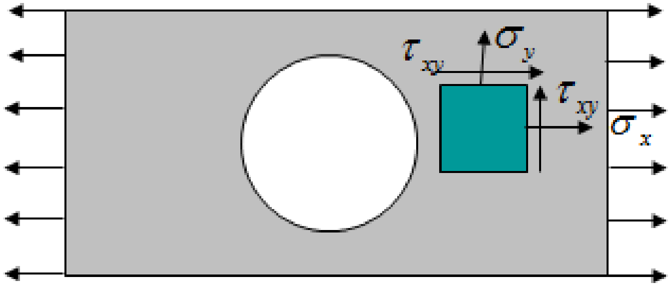


Assumptions:

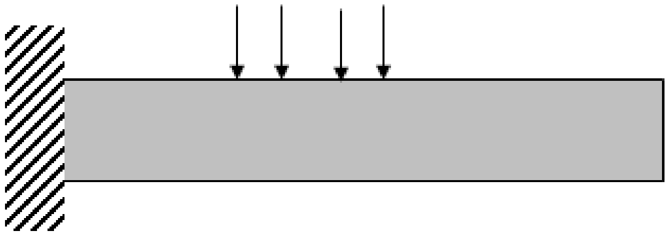
1. h<<D
2. Top and bottom surfaces are free from traction
3. X_c=0 and p_z=0

PLANE STRESS Examples:

1. Thin plate with a hole



2. Thin cantilever plate



PLANE STRESS

Nonzero **stresses**: $\sigma_x, \sigma_y, \tau_{xy}$
Nonzero **strains**: $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}$

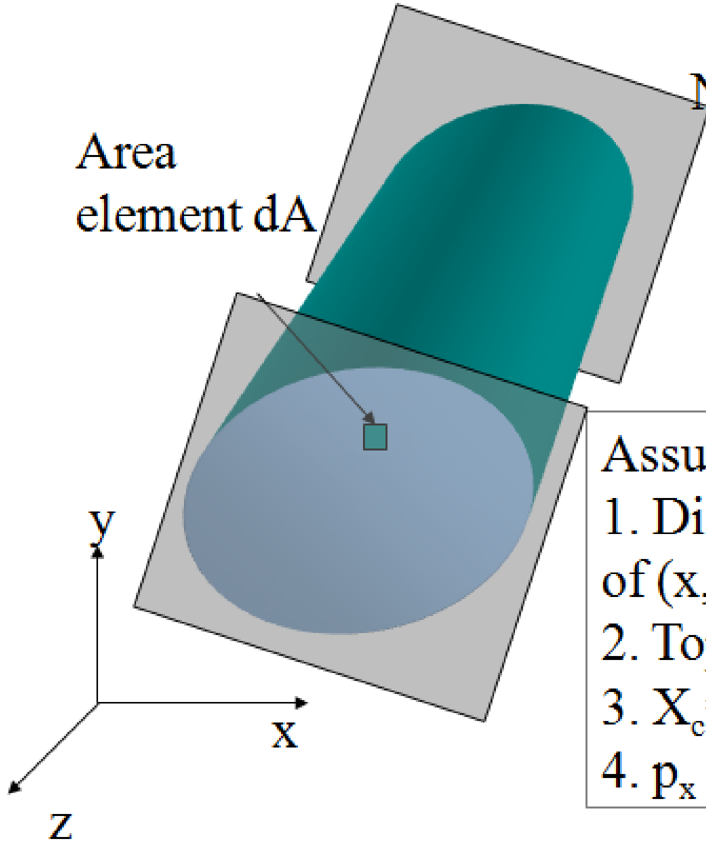
Isotropic linear elastic stress-strain law $\boxed{\underline{\sigma} = \underline{D}\,\underline{\varepsilon}}$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad \varepsilon_z = -\frac{\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right)$$

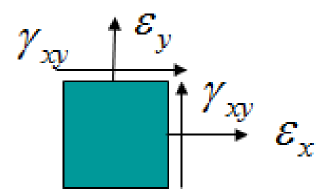
Hence, the $\underline{D}$ matrix for the **plane stress case** is

$$\underline{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

PLANE STRAIN: Only the in-plane strain components are nonzero



Nonzero **strain** components $\varepsilon_x, \varepsilon_y, \gamma_{xy}$

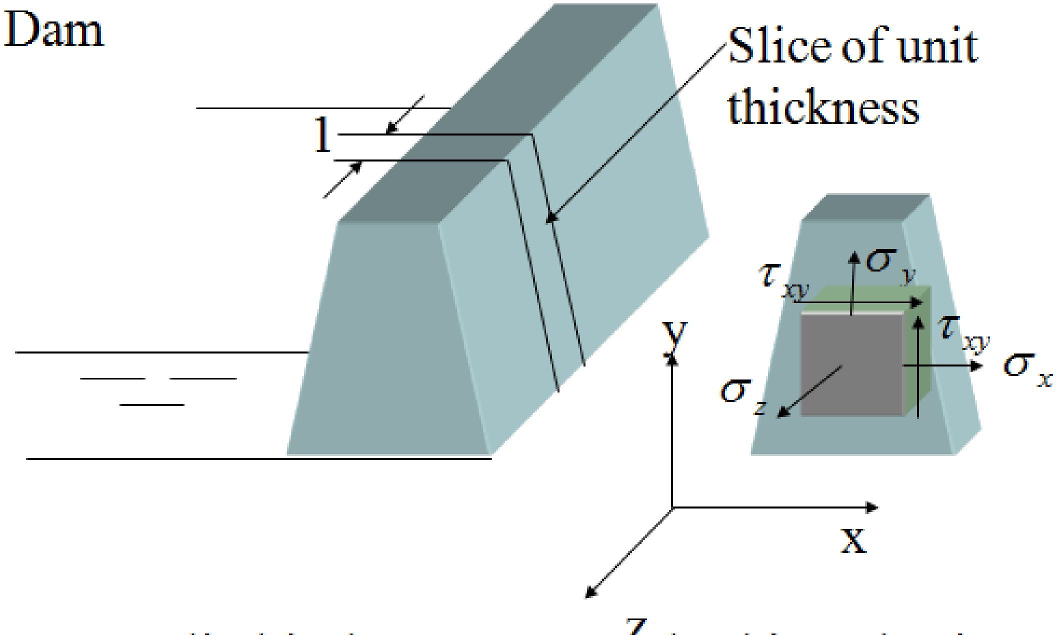


Assumptions:

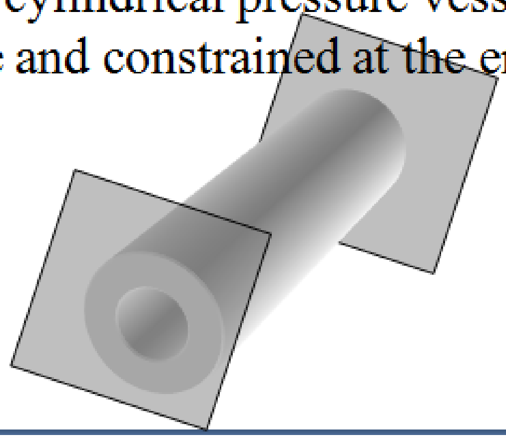
1. Displacement components u,v functions of (x,y) only and w=0
2. Top and bottom surfaces are fixed
3. $X_c=0$
4. p_x and p_y do not vary with z

PLANE STRAIN Examples:

1. Dam



2. Long cylindrical pressure vessel subjected to internal/external pressure and constrained at the ends



PLANE STRAIN

Nonzero **stress**: $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}$

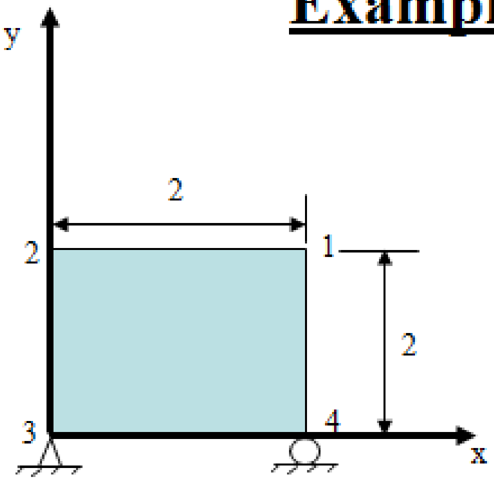
Nonzero **strain** components: $\varepsilon_x, \varepsilon_y, \gamma_{xy}$

Isotropic linear elastic stress-strain law $\boxed{\underline{\sigma} = \underline{D}\,\underline{\varepsilon}}$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad \sigma_z = \nu(\sigma_x + \sigma_y)$$

Hence, the D matrix for the **plane strain case** is

$$\underline{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$



Example problem

The square block is in **plane strain** and is subjected to the following strains

$$\varepsilon_x = 2xy$$

$$\varepsilon_y = 3xy^2$$

$$\gamma_{xy} = x^2 + y^3$$

Compute the displacement field (i.e., displacement components u(x,y) and v(x,y)) within the block

Solution

Recall from definition

$$\varepsilon_x = \frac{\partial u}{\partial x} = 2xy \qquad (1)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = 3xy^2 \qquad (2)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = x^2 + y^3 \qquad (3)$$

Arbitrary function of 'x'

Integrating (1) and (2)

$$u(x, y) = x^2 y + C_1(y) \qquad (4)$$

$$v(x, y) = xy^3 + C_2(x) \qquad (5)$$

Arbitrary function of 'y'

Plug expressions in (4) and (5) into equation (3)

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=x^2+y^3 \quad (3)$$

$$\Rightarrow \frac{\partial\left[x^2y+C_1(y)\right]}{\partial y}+\frac{\partial\left[xy^3+C_2(x)\right]}{\partial x}=x^2+y^3$$

$$\Rightarrow x^2+\frac{\partial C_1(y)}{\partial y}+y^3+\frac{\partial C_2(x)}{\partial x}=x^2+y^3$$

$$\Rightarrow \frac{\partial C_1(y)}{\partial y}+\frac{\partial C_2(x)}{\partial x}=0$$

Function of ‘y’

Function of ‘x’

Hence

$$\frac{\partial C_1(y)}{\partial y} = -\frac{\partial C_2(x)}{\partial x} = C \text{ (a constant)}$$

Integrate to obtain

$$C_1(y) = Cy + D_1$$
$$C_2(x) - Cx + D_2$$

D_1 and D_2 are two constants of integration

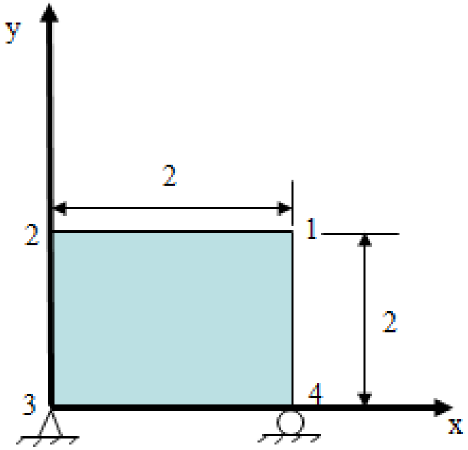
Plug these back into equations (4) and (5)

$$(4)\quad u(x,y) = x^2 y + Cy + D_1$$

$$(5)\quad v(x,y) = xy^3 - Cx + D_2$$

How to find C, D_1 and D_2?

Use the 3 **boundary conditions**



$$u(0,0) = 0$$

$$v(0,0) = 0$$

$$v(2,0) = 0$$

To obtain

$$C = 0$$

$$D_1 = 0$$

$$D_2 = 0$$

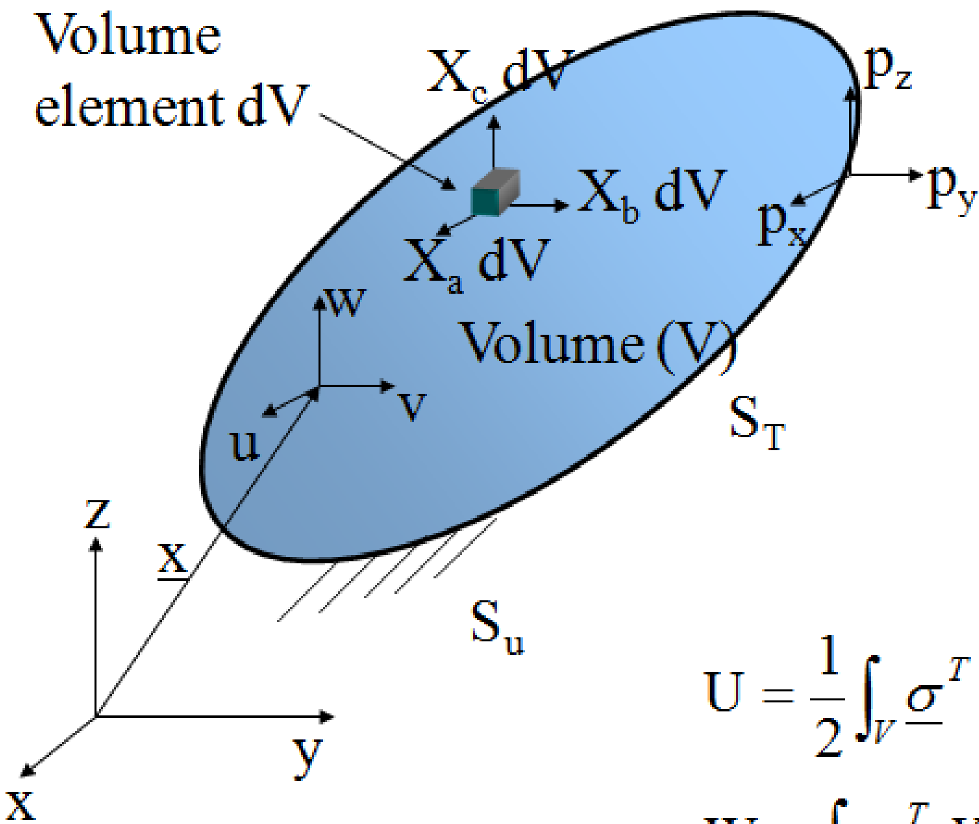
Hence the solution is

$$\boxed{\begin{aligned} u(x,y) &= x^2 y \\ v(x,y) &= xy^3 \end{aligned}}$$

Principle of Minimum Potential Energy

Definition: For a linear elastic body subjected to body forces $\underline{X}=[X_a,X_b,X_c]^T$ and surface tractions $\underline{T}_S=[p_x,p_y,p_z]^T$, causing displacements $u=[u,v,w]^T$ and strains $\underline{\varepsilon}$ and stresses $\underline{\sigma}$, the **potential energy** Π is defined as the strain energy minus the potential energy of the loads involving $\underline{X}$ and $\underline{T}_S$

$$\Pi=U-W$$



$$U = \frac{1}{2}\int_V \underline{\sigma}^T \underline{\varepsilon}\, dV$$

$$W = \int_V \underline{u}^T \underline{X}\, dV + \int_{S_T} \underline{u}^T \underline{T}_S\, dS$$

Strain energy of the elastic body

Using the stress-strain law $\boxed{\underline{\sigma} = \underline{D}\,\underline{\varepsilon}}$

$$U = \frac{1}{2}\int_V \underline{\sigma}^T \underline{\varepsilon}\, dV = \frac{1}{2}\int_V \underline{\varepsilon}^T \underline{D}\,\underline{\varepsilon}\, dV$$

<u>In 1D</u>

$$U = \frac{1}{2}\int_V \sigma\varepsilon\, dV = \frac{1}{2}\int_V E\varepsilon^2\, dV = \frac{1}{2}\int_{x=0}^{L} E\varepsilon^2\, A dx$$

<u>In 2D **plane stress** and **plane strain**</u>

$$U = \frac{1}{2}\int_V \left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy}\right) dV$$

Why?

Principle of minimum potential energy:

- Among all **admissible** displacement fields the one that satisfies the equilibrium equations also render the potential energy P a minimum.
- "admissible displacement field":
- 1. first derivative of the displacement components exist
- 2. satisfies the boundary conditions on S_u