INSTITUTE OF AERONAUTICAL ENGINEERING

(Autonomous)

Dundigal,Hyderabad-500043

CIVIL ENGINEERING

COURSE LECTURE NOTES

Course Name	**MATHEMATICAL TRANSFORM TECHNIQUES**
Course Code	AHSB11
Programme	B.Tech
Semester	II
Course Coordinator	Dr. S. Jagadha, Associate Professor
Course Faculty	Dr. P. Srilatha, Associate Professor Ms. L Indira, Assistant Professor Ms. C Rachana, Assistant Professor Ms. P Rajani, Assistant Professor Ms. B. Praveena, Assistant Professor
Lecture Numbers	1-63
Topic Covered	All

COURSE OBJECTIVES (COs):

The course should enable the students to:	
I	Enrich the knowledge of solving Algebraic and Transcendental equations and Differential equation by numerical methods.
II	Determine the Fourier transforms for various functions in a given period.
III	Determine the Laplace and Inverse Laplace transforms for various functions using standard types.
IV	Formulate to solve Partial differential equation.

COURSE LEARNING OUTCOMES (CLOs):

Students, who complete the course, will have demonstrated the ability to do the following:

AHSB11.01	Evaluate the real roots of algebraic and transcendental equations by Bisection method, False position and Newton -Raphson method.
AHSB11.02	Apply the nature of properties to Laplace transform and inverse Laplace transform of the given function.
AHSB11.03	Solving Laplace transforms of a given function using shifting theorems.
AHSB11.04	Evaluate Laplace transforms using derivatives of a given function.
AHSB11.05	Evaluate Laplace transforms using multiplication of a variable to a given function.
AHSB11.06	Apply Laplace transforms to periodic functions.
AHSB11.07	Apply the symbolic relationship between the operators using finite differences.
AHSB11.08	Apply the Newtons forward and Backward, Gauss forward and backward Interpolation

	method to determine the desired values of the given data at equal intervals, also unequal intervals.
AHSB11.09	Solving Laplace transforms and inverse Laplace transform using derivatives and integrals.
AHSB11.10	Evaluate inverse of Laplace transforms and inverse Laplace transform by the method of convolution.
AHSB11.11	Solving the linear differential equations using Laplace transform.
AHSB11.12	Understand the concept of Laplace transforms to the real-world problems of electrical circuits, harmonic oscillators, optical devices, and mechanical systems
AHSB11.13	Ability to curve fit data using several linear and non linear curves by method of least squares.
AHSB11.14	Understand the nature of the Fourier integral.
AHSB11.15	Ability to compute the Fourier transforms of the given function.
AHSB11.16	Ability to compute the Fourier sine and cosine transforms of the function
AHSB11.17	Evaluate the inverse Fourier transform, Fourier sine and cosine transform of the given function.
AHSB11.18	Evaluate finite and infinite Fourier transforms
AHSB11.19	Understand the concept of Fourier transforms to the real-world problems of circuit analysis, control system design
AHSB11.20	Apply numerical methods to obtain approximate solutions to Taylors, Eulers, Modified Eulers
AHSB11.21	Runge-Kutta methods of ordinary differential equations.
AHSB11.22	Understand the concept of order and degree with reference to partial differential equation
AHSB11.23	Formulate and solve partial differential equations by elimination of arbitrary constants and functions
AHSB11.24	Understand partial differential equation for solving linear equations by Lagrange method.
AHSB11.25	Apply the partial differential equation for solving non-linear equations by Charpit's method
AHSB11.26	Apply method of separation of variables, Solving the heat equation and wave equation in subject to boundary conditions
AHSB11.27	Understand the concept of partial differential equations to the real-world problems of electromagnetic and fluid dynamics

SYLLABUS

Module-I	ROOT FINDING TECHNIQUES AND LAPLACE TRANSFORMS	Classes: 09
ROOT FINDING TECHNIQUES: Root finding techniques: Solving algebraic and Transcendental equations by bisection method, Method of false position, Newton-Raphson method. **LAPLACE TRANSFORMS**:Definition of Laplace transform, Linearity property, Piecewise continuous function, existence of Laplace transform, Function of exponential order, First and second shifting theorems, change of scale property, Laplace transforms of derivatives and integrals, Multiplied by t, Divided by t, Laplace transform of periodic functions.		
Module-II	**INTERPOLATION AND INVERSE LAPLACE TRANSFORMS**	**Classes: 09**
INTERPOLATION:Interpolation: Finite differences, Forward differences, Backward differences and central differences; Symbolic relations; Newton's forward interpolation, Newton's backward interpolation; Gauss forward central difference formula, Gauss backward central difference formula; Interpolation of unequal intervals: Lagrange's interpolation. **INVERSE LAPLACE TRANSFORMS:** Inverse Laplace transform: Definition of Inverse Laplace transform, Linearity property, First and second shifting theorems, Change of scale property, Multiplied by		

s, divided by s; Convolution theorem and applications.		
Module-III	**CURVE FITTING AND FOURIER TRANSFORMS**	**Classes: 09**
CURVE FITTING: Fitting a straight line; Second degree curves; Exponential curve, Power curve by method of least squares. **FOURIERTRANSFORMS**:Fourier integral theorem, Fourier sine and cosine integrals; Fourier transforms; Fourier sine and cosine transform, Properties, Inverse transforms, Finite Fourier transforms.		
Module-IV	**NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS**	**Classes: 09**
STEP BY STEP METHOD: Taylor's series method; Euler's method, Modified Euler's method for first order differential equations. **MULTI STEP METHOD:** Runge-Kutta method for first order differential equations.		
Module-V	**PARTIAL DIFFERENTIAL EQUATIONS AND APPLICATIONS**	**Classes: 09**
PARTIAL DIFFERENTIAL EQUATIONS: Formation of partial differential equations by elimination of arbitrary constants and arbitrary functions, Solutions of first order linear equation by Lagrange method and nonlinear by Charpit method **APPLICATIONS:** Method of separation of variables; One dimensional heat and wave equations under initial and boundary conditions.		
Text Books:		
1. B.S. Grewal, Higher Engineering Mathematics, Khanna Publishers, 36th Edition, 2010. 2. N.P. Bali and Manish Goyal, A text book of Engineering Mathematics, Laxmi Publications, Reprint, 2008. 3. Ramana B.V., Higher Engineering Mathematics, Tata McGraw Hill New Delhi, 11th Reprint,2010.		
Reference Books:		
1. Erwin Kreyszig, Advanced Engineering Mathematics, 9th Edition, John Wiley & Sons, 2006. 2. Veerarajan T., Engineering Mathematics for first year, Tata McGraw-Hill, New Delhi, 2008. 3. D. Poole, Linear Algebra: A Modern Introduction, 2nd Edition, Brooks/Cole, 2005. 4. Dr. M Anita, Engineering Mathematics-I, Everest Publishing House, Pune, First Edition, 2016.		

MODULE-I

ROOT FINDING TECHNIQUES AND LAPLACE TRANSFORMS

Solutions of Algebraic and Transcendental equations:

1) **Polynomial function:** A function $f(x)$ is said to be a polynomial function if $f(x)$ is a polynomial in x.

ie, $f(x) = a_0x^n + a_1x^{n-1} + \cdots \dots \dots \dots . . + a_{n-1}x + a_n$

where $a_0 \neq 0$, the co-efficients $a_0, a_1 \dots\dots\dots a_n$ are real constants and n is a non-negative integer.

2) **Algebraic function:** A function which is a sum (or) difference (or) product of two polynomials is called an algebraic function. Otherwise, the function is called a transcendental (or) non-algebraic function.

Eg: $f(x) = c_1e^x + c_2e^{-x} = 0$

$$f(x) = e^{5x} - \frac{x^3}{2} + 3 = 0$$

3) **Root of an equation:** A number α is called a root of an equation $f(x) = 0$ if

$f(\alpha) = 0$. We also say that α is a zero of the function.

Note: The roots of an equation are the abscissae of the points where the graph $y = f(x)$ cuts the x-axis.

Methods to find the roots of $f(x) = 0$

Direct method: We know the solution of the polynomial equations such as linear equation $ax + b = 0$, and quadratic equation $ax^2 + bx + c = 0$,using direct methods or analytical methods. Analytical methods for the solution of cubic and quadratic equations are also available.

Bisection method: Bisection method is a simple iteration method to solve an equation. This method is also known as Bolzono method of successive bisection. Some times it is referred to as half-interval method. Suppose we know an equation of the form $f(x) = 0$

has exactly one real root between two real numbers x_0, x_1.The number is choosen such that $f(x_0)$ and $f(x_1)$will have opposite sign. Let us bisect the interval $[x_0, x_1]$ into two half intervals and find the mid point $x_2 = \frac{x_0 + x_1}{2}$. If $f(x_2) = 0$ then x_2 is a root.

If $f(x_1)$ and $f(x_2)$ have same sign then the root lies between x_0 and x_2. The interval is taken as$[x_0, x_2]$. Otherwise the root lies in the interval $[x_2, x_1]$.

PROBLEMS

1). Find a root of the equation $x^3 - 5x + 1 = 0$ using the bisection method in 5 – stages

Sol Let $f(x) = x^3 - 5x + 1$. We note that $\begin{matrix} f(0) > 0 \\ f(1) < 0 \end{matrix}$ *and*

$\therefore$ One root lies between 0 and 1

Consider $x_0 = 0$ *and* $x_1 = 1$

By Bisection method the next approximation is

$$x_2 = \frac{x_0 + x_1}{2} = \frac{1}{2}(0+1) = 0.5$$

$$\Rightarrow f(x_2) = f(0:5) = -1.375 < 0 \text{ and } f(0) > 0$$

We have the root lies between 0 and 0.5

Now $x_3 = \frac{0+0.5}{2} = 0.25$

We find $f(x_3) = -0.234375 < 0$ *and* $f(0) > 0$

Since $f(0) > 0$, we conclude that root lies between x_0 *and* x_3

The third approximation of the root is

$$x_4 = \frac{x_0 + x_3}{2} = \frac{1}{2}(0 + 0.25) = 0.125$$

We have $f(x_4) = 0.37495 > 0$

Since $f(x_4) > 0$ *and* $f(x_3) < 0$, the root lies between

$x_4 = 0.125$ *and* $x_3 = 0.25$

Considering the 4th approximation of the roots

$x_5 = \frac{x_3 + x_4}{2} = \frac{1}{2}(0.125 + 0.25) = 0.1875$

$f(x_5) = 0.06910 > 0$, since $f(x_5) > 0$ *and* $f(x_3) < 0$ the root must lie between $x_5 = 0.18758$ *and* $x_3 = 0.25$

Here the fifth approximation of the root is

$$x_6 = \frac{1}{2}(x_5 + x_3)$$
$$= \frac{1}{2}(0.1875 + 0.25)$$
$$= 0.21875$$

We are asked to do up to 5 stages

We stop here 0.21875 is taken as an approximate value of the root and it lies between 0 and 1

2) Find a root of the equation $x^3 - 4x - 9 = 0$ using bisection method in four stages

Sol Let $f(x) = x^3 - 4x - 9$

We note that $f(2) < 0$ and $f(3) > 0$

$\therefore$ One root lies between 2 and 3

Consider $x_0 = 2$ *and* $x_1 = 3$

By Bisection method $x_2 = \frac{x_0 + x_1}{2} = 2.5$

Calculating $f(x_2) = f(2.5) = -3.375 < 0$

$\therefore$ The root lies between x_2 *and* x_1

The second approximation is $x_3 = \frac{1}{2}(x_1 + x_2) = \frac{2.5+3}{2} = 2.75$

Now $f(x_3) = f(2.75) = 0.7969 > 0$

$\therefore$ The root lies between x_2 *and* x_3

Thus the third approximation to the root is

$x_4 = \frac{1}{2}(x_2 + x_3) = 2.625$

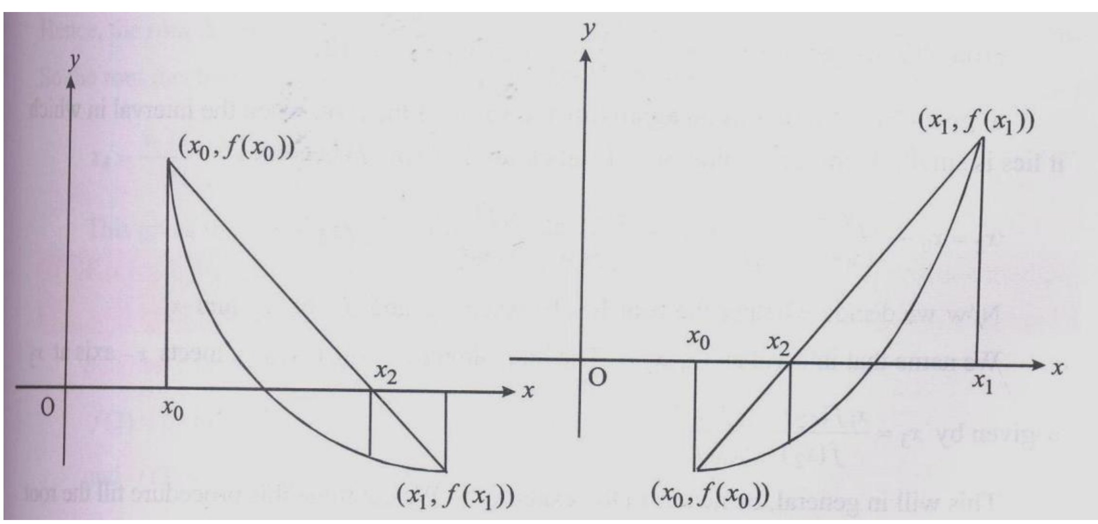
Again $f(x_4) = f(2.625) = -1.421 < 0$

$\therefore$ The root lies between x_3 *and* x_4

Fourth approximation is $x_5 = \frac{1}{2}(x_3 + x_4) = \frac{1}{2}(2.75 + 2.625) = 2.6875$

False Position Method (Regula – Falsi Method)

In the false position method we will find the root of the equation $f(x) = 0$ Consider two initial approximate values x_0 *and* x_1 near the required root so that $f(x_0)$ *and* $f(x_1)$ have different signs. This implies that a root lies between x_0 *and* x_1. The curve $f(x)$ crosses x- axis only once at the point x_2 lying between the points x_0 *and* x_1. Consider the point $A = (x_0, f(x_0))$ and $B = (x_1, f(x_1))$ on the graph and suppose they are connected by a straight line. Suppose this line cuts x-axis at x_2. We calculate the value of $f(x_2)$ at the point. If $f(x_0)$ *and* $f(x_2)$ are of opposite signs, then the root lies between x_0 *and* x_2 and value x_1 is replaced by x_2 Other wise the root lies between x_2 and x_1 and the value of x_0 is replaced by x_2. Another line is drawn by connecting the newly obtained pair of values. Again the point here cuts the x-axis is a closer approximation to the root. This process is repeated as many times as required to obtain the desired accuracy. It can be observed that the points $x_2, x_3, x_4, \ldots$ obtained converge to the expected root of the equation $y = f(x)$



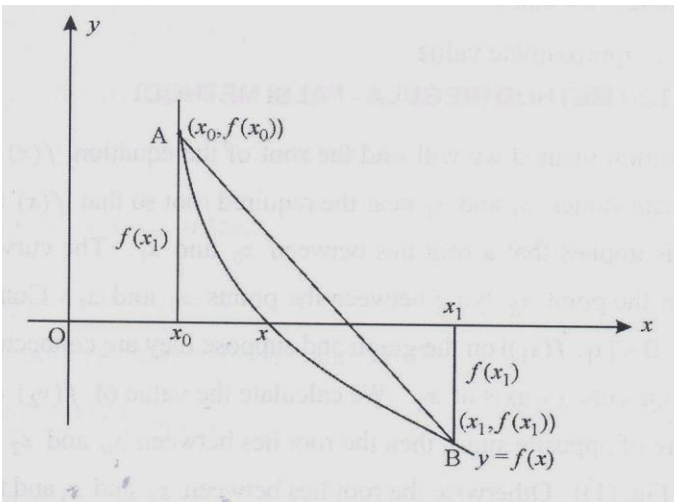
To Obtain the equation to find the next approximation to the root

Let $A=\left(x_0, f\left(x_0\right)\right)$ *and* $B=\left(x_1, f\left(x_1\right)\right)$ be the points on the curve $y=f\left(x\right)$ Then the equation to the chord AB is $\frac{y-f(x_0)}{x-x_0}=\frac{f(x_1)-f(x_0)}{x_1-x_0}$ ------(1)

At the point C where the line AB crosses the x – axis, where $f(x)=0$ $ie,\ y=0$

From (1), we get $x=x_0-\frac{x_1-x_0}{f\left(x_1\right)-f\left(x_0\right)} f\left(x_0\right)$ --------------(2)

x is given by (2) serves as an approximated value of the root, when the interval in which it lies is small. If the new value of x is taken as x_2 then (2) becomes



$$x_2 = x_0-\frac{\left(x_1-x_0\right)}{f\left(x_1\right)-f\left(x_0\right)} f\left(x_0\right)$$

$$=\frac{x_0 f\left(x_1\right)-x_1 f\left(x_0\right)}{f\left(x_1\right)-f\left(x_0\right)} \text{ -------------(3)}$$

Now we decide whether the root lies between

x_0 *and* x_2 $(or)\, x_2$ *and* x_1

We name that interval as $\left(x_1, x_2\right)$ The line joining$(x_1, y_1), (x_2, y_2)$ meets x – axis at x_3 is given by $x_3=\frac{x_1 f\left(x_2\right)-x_2 f\left(x_1\right)}{f\left(x_2\right)-f\left(x_1\right)}$

This will in general, be nearest to the exact root. We continue this procedure till the root is found to the desired accuracy

The iteration process based on (3) is known as the method of false position

The successive intervals where the root lies, in the above procedure are named as

$(x_0, x_1), (x_1, x_2), (x_2, x_3)$ etc

Where $x_i < x_{i+1}$ and $f(x_0), f(x_{i+1})$ are of opposite signs.

Also $x_{i+1} = \dfrac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$

PROBLEMS:

1. **By using Regula - Falsi method, find an approximate root of the equation** $x^4 - x - 10 = 0$ **that lies between 1.8 and 2. Carry out three approximations**

Sol. Let us take $f(x) = x^4 - x - 10$ and $x_0 = 1.8, x_1 = 2$

Then $f(x_0) = f(1.8) = -1.3 < 0$ and $f(x_1) = f(2) = 4 > 0$

Since $f(x_0)$ and $f(x_1)$ are of opposite signs,the equation $f(x) = 0$ has a root between x_0 *and* x_1

The first order approximation of this root is

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 1.8 - \frac{2 - 1.8}{4 + 1.3} \times (-1.3)$$

$$= 1.849$$

We find that $f(x_2) = -0.161$ so that $f(x_2)$ *and* $f(x_1)$ are of opposite signs. Hence the root lies between x_2 *and* x_1 and the second order approximation of the root is

$$x_3 = x_2 - \left[\frac{x_1 - x_2}{f(x_1) - f(x_2)}\right] . f(x_2)$$

$$= 1.8490 - \left[\frac{2 - 1.849}{0.159}\right] \times (-0.159)$$

$$= 1.8548$$

we find that $f(x_3) = f(1.8548)$

$$= -0.019$$

So that $f(x_3)$ *and* $f(x_2)$ are of the same sign. Hence, the root does not lie between x_2 *and* x_3.But $f(x_3)$ *and* $f(x_1)$ are of opposite signs. So the root lies between x_3 *and* x_1 and the third order approximate value of the root is $x_4 = x_3 - \left[\frac{x_1 - x_3}{f(x_1) - f(x_3)}\right] f(x_3)$

$$= 1.8548 - \frac{2 - 1.8548}{4 + 0.019} \times (-0.019)$$

$= 1.8557$

This gives the approximate value of x.

2. **Find out the roots of the equation** $x^3 - x - 4 = 0$ **using False position method**

Sol. Let $f(x) = x^3 - x - 4 = 0$

Then $f(0) = -4, f(1) = -4, f(2) = 2$

Since $f(1)$ *and* $f(2)$ have opposite signs the root lies between 1 and 2

By False position method $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

$$x_2 = \frac{(1 \times 2) - 2(-4)}{2 - (-4)}$$

$$= \frac{2+8}{6} = \frac{10}{6} = 1.666$$

$$f(1.666) = (1.666)^3 - 1.666 - 4$$
$$= -1.042$$

Now, the root lies between 1.666 and 2

$$x_3 = \frac{1.666 \times 2 - 2 \times (-1.042)}{2 - (-1.042)} = 1.780$$

$$f(1.780) = (1.780)^3 - 1.780 - 4$$
$$= -0.1402$$

Now, the root lies between 1.780 and 2

$$x_4 = \frac{1.780 \times 2 - 2 \times (-0.1402)}{2 - (-0.1402)} = 1.794$$

$$f(1.794) = (1.794)^3 - 1.794 - 4$$
$$= -0.0201$$

Now, the root lies between 1.794 and 2

$$x_5 = \frac{1.794 \times 2 - 2 \times (-0.0201)}{2 - (-0.0201)} = 1.796$$

$$f(1.796) = (1.796)^3 - 1.796 - 4 = -0.0027$$

Now, the root lies between 1.796 and 2

$$x_6 = \frac{1.796 \times 2 - 2 \times (-0.0027)}{2 - (-0.0027)} = 1.796$$

The root is 1.796

Newton- Raphson Method:-

The Newton- Raphson method is a powerful and elegant method to find the root of an equation. This method is generally used to improve the results obtained by the previous methods.

Let x_0 be an approximate root of $f(x) = 0$ and let $x_1 = x_0 + h$ be the correct root which implies that $f(x_1) = 0$. We use Taylor's theorem and expand $f(x_1) = f(x_0 + h) = 0$

$$\Rightarrow f(x_0) + hf^1(x_0) = 0$$

$$\Rightarrow h = -\frac{f(x_0)}{f^1(x_0)}$$

Substituting this in x_1 ,we get

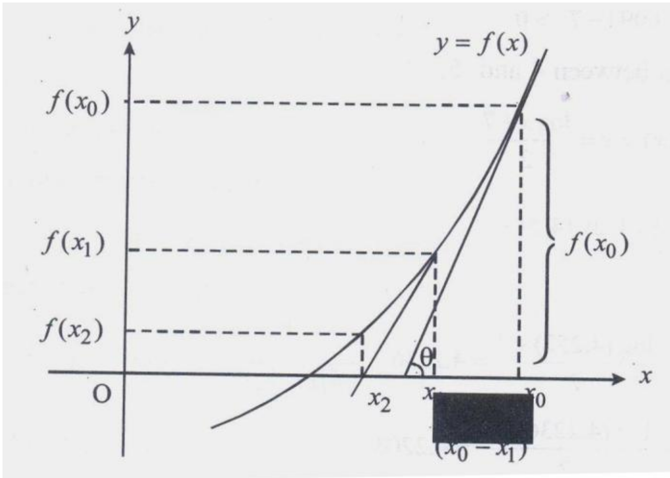
$$x_1 = x_0 + h$$

$$= x_0 - \frac{f(x_0)}{f^1(x_0)}$$

$\therefore x_1$ is a better approximation than x_0

Successive approximations are given by

$x_2, x_3 \ldots\ldots\ldots\ldots x_{n+1}$ where $x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)}$



Problems:

1. Apply Newton – Raphson method to find an approximate root, correct to three decimal places, of the equation $x^3 - 3x - 5 = 0,$ which lies near $x = 2$

<u>Sol:-</u> Here $f\left(x\right) = x^3 - 3x - 5 = 0$ *and* $f^1\left(x\right) = 3\left(x^2 - 1\right)$

$\therefore$ The Newton – Raphson iterative formula

$$x_{i+1} = x_i - \frac{x_i^3 - 3x_i - 5}{3\left(x_i^2 - 1\right)} = \frac{2x_i^3 + 5}{3\left(x_i^2 - 1\right)}, i = 0,1,2....(1)$$

To find the root near $x = 2$, we take $x_0 = 2$ then (1) gives

$$x_1 = \frac{2x_0^3 + 5}{3\left(x_0^2 - 1\right)} = \frac{16+5}{3(4-1)} = \frac{21}{9} = 2.3333$$

$$x_2 = \frac{2x_1^3 + 5}{3\left(x_1^2 - 1\right)} = \frac{2\times(2.3333)^3 + 5}{3\left[(2.3333)^2 - 1\right]} = 2.2806$$

$$x_3 = \frac{2x_2^3 + 5}{3(x_2^3 - 1)} = \frac{2 \times (2.2806)^3 + 5}{3[(2.2806)^2 - 1]} = 2.2790$$

$$x_4 = \frac{2 \times (2.2790)^3 + 5}{3[(2.2790)^2 - 1]} = 2.2790$$

Since x_3 and x_4 are identical up to 3 places of decimal, we take $x_4 = 2.279$ as the required root, correct to three places of the decimal

LAPLACE TRANSFORMS

Introduction

In mathematics the Laplace transform is an integral transform named after its discoverer Pierre-Simon Laplace . It takes a function of a positive real variable t (often time) to a function of a complex variable s (frequency).The Laplace transform is very similar to the Fourier transform. While the Fourier transform of a function is a complex function of a real variable (frequency), the Laplace transform of a function is a complex function of a complex variable. Laplace transforms are usually restricted to functions of t with $t > 0$. A consequence of this restriction is that the Laplace transform of a function is a holomorphic function of the variable s. Unlike the Fourier transform, the Laplace transform of a distribution is generally a well-behaved function. Also techniques of complex variables can be used directly to study Laplace transforms. As a holomorphic function, the Laplace transform has a power series representation. This power series expresses a function as a linear superposition of moments of the function. This perspective has applications in probability theory.

Introduction

Let f(t) be a given function which is defined for all positive values of t, if

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

exists, then F(s) is called Laplace transform of f(t) and is denoted by

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

The inverse transform, or inverse of L{f(t)} or F(s), is

$$f(t) = L^{-1}\{F(s)\}$$

where s is real or complex value.

Laplace Transform of Basic Functions

1. $\mathcal{L}\ [1]=\int_0^{\infty} e^{-st}dt=-\frac{1}{s}e^{-st}\Big|_0^{\infty}=\frac{1}{s}$

2. $\mathcal{L}\ [t^a]=\int_0^{\infty} t^a e^{-st}dt=\int_0^{\infty}(\frac{u}{s})^a e^{-u}\frac{du}{s}=\frac{1}{s^{a+1}}\int_0^{\infty}u^a e^{-u}du=\frac{\Gamma(a+1)}{s^{a+1}}$

3. $\mathcal{L}\ [e^{at}]=\int_0^{\infty} e^{at}e^{-st}dt=\frac{e^{-(s-a)t}}{-(s-a)}\Big|_0^{\infty}=\frac{1}{s-a}$

4. $\mathcal{L}\ [e^{iat}]=\frac{1}{s-ia}\Rightarrow \mathcal{L}\ [\cos at+i\sin at]=\frac{s}{s^2+a^2}+i\frac{a}{s^2+a^2}$

$\therefore \mathcal{L}\ [\cos at]=\frac{s}{s^2+a^2}$, and $\mathcal{L}\ [\sin at]=\frac{a}{s^2+a^2}$

5. $\mathcal{L}\ [\sinh at]=\mathcal{L}\ [\frac{e^{at}-e^{-at}}{2}]=\frac{1}{2}(\frac{1}{s-a}-\frac{1}{s+a})=\frac{a}{s^2-a^2}$

$\mathcal{L}\ [\cosh at]=\mathcal{L}\ [\frac{e^{at}+e^{-at}}{2}]=\frac{1}{2}(\frac{1}{s-a}+\frac{1}{s+a})=\frac{s}{s^2-a^2}$

1. Linearity

$\mathcal{L}\ [af(t)+bg(t)]=\int_0^{\infty}[af(t)+bg(t)]e^{-st}dt=a\int_0^{\infty}f(t)e^{-st}dt+b\int_0^{\infty}g(t)e^{-st}dt=aF(s)+bG(s)$

EX: Find the Laplace transform of $\cos^2 t$.

Solution : $\mathcal{L}[\cos^2 t]=\mathcal{L}[\frac{1+\cos 2t}{2}]=\frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+2^2}\right)=\frac{s^2+2}{s(s^2+4)}$

2. Shifting

$(a)\ \mathcal{L}[f(t-a)u(t-a)]=\int_0^{\infty}f(t-a)u(t-a)e^{-st}dt=\int_a^{\infty}f(t-a)e^{-st}dt$

Let $\tau=t-a$, then

$\mathcal{L}[f(t-a)u(t-a)]=\int_0^{\infty}f(\tau)e^{-s(\tau+a)}d\tau=e^{-sa}\int_0^{\infty}f(\tau)e^{-s\tau}d\tau=e^{-sa}F(s)$

$(b)\ F(s-a)=\int_0^{\infty}f(t)e^{-(s-a)t}dt=\int_0^{\infty}[e^{at}f(t)]e^{-st}dt=\mathcal{L}[e^{at}f(t)]$

EX: What is the Laplace transform of the function $f(t)=\begin{cases}0, & t<4\\ 2t^3, & t\ge 4\end{cases}$

Solution: $f(t)=2t^3u(t-4)$

L $[f(t)]=$L $\{2[(t-4)^3+12(t-4)^2+48(t-4)+64]u(t-4)\}$

$$=2e^{-4s}\left(\frac{3!}{s^4}+12\times\frac{2!}{s^3}+48\times\frac{1}{s^2}+\frac{64}{s}\right)=4e^{-4s}\left(\frac{3}{s^4}+\frac{12}{s^3}+\frac{24}{s^2}+\frac{32}{s}\right)$$

3. Scaling

$$\text{L}\ [f(at)]=\int_0^\infty f(at)e^{-st}dt$$

Let $\tau=at$, then

$$\text{L}\ [f(at)]=\int_0^\infty f(\tau)e^{-s\frac{\tau}{a}}d\frac{\tau}{a}=\frac{1}{a}\int_0^\infty f(\tau)e^{-\frac{s}{a}\tau}d\tau=\frac{1}{a}F(\frac{s}{a})$$

EX: Find the Laplace transform of cos2t.

$$\text{Solution}:\because\ \text{L}\ [\cos t]=\frac{s}{s^2+1}$$

$$\therefore\ \text{L}\ [\cos 2t]=\frac{1}{2}\frac{\frac{s}{2}}{(\frac{s}{2})^2+1}=\frac{s}{s^2+4}$$

4. Derivative

(a) Derivative of original function

$$\text{L}\ [f'(t)]=\int_0^\infty f'(t)e^{-st}dt=f(t)e^{-st}\Big|_0^\infty-(-s)\int_0^\infty f(t)e^{-st}dt$$

(1) If $f(t)$ is continuous, equation (2.1) reduces to

L $[f'(t)]=-f(0)+sF(s)=sF(s)-f(0)$

(2) If $f(t)$ is not continuous at $t=a$, equation reduces to

L $[f'(t)]=f(t)e^{-st}\Big|_0^{a^-}+f(t)e^{-st}\Big|_{a^+}^\infty+sF(s)=[f(a^-)e^{-sa}-f(0)]+[0-f(a^+)e^{-sa}]+sF(s)$

$=sF(s)-f(0)-e^{-sa}[f(a^+)-f(a^-)]$

(3) Similarly, if $f(t)$ is not continuous at $t=a_1, a_2, \cdots,\cdots,a_n$, equation reduces to

$$\text{L}\ [f'(t)]=sF(s)-f(0)-\sum_{i=1}^{n}e^{-sa_i}[f(a_i^+)-f(a_i^-)]$$

If $f(t)$, $f'(t)$, $f''(t)$, $\cdots$, $f^{(n-1)}(t)$ are continuous, and $f^{(n)}(t)$ is piecewise continuous, and all of them are exponential order functions, then

$$\mathcal{L}\,[f^{(n)}(t)] = s^n F(s) - \sum_{i=1}^{n} s^{n-i} f^{(i-1)}(0)$$

(b) Derivative of transformed function

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt = \int_0^\infty \frac{\partial}{\partial s}[f(t)e^{-st}]dt = \int_0^\infty (-t)f(t)e^{-st}dt = \mathcal{L}\,[(-t)f(t)]$$

[Deduction] $\dfrac{d^nF(s)}{ds^n} = \mathcal{L}\,[(-t)^n f(t)]$

EX: Find the Laplace transform of te^t.

$$\text{Solution}: \mathcal{L}\,(e^t) = \frac{1}{s-1} \Rightarrow \mathcal{L}\,(te^t) = -\frac{d}{ds}\left(\frac{1}{s-1}\right) = \frac{1}{(s-1)^2}$$

EX: $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases}$, find $\mathcal{L}\,[f'(t)]$.

$$\text{Solution}: f(t) = t^2[u(t) - u(t-1)]$$

$$\mathcal{L}\,[f(t)] = \mathcal{L}\,[t^2u(t)] - \mathcal{L}\,[t^2u(t-1)] = \frac{2!}{s^3} - \mathcal{L}\,\{[(t-1)+1]^2u(t-1)\}$$

$$= \frac{2}{s^3} - \mathcal{L}\,\{[(t-1)^2 + 2(t-1) + 1]u(t-1)\}$$

$$= \frac{2}{s^3} - e^{-s}(\frac{2}{s^3} + 2\frac{1}{s^2} + \frac{1}{s})$$

$$\mathcal{L}\,[f'(t)] = sF(s) - f(0) - e^{-s}[f(1^+) - f(1^-)]$$

$$= [\frac{2}{s^2} - e^{-s}(\frac{2}{s^2} + \frac{2}{s} + 1)] - 0 - e^{-s}(0-1) = \frac{2}{s^2} - e^{-s}(\frac{2}{s^2} + \frac{2}{s})$$

5. Integration

(a) Integral of original function

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \int_0^\infty \int_0^t f(\tau)d\tau e^{-st}dt$$

$$= \frac{1}{-s}\left[e^{-st}\int_0^t f(\tau)d\tau\Big|_0^\infty - \int_0^\infty f(t)e^{-st}dt \right] = \frac{1}{s}F(s)$$

$$\Rightarrow \mathcal{L}\left[\int_0^t\int_0^t ... \int_0^t f(t)dtdt\cdots dt\right] = \frac{1}{s^n}F(s)$$

(b) Integration of Laplace transform

$$\int_s^\infty F(s)ds = \int_s^\infty \int_0^\infty f(t)e^{-st}dtds = \int_0^\infty f(t)\int_s^\infty e^{-st}dsdt$$

$$= \int_0^\infty f(t)\frac{e^{-st}}{-t}\Big|_s^\infty dt = \int_0^\infty \frac{f(t)}{t}e^{-st}dt = \mathcal{L}\left[\frac{f(t)}{t}\right]$$

$$\Rightarrow \int_s^\infty \int_s^\infty \cdots \int_s^\infty F(s)dsds\cdots ds = \mathcal{L}\left[\frac{1}{t^n}f(t)\right]$$

EX: Find (a) $\mathcal{L}\left[\frac{1-e^{-t}}{t}\right]$ (b) $\mathcal{L}\left[\frac{1-e^{-t}}{t^2}\right]$.

Solution : (a) $\mathcal{L}[1-e^{-t}] = \frac{1}{s} - \frac{1}{s+1}$

$$\mathcal{L}\left[\frac{1-e^{-t}}{t}\right] = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s+1}\right)ds = \ln s - \ln(s+1)\Big|_s^\infty = \ln\frac{s}{s+1}\Big|_s^\infty$$

$$= 0 - \ln\frac{s}{s+1} = \ln\frac{s+1}{s}$$

$$(b)\,\mathcal{L}\left[\frac{1-e^{-t}}{t^2}\right] = \int_s^\infty \ln\frac{s+1}{s}ds = s\ln\frac{s+1}{s}\Big|_s^\infty - \int_s^\infty s\left(\frac{1}{s+1} - \frac{1}{s}\right)ds$$

$$= s\ln\frac{s+1}{s}\Big|_s^\infty + \int_s^\infty \frac{1}{s+1}ds = \left[s\ln\frac{s+1}{s} + \ln(s+1)\right]_s^\infty$$

$$= \left[(s+1)\ln(s+1) - s\ln s\right]_s^\infty = s\ln s - (s+1)\ln(s+1)$$

EX: Find (a) $\int_0^\infty \frac{\sin kt\, e^{-st}}{t}dt$ (b) $\int_{-\infty}^\infty \frac{\sin x}{x}dx$.

Solution : $(a)\int_0^{\infty} \frac{\sin kte^{-st}}{t}\,dt = \mathcal{L}\,[\frac{\sin kt}{t}]$

$$\because \mathcal{L}\,[\sin kt] = \frac{k}{s^2+k^2}$$

$$\mathcal{L}\,[\frac{\sin kt}{t}] = \int_s^{\infty} \frac{k}{s^2+k^2}\,ds = \frac{1}{k}\int_s^{\infty} \frac{1}{(\frac{s}{k})^2+1}\,ds$$

$$= \tan^{-1}\frac{s}{k}\Big|_s^{\infty} = \frac{\pi}{2} - \tan^{-1}\frac{s}{k}$$

$$(b)\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = 2\int_0^{\infty} \frac{\sin x}{x}\,dx$$

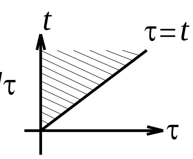
$$= 2\lim_{\substack{k\to 1\\ s\to 0}} \int_0^{\infty} \frac{\sin kte^{-st}}{t}\,dt$$

$$= 2\lim_{\substack{k\to 1\\ s\to 0}} (\frac{\pi}{2} - \tan^{-1}\frac{s}{k}) = \pi$$

6. Convolution theorem

$$\mathcal{L}\,[\int_0^t f(\tau)g(t-\tau)d\tau] = \int_0^{\infty}\int_0^t f(\tau)g(t-\tau)d\tau e^{-st}dt$$

$$= \int_0^{\infty}\int_{\tau}^{\infty} f(\tau)g(t-\tau)e^{-st}dtd\tau = \int_0^{\infty} f(\tau)\int_{\tau}^{\infty} g(t-\tau)e^{-st}dtd\tau$$



Let $u = t-\tau$, $du = dt$, then

$$\mathcal{L}\,[\int_0^t f(\tau)g(t-\tau)d\tau] = \int_0^{\infty} f(\tau)\int_0^{\infty} g(u)e^{-s(u+\tau)}dud\tau$$

$$= \int_0^{\infty} f(\tau)e^{-s\tau}d\tau\int_0^{\infty} g(u)e^{-su}du = F(s)G(s)$$

EX: Find the Laplace transform of $\int_0^t e^{t-\tau}\sin 2\tau\, d\tau$.

Solution : $\because \mathcal{L}\,[e^t] = \frac{1}{s-1}$, $\mathcal{L}\,[\sin 2t] = \frac{2}{s^2+4}$

$$\therefore \mathcal{L}\,[\int_0^t e^{t-\tau}\sin 2t\, d\tau] = \mathcal{L}\,[e^t * \sin 2t] = \mathcal{L}\,[e^t]\cdot\mathcal{L}\,[\sin 2t]$$

$$= \frac{1}{s-1}\cdot\frac{2}{s^2+4} = \frac{2}{(s-1)(s^2+4)}$$

7. Periodic Function: *f* (*t* + *T*) =*f* (*t*)

$$\mathcal{L}\,[f(t)] = \int_0^\infty f(t)e^{-st}dt = \int_0^T f(t)e^{-st}dt + \int_T^{2T} f(t)e^{-st}dt + \cdots\cdots$$

$$\text{and} \int_T^{2T} f(t)e^{-st}dt = \int_0^T f(u+T)e^{-s(u+T)}du = e^{-sT}\int_0^T f(u)e^{-su}du$$

Similarly,

$$\int_{2T}^{3T} f(t)e^{-st}dt = e^{-2sT}\int_0^T f(u)e^{-su}du$$

$$\therefore \mathcal{L}\,[f(t)] = (1 + e^{-sT} + e^{-2sT} + \cdots\cdots)\int_0^T f(t)e^{-st}dt$$

$$= \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt$$

EX: Find the Laplace transform of $f(t) = \frac{k}{p}t,\ 0 < t < p,\ f(t+p) = f(t)$.

$$\text{Solution}: \mathcal{L}\,[f(t)] = \frac{1}{1-e^{-ps}}\int_0^p \frac{k}{p}te^{-st}dt$$

$$= \frac{1}{1-e^{-ps}}\frac{k}{p}[\frac{1}{-s}(te^{-st}\Big|_0^p - \int_0^p e^{-st}dt)]$$

$$= \frac{-k}{ps(1-e^{-ps})}(te^{-st} + \frac{1}{s}e^{-st})\Big|_0^p$$

$$= \frac{-k}{ps(1-e^{-ps})}(pe^{-sp} + \frac{e^{-sp}}{s} - \frac{1}{s})$$

8. Initial Value Theorem:

$$\because \mathcal{L}\,[f'(t)] = sF(s) - f(0) \Rightarrow \lim_{s\to\infty}\int_0^\infty f'(t)e^{-st}dt = \lim_{s\to\infty} sF(s) - f(0) \Rightarrow 0 = \lim_{s\to\infty} sF(s) - f(0)$$

we get initial value theorem $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Deduce general initial value theorem : $\lim_{t\to 0}\frac{f(t)}{g(t)} = \lim_{s\to\infty}\frac{F(s)}{G(s)}$

9. Final Value Theorem:

$$\mathcal{L}[f'(t)] = sF(s) - f(0) \Rightarrow \lim_{s\to 0}\int_0^\infty f'(t)e^{-st}dt = \lim_{s\to 0} sF(s) - f(0) \Rightarrow$$

$$\lim_{t\to\infty} f(t) - f(0) = \lim_{s\to 0} sF(s) - f(0) \Rightarrow \text{final value theorem}: \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$\text{General final value theorem}: \lim_{t\to\infty}\frac{f(t)}{g(t)} = \lim_{s\to 0}\frac{F(s)}{G(s)}$$

EX: Find $\mathcal{L}[\int_0^t \frac{\sin x}{x} dx]$.

$$\text{Solution: Let } f(t) = \int_0^t \frac{\sin x}{x} dx \Rightarrow f'(t) = \frac{\sin t}{t},\ f(0) = 0$$

$$\mathcal{L}[tf'(t)] = \mathcal{L}[\sin t] = \frac{1}{s^2+1}$$

$$-\frac{d}{ds}\mathcal{L}[f'(t)] = \frac{1}{s^2+1}$$

$$-\frac{d}{ds}[sF(s) - f(0)] = \frac{1}{s^2+1} \Rightarrow \frac{d}{ds}[sF(s)] = -\frac{1}{s^2+1}$$

$$sF(s) = -\tan^{-1}s + C$$

From the initial value theorem, we get

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

$$0 = -\frac{\pi}{2} + C \quad \therefore C = \frac{\pi}{2}$$

$$sF(s) = \frac{\pi}{2} - \tan^{-1}s = \tan^{-1}\frac{1}{s}$$

$$F(s) = \frac{1}{s}\tan^{-1}\frac{1}{s}$$

EX: Find $\mathcal{L}\left[\int_t^\infty \frac{e^{-x}}{x} dx\right]$.

Solution : Let $f(t)=\int_x^{\infty}\frac{e^{-x}}{x}dx \Rightarrow f'(t)=-\frac{e^{-t}}{t}, \lim_{t\to\infty} f(t)=0$

$$L\,[tf'(t)]=L\,[-e^{-t}]=-\frac{1}{s+1}$$

$$-\frac{d}{ds}[sF(s)-f(0)]=-\frac{1}{s+1}$$

$$\frac{d}{ds}[sF(s)]=\frac{1}{s+1}$$

$$sF(s)=\ln(s+1)+C$$

From the final value theorem : $\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)$

$$0=0+C \Rightarrow C=0, \text{ and } F(s)=\frac{\ln(s+1)}{s}$$

Note: $\int_0^t \frac{\sin x}{x}dx$, and $\int_t^{\infty}\frac{e^{-x}}{x}dx$ are called sine, and exponential integral function, respectively.

Module-II

INTERPOLATION AND INVERSE LAPLACE TRANSFORMS

INTERPOLATION

Introduction:-

If we consider the statement $y=f(x)x_0\le x\le x_n$ we understand that we can find the value of y, corresponding to every value of x in the range $x_0\le x\le x_n$. If the function $f(x)$ is single valued and continuous and is known explicitly then the values of $f(x)$ for certain values of x like $x_0, x_1, ...x_n$ can be calculated. The problem now is if we are given the set of tabular values

$$x: x_0 \quad x_1 \quad x_2........x_n$$

$$y: y_0 \quad y_1 \quad y_2........y_n$$

Satisfying the relation $y=f(x)$ and the explicit definition of $f(x)$ is not known, then it is possible to find a simple function say $f(x)$ such that $f(x)$ and $\phi(x)$ agree at the set of tabulated points. This process to finding $\phi(x)$ is called interpolation. If $\phi(x)$

is a polynomial then the process is called polynomial interpolation and $\phi(x)$ is called interpolating polynomial. In our study we are concerned with polynomial interpolation

Errors in Polynomial Interpolation:-

Suppose the function $y(x)$ which is defined at the points $(x_i, y_i)\, i = 0,1,2,3----n$ is continuous and differentiable $(n+1)$ times let $\phi_n(x)$ be polynomial of degree not exceeding n such that $\phi_n(x_i) = y_i, i = 1,2---n \rightarrow (1)$ be the approximation of $y(x)$ using this $\phi_n(x_i)$ for other value of x, not defined by (1) the error is to be determined

since $y(x) - \phi_n(x) = 0 \;\; \mathit{for} \;\; x - x_0, x_1, ...x_n$ we put

$y(x) - \phi_n(x) = L\pi_{n+1}(x)$

Where $\pi_{n+1}(x) = (x - x_0)........(x - x_n) \rightarrow (3)$ and L to be determined such that the equation (2) holds for any intermediate value of x such as $x = x^1, x_0 < x^1 < x_n$

Clearly $L = \dfrac{y(x^1) - \phi_n(x^1)}{\pi_{n+1}(x^1)} \rightarrow (4)$

We construct a function $F(x)$ such that $F(x) = F(x_n) = F(x^1)$. Then $F(x)$ vanishes $(n+2)$ times in the interval $[x_0, x_n]$. Then by repeated application of Rolle's theorem. $F^1(x)$must be zero $(n+1)$ times, $F^{11}(x)$ must be zero n times…….. in the interval$[x_0, x_n]$. Also $F^{n+1}(x) = 0$ once in this interval. suppose this point is $x = \varepsilon$, $x_0 < \varepsilon < x_n$ differentiate (5) $(n+1)$ times with respect to x and putting $x = \varepsilon$, we get

$y^{n+1}(\varepsilon) - L(n+1)! = 0$ which implies that $L = \dfrac{y^{n+1}(\varepsilon)}{(n+1)!}$

Comparing (4) and (6) , we get

$$y(x^1) - \phi_n(x^1) = \frac{y^{n+1}(\varepsilon)}{(n+1)!}\pi_{n+1}(x^1)$$

Which can be written as $y(x)-\phi_n(x)=\frac{\pi_{n+1}(x)}{(n+1)!}y^{n+1}(\varepsilon)$

This given the required expression $x_0 < \varepsilon < x_n$ for error

<u>Finite Differences:-</u>

1.**Introduction:-**

In this chapter, we introduce what are called the forward, backward and central differences of a function $y=f(x)$. These differences and three standard examples of finite differences and play a fundamental role in the study of differential calculus, which is an essential part of numerical applied mathematics

2.**Forward Differences:-**

Consider a function $y=f(x)$ of an independent variable x. let $y_0, y_1, y_2, ...y_r$ be the values of y corresponding to the values $x_0, x_1, x_2....x_r$ of x respectively. Then the differences $y_1-y_0, y_2-y_1 -----$ are called the first forward differences of y, and we denote them by $\Delta y_0, \Delta y_1, ...$ that is

$$\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \Delta y_2 = y_3 - y_2.........$$

In general $\Delta y_r = y_{r+1} - y_r \therefore r = 0,1,2-----$

Here, the symbol Δ is called the forward difference operator

The first forward differences of the first forward differences are called second forward differences and are denoted by $\Delta^2 y_0, \Delta^2 y_1......$ that is

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$

$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

In general $\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r \quad r = 0,1,2.......$ similarly, the nth forward differences are defined by the formula.

$$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r \quad r = 0,1,2.......$$

While using this formula for $n=1$, use the notation $\Delta^0 y_r = y_r$ and we have $\Delta^n y_r = 0 \forall n = 1,2......$ *and* $r = 0,2,.........$ the symbol Δ^n is referred as the nth forward difference operator.

3. Forward Difference Table:-

The forward differences are usually arranged in tabular columns as shown in the following table called a forward difference table

Values of x	**Values of y**	**First differences**	**Second differences**	**Third differences**	**Fourth differences**
x_o	y_0				
		$\Delta y_0 = y_1 - y_0$			

x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - y_0$		
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
x_3	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
	y_4	$= y_4 - y_3$			

Example finite forward difference table for $y = x^3$

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		
		91			
6	216				

4. Backward Differences:-

As mentioned earlier, let $y_0, y_1 ......y_r......$ be the values of a function $y = f(x)$ corresponding to the values $x_0, x_1, x_2 ....x_r......$ of x respectively. Then, $\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \nabla y_3 = y_3 - y_2, ...$ are called the first backward differences

In general $\nabla y_r = y_r - y_{r-1},\ r = 1, 2, 3......... \rightarrow (1)$

The symbol ∇ is called the backward difference operator, like the operator Δ, this operator is also a linear operator

Comparing expression (1) above with the expression (1) of section we immediately note that $\nabla y_r = \nabla y_{r-1}, r = 0,1,2....... \rightarrow (2)$

The first backward differences of the first background differences are called second differences and are denoted by $\nabla^2 y_2, \nabla^2 y_3 --- \nabla^2{}_r ----$ i.e.,..

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1, \nabla^2 y_3 = \nabla y_3 - \nabla y_2 \ldots\ldots\ldots$$

In general $\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}, r = 2,3..... \rightarrow (3)$ similarly, the nth backward differences are defined by the formula $\nabla^n y_r = \nabla^{n-1} y_r - \nabla^{n-1} y_{r-1}, r = n, n+1..... \rightarrow (4)$ While using this formula, for n = 1 we employ the notation $\nabla^0 y_r = y_r$

If $y = f(x)$ is a constant function , then y = c is a constant, for all x, and we get $\nabla^n y_r = 0 \forall n$ the symbol ∇^n is referred to as the nth backward difference operator

5. Backward Difference Table:-

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
		∇y_1		
x_1	y_1		$\nabla^2 y_2$	
		∇y_2		$\Delta^2 y_3$
x_2	y_2		$\nabla^2 y_3$	
		∇y_3		
x_3	y_3			

6. Central Differences:-

With $y_0, y_1, y_2....y_r$ as the values of a function $y = f(x)$ corresponding to the values $x_1, x_2.....x_r....$ *of* x, we define the first central differences

$\delta y_{1/2}, \delta y_{3/2}, \delta y_{5/2} ----$ as follows

$$\delta y_{1/2} = y_1 - y_0, \delta y_{3/2} = y_2 - y_1, \delta y_{5/2} = y_3 - y_2 ----$$

$$\delta y_{r-1/2} = y_r - y_{r-1} \rightarrow (1)$$

The symbol δ is called the central differences operator. This operator is a linear operator

Comparing expressions (1) above with expressions earlier used on forward and backward differences we get

$$\delta y_{1/2} = \Delta y_0 = \nabla y_1, \delta y_{3/2} = \Delta y_1 = \nabla y_2.....$$

In general $\delta y_{n+1/2} = \Delta y_n = \nabla y_{n+1}, n = 0,1,2...... \rightarrow (2)$

The first central differences of the first central differences are called the second central differences and are denoted by $\delta^2 y_1, \delta^2 y_2 ...$

Thus $\delta^2 y_1 = \delta_{3/2} - \delta y_{1/2}, \delta^2 y_2 = \delta_{5/2} - \delta_{3/2}.......$

$\delta^2 y_n = \delta y_{n+1/2} - \delta y_{n-1/2} \rightarrow (3)$

Higher order central differences are similarly defined. In general the n[th] central differences are given by

i) for odd $n : \delta^n y_{r-1/2} = \delta^{n-1} y_r - \delta^{n-1} y_{r-1}, r = 1, 2.... \rightarrow (4)$

ii) for even $n : \delta^n y_r = \delta^{n-1} y_{r+1/2} - \delta^{n-1} y_{r-1/2}, r = 1, 2.... \rightarrow (5)$

while employing for formula (4) for $n = 1$, we use the notation $\delta^0 y_r = y_r$

If y is a constant function, that is if $y = c$ a constant, then $\delta^n y_r = 0 \;\; for \;\; all \;\; n \geq 1$

7. Central Difference Table

x_0	y_0	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
		$\delta y_{1/2}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{2/2}$		$\delta^3 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{7/2}$			
x_4	y_4				

Example: Given $f(-2) = 12, f(-1) = 16, f(0) = 15, f(1) = 18, f(2) = 20$ from the central difference table and write down the values of $\delta y_{3/2}, \delta^2 y_0 \;\; and \;\; \delta^3 y_{7/2}$ by taking $x_0 = 0$

Sol. The central difference table is

x	$y = f(x)$	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
-2	12				
		4			
-1	16		-5		
		-1		9	
0	15		4		-14
		3		-5	
1	18		-1		

		2			
2	20				

Symbolic Relations and Separation of symbols:

We will define more operators and symbols in addition to Δ, ∇ and δ already defined and establish difference formulae by symbolic methods

Definition:- The averaging operator μ is defined by the equation $\mu y_r = \frac{1}{2}\left[y_{r+1/2} + y_{r-1/2}\right]$

Definition:- The shift operator E is defined by the equation $Ey_r = y_{r+1}$. This shows that the effect of E is to shift the functional value y_r to the next higher value y_{r+1}. A second operation with E gives $E^2 y_r = E\left(Ey_r\right) = E\left(y_{r+1}\right) = y_{r+2}$

Generalizing $E^n y^r = y_{r+n}$

Relationship Between Δ *and* E

We have

$$\Delta y_0 = y_1 - y_0$$
$$= Ey_0 - y_0 = \left(E-1\right) y_0$$
$$\Rightarrow \Delta = E - y\left(or\right) E = 1 + \Delta$$

Some more relations

$$\Delta^3 y_0 = \left(E-1\right)^3 y_0 = \left(E^3 - 3E^2 + 3E - 1\right) y_0$$
$$= y_3 - 3y_2 + 3y_1 - y_0$$

Definition

Inverse operator E^{-1} is defined as $E^{-1} y_r = y_{r-1}$

In general $E^{-n} y_n = y_{r-n}$

We can easily establish the following relations

i) $\nabla \equiv 1 - E^{-1}$

ii) $\delta \equiv E^{1/2} - E^{-1/2}$

iii) $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$

iv) $\Delta = \nabla E = E^{1/2}$

v) $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$

Definition The operator D is defined as $Dy\left(x\right) = \frac{\partial}{\partial x}\left[y\left(x\right)\right]$

Relation Between The Operators D And E

Using Taylor's series we have, $y(x+h)=y(x)+hy^1(x)+\frac{h^2}{2!}y^{11}(x)+\frac{h^3}{3!}y^{111}(x)+----$

This can be written in symbolic form

$$Ey_x=\left[1+hD+\frac{h^2D^2}{2!}+\frac{h^3D^3}{3!}+----\right]y_x \quad =e^{hD}.y_x$$

We obtain in the relation $E=e^{hD}\rightarrow(3)$

- ❖ If $f(x)$ is a polynomial of degree n and the values of x are equally spaced then $\Delta^n f(x)$ is constant

Proof:

Let $f(x)=a_0x^n+a_1x^{n-1}+------+a_{n-1}x+a_n$ where $a_0,a_1,a_2....a_n$ are constants and $a_0\neq 0$. If h is the step- length, we know the formula for the first forward difference

$$\Delta f(x)=f(x+h)-f(x)=\left[a_0(x+h)^n+a_1(x+h)^{n-1}+----+a_{n-1}(x+h)+a_n\right]$$
$$-\left[a_0x^n+a_1x^{n-1}+----+a_{n-1}x+a_n\right]$$
$$=a_0\left[\left\{x^n+n.x^{n-1}h+\frac{n(n-1)}{2!}x^{n-2}.h^2+---\right\}-x^n\right]+$$
$$a_1\left[\left\{x^{n-1}+(n-1)x^{n-2}.h+\frac{(n-1)(n-2)}{2!}x^{n-3}.h^2+---\right\}-x^{n-1}\right]+$$
$$------+a_{n-1}h$$
$$=a_0nhx^{n-1}+b_2x^{n-2}+b_3x^{n-3}+----+b_{n-3}x+b_{n-2}$$

Where $b_2,b_3,.......b_{n-2}$ are constants. Here this polynomial is of degree $(n-1)$, thus, the first difference of a polynomial of n^th degree is a polynomial of degree $(n-1)$

Now

$$\Delta^2 f(x)=\Delta\left[\Delta f(x)\right]$$
$$=\Delta\left[a_0nh.x^{n-1}+b_2x^{n-2}+b_3x^{n-3}+----+b_{n-1}x+b_{n-2}\right]$$
$$=a_0nh\left[(x+h)^{n-1}-x^{n-1}\right]+b_2\left[(x+h)^{n-2}-x^{n-2}\right]+---+b_{n-1}\left[(x+h)-x\right]$$
$$=a_0n^{(n-1)}h^2x^{n-2}+c_3x^{n-3}+-----+c_{n-4}x+c_{n-3}$$

Where $c_3....c_{n-3}$ are constants. This polynomial is of degree $(n-2)$

Thus, the second difference of a polynomial of degree n is a polynomial of degree $(n-2)$ continuing like this we get $\Delta^n f(x)=a_0n(n-1)(n-2)-----2.1.h^n=a_0h^n(n!)$

$\therefore$ which is constant

Note:-

1. As $\Delta^n f(x)$ is a constant, it follows that $\Delta^{n+1} f(x)=0, \Delta^{n+2} f(x)=0, \ldots\ldots..$
2. The converse of above result is also true that is, if $\Delta^n f(x)$ is tabulated at equal spaced intervals and is a constant, then the function $f(x)$ is a polynomial of degree n

Example:-

1. Form the forward difference table and write down the values of $\Delta f(10)$, $\Delta^2 f(10), \Delta^3 f(15)$ *and* $\Delta^4 y(15)$

x	**10**	**15**	**20**	**25**	**30**	**35**
y	**19.97**	**21.51**	**22.47**	**23.52**	**24.65**	**25.89**

x	**Y**	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
10	**19.97**					
		1.54				
15	**21.51**		**- 0.58**			
		0.96		**0.67**		
20	**22.47**		**0.09**		**- 0.68**	
		1.05		**- 0.01**		**0.72**
25	**23.52**		**0.08**		**0.04**	
		1.13		**0.03**		
30	**24.65**		**0.11**			
		1.24				
35	**25.89**					

We note that the values of x are equally spaced with step- length h = 5

Note: - $\therefore x_0 = 10, x_1 = 15 ---- x_5 = 35$ and

$y_0 = f(x_0) = 19.97$

$y_1 = f(x_1) = 21.51$

$y_5 = f(x_5) = 25.89$

$y_5 = f(x_5) = 25.89$

From table

$$\Delta f\left(10\right)=\Delta y_0 = 1.54$$

$$\Delta^2 f\left(10\right)=\Delta^2 y_0 = -0.58$$

$$\Delta^3 f\left(15\right)=\Delta^3 y_1 = -0.01$$

$$\Delta^4 f\left(15\right)=\Delta^3 y_1 = 0.04$$

2. **Evaluate**

$(i)\,\Delta\cos x$

$(ii)\,\Delta^2\sin\left(px+q\right)$

$(iii)\,\Delta^n e^{ax+b}$

Sol. Let h be the interval of differencing

$$(i)\,\Delta\cos x=\cos\left(x+h\right)-\cos x$$

$$=-2\sin\left(x+\frac{h}{2}\right)\sin\frac{\mathrm{h}}{2}$$

$$(ii)\,\Delta\sin\left(px+q\right)=\sin\left[p\left(x+h\right)+q\right]-\sin\left(px+q\right)$$

$$=2\cos\left(px+q+\frac{ph}{2}\right)\sin\frac{ph}{2}$$

$$=2\sin\frac{ph}{2}\sin\left(\frac{\pi}{2}+px+q+\frac{ph}{2}\right)$$

$$\Delta^2\sin\left(px+q\right)=2\sin\frac{ph}{2}\Delta\left[\sin\left(px+q\right)+\frac{1}{2}\left(\pi+ph\right)\right]$$

$$=\left[2\sin\frac{ph}{2}\right]^2\sin\left[px+q+\frac{1}{2}\left(\pi+ph\right)\right]$$

$$(iii)\,\Delta e^{ax+b}=e^{a\left(x+h\right)+b}-e^{ax+b}$$

$$=e^{\left(ax+b\right)}\left(e^{ah-1}\right)$$

$$\Delta^2 e^{ax+b}=\Delta\left[\Delta\left(e^{ax+b}\right)\right]-\Delta\left[\left(e^{ah}-1\right)\left(e^{ax+b}\right)\right]$$

$$=\left(e^{ah}-1\right)^2\Delta\left(e^{ax+h}\right)$$

$$=\left(e^{ah}-1\right)^2 e^{ax+b}$$

Proceeding on, we get $\Delta^n\left(e^{ax+b}\right)=\left(e^{ah}-1\right)^n e^{ax+b}$

3. **Using the method of separation of symbols show that**

$$\Delta^n\mu_{x-n}=\mu_{x-n}-n\mu_{x-1}+\frac{n\left(n-1\right)}{2}\mu_{x-2}+----+\left(-1\right)^n\mu_{x-n}$$

Sol. To prove this result, we start with the right hand side. Thus

$$\mu x-n\mu x-1+\frac{n(n-1)}{2}\mu x-2+-----+(-1)^n\ \mu x-n$$

$$=\mu x-nE^{-1}\mu x+\frac{n(n-1)}{2}E^{-2}\mu x+-----+(-1)^n\ E^{-n}\mu x$$

$$=\left[1-nE^{-1}+\frac{n(n-1)}{2}E^{-2}+-----+(-1)^n\ E^{-n}\right]\mu x=\left(1-E^{-1}\right)^n\ \mu x$$

$$=\left(1-\frac{1}{E}\right)^n\ \mu n=\frac{(E-1)^n}{E}\ \mu n$$

$$=\frac{\Delta^n}{E^n}\ \mu x=\Delta^n E^{-n}\mu x$$

$=\Delta^n\mu_{x-n}$ which is left hand side

4. **Find the missing term in the following data**

x	0	1	2	3	4
y	1	3	9	-	81

Why this value is not equal to 3^3 . Explain

Sol. Consider $\Delta^4 y_0=0$

$\Rightarrow 4y_0-4y_3+5y_2-4y_1+y_0=0$

Substitute given values we get

$81-4y_3+54-12+1=0\Rightarrow y_3=31$

From the given data we can conclude that the given function is $y=3^x$. To find y_3 , we have to assume that y is a polynomial function, which is not so. Thus we are not getting $y=3^3=27$

Newton's Forward Interpolation Formula:-

Let $y=f(x)$ be a polynomial of degree n and taken in the following form

$$y=f(x)=b_0+b_1(x-x_0)+b_2(x-x_0)(x-x_1)+b_3(x-x_0)(x-x_1)(x-x_2)+---$$
$$+b_n(x-x_0)(x-x_1)----(x-x_{n-1})\rightarrow(1)$$

This polynomial passes through all the points $[xi;yi]$ for i = 0 to n. there fore, we can obtain the y_i's by substituting the corresponding x_i's as

$at\ \ x=x_0,y_0=b_0$

$at\ \ x=x_1,y_1=b_0+b_1(x_1-x_0)$

$at\ \ x=x_2,y_2=b_0+b_1(x_2-x_0)+b_2(x_2-x_0)(x_2-x_1)\rightarrow(1)$

Let 'h' be the length of interval such that x_i's represent

$x_0,x_0+h,x_0+2h,x_0+3h----x_0+xh$

This implies $x_1 - x_0 = h, x_2 - x_0 - 2h, x_3 - x_0 = 3h - - - - x_n - x_0 = nh \rightarrow (2)$

From (1) and (2), we get

$y_0 = b_0$

$y_1 = b_0 + b_1 h$

$y_2 = b_0 + b_1 2h + b_2 (2h) h$

$y_3 = b_0 + b_1 3h + b_2 (3h)(2h) + b_3 (3h)(2h) h$

....

....

$y_n = b_0 + b_1 (nh) + b_2 (nh)(n-1)h + - - - + b_n (nh)\left[(n-1)h\right]\left[(n-2)h\right] \rightarrow (3)$

Solving the above equations for $b_0, b_{11}, b_2 .....b_n$, we get $b_0 = y_0$

$$b_1 = \frac{y_1 - b_0}{h} = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}$$

$$b_2 = \frac{y_2 - b_0 - b_1 2h}{2h^2} = y_2 - y_0 - \frac{(y_1 - y_0)}{h} 2h$$

$$= \frac{y_2 - y_0 - 2y_1 - 2y_0}{2h^2} = \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2h^2}$$

$$\therefore b_2 = \frac{\Delta^2 y_0}{2!h^2}$$

Similarly, we can see that

$$b_3 = \frac{\Delta^3 y_0}{3!h^3}, b_4 = \frac{\Delta^4 y_0}{4!h^4} - - - - - b_n = \frac{\Delta^n y_0}{n!h^n}$$

$$\therefore y = f(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1)$$

$$+ \frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2) + - - - +$$

$$+ \frac{\Delta^n y_0}{n!h^n}(x - x_0)(x - x_1) - - - (x - x_{n-1}) \rightarrow (3)$$

If we use the relationship $x = x_0 + ph \Rightarrow x - x_0 = ph,\ where\ p = 0, 1, 2, ...n$

Then

$$x - x_1 = x - (x_0 + h) = (x - x_0) - h$$
$$= ph - h = (p-1)h$$
$$x - x_2 = x - (x_1 + h) = (x - x_1) - h$$
$$= (p-1)h - h = (p-2)h$$
....
$$x - x_i = (p - i)h$$
....
$$x - x_{n-1} = [p - (n-1)]h$$

Equation (3) becomes

$$y = f(x) = f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + ----+$$
$$\frac{p(p-1)(p-2)----(p-(n-1))}{n!}\Delta^n y_0 \rightarrow (4)$$

Newton's Backward Interpolation Formula:-

If we consider

$$y_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + a_3(x - x_n)(x - x_{n-1})(x - x_{n-2}) + ----(x - x_i)$$

and impose the condition that y and $y_n(x)$ should agree at the tabulated points

$x_n, x_n - 1, ...x_2, x_1, x_0$

We obtain

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2i}\nabla^2 y_n + ---$$
$$\frac{p(p+1)----[p+(n-1)]}{n!}\nabla^n y_n + ---- \rightarrow (6)$$

Where $p = \frac{x - x_n}{h}$

This uses tabular values of the left of y_n. Thus this formula is useful formula is useful for interpolation near the end of the table values

Formula for Error in Polynomial Interpolation:-

If $y = f(x)$ is the exact curve and $y = \phi_n(x)$ is the interpolating curve, then the error

in polynomial interpolation is given by

$$Error = f(x) - \phi_n(x) = \frac{(x - x_0)(x - x_1)----(x - x_n)}{(n+1)!} f^{n+1}(\varepsilon) \rightarrow (7)$$

for any x, where $x_0 < x < x_n$ *and* $x_0 < \varepsilon < x_n$

The error in Newton's forward interpolation formula is given by

$$f(x)-\phi_n(x)=\frac{p(p-1)(p-2).......(p-n)}{(n+1)!}\Delta^{n+1}f(\varepsilon)$$

Where $p=\frac{x-x_0}{h}$

The error in Newton's backward interpolation formula is given by

$$f(x)-\phi_n(x)=\frac{p(p+1)(p+2).......(p+n)}{(n+1)!}h^{n+1}y^{n+1}f(\varepsilon)\text{Where } p=\frac{x-x_n}{h}$$

Examples:-

1. Find the melting point of the alloy containing 54% of lead, using appropriate interpolation formula

Percentage of lead(p)	50	60	70	80
Temperature $(Q°c)$	205	225	248	274

Sol. The difference table is

x	Y	Δ	Δ^2	Δ^3
50	205			
		20		
60	225		3	
		23		0
70	248		3	
		26		
80	274			

Let temperature = $f(x)$

$$x_0+ph=24, x_0=50, h=10$$

$$50+p(10)=54\ (or)\ p=0.4$$

By Newton's forward interpolation formula

$$f(x_0+ph)=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2y_0+\frac{p(p-1)(p-2)}{n!}\Delta^3y_0+---$$

$$f(54)=205+0.4(20)+\frac{0.4(0.4-1)}{2!}(3)+\frac{(0.4)(0.4-1)(0.4-2)}{3!}(0)$$

$$=205+8-0.36$$

$$=212.64$$

Melting point = 212.64

2. Using Newton's forward interpolation formula, and the given table of values

X	1.1	1.3	1.5	1.7	1.9
$f(x)$	0.21	0.69	1.25	1.89	2.61

Obtain the value of $f(x)$ *when* $x = 1.4$

Sol.

x	$y = f(x)$	Δ	Δ^2	Δ^3	Δ^4
1.1	0.21				
		0.48			
1.3	0.69		0.08		
		0.56		0	
1.5	1.25		0.08		0
		0.64		0	
1.7	1.89		0.08		
		0.72			
1.9	2.61				

If we take $x_0 = 1.3$ *then* $y_0 = 0.69$,

$$\Delta y_0 = 0.56, \Delta^2 y_0 = 0.08, \Delta^3 y_0 = 0, L = 0.2, x = 1.3$$

$$x_0 + ph = 1.4 (or) 1.3 + p(0.2) = 1.4, p = \frac{1}{2}$$

Using Newton's interpolation formula

$$f(1.4) = 0.69 + \frac{1}{2} \times 0.56 + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!} \times 0.08$$

$$= 0.69 + 0.28 - 0.01 = 0.96$$

3. The population of a town in the decimal census was given below. Estimate the population for the 1895

Year x	1891	1901	1911	1921	1931
Population of y	46	66	81	93	101

Sol. Putting $L=10, x_0=1891, x=1895$ in the formula $x = x_0 + ph$ we obtain $p = 2/5 = 0.4$

X	Y	Δ	Δ^2	Δ^3	Δ^4
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
192	93		-4		
		8			
1931	101				

$$y(1895) = 46 + (0.4)(20) + \frac{(0.4)(0.4-1)}{6} - (-5)$$
$$+ \frac{(0.4-1)0.4(0.4-2)}{6}(2)$$
$$+ \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{24}$$
$$= 54.45 \textit{ thousands}$$

Gauss's Interpolation Formula:- We take x_0 as one of the specified of x that lies around the middle of the difference table and denote $x_0 - rh$ by $x - r$ and the corresponding value of y by $y-r$. Then the middle part of the forward difference table will appear as shown in the next page

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_{-4}	y_{-4}					
x_{-3}	y_{-3}	Δy_{-4}				
x_{-2}	y_{-2}	Δy_{-3}	$\Delta^2 y_{-4}$			
x_{-1}	y_{-1}	Δy_{-2}	$\Delta^2 y_{-3}$	$\Delta^3 y_{-4}$		
x_0	y_0	Δy_{-1}	$\Delta^2 y_{-2}$	$\Delta^3 y_{-3}$	$\Delta^4 y_{-4}$	
x_1	y_1	Δy_0	$\Delta^2 y_{-1}$	$\Delta^3 y_{-2}$	$\Delta^4 y_{-3}$	$\Delta^5 y_{-4}$
x_2	y_2	Δy_1	$\Delta^2 y_0$	$\Delta^3 y_{-1}$	$\Delta^4 y_{-2}$	$\Delta^5 y_{-3}$
x_3	y_3	Δy_2	$\Delta^2 y_1$	$\Delta^3 y_0$	$\Delta^4 y_{-1}$	$\Delta^5 y_{-2}$
x_4	y_4	Δy_3	$\Delta^2 y_2$	$\Delta^3 y_1$	$\Delta^4 y_0$	$\Delta^5 y_{-1}$

$$\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$$
$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$
$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$
$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} - - - - - (1) \quad and$$
$$\Delta y_{-1} = \Delta y_{-2} + \Delta^2 y_{-2}$$
$$\Delta^2 y_{-1} = \Delta^2 y_{-2} + \Delta^3 y_{-2}$$
$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$
$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2} - - - - - (2)$$

By using the expressions (1) and (2), we now obtain two versions of the following Newton's forward interpolation formula

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_0) + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + - - -] \quad \text{----------------3}$$

Here y_p is the value of y at $x = x_p = x_0 + ph$

Gauss Forward Interpolation Formula:-

Substituting for $\Delta^2 y_0, \Delta^3 y_0, ...$ from (1)in the formula (3), we get

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{-1}$$
$$+ \Delta^4 y_{-1} + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_{-1} + \Delta^5 y_{-1} + ----]$$

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1}) + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{p(p+1)(p-1)(p-2)}{4!}(\Delta^4 y_{-1}) + ----]$$

Substituting $\Delta^4 y_{-1}$ *from* (2), this becomes

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(p+1)(p-1)p(p-2)}{4!}(\Delta^4 y_{-2}) + ----] \quad \text{----------------4}$$

Note:- we observe from the difference table that

$\Delta y_0 = \delta y_{1/2}, \Delta^2 y_{-1} = \delta^2 y_0, \Delta^3 y_{-1} = \delta^3 y_{1/2}, \Delta^4 y_{-2} = \delta^4 y_0$ and so on. Accordingly the formula

(4) can be written in the notation of central differences as given below

$$y_p = [y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 y_{1/2}$$
$$+ \frac{(p+1)(p-1)p(p-2)}{4!}\delta^4 y_0 + ----] \quad \text{----------------5}$$

2. **Gauss's Backward Interpolation formula:-**

Let us substitute for $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ ----- from (1) in the formula (3), thus we obtain

$$y_p = [y_0 + p(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{(p-1)p(p-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) +$$
$$\frac{(p-1)(p-2)p(p-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + ----]$$

$$=[y_0+p(\Delta y_{-1})+\frac{(p+1)}{2!}p(\Delta^2 y_{-1})+\frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}+\frac{(p+1)p(p-1)(p-2)}{4!}(\Delta^4 y_{-1})+---]$$

Substituting for $\Delta^3 y_{-1}$ and $\Delta^4 y_{-1}$ from (2) this becomes

$$y_p=[y_0+p(\Delta y_{-1})+\frac{(p+1)p}{2!}\Delta^2 y_{-1}+\frac{(p+1)p(p-1)}{3!}(\Delta^3 y_{-1}+\Delta^4 y_{-2})$$
$$+\frac{(p+1)p(p-1)(p-2)}{4!}(\Delta^4 y_{-2}+\Delta^5 y_{-2})+---]$$

Lagrange's Interpolation Formula:-

Let $x_0,x_1,x_2,....\ x_n$ be the $(n+1)$ values of x which are not necessarily equally spaced. Let $y_0,y_1,y_2........y_n$ be the corresponding values of $y=f(x)$ let the polynomial of degree n for the function $y=f(x)$ passing through the $(n+1)$ points $(x_0,f(x_0)),(x_1,f(x_1))----(x_n,f(x_n))$ be in the following form

$$y=f(x)=a_0(x-x_1)(x-x_2)......(x-x_n)+a_1(x-x_0)(x-x_2).........(x-x_n)+$$
$$a_2(x-x_0)(x-x_1).......(x-x_n)+........+a_n(x-x_0)(x-x_1)......(x-x_{n-1})\to(1)$$

Where $a_0,a_1,a_2....$ a^n are constants

Since the polynomial passes through $(x_0,f(x_0)),(x_1,f(x_1))......(x_n,f(x_n))$. The constants can be determined by substituting one of the values of $x_0,x_1,.....x_n$ *for* x in the above equation

Putting $x=x_0$ in (1) we get, $f(x_0)=a_0(x-x_1)(x_0-x_2)(x_0-x_n)$

$$\Rightarrow a_0=\frac{f(x_0)}{(x-x_1)(x_0-x_2)....(x_0-x_n)}$$

Putting $x=x_1$ in (1) we get, $f(x_1)=a_1(x-x_0)(x_1-x_2)----(x_1-x_n)$

$$\Rightarrow a_1=\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)....(x_1-x_n)}$$

Similarly substituting $x=x_2$ in (1), we get

$$\Rightarrow a_2=\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)......(x_2-x_n)}$$

Continuing in this manner and putting $x=x_n$ in (1) we

get $a_n=\dfrac{f(x_n)}{(x_n-x_0)(x_n-x_1)----(x_n-x_{n-1})}$

Substituting the values of $a_0,a_1,a_2....a_n$, we get

$$f(x)=\frac{(x-x_1)(x-x_2).......(x-x_n)}{(x_0-x_1)(x_0-x_2).........(x_0-x_n)}f(x_0)+\frac{(x-x_0)(x-x_2).....(x-x_n)}{(x_1-x_0)(x_1-x_2)....(x_1-x_n)}$$

$$f(x_1)+\frac{(x-x_0)(x-x_1)(x-x_2).....(x-x_n)}{(x_2-x_0)(x_2-x_1)......(x_2-x_n)}+.....f(x_2)+\frac{(x-x_0)(x-x_1).....(x-x_{n-1})}{(x_n-x_1)(x_n-x_2).....(x_n-x_{n-1})}f(x_n)$$

Examples:-

1. Using Lagrange's formula calculate $f(3)$ from the following table

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

Sol. Given $x_0=0, x_1=1, x_2=2, x_3=4, x_5=6, x_4=5$

$f(x_0)=1, f(x_1)=14, f(x_2)=15, f(x_3)=5, f(x_4)=6, f(x_5)=19$

From langrange's interpolation formula

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)}f(x_0)$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}f(x_1)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)}f(x_2)$$

$$-----------------------$$

$$-----------------------$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}f(x_5)$$

Here $x=3$ then

$$f(3)=\frac{(3-1)(3-2)(3-4)(3-5)(3-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)}\times1+$$

$$\frac{(3-0)(3-2)(3-4)(3-5)(3-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)}\times14+$$

$$\frac{(3-0)(3-1)(3-4)(3-5)(3-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)}\times15+$$

$$\frac{(3-0)(3-1)(3-2)(3-5)(3-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)}\times 5+$$
$$\frac{(3-0)(3-1)(3-2)(3-4)(3-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)}\times 6+$$
$$\frac{(3-0)(3-1)(3-2)(3-4)(3-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)}\times 19$$
$$=\frac{12}{240}-\frac{18}{60}\times 14+\frac{36}{48}\times 15+\frac{36}{48}\times 5-\frac{18}{60}\times 6+\frac{12}{40}\times 19$$
$$=0.05-4.2+11.25+3.75-1.8+0.95$$
$$=10$$
$$f(x_3)=10$$

1) **Find** $f(3.5)$ **using lagrange method of** 2^{nd} **and** 3^{rd} **order degree polynomials.**

x	1	2	3	4
$f(x)$	1	2	9	28

Sol: By lagrange's interpolation formula

$$f(x)=\sum_{k=0}^{n} f(x_k)\frac{(x-x_0).......(x-x_{k-1})(x-x_{k+1})(x-x_n)}{(x_k-x_0).......(x_k-x_{k-1}).......(x_k-x_n)}$$

For $n=4$,we have

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+$$
$$\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)+$$
$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+$$
$$\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)+$$
$$\therefore f(3.5)=\frac{(3.5-2)(3.5-3)(3.5-4)}{(1-2)(1-3)(1-4)}(1)+\quad\frac{(3.5-1)(3.5-3)(3.5-4)}{(2-1)(2-3)(2-4)}(2)+$$

$$\frac{(3.5-1)(3.5-2)(3.5-4)}{(3-1)(3-2)(3-4)}(9)+$$

$$\frac{(3.5-1)(3.5-2)(3.5-3)}{(4-1)(4-2)(4-3)}(28)+ \quad ----$$

=0.0625+(-0.625)+8.4375+8.75

=16.625

$$f(x)=\frac{(x-2)(x-3)(x-4)}{-6}(1)+\frac{(x-1)(x-3)(x-4)}{2}(2)$$

$$+\frac{(x-1)(x-2)(x-4)}{(-2)}(9)+\frac{(x-1)(x-2)(x-3)}{6}(28)$$

$$=\frac{(x^2-5x+6)(x-4)}{-6}+(x^2-4x+3)(x-4)+\frac{(x^2-3x+2)}{-2}(x-4)(9)+\frac{(x^2-3x+2)}{6}(x-3)(28)$$

$$=\frac{x^3-9x^2+26x-24}{-6}+x^3-8x^2+9x-12+\frac{x^3-7x^2+14x-8}{-2}(9)+\frac{x^3-6x^2+11x-6}{6}(28)$$

$$=\frac{\left[-x^3+9x^2-26x+24+6x^3-48x^2+114x-72-27x^3+189x^2-378x+216+308x+28x^3-168x^2-168\right]}{6}$$

$$=\frac{6x^3-18x^2+18x}{6}\Rightarrow f(x)=x^3-3x^2+3x$$

$$\therefore f(3.5)=(3.5)^3-3(3.5)^2+3(3.5)=16.625$$

Example:

Find y(25), given that y_{20} = 24, y_{24} = 32, y_{28} = 35 y_{32} = 40 using Guass forward difference Formula :

Solution: Given

x	20	24	28	32
y	24	32	35	40

By Gauss Forward difference formula

$$y_p=[y_0+p(\Delta y_0)+\frac{p(p-1)}{2!}(\Delta^2 y_{-1}+\Delta^3 y_{-1})+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_{-1}$$

$$+\Delta^4 y_{-1}+\frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_{-1}+\Delta^5 y_{-1}+----]$$

We take x= 24 as origin.

X_0 = 24, h = 4, x = 25 p = x-x_0/ h, p = 25-24/4 = 2.5

Gauss Forward difference table is

X	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	24			
24	32	Δy_{-1}= 8		
28	35	Δy_0 = 3	$\Delta^2 y_{-1}$ = -5	
32	40	Δy_1 = 5	$\Delta^2 y_0$ = 2	$\Delta^3 y_{-1}$ = 7

By gauss Forward interpolation Formula

We y(25) = 32 +.25(3) + $(\frac{.25)(.25-1)}{2}(-5) + \frac{(.25+1)(.25)(.25-1)}{6}(7)$ = 32 + .75 + .46875 - .2734 = 32.945

Y(25) = 32.945.

Example:

Use Gauss Backward interpolation formula to find f(32) given that f(25) = .2707, f(30) = .3027, f(35) = .3386 f(40) = .3794.

Solution: let x_0 = 35 and difference table is

X	y	Δy	$\Delta^2 y$	$\Delta^3 y$
25	.2707			
30	.3027	.032		
35	.3386	.0359	.0039	
40	.3794	.0408	.0049	.0010

From the table y_0 = 0.3386

Δy_{-1} = 0.0359 , $\Delta^2 y_{-1}$ = 0.0049, $\Delta^3 y_{-2}$ = 0.0010, x_p = 32 p = x_p- x_0/h = 32-35/5 = -.6

By Gauss Backward difference formula

f(32) = .3386 + (-.6)(.0359) + (-.6)(-.6+1)(.0049)/2 + (-.6)(.36-1)(0.00010)/6 = .3165

INVERSE LAPLACE TRANSFORMS

I. Inversion from Basic Properties

1. Linearity

Ex. 1.

$(a) L^{-1}[\frac{2s+1}{s^2+4}]$ $(b) L^{-1}[\frac{4(s+1)}{s^2-16}]$.

Solution : $(a)\mathcal{L}^{-1}[\frac{2s+1}{s^2+4}] = \mathcal{L}^{-1}[2\frac{s}{s^2+2^2} + \frac{1}{2}\frac{2}{s^2+2^2}] = 2\cos 2t + \frac{1}{2}\sin 2t$

$(b)\mathcal{L}^{-1}[\frac{4(s+1)}{s^2-16}] = \mathcal{L}^{-1}[4\frac{s}{s^2-4^2} + \frac{4}{s^2-4^2}] = 4\cosh 4t + \sinh 4t$

2. Shifting

Ex. 2.

$(a)\ \mathcal{L}^{-1}[\frac{e^{-\pi s}}{s^2+2s+2}]$ $\qquad (b)\ \mathcal{L}^{-1}[\frac{2s+3}{s^2+3s+2}]$.

Solution : $(a)\ \mathcal{L}^{-1}[\frac{e^{-\pi s}}{s^2+2s+2}] = \mathcal{L}^{-1}[\frac{e^{-\pi s}}{(s+1)^2+1}]$

$\because \mathcal{L}^{-1}[\frac{1}{(s+1)^2+1}] = e^{-t}\sin t$

and $\mathcal{L}[f(t-a)u(t-a)] = e^{-as}F(s)$

$\therefore \mathcal{L}^{-1}[\frac{e^{-\pi s}}{(s+1)^2+1}] = e^{-(t-\pi)}\sin(t-\pi)u(t-\pi) = -e^{-(t-\pi)}\sin t u(t-\pi)$

$(b)\ \mathcal{L}^{-1}[\frac{2s+3}{s^2+3s+2}] = \mathcal{L}^{-1}[\frac{2(s+\frac{3}{2})}{(s+\frac{3}{2})^2-(\frac{1}{2})^2}] = 2e^{-\frac{3}{2}t}\cosh\frac{t}{2}$

3. Scaling

Ex. 3.

$\mathcal{L}^{-1}[\frac{4s}{16s^2-4}]$.

Solution : $\mathcal{L}^{-1}[\frac{4s}{16s^2-4}] = \mathcal{L}^{-1}[\frac{4s}{(4s)^2-2^2}] = \frac{1}{4}\cosh 2\cdot\frac{1}{4}t = \frac{1}{4}\cosh\frac{t}{2}$

4. Derivative

Ex. 4.

$(a)\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}]$ $\qquad (b)\mathcal{L}^{-1}[\ln\frac{s+a}{s+b}]$.

$$\text{solution}: (a) \mathcal{L}[\sin\omega t] = \frac{\omega}{s^2+\omega^2} \Rightarrow \mathcal{L}[t\sin\omega t] = -\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right) = \frac{2\omega s}{(s^2+\omega^2)^2}$$

$$\text{Let } F(t) = t\sin\omega t \Rightarrow \mathcal{L}[F'(t)] = s\cdot\frac{2\omega s}{(s^2+\omega^2)^2} - F(0)$$

$$\mathcal{L}[F'(t)] = 2\omega\frac{s^2}{(s^2+\omega^2)^2} = 2\omega\left[\frac{(s^2+\omega^2)-\omega^2}{(s^2+\omega^2)^2}\right] = 2\omega\left[\frac{1}{s^2+\omega^2} - \frac{\omega^2}{(s^2+\omega^2)^2}\right]$$

$$= 2\mathcal{L}[\sin\omega t] - \frac{2\omega^3}{(s^2+\omega^2)^2}$$

$$\frac{1}{(s^2+\omega^2)^2} = \frac{1}{2\omega^3}\cdot\mathcal{L}[2\sin\omega t - F'(t)]$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+\omega^2)^2}\right] = \frac{1}{2\omega^3}\cdot[2\sin\omega t - F'(t)] = \frac{1}{2\omega^3}(\sin\omega t - \omega t\cos\omega t)$$

$$(b) \text{ Let } \mathcal{L}[f(t)] = \ln\frac{s+a}{s+b} = \ln(s+a) - \ln(s+b)$$

$$\mathcal{L}[tf(t)] = -\frac{d}{ds}[\ln(s+a) - \ln(s+b)] = \frac{1}{s+b} - \frac{1}{s+a} = \mathcal{L}[e^{-bt} - e^{-at}]$$

$$\therefore f(t) = \frac{e^{-bt} - e^{-at}}{t}$$

5. Integration

Ex. 5.

$$(a) \mathcal{L}^{-1}\left[\frac{1}{s^2}\left(\frac{s-1}{s+1}\right)\right] \qquad (b) \mathcal{L}^{-1}\left[\ln\frac{s+a}{s+b}\right].$$

$$\text{Solution}: (a) \mathcal{L}^{-1}\left[\frac{1}{s^2}\left(\frac{s-1}{s+1}\right)\right] = \mathcal{L}^{-1}\left[\frac{1}{s(s+1)} - \frac{1}{s^2(s+1)}\right] = \int_0^t e^{-t}dt - \int_0^t\int_0^t e^{-t}dtdt$$

$$= -(e^{-t}-1) + \int_0^t (e^{-t}-1)dt = -(e^{-t}-1) - (e^{-t}-1) - t = 2 - 2e^{-t} - t$$

$$(b) \mathcal{L}[e^{-bt} - e^{-at}] = \frac{1}{s+b} - \frac{1}{s+a}$$

$$\mathcal{L}\left[\frac{e^{-bt} - e^{-at}}{t}\right] = \int_s^\infty \left(\frac{1}{s+b} - \frac{1}{s+a}\right)ds = \ln\frac{s+b}{s+a}\bigg|_s^\infty = \ln\frac{s+a}{s+b}$$

$$\therefore \mathcal{L}^{-1}\left[\ln\frac{s+a}{s+b}\right] = \frac{e^{-bt} - e^{-at}}{t}$$

6. Convolution

Ex. 6.

$$(a) \mathcal{L}^{-1}\left[\frac{1}{(s^2+\omega^2)^2}\right] \qquad (b) \mathcal{L}^{-1}\left[\frac{s}{(s^2+\omega^2)^2}\right].$$

Solution : $(a)\mathcal{L}\,[\sin\omega t]=\dfrac{\omega}{s^2+\omega^2}\Rightarrow\mathcal{L}\,[\dfrac{1}{\omega}\sin\omega t]=\dfrac{1}{s^2+\omega^2}$

$$\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}]=\frac{1}{\omega^2}\int_0^t \sin\omega\tau\sin\omega(t-\tau)d\tau$$

$$=\frac{1}{\omega^2}\int_0^t\frac{1}{2}[\cos(\omega\tau-\omega t+\omega\tau)-\cos(\omega\tau+\omega t-\omega\tau)]d\tau$$

$$=\frac{1}{2\omega^2}\int_0^t[\cos(2\omega\tau-\omega t)-\cos\omega t]d\tau=\frac{1}{2\omega^2}\left[\frac{1}{2\omega}\sin(2\omega\tau-\omega t)-\tau\cos\omega t\right]_0^t$$

$$=\frac{1}{2\omega^2}\{[\frac{1}{2\omega}(\sin\omega t-\sin(-\omega t)]-t\cos\omega t\}=\frac{1}{2\omega^3}(\sin\omega t-\omega t\cos\omega t)$$

$(b)\mathcal{L}\,[\dfrac{1}{\omega}\sin\omega t]=\dfrac{1}{s^2+\omega^2}\qquad \mathcal{L}\,[\cos\omega t]=\dfrac{s}{s^2+\omega^2}$

$$\mathcal{L}^{-1}[\frac{s}{(s^2+\omega^2)^2}]=\frac{1}{\omega}\int_0^t\sin\omega\tau\cos\omega(t-\tau)d\tau$$

$$=\frac{1}{\omega}\int_0^t\frac{1}{2}[\sin(\omega\tau+\omega t-\omega\tau)+\sin(\omega\tau-\omega t+\omega\tau)]d\tau$$

$$=\frac{1}{2\omega}\int_0^t[\sin\omega t+\sin(2\omega\tau-\omega t)]d\tau=\frac{1}{2\omega}\left[\tau\sin\omega t+\frac{-1}{2\omega}\cos(2\omega\tau-\omega t)\right]_0^t$$

$$=\frac{1}{2\omega}\{t\sin\omega t-\frac{1}{2\omega}[\cos\omega t-\cos(-\omega t)]\}=\frac{t}{2\omega}\sin\omega t$$

II. Partial Fraction

If $F(s)$□ $\dfrac{P(s)}{Q(s)}$, where deg[$P(s)$]<deg[$Q(s)$]

1.$Q(s)$□ 0 with unrepeated factorss□a_i

$$\frac{P(s)}{Q(s)}=\frac{A_1}{s-a_1}+\frac{A_2}{s-a_2}+\cdots+\frac{A_n}{s-a_n}$$

$$A_k=\lim_{S\to a_k}[\frac{P(s)}{Q(s)}(s-a_k)]=P(a_k)\lim_{S\to a_k}\frac{s-a_k}{Q(s)}$$

$$=P(a_k)\lim_{s\to a_k}\frac{1}{Q'(s)}=\frac{P(a_k)}{Q'(a_k)}$$

$$\frac{P(s)}{Q(s)}=\frac{P(a_1)/Q'(a_1)}{s-a_1}+\frac{P(a_2)/Q'(a_2)}{s-a_2}+\cdots\cdots+\frac{P(a_n)/Q'(a_n)}{s-a_n}$$

$$\mathcal{L}^{-1}[\frac{P(s)}{Q(s)}]=\frac{P(a_1)}{Q'(a_1)}e^{a_1t}+\frac{P(a_2)}{Q'(a_2)}e^{a_2t}+\cdots\cdots+\frac{P(a_n)}{Q'(a_n)}e^{a_nt}$$

Ex. 7.

$$\mathcal{L}^{-1}[\frac{s+1}{s^3+s^2-6s}].$$

Solution : $\frac{s+1}{s^3+s^2-6s}=\frac{s+1}{s(s-2)(s+3)}=\frac{A_1}{s}+\frac{A_2}{s-2}+\frac{A_3}{s+3}$

$$A_1=\lim_{s\to 0}\frac{s+1}{(s-2)(s+3)}=-\frac{1}{6}$$

$$A_2=\lim_{s\to 2}\frac{s+1}{s(s+3)}=\frac{3}{10}$$

$$A_3=\lim_{s\to -3}\frac{s+1}{s(s-2)}=\frac{-2}{15}$$

$$\mathcal{L}^{-1}[\frac{s+1}{s^3+s^2-6s}]=\frac{-\frac{1}{6}}{s}+\frac{\frac{3}{10}}{s-2}+\frac{\frac{-2}{15}}{s+3}=-\frac{1}{6}+\frac{3}{10}e^{2t}-\frac{2}{15}e^{-3t}$$

2. $Q(s)$ □ 0 with repeated factors $(s$ □ $a_k)^m$

$$\frac{P(s)}{Q(s)}=\frac{C_m}{(s-a_k)^m}+\frac{C_{m-1}}{(s-a_k)^{m-1}}+\cdots\cdots+\frac{C_1}{s-a_k}$$

$$\frac{P(s)}{Q(s)}(s-a_k)^m=C_m+C_{m-1}(s-a_k)+C_{m-2}(s-a_k)^2+\cdots+C_1(s-a_k)^{m-1}$$

$$C_m=\lim_{s\to a_k}[\frac{P(s)}{Q(s)}(s-a_k)^m]$$

$$C_{m-1}=\lim_{s\to a_k}\{\frac{d}{ds}[\frac{P(s)}{Q(s)}(s-a_k)^m]\}$$

$$C_{m-2}=\lim_{s\to a_k}\{\frac{d^2}{ds^2}[\frac{P(s)}{Q(s)}(s-a_k)^m]\}\frac{1}{2!}$$

....

$$C_1=\lim_{s\to a_k}\{\frac{d^{m-1}}{ds^{m-1}}[\frac{P(s)}{Q(s)}(s-a_k)^m]\}\frac{1}{(m-1)!}$$

$$\mathcal{L}^{-1}[\frac{P(s)}{Q(s)}]=e^{a_k t}[C_m\frac{t^{m-1}}{(m-1)!}+C_{m-1}\frac{t_{m-2}}{(m-2)!}+\cdots+C_2t+C_1]$$

Ex. 8.

$$\mathcal{L}^{-1}[\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-2)(s-3)}].$$

$$\text{Solution}: \frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-2)(s-3)} = \frac{C_2}{s^2}+\frac{C_1}{s}+\frac{A_1}{s-1}+\frac{A_2}{s-2}+\frac{A_3}{s-3}$$

$$C_2 = \lim_{s\to 0}\frac{s^4-7s^3+13s^2+4s-12}{(s-1)(s-2)(s-3)} = \frac{-12}{-6} = 2$$

$$C_1 = \lim_{s\to 0}\frac{d}{ds}[\frac{s^4-7s^3+13s^2+4s-12}{(s-1)(s-2)(s-3)}]$$

$$= \frac{4(-1)(-2)(-3)-(-12)[(-2)(-3)+(-1)(-3)+(-1)(-2)]}{[(-1)(-2)(-3)]^2} = \frac{-24+12\times 11}{6^2} = 3$$

$$A_1 = \lim_{s\to 1}\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-2)(s-3)} = \frac{-1}{2}$$

$$A_2 = \lim_{s\to 2}\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-3)} = \frac{8}{-4} = -2$$

$$A_3 = \lim_{s\to 3}\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-2)} = \frac{9}{18} = \frac{1}{2}$$

$$\mathcal{L}^{-1}[\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-2)(s-3)}] = 2t+3-\frac{1}{2}e^t-2e^{2t}+\frac{1}{2}e^{3t}$$

3. $Q(s)$□ 0 with unrepeated factor $(s$□ □$)^2$+□ , where □ >0

$$\frac{P(s)}{Q(s)} = \frac{As+B}{(s-\alpha)^2+\beta^2}$$

$$\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2] = As+B$$

$$\lim_{s\to\alpha+i\beta}\{\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2]\} = A(\alpha+i\beta)+B$$

$$R+iI = (A\alpha+\beta)+iA\beta$$

where R and I are the real and imaginary parts of $\lim_{s\to\alpha+i\beta}\{\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2]\}$, respectively

then, $\begin{cases} A\alpha+B=R \\ A\beta = I \end{cases}$, where we can get A and B, and

$$\mathcal{L}^{-1}[\frac{P(s)}{Q(s)}] = \mathcal{L}^{-1}[\frac{A(s-\alpha)+(A\alpha+B)}{(s-\alpha)^2+\beta^2}] = e^{\alpha t}\left(A\cos\beta t+\frac{A\alpha+B}{\beta}\sin\beta t\right)$$

Ex. 9.

$$\mathcal{L}^{-1}[\frac{s^2}{s^4+4}].$$

Solution : $\dfrac{s^2}{s^4+4}=\dfrac{s^2}{(s^2)^2+2\cdot s^2\cdot 2+2^2-2\cdot s^2\cdot 2}=\dfrac{s^2}{(s^2+2)^2-(2s)^2}$

$$=\frac{s^2}{(s^2+2s+2)(s^2-2s+2)}=\frac{A_1 s+B_1}{(s+1)^2+1}+\frac{A_2 s+B_2}{(s-1)^2+1}$$

$$\lim_{s\to -1+i}\frac{s^2}{(s-1)^2+1}=A_1(-1+i)+B_1\Rightarrow\frac{-2i}{4-4i}=(-A_1+B_1)+iA_1$$

$$\frac{8-8i}{32}=(-A_1+B_1)+iA_1\Rightarrow A_1=-\frac{1}{4},B_1=0$$

$$\lim_{s\to 1+i}\frac{s^2}{(s+1)^2+1}=A_2(1+i)+B_2\Rightarrow\frac{2i}{4+4i}=(A_2+B_2)+iA_2$$

$$\frac{8+8i}{32}=(A_2+B_2)+iA_2\Rightarrow A_2=\frac{1}{4},B_2=0$$

$$L^{-1}[\frac{s^2}{s^4+4}]=L^{-1}[\frac{-\frac{1}{4}(s+1)+\frac{1}{4}}{(s+1)^2+1}+\frac{\frac{1}{4}(s-1)+\frac{1}{4}}{(s-1)^2+1}]$$

$$=\frac{e^{-t}}{4}(-\cos t+\sin t)+\frac{e^{t}}{4}(\cos t+\sin t)$$

4. Q(s)□ 0 with repeated complex factor [(s□ □)^2+□ ^{2}]^2, where □ >0

$$\frac{P(s)}{Q(s)}=\frac{As+B}{[(s-\alpha)^2+\beta^2]^2}+\frac{Cs+D}{(s-\alpha)^2+\beta^2}$$

$$\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2]^2=As+B+(Cs+D)[(s-\alpha)^2+\beta^2]$$

$$\lim_{s\to\alpha+i\beta}\{\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2]^2\}=A(\alpha+i\beta)+B$$

$R_1+iI_1=(A\alpha+B)+iA\beta\Rightarrow\{\begin{matrix}A\alpha+B=R_1\\ A\beta=I_1\end{matrix}$, where A and B can be obtained

$$\lim_{s\to\alpha+i\beta}\frac{d}{ds}\{\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2]^2\}=A+[C(\alpha+i\beta)+D]\lim_{s\to\alpha+i\beta}\frac{d}{ds}[(s-\alpha)^2+\beta^2]$$

$$R_2+iI_2=A+[C(\alpha+i\beta)+D]2i\beta=(A-2C\beta^2)+i(2\alpha\beta C+2\beta D)$$

$\Rightarrow\{\begin{matrix}A-2C\beta^2=R_2\\ 2\alpha\beta C+2\beta D=I_2\end{matrix}$, where we get C and D, hence

$$L^{-1}[\frac{P(s)}{Q(s)}]=L^{-1}\{\frac{A(s-\alpha)+(A\alpha+B)}{[(s-\alpha)^2+\beta^2]^2}\}+L^{-1}[\frac{C(s-\alpha)+(C+D)}{(s-\alpha)^2+\beta^2}]$$

$$=e^{\alpha t}\{[\frac{At}{2\beta}\sin\beta t+(A\alpha+B)\frac{1}{2\beta^3}(\sin\beta t-\beta t\cos\beta t)]+[C\cos\beta t+(C\alpha+D)\frac{1}{\beta}\sin\beta t]\}$$

Ex. 10.

$$\mathcal{L}^{-1}[\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}].$$

Solution : $\dfrac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}=\dfrac{As+B}{[(s-1)^2+1]^2}+\dfrac{cs+D}{(s-1)^2+1}$

$$\lim_{s\to 1+i}(s^3-3s^2+6s+4)=A(1+i)+B$$

$$2i=(A+B)+iA \Rightarrow A=2, B=-2$$

$$\lim_{s\to 1+i}\frac{d}{ds}(s^3-3s^2+6s+4)=A+[c(1+i)+D]\lim_{s\to 1+i}\frac{d}{ds}[(s-1)^2+1]$$

$$0=A+(c+ic+D)2i=(A-2c)+2i(c+D)$$

$$c=1, D=-1$$

$$\mathcal{L}^{-1}[\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}]=\mathcal{L}^{-1}\{\frac{2(s-1)}{[(s-1)^2+1]^2}\}+\mathcal{L}^{-1}[\frac{s-1}{(s-1)^2+1}]$$

$$=e^t(2\cdot\frac{t}{2}\sin t+\cos t)=e^t(t\sin t+\cos t)$$

IV. Differentiation with Respect to a Number

Ex. 11.

$$\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}].$$

Solution : $\dfrac{d}{d\omega}(\dfrac{1}{s^2+\omega^2})=\dfrac{-2\omega}{(s^2+\omega^2)^2}\Rightarrow \mathcal{L}^{-1}[\dfrac{d}{d\omega}(\dfrac{1}{s^2+\omega^2})]=\mathcal{L}^{-1}[\dfrac{-2\omega}{(s^2+\omega^2)^2}]$

$$-2\omega\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}]=\frac{d}{d\omega}\mathcal{L}^{-1}[\frac{1}{s^2+\omega^2}]=\frac{d}{d\omega}(\frac{1}{\omega}\sin\omega t)=-\frac{1}{\omega^2}\sin\omega t+\frac{t}{\omega}\cos\omega t$$

$$\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}]=\frac{1}{2\omega^3}(\sin\omega t-\omega t\cos\omega t)$$

V. Method of Differential Equation

Ex. 12.

$$\mathcal{L}^{-1}[e^{-\sqrt{s}}].$$

Solution : $\bar{y} = e^{-\sqrt{s}} \Rightarrow \bar{y}' = -\dfrac{e^{-\sqrt{s}}}{2\sqrt{s}}, \ \bar{y}'' = \dfrac{e^{-\sqrt{s}}}{4s} + \dfrac{e^{-\sqrt{s}}}{4\sqrt{s^3}}$

we get the equation $4s\bar{y}'' + 2\bar{y}' - \bar{y} = 0 \Rightarrow 4\mathcal{L}\ [\dfrac{d}{dt}(t^2 y)] + 2\mathcal{L}\ [-ty] - \mathcal{L}\ [y] = 0$

$4\dfrac{d}{dt}(t^2 y) - 2ty - y = 0 \Rightarrow 4t^2 y' + (6t-1)y = 0 \Rightarrow \dfrac{dy}{y} + \dfrac{6t-1}{4t^2}dt = 0$

$\ln y + \dfrac{3}{2}\ln t + \dfrac{1}{4t} = c_1 \Rightarrow y = ct^{-\frac{3}{2}}e^{-\frac{1}{4t}}$

$\because \mathcal{L}\ [t^{-\frac{1}{2}}] = \dfrac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}} = \dfrac{\sqrt{\pi}}{\sqrt{s}}$, and $\mathcal{L}\ [ty] = \mathcal{L}\ [ct^{-\frac{1}{2}}e^{-\frac{1}{4t}}]$

while $\mathcal{L}\ [ty] = -\bar{y}' = \dfrac{e^{-\sqrt{s}}}{2\sqrt{s}} \Rightarrow \mathcal{L}\ [ct^{-\frac{1}{2}}e^{-\frac{1}{4t}}] = \dfrac{e^{-\sqrt{s}}}{2\sqrt{s}}$

Apply general final value theorem $\lim\limits_{t\to\infty} \dfrac{ct^{-\frac{1}{2}}e^{-\frac{1}{4t}}}{t^{-\frac{1}{2}}} = \lim\limits_{s\to 0} \dfrac{\frac{e^{-\sqrt{s}}}{2\sqrt{s}}}{\frac{\sqrt{\pi}}{\sqrt{s}}} \Rightarrow c = \dfrac{1}{2\sqrt{\pi}}$

$\therefore y = \dfrac{1}{2\sqrt{\pi}t^{3/2}}e^{-\frac{1}{4t}}$

Applied to Solve Differential Equations
I. Ordinary Differential Equations with Constant Coefficients

Ex. 1.

$y'' + y' + y = g(x),\ y(0) = 1,\ y'(0) = 0,$ where $g(x) = \begin{cases} 1 & 0 < x < 3 \\ 3 & x > 3 \end{cases}.$

Solution : $g(x) = u(x) + 2u(x-3)$

$$[s^2Y - sy(0) - y'(0)] + [sY - y(0)] + Y = \frac{1}{s} + 2\frac{e^{-3s}}{s}$$

$$(s^2 + s + 1)Y = s + 1 + \frac{1}{s} + 2\frac{e^{-3s}}{s}$$

$$Y = \frac{s+1}{s^2+s+1} + \frac{1}{s(s^2+s+1)} + \frac{2e^{-3s}}{s(s^2+s+1)}$$

$$= \frac{s+1}{s^2+s+1} + (\frac{1}{s} - \frac{s+1}{s^2+s+1}) + 2e^{-3s}(\frac{1}{s} - \frac{s+1}{s^2+s+1})$$

$$\frac{s+1}{s^2+s+1} = \frac{(s+\frac{1}{2}) + \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \Rightarrow \mathcal{L}^{-1}[\frac{s+1}{s^2+s+1}] = e^{-\frac{x}{2}}(\cos\frac{\sqrt{3}}{2}x + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}x)$$

$$y(x) = u(x) + 2u(x-3)\{1 - e^{-\frac{x-3}{2}}[\cos\frac{\sqrt{3}}{2}(x-3) + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}(x-3)]\}$$

Ex. 2.

$y'''(t) - 2y''(t) + 5y'(t) = 0,\ y(0) = 0,\ y'(0) = 1,\ y(\frac{\pi}{8}) = 1$.

Solution : $[s^3Y - s^2y(0) - sy'(0) - y''(0)] - 2[s^2Y - sy(0) - y'(0)] + 5[sY - y(0)] = 0$

$$y''(0) = c$$

$$Y = \frac{s+c-2}{s(s^2-2s+5)} = \frac{A}{s} + \frac{Ps+Q}{(s-1)^2+2^2}$$

$$A = \lim_{s\to 0}\frac{s+c-2}{s^2-2s+5} = \frac{c-2}{5}$$

$$P(1+2i) + Q = \lim_{s\to 1+2i}\frac{s+c-2}{s} = \frac{-1+c+2i}{1+2i} = \frac{c+3}{5} + \frac{4-2c}{5}i$$

$$P = \frac{2-c}{5}, \quad Q = \frac{2c+1}{5}$$

$$y(t) = \frac{c-2}{5} + e^t(\frac{2-c}{5}\cos 2t + \frac{c+3}{10}\sin 2t)$$

$$y(\frac{\pi}{8}) = 1 \Rightarrow 1 = \frac{c-2}{5} + e^{\frac{\pi}{8}}(\frac{2-c}{5}\frac{1}{\sqrt{2}} + \frac{c+3}{10}\frac{1}{\sqrt{2}}) \Rightarrow c = 7$$

$$\therefore y(t) = 1 + e^t(-\cos 2t + \sin 2t)$$

Module-III

CURVE FITTING AND FOURIER TRANSFORMS

Suppose that a data is given in two variables x & y the problem of finding an analytical expression of the form $y = f(x)$ which fits the given data is called curve fitting

Let $(x_1, y_1), (x_2, y_2)........(x_n, y_n)$ be the observed set of values in an experiment and $y = f(x)$ be the given relation $x \& y, Let\ E_1, E_2, ...E_x$ are the error of approximations then we have

$E_1 = y_1 - f(x_1)$

$E_2 = y_2 - f(x_2)$

$E_3 = y_3 - f(x_3)$

$E_n = y_n - f(x_n)$ where $f(x_1), f(x_2)..........f(x_n)$ are called the expected values of y corresponding to $x = x_1, x = x_2........x = x_n$

$y_1, y_2......y_n$ are called the observed values of y corresponding to $x = x_1, x = x_2........x = x_n$ the differences $E_1, E_2.....E_n$ between expected values of y and observed values of y are called the errors, of all curves approximating a given set of points, the curve for which

$E = E_1^2 + E_2^2 + ...E_n^2$ is a minimum is called the best fitting curve (or) the least square curve

This is called the method of least squares (or) principles of least squares

1. FITTING OF A STRAIGHT LINE:-

Let the straight line be $y = a + bx \rightarrow (1)$

Let the straight line (1) passes through the data points

$(x_1, y_1), (x_2, y_2)......(x_n, y_n) i.e., (x_i, y_i), i = 1, 2....n$

So we have $yi = a + bxi \rightarrow (2)$

The error between the observed values and expected values of $y = yi$ is defined as $Ei = y_1 - (a + bxi).i = 1, 2.......n \rightarrow (3)$

The sum of squares of these error is

$E = \sum_{i=1}^{n} Ei^2 = \sum_{i=1}^{n} [yi - (a + bxi)]^2$ now for E to be minimum

$\frac{\partial E}{\partial a} = 0; \frac{\partial E}{\partial b} = 0$

These equations will give normal equations

$$\sum_{i=1}^{n} yi = na + b\sum_{i=1}^{n} xi$$

$$\sum_{i=1}^{n} xiyi = a\sum_{i=1}^{n} xi + b\sum_{i=1}^{n} xi^2$$

The normal equations can also be written as

$$\sum y = na + b\sum x$$

$$\sum xy = a\sum x + b\sum x^2$$

Solving these equation for a, b substituting in (1) we get required line of best fit to the given data.

NON LINEAR CURVE FITTING

PARABOLA:-

2. Let the equation of the parabola to be fit

The parabola (1) passes through the data points

$(x_1, y_1), (x_2, y_2)............(x_n, y_n), i.e., (x_i, y_i); i = 1, 2......x$

We have $yi = a + bx_i + cx_i^2 \rightarrow (2)$

$y = a + bx + cx^2 \rightarrow (1)$

The error Ei between the observed an expected value of $y = y_i$ is defined as

$Ei = yi - \left(a + bxi + cxi^2\right), i = 1, 2, 3......n \rightarrow (3)$

The sum of the squares of these error is

$E = \varepsilon_{i=1}^{n}\ Ei^2 = \varepsilon_{i=1}^{n}\ \left(yi - a - bxi - cxi^2\right)^2 \rightarrow (4)$

For E to be minimum, we have

$$\frac{\partial E}{\partial a} = 0, \frac{\partial E}{\partial b} = 0, \frac{\partial E}{\partial c} = 0$$

The normal equations can also be written as

$\varepsilon y = na + b\varepsilon x + c\varepsilon x^2$

$\varepsilon xy = a\varepsilon x + b\varepsilon x^2 + c\varepsilon x^3$ $\quad$ $use\ \sum instead\ of\ \varepsilon$

$\varepsilon x^2 y = a\varepsilon x^2 + b\varepsilon x^3 + c\varepsilon x^4$

Solving these equations for a, b, c and satisfying (1) we get required parabola of best fit

3. **POWER CURVE:-**

The power curve is given by $y = ax^b \rightarrow (1)$

Taking logarithms on both sides

$\log_{10}{}^{y} = \log_{10}{}^{a} + b\log_{10}{}^{X}$

$(or)\ y = A + bX \rightarrow (2)$

where $y = \log_{10}{}^{y}, A = \log_{10}{}^{a}$ *and* $X = \log_{10}{}^{X}$

Equation (2) is a linear equation in X & y

$\therefore$ The normal equations are given by

$\varepsilon y = nA + b\varepsilon X$

$\varepsilon xy = A\varepsilon X + b\varepsilon X^2 \quad use\ \Sigma\, symbol$

From these equations, the values A and b can be calculated then a = antilog (A) substitute a & b in (1) to get the required curve of best fit

4. **<u>EXPONENTIAL CURVE</u> :-** $(1)\ y = ae^{bx}\quad (2)\ y = ab^x$

$y = ae^{bx} \rightarrow (1)$

Taking logarithms on both sides

$\log_{10} y = \log_{10} a + bx\log_{10} e$

$(or)\ y = A + BX \rightarrow (2)$

Where $y = \log_{10} y, A = \log_{10} a \,\&\, B = b\log_{10} e$

Equation (2) is a linear equation in X and Y

So the normal equation are given by

$\Sigma Y = nA + B\Sigma X$

$\Sigma xy = A\Sigma X + B\Sigma X^2$

Solving the equation for A & B, we can find

$$a = anti\log A \,\&\, b = \frac{B}{\log_{10} e}$$

Substituting the values of a and b so obtained in (1) we get

The curve of best fir to the given data.

2. $y = ab^x \rightarrow (1)$

Taking log on both sides

$\log_{10} y = \log_{10} a + x \log_{10} b \; (or) Y = A + Bx$

$Y = \log_{10} y, A = \log_{10} a, B = \log_{10} b$

The normal equation (2) are given by

$\Sigma y = nA + B\Sigma X$

$\Sigma xy = A\Sigma X + B\Sigma X^2$

Solving these equations for A and B we can find $a = anti\log A, b = anti\log B$

Substituting a and b in (1)

1. **By the method of least squares, find the straight line that best fits the following data**

X	1	2	3	4	5
Y	14	27	40	55	68

$y = a + bx$

Ans. The values of $\varepsilon x, \varepsilon y, \varepsilon x^2$ *and* εxy are calculated as follows

xi	yi	xi^2	$xiyi$
1	14	1	14
2	27	4	54
3	40	9	120
4	55	16	220
5	68	25	340

Replace xi, yi *by* x_i, y_i and use Σ *instead of* ε

$\varepsilon xi = 15; \varepsilon yi = 204, \varepsilon xi^2 = 55 \text{ and } \varepsilon xiyi = 748$

The normal equations are

$\varepsilon y = na + b\varepsilon x \to (1)$

$\varepsilon xy = a\varepsilon x + b\varepsilon x^2 \to (2)$

$204 = 15a + 5b$

$748 = 55a + 15b$

Solving we get $a = 0, b = 13.6$

Substituting these values a & b we get

$y = 0 + 13.6x \Rightarrow y = 13.6x$

2. Fit a second degree parabola to the following data

X	0	1	2	3	4
Y	1	5	10	22	38

$y = a + bx + cx^2$

Ans. Equation of parabola $y = a + bx + cx^2 \rightarrow (1)$

Normal equations $\varepsilon y = na + b\varepsilon x + c\varepsilon x^2$

$\varepsilon xy = a\varepsilon x + b\varepsilon x^2 + c\varepsilon x^3$

$\varepsilon x^2 y = a\varepsilon x^2 + b\varepsilon x^3 + c\varepsilon x^4 \rightarrow (2)$

x	y	xy	x^2	x^2y	x^3	x^4
0	**1**	**0**	**0**	**0**	**0**	**0**
1	**5**	**5**	**1**	**5**	**1**	**1**
2	**10**	**20**	**4**	**40**	**8**	**16**
3	**22**	**66**	**9**	**198**	**27**	**81**
4	**38**	**152**	**16**	**608**	**64**	**256**

$\varepsilon x = 10, \varepsilon y = 76, \varepsilon xy = 243, \varepsilon x^2 = 30, \varepsilon x^2 y = 851, \varepsilon x^3 = 100, \varepsilon x^4 = 354$

Normal equations

$76 = 5a + 10b + 30c$

$243 = 10a + 30b + 100c$

$851 = 30a + 100b + 354c$

Solving $a = 1.42, b = 0.26, c = 2.221$

Substitute in (1) $\Rightarrow y = 1.42 + 0.26x + 2.221x^2$

3. Fit a curve $y = ax^b$ to the following data

X	1	2	3	4	5	6
Y	2.98	4.26	5.21	6.10	6.80	7.50

Ans. Let the equation of the curve be $y = ax^b \rightarrow (1)$

Taking log on both sides

$\log y = \log a + b\log x$

$y = A + bX \rightarrow (2)$

$y = \log y, A = \log a, X = \log x$

$\varepsilon y = nA + b\varepsilon X$

$\varepsilon xy = A\varepsilon x + b\varepsilon x^2 \rightarrow (3)$

x	$X = \log x$	y	$y = \log y$	xy	x^2
1	**0**	**2.98**	**0.4742**	**0**	**0**
2	**0.3010**	**4.26**	**0.6294**	**0.1894**	**0.0906**
3	**0.4771**	**5.21**	**0.7168**	**0.3420**	**0.2276**
4	**0.6021**	**6.10**	**0.7853**	**0.4728**	**0.3625**
5	**0.6990**	**6.80**	**0.8325**	**0.5819**	**0.4886**

$\varepsilon x = 2.8574, \varepsilon y = 4.3133, \varepsilon xy = 2.2671, \varepsilon x^2 = 1.7749$

$4.3313 = 6A + 2.8574b$

$2.2671 = 2.8574A + 1.7749b$

solving $A = 0.4739$ $b = 0.5143$

$a = \text{anti}\log(A) = 2.978$

$\therefore y = 2.978.x^{0.5143}$

4. Fit a curve $y = ab^x \rightarrow (1)$

X	2	3	4	5	6
Y	144	172.8	207.4	248.8	298.5

$\log y = \log a + x\log b \rightarrow (I)$

$y = A + xB \rightarrow (2)$

Ans. $y = \log y, A = \log a, B = \log b$

$\Sigma y = nA + B\varepsilon x$

$\varepsilon xy = A\varepsilon x + B\varepsilon x^2 \rightarrow (3)$

x	y	x^2	$Y = \log y$	xy
2	**144.0**	**4**	**2.1584**	**4.3168**
3	**172.8**	**9**	**2.2375**	**6.7125**
4	**207.4**	**16**	**2.3168**	**9.2672**

5	**248.8**	**25**	**2.3959**	**11.9795**
6	**298.5**	**36**	**2.4749**	**14.8494**

Ans. Equation of parabola $y = a + bx + cx^2 \rightarrow (1)$

Normal equations $\varepsilon y = na + b\varepsilon x + c\varepsilon x^2$

$$\varepsilon xy = a\varepsilon x + b\varepsilon x^2 + c\varepsilon x^3 \ \& \qquad \varepsilon x^2 y = a\varepsilon x^2 + b\varepsilon x^3 + c\varepsilon x^4 \rightarrow (2)$$

x	y	xy	x^2	x^2y	x^3	x^4
0	**1**	**0**	**0**	**0**	**0**	**0**
1	**1.8**	**1.8**	**1**	**1.8**	**1**	**1**
2	**1.3**	**2.6**	**4**	**5.2**	**8**	**16**
3	**2.5**	**7.5**	**9**	**22.5**	**27**	**81**
4	**6.3**	**25.2**	**16**	**100.8**	**64**	**256**

$\sum x_i = 10$, $\sum y_i = 12.9$, $\sum x^2 = 30$, $\sum x_i^3 = 100$, $\sum x_i^4 = 354$, $\sum x_i^2 y_i = 130.3$

$\sum x_i\, y_i, = 37.1$

Normal equations

5a + 10b + 30c = 12.9

10a + 30b +100c = 37.1

30a + 100b +354c = 130.3

Solving a= 1.42 b = -1.07 c= .55

Substitute in (1) $y = 1.42 - 1.07x + .55x^2$

FOURIER TRANSFORMS

Introduction

The Fourier transform named after Joseph Fourier, is a mathematical transformation employed to transform signals between time (or spatial) domain and frequency domain, which has many applications in physics and engineering. It is reversible, being able to transform from either domain to the other. The term itself refers to both the transform operation and to the function it produces.

In the case of a periodic function over time (for example, a continuous but not necessarily sinusoidal musical sound), the Fourier transform can be simplified to the calculation of a discrete set of complex amplitudes, called Fourier series coefficients. They represent the frequency spectrum of the original time-domain signal. Also, when a time-domain function is sampled to facilitate storage or computer-processing, it is still possible to recreate a version of the original Fourier transform according to the Poisson summation formula, also known as discrete-time Fourier transform. See also Fourier analysis and List of Fourier-related transforms.

Integral Transform

The integral transform of a function f(x) is given by

$$I[f(x)] \text{ or } F(s) = \int_a^b f(x)k(s,x)dx$$

Where k(s, x) is a known function called **kernel of the transform**

s is called the **parameter of the transform**

f(x) is called the **inverse transform of F(s)**

Fourier transform

$$k(s,x) = e^{isx}$$

$$F[f(x)] = F(s) = \int_{-\infty}^{\infty} f(x)e^{isx}dx$$

Laplace transform

$$k(s,x) = e^{-sx}$$

$$L[f(x)] = F(s) = \int_0^{\infty} f(x)e^{-sx}dx$$

Henkel transform

$$k(s,x) = xJ_n(sx)$$

$$H[f(x)] = H(s) = \int_0^{\infty} f(x)xJ_n(sx)dx$$

Mellin transform

$$k(s,x) = x^{s-1}$$

$$M[f(x)] = M(s) = \int_0^{\infty} f(x)x^{s-1}dx$$

DIRICHLET'S CONDITION

A function f(x) is said to satisfy Dirichlet's conditions in the interval (a,b) if

1. f(x) defined and is single valued function except possibly at a finite number of points in the interval (a,b)
2. f(x) and $f^1(x)$ are piecewise continuous in (a,b)

Fourier integral theorem

If f(x) is a given function defined in (-*l*,*l*) and satisfies the Dirichlet conditions then

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\lambda(t-x)dtd\lambda$$

Proof:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos(\frac{n\pi x}{L}) + \sum_{n=1}^\infty b_n \sin(\frac{n\pi x}{L})$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos(\frac{n\pi x}{L})\,dt$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin(\frac{n\pi x}{L})\,dt$$

Substituting the values in f(x)

$$f(x) = \frac{1}{L}\int_{-L}^{L} f(t)[1 + 2\sum_{n=1}^\infty \cos(\frac{n\pi}{L})(t-x)]\,dt --------- (1)$$

But cosine functions are even functions

$$\sum_{n=-\infty}^\infty \cos(\frac{n\pi}{L})(t-x) = 1 + 2\sum_{n=1}^\infty \cos(\frac{n\pi}{L})(t-x) ------- (2)$$

Substituting equation (2) in (1)

$$f(x)=\frac{1}{2\pi}\int_{-L}^{L} f(t)\frac{\pi}{L}\sum_{n=-\infty}^{\infty}\cos(\frac{n\pi}{L})(t-x)\,dt$$

$$\frac{n\pi}{L}=\lambda$$

$$\underset{n\to\infty}{Lt}\frac{\pi}{L}\sum_{n=-\infty}^{\infty}\cos(\frac{n\pi}{L})(t-x)=\int_{-\infty}^{\infty}\cos\lambda(t-x)\,d\lambda=2\int_{0}^{\infty}\cos\lambda(t-x)\,d\lambda$$

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)[2\int_{0}^{\infty}\cos\lambda(t-x)\,d\lambda]dt$$

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,d\lambda dt$$

Fourier Sine Integral

If f (t) is an odd function

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}\sin\lambda x\int_{0}^{\infty} f(t)\sin\lambda t dt d\lambda$$

Fourier Cosine Integral

If f (t) is an even function

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}\cos\lambda x\int_{0}^{\infty} f(t)\cos\lambda t dt d\lambda$$

Problems

1 **Express** $f(x)=\{\begin{matrix}1,|x|<1\\0,|x|>1\end{matrix}$ **as a Fourier integral. Hence evaluate** $\int_{0}^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}$ **and also find the value of** $\int_{0}^{\infty}\frac{\sin\lambda}{\lambda}$

Sol

$$f(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\cos\lambda(t-x)\,d\lambda dt$$

$$f(x)=\frac{1}{\pi}\int_0^\infty\int_{-1}^{1}\cos\lambda(t-x)\,d\lambda dt$$

$$f(x)=\frac{1}{\pi}\int_0^\infty\frac{2}{\lambda}\sin\lambda\cos\lambda\, xd\lambda$$

$$\int_0^\infty\frac{\sin\lambda\cos\lambda\, x}{\lambda}d\lambda=\frac{\pi}{2}f(x)=\{\begin{matrix}\frac{\pi}{2},|x|<1\\ 0,|x|>1\end{matrix}$$

$$|x|=1$$

$$\int_0^\infty\frac{\sin\lambda\cos\lambda\, x}{\lambda}d\lambda=\frac{\pi}{2}\left[\frac{1+0}{2}\right]=\frac{\pi}{4}$$

$$x=0$$

$$\int_0^\infty\frac{\sin\lambda}{\lambda}d\lambda=\frac{\pi}{2}$$

2 **Using Fourier Integral show that** $e^{-x}\cos x=\frac{2}{\pi}\int_0^\infty\frac{\lambda^2+2}{\lambda^4+2}\cos\lambda xd\lambda$

Sol

$$f(x)=e^{-x}\cos x$$

$$f(x)=\frac{2}{\pi}\int_0^\infty\cos\lambda x[\int_0^\infty f(t)\cos\lambda\, t\,dt]\,d\lambda$$

$$f(x)=\frac{2}{\pi}\int_0^\infty\cos\lambda x[\int_0^\infty e^{-t}\cos t\cos\lambda\, t\,dt]\,d\lambda$$

$$f(x)=\frac{1}{\pi}\int_0^\infty\cos\lambda x[\int_0^\infty e^{-t}(\cos(\lambda+1)\,t+\cos(\lambda-1)\,t\,dt]\,d\lambda$$

$$f(x)=\frac{1}{\pi}\int_0^\infty\cos\lambda x[\frac{1}{(\lambda+1)^2+1}+\frac{1}{(\lambda-1)^2+1}]\,d\lambda$$

$$f(x)=\frac{2}{\pi}\int_0^\infty\frac{\lambda^2+2}{\lambda^4+2}\cos\lambda xd\lambda$$

FOURIER TRANSFORMS

The complex form of Fourier integral of any function f(x) is in the form

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\lambda x}\int_{-\infty}^\infty f(t)e^{i\lambda t}dtd\lambda$$

Replacing λ by s

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} ds \int_{-\infty}^{\infty} f(t) e^{ist} dt$$

Let

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{ist} dt$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

Here F(s) is called Fourier transform of f(x) and f(x) is called inverse Fourier transform of F(s)

Alternative Definitions

$$F[f(t)] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{ist} dt,\ f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-isx} dx,\ f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s) e^{isx} ds$$

Fourier Cosine Transform

Infinite

$$F_C[f(t)] = F_C(s) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(t)\cos st\,dt$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} F_C[f(t)]\cos sx\,ds$$

Finite

$$F_C[f(t)] = F_C(s) = \sqrt{\frac{2}{l}}\int_{0}^{l} f(t)\cos(\frac{n\pi t}{l})dt$$

$$f(x) = \frac{1}{l}F_C(0) + \sqrt{\frac{2}{l}}\sum_{n=1}^{\infty} F_C(s)\cos(\frac{n\pi x}{l})$$

Fourier Sine Transform

Infinite

$$F_S[f(t)] = F_S(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin st\,dt$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_S[f(t)]\sin sx\,ds$$

Finite

$$F_S[f(t)] = F_S(s) = \sqrt{\frac{2}{l}}\int_0^l f(t)\sin(\frac{n\pi t}{l})dt$$

$$f(x) = \sqrt{\frac{2}{l}}\sum_{n=1}^{\infty} F_S(s)\sin(\frac{n\pi x}{l})$$

Alternative Definitions:

$$1. F_C(s) = \int_0^\infty f(x)\cos sx\,dx, \quad f(x) = \frac{2}{\pi}\int_0^\infty F_C(s)\cos sx\,ds$$

$$2. F_S(s) = \int_0^\infty f(x)\sin sx\,dx, \quad f(x) = \frac{2}{\pi}\int_0^\infty F_S(s)\sin sx\,ds$$

Properties of Fourier Transforms

1 **Linear Property:** $F[af_1(x) + bf_2(x)] = aF_1(s) + bF_2(s)$

Proof

$$F[af_1(x) + bf_2(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[af_1(x) + bf_2(x)]e^{ist}dt$$

$$F[af_1(x) + bf_2(x)] = \frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(x)e^{ist}dt + \frac{b}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_2(x)e^{ist}dt$$

$$F[af_1(x) + bf_2(x)] = aF_1(s) + bF_2(s)$$

2 **Shifting Theorem:** (a) $F[f(x-a)] = e^{ias}F(s)$

(b) $F[e^{iax}f(x)] = F(s+a)$

Proof

(a) $$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t-a)e^{ist}dt$$

$t - a = z$

$dt = dz$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z)e^{isz}e^{ias}dz$$

$$F[f(x-a)] = e^{ias} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z)e^{isz}dz$$

$$F[f(x-a)] = e^{ias} F(s)$$

(b) $$F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{ist}e^{iat}dt$$

$$F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{i(a+s)t}dt$$

$$F[e^{iax} f(x)] = F(s+a)$$

3 **Change of scale property:** $F[f(ax)] = \frac{1}{a}F(\frac{s}{a})(a > 0)$

Proof $$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(at)e^{ist}dt$$

$at = z$

$dt = \frac{1}{a}dz$

$$F[f(ax)] = \frac{1}{a}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z)e^{i\left(\frac{s}{a}\right)z}dz$$

$$F[f(ax)] = \frac{1}{a}F(\frac{s}{a})$$

4 **Multiplication Property:** $F[x^n f(x)] = (-i)^n \frac{d^n F}{ds^n}$

Proof $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{ist}dt$

$$\frac{dF}{ds} = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t.f(t)e^{ist}dt$$

$$\frac{d^2F}{ds^2} = \frac{i^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^2.f(t)e^{ist}dt$$

continuing

$$\frac{d^nF}{ds^n} = \frac{i^n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^n.f(t)e^{ist}dt$$

$$F[x^n f(x)] = (-i)^n \frac{d^nF}{ds^n}$$

5 **Modulation Theorem:** $F[f(x)\cos ax] = \frac{1}{2}\left[F(s+a) + F(s-a)\right], F[s] = F[f(x)]$

Proof $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\cos at\, e^{ist}dt$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\left[\frac{e^{iat} + e^{-iat}}{2}\right]e^{ist}dt$$

$$F[f(x)] = \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{i(s+a)t}dt + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{i(s-a)t}dt\right]$$

$$F[f(x)\cos ax] = \frac{1}{2}\left[F(s+a) + F(s-a)\right]$$

Problems

1 **Find the Fourier transform of** $f(x) = \begin{cases} 1, |x| < 1 \\ 0, |x| > 1 \end{cases}$

Hence evaluate $\int_{0}^{\infty} \frac{\sin x}{x} dx$

Sol:

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F[f(x)] = \int_{-1}^{1} 1.e^{isx}dx$$

$$F[f(x)] = \left|\frac{e^{isx}}{is}\right|_{-1}^{1}$$

$$F[f(x)] = \frac{e^{is} - e^{-is}}{is} = 2\frac{\sin s}{s}$$

$$f(x) = \frac{1}{2\Pi}\int_{-\infty}^{\infty} F[s]e^{-isx}ds$$

$$f(x) = \frac{1}{2\Pi}\int_{-\infty}^{\infty} 2\frac{\sin s}{s}e^{-isx}ds$$

$$f(x) = \frac{1}{\Pi}\int_{-\infty}^{\infty} \frac{\sin s}{s}e^{-isx}ds$$

$$\frac{1}{\Pi}\int_{-\infty}^{\infty} \frac{\sin s}{s}e^{-isx}ds = \{\begin{matrix} 1, |x| < 1 \\ 0, |x| > 1 \end{matrix}$$

$$x = 0$$

$$\int_{-\infty}^{\infty} \frac{\sin s}{s}ds = \Pi$$

$$\int_{0}^{\infty} \frac{\sin s}{s}ds = \frac{\Pi}{2}$$

2 **Find the Fourier transform of** $f(x) = \{\begin{matrix} 1 - x^2, |x| \le 1 \\ 0, |x| > 1 \end{matrix}$

Hence evaluate $\int_{0}^{\infty} \frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}dx$

Sol:

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F[f(x)] = \int_{-1}^{1} (1-x^2)e^{isx}dx$$

$$F[f(x)] = \left|(1-x^2)\frac{e^{isx}}{is} - 2x\frac{e^{isx}}{(is)^2} + 2\frac{e^{isx}}{(is)^3}\right|_{-1}^{1}$$

$$F[f(x)] = 2\left(\frac{e^{is}+e^{-is}}{-s^2}\right) - 2\left(\frac{e^{is}-e^{-is}}{-is^3}\right)$$

$$F[f(x)] = \frac{-4}{s^3}(s\cos s - \sin s)$$

$$f(x) = \frac{1}{2\Pi}\int_{-\infty}^{\infty} F[s]e^{-isx}ds$$

$$f(x) = \frac{1}{2\Pi}\int_{-\infty}^{\infty} \frac{-4}{s^3}(s\cos s - \sin s)e^{-isx}ds$$

$$\frac{1}{2\Pi}\int_{-\infty}^{\infty} \frac{-4}{s^3}(s\cos s - \sin s)e^{-isx}ds = \{\begin{matrix}1-x^2, |x| \le 1\\ 0, |x| > 1\end{matrix}$$

$$x = 1/2$$

$$\frac{1}{2\Pi}\int_{-\infty}^{\infty} \frac{-4}{s^3}(s\cos s - \sin s)e^{-isx}ds = \frac{3}{4}$$

$$\int_{-\infty}^{\infty} \frac{(s\cos s - \sin s)}{s^3}[\cos\frac{s}{2} - i\sin\frac{s}{2}]ds = -\frac{3\Pi}{8}$$

$$\int_{0}^{\infty} \frac{(s\cos s - \sin s)}{s^3}\cos\frac{s}{2}ds = -\frac{3\Pi}{16}$$

3 **Find the Fourier transform of $e^{-a^2x^2}$ .Hence deduce that $e^{x^2/2}$ is self-reciprocal in respect of Fourier transform**

Sol:

$$F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F[f(x)] = \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} dx$$

$$F[f(x)] = \int_{-\infty}^{\infty} e^{-a^2(x^2 - isx/a^2)} dx$$

$$F[f(x)] = \int_{-\infty}^{\infty} e^{-a^2(x - isx/2a^2)^2} e^{-s^2/4a^2} dx$$

$$t = a(x - isx/2a^2)$$

$$dx = dt/a$$

$$F[f(x)] = \int_{-\infty}^{\infty} e^{-t^2} e^{-s^2/4a^2} \frac{dt}{a}$$

$$F[f(x)] = \frac{e^{-s^2/4a^2}}{a} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$F[f(x)] = \frac{e^{-s^2/4a^2}}{a} \sqrt{\Pi}$$

$$F[f(x)] = \frac{\sqrt{\Pi}}{a} e^{-s^2/4a^2}$$

$$a^2 = 1/2$$

$$F[e^{-x^2/2}] = \sqrt{2\Pi} e^{-s^2/2}$$

Hence $e^{x^2/2}$ is self-reciprocal in respect of Fourier transform

4 **Find the Fourier cosine transform** e^{-x^2} **.**

Sol:
$$F_c(e^{-x^2}) = \int_0^{\infty} e^{-x^2}\cos sx dx = I$$

$$\frac{dI}{ds} = -\int_0^{\infty} xe^{-x^2}\sin s\, x dx = \frac{1}{2}\int_0^{\infty}(-2xe^{-x^2})\sin s\, x dx$$

$$\frac{dI}{ds} = \frac{-s}{2}\int_0^{\infty} e^{-x^2}\cos sx dx = \frac{-s}{2} I$$

$$\frac{dI}{I} = \frac{-s}{2} ds$$

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$$\log I = \int \frac{-s}{2} ds + \log c = \frac{-s^2}{4} + \log c = \log(ce^{-s^2/4})$$

$$I = ce^{-s^2/4}$$

$$\int_0^{\infty} e^{-x^2}\cos sx dx = ce^{-s^2/4}$$

$$s = 0$$

$$c = \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\Pi}}{2}$$

$$\int_0^{\infty} e^{-x^2}\cos sx dx = \frac{\sqrt{\Pi}}{2} e^{-s^2/4}$$

5 Find the Fourier sine transform $e^{-|x|}$ **.Hence show that**

$$\int_0^{\infty} \frac{x\sin mx}{1+x^2} dx = \frac{\Pi e^{-m}}{2}$$ **,m>0**

Sol: x being positive in the interval $(0,\infty)$

$$e^{-|x|} = e^{-x}$$

$$F_s(e^{-x}) = \int_0^{\infty} e^{-x}\sin sx dx = \frac{s}{1+s^2}$$

$$f(x) = \frac{2}{\Pi}\int_0^{\infty} F_s(e^{-x})\sin sx ds$$

$$f(x) = \int_0^{\infty} \frac{s}{1+s^2}\sin sx ds$$

$$e^{-x} = \int_0^{\infty} \frac{s}{1+s^2}\sin sx ds$$

Replace x by m

$$e^{-m} = \frac{2}{\Pi}\int_0^\infty \frac{s}{1+s^2}\sin smds$$

$$\int_0^\infty \frac{s}{1+s^2}\sin smds = \frac{\Pi}{2}e^{-m}$$

$$\int_0^\infty \frac{x}{1+x^2}\sin mxds = \frac{\Pi}{2}e^{-m}$$

7

6 **Find the Fourier cosine transform** $f(x) = \begin{cases} x, 0 < x < 1 \\ 2-x, 1 < x < 2 \\ 0, x > 2 \end{cases}$ **.**

Sol:

$$F_c(f(x)) = \int_0^\infty f(x)\cos sxdx$$

$$F_c(f(x)) = \int_0^1 x\cos sxdx + \int_1^2 (2-x)\cos sxdx + \int_2^\infty 0.\cos sxdx$$

$$F_c(f(x)) = \left(\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2}\right) + \left(-\frac{\sin s}{s} - \frac{\cos 2s}{s^2} + \frac{\cos s}{s^2}\right)$$

$$F_c(f(x)) = \frac{2\cos s}{s^2} - \frac{1}{s^2} - \frac{\cos 2s}{s^2}$$

7 **If the Fourier sine transform of** $f(x) = \frac{1-\cos n\Pi}{(n\Pi)^2}$ **then find f(x).**

Sol:

$$f(x) = \frac{2}{\Pi}\sum_{n=1}^{\infty} F_s(n)\sin nx$$

$$F_s(n) = \frac{1-\cos n\Pi}{(n\Pi)^2}$$

$$f(x) = \frac{2}{\Pi}\sum_{n=1}^{\infty} \frac{1-\cos n\Pi}{(n\Pi)^2}\sin nx$$

$$f(x) = \frac{2}{\Pi^3}\sum_{n=1}^{\infty} \frac{1-\cos n\Pi}{n^2}\sin nx$$

MODULE –IV

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

The important methods of solving ordinary differential equations of first order numerically are as follows

1) Taylors series method
2) Euler's method
3) Modified Euler's method of successive approximations
4) Runge- kutta method

To describe various numerical methods for the solution of ordinary differential eqn's,we consider the general 1^{st} order differential eqn

dy/dx=f(x,y)-------(1)

with the initial condition $y(x_0)=y_0$

The methods will yield the solution in one of the two forms:

i) A series for y in terms of powers of x,from which the value of y can be obtained by direct substitution.

ii) A set of tabulated values of y corresponding to different values of x

The methods of Taylor and picard belong to class(i)

The methods of Euler, Runge - kutta method, Adams, Milne etc, belong to class (ii)

TAYLOR'S SERIES METHOD

To find the numerical solution of the differential equation

$$\frac{dy}{dx} = f(x,y) \rightarrow (1)$$

With the initial condition $y(x_0) = y_0 \rightarrow (2)$

$y(x)$ can be expanded about the point x_0 in a Taylor's series in powers of $(x-x_0)$ as

$$y(x) = y(x_0) + \frac{(x-x_0)}{1} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \ldots\ldots\ldots\ldots + \frac{(x-x_0)^n}{n!} y^n(x_0) \rightarrow (3)$$

In equ3, $y(x_0)$ is known from I.C equ2. The remaining coefficients $y'(x_0), y''(x_0), \ldots\ldots\ldots y^n(x_0)$ etc are obtained by successively differentiating equ1 and evaluating at x_0. Substituting these values in equ3, $y(x)$ at any point can be calculated from equ3. Provided $h = x - x_0$ is small.

When $x_0 = 0$, then Taylor's series equ3 can be written as

$$y(x) = y(0) + x.y'(0) + \frac{x^2}{2!} y''(0) + \ldots\ldots + \frac{x^n}{n!} y^n(0) + \ldots\ldots\ldots \rightarrow (4)$$

1. Using Taylor's expansion evaluate the integral of $y' - 2y = 3e^x, y(0) = 0$**, at a)** $x = 0.2$

b) compare the numerical solution obtained with exact solution .

Sol: Given equation can be written as $2y+3e^x = y', y(0)=0$

Differentiating repeatedly w.r.t to 'x' and evaluating at $x=0$

$$y'(x) = 2y+3e^x, y'(0) = 2y(0)+3e^0 = 2(0)+3(1) = 3$$
$$y''(x) = 2y'+3e^x, y''(0) = 2y'(0)+3e^0 = 2(3)+3 = 9$$
$$y'''(x) = 2.y''(x)+3e^x, y'''(0) = 2y''(0)+3e^0 = 2(9)+3 = 21$$
$$y^{iv}(x) = 2.y'''(x)+3e^x, y^{iv}(0) = 2(21)+3e^0 = 45$$
$$y^{v}(x) = 2.y^{iv}+3e^x, y^{v}(0) = 2(45)+3e^0 = 90+3 = 93$$

In general, $y^{(n+1)}(x) = 2.y^{(n)}(x)+3e^x$ or $y^{(n+1)}(0) = 2.y^{(n)}(0)+3e^0$

The Taylor's series expansion of $y(x)$ about $x_0 = 0$ is

$$y(x) = y(0)+xy'(0)+\frac{x^2}{2!}y''(0)+\frac{x^3}{3!}y'''(0)+\frac{x^4}{4!}y''''(0)+\frac{x^5}{5!}y'''''(0)+....$$

Substituting the values of $y(0), y'(0), y''(0), y'''(0),..........$

$$y(x) = 0+3x+\frac{9}{2}x^2+\frac{21}{6}x^3+\frac{45}{24}x^4+\frac{93}{120}x^5+........$$

$$y(x) = 3x+\frac{9}{2}x^2+\frac{7}{2}x^3+\frac{15}{8}x^4+\frac{31}{40}x^5+........\rightarrow \text{equ1}$$

Now put $x=0.1$ in equ1

$$y(0.1) = 3(0.1)+\frac{9}{2}(0.1)^2+\frac{7}{2}(0.1)^3+\frac{15}{8}(0.1)^4+\frac{31}{40}(0.1)^5 = 0.34869$$

Now put $x=0.2$ in equ1

$$y(0.2) = 3(0.2)+\frac{9}{2}(0.2)^2+\frac{7}{2}(0.2)^3+\frac{15}{8}(0.2)^4+\frac{31}{40}(0.2)^5 = 0.811244$$

$$y(0.3) = 3(0.3)+\frac{9}{2}(0.3)^2+\frac{7}{2}(0.3)^3+\frac{15}{8}(0.3)^4+\frac{31}{40}(0.3)^5 = 1.41657075$$

Analytical Solution:

The exact solution of the equ $\frac{dy}{dx} = 2y+3e^x$ with $y(0)=0$ can be found as follows

$\frac{dy}{dx}-2y = 3e^x$ Which is a linear in y.

Here $P=-2, Q=3e^x$

I.F = $\int_e^{pdx} = \int_e^{-2dx} = e^{-24}$

General solution is $y.e^{-2x} = \int 3e^x.e^{-2x}dx + c = -3e^{-x} + c$

$\therefore y = -3e^x + ce^{2x}$ where $x = 0, y = 0 \quad 0 = -3 + c \Rightarrow c = 3$

The particular solution is $y = 3e^{2x} - 3e^x$ or $y(x) = 3e^{2x} - 3e^x$

Put $x = 0.1$ in the above particular solution,

$$y = 3.e^{0.2} - 3e^{0.1} = 0.34869$$

Similarly put $x = 0.2$

$$y = 3e^{0.4} - 3e^{0.2} = 0.811265$$

put $x = 0.3$

$$y = 3e^{0.6} - 3e^{0.3} = 1.416577$$

2. Using Taylor's series method, solve the equation $\frac{dy}{dx} = x^2 + y^2$ for $x = 0.4$ given that $y = 0$ when $x = 0$

Sol: Given that $\frac{dy}{dx} = x^2 + y^2$ and $y = 0$ when $x = 0$ i.e. $y(0) = 0$

Here $y_0 = 0$, $x_0 = 0$

Differentiating repeatedly w.r.t 'x' and evaluating at $x = 0$

$$y'(x) = x^2 + y^2, y'(0) = 0 + y^2(0) = 0 + 0 = 0$$
$$y''(x) = 2x + y'.2y, y''(0) = 2(0) + y'(0)2.y = 0$$
$$y'''(x) = 2 + 2yy'' + 2y'.y', y'''(0) = 2 + 2.y(0).y''(0) + 2.y'(0)^2 = 2$$
$$y''''(x) = 2.y.y''' + 2.y''.y' + 4.y''.y', y''''(0) = 0$$

The Taylor's series for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y''''(0) + ...$$

Substituting the values of $y(0), y'(0), y''(0), ...$

$$y(x) = 0 + x(0) + 0 + \frac{2x^3}{3!} + 0 + ... = \frac{x^3}{3} + \text{(Higher order terms are neglected)}$$

$$\therefore y(0.4) = \frac{(0.4)^3}{3} = \frac{0.064}{3} = 0.02133$$

3. Solve $y' = x - y^2, y(0) = 1$ **using Taylor's series method and compute y(0.1),y(0.2)**

Sol: Given that $y' = x - y^2, y(0) = 1$

Here $y_0 = 1$, $x_0 = 0$

Differentiating repeatedly w.r.t 'x' and evaluating at x=0

$$y'(x) = x - y^2, y'(0) = 0 - y(0)^2 = 0 - 1 = -1$$
$$y''(x) = 1 - 2y.y', y''(0) = 1 - 2.y(0)y'(0) = 1 - 2(-1) = 3$$
$$y'''(x) = 1 - 2yy' - 2(y')^2, y'''(0) = -2.y(0).y''(0) - 2.(y'(0))^2 = -6 - 2 = -8$$
$$y''''(x) = -2.y.y''' - 2.y''.y' - 4.y''.y', y''''(0) = -2.y(0).y'''(0) - 6.y''(0).y'(0) = 16 + 18 = 34$$

The Taylor's series for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + \frac{x}{1!} y^1(0) + \frac{x^2}{2!} y^{11}(0) + \frac{x^3}{3!} y^{111}(0) + \ldots..$$

Substituting the value of y(0), $y^1(0)$, $y^{11}(0)$,…..

$$y(x) = 1 - x + \frac{3}{2} x^2 - \frac{8}{6} x^3 + \frac{34}{24} x^4 + \ldots..$$

$$y(x) = 1 - x + \frac{3}{2} x^2 - \frac{4}{3} x^3 + \frac{17}{12} x^4 + \ldots.. \quad \rightarrow (1)$$

now put x = 0.1 in (1)

$$y(0.1) = 1 - 0.1 + \frac{3}{2}(0.1)^2 + \frac{4}{3}(0.1)^3 + \frac{17}{12}(0.1)^4 + \ldots..$$

$$= 0.91380333 \simeq 0.91381$$

Similarly put x = 0.2 in (1)

$$y(0.2) = 1 - 0.2 + \frac{3}{2}(0.2)^2 - \frac{4}{3}(0.2)^3 + \frac{17}{12}(0.2)^4 + \ldots..$$

$$= 0.8516.$$

4. Solve $y^1 = x^2 - y$, y(0) = 1, using Taylor's series method and compute y(0.1), y(0.2), y(0.3) and y(0.4) (correct to 4 decimal places).

Sol. Given that $y^I = x^2 - y$ and y(0) = 1

Here $x_0 = 0$, $y_0 = 1$ or y =1 when x=0

Differentiating repeatedly w.r.t 'x' and evaluating at x = 0.

$Y^{I}(x) = x^2 - y$, $y^{I}(0) = 0 - 1 = -1$

$y^{II}(x) = 2x - y^{I}$, $y^{II}(0) = 2(0) - y^{I}(0) = 0 - (-1) = 1$

$y^{III}(x) = 2 - y^{II}$, $y^{III}(0) = 2 - y^{II}(0) = 2 - 1 = 1$,

$y^{IV}(x) = - y^{III}$, $y^{IV}(0) = -y^{III}(0) = -1$.

The Taylor's servies for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + \frac{x}{1!}y^{I}(0) + \frac{x^2}{2!}y^{II}(0) + \frac{x^3}{3!}y^{III}(0) + \frac{x^4}{4!}y^{IV}(0) + \ldots\ldots$$

substituting the values of $y(0)$, $y^{1}(0)$, $y^{11}(0)$, $y^{111}(0)$, ……

$$y(x) = 1 + x(-1) + \frac{x^2}{2}(1) + \frac{x^3}{6}(1) + \frac{x^4}{24}(-1) + \ldots\ldots$$

$$y(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + \ldots\ldots \quad \rightarrow (1)$$

Now put x = 0.1 in (1),

$$y(0.1) = 1 - 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{6} - \frac{(0.1)^4}{24} + \ldots.$$

$$= 1 - 0.1 + 0.005 + 0.01666 - 0.0000416 - 0.905125 \sim 0.9051$$

(4 decimal places)

Now put x = 0.2 in eq (1),

$$y(0.2) = 1 - 0.2 + \frac{(0.2)^2}{2} + \frac{(0.2)^3}{6} - \frac{(0.2)^4}{64}$$

$$= 1 - 0.2 + 0.02 + 0.001333 - 0.000025$$

$$= 1.021333 - 0.200025$$

$$= 0.821308 \sim 0.8213 \text{ (4 decimals)}$$

Similarly y(0.3) = 0.7492 and y (0.4) = 0.6897 (4 decimal places).

5. Solve $\frac{dy}{dx} - 1 = xy$ and y(0) = 1 using Taylor's series method and compute y(0.1).

Sol. Given that $\frac{dy}{dx} - 1 = xy$ and y(0) = 1

Here $\frac{dy}{dx} = 1 + xy$ and $y_0 = 1$, $x_0 = 0$.

Differentiating repeatedly w.r.t 'x' and evaluating at $x_0 = 0$

$y^I(x) = 1 + xy,$ $\qquad$ $y^I(0) = 1+0(1) = 1$.

$y^{II}(x) = x.y' + y,$ $\qquad$ $y^{II}(0) = 0+1=1$

$y^{III}(x) = x.y'' + y^I + y^I,$ $\qquad$ $y^{III}(0) = 0.(1) + 2\,\square(1) =2$

$y^{IV}(x) = xy^{III} + y^{II} + 2y^{II},$ $\qquad$ $y^{IV}(0) = 0+3(1) =3.$

$y^V(x) = xy^{IV} + y^{III} +2y^{III},$ $\qquad$ $y^V(0) = 0 + 2 + 2(3) = 8$

The Taylor series for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + x.y^I(0) + \frac{x^2}{2!}\, y^{II}(0) + \frac{x^3}{3!}\, y^{III}(0) + \frac{x^4}{4!}\, y^{IV}(0) + \frac{x^5}{5!}\, y^V(0)+\ldots..$$

Substituting the values of $y(0)$, $y^I(0)$, $y^{II}(0)$, ….

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}(2) + \frac{x^4}{24}(3) + \frac{x^5}{120}(8) + \ldots.$$

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \ldots. \qquad \rightarrow (1)$$

Now put x = 0.1 in equ (1),

$$y(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{8} + \frac{(0.1)^5}{15} + \ldots..$$

$$= 1 + 0.1 +0.005 + 0.000333 + 0.0000125 + 0.0000006$$

$$= 1.1053461$$

H.W

6. Given the differential equ $y^1 = x^2 + y^2$, y(0) = 1.Obtain y(0.25), and y(0.5) by Taylor's Series method.

Ans: 1.3333, 1.81667

7. Solve $y^1 = xy^2 + y$, y(0) =1 using Taylor's series method and compute y(0.1) and y(0.2).

Ans: 1.111, 1.248.

Note: We know that the Taylor's expansion of y(x) about the point x_0 in a power of $(x - x_0)$is.

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y^{I}(x_0) + \frac{(x-x_0)^2}{2!} y^{II}(x_0) + \frac{(x-x_0)^3}{3!} y^{III}(x_0) + \ldots \rightarrow (1)$$

Or

$$y(x) = y_0 + \frac{(x-x_0)}{1!} y_0^{I} + \frac{(x-x_0)^2}{2!} y_0^{II} + \frac{(x-x_0)^3}{3!} y_0^{III} + \ldots..$$

If we let $x - x_0 = h$. (i.e. $x = x_0 + h = x_1$) we can write the Taylor's series as

$$y(x) = y(x_1) = y_0 + \frac{h}{1!} y_0^{I} + \frac{h^2}{2!} y_0^{II} + \frac{h^3}{3!} y_0^{III} + \frac{h^4}{4!} y_0^{IV} + \ldots.$$

$$\text{i.e. } y_1 = y_0 + \frac{h}{1!} y_0^{I} + \frac{h^2}{2!} y_0^{II} + \frac{h^3}{3!} y_0^{III} + \frac{h^{IV}}{4!} y_0^{IV} + \ldots.. \rightarrow (2)$$

Similarly expanding y(x) in a Taylor's series about $x = x_1$. We will get.

$$y_2 = y1 + \frac{h}{1!} y_1^{I} + \frac{h^2}{2!} y_1^{II} + \frac{h^3}{3!} y_1^{III} + \frac{h^4}{4!} y_1^{IV} + \ldots\ldots \rightarrow (3)$$

Similarly expanding y(x) in a Taylor's series about $x = x_2$ We will get.

$$y_3 = y_2 + \frac{h}{1!} y_2^{I} + \frac{h^2}{2!} y_2^{II} + \frac{h^3}{3!} y_2^{III} + \frac{h^4}{4!} y_2^{IV} + \ldots... \rightarrow (4)$$

In general, Taylor's expansion of y(x) at a point $x= x_n$ is

$$y_{n+1} = y_n + \frac{h}{1!} y_n^{I} + \frac{h^2}{2!} y_n^{II} + \frac{h^3}{3!} y_n^{III} + \frac{h^4}{4!} y_n^{IV} + \ldots.. \rightarrow (5)$$

8. Solve $y^1 = x-y^2$, y(0) = 1 using Taylor's series method and evaluate y(0.1), y(0.2).

Sol: Given $y^1 = x - y^2$ $\rightarrow (1)$

and y(0) = 1 $\rightarrow (2)$

Here $x_0 = 0$, $y_0 = 1$.

Differentiating (1) w.r.t 'x', we get.

$y^{II} = 1 - 2yy^{I} \rightarrow (3)$

$y^{III} = -2(y.\ y^{II} + (y^{I})^2)$ $\rightarrow (4)$

$y^{IV} = -2[y.\ y^{III} + y.\ y^{II} + 2y^{I}.\ y^{II}]$ $\rightarrow (5)$

$= -2(3y^{I}.\ y^{II} + y.\ y^{III})$ ……

Put $x_0 = 0$, $y_0 = 1$ in (1),(3),(4) and (5),

We get

$$y_0^{I} = 0-1 = -1,$$

$$y_0^{II} = 1-2(1)(-1) = 3,$$

$$y_0^{III} = -2[(-1)^2) + (1)(3)] = -8$$

$$y_0^{IV} = -2[3(-1)(3) + (1)(-8)] = -2(-9 -8) = 34.$$

Take h=0.1

<u>Step1</u>: By Taylor's series, we have

$$y_1 = y_0 + \frac{h}{1!}y_0^{I} + \frac{h^2}{2!}y_0^{II} + \frac{h^3}{3!}y_0^{III} + \frac{h^4}{4!}y_0^{IV} + \ldots \qquad \rightarrow \textbf{(6)}$$

on substituting the values of y_0, y_0^{I}, y_0^{II}, etc in equ (6) we get

$$y(0.1) = y_1 = 1 + \frac{0.1}{1}(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-8) + \frac{(0.1)^4}{24}(34) + \ldots$$

$$= 1 - 0.1 + 0.015 - 0.00133 + 0.00014 + \ldots$$

$$= 0.91381$$

<u>Step2</u>: Let us find y(0.2), we start with (x_1,y_1) as the starting value.

Here $x_1 = x_0 + h = 0+0.1 = 0.1$ and $y_1 = 0.91381$

Put these values of x_1 and y_1 in (1),(3),(4) and (5),we get

$$y_1^{I} = x_1 - y_1^2 = 0.1 - (0.91381)^2 = 0.1 - 0.8350487 = -0.735$$

$$y_1^{II} = 1 - 2y_1 \cdot y_1^{I} = 1- 2(0.91381)(-0.735) = 1 + 1.3433 = 2.3433$$

$$y_1^{III} = -2[(y_1^{I})^2 + y_1 \cdot y_1^{II}] = -2[(-0.735)^2 + (0.91381)(2.3433)] = -5.363112$$

$$y_1^{IV} = -2[3.\ y_1^{I}\ y_1^{II} + y_1\ y_1^{III}] = -2[3.(-0.735)(2.3433) + (0.91381)(-5.363112)]$$

$$= -2[(-5.16697) - 4.9] = 20.133953$$

By Taylor's series expansion,

$$y_2 = y_1 + \frac{h}{1!}y_1^{I} + \frac{h^2}{2!}y_1^{II} + \frac{h^3}{3!}y_1^{III} + \frac{h^4}{4!}y_1^{IV} + \ldots$$

$$\therefore y(0.2) = y_2 = 0.91381 + (0.1)(-0.735) + \frac{(0.1)^2}{2}(2.3433) +$$

$$\frac{(0.1)^3}{6}(-5.363112) + \frac{(0.1)^4}{24}(20.133953) + \ldots.$$

y(0.2) = 0.91381 – 0.0735 + 0.0117 – 0.00089 + 0.00008

= 0.8512

9. Tabulate y(0.1), y(0.2) and y(0.3) using Taylor's series method given that $y^1 = y^2 + x$ and y(0) = 1

Sol: Given $y^1 = y^2 + x$ →(1)

and y(0) = 1 →(2)

Here $x_0 = 0$, $y_0 = 1$.

Differentiating (1) w.r.t 'x', we get

$y^{II} = 2y \cdot y^I + 1$ →(3)

$y^{III} = 2[y \cdot y^{II} + (y^I)^2]$ →(4)

$y^{IV} = 2[y \cdot y^{III} + y^I y^{II} + 2 y^I y^{II}]$

$= 2[y \cdot y^{III} + 3 y^I y^{II}]$ →(5)

Put $x_0 = 0$, $y_0 = 1$ in (1), (3), (4) and (5), we get

$y_0^I = (1)^2 + 0 = 1$

$y_0^{II} = 2(1)(1) + 1 = 3,$

$y_0^{III} = 2((1)(3) + (1)^2) = 8$

$y_0^{IV} = 2[(1)(8) + 3(1)(3)]$

$= 34$

Take h = 0.1.

<u>**Step1**</u>: By Taylor's series expansion, we have

$$y(x_1) = y_1 = y_0 + \frac{h}{1!}y_0^I + \frac{h^2}{2!}y_0^{II} + \frac{h^3}{3!}y_0^{III} + \frac{h^4}{4!}y_0^{IV} + \ldots. \quad \rightarrow(6)$$

on substituting the values of y_o, y_0^I, y_0^{II} etc in (6),we get

$$y(0.1) = y_1 = 1 + (0.1)(1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(8) + \frac{(0.1)^4}{24}(34) + \ldots.$$

= 1 + 0.1 + 0.015 + 0.001333 + 0.000416

$y_1 = 1.116749$

Step2: Let us find y(0.2),we start with (x_1,y_1) as the starting values

Here $x_1 = x_0 + h = 0 + 0.1 = 0.1$ and $y_1 = 1.116749$

Putting these values in (1),(3),(4) and (5), we get

$$y_1^I = y_1^2 + x_1 = (1.116749)^2 + 0.1 = 1.3471283$$

$$y_1^{II} = 2y_1\, y_1^I + 1 = 2(10116749)\,(1.3471283) + 1 = 4.0088$$

$$y_1^{III} = 2(y_1\, y_1^{II} + (y_1^I)^2) = 2((1.116749)\,(4.0088) + (1.3471283)^2] = 12.5831$$

$$y_1^{IV} = 2y_1\, y_1^{III} + 6\, y_1^I\, y_1^{II} = 2(1.116749)\,(12.5831) + 6(1.3471283)\,(4.0088) = 60.50653$$

By Taylor's expansion

$$y(x_2) = y_2 = y_1 + \frac{h}{1!} y_1^I + \frac{h^2}{2!} y_1^{II} + \frac{h^3}{3!} y_1^{III} + \frac{h^4}{4!} y_1^{IV} + \ldots.$$

$$\therefore y(0.2) = y_2 = 1.116749 + (0.1)\,(1.3471283) + \frac{(0.1)^2}{2}\,(4.0088) + \frac{(0.1)^3}{6}\,(12.5831)$$

$$+ \frac{(0.1)^4}{24}\,(60.50653)$$

$$y_2 = 1.116749 + 0.13471283 + 0.020044 + 0.002097 + 0.000252$$

$$= 1.27385$$

$$y(0.2) = 1.27385$$

Step3: Let us find y(0.3),we start with (x_2,y_2) as the starting value.

Here $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$ and $y_2 = 1.27385$

Putting these values of x_2 and y_2 in eq (1), (3), (4) and (5), we get

$$y_2^I = y_2^2 + x_2 = (1.27385)^2 + 0.2 = 1.82269$$

$$y_2^{II} = 2y_2\, y_2^1 + 1 = 2(1.27385)\,(1.82269) + 1 = 5.64366$$

$$y_2^{III} = 2[y_2\, y_2^{II} + (y_2^I)^2] = 2[(1.27385)\,(5.64366) + (1.82269)^2]$$

$$= 14.37835 + 6.64439 = 21.02274$$

$$y_2^{IV} = 2y_2 + y_2^{III} + 6\, y_2^I \cdot y_1^{II} = 2(1.27385)\,(21.00274) + 6(1.82269)\,(5.64366)$$

$$= 53.559635 + 61.719856 = 115.27949$$

By Taylor's expansion,

$y(x_3) = y_3 = y_2 + \frac{h}{1!}y_2^{I} + \frac{h^2}{2!}y_2^{II} + \frac{h^3}{3!}y_2^{III} + \frac{h^4}{4!}y_2^{IV} + \ldots..$

$y(0.3) = y_3 = 1.27385 + (0.1)(1.82269) + \frac{(0.1)^2}{2}(5.64366) + \frac{(0.1)^3}{6}(21.02274)$

$+ \frac{(0.1)^4}{24}(115.27949)$

$= 1.27385 + 0.182269 + 0.02821 + 0.0035037 + 0.00048033$

$= 1.48831$

$y(0.3) = 1.48831$

10. Solve $y^1 = x^2 - y$, $y(0) = 1$ using Taylor's series method and evaluate

y(0.1),y(0.2),y(0.3) and y(0.4) (correct to 4 decimal places)

Sol: Given $y^1 = x^2 - y$ →(1)

and $y(0) = 1$ →(2)

Here $x_0 = 0$, $y_0 = 1$

Differentiating (1) w.r.t 'x', we get

$y^{II} = 2x - y^1$ →(3)

$y^{III} = 2 - y^{II}$ →(4)

$y^{IV} = -y^{III}$ →(5)

put $x_0 = 0$, $y_0 = 1$ in (1),(3),(4) and (5), we get

$y_0^I = x_0^2 - y_0 = 0-1 = -1,$

$y_0^{II} = 2x_0 - y_0^I = 2(0) - (-1) = 1$

$y_0^{III} = 2 - y_0^{II} = 2-1 = 1,$

$y_0^{IV} = - y_0^{III} = -1$ Take h = 0.1

<u>**Step1**</u>: by Taylor's series expansion

$y(x_1) = y_1 = y_0 + \frac{h}{1!}y_0^I + \frac{h^2}{2!}y_0^{II} + \frac{h^3}{3!}y_0^{III} + \frac{h^4}{4!}y_0^{IV} + \ldots.$ →(6)

On substituting the values of y_0, y_0^I, y_0^{II} etc in (6), we get

$y(0.1) = y_1 = 1 + (0.1)(-1) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(1) + \frac{(0.1)^4}{24}(-1) + \ldots.$

$= 1 - 0.1 + 0.005 + 0.01666 – 0.0000416$

$= 0.905125 \underset{\sim}{} 0.9051$ (4 decimal place).

Step2: Let us find y(0.2) we start with (x_1, y_1) as the starting values

Here $x = x_0 + h = 0 + 0.1 = 0.1$ and $y_1 = 0.905125$,

Putting these values of x_1 and y_1 in (1), (3), (4) and (5), we get

$y_1^1 = x_1^2 - y_1 = (0.1)^2 – 0.905125 = -0.895125$

$y_1^{II} = 2x_1 - y_1^1 = 2(0.1) – (-0.895125) = 1.095125$,

$y_1^{III} = 2 - y_1^{II} = 2 – 1.095125 = 0.90475$,

$y_1^{IV} = - y_1^{III} = -0.904875$,

By Taylor's series expansion,

$$y(x_2) = y_2 = y_1 + \frac{h}{1!} y_1^I + \frac{h^2}{2!} y_1^{II} + \frac{h^3}{3!} y_1^{III} + \frac{h^4}{4!} y_1^{IV} + \ldots$$

$$y(0.2) = y_2 = 0.905125 + (0.1)(-0.895125) + + \frac{(0.1)^2}{2}(1.09125)$$

$$+ \frac{(0.1)^3}{6}(1.095125) + \frac{(0.1)^4}{24}(-0.904875) + \ldots$$

$y(0.2) = y_2 = 0.905125 – 0.0895125 + 0.00547562 + 0.000150812 – 0.0000377$

$= 0.8212351 \underset{\sim}{} 0.8212$ (4 decimal places)

Step3: Let us find y(0.3), we start with (x_2, y_2) as the starting value

Here $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$ and $y_2 = 0.8212351$

Putting these values of x_2 and y_2 in (1),(3),(4), and (5) we get

$y_2^1 = x_2^2 - y_2 = (0.2)^2 – 0.8212351 = 0.04 – 0.8212351 = - 0.7812351$

$y_2^{II} = 2x_2 - y_2^1 = 2(0.2) + (0.7812351) = 1.1812351$,

$y_2^{III} = 2 - y_2^{II} = 2 – 1.\ 1812351 = 0.818765$,

$y_2^{IV} = - y_2^{III} = -0.818765$,

By Taylor's series expansion,

$$y(x_3) = y_3 = y_2 + \frac{h}{1!} y_2^I + \frac{h^2}{2!} y_2^{II} + \frac{h^3}{3!} y_2^{III} + \frac{h^4}{4!} y_2^{IV} + \ldots$$

$$y(0.3) = y_3 = 0.8212351 + (0.1)(-0.7812351) + \frac{(0.1)^2}{2}(1.1812351) + \frac{(0.1)^3}{6}(0.818765) + \frac{(0.1)^4}{24}(-0.818765) + \ldots$$

$y(0.3) = y_3 = 0.8212351 - 0.07812351 + 0.005906 + 0.000136 - 0.0000034$

$= 0.749150 \simeq 0.7492$ (4 decimal places)

Step4: Let us find y(0.4), we start with (x_3, y_3) as the starting value

Here $x_3 = x_2 + h = 0.2 + 0.1 = 0.3$ and $y_3 = 0.749150$

Putting these values of x_3 and y_3 in (1),(3),(4), and (5) we get

$y_3^1 = x_3^2 - y_3 = (0.3)^2 - 0.749150 = -0.65915$,

$y_3^{II} = 2x_3 - y_3^1 = 2(0.3) + (0.65915) = 1.25915$,

$y_3^{III} = 2 - y_3^{II} = 2 - 1.25915 = 0.74085$,

$y_3^{IV} = -y_3^{III} = -0.74085$,

By Taylor's series expansion,

$$y(x_4) = y_4 = y_3 + \frac{h}{1!} y_3^I + \frac{h^2}{2!} y_3^{II} + \frac{h^3}{3!} y_3^{III} + \frac{h^4}{4!} y_3^{IV} + \ldots$$

$$y(0.4) = y_4 = 0.749150 + (0.1)(-0.65915) + \frac{(0.1)^2}{2}(1.25915) + \frac{(0.1)^3}{6}(0.74085) + \frac{(0.1)^4}{24}(-0.74085) + \ldots$$

$y(0.4) = y_4 = 0.749150 - 0.065915 + 0.0062926 + 0.000123475 - 0.0000030$

$= 0.6896514 \simeq 0.6896$ (4 decimal places)

11. Solve $y^1 = x^2 - y$, $y(0) = 1$using T.S.M and evaluate y(0.1),y(0.2),y(0.3) and y(0.4) (correct to 4 decimal place) 0.9051, 0.8212, 07492, 0.6896

12. Given the differentiating equation $y^1 = x^1 + y^2$, $y(0) = 1$. Obtain y(0.25) and y(0.5) by T.S.M.

Ans: 1.3333, 1.81667

13. Solve $y^1 = xy^2 + y$, $y(0) = 1$ using Taylor's series method and evaluate y(0.1) and y(0.2)

Ans: 1.111, 1.248.

EULER'S METHOD

It is the simplest one-step method and it is less accurate. Hence it has a limited application.

Consider the differential equation $\frac{dy}{dx} = f(x,y)$ →(1)

With $y(x_0) = y_0$ →(2)

Consider the first two terms of the Taylor's expansion of y(x) at $x = x_0$

$$y(x) = y(x_0) + (x - x_0)\, y^1(x_0) \qquad \rightarrow(3)$$

from equation (1) $y^1(x_0) = f(x_0,y(x_0)) = f(x_0,y_0)$

Substituting in equation (3)

$\therefore y(x) = y(x_0) + (x - x_0)\, f(x_0,y_0)$

At $x = x_1$, $y(x_1) = y(x_0) + (x_1 - x_0)\, f(x_0,y_0)$

$\therefore y_1 = y_0 + h\, f(x_0,y_0)$ where $h = x_1 - x_0$

Similarly at $x = x_2$, $y_2 = y_1 + h\, f(x_1,y_1)$,

Proceeding as above, $y_{n+1} = y_n + h\, f(x_n,y_n)$

This is known as Euler's Method

1. **Using Euler's method solve for x = 2 from $\frac{dy}{dx} = 3x^2 + 1$, y(1) = 2, taking step size** (I) h = 0.5 and (II) h=0.25

Sol: here $f(x,y) = 3x^2 + 1$, $x_0 = 1, y_0 = 2$

Euler's algorithm is $y_{n+1} = y_n + h\ f(x_n,y_n)$, n = 0,1,2,3,….. →(1)

h = 0.5 $\therefore x_1 = x_0 + h = 1+0.5 = 1.5$

Taking n = 0 in (1) , we have $x_2 = x_1 + h = 1.5 + 0.5 = 2$

$y_1 = y_0 + h\, f(x_0,y_0)$

i.e. $y_1 = y(0.5) = 2 + (0.5)\, f(1,2) = 2 + (0.5)\,(3 + 1) = 2 + (0.5)(4)$

Here $x_1 = x_0 + h = 1 + 0.5 = 1.5$

$\therefore y(1.5) = 4 = y_1$

Taking n = 1 in (1),we have

$y_2 = y_1 + h\, f(x_1,y_1)$

i.e. $y(x_2) = y_2 = 4 + (0.5)\ f(1.5,4) = 4 + (0.5)[3(1.5)^2 + 1] = 7.875$

Here $x_2 = x_4 + h = 1.5 + 0.5 = 2$

$\therefore y(2) = 7.875$

(I) $h = 0.25$ $\therefore x_1 = 1.25,\ x_2 = 1.50,\ x_3 = 1.75,\ x_4 = 2$

Taking n = 0 in (1), we have

$y_1 = y_0 + h\ f(x_0,y_0)$

i.e. $y(x_1) = y_1 = 2 + (0.25)\ f(1,2) = 2 + (0.25)\ (3 + 1) = 3$

$y(x_2) = y_2 = y_1 + h\ f(x_1,y_1)$

i.e. $y(x_2) = y_2 = 3 + (0.25)\ f(1.25,3)$

$= 3 + (0.25)[3(1.25)^2 + 1]$

$= 4.42188$

Here $x_2 = x_1 + h = 1.25 + 0.25 = 1.5$

$\therefore y(1.5) = 5.42188$

Taking n = 2 in (1), we have

i.e. $y(x_3) = y_3 = h\ f(x_2,y_2)$

$= 5.42188 + (0.25)\ f(1.5,2)$

$= 5.42188 + (0.25)\ [3(1.5)^2 + 1]$

$= 6.35938$

Here $x_3 = x_2 + h = 1.5 + 0.25 = 1.75$

$\therefore y(1.75) =7.\ 35938$

Taking n = 4 in (1),we have

$y(x_4) = y_4 = y_3 + h\ f(x_3,y_3)$

i.e. $y(x_4) = y_4 = 7.35938 + (0.25)\ f(1.75,2)$

$= 7.35938 + (0.25)[3(1.75)^2 + 1]$

$= 8.90626$

Note that the difference in values of y(2) in both cases
(i.e. when h = 0.5 and when h = 0.25).The accuracy is improved significantly when h is reduced to 0.25 (Example significantly of the equ is $y = x^3 + x$ and with this $y(2) = y_2 = 10$

2. **Solve by Euler's method,$y^1 = x + y$, y(0) = 1 and find y(0.3) taking step size h = 0.1. compare the result obtained by this method with the result obtained by analytical solution**

Sol: $y_1 = 1.1 = y(0.1)$,

$y_2 = y(0.2) = 1.22$

$y_3 = y(0.3) = 1.362$

Particular solution is $y = 2e^x - (x + 1)$

Hence y(0.1) = 1.11034, y(0.2) = 1.3428, y(0.3) = 1.5997

We shall tabulate the result as follows

X	0	0.1	0.2	0.3
Euler y	1	1.1	1.22	1.362
Euler y	1	1.11034	1.3428	1.3997

The value of y deviate from the execute value as x increases. This indicate that the method is not accurate

3. **Solve by Euler's method $y^1 + y = 0$ given y(0) = 1 and find y(0.04) taking step size**

h = 0.01 Ans: 0.9606

4. **Using Euler's method, solve y at x = 0.1 from $y^1 = x + y + xy$, y()) = 1 taking step size h = 0.025.**

5. **Given that $\frac{dy}{dx}$ = xy ,y(0) = 1 determine y(0.1),using Euler's method.** h =0.1

Sol: The given differentiating equation is $\frac{dy}{dx}$ = xy, y(0) = 1

a = 0

Here f(x,y) = xy , $x_0 = 0$ and $y_0 = 1$

Since h is not given much better accuracy is obtained by breaking up the interval (0,0.1) in to five steps.

i.e. $h = \frac{b-a}{5} = \frac{0.1}{5} = 0.02$

Euler's algorithm is $y_{n+1} = y_n + h\, f(x_n, y_n)$ →(1)

∴ From (1) form = 0, we have

$y_1 = y_0 + h\, f(x0, y_0)$

$= 1 + (0.02)\, f(0,1)$

$= 1 + (0.02)\,(0)$

$= 1$

Next we have $x_1 = x_0 + h = 0 + 0.02 = 0.02$

∴ From (1), form = 1,we have

$y_2 = y_1 + h\, f(x_1, y_1)$

$= 1 + (0.02)\, f(0.02,1)$

$= 1 + (0.02)\,(0.02)$

$= 1.0004$

Next we have $x_2 = x_1 + h = 0.02 + 0.02 = 0.04$

∴ From (1), form = 2,we have

$y_3 = y_2 + h\, f(x_2, y_2)$

$= 1.004 + (0.02)\,(0.04)\,(1.0004)$

$= 1.0012$

Next we have $x_3 = x_2 + h = 0.04 + 0.02 = 0.06$

∴ From (1), form = 3,we have

$y_4 = y_3 + h\, f(x_3, y_3)$

$= 1.0012 + (0.02)\,(0.06)\,(1.00012)$

$= 1.0024.$

Next we have $x_4 = x_3 + h = 0.06 + 0.02 = 0.08$

∴ From (1), form = 4,we have

$y_5 = y_4 + h\, f(x_4, y_4)$

$= 1.0024 + (0.02)\,(0.08)\,(1.00024)$

$= 1.0040.$

Next we have $x_5 = x_4 + h = 0.08 + 0.02 = 0.1$

When $x = x_5$, $y \simeq y_5$

$\therefore y = 1.0040$ when $x = 0.1$

6. **Solve by Euler's method $y^1 = \frac{2y}{x}$ given y(1) = 2 and find y(2).**

7. **Given that $\frac{dy}{dx} = 3x^2 + y$, y(0) = 4.Find y(0.25) and y(0.5) using Euler's method**

Sol: given $\frac{dy}{dx} = 3x^2 + y$ and y(1) = 2.

Here $f(x,y) = 3x^2 + y$, $x_0 = (1)$, $y_0 = 4$

Consider h = 0.25

Euler's algorithm is $y_{n+1} = y_n + h\, f(x_n,y_n)$ →(1)

$\therefore$ From (1), for n = 0, we have

$y_1 = y_0 + h\, f(x_0,y_0)$

$= 2 + (0.25)[0 + 4]$

$= 2 + 1$

$= 3$

Next we have $x_1 = x_0 + h = 0 + 0.25 = 0.25$

When $x = x_1$, $y_1 \simeq y$

$\therefore y = 3$ when $x = 0.25$

$\therefore$ From (1), for n = 1, we have

$y_2 = y_1 + h\, f(x_1,y_1)$

$= 3 + (0.25)[3.(0.25)^2 + 3]$

$= 3.7968$

Next we have $x_2 = x_1 + h = 0.25 + 0.25 = 0.5$

When $x = x_2$, $y \simeq y_2$

$\therefore y = 3.7968$ when $x = 0.5$.

a. **Solve first order diff equation $\frac{dy}{dx} = \frac{y-x}{y+x}$, y(0) = 1 and estimate y(0.1) using Euler's method (5 steps)** Ans: 1.0928

b. Use Euler's method to find approximate value of solution of $\frac{dy}{dx}$ = y-x + 5 at x = 2-1 and 2-2with initial contention y(0.2) = 1

<u>Modified Euler's method</u>

It is given by $y^{(i)}{}_{k+1} = y_k + h/2\, f\left[\left(x_k, y_k\right) + f\left(x_{k+1}, 1\right)_{k+1}{}^{(i-1)}\right], i = 1, 2....., ki = 0, 1.....$

Working rule :

i)Modified Euler's method

$y^{(i)}_{k+1} = y_k + h/2 f\left[\left(x_k, y_k\right) + f\left(x_{k+1}, 1\right)_{k+1}^{(i-1)}\right], i = 1, 2....., ki = 0, 1.....$

ii) When $i = 1$ $y^0{}_{k+1}$ can be calculated from Euler's method

iii) K=0, 1……… gives number of iteration. $i = 1, 2...$

gives number of times, a particular iteration k is repeated

Suppose consider dy/dx=f(x, y) -------- (1) with $y(x_0) = y_0$----------- (2)

To find $y(x_1) = y_1$ at $x = x_1 = x_0 + h$

Now take k=0 in modified Euler's method

We get $y_1^{(1)} = y_0 + h/2\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(i-1)}\right)\right]$……………………… (3)

Taking i=1, 2, 3...k+1 in eqn (3), we get

$y_1^{(0)} = y_0 + h/2\left[f\left(x_0, y_0\right)\right]$ (By Euler's method)

$y_1^{(1)} = y_0 + h/2\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(0)}\right)\right]$

$y_1^{(2)} = y_0 + h/2\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(1)}\right)\right]$

$y_1^{(k+1)} = y_0 + h/2\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(k)}\right)\right]$

If two successive values of $y_1^{(k)}, y_1^{(k+1)}$ are sufficiently close to one another, we will take the common value as $y_2 = y\left(x_2\right) = y\left(x_1 + h\right)$

We use the above procedure again

1) using modified Euler's method find the approximate value of x when $x = 0.3$ given that $dy/dx = x + y$ *and* $y\left(0\right) = 1$

sol: Given $dy/dx = x + y$ *and* $y\left(0\right) = 1$

Here $f\left(x, y\right) = x + y, x_0 = 0,$ *and* $y_0 = 1$

Take h = 0.1 which is sufficiently small

Here $x_0 = 0, x_1 = x_0 + h = 0.1, x_2 = x_1 + h = 0.2, x_3 = x_2 + h = 0.3$

The formula for modified Euler's method is given by

$y_{k+1}^{(i)} = y_k + h/2\left[f\left(x_k + y_k\right) + f\left(x_{k+1}, y_{k+1}^{(i-1)}\right)\right] \rightarrow (1)$

Step1: To find $y_1 = y(x_1) = y\ (0.1)$

Taking k = 0 in eqn(1)

$y_{k+1}^{(i)} = y_0 + h/2\left[f\left(x_0 + y_0\right) + f\left(x_1, y_1^{(i-1)}\right)\right] \rightarrow (2)$

when i = 1 in eqn (2)

$$y_1^{(i)} = y_0 + h/2\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(0)}\right)\right]$$

First apply Euler's method to calculate $y_1^{(0)}$ = y_1

$$\therefore\ y_1^{(0)} = y_0 + h\, f\left(x_0, y_0\right)$$

= 1+(0.1)f(0.1)

= 1+(0.1)

= 1.10

$$now\left[x_0 = 0, y_0 = 1, x_1 = 0.1, y_1(0) = 1.10\right]$$

$$\therefore\ y_1^{(1)} = y_0 + 0.1/2\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(0)}\right)\right]$$

= 1+0.1/2[f(0,1) + f(0.1,1.10)

= 1+0.1/2[(0+1)+(0.1+1.10)]

= 1.11

When i=2 in eqn (2)

$$y_1^{(2)} = y_0 + h/2\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(1)}\right)\right]$$

= 1+0.1/2[f(0.1)+f(0.1,1.11)]

= 1 + 0.1/2[(0+1)+(0.1+1.11)]

= 1.1105

$$y_1^{(3)} = y_0 + h/2\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(2)}\right)\right]$$

= 1+0.1/2[f(0,1)+f(0.1 , 1.1105)]

= 1+0.1/2[(0+1)+(0.1+1.1105)]

= 1.1105

Since $y_1^{(2)} = y_1^{(3)}$

$\therefore$ y_1 = 1.1105

<u>Step:2</u> To find $y_2 = y(x_2) = y(0.2)$

Taking k = 1 in eqn (1) , we get

$$y_2^{(i)} = y_1 + h/2\left[f\left(x_1, y_1\right) + f\left(x_2, y_2^{(i-1)}\right)\right] \rightarrow (3)$$

i = 1,2,3,4,…..

For i = 1

$$y_2^{(1)} = y_1 + h/2\left[f\left(x_1, y_1\right) + f\left(x_2, y_2^{(0)}\right)\right]$$

$y_2^{(0)}$ is to be calculate from Euler's method

$y_2^{(0)} = y_1 + h\ f(x_1, y_1)$

= 1.1105 + (0.1) f(0.1 , 1.1105)

= 1.1105+(0.1)[0.1+1.1105]

= 1.2316

$\therefore\ y_2^{(1)}$ = $1.1105 + 0.1/2\left[f(0.1, 1.1105) + f(0.2, 1.2316)\right]$

= 1.1105 +0.1/2[0.1+1.1105+0.2+1.2316]

= 1.2426

$y_2^{(2)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2 y_2^{(1)}\right)\right]$

= 1.1105 + 0.1/2[f(0.1 , 1.1105) , f(0.2 . 1.2426)]

= 1.1105 + 0.1/2[1.2105 + 1.4426]

= 1.1105 + 0.1(1.3266)

= 1.2432

$y_2^{(3)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2 y_2^{(2)}\right)\right]$

= 1.1105+0.1/2[f(0.1,1.1105)+f(0.2 , 1.2432)]

= 1.1105+0.1/2[1.2105+1.4432)]

= 1.1105 + 0.1(1.3268)

= 1.2432

Since $y_2^{(3)} = y_2^{(3)}$

Hence y_2 = 1.2432

Step:3

To find $y_3 = y(x_3)$ = y y(0.3)

Taking k =2 in eqn (1) we get

$$y_3^{(1)} = y_2 + h/2\left[f(x_2, y_2) + f(x_3, y_3^{(i-1)})\right] \rightarrow (4)$$

For i = 1 ,

$$y_3^{(1)} = y_2 + h/2\left[f(x_2, y_2) + f\left(x_3, y_3^{(0)}\right)\right]$$

$y_3^{(0)}$ is to be evaluated from Euler's method .

$$y_3^{(0)} = y_2 + h\, f(x_2, y_2)$$

= 1.2432 +(0.1) f(0.2 , 1.2432)

= 1.2432+(0.1)(1.4432)

= 1.3875

$\therefore\ y_3^{(1)}$ = 1.2432+0.1/2[f(0.2 , 1.2432)+f(0.3, 1.3875)]

= 1.2432 + 0.1/2[1.4432+1.6875]

= 1.2432+0.1(1.5654)

= 1.3997

$$y_3^{(2)} = y_2 + h/2\left[f(x_2, y_2) + f\left(x_3, y_3^{(1)}\right)\right]$$

= 1.2432+0.1/2[1.4432+(0.3+1.3997)]

= 1.2432+ (0.1) (1.575)

= 1.4003

$$y_3^{(3)} = y_2 + h/2\left[f(x_2, y_2) + f\left(x_3, y_3^{(2)}\right)\right]$$

= 1.2432+0.1/2[f(0.2 , 1.2432)+f(0.3 , 1.4003)]

= 1.2432 + 0.1(1.5718)

= 1.4004

$$y_3^{(4)} = y_2 + h/2\left[f(x_2, y_2) + f\left(x_3, y_3^{(3)}\right)\right]$$

= 1.2432 + 0.1/2[1.4432+1.7004]

= 1.2432+(0.1)(1.5718)

= 1.4004

Since $y_3^{(3)} = y_3^{(4)}$

Hence $y_3 = 1.4004$ $\therefore$ The value of y at x = 0.3 is 1.4004

2 . Find the solution of $\frac{dy}{dx}$ = x-y , y(0)=1 at x =0.1 , 0.2 ,0.3 , 0.4 and 0.5 . Using modified Euler's method

Sol . Given $\frac{dy}{dx}$ = x-y and y(0) = 1

Here f(x,y) = x-y , $x_0 = 0$ and $y_0 = 1$

Consider h = 0.1 so that

x = 0.1 , $x_2 = 0.2$, $x_3 = 0.3$, $x_4 = 0.4$ and $x_5 = 0.5$

The formula for modified Euler's method is given by

$$y_{k+1}^{(i)} = y_k + h/2 \; f\left(x_k, y_k\right) + f x_{k+1}, y_{k+1}^{(i-1)} \rightarrow (1)$$

Where k = 0,1, 2, 3,….. i = 1, 2, 3,…..

3. Find y(0.1) and y(0.2) using modified Euler's formula given that dy/dx=x^2-y,y(0)=1

[consider h=0.1,y_1=0.90523,y_2=0.8214]

4. Given $dy/dx = -xy^2, y(0) = 2$ compute y(0.2) in steps of 0.1

Using modified Euler's method

[h=0.1, y_1=1.9804, y_2=1.9238]

5. Given y^1 = x+siny, y(0)=1 compute y(0.2) and y(0.4) with h=0.2 using modified Euler's method

[y_1=1.2046, y_2=1.4644]

<u>Runge – Kutta Method</u>

I. Second order R-K Formula

$y_{i+1} = y_i + 1/2\ (K_1 + K_2)$,

Where $K_1 = h\ (x_i, y_i)$

$K_2 = h\ (x_i + h, y_i + k_1)$

For i= 0,1,2-------

II. Third order R-K Formula

$y_{i+1} = y_i + 1/6\ (K_1 + 4K_2 + K_3)$,

Where $K_1 = h\ (x_i, y_i)$

$K_2 = h\ (x_i + h/2, y_0 + k_1/2)$

$K_3 = h\ (x_i + h, y_i + 2k_2 - k_1)$

For i= 0,1,2-------

III. Fourth order R-K Formula

$y_{i+1} = y_i + 1/6\ (K_1 + 2K_2 + 2K_3 + K_4)$,

Where $K_1 = h\ (x_i, y_i)$

$K_2 = h\ (x_i + h/2, y_i + k_1/2)$

$K_3 = h\ (x_i + h/2, y_i + k_2/2)$

$K_4 = h\ (x_i + h, y_i + k_3)$

For i= 0,1,2-------

1. Using Runge-Kutta method of second order, find $y(2.5)$ from $\frac{dy}{dx} = \frac{x+y}{x}$, y(2)=2 , h = 0.25 .

Sol: Given $\frac{dy}{dx} = \frac{x+y}{x}$, y(2) = 2 .

Here $f(x, y) = \frac{x+y}{x}$, $x_0 = 0$, $y_0 = 2$ and h = 0.25

$x_1 = x_0+h = 2+0.25 = 2.25$, $x_2 = x_1+h = 2.25+0.25 = 2.5$

By R-K method of second order,

$$y_{i+1} = y_i + 1/2\left(k_1+k_2\right), k_1 - hf\left(x_i+h, y_i+k_1\right), i = 0,1.... \to (1)$$

Step -1:-

To find $y(x_1)$i.e y(2.25) by second order R - K method taking i=0 in eqn(i)

We have $y_1 = y_0 + \frac{1}{2}\left(k_1+k_2\right)$

Where k_1= hf (x_0,y_0), k_2= hf (x_0+h,y_0+k_1)

f (x_0,y_0)=f(2,2)=2+2/2=2

k_1=hf (x_0,y_0)=0.25(2)=0.5

k_2= hf (x_0+h,y_0+k_1)=(0.25)f(2.25,2.5)

=(0.25)(2.25+2.5/2.25)=0.528

$\therefore$ y_1=y(2.25)=2+1/2(0.5+0.528)

=2.514

Step2:

To find $y(x_2)$ i.e., y(2.5)

i=1 in (1)

x_1=2.25,y_1=2.514,and h=0.25

$y_2=y_1+1/2(k_1+k_2)$

where k_1=h f((x_1,y_1)=(0.25)f(2.25,2.514)

=(0.25)[2.25+2.514/2.25]=0.5293

$$k_2 = h\,f\left(x_0+h, y_0+k_1\right) = (0.1)\,f\left(0.1, 1-0.1\right) = (0.1)(-0.9) = -0.09$$

=(0.25)[2.5+2.514+0.5293/2.5]

=0.55433

$y_2 = y$ (2.5)=2.514+1/2(0.5293+0.55433)

=3.0558

$\therefore$ y =3.0558 when x = 2.5

Obtain the values of y at x=0.1,0.2 using R-K method of

(i)second order (ii)third order (iii)fourth order for the diff eqn y^1+y=0,y(0)=1

Sol: Given dy/dx = -y, y(0)=1

f(x,y) = -y, $x_0 = 0$, $y_0 = 1$

Here f (x,y) = -y, $x_0 = 0$, $y_0 = 1$ take h = 0.1

$\therefore$ $x_1 = x_0+h = 0.1$,

$x_2 = x_1+h = 0.2$

<u>Second order</u>:

step1: To find $y(x_1)$ i.e y(0.1) or y_1

by second-order R-K method,we have

$y_1 = y_0+1/2(k_1+k_2)$

where $k_1=hf(x_0,y_0)=(0.1)\ f(0,1) = (0.1)(-1)= -\ 0.1$

$k_2= hf\ (x_0+h,\ y_0+k_1)= (0.1)\ f\ (0.1,\ 1-0.1) = (0.1)(-0.9) = -0.09$

$y_1=y(0.1)=1+1/2(-0.1-0.09)=1-0.095=0.905$

$\therefore$ y =0.905 when x=0.1

Step2:

To find y_2 i.e $y(x_2)$ i.e y(0.2)

Here $x_1 = 0.1$, $y_1 = 0.905$ and h=0.1

By second-order R-K method, we have

$y_2 = y(x_2) = y_1 + 1/2(k_1 + k_2)$

Where $k_1 = h\ f\left(x_1, y_1\right)$ =(0.1)f(0.1,0.905)=(0.1)(-0.905)=-0.0905

$$k_2 = h\ f\left(x_1 + h, y_1 + k_1\right) = \left(0.1\right) f\left(0.2, 0.905 - 0.0905\right)$$
$$= \left(0.1\right) f\left(0.2, 0.8145\right) = \left(0.1\right)\left(-0.8145\right)$$
$$= -0.08145$$

$y_2 =$ y(0.2)=0.905+1/2(-0.0905-0.08145)

= 0.905- 0.085975 = 0819025

Third order

Step1:

To find y_1 i.e $y(x_1)$= y(0.1)

By Third order Runge kutta method

$$y_1 = y_0 + 1/6\left(k_1 + 4k_2 + k_3\right)$$

where k_1 = h f(x_0, y_0) = (0.1) f (0.1) = (0.1) (-1) = -0.1

$$k_2 = h\ f\left(x_0 + h/2, y_0 + k_1/2\right) = \left(0.1\right) f\left(0.1/2, 1 - 0.1/2\right) = \left(0.1\right) f\left(0.05, 0.95\right)$$
$$= \left(0.1\right)\left(-0.95\right) = -0.095$$

and k_3 = h f((x_0+h,y_0+2k_2-k_1)

(0.1) f (0.1,1+2(-0.095)+0.1)= -0.905

Hence y_1 = 1+1/6(-0.1+4(-0.095)-0.09) = 1+1/6 (-0.57) = 0.905

y_1=0.905 i.e y(0.1)= 0.905

Step2:

To find y_2,i.e $y(x_2)$= y(0.2)

Here x_1=0.1,y_1=0.905 and h = 0.1

Again by 2^{nd} order R-K method

$y_2 = y_1+1/6(k_1+4k_2+k_3)$

Where k_1=h f(x_1, y_1) = (0.1)f (0.1,0.905)= -0.0905

k_2 = h f (x_1+h/2,y_1+k_1/2)=(0.1)f(0.1+0.2,0.905 - 0.0905)= -(0.1) f (0.15, 0.85975)= (0.1) (-0.85975)

and k_3 = h f((x_1+h,y_1+2k_2-k_1)=(0.1)f(0.2,0.905+2(0.08975)+0.0905= -0.082355

hence y_2 = 0.905+1/6(-0.0905+4(-0.085975)-0.082355)=0.818874

$\therefore$ y = 0.905 when x = 0.1

And y =0.818874 when x =0.2

fourth order:

step1:

x_0=0,y_0=1,h=0.1 To find y_1 i.e y(x_1)=y(0.1)
By 4^{th} order R-K method, we have
$y_1 = y_0+1/6(k_1+2k_2+2k_3+k_4)$
Where k_1=h f(x_0,y_0)=(0.1)f(0.1)= -0.1
k_2= h f (x_0+h/2, y_0+k1/2) = -0.095
and k_3= h f((x_0+h/2,y_0+k_2/2)=(0.1)f (0.1/2,1-0.095/2)
= (0.1)f(0.05,0.9525)
= -0.09525
and k_4= h f(x_0+h,y_0+k_3)
= (0.1) f(0.1,1-0.09525)=(0.1)f(0.1,0.90475)

=-0.090475

Hence y_1=1+1/6(-0.1)+2(-0.095)+2(0.09525)-0.090475)

=1+1/6(-0.570975)+1-0.951625 = 0.9048375

Step2:

To find $y_2, i.e., y\left(x_2\right) = y\left(0.2\right), y_1 = 0.9048375, i.e., y\left(0.1\right) = 0.9048375$

Here $x_1 = 0.1$, $y_1=0.9048375$ and $h = 0.1$

Again by 4th order R-K method, we have

$y_2 = y_1+1/6(k_1+2k_2+2k_3+k_4)$

Where $k_1=h\ f(x_1,y_1)=(0.1)f(0.1,0.9048375)=-0.09048375$

$k_2= hf\ (x_1+h/2,y_1+k_1/2)=(0.1)f(0.1+0.1/2,0.9048375\ -0.09048375\ /2)=-0.08595956$

and $k_3=hf(x_1+h/2,\ y_1+k_2/2)=(0.1)f(0.15,0.\ 8618577)= -0.08618577$

$k_4 =h\ f(x_1+h,y_1+k_3)=(0.1)f(0.2,0.86517)$

$= -0.08186517$

Hence $y_2 = 0.09048375+1/6(-0.09048375-2(0.08595956)-2(0.08618577)- 0.08186517$

$=0.9048375-0.0861065$

$=0.818731$

$y = 0.9048375$ when $x =0.1$ and $y =0.818731$

3. Apply the 4th order R-K method to find an approximate value of y when x=1.2 in steps of 0.1,given that

$y^1 = x^2+y^2, y\ (1)=1.5$

sol. Given $y^1= x^2+y^2$,and $y(1)=1.5$

Here $f(x,y)= x^2+y^{2,}\ y_0 =1.5$ and $x_0=1,h=0.1$

So that $x_1=1.1$ and $x_2=1.2$

Step1:

To find $y_{1\text{ i.e.,}}\ y(x_1)$

by 4th order R-K method we have

$y_1=y_0+1/6\ (k_1+2k_2+2k_3+k_4)$

$k_1=hf(x_0,y_0)=(0.1)f(1,1.5)=(0.1)\ [1^2+(1.5)^2]=0.325$

k_2= hf (x_0+h/2,y_0+k_1/2)=(0.1)f(1+0.05,1.5+0.325)=0.3866

and k_3=hf((x_0+h/2,y_0+k_2/2)=(0.1)f(1.05,1.5+0. 3866/2)=(0.1)[$(1.05)^2$+$(1.6933)^2$]

=0.39698

k_4=hf(x_0+h,y_0+k_3)=(0.1)f(1.0,1.89698)

=0.48085

Hence

$$y_1 = 1.5 + \frac{1}{6}\left[0.325 + 2(0.3866) + 2(0.39698) + 0.48085\right]$$
$$= 1.8955$$

Step2:

To find y_2, i.e., $y(x_2) = y(1.2)$

Here x_1=0.1,y_1=1.8955 and h=0.1

by 4th order R-K method we have

$y_2 = y_1+1/6(k_1+2k_2+2k_3+k_4)$

k_1=hf(x_1,y_1)=(0.1)f(0.1,1.8955)=(0.1) [1^2+$(1.8955)^2$]=0.48029

k_2= hf (x_1+h/2,y_1+k_1/2)=(0.1)f(1.1+0.1,1.8937+0.4796) =0.58834

and k_3=hf((x_1+h/2,y_1+k_2/2)=(0.1)f(1.5,1.8937+0.58743) =(0.1)[$(1.05)^2$+$(1.6933)^2$]

=0.611715

k_4=hf(x_1+h,y_1+k_3)=(0.1)f(1.2,1.8937+0.610728)

=0.77261

Hence y_2=1.8937+1/6(0.4796+2(0.58834)+2(0.611715)+0.7726) =2.5043

$\therefore$ y =2.5043 where $x = 0.2$

4. using R-K method, find y(0.2) for the eqn dy/dx=y-x,y(0)=1,take h=0.2

Ans:1.15607

5.Given that y^1=y-x,y(0)=2 find y(0.2) using R- K method take h=0.1

Ans: 2.4214

6. Apply the 4th order R-K method to find $y(0.2)$ *and* $y(0.4)$ for one equation

$10\frac{dy}{dx}=x^2+y^2, y(0)=1$ **take h = 0.1** Ans. 1.0207, 1.038

7. using R-K method, estimate y(0.2) and y(0.4) for the eqn dy/dx=y^2-x^2/ y^2+x^2,y(0)=1,h=0.2

Ans:1.19598,1.3751

8. use R-K method, to approximate y when x=0.2 given that y^1=x+y,y(0)=1

Sol: Here f(x,y)=x+y,y_0=1,x_0=0

Since h is not given for better approximation of y

Take h=0.1

$\therefore$ x_1=0.1, x_2=0.2

Step1

To find y_1 i.e y(x_1)=y(0.1)

By R-K method,we have

$y_1=y_0+1/6\ (k_1+2k_2+2k_3+k_4)$

Where $k_1=hf(x_0,y_0)=(0.1)f(0,1)=(0.1)\ (1)=0.1$

$k_2= hf\ (x_0+h/2,y_0+k_1/2)=(0.1)f(0.05,1.05)=0.11$

and $k_3=hf((x_0+h/2,y_0+k_2/2)=(0.1)f(0.05,1+0.\ 11/2)=(0.1)[(0.05)\ +(4.0.11/2)]$

=0.1105

$k_4=h\ f\ (x_0+h,y_0+k_3)=(0.1)f(0.1,1.1105)=(0.1)[0.1+1.1105]$

=0.12105

Hence

$$\therefore\ y_1 = y(0.1) = 1 + \frac{1}{6}(0.1 + 0.22 + 0.240 + 0.12105)$$

y = 1.11034

<u>Step2:</u>

To find y_2 i.e $y(x_2) = y(0.2)$

Here x_1=0-1, y_1=1.11034 and h=0.1

Again By R-K method,we have

$y_2=y_1+1/6(k_1+2k_2+2k_3+k_4)$

k_1=h $f(x_1,y_1)$=(0.1)f(0.1,1.11034)=(0.1) [1.21034]=0.121034

k_2= h f $(x_1+h/2,\ y_1+k_1/2)$=(0.1)f(0.1+0.1/2,1.11034+0.121034/2)

=0.1320857

and k_3=h f($(x_1+h/2,y_1+k_2/2)$=(0.1)f(0.15,1.11034+0.1320857/2)

=0.1326382

k_4=h $f(x_1+h,y_1+k_3)$=(0.1)f(0.2,1.11034+0.1326382)

(0.1)(0.2+1.2429783)=0.1442978

Hence y_2=1.11034+1/6(0.121034+0.2641714+0.2652764+0.1442978

=1.11034+0.1324631 =1.242803

$\therefore$ y =1.242803 when x=0.2

9.using Runge-kutta method of order 4,compute y(1.1) for the eqn $y^1=3x+y^2$,y(1)=1.2 h = 0.05

Ans:1.7278

10. using Runge-kutta method of order 4,compute y(2.5) for the eqn dy/dx = x+y/x, y(2)=2 [hint h = 0.25(2 steps)]

Ans:3.058

MODULE – V

PARTIAL DIFFERENTIAL EQUATIONS AND APPLICATIONS

Introduction

The concept of a differential equation to include equations that involve partial derivatives, not just ordinary ones. Solutions to such equations will involve functions not just of one variable, but of several variables. Such equations arise naturally, for example, when one is working with situations that involve positions in space that vary over time. To model such a situation, one needs to use functions that have several variables to keep track of the spatial dimensions and an additional variable for time.

Examples of some important PDEs:

(1) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ *One-dimensional wave equation*

(2) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ *One-dimensional heat equation*

(3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ *Two-dimensional Laplace equation*

(4) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ *Two-dimensional Poisson equation*

Partial differential equations: An equation involving partial derivatives of one dependent variable with respective more than one independent variables.

Notations which we use in this unit:

$p = \frac{\partial z}{\partial x}$,$q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $= \frac{\partial^2 z}{\partial x \, \partial y}$, t= $\frac{\partial^2 z}{\partial y^2}$,

Formation of partial differential equation:

A partial differential equation of given curve can be formed in two ways

1. By eliminating arbitrary constants
2. By eliminating arbitrary functions

Problems

1 Form a partial differential equation by eliminating a,b,c from

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Sol Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Differentiating partially w.r.to x and y, we have

$\frac{1}{a^2}(2x) + \frac{1}{c^2}(2z)\frac{\partial z}{\partial x}$=o

$\frac{1}{a^2}(x) + \frac{1}{c^2}(z)\,p$ =o ________(1)

And $\frac{1}{b^2}(2x) + \frac{1}{c^2}(2z)\frac{\partial z}{\partial x}$=o

$\frac{1}{b^2}(y) + \frac{1}{c^2}(z)$ q = o ________(2)

Diff (1) partially w.r.to x, we have

$\frac{1}{a^2} + \frac{p}{c^2}\frac{\partial z}{\partial x} + \frac{z}{c^2}\frac{\partial p}{\partial x}$=o _______(3)

$\frac{1}{a^2} + \frac{p^2}{c^2} + \frac{z}{c^2}r$ =O

Multiply this equation by x and then subtracting (1) from it

$$\frac{1}{c^2}(xzr + xp^2 - pz) = 0$$

2 **Form a partial differential equation by eliminating the constants from $(x-a)^2 + (y-b)^2 = z^2cot^2$α, where α is a parameter**

Sol Given $(x-a)^2 + (y-b)^2 = z^2cot^2$α ____________(1)

Differentiating partially w.r.to x and y, we have

2 $(x-a)$+0 = 2 z p$cot^2\alpha$

$(x-a)$ = Zp$cot^2\alpha$

And 0+2(y-b) = 2zq$cot^2\alpha$

(Y-b) = zq$cot^2\alpha$

Substituting the values of (x-a) and (y-b) in (1),we get

$$(zpcot^2\alpha)^2 + (zqcot^2\alpha)^2 = z^2cot^2\alpha$$

$$(p^2+q^2)(cot^2\alpha)^2 = cot^2\alpha$$

$$p^2+q^2 = tan^2\alpha$$

3 **Form the partial differential equation by eliminating a and b from log (az-1)=x+ay+b**

Sol Given equation is

Log (az-1)=x+ay+b

Differentiating partially w.r.t. x and y ,we get

$\frac{1}{az-1}(ap) = 1 \Longrightarrow ap = az - 1$ --------------- (1)

$\frac{1}{az-1}(aq) = a \Longrightarrow aq = a(az - 1)$ ------------- (2)

(2)/(1) gives

$\frac{q}{p} = a \;\; or \; ap = q$---------------- (3)

Substituting (3) in (1), we get

$$q = \frac{q}{p}.(z - 1)$$

i.e. pq=qz-p

$$p(q + 1) = qz$$

4 **Find the differential equation of all spheres whose centers lie on z-axis with a given radius r.**

Sol The equation of the family of spheres having their centers on z-axis and having radius r is

$$x^2 + y^2 + (z - c)^2 = r^2$$

Where c and r are arbitrary constants

Differentiating this eqn partially w.r.t. x and y ,we get

$2x + 2(z - c)\frac{\partial z}{\partial x} = 0 \Rightarrow x + (z - c)p = 0$ _________(1)

$2y + 2(z - c)\frac{\partial z}{\partial y} = 0 \Longrightarrow y + (z - c)q = 0$_________(2)

From (1),$(z - c) = -\frac{x}{p}$ ___________(3)

From (2),$(z - c) = -\frac{y}{q}$ ___________(4)

From (3) and (4)

We get $-\frac{x}{p} = -\frac{y}{q}$

i.e. $xq - yp = 0$

Linear partial differential equations of first order :

Lagrange's linear equation: An equation of the form Pp + Qq = R is called Lagrange's linear equation.

To solve Lagrange's linear equation consider auxiliary equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Problems

1 **solve** $(\boldsymbol{x^2 - y^2 - yz})\boldsymbol{p} + (\boldsymbol{x^2 - y^2 - zx})\boldsymbol{q} = \boldsymbol{z}(\boldsymbol{x - y})$

Sol Here

$$P=(x^2 - y^2 - yz), Q = (x^2 - y^2 - zx), R = z(x - y)$$

The subsidiary equations are $\frac{dx}{(x^2-y^2-yz)} = \frac{dy}{(x^2-y^2-zx)} = \frac{dz}{z(x-y)}$

Using 1,-1,0 and x,-y,0 as multipliers , we have

. $\frac{dz}{z(x-y)} = \frac{dx-dy}{z(x-y)} = \frac{x\,dx-ydy}{(x^2-y^2)(x-y)}$

From the first two rations 0f ,we have

dz= dx-dy

integrating , z=x-y-c_1 or x-y-z = c_1

now taking first and last ratios in (2) ,we get

$$\frac{dz}{z} = \frac{x\,dx - y\,dy}{x^2 - y^2} \quad \text{or} \frac{2dz}{z} = \frac{2x\,dx - 2y\,dy}{x^2 - y^2}$$

Integrating ,2 log z = $\log(x^2 - y^2) - \log c_2$

$$\Rightarrow \frac{x^2 - y^2}{z^2} = c_2$$

The required general solution is $f\left(x - y - z\,, \frac{x^2-y^2}{z^2}\right) = 0$

5 **Find the general solution of the first-order linear partial differential equation with the constant coefficients: $4u_x+u_y=x^2y$**

Sol The auxiliary system of equations is

$$\frac{dx}{4} = \frac{dy}{1} = \frac{du}{x^2y}$$

From here we get

$\frac{dx}{4} = \frac{dy}{1}$ or dx-4dy=0. Integrating both sides

we get x-4y=c. Also $\frac{dx}{4} = \frac{du}{x^2y}$ or x^2y dx=4du

or $x^2(\frac{x-c}{4})\,dx$=4du or

$\frac{1}{16}$ $(x^3 - cx^2)$ dx = du

Integrating both sides we get

u=c_1+ $\frac{3x^4 - 4cx^3}{192}$

= f(c)+ $\frac{3x^4 - 4cx^3}{192}$

After replacing c by x-4y, we get the general solution

u=f(x-4y)+ $\frac{3x^4 - 4(x-4y)x^3}{192}$

$$=f(x-4y)-\frac{x^4}{192}+\frac{x^3y}{12}$$

6 **Find the general solution of the partial differential equation $y^2up + x^2uq = y^2x$**

Sol The auxiliary system of equations is

$$\frac{dx}{y^2u}=\frac{dy}{x^2u}=\frac{du}{xy^2}$$

Taking the first two members we have $x^2dx = y^2dy$ which on integration given $x^3-y^3 = c_1$. Again taking the first and third members,

we have x dx = u du

which on integration given $x^2-u^2 = c_2$

Hence, the general solution is

$F(x^3-y^3,x^2-u^2) = 0$

7 **Find the general solution of the partial differential equation.**

$$\left(\frac{\partial u}{\partial x}\right)^2 x+\left(\frac{\partial u}{\partial y}\right)^2 y-u=0$$

Sol : Let $p=\frac{\partial u}{\partial x}$, $q=\frac{\partial u}{\partial y}$

The auxiliary system of equations is

$$\frac{dx}{2px}=\frac{dy}{2qy}=\frac{du}{2(p^2x+q^2y)}=\frac{dp}{p-p^2}=\frac{dq}{q-q^2}$$

which we obtain from putting values of

$$\frac{\partial F}{\partial p}=2px\ ,\frac{\partial F}{\partial q}=2qy,\frac{\partial F}{\partial x}=p^2,\frac{\partial F}{\partial u}=-1,\frac{\partial F}{\partial y}=q^2$$

and multiplying by -1 throughout the auxiliary system. From first and 4th expression in (11.38) we get

$dx=\frac{p^2dx+2pxdp}{py}$. From second and 5th expression

$$dy=\frac{q^2dy+2qydq}{qy}$$

Using these values of dx and dy we get

$$\frac{p^2dx+2pxdp}{p^2x}=\frac{q^2dy+2qydq}{q^2y}$$

or $\frac{dx}{x}+\frac{2}{p}dp=\frac{dy}{y}+\frac{2dq}{q}$

Taking integral of all terms we get

$\ln|x| + 2\ln|p| = \ln|y|+2\ln|q|+\ln c$

or $\ln|x|\, p^2 = \ln|y| q^2 c$

or $p^2x=cq^2y$, where c is an arbitrary constant.

Solving for p and q we get $cq^2y+q^2y - u=0$

$(c+1)q^2y=u$

$$q=\left\{\frac{u}{(c+1)y}\right\}^{1/2}$$

$$p=\left\{\frac{cu}{(c+1)x}\right\}^{1/2}$$

$$du=\left\{\frac{cu}{(c+1)x}\right\}^{1/2}dx+\left\{\frac{u}{(c+1)y}\right\}^{1/2}dy$$

$$\text{or } \left(\frac{1+c}{u}\right)^{1/2}du=\left(\frac{c}{x}\right)^{1/2}dx+\left(\frac{i}{y}\right)^{1/2}dy$$

By integrating this equation we obtain $((1+c)u)^{1/2}=(cx)^{1/2}+(y)^{1/2}+c_1$

This is a complete solution.

8 Solve $p^2+q^2=1$

Sol The auxiliary system of equation is

$$-\frac{dx}{-2p}=\frac{dy}{2q}=\frac{du}{-2p^2-2q^2}=\frac{dp}{0}=\frac{dq}{0}$$

$$\text{or} \quad \frac{dx}{p}=\frac{dy}{q}=\frac{du}{p^2+q^2}=\frac{dp}{0}=\frac{dq}{0}$$

Using dp =0, we get p=c and q= $\sqrt{1-c^2}$, and these two combined with du =pdx+qdy yield

$u=cx+y\sqrt{1-c^2} + c_1$ which is a complete solution.

Using $\frac{dx}{du}$ = p , we get du = $\frac{dx}{c}$ where p= c

Integrating the equation we get u = $\frac{x}{c}$ + c_1

Also du = $\frac{dy}{q}$, where q = $\sqrt{1-p^2}$ = $\sqrt{1-c^2}$

or du = $\frac{dy}{\sqrt{1-c^2}}$. Integrating this equation we get u = $\frac{1}{\sqrt{1-c^2}}$ y +c_2

This cu = x+cc_1 and $u\sqrt{1-c^2}$ = y + $c_2\sqrt{1-c^2}$

Replacing cc_1 and $c_2\sqrt{1-c^2}$ by $-\alpha$ and $-\beta$ respectively, and eliminating c, we get

$$u^2 = (x-\alpha)^2 + (y-\beta)^2$$

9 **Solve $u^2+pq - 4 = 0$**

Sol The auxiliary system of equations is

$$\frac{dx}{q} = \frac{dy}{p} = \frac{du}{2pq} = \frac{dp}{-2up} = \frac{dq}{-2uq}$$

The last two equations yield $p = a^2q$.

Substituting in $u^2+pq - 4 = 0$ gives

$q = \pm\frac{1}{a}\sqrt{4-u^2}$ and $p = \pm a\sqrt{4-u^2}$

Then du = pdx+qdy yields

$$du = \pm\sqrt{4-u^2}\left(adx + \frac{1}{a}dy\right)$$

or $$\frac{du}{\sqrt{4-u^2}} = \pm adx + \frac{1}{a}dy$$

Integrating we get $\sin^{--1}\frac{u}{2} = \pm\left(adx + \frac{1}{a}y + c\right)$

or $u = \pm 2\sin\left(ax + \frac{1}{a}y + c\right)$

10 **Solve $p^2(1-x^2)-q^2(4-y^2) = 0$**

Sol Let $p^2(1-x^2) = q^2(4-y^2) = a^2$

This gives $p = \frac{a}{\sqrt{1-x^2}}$ and $q = \frac{a}{\sqrt{4-y^2}}$

(neglecting the negative sign).

Substituting in du = pdx + q dy we have

$$du = \frac{a}{\sqrt{1-x^2}}dx + \frac{a}{\sqrt{4-y^2}}dy$$

Integration gives $u = a\left(\sin' x + \sin'\frac{y}{2}\right) + c$.

Wave Equation

For the rest of this introduction to PDEs we will explore PDEs representing some of the basic types of linear second order PDEs: heat conduction and wave propagation. These represent two entirely different physical processes: the process of diffusion, and the process of oscillation, respectively. The field of PDEs is extremely large, and there is still a considerable amount of undiscovered territory in it, but these two basic types of PDEs represent the ones that are in some sense, the best understood and most developed of all of the PDEs. Although there is no one way to solve all PDEs explicitly, the main technique that we will use to solve these various PDEs represents one of the most important techniques used in the field of PDEs, namely separation of variables (which we saw in a different form while studying ODEs). The essential manner of using separation of variables is to try to break up a differential equation involving several partial derivatives into a series of simpler, ordinary differential equations.

We start with the wave equation. This PDE governs a number of similarly related phenomena, all involving oscillations. Situations described by the wave equation include acoustic waves, such as vibrating guitar or violin strings, the vibrations of drums, waves in fluids, as well as waves generated by electromagnetic fields, or any other physical situations involving oscillations, such as vibrating power lines, or even suspension bridges in certain circumstances. In short, this one type of PDE covers a lot of ground.

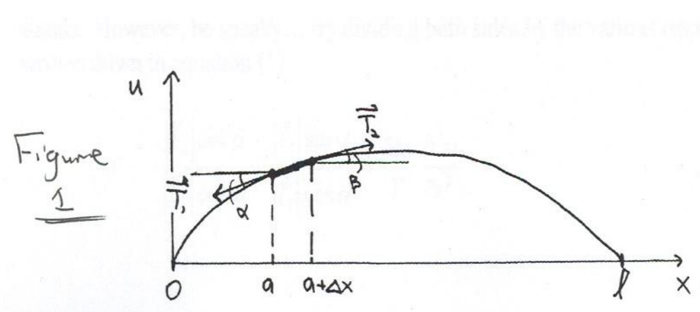
We begin by looking at the simplest example of a wave PDE, the one-dimensional wave equation. To get at this PDE, we show how it arises as we try to model a simple vibrating string, one that is held in place between two secure ends. For instance, consider plucking a guitar string and watching (and listening) as it vibrates. As is typically the case with modeling, reality is quite a bit more complex than we can deal with all at once, and so we need to make some simplifying assumptions in order to get started.

First off, assume that the string is stretched so tightly that the only real force we need to consider is that due to the string's tension. This helps us out as we only have to deal with one force, i.e. we can safely ignore the effects of gravity if the tension force is orders of magnitude greater than that of gravity. Next we assume that the string is as uniform, or homogeneous, as possible, and that it is perfectly elastic. This makes it possible to predict the motion of the string more readily since we don't need to keep track of kinks that might occur if the string wasn't uniform. Finally, we'll assume that the vibrations are pretty minimal in relation to the overall length of the string, i.e. in terms of displacement, the amount that the string bounces up and down is pretty small. The reason this will help us out is that we can concentrate on the simple up and down motion of the string, and not worry about any possible side to side motion that might occur.

Now consider a string of a certain length, l, that's held in place at both ends. First off, what exactly are we trying to do in "modeling the string's vibrations"? What kind of function do we want to solve for to keep track of the motion of string? What will it be a function of? Clearly if the string is vibrating, then its motion changes over time, so *time* is one variable we will want to keep track of. To keep track of the actual motion of the string we will need to have a function that tells us the shape of the string at any particular time. One way we can do this is by looking for a function that tells us the *vertical displacement* (positive up, negative down) that exists at any point along the string – how far away any particular point on the string is from the undisturbed resting position of the string, which is just a straight line. Thus, we would like to find a function $u(x,t)$ of two variables. The variable *x* can measure distance along the string, measured away from one chosen end of the string (i.e. $x = 0$ is one of the tied down endpoints of the string), and *t* stands for time. The function $u(x,t)$ then gives the vertical displacement of the string at any point, *x*, along the string, at any particular time *t*.

As we have seen time and time again in calculus, a good way to start when we would like to study a surface or a curve or arc is to break it up into a series of very small pieces. At the end of our study of one little segment of the vibrating string, we will think about what happens as the length of the little segment goes to zero, similar to the type of limiting process we've seen as we progress from Riemann Sums to integrals.

Suppose we were to examine a very small length of the vibrating string as shown in figure 1:



Now what? How can we figure out what is happening to the vibrating string? Our best hope is to follow the standard path of modeling physical situations by studying all of the forces involved and then turning to Newton's classic equation $F = ma$. It's not a surprise that this will help us, as we have already pointed out that this equation is itself a differential equation (acceleration being the second derivative of position with respect to time). Ultimately, all we will be doing is substituting in the particulars of our situation into this basic differential equation.

Because of our first assumption, there is only one force to keep track of in our situation, that of the string tension. Because of our second assumption, that the string is perfectly elastic with no kinks, we can assume that the force due to the tension of the string is tangential to the ends of the small string segment, and so we need to keep track of the string tension forces T_1 and T_2 at each end of the string segment. Assuming that the string is only vibrating up and down means that the horizontal components of the tension forces on each end of the small segment must perfectly balance each other out. Thus

(1) $\left|\vec{T}_1\right|\cos\alpha = \left|\vec{T}_2\right|\cos\beta = T$

where$T$ is a string tension constant associated with the particular set-up (depending, for instance, on how tightly strung the guitar string is). Then to keep track of all of the forces involved means just summing up the vertical components of T_1 and T_2. This is equal to

(2) $\left|\vec{T}_2\right|\sin\beta - \left|\vec{T}_1\right|\sin\alpha$

where we keep track of the fact that the forces are in opposite direction in our diagram with the appropriate use of the minus sign. That's it for "Force," now on to "Mass" and "Acceleration." The mass of the string is simple, just $\delta\Delta x$, where δ is the mass per unit length of the string, and Δx is (approximately) the length of the little segment. Acceleration is the second derivative of position with respect to time. Considering that the position of the string segment at a particular time is just $u(x,t)$, the function we're trying to find, then the acceleration for the little segment is $\frac{\partial^2 u}{\partial t^2}$ (computed at some point between a and $a+\Delta x$). Putting all of this together, we find that:

(3) $\left|\vec{T}_2\right|\sin\beta - \left|\vec{T}_1\right|\sin\alpha = \delta\Delta x\frac{\partial^2 u}{\partial t^2}$

Now what? It appears that we've got nowhere to go with this – this looks pretty unwieldy as it stands. However, be sneaky… try dividing both sides by the various respective equal parts written down in equation (1):

(4) $\frac{\left|\vec{T}_2\right|\sin\beta}{\left|\vec{T}_2\right|\cos\beta} - \frac{\left|\vec{T}_1\right|\sin\alpha}{\left|\vec{T}_1\right|\cos\alpha} = \frac{\delta\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$

or more simply:

(5) $\tan\beta - \tan\alpha = \frac{\delta\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$

Now, finally, note that $\tan\alpha$ is equal to the slope at the left-hand end of the string segment, which is just $\frac{\partial u}{\partial x}$ evaluated at a, i.e. $\frac{\partial u}{\partial x}(a,t)$ and similarly $\tan\beta$ equals $\frac{\partial u}{\partial x}(a+\Delta x,t)$, so (5) becomes…

(6) $$\frac{\partial u}{\partial x}(a+\Delta x,t)-\frac{\partial u}{\partial x}(a,t)=\frac{\delta\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$$

or better yet, dividing both sides by Δx …

(7) $$\frac{1}{\Delta x}\left(\frac{\partial u}{\partial x}(a+\Delta x,t)-\frac{\partial u}{\partial x}(a,t)\right)=\frac{\delta}{T}\frac{\partial^2 u}{\partial t^2}$$

Now we're ready for the final push. Let's go back to the original idea – start by breaking up the vibrating string into little segments, examine each such segment using Newton's $F = ma$ equation, and finally figure out what happens as we let the length of the little string segment dwindle to zero, i.e. examine the result as Δx goes to 0. Do you see any limit definitions of derivatives kicking around in equation (7)? As Δx goes to 0, the left-hand side of the equation is in fact just equal to $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial^2 u}{\partial x^2}$, so the whole thing boils down to:

(8) $$\frac{\partial^2 u}{\partial x^2}=\frac{\delta}{T}\frac{\partial^2 u}{\partial t^2}$$

which is often written as

(9) $$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$$

by bringing in a new constant $c^2=\frac{T}{\delta}$ (typically written with c^2, to show that it's a positive constant).

This equation, which governs the motion of the vibrating string over time, is called the ***one-dimensional wave equation***. It is clearly a second order PDE, and it's linear and homogeneous.

Solution of the Wave Equation by Separation of Variables

There are several approaches to solving the wave equation. The first one we will work with, using a technique called separation of variables, again, demonstrates one of the most widely used solution techniques for PDEs. The idea behind it is to split up the original PDE into a series of simpler ODEs, each of which we should be able to solve readily using tricks already learned. The second technique, which we will see in the next section, uses a transformation trick that also reduces the complexity of the original PDE, but in a very different manner. This second solution is due to Jean Le Rond D'Alembert (an 18th century French mathematician), and is called D'Alembert's solution, as a result.

First, note that for a specific wave equation situation, in addition to the actual PDE, we will also have boundary conditions arising from the fact that the endpoints of the string are attached solidly, at the left end of the string, when $x = 0$ and at the other end of the string, which we suppose has overall length l. Let's start the process of solving the PDE by first figuring out what these boundary conditions imply for the solution function, $u(x,t)$.

Answer: for all values of t, the time variable, it must be the case that the vertical displacement at the endpoints is 0, since they don't move up and down at all, so that

(1) $u(0,t) = 0$ and $u(l,t) = 0$ for all values of t

are the ***boundary conditions*** for our wave equation. These will be key when we later on need to sort through possible solution functions for functions that satisfy our particular vibrating string set-up.

You might also note that we probably need to specify what the shape of the string is right when time $t = 0$, and you're right - to come up with a particular solution function, we would need to know $u(x,0)$. In fact we would also need to know the initial velocity of the string, which is just $u_t(x,0)$. These two requirements are called the ***initial conditions*** for the wave equation, and are also necessary to specify a particular vibrating string solution. For instance, as the simplest example of initial conditions, if no one is plucking the string, and it's perfectly flat to start with, then the initial conditions would just be $u(x,0) = 0$ (a perfectly flat string) with initial velocity, $u_t(x,0) = 0$. Here, then, the solution function is pretty unenlightening – it's just $u(x,t) = 0$, i.e. no movement of the string through time.

To start the separation of variables technique we make the key assumption that whatever the solution function is, that it can be written as the product of two independent functions, each one of which depends on just one of the two variables, x or t. Thus, imagine that the solution function, $u(x,t)$ can be written as

(2) $u(x,t) = F(x)G(t)$

where$F$, and G, are single variable functions of x and t respectively. Differentiating this equation for $u(x,t)$ twice with respect to each variable yields

(3) $\dfrac{\partial^2 u}{\partial x^2} = F''(x)G(t)$ and $\dfrac{\partial^2 u}{\partial t^2} = F(x)G''(t)$

Thus when we substitute these two equations back into the original wave equation, which is

(4) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

then we get

(5) $$\frac{\partial^2 u}{\partial t^2} = F(x)G''(t) = c^2 \frac{\partial^2 u}{\partial x^2} = c^2 F''(x)G(t)$$

Here's where our separation of variables assumption pays off, because now if we separate the equation above so that the terms involving *F* and its second derivative are on one side, and likewise the terms involving *G* and its derivative are on the other, then we get

(6) $$\frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)}$$

Now we have an equality where the left-hand side just depends on the variable *t*, and the right-hand side just depends on *x*. Here comes the critical observation - how can two functions, one just depending on *t*, and one just on *x*, be equal for all possible values of *t* and *x*? The answer is that they must each be constant, for otherwise the equality could not possibly hold for all possible combinations of *t* and *x*. Aha! Thus we have

(7) $$\frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k$$

where*k* is a constant. First let's examine the possible cases for *k*.

Case One: k = 0

Suppose *k* equals 0. Then the equations in (7) can be rewritten as

(8) $G''(t) = 0 \cdot c^2 G(t) = 0$ and $F''(x) = 0 \cdot F(x) = 0$

yielding with very little effort two solution functions for *F* and *G:*

(9) $G(t) = at + b$ and $F(x) = px + r$

where*a*,*b*, *p* and *r*, are constants (note how easy it is to solve such simple ODEs versus trying to deal with two variables at once, hence the power of the separation of variables approach).

Putting these back together to form $u(x,t) = F(x)G(t)$, then the next thing we need to do is to note what the boundary conditions from equation (1) force upon us, namely that

(10) $u(0,t) = F(0)G(t) = 0$ and $u(l,t) = F(l)G(t) = 0$ for all values of t

Unless $G(t) = 0$ (which would then mean that $u(x,t) = 0$, giving us the very dull solution equivalent to a flat, unplucked string) then this implies that

(11) $F(0) = F(l) = 0$.

But how can a linear function have two roots? Only by being identically equal to 0, thus it must be the case that $F(x) = 0$. Sigh, then we still get that $u(x,t) = 0$, and we end up with the dull solution again, the only possible solution if we start with $k = 0$.

So, let's see what happens if…

Case Two: k > 0

So now if k is positive, then from equation (7) we again start with

(12) $G''(t) = kc^2 G(t)$ and
(13) $F''(x) = kF(x)$

Try to solve these two ordinary differential equations. You are looking for functions whose second derivatives give back the original function, multiplied by a positive constant. Possible candidate solutions to consider include the exponential and sine and cosine functions. Of course, the sine and cosine functions don't work here, as their second derivatives are *negative* the original function, so we are left with the exponential functions.

Let's take a look at (13) more closely first, as we already know that the boundary conditions imply conditions specifically for $F(x)$, i.e. the conditions in (11). Solutions for $F(x)$ include anything of the form

(14) $F(x) = Ae^{\omega x}$

where $\omega^2 = k$ and A is a constant. Since ω could be positive or negative, and since solutions to (13) can be added together to form more solutions (note (13) is an example of a second order linear homogeneous ODE, so that the superposition principle holds), then the general solution for (13) is

(14) $F(x) = Ae^{\omega x} + Be^{-\omega x}$

where now A and B are constants and $\omega = \sqrt{k}$. Knowing that $F(0) = F(l) = 0$, then unfortunately the only possible values of A and B that work are $A = B = 0$, i.e. that $F(x) = 0$. Thus, once again we end up with $u(x,t) = F(x)G(t) = 0 \cdot G(t) = 0$, i.e. the dull solution once more. Now we place all of our hope on the third and final possibility for *k*, namely…

Case Three: k < 0

So now we go back to equations (12) and (13) again, but now working with *k* as a negative constant. So, again we have

(12) $G''(t) = kc^2 G(t)$

and

(13) $F''(x) = kF(x)$

Exponential functions won't satisfy these two ODEs, but now the sine and cosine functions will. The general solution function for (13) is now

(15) $F(x) = A\cos(\omega x) + B\sin(\omega x)$

where again *A* and *B* are constants and now we have $\omega^2 = -k$. Again, we consider the boundary conditions that specified that $F(0) = F(l) = 0$. Substituting in 0 for *x* in (15) leads to

(16) $F(0) = A\cos(0) + B\sin(0) = A = 0$

so that $F(x) = B\sin(\omega x)$. Next, consider $F(l) = B\sin(\omega l) = 0$. We can assume that *B* isn't equal to 0, otherwise $F(x) = 0$ which would mean that $u(x,t) = F(x)G(t) = 0 \cdot G(t) = 0$, again, the trivial unplucked string solution. With $B \neq 0$, then it must be the case that $\sin(\omega l) = 0$ in order to have $B\sin(\omega l) = 0$. The only way that this can happen is for ωl to be a multiple of π. This means that

(17) $\omega l = n\pi$ or $\omega = \dfrac{n\pi}{l}$ (where *n* is an integer)

This means that there is an infinite set of solutions to consider (letting the constant *B* be equal to 1 for now), one for each possible integer *n*.

(18) $F(x) = \sin\left(\frac{n\pi}{l}x\right)$

Well, we would be done at this point, except that the solution function $u(x,t) = F(x)G(t)$ and we've neglected to figure out what the other function, $G(t)$, equals. So, we return to the ODE in (12):

(12) $G''(t) = kc^2G(t)$

where, again, we are working with *k*, a negative number. From the solution for $F(x)$ we have determined that the only possible values that end up leading to non-trivial solutions are with $k = -\omega^2 = -\left(\frac{n\pi}{l}\right)^2$ for*n* some integer. Again, we get an infinite set of solutions for (12) that can be written in the form

(19) $G(t) = C\cos(\lambda_n t) + D\sin(\lambda_n t)$

where*C* and *D* are constants and $\lambda_n = c\sqrt{-k} = c\omega = \frac{cn\pi}{l}$, where *n* is the same integer that showed up in the solution for $F(x)$ in (18) (we're labeling λ with a subscript "*n*" to identify which value of *n* is used).

Now we really are done, for all we have to do is to drop our solutions for $F(x)$ and $G(t)$ into $u(x,t) = F(x)G(t)$, and the result is

(20) $u_n(x,t) = F(x)G(t) = \left(C\cos(\lambda_n t) + D\sin(\lambda_n t)\right)\sin\left(\frac{n\pi}{l}x\right)$

where the integer *n* that was used is identified by the subscript in $u_n(x,t)$ and λ_n, and *C* and *D* are arbitrary constants.

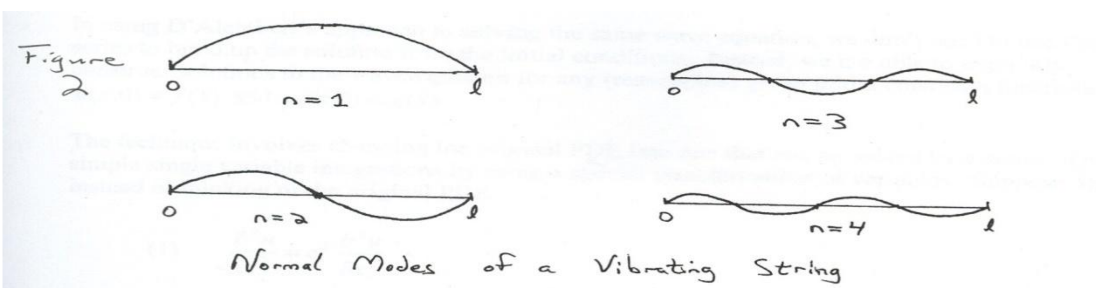
At this point you should be in the habit of immediately checking solutions to differential equations. Is (20) really a solution for the original wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

and does it actually satisfy the boundary conditions $u(0,t)=0$ and $u(l,t)=0$ for all values of t

The solution given in the last section really does satisfy the one-dimensional wave equation. To think about what the solutions look like, you could graph a particular solution function for varying values of time, *t*, and then examine how the string vibrates over time for solution functions with different values of *n* and constants *C* and *D*. However, as the functions involved are fairly simple, it's possible to make sense of the solution $u_n(x,t)$ functions with just a little more effort.

For instance, over time, we can see that the $G(t)=\left(C\cos(\lambda_n t)+D\sin(\lambda_n t)\right)$ part of the function is periodic with period equal to $\frac{2\pi}{\lambda_n}$. This means that it has a frequency equal to $\frac{\lambda_n}{2\pi}$ cycles per unit time. In music one cycle per second is referred to as one *hertz*. Middle C on a piano is typically 263 hertz (i.e. when someone presses the middle C key, a piano string is struck that vibrates predominantly at 263 cycles per second), and the A above middle C is 440 hertz. The solution function when *n* is chosen to equal 1 is called the ***fundamental mode*** (for a particular length string under a specific tension). The other ***normal modes*** are represented by different values of *n*. For instance one gets the 2nd and 3rd normal modes when *n* is selected to equal 2 and 3, respectively. The fundamental mode, when *n* equals 1 represents the simplest possible oscillation pattern of the string, when the whole string swings back and forth in one wide swing. In this fundamental mode the widest vibration displacement occurs in the center of the string (see the figures below).



Thus suppose a string of length *l*, and string mass per unit length δ, is tightened so that the values of *T*, the string tension, along the other constants make the value of $\lambda_1=\frac{\sqrt{T}}{2l\sqrt{\delta}}$ equal to 440. Then if the string is made to vibrate by striking or plucking it, then its fundamental (lowest) tone would be the A above middle C.

Now think about how different values of n affect the other part of $u_n(x,t) = F(x)G(t)$, namely $F(x) = \sin\left(\frac{n\pi}{l}x\right)$. Since $\sin\left(\frac{n\pi}{l}x\right)$ function vanishes whenever x equals a multiple of $\frac{l}{n}$, then selecting different values of n higher than 1 has the effect of identifying which parts of the vibrating string do not move. This has the affect musically of producing *overtones*, which are musically pleasing higher tones relative to the fundamental mode tone. For instance picking n = 2 produces a vibrating string that appears to have two separate vibrating sections, with the middle of the string standing still. This mode produces a tone exactly an octave above the fundamental mode. Choosing n = 3 produces the 3rd normal mode that sounds like an octave and a fifth above the original fundamental mode tone, then 4th normal mode sounds an octave plus a fifth plus a major third, above the fundamental tone, and so on.

It is this series of fundamental mode tones that gives the basis for much of the tonal scale used in Western music, which is based on the premise that the lower the fundamental mode differences, down to octaves and fifths, the more pleasing the relative sounds. Think about that the next time you listen to some Dave Matthews!

Finally note that in real life, any time a guitar or violin string is caused to vibrate, the result is typically a combination of normal modes, so that the vibrating string produces sounds from many different overtones. The particular combination resulting from a particular set-up, the type of string used, the way the string is plucked or bowed, produces the characteristic tonal quality associated with that instrument. The way in which these different modes are combined makes it possible to produce solutions to the wave equation with different initial shapes and initial velocities of the string. This process of combination involves ***Fourier Series*** which will be covered at the end of Math 21b (come back to see it in action!)

Finally, finally, note that the solutions to the wave equations also show up when one considers acoustic waves associated with columns of air vibrating inside pipes, such as in organ pipes, trombones, saxophones or any other wind instruments (including, although you might not have thought of it in this way, your own voice, which basically consists of a vibrating wind-pipe, i.e. your throat!). Thus the same considerations in terms of fundamental tones, overtones and the characteristic tonal quality of an instrument resulting from solutions to the wave equation also occur for any of these instruments as well. So, the wave equation gets around quite a bit musically!

D'Alembert's Solution of the Wave Equation

As was mentioned previously, there is another way to solve the wave equation, found by Jean Le Rond D'Alembert in the 18th century. In the last section on the solution to the wave equation using the separation of variables technique, you probably noticed that although we made use of the boundary conditions in finding the solutions to the PDE, we glossed over the issue of the initial conditions, until the very end when we claimed that one could make use of something called Fourier Series to build up combinations of solutions. If you recall, being given specific initial conditions meant being given both the shape of the string at time t = 0, i.e. the function $u(x,0) = f(x)$, as well as the initial velocity, $u_t(x,0) = g(x)$ (note that these two initial condition functions are functions of x alone, as t is set equal to 0). In the separation of variables solution, we ended up with an infinite set, or family, of solutions, $u_n(x,t)$ that we said could be combined in such a way as to satisfy any reasonable initial conditions.

In using D'Alembert's approach to solving the same wave equation, we don't need to use Fourier series to build up the solution from the initial conditions. Instead, we are able to explicitly construct solutions to the wave equation for any (reasonable) given initial condition functions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$.

The technique involves changing the original PDE into one that can be solved by a series of two simple single variable integrations by using a special transformation of variables. Suppose that instead of thinking of the original PDE

(1) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

in terms of the variables x, and t, we rewrite it to reflect two new variables

(2) $v = x + ct$ and $z = x - ct$

This then means that u, originally a function of x, and t, now becomes a function of v and z, instead. How does this work? Note that we can solve for x and t in (2), so that

(3) $$x = \frac{1}{2}(v+z) \text{ and } t = \frac{1}{2c}(v-z)$$

Now using the chain rule for multivariable functions, you know that

(4) $$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} = c\frac{\partial u}{\partial v} - c\frac{\partial u}{\partial z}$$

since $\frac{\partial v}{\partial t} = c$ and $\frac{\partial z}{\partial t} = -c$, and that similarly

(5) $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial u}{\partial v} + \frac{\partial u}{\partial z}$$

since $\frac{\partial v}{\partial x} = 1$ and $\frac{\partial z}{\partial x} = 1$. Working up to second derivatives, another, more involved application of the chain rule yields that

(6) $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(c\frac{\partial u}{\partial v} - c\frac{\partial u}{\partial z} \right) = c\left(\frac{\partial^2 u}{\partial v^2}\frac{\partial v}{\partial t} + \frac{\partial^2 u}{\partial z \partial v}\frac{\partial z}{\partial t} \right) - c\left(\frac{\partial^2 u}{\partial z^2}\frac{\partial z}{\partial t} + \frac{\partial^2 u}{\partial v \partial z}\frac{\partial v}{\partial t} \right)$$

$$= c^2\left(\frac{\partial^2 u}{\partial v^2} - \frac{\partial^2 u}{\partial z \partial v} \right) + c^2\left(\frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial v \partial z} \right) = c^2\left(\frac{\partial^2 u}{\partial v^2} - 2\frac{\partial^2 u}{\partial z \partial v} + \frac{\partial^2 u}{\partial z^2} \right)$$

Another almost identical computation using the chain rule results in the fact that

(7) $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial z} \right) = \left(\frac{\partial^2 u}{\partial v^2}\frac{\partial v}{\partial x} + \frac{\partial^2 u}{\partial z \partial v}\frac{\partial z}{\partial x} \right) + \left(\frac{\partial^2 u}{\partial z^2}\frac{\partial z}{\partial x} + \frac{\partial^2 u}{\partial v \partial z}\frac{\partial v}{\partial x} \right)$$

$$= \frac{\partial^2 u}{\partial v^2} + 2\frac{\partial^2 u}{\partial z \partial v} + \frac{\partial^2 u}{\partial z^2}$$

Now we revisit the original wave equation

(8) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

and substitute in what we have calculated for $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^2 u}{\partial x^2}$ in terms of $\frac{\partial^2 u}{\partial v^2}$, $\frac{\partial^2 u}{\partial z^2}$ and $\frac{\partial^2 u}{\partial z \partial v}$. Doing this gives the following equation, ripe with cancellations:

(9) $$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial v^2} - 2\frac{\partial^2 u}{\partial z \partial v} + \frac{\partial^2 u}{\partial z^2} \right) = c^2\frac{\partial^2 u}{\partial x^2} = c^2\left(\frac{\partial^2 u}{\partial v^2} + 2\frac{\partial^2 u}{\partial z \partial v} + \frac{\partial^2 u}{\partial z^2} \right)$$

Dividing by c^2 and canceling the terms involving $\frac{\partial^2 u}{\partial v^2}$ and $\frac{\partial^2 u}{\partial z^2}$ reduces this series of equations to

(10) $$-2\frac{\partial^2 u}{\partial z \partial v} = +2\frac{\partial^2 u}{\partial z \partial v}$$

which means that

(11) $$\frac{\partial^2 u}{\partial z \partial v} = 0$$

So what, you might well ask, after all, we still have a second order PDE, and there are still several variables involved. But wait, think about what (11) implies. Picture (11) as it gives you information about the partial derivative of a partial derivative:

(12) $$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial v}\right) = 0$$

In this form, this implies that $\frac{\partial u}{\partial v}$ considered as a function of *z* and *v* is a constant in terms of the variable *z*, so that $\frac{\partial u}{\partial v}$ can only depend on *v*, i.e.

(13) $$\frac{\partial u}{\partial v} = M(v)$$

Now, integrating this equation with respect to *v* yields that

(14) $$u(v, z) = \int M(v)dv$$

This, as an indefinite integral, results in a constant of integration, which in this case is just constant from the standpoint of the variable *v*. Thus, it can be any arbitrary function of *z* alone, so that actually

(15) $$u(v, z) = \int M(v)dv + N(z) = P(v) + N(z)$$

where $P(v)$ is a function of *v* alone, and $N(z)$ is a function of *z* alone, as the notation indicates.

Substituting back the original change of variable equations for *v* and *z* in (2) yields that

(16) $$u(x, t) = P(x + ct) + N(x - ct)$$

where*P* and *N* are arbitrary single variable functions. This is called D’Alembert’s solution to the wave equation. Except for the somewhat annoying but easy enough chain rule computations, this was a pretty straightforward solution technique. The reason it worked so well in this case was the fact that the change of variables used in (2) were carefully selected so as to turn the original PDE into one in which the variables basically had no interaction, so that the original second order PDE could be solved by a series of two single variable integrations, which was easy to do.

Check out that D'Alembert's solution really works. According to this solution, you can pick any functions for P and N such as $P(v) = v^2$ and $N(v) = v + 2$. Then

(17) $$u(x,t) = (x+ct)^2 + (x-ct) + 2 = x^2 + x + ct + c^2t^2 + 2$$

Now check that

(18) $$\frac{\partial^2 u}{\partial t^2} = 2c^2$$

and that

(19) $$\frac{\partial^2 u}{\partial x^2} = 2$$

so that indeed

(20) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

and so this is in fact a solution of the original wave equation.

This same transformation trick can be used to solve a fairly wide range of PDEs. For instance one can solve the equation

(21) $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y^2}$$

by using the transformation of variables

(22) $v = x$ and $z = x + y$

(Try it out! You should get that $u(x, y) = P(x) + N(x + y)$ with arbitrary functions P and N)

Note that in our solution (16) to the wave equation, nothing has been specified about the initial and boundary conditions yet, and we said we would take care of this time around. So now we take a look at what these conditions imply for our choices for the two functions *P* and *N*.

If we were given an initial function $u(x,0) = f(x)$ along with initial velocity function $u_t(x,0) = g(x)$ then we can match up these conditions with our solution by simply substituting in $t = 0$ into (16) and follow along. We start first with a simplified set-up, where we assume that we are given the initial displacement function $u(x,0) = f(x)$, and that the initial velocity function $g(x)$ is equal to 0 (i.e. as if someone stretched the string and simply released it without imparting any extra velocity over the string tension alone).

Now the first initial condition implies that

$$(23) \qquad u(x,0) = P(x + c \cdot 0) + N(x - c \cdot 0) = P(x) + N(x) = f(x)$$

We next figure out what choosing the second initial condition implies. By working with an initial condition that $u_t(x,0) = g(x) = 0$, we see that by using the chain rule again on the functions *P* and *N*

$$(24) \qquad u_t(x,0) = \frac{\partial}{\partial t}\big(P(x+ct) + N(x-ct)\big) = cP'(x+ct) - cN'(x-ct)$$

(remember that *P* and *N* are just single variable functions, so the derivative indicated is just a simple single variable derivative with respect to their input). Thus in the case where $u_t(x,0) = g(x) = 0$, then

$$(25) \qquad cP'(x+ct) - cN'(x+ct) = 0$$

Dividing out the constant factor *c* and substituting in $t = 0$

$$(26) \qquad P'(x) = N'(x)$$

and so $P(x) + k = N(x)$ for some constant *k*. Combining this with the fact that $P(x) + N(x) = f(x)$, means that $2P(x) + k = f(x)$, so that $P(x) = \big(f(x) - k\big)/2$ and likewise $N(x) = \big(f(x) + k\big)/2$. Combining these leads to the solution

(27) $u(x,t) = P(x+ct) + N(x-ct) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right)$

To make sure that the boundary conditions are met, we need

(28) $u(0,t) = 0$ and $u(l,t) = 0$ for all values of t

The first boundary condition implies that

(29) $u(0,t) = \frac{1}{2}\left(f(ct) + f(-ct)\right) = 0$

or

(30) $f(-ct) = -f(ct)$

so that to meet this condition, then the initial condition function f must be selected to be an odd function. The second boundary condition that $u(l,t) = 0$ implies

(31) $u(l,t) = \frac{1}{2}\left(f(l+ct) + f(l-ct)\right) = 0$

so that $f(l+ct) = -f(l-ct)$. Next, since we've seen that f has to be an odd function, then $-f(l-ct) = f(-l+ct)$. Putting this all together this means that

(32) $f(l+ct) = f(-l+ct)$ for all values of t

which means that f must have period $2l$, since the inputs vary by that amount. Remember that this just means the function repeats itself every time $2l$ is added to the input, the same way that the sine and cosine functions have period 2π.

What happens if the initial velocity isn't equal to 0? Thus suppose $u_t(x,0) = g(x) \neq 0$. Tracing through the same types of arguments as the above leads to the solution function

(33) $u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$

In the next installment of this introduction to PDEs we will turn to the ***Heat Equation***.

Heat Equation

For this next PDE, we create a mathematical model of how heat spreads, or diffuses through an object, such as a metal rod, or a body of water. To do this we take advantage of our knowledge of vector calculus and the divergence theorem to set up a PDE that models such a situation. Knowledge of this particular PDE can be used to model situations involving many sorts of diffusion processes, not just heat. For instance the PDE that we will derive can be used to model the spread of a drug in an organism, of the diffusion of pollutants in a water supply.

The key to this approach will be the observation that heat tends to flow in the direction of decreasing temperature. The bigger the difference in temperature, the faster the heat flow, or heat loss (remember Newton's heating and cooling differential equation). Thus if you leave a hot drink outside on a freezing cold day, then after ten minutes the drink will be a lot colder than if you'd kept the drink inside in a warm room - this seems pretty obvious!

If the function $u(x, y, z, t)$ gives the temperature at time *t* at any point (*x, y, z*) in an object, then in mathematical terms the direction of fastest decreasing temperature away from a specific point (*x, y, z*), is just the gradient of *u* (calculated at the point (*x, y, z*) and a particular time *t)*. Note that here we are considering the gradient of *u* as just being with respect to the spatial coordinates *x, y* and *z,* so that we write

$$(1) \qquad grad(u) = \nabla u = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}$$

Thus the rate at which heat flows away (or toward) the point is proportional to this gradient, so that if **F** is the vector field that gives the velocity of the heat flow, then

$$(2) \qquad \mathbf{F} = -k(grad(u))$$

(negative as the flow is in the direction of fastest *decreasing* temperature).

The constant, *k,* is called the *thermal conductivity* of the object, and it determines the rate at which heat is passed through the material that the object is made of. Some metals, for instance, conduct heat quite rapidly, and so have high values for *k*, while other materials act more like insulators, with a much lower value of *k* as a result.

Now suppose we know the temperature function, $u(x, y, z, t)$, for an object, but just at an initial time, when *t* = 0, i.e. we just know $u(x, y, z, 0)$. Suppose we also know the thermal conductivity of the material. What we would like to do is to figure out how the temperature of the object,

$u(x, y, z, t)$, changes over time. The goal is to use the observation about the rate of heat flow to set up a PDE involving the function $u(x, y, z, t)$ (i.e. the Heat Equation), and then solve the PDE to find $u(x, y, z, t)$.

Deriving the Heat Equation

To get to a PDE, the easiest route to take is to invoke something called the Divergence Theorem. As this is a multivariable calculus topic that we haven't even gotten to at this point in the semester, don't worry! (It will be covered in the vector calculus section at the end of the course in Chapter 13 of Stewart). It's such a neat application of the use of the Divergence Theorem, however, that at this point you should just skip to the end of this short section and take it on faith that we will get a PDE in this situation (i.e. skip to equation (10) below. Then be sure to come back and read through this section once you've learned about the divergence theorem.

First notice if *E* is a region in the body of interest (the metal bar, the pool of water, etc.) then the amount of heat that leaves *E* per unit time is simply a surface integral. More exactly, it is the flux integral over the surface of *E* of the heat flow vector field, **F**. Recall that **F** is the vector field that gives the velocity of the heat flow - it's the one we wrote down as $\mathbf{F} = -k\nabla u$ in the previous section. Thus the amount of heat leaving *E* per unit time is just

(1) $$\iint_{\mathbf{S}} \mathbf{F} \cdot d\mathbf{S}$$

where*S* is the surface of *E*. But wait, we have the highly convenient divergence theorem that tells us that

(2) $$\iint_{\mathbf{S}} \mathbf{F} \cdot d\mathbf{S} = -k \iiint_{E} div(grad(u))dV$$

Okay, now what is *div(grad(u))*? Given that

(3) $$grad(u) = \nabla u = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}$$

then*div(grad(u))* is just equal to

(4) $$div(grad(u)) = \nabla \cdot (\nabla u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Incidentally, this combination of divergence and gradient is used so often that it's given a name, the *Laplacian*. The notation $div(grad(u) = \nabla \cdot (\nabla u)$ is usually shortened up to simply $\nabla^2 u$. So we could rewrite (2), the heat leaving region *E* per unit time as

$$\text{(5)} \qquad \iint_{\mathbf{S}} \mathbf{F} \cdot d\mathbf{S} = -k \iiint_E (\nabla^2 u) dV$$

On the other hand, we can calculate the total amount of heat, *H*, in the region, *E*, at a particular time, *t*, by computing the triple integral over *E*:

$$\text{(6)} \qquad H = \iiint_E (\sigma\delta) u(x, y, z, t) dV$$

where δ is the *density* of the material and the constant σ is the *specific heat* of the material (don't worry about all these extra constants for now - we will lump them all together in one place in the end). How does this relate to the earlier integral? On one hand (5) gives the rate of heat leaving *E* per unit time. This is just the same as $-\frac{\partial H}{\partial t}$, where *H* gives the total amount of heat in *E*. This means we actually have two ways to calculate the same thing, because we can calculate $\frac{\partial H}{\partial t}$ by differentiating equation (6) giving *H*, i.e.

$$\text{(7)} \qquad -\frac{\partial H}{\partial t} = -\iiint_E (\sigma\delta) \frac{\partial u}{\partial t} dV$$

Now since both (5) and (7) give the rate of heat leaving *E* per unit time, then these two equations must equal each other, so…

$$\text{(8)} \qquad -\frac{\partial H}{\partial t} = -\iiint_E (\sigma\delta) \frac{\partial u}{\partial t} dV = -k \iiint_E (\nabla^2 u) dV$$

For these two integrals to be equal means that their two integrands must equal each other (since this integral holds over any arbitrary region *E* in the object being studied), so…

$$\text{(9)} \qquad (\sigma\delta) \frac{\partial u}{\partial t} = k(\nabla^2 u)$$

or, if we let $c^2 = \frac{k}{\sigma\delta}$, and write out the Laplacian, $\nabla^2 u$, then this works out simply as

(10) $$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

This, then, is the PDE that models the diffusion of heat in an object, i.e. the Heat Equation! This particular version (10) is the ***three-dimensional heat equation.***

Solving the Heat Equation in the one-dimensional case

We simplify our heat diffusion modeling by considering the specific case of heat flowing in a long thin bar or wire, where the cross-section is very small, and constant, and insulated in such a way that the heat flow is just along the length of the bar or wire. In this slightly contrived situation, we can model the heat flow by keeping track of the temperature at any point along the bar using just one spatial dimension, measuring the position along the bar.

This means that the function, u, that keeps track of the temperature, just depends on x, the position along the bar, and t, time, and so the heat equation from the previous section becomes the so-called ***one-dimensional heat equation***:

(1) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

One of the interesting things to note at this point is how similar this PDE appears to the wave equation PDE. However, the resulting solution functions are remarkably different in nature. Remember that the solutions to the wave equation had to do with oscillations, dealing with vibrating strings and all that. Here the solutions to the heat equation deal with temperature flow, not oscillation, so that means the solution functions will likely look quite different. If you're familiar with the solution to Newton's heating and cooling differential equations, then you might expect to see some type of exponential decay function as part of the solution function.

Before we start to solve this equation, let's mention a few more conditions that we will need to know to nail down a specific solution. If the metal bar that we're studying has a specific length, l, then we need to know the temperatures at the ends of the bars. These temperatures will give us boundary conditions similar to the ones we worked with for the wave equation. To make life a bit simpler for us as we solve the heat equation, let's start with the case when the ends of the bar, at $x = 0$ and $x = l$ both have temperature equal to 0 for all time (you can picture this situation as a metal bar with the ends stuck against blocks of ice, or some other cooling apparatus keeping the ends exactly at 0 degrees). Thus we will be working with the same boundary conditions as before, namely

(2) $u(0,t) = 0$ and $u(l,t) = 0$ for all values of t

Finally, to pick out a particular solution, we also need to know the initial starting temperature of the entire bar, namely we need to know the function $u(x,0)$. Interestingly, that's all we would need for an initial condition this time around (recall that to specify a particular solution in the wave equation we needed to know two initial conditions, $u(x,0)$ and $u_t(x,0)$).

The nice thing now is that since we have already solved a PDE, then we can try following the same basic approach as the one we used to solve the last PDE, namely separation of variables. With any luck, we will end up solving this new PDE. So, remembering back to what we did in that case, let's start by writing

(3) $u(x,t) = F(x)G(t)$

where*F*, and *G*, are single variable functions. Differentiating this equation for $u(x,t)$ with respect to each variable yields

(4) $\dfrac{\partial^2 u}{\partial x^2} = F''(x)G(t)$ and $\dfrac{\partial u}{\partial t} = F(x)G'(t)$

When we substitute these two equations back into the original heat equation

(5) $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

we get

(6) $\dfrac{\partial u}{\partial t} = F(x)G'(t) = c^2 \dfrac{\partial^2 u}{\partial x^2} = c^2 F''(x)G(t)$

If we now separate the two functions *F* and *G* by dividing through both sides, then we get

(7) $\dfrac{G'(t)}{c^2 G(t)} = \dfrac{F''(x)}{F(x)}$

Just as before, the left-hand side only depends on the variable *t*, and the right-hand side just depends on *x*. As a result, to have these two be equal can only mean one thing, that they are both equal to the same constant, *k*:

(8) $$\frac{G'(t)}{c^2G(t)} = \frac{F''(x)}{F(x)} = k$$

As before, let's first take a look at the implications for $F(x)$ as the boundary conditions will again limit the possible solution functions. From (8) we get that $F(x)$ has to satisfy

(9) $$F''(x) - kF(x) = 0$$

Just as before, one can consider the various cases with *k* being positive, zero, or negative. Just as before, to meet the boundary conditions, it turns out that k must in fact be negative (otherwise $F(x)$ ends up being identically equal to 0, and we end up with the trivial solution $u(x,t) = 0$). So skipping ahead a bit, let's assume we have figured out that *k* must be negative (you should check the other two cases just as before to see that what we've just written is true!). To indicate this, we write, as before, that $k = -\omega^2$, so that we now need to look for solutions to

(10) $$F''(x) + \omega^2 F(x) = 0$$

These solutions are just the same as before, namely the general solution is:

(11) $$F(x) = A\cos(\omega x) + B\sin(\omega x)$$

where again *A* and *B* are constants and now we have $\omega = \sqrt{-k}$. Next, let's consider the boundary conditions $u(0,t) = 0$ and $u(l,t) = 0$. These are equivalent to stating that $F(0) = F(l) = 0$. Substituting in 0 for *x* in (11) leads to

(12) $$F(0) = A\cos(0) + B\sin(0) = A = 0$$

so that $F(x) = B\sin(\omega x)$. Next, consider $F(l) = B\sin(\omega l) = 0$. As before, we check that *B* can't equal 0, otherwise $F(x) = 0$ which would then mean that $u(x,t) = F(x)G(t) = 0 \cdot G(t) = 0$, the trivial solution, again. With $B \neq 0$, then it must be the case that $\sin(\omega l) = 0$ in order to have $B\sin(\omega l) = 0$. Again, the only way that this can happen is for ωl to be a multiple of π. This means that once again

(13) $$\omega l = n\pi \text{ or } \omega = \frac{n\pi}{l}$$ (where *n* is an integer)

and so

(14) $$F(x) = \sin\left(\frac{n\pi}{l}x\right)$$

where*n* is an integer. Next we solve for $G(t)$, using equation (8) again. So, rewriting (8), we see that this time

(15) $$G'(t) + \lambda_n{}^2 G(t) = 0$$

where $\lambda_n = \frac{cn\pi}{l}$, since we had originally written $k = -\omega^2$, and we just determined that $\omega = \frac{n\pi}{l}$ during the solution for $F(x)$. The general solution to this first order differential equation is just

(16) $$G(t) = Ce^{-\lambda_n{}^2 t}$$

So, now we can put it all together to find out that

(17) $$u(x,t) = F(x)G(t) = C\sin\left(\frac{n\pi}{l}x\right)e^{-\lambda_n{}^2 t}$$

Where*n* is an integer, *C* is an arbitrary constant, and $\lambda_n = \frac{cn\pi}{l}$. As is always the case, given a supposed solution to a differential equation, you should check to see that this indeed is a solution to the original heat equation, and that it satisfies the two boundary conditions we started with.

The next question is how to get from the general solution to the heat equation

(1) $$u(x,t) = C\sin\left(\frac{n\pi}{l}x\right)e^{-\lambda_n{}^2 t}$$

that we found in the last section, to a specific solution for a particular situation. How can one figure out which values of *n* and *C*are needed for a specific problem? The answer lies not in choosing one such solution function, but more typically it requires setting up an infinite series of such solutions. Such an infinite series, because of the principle of superposition, will still be a solution function to the equation, because the original heat equation PDE was linear and homogeneous. Using the superposition principle, and by summing together various solutions with carefully chosen values of *C*, then it is possible to create a specific solution function that will match any (reasonable) given starting temperature function $u(x,0)$.