Finite Element Method

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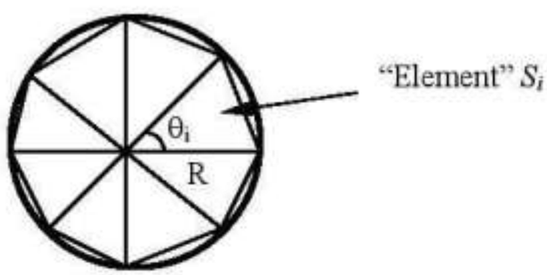
Chapter 1. Introduction

I. Basic Concepts

The *finite element method* (FEM), or *finite element analysis* (FEA), is based on the idea of building a complicated object with simple blocks, or, dividing a complicated object into small and manageable pieces. Application of this simple idea can be found everywhere in everyday life, as well as in engineering.

Examples:

- Lego (kids' play)
- Buildings
- Approximation of the area of a circle:



Area of one triangle: $S_i = \frac{1}{2} R^2 \sin\theta_i$

Area of the circle: $S_N = \sum_{i=1}^{N} S_i = \frac{1}{2} R^2 N \sin\left(\frac{2\pi}{N}\right) \rightarrow \pi R^2 \; as \; N \rightarrow \infty$

where N = total number of triangles (elements).

Observation: Complicated or smooth objects can be represented by geometrically simple pieces (elements).

Why Finite Element Method?

- *Design analysis*: hand calculations, experiments, and computer simulations
- FEM/FEA is the most widely applied computer simulation method in engineering
- Closely integrated with CAD/CAM applications
- ...

Applications of FEM in Engineering

- Mechanical/Aerospace/Civil/Automobile Engineering
- Structure analysis (static/dynamic, linear/nonlinear)
- Thermal/fluid flows
- Electromagnetics
- Geomechanics
- Biomechanics
- ...

Modeling of gear coupling

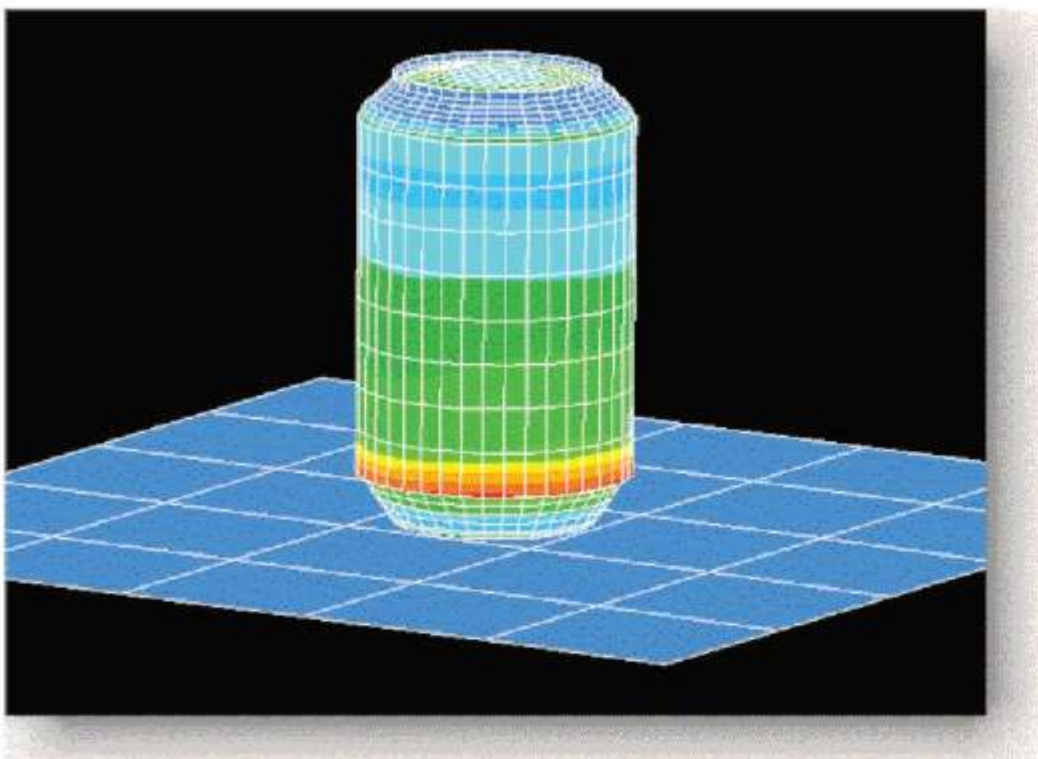
Examples:

...

A Brief History of the FEM

- 1943 ----- Courant (Variational methods)
- 1956 ----- Turner, Clough, Martin and Topp (Stiffness)
- 1960 ----- Clough ("Finite Element", plane problems)

- 1970s ----- Applications on mainframe computers
- 1980s ----- Microcomputers, pre- and postprocessors
- 1990s ----- Analysis of large structural systems

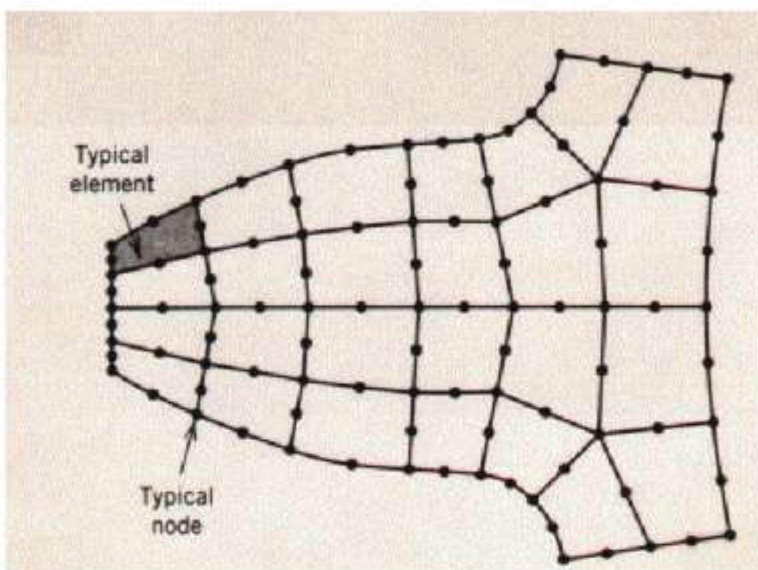


Can Drop Test (Click for more information and an animation)

FEM in Structural Analysis (The Procedure)

- Divide structure into pieces (elements with nodes)
- Describe the behavior of the physical quantities on each element
- Connect (assemble) the elements at the nodes to form an approximate system of equations for the whole structure
- Solve the system of equations involving unknown quantities at the nodes (e.g., displacements)
- Calculate desired quantities (e.g., strains and stresses) at selected elements

Example:



FEM model for a gear tooth (From Cook's book, p.2).

Computer Implementations

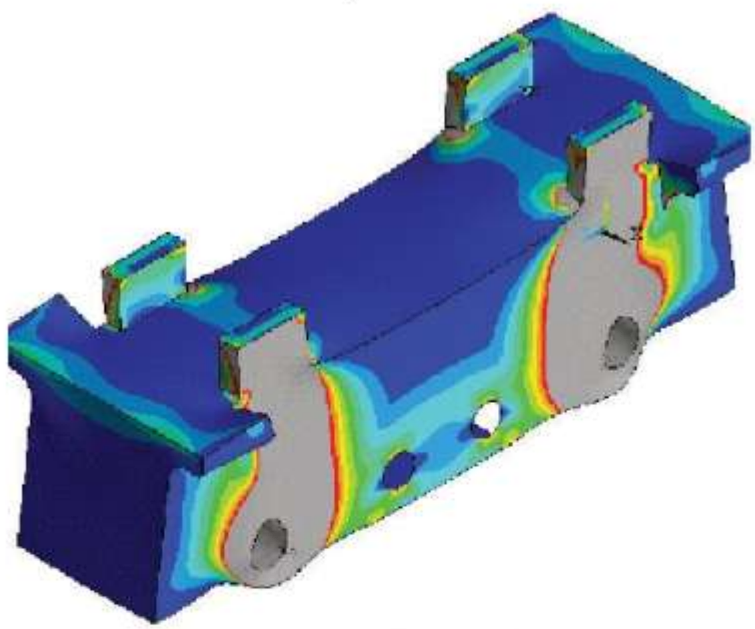
- Preprocessing (build FE model, loads and constraints)
- FEA solver (assemble and solve the system of equations)
- Postprocessing (sort and display the results)

Available Commercial FEM Software Packages

- *ANSYS* (General purpose, PC and workstations)
- *SDRC/I-DEAS* (Complete CAD/CAM/CAE package)
- *NASTRAN* (General purpose FEA on mainframes)
- *ABAQUS* (Nonlinear and dynamic analyses)
- *COSMOS* (General purpose FEA)
- *ALGOR* (PC and workstations)
- *PATRAN* (Pre/Post Processor)
- *HyperMesh* (Pre/Post Processor)
- *Dyna-3D* (Crash/impact analysis)
- ...

Objectives of This FEM Course

- Understand the fundamental ideas of the FEM
- Know the behavior and usage of each type of elements covered in this course
- Be able to prepare a suitable FE model for given problems
- Can interpret and evaluate the quality of the results (know the physics of the problems)
- Be aware of the limitations of the FEM (don't misuse the FEM - a numerical tool)



FEA of an Unloader Trolley (Click for more info)
By Jeff Badertscher (ME Class of 2001, UC)

See more examples in:

Showcase: Finite Element Analysis in Actions

II. Review of Matrix Algebra

Linear System of Algebraic Equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\ldots\ldots\ldots & \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n
\end{aligned}
\tag{1}
$$

where $x_1, x_2, \ldots, x_n$ are the unknowns.

In *matrix form*:

$$
\mathbf{Ax} = \mathbf{b} \tag{2}
$$

where

$$
\begin{gathered}
\mathbf{A} = \left[a_{ij}\right] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \\
\mathbf{x} = \{x_i\} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \qquad \mathbf{b} = \{b_i\} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix}
\end{gathered}
\tag{3}
$$

A is called a $n \times n$ (square) matrix, and **x** and **b** are (column) vectors of dimension n.

Row and Column Vectors

$$\mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \qquad \mathbf{w} = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}$$

Matrix Addition and Subtraction

For two matrices $\mathbf{A}$ and $\mathbf{B}$, both of the *same size* $(m \times n)$, the addition and subtraction are defined by

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \quad \text{with} \quad c_{ij} = a_{ij} + b_{ij}$$
$$\mathbf{D} = \mathbf{A} - \mathbf{B} \quad \text{with} \quad d_{ij} = a_{ij} - b_{ij}$$

Scalar Multiplication

$$\lambda \mathbf{A} = \left[\lambda a_{ij} \right]$$

Matrix Multiplication

For two matrices $\mathbf{A}$ (of size $l \times m$) and $\mathbf{B}$ (of size $m \times n$), the product of $\mathbf{AB}$ is defined by

$$\mathbf{C} = \mathbf{AB} \quad \text{with } c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

where $i = 1, 2, \ldots, l$; $j = 1, 2, \ldots, n$.

Note that, in general, $\mathbf{AB} \neq \mathbf{BA}$, but $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ (associative).

Transpose of a Matrix

If $\mathbf{A} = [a_{ij}]$, then the transpose of $\mathbf{A}$ is

$$\mathbf{A}^T = \left[a_{ji}\right]$$

Notice that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Symmetric Matrix

A *square* $(n \times n)$ matrix $\mathbf{A}$ is called symmetric, if

$$\mathbf{A} = \mathbf{A}^T \qquad \text{or} \qquad a_{ij} = a_{ji}$$

Unit (Identity) Matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Note that $\mathbf{AI} = \mathbf{A}$, $\mathbf{Ix} = \mathbf{x}$.

Determinant of a Matrix

The determinant of *square* matrix $\mathbf{A}$ is a scalar number denoted by $\det \mathbf{A}$ or $|\mathbf{A}|$. For 2×2 and 3×3 matrices, their determinants are given by

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

and

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13}$$
$$- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{23}a_{32}a_{11}$$

Singular Matrix

A *square* matrix **A** is *singular* if det **A** = 0, which indicates problems in the systems (nonunique solutions, degeneracy, etc.)

Matrix Inversion

For a *square* and *nonsingular* matrix **A** ($\det \mathbf{A} \neq 0$), its *inverse* $\mathbf{A}^{-1}$ is constructed in such a way that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

The *cofactor matrix* **C** of matrix **A** is defined by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

where M_{ij} is the determinant of the smaller matrix obtained by eliminating the *i*th row and *j*th column of **A**.

Thus, the inverse of **A** can be determined by

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \mathbf{C}^T$$

We can show that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Examples:

(1) $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{}{(ad-bc)}\begin{bmatrix} & \cdot b \\ -c & a \end{bmatrix}$$

Checking,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(2) $$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}^{-1} = \frac{1}{(4-2-1)}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{T} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Checking,

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If det $\mathbf{A} = 0$ (i.e., $\mathbf{A}$ is singular), then $\mathbf{A}^{-1}$ does not exist!

The solution of the linear system of equations (Eq.(1)) can be expressed as (assuming the coefficient matrix $\mathbf{A}$ is nonsingular)

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Thus, the main task in solving a linear system of equations is to found the inverse of the coefficient matrix.

Solution Techniques for Linear Systems of Equations

- Gauss elimination methods
- Iterative methods

Positive Definite Matrix

A *square* $(n \times n)$ matrix $\mathbf{A}$ is said to be *positive definite*, if for all nonzero vector $\mathbf{x}$ of dimension n,

$$\mathbf{x}^T \mathbf{A} \mathbf{x} > 0$$

Note that positive definite matrices are nonsingular.

Differentiation and Integration of a Matrix

Let

$$\mathbf{A}(t) = \left[a_{ij}(t) \right]$$

then the differentiation is defined by

$$\frac{d}{dt}\mathbf{A}(t) = \left[\frac{da_{ij}(t)}{dt} \right]$$

and the integration by

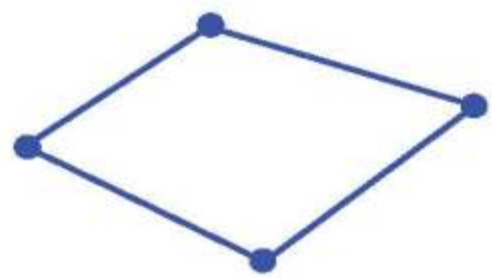
$$\int \mathbf{A}(t)dt = \left[\int a_{ij}(t)dt \right]$$

Types of Finite Elements

1-D (Line) Element

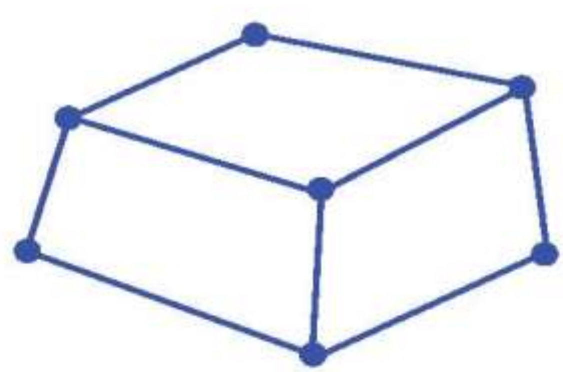
(Spring, truss, beam, pipe, etc.)

2-D (Plane) Element



(Membrane, plate, shell, etc.)

3-D (Solid) Element

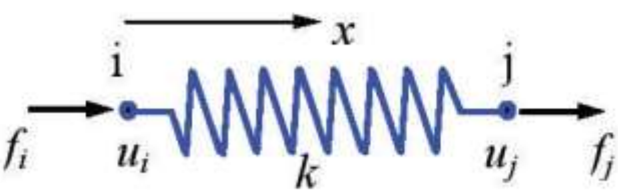


(3-D fields - temperature, displacement, stress, flow velocity)

III. Spring Element

"Everything important is simple."

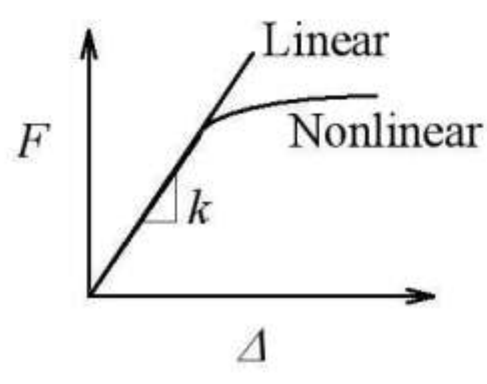
One Spring Element



Two nodes:	i, j
Nodal displacements:	u_i, u_j (in, m, mm)
Nodal forces:	f_i, f_j (lb, Newton)
Spring constant (stiffness):	k (lb/in, N/m, N/mm)

Spring force-displacement relationship:

$$F = k\Delta \qquad \text{with } \Delta = u_j - u_i$$



$k = F / \Delta$ (> 0) is the force needed to produce a unit stretch.

We only consider ***li*** s in this introductory course.

Consider the equilibrium of forces for the spring. At node i, we have

$$f_i = -F = -k(u_j - u_i) = ku_i - ku_j$$

and at node j,

$$f_j = F = k(u_j - u_i) = -ku_i + ku_j$$

In matrix form,

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_j \end{Bmatrix}$$

or,

$$\mathbf{ku} = \mathbf{f}$$

where

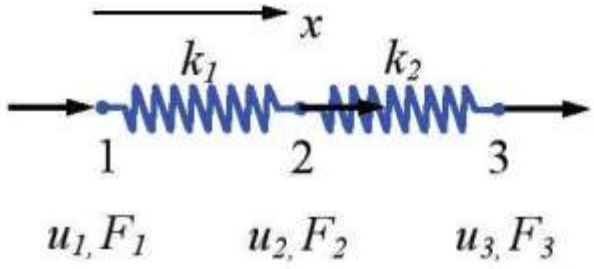
$\mathbf{k}$ = (element) stiffness matrix

$\mathbf{u}$ = (element nodal) displacement vector

$\mathbf{f}$ = (element nodal) force vector

Note that $\mathbf{k}$ is symmetric. Is $\mathbf{k}$ singular or nonsingular? That is, can we solve the equation? If not, why?

Spring System



For element 1,

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1 \end{Bmatrix}$$

element 2,

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^2 \\ f_2^2 \end{Bmatrix}$$

where f_i^m is the (internal) force acting on *local* node i of element m ($i = 1, 2$).

Assemble the stiffness matrix for the whole system:

Consider the equilibrium of forces at node 1,

$$F_1 = f_1^1$$

at node 2,

$$F_2 = f_2^1 + f_1^2$$

and node 3,

$$F_3 = f_2^2$$

That is,

$$
\begin{aligned}
F_1 &= k_1 u_1 - k_1 u_2 \\
F_2 &= -k_1 u_1 + (k_1 + k_2) u_2 - k_2 u_3 \\
F_3 &= -k_2 u_2 + k_2 u_3
\end{aligned}
$$

In matrix form,

$$
\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}
$$

or

$$
\mathbf{KU} = \mathbf{F}
$$

K is the stiffness matrix (structure matrix) for the spring system.

An alternative way of assembling the whole stiffness matrix:

"Enlarging" the stiffness matrices for elements 1 and 2, we have

$$
\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1 \\ 0 \end{Bmatrix}
$$

$$
\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f_1^2 \\ f_2^2 \end{Bmatrix}
$$

Adding the two matrix equations (*superposition*), we have

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ f_2^2 \end{Bmatrix}$$

This is the same equation we derived by using the force equilibrium concept.

Boundary and load conditions:

Assuming, $u_1 = 0$ and $F_2 = F_3 = P$

we have

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \\ P \end{Bmatrix}$$

which reduces to

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P \\ P \end{Bmatrix}$$

and

$$F_1 = -k_1 u_2$$

Unknowns are

$$\mathbf{U} = \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$ and the reaction force F_1 (if desired).

Solving the equations, we obtain the displacements

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 2P/k_1 \\ 2P/k_1 + P/k_2 \end{Bmatrix}$$

and the reaction force

$$F_1 = -2P$$

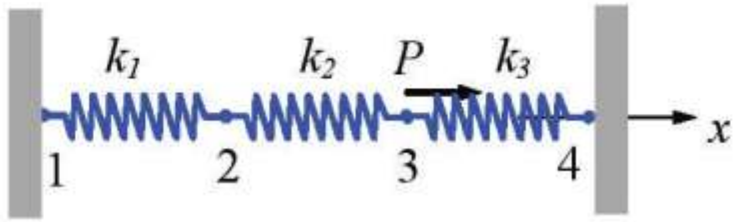
Checking the Results

- Deformed shape of the structure
- Balance of the external forces
- Order of magnitudes of the numbers

Notes About the Spring Elements

- Suitable for stiffness analysis
- Not suitable for stress analysis of the spring itself
- Can have spring elements with stiffness in the lateral direction, spring elements for torsion, etc.

Example 1.1



Given: For the spring system shown above,

$$k_1 = 100\ \mathrm{N/mm},\ \ k_2 = 200\ \mathrm{N/mm},\ \ k_3 = 100\ \mathrm{N/mm}$$

$$P = 500\ \mathrm{N},\ \ \mathrm{u}_1 = u_4 = 0$$

Find: (a) the global stiffness matrix

(b) displacements of nodes 2 and 3

(c) the reaction forces at nodes 1 and 4

(d) the force in the spring 2

Solution:

(a) The element stiffness matrices are

$$\mathbf{k}_1 = \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} \quad (\mathrm{N/mm}) \tag{1}$$

$$\mathbf{k}_2 = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \quad (\mathrm{N/mm}) \tag{2}$$

$$\mathbf{k}_3 = \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} \quad (\mathrm{N/mm}) \tag{3}$$

Applying the superposition concept, we obtain the global stiffness matrix for the spring system as

$$\mathbf{K} = \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 100+200 & -200 & 0 \\ 0 & -200 & 200+100 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix} \end{array}$$

or

$$\mathbf{K} = \begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 300 & -200 & 0 \\ 0 & -200 & 300 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix}$$

which is *symmetric* and ***banded***.

Equilibrium (FE) equation for the whole system is

$$\begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 300 & -200 & 0 \\ 0 & -200 & 300 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \\ P \\ F_4 \end{Bmatrix} \tag{4}$$

(b) Applying the BC ($u_1 = u_4 = 0$) in Eq(4), or deleting the 1st and 4th rows and columns, we have

$$\begin{bmatrix} 300 & -200 \\ -200 & 300 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix} \tag{5}$$

Solving Eq.(5), we obtain

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P/250 \\ 3P/500 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 3 \end{Bmatrix} \text{ (mm)} \tag{6}$$

(c) From the 1st and 4th equations in (4), we get the reaction forces

$$F_1 = -100u_2 = -200 \text{ (N)}$$

$$F_4 = -100u_3 = -300 \text{ (N)}$$

(d) The FE equation for spring (element) 2 is

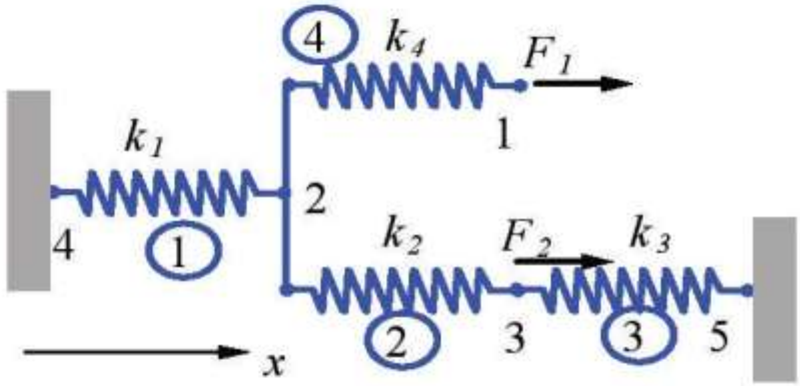
$$\begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_j \end{Bmatrix}$$

Here $i = 2, j = 3$ for element 2. Thus we can calculate the spring force as

$$\begin{aligned} F = f_j = -f_i &= \begin{bmatrix} -200 & 200 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \\ &= \begin{bmatrix} -200 & 200 \end{bmatrix} \begin{Bmatrix} 2 \\ 3 \end{Bmatrix} \\ &= 200 \text{ (N)} \end{aligned}$$

Check the results!

Example 1.2



Problem: For the spring system with arbitrarily numbered nodes and elements, as shown above, find the global stiffness matrix.

Solution:

First we construct the following

Element Connectivity Table

Element	*Node i (1)*	*Node j (2)*
1	4	2
2	2	3
3	3	5
4	2	1

which specifies the *global* node numbers corresponding to the *local* node numbers for each element.

Then we can write the element stiffness matrices as follows

$$
\begin{array}{cc} & \begin{array}{cc} u_4 & u_2 \end{array} \\ \mathbf{k}_1 = & \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \end{array}
\qquad
\begin{array}{cc} & \begin{array}{cc} u_2 & u_3 \end{array} \\ \mathbf{k}_2 = & \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \end{array}
$$

$$
\begin{array}{cc} & \begin{array}{cc} u_3 & u_5 \end{array} \\ \mathbf{k}_3 = & \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \end{array}
\qquad
\begin{array}{cc} & \begin{array}{cc} u_2 & u_1 \end{array} \\ \mathbf{k}_4 = & \begin{bmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{bmatrix} \end{array}
$$

Finally, applying the superposition method, we obtain the global stiffness matrix as follows

$$
\begin{array}{cc} & \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \mathbf{K} = & \begin{bmatrix} k_4 & -k_4 & 0 & 0 & 0 \\ -k_4 & k_1+k_2+k_4 & -k_2 & -k_1 & 0 \\ 0 & -k_2 & k_2+k_3 & 0 & -k_3 \\ 0 & -k_1 & 0 & k_1 & 0 \\ 0 & 0 & -k_3 & 0 & k_3 \end{bmatrix} \end{array}
$$

The matrix is *symmetric*, *banded*, but *singular*.

Chapter 2. Bar a Elements

I. Linear Static Analysis

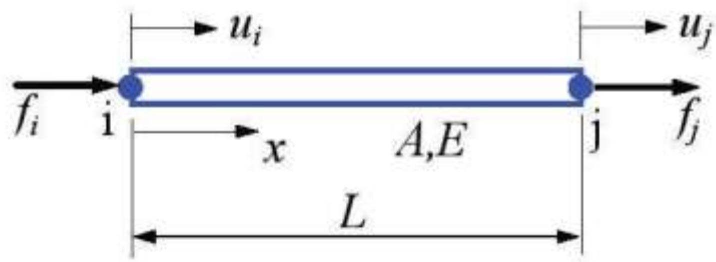
Most structural analysis problems can be treated as *linear static* problems, based on the following assumptions

1. *Small deformations* (loading pattern is not changed due to the deformed shape)
2. *Elastic materials* (no plasticity or failures)
3. *Static loads* (the load is applied to the structure in a slow or steady fashion)

Linear analysis can provide most of the information about the behavior of a structure, and can be a good approximation for many analyses. It is also the bases of nonlinear analysis in most of the cases.

II. Bar Element

Consider a uniform prismatic bar:



L	length
A	cross-sectional area
E	elastic modulus
$u = u(x)$	displacement
$\varepsilon = \varepsilon(x)$	strain
$\sigma = \sigma(x)$	stress

Strain-displacement relation:

$$\varepsilon = \frac{du}{dx} \tag{1}$$

Stress-strain relation:

$$\sigma = E\varepsilon \tag{2}$$

Stiffness Matrix --- Direct Method

Assuming that the displacement u is *varying linearly* along the axis of the bar, i.e.,

$$u(x) = \left(1 - \frac{x}{L}\right)u_i + \frac{x}{L}u_j \tag{3}$$

we have

$$\varepsilon = \frac{u_j - u_i}{L} = \frac{\Delta}{L} \qquad (\Delta = \text{elongation}) \tag{4}$$

$$\sigma = E\varepsilon = \frac{E\Delta}{L} \tag{5}$$

We also have

$$\sigma = \frac{F}{A} \qquad (F = \text{force in bar}) \tag{6}$$

Thus, (5) and (6) lead to

$$F = \frac{EA}{L}\Delta = k\Delta \tag{7}$$

where $k = \dfrac{EA}{L}$ is the stiffness of the bar.

The bar is acting like a spring in this case and we conclude that element stiffness matrix is

$$\mathbf{k} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix}$$

or

$$\mathbf{k} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{8}$$

This can be verified by considering the equilibrium of the forces at the two nodes.

Element equilibrium equation is

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_j \end{Bmatrix} \tag{9}$$

Degree of Freedom (dof)

Number of components of the displacement vector at a node.

For 1-D bar element: one dof at each node.

Physical Meaning of the Coefficients in **k**

The jth column of **k** (here $j = 1$ or 2) represents the forces applied to the bar to maintain a deformed shape with unit displacement at node j and zero displacement at the other node.

Stiffness Matrix --- A Formal Approach

We derive the same stiffness matrix for the bar using a formal approach which can be applied to many other more complicated situations.

Define two *linear shape functions* as follows

$$N_i(\xi) = 1 - \xi, \qquad N_j(\xi) = \xi \tag{10}$$

where

$$\xi = \frac{x}{L}, \qquad 0 \le \xi \le 1 \tag{11}$$

From (3) we can write the displacement as

$$u(x) = u(\xi) = N_i(\xi)u_i + N_j(\xi)u_j$$

or

$$u = \begin{bmatrix} N_i & N_j \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \mathbf{Nu} \tag{12}$$

Strain is given by (1) and (12) as

$$\varepsilon = \frac{du}{dx} = \left[\frac{d}{dx}\mathbf{N}\right]\mathbf{u} = \mathbf{Bu} \tag{13}$$

where **B** is the element *strain-displacement matrix*, which is

$$\mathbf{B} = \frac{d}{dx}\begin{bmatrix} N_i(\xi) & N_j(\xi) \end{bmatrix} = \frac{d}{d\xi}\begin{bmatrix} N_i(\xi) & N_j(\xi) \end{bmatrix} \bullet \frac{d\xi}{dx}$$

i.e.,

$$\mathbf{B} = \begin{bmatrix} -1/L & 1/L \end{bmatrix} \tag{14}$$

Stress can be written as

$$\sigma = E\varepsilon = E\mathbf{Bu} \tag{15}$$

Consider the *strain energy* stored in the bar

$$\begin{aligned} U &= \frac{1}{2}\int_V \sigma^{\mathrm{T}} \varepsilon dV = \frac{1}{2}\int_V \left(\mathbf{u}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}} E\mathbf{Bu}\right) dV \\ &= \frac{1}{2}\mathbf{u}^{\mathrm{T}}\left[\int_V \left(\mathbf{B}^{\mathrm{T}} E\mathbf{B}\right) dV\right]\mathbf{u} \end{aligned} \tag{16}$$

where (13) and (15) have been used.

The *work* done by the two nodal forces is

$$W = \frac{1}{2} f_i u_i + \frac{1}{2} f_j u_j = \frac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{f} \tag{17}$$

For conservative system, we state that

$$U = W \tag{18}$$

which gives

$$\frac{1}{2}\mathbf{u}^{\mathrm{T}}\left[\int_V \left(\mathbf{B}^{\mathrm{T}} E\mathbf{B}\right) dV\right]\mathbf{u} = \frac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{f}$$

We can conclude that

$$\left[\int_V \left(\mathbf{B}^{\mathrm{T}} E\mathbf{B}\right) dV\right]\mathbf{u} = \mathbf{f}$$

or

$$\mathbf{k}\mathbf{u} = \mathbf{f} \tag{19}$$

where

$$\mathbf{k} = \int_V \left(\mathbf{B}^{\mathrm{T}} E \mathbf{B}\right) dV \tag{20}$$

is the *element stiffness matrix*.

Expression (20) is a general result which can be used for the construction of other types of elements. This expression can also be derived using other more rigorous approaches, such as the *Principle of Minimum Potential Energy*, or the *Galerkin's Method*.

Now, we evaluate (20) for the bar element by using (14)

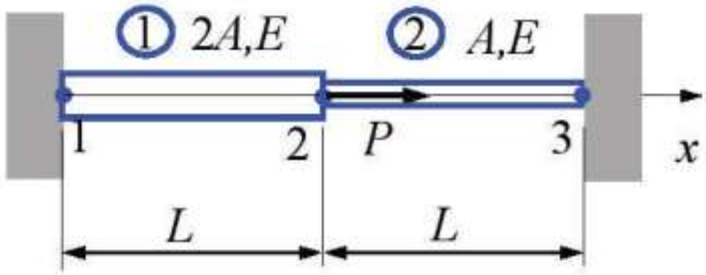
$$\mathbf{k} = \int_0^L \begin{Bmatrix} -1/L \\ 1/L \end{Bmatrix} E \begin{bmatrix} -1/L & 1/L \end{bmatrix} A dx = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

which is the same as we derived using the direct method.

Note that from (16) and (20), the strain energy in the element can be written as

$$U = \frac{1}{2} \mathbf{u}^{\mathrm{T}} \mathbf{k} \mathbf{u} \tag{21}$$

Example 2.1



Problem: Find the stresses in the two bar assembly which is loaded with force P, and constrained at the two ends, as shown in the figure.

Solution: Use two 1-D bar elements.

Element 1,

$$\begin{array}{cc} & \begin{array}{cc} u_1 & u_2 \end{array} \\ \mathbf{k}_1 = \dfrac{2EA}{L} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

Element 2,

$$\begin{array}{cc} & \begin{array}{cc} u_2 & u_3 \end{array} \\ \mathbf{k}_2 = \dfrac{EA}{L} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

Imagine a frictionless pin at node 2, which connects the two elements. We can assemble the global FE equation as follows,

$$\frac{EA}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Load and boundary conditions (BC) are,

$$u_1 = u_3 = 0, \qquad F_2 = P$$

FE equation becomes,

$$\frac{EA}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \\ F_3 \end{Bmatrix}$$

Deleting the 1st row and column, and the 3rd row and column, we obtain,

$$\frac{EA}{L}[3]\{u_2\} = \{P\}$$

Thus,

$$u_2 = \frac{PL}{3EA}$$

and

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{PL}{3EA}\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

Stress in element 1 is

$$\sigma_1 = E\varepsilon_1 = E\mathbf{B}_1\mathbf{u}_1 = E\begin{bmatrix} -1/L & 1/L \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$
$$= E\frac{u_2 - u_1}{L} = \frac{E}{L}\left(\frac{PL}{3EA} - 0\right) = \frac{P}{3A}$$

Similarly, stress in element 2 is

$$\sigma_2 = E\varepsilon_2 = E\mathbf{B}_2\mathbf{u}_2 = E\begin{bmatrix} -1/L & 1/L \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$
$$= E\frac{u_3 - u_2}{L} = \frac{E}{L}\left(0 - \frac{PL}{3EA}\right) = -\frac{P}{3A}$$

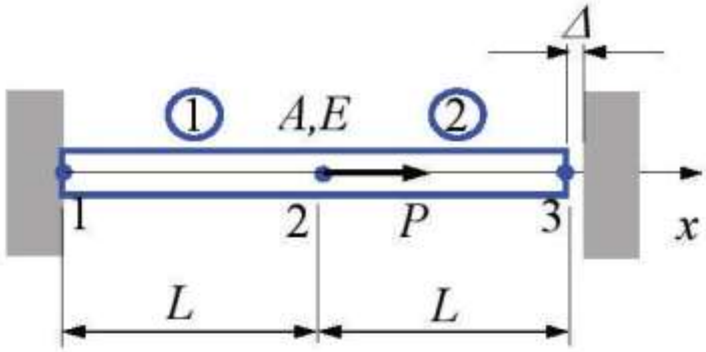
which indicates that bar 2 is in compression.

Check the results!

Notes:

- In this case, the calculated stresses in elements 1 and 2 are exact within the linear theory for 1-D bar structures. It will not help if we further divide element 1 or 2 into smaller finite elements.
- For tapered bars, averaged values of the cross-sectional areas should be used for the elements.
- We need to find the displacements first in order to find the stresses, since we are using the *displacement based FEM*.

Example 2.2



Problem: Determine the support reaction forces at the two ends of the bar shown above, given the following,

$$P = 6.0 \times 10^4 \text{ N}, \quad E = 2.0 \times 10^4 \text{ N/mm}^2,$$

$$A = 250 \text{ mm}^2, \quad L = 150 \text{ mm}, \quad \Delta = 1.2 \text{ mm}$$

Solution:

We first check to see if or not the contact of the bar with the wall on the right will occur. To do this, we imagine the wall on the right is removed and calculate the displacement at the right end,

$$\Delta_0 = \frac{PL}{EA} = \frac{(6.0 \times 10^4)(150)}{(2.0 \times 10^4)(250)} = 1.8 \text{ mm} > \Delta = 1.2 \text{ mm}$$

Thus, contact occurs.

The global FE equation is found to be,

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

The load and boundary conditions are,

$$F_2 = P = 6.0 \times 10^4 \text{ N}$$
$$u_1 = 0, \qquad\qquad u_3 = \Delta = 1.2 \text{ mm}$$

FE equation becomes,

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ \Delta \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \\ F_3 \end{Bmatrix}$$

The 2nd equation gives,

$$\frac{EA}{L}\begin{bmatrix} 2 & -1 \end{bmatrix}\begin{Bmatrix} u_2 \\ \Delta \end{Bmatrix} = \{P\}$$

that is,

$$\frac{EA}{L}[2]\{u_2\} = \left\{P + \frac{EA}{L}\Delta\right\}$$

Solving this, we obtain

$$u_2 = \frac{1}{2}\left(\frac{PL}{EA} + \Delta\right) = 1.5 \text{ mm}$$

and

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1.5 \\ 1.2 \end{Bmatrix} (\text{mm})$$

To calculate the support reaction forces, we apply the 1st and 3rd equations in the global FE equation.

The 1st equation gives,

$$F_1 = \frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{EA}{L}(-u_2) = -5.0 \times 10^4 \text{ N}$$

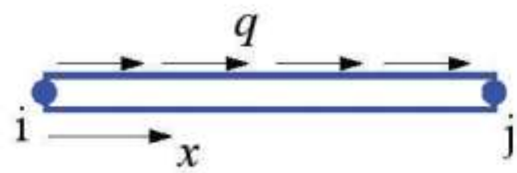
and the 3rd equation gives,

$$F_3 = \frac{EA}{L}\begin{bmatrix} 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{EA}{L}(-u_2 + u_3)$$

$$= -1.0 \times 10^4 \text{ N}$$

Check the results.!

Distributed Load



Uniformly distributed axial load q (N/mm, N/m, lb/in) can be converted to two equivalent nodal forces of magnitude $qL/2$. We verify this by considering the work done by the load q,

$$W_q = \int_0^L \frac{1}{2} uq dx = \frac{1}{2}\int_0^1 u(\xi)q(Ld\xi) = \frac{qL}{2}\int_0^1 u(\xi)d\xi$$

$$= \frac{qL}{2}\int_0^1 \begin{bmatrix} N_i(\xi) & N_j(\xi) \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} d\xi$$

$$= \frac{qL}{2}\int_0^1 \begin{bmatrix} 1-\xi & \xi \end{bmatrix} d\xi \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} \frac{qL}{2} & \frac{qL}{2} \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} u_i & u_j \end{bmatrix} \begin{Bmatrix} qL/2 \\ qL/2 \end{Bmatrix}$$

that is,

$$W_q = \frac{1}{2}\mathbf{u}^T\mathbf{f}_q \qquad \text{with } \mathbf{f}_q = \begin{Bmatrix} qL/2 \\ qL/2 \end{Bmatrix} \tag{22}$$

Thus, from the U=W concept for the element, we have

$$\frac{1}{2}\mathbf{u}^T\mathbf{k}\mathbf{u} = \frac{1}{2}\mathbf{u}^T\mathbf{f} + \frac{1}{2}\mathbf{u}^T\mathbf{f}_q \tag{23}$$

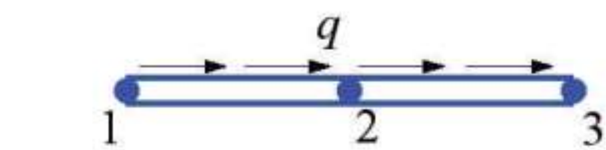
which yields

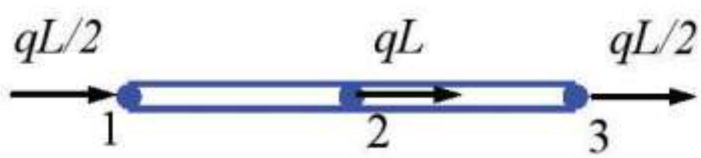
$$\mathbf{k}\mathbf{u} = \mathbf{f} + \mathbf{f}_q \tag{24}$$

The new nodal force vector is

$$\mathbf{f} + \mathbf{f}_q = \begin{Bmatrix} f_i + qL/2 \\ f_j + qL/2 \end{Bmatrix} \tag{25}$$

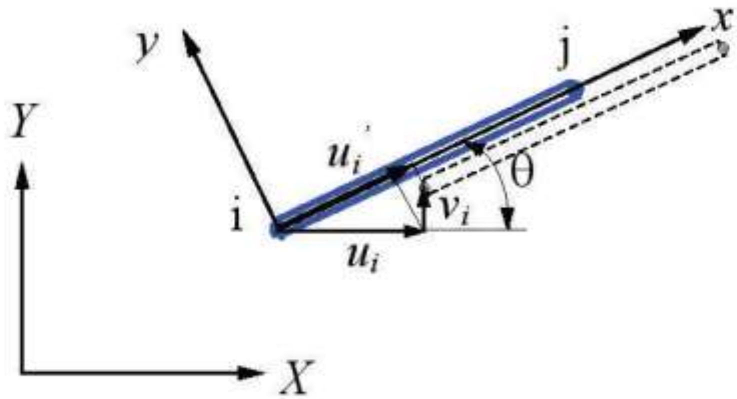
In an assembly of bars,





Bar Elements in 2-D and 3-D Space

2-D Case



Local	*Global*
x, y	X, Y
u_i', v_i'	u_i, v_i
1 dof at a node	2 dof's at a node

Note: Lateral displacement v_i' does not contribute to the stretch of the bar, within the linear theory.

Transformation

$$u_i' = u_i \cos\theta + v_i \sin\theta = \begin{bmatrix} l & m \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

$$v_i' = -u_i \sin\theta + v_i \cos\theta = \begin{bmatrix} -m & l \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

where $l = \cos\theta$, $m = \sin\theta$.

In matrix form,

$$\begin{Bmatrix} u_i' \\ v_i' \end{Bmatrix} = \begin{bmatrix} l & m \\ -m & l \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix} \tag{26}$$

or,

$$\mathbf{u}_i' = \widetilde{\mathbf{T}}\mathbf{u}_i$$

where the *transformation matrix*

$$\widetilde{\mathbf{T}} = \begin{bmatrix} l & m \\ -m & l \end{bmatrix} \tag{27}$$

is *orthogonal*, that is, $\widetilde{\mathbf{T}}^{-1} = \widetilde{\mathbf{T}}^{T}$.

For the two nodes of the bar element, we have

$$\begin{Bmatrix} u_i' \\ v_i' \\ u_j' \\ v_j' \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ -m & l & 0 & 0 \\ 0 & 0 & l & m \\ 0 & 0 & -m & l \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix} \tag{28}$$

or,

$$\mathbf{u}' = \mathbf{T}\mathbf{u} \qquad \text{with } \mathbf{T} = \begin{bmatrix} \widetilde{\mathbf{T}} & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathbf{T}} \end{bmatrix} \tag{29}$$

The nodal forces are transformed in the same way,

$$\mathbf{f}' = \mathbf{T}\mathbf{f} \tag{30}$$

Stiffness Matrix in the 2-D Space

In the local coordinate system, we have

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_i' \\ u_j' \end{Bmatrix} = \begin{Bmatrix} f_i' \\ f_j' \end{Bmatrix}$$

Augmenting this equation, we write

$$\frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_i' \\ v_i' \\ u_j' \\ v_j' \end{Bmatrix} = \begin{Bmatrix} f_i' \\ 0 \\ f_j' \\ 0 \end{Bmatrix}$$

or,

$$\mathbf{k}'\mathbf{u}' = \mathbf{f}'$$

Using transformations given in (29) and (30), we obtain

$$\mathbf{k}'\mathbf{T}\mathbf{u} = \mathbf{T}\mathbf{f}$$

Multiplying both sides by $\mathbf{T}^T$ and noticing that $\mathbf{T}^T\mathbf{T} = \mathbf{I}$, we obtain

$$\mathbf{T}^T\mathbf{k}'\mathbf{T}\mathbf{u} = \mathbf{f} \qquad (31)$$

Thus, the element stiffness matrix **k** in the global coordinate system is

$$\mathbf{k} = \mathbf{T}^T\mathbf{k}'\mathbf{T} \qquad (32)$$

which is a 4×4 symmetric matrix.

Explicit form,

$$\begin{array}{cc} & \begin{array}{cccc} u_i & v_i & u_j & v_j \end{array} \\ \mathbf{k} = \dfrac{EA}{L} & \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} \end{array} \tag{33}$$

Calculation of the *directional cosines* l and m:

$$l = \cos\theta = \frac{X_j - X_i}{L}, \qquad m = \sin\theta = \frac{Y_j - Y_i}{L} \tag{34}$$

The structure stiffness matrix is assembled by using the element stiffness matrices in the usual way as in the 1-D case.

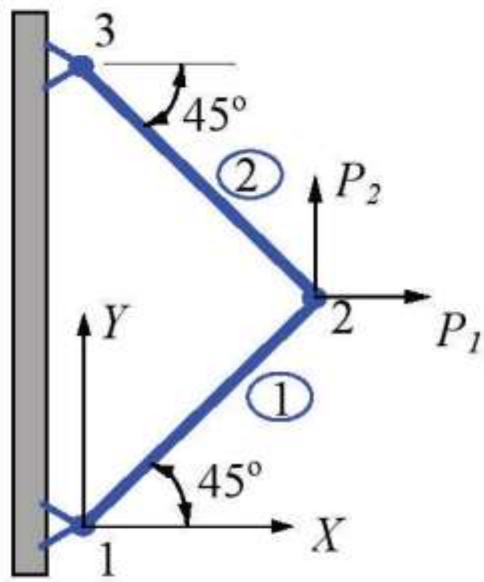
Element Stress

$$\sigma = E\varepsilon = E\mathbf{B}\begin{Bmatrix} u_i' \\ u_j' \end{Bmatrix} = E\begin{bmatrix} -\dfrac{1}{L} & \dfrac{1}{L} \end{bmatrix}\begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}$$

That is,

$$\sigma = \frac{E}{L}\begin{bmatrix} -l & -m & l & m \end{bmatrix}\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix} \tag{35}$$

Example 2.3



A simple plane truss is made of two identical bars (with E, A, and L), and loaded as shown in the figure. Find

1) displacement of node 2;

2) stress in each bar.

Solution:

This simple structure is used here to demonstrate the assembly and solution process using the bar element in 2-D space.

In local coordinate systems, we have

$$\mathbf{k}_1' = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \mathbf{k}_2'$$

These two matrices cannot be assembled together, because they are in different coordinate systems. We need to convert them to global coordinate system OXY.

Element 1:

$$\theta = 45^o, \quad l = m = \frac{\sqrt{2}}{2}$$

Using formula (32) or (33), we obtain the stiffness matrix in the global system

$$\begin{array}{c} \\ \mathbf{k}_1 = \mathbf{T}_1^T \mathbf{k}_1^{'} \mathbf{T}_1 = \dfrac{EA}{2L} \end{array} \begin{array}{c} \begin{matrix} u_1 & v_1 & u_2 & v_2 \end{matrix} \\ \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \end{array}$$

Element 2:

$$\theta = 135^o,\ l = -\frac{\sqrt{2}}{2},\ \ m = \frac{\sqrt{2}}{2}$$

We have,

$$\begin{array}{c} \\ \mathbf{k}_2 = \mathbf{T}_2^T \mathbf{k}_2^{'} \mathbf{T}_2 = \dfrac{EA}{2L} \end{array} \begin{array}{c} \begin{matrix} u_2 & v_2 & u_3 & v_3 \end{matrix} \\ \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \end{array}$$

Assemble the structure FE equation,

$$\begin{array}{c} \\ \dfrac{EA}{2L} \end{array} \begin{array}{c} \begin{matrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{matrix} \\ \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & -1 & 1 \\ -1 & -1 & 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix} \end{array} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} F_{1X} \\ F_{1Y} \\ F_{2X} \\ F_{2Y} \\ F_{3X} \\ F_{3Y} \end{Bmatrix}$$

Load and boundary conditions (BC):

$$u_1 = v_1 = u_3 = v_3 = 0, \qquad F_{2X} = P_1, \; F_{2Y} = P_2$$

Condensed FE equation,

$$\frac{EA}{2L}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

Solving this, we obtain the displacement of node 2,

$$\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \frac{L}{EA}\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

Using formula (35), we calculate the stresses in the two bars,

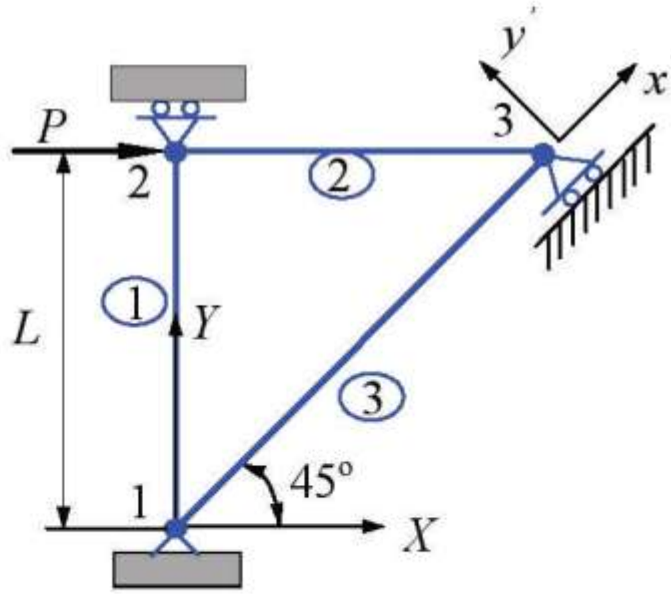
$$\sigma_1 = \frac{E}{L}\frac{\sqrt{2}}{2}\begin{bmatrix} -1 & -1 & 1 & 1 \end{bmatrix}\frac{L}{EA}\begin{Bmatrix} 0 \\ 0 \\ P_1 \\ P_2 \end{Bmatrix} = \frac{\sqrt{2}}{2A}(P_1 + P_2)$$

$$\sigma_2 = \frac{E}{L}\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}\frac{L}{EA}\begin{Bmatrix} P_1 \\ P_2 \\ 0 \\ 0 \end{Bmatrix} = \frac{\sqrt{2}}{2A}(P_1 - P_2)$$

Check the results:

Look for the equilibrium conditions, symmetry, antisymmetry, etc.

Example 2.4 (Multipoint Constraint)



For the plane truss shown above,

$$P = 1000 \text{ kN}, \quad L = 1 m, \quad E = 210 GPa,$$

$A = 6.0 \times 10^{-4} m^2$ for elements 1 and 2,

$A = 6\sqrt{2} \times 10^{-4} m^2$ for element 3.

Determine the displacements and reaction forces.

Solution:

We have an inclined roller at node 3, which needs special attention in the FE solution. We first assemble the global FE equation for the truss.

Element 1:

$$\theta = 90^o, \quad l = 0,$$

$$\mathbf{k}_1 = \frac{(210\times10^9)(6.0\times10^{-4})}{1} \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} (\mathrm{N/m})$$

Element 2:

$$\theta = 0^o, \quad l = 1, \quad m = 0$$

$$\mathbf{k}_2 = \frac{(210\times10^9)(6.0\times10^{-4})}{1} \begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} (\mathrm{N/m})$$

Element 3:

$$\theta = 45^o, \quad l = \frac{1}{\sqrt{2}}, \quad m = \frac{1}{\sqrt{2}}$$

$$\mathbf{k}_3 = \frac{(210\times10^9)(6\sqrt{2}\times10^{-4})}{\sqrt{2}} \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \end{array}$$

$$(\mathrm{N/m})$$

The global FE equation is,

$$1260 \times 10^5 \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.5 & -0.5 \\ & 1.5 & 0 & -1 & -0.5 & -0.5 \\ & & 1 & 0 & -1 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1.5 & 0.5 \\ Sym. & & & & & 0.5 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} F_{1X} \\ F_{1Y} \\ F_{2X} \\ F_{2Y} \\ F_{3X} \\ F_{3Y} \end{Bmatrix}$$

Load and boundary conditions (BC):

$$u_1 = v_1 = v_2 = 0, \text{ and } v_3' = 0,$$
$$F_{2X} = P, \; F_{3x'} = 0.$$

From the transformation relation and the BC, we have

$$v_3' = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix} = \frac{\sqrt{2}}{2}(-u_3 + v_3) = 0,$$

that is,

$$u_3 - v_3 = 0$$

This is a *multipoint constraint* (MPC).

Similarly, we have a relation for the force at node 3,

$$F_{3x'} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{Bmatrix} F_{3X} \\ F_{3Y} \end{Bmatrix} = \frac{\sqrt{2}}{2}(F_{3X} + F_{3Y}) = 0,$$

that is,

$$F_{3X} + F_{3Y} = 0$$

Applying the load and BC's in the structure FE equation by 'deleting' 1st, 2nd and 4th rows and columns, we have

$$1260 \times 10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} P \\ F_{3X} \\ F_{3Y} \end{Bmatrix}$$

Further, from the MPC and the force relation at node 3, the equation becomes,

$$1260 \times 10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P \\ F_{3X} \\ -F_{3X} \end{Bmatrix}$$

which is

$$1260 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P \\ F_{3X} \\ -F_{3X} \end{Bmatrix}$$

The 3rd equation yields,

$$F_{3X} = -1260 \times 10^5 u_3$$

Substituting this into the [illegible] nd rearranging, we have

$$1260 \times 10^5 \begin{bmatrix} 1 & \\ -1 & \end{bmatrix} \quad \begin{matrix} P \\ 0 \end{matrix}\Bigg\}$$

Solving this, we obtain the displacements,

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{1}{2520 \times 10^5} \begin{Bmatrix} 3P \\ P \end{Bmatrix} = \begin{Bmatrix} 0.01191 \\ 0.003968 \end{Bmatrix} \quad \text{(m)}$$

From the global FE equation, we can calculate the reaction forces,

$$\begin{Bmatrix} F_{1X} \\ F_{1Y} \\ F_{2Y} \\ F_{3X} \\ F_{3Y} \end{Bmatrix} = 1260 \times 10^5 \begin{bmatrix} 0 & -0.5 & -0.5 \\ 0 & -0.5 & -0.5 \\ 0 & 0 & 0 \\ -1 & 1.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} -500 \\ -500 \\ 0.0 \\ -500 \\ 500 \end{Bmatrix} \text{(kN)}$$

Check the results!

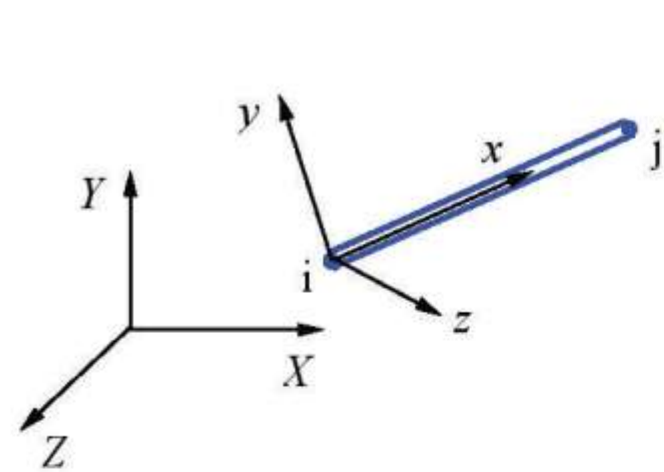
A general *multipoint constraint* (MPC) can be described as,

$$\sum_j A_j u_j = 0$$

where A_j's are constants and u_j's are nodal displacement components. In the FE software, such as *MSC/NASTRAN*, users only need to specify this relation to the software. The software will take care of the solution.

Penalty Approach for Hc and MPC's

3-D Case



Local	***Global***
x, y, z	X, Y, Z
$u_i^{'}, v_i^{'}, w_i^{'}$	u_i, v_i, w_i
1 dof at node	3 dof's at node

Element stiffness matrices are calculated in the local coordinate systems and then transformed into the global coordinate system (X, Y, Z) where they are assembled.

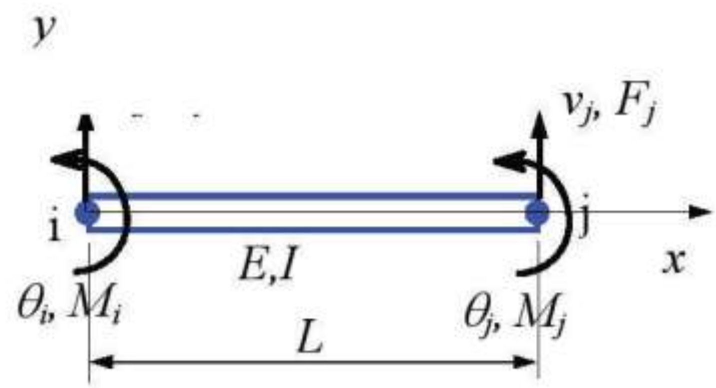
FEA software packages will do this transformation automatically.

Input data for bar elements:

- (X, Y, Z) for each node
- E and A for each element

III. Beam Elen

Simple Plane Be



L	length
I	moment of inertia of the cross-sectional area
E	elastic modulus
$v = v(x)$	deflection (lateral displacement) of the neutral axis
$\theta = \dfrac{dv}{dx}$	rotation about the z-axis
$F = F(x)$	shear force
$M = M(x)$	moment about z-axis

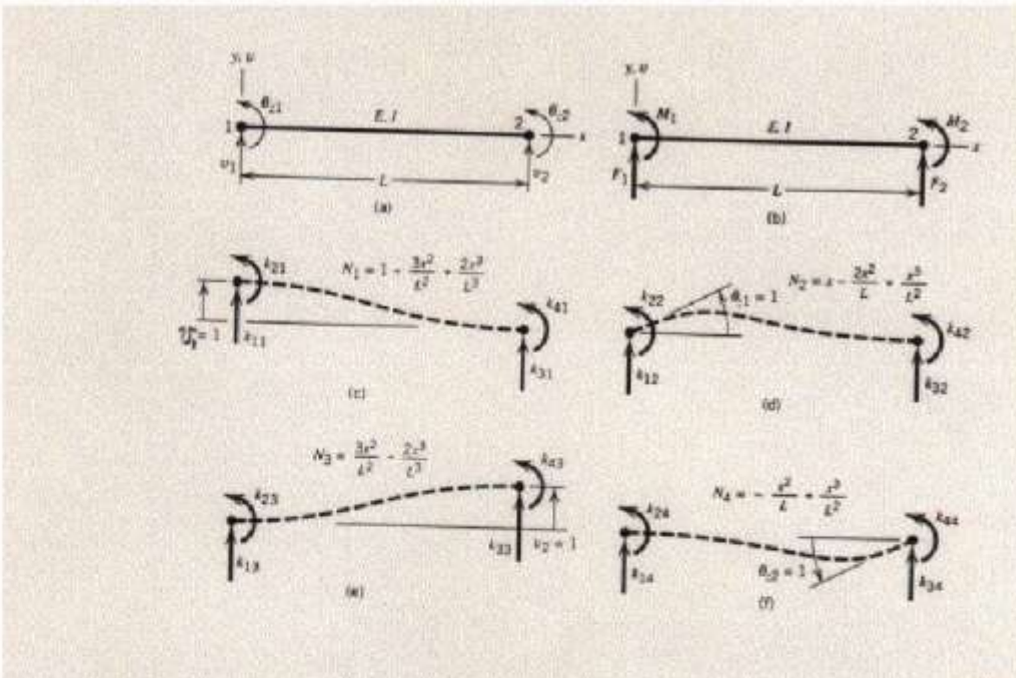
Elementary Beam Theory:

$$EI\frac{d^2v}{dx^2} = M(x) \tag{36}$$

$$\sigma = -\frac{My}{I} \tag{37}$$

Direct Method

Using the results from elementary beam theory to compute each column of the stiffness matrix.



Element stiffness equation (*local node*: i, j or 1, 2):

(Fig. 2.3-1. on Page 21 of Cook's Book)

Element stiffness equation (*local node*: i, j or 1, 2):

$$
\begin{array}{cc}
 & \begin{array}{cccc} v_i & \theta_i & v_j & \theta_j \end{array} \\
\dfrac{EI}{L^3} & \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}
\end{array}
\begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix} = \begin{Bmatrix} F_i \\ M_i \\ F_j \\ M_j \end{Bmatrix} \qquad (38)
$$

Formal Approach

Apply the formula,

$$\mathbf{k} = \int_0^L \mathbf{B}^T EI \mathbf{B} dx \tag{39}$$

To derive this, we introduce the shape functions,

$$\begin{aligned} N_1(x) &= 1 - 3x^2 / L^2 + 2x^3 / L^3 \\ N_2(x) &= x - 2x^2 / L + x^3 / L^2 \\ N_3(x) &= 3x^2 / L^2 - 2x^3 / L^3 \\ N_4(x) &= -x^2 / L + x^3 / L^2 \end{aligned} \tag{40}$$

Then, we can represent the deflection as,

$$\begin{aligned} v(x) &= \mathbf{Nu} \\ &= \begin{bmatrix} N_1(x) & N_2(x) & N_3(x) & N_4(x) \end{bmatrix} \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix} \end{aligned} \tag{41}$$

which is a cubic function. Notice that,

$$\begin{aligned} N_1 + N_3 &= 1 \\ N_2 + N_3 L + N_4 &= x \end{aligned}$$

which implies that the rigid body motion is represented by the assumed deformed shape of the beam.

Curvature of the beam is,

$$\frac{d^2v}{dx^2} = \frac{d^2}{dx^2}\mathbf{N}\mathbf{u} = \mathbf{B}\mathbf{u} \tag{42}$$

where the strain-displacement matrix **B** is given by,

$$\begin{aligned} \mathbf{B} &= \frac{d^2}{dx^2}\mathbf{N} = \begin{bmatrix} N_1''(x) & N_2''(x) & N_3''(x) & N_4''(x) \end{bmatrix} \\ &= \begin{bmatrix} -\frac{6}{L^2}+\frac{12x}{L^3} & -\frac{4}{L}+\frac{6x}{L^2} & \frac{6}{L^2}-\frac{12x}{L^3} & -\frac{2}{L}+\frac{6x}{L^2} \end{bmatrix} \end{aligned} \tag{43}$$

Strain energy stored in the beam element is

$$\begin{aligned} U &= \frac{1}{2}\int_V \sigma^T \varepsilon dV = \frac{1}{2}\int_0^L\int_A \left(-\frac{My}{I}\right)^T \frac{1}{E}\left(-\frac{My}{I}\right) dA dx \\ &= \frac{1}{2}\int_0^L M^T \frac{1}{EI} M dx = \frac{1}{2}\int_0^L \left(\frac{d^2v}{dx^2}\right)^T EI \left(\frac{d^2v}{dx^2}\right) dx \\ &= \frac{1}{2}\int_0^L (\mathbf{Bu})^T EI (\mathbf{Bu}) dx \\ &= \frac{1}{2}\mathbf{u}^T\left(\int_0^L \mathbf{B}^T EI \mathbf{B} dx\right)\mathbf{u} \end{aligned}$$

We conclude that the stiffness matrix for the simple beam element is

$$\mathbf{k} = \int_0^L \mathbf{B}^T EI \mathbf{B} dx$$

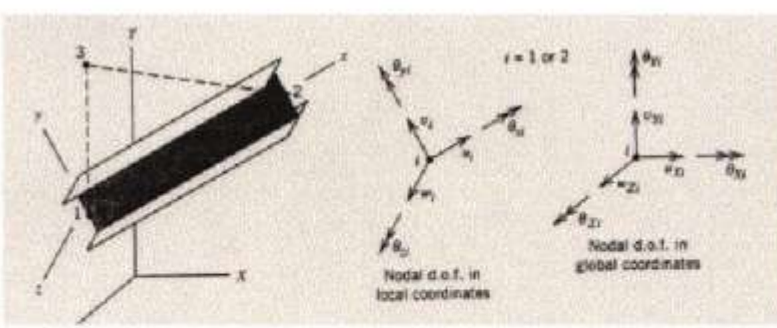
Applying the result in (43) and carrying out the integration, we arrive at the same stiffness matrix as given in (38).

Combining the axial stiffness (bar element), we obtain the stiffness matrix of a *general 2-D beam element*,

$$
\begin{array}{cc}
 & \begin{array}{cccccc} u_i & v_i & \theta_i & u_j & v_j & \theta_j \end{array} \\
\mathbf{k} = & \begin{bmatrix}
\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\
0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
-\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\
0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
\end{array}
$$

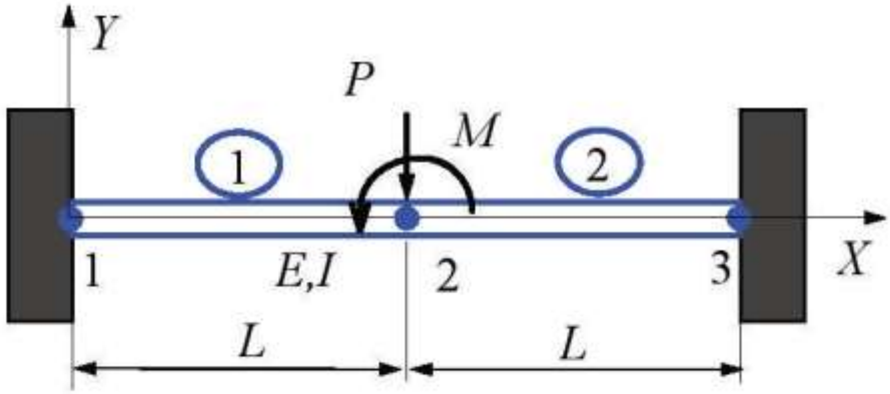
3-D Beam Element

The element stiffness matrix is formed in the local (2-D) coordinate system first and then transformed into the global (3-D) coordinate system to be assembled.



(Fig. 2.3-2. On Page 24 of Cook's book)

Example 2.5



Given: The beam shown above is clamped at the two ends and acted upon by the force P and moment M in the mid-span.

Find: The deflection and rotation at the center node and the reaction forces and moments at the two ends.

Solution: Element stiffness matrices are,

$$
\begin{array}{cc}
 & \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\
\mathbf{k}_1 = \dfrac{EI}{L^3} & \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}
\end{array}
$$

$$
\begin{array}{cc}
 & \begin{array}{cccc} v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\
\mathbf{k}_2 = \dfrac{EI}{L^3} & \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}
\end{array}
$$

Global FE equation is,

$$
\begin{array}{cc}
 & \begin{array}{cccccc} v_1 & \theta_1 & v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\
\dfrac{EI}{L^3} & \left[\begin{array}{cc|cc|cc}
12 & 6L & -12 & 6L & 0 & 0 \\
6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\
\hline
-12 & -6L & 24 & 0 & -12 & 6L \\
6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\
\hline
0 & 0 & -12 & -6L & 12 & -6L \\
0 & 0 & 6L & 2L^2 & -6L & 4L^2
\end{array}\right]
\end{array}
\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}
=
\begin{Bmatrix} F_{1Y} \\ M_1 \\ F_{2Y} \\ M_2 \\ F_{3Y} \\ M_3 \end{Bmatrix}
$$

Loads and constraints (BC's) are,

$$F_{2Y} = -P, \qquad M_2 = M,$$
$$v_1 = v_3 = \theta_1 = \theta_3 = 0$$

Reduced FE equation,

$$\frac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -P \\ M \end{Bmatrix}$$

Solving this we obtain,

$$\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{L}{24EI}\begin{Bmatrix} -PL^2 \\ 3M \end{Bmatrix}$$

From global FE equation, we obtain the reaction forces and moments,

$$\begin{Bmatrix} F_{1Y} \\ M_1 \\ F_{3Y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} -12 & 6L \\ -6L & 2L^2 \\ -12 & -6L \\ 6L & 2L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{1}{4}\begin{Bmatrix} 2P+3M/L \\ PL+M \\ 2P-3M/L \\ -PL+M \end{Bmatrix}$$

Stresses in the beam at the two ends can be calculated using the formula,

$$\sigma = \sigma_x = -\frac{My}{I}$$

Note that the FE solution is exact according to the simple beam theory, since no distributed load is present between the nodes. Recall that,

$$EI\frac{d^2v}{dx^2} = M(x)$$

and

$$\frac{dM}{dx} = V \quad (V \text{ - shear force in the beam})$$

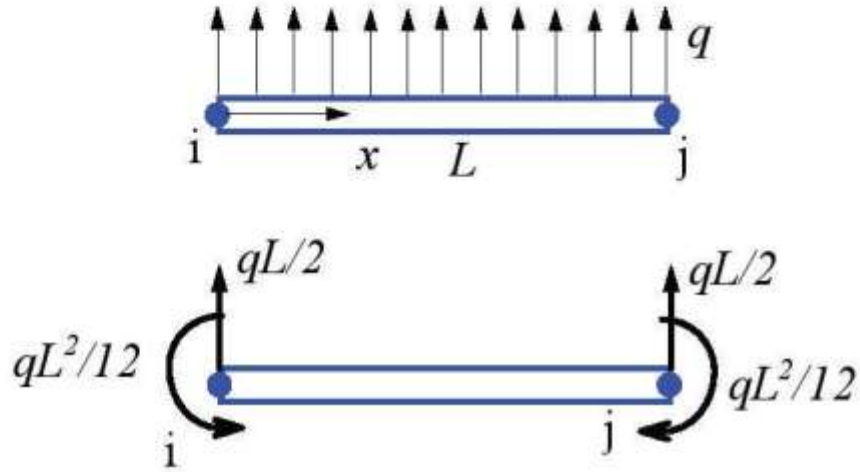
$$\frac{dV}{dx} = q \quad (q \text{ - distributed load on the beam})$$

Thus,

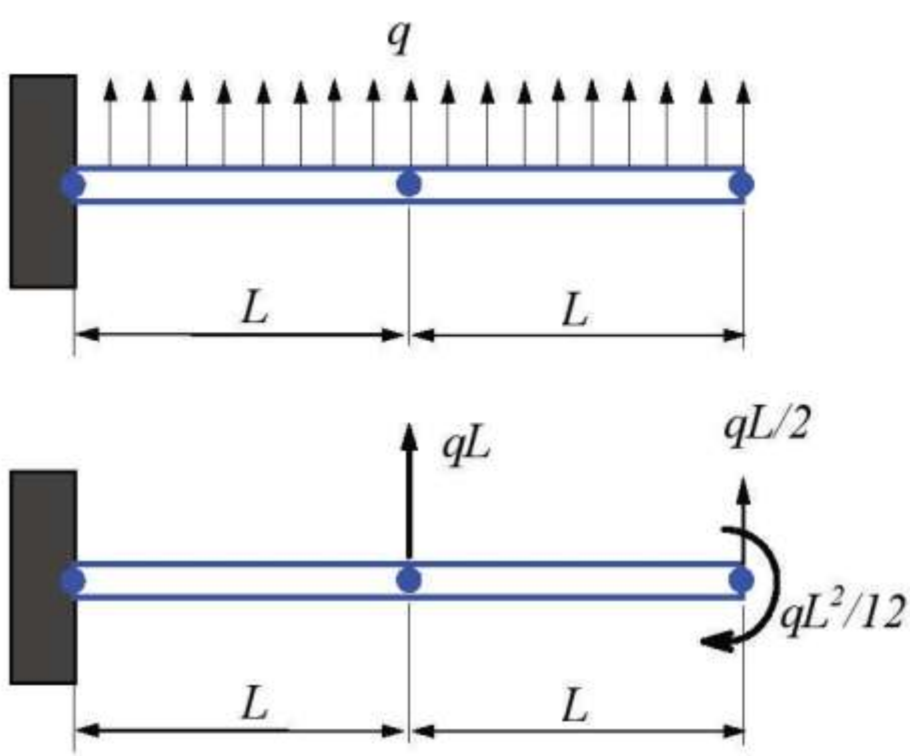
$$EI\frac{d^4v}{dx^4} = q(x)$$

If $q(x)$=0, then exact solution for the deflection v is a cubic function of x, which is what described by our shape functions.

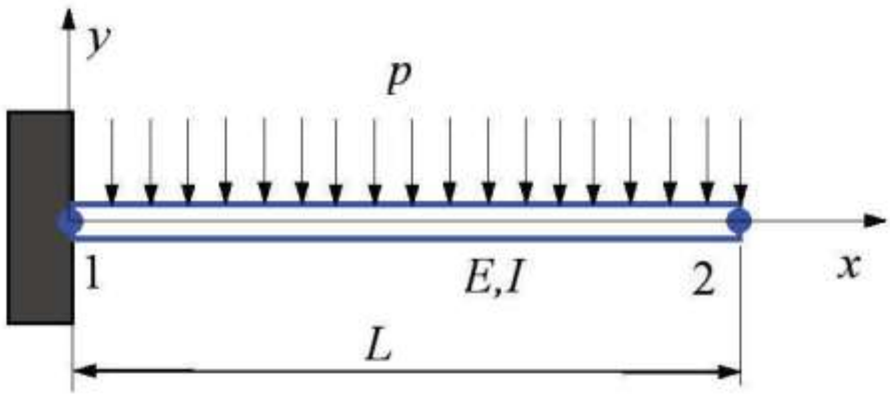
Equivalent Nodal Loads of Distributed Transverse Load



This can be verified by considering the work done by the distributed load q.



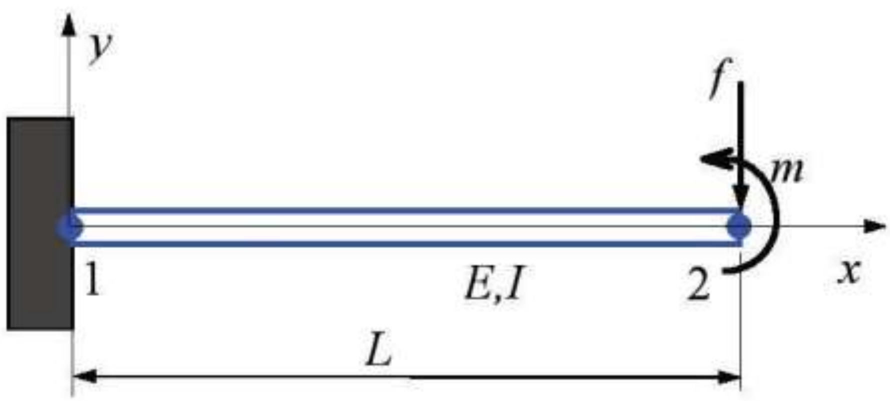
Example 2.6



Given: A cantilever beam with distributed lateral load p as shown above.

Find: The deflection and rotation at the right end, the reaction force and moment at the left end.

Solution: The work-equivalent nodal loads are shown below,



where

$$f = pL/2, \qquad m = pL^2/12$$

Applying the FE equation, we have

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_{1Y} \\ M_1 \\ F_{2Y} \\ M_2 \end{Bmatrix}$$

Load and constraints (BC's) are,

$$F_{2Y} = -f, \qquad M_2 = m$$
$$v_1 = \theta_1 = 0$$

Reduced equation is,

$$\frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -f \\ m \end{Bmatrix}$$

Solving this, we obtain,

$$\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{L}{6EI}\begin{Bmatrix} -2L^2 f + 3Lm \\ -3Lf + 6m \end{Bmatrix} = \begin{Bmatrix} -pL^4/8EI \\ -pL^3/6EI \end{Bmatrix} \qquad \text{(A)}$$

These nodal values are the same as the exact solution. Note that the deflection $v(x)$ (for $0 < x < 0$) in the beam by the FEM is, however, different from that by the exact solution. The exact solution by the simple beam theory is a 4th order polynomial of x, while the FE solution of v is only a 3rd order polynomial of x.

If the equivalent moment m is ignored, we have,

$$\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{L}{6EI}\begin{Bmatrix} -2L^2 f \\ -3Lf \end{Bmatrix} = \begin{Bmatrix} -pL^4/6EI \\ -pL^3/4EI \end{Bmatrix} \qquad \text{(B)}$$

The errors in (B) will decrease if more elements are used. The equivalent moment m is often ignored in the FEM applications. The FE solutions still converge as more elements are applied.

From the FE equation, we can calculate the reaction force and moment as,

$$\begin{Bmatrix} F_{1Y} \\ M_1 \end{Bmatrix} = \frac{L^3}{EI} \begin{bmatrix} -12 & 6L \\ -6L & 2L^2 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} pL/2 \\ 5pL^2/12 \end{Bmatrix}$$

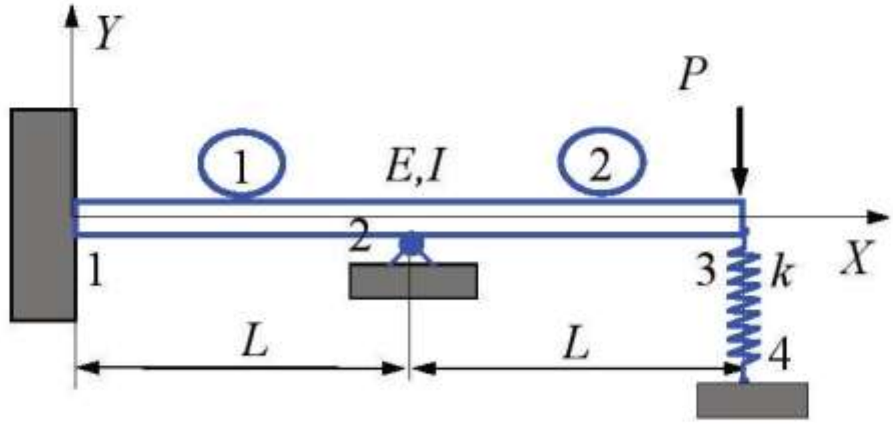
where the result in (A) is used. This force vector gives the total *effective nodal forces* which include the equivalent nodal forces for the distributed lateral load p given by,

$$\begin{Bmatrix} -pL/2 \\ -pL^2/12 \end{Bmatrix}$$

The correct *reaction forces* can be obtained as follows,

$$\begin{Bmatrix} F_{1Y} \\ M_1 \end{Bmatrix} = \begin{Bmatrix} pL/2 \\ 5pL^2/12 \end{Bmatrix} - \begin{Bmatrix} -pL/2 \\ -pL^2/12 \end{Bmatrix} = \begin{Bmatrix} pL \\ pL^2/2 \end{Bmatrix}$$

Example 2.7



Given: P = 50 kN, k = 200 kN/m, L = 3 m,

E = 210 GPa, $I = 2\times10^{-4}$ m^4.

Find: Deflections, rotations and reaction forces.

Solution:

The beam has a roller (or hinge) support at node 2 and a spring support at node 3. We use two beam elements and one spring element to solve this problem.

The spring stiffness matrix is given by,

$$\begin{array}{cc} & \begin{array}{cc} v_3 & v_4 \end{array} \\ \mathbf{k}_s = & \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \end{array}$$

Adding this stiffness matrix to the global FE equation (see *Example 2.5*), we have

$$\frac{EI}{L^3}\begin{array}{c}\begin{array}{ccccccc} v_1 & \theta_1 & v_2 & \theta_2 & v_3 & \theta_3 & v_4 \end{array}\\ \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 & 0 \\ & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 \\ & & 24 & 0 & -12 & 6L & 0 \\ & & & 8L^2 & -6L & 2L^2 & 0 \\ & & & & 12+k' & -6L & -k' \\ & & & & & 4L^2 & 0 \\ & \mathit{Symmetry} & & & & & k' \end{bmatrix}\end{array}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} F_{1Y} \\ M_1 \\ F_{2Y} \\ M_2 \\ F_{3Y} \\ M_3 \\ F_{4Y} \end{Bmatrix}$$

in which

$$k' = \frac{L^3}{EI} k$$

is used to simply the notation.

We now apply the boundary conditions,

$$v_1 = \theta_1 = v_2 = v_4 = 0,$$
$$M_2 = M_3 = 0, \qquad F_{3Y} = -P$$

'Deleting' the first three and seventh equations (rows and columns), we have the following reduced equation,

$$\frac{EI}{L^3}\begin{bmatrix} 8L^2 & -6L & 2L^2 \\ -6L & 12+k' & -6L \\ 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \end{Bmatrix}$$

Solving this equation, we obtain the deflection and rotations at node 2 and node 3,

$$\begin{Bmatrix} \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = -\frac{PL^2}{EI(12+7k')} \begin{Bmatrix} 3 \\ 7L \\ 9 \end{Bmatrix}$$

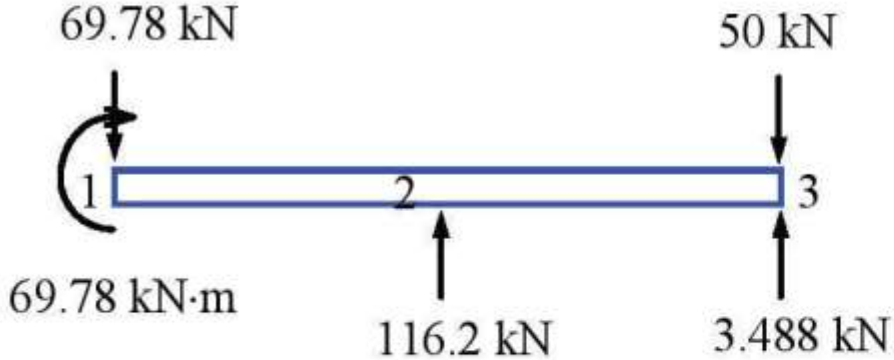
The influence of the spring k is easily seen from this result. Plugging in the given numbers, we can calculate

$$\begin{Bmatrix} \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} -0.002492 \text{ rad} \\ -0.01744 \text{ m} \\ -0.007475 \text{ rad} \end{Bmatrix}$$

From the global FE equation, we obtain the nodal reaction forces as,

$$\begin{Bmatrix} F_{1Y} \\ M_1 \\ F_{2Y} \\ F_{4Y} \end{Bmatrix} = \begin{Bmatrix} -69.78 \text{ kN} \\ -69.78 \text{ kN} \cdot \text{m} \\ 116.2 \text{ kN} \\ 3.488 \text{ kN} \end{Bmatrix}$$

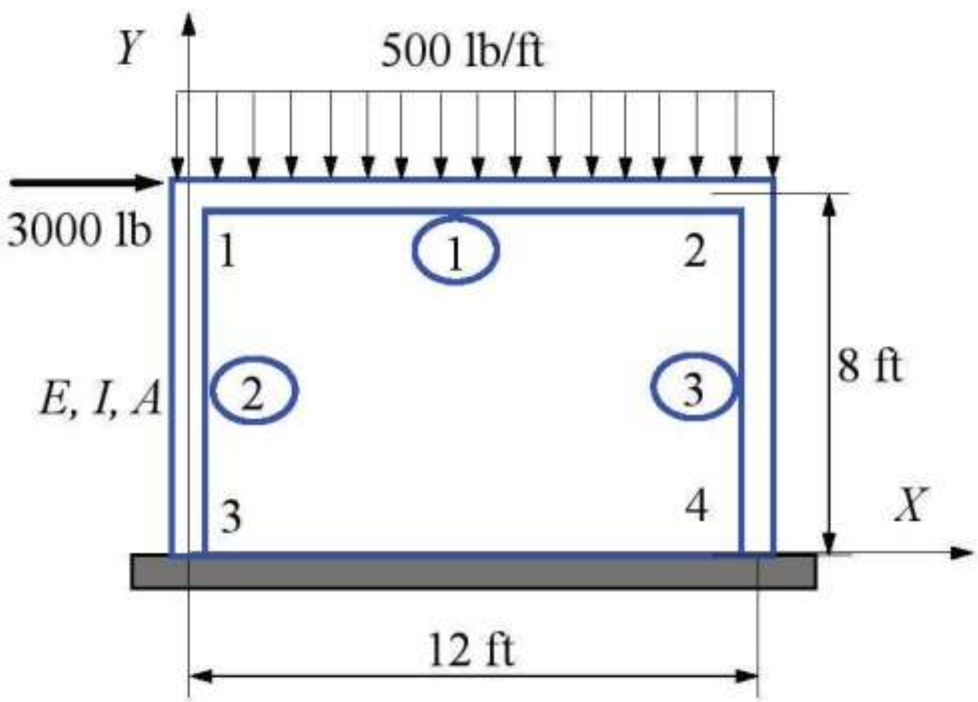
Checking the results: Draw *free body diagram* of the beam



FE Analysis of Frame Structures

Members in a frame are considered to be rigidly connected. Both forces and moments can be transmitted through their joints. We need the *general beam element* (combinations of bar and simple beam elements) to model frames.

Example 2.8

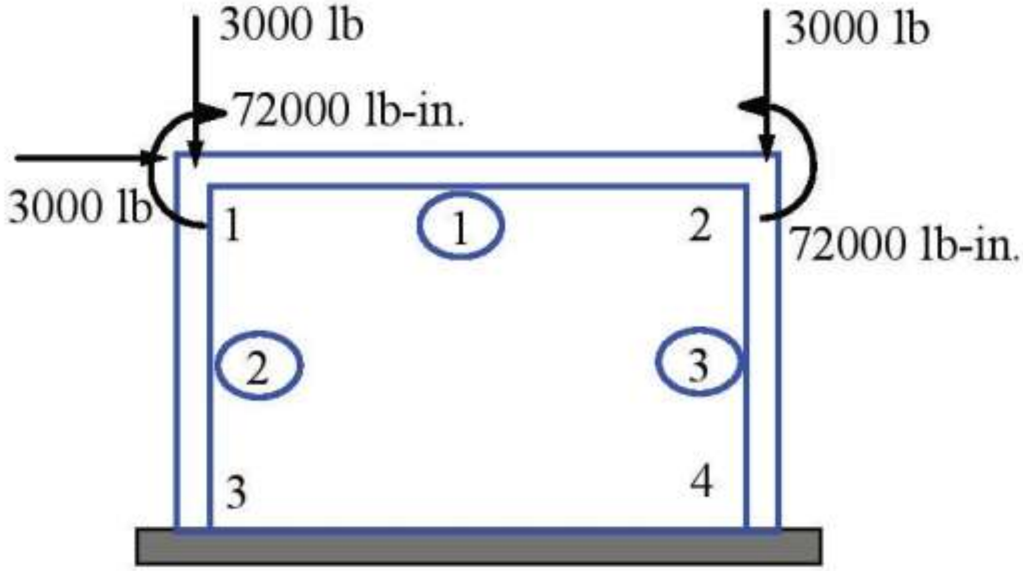


Given: $E = 30 \times 10^6$ psi, $I = 65$ in.^{4}, $A = 6.8$ in.^{2}

Find: Displacements and rotations of the two joints 1 and 2.

Solution:

For this example, we first convert the distributed load to its equivalent nodal loads.



In *local coordinate system*, the stiffness matrix for a general 2-D beam element is

$$
\begin{array}{cc}
 & \begin{array}{cccccc} u_i & v_i & \theta_i & u_j & v_j & \theta_j \end{array} \\
\mathbf{k} = & \begin{bmatrix}
\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\
0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
-\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\
0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
\end{array}
$$

Element Connectivity Table

Element	*Node i (1)*	*Node j (2)*
1	1	2
2	3	1
3	4	2

For element 1, we have

$$\begin{array}{cc} & \begin{array}{cccccc} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{array} \\ \mathbf{k}_1 = \mathbf{k}_1' = 10^4 \times & \begin{bmatrix} 141.7 & 0 & 0 & -141.7 & 0 & 0 \\ 0 & 0.784 & 56.4 & 0 & -0.784 & 56.4 \\ 0 & 56.4 & 5417 & 0 & -56.4 & 2708 \\ -141.7 & 0 & 0 & 141.7 & 0 & 0 \\ 0 & -0.784 & -56.4 & 0 & 0.784 & -56.4 \\ 0 & 56.4 & 2708 & 0 & -56.4 & 5417 \end{bmatrix} \end{array}$$

For elements 2 and 3, the stiffness matrix in *local system* is,

$$\begin{array}{cc} & \begin{array}{cccccc} u_i' & v_i' & \theta_i' & u_j' & v_j' & \theta_j' \end{array} \\ \mathbf{k}_2' = \mathbf{k}_3' = 10^4 \times & \begin{bmatrix} 212.5 & 0 & 0 & -212.5 & 0 & 0 \\ 0 & 2.65 & 127 & 0 & -2.65 & 127 \\ 0 & 127 & 8125 & 0 & -127 & 4063 \\ -212.5 & 0 & 0 & 212.5 & 0 & 0 \\ 0 & -2.65 & -127 & 0 & 2.65 & -127 \\ 0 & 127 & 4063 & 0 & -127 & 8125 \end{bmatrix} \end{array}$$

where i=3, j=1 for element 2 and i=4, j=2 for element 3.

In general, the transformation matrix **T** is,

$$\mathbf{T} = \begin{bmatrix} l & m & 0 & 0 & 0 & 0 \\ -m & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & 0 \\ 0 & 0 & 0 & -m & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We have

$$l = 0, \; m = 1$$

for both elements 2 and 3. Thus,

$$\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Using the transformation relation,

$$\mathbf{k} = \mathbf{T}^T \mathbf{k}' \mathbf{T}$$

we obtain the stiffness matrices in the *global coordinate system* for elements 2 and 3,

$$\mathbf{k}_2 = 10^4 \times \begin{array}{c} \begin{matrix} u_3 & v_3 & \theta_3 & u_1 & v_1 & \theta_1 \end{matrix} \\ \begin{bmatrix} 2.65 & 0 & -127 & -2.65 & 0 & -127 \\ 0 & 212.5 & 0 & 0 & -212.5 & 0 \\ -127 & 0 & 8125 & 127 & 0 & 4063 \\ -2.65 & 0 & 127 & 2.65 & 0 & 127 \\ 0 & -212.5 & 0 & 0 & 212.5 & 0 \\ -127 & 0 & 4063 & 127 & 0 & 8125 \end{bmatrix} \end{array}$$

and

$$\mathbf{k}_3 = 10^4 \times \begin{array}{c} \begin{matrix} u_4 & v_4 & \theta_4 & u_2 & v_2 & \theta_2 \end{matrix} \\ \begin{bmatrix} 2.65 & 0 & -127 & -2.65 & 0 & -127 \\ 0 & 212.5 & 0 & 0 & -212.5 & 0 \\ -127 & 0 & 8125 & 127 & 0 & 4063 \\ -2.65 & 0 & 127 & 2.65 & 0 & 127 \\ 0 & -212.5 & 0 & 0 & 212.5 & 0 \\ -127 & 0 & 4063 & 127 & 0 & 8125 \end{bmatrix} \end{array}$$

Assembling the global FE equation and noticing the following boundary conditions,

$$\begin{aligned} & u_3 = v_3 = \theta_3 = u_4 = v_4 = \theta_4 = 0 \\ & F_{1X} = 3000\text{lb},\ F_{2X} = 0,\ F_{1Y} = F_{2Y} = -3000\text{lb}, \\ & M_1 = -72000\text{lb}\cdot\text{in},\ M_2 = 72000\text{lb}\cdot\text{in.} \end{aligned}$$

we obtain the condensed

$$10^4 \times \begin{bmatrix} 144.3 & 0 & 127 & -141.7 & 0 & 0 \\ 0 & 213.3 & 56.4 & 0 & -0.784 & 56.4 \\ 127 & 56.4 & 13542 & 0 & -56.4 & 2708 \\ -141.7 & 0 & 0 & 144.3 & 0 & 127 \\ 0 & -0.784 & -56.4 & 0 & 213.3 & -56.4 \\ 0 & 56.4 & 2708 & 127 & -56.4 & 13542 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 3000 \\ -3000 \\ -72000 \\ 0 \\ -3000 \\ 72000 \end{Bmatrix}$$

Solving this, we get

$$\begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0.0 \\ -0.00104\,\text{in.} \\ -0.00139\,\text{rad} \\ 0.0901\,\text{in.} \\ -0.0018\,\text{in.} \\ -3.88 \times 10^{-5}\,\text{rad} \end{Bmatrix}$$

To calculate the reaction forces and moments at the two ends, we employ the element FE equations for element 2 and element 3. We obtain,

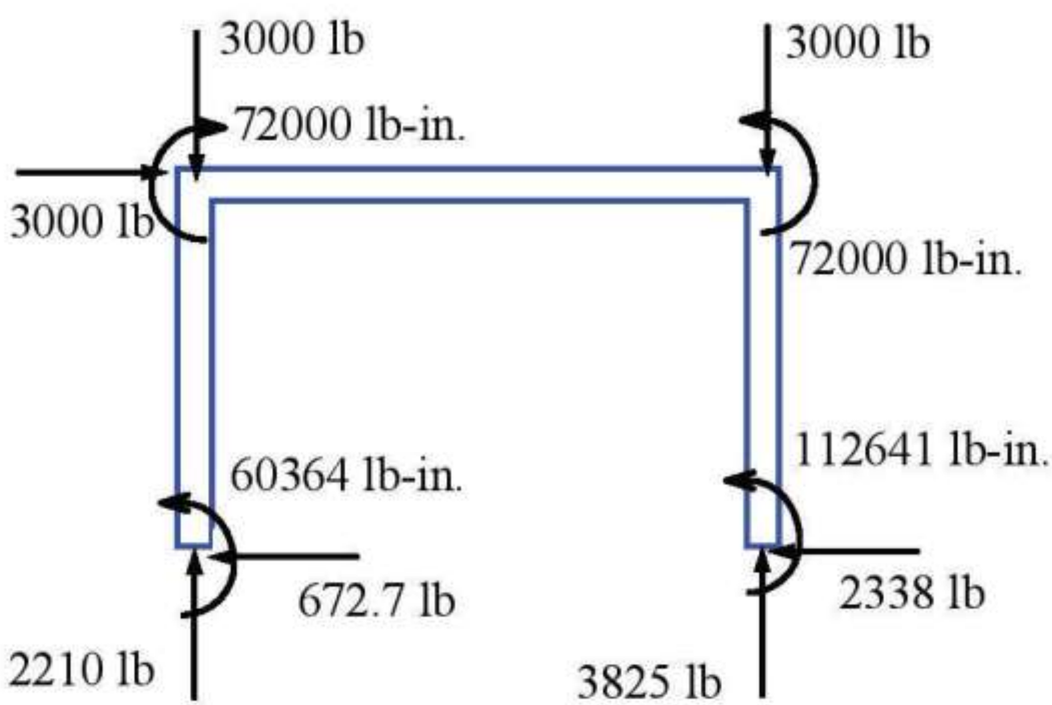
$$\begin{Bmatrix} F_{3X} \\ F_{3Y} \\ M_3 \end{Bmatrix} = \begin{Bmatrix} -672.7\,\text{lb} \\ 2210\ \text{lb} \\ 60364\ \text{lb}\cdot\text{in.} \end{Bmatrix}$$

and

$$\begin{Bmatrix} F_{4X} \\ F_{4Y} \\ M_4 \end{Bmatrix} = \begin{Bmatrix} -2338\,\text{lb} \\ 3825\ \text{lb} \\ 112641\ \text{lb}\cdot\text{in.} \end{Bmatrix}$$

Check the results:

Draw the free-body diagram of the frame. Equilibrium is maintained with the calculated forces and moments.



Chapter 3. Two-Dimensional Problems

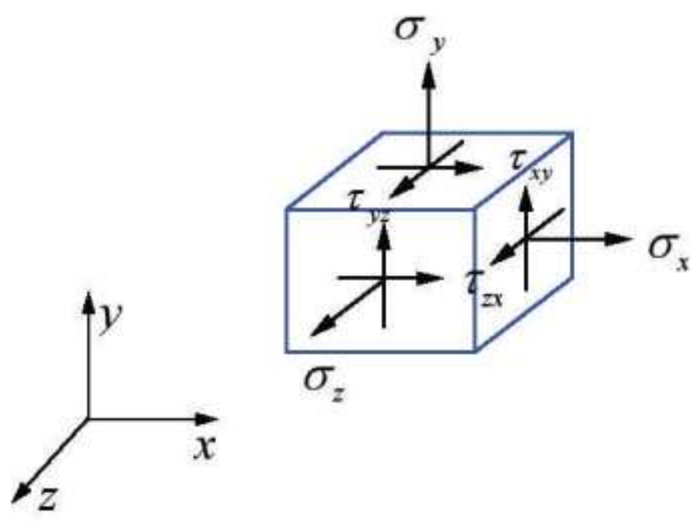
I. Review of the Basic Theory

In general, the stresses and strains in a structure consist of six components:

$$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx} \qquad \text{for stresses,}$$

and

$$\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx} \qquad \text{for strains.}$$



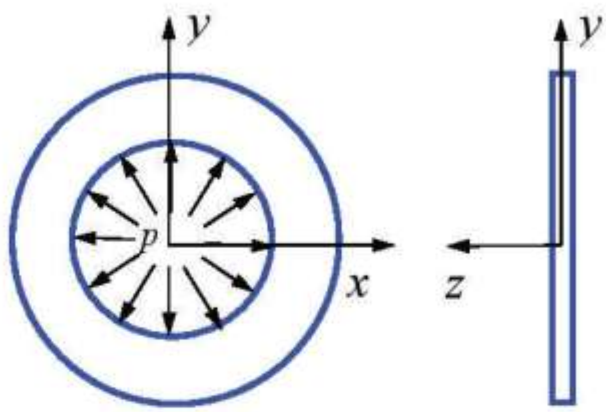
Under contain conditions, the state of stresses and strains can be simplified. A general 3-D structure analysis can, therefore, be reduced to a 2-D analysis.

Plane (2-D) Problems

- *Plane stress*:

$$\sigma_z = \tau_{yz} = \tau_{zx} = 0 \qquad (\varepsilon_z \neq 0) \tag{1}$$

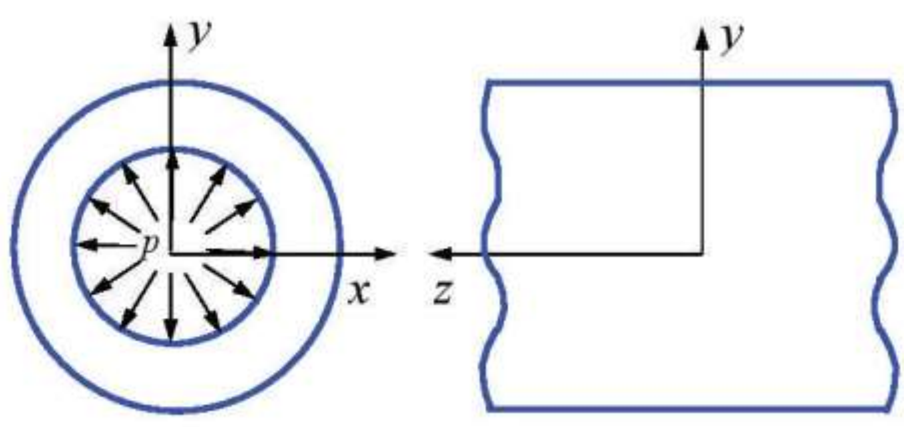
A thin planar structure with constant thickness and loading within the plane of the structure (*xy*-plane).



- *Plane strain*:

$$\varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0 \qquad (\sigma_z \neq 0) \tag{2}$$

A long structure with a uniform cross section and transverse loading along its length (*z*-direction).



Stress-Strain-Temperature (Constitutive) Relations

For elastic and isotropic materials, we have,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & 0 \\ -\nu/E & 1/E & 0 \\ 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} + \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix} \quad (3)$$

or,

$$\varepsilon = \mathbf{E}^{-1}\sigma + \varepsilon_0$$

where ε_0 is the initial strain, E the Young's modulus, ν the Poisson's ratio and G the shear modulus. Note that,

$$G = \frac{E}{2(1+\nu)} \quad (4)$$

which means that there are only two independent materials constants for *homogeneous* and *isotropic* materials.

We can also express stresses in terms of strains by solving the above equation,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix} \right) \quad (5)$$

or,

$$\sigma = \mathbf{E}\varepsilon + \sigma_0$$

where $\sigma_0 = -\mathbf{E}\varepsilon_0$ is the initial stress.

The above relations are valid for *plane stress* case. For *plane strain* case, we need to replace the material constants in the above equations in the following fashion,

$$
\begin{aligned}
&E \rightarrow \frac{E}{1-\nu^2} \\
&\nu \rightarrow \frac{\nu}{1-\nu} \\
&G \rightarrow G
\end{aligned}
\tag{6}
$$

For example, the stress is related to strain by

$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix} \right)
$$

in the *plane strain* case.

Initial strains due to *temperature change* (thermal loading) is given by,

$$
\begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix} = \begin{Bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{Bmatrix} \tag{7}
$$

where α is the coefficient of thermal expansion, ΔT the change of temperature. Note that if the structure is free to deform under thermal loading, there will be no (elastic) stresses in the structure.

Strain and Displacement Relations

For small strains and small rotations, we have,

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

In matrix form,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \partial / \partial x & 0 \\ 0 & \partial / \partial y \\ \partial / \partial y & \partial / \partial x \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}, \quad \text{or} \quad \varepsilon = \mathbf{Du} \tag{8}$$

From this relation, we know that the strains (and thus stresses) are one order lower than the displacements, if the displacements are represented by polynomials.

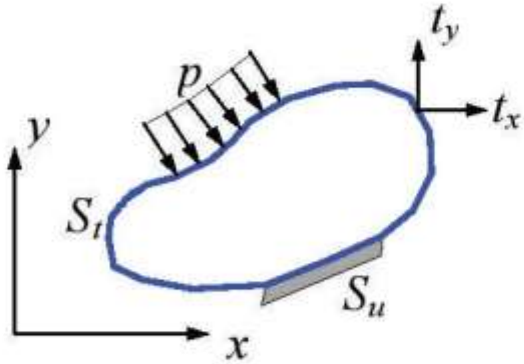
Equilibrium Equations

In elasticity theory, the stresses in the structure must satisfy the following equilibrium equations,

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + f_x &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y &= 0 \end{aligned} \tag{9}$$

where f_x and f_y are body forces (such as gravity forces) per unit volume. In FEM, these equilibrium conditions are satisfied in an approximate sense.

Boundary Conditions



The boundary S of the body can be divided into two parts, S_u and S_t. The boundary conditions (BC's) are described as,

$$\begin{aligned} & u = \bar{u}, \quad v = \bar{v}, && \text{on } S_u \\ & t_x = \bar{t}_x, \quad t_y = \bar{t}_y, && \text{on } S_t \end{aligned} \tag{10}$$

in which t_x and t_y are traction forces (stresses on the boundary) and the barred quantities are those with known values.

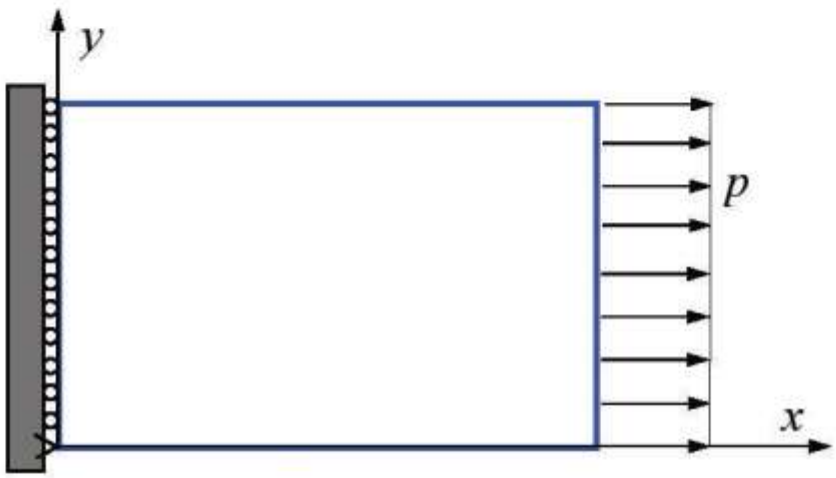
In FEM, all types of loads (distributed surface loads, body forces, concentrated forces and moments, etc.) are converted to point forces acting at the nodes.

Exact Elasticity Solution

The exact solution (displacements, strains and stresses) of a given problem must satisfy the equilibrium equations (9), the given boundary conditions (10) and compatibility conditions (structures should deform in a continuous manner, no cracks or overlaps in the obtained displacement fields).

Example 3.1

A plate is supported and loaded with distributed force p as shown in the figure. The material constants are E and ν.



The exact solution for this simple problem can be found easily as follows,

Displacement:

$$u = \frac{p}{E}x, \qquad v = -\nu\frac{p}{E}y$$

Strain:

$$\varepsilon_x = \frac{p}{E}, \qquad \varepsilon_y = -\nu\frac{p}{E}, \qquad \gamma_{xy} = 0$$

Stress:

$$\sigma_x = p, \qquad \sigma_y = 0, \qquad \tau_{xy} = 0$$

Exact (or analytical) solutions for *simple* problems are numbered (suppose there is a hole in the plate!). That is why we need FEM!

II. Finite Elements for 2-D Problems

A General Formula for the Stiffness Matrix

Displacements (u, v) in a plane element are interpolated from nodal displacements (u_i, v_i) using shape functions N_i as follows,

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots \\ 0 & N_1 & 0 & N_2 & \cdots \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \end{Bmatrix} \quad \text{or} \quad \mathbf{u} = \mathbf{Nd} \quad (11)$$

where **N** is the *shape function matrix*, **u** the displacement vector and **d** the *nodal* displacement vector. Here we have assumed that u depends on the nodal values of u only, and v on nodal values of v only.

From strain-displacement relation (Eq.(8)), the strain vector is,

$$\varepsilon = \mathbf{Du} = \mathbf{DNd}, \qquad \text{or} \qquad \varepsilon = \mathbf{Bd} \qquad (12)$$

where $\mathbf{B} = \mathbf{DN}$ is the *strain-displacement matrix*.

Consider the strain energy stored in an element,

$$
\begin{aligned}
U &= \frac{1}{2}\int_V \sigma^T \varepsilon \, dV = \frac{1}{2}\int_V \left(\sigma_x \, \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy}\gamma_{xy}\right) dV \\
&= \frac{1}{2}\int_V (\mathbf{E}\varepsilon)^T \varepsilon \, dV = \frac{1}{2}\int_V \varepsilon^T \mathbf{E} \varepsilon \, dV \\
&= \frac{1}{2}\mathbf{d}^T \int_V \mathbf{B}^T \mathbf{E} \mathbf{B} \, dV \mathbf{d} \\
&= \frac{1}{2}\mathbf{d}^T \mathbf{k} \mathbf{d}
\end{aligned}
$$

From this, we obtain the general formula for the *element stiffness matrix*,

$$
\mathbf{k} = \int_V \mathbf{B}^T \mathbf{E} \mathbf{B} \, dV \tag{13}
$$

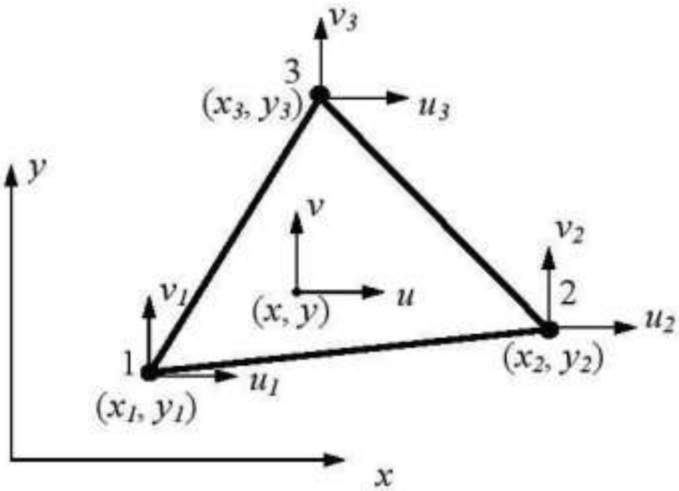
Note that unlike the 1-D cases, **E** here is a *matrix* which is given by the stress-strain relation (e.g., Eq.(5) for plane stress).

The stiffness matrix **k** defined by (13) is symmetric since **E** is symmetric. Also note that given the material property, the behavior of **k** depends on the **B** matrix only, which in turn on the shape functions. Thus, the quality of finite elements in representing the behavior of a structure is entirely determined by the choice of shape functions.

Most commonly employed 2-D elements are linear or quadratic triangles and quadrilaterals.

Constant Strain Triangle (CST or T3)

This is the simplest 2-D element, which is also called ***linear triangular element***.



Linear Triangular Element

For this element, we have three nodes at the vertices of the triangle, which are numbered around the element in the counterclockwise direction. Each node has two degrees of freedom (can move in the *x* and *y* directions). The displacements *u* and *v* are assumed to be linear functions within the element, that is,

$$u = b_1 + b_2 x + b_3 y, \quad v = b_4 + b_5 x + b_6 y \tag{14}$$

where b_i (i = 1, 2, ..., 6) are constants. From these, the strains are found to be,

$$\varepsilon_x = b_2, \quad \varepsilon_y = b_6, \quad \gamma_{xy} = b_3 + b_5 \tag{15}$$

which are constant throughout the element. Thus, we have the name "constant strain triangle" (CST).

Displacements given by (14) should satisfy the following six equations,

$$
\begin{aligned}
u_1 &= b_1 + b_2 x_1 + b_3 y_1 \\
u_2 &= b_1 + b_2 x_2 + b_3 y_2 \\
&\vdots \\
v_3 &= b_4 + b_5 x_3 + b_6 y_3
\end{aligned}
$$

Solving these equations, we can find the coefficients b_1, b_2, ..., and b_6 in terms of nodal displacements and coordinates. Substituting these coefficients into (14) and rearranging the terms, we obtain,

$$
\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \tag{16}
$$

where the shape functions (linear functions in x and y) are

$$
\begin{aligned}
N_1 &= \frac{1}{2A}\{(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x + (x_3 - x_2)y\} \\
N_2 &= \frac{1}{2A}\{(x_3 y_1 - x_1 y_3) + (y_3 - y_1)x + (x_1 - x_3)y\} \\
N_3 &= \frac{1}{2A}\{(x_1 y_2 - x_2 y_1) + (y_1 - y_2)x + (x_2 - x_1)y\}
\end{aligned} \tag{17}
$$

and

$$A = \frac{1}{2}\det\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \tag{18}$$

is the area of the triangle (Prove this!).

Using the strain-displacement relation (8), results (16) and (17), we have,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \mathbf{Bd} = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \tag{19}$$

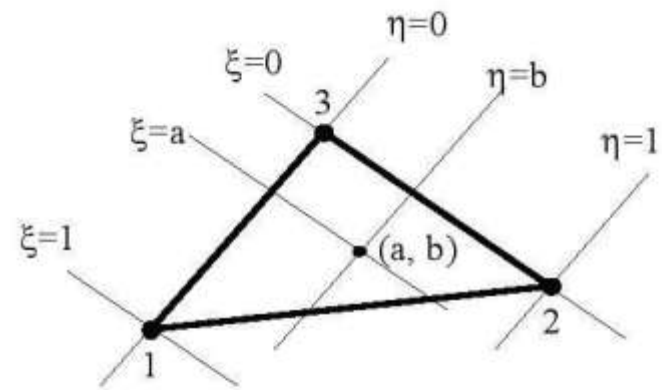
where $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$ $(i, j = 1, 2, 3)$. Again, we see constant strains within the element. From stress-strain relation (Eq.(5), for example), we see that stresses obtained using the CST element are also constant.

Applying formula (13), we obtain the element stiffness matrix for the CST element,

$$\mathbf{k} = \int_V \mathbf{B}^T\mathbf{E}\mathbf{B}\,dV = tA(\mathbf{B}^T\mathbf{E}\mathbf{B}) \tag{20}$$

in which t is the thickness of the element. Notice that $\mathbf{k}$ for CST is a 6 by 6 *symmetric* matrix. The matrix multiplication in (20) can be carried out by a computer program.

Both the expressions of the shape functions in (17) and their derivations are lengthy and offer little insight into the behavior of the element.



The Natural Coordinates

We introduce the *natural coordinates* (ξ,η) on the triangle, then *the shape functions* can be represented simply by,

$$N_1 = \xi, \quad N_2 = \eta, \quad N_3 = 1 - \xi - \eta \tag{21}$$

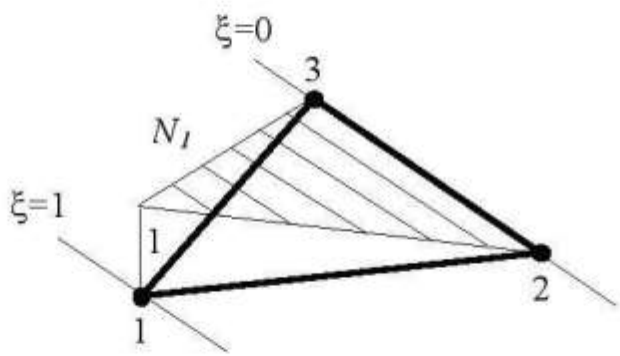
Notice that,

$$N_1 + N_2 + N_3 = 1 \tag{22}$$

which ensures that the rigid body translation is represented by the chosen shape functions. Also, as in the 1-D case,

$$N_i = \begin{cases} 1, & \text{at node i;} \\ 0, & \text{at the other nodes} \end{cases} \tag{23}$$

and varies linearly within the element. The plot for shape function N_1 is shown in the following figure. N_2 and N_3 have similar features.



Shape Function N_1 for CST

We have two coordinate systems for the element: the global coordinates (x, y) and the natural coordinates (ξ, η). The relation between the two is given by

$$\begin{aligned} x &= N_1 x_1 + N_2 x_2 + N_3 x_3 \\ y &= N_1 y_1 + N_2 y_2 + N_3 y_3 \end{aligned} \tag{24}$$

or,

$$\begin{aligned} x &= x_{13}\xi + x_{23}\eta + x_3 \\ y &= y_{13}\xi + y_{23}\eta + y_3 \end{aligned} \tag{25}$$

where $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$ $(i, j = 1, 2, 3)$ as defined earlier.

Displacement u or v on the element can be viewed as functions of (x, y) or (ξ, η). Using the chain rule for derivatives, we have,

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} \tag{26}$$

where $\mathbf{J}$ is called the *Jacobian matrix* of the transformation.

From (25), we calculate,

$$\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}, \qquad \mathbf{J}^{-1} = \frac{1}{2A}\begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \tag{27}$$

where $\det \mathbf{J} = x_{13}y_{23} - x_{23}y_{13} = 2A$ has been used (A is the area of the triangular element. Prove this!).

From (26), (27), (16) and (21) we have,

$$\begin{aligned} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} &= \frac{1}{2A}\begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} \\ &= \frac{1}{2A}\begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} u_1 - u_3 \\ u_2 - u_3 \end{Bmatrix} \end{aligned} \tag{28}$$

Similarly,

$$\begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} v_1 - v_3 \\ v_2 - v_3 \end{Bmatrix} \tag{29}$$

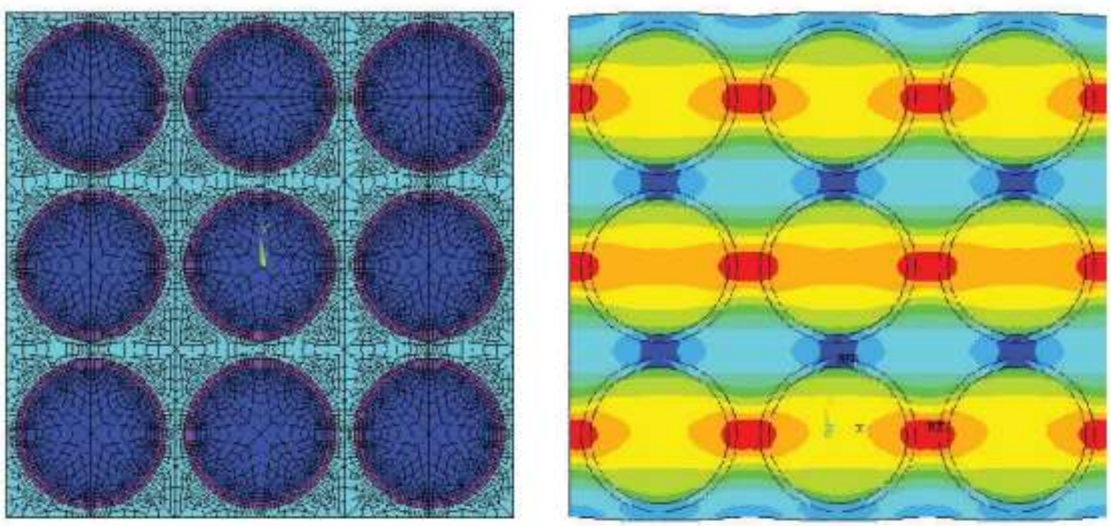
Using the results in (28) and (29), and the relations $\varepsilon = \mathbf{Du} = \mathbf{DNd} = \mathbf{Bd}$, we obtain the strain-displacement matrix,

$$\mathbf{B} = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \tag{30}$$

which is the same as we derived earlier in (19).

Applications of the CST Element:

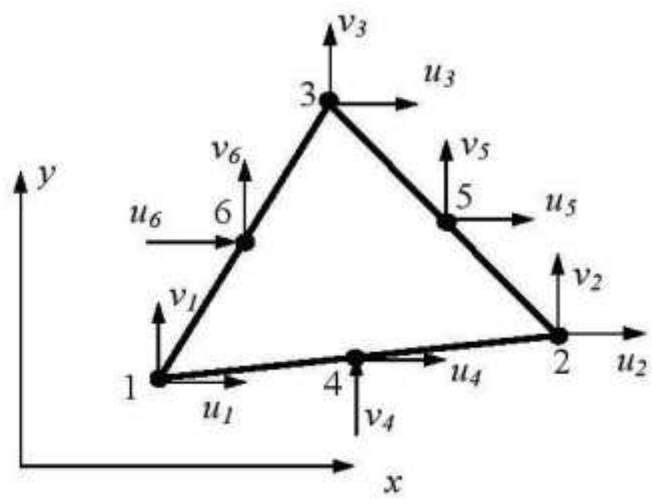
- Use in areas where the strain gradient is small.
- Use in mesh transition areas (fine mesh to coarse mesh).
- Avoid using CST in stress concentration or other crucial areas in the structure, such as edges of holes and corners.
- Recommended for quick and preliminary FE analysis of 2-D problems.



Analysis of composite materials (for which the CST is NOT appropriate!)

Linear Strain Triangle (LST or T6)

This element is also called ***quadratic triangular element***.



Quadratic Triangular Element

There are six nodes on this element: three corner nodes and three midside nodes. Each node has two degrees of freedom (DOF) as before. The displacements (u, v) are assumed to be quadratic functions of (x, y),

$$\begin{aligned} u &= b_1 + b_2 x + b_3 y + b_4 x^2 + b_5 xy + b_6 y^2 \\ v &= b_7 + b_8 x + b_9 y + b_{10} x^2 + b_{11} xy + b_{12} y^2 \end{aligned} \tag{31}$$

where b_i (i = 1, 2, ..., 12) are constants. From these, the strains are found to be,

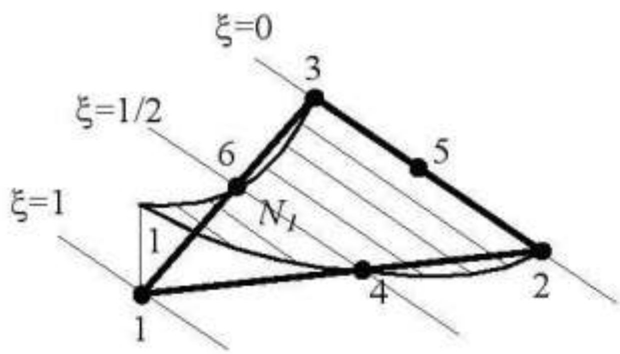
$$\begin{aligned} \varepsilon_x &= b_2 + 2b_4 x + b_5 y \\ \varepsilon_y &= b_9 + b_{11} x + 2b_{12} y \\ \gamma_{xy} &= (b_3 + b_8) + (b_5 + 2b_{10})x + (2b_6 + b_{11})y \end{aligned} \tag{32}$$

which are linear functions. Thus, we have the "linear strain triangle" (LST), which provides better results than the CST.

In the natural coordinate system we defined earlier, the six shape functions for the LST element are,

$$
\begin{aligned}
N_1 &= \xi(2\xi - 1) \\
N_2 &= \eta(2\eta - 1) \\
N_3 &= \zeta(2\zeta - 1) \\
N_4 &= 4\xi\eta \\
N_5 &= 4\eta\zeta \\
N_6 &= 4\zeta\xi
\end{aligned}
\tag{33}
$$

in which $\zeta = 1 - \xi - \eta$. Each of these six shape functions represents a quadratic form on the element as shown in the figure.



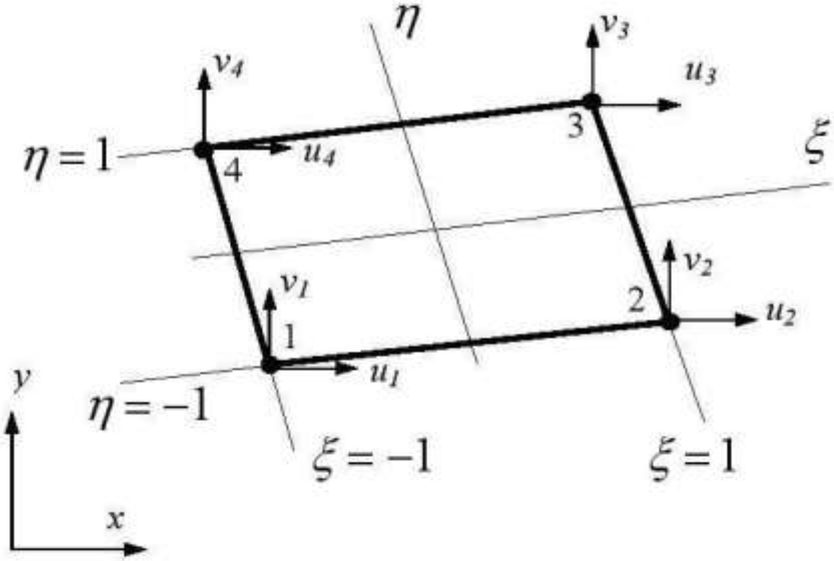
Shape Function N_1 for LST

Displacements can be written as,

$$
u = \sum_{i=1}^{6} N_i u_i, \qquad v = \sum_{i=1}^{6} N_i v_i \tag{34}
$$

The element stiffness matrix is still given by $\mathbf{k} = \int_V \mathbf{B}^T \mathbf{E} \mathbf{B} \, dV$, but here $\mathbf{B}^T\mathbf{E}\mathbf{B}$ is quadratic in x and y. In general, the integral has to be computed numerically.

Linear Quadrilateral Element (Q4)



Linear Quadrilateral Element

There are four nodes at the corners of the quadrilateral shape. In the natural coordinate system (ξ, η), the four shape functions are,

$$\begin{aligned} N_1 &= \frac{1}{4}(1-\xi)(1-\eta), \qquad N_2 = \frac{1}{4}(1+\xi)(1-\eta) \\ N_3 &= \frac{1}{4}(1+\xi)(1+\eta), \qquad N_4 = \frac{1}{4}(1-\xi)(1+\eta) \end{aligned} \tag{35}$$

Note that $\sum_{i=1}^{4} N_i = 1$ at any point inside the element, as expected.

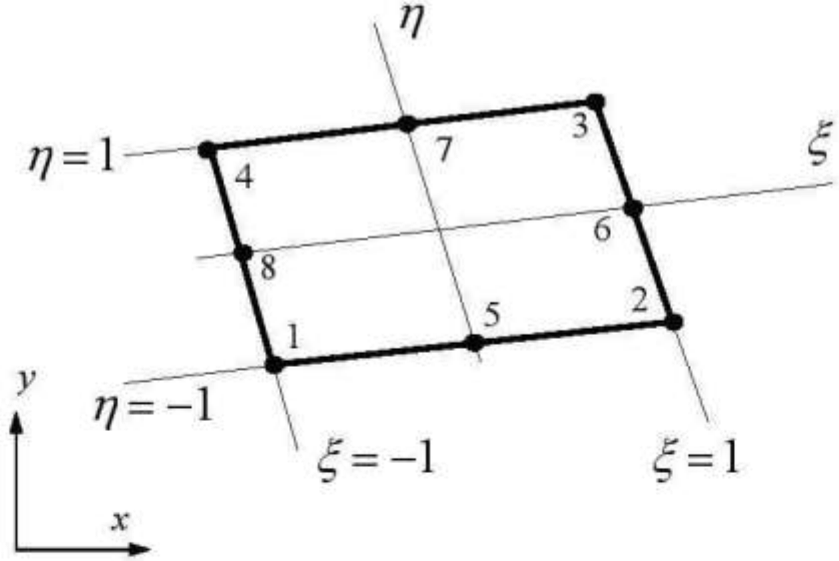
The displacement field is given by

$$u = \sum_{i=1}^{4} N_i u_i, \qquad v = \sum_{i=1}^{4} N_i v_i \tag{36}$$

which are bilinear functions over the element.

Quadratic Quadrilateral Element (Q8)

This is the most widely used element for 2-D problems due to its high accuracy in analysis and flexibility in modeling.



Quadratic Quadrilateral Element

There are eight nodes for this element, four corners nodes and four midside nodes. In the natural coordinate system (ξ, η), the eight shape functions are,

$$
\begin{aligned}
N_1 &= \frac{1}{4}(1-\xi)(\eta-1)(\xi+\eta+1) \\
N_2 &= \frac{1}{4}(1+\xi)(\eta-1)(\eta-\xi+1) \\
N_3 &= \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1) \\
N_4 &= \frac{1}{4}(\xi-1)(\eta+1)(\xi-\eta+1)
\end{aligned}
\qquad (37)
$$

$$N_5 = \frac{1}{2}(1-\eta)(1-\xi^2)$$

$$N_6 = \frac{1}{2}(1+\xi)(1-\eta^2)$$

$$N_7 = \frac{1}{2}(1+\eta)(1-\xi^2)$$

$$N_8 = \frac{1}{2}(1-\xi)(1-\eta^2)$$

Again, we have $\sum_{i=1}^{8} N_i = 1$ at any point inside the element.

The displacement field is given by

$$u = \sum_{i=1}^{8} N_i u_i, \qquad v = \sum_{i=1}^{8} N_i v_i \tag{38}$$

which are quadratic functions over the element. Strains and stresses over a quadratic quadrilateral element are linear functions, which are better representations.

Notes:

- Q4 and T3 are usually used together in a mesh with linear elements.
- Q8 and T6 are usually applied in a mesh composed of quadratic elements.
- Quadratic elements are preferred for stress analysis, because of their high accuracy and the flexibility in modeling complex geometry, such as curved boundaries.