LECTURE NOTES

ON

COMPUTATIONAL MATHEMATICS AND INTEGRAL CALCULUS

I B. Tech I semester

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UNIT-I

Solution of algebraic and Transcendental equations and Interpolation

Solutions of Algebraic and Transcendental equations:

1) **Polynomial function:** A function $f(x)$ is said to be a polynomial function

if $f(x)$ is a polynomial in x.

ie, $f(x) = a_0x^n + a_1x^{n-1} + \cdots \ldots\ldots\ldots.. + a_{n-1}x + a_n$

where $a_0 \neq 0$, the co-efficients $a_0, a_1 \ldots\ldots\ldots. a_n$ are real constants and n is a non-negative integer.

2) **Algebraic function:** A function which is a sum (or) difference (or) product of two polynomials is called an algebraic function. Otherwise, the function is called a transcendental (or) non-algebraic function.

Eg: $f(x) = c_1e^x + c_2e^{-x} = 0$

$$f(x) = e^{5x} - \frac{x^3}{2} + 3 = 0$$

3) **Root of an equation:** A number α is called a root of an equation $f(x) = 0$ if $f(\alpha) = 0$. We also say that α is a zero of the function.

Note: The roots of an equation are the abscissae of the points where the graph $y = f(x)$ cuts the x-axis.

Methods to find the roots of $f(x) = 0$

Direct method:

We know the solution of the polynomial equations such as linear equation $ax + b = 0$, and quadratic equation $ax^2 + bx + c = 0$, using direct methods or analytical methods. Analytical methods for the solution of cubic and quadratic equations are also available.

1) **Bisection method:** Bisection method is a simple iteration method to solve an equation. This method is also known as Bolzono method of successive bisection. Some times it is referred to as half-interval method. Suppose we know an equation of the form $f(x) = 0$ has exactly one real root between two real numbers x_0, x_1. The number is choosen such that $f(x_0)$ and $f(x_1)$ will have opposite sign. Let us bisect the interval $[x_0, x_1]$ into two half intervals and find the mid point $x_2 = \frac{x_0 + x_1}{2}$. If $f(x_2) = 0$ then x_2 is a root. If $f(x_1)$ and $f(x_2)$ have same sign then the root lies between x_0 and x_2. The interval is taken as $[x_0, x_2]$. Otherwise the root lies in the interval $[x_2, x_1]$.

PROBLEMS

1). Find a root of the equation $x^3 - 5x + 1 = 0$ using the bisection method in 5 – stages

Sol Let $f(x) = x^3 - 5x + 1$. We note that $f(0) > 0$ and $f(1) < 0$

$\therefore$ One root lies between 0 and 1

Consider $x_0 = 0$ *and* $x_1 = 1$

By Bisection method the next approximation is

$$x_2 = \frac{x_0 + x_1}{2} = \frac{1}{2}(0+1) = 0.5$$

$$\Rightarrow f(x_2) = f(0:5) = -1.375 < 0 \text{ and } f(0) > 0$$

We have the root lies between 0 and 0.5

Now $x_3 = \dfrac{0+0.5}{2} = 0.25$

We find $f(x_3) = -0.234375 < 0$ *and* $f(0) > 0$

Since $f(0) > 0$, we conclude that root lies between x_0 *and* x_3

The third approximation of the root is

$$x_4 = \frac{x_0 + x_3}{2} = \frac{1}{2}(0 + 0.25) = 0.125$$

We have $f(x_4) = 0.37495 > 0$

Since $f(x_4) > 0$ *and* $f(x_3) < 0$, the root lies between

$x_4 = 0.125$ *and* $x_3 = 0.25$

Considering the 4th approximation of the roots

$$x_5 = \frac{x_3 + x_4}{2} = \frac{1}{2}(0.125 + 0.25) = 0.1875$$

$f(x_5) = 0.06910 > 0$, since $f(x_5) > 0$ *and* $f(x_3) < 0$ the root must lie between

$$x_5 = 0.18758 \text{ and } x_3 = 0.25$$

Here the fifth approximation of the root is

$$x_6 = \frac{1}{2}(x_5 + x_3)$$
$$= \frac{1}{2}(0.1875 + 0.25)$$
$$= 0.21875$$

We are asked to do up to 5 stages

We stop here 0.21875 is taken as an approximate value of the root and it lies between 0 and 1

2) Find a root of the equation $x^3 - 4x - 9 = 0$ using bisection method in four stages

Sol Let $f(x) = x^3 - 4x - 9$

We note that $f(2) < 0$ and $f(3) > 0$

$\therefore$ One root lies between 2 and 3

Consider $x_0 = 2$ *and* $x_1 = 3$

By Bisection method $x_2 = \frac{x_0 + x_1}{2} = 2.5$

Calculating $f(x_2) = f(2.5) = -3.375 < 0$

$\therefore$ The root lies between x_2 *and* x_1

The second approximation is $x_3 = \frac{1}{2}(x_1 + x_2) = \frac{2.5+3}{2} = 2.75$

Now $f(x_3) = f(2.75) = 0.7969 > 0$

$\therefore$ The root lies between x_2 *and* x_3

Thus the third approximation to the root is

$$x_4 = \frac{1}{2}(x_2 + x_3) = 2.625$$

Again $f(x_4) = f(2.625) = -1.421 < 0$

$\therefore$ The root lies between x_3 *and* x_4

Fourth approximation is$x_5 = \frac{1}{2}(x_3 + x_4) = \frac{1}{2}(2.75 + 2.625) = 2.6875$

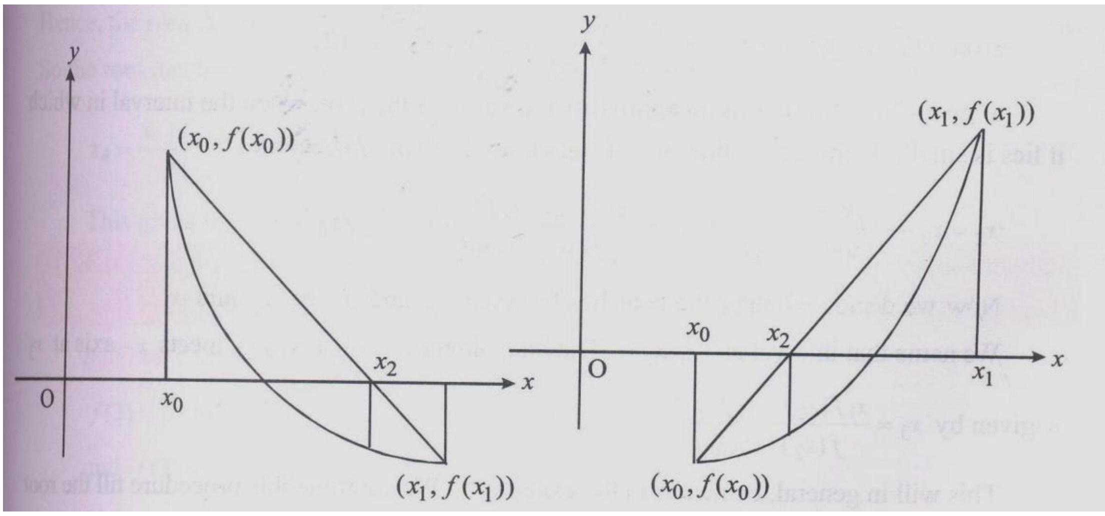
False Position Method (Regula – Falsi Method)

In the false position method we will find the root of the equation $f(x) = 0$ Consider two initial approximate values x_0 *and* x_1 near the required root so that $f(x_0)$ *and* $f(x_1)$ have different signs. This implies that a root lies between x_0 *and* x_1. The curve $f(x)$ crosses x-axis only once at the point x_2 lying between the points x_0 andx_1. Consider the point $A = (x_0, f(x_0))$ and $B = (x_1, f(x_1))$ on the graph and suppose they are connected by a straight line. Suppose this line cuts x-axis atx_2. We calculate the value of $f(x_2)$ at the point. If $f(x_0)$ *and* $f(x_2)$ are of opposite signs, then the root lies between x_0 *and* x_2 and value x_1 is replaced by x_2

Other wise the root lies between x_2 and x_1 and the value of x_0 is replaced byx_2.

Another line is drawn by connecting the newly obtained pair of values.

Again the point here cuts the x-axis is a closer approximation to the root. This process is repeated as many times as required to obtain the desired accuracy. It can be observed that the points $x_2, x_3, x_4, \ldots$obtained converge to the expected root of the equation $y = f(x)$



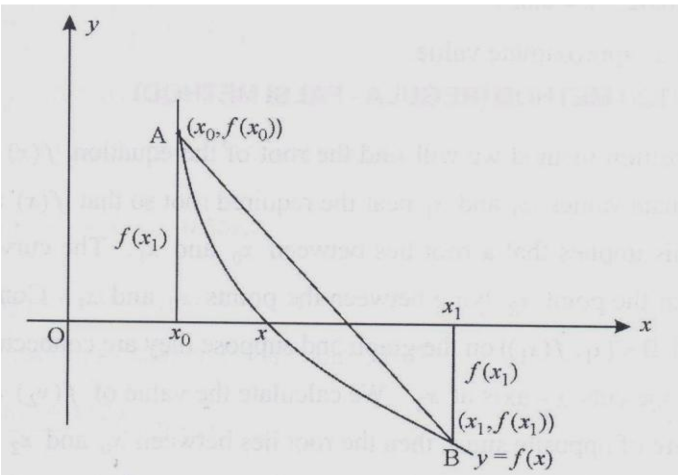
To Obtain the equation to find the next approximation to the root

Let $A=(x_0, f(x_0))$ *and* $B=(x_1, f(x_1))$ be the points on the curve $y=f(x)$ Then the equation to the chord AB is $\frac{y-f(x_0)}{x-x_0}=\frac{f(x_1)-f(x_0)}{x_1-x_0}$ ------(1)

At the point C where the line AB crosses the x – axis, where $f(x)=0$ $ie,$ $y=0$

From (1), we get $x=x_0-\frac{x_1-x_0}{f(x_1)-f(x_0)}f(x_0)$ --------------(2)

x is given by (2) serves as an approximated value of the root, when the interval in which it lies is small. If the new value of x is taken as x_2 then (2) becomes



$$x_2=x_0-\frac{(x_1-x_0)}{f(x_1)-f(x_0)}f(x_0)$$

$$=\frac{x_0 f(x_1)-x_1 f(x_0)}{f(x_1)-f(x_0)} \text{ -------------(3)}$$

Now we decide whether the root lies between

x_0 *and* x_2 (or) x_2 *and* x_1

We name that interval as (x_1, x_2) The line joining $(x_1, y_1), (x_2, y_2)$ meets x – axis at x_3 is given by $x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

This will in general, be nearest to the exact root. We continue this procedure till the root is found to the desired accuracy

The iteration process based on (3) is known as the method of false position

The successive intervals where the root lies, in the above procedure are named as

$(x_0, x_1), (x_1, x_2), (x_2, x_3)$ etc

Where $x_i < x_{i+1}$ and $f(x_0), f(x_{i+1})$ are of opposite signs.

Also $x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$

PROBLEMS:

1. **By using Regula - Falsi method, find an approximate root of the equation** $x^4 - x - 10 = 0$ **that lies between 1.8 and 2. Carry out three approximations**

Sol.Let us take $f(x) = x^4 - x - 10$ and $x_0 = 1.8, x_1 = 2$

Then $f(x_0) = f(1.8) = -1.3 < 0$ and $f(x_1) = f(2) = 4 > 0$

Since $f(x_0)$ and $f(x_1)$ are of opposite signs,the equation $f(x) = 0$ has a root between x_0 *and* x_1

The first order approximation of this root is

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 1.8 - \frac{2 - 1.8}{4 + 1.3} \times (-1.3)$$

$$= 1.849$$

We find that $f(x_2) = -0.161$ so that $f(x_2)$ *and* $f(x_1)$ are of opposite signs. Hence the root lies between x_2 *and* x_1 and the second order approximation of the root is

$$x_3 = x_2 - \left[\frac{x_1 - x_2}{f(x_1) - f(x_2)}\right] . f(x_2)$$

$$= 1.8490 - \left[\frac{2 - 1.849}{0.159}\right] \times (-0.159)$$

$$= 1.8548$$

We find that $f(x_3) = f(1.8548)$

$$= -0.019$$

So that $f(x_3)$ *and* $f(x_2)$ are of the same sign. Hence, the root does not lie between x_2 *and* x_3.But $f(x_3)$ *and* $f(x_1)$ are of opposite signs. So the root lies between x_3 *and* x_1 and the third order approximate value of the root is $x_4 = x_3 - \left[\frac{x_1 - x_3}{f(x_1) - f(x_3)}\right] f(x_3)$

$$= 1.8548 - \frac{2 - 1.8548}{4 + 0.019} \times (-0.019)$$

$= 1.8557$

This gives the approximate value of x.

2. **Find out the roots of the equation $x^3 - x - 4 = 0$ using False position method**

Sol. Let $f(x) = x^3 - x - 4 = 0$

Then $f(0) = -4, f(1) = -4, f(2) = 2$

Since $f(1)$ *and* $f(2)$ have opposite signs the root lies between 1 and 2

By False position method $x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

$$x_2 = \frac{(1 \times 2) - 2(-4)}{2 - (-4)}$$

$$= \frac{2+8}{6} = \frac{10}{6} = 1.666$$

$$f(1.666) = (1.666)^3 - 1.666 - 4$$
$$= -1.042$$

Now, the root lies between 1.666 and 2

$$x_3 = \frac{1.666 \times 2 - 2 \times (-1.042)}{2 - (-1.042)} = 1.780$$

$$f(1.780) = (1.780)^3 - 1.780 - 4$$
$$= -0.1402$$

Now, the root lies between 1.780 and 2

$$x_4 = \frac{1.780 \times 2 - 2 \times (-0.1402)}{2 - (-0.1402)} = 1.794$$

$$f(1.794) = (1.794)^3 - 1.794 - 4$$
$$= -0.0201$$

Now, the root lies between 1.794 and 2

$$x_5 = \frac{1.794 \times 2 - 2 \times (-0.0201)}{2 - (-0.0201)} = 1.796$$

$$f(1.796) = (1.796)^3 - 1.796 - 4 = -0.0027$$

Now, the root lies between 1.796 and 2

$$x_6 = \frac{1.796 \times 2 - 2 \times (-0.0027)}{2 - (-0.0027)} = 1.796 \qquad \text{The root is } 1.796$$

Newton- Raphson Method:-

The Newton- Raphson method is a powerful and elegant method to find the root of an equation. This method is generally used to improve the results obtained by the previous methods.

Let x_0 be an approximate root of $f(x) = 0$ and let $x_1 = x_0 + h$ be the correct root which implies that $f(x_1) = 0$. We use Taylor's theorem and expand $f(x_1) = f(x_0 + h) = 0$

$$\Rightarrow f(x_0) + hf^1(x_0) = 0$$

$$\Rightarrow h = -\frac{f(x_0)}{f^1(x_0)}$$

Substituting this in x_1 ,we get

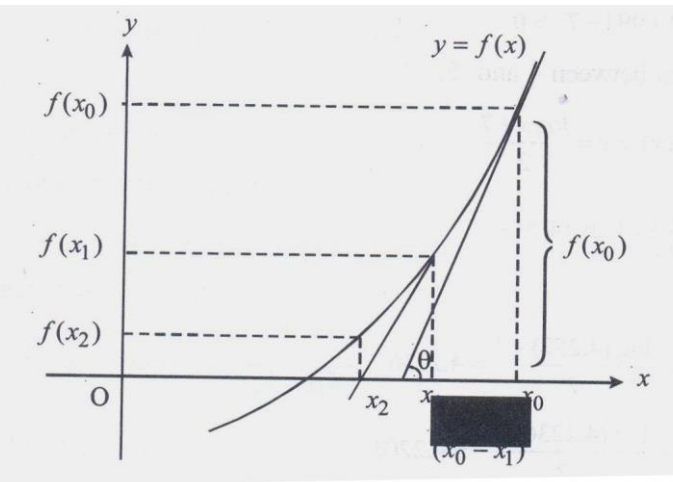
$$x_1 = x_0 + h$$

$$= x_0 - \frac{f(x_0)}{f^1(x_0)}$$

$\therefore x_1$ is a better approximation than x_0

Successive approximations are given by

$x_2, x_3 \ldots \ldots \ldots \ldots x_{n+1}$ where $x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)}$



Problems:

1. Apply Newton – Raphson method to find an approximate root, correct to three decimal places, of the equation $x^3 - 3x - 5 = 0,$ which lies near $x = 2$

Sol:- Here $f(x) = x^3 - 3x - 5 = 0$ *and* $f^1(x) = 3(x^2 - 1)$

$\therefore$ The Newton – Raphson iterative formula

$$x_{i+1} = x_i - \frac{x_i^3 - 3x_i - 5}{3\left(x_i^2 - 1\right)} = \frac{2x_i^3 + 5}{3\left(x_i^2 - 1\right)}, i = 0,1,2....(1)$$

To find the root near $x = 2$, we take $x_0 = 2$ then (1) gives

$$x_1 = \frac{2x_0^3 + 5}{3\left(x_0^2 - 1\right)} = \frac{16 + 5}{3(4 - 1)} = \frac{21}{9} = 2.3333$$

$$x_2 = \frac{2x_1^3 + 5}{3\left(x_1^2 - 1\right)} = \frac{2 \times (2.3333)^3 + 5}{3\left[(2.3333)^2 - 1\right]} = 2.2806$$

$$x_3 = \frac{2x_2^3 + 5}{3(x_2^3 - 1)} = \frac{2 \times (2.2806)^3 + 5}{3[(2.2806)^2 - 1]} = 2.2790$$

$$x_4 = \frac{2 \times (2.2790)^3 + 5}{3[(2.2790)^2 - 1]} = 2.2790$$

Since x_3 and x_4 are identical up to 3 places of decimal, we take $x_4 = 2.279$ as the required root, correct to three places of the decimal

2. Using Newton – Raphson method

a) Find square root of a number

b) Find reciprocal of a number

Sol. a) **Square root:-**

Let $f(x) = x^2 - N = 0$, where N is the number whose square root is to be found. The solution to $f(x)$ is then $x = \sqrt{N}$

Here $f'(x) = 2x$

By Newton-Raphson technique

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)} = x_i - \frac{x_i^2 - N}{2x_i}$$

$$\Rightarrow x_{i+1} = \frac{1}{2}\left[x_i + \frac{N}{x_i}\right]$$

Using the above iteration formula the square root of any number N can be found to any desired accuracy. For example, we will find the square root of $N = 24$.

Let the initial approximation be $x_0 = 4.8$

$$x_1 = \frac{1}{2}\left(4.8 + \frac{24}{4.8}\right) = \frac{1}{2}\left(\frac{23.04 + 24}{4.8}\right) = \frac{47.04}{9.6} = 4.9$$

$$x_2 = \frac{1}{2}\left(4.9 + \frac{24}{4.9}\right) = \frac{1}{2}\left(\frac{24.01 + 24}{4.9}\right) = \frac{48.01}{9.8} = 4.898$$

$$x_3 = \frac{1}{2}\left(4.898 + \frac{24}{4.898}\right) = \frac{1}{2}\left(\frac{23.9904 + 24}{4.898}\right) = \frac{47.9904}{9.796} = 4.898$$

Since $x_2 = x_3$, there fore the solution to $f(x) = x^2 - 24 = 0$ is 4.898. That means, the square root of 24 is 4.898

b) Reciprocal:-

Let $f(x)=\frac{1}{x}-N=0$ where N is the number whose reciprocal is to be found

The solution to $f(x)$ is then $x=\frac{1}{N}$. Also, $f^1(x)=\frac{-1}{x^2}$

To find the solution for $f(x)=0$, apply Newton – Raphson method

$$x_{i+1}=x_i-\frac{\left(\frac{1}{x_i}-N\right)}{-1/x_i^2}=x_i(2-x_iN)$$

For example, the calculation of reciprocal of 22 is as follows

Assume the initial approximation be $x_0=0.045$

$$\therefore x_1=0.045(2-0.045\times 22)$$
$$=0.045(2-0.99)$$
$$=0.0454(1.01)=0.0454$$
$$x_2=0.0454(2-0.0454\times 22)$$
$$=0.0454(2-0.9988)$$
$$=0.0454(1.0012)=0.04545$$
$$x_3=0.04545(2-0.04545\times 22)$$
$$=0.04545(1.0001)=0.04545$$
$$x_4=0.04545(2-0.04545\times 22)$$
$$=0.04545(2-0.99998)$$
$$=0.04545(1.00002)$$
$$=0.0454509$$

$\therefore$ The reciprocal of 22 is 0.04545

3. Find by Newton's method, the real root of the equation $xe^x-2=0$ correct to three decimal places.

Sol. Let $f(x)=xe^x-2\rightarrow(1)$

Then $f(0)=-2$ and $f(1)=e-2=0.7183$

So root of $f(x)$ lies between 0 and 1

It is near to 1. So we take $x_0=1$ and $f^1(x)=xe^x+e^x$ *and* $f^1(1)=e+e=5.4366$

$\therefore$ By Newton's Rule

First approximation $x_1=x_0-\frac{f(x_0)}{f^1(x_0)}$

$$=1-\frac{0.7183}{5.4366}=0.8679$$

$\therefore f(x_1) = 0.0672 \qquad f^1(x_1) = 4.4491$

The second approximation $x_2 = x_1 - \dfrac{f(x_1)}{f^1(x_1)}$

$$= 0.8679 - \frac{0.0672}{4.4491}$$
$$= 0.8528$$

$\therefore$ Required root is 0.853 correct to 3 decimal places.

Interpolation

Introduction:-

If we consider the statement $y = f(x) x_0 \le x \le x_n$ we understand that we can find the value of y, corresponding to every value of x in the range $x_0 \le x \le x_n$. If the function $f(x)$ is single valued and continuous and is known explicitly then the values of $f(x)$ for certain values of x like $x_0, x_1, ...x_n$ can be calculated. The problem now is if we are given the set of tabular values

$$x : x_0 \quad x_1 \quad x_2........x_n$$
$$y : y_0 \quad y_1 \quad y_2........y_n$$

Satisfying the relation $y = f(x)$ and the explicit definition of $f(x)$ is not known, then it is possible to find a simple function say $f(x)$ such that $f(x)$ and $\phi(x)$ agree at the set of tabulated points. This process to finding $\phi(x)$ is called interpolation. If $\phi(x)$ is a polynomial then the process is called polynomial interpolation and $\phi(x)$ is called interpolating polynomial. In our study we are concerned with polynomial interpolation

Errors in Polynomial Interpolation:-

Suppose the function $y(x)$ which is defined at the points $(x_i, y_i) i = 0,1,2,3 - - - - n$ is continuous and differentiable $(n+1)$ times let $\phi_n(x)$ be polynomial of degree not exceeding n such that $\phi_n(x_i) = y_i, i = 1,2 - - - n \to (1)$ be the approximation of $y(x)$ using this $\phi_n(x_i)$ for other value of x, not defined by (1) the error is to be determined

since $y(x) - \phi_n(x) = 0 \;\; for \;\; x - x_0, x_1, ...x_n$ we put

$$y(x) - \phi_n(x) = L\pi_{n+1}(x)$$

Where $\pi_{n+1}(x)=(x-x_0).........(x-x_n)\rightarrow(3)$ and L to be determined such that the equation (2) holds for any intermediate value of x such as $x=x^1, x_0<x^1<x_n$

Clearly $L=\frac{y(x^1)-\phi_n(x^1)}{\pi_{n+1}(x^1)}\rightarrow(4)$

We construct a function $F(x)$ such that $F(x)=F(x_n)=F(x^1)$. Then $F(x)$ vanishes $(n+2)$ times in the interval $[x_0,x_n]$. Then by repeated application of Rolle's theorem. $F^1(x)$must be zero $(n+1)$ times, $F^{11}(x)$ must be zero n times…….. in the interval$[x_0,x_n]$. Also $F^{n+1}(x)=0$ once in this interval. suppose this point is $x=\varepsilon$, $x_0<\varepsilon<x_n$ differentiate (5) $(n+1)$ times with respect to x and putting $x=\varepsilon$, we get

$$y^{n+1}(\varepsilon)-L(n+1)!=0 \text{ which implies that } L=\frac{y^{n+1}(\varepsilon)}{(n+1)!}$$

Comparing (4) and (6) , we get

$$y(x^1)-\phi_n(x^1)=\frac{y^{n+1}(\varepsilon)}{(n+1)!}\pi_{n+1}(x^1)$$

Which can be written as $y(x)-\phi_n(x)=\frac{\pi_{n+1}(x)}{(n+1)!}y^{n+1}(\varepsilon)$

This given the required expression $x_0<\varepsilon<x_n$ for error

Finite Differences:-

1.**Introduction:-**

In this chapter, we introduce what are called the forward, backward and central differences of a function $y=f(x)$. These differences and three standard examples of finite differences and play a fundamental role in the study of differential calculus, which is an essential part of numerical applied mathematics

2.**Forward Differences:-**

Consider a function $y=f(x)$of an independent variable x. let $y_0,y_1,y_2,....y_r$ be the values of y corresponding to the values $x_0,x_1,x_2....x_r$ of x respectively. Then the differences y_1-y_0,y_2-y_1----- are called the first forward differences of y, and we denote them by $\Delta y_0,\Delta y_1,.......$ that is

$$\Delta y_0=y_1-y_0,\Delta y_1=y_2-y_1,\Delta y_2=y_3-y_2.........$$

In general $\Delta y_r=y_{r+1}-y_r \therefore r=0,1,2-----$

Here, the symbol Δ is called the forward difference operator

The first forward differences of the first forward differences are called second forward differences and are denoted by $\Delta^2 y_0, \Delta^2 y_1......$ that is

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$
$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

In general $\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r \quad r = 0,1,2.......$ similarly, the nth forward differences are defined by the formula.

$$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r \quad r = 0,1,2.......$$

While using this formula for $n = 1$, use the notation $\Delta^0 y_r = y_r$ and we have $\Delta^n y_r = 0 \forall n = 1,2...... \ and \ r = 0,2,.........$ the symbol Δ^n is referred as the nth forward difference operator.

3.Forward Difference Table:-

The forward differences are usually arranged in tabular columns as shown in the following table called a forward difference table

Values of x	Values of y	First differences	Second differences	Third differences	Fourth differences
x_o	y_0				
		$\Delta y_0 = y_1 - y_0$			
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - y_0$		
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
x_3	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
	y_4	$= y_4 - y_3$			

Example finite forward difference table for $y = x^3$

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		
		91			
6	216				

4. Backward Differences:-

As mentioned earlier, let $y_0, y_1......y_r......$ be the values of a function $y = f(x)$ corresponding to the values $x_0, x_1, x_2.............x_r......$ of x respectively. Then, $\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \nabla y_3 = y_3 - y_2,....$ are called the first backward differences

In general $\nabla y_r = y_r - y_{r-1},\ r = 1,2,3......... \rightarrow (1)$

The symbol ∇ is called the backward difference operator, like the operator Δ, this operator is also a linear operator

Comparing expression (1) above with the expression (1) of section we immediately note that $\nabla y_r = \nabla y_{r-1}, r = 0,1,2....... \rightarrow (2)$

The first backward differences of the first background differences are called second differences and are denoted by $\nabla^2 y_2, \nabla^2 y_3 --- \nabla^2{}_r ----$ i.e.,..

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1, \nabla^2 y_3 = \nabla y_3 - \nabla y_2 \ldots\ldots\ldots$$

In general $\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}, r = 2,3..... \rightarrow (3)$ similarly, the n^th backward differences are defined by the formula $\nabla^n y_r = \nabla^{n-1} y_r - \nabla^{n-1} y_{r-1}, r = n, n+1..... \rightarrow (4)$ While using this formula, for n = 1 we employ the notation $\nabla^0 y_r = y_r$

If $y = f(x)$ is a constant function , then y = c is a constant, for all x, and we get $\nabla^n y_r = 0 \forall n$ the symbol ∇^n is referred to as the n^th backward difference operator

5. Backward Difference Table:-

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
		∇y_1		
x_1	y_1		$\nabla^2 y_2$	
		∇y_2		$\Delta^2 y_3$
x_2	y_2		$\nabla^2 y_3$	
		∇y_3		
x_3	y_3			

6. Central Differences:-

With $y_0, y_1, y_2 .... y_r$ as the values of a function $y = f(x)$ corresponding to the values $x_1, x_2 ..... x_r ....$ *of* x, we define the first central differences

$\delta y_{1/2}, \delta y_{3/2}, \delta y_{5/2} ----$ as follows

$\delta y_{1/2} = y_1 - y_0, \delta y_{3/2} = y_2 - y_1, \delta y_{5/2} = y_3 - y_2 ----$

$\delta y_{r-1/2} = y_r - y_{r-1} \rightarrow (1)$

The symbol δ is called the central differences operator. This operator is a linear operator

Comparing expressions (1) above with expressions earlier used on forward and backward differences we get

$\delta y_{1/2} = \Delta y_0 = \nabla y_1, \delta y_{3/2} = \Delta y_1 = \nabla y_2 .....$

In general $\delta y_{n+1/2} = \Delta y_n = \nabla y_{n+1}, n = 0,1,2...... \rightarrow (2)$

The first central differences of the first central differences are called the second central differences and are denoted by $\delta^2 y_1, \delta^2 y_2 ...$

Thus $\delta^2 y_1 = \delta_{3/2} - \delta y_{1/2}, \delta^2 y_2 = \delta_{5/2} - \delta_{3/2} ....$

$\delta^2 y_n = \delta y_{n+1/2} - \delta y_{n-1/2} \rightarrow (3)$

Higher order central differences are similarly defined. In general the nth central differences are given by

i) for odd $n : \delta^n y_{r-1/2} = \delta^{n-1} y_r - \delta^{n-1} y_{r-1}, r = 1, 2.... \rightarrow (4)$

ii) for even $n : \delta^n y_r = \delta^{n-1} y_{r+1/2} - \delta^{n-1} y_{r-1/2}, r = 1, 2.... \rightarrow (5)$

while employing for formula (4) for $n = 1$, we use the notation $\delta^0 y_r = y_r$

If y is a constant function, that is if $y = c$ a constant, then $\delta^n y_r = 0 \quad for \quad all \quad n \geq 1$

7. Central Difference Table

x_0	y_0	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
		$\delta y_{1/2}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{2/2}$		$\delta^3 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{7/2}$			
x_4	y_4				

Example: Given $f(-2)=12, f(-1)=16, f(0)=15, f(1)=18, f(2)=20$ from the central difference table and write down the values of $\delta y_{3/2}, \delta^2 y_0$ *and* $\delta^3 y_{7/2}$ by taking $x_0 = 0$

Sol. The central difference table is

x	$y = f(x)$	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
-2	12				
		4			
-1	16		-5		
		-1		9	
0	15		4		-14
		3		-5	
1	18		-1		
		2			
2	20				

Symbolic Relations and Separation of symbols:

We will define more operators and symbols in addition to Δ, ∇ and δ already defined and establish difference formulae by symbolic methods

Definition:- The averaging operator μ is defined by the equation $\mu y_r = \frac{1}{2}[y_{r+1/2} + y_{r-1/2}]$

Definition:- The shift operator E is defined by the equation $Ey_r = y_{r+1}$. This shows that the effect of E is to shift the functional value y_r to the next higher value y_{r+1}. A second operation with E gives $E^2 y_r = E(Ey_r) = E(y_{r+1}) = y_{r+2}$

Generalizing $E^n y^r = y_{r+n}$

Relationship Between Δ *and* E

We have

$$\Delta y_0 = y_1 - y_0$$
$$= Ey_0 - y_0 = (E-1)\,y_0$$
$$\Rightarrow \Delta = E - y\,(or)\,E = 1 + \Delta$$

Some more relations

$$\Delta^3 y_0 = (E-1)^3\, y_0 = \left(E^3 - 3E^2 + 3E - 1\right) y_0$$
$$= y_3 - 3y_2 + 3y_1 - y_0$$

Definition

Inverse operator E^{-1} is defined as $E^{-1} y_r = y_{r-1}$

In general $E^{-n} y_n = y_{r-n}$

We can easily establish the following relations

i) $\nabla \equiv 1 - E^{-1}$

ii) $\delta \equiv E^{1/2} - E^{-1/2}$

iii) $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$

iv) $\Delta = \nabla E = E^{1/2}$

v) $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$

Definition The operator D is defined as $Dy(x) = \frac{\partial}{\partial x}\left[y(x)\right]$

Relation Between The Operators D And E

Using Taylor's series we have, $y(x+h) = y(x) + hy^1(x) + \frac{h^2}{2!} y^{11}(x) + \frac{h^3}{3!} y^{111}(x) + ----$

This can be written in symbolic form

$$Ey_x = \left[1 + hD + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + ----\right] y_x = e^{hD}.y_x$$

We obtain in the relation $E = e^{hD} \rightarrow (3)$

❖ If $f(x)$ is a polynomial of degree n and the values of x are equally spaced then $\Delta^n f(x)$ is constant

Proof:

Let $f(x)=a_0x^n+a_1x^{n-1}+-----+a_{n-1}x+a_n$ where $a_0,a_1,a_2....a_n$ are constants and $a_0\neq 0$. If h is the step- length, we know the formula for the first forward difference

$$\Delta f(x)= f(x+h)-f(x)=\left[a_0(x+h)^n+a_1(x+h)^{n-1}+----+a_{n-1}(x+h)+a_n\right]$$
$$-\left[a_0x^n+a_1x^{n-1}+----+a_{n-1}x+a_n\right]$$
$$=a_0\left[\left\{x^n+n.x^{n-1}h+\frac{n(n-1)}{2!}x^{n-2}.h^2+---\right\}-x^n\right]+$$
$$a_1\left[\left\{x^{n-1}+(n-1)x^{n-2}.h+\frac{(n-1)(n-2)}{2!}x^{n-3}.h^2+---\right\}-x^{n-1}\right]+$$
$$------+a_{n-1}h$$
$$=a_0nhx^{n-1}+b_2x^{n-2}+b_3x^{n-3}+----+b_{n-3}x+b_{n-2}$$

Where $b_2,b_3,.......b_{n-2}$ are constants. Here this polynomial is of degree $(n-1)$, thus, the first difference of a polynomial of n[th] degree is a polynomial of degree $(n-1)$

Now

$$\Delta^2 f(x)=\Delta\left[\Delta f(x)\right]$$
$$=\Delta\left[a_0nh.x^{n-1}+b_2x^{n-2}+b_3x^{n-3}+----+b_{n-1}x+b_{n-2}\right]$$
$$=a_0nh\left[(x+h)^{n-1}-x^{n-1}\right]+b_2\left[(x+h)^{n-2}-x^{n-2}\right]+---+b_{n-1}\left[(x+h)-x\right]$$
$$=a_0n^{(n-1)}h^2x^{n-2}+c_3x^{n-3}+------+c_{n-4}x+c_{n-3}$$

Where $c_3....c_{n-3}$ are constants. This polynomial is of degree $(n-2)$

Thus, the second difference of a polynomial of degree n is a polynomial of degree $(n-2)$ continuing like this we get $\Delta^n f(x)=a_0n(n-1)(n-2)-----2.1.h^n=a_0h^n(n!)$

$\therefore$ which is constant

<u>Note:-</u>

1. As $\Delta^n f(x)$ is a constant, it follows that $\Delta^{n+1} f(x)=0,\Delta^{n+2} f(x)=0,........$
2. The converse of above result is also true that is, if $\Delta^n f(x)$ is tabulated at equal spaced intervals and is a constant, then the function $f(x)$ is a polynomial of degree n

Example:-

1. Form the forward difference table and write down the values of $\Delta f(10)$, $\Delta^2 f(10),\Delta^3 f(15)$ *and* $\Delta^4 y(15)$

x	**10**	**15**	**20**	**25**	**30**	**35**
y	**19.97**	**21.51**	**22.47**	**23.52**	**24.65**	**25.89**

x	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
10	**19.97**					
		1.54				
15	**21.51**		**- 0.58**			
		0.96		**0.67**		
20	**22.47**		**0.09**		**- 0.68**	
		1.05		**- 0.01**		**0.72**
25	**23.52**		**0.08**		**0.04**	
		1.13		**0.03**		
30	**24.65**		**0.11**			
		1.24				
35	**25.89**					

We note that the values of x are equally spaced with step- length h = 5

Note: - $\therefore x_0 = 10, x_1 = 15 ---- x_5 = 35$ and

$y_0 = f(x_0) = 19.97$

$y_1 = f(x_1) = 21.51$

$y_5 = f(x_5) = 25.89$

$y_5 = f(x_5) = 25.89$

From table

$\Delta f(10) = \Delta y_0 = 1.54$

$\Delta^2 f(10) = \Delta^2 y_0 = -0.58$

$\Delta^3 f(15) = \Delta^3 y_1 = -0.01$

$\Delta^4 f(15) = \Delta^3 y_1 = 0.04$

2. **Evaluate**

$(i)\Delta \cos x$

$(ii)\Delta^2 \sin(px+q)$

$(iii)\Delta^n e^{ax+b}$

Sol. Let h be the interval of differencing

$$(i)\,\Delta \cos x = \cos(x+h) - \cos x$$

$$= -2\sin\left(x+\frac{h}{2}\right)\sin\frac{\mathrm{h}}{2}$$

$$(ii)\,\Delta \sin(px+q) = \sin\left[p(x+h)+q\right] - \sin(px+q)$$

$$= 2\cos\left(px+q+\frac{ph}{2}\right)\sin\frac{ph}{2}$$

$$= 2\sin\frac{ph}{2}\sin\left(\frac{\pi}{2}+px+q+\frac{ph}{2}\right)$$

$$\Delta^2 \sin(px+q) = 2\sin\frac{ph}{2}\Delta\left[\sin(px+q)+\frac{1}{2}(\pi+ph)\right]$$

$$= \left[2\sin\frac{ph}{2}\right]^2 \sin\left[px+q+\frac{1}{2}(\pi+ph)\right]$$

$$(iii)\,\Delta e^{ax+b} = e^{a(x+h)+b} - e^{ax+b}$$

$$= e^{(ax+b)}\left(e^{ah-1}\right)$$

$$\Delta^2 e^{ax+b} = \Delta\left[\Delta\left(e^{ax+b}\right)\right] - \Delta\left[\left(e^{ah}-1\right)\left(e^{ax+b}\right)\right]$$

$$= \left(e^{ah}-1\right)^2 \Delta\left(e^{ax+h}\right)$$

$$= \left(e^{ah}-1\right)^2 e^{ax+b}$$

Proceeding on, we get $\Delta^n\left(e^{ax+b}\right) = \left(e^{ah}-1\right)^n e^{ax+b}$

3. **Using the method of separation of symbols show that**

$$\Delta^n \mu_{x-n} = \mu_{x-n} - n\mu_{x-1} + \frac{n(n-1)}{2}\mu_{x-2} + ----+(-1)^n \mu_{x-n}$$

Sol. To prove this result, we start with the right hand side. Thus

$$\mu x - n\mu x - 1 + \frac{n(n-1)}{2}\mu x - 2 + ------+(-1)^n \mu x - n$$

$$= \mu x - nE^{-1}\mu x + \frac{n(n-1)}{2}E^{-2}\mu x + ------+(-1)^n E^{-n}\mu x$$

$$= \left[1 - nE^{-1} + \frac{n(n-1)}{2}E^{-2} + ------+(-1)^n E^{-n}\right]\mu x = \left(1-E^{-1}\right)^n \mu x$$

$$= \left(1-\frac{1}{E}\right)^n \mu n = \frac{(E-1)^n}{E}\mu n$$

$$= \frac{\Delta^n}{E^n}\mu x = \Delta^n E^{-n}\mu x$$

$= \Delta^n \mu_{x-n}$ which is left hand side

4. **Find the missing term in the following data**

x	0	1	2	3	4
y	1	3	9	-	81

Why this value is not equal to 3^3. Explain

Sol. Consider $\Delta^4 y_0 = 0$

$$\Rightarrow 4y_0 - 4y_3 + 5y_2 - 4y_1 + y_0 = 0$$

Substitute given values we get

$$81 - 4y_3 + 54 - 12 + 1 = 0 \Rightarrow y_3 = 31$$

From the given data we can conclude that the given function is $y = 3^x$. To find y_3, we have to assume that y is a polynomial function, which is not so. Thus we are not getting $y = 3^3 = 27$

Newton's Forward Interpolation Formula:-

Let $y = f(x)$ be a polynomial of degree n and taken in the following form

$$y = f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2) + ---$$
$$+ b_n(x - x_0)(x - x_1) ---- (x - x_{n-1}) \to (1)$$

This polynomial passes through all the points $[xi; yi]$ for i = 0 to n. there fore, we can obtain the $y_i's$ by substituting the corresponding $x_i's$ as

$$at \ \ x = x_0, y_0 = b_0$$
$$at \ \ x = x_1, y_1 = b_0 + b_1(x_1 - x_0)$$
$$at \ \ x = x_2, y_2 = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1) \to (1)$$

Let 'h' be the length of interval such that $x_i's$ represent

$$x_0, x_0 + h, x_0 + 2h, x_0 + 3h ---- x_0 + xh$$

This implies $x_1 - x_0 = h, x_2 - x_0 - 2h, x_3 - x_0 = 3h ---- x_n - x_0 = nh \to (2)$

From (1) and (2), we get

$$y_0 = b_0$$
$$y_1 = b_0 + b_1 h$$
$$y_2 = b_0 + b_1 2h + b_2(2h)h$$
$$y_3 = b_0 + b_1 3h + b_2(3h)(2h) + b_3(3h)(2h)h$$

....

....

$$y_n = b_0 + b_1(nh) + b_2(nh)(n-1)h + ---- + b_n(nh)[(n-1)h][(n-2)h] \to (3)$$

Solving the above equations for $b_0, b_{11}, b_2 .....b_n$, we get $b_0 = y_0$

$$b_1 = \frac{y_1 - b_0}{h} = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}$$

$$b_2 = \frac{y_2 - b_0 - b_1 2h}{2h^2} = y_2 - y_0 - \frac{(y_1 - y_0)}{h} 2h$$

$$= \frac{y_2 - y_0 - 2y_1 - 2y_0}{2h^2} = \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2h^2}$$

$$\therefore b_2 = \frac{\Delta^2 y_0}{2!h^2}$$

Similarly, we can see that

$$b_3 = \frac{\Delta^3 y_0}{3!h^3}, b_4 = \frac{\Delta^4 y_0}{4!h^4} - - - - - b_n = \frac{\Delta^n y_0}{n!h^n}$$

$$\therefore y = f(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1)$$

$$+ \frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2) + - - - +$$

$$+ \frac{\Delta^n y_0}{n!h^n}(x - x_0)(x - x_1) - - - (x - x_{n-1}) \to (3)$$

If we use the relationship $x = x_0 + ph \Rightarrow x - x_0 = ph,\ where\ p = 0,1,2,.....n$

Then

$$x - x_1 = x - (x_0 + h) = (x - x_0) - h$$

$$= ph - h = (p - 1)h$$

$$x - x_2 = x - (x_1 + h) = (x - x_1) - h$$

$$= (p - 1)h - h = (p - 2)h$$

....

$$x - x_i = (p - i)h$$

....

$$x - x_{n-1} = [p - (n - 1)]h$$

Equation (3) becomes

$$y = f(x) = f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + - - - - - +$$

$$\frac{p(p-1)(p-2) - - - - (p - (n-1))}{n!}\Delta^n y_0 \to (4)$$

Newton's Backward Interpolation Formula:-

If we consider

$$y_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + a_3(x - x_n)(x - x_{n-1})(x - x_{n-2}) + - - - - (x - x_i)$$

and impose the condition that y and $y_n(x)$ should agree at the tabulated points

$x_n, x_n - 1, ...x_2, x_1, x_0$

We obtain

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2i}\nabla^2 y_n + ---$$
$$\frac{p(p+1)----[p+(n-1)]}{n!}\nabla^n y_n + --- \rightarrow (6)$$

Where $p = \frac{x - x_n}{h}$

This uses tabular values of the left of y_n. Thus this formula is useful formula is useful for interpolation near the end of the table values

Formula for Error in Polynomial Interpolation:-

If $y = f(x)$ is the exact curve and $y = \phi_n(x)$ is the interpolating curve, then the error

in polynomial interpolation is given by

$$Error = f(x) - \phi_n(x) = \frac{(x-x_0)(x-x_1)----(x-x_n)}{(n+1)!} f^{n+1}(\varepsilon) \rightarrow (7)$$

for any x, where $x_0 < x < x_n$ *and* $x_0 < \varepsilon < x_n$

The error in Newton's forward interpolation formula is given by

$$f(x) - \phi_n(x) = \frac{p(p-1)(p-2).......(p-n)}{(n+1)!}\Delta^{n+1} f(\varepsilon)$$

Where $p = \frac{x - x_0}{h}$

The error in Newton's backward interpolation formula is given by

$$f(x) - \phi_n(x) = \frac{p(p+1)(p+2).......(p+n)}{(n+1)!} h^{n+1} y^{n+1} f(\varepsilon) \text{ Where } p = \frac{x - x_n}{h}$$

Examples:-

1. Find the melting point of the alloy containing 54% of lead, using appropriate interpolation formula

Percentage of lead(p)	50	60	70	80
Temperature $(Q°c)$	205	225	248	274

Sol. The difference table is

x	Y	Δ	Δ^2	Δ^3
50	205			
		20		
60	225		3	
		23		0
70	248		3	
		26		
80	274			

Let temperature = $f(x)$

$$x_0 + ph = 24, x_0 = 50, h = 10$$
$$50 + p(10) = 54 \ (or) \ p = 0.4$$

By Newton's forward interpolation formula

$$f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{n!}\Delta^3 y_0 + ---$$
$$f(54) = 205 + 0.4(20) + \frac{0.4(0.4-1)}{2!}(3) + \frac{(0.4)(0.4-1)(0.4-2)}{3!}(0)$$
$$= 205 + 8 - 0.36$$
$$= 212.64$$

Melting point = 212.64

2. Using Newton's forward interpolation formula, and the given table of values

X	1.1	1.3	1.5	1.7	1.9
$f(x)$	0.21	0.69	1.25	1.89	2.61

Obtain the value of $f(x)$ *when* $x = 1.4$

Sol.

x	$y = f(x)$	Δ	Δ^2	Δ^3	Δ^4
1.1	0.21				
		0.48			
1.3	0.69		0.08		
		0.56		0	
1.5	1.25		0.08		0
		0.64		0	
1.7	1.89		0.08		
		0.72			
1.9	2.61				

If we take $x_0 = 1.3$ *then* $y_0 = 0.69$,

$$\Delta y_0 = 0.56, \Delta^2 y_0 = 0.08, \Delta^3 y_0 = 0, L = 0.2, x = 1.3$$

$$x_0 + ph = 1.4 (or) 1.3 + p(0.2) = 1.4, p = \frac{1}{2}$$

Using Newton's interpolation formula

$$f(1.4) = 0.69 + \frac{1}{2} \times 0.56 + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!} \times 0.08$$

$$= 0.69 + 0.28 - 0.01 = 0.96$$

3. The population of a town in the decimal census was given below. Estimate the population for the 1895

Year x	1891	1901	1911	1921	1931
Population of y	46	66	81	93	101

Sol. Putting $L = 10, x_0 = 1891, x = 1895$ in the formula $x = x_0 + ph$ we obtain $p = 2/5 = 0.4$

X	Y	Δ	Δ^2	Δ^3	Δ^4
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
192	93		-4		
		8			
1931	101				

$$y(1895) = 46 + (0.4)(20) + \frac{(0.4)(0.4-1)}{6} - (-5)$$

$$+ \frac{(0.4-1)0.4(0.4-2)}{6}(2)$$

$$+ \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{24}$$

$$= 54.45 \text{ } thousands$$

Gauss's Interpolation Formula:- We take x_0 as one of the specified of x that lies around the middle of the difference table and denote $x_0 - rh$ by $x - r$ and the corresponding value of y by $y - r$. Then the middle part of the forward difference table will appear as shown in the next page

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_{-4}	y_{-4}					
x_{-3}	y_{-3}	Δy_{-4}				
x_{-2}	y_{-2}	Δy_{-3}	$\Delta^2 y_{-4}$			
x_{-1}	y_{-1}	Δy_{-2}	$\Delta^2 y_{-3}$	$\Delta^3 y_{-4}$		
x_0	y_0	Δy_{-1}	$\Delta^2 y_{-2}$	$\Delta^3 y_{-3}$	$\Delta^4 y_{-4}$	
x_1	y_1	Δy_0	$\Delta^2 y_{-1}$	$\Delta^3 y_{-2}$	$\Delta^4 y_{-3}$	$\Delta^5 y_{-4}$
x_2	y_2	Δy_1	$\Delta^2 y_0$	$\Delta^3 y_{-1}$	$\Delta^4 y_{-2}$	$\Delta^5 y_{-3}$
x_3	y_3	Δy_2	$\Delta^2 y_1$	$\Delta^3 y_0$	$\Delta^4 y_{-1}$	$\Delta^5 y_{-2}$
x_4	y_4	Δy_3	$\Delta^2 y_2$	$\Delta^3 y_1$	$\Delta^4 y_0$	$\Delta^5 y_{-1}$

$$\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$$
$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$
$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$
$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} -----(1) \quad and$$
$$\Delta y_{-1} = \Delta y_{-2} + \Delta^2 y_{-2}$$
$$\Delta^2 y_{-1} = \Delta^2 y_{-2} + \Delta^3 y_{-2}$$
$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$
$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2} -----(2)$$

By using the expressions (1) and (2), we now obtain two versions of the following Newton's forward interpolation formula

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_0) + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$
$$+ \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + ---] \quad \text{-----------------3}$$

Here y_p is the value of y at $x = x_p = x_0 + ph$

Gauss Forward Interpolation Formula:-

Substituting for $\Delta^2 y_0, \Delta^3 y_0, ...$ from (1)in the formula (3), we get

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{-1}$$
$$+ \Delta^4 y_{-1} + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_{-1} + \Delta^5 y_{-1} + ---]$$

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1}) + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{p(p+1)(p-1)(p-2)}{4!}(\Delta^4 y_{-1}) + ----]$$

Substituting $\Delta^4 y_{-1}$ *from* (2), this becomes

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(p+1)(p-1)p(p-2)}{4!}(\Delta^4 y_{-2}) + ----] \quad ----------------4$$

Note:- we observe from the difference table that

$\Delta y_0 = \delta y_{1/2}, \Delta^2 y_{-1} = \delta^2 y_0, \Delta^3 y_{-1} = \delta^3 y_{1/2}, \Delta^4 y_{-2} = \delta^4 y_0$ and so on. Accordingly the formula (4) can be written in the notation of central differences as given below

$$y_p = [y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 y_{1/2}$$
$$+ \frac{(p+1)(p-1)p(p-2)}{4!}\delta^4 y_0 + ----] \quad ----------------5$$

2. **Gauss's Backward Interpolation formula:-**

Let us substitute for $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ ----- from (1) in the formula (3), thus we obtain

$$y_p = [y_0 + p(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{(p-1)p(p-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) +$$
$$\frac{(p-1)(p-2)p(p-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + ----]$$

$$= [y_0 + p(\Delta y_{-1}) + \frac{(p+1)}{2!}p(\Delta^2 y_{-1}) + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}(\Delta^4 y_{-1}) + ----]$$

Substituting for $\Delta^3 y_{-1}$ and $\Delta^4 y_{-1}$ from (2) this becomes

$$y_p = [y_0 + p(\Delta y_{-1}) + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-2})$$
$$+ \frac{(p+1)p(p-1)(p-2)}{4!}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + ----]$$

Lagrange's Interpolation Formula:-

Let $x_0, x_1, x_2, ...\ x_n$ be the $(n+1)$ values of x which are not necessarily equally spaced. Let $y_0, y_1, y_2........y_n$ be the corresponding values of $y = f(x)$ let the polynomial of degree n for the function $y = f(x)$ passing through the $(n+1)$ points $(x_0, f(x_0)), (x_1, f(x_1)) ---- (x_n, f(x_n))$ be in the following form

$$y = f(x) = a_0(x-x_1)(x-x_2)......(x-x_n) + a_1(x-x_0)(x-x_2).........(x-x_n) +$$
$$a_2(x-x_0)(x-x_1).......(x-x_n) + ... + a_n(x-x_0)(x-x_1)......(x-x_{n-1}) \rightarrow (1)$$

Where $a_0, a_1, a_2....$ a^n are constants

Since the polynomial passes through $(x_0, f(x_0)), (x_1, f(x_1))......(x_n, f(x_n))$. The constants can be determined by substituting one of the values of $x_0, x_1, ...x_n$ *for* x in the above equation

Putting $x = x_0$ in (1) we get, $f(x_0) = a_0(x - x_1)(x_0 - x_2)(x_0 - x_n)$

$$\Rightarrow a_0 = \frac{f(x_0)}{(x - x_1)(x_0 - x_2)....(x_0 - x_n)}$$

Putting $x = x_1$ in (1) we get, $f(x_1) = a_1(x - x_0)(x_1 - x_2) - - - - (x_1 - x_n)$

$$\Rightarrow a_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)....(x_1 - x_n)}$$

Similarly substituting $x = x_2$ in (1), we get

$$\Rightarrow a_2 = \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)......(x_2 - x_n)}$$

Continuing in this manner and putting $x = x_n$ in (1) we

get $a_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_1) - - - - (x_n - x_{n-1})}$

Substituting the values of $a_0, a_1, a_2....a_n$, we get

$$f(x) = \frac{(x - x_1)(x - x_2).......(x - x_n)}{(x_0 - x_1)(x_0 - x_2).........(x_0 - x_n)} f(x_0) + \frac{(x - x_0)(x - x_2).....(x - x_n)}{(x_1 - x_0)(x_1 - x_2)....(x_1 - x_n)}$$

$$f(x_1) + \frac{(x - x_0)(x - x_1)(x - x_2).....(x - x_n)}{(x_2 - x_0)(x_2 - x_1)......(x_2 - x_n)} + ...f(x_2) + \frac{(x - x_0)(x - x_1).....(x - x_{n-1})}{(x_n - x_1)(x_n - x_2).....(x_n - x_{n-1})} f(x_n)$$

Examples:-

1. Using Lagrange's formula calculate $f(3)$ from the following table

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

Sol. Given $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 4, x_5 = 6, x_4 = 5$

$$f(x_0) = 1, f(x_1) = 14, f(x_2) = 15, f(x_3) = 5, f(x_4) = 6, f(x_5) = 19$$

From langrange's interpolation formula

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)}f(x_0)$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}f(x_1)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)}f(x_2)$$

−−−−−−−−−−−−−−−−−−−−

−−−−−−−−−−−−−−−−−−−−

$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}f(x_5)$$

Here $x=3$ then

$$f(3)=\frac{(3-1)(3-2)(3-4)(3-5)(3-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)}\times 1+$$

$$\frac{(3-0)(3-2)(3-4)(3-5)(3-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)}\times 14+$$

$$\frac{(3-0)(3-1)(3-4)(3-5)(3-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)}\times 15+$$

$$\frac{(3-0)(3-1)(3-2)(3-5)(3-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)}\times 5+$$

$$\frac{(3-0)(3-1)(3-2)(3-4)(3-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)}\times 6+$$

$$\frac{(3-0)(3-1)(3-2)(3-4)(3-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)}\times 19$$

$$=\frac{12}{240}-\frac{18}{60}\times 14+\frac{36}{48}\times 15+\frac{36}{48}\times 5-\frac{18}{60}\times 6+\frac{12}{40}\times 19$$

$$=0.05-4.2+11.25+3.75-1.8+0.95$$

$$=10$$

$$f(x_3)=10$$

1) Find $f(3.5)$ using lagrange method of 2^{nd} and 3^{rd} order degree polynomials.

x	1	2	3	4
$f(x)$	1	2	9	28

Sol: By lagrange's interpolation formula

$$f(x)=\sum_{k=0}^{n}f(x_k)\frac{(x-x_0).......(x-x_{k-1})(x-x_{k+1})(x-x_n)}{(x_k-x_0).......(x_k-x_{k-1}).......(x_k-x_n)}$$

For $n=4$,we have

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+$$
$$\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)+$$
$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+$$
$$\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)+$$

$$\therefore f(3.5)=\frac{(3.5-2)(3.5-3)(3.5-4)}{(1-2)(1-3)(1-4)}(1)+\quad\frac{(3.5-1)(3.5-3)(3.5-4)}{(2-1)(2-3)(2-4)}(2)+$$

$$\frac{(3.5-1)(3.5-2)(3.5-4)}{(3-1)(3-2)(3-4)}(9)+$$
$$\frac{(3.5-1)(3.5-2)(3.5-3)}{(4-1)(4-2)(4-3)}(28)+ \quad - - - -$$

=0.0625+(-0.625)+8.4375+8.75

=16.625

$$f(x)=\frac{(x-2)(x-3)(x-4)}{-6}(1)+\frac{(x-1)(x-3)(x-4)}{2}(2)$$
$$+\frac{(x-1)(x-2)(x-4)}{(-2)}(9)+\frac{(x-1)(x-2)(x-3)}{6}(28)$$

$$=\frac{(x^2-5x+6)(x-4)}{-6}+(x^2-4x+3)(x-4)+\frac{(x^2-3x+2)}{-2}(x-4)(9)+\frac{(x^2-3x+2)}{6}(x-3)(28)$$

$$=\frac{x^3-9x^2+26x-24}{-6}+x^3-8x^2+9x-12+\frac{x^3-7x^2+14x-8}{-2}(9)+\frac{x^3-6x^2+11x-6}{6}(28)$$

$$=\frac{\left[-x^3+9x^2-26x+24+6x^3-48x^2+114x-72-27x^3+189x^2-378x+216+308x+28x^3-168x^2-168\right]}{6}$$

$$=\frac{6x^3-18x^2+18x}{6}\Rightarrow f(x)=x^3-3x^2+3x$$

$$\therefore f(3.5)=(3.5)^3-3(3.5)^2+3(3.5)=16.625$$

Example:

Find y(25), given that y_{20} = 24,y_{24} = 32, y_{28} = 35 y_{32} = 40 using Guass forward difference Formula :

Solution: Given

X	20	24	28	32
Y	24	32	35	40

By Gauss Forward difference formula

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{-1}$$
$$+ \Delta^4 y_{-1} + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_{-1} + \Delta^5 y_{-1} + ----]$$

We take x= 24 as origin.

X_0 = 24, h = 4, x = 25 p = x-x_0/ h, p = 25-24/4 = 2.5

Gauss Forward difference table is

X	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	24			
24	32	Δy_{-1} = 8		
28	35	Δy_0 = 3	$\Delta^2 y_{-1}$ = - 5	
32	40	Δy_1 = 5	$\Delta^2 y_0$ = 2	$\Delta^3 y_{-1}$ = 7

By gauss Forward interpolation Formula

We y(25) = 32 +.25(3) + ($\frac{.25)(.25-1)}{2}(-5)$ + $\frac{(.25+1)(.25)(.25-1)}{6}(7)$ = 32 + .75 + .46875 - .2734 = 32.945

Y(25) = 32.945.

Example:

Use Gauss Backward interpolation formula to find f(32) given that f(25) = .2707, f(30) = .3027, f(35) = .3386 f(40) = .3794.

Solution: let x_0 = 35 and difference table is

X	y	Δy	$\Delta^2 y$	$\Delta^3 y$
25	.2707			
30	.3027	.032		
35	.3386	.0359	.0039	
40	.3794	.0408	.0049	.0010

From the table y_0 = 0.3386

Δy_{-1} = 0.0359 , $\Delta^2 y_{-1}$ = 0.0049, $\Delta^3 y_{-2}$ = 0.0010, x_p = 32 p = x_p- x_0/h = 32-35/5 = -.6

By Gauss Backward difference formula

f(32) = .3386 + (-.6)(.0359) + (-.6)(-.6+1)(.0049)/2 + (-.6)(.36-1)(0.00010)/6 = .3165

UNIT-II

CURVE FITTING

Curve fitting

Suppose that a data is given in two variables x & y the problem of finding an analytical expression of the form $y = f(x)$ which fits the given data is called curve fitting

Let $(x_1, y_1), (x_2, y_2)........(x_n, y_n)$ be the observed set of values in an experiment and $y = f(x)$ be the given relation $x \& y, Let\ E_1, E_2, ...E_x$ are the error of approximations then we have

$$E_1 = y_1 - f(x_1)$$
$$E_2 = y_2 - f(x_2)$$
$$E_3 = y_3 - f(x_3)$$

$E_n = y_n - f(x_n)$ where $f(x_1), f(x_2)..........f(x_n)$ are called the expected values of y corresponding to $x = x_1, x = x_2........x = x_n$

$y_1, y_2......y_n$ are called the observed values of y corresponding to $x = x_1, x = x_2........x = x_n$ the differences $E_1, E_2.....E_n$ between expected values of y and observed values of y are called the errors, of all curves approximating a given set of points, the curve for which

$E = E_1{}^2 + E_2{}^2 + ...E_n{}^2$ is a minimum is called the best fitting curve (or) the least square curve

This is called the method of least squares (or) principles of least squares

1. FITTING OF A STRAIGHT LINE:-

Let the straight line be $y = a + bx \to (1)$

Let the straight line (1) passes through the data points

$(x_1, y_1), (x_2, y_2)......(x_n, y_n) i.e., (x_i, y_i), i = 1, 2....n$

So we have $yi = a + bxi \to (2)$

The error between the observed values and expected values of $y = yi$ is defined as

$Ei = y_1 - (a + bxi).i = 1, 2.......n \to (3)$

The sum of squares of these error is

$E = \sum_{i=1}^{n} Ei^2 = \sum_{i=1}^{n} \left[yi - (a + bxi) \right]^2$ now for E to be minimum

$$\frac{\partial E}{\partial a} = 0; \frac{\partial E}{\partial b} = 0$$

These equations will give normal equations

$$\sum_{i=1}^{n} yi = na + b\sum_{i=1}^{n} xi$$

$$\sum_{i=1}^{n} xiyi = a\sum_{i=1}^{n} xi + b\sum_{i=1}^{n} xi^2$$

The normal equations can also be written as

$$\sum y = na + b\sum x$$

$$\sum xy = a\sum x + b\sum x^2$$

Solving these equation for a, b substituting in (1) we get required line of best fit to the given data.

NON LINEAR CURVE FITTING

PARABOLA:-

2. Let the equation of the parabola to be fit

The parabola (1) passes through the data points

$(x_1, y_1), (x_2, y_2)............(x_n, y_n), i.e., (x_i, y_i); i = 1, 2......x$

We have $yi = a + bx_i + cx_i^2 \rightarrow (2)$

$y = a + bx + cx^2 \rightarrow (1)$

The error Ei between the observed an expected value of $y = y_i$ is defined as

$Ei = yi - (a + bxi + cxi^2), i = 1, 2, 3......n \rightarrow (3)$

The sum of the squares of these error is

$E = \varepsilon_{i=1}^{n} \; Ei^2 = \varepsilon_{i=1}^{n} \; (yi - a - bxi - cxi^2)^2 \rightarrow (4)$

For E to be minimum, we have

$$\frac{\partial E}{\partial a} = 0, \frac{\partial E}{\partial b} = 0, \frac{\partial E}{\partial c} = 0$$

The normal equations can also be written as

$\varepsilon y = na + b\varepsilon x + c\varepsilon x^2$

$\varepsilon xy = a\varepsilon x + b\varepsilon x^2 + c\varepsilon x^3$ $\quad$ *use* $\sum$ *instead of* ε

$\varepsilon x^2 y = a\varepsilon x^2 + b\varepsilon x^3 + c\varepsilon x^4$

Solving these equations for a, b, c and satisfying (1) we get required parabola of best fit

3. **POWER CURVE:-**

The power curve is given by $y = ax^b \rightarrow (1)$

Taking logarithms on both sides

$$\log_{10}{}^{y} = \log_{10}{}^{a} + b\log_{10}{}^{X}$$
$$(or)\ y = A + bX \rightarrow (2)$$
$$where\ \ y = \log_{10}{}^{y}, A = \log_{10}{}^{a}\ \ and\ \ X = \log_{10}{}^{X}$$

Equation (2) is a linear equation in X & y

$\therefore$ The normal equations are given by

$$\varepsilon y = nA + b\varepsilon X$$
$$\varepsilon xy = A\varepsilon X + b\varepsilon X^2 \qquad use\ \ \Sigma\, symbol$$

From these equations, the values A and b can be calculated then a = antilog (A) substitute a & b in (1) to get the required curve of best fit

4. **EXPONENTIAL CURVE :-** $(1)\ y = ae^{bx}\ \ (2)\ y = ab^{x}$

1. $y = ae^{bx} \rightarrow (1)$

Taking logarithms on both sides

$$\log_{10} y = \log_{10} a + bx\log_{10} e$$
$$(or)\ y = A + BX \rightarrow (2)$$

Where $y = \log_{10} y, A = \log_{10} a \,\&\, B = b\log_{10} e$

Equation (2) is a linear equation in X and Y

So the normal equation are given by

$$\Sigma Y = nA + B\Sigma X$$
$$\Sigma xy = A\Sigma X + B\Sigma X^2$$

Solving the equation for A & B, we can find

$$a = anti\log A \,\&\, b = \frac{B}{\log_{10} e}$$

Substituting the values of a and b so obtained in (1) we get

The curve of best fir to the given data.

2. $y = ab^{x} \rightarrow (1)$

Taking log on both sides

$$\log_{10} y = \log_{10} a + x\log_{10} b\ \ (or) Y = A + Bx$$
$$Y = \log_{10} y, A = \log_{10} a, B = \log_{10} b$$

The normal equation (2) are given by

$$\Sigma y = nA + B\Sigma X$$
$$\Sigma xy = A\Sigma X + B\Sigma X^2$$

Solving these equations for A and B we can find $a = anti\log A, b = anti\log B$

Substituting a and b in (1)

1. By the method of least squares, find the straight line that best fits the following data

X	1	2	3	4	5
Y	14	27	40	55	68

$y = a + bx$

Ans. The values of $\varepsilon x, \varepsilon y, \varepsilon x^2$ *and* εxy are calculated as follows

xi	yi	xi^2	$xiyi$
1	14	1	14
2	27	4	54
3	40	9	120
4	55	16	220
5	68	25	340

Replace xi, yi *by* x_i, y_i and use Σ *instead of* ε

$\varepsilon xi = 15; \varepsilon yi = 204, \varepsilon xi^2 = 55$ *and* $\varepsilon xiyi = 748$

The normal equations are

$\varepsilon y = na + b\varepsilon x \rightarrow (1)$

$\varepsilon xy = a\varepsilon x + b\varepsilon x^2 \rightarrow (2)$

$204 = 15a + 5b$

$748 = 55a + 15b$

Solving we get $a = 0, b = 13.6$

Substituting these values a & b we get

$y = 0 + 13.6x \Rightarrow y = 13.6x$

2. Fit a second degree parabola to the following data

x	0	1	2	3	4
y	1	5	10	22	38

$y = a + bx + cx^2$

Ans. Equation of parabola $y = a + bx + cx^2 \rightarrow (1)$

Normal equations $\varepsilon y = na + b\varepsilon x + c\varepsilon x^2$

$\varepsilon xy = a\varepsilon x + b\varepsilon x^2 + c\varepsilon x^3$

$\varepsilon x^2 y = a\varepsilon x^2 + b\varepsilon x^3 + c\varepsilon x^4 \rightarrow (2)$

x	y	xy	x^2	x^2y	x^3	x^4
0	**1**	**0**	**0**	**0**	**0**	**0**
1	**5**	**5**	**1**	**5**	**1**	**1**
2	**10**	**20**	**4**	**40**	**8**	**16**
3	**22**	**66**	**9**	**198**	**27**	**81**
4	**38**	**152**	**16**	**608**	**64**	**256**

$\varepsilon x = 10, \varepsilon y = 76, \varepsilon xy = 243, \varepsilon x^2 = 30, \varepsilon x^2 y = 851, \varepsilon x^3 = 100, \varepsilon x^4 = 354$

Normal equations

$76 = 5a + 10b + 30c$

$243 = 10a + 30b + 100c$

$851 = 30a + 100b + 354c$

Solving $a = 1.42, b = 0.26, c = 2.221$

Substitute in (1) $\Rightarrow y = 1.42 + 0.26x + 2.221x^2$

3. Fit a curve $y = ax^b$ to the following data

x	1	2	3	4	5	6
y	2.98	4.26	5.21	6.10	6.80	7.50

Ans. Let the equation of the curve be $y = ax^b \rightarrow (1)$

Taking log on both sides

$\log y = \log a + b \log x$

$y = A + bX \rightarrow (2)$

$y = \log y, A = \log a, X = \log x$

$\varepsilon y = nA + b\varepsilon X$

$\varepsilon xy = A\varepsilon x + b\varepsilon x^2 \rightarrow (3)$

x	$X = \log x$	y	$y = \log y$	xy	x^2
1	**0**	**2.98**	**0.4742**	**0**	**0**
2	**0.3010**	**4.26**	**0.6294**	**0.1894**	**0.0906**
3	**0.4771**	**5.21**	**0.7168**	**0.3420**	**0.2276**
4	**0.6021**	**6.10**	**0.7853**	**0.4728**	**0.3625**
5	**0.6990**	**6.80**	**0.8325**	**0.5819**	**0.4886**

$\varepsilon x = 2.8574, \varepsilon y = 4.3133, \varepsilon xy = 2.2671, \varepsilon x^2 = 1.7749$

$4.3313 = 6A + 2.8574b$

$2.2671 = 2.8574A + 1.7749b$

solving $A = 0.4739$ $b = 0.5143$

$a = \text{anti}\log(A) = 2.978$

$\therefore y = 2.978.x^{0.5143}$

4. Fit a curve $y = ab^x \rightarrow (1)$

x	2	3	4	5	6
y	144	172.8	207.4	248.8	298.5

$\log y = \log a + x \log b \rightarrow (I)$

$y = A + xB \rightarrow (2)$

Ans. $y = \log y, A = \log a, B = \log b$

$\Sigma y = nA + B\varepsilon x$

$\varepsilon xy = A\varepsilon x + B\varepsilon x^2 \rightarrow (3)$

x	y	x^2	$Y = \log y$	xy
2	**144.0**	**4**	**2.1584**	**4.3168**
3	**172.8**	**9**	**2.2375**	**6.7125**
4	**207.4**	**16**	**2.3168**	**9.2672**
5	**248.8**	**25**	**2.3959**	**11.9795**
6	**298.5**	**36**	**2.4749**	**14.8494**

5. **Fit a second degree parabola to the following data by the method of least squares.**

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

Ans. Equation of parabola $y = a + bx + cx^2 \rightarrow (1)$

Normal equations $\varepsilon y = na + b\varepsilon x + c\varepsilon x^2$

$\varepsilon xy = a\varepsilon x + b\varepsilon x^2 + c\varepsilon x^3$ & $\varepsilon x^2 y = a\varepsilon x^2 + b\varepsilon x^3 + c\varepsilon x^4 \rightarrow (2)$

x	y	xy	x^2	$x^2 y$	x^3	x^4
0	**1**	**0**	**0**	**0**	**0**	**0**
1	**1.8**	**1.8**	**1**	**1.8**	**1**	**1**
2	**1.3**	**2.6**	**4**	**5.2**	**8**	**16**
3	**2.5**	**7.5**	**9**	**22.5**	**27**	**81**
4	**6.3**	**25.2**	**16**	**100.8**	**64**	**256**

$\sum x_i = 10, \sum y_i = 12.9, \sum x^2 = 30, \sum x_i^3 = 100, \sum x_i^4 = 354, \sum x_i^2 y_i = 130.3$
$\sum x_i y_i, = 37.1$

Normal equations

5a + 10b + 30c = 12.9

10a + 30b +100c = 37.1

30a + 100b +354c = 130.3
Solving a= 1.42 b = -1.07 c= .55

Substitute in (1) $y = 1.42- 1.07x+.55x^2$

Numerical solutions of ordinary differential equations

1. The important methods of solving ordinary differential equations of first order numerically are as follows
   1) Taylors series method
   2) Euler's method
   3) Modified Euler's method of successive approximations
   4) Runge- kutta method

To describe various numerical methods for the solution of ordinary differential eqn's,we consider the general 1st order differential eqn

dy/dx=f(x,y)-------(1)

with the initial condition $y(x_0)=y_0$

The methods will yield the solution in one of the two forms:

i) A series for y in terms of powers of x,from which the value of y can be obtained by direct substitution.

ii) A set of tabulated values of y corresponding to different values of x

The methods of Taylor and picard belong to class(i)

The methods of Euler, Runge - kutta method, Adams, Milne etc, belong to class (ii)

TAYLOR'S SERIES METHOD

To find the numerical solution of the differential equation

$$\frac{dy}{dx} = f(x,y) \rightarrow (1)$$

With the initial condition $y(x_0) = y_0 \rightarrow (2)$

$y(x)$ can be expanded about the point x_0 in a Taylor's series in powers of $(x-x_0)$ as

$$y(x) = y(x_0) + \frac{(x-x_0)}{1} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + ... + \frac{(x-x_0)^n}{n!} y^n(x_0) \rightarrow (3)$$

In equ3, $y(x_0)$ is known from I.C equ2. The remaining coefficients $y'(x_0), y''(x_0), ...y^n(x_0)$ etc are obtained by successively differentiating equ1 and evaluating at x_0. Substituting these values in equ3, $y(x)$ at any point can be calculated from equ3. Provided $h = x - x_0$ is small.

When $x_0 = 0$, then Taylor's series equ3 can be written as

$$y(x) = y(0) + x.y'(0) + \frac{x^2}{2!} y''(0) + ... + \frac{x^n}{n!} y^n(0) + ... \rightarrow (4)$$

1. Using Taylor's expansion evaluate the integral of $y' - 2y = 3e^x, y(0) = 0$**, at a)** $x = 0.2$

b) compare the numerical solution obtained with exact solution .

Sol: Given equation can be written as $2y+3e^{x}=y', y(0)=0$

Differentiating repeatedly w.r.t to 'x' and evaluating at $x=0$

$$y'(x)=2y+3e^{x}, y'(0)=2y(0)+3e^{0}=2(0)+3(1)=3$$
$$y''(x)=2y'+3e^{x}, y''(0)=2y'(0)+3e^{0}=2(3)+3=9$$
$$y'''(x)=2.y''(x)+3e^{x}, y'''(0)=2y''(0)+3e^{0}=2(9)+3=21$$
$$y^{iv}(x)=2.y'''(x)+3e^{x}, y^{iv}(0)=2(21)+3e^{0}=45$$
$$y^{v}(x)=2.y^{iv}+3e^{x}, y^{v}(0)=2(45)+3e^{0}=90+3=93$$

In general, $y^{(n+1)}(x)=2.y^{(n)}(x)+3e^{x}$ or $y^{(n+1)}(0)=2.y^{(n)}(0)+3e^{0}$

The Taylor's series expansion of $y(x)$ about $x_0=0$ is

$$y(x)=y(0)+xy'(0)+\frac{x^2}{2!}y''(0)+\frac{x^3}{3!}y'''(0)+\frac{x^4}{4!}y''''(0)+\frac{x^5}{5!}y'''''(0)+....$$

Substituting the values of $y(0), y'(0), y''(0), y'''(0),..........$

$$y(x)=0+3x+\frac{9}{2}x^2+\frac{21}{6}x^3+\frac{45}{24}x^4+\frac{93}{120}x^5+........$$

$$y(x)=3x+\frac{9}{2}x^2+\frac{7}{2}x^3+\frac{15}{8}x^4+\frac{31}{40}x^5+........\rightarrow \text{equ1}$$

Now put $x=0.1$ in equ1

$$y(0.1)=3(0.1)+\frac{9}{2}(0.1)^2+\frac{7}{2}(0.1)^3+\frac{15}{8}(0.1)^4+\frac{31}{40}(0.1)^5=0.34869$$

Now put $x=0.2$ in equ1

$$y(0.2)=3(0.2)+\frac{9}{2}(0.2)^2+\frac{7}{2}(0.2)^3+\frac{15}{8}(0.2)^4+\frac{31}{40}(0.2)^5=0.811244$$

$$y(0.3)=3(0.3)+\frac{9}{2}(0.3)^2+\frac{7}{2}(0.3)^3+\frac{15}{8}(0.3)^4+\frac{31}{40}(0.3)^5=1.41657075$$

Analytical Solution:

The exact solution of the equ $\frac{dy}{dx}=2y+3e^{x}$ with $y(0)=0$ can be found as follows

$\frac{dy}{dx}-2y=3e^{x}$ Which is a linear in y.

Here $P=-2, Q=3e^{x}$

I.F = $\int_e^{pdx} = \int_e^{-2dx} = e^{-24}$

General solution is $y.e^{-2x}=\int 3e^{x}.e^{-2x}dx+c=-3e^{-x}+c$

$\therefore y=-3e^{x}+ce^{2x}$ where $x=0, y=0 \quad 0=-3+c \Rightarrow c=3$

The particular solution is $y=3e^{2x}-3e^{x}$ or $y(x)=3e^{2x}-3e^{x}$

Put $x=0.1$ in the above particular solution,

$$y=3.e^{0.2}-3e^{0.1}=0.34869$$

Similarly put $x = 0.2$

$$y = 3e^{0.4} - 3e^{0.2} = 0.811265$$

put $x = 0.3$

$$y = 3e^{0.6} - 3e^{0.3} = 1.416577$$

2. Using Taylor's series method, solve the equation $\frac{dy}{dx} = x^2 + y^2$ **for** $x = 0.4$ **given that** $y = 0$ **when** $x = 0$

Sol: Given that $\frac{dy}{dx} = x^2 + y^2$ and $y = 0$ when $x = 0$ i.e. $y(0) = 0$

Here $y_0 = 0$, $x_0 = 0$

Differentiating repeatedly w.r.t 'x' and evaluating at $x = 0$

$$y'(x) = x^2 + y^2, y'(0) = 0 + y^2(0) = 0 + 0 = 0$$
$$y''(x) = 2x + y'.2y, y''(0) = 2(0) + y'(0)2.y = 0$$
$$y'''(x) = 2 + 2yy'' + 2y'.y', y'''(0) = 2 + 2.y(0).y''(0) + 2.y'(0)^2 = 2$$
$$y''''(x) = 2.y.y''' + 2.y''.y' + 4.y''.y', y''''(0) = 0$$

The Taylor's series for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y''''(0) + ...$$

Substituting the values of $y(0), y'(0), y''(0), ...$

$$y(x) = 0 + x(0) + 0 + \frac{2x^3}{3!} + 0 + ... = \frac{x^3}{3} + \text{(Higher order terms are neglected)}$$

$$\therefore y(0.4) = \frac{(0.4)^3}{3} = \frac{0.064}{3} = 0.02133$$

3. Solve $y' = x - y^2, y(0) = 1$ **using Taylor's series method and compute y(0.1),y(0.2)**

Sol: Given that $y' = x - y^2, y(0) = 1$

Here $y_0 = 1$, $x_0 = 0$

Differentiating repeatedly w.r.t 'x' and evaluating at x=0

$$y'(x) = x - y^2, y'(0) = 0 - y(0)^2 = 0 - 1 = -1$$
$$y''(x) = 1 - 2y.y', y''(0) = 1 - 2.y(0)y'(0) = 1 - 2(-1) = 3$$
$$y'''(x) = 1 - 2yy' - 2(y')^2, y'''(0) = -2.y(0).y''(0) - 2.(y'(0))^2 = -6 - 2 = -8$$
$$y''''(x) = -2.y.y''' - 2.y''.y' - 4.y''.y', y''''(0) = -2.y(0).y'''(0) - 6.y''(0).y'(0) = 16 + 18 = 34$$

The Taylor's series for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + \frac{x}{1!}y^1(0) + \frac{x^2}{2!}y^{11}(0) + \frac{x^3}{3!}y^{111}(0) + ...$$

Substituting the value of $y(0), y^1(0), y^{11}(0), ...$

$$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{8}{6}x^3 + \frac{34}{24}x^4 + ...$$

$$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 + \frac{17}{12}x^4 + \ldots.. \quad \rightarrow(1)$$

now put x = 0.1 in (1)

$$y(0.1) = 1 - 0.1 + \frac{3}{2}(0.1)^2 + \frac{4}{3}(0.1)^3 + \frac{17}{12}(0.1)^4 + \ldots..$$

$$= 0.91380333 \underset{\sim}{} 0.91381$$

Similarly put x = 0.2 in (1)

$$y(0.2) = 1 - 0.2 + \frac{3}{2}(0.2)^2 - \frac{4}{3}(0.2)^3 + \frac{17}{12}(0.2)^4 + \ldots..$$

$$= 0.8516.$$

4. Solve $y^1 = x^2 - y$, y(0) = 1, using Taylor's series method and compute y(0.1), y(0.2), y(0.3) and y(0.4) (correct to 4 decimal places).

Sol. Given that $y^I = x^2 - y$ and y(0) = 1

Here $x_0 = 0$, $y_0 = 1$ or y =1 when x=0

Differentiating repeatedly w.r.t 'x' and evaluating at x = 0.

$Y^I(x) = x^2 - y$, $y^I(0) = 0 - 1 = -1$

$y^{II}(x) = 2x - y^I$, $\quad y^{II}(0) = 2(0) - y^I(0) = 0 - (-1) = 1$

$y^{III}(x) = 2 - y^{II}$, $\quad y^{III}(0) = 2 - y^{II}(0) = 2 - 1 = 1$,

$y^{IV}(x) = - y^{III}$, $\quad y^{IV}(0) = -y^{III}(0) = -1$.

The Taylor's servies for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + \frac{x}{1!}y^I(0) + \frac{x^2}{2!}y^{II}(0) + \frac{x^3}{3!}y^{III}(0) + \frac{x^4}{4!}y^{IV}(0) + \ldots\ldots$$

substituting the values of $y(0)$, $y^1(0)$, $y^{11}(0)$, $y^{111}(0)$, ……

$$y(x) = 1 + x(-1) + \frac{x^2}{2}(1) + \frac{x^3}{6}(1) + \frac{x^4}{24}(-1) + \ldots\ldots$$

$$y(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + \ldots\ldots \quad \rightarrow(1)$$

Now put x = 0.1 in (1),

$$y(0.1) = 1 - 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{6} - \frac{(0.1)^4}{24} + \ldots.$$

$$= 1 - 0.1 + 0.005 + 0.01666 - 0.0000416 - 0.905125 \sim 0.9051$$

(4 decimal places)

Now put x = 0.2 in eq (1),

$$y(0.2) = 1 - 0.2 + \frac{(0.2)^2}{2} + \frac{(0.2)^3}{6} - \frac{(0.2)^4}{64}$$

$$= 1 - 0.2 + 0.02 + 0.001333 - 0.000025$$

$= 1.021333 – 0.200025$

$= 0.821308 \sim 0.8213$ (4 decimals)

Similarly y(0.3) = 0.7492 and y (0.4) = 0.6897 (4 decimal places).

5. Solve $\frac{dy}{dx}$ -1 = xy and y(0) = 1 using Taylor's series method and compute y(0.1).

Sol. Given that $\frac{dy}{dx} - 1 = xy$ and y(0) = 1

Here $\frac{dy}{dx} = 1 + xy$ and $y_0 = 1$, $x_0 = 0$.

Differentiating repeatedly w.r.t 'x' and evaluating at $x_0 = 0$

$y^I(x) = 1 + xy$, $y^I(0) = 1+0(1) = 1$.

$y^{II}(x) = x.y' + y$, $y^{II}(0) = 0+1=1$

$y^{III}(x) = x.y'' + y^I + y^I$, $y^{III}(0) = 0.(1) + 2 \square(1) = 2$

$y^{IV}(x) = xy^{III} + y^{II} + 2y^{II,}$ $y^{IV}(0) = 0+3(1) = 3$.

$y^V(x) = xy^{IV} + y^{III} + 2y^{III}$, $y^V(0) = 0 + 2 + 2(3) = 8$

The Taylor series for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + x.y^I(0) + \frac{x^2}{2!} y^{II}(0) + \frac{x^3}{3!} y^{III}(0) + \frac{x^4}{4!} y^{IV}(0) + \frac{x^5}{5!} y^V(0) + \ldots..$$

Substituting the values of y(0) , $y^I(0)$, $y^{II}(0)$, ….

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}(2) + \frac{x^4}{24}(3) + \frac{x^5}{120}(8) + \ldots.$$

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \ldots. \qquad \rightarrow (1)$$

Now put x = 0.1 in equ (1),

$$y(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{8} + \frac{(0.1)^5}{15} + \ldots..$$

$= 1 + 0.1 + 0.005 + 0.000333 + 0.0000125 + 0.0000006$

$= 1.1053461$

<u>H.W</u>

6. Given the differential equ $y^1 = x^2 + y^2$, y(0) = 1.Obtain y(0.25), and y(0.5) by Taylor's Series method.

Ans: 1.3333, 1.81667

7. Solve $y^1 = xy^2 + y$, y(0) =1 using Taylor's series method and compute y(0.1) and y(0.2).

Ans: 1.111, 1.248.

Note: We know that the Taylor's expansion of y(x) about the point x_0 in a power of $(x - x_0)$is.

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y^{I}(x_0) + \frac{(x-x_0)^2}{2!} y^{II}(x_0) + \frac{(x-x_0)^3}{3!} y^{III}(x_0) + \ldots \rightarrow (1)$$

Or

$$y(x) = y_0 + \frac{(x-x_0)}{1!}\ y_0^{I} + \frac{(x-x_0)^2}{2!}\ y_0^{II} + \frac{(x-x_0)^3}{3!}\ y_0^{III} + \ldots..$$

If we let $x - x_0 = h$. (i.e. $x = x_0 + h = x_1$) we can write the Taylor's series as

$$y(x) = y(x_1) = y_0 + \frac{h}{1!}\ y_0^{I} + \frac{h^2}{2!}\ y_0^{II} + \frac{h^3}{3!}\ y_0^{III} + \frac{h^4}{4!}\ y_0^{IV} + \ldots.$$

$$\text{i.e. } y_1 = y_0 + \frac{h}{1!}\ y_0^{I} + \frac{h^2}{2!}\ y_0^{II} + \frac{h^3}{3!}\ y_0^{III} + \frac{h^{IV}}{4!} y_0^{IV} + \ldots.. \rightarrow (2)$$

Similarly expanding y(x) in a Taylor's series about $x = x_1$. We will get.

$$y_2 = y1 + \frac{h}{1!} y_1^{I} + \frac{h^2}{2!} y_1^{II} + \frac{h^3}{3!} y_1^{III} + \frac{h^4}{4!} y_1^{IV} + \ldots\ldots. \rightarrow (3)$$

Similarly expanding y(x) in a Taylor's series about $x = x_2$ We will get.

$$y_3 = y_2 + \frac{h}{1!} y_2^{I} + \frac{h^2}{2!} y_2^{II} + \frac{h^3}{3!} y_2^{III} + \frac{h^4}{4!} y_2^{IV} + \ldots... \rightarrow (4)$$

In general, Taylor's expansion of y(x) at a point $x= x_n$ is

$$y_{n+1} = y_n + \frac{h}{1!} y_n^{I} + \frac{h^2}{2!} y_n^{II} + \frac{h^3}{3!} y_n^{III} + \frac{h^4}{4!} y_n^{IV} + \ldots.. \rightarrow (5)$$

8. Solve $y^1 = x-y^2$, y(0) = 1 using Taylor's series method and evaluate y(0.1), y(0.2).

Sol: Given $y^1 = x - y^2$ $\rightarrow$(1)

and y(0) = 1 $\rightarrow$(2)

Here $x_0 = 0$, $y_0 = 1$.

Differentiating (1) w.r.t 'x', we get.

$y^{II} = 1 - 2yy^{I} \rightarrow (3)$

$y^{III} = -2(y.\ y^{II} + (y^{I})^2)$ $\rightarrow$ (4)

$y^{IV} = -2[y.\ y^{III} + y.\ y^{II} + 2y^{I}.\ y^{II}]$ $\rightarrow$(5)

$= -2(3y^{I}.\ y^{II} + y.\ y^{III})$ ……

Put $x_0 = 0$, $y_0 = 1$ in (1),(3),(4) and (5),

We get

$y_0^{I} = 0-1 = -1,$

$y_0^{II} = 1-2(1)(-1) = 3,$

$y_0^{III} = -2[(-1)^2) + (1)(3)] = -8$

$y_0^{IV} = -2[3(-1)(3) + (1)(-8)] = -2(-9 -8) = 34.$

Take h=0.1

Step1: By Taylor's series, we have

$y_1 = y_0 + \frac{h}{1!} y_0^I + \frac{h^2}{2!} y_0^{II} + \frac{h^3}{3!} y_0^{III} + \frac{h^4}{4!} y_0^{IV} + \ldots..$ →**(6)**

on substituting the values of y_0, y_0^I, y_0^{II}, etc in equ (6) we get

$$y(0.1) = y_1 = 1 + \frac{0.1}{1}(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-8) + \frac{(0.1)^4}{24}(34) + \ldots.$$

$= 1 – 0.1 + 0.015 – 0.00133 + 0.00014 + \ldots$

$= 0.91381$

<u>Step2</u>: Let us find y(0.2), we start with (x_1,y_1) as the starting value.

Here $x_1 = x_0 + h = 0+0.1 = 0.1$ and $y_1 = 0.91381$

Put these values of x_1 and y_1 in (1),(3),(4) and (5),we get

$$y_1^I = x_1 - y_1^2 = 0.1 – (0.91381)^2 = 0.1 – 0.8350487 = -0.735$$

$$y_1^{II} = 1 – 2y_1 \cdot y_1^I = 1- 2(0.91381)(-0.735) = 1 + 1.3433 = 2.3433$$

$$y_1^{III} = - 2[(y_1^I)^2 + y_1 \cdot y_1^{II}] = - 2[(-0.735)^2 + (0.91381)(2.3433)] = -5.363112$$

$$y_1^{IV} = - 2[3.\ y_1^I\ y_1^{II} + y_1\ y_1^{III}] = - 2[3.(-0.735)(2.3433) + (0.91381)(-5.363112)]$$

$$= -2[(-5.16697) – 4.9] = 20.133953$$

By Taylor's series expansion,

$$y_2 = y_1 + \frac{h}{1!} y_1^I + \frac{h^2}{2!} y_1^{II} + \frac{h^3}{3!} y_1^{III} + \frac{h^4}{4!} y_1^{IV} + \ldots.$$

$$\therefore y(0.2) = y_2 = 0.91381 + (0.1)(-0.735) + \frac{(0.1)^2}{2}(2.3433) +$$

$$\frac{(0.1)^3}{6}(-5.363112) + \frac{(0.1)^4}{24}(20.133953) + \ldots.$$

$y(0.2) = 0.91381 – 0.0735 + 0.0117 – 0.00089 + 0.00008$

$= 0.8512$

9. Tabulate y(0.1), y(0.2) and y(0.3) using Taylor's series method given that $y^1 = y^2 + x$ and y(0) = 1

Sol: Given $y^1 = y^2 + x$ →(1)

and $y(0) = 1$ →(2)

Here $x_0 = 0$, $y_0 = 1$.

Differentiating (1) w.r.t 'x', we get

$y^{II} = 2y \cdot y^I + 1$ →(3)

$y^{III} = 2[y \cdot y^{II} + (y^I)^2]$ →(4)

$y^{IV} = 2[y \cdot y^{III} + y^I y^{II} + 2 y^I y^{II}]$

$= 2[y \cdot y^{III} + 3 y^I y^{II}]$ →(5)

Put $x_0 = 0$, $y_0 = 1$ in (1), (3), (4) and (5), we get

$y_0^I = (1)^2 + 0 = 1$

$y_0^{II} = 2(1)(1) + 1 = 3,$

y_0^{III} = 2((1) (3) + (1)^2) = 8

y_0^{IV} = 2[(1)(8) + 3(1)(3)]

= 34

Take h = 0.1.

Step1: By Taylor's series expansion, we have

$$y(x_1) = y_1 = y_0 + \frac{h}{1!}y_0^I + \frac{h^2}{2!}y_0^{II} + \frac{h^3}{3!}y_0^{III} + \frac{h^4}{4!}y_0^{IV} + \ldots \quad \rightarrow(6)$$

on substituting the values of y_o, y_0^I , y_0^{II} etc in (6),we get

$$y(0.1) = y_1 = 1 + (0.1)(1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(8) + \frac{(0.1)^4}{24}(34) + \ldots$$

= 1 + 0.1 + 0.015 + 0.001333 + 0.000416

y_1 = 1.116749

Step2: Let us find y(0.2),we start with (x_1,y_1) as the starting values

Here $x_1 = x_0 + h = 0 + 0.1 = 0.1$ and y_1 = 1.116749

Putting these values in (1),(3),(4) and (5), we get

$y_1^I = y_1^2 + x_1 = (1.116749)^2 + 0.1 = 1.3471283$

$y_1^{II} = 2y_1\, y_1^I + 1$ = 2(10116749) (1.3471283) + 1 = 4.0088

$y_1^{III} = 2(y_1\, y_1^{II} + (y_1^I)^2)$ = 2((1.116749) (4.0088) + (1.3471283)^2] = 12.5831

$y_1^{IV} = 2y_1\, y_1^{III} + 6\, y_1^I\, y_1^{II}$ = 2(1.116749) (12.5831) + 6(1.3471283) (4.0088) = 60.50653

By Taylor's expansion

$$y(x_2) = y_2 = y_1 + \frac{h}{1!}y_1^I + \frac{h^2}{2!}y_1^{II} + \frac{h^3}{3!}y_1^{III} + \frac{h^4}{4!}y_1^{IV} + \ldots$$

$$\therefore y(0.2) = y_2 = 1.116749 + (0.1)(1.3471283) + \frac{(0.1)^2}{2}(4.0088) + \frac{(0.1)^3}{6}(12.5831)$$

$$+ \frac{(0.1)^4}{24}(60.50653)$$

y_2 = 1.116749 + 0.13471283 + 0.020044 + 0.002097 + 0.000252

= 1.27385

y(0.2) = 1.27385

Step3: Let us find y(0.3),we start with (x_2,y_2) as the starting value.

Here $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$ and y_2 = 1.27385

Putting these values of x_2 and y_2 in eq (1), (3), (4) and (5), we get

$y_2^I = y_2^2 + x_2 = (1.27385)^2 + 0.2 = 1.82269$

$y_2^{II} = 2y_2\, y_2^1 + 1$ = 2(1.27385) (1.82269) + 1 = 5.64366

$y_2^{III} = 2[y_2\, y_2^{II} + (y_2^I)^2]$ = 2[(1.27385) (5.64366) + (1.82269)^2]

= 14.37835 + 6.64439 = 21.02274

$y_2^{IV} = 2y_2 + y_2^{III} + 6\, y_2^I \cdot y_1^{II}$ = 2(1.27385) (21.00274) + 6(1.82269) (5.64366)

$= 53.559635 + 61.719856 \quad = 115.27949$

By Taylor's expansion,

$$y(x_3) = y_3 = y_2 + \frac{h}{1!}y_2^I + \frac{h^2}{2!}y_2^{II} + \frac{h^3}{3!}y_2^{III} + \frac{h^4}{4!}y_2^{IV} + \ldots..$$

$$y(0.3) = y_3 = 1.27385 + (0.1)(1.82269) + \frac{(0.1)^2}{2}(5.64366) + \frac{(0.1)^3}{6}(21.02274)$$

$$+ \frac{(0.1)^4}{24}(115.27949)$$

$= 1.27385 + 0.182269 + 0.02821 + 0.0035037 + 0.00048033$

$= 1.48831$

$y(0.3) = 1.48831$

10. Solve $y^1 = x^2 - y$, $y(0) = 1$ using Taylor's series method and evaluate

$y(0.1), y(0.2), y(0.3)$ and $y(0.4)$ (correct to 4 decimal places)

Sol: Given $y^1 = x^2 - y$ →(1)

and $y(0) = 1$ →(2)

Here $x_0 = 0$, $y_0 = 1$

Differentiating (1) w.r.t 'x', we get

$y^{II} = 2x - y^1$ →(3)

$y^{III} = 2 - y^{II}$ →(4)

$y^{IV} = -y^{III}$ →(5)

put $x_0 = 0$, $y_0 = 1$ in (1),(3),(4) and (5), we get

$y_0^I = x_0^2 - y_0 = 0-1 = -1,$

$y_0^{II} = 2x_0 - y_0^I = 2(0) - (-1) = 1$

$y_0^{III} = 2 - y_0^{II} = 2-1 = 1,$

$y_0^{IV} = -y_0^{III} = -1$ Take $h = 0.1$

<u>Step1</u>: by Taylor's series expansion

$$y(x_1) = y_1 = y_0 + \frac{h}{1!}y_0^I + \frac{h^2}{2!}y_0^{II} + \frac{h^3}{3!}y_0^{III} + \frac{h^4}{4!}y_0^{IV} + \ldots. \quad \rightarrow(6)$$

On substituting the values of y_0, y_0^I, y_0^{II} etc in (6), we get

$$y(0.1) = y_1 = 1 + (0.1)(-1) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(1) + \frac{(0.1)^4}{24}(-1) + \ldots.$$

$= 1 - 0.1 + 0.005 + 0.01666 - 0.0000416$

$= 0.905125 \simeq 0.9051$ (4 decimal place).

<u>Step2</u>: Let us find $y(0.2)$ we start with (x_1, y_1) as the starting values

Here $x = x_0 + h = 0 + 0.1 = 0.1$ and $y_1 = 0.905125$,

Putting these values of x_1 and y_1 in (1), (3), (4) and (5), we get

$y_1^1 = x_1^2 - y_1 = (0.1)^2 - 0.905125 = -0.895125$

$y_1^{II} = 2x_1 - y_1^1 = 2(0.1) - (-0.895125) = 1.095125,$

y_1^{III} = 2 - y_1^{II} = 2 – 1.095125 = 0.90475,

y_1^{IV} = - y_1^{III} = -0.904875,

By Taylor's series expansion,

$$y(x_2) = y_2 = y_1 + \frac{h}{1!} y_1^I + \frac{h^2}{2!} y_1^{II} + \frac{h^3}{3!} y_1^{III} + \frac{h^4}{4!} y_1^{IV} + \ldots$$

$$y(0.2) = y_2 = 0.905125 + (0.1)(-0.895125) + + \frac{(0.1)^2}{2}(1.09125) + \frac{(0.1)^3}{6}(1.095125) + \frac{(0.1)^4}{24}(-0.904875) + \ldots$$

y(0.2) = y_2 = 0.905125 – 0.0895125 + 0.00547562 + 0.000150812 – 0.0000377

= 0.8212351 ~ 0.8212 (4 decimal places)

Step3: Let us find y(0.3), we start with (x_2,y_2) as the starting value

Here $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$ and $y_2 = 0.8212351$

Putting these values of x_2 and y_2 in (1),(3),(4), and (5) we get

$y_2^1 = x_2^2 - y_2 = (0.2)^2$ – 0.8212351= 0.04 – 0.8212351 = - 0.7812351

y_2^{II} = 2x_2 - y_2^1 = 2(0.2) + (0.7812351) = 1.1812351,

y_2^{III} = 2 - y_2^{II} = 2 – 1. 1812351 = 0.818765,

y_2^{IV} = - y_2^{III} = -0.818765,

By Taylor's series expansion,

$$y(x_3) = y_3 = y_2 + \frac{h}{1!} y_2^I + \frac{h^2}{2!} y_2^{II} + \frac{h^3}{3!} y_2^{III} + \frac{h^4}{4!} y_2^{IV} + \ldots$$

$$y(0.3) = y_3 = 0.\ 8212351 + (0.1)(-0.7812351) + \frac{(0.1)^2}{2}(1.1812351) + \frac{(0.1)^3}{6}(0.818765) + \frac{(0.1)^4}{24}(-0.818765) + \ldots$$

y(0.3) = y_3 = 0. 8212351– 0.07812351+ 0.005906 + 0.000136 – 0.0000034

= 0.749150 ~ 0.7492 (4 decimal places)

Step4: Let us find y(0.4), we start with (x_3,y_3) as the starting value

Here $x_3 = x_2 + h = 0.2 + 0.1 = 0.3$ and $y_3 = 0.749150$

Putting these values of x_3 and y_3 in (1),(3),(4), and (5) we get

$y_3^1 = x_3^2 - y_3 = (0.3)^2$ – 0.749150= -0.65915,

y_3^{II} = 2x_3 - y_3^1 = 2(0.3) + (0.65915) = 1.25915,

y_3^{III} = 2 - y_3^{II} = 2 – 1. 25915 = 0.74085,

y_3^{IV} = - y_3^{III} = -0.74085,

By Taylor's series expansion,

$y(x_4) = y_4 = y_3 + \frac{h}{1!} y_3^{I} + \frac{h^2}{2!} y_3^{II} + \frac{h^3}{3!} y_3^{III} + \frac{h^4}{4!} y_3^{IV} + \ldots.$

$y(0.4) = y_4 = 0.749150 + (0.1)(-0.65915) + \frac{(0.1)^2}{2}(1.25915) +$

$\frac{(0.1)^3}{6}(0.74085) + \frac{(0.1)^4}{24}(-0.74085) + \ldots.$

$y(0.4) = y_4 = 0.749150 – 0.065915 + 0.0062926 + 0.000123475 – 0.0000030$

$= 0.6896514 \simeq 0.6896$ (4 decimal places)

11. Solve $y^1 = x^2 – y$, $y(0) = 1$using T.S.M and evaluate y(0.1),y(0.2),y(0.3) and y(0.4) (correct to 4 decimal place) 0.9051, 0.8212, 07492, 0.6896

12. Given the differentiating equation $y^1 = x^1 + y^2$, $y(0) = 1$. Obtain y(0.25) and y(0.5) by T.S.M.

Ans: 1.3333, 1.81667

13. Solve $y^1 = xy^2 + y$, $y(0) = 1$ using Taylor's series method and evaluate y(0.1) and y(0.2)

Ans: 1.111, 1.248.

EULER'S METHOD

It is the simplest one-step method and it is less accurate. Hence it has a limited application.

Consider the differential equation $\frac{dy}{dx} = f(x,y)$ →(1)

With $y(x_0) = y_0$ →(2)

Consider the first two terms of the Taylor's expansion of y(x) at $x = x_0$

$y(x) = y(x_0) + (x – x_0)\, y^1(x_0)$ →(3)

from equation (1) $y^1(x_0) = f(x_0,y(x_0)) = f(x_0,y_0)$

Substituting in equation (3)

$\therefore y(x) = y(x_0) + (x – x_0) f(x_0,y_0)$

At $x = x_1$, $y(x_1) = y(x_0) + (x_1 – x_0) f(x_0,y_0)$

$\therefore y_1 = y_0 + h f(x_0,y_0)$ where $h = x_1 – x_0$

Similarly at $x = x_2$, $y_2 = y_1 + h f(x_1,y_1)$,

Proceeding as above, $y_{n+1} = y_n + h f(x_n,y_n)$

This is known as Euler's Method

1. **Using Euler's method solve for x = 2 from $\frac{dy}{dx} = 3x^2 + 1, y(1) = 2$,taking step size** (I) h = 0.5 and (II) h=0.25

Sol: here $f(x,y) = 3x^2 + 1$, $x_0 = 1, y_0 = 2$

Euler's algorithm is $y_{n+1} = y_n + h\, f(x_n,y_n)$, n = 0,1,2,3,….. →(1)

(I) h = 0.5 $\therefore x_1 = x_0 + h = 1+0.5 = 1.5$

Taking n = 0 in (1) , we have $x_2 = x_1 + h = 1.5 + 0.5 = 2$

$y_1 = y_0 + h\, f(x_0,y_0)$

i.e. $y_1 = y(0.5) = 2 + (0.5)\, f(1,2) = 2 + (0.5)\,(3 + 1) = 2 + (0.5)(4)$

Here $x_1 = x_0 + h = 1 + 0.5 = 1.5$

$\therefore y(1.5) = 4 = y_1$

Taking n = 1 in (1),we have

$y_2 = y_1 + h\, f(x_1,y_1)$

i.e. $y(x_2) = y_2 = 4 + (0.5)\, f(1.5,4) = 4 + (0.5)[3(1.5)^2 + 1] = 7.875$

Here $x_2 = x_4 + h = 1.5 + 0.5 = 2$

$\therefore y(2) = 7.875$

(II) h = 0.25 $\therefore x_1 = 1.25,\ x_2 = 1.50,\ x_3 = 1.75,\ x_4 = 2$

Taking n = 0 in (1), we have

$y_1 = y_0 + h\, f(x_0,y_0)$

i.e. $y(x_1) = y_1 = 2 + (0.25)\, f(1,2) = 2 + (0.25)\,(3 + 1) = 3$

$y(x_2) = y_2 = y_1 + h\, f(x_1,y_1)$

i.e. $y(x_2) = y_2 = 3 + (0.25)\, f(1.25,3)$

$= 3 + (0.25)[3(1.25)^2 + 1]$

$= 4.42188$

Here $x_2 = x_1 + h = 1.25 + 0.25 = 1.5$

$\therefore y(1.5) = 5.42188$

Taking n = 2 in (1), we have

i.e. $y(x_3) = y_3 = h\, f(x_2,y_2)$

$= 5.42188 + (0.25)\, f(1.5,2)$

$= 5.42188 + (0.25)\,[3(1.5)^2 + 1]$

$= 6.35938$

Here $x_3 = x_2 + h = 1.5 + 0.25 = 1.75$

$\therefore y(1.75) = 7.\ 35938$

Taking n = 4 in (1),we have

$y(x_4) = y_4 = y_3 + h\, f(x_3,y_3)$

i.e. $y(x_4) = y_4 = 7.35938 + (0.25)\, f(1.75,2)$

$= 7.35938 + (0.25)[3(1.75)^2 + 1]$

$= 8.90626$

Note that the difference in values of y(2) in both cases

(i.e. when h = 0.5 and when h = 0.25).The accuracy is improved significantly when h is reduced to 0.25 (Example significantly of the equ is $y = x^3 + x$ and with this $y(2) = y_2 = 10$

2. **Solve by Euler's method, $y^1 = x + y$, y(0) = 1 and find y(0.3) taking step size h = 0.1. compare the result obtained by this method with the result obtained by analytical solution**

Sol: $y_1 = 1.1 = y(0.1)$,

$y_2 = y(0.2) = 1.22$

$y_3 = y(0.3) = 1.362$

Particular solution is $y = 2e^x - (x + 1)$

Hence y(0.1) = 1.11034, y(0.2) = 1.3428, y(0.3) = 1.5997

We shall tabulate the result as follows

X	0	0.1	0.2	0.3
Euler y	1	1.1	1.22	1.362
Euler y	1	1.11034	1.3428	1.3997

The value of y deviate from the execute value as x increases. This indicate that the method is not accurate

3. **Solve by Euler's method $y^1 + y = 0$ given y(0) = 1 and find y(0.04) taking step size**

h = 0.01 Ans: 0.9606

4. **Using Euler's method, solve y at x = 0.1 from $y^1 = x + y + xy$, y()) = 1 taking step size h = 0.025.**

5. **Given that $\frac{dy}{dx}$ = xy ,y(0) = 1 determine y(0.1),using Euler's method.** h =0.1

Sol: The given differentiating equation is $\frac{dy}{dx} = xy$, y(0) = 1

a = 0

Here f(x,y) = xy , $x_0 = 0$ and $y_0 = 1$

Since h is not given much better accuracy is obtained by breaking up the interval (0,0.1) in to five steps.

i.e. $h = \frac{b-a}{5} = \frac{0.1}{5} = 0.02$

Euler's algorithm is $y_{n+1} = y_n + h\, f(x_n, y_n)$ →(1)

∴ From (1) form = 0, we have

$y_1 = y_0 + h\, f(x0, y_0)$

$= 1 + (0.02)\, f(0,1)$

$= 1 + (0.02)\,(0)$

$= 1$

Next we have $x_1 = x_0 + h = 0 + 0.02 = 0.02$

$\therefore$ From (1), form = 1,we have

$y_2 = y_1 + h\, f(x_1, y_1)$

$= 1 + (0.02)\, f(0.02,1)$

$= 1 + (0.02)\,(0.02)$

$= 1.0004$

Next we have $x_2 = x_1 + h = 0.02 + 0.02 = 0.04$

$\therefore$ From (1), form = 2,we have

$y_3 = y_2 + h\, f(x_2, y_2)$

$= 1.004 + (0.02)\,(0.04)\,(1.0004)$

$= 1.0012$

Next we have $x_3 = x_2 + h = 0.04 + 0.02 = 0.06$

$\therefore$ From (1), form = 3,we have

$y_4 = y_3 + h\, f(x_3, y_3)$

$= 1.0012 + (0.02)\,(0.06)\,(1.00012)$

$= 1.0024.$

Next we have $x_4 = x_3 + h = 0.06 + 0.02 = 0.08$

$\therefore$ From (1), form = 4,we have

$y_5 = y_4 + h\, f(x_4, y_4)$

$= 1.0024 + (0.02)\,(0.08)\,(1.00024)$

$= 1.0040.$

Next we have $x_5 = x_4 + h = 0.08 + 0.02 = 0.1$

When $x = x_5$, $y \simeq y_5$

$\therefore y = 1.0040$ when $x = 0.1$

6. Solve by Euler's method $y^1 = \frac{2y}{x}$ given y(1) = 2 and find y(2).

7. Given that $\frac{dy}{dx} = 3x^2 + y$, y(0) = 4.Find y(0.25) and y(0.5) using Euler's method

Sol: given $\frac{dy}{dx} = 3x^2 + y$ and y(1) = 2.

Here $f(x,y) = 3x^2 + y$, $x_0 = (1)$, $y_0 = 4$

Consider h = 0.25

Euler's algorithm is $y_{n+1} = y_n + h\, f(x_n, y_n)$ → (1)

$\therefore$ From (1), for n = 0, we have

$y_1 = y_0 + h\, f(x_0, y_0)$

$= 2 + (0.25)[0 + 4]$

$= 2 + 1$

= 3

Next we have $x_1 = x_0 + h = 0 + 0.25 = 0.25$

When $x = x_1$, $y_1 \simeq y$

$\therefore$ y = 3 when x = 0.25

$\therefore$ From (1), for n = 1, we have

$y_2 = y_1 + h\, f(x_1,y_1)$

$= 3 + (0.25)[3.(0.25)^2 + 3]$

$= 3.7968$

Next we have $x_2 = x_1 + h = 0.25 + 0.25 = 0.5$

When $x = x_2$, $y \simeq y_2$

$\therefore$ y = 3.7968 when x = 0.5.

8. **Solve first order diff equation $\frac{dy}{dx} = \frac{y-x}{y+x}$, y(0) = 1 and estimate y(0.1) using Euler's method (5 steps)** Ans: 1.0928

9. **Use Euler's method to find approximate value of solution of $\frac{dy}{dx}$ = y-x + 5 at x = 2-1 and 2-2with initial contention y(0.2) = 1**

Modified Euler's method

It is given by $y^{(i)}{}_{k+1} = y_k + h/2\, f\left[\left(x_k, y_k\right) + f\left(x_{k+1}, 1\right)_{k+1}{}^{(i-1)}\right], i = 1, 2....., ki = 0, 1.....$

Working rule :

i)Modified Euler's method

$y^{(i)}{}_{k+1} = y_k + h/2\, f\left[\left(x_k, y_k\right) + f\left(x_{k+1}, 1\right)_{k+1}{}^{(i-1)}\right], i = 1, 2....., ki = 0, 1.....$

ii) When $i = 1$ $y^0{}_{k+1}$ can be calculated from Euler's method

iii) K=0, 1……… gives number of iteration. $i = 1, 2...$

gives number of times, a particular iteration k is repeated

Suppose consider dy/dx=f(x, y) -------- (1) with $y(x_0) = y_0$----------- (2)

To find $y(x_1) = y_1$ at $x = x_1 = x_0 + h$

Now take k=0 in modified Euler's method

We get $y_1^{(1)} = y_0 + h/2\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(i-1)}\right)\right]$……………………… (3)

Taking i=1, 2, 3...k+1 in eqn (3), we get

$y_1^{(0)} = y_0 + h/2\left[f\left(x_0, y_0\right)\right]$ (By Euler's method)

$y_1^{(1)} = y_0 + h/2\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(0)}\right)\right]$

$y_1^{(2)} = y_0 + h/2\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(1)}\right)\right]$

$y_1^{(k+1)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(k)}\right)\right]$

If two successive values of $y_1^{(k)}, y_1^{(k+1)}$ are sufficiently close to one another, we will take the common value as $y_2 = y(x_2) = y(x_1 + h)$

We use the above procedure again

1) using modified Euler's method find the approximate value of x when $x = 0.3$ given that $dy/dx = x + y \; and \; y(0) = 1$

sol: Given $dy/dx = x + y \; and \; y(0) = 1$

Here $f(x, y) = x + y, x_0 = 0, \; and \; y_0 = 1$

Take h = 0.1 which is sufficiently small

Here $x_0 = 0, x_1 = x_0 + h = 0.1, x_2 = x_1 + h = 0.2, x_3 = x_2 + h = 0.3$

The formula for modified Euler's method is given by

$$y_{k+1}^{(i)} = y_k + h/2\left[f(x_k + y_k) + f\left(x_{k+1}, y_{k+1}^{(i-1)}\right)\right] \rightarrow (1)$$

Step1: To find $y_1 = y(x_1) = y(0.1)$

Taking k = 0 in eqn(1)

$$y_{k+1}^{(i)} = y_0 + h/2\left[f(x_0 + y_0) + f\left(x_1, y_1^{(i-1)}\right)\right] \rightarrow (2)$$

when i = 1 in eqn (2)

$$y_1^{(i)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$

First apply Euler's method to calculate $y_1^{(0)}$ = y_1

$\therefore \; y_1^{(0)} = y_0 + h\, f(x_0, y_0)$

= 1+(0.1)f(0.1)

= 1+(0.1)

= 1.10

$now\left[x_0 = 0, y_0 = 1, x_1 = 0.1, y_1(0) = 1.10\right]$

$\therefore \; y_1^{(1)} = y_0 + 0.1/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$

= 1+0.1/2[f(0,1) + f(0.1,1.10)

= 1+0.1/2[(0+1)+(0.1+1.10)]

= 1.11

When i=2 in eqn (2)

$$y_1^{(2)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

= 1+0.1/2[f(0.1)+f(0.1,1.11)]

= 1 + 0.1/2[(0+1)+(0.1+1.11)]

= 1.1105

$$y_1^{(3)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right]$$

= 1+0.1/2[f(0,1)+f(0.1 , 1.1105)]

= 1+0.1/2[(0+1)+(0.1+1.1105)]

= 1.1105

Since $y_1^{(2)} = y_1^{(3)}$

$\therefore$ y_1 = 1.1105

Step:2 To find $y_2 = y(x_2) = y(0.2)$

Taking k = 1 in eqn (1) , we get

$$y_2^{(i)} = y_1 + h/2\left[f\left(x_1, y_1\right) + f\left(x_2, y_2^{(i-1)}\right)\right] \rightarrow (3)$$

i = 1,2,3,4,…..

For i = 1

$$y_2^{(1)} = y_1 + h/2\left[f\left(x_1, y_1\right) + f\left(x_2, y_2^{(0)}\right)\right]$$

$y_2^{(0)}$ is to be calculate from Euler's method

$$y_2^{(0)} = y_1 + h\ f\left(x_1, y_1\right)$$

= 1.1105 + (0.1) f(0.1 , 1.1105)

= 1.1105+(0.1)[0.1+1.1105]

= 1.2316

$$\therefore\ y_2^{(1)} = 1.1105 + 0.1/2\left[f\left(0.1, 1.1105\right) + f\left(0.2, 1.2316\right)\right]$$

= 1.1105 +0.1/2[0.1+1.1105+0.2+1.2316]

= 1.2426

$$y_2^{(2)} = y_1 + h/2\left[f\left(x_1, y_1\right) + f\left(x_2 y_2^{(1)}\right)\right]$$

= 1.1105 + 0.1/2[f(0.1 , 1.1105) , f(0.2 . 1.2426)]

= 1.1105 + 0.1/2[1.2105 + 1.4426]

= 1.1105 + 0.1(1.3266)

= 1.2432

$$y_2^{(3)} = y_1 + h/2\left[f\left(x_1, y_1\right) + f\left(x_2 y_2^{(2)}\right)\right]$$

= 1.1105+0.1/2[f(0.1,1.1105)+f(0.2 , 1.2432)]

= 1.1105+0.1/2[1.2105+1.4432)]

= 1.1105 + 0.1(1.3268)

= 1.2432

Since $y_2^{(3)} = y_2^{(3)}$

Hence y_2 = 1.2432

Step:3

To find y_3 = y(x_3) = y y(0.3)

Taking k =2 in eqn (1) we get

$$y_3^{(1)} = y_2 + h/2\left[f\left(x_2, y_2\right) + f\left(x_3, y_3^{(i-1)}\right)\right] \rightarrow (4)$$

For i = 1 ,

$$y_3^{(1)} = y_2 + h/2\left[f\left(x_2, y_2\right) + f\left(x_3, y_3^{(0)}\right)\right]$$

$y_3^{(0)}$ is to be evaluated from Euler's method .

$$y_3^{(0)} = y_2 + h\, f\left(x_2, y_2\right)$$

= 1.2432 +(0.1) f(0.2 , 1.2432)

= 1.2432+(0.1)(1.4432)

= 1.3875

$\therefore\ y_3^{(1)}$ = 1.2432+0.1/2[f(0.2 , 1.2432)+f(0.3, 1.3875)]

= 1.2432 + 0.1/2[1.4432+1.6875]

= 1.2432+0.1(1.5654)

= 1.3997

$$y_3^{(2)} = y_2 + h/2\left[f\left(x_2, y_2\right) + f\left(x_3, y_3^{(1)}\right)\right]$$

= 1.2432+0.1/2[1.4432+(0.3+1.3997)]

= 1.2432+ (0.1) (1.575)

= 1.4003

$$y_3^{(3)} = y_2 + h/2\left[f\left(x_2, y_2\right) + f\left(x_3, y_3^{(2)}\right)\right]$$

= 1.2432+0.1/2[f(0.2 , 1.2432)+f(0.3 , 1.4003)]

= 1.2432 + 0.1(1.5718)

= 1.4004

$$y_3^{(4)} = y_2 + h/2\left[f\left(x_2, y_2\right) + f\left(x_3, y_3^{(3)}\right)\right]$$

= 1.2432 + 0.1/2[1.4432+1.7004]

= 1.2432+(0.1)(1.5718)

= 1.4004

Since $y_3^{(3)} = y_3^{(4)}$

Hence $y_3 = 1.4004$ $\therefore$ The value of y at x = 0.3 is 1.4004

2 . Find the solution of $\frac{dy}{dx}$ = x-y , y(0)=1 at x =0.1 , 0.2 ,0.3 , 0.4 and 0.5 . Using modified Euler's method

Sol . Given $\frac{dy}{dx}$ = x-y and y(0) = 1

Here f(x,y) = x-y , $x_0 = 0$ and $y_0 = 1$

Consider h = 0.1 so that

x = 0.1 , $x_2 = 0.2$, $x_3 = 0.3$, $x_4 = 0.4$ and $x_5 = 0.5$

The formula for modified Euler's method is given by

$$y_{k+1}^{(i)} = y_k + h/2\ f(x_k, y_k) + f x_{k+1}, y_{k+1}^{(i-1)} \rightarrow (1)$$

Where k = 0,1, 2, 3,….. i = 1, 2, 3,…..

x	$f(x_k, y_k) = x_k - y_k$	$\frac{1}{2}\left[f(x_k, y_k) + f\left(x_{k+1}, y_{k+1}^{(i-1)}\right)\right]$	$y_{k+1}^{(i)} = y_k + h/2\left[f(x_k, y_k) + f\left(x_{k+1}, y_{k+1}^{(i-1)}\right)\right]$
K = 0			
0.0	0-1=-1	-	1+(0.1)(-1)=0.9 = $y_1^{(0)}$

0.1(i=1)	0-1=-1	½(-1-0.8) = -0.9	1+(0.1)(-0.9)=0.91
0.1(i=2)	0-1=-1	½(-1-0.81)= -0.905	1+(0.1)(-0.905)=0.9095
0.1(i=3)	0-1=-1	½(-1-0.80.95)= -0.90475	1+(0.1)(-0.90475)=0.9095
K=1			
0.1	0.1-0.9095= -0.8095	-	0.9095+(0.1)(-0.8095)=0.82855
0.2(i=1)	-0.8095	½(-0.8095-0.62855)	0.9095+(0.1)(-0.719025)=0.8376
0.2(i=2)	-0.8095	½(-0.8095-0.6376)	0.9095+(0.1)(-0.72355)=0.8371
0.2(i=3)	-0.8095	½(-0.8095-0.6371)	0.9095+(0.1)(-0.7233)=0.8372

0.2(i=4)	-0.8095	½(-0.8095-0.6372)	0.9095+(0.1)(-0.72355)=0.8371
K=2			
0.2	0.2-0.8371=-0.6371	-	0.8371+(0.1)(-0.6371)=0.7734
0.3(i=1)	= -0.6371	½(-0.6371-0.4734)	0.8371+(0.1)(-0.555)=0.7816
0.3(i=2)	= -0.6371	½(-0.6371-0.4816)	0.8371-0.056=0.7811
0.3(i=3)	= -0.6371	½(-0.6371-0.4811)	0.8371-0.05591=0.7812
0.3(i=4)	= -0.6371	½(-0.6371-0.4812)	0.8371-0.055915 = 0.7812
K =3			
0.3(i=1)	0.3-0.7812	-	0.7812+(0.1)(-0.4812) = 0.7331
0.4(i=1)	-0.4812	½(-0.4812-0.4311)	0.7812-0.0457 = 0.7355
0.4(i=2)	-0.4812	½(-0.4812-0.4355)	0.7812-0.0458 = 0.7354
0.4(i=3)	-0.4812	½(-0.4812-0.4354)	0.7812-0.0458 = 0.7354
K=4			
0.4	-0.3354	-	0.7354-0.03354 = 0.70186
0.5	-0.3354	½(-0.3354-0.301816)	0.7354-0.03186 = 0.7035
0.5	-0.3354	½(-0.3354-0.30354)	0.7354-0.0319 = 0.7035

3. Find y(0.1) and y(0.2) using modified Euler's formula given that $dy/dx=x^2-y, y(0)=1$

[consider $h=0.1, y_1=0.90523, y_2=0.8214$]

4. Given $dy/dx = -xy^2, y(0) = 2$ compute y(0.2) in steps of 0.1

Using modified Euler's method

[$h=0.1$, $y_1=1.9804$, $y_2=1.9238$]

5. Given $y^1 = x+\sin y$, $y(0)=1$ compute y(0.2) and y(0.4) with h=0.2 using modified Euler's method

[$y_1=1.2046$, $y_2=1.4644$]

Runge – Kutta Methods

I. Second order R-K Formula

$y_{i+1} = y_i+1/2\ (K_1+K_2)$,

Where $K_1 = h\ (x_i, y_i)$

$K_2 = h\ (x_i+h, y_i+k_1)$

For i= 0,1,2-------

II. Third order R-K Formula

$y_{i+1} = y_i+1/6\ (K_1+4K_2+ K_3)$,

Where $K_1 = h\ (x_i, y_i)$

$K_2 = h\ (x_i+h/2, y_0+k_1/2)$

$K_3 = h\ (x_i+h, y_i+2k_2-k_1)$

For i= 0,1,2-------

III. Fourth order R-K Formula

$y_{i+1} = y_i+1/6\ (K_1+2K_2+ 2K_3+K_4)$,

Where $K_1 = h\ (x_i, y_i)$

$K_2 = h\ (x_i+h/2, y_i+k_1/2)$

$K_3 = h\ (x_i+h/2, y_i+k_2/2)$

$K_4 = h\ (x_i+h, y_i+k_3)$

For i= 0,1,2-------

1. Using Runge-Kutta method of second order, find $y(2.5)$ from $\frac{dy}{dx} = \frac{x+y}{x}$, y(2)=2 , h = 0.25 .

Sol: Given $\frac{dy}{dx} = \frac{x+y}{x}$, y(2) = 2 .

Here $f(x, y) = \frac{x+y}{x}$, $x_0 = 0$, $y_0=2$ and h = 0.25

$\therefore$ $x_1 = x_0+h = 2+0.25 = 2.25$, $x_2 = x_1+h = 2.25+0.25 = 2.5$

By R-K method of second order,

$y_{i+1} = y_i + 1/2(k_1 + k_2), k_1 - hf(x_i + h, y_i + k_1), i = 0,1.... \to (1)$

Step -1:-

To find $y(x_1)$i.e y(2.25) by second order R - K method taking i=0 in eqn(i)

We have $y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

Where k_1= hf (x_0,y_0), k_2= hf (x_0+h,y_0+k_1)

f (x_0,y_0)=f(2,2)=2+2/2=2

k_1=hf (x_0,y_0)=0.25(2)=0.5

k_2= hf (x_0+h,y_0+k_1)=(0.25)f(2.25,2.5)

=(0.25)(2.25+2.5/2.25)=0.528

$\therefore$ y_1=y(2.25)=2+1/2(0.5+0.528)

=2.514

Step2:

To find $y(x_2)$ i.e., y(2.5)

i=1 in (1)

x_1=2.25,y_1=2.514,and h=0.25

$y_2=y_1+1/2(k_1+k_2)$

where k_1=h f((x_1,y_1)=(0.25)f(2.25,2.514)

=(0.25)[2.25+2.514/2.25]=0.5293

$k_2 = h\, f(x_0 + h, y_0 + k_1) = (0.1)\, f(0.1, 1 - 0.1) = (0.1)(-0.9) = -0.09$

=(0.25)[2.5+2.514+0.5293/2.5]

=0.55433

$y_2 = y$ (2.5)=2.514+1/2(0.5293+0.55433)

=3.0558

$\therefore$ y =3.0558 when x = 2.5

Obtain the values of y at x=0.1,0.2 using R-K method of

(i)second order (ii)third order (iii)fourth order for the diff eqn y^1+y=0,y(0)=1

Sol: Given dy/dx = -y, y(0)=1

f(x,y) = -y, x_0 = 0, y_0 = 1

Here f (x,y) = -y, x_0 = 0, y_0 = 1 take h = 0.1

$\therefore$ $x_1 = x_0+h = 0.1$,

$x_2 = x_1+h = 0.2$

Second order:

step1: To find $y(x_1)$ i.e $y(0.1)$ or y_1

by second-order R-K method,we have

$y_1 = y_0+1/2(k_1+k_2)$

where $k_1=hf(x_0,y_0)=(0.1)\ f(0,1) = (0.1)(-1)= -\ 0.1$

$k_2= hf\ (x_0+h,\ y_0+k_1)= (0.1)\ f\ (0.1,\ 1-0.1) = (0.1)(-0.9) = -0.09$

$y_1=y(0.1)=1+1/2(-0.1-0.09)=1-0.095=0.905$

$\therefore$ y =0.905 when x=0.1

Step2:

To find y_2 i.e $y(x_2)$ i.e $y(0.2)$

Here $x_1 = 0.1$, $y_1 = 0.905$ and h=0.1

By second-order R-K method, we have

$y_2 = y(x_2)= y_1+1/2(k_1+k_2)$

Where $k_1 = h\ f\left(x_1, y_1\right)$ =(0.1)f(0.1,0.905)=(0.1)(-0.905)=-0.0905

$$\begin{aligned} k_2 &= h\ f\left(x_1+h, y_1+k_1\right) = \left(0.1\right) f\left(0.2, 0.905-0.0905\right) \\ &= \left(0.1\right) f\left(0.2, 0.8145\right) = \left(0.1\right)\left(-0.8145\right) \\ &= -0.08145 \end{aligned}$$

$y_2 =\ \ y(0.2)=0.905+1/2(-0.0905-0.08145)$

= 0.905- 0.085975 = 0819025

Third order

Step1:

To find y_1 i.e $y(x_1)= y(0.1)$

By Third order Runge kutta method

$$y_1 = y_0 + 1/6\left(k_1 + 4k_2 + k_3\right)$$

where $k_1 = h\ f(x_0,\ y_0) = (0.1)\ f\ (0.1) =\ (0.1)\ (-1) = -0.1$

$$\begin{aligned} k_2 = h\ f\left(x_0 + h/2, y_0 + k_1/2\right) = \left(0.1\right) f\left(0.1/2, 1-0.1/2\right) &= \left(0.1\right) f\left(0.05, 0.95\right) \\ &= \left(0.1\right)\left(-0.95\right) = -0.095 \end{aligned}$$

and $k_3 = h\ f((x_0+h,y_0+2k_2-k_1)$

(0.1) f (0.1,1+2(-0.095)+0.1)= -0.905

Hence y_1 = 1+1/6(-0.1+4(-0.095)-0.09) = 1+1/6 (-0.57) = 0.905

y_1=0.905 i.e y(0.1)= 0.905

Step2:

To find y_2,i.e $y(x_2)$= y(0.2)

Here x_1=0.1,y_1=0.905 and h = 0.1

Again by 2nd order R-K method

$y_2 = y_1+1/6(k_1+4k_2+k_3)$

Where k_1=h $f(x_1, y_1)$ = (0.1)f (0.1,0.905)= -0.0905

k_2 = h f $(x_1+h/2,y_1+k_1/2)$=(0.1)f(0.1+0.2,0.905 - 0.0905)= -(0.1) f (0.15, 0.85975)= (0.1) (-0.85975)

and k_3 = h $f((x_1+h,y_1+2k_2-k_1)$=(0.1)f(0.2,0.905+2(0.08975)+0.0905= -0.082355

hence y_2 = 0.905+1/6(-0.0905+4(-0.085975)-0.082355)=0.818874

$\therefore$ y = 0.905 when x = 0.1

And y =0.818874 when x =0.2

fourth order:

step1:

x_0=0,y_0=1,h=0.1 To find y_1 i.e $y(x_1)$=y(0.1)
By 4th order R-K method, we have
$y_1 = y_0+1/6(k_1+2k_2+2k_3+k_4)$
Where k_1=h $f(x_0,y_0)$=(0.1)f(0.1)= -0.1
k_2= h f $(x_0+h/2, y_0+k1/2)$ = -0.095
and k_3= h $f((x_0+h/2,y_0+k_2/2)$=(0.1)f (0.1/2,1-0.095/2)
= (0.1)f(0.05,0.9525)
= -0.09525
and k_4= h $f(x_0+h,y_0+k_3)$
= (0.1) f(0.1,1-0.09525)=(0.1)f(0.1,0.90475)

=-0.090475

Hence y_1=1+1/6(-0.1)+2(-0.095)+2(0.09525)-0.090475)

=1+1/6(-0.570975)+1-0.951625 = 0.9048375

Step2:

To find $y_2, i.e., y(x_2) = y(0.2), y_1 = 0.9048375, i.e., y(0.1) = 0.9048375$

Here $x_1 = 0.1$, $y_1=0.9048375$ and $h = 0.1$

Again by 4th order R-K method, we have

$y_2 = y_1+1/6(k_1+2k_2+2k_3+k_4)$

Where $k_1=h\ f(x_1,y_1)=(0.1)f(0.1,0.9048375)=-0.09048375$

$k_2= hf\ (x_1+h/2,y_1+k_1/2)=(0.1)f(0.1+0.1/2,0.9048375\ -0.09048375\ /2)=-0.08595956$

and $k_3=hf(x_1+h/2,\ y_1+k_2/2)=(0.1)f(0.15,0.\ 8618577)=\ -0.08618577$

$k_4 =h\ f(x_1+h,y_1+k_3)=(0.1)f(0.2,0.86517)$

$= -0.08186517$

Hence $y_2 = 0.09048375+1/6(-0.09048375-2(0.08595956)-2(0.08618577)-\ 0.08186517$

$=0.9048375-0.0861065$

$=0.818731$

y = 0.9048375 when x =0.1 and y =0.818731

3. Apply the 4th order R-K method to find an approximate value of y when x=1.2 in steps of 0.1,given that

$y^1 = x^2+y^2$,y (1)=1.5

sol. Given $y^1= x^2+y^2$,and $y(1)=1.5$

Here $f(x,y)= x^2+y^2$, $y_0 =1.5$ and $x_0=1,h=0.1$

So that $x_1=1.1$ and $x_2=1.2$

Step1:

To find $y_{1\text{ i.e.,}}$ $y(x_1)$

by 4th order R-K method we have

$y_1=y_0+1/6\ (k_1+2k_2+2k_3+k_4)$

$k_1=hf(x_0,y_0)=(0.1)f(1,1.5)=(0.1)\ [1^2+(1.5)^2]=0.325$

$k_2= hf\ (x_0+h/2,y_0+k_1/2)=(0.1)f(1+0.05,1.5+0.325)=0.3866$

and $k_3=hf((x_0+h/2,y_0+k_2/2)=(0.1)f(1.05,1.5+0.\ 3866/2)=(0.1)[(1.05)^2+(1.6933)^2]$

$=0.39698$

$k_4=hf(x_0+h,y_0+k_3)=(0.1)f(1.0,1.89698)$

$=0.48085$

Hence

$$y_1 = 1.5 + \frac{1}{6}\left[0.325 + 2\left(0.3866\right) + 2\left(0.39698\right) + 0.48085\right]$$
$$= 1.8955$$

Step2:

To find y_2, i.e., $y\left(x_2\right) = y\left(1.2\right)$

Here x_1=0.1,y_1=1.8955 and h=0.1

by 4^{th} order R-K method we have

$y_2 = y_1+1/6(k_1+2k_2+2k_3+k_4)$

k_1=hf(x_1,y_1)=(0.1)f(0.1,1.8955)=(0.1) [1^2+$(1.8955)^2$]=0.48029

k_2= hf (x_1+h/2,y_1+k_1/2)=(0.1)f(1.1+0.1,1.8937+0.4796) =0.58834

and k_3=hf((x_1+h/2,y_1+k_2/2)=(0.1)f(1.5,1.8937+0.58743) =(0.1)[$(1.05)^2$+$(1.6933)^2$]

=0.611715

k_4=hf(x_1+h,y_1+k_3)=(0.1)f(1.2,1.8937+0.610728)

=0.77261

Hence y_2=1.8937+1/6(0.4796+2(0.58834)+2(0.611715)+0.7726) =2.5043

$\therefore$ y =2.5043 where $x = 0.2$

4. using R-K method, find y(0.2) for the eqn dy/dx=y-x,y(0)=1,take h=0.2

Ans:1.15607

5.Given that y^1=y-x,y(0)=2 find y(0.2) using R- K method take h=0.1

Ans: 2.4214

6. Apply the 4^{th} order R-K method to find $y\left(0.2\right)$ *and* $y\left(0.4\right)$ for one equation
$10\frac{dy}{dx} = x^2 + y^2, y\left(0\right) = 1$ **take h = 0.1** Ans. 1.0207, 1.038

7. using R-K method, estimate y(0.2) and y(0.4) for the eqn dy/dx=y^2-x^2/ y^2+x^2,y(0)=1,h=0.2

Ans:1.19598,1.3751

8. use R-K method, to approximate y when x=0.2 given that y^1=x+y,y(0)=1

Sol: Here f(x,y)=x+y,y_0=1,x_0=0

Since h is not given for better approximation of y

Take h=0.1

$\therefore$ x_1=0.1, x_2=0.2

Step1

To find y_1 i.e $y(x_1)$=y(0.1)

By R-K method,we have

$y_1=y_0+1/6\ (k_1+2k_2+2k_3+k_4)$

Where $k_1=hf(x_0,y_0)=(0.1)f(0,1)=(0.1)\ (1)=0.1$

$k_2=\ hf\ (x_0+h/2,y_0+k_1/2)=(0.1)f(0.05,1.05)=0.11$

and $k_3=hf((x_0+h/2,y_0+k_2/2)=(0.1)f(0.05,1+0.\ 11/2)=(0.1)[(0.05)\ +(4.0.11/2)]$

=0.1105

$k_4=h\ f\ (x_0+h,y_0+k_3)=(0.1)f(0.1,1.1105)=(0.1)[0.1+1.1105]$

=0.12105

Hence $\therefore\ y_1 = y\left(0.1\right) = 1+\frac{1}{6}\left(0.1+0.22+0.240+0.12105\right)$

y = 1.11034

Step2:

To find y_2 i.e $y(x_2)$ = y(0.2)

Here x_1=0-1, y_1=1.11034 and h=0.1

Again By R-K method,we have

$y_2=y_1+1/6(k_1+2k_2+2k_3+k_4)$

$k_1=h\ f(x_1,y_1)=(0.1)f(0.1,1.11034)=(0.1)\ [1.21034]=0.121034$

$k_2=\ h\ f\ (x_1+h/2,\ y_1+k_1/2)=(0.1)f(0.1+0.1/2,1.11034+0.121034/2)$

=0.1320857

and $k_3=h\ f((x_1+h/2,y_1+k_2/2)=(0.1)f(0.15,1.11034+0.1320857/2)$

=0.1326382

$k_4=h\ f(x_1+h,y_1+k_3)=(0.1)f(0.2,1.11034+0.1326382)$

(0.1)(0.2+1.2429783)=0.1442978

Hence y_2=1.11034+1/6(0.121034+0.2641714+0.2652764+0.1442978

=1.11034+0.1324631 =1.242803

$\therefore$ y =1.242803 when x=0.2

9.using Runge-kutta method of order 4,compute y(1.1) for the eqn $y^1=3x+y^2$,y(1)=1.2 h = 0.05

Ans:1.7278

10. using Runge-kutta method of order 4,compute y(2.5) for the eqn dy/dx = x+y/x, y(2)=2 [hint h = 0.25(2 steps)]

Ans:3.058

UNIT-III

Multiple Integrals

Double Integral :

I. When y_1, y_2 are functions of x and x_1 and x_2 are constants. f(x,y)is first integrated w.r.t y keeping 'x' fixed between limits y_1, y_2 and then the resulting expression is integrated w.r.t 'x' with in the limits x_1, x_2 i.e.,

$$\iint_R f(x,y)\,dx\,dy = \int_{x=x_1}^{x=x_2} \int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x,y)\,dy\,dx$$

II. When x_1, x_2 are functions of y and y_1, y_2 are constants, f(x,y)is first integrated w.r.t 'x' keeping 'y' fixed, with in the limits x_1, x_2 and then resulting expression is integrated w.r.t 'y' between the limits y_1, y_2 i.e.,

$$\iint_R f(x,y)\,dx\,dy = \int_{y=y_1}^{y=y_2} \int_{x=\phi_1(y)}^{x=\phi_2(y)} f(x,y)\,dx\,dy$$

III. When x_1, x_2, y_1, y_2 are all constants. Then

$$\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x,y)\,dx\,dy = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y)\,dy\,dx$$

Problems

1. Evaluate $\int_1^2 \int_1^3 xy^2\,dx\,dy$

Sol. $\int_1^2 \left[\int_1^3 xy^2\,dx \right] dy$

$$= \int_1^2 \left[y^2 . \frac{x^2}{2} \right]_1^3 dy = \int_1^2 \frac{y^2}{2}\,dy\,[9-1]$$

$$= \frac{8}{2} \int_1^2 y^2\,dy = 4.\int_1^2 y^2\,dy$$

$$= 4.\left[{y^3}/{3} \right]_1^2 = \frac{4}{3}[8-1] = \frac{4.7}{3} = \frac{28}{3}$$

2. Evaluate $\int_0^2 \int_0^x y\,dy\,dx$

Sol. $\int_{x=0}^{2} \int_{y=0}^{x} y\,dy\,dx = \int_{x=0}^{2} \left[\int_{y=0}^{x} y\,dy \right] dx$

$$= \int_{x=0}^{2} \left[\frac{y^2}{2} \right]_0^x dx = \int_{x=0}^{2} \frac{1}{2}\left(x^2 - 0\right) dx = \frac{1}{2} \int_{x=0}^{2} x^2 dx = \frac{1}{2}\left[\frac{x^3}{3} \right]_0^2 = \frac{1}{6}(8-0) = \frac{8}{6} = \frac{4}{3}$$

3. Evaluate $\int_0^5 \int_0^{x^2} x\left(x^2+y^2\right)dx\,dy$

Sol.

$$\int_{x=0}^{5}\int_{y=0}^{x^2} x\left(x^2+y^2\right)dy\,dx=\int_{x=0}^{5}\left[x^3y+\frac{xy^3}{3}\right]_{y=0}^{x^2}dx$$

$$=\int_{x=0}^{5}\left[x^3.x^2+\frac{x(x^2)^3}{3}\right]dx=\int_{x=0}^{5}\left(x^5+\frac{x^7}{3}\right)dx=\left[\frac{x^6}{6}+\frac{1}{3}.\frac{x^8}{8}\right]_0^5=\frac{5^6}{6}+\frac{5^8}{24}$$

4. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2}$

Sol: $$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2}=\int_{x=0}^{1}\left[\int_{y=0}^{\sqrt{1+x^2}}\frac{1}{\left(1+x^2\right)+y^2}dy\right]dx$$

$$=\int_{x=0}^{1}\left[\int_{y=0}^{\sqrt{1+x^2}}\frac{1}{\left(\sqrt{1+x^2}\right)^2+y^2}dy\right]dx=\int_{x=0}^{1}\frac{1}{\sqrt{1+x^2}}\left[Tan^{-1}\frac{y}{\sqrt{1+x^2}}\right]_{y=0}^{\sqrt{1+x^2}}dx\ [\because \int\frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}(x/a)]$$

$$=\int_{x=0}^{1}\frac{1}{\sqrt{1+x^2}}\left[Tan^{-1}1-Tan^{-1}0\right]dx\ \ or\ \ \frac{\pi}{4}(\sinh^{-1}x)_0^1=\frac{\pi}{4}(\sinh^{-1}1)$$

$$=\frac{\pi}{4}\int_{x=0}^{1}\frac{1}{\sqrt{1+x^2}}dx=\frac{\pi}{4}\left[\log(x+\sqrt{x^2+1})\right]_{x=0}^{1}$$

$$=\frac{\pi}{4}\log(1+\sqrt{2})$$

5. Evaluate $\int_0^4 \int_0^{x^2} e^{y/x}dydx$

Ans: $3e^4-7$

6. Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)dxdy$

Ans: 3/35

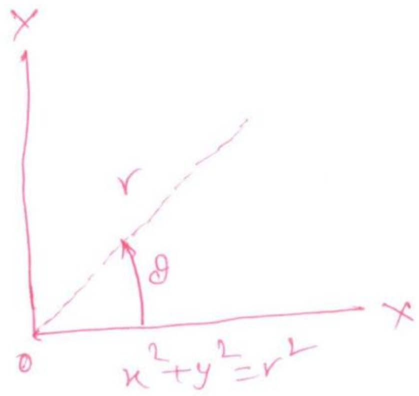
7. Evaluate $\int_0^2 \int_0^{x} e^{(x+y)}dydx$

Ans: $\frac{e^4-e^2}{2}$

8. Evaluate $\int_0^{\frac{\pi}{2}} \int_{-1}^{1} x^2y^2dxdy$

Ans: $\frac{\pi^3}{36}$

9. Evaluate $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy$



Sol: $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy = \int_0^\infty e^{-y^2}\left[\int_0^\infty e^{-x^2}dx\right]dy$

$= \int_0^\infty e^{-y^2}\frac{\sqrt{\pi}}{2}dy \qquad \because \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

$= \frac{\sqrt{\pi}}{2}\int_0^\infty e^{-y^2}dy = \frac{\sqrt{\pi}}{2}.\frac{\sqrt{\pi}}{2} = \frac{\pi}{4}$

Alter:

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy = \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{\infty} e^{-r^2}rdrd\theta \qquad (\because x^2+y^2=r^2)$$

(changing to polar coordinates taking $x = r\cos\theta, y = r\sin\theta$)

$= \int_0^{\pi/2}\left[\frac{e^{-r^2}}{-2}\right]_0^\infty d\theta = \int_0^{\pi/2}\left[\frac{0-1}{-2}\right]d\theta$

$= \frac{1}{2}(\theta)_0^{\pi/2} = \frac{1}{2}\left(\pi/2 - 0\right)$

$= \frac{\pi}{4}$

10. Evaluate $\iint xy(x+y)dxdy$ over the region R bounded by y=x^2 and y=x

Sol: y= x^2 is a parabola through (0,0) symmetric about y-axis y=x is a straight line through (0,0) with slope1.

Let us find their points of intersection solving y= x^2, y=x we get x^2=x $\Rightarrow$ x=0,1Hence y=0,1

$\therefore$ The point of intersection of the curves are (0,0), (1,1)

Consider $\iint_R xy(x+y)dxdy$

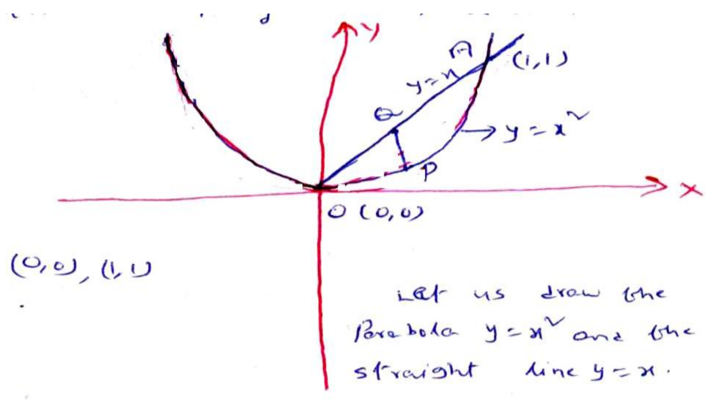
For the evaluation of the integral, we first integrate w.r.t 'y' from y=x^2 to y=x and then w.r.t. 'x' from x=0 to x=1

$$\int_{x=0}^1\left[\int_{y=x^2}^x xy(x+y)dy\right]dx = \int_{x=0}^1\left[\int_{y=x^2}^x\left(x^2y+xy^2\right)dy\right]dx$$

$$= \int_{x=0}^1\left(x^2\frac{y^2}{2}+\frac{xy^3}{3}\right)_{y=x^2}^x dx$$

$$= \int_{x=0}^1\left(\frac{x^4}{2}+\frac{x^4}{3}-\frac{x^6}{2}-\frac{x^7}{3}\right)dx$$

$$= \int_{x=0}^1\left(\frac{5x^4}{6}-\frac{x^6}{2}-\frac{x^7}{3}\right)dx$$



$$=\left(\frac{5}{6}.\frac{x^5}{5}-\frac{x^7}{14}-\frac{x^8}{24}\right)_0^1$$

$$=\frac{1}{6}-\frac{1}{14}-\frac{1}{24}=\frac{28-12-7}{168}=\frac{28-19}{168}=\frac{9}{168}=\frac{3}{56}$$

11. Evaluate $\iint_R xydxdy$ where R is the region bounded by x-axis and x=2a and the curve $x^2=4ay$.

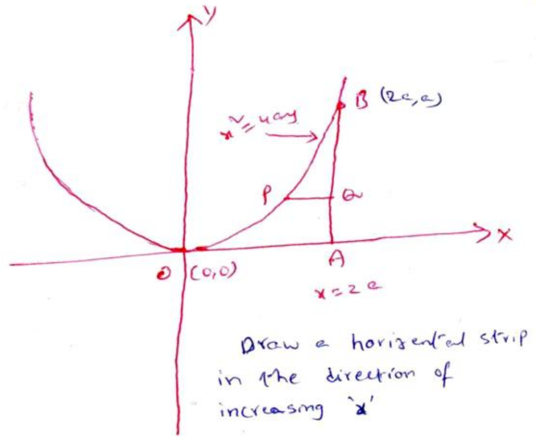
Sol. The line x=2a and the parabola $x^2=4ay$ intersect at B(2a,a)

∴The given integral = $\iint_R xy\ dx\ dy$

Let us fix 'y'

For a fixed 'y', x varies from $2\sqrt{ay}$ to 2a. Then y varies from 0 to a.

Hence the given integral can also be written as



$$\int_{y=0}^{a}\int_{x=2\sqrt{ay}}^{x=2a} xy\,dx\,dy=\int_{y=0}^{a}\left[\int_{x=2\sqrt{ay}}^{x=2a} xdx\right]ydy$$

$$=\int_{y=0}^{a}\left[\frac{x^2}{2}\right]_{x=2\sqrt{ay}}^{2a} ydy$$

$$=\int_{y=0}^{a}\left[2a^2-2ay\right]y\,dy$$

$$=\left[\frac{2a^2y^2}{2}-\frac{2ay^3}{3}\right]_0^a=a^4-\frac{2a^4}{3}=\frac{3a^4-2a^4}{3}=\frac{a^4}{3}$$

12. Evaluate $\int_0^1\int_0^{\pi/2} r\sin\theta d\theta\, dr$

Sol. $\int_{r=0}^{1} r\left[\int_{\theta=0}^{\pi/2}\sin\theta d\theta\right]dr$

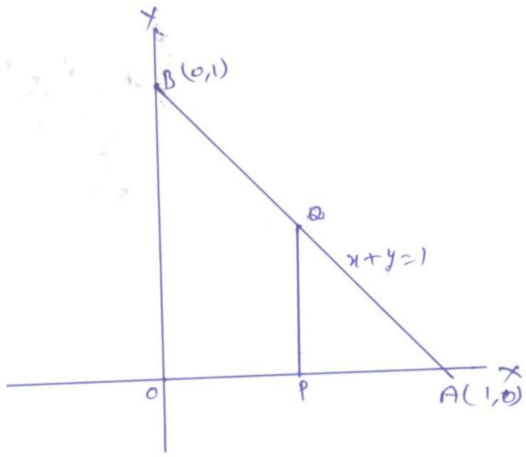
$$=\int_{r=0}^{1} r\left(-\cos\theta\right)_{\theta=0}^{\pi/2} dr$$

$$=\int_{r=0}^{1} -r\left(\cos\pi/2-\cos 0\right)dr$$

$$=\int_{r=0}^{1} -r\left(0-1\right)dr=\int_0^1 rdr=\left(\frac{r^2}{2}\right)_0^1=\frac{1}{2}-0=\frac{1}{2}$$

13. Evaluate $\iint\left(x^2+y^2\right)dx\,dy$ in the positive quardrant for

which $x+y\le 1$



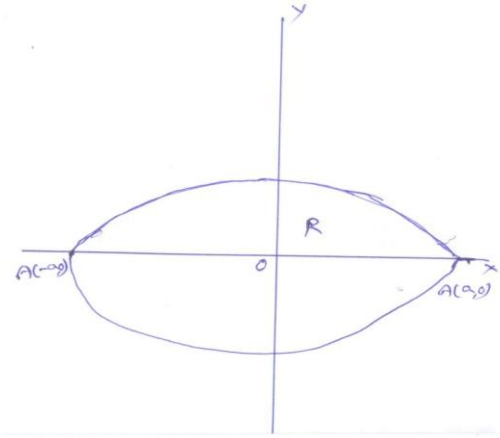
Sol. $\iint_R\left(x^2+y^2\right)dx\,dy=\int_{x=0}^{1} dx\int_{y=0}^{y=1-x}\left(x^2+y^2\right)dy$

$$=\int_{x=0}^{1}\left(x^2y+\frac{y^3}{3}\right)_0^{1-x} dx$$

$$=\int_{x=0}^{1}\left(x^2-x^3+\frac{1}{3}(1-x)^3\right)dx=\left[\frac{x^3}{3}-\frac{x^4}{4}-\frac{1}{12}(1-x)^4\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{4} - 0 + \frac{1}{12} = \frac{1}{6}$$

14. Evaluate $\iint (x^2 + y^2)\,dx\,dy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Sol. Given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$i.e., \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{1}{a^2}\left(a^2 - x^2\right) (or)\; y^2 = \frac{b^2}{a^2}\left(a^2 - x^2\right)$$

$$\therefore y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

Hence the region of integration R can be expressed as

$$-a \le x \le a, \frac{-b}{a}\sqrt{a^2 - x^2} \le y \le \frac{b}{a}\sqrt{a^2 - x^2}$$

$$\therefore \iint_R \left(x^2 + y^2\right) dx\,dy = \int_{x=-a}^{a} \int_{y=-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} \left(x^2 + y^2\right) dx\,dy$$

$$= 2\int_{x=-a}^{a} \int_{y=0}^{b/a\sqrt{a^2-x^2}} \left(x^2 + y^2\right) dx\,dy = 2\int_{-a}^{a} \left(x^2 y + y^3/3\right)_0^{b/a\sqrt{a^2-x^2}}$$

$$= 2\int_{-a}^{a} \left[x^2 . b/a \sqrt{a^2 - x^2} + \frac{b^3}{3a^3}\left(a^2 - x^2\right)^{3/2}\right] dx$$

$$= 4\int_{0}^{a} \left[b/a\, x^2 \sqrt{a^2 - x^2} + \frac{b^3}{3a^3}\left(a^2 - x^2\right)^{3/2}\right] dx$$

Changing to polar coordinates

$putting\; x = a\sin\theta$

$dx = a\cos\theta d\theta$

$\frac{x}{a} = \sin\theta \Rightarrow \theta = \sin^{-1}\frac{x}{a}$

$x \to 0, \theta \to 0$

$x \to a, \theta \to \frac{\pi}{2}$

$$= 4\int_{0}^{\pi/2} \left[b/a . a^2 \sin^2\theta . a\cos\theta + \frac{b^3}{3a^3} . a^3 \cos^3\theta\right] a\cos\theta d\theta$$

$$= 4\int_{0}^{\pi/2} \left[a^3 b \sin^2\theta \cos^2\theta + \frac{ab^3}{3}\cos^4\theta\right] d\theta = 4\left[a^3 b . \frac{1}{4}\frac{1}{2}.\frac{\pi}{2} + \frac{ab^3}{3}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}\right]$$

$$\left[\because \int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta d\theta = \frac{n-1}{m+n}.\frac{n-3}{m+n-2}.........\frac{\frac{1}{2}.\frac{\pi}{2}}{m}\right]$$

$$= \frac{4\pi}{16}\left(a^3 b + ab^3\right) = \frac{\pi ab}{4}\left(a^2 + b^2\right)$$

Double integrals in polar co-ordinates:

1. Evaluate $\int_0^{\pi/4}\int_0^{a\sin\theta}\frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}$

Sol. $\int_0^{\pi/4}\int_0^{a\sin\theta}\frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}=\int_0^{\pi/4}\left\{\int_0^{a\sin\theta}\frac{r}{\sqrt{a^2-r^2}}dr\right\}d\theta=-\tfrac{1}{2}\int_0^{\pi/4}\left\{\int_0^{a\sin\theta}\frac{-2r}{\sqrt{a^2-r^2}}dr\right\}d\theta$

$=\frac{-1}{2}\int_0^{\pi/4}2\left(\sqrt{a^2-r^2}\right)_0^{a\sin\theta}d\theta=(-1)\int_0^{\pi/4}2\left[\sqrt{a^2-a^2\sin^2\theta}-\sqrt{a^2-0}\right]d\theta$

$=(-a)\int_0^{\pi/4}(\cos\theta-1)\,d\theta=(-a)(\sin\theta-\theta)_0^{\pi/4}$

$=(-a)\left[\left[\sin\pi/4-\pi/4\right]-(0-0)\right]$

$=(-a)\left[1/\sqrt{2}-\pi/4\right]=2\left[\pi/4-1/\sqrt{2}\right]$

2. Evaluate $\int_0^{\pi}\int_0^{a\sin\theta}r\,dr\,d\theta$ Ans: $\frac{a^2\pi}{4}$

3. Evaluate $\int_0^{\infty}\int_0^{\pi/2}e^{-r^2}r\,d\theta\,dr$ Ans: $\frac{\pi}{4}$

4. Evaluate $\int_0^{\pi}\int_0^{a(1+\cos\theta)}r\,dr\,d\theta$ Ans: $\frac{3\pi a^2}{4}$

Change of order of Integration:

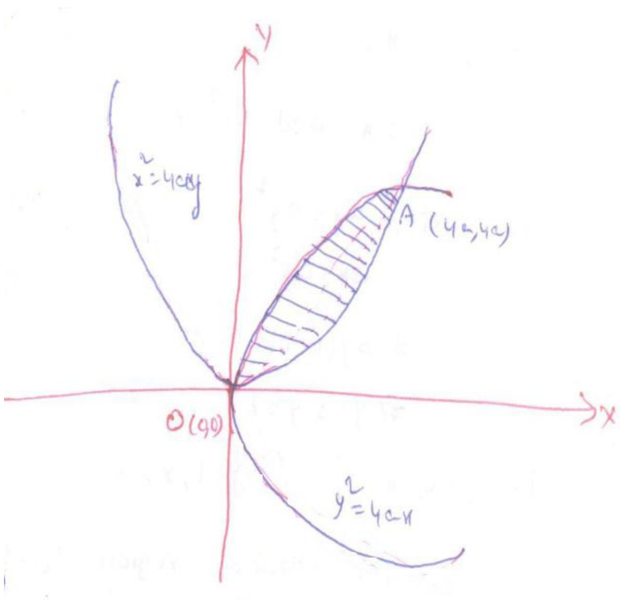
1. Change the order of Integration and evaluate $\int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}}dy\,dx$

Sol. In the given integral for a fixed x, y varies from $\frac{x^2}{4a}$ to $2\sqrt{ax}$ and then x varies from 0 to 4a. Let us draw the curves $y=\frac{x^2}{4a}$ and $y=2\sqrt{ax}$

The region of integration is the shaded region in diagram.

The given integral is $=\int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}}dy\,dx$

Changing the order of integration, we must fix y first,for a fixed y, x varies from $\frac{y^2}{4a}$ to $\sqrt{4ay}$ and then y varies from 0 to 4a.



Hence the integral is equal to

$\int_{y=0}^{4a}\int_{x=y^2/4a}^{2\sqrt{ay}}dx\,dy=\int_{y=0}^{4a}\left[\int_{x=y^2/4a}^{2\sqrt{ay}}dx\right]dy$

$=\int_{y=0}^{4a}[x]_{x=y^2/4a}^{2\sqrt{ay}}dy=\int_{y=0}^{4a}\left[2\sqrt{ay}-y^2/4a\right]dy$

$$= \left[2\sqrt{a}.\frac{y^{3/2}}{3/2} - \frac{1}{4a}.\frac{y^3}{3} \right]_0^{4a}$$

$$= \frac{4}{3}.\sqrt{a}.4a\sqrt{4a} - \frac{1}{12a}.64a^3$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2$$

2. Change the order of integration and evaluate $= \int_0^a \int_{x/a}^{\sqrt{x/a}} \left(x^2 + y^2\right) dx\, dy$

Sol. In the given integral for a fixed x, y varies from $\frac{x}{a}$ to $\sqrt{\frac{x}{a}}$ and then x varies from 0 to a

Hence we shall draw the curves $y = \frac{x}{a}$ and $y = \sqrt{\frac{x}{a}}$

i.e. ay=x and ay^2=x

we get $ay = ay^2$

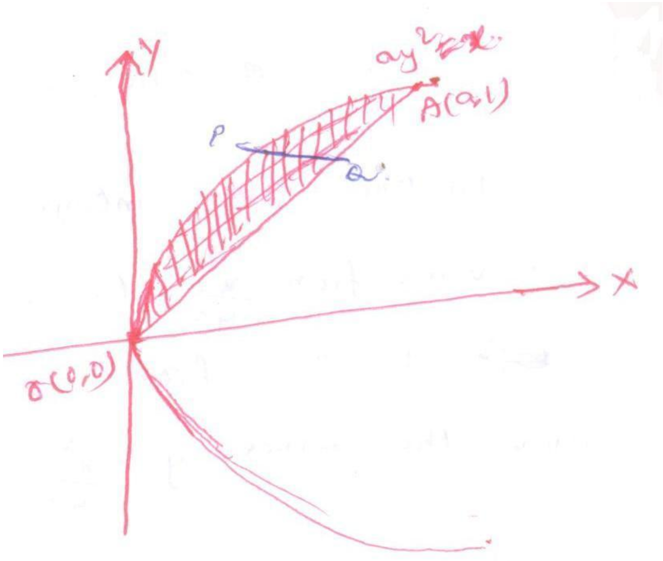
$\Rightarrow ay - ay^2 = 0$

$\Rightarrow ay\left(1 - y\right) = 0$

$\Rightarrow y = 0, y = 1$

If y=0, x=0 if y=1, x=a

The shaded region is the region of integration.



The given integral is $\int_{x=0}^{a} \int_{y=x/a}^{y=\sqrt{x/a}} \left(x^2 + y^2\right) dx\, dy$

Changing the order of integration, we must fix y first. For a fixed y, x varies from ay^2 to ay and then y varies from 0 to 1.

Hence the given integral, after change of the order of integration becomes

$$\int_{y=0}^{1} \int_{x=ay^2}^{ay} \left(x^2 + y^2\right) dx\, dy$$

$$= \int_{y=0}^{1} \left[\int_{x=ay^2}^{ay} \left(x^2 + y^2\right) dx \right] dy$$

$$= \int_{y=0}^{1} \left(\frac{x^3}{3} + xy^2 \right)_{x=ay^2}^{ay} dy$$

$$= \int_{y=0}^{1} \left(\frac{a^3y^3}{3} + ay^3 - \frac{a^3y^6}{3} - ay^4 \right) dy$$

$$= \left(\frac{a^3y^4}{12} + \frac{ay^4}{4} - \frac{a^3y^7}{21} - \frac{ay^5}{5} \right)_{y=0}^{1}$$

$$= \frac{a^3}{12} + \frac{a}{4} - \frac{a^3}{21} - \frac{a}{5} = \frac{a^3}{28} + \frac{a}{20}$$

3.Change the order of integration in $\int_0^1 \int_{x^2}^{2-x} xydxdy$ and hence evaluate the double integral.

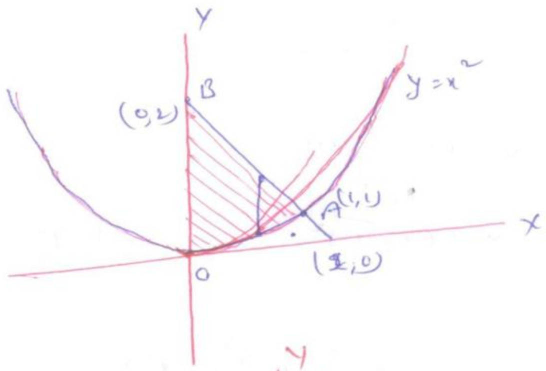
Sol. In the given integral for a fixed x,y varies from x^2 to 2-x and then x varies from 0 to 1.

Hence we shall draw the curves $y=x^2$ and y=2-x.

The line y=2-x passes through (0,2), (2,0)

Solving $y=x^2$,y=2-x

Then we get $x^2 = 2 - x$



$$\Rightarrow x^2 + x - 2 = 0$$

$$\Rightarrow x^2 + 2x - x - 2 = 0$$

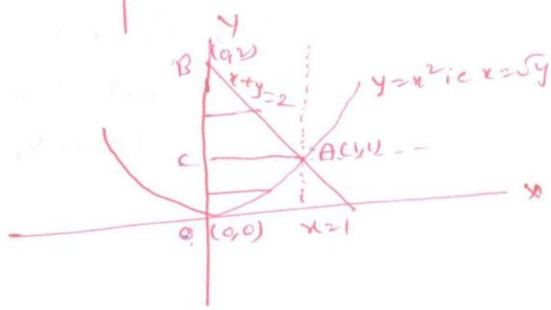
$$\Rightarrow x(x+2) - 1(x+2) = 0$$

$$\Rightarrow (x-1)(x+2) = 0$$

$$\Rightarrow x = 1, -2$$

$$If\ x = 1, y = 1$$

$$If\ x = -2, y = 4$$



Hence the points of intersection of the curves are (-2,4) (1,1)

The Shaded region in the diagram is the region of intersection.

Changing the order of integration, we must fix y, for the region with in OACO for a fixed y, x varies from 0 to $\sqrt{y}$

Then y varies from 0 to 1

For the region within CABC, for a fixed y, x varies from 0 to 2-y ,then y varies from 1 to 2

Hence $\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx = \iint_{OACO} xy\,dx\,dy + \iint_{CABC} xy\,dx\,dy$

$$= \int_{y=0}^{1} \left[\int_{x=0}^{\sqrt{y}} xdx \right] y\,dy + \int_{y=1}^{2} \left[\int_{x=0}^{2-y} x\,dx \right] y\,dy$$

$$= \int_{y=0}^{1} \left(\frac{x^2}{2} \right)_{x=0}^{\sqrt{y}} y\,dy + \int_{y=1}^{2} \left(\frac{x^2}{2} \right)_{x=0}^{2-y} y\,dy$$

$$= \int_{y=0}^{1} \frac{y}{2}.y\,dy + \int_{y=1}^{2} \frac{(2-y)^2}{2} y\,dy$$

$$= \frac{1}{2} \int_{y=0}^{1} y^2 dy + \frac{1}{2}.\int_{y=1}^{2} \left(4y - 4y^2 + y^3\right) dy$$

$$=\frac{1}{2}.\left(\frac{y^3}{3}\right)_0^1+\frac{1}{2}.\left[\frac{4y^2}{2}-\frac{4y^3}{3}+\frac{y^4}{4}\right]_1^2$$

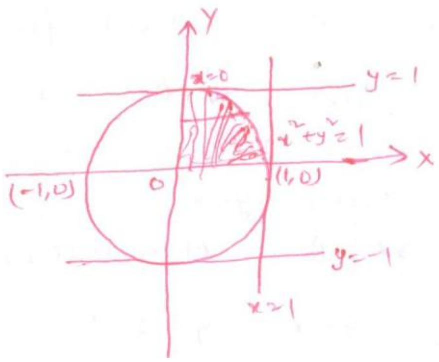
$$=\frac{1}{2}.\frac{1}{3}+\frac{1}{2}\left[2.4-2.1-\tfrac{4}{3}(8-1)+\tfrac{1}{4}(16-1)\right]$$

$$=\frac{1}{6}+\frac{1}{2}\left[6-\frac{28}{3}+\frac{15}{4}\right]=\frac{1}{6}+\frac{1}{2}\left[\frac{72-112+45}{12}\right]=\frac{1}{6}+\frac{1}{2}\left[\frac{5}{12}\right]=\frac{4+5}{24}=\frac{9}{24}=\frac{3}{8}$$

4. Changing the order of integration $\int_0^a\int_{x^2/a}^{2a-x} xy^2\,dy\,dx$

5. Change of the order of integration $\int_0^1\int_0^{\sqrt{1-x^2}} y^2\,dx\,dy \qquad Ans: \frac{\pi}{16}$

Hint : Now limits are $y=0\,to\,1$ and $x=0\,to\,\sqrt{1-y^2}$



$put\ y=\sin\theta$

$\sqrt{1-y^2}=\cos\theta$

$dy=\cos\theta d\theta$

$$=\int_0^1 y^2\sqrt{1-y^2}\,dy$$

$$=\int_0^{\pi/2}\sin^2\theta\cos^2\theta d\theta=\int_0^{\pi/2}\sin^2\theta d\theta-\int_0^{\pi/2}\sin^4\theta d\theta$$

$$=\frac{1}{2}\left(\pi/2\right)-\frac{3}{4}.\frac{1}{2}\left(\pi/2\right)=\pi/16$$

Change of variables:

The variables x,y in $\iint_R f(x,y)dx\,dy$ are changed to u,v with the help of the relations

$x=f_1(u,v), y=f_2(u,v)$ then the double integral is transferred into

$$\iint_{R^1} f\left[f_1(u,v),f_2(u,v)\right]\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$$

Where $R^1$is the region in the uv plane, corresponding to the region R in the xy-plane.

<u>Changing from Cartesian to polar co-ordinates</u>

$x=r\cos\theta, y=r\sin\theta$

$$\partial\left(\frac{(x,y)}{(r,\theta)}\right)=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}$$

$$=r\left(\cos^2\theta+\sin^2\theta\right)=r$$

$$\therefore\iint_R f(x,y)dx\,dy=\iint_{R_1} f(r\cos\theta,r\sin\theta)r\,dr\ d\theta$$

Note : In polar form dx dy is replaced by $r\,dr\;d\theta$

Problems:

1. Evaluate the integral by changing to polar co-ordinates $\int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)} dx\,dy$

Sol. The limits of x and y are both from 0 to ∞.

$\therefore$ The region is in the first quadrant where r varies from 0 to ∞ and θ varies from 0 to $\pi/2$

Substituting $x = r\cos\theta, y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$

Hence $\int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)} dx\,dy = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} r\,dr\,d\theta$

$Put\, r^2 = t$
$\Rightarrow 2rdr = dt$
$\Rightarrow r\,dr = dt/2$

$Where\ r = 0 \Rightarrow t = 0\ and\ \ r = \infty \Rightarrow t = \infty$

$\therefore \int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)} dx\,dy = \int_{\theta=0}^{\pi/2} \int_{t=0}^{\infty} \frac{1}{2} e^{-t} dt\,d\theta$

$= \int_0^{\pi/2} \frac{-1}{2}\left(e^{-t}\right)_0^\infty d\theta$

$= \frac{-1}{2}\int_0^{\pi/2} (0-1)\,d\theta \Rightarrow \frac{1}{2}(\theta)_0^{\pi/2} = \frac{1}{2}\pi/2 = \pi/4$

2. Evaluate the integral by changing to polar co-ordinates $\int_0^a \int_0^{\sqrt{a^2-y^2}} \left(x^2+y^2\right) dx\,dy$

Sol. The limits for x are x=0 to $x = \sqrt{a^2-y^2}$
$\Rightarrow x^2 + y^2 = a^2$

$\therefore$ The given region is the first quadrant of the circle.
By changing to polar co-ordinates
$x = r\cos\theta, y = r\sin\theta, dx\,dy = r\,dr\,d\theta$

Here 'r' varies from 0 to a and $'\theta'$ varies from 0 to $\pi/2$

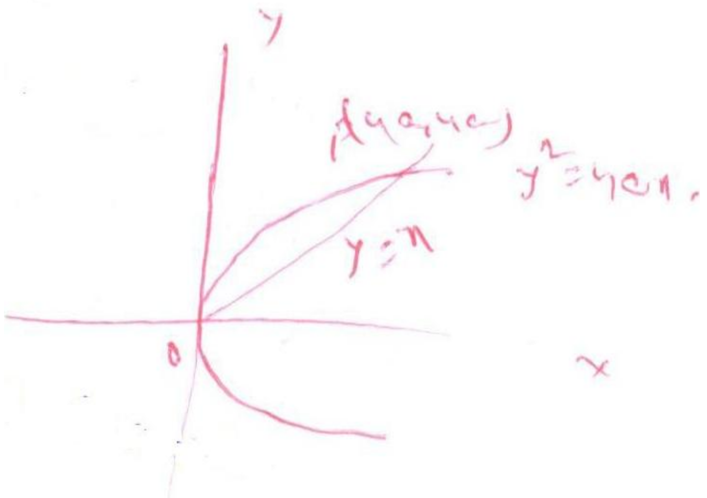
$\therefore \int_0^a \int_0^{\sqrt{a^2-y^2}} \left(x^2+y^2\right) dx\,dy = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^2 r dr d\theta$

$= \int_0^{\pi/2} \left(\frac{r^4}{4}\right)_0^a d\theta = \frac{a^4}{4}(\theta)_0^{\pi/2}$

$= \pi/8\, a^4$

3. Show that $\int_0^{4a} \int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2} dx\,dy = 8a^2\left(\frac{\pi}{2} - \frac{5}{3}\right)$

4. Sol. The region of integration is given by $x = y^2/4a$, $x = y$ and y=0, y=4a.



i.e., The region is bounded by the parabola y^2=4ax and the straight line x=y.

Let $x = r\cos\theta, y = r\sin\theta. Then\, dx\,dy = r\,dr\,d\theta$

The limits for r are r=0 at O and for P on the parabola

$$r^2\sin^2\theta = 4a(r\cos\theta) \Rightarrow r = \frac{4a\cos\theta}{\sin^2\theta}$$

For the line y=x, slope m=1 i.e., $Tan\theta = 1, \theta = \pi/4$

The limits for $\theta : \pi/4 \to \pi/2$

Also $x^2 - y^2 = r^2(\cos^2\theta - \sin^2\theta)\, and\, x^2 + y^2 = r^2$

$$\therefore \int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2 - y^2}{x^2 + y^2}\,dx\,dy = \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{4a\cos\theta/\sin^2\theta} (\cos^2\theta - \sin^2\theta)\,r\,dr\,d\theta$$

$$= \int_{\theta=\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta)\left(\frac{r^2}{2}\right)_0^{4a\cos\theta/\sin^2\theta} d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta)\frac{\cos^2\theta}{\sin^4\theta}\,d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2} (\cos^4\theta - \cot^2\theta)\,d\theta = 8a^2\left[\frac{3\pi - 8}{12} + \frac{\pi}{4} - 1\right] = 8a^2\left(\frac{\pi}{2} - \frac{5}{3}\right)$$

Triple integrals :

If x_1, x_2 are constants. y_1, y_2 are functions of x and z_1, z_2 are functions of x and y, then f(x,y,z) is first integrated w.r.t. ‘z’ between the limits z_1 and z_2 keeping x and y fixed. The resulting expression is integrated w.r.t ‘y’ between the limits y_1 and y_2 keeping x constant. The resulting expression is integrated w.r.t. ‘x’ from x_1 to x_2

$$i.e. \iiint_V f(x, y, z)\,dx\,dy\,dz =$$

$$\int_{x=a}^{b}\int_{y=g_1(x)}^{y=g_2(x)}\int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x, y, z)\,dz\ dy\ dx$$

Problems

1. Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dx\,dy\,dz$

Sol. $\int_{x=0}^1\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\,dx\,dy\,dz$

$$=\int_{x=0}^1 dx\int_{y=0}^{\sqrt{1-x^2}} dy\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\,dz$$

$$=\int_{x=0}^1 dx\int_{y=0}^{\sqrt{1-x^2}} xy\left(\frac{z^2}{2}\right)_{z=0}^{\sqrt{1-x^2-y^2}} dy$$

$$=\frac{1}{2}\int_{x=0}^1 dx\int_{y=0}^{\sqrt{1-x^2}} xy\left(1-x^2-y^2\right)dy$$

$$=\frac{1}{2}\int_{x=0}^1 dx\int_{y=0}^{\sqrt{1-x^2}} x\left[\left(1-x^2\right)y-y^3\right]dy$$

$$=\frac{1}{2}\int_{x=0}^1 x\left[\left(1-x^2\right)\frac{y^2}{2}-\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$=\frac{1}{2}.\int_{x=0}^1 x\left[\frac{y^2}{2}-\frac{x^2y^2}{2}-\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$=\frac{1}{8}.\int_{x=0}^1 x\left[2\left(1-x^2\right)-2x^2\left(1-x^2\right)-\left(1-x^2\right)^2\right]dx$$

$$=\frac{1}{8}\int_{x=0}^1\left(x-2x^3+x^5\right)dx=\frac{1}{8}\left[\frac{x^2}{2}-\frac{2x^4}{4}+\frac{x^6}{6}\right]_0^1$$

$$=\frac{1}{8}\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right)=\frac{1}{48}$$

2. Evaluate $\int_{-1}^1\int_0^z\int_{x-z}^{x+z}\left(x+y+z\right)dx\,dy\,dz$

$$Sol:\quad \int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}\left(x+y+y\right)dx\,dydz$$

$$=\int_{-1}^1\int_0^z\left[\left(xy+{y^2}/{2}+zy\right)_{x-z}^{x+z}\right]dx\,dz$$

$$=\int_{-1}^1\int_0^z x\left(x+z\right)-x\left(x-z\right)+\left[\frac{x+z}{2}\right]^2-\left[\frac{x-z}{2}\right]^2+z(x+z)-z(x-z)dx\,dz$$

$$=\int_{-1}^1\int_0^z\left[2z(x+z)+\frac{1}{2}4xz\right]dx\,dz$$

$$=2\int_{-1}^1\left[z.\frac{x^2}{2}+z^2x+z.\frac{x^2}{2}\right]_0^z dz$$

$$= 2.\int_{-1}^{1}\left[\frac{z^3}{2} + z^3 + \frac{z^3}{2}\right]dz = 4.\left(\frac{z^4}{4}\right)_{-1}^{1} = 0$$

UNIT-IV

<u>Vector Calculus and Vector Operators</u>

INTRODUCTION

In this chapter, vector differential calculus is considered, which extends the basic concepts of differential calculus, such as, continuity and differentiability to vector functions in a simple and natural way. Also, the new concepts of gradient, divergence and curl are introduced.

<u>DIFFERENTIATION OF A VECTOR FUNCTION</u>

Let S be a set of real numbers. Corresponding to each scalar t ε S, let there be associated a unique vector $\bar{f}$. Then $\bar{f}$ is said to be a vector (vector valued) function. S is called the domain of $\bar{f}$. We write $\bar{f} = \bar{f}$ (t).

Let $\bar{i}, \bar{j}, \bar{k}$ be three mutually perpendicular unit vectors in three dimensional space. We can write $\bar{f} = \bar{f}$ (t)= $f_1(t)\bar{i} + f_2(t)\,\bar{j} + f_3(t)\bar{k}$, where $f_1(t)$, $f_2(t)$, $f_3(t)$ are real valued functions (which are called components of $\bar{f}$). (we shall assume that $\bar{i}, \bar{j}, \bar{k}$ are constant vectors).

1. Derivative:

Let $\bar{f}$ be a vector function on an interval *I* and a ϵ *I*. Then $Lt_{t\to a}\dfrac{\bar{f}(t)-\bar{f}(a)}{t-a}$, if exists, is called the derivative of $\bar{f}$ at a and is denoted by $\bar{f}^{1}$(a) or $\left(\dfrac{d\bar{f}}{dt}\right)$ at t = a. We also say that $\bar{f}$ is differentiable at t =a if $\bar{f}^{1}$(a) exists.

2. Higher order derivatives

Let $\bar{f}$ be differentiable on an interval *I* and $\bar{f}^{1}= \dfrac{d\bar{f}}{dt}$ be the derivative of $\bar{f}$. If $Lt_{t\to a}\dfrac{\bar{f}^1(t)-\bar{f}^1(a)}{t-a}$ exists for every a Є $I_1 \subset I$. It is denoted by $\bar{f}^{11}= \dfrac{d^2\bar{f}}{dt^2}$.

Similarly we can define $\bar{f}^{111}(t)$ etc.

We now state some properties of differentiable functions (without proof)

(1) Derivative of a constant vector is $\bar{a}$.

If $\bar{a}$ and $\bar{b}$ are differentiable vector functions, then

(2). $\dfrac{d}{dt}(\bar{a}\pm\bar{b}) = \dfrac{d\bar{a}}{dt}\pm\dfrac{d\bar{b}}{dt}$

(3). $\dfrac{d}{dt}(\bar{a}.\bar{b}) = \dfrac{d\bar{a}}{dt}.\bar{b}+\bar{a}.\dfrac{d\bar{b}}{dt}$

(4). $\dfrac{d}{dt}(\bar{a}\times\bar{b}) = \dfrac{d\bar{a}}{dt}\times\bar{b}+\bar{a}\times\dfrac{d\bar{b}}{dt}$

(5). If $\bar{f}$ is a differentiable vector function and ϕ is a scalar differential function, then $\frac{d}{dt}(\phi\,\bar{f}) = \phi\frac{d\bar{f}}{dt} + \frac{d\phi}{dt}\bar{f}$

(6). If $\bar{f} = f_1(t)\bar{i} + f_2(t)\,\bar{j} + f_3(t)\bar{k}$ where $f_1(t)$, $f_2(t)$, $f_3(t)$ are cartesian components of the vector $\bar{f}$, then $\frac{d\bar{f}}{dt} = \frac{df_1}{dt}\bar{i} + \frac{df_2}{dt}\bar{j} + \frac{df_3}{dt}\bar{k}$

(7). The necessary and sufficient condition for $\bar{f}$ (t) to be constant vector function is $\frac{d\bar{f}}{dt} = \bar{0}$.

3. Partial Derivatives

Partial differentiation for vector valued functions can be introduced as was done in the case of functions of real variables. Let $\bar{f}$ be a vector function of scalar variables *p, q, t*. Then we write $\bar{f} = \bar{f}$ (*p,q,t*). Treating *t* as a variable and *p,q* as constants, we define

$$Lt_{\delta t\to 0}\frac{\bar{f}(p,q,t+\delta t) - \bar{f}(p,q,t)}{\delta t}$$

if exists, as partial derivative of $\bar{f}$ w.r.t. *t* and is denot by $\frac{\partial\bar{f}}{\partial t}$

Similarly, we can define $\frac{\partial\bar{f}}{\partial p}, \frac{\partial\bar{f}}{\partial q}$ also. The following are some useful results on partial differentiation.

4. Properties

1) $\frac{\partial}{\partial t}(\phi\bar{a}) = \frac{\partial\phi}{\partial t}\bar{a} + \phi\frac{\partial\bar{a}}{\partial t}$

2). If λ is a constant, then $\frac{\partial}{\partial t}(\lambda\bar{a}) = \lambda\frac{\partial\bar{a}}{\partial t}$

3). If $\bar{c}$ is a constant vector, then $\frac{\partial}{\partial t}(\phi\bar{c}) = \bar{c}\frac{\partial\phi}{\partial t}$

4). $\frac{\partial}{\partial t}(\bar{a} \pm \bar{b}) = \frac{\partial\bar{a}}{\partial t} \pm \frac{\partial\bar{b}}{\partial t}$

5). $\frac{\partial}{\partial t}(\bar{a}.\bar{b}) = \frac{\partial\bar{a}}{\partial t}.\bar{b} + \bar{a}.\frac{\partial\bar{b}}{\partial t}$

6). $\frac{\partial}{\partial t}(\bar{a}\times\bar{b}) = \frac{\partial\bar{a}}{\partial t}\times\bar{b} + \bar{a}\times\frac{\partial\bar{b}}{\partial t}$

7). Let $\bar{f} = f_1\bar{i} + f_2\,\bar{j} + f_3\bar{k}$, where f_1, f_2, $f_3$are differential scalar functions of more then one variable, Then $\frac{\partial\bar{f}}{\partial t} = \bar{i}\frac{\partial f_1}{\partial t} + \bar{j}\frac{\partial f_2}{\partial t} + \bar{k}\frac{\partial f_3}{\partial t}$ (treating $\bar{i}, \bar{j}, \bar{k}$ as fixed directions)

5. Higher order partial derivatives

Let $\bar{f} = \bar{f}$ (*p,q,t*). Then $\frac{\partial^2\bar{f}}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial\bar{f}}{\partial t}\right), \frac{\partial^2\bar{f}}{\partial p\partial t} = \frac{\partial}{\partial p}\left(\frac{\partial\bar{f}}{\partial t}\right) etc.$

6.Scalar and vector point functions: Consider a region in three dimensional space. To each point *p*(*x,y,z*), suppose we associate a unique real number (called scalar) say ϕ. This ϕ(*x,y,z*) is called a scalar point function. Scalar point function defined on the region. Similarly if to each point $p(x,y,z)$we associate a unique vector $\bar{f}\ (x,y,z)$, $\bar{f}$ is called a ***vector point function.***

Examples:

For example take a heated solid. At each point p(x,y,z)of the solid, there will be temperature T(x,y,z). This T is a scalar point function.

Suppose a particle (or a very small insect) is tracing a path in space. When it occupies a position p(x,y,z) in space, it will be having some speed, say, v. This **speed**v is a scalar point function.

Consider a particle moving in space. At each point P on its path, the particle will be having a velocity $\bar{v}$ which is vector point function. Similarly, the acceleration of the particle is also a vector point function.

In a magnetic field, at any point P(x,y,z) there will be a magnetic force $\bar{f}\ (x,y,z)$. This is called magnetic force field. This is also an example of a vector point function.

7. Tangent vector to a curve in space.

Consider an interval [a,b].

Let x = x(t),y=y(t),z=z(t)be continuous and derivable for a≤ t ≤b.

Then the set of all points (x(t),y(t),z(t)) is called a curve in a space.

Let A = (x(a),y(a),z(a)) and B = (x(b),y(b),z(b)). These A,B are called the end points of the curve. If A =B, the curve in said to be a closed curve.

Let P and Q be two neighbouring points on the curve.

Let $\overline{OP} = \bar{r}(t), \overline{OQ} = \bar{r}(t+\delta t) = \bar{r} + \delta\bar{r}. Then\ \delta\bar{r} = \overline{OQ} - \overline{OP} = \overline{PQ}$

Then $\frac{\delta\bar{r}}{\delta t}$ is along the vector $\overline{PQ}$. As Q→P, $\overline{PQ}$ and hence $\frac{\overline{PQ}}{\delta t}$ tends to be along the tangent to the curve at P.

Hence $\lim_{\delta t\to 0}\frac{\delta\bar{r}}{\delta t} = \frac{d\bar{r}}{dt}$ will be a tangent vector to the curve at P. (This $\frac{d\bar{r}}{dt}$ may not be a unit vector)

Suppose arc length AP = s. If we take the parameter as the arc length parameter, we can observe that $\frac{d\bar{r}}{ds}$ is unit tangent vector at P to the curve.

<u>VECTOR DIFFERENTIAL OPERATOR</u>

Def. The vector differential operator ∇(read as del) is defined as

$\nabla \equiv \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$. This operator possesses properties analogous to those of ordinary vectors as well as differentiation operator. We will define now some quantities known as "**gradient",**

"divergence" and **"curl"** involving this operator ∇. We must note that this operator has no meaning by itself unless it operates on some function suitably.

GRADIENT OF A SCALAR POINT FUNCTION

Let $\phi(x,y,z)$ be a scalar point function of position defined in some region of space. Then the vector function $\bar{i}\frac{\partial \phi}{\partial x}+\bar{j}\frac{\partial \phi}{\partial y}+\bar{k}\frac{\partial \phi}{\partial z}$ is known as the gradient of φ or ∇φ

$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\phi = \bar{i}\frac{\partial \phi}{\partial x}+\bar{j}\frac{\partial \phi}{\partial y}+\bar{k}\frac{\partial \phi}{\partial z}$$

Properties:

(1) If f and g are two scalar functions then grad(f ±g)= grad f ± grad g

(2) The necessary and sufficient condition for a scalar point function to be constant is that $\nabla f = \bar{0}$

(3) grad(fg) = *f*(grad g)+g(grad f)

(4) If c is a constant, grad (c*f*) = *c*(grad *f*)

(5) grad $\left(\frac{f}{g}\right)=\frac{g(grad\ f)-f(grad\ g)}{g^2}$, $(g \neq 0)$

(6) Let $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$. Then $d\bar{r}=dx\bar{i}+dy\bar{j}+dz\bar{k}$ if φ is any scalar point function, then

$$d\phi=\frac{\partial \phi}{\partial x}dx+\frac{\partial \phi}{\partial y}dy+\frac{\partial \phi}{\partial z}dz=\left(\bar{i}\frac{\partial \Phi}{\partial x}+\bar{j}\frac{\partial \Phi}{\partial y}+\bar{k}\frac{\partial \Phi}{\partial z}\right).\left(\bar{i}dx+\bar{j}dy+\bar{k}dz\right)=\nabla\Phi.d\bar{r}$$

DIRECTIONAL DERIVATIVE

Let φ(x,y,z) be a scalar function defined throughout some region of space. Let this function have a value φ at a point P whose position vector referred to the origin O is $\overline{OP} = \bar{r}$. Let φ+Δφ be the value of the function at neighbouring point Q. If $\overline{OQ} = \bar{r} + \Delta\bar{r}$. Let Δr be the length of $\Delta\bar{r}$

$\frac{\Delta\phi}{\Delta r}$ gives a measure of the rate at which φ change when we move from P to Q. The limiting value of $\frac{\Delta\phi}{\Delta r}$ $as\ \Delta r \to 0$ is called the derivative of φ in the direction of $\overline{PQ}$ or simply directional derivative of φ at P and is denoted by dφ/dr.

Theorem 1: The directional derivative of a scalar point function φ at a point P(x,y,z) in the direction of a unit vector $\bar{e}$ is equal to $\bar{e}$. grad φ=$\bar{e}$. ∇φ.

Level Surface

If a surface φ(x,y,z)= c be drawn through any point $P(\bar{r})$, such that at each point on it, function has the same value as at P, then such a surface is called a level surface of the function φ through *P*.

e.g : equipotential or isothermal surface.

Theorem 2: ∇φ at any point is a vector normal to the level surface φ(x,y,z)=c through that point, where *c* is a constant.

The physical interpretation of ∇ϕ

The gradient of a scalar function ϕ(x,y,z) at a point *P*(x,y,z) is a vector along the normal to the level surface ϕ(x,y,z) = c at *P* and is in increasing direction. Its magnitude is equal to the greatest rate of increase of **ϕ**. Greatest value of directional derivative of $\overline{\Phi}$ **at a point P = |grad ϕ|** at that point.

SOLVED PROBLEMS

1: If a=x+y+z, b= $x^2+y^2+z^2$, c = xy+yz+zx, prove that [grad a, grad b, grad c] = 0.

Sol:- Given a=x+y+z

There fore $\frac{\partial a}{\partial x}=1, \frac{\partial a}{\partial y}=1, \frac{\partial a}{\partial z}=1$

Grad a = ∇a = $\sum \bar{i}\frac{\partial a}{\partial x}=\bar{i}+\bar{j}+\bar{k}$

Given b= $x^2+y^2+z^2$

Therefore $\frac{\partial b}{\partial x}=2x, \frac{\partial b}{\partial y}=2y, \frac{\partial b}{\partial z}=2z$

Grad b = ∇b = $\bar{i}\frac{\partial b}{\partial x}+\bar{j}\frac{\partial b}{\partial y}+\bar{z}\frac{\partial b}{\partial z}=2x\bar{i}+2y\bar{j}+2z\bar{k}$

Again c = xy+yz+zx

Therefore $\frac{\partial c}{\partial x}=y+z, \frac{\partial c}{\partial y}=z+x, \frac{\partial c}{\partial z}=y+x$

Grad c = $\bar{i}\frac{\partial c}{\partial x}+\bar{j}\frac{\partial c}{\partial y}+\bar{z}\frac{\partial c}{\partial z}=(y+z)\bar{i}+(z+x)\bar{j}+(x+y)\bar{k}$

[grad a, grad b, grad c] = $\begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix} = 0, (on\, simplification)$

[grad a, grad b, grad c] =0

2: Show that ∇[f(r)] = $\frac{f^{i}(r)}{r}\bar{r}$ where $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$.

Sol:- Since $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$, we have $r^2=x^2+y^2+z^2$

Differentiating w.r.t. 'x' partially, we get

$2r\frac{\partial r}{\partial x}=2x \Rightarrow \frac{\partial r}{\partial x}=\frac{x}{r}.Similarly\ \frac{\partial r}{\partial y}=\frac{y}{r}\ ,\ \frac{\partial r}{\partial z}=\frac{z}{r}$

$$\nabla[f(r)]=\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)f(r)=\sum \bar{i}f^{1}(r)\frac{\partial r}{\partial x}=\sum \bar{i}f^{1}(r)\frac{x}{r}$$

$$=\frac{f^{1}(r)}{r}\sum \bar{i}x=\frac{f^{1}(r)}{r}.\bar{r}$$

Note : From the above result, ∇(log r) = $\frac{1}{r^2}\bar{r}$

3: Prove that $\nabla(r^n)= nr^{n-2}\bar{r}$.

Sol:- Let $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$ and r = $|\bar{r}|$. Then we have $r^2 = x^2+y^2+z^2$ Differentiating w.r.t. x partially, we have

$2r\frac{\partial r}{\partial x}=2x \Rightarrow \frac{\partial r}{\partial x}=\frac{x}{r}.Similarly=\frac{\partial r}{\partial y}=\frac{y}{r}\ and\ \frac{\partial r}{\partial z}=\frac{z}{r}$

$\nabla(r^n) = \sum \bar{i}\frac{\partial}{\partial x}(r^n) = \sum \bar{i} n r^{n-1}\frac{\partial r}{\partial x} = \sum \bar{i} n r^{n-1}\frac{x}{r} = n r^{n-2}\sum \bar{i}x = n r^{n-2}(\bar{r})$

Note : From the above result, we can have

(1). $\nabla\left(\frac{1}{r}\right) = -\frac{\bar{r}}{r^3}$, taking n = -1 (2) grad r = $\frac{\bar{r}}{r}$, taking n = 1

4: Find the directional derivative of $f = xy+yz+zx$ in the direction of vector $\bar{i}+2\bar{j}+2\bar{k}$ at the point (1,2,0).

Sol:- Given $f = xy+yz+zx$.

$$\text{Grad f} = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{z}\frac{\partial f}{\partial z} = (y+z)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}$$

If $\bar{e}$ is the unit vector in the direction of the vector $\bar{i}+2\bar{j}+2\bar{k}$, then

$$\bar{e} = \frac{\bar{i}+2\bar{j}+2\bar{k}}{\sqrt{1^2+2^2+2^2}} = \frac{1}{3}(\bar{i}+2\bar{j}+2\bar{k})$$

Directional derivative of f along the given direction $= \bar{e}.\nabla f$

$= \frac{1}{3}(\bar{i}+2\bar{j}+2\bar{k}).[(y+z)\bar{i}+(z+x)\bar{j}+(x+y\bar{k})] at (1,2,0)$

$= \frac{1}{3}[(y+z)+2(z+x)+2(x+y)](1,2,0) = \frac{10}{3}$

5: Find the directional derivative of the function $xy^2+yz^2+zx^2$ along the tangent to the curve x =t, y = t^2, z = t^3 at the point (1,1,1).

Sol: - Here $f = xy^2+yz^2+zx^2$

$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = (y^2+2xz)\bar{i} + (z^2+2xy)\bar{j} + (x^2+2yz)\bar{k}$$

At (1,1,1) , $\nabla f = 3\bar{i}+3\bar{j}+3\bar{k}$

Let $\bar{r}$ be the position vector of any point on the curve x =t , y = t^2, z = t^3. then

$$\bar{r} = x\bar{i}+y\bar{j}+z\bar{k} = t\bar{i}+t^2\bar{j}+t^3\bar{k}$$

$$\frac{\partial \bar{r}}{\partial t} = \bar{i}+2t\bar{j}+3t^2\bar{k} = (\bar{i}+2\bar{j}+3\bar{k}) \text{ at } (1,1,1)$$

We know that $\frac{\partial \bar{r}}{\partial t}$ is the vector along the tangent to the curve.

Unit vector along the tangent $= \bar{e} = \frac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{1+2^2+3^2}} = \frac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{14}}$

Directional derivative along the tangent = $\nabla f.\bar{e}$

$$= \frac{1}{\sqrt{14}}(\bar{i}+2\bar{j}+3\bar{k}).3(\bar{i}+\bar{j}+\bar{k})\ \frac{3}{\sqrt{14}}(1+2+3) = \frac{18}{\sqrt{14}}$$

6: Find the directional derivative of the function f = x^2-y^2+$2z^2$ at the point P =(1,2,3) in the direction of the line $\overline{PQ}$ where Q = (5,0,4).

Sol:- The position vectors of P and Q with respect to the origin are $\overline{OP} = \bar{i}+2\bar{j}+3\bar{k}$ and $\overline{OQ} = 5\bar{i}+4\bar{k}$

$$\overline{PQ} = \overline{OQ} - \overline{OP} = 4\bar{i}-2\bar{j}+\bar{k}$$

Let $\bar{e}$ be the unit vector in the direction of $\overline{PQ}$. Then $\bar{e} = \dfrac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{21}}$

grad $f = \bar{i}\dfrac{\partial f}{\partial x} + \bar{j}\dfrac{\partial f}{\partial y} + \bar{k}\dfrac{\partial f}{\partial z} = 2x\bar{i} - 2y\bar{j} + 4z\bar{k}$

The directional derivative of $\bar{f}$ at P (1,2,3) in the direction of $\overline{PQ} = \bar{e}.\nabla f$

$= \dfrac{1}{\sqrt{21}}(4\bar{i} - 2\bar{j} + \bar{k}).(2x\bar{i} - 2y\bar{j} + 4z\bar{k})\ \dfrac{1}{\sqrt{21}}(8x + 4y + 4z)_{at(1,2,3)} = \dfrac{1}{\sqrt{21}}(28)$

7: Find the greatest value of the directional derivative of the function f = x^2yz^3 at (2,1,-1).
Sol: we have

grad $f = \bar{i}\dfrac{\partial f}{\partial x} + \bar{j}\dfrac{\partial f}{\partial y} + \bar{k}\dfrac{\partial f}{\partial z} = 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k} = -4\bar{i} - 4\bar{j} + 12\bar{k}$ at (2,1,-1).

Greatest value of the directional derivative of $f = |\nabla f| = \sqrt{16+16+144} = 4\sqrt{11}.$

8: Find the directional derivative of xyz^2+xz at (1, 1 ,1) in a direction of the normal to the surface $3xy^2+y= z$ at (0,1,1).

Sol:- Let f(x, y, z) ≡ $3xy^2+y- z = 0$

Let us find the unit normal e to this surface at (0,1,1). Then

$\dfrac{\partial f}{\partial x} = 3y^2,\ \dfrac{\partial f}{\partial y} = 6xy + 1, \dfrac{\partial f}{\partial z} = -1.$

$\nabla f = 3y^2i+(6xy+1)j-k$

$(\nabla f)_{(0,1,1)} = 3i+j-k = \bar{n}$

$\bar{e} = \dfrac{\bar{n}}{|\bar{n}|} = \dfrac{3i + j - k}{\sqrt{9+1+1}} = \dfrac{3i + j - k}{\sqrt{11}}$

Let $g(x,y,z) = xyz^2+xz$,then

$\dfrac{\partial g}{\partial x} = yz^2 + z,\ \dfrac{\partial g}{\partial y} = xz^2, \dfrac{\partial g}{\partial z} = 2xy + x$

$\nabla g=(yz^2+z)i+xz^2j+(2xyz+x)k$

And $[\nabla g]_{(1,1,1)} = 2i+j+3k$

Directional derivative of the given function in the direction of $\bar{e}$ at (1,1,1) = $\nabla g.\bar{e}$

$=(2i+j+3k).\left(\dfrac{3i + j - k}{\sqrt{11}}\right) = \dfrac{6+1-3}{\sqrt{11}} = \dfrac{4}{\sqrt{11}}$

9: Find the directional derivative of $2xy+z^2$ at (1,-1,3) in the direction of $\bar{i} + 2\bar{j} + 3\bar{k}$.

Sol: Let $f = 2xy+z^2$ then $\dfrac{\partial f}{\partial x} = 2y,\ \dfrac{\partial f}{\partial y} = 2x, \dfrac{\partial f}{\partial z} = 2z.$

grad f = $\sum \bar{i}\dfrac{\partial f}{\partial x} = 2y\bar{i} + 2x\bar{j} + 2z\bar{k}$ and (grad f)at (1,-1,3)= $-2\bar{i} + 2\bar{j} + 6\bar{k}$

given vector is $\bar{a} = \bar{i} + 2\bar{j} + 3\bar{k} \Rightarrow |\bar{a}| = \sqrt{1+4+9} = \sqrt{14}$

Directional derivative of f in the direction of $\bar{a}$ is

$\dfrac{\bar{a}.\nabla f}{|\bar{a}|} = \dfrac{(\bar{i} + 2\bar{j} + 3\bar{k})(-2\bar{i} + 2\bar{j} + 6\bar{k}).}{\sqrt{14}} = \dfrac{-2+4+18}{\sqrt{14}} = \dfrac{20}{\sqrt{14}}$

10: Find the directional derivative of $\phi = x^2yz+4xz^2$ at (1,-2,-1) in the direction 2i-j-2k.

Sol:- Given $\phi = x^2yz+4xz^2$

$$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2,\ \frac{\partial\phi}{\partial y} = x^2 z, \frac{\partial\phi}{\partial z} = x^2 y + 8xz.$$

Hence $\nabla\phi = \sum \bar{i}\,\frac{\partial\phi}{\partial x} = \bar{i}(2xyz + 4z^2) + \bar{j}x^2 z + \bar{k}(x^2 y + 8xz)$

$\nabla\phi$ at (1,-2,-1) = i(4+4)+j(-1)+k(-2-8)= 8i-j-10k.

The unit vector in the direction 2i-j-2k is

$$\bar{a} = \frac{2i - j - 2k.}{\sqrt{4+1+4}} = \frac{1}{3}(2i - j - 2k)$$

Required directional derivative along the given direction = $\nabla\phi.\ \bar{a}$

= (8i-j-10k). 1/3 (2i-j-2k)

= 1/3(16+1+20) = 37/3.

11: If the temperature at any point in space is given by t = xy+yz+zx, find the direction in which temperature changes most rapidly with distance from the point (1,1,1) and determine the maximum rate of change.

Sol:- The greatest rate of increase of t at any point is given in magnitude and direction by ∇t.

We have $\nabla t = \left(\bar{i}\,\frac{\partial}{\partial x} + \bar{j}\,\frac{\partial}{\partial y} + \bar{k}\,\frac{\partial}{\partial z}\right)(xy + yz + zx)$

$= \bar{i}(y+z) + \bar{j}(z+x) + \bar{k}(x+y) = 2\bar{i} + 2\bar{j} + 2\bar{k}$ at (1,1,1)

Magnitude of this vector is $\sqrt{2^2 + 2^2 + 2^2} = \sqrt{12} = 2\sqrt{3}$

Hence at the point (1,1,1) the temperature changes most rapidly in the direction given by the vector $2\bar{i} + 2\bar{j} + 2\bar{k}$ and greatest rate of increase = $2\sqrt{3}$.

12: Findthe directional derivative of $\phi(x,y,z) = x^2yz+4xz^2$ at the point (1,-2,-1) in the direction of the normal to the surface $f(x,y,z) = x \log z-y^2$ at (-1,2,1).

Sol:- Given $\phi(x,y,z) = x^2yz+4xz^2$ at (1,-2,-1) and $f(x,y,z) = x \log z-y^2$ at (-1,2,1)

Now $\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} \mp \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$

$= (2xyz + 4z^2)\bar{i} + (x^2 z)\bar{j} + (x^2 y + 8xz)\bar{k}$

$(\nabla\phi)_{(1,-2,-1)} = [2(1)(-2)(-1) + 4(-1)^2]\bar{i} + [(1)^2(-1)\bar{j}] + [(1^2)(-2) + 8(-1)]\bar{k} ----(1)$

$= 8\bar{i} - \bar{j} - 10\bar{k}$

Unit normal to the surface

$f(x,y,z)= x \log z- y^2$ is $\frac{\nabla f}{|\nabla f|}$

Now $\nabla f = \bar{i}\,\frac{\partial f}{\partial x} + \bar{j}\,\frac{\partial f}{\partial y} + \bar{k}\,\frac{\partial f}{\partial z} = \log z\,\bar{i} + (-2y)\bar{j} + \frac{x}{z}\bar{k}$

At (-1,2,1), $\nabla f = \log(1)\bar{i} - 2(2)\bar{j} + \frac{-1}{1}\bar{k} = -4\bar{j} - \bar{k}$

$$\frac{\nabla f}{|\nabla f|} = \frac{-4\bar{j}-\bar{k}.}{\sqrt{16+1}} = \frac{-4\bar{j}-\bar{k}.}{\sqrt{17}}$$

Directional derivative = $\nabla\phi . \frac{\nabla f}{|\nabla f|}$

$$= (8\bar{i} - \bar{j} - 10\bar{k}). \frac{-4\bar{j}-\bar{k}.}{\sqrt{17}} = \frac{4+10}{\sqrt{17}} = \frac{14}{\sqrt{17}}.$$

13: Find a unit normal vector to the given surface $x^2y+2xz = 4$ at the point (2,-2,3).

Sol:- Let the given surface be $f = x^2y+2xz - 4$

On differentiating,

$$\frac{\partial f}{\partial x} = 2xy + 2z,\ \frac{\partial f}{\partial y} = x^2, \frac{\partial f}{\partial z} = 2x.$$

grad $f = \sum \bar{i}\frac{\partial f}{\partial x} = \bar{i}(2xy+2z) + \bar{j}x^2 + 2x\bar{k}$

(grad f) at (2,-2,3) $= \bar{i}(-8+6) + 4\bar{j} + 4\bar{k} = 2\bar{i} + 4\bar{j} + 4\bar{k}$

grad (f) is the normal vector to the given surface at the given point.

Hence the required unit normal vector $\frac{\nabla f}{|\nabla f|} = \frac{2(-\bar{i}+2\bar{j}+2\bar{k}).}{2\sqrt{1+2^2+2^2}} = \frac{-\bar{i}+2\bar{j}+2\bar{k}}{3}$

14: Evaluate the angle between the normal to the surface $xy= z^2$ at the points (4,1,2) and (3,3,-3).

Sol:- Given surface is $f(x,y,z) = xy- z^2$

Let $\bar{n}_1$ and $\bar{n}_2$ be the normal to this surface at (4,1,2) and (3,3,-3) respectively.

Differentiating partially, we get

$$\frac{\partial f}{\partial x} = y,\ \frac{\partial f}{\partial y} = x, \frac{\partial f}{\partial z} = -2z.$$

grad f = $y\bar{i} + x\bar{j} - 2z\bar{k}$

$\bar{n}_1$ = (grad f) at (4,1,2) $= \bar{i} + 4\bar{j} - 4\bar{k}$

$\bar{n}_2$ = (grad f) at (3,3,-3) = $3\bar{i} + 3\bar{j} + 6\bar{k}$

Let θ be the angle between the two normal.

$$\cos\theta = \frac{\overline{n_1}.\overline{n_2}}{|\overline{n_1}||\overline{n_2}|} = \frac{(i+4j-4k)}{\sqrt{1+16+16}}.\frac{(3i+3j+6k)}{\sqrt{9+9+36}}$$

$$\frac{(3+12-24)}{\sqrt{33}\sqrt{54}} = \frac{-9}{\sqrt{33}\sqrt{54}}$$

15: Find a unit normal vector to the surface $x^2+y^2+2z^2 = 26$ at the point (2, 2 ,3).

Sol:- Let the given surface be $f(x,y,z) \equiv x^2+y^2+2z^2 - 26=0$. Then

$$\frac{\partial f}{\partial x} = 2x,\ \frac{\partial f}{\partial y} = 2y, \frac{\partial f}{\partial z} = 4z.$$

grad f = $\sum \bar{i}\,\frac{\partial f}{\partial x} = $ 2xi+2yj+4zk

Normal vector at(2,2,3) = $[\nabla f]_{(2,2,3)}$ = $4\bar{i} + 4\bar{j} + 12\bar{k}$

Unit normal vector = $\frac{\nabla f}{|\nabla f|} = \frac{4(\bar{i}+\bar{j}+3\bar{k})}{4\sqrt{11}} = \frac{\bar{i}+\bar{j}+3\bar{k}}{\sqrt{11}}$

16: Find the values of a and b so that the surfaces ax^2-byz = (a+2)x and $4x^2y+z^3$= 4 may intersect orthogonally at the point (1, -1,2).

(or) Find the constants a and b so that surface ax^2-byz=(a+2)x will orthogonal to $4x^2y+z^3$=4 at the point (1,-1,2).

Sol:- Let the given surfaces be f(x,y,z) = ax^2-byz - (a+2)x-------------(1)

And g(x,y,z) = $4x^2y+z^3$- 4------------(2)

Given the two surfaces meet at the point (1,-1,2).

Substituting the point in (1), we get

a+2b-(a+2) = 0 ⇒ b=1

Now $\frac{\partial f}{\partial x} = 2ax-(a+2),\ \frac{\partial f}{\partial y} = -bz\ and\ \frac{\partial f}{\partial z} = -by.$

∇f = $\sum \bar{i}\,\frac{\partial f}{\partial x} = $ [(2ax-(a+2)]i-bz+bk = (a-2)i-2bj+bk

= (a-2)i-2j+k = $\bar{n}_1$, normal vector to surface 1.

Also $\frac{\partial g}{\partial x} = 8xy,\ \frac{\partial g}{\partial y} = 4x^2, \frac{\partial g}{\partial z} = 3z^2.$

∇g = $\sum \bar{i}\,\frac{\partial g}{\partial x} = 8xyi+4x^2j+3z^2k$

$(\nabla g)_{(1,-1,2)}$ = -8i+4j+12k = $\bar{n}_2$, normal vector to surface 2.

Given the surfaces f(x,y,z), g(x,y,z) are orthogonal at the point (1,-1,2).

$[\nabla f].[\nabla g] = 0 \Rightarrow$ ((a-2)i-2j+k). (-8i+4j+12k)=0

⇒-8a+16-8+12 ⇒ a =5/2

Hence a = 5/2 and b=1.

17: Find a unit normal vector to the surface z= x^2+y^2 at (-1,-2,5)

Sol:- Let the given surface be f = x^2+y^2-z

$\frac{\partial f}{\partial x} = 2x,\ \frac{\partial f}{\partial y} = 2y, \frac{\partial f}{\partial z} = -1.$

grad f = $\nabla f = \sum \bar{i}\,\frac{\partial f}{\partial x} = $ 2xi+2yj-k

(∇f) at (-1,-2,5)= -2i-4j-k

∇f is the normal vector to the given surface.

Hence the required unit normal vector = $\dfrac{\nabla f}{|\nabla f|}$ =

$$\frac{-2i-4j-k}{\sqrt{(-2)^2+(-4)^2+(-1)^2}} = \frac{-2i-4j-k}{\sqrt{21}} = -\frac{1}{\sqrt{21}}(2i+4j+k)$$

18: Find the angle of intersection of the spheres $x^2+y^2+z^2=29$ and $x^2+y^2+z^2+4x-6y-8z-47=0$ at the point (4,-3,2).

Sol:- Let f = $x^2+y^2+z^2-29$ and g = $x^2+y^2+z^2+4x-6y-8z-47$

Then grad f= $\bar{i}\dfrac{\partial f}{\partial x}+\bar{j}\dfrac{\partial f}{\partial y}+\bar{k}\dfrac{\partial f}{\partial z} = 2x\bar{i}+2y\bar{j}+2z\bar{k}$ and

grad g = $(2x+4)\bar{i}+(2y-6)\bar{j}+(2z-8)\bar{k}$

The angle between two surfaces at a point is the angle between the normal to the surfaces at that point.

Let $\bar{n}_1$ = (grad *f*) at (4,-3,2) $=8\bar{i}-6\bar{j}+4\bar{k}$

$\bar{n}_2$ = (grad *f*) at (4,-3,2) = $12\bar{i}-12\bar{j}-4\bar{k}$

The vectors $\bar{n}_1$ and $\bar{n}_2$ are along the normal to the two surfaces at (4,-3,2). Let θ be the angle between the surfaces. Then

$$\text{Cos }\theta = \frac{\overline{n_1}.\overline{n_2}}{|\overline{n_1}||\overline{n_2}|} = .\frac{152}{\sqrt{116}\sqrt{304}}$$

$$\therefore \theta = \cos^{-1}\left(\sqrt{\frac{19}{29}}\right)$$

19: Find the angle between the surfaces $x^2+y^2+z^2=9$, and $z = x^2+y^2-3$ at point (2,-1,2).

Sol:- Let $\phi_1 = x^2+y^2+z^2-9=0$ and $\phi_2= x^2+y^2-z-3=0$ be the given surfaces. Then

$\nabla\phi_1$= 2xi+2yj+2zk and $\nabla\phi_2$ = 2xi+2yj-k

Let $\bar{n}_1 = \nabla\phi_1$ at(2,-1,2)= 4i-2j+4k and

$\bar{n}_2 = \nabla\phi_2$ at (2,-1,2) = 4i-2j-k

The vectors $\bar{n}_1$ and $\bar{n}_2$ are along the normals to the two surfaces at the point (2,-1,2). Let θ be the angle between the surfaces. Then

$$\text{Cos }\theta = \frac{\bar{n}_1\,\bar{n}_2}{|\bar{n}_1\,\bar{n}_2|} = \frac{(4i-2j+4k)}{\sqrt{16+4+16}}.\frac{(4i-2j-k)}{\sqrt{16+4+16}} = \frac{16+4-4}{6\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

20: If $\bar{a}$ is constant vector then prove that grad ($\bar{a}\,.\bar{r}$)= $\bar{a}$

Sol: Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, where a_1,a_2,a_3 are constants.

$\bar{a} . \bar{r} = (a_1\bar{i} + a_2\bar{j} + a_3\bar{k}).(x\bar{i} + y\bar{j} + z\bar{k}) = a_1x + a_2y + a_3z$

$\frac{\partial}{\partial x}(\bar{a}.\bar{r}) = a_1, \frac{\partial}{\partial y}(\bar{a}.\bar{r}) = a_2, \frac{\partial}{\partial z}(\bar{a}.\bar{r}) = a_3$

grad ($\bar{a} . \bar{r}$)= $a_1\bar{i} + a_2\bar{j} + a_3\bar{k} = \bar{a}$

21: If $\nabla\phi = yz\bar{i} + zx\bar{j} + xy\bar{k}$, find ϕ.

Sol:- We know that $\nabla\phi = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

Given that $\nabla\phi = yz\bar{i} + zx\bar{j} + xy\bar{k}$

Comparing the corresponding coefficients, we have $\frac{\partial\phi}{\partial x} = yz, \frac{\partial\phi}{\partial y} = zx, \frac{\partial\phi}{\partial z} = xy$

Integrating partially w.r.t. x,y,z, respectively, we get

ϕ= xyz + a constant independent of x.

ϕ= xyz + a constant independent of y.

ϕ= xyz + a constant independent of z.

Here a possible form of ϕ is ϕ= xyz+a constant.

DIVERGENCE OF A VECTOR

Let $\bar{f}$ be any continuously differentiable vector point function. Then $\bar{i}.\frac{\partial\bar{f}}{\partial x} + \bar{j}.\frac{\partial\bar{f}}{\partial y} + \bar{k}.\frac{\partial\bar{f}}{\partial z}$ is called the divergence of $\bar{f}$ and is written as div $\bar{f}$.

i.e., div $\bar{f} = \bar{i}.\frac{\partial\bar{f}}{\partial x} + \bar{j}.\frac{\partial\bar{f}}{\partial y} + \bar{k}.\frac{\partial\bar{f}}{\partial z} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).\bar{f}$

Hence we can write div $\bar{f}$ as

div $\bar{f} = \nabla . \bar{f}$

This is a scalar point function.

Theorem 1: If the vector $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$, then div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

Prof: Given $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$\frac{\partial\bar{f}}{\partial x} = \bar{i}\frac{\partial f_1}{\partial x} + \bar{j}\frac{\partial f_2}{\partial x} + \bar{k}\frac{\partial f_3}{\partial x}$

Also $\bar{i}.\frac{\partial\bar{f}}{\partial x} = \frac{\partial f_1}{\partial x}$. Similarly $\bar{j}.\frac{\partial\bar{f}}{\partial y} = \frac{\partial f_2}{\partial y}$ and $\bar{k}.\frac{\partial\bar{f}}{\partial z} = \frac{\partial f_3}{\partial z}$

We have div $\bar{f} = \sum\bar{i}.\left(\frac{\partial\bar{f}}{\partial x}\right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

Note : If $\bar{f}$ is a constant vector then $\frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial y}, \frac{\partial f_3}{\partial z}$ are zeros.

$\therefore$ div $\bar{f}$ =0 for a constant vector $\bar{f}$.

Theorem 2: div ($\bar{f} \pm \bar{g}$) = $div\,\bar{f} \pm div\,\bar{g}$

Proof: div ($\bar{f} \pm \bar{g}$)= $\sum \bar{i}.\frac{\partial}{\partial x}\left(\bar{f} \pm \bar{g}\right) = \sum \bar{i}.\frac{\partial}{\partial x}\left(\bar{f}\right) \pm \sum \bar{i}.\frac{\partial}{\partial x}\left(\bar{g}\right) = div\,\bar{f} \pm div\,\bar{g}$.

Note: If ϕ is a scalar function and $\bar{f}$ is a vector function, then

(i). $(\bar{a}.\nabla)\phi = \left[\bar{a}.\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\right]\phi$

$$= \left[(\bar{a}.\bar{i})\frac{\partial}{\partial x} + (\bar{a}.\bar{j})\frac{\partial}{\partial y} + (\bar{a}.\bar{k})\frac{\partial}{\partial z}\right]\phi$$

$$= \left[(\bar{a}.\bar{i})\frac{\partial \phi}{\partial x} + (\bar{a}.\bar{j})\frac{\partial \phi}{\partial y} + (\bar{a}.\bar{k})\frac{\partial \phi}{\partial z}\right]$$

$= \sum (\bar{a}.\bar{i})\frac{\partial \phi}{\partial x}$. and

(ii). $(\bar{a}.\nabla)\bar{f} = \sum (\bar{a}.\bar{i})\frac{\partial \bar{f}}{\partial x}$. by proceeding as in (i) [simply replace ϕ by $\bar{f}$ in (i)].

SOLENOIDAL VECTOR

A vector point function $\bar{f}$ is said to be solenoidal if div $\bar{f}$ =0.

Physical interpretation of divergence:

Depending upon $\bar{f}$ in a physical problem, we can interpret div $\bar{f}$ (= $\nabla.\bar{f}$).

Suppose $\bar{F}$ (x,y,z,t) is the velocity of a fluid at a point(x,y,z) and time 't'. Though time has no role in computing divergence, it is considered here because velocity vector depends on time.

Imagine a small rectangular box within the fluid as shown in the figure. We would like to measure the rate per unit volume at which the fluid flows out at any given time. The divergence of $\bar{F}$ measures the outward flow or expansions of the fluid from their point at any time. This gives a physical interpretation of the divergence.

Similar meanings are to be understood with respect to divergence of vectors $\bar{f}$ from other branches. A detailed elementary interpretation can be seen in standard books on fluid dynamics, electricity and magnetism etc.

SOLVED PROBLEMS

1: If $\bar{f}$ **=** $xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ find div $\bar{f}$ at(1, -1, 1).

Sol:- Given $\bar{f}$ = $xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$.

Thendiv $\bar{f}$ = $\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2) = y^2+2x^2z-6yz$

(div $\bar{f}$) at (1, -1, 1) = 1+2+6 =9

2: Find div $\bar{f}$ when grad($x^3+y^3+z^3-3xyz$)

Sol:- Let $\phi = x^3+y^3+z^3-3xyz$.

Then $\frac{\partial \phi}{\partial x} = 3x^2 - 3yz,\ \frac{\partial \phi}{\partial y} = 3y^2 - 3zx,\ \frac{\partial \phi}{\partial z} = 3z^2 - 3xy$

grad $\phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z} = 3[(x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}]$

div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}[3(x^2 - yz)] + \frac{\partial}{\partial y}[3(y^2 - zx)] + \frac{\partial}{\partial z}[3(z^2 - xy)]$

$= 3(2x)+3(2y)+3(2z) = 6(x+y+z)$

3: If $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$ is solenoidal, find P.

Sol:- Let $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

We have $\frac{\partial f_1}{\partial x} = 1,\ \frac{\partial f_2}{\partial y} = 1,\ \frac{\partial f_3}{\partial z} = p$

div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 1+1+p = 2+p$

since $\bar{f}$ is solenoidal, we have div $\bar{f} = 0 \Rightarrow 2+p=0 \Rightarrow p=-2$

4: Find div $\bar{f} = r^n\bar{r}$. Find n if it is solenoidal?

Sol: Given $\bar{f} = r^n\bar{r}$. where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$

We have $r^2 = x^2+y^2+z^2$

Differentiating partially w.r.t. x , we get

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$\bar{f} = r^n(x\bar{i} + y\bar{j} + z\bar{k})$

div $\bar{f} = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$

$= nr^{n-1}\frac{\partial r}{\partial x}x + r^n + nr^{n-1}\frac{\partial r}{\partial y}y + r^n + nr^{n-1}\frac{\partial r}{\partial z}z + r^n$

$= nr^{n-1}\left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right] + 3r^n = nr^{n-1}\frac{(r^2)}{r} + 3r^n = nr^n + 3r^n = (n+3)r^n$

Let $\bar{f} = r^n\bar{r}$ be solenoidal. Then div $\bar{f} = 0$

$(n+3)r^n = 0 \Rightarrow n = -3$

5: Evaluate $\nabla.\left(\frac{\bar{r}}{r^3}\right)$ where $\bar{r} = xi + yj + zk$ and $r = |\bar{r}|$.

Sol:- We have

$\bar{r} = xi+yj+zk$ and $r = \sqrt{x^2+y^2+z^2}$

$$\frac{\partial r}{\partial x}=\frac{x}{r},\ \frac{\partial r}{\partial y}=\frac{y}{r},\ and\ \frac{\partial r}{\partial z}=\frac{z}{r}$$

$\therefore \frac{\bar{r}}{r^3}=\bar{r}.\ r^{-3} = r^{-3}xi+r^{-3}yj+r^{-3}zk = f_1i+f_2j+f_3k$

Hence $\nabla.\left(\frac{\bar{r}}{r^3}\right)=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$

We have $f_1= r^{-3}x \Rightarrow \frac{\partial f_1}{\partial x}=r^{-3}.1+x(-3)r^{-4}.\frac{\partial r}{\partial x}$

$$\therefore \frac{\partial f_1}{\partial x}=r^{-3}-3xr^{-4}\frac{x}{y}=r^{-3}-3x^2r^{-5}$$

$$\nabla.\left(\frac{\bar{r}}{r^3}\right)=\sum \frac{\partial f_1}{\partial x}=3r^{-3}-3r^{-5}\sum x^2$$

$= 3r^{-3}-3r^{-5}r^2 = 3r^{-3}-3r^{-3} =0$

6: Find div $\bar{r}$ where $\bar{r}= x\bar{i}+y\bar{j}+z\bar{k}$

Sol:- We have $\bar{r}= x\bar{i}+y\bar{j}+z\bar{k} = f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$

$$\text{div } \bar{r}=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)=1+1+1=3$$

CURL OF A VECTOR

Def: Let $\bar{f}$ be any continuously differentiable vector point function. Then the vector function defined by $\bar{i}\times\frac{\partial \bar{f}}{\partial x}+\bar{j}\times\frac{\partial \bar{f}}{\partial y}+\bar{k}\times\frac{\partial \bar{f}}{\partial z}$ is called curl of $\bar{f}$ and is denoted by curl $\bar{f}$ or (∇x $\bar{f}$).

$$\text{Curl } \bar{f} = \bar{i}\times\frac{\partial \bar{f}}{\partial x}+\bar{j}\times\frac{\partial \bar{f}}{\partial y}+\bar{k}\times\frac{\partial \bar{f}}{\partial z}=\sum\left(\bar{i}\times\frac{\partial \bar{f}}{\partial x}\right)$$

Theorem 1: If $\bar{f}$ is differentiable vector point function given by $\bar{f}=f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$ then curl $\bar{f}$ = $\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)\bar{i}+\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)\bar{j}+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)\bar{k}$

Proof : curl $\bar{f} = \sum\bar{i}\times\frac{\partial}{\partial x}(\bar{f})=\sum\bar{i}\times\frac{\partial}{\partial x}(f_1\bar{i}+f_2\bar{j}+f_3\bar{k})=\sum\left(\frac{\partial f_2}{\partial x}\bar{k}-\frac{\partial f_3}{\partial x}\bar{j}\right)$

$$=\left(\frac{\partial f_2}{\partial x}\bar{k}-\frac{\partial f_3}{\partial x}\bar{j}\right)+\left(\frac{\partial f_3}{\partial y}\bar{i}-\frac{\partial f_1}{\partial y}\bar{k}\right)+\left(\frac{\partial f_1}{\partial z}\bar{j}-\frac{\partial f_2}{\partial z}\bar{i}\right)$$

$$=\bar{i}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)+\bar{j}\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)+\bar{k}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)$$

Note : (1) The above expression for curl $\bar{f}$ can be remembered easily through the representation.

$$\textbf{curl } \bar{f} = \begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_1 & f_2 & f_3\end{vmatrix} = \nabla \mathbf{x}\ \bar{f}$$

Note (2) : If $\bar{f}$ is a constant vector then curl $\bar{f}=\bar{o}$.

Theorem 2: curl $(\bar{a} \pm \bar{b}) = curl\,\bar{a} \pm curl\,\bar{b}$

Proof: curl$(\bar{a} \pm \bar{b}) = \sum \bar{i} \times \frac{\partial}{\partial x}(\bar{a} \pm \bar{b})$

$$= \sum i \times \left(\frac{\partial \bar{a}}{\partial x} \pm \frac{\partial \bar{b}}{\partial x} \right) = \sum i \times \frac{\partial \bar{a}}{\partial x} \pm \sum ix \frac{\partial \bar{b}}{\partial x}$$

$$= curl\,\bar{a} \pm curl\,\bar{b}$$

1. Physical Interpretation of curl

If $\bar{w}$ is the angular velocity of a rigid body rotating about a fixed axis and $\bar{v}$ is the velocity of any point P(x,y,z) on the body, then $\bar{w}$ = ½ curl $\bar{v}$. Thus the angular velocity of rotation at any point is equal to half the curl of velocity vector. This justifies the use of the word "curl of a vector".

2. Irrotational Motion, Irrotational Vector

Any motion in which curl of the velocity vector is a null vector i.e curl $\bar{v} = \bar{0}$ is said to be Irrotational.

Def: A vector $\bar{f}$ is said to be Irrotational if curl $\bar{f}$ = $\bar{0}$.

If $\bar{f}$ is Irrotational, there will always exist a scalar function φ(x,y,z) such that $\bar{f}$ =grad ϕ. Thisϕ is called scalar potential of $\bar{f}$.

It is easy to prove that, if $\bar{f}$ = grad ϕ, then curl $\bar{f}$ = 0.

Hence ∇x $\bar{f}$ = 0 ⇔ there exists a scalar function ϕ such that $\bar{f}$ = ∇ϕ.

This idea is useful when we study the "work done by a force" later.

SOLVED PROBLEMS

1: If $\bar{f} = xy^2\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$ find curl $\bar{f}$ at the point (1,-1,1).

Sol:- Let $\bar{f} = xy^2\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$. Then

$$\text{curl } \bar{f} = \nabla \text{x } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz) \right) + \bar{j}\left(\frac{\partial}{\partial z}(xy^2) - \frac{\partial}{\partial x}(-3yz^2) \right) + \bar{k}\left(\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2) \right)$$

$$= \bar{i}(-3z^2 - 2x^2z) + \bar{j}(0-0) + \bar{k}(4xyz - 2xy) = -(3z^2 + 2x^2y)\bar{i} + (4xyz - 2xy)\bar{k}$$

= curl $\bar{f}$ at (1,-1,1) = $-\bar{i} - 2\bar{k}$.

2: Find curl $\bar{f}$ where $\bar{f}$ = grad($x^3+y^3+z^3-3xyz$)

Sol:- Let ϕ= $x^3+y^3+z^3-3xyz$. Then

grad ϕ= $\sum \bar{i}\frac{\partial \phi}{\partial x} = 3(x^2 - yz)\bar{i} + 3(y^2 - zx)\bar{j} + 3(z^2 - xy)\bar{k}$

curl grad ϕ= ∇x grad ϕ= $3\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$

$= 3[\bar{i}(-x + x) - \bar{j}(-y + y) + \bar{k}(-z + z)] = \bar{0}$

$\therefore$ curl $\bar{f} = \bar{0}$.

Note: We can prove in general that curl (grad ϕ)= $\bar{0}$.(i.e) grad ϕ is always irrotational.

3: Prove that if $\bar{r}$ is the position vector of an point in space, then $r^n\bar{r}$ is Irrotational. (or) Show that curl$(r^n\bar{r}) = 0$

Sol:- Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and r = $|\bar{r}|$ $\therefore r^2 = x^2+y^2+z^2$.

Differentiating partially w.r.t. 'x', we get

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}, and \frac{\partial r}{\partial z} = \frac{z}{r}$

We have $r^n\bar{r} = r^n(x\bar{i} + y\bar{j} + z\bar{k})$

$\nabla \times (r^n\bar{r}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xr^n & yr^n & zr^n \end{vmatrix}$

$= \bar{i}\left(\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right) + \bar{j}\left(\frac{\partial}{\partial z}(r^n x) - \frac{\partial}{\partial x}(r^n z)\right) + \bar{k}\left(\frac{\partial}{\partial x}(r^n y) - \frac{\partial}{\partial y}(r^n x)\right)$

$= \sum \bar{i}\left\{znr^{n-1}\frac{\partial r}{\partial y} - ynr^{n-1}\frac{\partial r}{\partial z}\right\} = nr^{n-1}\sum \bar{i}\left\{z\left(\frac{y}{r}\right) - y\left(\frac{z}{r}\right)\right\}$

$= nr^{n-2}[(zy - yz)\bar{i} + (xz - zx)\bar{j} + (xy - yz)\bar{k}]$

$= nr^{n-2}[0\bar{i} + 0j + 0\bar{k}] = nr^{n-2}[\bar{0}] = \bar{0}$

Hence $r^n\bar{r}$ is Irrotational.

4: Prove that curl $\bar{r} = \bar{0}$

Sol:- Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

curl $\bar{r} = \sum \bar{i} \times \frac{\partial}{\partial x}(\bar{r}) = \sum(\bar{i} x \bar{i}) = \bar{0} + \bar{0} + \bar{0} = \bar{0}$

$\therefore \bar{r}$ is Irrotational vector.

5: If $\bar{a}$ is a constant vector, prove that curl $\left(\frac{\bar{a} x \bar{r}}{r^3}\right) = -\frac{\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{a}.\bar{r})$.

Sol:- We have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\frac{\partial \bar{r}}{\partial x} = \bar{i}\, , \frac{\partial \bar{r}}{\partial y} = \bar{j}\, , \frac{\partial \bar{r}}{\partial z} = \bar{k}$

If $|\bar{r}|$ = r then $r^2 = x^2+y^2+z^2$

$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, and\, \frac{\partial r}{\partial z} = \frac{z}{r}$

curl $\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = \sum \bar{i}\times\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right)$

Now $\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = \bar{a}\times\frac{\partial}{\partial x}\left(\frac{\bar{r}}{r^3}\right) = \bar{a}\times\left[\frac{1}{r^3}\frac{\partial \bar{r}}{\partial x} - \frac{3}{r^4}\frac{\partial r}{\partial x}\bar{r}\right]$

$$= \bar{a}\times\left[\frac{1}{r^3}\bar{i} - \frac{3}{r^5}x\bar{r}\right] = \frac{\bar{a}\times\bar{i}}{r^3} - \frac{3x(\bar{a}.\times\bar{r})}{r^5}.$$

$$\therefore\ i\times\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = \bar{i}\times\left[\frac{\bar{a}\times\bar{i}}{r^3} - \frac{3x}{r^5}(\bar{a}\times\bar{r})\right] = \frac{\bar{i}\times(\bar{a}\times\bar{i})}{r^3} - \frac{3x}{r^5}\bar{i}\times(\bar{a}\times\bar{r})$$

$$= \frac{(\bar{i}.\bar{i})\bar{a} - (\bar{i}.\bar{a})i}{r^3} - \frac{3x}{r^5}[(\bar{i}.\bar{r})\bar{a} - (i.a)\bar{r}]$$

Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$. Then $\bar{i}\,.\,\bar{a} = a_1$, etc.

$$\therefore\ i\times\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = \sum\frac{(\bar{a} - a_1 i)}{r^3} - \frac{3x}{r^3}(x\bar{a} - a_1\bar{r})$$

$$\therefore\quad \sum i\times\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = \sum\frac{\bar{a} - a_1\bar{i}}{r^3} - \frac{3}{r^5}\sum(x^2\bar{a} - a_1 x\bar{r})$$

$$= \frac{3\bar{a} - \bar{a}}{r^3} - \frac{3\bar{a}}{r^5}(r^2) + \frac{3\bar{r}}{r^5}(a_1x + a_2y + a_3z)$$

$$= \frac{2\bar{a}}{r^3} - \frac{3\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{r}.\bar{a}) = -\frac{\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{r}.\bar{a})$$

6: Show that the vector $(x^2 - yz)\bar{i} + (y^2 - zx)\,\bar{j} + (z^2 - xy)\,\bar{k}$ is irrotational and find its scalar potential.

Sol: let $\bar{f} = (x^2 - yz)\bar{i} + (y^2 - zx)\,\bar{j} + (z^2 - xy)\,\bar{k}$

$$\text{Then curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix} = \sum \bar{i}(-x + x) = \bar{0}$$

$\therefore\ \bar{f}$ is Irrotational. Then there exists ϕ such that $\bar{f} = \nabla\phi$.

$\Rightarrow\ \bar{i}\,\frac{\partial\phi}{\partial x} + \bar{j}\,\frac{\partial\phi}{\partial y} + \bar{k}\,\frac{\partial\phi}{\partial z} = (x^2 - yz)\bar{i} + (y^2 - zx)\,\bar{j} + (z^2 - xy)\,\bar{k}$

Comparing components, we get

$$\frac{\partial\phi}{\partial x} = x^2 - yz \Rightarrow \phi = \int\left(x^2 - yz\right)dx = \frac{x^3}{3} - xyz + f_1(y,z)......(1)$$

$$\frac{\partial\phi}{\partial y} = y^2 - zx \Rightarrow \phi = \frac{y^3}{3} - xyz + f_2(z,x)......(2)$$

$$\frac{\partial\phi}{\partial z} = z^2 - xy \Rightarrow \phi = \frac{z^3}{3} - xyz + f_3(x, y)......(3)$$

From (1), (2),(3), $\phi = \dfrac{x^3 + y^3 + z^3}{3} - xyz$

$$\therefore \quad \phi = \frac{1}{3}(x^3 + y^3 + z^3) - xyz + constant$$

Which is the required scalar potential.

7: Find constants a,b and c if the vector $\bar{f} = (2x+3y+az)\bar{i} + (bx+2y+3z)\,\bar{j} + (2x+cy+3z)\bar{k}$ is Irrotational.

Sol:- Given $\bar{f} = (2x+3y+az)\bar{i} + (bx+2y+3z)\,\bar{j} + (2x+cy+3z)\bar{k}$

$$\text{Curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix} = (c-3)\bar{i} - (2-a)\,\bar{j} + (b-3)\bar{k}$$

If the vector is Irrotational then curl $\bar{f} = \bar{0}$

$$\therefore 2-a=0 \Rightarrow a=2, b-3=0 \Rightarrow b=3, c-3=0 \Rightarrow c=3$$

8: If f(r) is differentiable, show that curl { $\bar{r}$ f(r)} = $\bar{0}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

Sol: r = $\bar{r} = \sqrt{x^2 + y^2 + z^2}$

$\therefore r^2 = x^2+y^2+z^2$

$$\Rightarrow 2r\,\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \text{ similarly } \frac{\partial r}{\partial y} = \frac{y}{r}, and \frac{\partial r}{\partial z} = \frac{z}{r}$$

curl{ $\bar{r}$ f(r)}= curl{f(r)($x\bar{i} + y\bar{j} + z\bar{k}$)}= curl $(x.f(r)\bar{i} + y.f(r)\,\bar{j} + z.f(r)\bar{k})$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix} = \sum \bar{i}\left[\frac{\partial}{\partial y}[zf(r)] - \frac{\partial}{\partial z}[yf(r)]\right]$$

$$\sum \bar{i}\left[zf^1(r)\frac{\partial r}{\partial y} - yf^1(r)\frac{\partial r}{\partial z}\right] = \sum \bar{i}\left[zf^1(r)\frac{y}{r} - yf^1(r)\frac{z}{r}\right]$$

= $\bar{0}$.

9: If $\bar{A}$ is irrotational vector, evaluate div($\bar{A}$ x $\bar{r}$) where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

Sol: We have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Given $\bar{A}$ is an irrotational vector

∇x $\bar{A}$ = $\bar{0}$

div ($\bar{A}$ x $\bar{r}$) = ∇.($\bar{A}$ x $\bar{r}$)

= $\bar{r}$.(∇x $\bar{A}$)- $\bar{A}$.(∇x $\bar{r}$)

$= \bar{r}.(\bar{0}) - \bar{A}.(\nabla \times \bar{r})$ [using (1)]
$= - \bar{A}.(\nabla \times \bar{r})$…..(2)

Now $\nabla \times \bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \bar{i}\left(\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y\right) - \bar{j}\left(\frac{\partial}{\partial x}z - \frac{\partial}{\partial z}x\right) + \bar{k}\left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right) = \bar{0}$

$\therefore \bar{A}.(\nabla \times \bar{r})=0$ …(3)

Hence div $(\bar{A} \times \bar{r})=0$. [using (2) and (3)]

10: Find constants a,b,c so that the vector $\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\,\bar{j} + (4x+cy+2z)\,\bar{k}$ is Irrotational. Also find ϕ such that $\bar{A} = \nabla\phi$.

Sol: Given vector is $\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\,\bar{j} + (4x+cy+2z)\,\bar{k}$

Vector $\bar{A}$ is Irrotational $\Rightarrow$ curl $\bar{A} = \bar{0}$

$$\Rightarrow \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \bar{0}$$

$\Rightarrow (c+1)\bar{i} + (a-4)\,\bar{j} + (b-2)\bar{k} = \bar{0}$

$\Rightarrow (c+1)\bar{i} + (a-4)\,\bar{j} + (b-2)\bar{k} = 0\bar{i} + 0\,\bar{j} + 0\bar{k}$

Comparing both sides,

c+1=0, a-4=0, b-2=0

c= -1, a=4,b=2

Now $\bar{A} = (x+2y+4z)\bar{i} + (2x-3y-z)\,\bar{j} + (4x-y+2z)\bar{k}$, on substituting the values of a,b,c

we have $\bar{A} = \nabla\phi$.

$\Rightarrow \bar{A} = (x+2y+4z)\bar{i} + (2x-3y-z)\,\bar{j} + (4x-y+2z)\bar{k} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

Comparing both sides, we have

$\frac{\partial\phi}{\partial x} = x+2y+4z \Rightarrow \phi = x^2/2+2xy+4zx+f_1(y,z)$

$\frac{\partial\phi}{\partial y} = 2x-3y-z \Rightarrow \phi = 2xy-3y^2/2-yz+f_2(z,x)$

$\frac{\partial\phi}{\partial z} = 4x-y+2z \Rightarrow \phi = 4xz-yz+z^2+f_3(x,y)$

Hence $\phi = x^2/2 - 3y^2/2 + z^2 + 2xy + 4zx - yz + c$

11: If ω is a constant vector, evaluate curl V where $V = \omega \times \bar{r}$.

Sol: curl (ωx$\bar{r}$) = $\sum \bar{i}\times\frac{\partial}{\partial x}(\varpi\times\bar{r}) = \sum \bar{i}\times\left[\frac{\partial\varpi}{\partial x}\times\bar{r}+\varpi\times\frac{\partial\bar{r}}{\partial x}\right]$

$$=\sum \bar{i}\times[\bar{0}+\omega\times\bar{i}]\ \ [\therefore \bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}.\bar{c})\bar{b}-(\bar{a}.\bar{b}).\bar{c}]$$
$$=\sum \bar{i}\times(\omega\times\bar{i}) = \sum[(\bar{i}.\bar{i})\omega-(\bar{i}.\omega)\bar{i}] = \sum\omega-\sum(\bar{i}.\omega)\bar{i} = 3\omega-\omega = 2\omega$$

Assignments

1. If $\bar{f}$ = $e^{x+y+z}(\bar{i}+\bar{j}+\bar{k})$ find curl $\bar{f}$.

2. Prove that $\bar{f}$ = $(y+z)\bar{i}+(z+x)\bar{j}+(x+y)\bar{k}$ is irrotational.

3. Prove that $\nabla.(\bar{a}\times\bar{f}) = -\bar{a}$. curl $\bar{f}$ where $\bar{a}$ is a constant vector.

4. Prove that curl ($\bar{a}\times\bar{r}$)=2$\bar{a}$ where $\bar{a}$ is a constant vector.

5. If $\bar{f}$ = $x^2y\bar{i}-2zx\bar{j}+2yz\bar{k}$ find (i) curl $\bar{f}$ (ii) curl curl $\bar{f}$.

OPERATORS

Vector differential operator ∇

The operator ∇ = $\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}$ is defined such that ∇ϕ= $\bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}$ where ϕ is a scalar point function.

Note: If ϕ is a scalar point function then ∇ϕ= grad ϕ= $\sum i\frac{\partial\phi}{\partial x}$

(2) Scalar differential operator $\bar{a}$.∇

The operator $\bar{a}$.∇ = $(\bar{a}.\bar{i})\frac{\partial\phi}{\partial x}+(\bar{a}.\bar{j})\frac{\partial\phi}{\partial y}+(\bar{a}.\bar{k})\frac{\partial\phi}{\partial z}$ is defined such that

($\bar{a}$.∇)ϕ= $(\bar{a}.\bar{i})\frac{\partial\phi}{\partial x}+(\bar{a}.\bar{j})\frac{\partial\phi}{\partial y}+(\bar{a}.\bar{k})\frac{\partial\phi}{\partial z}$

And ($\bar{a}$.∇) $\bar{f} = (\bar{a}.\bar{i})\frac{\partial\bar{f}}{\partial x}+(\bar{a}.\bar{j})\frac{\partial\bar{f}}{\partial y}+(\bar{a}.\bar{k})\frac{\partial\bar{f}}{\partial z}$

(3). Vector differential operator $\bar{a}$ x∇

The operator $\bar{a}$ x∇= $(\bar{a}\times\bar{i})\frac{\partial}{\partial x}+(\bar{a}\times\bar{j})\frac{\partial}{\partial y}+(\bar{a}\times\bar{k})\frac{\partial}{\partial z}$ is defined such that

(i). ($\bar{a}$ x∇)ϕ= $(\bar{a}\times\bar{i})\frac{\partial\phi}{\partial x}+(\bar{a}\times\bar{j})\frac{\partial\phi}{\partial y}+(\bar{a}\times\bar{k})\frac{\partial\phi}{\partial z}$

(ii). ($\bar{a}$ x∇). $\bar{f} = (\bar{a}\times\bar{i}).\frac{\partial\bar{f}}{\partial x}+(\bar{a}\times\bar{j}).\frac{\partial\bar{f}}{\partial y}+(\bar{a}\times\bar{k}).\frac{\partial\bar{f}}{\partial z}$

(iii). ($\bar{a}$ x∇)x $\bar{f} = (\bar{a}\times\bar{i})\times\frac{\partial\bar{f}}{\partial x}+(\bar{a}\times\bar{j})\times\frac{\partial\bar{f}}{\partial y}+(\bar{a}\times\bar{k})\times\frac{\partial\bar{f}}{\partial z}$

(4). Scalar differential operator ∇.

The operator ∇ = $\bar{i}.\frac{\partial}{\partial x}+\bar{j}.\frac{\partial}{\partial y}+\bar{k}.\frac{\partial}{\partial z}$ is defined such that ∇. $\bar{f}$ = $\bar{i}.\frac{\partial\bar{f}}{\partial x}+\bar{j}.\frac{\partial\bar{f}}{\partial y}+\bar{k}.\frac{\partial\bar{f}}{\partial z}$

Note: ∇. $\bar{f}$ is defined as div $\bar{f}$. It is a scalar point function.

(5). Vector differential operator ∇ x

The operator ∇ x = $\bar{i}\times\frac{\partial}{\partial x}+\bar{j}\times\frac{\partial}{\partial y}+\bar{k}\times\frac{\partial}{\partial z}$ is defined such that

$$\nabla\text{x } \bar{f} = \bar{i}\times\frac{\partial\bar{f}}{\partial x}+\bar{j}\times\frac{\partial\bar{f}}{\partial y}+\bar{k}\times\frac{\partial\bar{f}}{\partial z}$$

Note : ∇x $\bar{f}$ is defined as curl $\bar{f}$. It is a vector point function.

(6). Laplacian Operator ∇^2

$$\nabla.\nabla\phi=\sum\bar{i}.\frac{\partial}{\partial x}\left(\bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}\right)=\sum\frac{\partial^2\phi}{\partial x^2}=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi=\nabla^2\phi$$

Thus the operator $\nabla^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is called Laplacian operator.

Note : (i). $\nabla^2\phi= \nabla.(\nabla\phi)$ = div(grad ϕ)

(ii). If $\nabla^2\phi=0$ then ϕ is said to satisfy Laplacian equation. This ϕ is called a harmonic function.

SOLVED PROBLEMS

1: Prove that div.(grad r^m)= $m(m+1)r^{m-2}$ (or) $\nabla^2(r^m) = m(m+1)r^{m-2}$ (or) $\nabla^2(r^n) = n(n+1)r^{n-2}$

Sol: Let $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ and r = $|\bar{r}|$ then $r^2 = x^2+y^2+z^2$.

Differentiating w.r.t. 'x' partially, wet get $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$.

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now grad(r^m) = $\sum\bar{i}\frac{\partial}{\partial x}(r^m)=\sum\bar{i}mr^{m-1}\frac{\partial r}{\partial x}=\sum\bar{i}mr^{m-1}\frac{x}{r}=\sum\bar{i}mr^{m-2}x$

∴ div (grad r^m) = $\sum\frac{\partial}{\partial x}[mr^{m-2}x]$=m$\sum\left[(m-2)r^{m-3}\frac{\partial r}{\partial x}x+r^{m-2}\right]$

$$=m\sum\left[(m-2)r^{m-4}x^2+r^{m-2}\right]=m\left[(m-2)r^{m-4}\sum x^2+\sum r^{m-2}\right]$$

$$= m[(m-2)r^{m-4}(r^2)+3r^{m-2}]$$

$$= m[(m-2)\, r^{m-2}+3r^{m-2}]= m[(m-2+3)r^{m-2}]= m(m+1)r^{m-2}.$$

Hence $\nabla^2(r^m) = m(m+1)r^{m-2}$

2: Show that $\nabla^2[f(r)]= \frac{d^2f}{dr^2}+\frac{2}{r}\frac{df}{dr}=f^{11}(r)+\frac{2}{r}f^1(r)$ where r = $|\bar{r}|$.

Sol: grad [f(r)] = ∇*f*(r)= $\sum i\frac{\partial}{\partial x}[f(r)]=\sum i\, f^1(r)\frac{\partial r}{\partial x}=\sum i\, f^1(r)\frac{x}{r}$

∴ div [grad *f*(r)] = $\nabla^2[f(r)] = \nabla.\nabla f(r)$= $\sum\frac{\partial}{\partial x}\left[f^1(r)\frac{x}{r}\right]$

$$= \sum\frac{r\frac{\partial}{\partial x}[f^1(r)x]-f^1(r)x\frac{\partial}{\partial x}(r)}{r^2}$$

$$= \sum \frac{r\left(f^{11}(r)\frac{\partial r}{\partial x}x + f^{1}(r)\right) - f^{1}(r)x\left(\frac{x}{r}\right)}{r^2}$$

$$= \sum \frac{rf^{11}(r)\frac{x}{r}x + rf^{1}(r) - f^{1}(r)x\left(\frac{x}{r}\right)}{r^2}$$

$$= \frac{\sum rf^{11}(r)\frac{x}{r}x + rf^{1}(r) - x^2}{r^2} \cdot \frac{f^{1}(r)}{r}$$

$$= \frac{f^{11}(r)}{r^2}\sum x^2 + \frac{1}{r}\sum f^{1}(r) - \frac{1}{r^3}f^{1}(r)\sum x^2$$

$$= \frac{f^{11}(r)}{r^2}(r^2) + \frac{3}{r}f^{1}(r) - \frac{1}{r^3}f^{1}(r)r^2$$

$$= f^{11}(r) + \frac{2}{r}f^{1}(r)$$

3: If ϕ satisfies Laplacian equation, show that $\nabla\phi$ is both solenoidal and irrotational.

Sol: Given $\nabla^2\phi = 0 \Rightarrow$ div(grad ϕ)= 0 $\Rightarrow$ grad ϕ is solenoidal

We know that curl (grad ϕ) = $\bar{0} \Rightarrow$ grad ϕ is always irrotational.

4: Show that (i) $(\bar{a}.\nabla)\phi = \bar{a}.\nabla\phi$ (ii) $(\bar{a}.\nabla)\bar{r} = \bar{a}$.

Sol: (i). Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$. Then

$$\bar{a}.\nabla = (a_1\bar{i} + a_2\bar{j} + a_3\bar{k}).\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) = a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}$$

$$\therefore (\bar{a}.\nabla)\phi = a_1\frac{\partial\phi}{\partial x} + a_2\frac{\partial\phi}{\partial y} + a_3\frac{\partial\phi}{\partial z}$$

Hence $(\bar{a}.\nabla)\phi = \bar{a}.\nabla\phi$

(ii). $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\therefore \frac{\partial\bar{r}}{\partial x} = \bar{i},\ \frac{\partial\bar{r}}{\partial y} = \bar{j},\ \frac{\partial\bar{r}}{\partial z} = \bar{k}$$

$$(\bar{a}.\nabla)\bar{r} = \sum a_1\frac{\partial}{\partial x}(\bar{r}) = \sum a_1\frac{\partial\bar{r}}{\partial x} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k} = \bar{a}$$

5: Prove that (i) $(\bar{f}\times\nabla).\bar{r} = 0$ (ii). $(\bar{f}\times\nabla)\times\bar{r} = -2\bar{f}$

Sol: (i) $(\bar{f}\times\nabla).\bar{r} = \sum(\bar{f}\times\bar{i}).\frac{\partial\bar{r}}{\partial x} = \sum(\bar{f}\times\bar{i}).\bar{i} = 0$

(ii) $(\bar{f}\times\nabla) = (\bar{f}\times\bar{i})\frac{\partial}{\partial x}\times(\bar{f}\times\bar{j})\frac{\partial}{\partial y}\times(\bar{f}\times\bar{k})\frac{\partial}{\partial z}$

$$(\bar{f}\times\nabla)\times\bar{r} = (\bar{f}\times\bar{i})\times\frac{\partial\bar{r}}{\partial x} + (\bar{f}\times\bar{j})\times\frac{\partial\bar{r}}{\partial y} + (\bar{f}\times\bar{k})\times\frac{\partial\bar{r}}{\partial z} = \sum(\bar{f}\times\bar{i})\times\bar{i} = \sum\left[(\bar{f}.i)i - \bar{f}\right]$$

$$= (\bar{f}.i)\bar{i} + (\bar{f}.\bar{j})\bar{j} + (\bar{f}.\bar{k})\bar{k} - 3\bar{f} = \overline{f} - 3\bar{f} = -2\overline{f}$$

6: Find div $\bar{F}$, where $\bar{F}$ = grad $(x^3+y^3+z^3-3xyz)$

Sol: Let $\phi= x^3+y^3+z^3-3xyz$. Then

$\bar{F}$ = grad ϕ

$$= \sum i \frac{\partial \phi}{\partial x} = 3(x^2 - yz)\bar{i} + 3(y^2 - zx)\,\bar{j} + 3(x^2 - xy)\,\bar{k} = F_1 i + F_2\, j + F_3 k\ \ (say)$$

$$\therefore \text{div } \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 6x+6y+6z= 6(x+y+z)$$

i.e div[grad$(x^3+y^3+z^3-3xyz)$]= $\nabla^2(x^3+y^3+z^3-3xyz)= 6(x+y+z)$.

7: If $f= (x^2+y^2+z^2)^{-n}$ then find div grad f and determine n if div grad $f= 0$.

Sol: Let $f= (x^2+y^2+z^2)^{-n}$ and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$r = |\bar{r}| \Rightarrow r^2 = x^2+y^2+z^2$

$\Rightarrow f(r) = (r^2)^{-n} = r^{-2n}$

$\therefore f^1(r)= -2n\, r^{-2n-1}$

and $f^{11}(r) = (-2n)(-2n-1)r^{-2n-2}= 2n(2n+1)r^{-2n-2}$

We have div grad f = $\nabla^2 f(r)= f^{11}(r)+{}^2/_r f^1(r)= (2n)(2n+1)r^{-2n-2} -4n\, r^{-2n-2}$

$= r^{-2n-2}[2n(2n+1-2)]= (2n)(2n-1)r^{-2n-2}$

If div grad $f(r)$ is zero, we get n = 0 or n = ½ .

8: Prove that $\nabla \text{x}\left(\frac{\bar{A}\times\bar{r}}{r^n}\right) = \frac{(2-n)\bar{A}}{r^n} + \frac{n(\bar{r}.\bar{A})\bar{r}}{r^{n+2}}$.

Sol: We have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$

$\therefore \frac{\partial \bar{r}}{\partial x} = \bar{i},\ \frac{\partial \bar{r}}{\partial y} = \bar{j},\ \frac{\partial \bar{r}}{\partial z} = \bar{k}$ and

$r^2 = x^2+y^2+z^2$…..(1)

Diff. (1) partially,

$2r\,\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$, similarly $\frac{\partial r}{\partial y} = \frac{y}{r}\ and\ \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\nabla \text{x}\left(\frac{\bar{A}\times\bar{r}}{r^n}\right) = \sum \bar{i} \times \frac{\partial}{\partial x}\left(\frac{(\bar{A}\times\bar{r})}{r^n}\right)$$

Now $\frac{\partial}{\partial x}\left(\frac{(\bar{A}\times\bar{r})}{r^n}\right) = \bar{A}\times\frac{\partial}{\partial x}\left(\frac{\bar{r}}{r^n}\right) = \bar{A}\times\left[\frac{r^n\bar{i} - \bar{r}\, n\, r^{n-1}}{r^{2n}}\right]\frac{\partial r}{\partial x}$

$$= \bar{A}\times\left[\frac{r^n\bar{i} - n\, r^{n-2} x\bar{r}}{r^{2n}}\right] = \bar{A}\times\left[\frac{1}{r^n}\bar{i} - \frac{n}{r^{n+2}}.x\bar{r}\right]$$

$$= \frac{\overline{A}\times\overline{i}}{r^n} - \frac{n}{r^{n+2}}.x(\overline{A}\times\overline{r})$$

$$\therefore\ \overline{i}\times\frac{\partial}{\partial x}\left(\frac{(\overline{A}\times\overline{r})}{r^n}\right) = \frac{\overline{i}\times(\overline{A}\times\overline{i})}{r^n} - \frac{nx}{r^{n+2}}.\overline{i}\times(\overline{A}\times\overline{r})$$

$$= \frac{(\overline{i}.\overline{i})\overline{A}-(\overline{i}.\overline{A})\overline{i}}{r^n} - \frac{nx}{r^{n+2}}[(\overline{i}.\overline{r})\overline{A}-(\overline{i}.\overline{A})\overline{r}]$$

Let $A_1\overline{i}+A_2\overline{j}+A_3\overline{k}$. Then $\overline{i}.\overline{A}=A_1$

$$\therefore\ \overline{i}\times\frac{\partial}{\partial x}\left(\frac{(\overline{A}\times\overline{r})}{r^n}\right) = \left(\frac{\overline{A}-A_1\overline{i}}{r^n}\right) - \frac{nx}{r^{n+2}}[x\overline{A}-A_1\overline{r}]$$

and $\sum\overline{i}\times\frac{\partial}{\partial x}\left(\frac{(\overline{A}\times\overline{r})}{r^n}\right) = \sum\left(\frac{\overline{A}-A_1\overline{i}}{r^n}\right) - \frac{nx}{r^{n+2}}[x\overline{A}-A_1\overline{r}]$

$$= \frac{3\overline{A}-\overline{A}}{r^n} - \frac{n}{r^{n+2}}[r^2\overline{A}] + \frac{n\overline{r}}{r^{n+2}}(A_1x+A_2y+A_3z)$$

$$= \frac{2\overline{A}}{r^n} - \frac{n}{r^n}\overline{A} + \frac{n\overline{r}}{r^{n+2}}(\overline{A}.\overline{r}) = \frac{(2-n)\overline{A}}{r^n} + \frac{n\overline{r}}{r^{n+2}}(\overline{A}.\overline{r})$$

Hence the result.

VECTOR IDENTITIES

Theorem 1: If $\overline{a}$ is a differentiable function and φ is a differentiable scalar function, then prove that div(φ$\overline{a}$)= (grad φ).$\overline{a}$ +φ div $\overline{a}$ or ∇.(φ$\overline{a}$)= (∇φ).$\overline{a}$ +φ(∇.$\overline{a}$)

Proof: div(φ$\overline{a}$)=∇.(φ$\overline{a}$)= $\sum i.\frac{\partial}{\partial x}(\phi\overline{a})$

$$= \sum\overline{i}.\left(\frac{\partial\phi}{\partial x}\overline{a}+\phi\frac{\partial\overline{a}}{\partial x}\right) = \sum\left(i.\frac{\partial\phi}{\partial x}\overline{a}\right) + \sum\left(i\frac{\partial\overline{a}}{\partial x}\right)\phi$$

$$= \sum\left(\overline{i}\frac{\partial\phi}{\partial x}\right).\overline{a} + \left(\sum\overline{i}\frac{\partial a}{\partial x}\right)\phi = (\nabla\phi).\overline{a} + \phi(\nabla.\overline{a})$$

Theorem 2: Prove that curl (φ$\overline{a}$)= (grad φ)x$\overline{a}$ +φ curl $\overline{a}$

Proof : curl (φ$\overline{a}$)=∇x(φ$\overline{a}$)= $\sum i\times\frac{\partial}{\partial x}(\phi\overline{a})$

$$= \sum\overline{i}\times\left(\frac{\partial\phi}{\partial x}\overline{a}+\phi\frac{\partial\overline{a}}{\partial x}\right) = \sum\left(\overline{i}\frac{\partial\phi}{\partial x}\right)\times\overline{a} + \sum\left(\overline{i}\times\frac{\partial\overline{a}}{\partial x}\right)\phi$$

= ∇φx$\overline{a}$ +(∇x$\overline{a}$)φ=(grad φ)x$\overline{a}$ +φ curl $\overline{a}$

Theorem 3: Prove that grad ($\overline{a}$. $\overline{b}$)= $(\overline{b}.\nabla)\overline{a}+(\overline{a}.\nabla)\overline{b}+\overline{b}\times curl\,\overline{a}+\overline{a}\times curl\,\overline{b}$

Proof: Consider

$$\overline{a}\times curl\left(\overline{b}\right) = \overline{a}\times\left(\nabla\times\overline{b}\right) = a\times\sum\left(\overline{i}\times\frac{\partial\overline{b}}{\partial x}\right)$$

$$= \sum \bar{a} \times \left(\bar{i} \times \frac{\partial \bar{b}}{\partial x} \right)$$

$$= \sum \left\{ \left(\bar{a} . \frac{\partial \bar{b}}{\partial x} \right) \bar{i} - (\bar{a}.\bar{i}) \frac{\partial \bar{b}}{\partial x} \right\} = \sum \bar{i} \left\{ \bar{a} . \frac{\partial \bar{b}}{\partial x} \right\} - \left\{ \bar{a} . \sum i \frac{\partial}{\partial x} \right\} \bar{b}$$

$$\therefore \bar{a} \times curl\, \bar{b} = \sum \bar{i} \left(\bar{a} . \frac{\partial \bar{b}}{\partial x} \right) - (\bar{a}.\nabla) \bar{b} \quad \ldots.(1)$$

Similarly, $\bar{b} \times curl\, \bar{b} = \sum \bar{i} \left(\bar{b} . \frac{\partial \bar{a}}{\partial x} \right) - (\bar{b}.\nabla) \bar{a}$ …………………….(2)

(1)+(2) gives

$$\bar{a} \times curl\, \bar{b} + \bar{b} \times curl\, \bar{a} = \sum \bar{i} \left(\bar{a} . \frac{\partial \bar{b}}{\partial x} \right) - (\bar{a}.\nabla) \bar{b} + \sum \bar{i} \left(\bar{b} . \frac{\partial \bar{a}}{\partial x} \right) - (\bar{b}.\nabla) \bar{a}$$

$$\Rightarrow \bar{a} \times curl\, \bar{b} + \bar{b} \times curl\, \bar{a} + (\bar{a}.\nabla)\bar{b} + (\bar{b}.\nabla)\bar{a} = \sum \bar{i} \left(\bar{a} . \frac{\partial \bar{b}}{\partial x} + \bar{b} . \frac{\partial \bar{a}}{\partial x} \right)$$

$$= \sum \bar{i} \frac{\partial}{\partial x} (\bar{a}.\bar{b})$$

$$= \nabla(\bar{a} . \bar{b}) = \text{grad} (\bar{a} . \bar{b})$$

Theorem 4: Prove that div $(\bar{a} \times \bar{b}) = \bar{b}.curl\, \bar{a} - \bar{a}.curl\, \bar{b}$

Proof: div $(\bar{a} \times \bar{b}) = \sum \bar{i} . \frac{\partial}{\partial x} (\bar{a} \times \bar{b}) = \sum \bar{i} . \left(\frac{\partial \bar{a}}{\partial x} \times \bar{b} + \bar{a} \times \frac{\partial \bar{b}}{\partial x} \right)$

$$= \sum \bar{i} . \left(\frac{\partial \bar{a}}{\partial x} \times \bar{b} \right) + \sum \bar{i} . \left(\bar{a} \times \frac{\partial \bar{b}}{\partial x} \right) = \sum \left(\bar{i} \times \frac{\partial \bar{a}}{\partial x} \right) . \bar{b} - \sum \left(\bar{i} \times \frac{\partial \bar{b}}{\partial x} \right) . \bar{a}$$

$$= (\nabla \times \bar{a}).\bar{b} - (\nabla \times \bar{b}).\bar{a} = \bar{b}.curl\, \bar{a} - \bar{a}.curl\, \bar{b}$$

Theorem 5 : Prove that $curl\, (\bar{a} \times \bar{b}) = \bar{a}\, div \bar{b} - \bar{b} div\, \bar{a} + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$

$Proof : curl\, (\bar{a} \times \bar{b}) = \sum \bar{i} \times \frac{\partial}{\partial x} (\bar{a} \times \bar{b}) = \sum \bar{i} \times \left[\frac{\partial \bar{a}}{\partial x} \times \bar{b} + \bar{a} \times \frac{\partial \bar{b}}{\partial x} \right]$

$$\sum \bar{i} \times \left(\frac{\partial \bar{a}}{\partial x} \times \bar{b} \right) + \sum \bar{i} \times \left(\bar{a} \times \frac{\partial \bar{b}}{\partial x} \right)$$

$$= \sum \left\{ (\bar{i}.\bar{b}) \frac{\partial \bar{a}}{\partial x} - \left(\bar{i} . \frac{\partial \bar{a}}{\partial x} \right) \bar{b} \right\} + \sum \left\{ \left(\bar{i} . \frac{\partial \bar{b}}{\partial x} \right) \bar{a} - (\bar{i}.\bar{a}) \frac{\partial \bar{b}}{\partial x} \right\}$$

$$= \sum (\bar{b}.\bar{i}) \frac{\partial \bar{a}}{\partial x} - \sum \left(\bar{i} . \frac{\partial \bar{a}}{\partial x} \right) \bar{b} + \sum \left(\bar{i} . \frac{\partial \bar{b}}{\partial x} \right) \bar{a} - \left(\bar{a} . \sum \bar{i} \frac{\partial}{\partial x} \right) \bar{b}$$

$$= (\bar{b}.\nabla)\bar{a} - (\nabla.\bar{a})\bar{b} + (\nabla.\bar{b})\bar{a} - (\bar{a}.\nabla)\bar{b}$$

$$= (\nabla.\bar{b})\bar{a} - (\nabla.\bar{a})\bar{b} + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$$

$$= \bar{a}\,div\,\bar{b} - \bar{b}\,div\,\bar{a} + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$$

Theorem 6: Prove that curl grad $\phi = 0$.

Proof: Let ϕ be any scalar point function. Then

$$.grad\ \phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$curl(grad\phi) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) - \bar{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right) - \bar{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right) = \bar{0}$$

Note : Since $Curl(grad\phi) = \bar{0}$, we have $grad\ \phi$ is always irrotational.

7. Prove that div $curl\,\bar{f} = 0$

$\Pr oof : Let\ \bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\therefore curl\ \bar{f} = \nabla\times\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\bar{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\bar{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\bar{k}$$

$$\therefore \quad div\ \ curl\ \bar{f} = \nabla.(\nabla\times\bar{f}) = \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \frac{\partial^2 f_3}{\partial x\partial y} - \frac{\partial^2 f_2}{\partial x\partial z} - \frac{\partial^2 f_3}{\partial y\partial x} + \frac{\partial^2 f_1}{\partial y\partial z} + \frac{\partial^2 f_2}{\partial z\partial x} - \frac{\partial^2 f_1}{\partial z\partial y} = 0$$

Note : Since $div(curl\,\bar{f}) = 0,$ we have $curl\,\bar{f}$ is always solenoidal.

Theorem 8: If f and g are two scalar point functions, prove that $div(f\nabla g) = f\nabla^2 g + \nabla f.\nabla g$

Sol: Let f and g be two scalar point functions. Then

$$\nabla g = \bar{i}\frac{\partial g}{\partial x} + \bar{j}\frac{\partial g}{\partial y} + \bar{k}\frac{\partial g}{\partial z}$$

Now $f\nabla g = \bar{i}f\,\frac{\partial g}{\partial x} + \bar{j}f\,\frac{\partial g}{\partial y} + \bar{k}f\,\frac{\partial g}{\partial z}$

$$\therefore \nabla.(f\nabla g) = \frac{\partial}{\partial x}\left(f\,\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\,\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(f\,\frac{\partial g}{\partial z}\right)$$

$$= f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial x^2}\right) + \left(\frac{\partial f}{\partial x}.\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}.\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}.\frac{\partial g}{\partial z}\right)$$

$$= f\nabla^2 g + \left(\bar{i}\,\frac{\partial f}{\partial x} + \bar{j}\,\frac{\partial f}{\partial y} + \bar{k}\,\frac{\partial f}{\partial z}\right).\left(\bar{i}\,\frac{\partial g}{\partial x} + \bar{j}\,\frac{\partial g}{\partial y} + \bar{k}\,\frac{\partial g}{\partial z}\right).$$

$$= f\nabla^2 g + \nabla f.\,\nabla g$$

Theorem 9: Prove that $\nabla \text{x}(\nabla \text{x}\,\bar{a}) = \nabla(\nabla.\,\bar{a}) - \nabla^2\bar{a}$.

Proof: $\nabla \text{x}(\nabla \text{x}\,\bar{a}) = \sum \bar{i} \times \frac{\partial}{\partial x}(\nabla \times \bar{a})$

Now $\bar{i} \times \frac{\partial}{\partial x}(\nabla \times \bar{a}) = i \times \frac{\partial}{\partial x}\left(\bar{i} \times \frac{\partial \bar{a}}{\partial x} + \bar{j} \times \frac{\partial \bar{a}}{\partial y} + \bar{k} \times \frac{\partial \bar{a}}{\partial z}\right)$

$$= \bar{i} \times \left(\bar{i} \times \frac{\partial^2 \bar{a}}{\partial x^2} + \bar{j} \times \frac{\partial^2 \bar{a}}{\partial x \partial y} + \bar{k} \times \frac{\partial^2 \bar{a}}{\partial x \partial z}\right)$$

$$= \bar{i} \times \left(\bar{i} \times \frac{\partial^2 \bar{a}}{\partial x^2}\right) + \bar{i} \times \left(\bar{j} \times \frac{\partial^2 \bar{a}}{\partial x \partial y}\right) + \bar{i} \times \left(\bar{k} \times \frac{\partial^2 \bar{a}}{\partial x \partial z}\right)$$

$$= \left(\bar{i}.\frac{\partial^2 \bar{a}}{\partial x^2}\right)\bar{i} - \frac{\partial^2 \bar{a}}{\partial x^2} + \left(\bar{i}.\frac{\partial^2 \bar{a}}{\partial x \partial y}\right)\bar{j} + \left(\bar{i}.\frac{\partial^2 \bar{a}}{\partial x \partial z}\right)\bar{k} \quad [\because i.i = 1, i.j = i.k = 0]$$

$$= \bar{i}\,\frac{\partial}{\partial x}\left(\bar{i}.\frac{\partial \bar{a}}{\partial x}\right) + j\,\frac{\partial}{\partial y}\left(\bar{i}.\frac{\partial \bar{a}}{\partial y}\right) + k\,\frac{\partial}{\partial z}\left(\bar{i}.\frac{\partial \bar{a}}{\partial x}\right) - \frac{\partial^2 \bar{a}}{\partial x^2} = \nabla\left(\bar{i}.\frac{\partial \bar{a}}{\partial x}\right) - \frac{\partial^2 \bar{a}}{\partial x^2}$$

$$\therefore \sum \bar{i} \times \frac{\partial}{\partial x}(\nabla \times \bar{a}) = \nabla \sum \bar{i}.\frac{\partial \bar{a}}{\partial x} - \sum \frac{\partial^2 \bar{a}}{\partial x^2} = \nabla(\nabla.\bar{a}) - \left(\frac{\partial^2 \bar{a}}{\partial x^2} + \frac{\partial^2 \bar{a}}{\partial y^2} + \frac{\partial^2 \bar{a}}{\partial z^2}\right)$$

$$\therefore \nabla \text{x}(\nabla \text{x}\,\bar{a}) = \nabla(\nabla.\,\bar{a}) - \nabla^2\bar{a}$$

i.e., $curl\,curl\,\bar{a} = grad\ div\,\bar{a} - \nabla^2\bar{a}$

SOLVED PROBLEMS

1: Prove that $(\nabla f \text{ x} \nabla g)$is solenoidal.

Sol: We know that div $(\bar{a} \times \bar{b}) = \bar{b}.curl\,\bar{a} - \bar{a}.curl\,\bar{b}$

Take $\bar{a} = \nabla f$ and $\bar{b} = \nabla g$

Then div (∇f x ∇g) = ∇g. curl (∇f) - ∇f. curl (∇g)=0$\left[\because curl(\nabla f) = \bar{0} = curl(\nabla g)\right]$

$\therefore \nabla f \times \nabla g$ is solenoidal.

2:Prove that (i) $div\left\{(\bar{r}\times\bar{a})\bar{b}\right\}=-2(\bar{b}.\bar{a})$ (ii) $curl\left\{(\bar{r}.\bar{a})\times b\right\}=\bar{b}\times\bar{a}$ where $\bar{a}$ and $\bar{b}$ are constant vectors.

Sol: (i)

$$div\left\{\left(\bar{r}\times\bar{a}\right)\times\bar{b}\right\}=div[(\bar{r}.\bar{b})\bar{a}-(\bar{a}..\bar{b})\bar{r}]$$

$$=div(\bar{r}.\bar{b})\bar{a}-(\bar{a}..\bar{b})\bar{r}$$

$$=\left[\left(\bar{r}.\bar{b}\right)div\,\bar{a}+\bar{a}.grad\left(\bar{r}.\bar{b}\right)\right]-\left[\left(\bar{a}.\bar{b}\right)div\,\bar{r}+\bar{r}.grad\left(\bar{a}.\bar{b}\right)\right]$$

$We\,have\,div\;\bar{a}=0, div\,\bar{r}=3, grad\,(\bar{a}.\bar{b})=0$

$div\left\{\left(\bar{r}\times\bar{a}\right)\times\bar{b}\right\}=0+\bar{a}.grad\left(\bar{r}.\bar{a}\right)-3\left(\bar{a}.\bar{a}\right)$

$=\bar{a}.\sum\frac{i\partial}{\partial x}\left(\bar{r}.\bar{b}\right)-3\left(\bar{a}.\bar{b}\right)$

$=\bar{a}.\sum\bar{i}\frac{\partial\bar{r}}{\partial x}.\bar{b}-3\left(\bar{a}.\bar{b}\right)$

$=\bar{a}.\sum\bar{i}\left(\bar{i}.\bar{b}\right)-3\left(\bar{a}.\bar{b}\right)$

$=\bar{a}.\bar{b}-3\left(\bar{a}.\bar{b}\right)=-2\left(\bar{a}.\bar{b}\right)$

$=-2\left(\bar{b}.\bar{a}\right)$

(ii) $curl\left\{\left(\bar{r}\times\bar{a}\right)\times\bar{b}\right\}=curl\left[\left(\bar{r}.\bar{b}\right)\bar{a}-\left(\bar{a}.\bar{b}\right)\bar{r}\right]$

$=curl\left(\bar{r}.\bar{b}\right)\bar{a}-curl\left(\bar{a}.\bar{b}\right)\bar{r}$

$=\left(\bar{r}.\bar{b}\right)curl\,\bar{a}+grad\left(\bar{r}.\bar{b}\right)\times\bar{a}$

$=\bar{0}+\nabla\left(\bar{r}.\bar{b}\right)\times\bar{a}\left(\because curl\,\bar{a}=\bar{0}\right)$

$=\bar{b}\times\bar{a}$ Since $grad\left(\bar{r}.\bar{b}\right)=\bar{b}$

3: Prove that $\nabla\left[\nabla.\frac{\bar{r}}{r}\right]=\frac{-2}{r^3}\bar{r}.$

Sol: We have $\nabla.\left(\frac{\bar{r}}{r}\right)=\sum i.\frac{\partial}{\partial x}\left(\frac{\bar{r}}{r}\right)$

$$=\sum i.\left[\frac{1}{r}\frac{\partial\bar{r}}{\partial x}+\bar{r}\left(\frac{-1}{r^2}\right)\left(\frac{x}{r}\right)\right]=\sum i.\left(\frac{1}{r}i-\frac{\bar{r}}{r^3}x\right)$$

$$=\frac{1}{r}\sum i.i-\frac{1}{r^3}r^2=\frac{3}{r}-\frac{1}{r}=\frac{2}{r}$$

$$\therefore\nabla\left[\nabla.\left(\frac{\bar{r}}{r}\right)\right]=\sum i\left(\frac{\partial}{\partial x}\left(\frac{2}{r}\right)\right)=\sum i\left(\frac{-2}{r^2}\right)\left(\frac{x}{r}\right)=\frac{-2}{r^3}\sum xi=\frac{-2\bar{r}}{r^3}.$$

4: Find $(A\times\nabla)\phi$, if $A = yz^2\bar{i} - 3xz^2\bar{j} + 2xyz\,\bar{k}$ and $\phi = xyz$.

Sol : We have

$$A\times\nabla = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ yz^2 & -3xz^2 & 2xyz \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial x}(-3xz^2) - \frac{\partial}{\partial y}(2xyz)\right] - \bar{j}\left[\frac{\partial}{\partial z}(yz^2) - \frac{\partial}{\partial x}(2xyz)\right] + \bar{k}\left[\frac{\partial}{\partial y}(yz^2) - \frac{\partial}{\partial x}(-3xz^2)\right]$$

$= \bar{i}(-6xz-2xz) - \bar{j}(2yz-2yz) + \bar{k}(z^2+3z^2) = -8xz\,\bar{i} - 0\,\bar{j} + 4z^2\bar{k}$

$\therefore (A\times\nabla)\phi, = (-8xz\,\bar{i} + 4z^2\bar{k})xyz = -8x^2yz^2\bar{i} + 4xyz^3\bar{k}$

Vector Integration

Line integral:- (i) $\int_c \bar{F}.d\bar{r}$ is called Line integral of $\bar{F}$ along c

Note : Work done by $\bar{F}$ along a curve c is $\int_c \bar{F}.d\bar{r}$

PROBLEMS

1. If $\bar{F}$ $(x^2-27)\,\bar{i} - 6yz\,\bar{j} + 8xz^2\,\bar{k}$, evaluate $\int \bar{F}.d\bar{r}$ from the point (0,0,0) to the point (1,1,1) along the Straight line from (0,0,0) to (1,0,0), (1,0,0) to (1,1,0) and (1,1,0) to (1,1,1).

Solution : Given $\bar{F} = (x^2-27)\bar{i} - 6yz\,\bar{j} + 8xz^2\bar{k}$

Now $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\therefore \quad \bar{F}.d\bar{r} = (x^2-27)dx - (6yz)dy + 8xz^2dz$

(i) Along the straight line from O = (0,0,0) to A = (1,0,0)

Here y =0 =z and dy=dz=0. Also x changes from 0 to 1.

$$\therefore \int_{OA} \bar{F}.d\bar{r} = \int_o^1 (x^2-27)dx = \left[\frac{x^3}{3} - 27x\right]_0^1 = \frac{1}{3} - 27 = \frac{-80}{3}$$

(ii) Along the straight line from A = (1,0,0) to B = (1,1,0)

Here x =1, z=0 $\Rightarrow$ dx=0, dz=0. y changes from 0 to 1.

$$\therefore \int_{AB} \bar{F} \cdot d\bar{r} = \int_{y=0}^{1} (-6yz)dy = 0$$

(iii) Along the straight line from B = (1,1,0) to C = (1,1,1)

x =1 =y $\Rightarrow$ dx=dy=0 and z changes from 0 to 1.

$$\therefore \int_{BC} \bar{F} \cdot d\bar{r} = \int_{z=0}^{1} 8xz^2dz = \int_{z=0}^{1} 8xz^2dz = \left[\frac{8z^3}{3}\right]_0^1 = \frac{8}{3}$$

$$(i)+(ii)+(iii) \Rightarrow \int_C \bar{F} \cdot d\bar{r} = \frac{88}{3}$$

2. If $\bar{F}$ =(5xy-6x^2)$\bar{i}$ +(2y-4x)$\bar{j}$, evaluate $\int_C \bar{F} \cdot d\bar{r}$ along the curve C in xy-plane y=x^{3}from (1,1) to (2,8).

Solution : Given $\bar{F}$ =(5xy-6x^2)$\bar{i}$ +(2y-4x)$\bar{j}$,-------(1)

Along the curve y=x^3, dy =3x^2 dx

$\therefore$ $\bar{F}$ =(5x^4-6x^2)$\bar{i}$ +(2x^3-4x)$\bar{j}$, [Putting y=x^3 in (1)]

$d\bar{r} = dx\bar{i} + dy\bar{j} = dx\bar{i} + 3x^2dx\,\bar{j}$

$\therefore$ $\bar{F} \cdot d\bar{r}$ = [(5x^4-6x^2)$\bar{i}$ +(2x^3-4x)$\bar{j}$].$\left[dx\,\bar{i} + 3x^2dx\,\bar{j}\right]$

= (5x^4 – 6x^2) dx+(2x^3 – 4x)3x^{3}dx

= (6x^5+5x^4-12x^3 -6x^2)dx

Hence $\int_{y=x^3} \bar{F} \cdot d\bar{r} = \int_1^2 (6x^5+5x^4-12x^3-6x^2)dx$

$$= \left(6.\frac{x^6}{6}+5.\frac{x^5}{5}-12.\frac{x^4}{4}-6.\frac{x^3}{4}\right) = \left(x^6+x^5-3x^4-2x^3\right)_1^2$$

= 16(4+2-3-1) – (1+1-3-2) = 32+3 = 35

3. Find the work done by the force $\bar{F} = z\bar{i} + x\bar{j} + y\bar{k}$, when it moves a particle along the arc of the curve $\bar{r}$ = cost $\bar{i}$ + sint $\bar{j}$ -t $\bar{k}$ from t = 0 to t = 2π

Solution : Given force $\bar{F}$ = z$\bar{i}$+ x $\bar{j}$ +y $\bar{k}$ and the arc is $\bar{r}$ = cost $\bar{i}$ + sin t $\bar{j}$ -t$\bar{k}$

i.e., x = cost, y= sin t, z = -t

$\therefore d\bar{r}$ = (-sin t $\bar{i}$ +cost $\bar{j}$ - $\bar{k}$)dt

$\therefore \bar{F} \cdot d\bar{r}$ = (-t $\bar{i}$+cost $\bar{j}$ +sin t $\bar{k}$). (-sin t $\bar{i}$ + cost $\bar{j}$ - $\bar{k}$)dt = (t sin t + cos^2 t – sin t)dt

Hence work done = $\int_0^{2\pi} \bar{F}.d\bar{r} = \int_0^{2\pi} (t \sin t + \cos^2 t - \sin t)\, dt$

$$= [t(-\cos t)]_0^{2\pi} - \int_0^{2\pi} (-\sin t)dt + \int_0^{2\pi} \frac{1+\cos 2t}{2} dt - \int_0^{2\pi} \sin t \, dt$$

$$= -2\pi - (\cos t)_0^{2\pi} + \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right)_0^{2\pi} + (\cos t)_0^{2\pi}$$

$$= -2\pi - (1-1) + \frac{1}{2}(2\pi) + (1-1) = -2\pi + \pi = -\pi$$

PROBLEMS

1 : Evaluate $\int \bar{F}.ndS$ where $\bar{F}$ = zi + xj – 3y²zk and S is the surface $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5.

Sol. The surface S is $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5.

Let $\phi = x^2 + y^2 = 16$

Then $\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = 2x\bar{i} + 2y\bar{j}$

$\therefore$ unit normal $\bar{n} = \frac{\nabla\phi}{|\nabla\varphi|} = \frac{x\bar{i} + y\bar{j}}{4}$ $(\because x^2 + y^2 = 16)$

Let R be the projection of S on yz-plane

Then $\int_S \bar{F}.ndS = \iint_R \bar{F}.\bar{n}\frac{dydz}{|\bar{n}\,.\,\bar{i}|}$ .... *

Given $\bar{F} = zi + xj - 3y^2zk$

$\therefore$ $\bar{F}\,.\,\bar{n} = \frac{1}{4}(xz + xy)$

and $\bar{n}\,.\,\bar{i} = \frac{x}{4}$

In yz-plane, x = 0, y = 4

In first octant, y varies from 0 to 4 and z varies from 0 to 5.

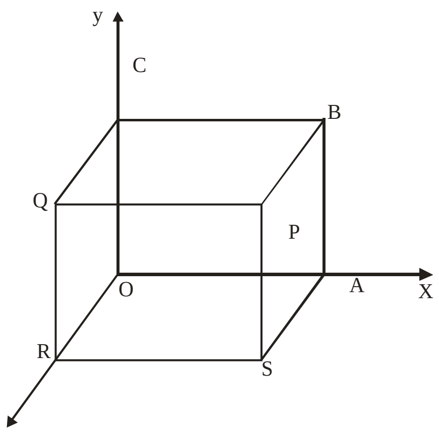
$$\int_S \bar{F}.ndS = \int_{y=0}^{4}\int_{z=0}^{5}\left(\frac{xz + xy}{4}\right)\frac{dydz}{\left|\frac{x}{4}\right|}$$

$$= \int_{y=0}^{4}\int_{z=0}^{5}(y+z)dz\,dy$$

= 90.

2 : If $\bar{F} = zi + xj - 3y^2zk$, evaluate $\int_S \bar{F}.\bar{n}dS$ where S is the surface of the cube bounded by x = 0, x = a, y = 0, y= a, z = 0, z = a.

Sol. Given that S is the surface of the x = 0, x = a, y = 0, y = a, z = 0, z = a, and $\bar{F} = zi + xj - 3y^2zk$ we need to evaluate $\int_S \bar{F}.\bar{n}dS$.



(i) For OABC

Eqn is z = 0 and dS = dxdy

$$\bar{n} = -\bar{k}$$

$$\int_{S_1} \bar{F}.\bar{n}dS = -\int_{x=0}^{a} -\int_{y=0}^{a} (yz)\, dxdy = 0$$

(ii) For PQRS

Eqn is z = a and dS = dxdy

$$\bar{n} = \bar{k}$$

$$\int_{S_2} \bar{F}.ndS = \int_{x=0}^{a}\left(\int_{y=0}^{a} y(a)dy\right) dx = \frac{a^4}{2}$$

(iii) For OCQR

Eqn is x = 0, and $\bar{n} = -\bar{i}$, dS = dydz

$$\int_{S_3} \bar{F}.ndS = \int_{y=0}^{a} \int_{z=0}^{a} 4xzdydz = 0$$

(iv) For ABPS

Eqn is x = a, and $\bar{n} = -\bar{i}$, dS = dydz

$$\int_{S_3} \bar{F}.\bar{n}dS = \int_{y=0}^{a}\left(\int_{z=0}^{a} 4azdz\right)dy = 2a^4$$

(v) For OASR

Eqn is y = 0, and $\bar{n} = -\bar{j}$, dS = dxdz

$$\int_{S_5} \overline{F}.ndS = \int_{y=0}^{a} \int_{z=0}^{a} y^2 dzdx = 0$$

(vi) For PBCQ

Eqn is y = a, and $\overline{n} = -\overline{j}$, dS = dxdz

$$\int_{S_6} \overline{F}.\overline{n}dS = -\int_{y=0}^{a} \int_{z=0}^{a} y^2 dzdx = 0$$

From (i) – (vi) we get

$$\int_{S_6} \overline{F}.\overline{n}dS = 0 + \frac{a^4}{2} + 0 + 2a^4 + 0 - a4 = \frac{3a^4}{2}$$

VOLUME INTEGRALS

Let V be the volume bounded by a surface $\overline{r} = \overline{f}$ (u,v). Let $\overline{F}(\overline{r})$ be a vector point function define over V. Divide V into m sub-regions of volumes $\delta V_1, \delta V_2, \ldots\delta V_p \ldots\delta V_m$

Let P_i ($\overline{r}_i$) be a point in δV_i .Then form the sum $I_m = \sum_{i=1}^{m} \overline{F}(r_i)\delta V_i$. Let m $\to \infty$ in such a way that δV_i shrinks to a point,. The limit of I_m if it exists, is called the volume integral of $\overline{F}(\overline{r})$ in the region V is denoted by $\int_V \overline{F}(\overline{r})\,dv$ or $\int_V \overline{F}dv$.

Cartesian form : Let $\overline{F}(r) = F_1\overline{i} + F_2\overline{i} + F_3\overline{k}$ where F_1, F_2, F_3 are functions of x,y,z. We know that dv = dx dy dz. The volume integral given by

$$\int_v \overline{F}dv = \iiint (F_1\overline{i} + F_2\overline{i} + F_3\overline{k})\,dx\,dy\,dz = \overline{i}\iiint F_1\,dxdydz + \overline{j}\iiint F_2\,dxdydz + \overline{k}\iiint F_3\,dxdydz$$

SOLVED EXAMPLES

Example 1 : If $\overline{F} = 2xz\overline{i} - x\overline{j} + y^2\overline{k}$ evaluate $\int_V \overline{F} dv$ where V is the region bounded by the surfaces $x=0, x=2, y=0, y=6, z=x^2, z=4$.

Solution : Given $\overline{F} = 2xz\overline{i} - x\overline{j} + y^2\overline{k}$. $\therefore$ The volume integral is

$$\int_V \overline{F} dv = \iiint (2xz\overline{i} - x\overline{j} + y^2\overline{k}) dx\,dy\,dz$$

$$= \overline{i}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} 2xz\,dx\,dy\,dz - \overline{j}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} x\,dx\,dy\,dz + \overline{k}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} y^2\,dx\,dy\,dz$$

$$= \overline{i}\int_{x=0}^{2}\int_{y=0}^{6} [xz^2]_{x^2}^{4}\,dx\,dy - \overline{j}\int_{x=0}^{2}\int_{y=0}^{6} (xz)_{x^2}^{4}\,dx\,dy + \overline{k}\int_{x=0}^{2}\int_{y=0}^{6} y^2(z)_{x^2}^{4}\,dx\,dy$$

$$= \overline{i}\int_{x=0}^{2}\int_{y=0}^{6} x(16-x^4)dx\,dy - \overline{j}\int_{x=0}^{2}\int_{y=0}^{6} x(4-x^2)dx\,dy - \overline{k}\int_{x=0}^{2}\int_{y=0}^{2} y^2(x^2-4)dx\,dy$$

$$= \overline{i}\int_{x=0}^{2}(16x-x^5)(y)_0^6 dx - \overline{j}\int_{x=0}^{2}(4x-x^3)(y)_0^6 dx - \overline{k}\int_{x=0}^{2}(x^2-4)\left(\frac{y^3}{3}\right)_0^6 dx$$

$$= \overline{i}\left(8x^2 - \frac{x^6}{6}\right)_0^2 (6) - \overline{j}\left(2x^2 - \frac{x^4}{4}\right)_0^2 (6) - \overline{k}\left(4x - \frac{x^3}{3}\right)_0^2 \left(\frac{211}{3}\right)$$

$$= 128\overline{i} - 24\overline{j} - 384\overline{k}$$

Vector Integral Theorems

Introduction

In this chapter we discuss three important vector integral theorems: (i) Gauss divergence theorem, (ii) Green's theorem in plane and (iii) Stokes theorem. These theorems deal with conversion of

(i) $\int_S \overline{F}.\overline{n}$ ds into a volume integral where S is a closed surface.

(ii) $\int_C \overline{F}.d\overline{r}$ into a double integral over a region in a plane when C is a closed curve in the plane and.

(iii) $\int_S (\nabla \times \bar{A}) \cdot \bar{n}\, ds$ into a line integral around the boundary of an open two sided surface.

I. GAUSS'S DIVERGENCE THEOREM
(Transformation between surface integral and volume integral)

Let S be a closed surface enclosing a volume V. If $\bar{F}$ is a continuously differentiable vector point function, then

$$\int_V div\, F dv = \int_s \bar{F} \cdot \bar{n}\ dS$$

When $\bar{n}$ is the outward drawn normal vector at any point of S.

SOLVED PROBLEMS

1) Verify Gauss Divergence theorem for $\bar{F} = (x^3 - yz)\bar{\imath} - 2x^2 y\bar{\jmath} + z\bar{k}$ taken over the surface of the cube bounded by the planes x = y = z = a and coordinate planes.

Sol: By Gauss Divergence theorem we have

$$\int_S \bar{F}.\bar{n} dS = \int_V div \bar{F} dv$$

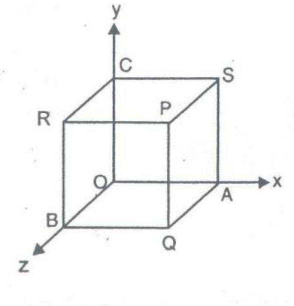
$$RHS = \int_0^a \int_0^a \int_0^a (3x^2 - 2x^2 + 1) dx\ dy\ dz = \int_0^a \int_0^a \int_0^a (x^2 + 1)\, dx\ dy\ dz = \int_0^a \int_0^a \left(\frac{x^3}{3} + x\right)_0^a dy\ dz$$

$$\int_0^a \int_0^a \left[\frac{a^3}{3} + a\right] dydz = \int_0^a \left[\frac{a^3}{3} + a\right] (y)_0^a dz = \left(\frac{a^3}{3} + a\right) a \int_0^a dz = \left(\frac{a^3}{3} + a\right)\left(a^2\right) = \frac{a^5}{3} + a^3 \ldots\ldots(1)$$

Verification: We will calculate the value of $\int_S \bar{F}.\bar{n} dS$ over the six faces of the cube.

(i) For S_1 = PQAS; unit outward drawn normal $\bar{n} = \bar{\imath}$

x=a; ds=dy dz; 0≤y≤a, 0≤z≤a



$$\therefore \bar{F}.\bar{n} = x^3 - yz = a^3 - yz \sin ce\, x = a$$

$$\therefore \iint_{S_1} \bar{F}.\bar{n} dS = \int_{z=0}^a \int_{y=0}^a (a^3 - yz) dydz$$

$$= \int_{z=0}^a \left[a^3 y - \frac{y^2}{2} z\right]_{y=0}^a dz$$

$$= \int_{z=0}^a \left(a^4 - \frac{a^2}{2} z\right) dz$$

$$= a^5 - \frac{a^4}{4} \ldots(2)$$

(ii) For S_2 = OCRB; unit outward drawn normal $\bar{n} = -\bar{i}$

x=0; ds=dy dz; 0≤y≤a, y≤z≤a
$\bar{F}.\bar{n} = -(x^3 - yz) = yz \; since \; x = 0$

$$\int_{S_3}\int \bar{F}.\bar{n}dS = \int_{z=0}^{a}\int_{y=0}^{a} yz\, dy\, dz = \int_{z=0}^{a}\left[\frac{y^2}{2}\right]_{y=0}^{a} zdz$$

$$= \frac{a^2}{2}\int_{z=0}^{a} zdz = \frac{a^4}{4} \dots (3)$$

(iii) For S_3 = RBQP; Z = a; ds = dxdy; $\bar{n} = \bar{k}$

0≤x≤a, 0≤y≤a

$\bar{F}.\bar{n} = z = a \;\; since \; z = a$

$$\therefore \int_{S_3}\int \overline{F}\,\overline{n}dS = \int_{y=0}^{a}\int_{x=0}^{a} adxdy = a^3 \dots\dots(4)$$

(iv) For S_4 = OASC; z = 0; $\bar{n} = -\bar{k}$, ds = dxdy;

0≤x≤a, 0≤y≤a

$\bar{F}.\bar{n} = -z = 0 \;\; since \; z = 0$

$$\int_{S_4}\int \bar{F}.\bar{n}dS = 0 \dots (5)$$

(v) For S_5 = PSCR; y = a; $\bar{n} = \bar{j}$, ds = dzdx;

0≤x≤a, 0≤z≤a

$\bar{F}.\bar{n} = -2x^2y = -2ax^2 \;\; since \; y = a$

$$\int_{S_5}\int \bar{F}.\bar{n}dS = \int_{x=0}^{a}\int_{z=0}^{a}(-2ax^2)dzdx$$

$$\int_{x=0}^{a}(-2ax^2z)_{z=0}^{a}dx$$

$$= -2a^2\left(\frac{x^3}{3}\right)_0^a = \frac{-2a^5}{3} \dots (6)$$

(vi) For S_6 = OBQA; y = 0; $\bar{n} = -\bar{j}$, ds = dzdx;

0≤x≤a, 0≤y≤a

$\bar{F}.\bar{n} = 2x^2y = 0 \; since \; y = 0$

$$\int_{S_6}\int \bar{F}.\bar{n}dS = 0$$

$$\int_{S}\int \bar{F}.\bar{n}dS = \int_{S_1}\int + \int_{S_2}\int + \int_{S_3}\int + \int_{S_4}\int + \int_{S_5}\int + \int_{S_6}\int$$

$$= a^5 - \frac{a^4}{4} - \frac{a^4}{4} + a^3 + 0 - \frac{2a^5}{3} + 0$$

$$= \frac{a^5}{3} + a^3 = \int\int_V\int \bar{\nabla}.\bar{F}\, dv \; using\ (1)$$

Hence Gauss Divergence theorem is verified

2. Compute $\int (\boldsymbol{ax^2 + by^2 + cz^2}) \boldsymbol{dS}$ over the surface of the sphere $x^2+y^2+z^2 = 1$

Sol: By divergence theorem $\int_S \bar{F}.\bar{n} dS = \int_V \bar{\nabla}.\bar{F}\ dv$

$Given\ \bar{F}.\bar{n} = ax^2 + by^2 + cz^2. Let\ \phi = x^2 + y^2 + z^2 - 1$

$\therefore$ *Normal vector* $\bar{n}$ *to the surface* ϕ *is*

$$\bar{\nabla}\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial y}\right)(x^2 + y^2 + z^2 - 1) = 2(x\bar{i} + y\bar{j} + z\bar{k})$$

$\therefore$ Unit normalvector $= \bar{n} = \dfrac{2(x\bar{i} + y\bar{j} + z\bar{k})}{2\sqrt{x^2 + y^2 + z^2}} = x\bar{i} + y\bar{j} + z\bar{k}$ Since $x^2 + y^2 + z^2 = 1$

$$\therefore \bar{F}.\bar{n} = \bar{F}.(x\bar{i} + y\bar{j} + z\bar{k}) = (ax^2 + by^2 + cz^2) = (a\,x\bar{i} + by\bar{j} + cz\bar{k}).(x\bar{i} + y\bar{j} + z\bar{k})$$

i.e., $\bar{F} = ax\bar{i} + by\bar{j} + cz\bar{k}$ $\nabla.\bar{F} = a + b + c$

Hence by Gauss Divergence theorem,

$$\int_S (ax^2 + by^2 + cz^2) dS = \int_V (a + b + c) dv = (a + b + c)V = \frac{4\pi}{3}(a + b + c)$$

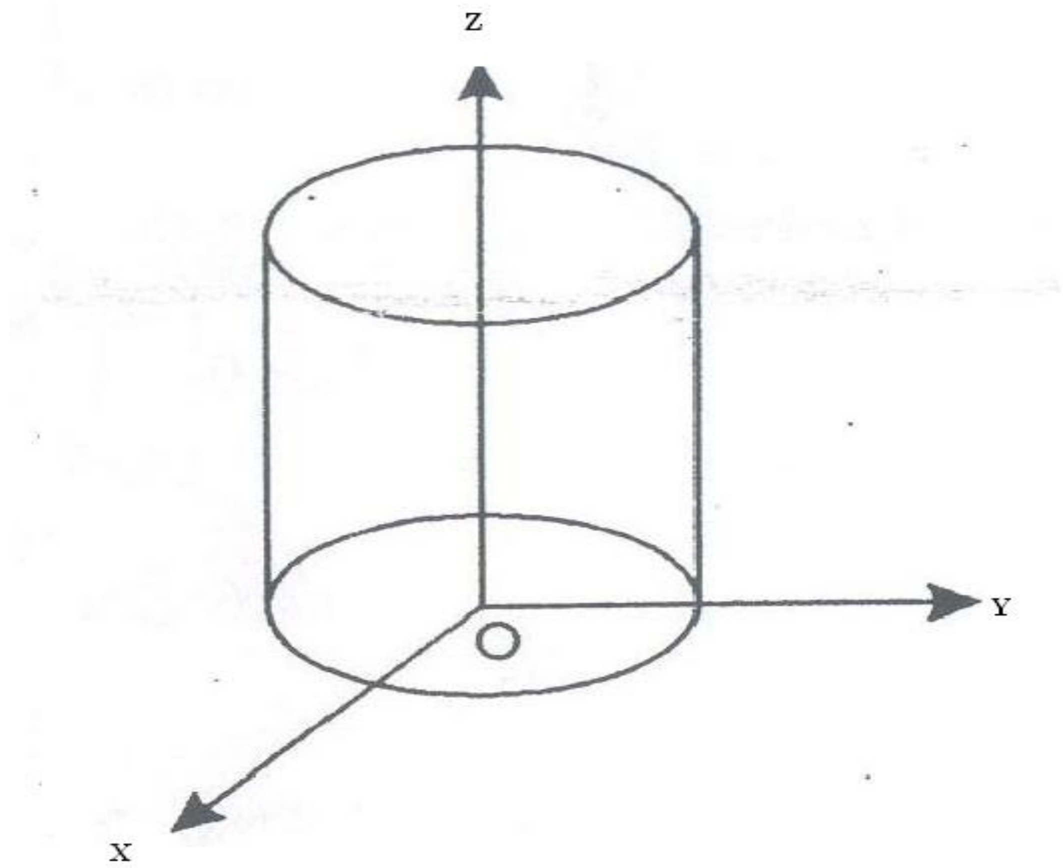
$$\left[SInce\ V = \frac{4\pi}{3}\ is\ the\ volume\ of\ the\ sphere\ of\ unit\ radius\right]$$

3) By transforming into triple integral, evaluate $\int\int \boldsymbol{x^3\ dy\ dz + x^2 y\ dz\ dx + x^2 dx\ dy}$ where S is the closed surface consisting of the cylinder $x^2+y^2 = a^2$ and the circular discs z= 0 , z= b.

Sol: Here $F_1 = x^3, F_2 = x^2 y, F_3 = x^2 z\ and\ \bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

$$\frac{\partial F_1}{\partial x} = 3x^2, \frac{\partial F_2}{\partial y} = x^2, \frac{\partial F_3}{\partial z} = x^2$$

$$\nabla.\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3x^2 + x^2 + x^2 = 5x^2$$



By Gauss Divergence theorem,

$$\iint F_1 dydz + F_2 dzdx + F_3 dxdy = \iiint \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dxdydz$$

$$\therefore \iint_s (x^3 dydz + x^2 ydzdx + x^2 zdxdy = \iiint 5x^2 dxdydz$$

$$= 5 \int_{-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=0}^{b} x^2 dxdydz$$

$$= 20 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_{z=0}^{b} x^2 dxdydz \text{ [Integrand is even function]}$$

$$= 20 \int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 (z)_0^b dxdy = 20b \int_{x=0}^{a} \int_0^{\sqrt{a^2-x^2}} x^2 dxdy$$

$$= 20b \int_{x=0}^{a} x^2 (y)_0^{\sqrt{a^2-x^2}} dx = 20b \int_0^a x^2 \sqrt{a^2 - x^2}\, dx$$

$$= 20b \int_0^{\frac{\pi}{2}} a^2 \sin^2\theta \sqrt{a^2 - a^2 \sin^2\theta}\, (a\cos\theta d\theta)$$

[Put $x = a\sin\theta \Longrightarrow dx = a\cos\theta d\theta$ when $x = a \Rightarrow \theta = \frac{\pi}{2}$ and $x = 0 \Rightarrow \theta = 0$]

$= 20a^4 b \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta d\theta = 5a^4 b \int_0^{\frac{\pi}{2}} (2\sin\theta\cos\theta)^2 \, d\theta = 5a^4 b \int_0^{\frac{\pi}{2}} \frac{1-\cos 4\theta}{2} d\theta$

$= \frac{5a^4 b}{2}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2} = \frac{5a^4 b}{2}\left[\frac{\pi}{2}\right] = \frac{5}{4}\pi a^4 b$

4: Applying Gauss divergence theorem, Prove that $\int \bar{r}\,.\bar{n}dS = 3V$ or $\int \bar{r}\,.d\bar{s} = 3V$

Sol: Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ we know that div $\bar{r} = 3$

By Gauss divergence theorem, $\int \bar{F}.\bar{n}dS = \int_v div\bar{F}dv$

$Take\ \bar{F} = \bar{r} \Rightarrow \int_S \bar{r}.\bar{n}dS = \int_V 3\, dV = 3V. Hence\ the\ result$

5: Show that $\int_S (ax\bar{i} + by\bar{j} + cz\bar{k}).\bar{n}dS = \frac{4\pi}{3}(a+b+c)$, where S is the surface of the sphere $x^2+y^2+z^2=1$.

Sol: Take $\bar{F} = ax\bar{i} + by\bar{j} + cz\bar{k}$

$$div\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = a+b+c$$

By Gauss divergence theorem, $\int_S \bar{F}.\bar{n}dS = \int_V \bar{\nabla}.\bar{F}dV = (a+b+c)\int_V dV = (a+b+c)V$

$We\ have\ V = \frac{4}{3}\pi r^3\ for\ the\ sphere. Here\ r = 1$

$$\therefore \int_s \bar{F}.\bar{n}dS = (a+b+c)\frac{4\pi}{3}$$

6: Using Divergence theorem, evaluate

$\int\int_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy), where\ S{:}x^2+y^2+z^2=a^2$

Sol: We have by Gauss divergence theorem, $\int_s \bar{F}.\bar{n}dS = \int_v div\bar{F}dv$

L.H.S can be written as $\int(F_1 dydz + F_2 dzdx + F_3 dxdy)$ in Cartesian form

Comparing with the given expression, we have $F_1=x$, $F_2=y$, $F_3=z$

Then $div\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3$

$$\therefore \int_v div\bar{F}dv = \int_v 3dv = 3V$$

Here V is the volume of the sphere with radius a.

$$\therefore V = \frac{4}{3}\pi a^3$$

Hence $\int\int(x\,dy\,dz + y\,dz\,dx + z\,dx\,dy) = 4\pi a^3$

7: Apply divergence theorem to evaluate $\iint_s (x+z)dydz+(y+z)dzdx+(x+y)dxdy$ S is the surface of the sphere $x^2+y^2+z^2=4$

Sol: Given $\iint_s (x+z)dydz+(y+z)dzdx+(x+y)dxdy$

Here $F_1 = x+z$, $F_2 = y+z$, $F_3 = x+y$

$\frac{\partial F_1}{\partial x}=1, \frac{\partial F_2}{\partial y}=1, \frac{\partial F_3}{\partial z}=0$ and $\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}=1+1+0=2$

By Gauss Divergence theorem,

$$\iint_s F_1dydz+F_2dzdx+F_3dxdy=\iiint_V\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)dxdydz$$

$$=\int\int\int 2dxdydz = 2\int_V dv = 2V$$

$$= 2\left[\frac{4}{3}\pi(2)^3\right] = \frac{64\pi}{3} \; [for\ the\ sphere, radius = 2]$$

8: Evaluate $\int_S \bar{F}.\bar{n}ds, if\ F = xy\bar{i} + z^2\bar{j} + 2yz\bar{k}$ over the tetrahedron bounded by x=0, y=0, z=0 and the plane x+y+z=1.

Sol: Given F = $xy\bar{i} + z^2\bar{j} + 2yz\bar{k}$, then div. F = y+2y = 3y

$$\therefore \int_s \bar{F}.\bar{n}dS = \int_v div\bar{F}dv = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} 3ydxdydz$$

$$= 3\int_{x=0}^{1}\int_{y=0}^{1-x} y[z]_0^{1-x-y}dxdy = 3\int_{x=0}^{1}\int_{y=0}^{1-x} y(1-x-y)dx\,dy$$

$$= 3\int_{x=0}^{1}\left[\frac{y^2}{2}-\frac{xy^2}{2}-\frac{y^3}{3}\right]_0^{1-x}dx = 3\int_0^1\left[\frac{(1-x)^2}{2}-\frac{x(1-x)^2}{2}-\frac{(1-x)^3}{3}\right]dx$$

$$= 3\int_0^1\left[\frac{(1-x)^3}{2}-\frac{(1-x)^3}{3}\right]dx = 3\int_0^1\frac{(1-x)^3}{6}dx = \frac{3}{6}\left[\frac{-(1-x)^4}{4}\right]_0^1 = \frac{1}{8}$$

9: Use divergence theorem to evaluate $\iint_s \bar{F}.d\bar{S}$ where $\bar{F} = x^3i+y^3j+z^3k$ and S is the surface of the sphere $x^2+y^2+z^2 = r^2$

Sol: We have

$$\bar{V}.\bar{F} = \frac{\partial}{\partial x}(x^3)+\frac{\partial}{\partial y}(y^3)+\frac{\partial}{\partial z}(z^3) = 3(x^2+y^2+z^2)$$

∴By divergence theorem,

$\bar{\nabla}.\bar{F}\,dV = \int\int_V \int \bar{\nabla}.\bar{F}\,dV = \iiint_v 3(x^2+y^2+z^2)dxdydz$

$$= 3 \int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2(r^2 \sin\theta\, dr\, d\theta\, d\phi)$$

$[Changing\ into\ spherical\ polar\ coordinates\ x = r\sin\theta\cos\phi\,, y = r\sin\theta\sin\phi\,, z = r\cos\theta]$

$$\int_S\int \bar{F}.dS = 3\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^4\sin\theta\left[\int_{\phi=0}^{2\pi} d\phi\right] dr\, d\theta$$

$$= 3\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^4\sin\theta(2\pi-0)drd\theta = 6\pi\int_{r=0}^{a} r^4\left[\int_0^{\pi}\sin\theta d\theta\right]dr$$

$$= 6\pi\int_{r=0}^{a} r^4(-\cos\theta)_0^{\pi}\, dr = -6\pi\int_0^a r^4(\cos\pi - \cos 0)\, dr$$

$$= 12\pi\int_0^a r^4 dr = 12\pi\left[\frac{r^5}{5}\right]_0^a = \frac{12\pi a^5}{5}$$

10: Use divergence theorem to evaluate $\int\int_S \bar{F}.dS$ where $\bar{F} = 4xi - 2y^2j + z^2k$ and S is the surface bounded by the region $x^2+y^2=4$, z=0 and z=3.

Sol: We have

$$div\bar{F} = \nabla.\bar{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2) = 4 - 4y + 2z$$

By divergence theorm,

$$\int_S\int \bar{F}.dS = \int\int_V\int \bar{\nabla}.\bar{F}\,dV$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3}(4-4y+2z)dx\,dy\,dz$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}[(4-4y)z+z^2]_0^3\,dx\,dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}[12(1-y)+9]\,dx\,dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(21-12y)dx\,dy$$

$$= \int_{-2}^{2}\left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 21\, dy - 12\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y dy\right] dx$$

$$= \int_{-2}^{2}\left[21\times 2\int_{0}^{\sqrt{4-x^2}} dy - 12(0)\right] dx$$

[Since the integrans in forst integral is even and in 2nd integral it is on add function]

$$= 42\int_{-2}^{2} (y)_0^{\sqrt{4-x^2}}\, dx$$

$$= 42\int_{-2}^{2}\sqrt{4-x^2}\,dx = 42\times 2\int_{0}^{2}\sqrt{4-x^2}\,dx$$

$$= 84\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 84\left[0 + 2.\frac{\pi}{2} - 0\right] = 84\pi$$

11: Verify divergence theorem for $\boldsymbol{\bar{F} = x^2 i + y^2 j + z^2 k}$ over the surface S of the solid cut off by the plane x+y+z=a in the first octant.

Sol; By Gauss theorem, $\int_s \bar{F}.\bar{n}dS = \int_v div\bar{F}dv$

$Let\ \phi = x + y + z - a\ be\ the\ given\ plane\ then$

$$\frac{\partial\phi}{\partial x} = 1, \frac{\partial\phi}{\partial y} = 1, \frac{\partial\phi}{\partial z} = 1$$

$$\therefore grad\phi = \sum \bar{i}\frac{\partial\phi}{\partial x} = \bar{i} + \bar{j} + \bar{k}$$

$$Unit\ normal = \frac{grad\ \phi}{|grad\ \phi|} = \frac{\bar{\imath} + \bar{\jmath} + \bar{k}}{\sqrt{3}}$$

Let R be the projection of S on xy-plane

Then the equation of the given plane will be x+y=a $\Rightarrow$ y=a-x

Also when y=0, x=a

$$\therefore \int_s \bar{F}.\bar{n}dS = \iint_R \frac{\bar{F}.\bar{n}dxdy}{|\bar{n}.\bar{k}|}$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x} \frac{\frac{x^2+y^2+z^2}{\sqrt{3}}}{1/\sqrt{3}}\,dx\,dy \quad = \int_0^a\int_{y=0}^{a-x}[x^2 + y^2 + (a-x-y)^2]dx\,dy\ [\,since\ x+y+z=a]$$

$$= \int_0^a\int_0^{a-x}[2x^2 + 2y^2 - 2ax + 2xy - 2ay + a^2]dx\,dy$$

$$= \int_{x=0}^{a}\left[2x^2y + \frac{2y^3}{3} + xy^2 - 2axy - ay^2 + a^2y\right]_0^{a-x} dx$$

$$= \int_{x=0}^{a} [2x^2(a-x) + \frac{2}{3}(a-x)^3 + x(a-x)^2 - 2ax(a-x) - a(a-x)^2 + a^2(a-x)dx$$

$$\therefore \int_s \overline{F}.\overline{n}dS = \int_0^a \left(-\frac{5}{3}x^3 + 3ax^2 - 2a^2x + \frac{2}{3}a^3\right)dx = \frac{a^4}{4}, \text{ on simplification}\ldots(1)$$

Given $\overline{F} = x^2\overline{i} + y^2\overline{j} + z^2\overline{k}$

$$\therefore div\ \overline{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) = 2(x+y+z)$$

$$Now \iiint div\overline{F}.dv = 2\int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y}(x+y+z)dxdydz$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{a-x}\left[z(x+y) + \frac{z^2}{2}\right]_0^{a-x-y} dx\ dy$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{a-x}(a-x-y)\left[x+y+\frac{a-x-y}{2}\right]dx\ dy$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x}(a-x-y)[a+x+y]dx\ dy$$

$$= \int_0^a\int_0^{a-x}[a^2-(x+y)^2]\ dy\ dx = \int_0^a\int_0^{a-x}(a^2-x^2-y^2-2xy)dx\ dy$$

$$= \int_0^a [a^2y - x^2y - \frac{y^3}{3} - xy^2]_0^{a-x}\ dx$$

$$= \int_0^a (a-x)(2a^2-x^2-ax)dx = \frac{a^4}{4}\ldots\ldots(2)$$

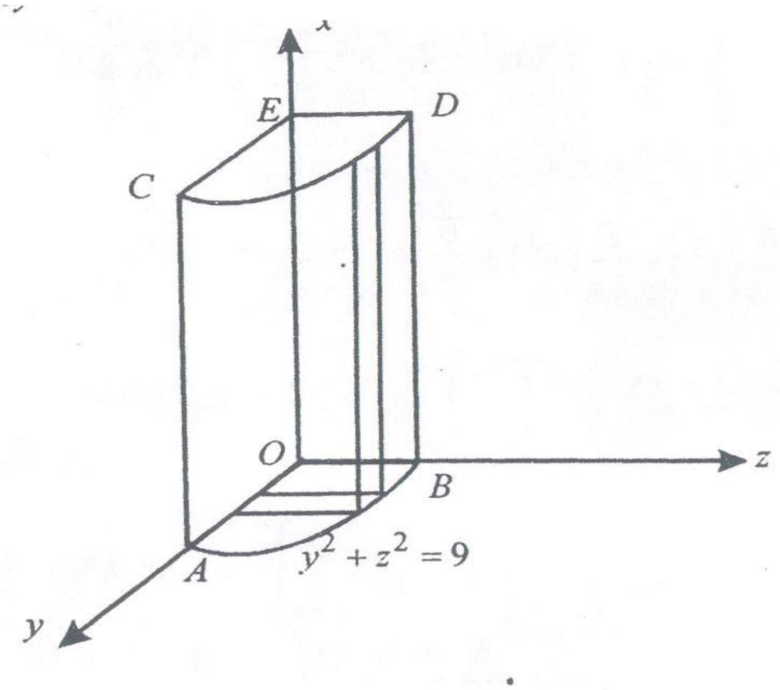
Hence from (1) and (2), the Gauss Divergence theorem is verified.

12: Verify divergence theorem for $2x^2y\overline{i} - y^2\overline{j} + 4xz^2\overline{k}$ taken over the region of first octant of the cylinder $y^2+z^2=9$ and $x=2$.

(or) Evaluate $\iint_s \overline{F}.\overline{n}dS$, where $\overline{F} = 2x^2y\overline{i} - y^2\overline{j} + 4xz^2\overline{k}$ and S is the closed surface of the region in the first octant bounded by the cylinder $y^2+z^2 = 9$ and the planes x=0, x=2, y=0, z=0

Sol: Let $\overline{F} = 2x^2y\overline{i} - y^2\overline{j} + 4xz^2\overline{k}$

$$\therefore \nabla.\ \overline{F} = \frac{\partial}{\partial x}(2x^2) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(4xz^2) = 4xy - 2y + 8xz$$



$$\iiint_V \bar{\nabla}.\bar{F}\,dv = \int_{x=0}^{2}\int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} (4xy-2y+8xz)dz\,dy\,dx$$

$$= \int_0^2\int_0^3 \left[(4xy-2y)z+8x\,\frac{z^2}{2}\right]_{z=0}^{\sqrt{9-y^2}} dy\,dx$$

$$= \int_0^2\int_0^3 \left[(4xy-2y)\sqrt{9-y^2}+4x(9-y^2)\right]dy\,dx$$

$$= \int_0^2\int_0^3 [(1-2x)(-2y)\sqrt{9-y^2}+4x(9-y^2)]\,dy\,dx$$

$$= \int_0^2\left\{\left[(1-2x)\frac{(9-y^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3+4x\left(9y-\frac{y^3}{3}\right)_0^3\right\}dx$$

$$= \int_0^2 \{\tfrac{2}{3}(1-2x)[0-27]+4x[27-9]\}\,dx = \int_0^2 [-18(1-2x)+72x]dx$$

$$\left[-18(x-x^2)+72\frac{x^2}{2}\right]_0^2 = -18(2-4)+36(4)=36+144=180\ldots(1)$$

Now we sall calculate $\int_S \bar{F}.\bar{n}\,ds$ *for all the five faces.*

$$\int_S \bar{F}.\bar{n}dS = \int_{S_1}\bar{F}.\bar{n}dS+\int_{S_2}\bar{F}.\bar{n}dS+\ldots\ldots+\int_{S_5}\bar{F}.\bar{n}dS$$

Where S_1 is the face OAB, S_2 is the face CED, S_3 is the face OBDE, S_4 is the face OACE and S_5 is the curved surface ABDC.

(i) On $S_1 : x=0, \bar{n}=-i \therefore \bar{F}.\bar{n}=0$ *Hence* $\int_{s_1} \bar{F}.\bar{n}dS$

(ii) On $S_2 : x=2, \bar{n}=i \therefore \bar{F}.\bar{n}=8y$

$$\therefore \int_{s_2} \bar{F}.\bar{n}dS = \int_0^3 \int_0^{\sqrt{9-z^2}} 8y\,dy\,dz = \int_0^3 8\left(\frac{y^2}{2}\right)_0^{\sqrt{9-z^2}} dz$$

$$= 4\int_0^3 (9-z^2)dz = 4\left(9z-\frac{z^3}{3}\right)_0^3 = 4(27-9) = 72$$

(iii) On $S_3 : y=0, \bar{n}=-j. \therefore \bar{F}.\bar{n}=0$ *Hence* $\int_{s_3} \bar{F}.\bar{n}dS$

$(iv) On\ S_4{:}z=0, \bar{n}=-k. \quad \bar{F}.\bar{n}=0. \quad Hence \int_{S_4} \bar{F}.\bar{n}ds = 0$

(v) On $S_5 : y^2+z^2=9, \bar{n}=\dfrac{\nabla(y^2+z^2)}{|\nabla(y^2+z^2)|}=\dfrac{2y\bar{j}+2z\bar{k}}{\sqrt{4y^2+4z^2}}=\dfrac{y\bar{j}+z\bar{k}}{\sqrt{4\times 9}}=\dfrac{y\bar{j}+z\bar{k}}{3}$

$\bar{F}.\bar{n}=\dfrac{-y^3+4xz^3}{3}$ and $\bar{n}.\bar{k}=\dfrac{z}{3}=\dfrac{1}{3}\sqrt{9-y^2}$

Hence $\int_{S_5} \bar{F}.\bar{n}ds = \int\int_R \bar{F}.\bar{n}\frac{dx\,dy}{|\bar{n}.\bar{k}|}$ *Where R is the projection of* S_5 *on* $xy-plane.$

$$= \int_R\int \frac{4xz^3-y^3}{\sqrt{9-y^2}}dx\,dy = \int_{x=0}^{2}\int_{y=0}^{3}\left[4x(9-y^2)-y^3\left(9-y^2\right)^{-\frac{1}{2}}\right]dy\,dx$$

$$= \int_0^2 72x\,dx - 18\int_0^2 dx = 72\left(\frac{x^2}{2}\right)_0^2 - 18(x)_0^2 = 144-36 = 108$$

$Thus \int_S \bar{F}.\bar{n}ds = 0+72+0+0+108 = 180 \ldots\ldots (2)$

Hence the Divergence theorem is verified from the equality of (1) and (2).

13: Use Divergence theorem to evaluate $\int\int\left(x\bar{i}+y\bar{j}+z^2\bar{k}\right).\bar{n}ds.$ Where S is the surface bounded by the cone $x^2+y^2=z^2$ in the plane z = 4.

Sol: Given $\int\int\left(x\bar{i}+y\bar{j}+z^2\bar{k}\right).\bar{n}.ds$ *Where* S is the surface bounded by the cone $x^2+y^2=z^2$ in the plane z = 4.

Let $\bar{F} = x\bar{i} + y\bar{j} + z^2\bar{k}$

By Gauss Divergence theorem, we have

$$\int\int (x\bar{i} + y\bar{j} + z^2\bar{k}).\bar{n}.ds = \int\int_V\int \bar{\nabla}.\bar{F}\,dv$$

$$Now\ \nabla.\bar{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z^2) = 1+1+2z = 2(1+z)$$

On the cone, $x^2 + y^2 = z^2$ and z=4 $\Rightarrow x^2 + y^2 = 16$

The limits are $z = 0\ to 4, y = o\ to\ \sqrt{16-x^2}, x = 0\ to\ 4.$

$$\int\int_V\int \bar{\nabla}.\bar{F}\,dv = \int_0^4\int_0^{\sqrt{16-x^2}}\int_0^4 2(1+z)dx\,dy\,dz$$

$$= 2\int_0^4\int_0^{\sqrt{16-x^2}} \left\{[z]_0^4 + \left[\frac{z^2}{2}\right]_0^4\right\} dx\,dy$$

$$= 2\int_0^4\int_0^{\sqrt{16-x^2}} [4+8]dxdy = 2\times 12\int_0^4 [y]_0^{\sqrt{16-x^2}} dx$$

$$= 24\int_0^4 \sqrt{16-x^2}\,dx = 24\int_0^{\frac{\pi}{2}} \sqrt{16-16\sin^2\theta}\ .4\cos\theta d\theta$$

$$[put x = 4\sin\theta \Rightarrow dx = 4\cos\theta d\theta.\ Also\ x = 0 \Rightarrow \theta = 0\ and\ x = 4 \Rightarrow \theta = \frac{\pi}{2}]$$

$$\therefore \int\int_V\int \nabla.\bar{F}dv = 96\times 4\int_0^{\frac{\pi}{2}} 4\sqrt{1-\sin^2\theta}\cos\theta d\theta = 96\times 4\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta$$

$$\int\int_V\int \bar{\nabla}.\bar{F}\,dv = 96\ X\ 4\int_0^{\frac{\pi}{2}} 4\sqrt{1-\sin^2\theta}\ \cos\theta d\theta = 96\ X\ 4\int_0^{\frac{\pi}{2}} \cos^2\theta\ d\theta$$

$$= 96\ X\ 4\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} d\theta = 96X4\int_0^{\frac{\pi}{2}} \left[\frac{1}{2} + \frac{\cos 2\theta}{2}\right] d\theta$$

$$= 384\left[\frac{1}{2}\theta + \frac{1}{2}\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} = 96\pi$$

14: Use Gauss Divergence theorem to evaluate $\int\int_S (yz^2\bar{i} + zx^2\bar{j} + 2z^2\bar{k}).ds, where$ S is the closed surface bounded by the xy-plane and the upper half of the sphere $x^2+y^2+z^2=a^2$ above this plane.

Sol: Divergence theorem states that

$$\int\int_S \bar{F}.ds = \int\int_V\int \bar{\nabla}.\bar{F}\ dv$$

Here $\nabla.\overline{F} = \frac{\partial}{\partial x}(yz^2) + \frac{\partial}{\partial y}(zx^2) + \frac{\partial}{\partial z}(2z^2) = 4z$

$$\therefore \int\int_s \overline{F}.ds = \int\int_V\int 4z dxdydz$$

Introducing spherical polar coordinates $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi,$

$z = r\cos\theta$ then $dxdydz = r^2 dr d\theta d\phi$

$$\therefore \int\int_s \overline{F}.ds = 4\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}(r\cos\theta)(r^2\sin\theta dr d\theta d\phi)$$

$$= 4\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^3\sin\theta\cos\theta\left[\int_{\phi=0}^{2\pi} d\phi\right] dr\ d\theta$$

$$= 4.\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^3\sin\theta\cos\theta\ (2\pi - 0)dr\ d\theta$$

$$= 4\pi\int_{r=0}^{a} r^3\left[\int_0^{\pi}\sin 2\theta\ d\theta\right]dr = 4\pi\int_{r=0}^{a} r^3\left(-\frac{\cos 2\theta}{2}\right)_0^{\pi} dr$$

$$= (-2\pi)\int_0^a r^3(1-1)dr = 0$$

15: Verify Gauss divergence theorem for $\bar{F} = x^3\bar{i} + y^3\bar{j} + z^3\bar{k}$ taken over the cube bounded by x = 0, x = a, y= 0, y = a, z = 0, z = a.

Sol: We have $\bar{F} = x^3\bar{i} + y^3\bar{j} + z^3\bar{k}$

$$\nabla.\overline{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = 3x^2 + 3y^2 + 3z^2$$

$$\int\int_V\int \bar{\nabla}.\bar{F}\ dv = \int\int_V\int(3x^2 + 3y^2 + 3z^2)dx\ dy\ dz$$

$$= 3\int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a}(x^2 + y^2 + z^2)dx\ dy\ dz$$

$$= 3\int_{z=0}^{a}\int_{y=0}^{a}\left(\frac{x^3}{3} + xy^2 + z^2x\right)_0^a dy\ dz$$

$$= 3\int_{z=0}^{a}\int_{y=0}^{a}\left(\frac{a^3}{a} + ay^2 + az^2\right)dy\ dz$$

$$= 3\int_{z=0}^{a}\left(\frac{a^3}{3}y + a\frac{y^3}{3} + az^2y\right)_0^a dz$$

$$= 3\int_0^a \left(\frac{a^4}{3}+\frac{a^4}{3}+a^2z^2\right)dz = 3\int_0^a \left(\frac{2}{3}a^4+a^2z^2\right)dz$$

$$= 3\left(\frac{2}{3}a^4z + a^2.\frac{z^3}{3}\right)_0^a = 3\left(\frac{2}{3}a^5+\frac{1}{3}a^5\right)$$

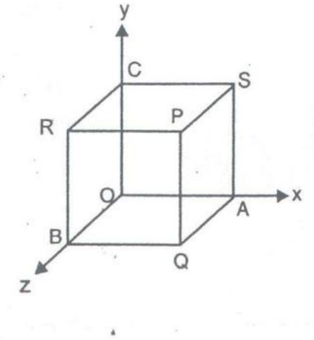
$$= 3a^5$$

To evaluate the surface integral divide the closed surface S of the cube into 6 parts.

i.e., S_1 : The face DEFA ; S_4 : The face OBDC

S_2 : The face AGCO ; S_5 : The face GCDE

S_3 : The face AGEF ; S_6: The face AFBO



$$\iint_S \bar{F}.\bar{n}ds = \iint_{S_1} \bar{F}.\bar{n}ds + \iint_{S_2} \bar{F}.\bar{n}ds + \cdots + \iint_{S_6} \bar{F}.\bar{n}ds$$

$On\ S_1, we\ have\ \bar{n} = \bar{i}, x = a$

$$\therefore \iint_{S_1} \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{y=0}^{a}\left(a^3\bar{i}+y^3\bar{j}+z^3\bar{k}\right).\bar{i}dydz$$

$$\iint_{S_1} \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{y=0}^{a}\left(a^3\bar{i}+y^3\bar{j}+z^3\bar{k}\right).\bar{i}\ dy\ dz$$

$$= \int_{z=0}^{a}\int_{y=0}^{a} a^3dy\ dz = a^3\int_0^a (y)_0^a\ dz$$

$$= a^4(z)_0^a = a^5$$

$On\ S_2, we\ have\ \bar{n} = -\bar{i}, x = 0$

$$\iint_{S_2} \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{y=0}^{a}\left(y^3\bar{j}+z^3\bar{k}\right).\left(-\bar{i}\right)dydz = 0$$

$On\ S_3, we\ have\ \bar{n} = \bar{j}, y = a$

$$\iint_{S_3} \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{x=0}^{a}\left(x^3\bar{i}+a^3\bar{j}+z^3\bar{k}\right).\bar{j}dxdz = a^3\int_{z=0}^{a}\int_{x=0}^{a} dxdz = a^3\int_0^a adz = a^4\left(z\right)_0^a$$

$$= a^5$$

$On\ S_4, we\ have\ \bar{n} = -\bar{j}, y = 0$

$$\iint_{S_4} \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{x=0}^{a}\left(x^3\bar{i}+z^3\bar{k}\right).(-\bar{j})dx\ dz = 0$$

$On\ S_5, we\ have\ \bar{n} = \bar{k}, z = a$

$$\iint_{S_5} \bar{F}.\bar{n}ds = \int_{y=0}^{a}\int_{x=0}^{a}\left(x^3\bar{i}+y^3\bar{j}+a^3\bar{k}\right).\bar{k}\ dx\ dy$$

$$= \int_{y=0}^{a}\int_{x=0}^{a} a^3 dx\, dy = a^3 \int_0^a (x)_0^a dy = a^4 (y)_0^a = a^5$$

$On\ S_6, we\ have\ \bar{n} = -\bar{k}, z = 0$

$$\int_{S_6}\int \bar{F}.\bar{n}ds = \int_{y=0}^{a}\int_{x=0}^{a} (x^3\bar{\imath} + y^3\bar{\jmath}).(-\bar{k})dx\, dy = 0$$

$$Thus \int_S\int \bar{F}.\bar{n}ds = a^5 + 0 + a^5 + 0 + a^5 + 0 = 3a^5$$

$$Hence \int_S\int \bar{F}.\bar{n}ds = \int_V\int \bar{\nabla}.\bar{F}\, dv$$

$\therefore$ *The* Gauss divergence theorem is verified.

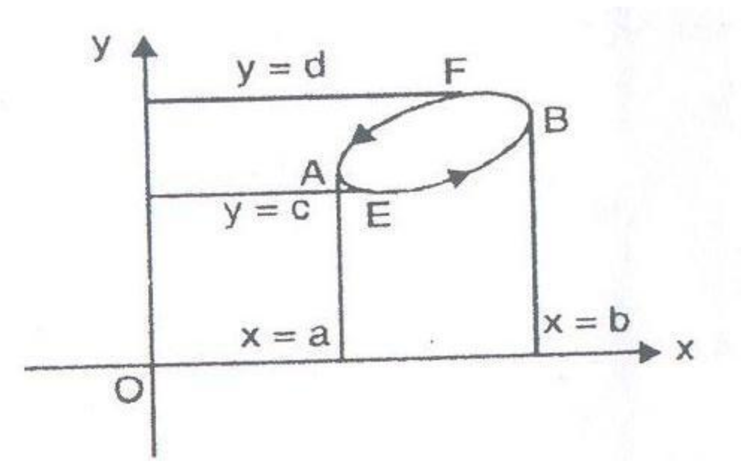
II. GREEN'S THEOREM IN A PLANE

(Transformation Between Line Integral and Surface Integral) [JNTU 2001S].

If S is Closed region in xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy.$$

Where C is traversed in the positive(anti clock-wise) direction

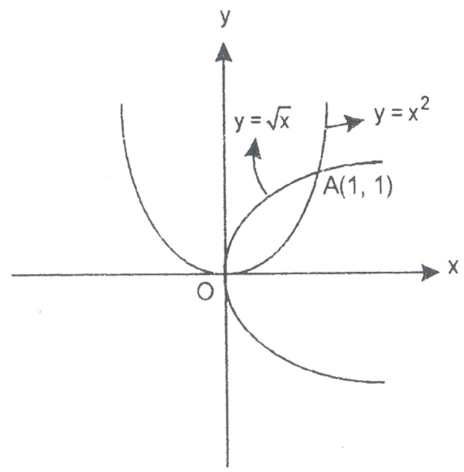


SOLVED PROBLEMS

1: Verify Green's theorem in plane for $\oint(3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the region bounded by y=$\sqrt{x}$ and y=x^2 .

Solution: Let M=$3x^2$-$8y^2$ and N=4y-6xy. Then

$\frac{\partial M}{\partial y} = -16y, \frac{\partial N}{\partial x} = -6y$



We have by Green's theorem,

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$$

Now $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R (16y - 6y) dxdy$

$$=10\iint_R ydxdy = 10\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} ydydx = 10\int_{x=0}^{1}\left(\frac{y^2}{2}\right)_{x^2}^{\sqrt{x}} dx$$

$$=5\int_0^1 (x - x^4)dx = 5\left(\frac{x^2}{2} - \frac{x^5}{5}\right)_0^1 = 5\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{2} \qquad ....(1)$$

Verification:

We can write the line integral along c

=[line integral along y=x^2(from O to A) + [line integral along y^2=x(from A to O)]

=I_1+I_2(say)

Now $I_1 = \int_{x=0}^{1}\{[3x^2 - 8(x^2)^2]dx + [4x^2 - 6x(x^2)]2xdx\}$ $\left[\because y = x^2 \Rightarrow \frac{dy}{dx} = 2x\right]$

$= \int_0^1 (3x^3 + 8x^3 - 20x^4)dx = -1$

And $I_2 = \int_1^0 \left[(3x^2 - 8x)dx + \left(4\sqrt{x} - 6x^{3/2}\right)\frac{1}{2\sqrt{x}}dx\right] = \int_1^0 (3x^2 - 11x + 2)dx = \frac{5}{2}$

$\therefore I_1 + I_2 = -1 + 5/2 = 3/2.$

From(1) and (2), we have $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$

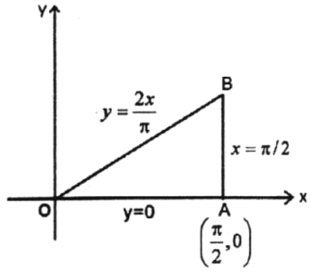
Hence the verification of the Green's theorem.

2: Evaluate by Green's theorem $\int_c (y - \sin x)dx + \cos x\, dy$ where C is the triangle enclosed by the lines y=0, x=$\frac{\pi}{2}$, $\pi y = 2x$.

<u>Solution :</u> Let M=y-sin x $and\ N = \cos x$ Then

$\frac{\partial M}{\partial y}$=1 and $\frac{\partial N}{\partial x}$=-sin x

$\therefore$ By Green's theorem $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$



$$\Rightarrow \int_c (y - \sin x)dx + \cos x dy = \iint_R (-1 - \sin x)dxdy$$

$$=-\int_{x=0}^{\pi/2} \int_{y=0}^{\frac{2x}{\pi}} (1 + \sin x)\, dxdy$$

$$= - \int_{x=0}^{\pi/2} (\sin x + 1)\, [y]_0^{2x/\pi}\, dx$$

$$= \frac{-2}{\pi} \int_{x=0}^{\pi/2} x(\sin x + 1) dx$$

$$= \frac{-2}{\pi}\left[x(-\cos x + x)\right]_0^{\pi} - \int_0^{\pi/2} 1(-\cos x + x)dx$$

$$= \frac{-2}{\pi}\left[x(-\cos x + x) + \sin x - \frac{x^2}{2}\right]_0^{\pi/2}$$

$$=$$

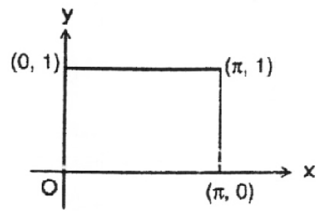
$$\frac{-2}{\pi}\left[-x\cos x + \frac{x^2}{2} + \sin x\right]_0^{\pi/2} = \frac{-2}{\pi}\left[\frac{\pi^2}{8} + 1\right] = -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

3: Evaluate by Green's theorem for $\oint_c (x^2 - \cosh y)dx + (y + \sin x)dy$ where C is the rectangle with vertices $(0,0), (\pi, 0), (\pi, 1), (0,1)$.

Solution: Let M=$x^2 - \cosh y$,$N = y + \sin x$

$$\therefore \frac{\partial M}{\partial y} = -\sinh y \;\; and \;\; \frac{\partial N}{\partial x} = \cos x$$

By Green's theorem, $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$



$$\Rightarrow \oint_c (x^2 - \cosh y)dx + (y + \sin x)dy = \int_R\int (\cos x + \sinh y)dxdy$$

$$\Rightarrow \oint_c (x^2 - \cosh y)dx + (y + \sin x)dy = \int_s\int (\cos x + \sinh$$

$$= \int_{x=0}^{\pi}\int_{y=0}^{1} (\cos x + \sinh y)dydx = \int_{x=0}^{\pi} (y\cos x + \cosh y)_0^1 dx$$

$$= \int_{x=0}^{\pi} (\cos x + \cosh 1 - 1)dx$$

$$= \pi(\cosh 1 - 1)$$

4: A Vector field is given by $\overline{F} = (\sin y)\bar{i} + x(1 + \cos y)\bar{j}$

Evaluate the line integral over the circular path $x^2 + y^2 = a^2$, z=0

(i) Directly (ii) By using Green's theorem

Solution : (i) Using the line integral

$$\oint_c \overline{F}.d\overline{r} = \oint_c F_1\,dx + F_2\,dy = \oint_c \sin y\,dx + x(1+\cos y)\,dy$$

$$= \oint_c \sin y\,dx + x\cos y\,dy + x\,dy = \oint_c d(x\sin y) + x\,dy$$

Given Circle is $x^2+y^2 = a^2$. Take x=a $\cos\theta$ and y=a $\sin\theta$ so that dx=-a $\sin\theta\ d\theta$ and dy=a $\cos\theta$ dθ and $\theta = 0 \to 2\pi$

$$\therefore\ \oint \overline{F}.d\overline{r} = \int_0^{2\pi} d[a\ \cos\theta \sin(a\ \sin\theta)] + \int_0^{2\pi} a(\cos\theta)a\ \cos\theta\, d\theta$$

$$=[a\ \cos\theta \sin(a\ \sin\theta)]_0^{2x} + 4a^2 \int_0^{\pi/2} cos^2\theta\, d\theta$$

$$=0+4a^2.\frac{1}{2}.\frac{\pi}{2} = \pi a^2$$

(ii)Using Green's theorem

Let M=$\sin y$ and N=x$(1+\cos y)$. Then

$\frac{\partial M}{\partial y}$=$\cos y$ and $\frac{\partial N}{\partial x}$=$(1+\cos y)$

By Green's theorem,

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

$$\therefore \oint_c \sin y\,dx + x(1+\cos y)dy = \iint_R (-\cos y + 1 + \cos y)dxdy == \iint dxdy$$

$$= \int\int_R dA = A = \pi a^2 (\because area\ \text{of circle}=\pi a^2)$$

We observe that the values obtained in (i) and (ii) are same to that Green's theorem is verified.

5: Show that area bounded by a simple closed curve C is given by $\frac{1}{2}\oint xdy - ydx$ and hence find the area of

(i)The ellipse x=$a\cos\theta, y = bsin\theta\ (i.e) \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(ii)The Circle x=a$\cos\theta$, $y = a\sin\theta$ $(i.e) x^2 + y^2 = a^2$

<u>Solution:</u> We have by Green's theorem $\oint_C Mdx + Ndy = \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$

Here M=-y and N=x so that $\frac{\partial M}{\partial y} = -1\ and \frac{\partial N}{\partial x} = 1$

$\oint_c xdy - ydx = 2\int_R dxdy = 2A$ where A is the area of the surface.

$\therefore \frac{1}{2}\int xdy - ydx = A$

(i)For the ellipse x=a$\cos\theta$ and y=b$\sin\theta$ and $\theta = 0 \to 2\pi$

$$\therefore Area, A=\frac{1}{2}\oint xdy - ydx = \frac{1}{2}\int_0^{2\pi}[(a\ \cos\theta)(b\cos\theta) - (b\ \sin\theta\ (-a\ \sin\theta))]d\,\theta$$

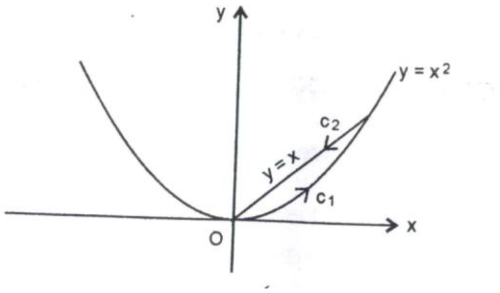
$$=\frac{1}{2}ab\int_0^{2\pi}(\cos^2\theta + sin^2\theta)\,d\theta = \frac{1}{2}ab(\theta)_0^{2\pi} = \frac{ab}{2}(2\pi - 0) = \pi ab$$

(ii)Put a=b to get area of the circle A=πa^2

6: Verify Green's theorem for $\int_c [(xy+y^2)dx + x^2 dy]$, where C is bounded by y=x and y=x^2

Solution: By Green's theorem, we have $\oint_C Mdx+Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$

Here M=xy +y^2 and N=x^2



The line y=x and the parabola y=x^2 intersect at O$(0,0)$ and A$(1,1)$

Now $\oint_c Mdx+Ndy = \int_{c_1} Mdx+Ndy + \int_{c_2} Mdx+Ndy$......(1) .....(1)

Along C_1 $(i.e. y = x^2)$, the line integral is

$$\int_{c_1} Mdx+Ndy = \int_{c_1}[x(x^2)+x^4]dx + x^2 d(x^2) \int_c (x^3+x^4+2x^3)dx = \int_0^1 (3x^3+x^4)dx$$

$$=\left(3.\frac{x^4}{4}+\frac{x^5}{5}\right)_0^1 = \frac{3}{4}+\frac{1}{5} = \frac{19}{20}$$ ....(2)

Along C_2 $(i.e. y = x)$ from $(1,1)$ to $(0,0)$, the line integral is

$$\int_{c_2} Mdx+Ndy = \int_{c_2}(x.x+x^2)dx + x^2dx \; [\because dy = dx]$$

$$=\int_{c_2} 3x^2 dx = 3\int_1^0 x^2 dx = 3\left(\frac{x^3}{3}\right)_1^0 = (x^3)_1^0 = 0-1=-1$$ ....(3)

From (1), (2) and (3), we have

$\int_c Mdx + Ndy = \frac{19}{20} - 1 = \frac{-1}{20}$

...(4)

Now

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \iint_R (2x - x - 2y)dxdy$$

$$=\int_0^1 [(x^2-x^2)-(x^3-x^4)]dx = \int_0^1 (x^4-x^3)dx$$

$$=\left(\frac{x^5}{5}+\frac{x^4}{4}\right)_0^1=\frac{1}{5}-\frac{1}{4}=\frac{-1}{20}$$

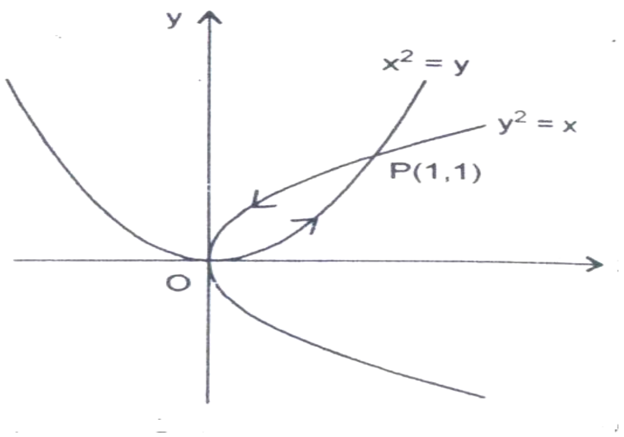
….(5)

From(4) and (5), We have $\oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$dxdy

Hence the verification of the Green's theorem.

7: Using Green's theorem evaluate $\int_c (2xy - x^2)dx + (x^2+y^2)dy$, Where "C" is the closed curve of the region bounded by y=x^2 and $y^2 = x$

Solution:



The two parabolas $y^2 = x$ and $y = x^2$ are intersecting at O$(0,0)$, and P(1,1)

Here M=2xy-x^2 and N=x^2 +y^2

$\therefore \frac{\partial M}{\partial y} = 2x$ and $\frac{\partial N}{\partial x} = 2x$

Hence $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - 2x = 0$

By Green's theorem $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$dxdy

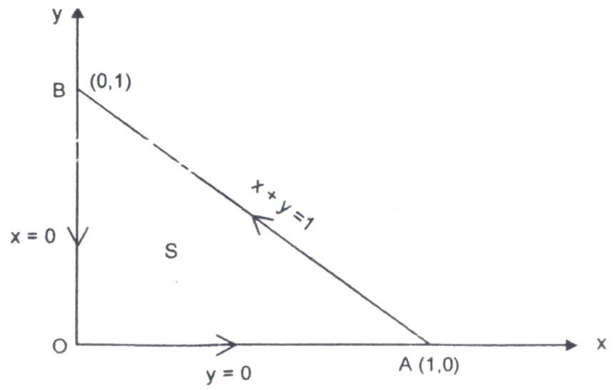
i.e., $\int_c (2xy - x^2)dx + (x^2+y^2)dy = \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} (0)dxdy = 0$

8: Verify Green's theorem for $\int_c [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$ where c is the region bounded by x=0, y=0 and x+y=1.

Solution : By Green's theorem, we have

$$\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

Here M=3$x^2 - 8y^2$ and N=4y-6xy



$\therefore \frac{\partial M}{\partial y} = -16y$ and $\frac{\partial N}{\partial x} = -6y$

Now $\int_c Mdx + Ndy = \int_{OA} Mdx + Ndy + \int_{AB} Mdx + Ndy + \int_{BC} Mdx + Ndy ...(1)$

Along OA, y=0 $\quad \therefore dy = 0$

$\int_{OA} Mdx + Ndy = \int_0^1 3x^2 dx = \left(\frac{x^3}{3}\right)_0^1 = 1$

Along AB, x+y=1 $\therefore dy = -dx$ and x=1-y and y varies from 0 to 1.

$\int_{AB} Mdx + Ndy = \int_0^1 [3(y-1)^2 - 8y^2](-dy) + [4y + 6y(y-1)]dy$

$= \int_0^1 (-5y^2 - 6y + 3)(-dy) + (6y^2 - 2y)dy$

$= \int_0^1 (11y^2 + 4y - 3)dy = \left(11\frac{y^3}{3} + 4\frac{y^2}{2} - 3y\right)_0^1$

$= \frac{11}{3} + 2 - 3 = \frac{8}{3}$

Along BO, x=0 $\therefore dx = 0$ and limits of y are from 1 to 0

. $\int_{BO} Mdx + Ndy = \int_1^0 4ydy = \left(4\frac{y^2}{2}\right)_1^0 = (2y^2)_0^1 = -2$

from (1), we have $\int_c Mdx + Ndy = 1 + \frac{8}{3} - 2 = \frac{5}{3}$

Now $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_{x=0}^{1} \int_{y=0}^{1-x} (-6y + 16y)dxdy$

$= 10 \int_{x=0}^1 \left[\int_{y=0}^{1-x} ydy\right] dx = 10 \int_0^1 \left(\frac{y^2}{2}\right)_0^{1-x} dx$

$= 5 \int_0^1 (1-x)^2 dx = 5\left[\frac{(1-x)^3}{-3}\right]_0^1$

$= -\frac{5}{3}[(1-1)^3 - (1-0)^3] = \frac{5}{3}$

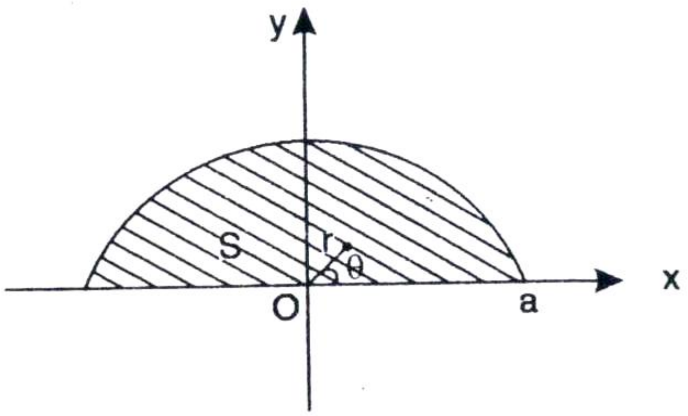
From (2) and (3), we have $\int_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

Hence the verification of the Green's Theorem.

9: Apply Green's theorem to evaluate $\oint_c (2x^2 - y^2)dx + (x^2 + y^2)\,dy, where\ c\ is$ the boundary of the area enclosed by the x-axis and upper half of the circle $x^2 + y^2 = a^2$

Solution : Let M=$2x^2 - y^2$ and N=$x^2 + y^2$ Then

$\frac{\partial M}{\partial y} = -2y\ and\ \frac{\partial N}{\partial x} = 2x$



Figure

$\therefore\ By\ Green's Theorem, \int_c M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

$\oint_c [(2x^2 - y^2)dx + (x^2 + y^2)dy] = \iint_R (2x + 2y)dxdy$

$=2\iint_R (x + y)dy$

$$=2\int_0^a \int_0^\pi r(\cos\theta + \sin\theta).rd\theta dr$$

[Changing to polar coordinates (r,θ), r varies from 0 to a and θ varies from 0 to π]

$\therefore \oint_c [(2x^2 - y^2)dx + (x^2 + y^2)dy] = 2\int_0^a r^2 dr \int_0^\pi (\cos\theta + \sin\theta)d\theta$

$=2.\frac{a^3}{3}(1+1) = \frac{4a^3}{3}$

10: Find the area of the Folium of Descartes $x^3 + y^3 = 3axy(a > 0) using\ Green's$ Theorem.

Solution: from Green's theorem, we have

$\int Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$

By Green's theorem, Area = $\frac{1}{2}\oint (xdy - ydx)$

Considering the loop of folium Descartes(a>0)

Let x$=\frac{3at}{1+t^3}$, $y=\frac{3at^2}{1+t^3}$, Then $dx=\left[\frac{d}{dt}\left(\frac{3at}{1+t^3}\right)\right]dt$ and $dy=\left[\frac{d}{dt}\left(\frac{3at^2}{1+t^3}\right)\right]dt$

The point of intersection of the loop is $\left(\frac{3a}{2},\frac{3a}{2}\right)\Rightarrow t=1$

Along OA, t varies from 0 to1.

$$\therefore \frac{1}{2}\oint(xdy-ydx)=\frac{1}{2}\int_0^1\left(\frac{3at}{1+t^3}\right)\left[\frac{d}{dt}\left(\frac{3at^2}{1+t^3}\right)\right]dt-\left(\frac{3at^2}{1+t^3}\right)\left[\frac{d}{dt}\left(\frac{3at}{1+t^3}\right)\right]dt$$

$$=\frac{1}{2}\int_0^1\left\{\frac{3at}{1+t^3}\left[\frac{3at(2-t^3)}{\left(1+t^3\right)^2}\right]-\frac{3at^2}{1+t^3}\left[\frac{3a(1-2t^3)}{\left(1+t^3\right)^2}\right]\right\}dt$$

$$=\frac{9a^2}{2}\int_0^1\left[\frac{t^2(2-t^3)}{(1+t^3)^3}-\frac{t^2(1-2t^3)}{(1+t^3)^3}\right]dt=\frac{9a^2}{2}\int_0^1\frac{2t^2-t^5-t^2+2t^5}{(1+t^3)^3}dt$$

$$=\frac{9a^2}{2}\int_0^1\frac{t^2+t^5}{(1+t^3)^3}dt=\frac{9a^2}{2}\int_0^1\frac{t^2(1+t^3)}{(1+t^3)^3}dt$$

$$=\frac{9a^2}{2}\int_0^1\frac{t^2}{(1+t^3)^2}dt$$ [Put 1+$t^3=x\Longrightarrow 3t^2\,dt=dx$

L.L. : x=1, U.L.:x=2]

$$=\frac{9a^2}{2}\int_1^2\frac{t^2}{x^2}.\frac{dx}{3t^2}=\frac{9a^2}{6}\int_1^2\frac{1}{x^2}dx=\frac{3a^2}{4}sq.\text{units(a>0).}$$

11: Verify Green's theorem in the plane for $\int_C(x^2-xy^3)\,dx+(y^2-2xy)dy$

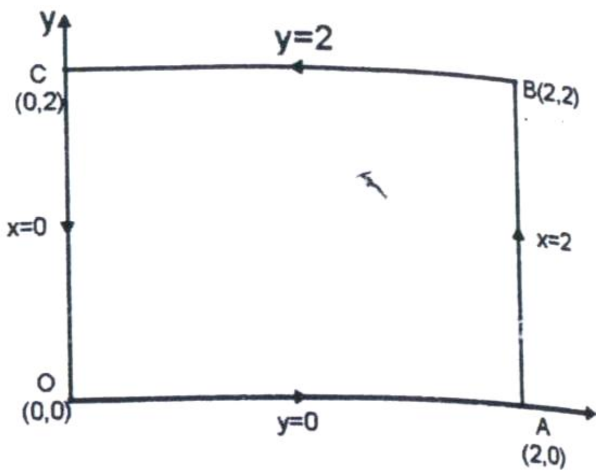
Where C is square with vertices (0,0), (2,0), (2,2), (0,2).

Solution: The Cartesian form of Green's theorem in the plane is

$$\int_c Mdx+Ndy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$$

Here M=x^2-xy^3 and N=y^2-2xy

$\therefore\ \frac{\partial M}{\partial y}$ = - 3 xy^2 and $\frac{\partial N}{\partial x}=-2y$



Evaluation of $\int_C(\boldsymbol{Mdx}+\boldsymbol{Ndy})$

To Evaluate $\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy$, we shall take C in four different segments viz (i) along OA(y=0) (ii) along AB(x=2) (iii) along BC(y=2) (iv) along CO(x=0).

(i)Along OA(y=0)

$$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_0^2 x^2 dx = \left(\frac{x^3}{3}\right)_0^2 = \frac{8}{3} \quad .....(1)$$

(ii)Along AB(x=2)

$$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_0^2 (y^2 - 4y)\,dy \quad [\because x = 2, dx = 0]$$

$$= \left(\frac{y^3}{3} - 2y^2\right)_0^2 = \left(\frac{8}{3} - 8\right) = 8\left(-\frac{2}{3}\right) = -\frac{16}{3}$$

....(2)

(iii)Along BC(y=2)

$$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_2^0 (x^2 - 8x)\,dx \quad [\because y = 2, dy = 0]$$

$$= \left(\frac{x^3}{3} - 4x^2\right)_0^2 = -\left(\frac{8}{3} - 16\right) = \frac{40}{3} \quad ......(3)$$

(iv)Along CO(x=0)

$$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_2^0 y^2\,dx \quad [\because x = 0, dx = 0] = \left(\frac{y^3}{3}\right)_2^0 = -\frac{8}{3} \quad .....(4)$$

Adding(1),(2),(3) and (4), we get

$$\int_c \left(x^2 - xy^3\right)dx + \left(y^2 - 2xy\right)dy = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = \frac{24}{3} = 8 \quad ...(5)$$

Evaluation of $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

Here x ranges from 0 to 2 and y ranges from 0 to 2.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_0^2 \int_0^2 (-2y + 3xy^2)\,dxdy$$

$$= \int_0^2 \left(-2xy + \frac{3x^2}{2} y^2\right)_0^2 dy$$

$$= \int_0^2 (-4y + 6y^2)\,dy = \left(-2y^2 + 2y^3\right)_0^2$$

$$= -8 + 16 = 8 \quad ...(6)$$

From (5) and (6), we have

$$\int_c M\,dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

Hence the Green's theorem is verified.

III. STOKE'S THEOREM

(Transformation between Line Integral and Surface Integral) [JNTU 2000]

Let S be a open surface bounded by a closed, non intersecting curve C. If $\bar{F}$ is any differentieable vector point function then $\oint_C \bar{F}.d\,\bar{r} =$ $\int_S curl\ \ \bar{F}.\bar{n}\, ds$ *where c is traversed in the positive* direction and $\bar{n}$ *is unit outward drawn normal at any point of the surface.*

PROBLEMS:

1: Prove by Stokes theorem, Curl grad $\phi = \bar{0}$

Solution: Let S be the surface enclosed by a simple closed curve C.

$\therefore$ *By Stokes theorem*

$$\int_s (curl\ grand\phi).\bar{n}\ ds = \int_s (\nabla x \nabla\phi).\bar{n}\ dS = \oint_C \nabla\phi.d\bar{r} = \oint_C \nabla\phi.d\bar{r}$$

$$= \oint_c \left(\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} \right).\left(\bar{i}dx + \bar{j}dy + \bar{k}dz\right)$$

$$= \oint_c \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \right) = \int d\phi = [\phi]_p$$ where P is any point on C.

$$\therefore \int curl\ grad\phi.\bar{n}\ ds = \bar{0} \Rightarrow curl\ grad\phi = \bar{0}$$

2: prove that $\int_s \phi curl\ \ \bar{f}.dS = \int_c \phi\bar{f}.d\bar{r} - \int_s curl\ grad\phi \times \bar{f} dS$

Solution: Applying Stokes theorem to the function $\phi\ \bar{f}$

$$\int_c \phi\bar{f}.d\bar{r} = \int curl\left(\phi\bar{f}\right).\bar{n}ds = \int_s \left(grad\phi \times \bar{f} + \phi curl\,\bar{f}\right)ds$$

$$\therefore \int_c \phi curl\,\bar{f}.ds = \int_c \phi\bar{f}.d\bar{r} - \int \nabla\phi \times \bar{f}.ds$$

3: Prove that $\oint_c f\nabla f.d\bar{r} = 0.$

Solution: By Stokes Theorem,

$$\oint_c (f\nabla f).d\bar{r} = \int_s curl f\nabla\bar{f}.\bar{n}\ ds = \int_s [\,f curl\nabla f + \nabla f \times \nabla f\,].\bar{n}\ ds$$

$$= \int \bar{0}.\bar{n}ds = 0[\because curl\nabla f = \bar{0}\,\text{and}\ \nabla f \times \nabla f = \bar{0}]$$

4: Prove that $\oint_c f\nabla g.d\bar{r} = \int (\nabla f \times \nabla g).\bar{n}ds$

Solution: By Stokes Theorem,

$$\oint_c \left(f\nabla g.d\bar{r}\right) = \int_s [\nabla \times (f\nabla g)]\bar{n}ds = \int_s [\nabla f \times \nabla g + f cur\lg radg].\bar{n}ds$$

$$= \int [\nabla f \times \nabla g].\bar{n}ds \left[\because curl(gradg) = \bar{0}\right]$$

5: Verify Stokes theorem for $\bar{F} = -y^3\bar{\imath} + x^3\bar{\jmath}$, Where S is the circular disc $x^2 + y^2 \leq 1, z = 0$.

Solution: Given that $\bar{F} = -y^3\bar{\imath} + x^3\bar{\jmath}$. The boundary of C of S is a circle in xy plane. $x^2 + y^2 \leq 1, z = 0$. We use the parametric co-ordinates x=cosθ, $y = sin\theta, z = 0, 0 \leq \theta \leq 2\pi$;

dx=-sinθ $d\theta$ and dy =cosθ $d\theta$

$$\therefore \oint_c \bar{F}.dr = \int_c F_1 dx + F_2 dy + F_3 dz = \int_c -y^3 dx + x^3 dy$$

$$= \int_0^{2\pi} [-sin^3\theta(-sin\theta) + cos^3\theta cos\theta] d\theta = \int_0^{2\pi} (cos^4\theta + sin^4\theta) d\theta$$

$$= \int_0^{2\pi} (1 - 2sin^2\theta\ cos^2\theta) d\theta = \int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi} (2sin\theta\ cos\theta)^2\, d\theta$$

$$= \int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi} sin^2 2d\theta = (2\pi - 0) - \frac{1}{4}\int_0^{2\pi} (1 - cos4\theta) d\theta$$

$$= 2\pi + \left[-\frac{1}{4}\theta + \frac{1}{16}sin4\theta\right]_0^{2\pi} = 2\pi - \frac{2\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$$

$$\text{Now} \nabla \times \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = \bar{k}(3x^2 + 3y^2)$$

$$\therefore \int_s (\nabla \times \bar{F}).\bar{n}ds = 3\int_s (x^2 + y^2)\bar{k}.\bar{n}ds$$

We have $(\bar{k}.\bar{n})ds = dxdy$ and R is the region on xy-plane

$$\therefore \iint_s (\nabla \times \bar{F}).\bar{n}ds = 3\iint_R (x^2 + y^2)\, dx\, dy$$

Put x=r cos$\emptyset$, $y = r\ sin\emptyset \therefore\ dxdy = rdr\ d\emptyset$

r is varying from 0 to 1 and 0$\leq \emptyset \leq 2\pi$.

$$\therefore \int (\nabla \times \bar{F}).\,\bar{n}ds = 3\int_{\emptyset=0}^{2\pi}\int_{r=0}^{1} r^2 .\text{rdr } d\emptyset = \frac{3\pi}{2}$$

L.H.S=R.H.S.Hence the theorem is verified.

6: If $\bar{F} = y\bar{i} + (x - 2xz)\bar{j} - xy\bar{k}$, evaluate $\int_s (\nabla \times F).\bar{n}ds$. Where S is the surface of sphere $x^2 + y^2 + z^2 = a^2$, *above the xy − plane.*

Solution: Given $\bar{F} = y\bar{\imath} + (x - 2xz)\bar{\jmath} - xy\bar{k}$.

By Stokes Theorem,

$$\int_s (\nabla \times \bar{F}).\bar{n}ds = \int_c \bar{F}.d\bar{r} = \int_c F_1 dx + F_2 dy + F_3 dz = \int_c ydx + (x - 2xz)dy - xydz$$

Above the xy plane the sphere is $x^2 + y^2 += a^2, z = 0$

$$\therefore \int_c \bar{F}.d\bar{r} = \int_c ydx + xdy.$$

Put x=a cos θ,y=asinθ *so that* $dx = -a\ sin\theta d\theta, dy = acos\theta d\theta$ *and* $\theta = 0 \to 2\pi$

$$\int_c \bar{F}.d\bar{r} = \int_0^{2\pi} (a\ sin\theta)(-a\ sin\theta)\, d\theta + (acos\theta)(acos\theta) d\theta$$

$$=a^2\int_0^{2\pi}\cos 2\theta\, d\theta = a^2\left[\frac{\sin 2\theta}{2}\right]_0^{2\pi} = \frac{a^2}{2}(0) = 0$$

7: Verify Stokes theorem for $\bar{F} = (2x-y)\bar{\imath} - yz^2\bar{\jmath} - y^2z\bar{k}$ over the upper half surface of the sphere $x^2+y^2+z^2=1$ bounded by the projection of the xy-plane.

Solution: The boundary C of S is a circle in xy plane i.e $x^2+y^2=1$, z=0

The parametric equations are x=$\cos\theta, y=\sin\theta, \theta = 0 \to 2\pi$

$\therefore\ dx = -\sin\theta\, d\theta, dy = \cos\theta\, d\theta$

$$\int_c \bar{F}.d\bar{r} = \int_c \bar{F_1}dx + \bar{F_2}dy + \bar{F_3}dz = \int_c (2x-y)dx - yz^2dy - y^2zdz$$

$$=\int_c (2x-y)dx(since\ z=0\ and\ dz=0)$$

$$=-\int_0^{2\pi}(2\cos\theta - \sin\theta)\sin\theta d\theta = \int_0^{2\pi}\sin^2\theta d\theta - \int_0^{2\pi}\sin 2\theta d\theta$$

$$=\int_{\theta=0}^{2\pi}\frac{1-\cos 2\theta}{2}d\theta - \int_0^{2\pi}\sin 2\theta\, d\theta = \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{2}.\cos 2\theta\right]_0^{2\pi}$$

$$=\frac{1}{2}(2\pi - 0) + 0 + \frac{1}{2}.(\cos 4\pi - \cos 0) = \pi$$

Again $\nabla\times\bar{F}=\begin{vmatrix}\bar{\imath} & \bar{\jmath} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x-y & -yz^2 & -y^2z\end{vmatrix} = \bar{\imath}(-2yz+2yz) - \bar{\jmath}(0-0) + \bar{k}(0+1) = \bar{k}$

$\therefore \int_S(\nabla\times\bar{F}).\bar{n}ds = \int_S \bar{k}.\bar{n}ds = \int_R\int dxdy$

Where R is the projection of S on xy plane and $\bar{k}.\bar{n}ds = dxdy$

Now $\int\int_R dxdy = 4\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} dydx = 4\int_{x=0}^{1}\sqrt{1-x^2}\,dx = 4\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1$

$$=4\left[\frac{1}{2}\sin^{-1}1\right] = 2\frac{\pi}{2} = \pi$$

$\therefore$ *The* Stokes theorem is verified.

8: Verify Stokes theorem for the function $\bar{F} = x^2\,\bar{\imath} + xy\,\bar{\jmath}$ integrated round the square in the plan z=0 whose sides are along the lines x=0, y=0, x=a, y=a.

Solution: Given $\bar{F} = x^2\,\bar{\imath} + xy\,\bar{\jmath}$

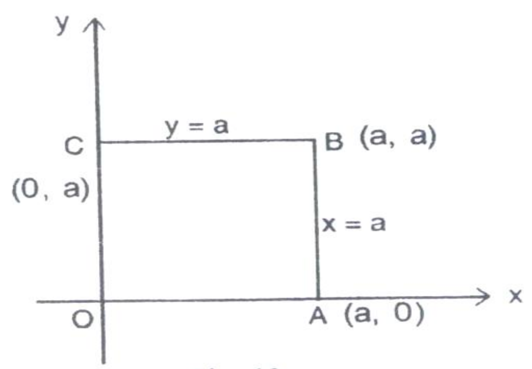


Fig. 13

By Stokes Theorem, $\int_S (\nabla \times \bar{F}).\bar{n}ds = \int_c \bar{F}.d\bar{r}$

Now $\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix} = \bar{k}y$

L.H.S.$= \int_S (\nabla \times \bar{F}).\bar{n}ds = \int_s y(\bar{n}.\bar{k})ds = \int_s ydxdy$

$\therefore \bar{n}.\bar{k}.ds = dxdy$ and R is the region bounded for the square.

$\therefore \int_S (\nabla \times \bar{F}).\bar{n}ds = \int_0^a \int_0^a ydydx = \frac{a^3}{2}$

R.H.S. $= \int_C \bar{F}.d\bar{r} = \int_C (x^2dx + xydy)$

But $\int \bar{F}.d\bar{r} = \int_{OA} \bar{F}.d\bar{r} + \int_{AB} \bar{F}.d\bar{r} + \int_{BC} \bar{F}.d\bar{r} + \int_{CO} \bar{F}.d\bar{r}$

(i)Along OA: y=0, z=0, dy=0, dz=0

$\therefore \int_{OA} \bar{F}.d\bar{r} = \int_0^a x^2dx = \frac{a^3}{3}$

(ii)Along AB:x=a, z=0,dx=0,dz=0

$\int_{AB} \bar{F}.d\bar{r} = \int_0^a aydy = \frac{1}{2}a^3$

(iii)Along BC: y=a,z=0,dy=0,dz=0

$\therefore \int_{BC} \bar{F}.d\bar{r} = \int_a^0 0dx = \frac{1}{3}a^3$

(iv)Along CO: x=0, z=0, dx=0, dz=0

$\therefore \int_{CO} \bar{F}.d\bar{r} = \int_a^0 0dy = 0$

Adding $\int_c \bar{F}.d\bar{r} = \frac{1}{3}a^3 + \frac{1}{2}a^3 + \frac{1}{3}a^3 + 0 = \frac{1}{2}a^3$

Hence the verification.

9: Apply Stokes theorem, to evaluate $\oint_c (ydx + zdy + xdz)$ where c is the curve of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and x+z=a.

Solution : The intersection of the sphere $x^2 + y^2 + z^2 = a^2$ *and* the plane x+z=a. is a circle in the plane x+z=a. with AB as diameter.

Equation of the plane is x+z=a$\Longrightarrow \frac{x}{a} + \frac{z}{a} = 1$

$\therefore OA = OB = a$ i.e., $A = (a,0,0)$ and B=(0,0,a)

$\therefore$ Length of the diameter AB$= \sqrt{a^2 + a^2 + 0}$ =a$\sqrt{2}$

Radius of the circle, r=$\frac{a}{\sqrt{2}}$

Let $\bar{F}.d\bar{r} = ydx + zdy + xdz \Longrightarrow \bar{F}.d\bar{r} = \bar{F}.(\bar{i}dx + \bar{j}dy + \bar{k}dz) = ydx + zdy + xdz$

$\Longrightarrow \bar{F} = y\,\bar{\imath} + z\,\bar{\jmath} + x\,\bar{k}$

$$\therefore \; curl\;\bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\left(\bar{\imath} + \bar{\jmath} + \bar{k}\right)$$

Let $\bar{n}$ be the unit normal to this surface. $\bar{n} = \frac{\nabla S}{|\nabla S|}$

Then s=x+z-a, $\nabla S = \bar{i} + \bar{k} \therefore \; \bar{n} = \frac{\nabla S}{|\nabla S|} = \frac{\bar{\imath}+\bar{k}}{\sqrt{2}}$

Hence $\oint_c \bar{F}.d\bar{r} = \int curl\;\bar{F}.\bar{n}\,ds\;(by\;Stokes\;Theorem)$

$$= -\int\left(\bar{\imath} + \bar{\jmath} + \bar{k}\right).\left(\frac{\bar{\imath}+\bar{k}}{\sqrt{2}}\right)ds = -\int\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)ds$$

$$= -\sqrt{2}\int_S ds = -\sqrt{2}S = -\sqrt{2}\left(\frac{\pi a^2}{2}\right) = \frac{\pi a^2}{\sqrt{2}}$$

10: Apply the Stoke's theorem and show that $\int_S \int curl\;\bar{F}.\bar{n}d\bar{s} = 0\;where\;\bar{F}$ is any vector and S = $x^2 + y^2 + z^2 = 1$

Solution: Cut the surface if the Sphere $x^2 + y^2 + z^2 = 1\;by\;any\;plane,$ Let $S_1\;and\;S_2$denotes its upper and lower portions a C, be the common curve bounding both these portions.

$$\therefore \int_s curl\bar{F}.d\bar{s} = \int_{s_1} \bar{F}.d\bar{s} + \int_{s_2} \bar{F}.d\bar{s}$$

Applying Stoke's theorem,

$$\int_s curl\bar{F}.d\bar{s} = \int_{s_1} \bar{F}.d\bar{R} + \int_{s_2} \bar{F}.d\bar{R} = 0$$

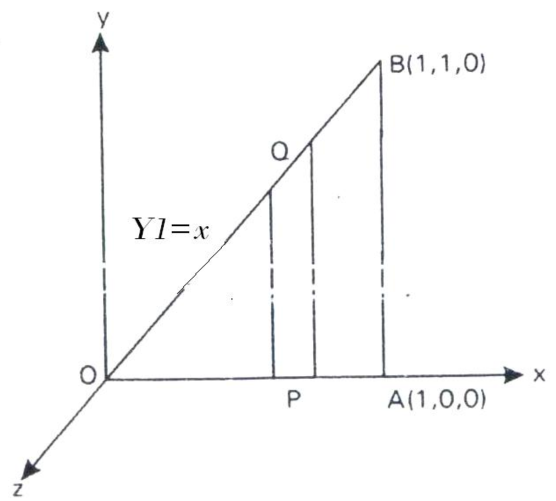
The 2^{nd} integral curl $\bar{F}.d\bar{s}$ is negative because it is traversed in opposite direction to first integral. The above result is true for any closed surface S.

11: Evaluate by Stokes theorem $\oint_c (x+y)dx + (2x-z)dy + (y+z)dz$ where C is the boundary of the triangle with vertices (0,0,0), (1,0,0) and (1,1,0).

Solution: Let $\bar{F}.d\bar{r} = \bar{F}.\left(\bar{\imath}dx + \bar{\jmath}dy + \bar{k}dz\right) = (x+y)dx + (2x-z)dy + (y+z)dz$

Then $\bar{F} = (x+y)\bar{\imath} + (2x-z)\bar{\jmath} + (y+z)\bar{k}$

By Stokes theorem, $\oint_C \bar{F}.d\bar{r} = \int\int_S curl\;\bar{F}.\bar{n}\,ds$



Where S is the surface of the triangle OAB which lies in the xy plane. Since the z Co-ordinates of O,A and B Are zero. Therefore $\bar{n} = \bar{k}$. Equation of OA is y=0 and that of OB, y=x in the xy plane.

$$\therefore \ curl\ \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = 2\bar{\imath} + \bar{k}$$

$\therefore \ curl\ \bar{F}.\bar{n}ds = curl\ \bar{F}.\bar{K}\ dx\ dy = dx\ dy$

$\therefore \oint_c \bar{F}.d\bar{r} = \int\int_s dx\ dy = \int\int_s dA = A = area\ of$ the $\Delta\ OAB$

$$= \tfrac{1}{2}OA \times AB = \frac{1}{2}\times 1\times 1 = \frac{1}{2}$$

12: Use Stoke's theorem to evaluate $\int\int_S curl\ \bar{F}.\bar{n}\ dS$ over the surface of the paraboloid $z + x^2 + y^2 = 1, z \geq 0$ where $\bar{F} = y\ \bar{\imath} + z\ \bar{\jmath} + x\ \bar{k}$.

Solution : By Stoke's theorem

$$\int_s curl\bar{F}.d\bar{s} = \oint_c \bar{F}.d\bar{r} = \int_c (y\bar{i} + z\bar{j} + x\bar{k}).(\bar{i}dx + \bar{j}dy + \bar{k}dz)$$

$$= \int_c ydx \text{ (Since z=0,dz=0) ......(1)}$$

Where C is the circle $x^2 + y^2 = 1$

The parametric equations of the circle are x=$cos\theta, y = sin\theta$

$\therefore dx = -\ sin\theta\ d\theta$

Hence (1) becomes

$$\int_s curl\bar{F}.ds = \int_{\theta=0}^{2\pi} \sin\theta(-\sin\theta)d\theta = -\int_{\theta=0}^{2\pi} \sin^2\theta d\theta = -4\int_0^{\frac{\pi}{2}} \sin^2\theta d\theta = -4\times\frac{1}{2}\times\frac{\pi}{2} = -\pi$$

13: Verify Stoke's theorem for $\bar{F} = (x^2 + y^2)\bar{\imath} - 2xy\bar{\jmath}$ taken round the rectangle bounded by the lines x=$\pm a, y = 0, y = b$.

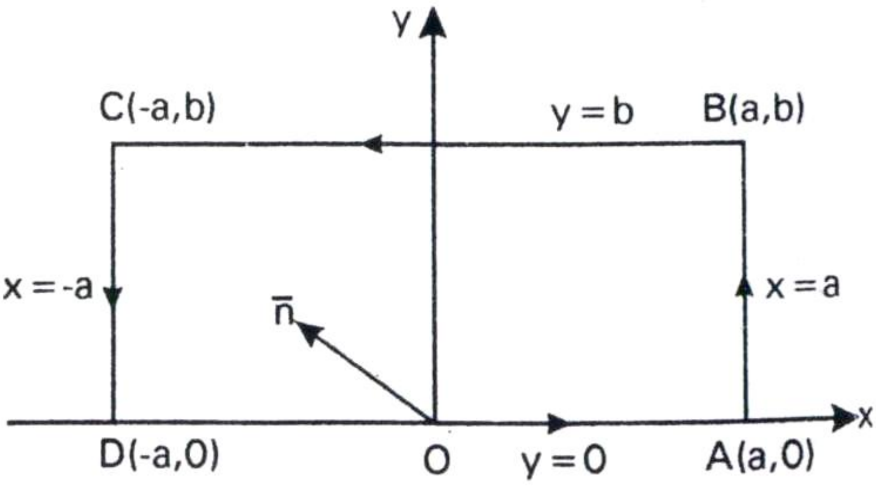
Solution: Let ABCD be the rectangle whose vertices are (a,0), (a,b), (-a,b) and (-a,0). Equations of AB, BC, CD and DA are x=a, y=b, x=-a and y=0.

We have to prove that $\oint_c \bar{F}.d\bar{r} = \int_S curl\ \bar{F}.\bar{n}ds$

$\oint_c \bar{F}.d\bar{r} = \oint_c \{(x^2 + y^2)\bar{\imath} - 2xy\bar{\jmath}\}.\{\bar{\imath}dx + \bar{\jmath}dy\}$

$$= \oint_C (x^2 + y^2)\ dx - 2xydy$$

$$= \int_{AB} + \int_{Bc} + \int_{CD} + \int_{DA} \quad .....(1)$$



(i) Along AB, x=a, dx=0

from (1), $\int_{AB} = \int_{y=0}^{b} -2ay\,dy = -2a\left[\frac{y^2}{2}\right]_0^b = -ab^2$

(ii)Along BC, y=b, dy=0

from (1), $\int_{BC} = \int_{x=a}^{x=-a} (x^2+b^2)dx = \left[\frac{x^3}{3}+b^2x\right]_{x=a}^{-a} = \frac{-2a^3}{3} - 2ab^2$

(iii) Along CD, x=-a, dx=0

from (1), $\int_{CD} = \int_{y=b}^{0} 2aydy = \cancel{2}a\left[\frac{y^2}{\cancel{2}}\right]_{y=b}^{0} = -ab^2$

(iv)Along DA, y=0, dy=0

from (1), $\int_{DA} = \int_{x=-a}^{x=a} x^2dx = \left[\frac{x^3}{3}\right]_{x=-a}^{a} = \frac{2a^3}{3}$

(i)+(ii)+(iii)+(iv) gives

$\therefore \oint_c \bar{F}.d\bar{r} = -ab^2 - \frac{-2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3} = -4ab^2$ ….(2)

Consider $\int_S curl\ \bar{F}.\bar{n}\,dS$

Vector Perpendicular to the xy-plane is $\bar{n} = k$

$$\therefore\ curl\ \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2+y^2) & -2xy & 0 \end{vmatrix} = 4y\bar{k}$$

Since the rectangle lies in the xy plane,

$\bar{n} = \bar{k}$ and ds =dx dy

$\int_S curl\ \bar{F}.\bar{n}dS = \int_S -4y\bar{k}.\bar{k}dx\,dy = \int_{x=-a}^{a}\int_{y=0}^{b} -4y\,dx\,dy$

$$= \int_{y=0}^{b}\int_{x=-a}^{a} -4y\,dx\,dy = 4\int_{y=0}^{b} y[x]_{-a}^{a}\,dy = -4\int_{y=0}^{b} 2aydy$$

$$=-4a[y^2]_{y=0}^{b} = -4ab^2 \qquad \ldots..(3)$$

Hence from (2) and (3), the Stoke's theorem is verified.

14: Verify Stoke's theorem for $\bar{F} = (y - z + 2)\bar{\imath} + (yz + 4)\bar{\jmath} - xz\bar{k}$ where S is the surface of the cube x =0, y=0, z=0, x=2, y=2,z=2 above the xy plane.

Solution: Given $\bar{F} = (y - z + 2)\bar{\imath} + (yz + 4)\bar{\jmath} - xz\bar{k}$ where S is the surface of the cube.

x=0, y=0, z=0, x=2, y=2, z=2 above the xy plane.

By Stoke's theorem, we have $\int curl\ \bar{F}.\bar{n}ds = \int \bar{F}.d\bar{r}$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z + 2 & y + 4 & -xz \end{vmatrix} = \bar{\imath}(0 + y) - \bar{\jmath}(-z + 1) + \bar{k}(0 - 1) = y\bar{\imath} - (1 - z)\bar{\jmath} - \bar{k}$$

$$\therefore\ \nabla \times \bar{F}.\bar{n} = \nabla \times \bar{F}.k = \left(y\bar{\imath} - (1 - z)\bar{\jmath} - \bar{k}\right).k = -1$$

$$\therefore \int \nabla \times \bar{F}.\bar{n}.ds = \int_0^2 \int_0^2 -1\ dx\ dy \ \ (\because z = 0, dz = 0) = -4 \qquad \ldots..(1)$$

To find $\int \bar{F}.d\bar{r}$

$$\int \bar{F}.d\bar{r} = \int \left((y - z + 2)\bar{\imath} + (yz + 4)\bar{\jmath} - xz\bar{k}\right).(dx\bar{\imath} + dy\bar{\jmath} + dz\bar{k})$$

$$= \int [(y - z + 2)dx + (yz + 4)dy - (xz)dz]$$

Sis the surface of the cube above the xy-plane

$\therefore z = 0 \Rightarrow dz = 0$

$\therefore \int \bar{F}.d\bar{r} = \int (y + 2)dx + \int 4dy$

Along $\overline{OA}, y = 0, z = 0, dy = 0, dz = 0, x\ change\ from\ 0\ to\ 2.$

$\int_0^2 2dx = 2[x]_0^2 = 4$ ……..(2)

Along $\overline{BC}, y = 2, z = 0, dy = 0, dz = 0, x\ change\ from\ 2\ to\ 0.$

$\int_2^0 4dx = 4[x]_2^0 = -8$ ……. .(3)

Along $\overline{AB}, x = 2, z = 0, dx = 0, dz = 0, y\ change\ from\ 0\ to\ 2.$

$\int \bar{F}.d\bar{r} = \int_0^2 4dy = [4y]_0^2 = 8$ ……(4)

Along $\overline{CO}, x = 0, z = 0, dx = 0, dz = 0, y\ change\ from\ 2\ to\ 0.$

.$\int_2^0 4dy = -8$ …..(5)

Above the surface When z=2

Along $0'A'$, $\int_0^2 \bar{F}\ \overline{dr} = 0$ ….(6)

Along $A'B', x = 2, z = 2, dx = 0, dz = 0,$ y changes from 0 to 2

$$\int_0^2 \bar{F}.d\bar{r} = \int_0^2 (2y + 4)dy = 2\left[\frac{y^2}{2}\right]_0^2 + 4[y]_0^2 = 4 + 8 = 12 \qquad \ldots.(7)$$

Along $B'C', y = 2, z = 2, dy = 0, dz = 0,$ x changes from 2 to 0

$\int_0^2 \bar{F}.\overline{dr} = 0$ ….(8)

Along $C'D', x = 0, z = 2, dx = 0, dz = 0,$ y changes from 2 to 0.

$$\int_2^0 (2y+4) = 2\left[\frac{y^2}{2}\right]_2^0 + 4[y]_2^0 = -12 \qquad \text{.....(9)}$$

(2)+(3)+(4)+(5)+(6)+(7)+(8)+(9) gives

$$\int_C \bar{F}.d\bar{r} = 4 - 8 + 8 - 8 + 0 + 12 + 0 - 12 = -4 \qquad \text{.....(10)}$$

By Stokes theorem, We have

$\int \bar{F}.d\bar{r} = \int curl\ \bar{F}.\bar{n}ds = -4$

Hence Stoke's theorem is verified.

15: Verify the Stoke's theorem for $\bar{F} = y\bar{i} + z\bar{j} + x\bar{k}$ and surface is the part of the sphere $x^2 + y^2 + z^2 = 1\ above\ the\ xy\ plane.$

Solution: Given $\bar{F} = y\bar{i} + z\bar{j} + x\bar{k}$ over the surface $x^2 + y^2 + z^2 = 1\ is\ xy\ plane.$

We have to prove $\int_C \bar{F}.d\bar{r} = \int\int_s Curl\ \bar{F}.\bar{n}ds$

$\bar{F}.d\bar{r} = .(y\bar{i} + z\bar{j} + x\bar{k}).(dx\bar{i} + dy\bar{j} + dz\bar{k}) = ydx + zdy + xdz$

$\int_C (ydx + zdy + xdz) = \int ydx \ \ (in\ xy\ plane\ z = 0, dz = 0)$

Let $x = cos\theta, y = sin\theta \Longrightarrow dx = -sin\theta\ d\theta, dy = cos\theta\ d\theta$

$\therefore \int_C \bar{F}.d\bar{r} = \int_C y.dx = \int_0^{2\pi} ydx \qquad [\because x^2 + y^2 = 1, z = 0]$

$$= \int_0^{2\pi} sin\theta\,(-sin\theta)d\theta = -4\int_0^{\pi/2} sin^2\theta\, d\theta$$

$$= -4\int_0^{\pi/2} \frac{1-cos2\theta}{2} d\theta = -4\left[\left(\frac{1}{2}.\frac{\pi}{2}\right) - \frac{1}{4}(sin\pi)\right]$$

$$= -4\left[\left(\frac{1}{2}.\frac{\pi}{2}\right) - 0\right] = -4\left[\frac{\pi}{4}\right] = -\pi \qquad \text{.....(1)}$$

$$Curl\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = -(\bar{i} + \bar{j} + \bar{k})$$

Unit normal vector $\bar{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\bar{i}+2y\bar{j}+2z\bar{k}}{\sqrt{4x^2+4y^2+4z^2}} = x\bar{i} + y\bar{j} + z\bar{k}$

Substituting the spherical polar coordinates, we get

$\bar{n} = sin\theta\cos\phi\,\bar{i} + sin\theta\sin\phi\,\bar{j} + cos\theta\,\bar{k}$

$\therefore Curl\ \bar{F}.\bar{n} = -(sin\theta\cos\phi + sin\theta\sin\phi + cos\theta)$

$$\int\int curl\bar{F}.\bar{n}ds = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} (\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta)\sin\theta d\theta d\phi$$

$$= -\int_0^{\pi/2} [sin\theta\sin\phi - sin\theta\cos\phi + \phi cos\theta]_0^{2\pi}\ sin\theta d\theta$$

$$= -2\pi\int_0^{\pi/2} cos\theta sin\theta d\theta = -\pi\int_0^{\pi/2} sin2\theta d\theta = (-\pi)\left[\frac{-cos2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{\pi}{2}(-1-1) = -\pi \qquad \text{.....(2)}$$

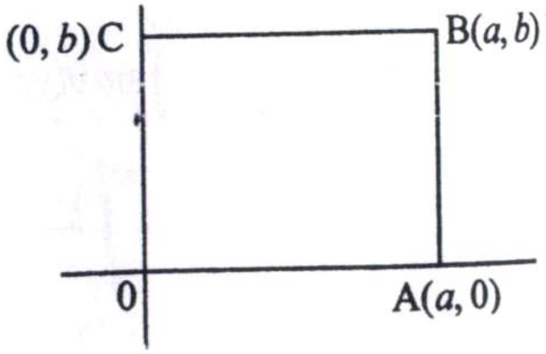
From (1) and (2), we have

$\int_C \bar{F}.d\bar{r} = \int\int_s Curl\,\bar{F}.\bar{n}ds \;= -\pi$

$\therefore$ *Stoke*'s theorem is verified.

16: Verify Stoke's theorem for $\bar{F} = \left(x^2 - y^2\right)\bar{i} + 2xy\bar{j}$ over the box bounded by the planes x=0,x=a,y=0,y=b.

Solution :



Stoke"s theorem states that $\int_c \bar{F}.d\bar{r} = \int_s Curl\bar{F}.\bar{n}ds$

Given $\bar{F} = (x^2 - y^2)\bar{\imath} + 2xy\bar{\jmath}$

Curl$\bar{F}$= $\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = \bar{i}(0,0) - \bar{j}(0,0) + \bar{k}(2y + 2y) = 4y\bar{k}$

R.H.S= $\int_s Curl\bar{F}.\bar{n}ds = \int_s 4y\left(\bar{k}.\bar{n}\right)ds$

Let R be the region bounded by the rectangle

$(\bar{k}.\bar{n})ds = dx\,dy$

$\int_s Curl\bar{F}.\bar{n}ds = \int_{x=0}^{a}\int_{y=0}^{b} 4ydxdy = \int_{x=0}^{a}\left[4\frac{y^2}{2}\right]_0^b dx = 2b^2\int_{x=0}^{a} 1dx$

$=2b^2(x)_0^a = 2ab^2$

To Calculate L.H.S

$\bar{F}.d\bar{r} = (x^2 - y^2)dx + 2xy\,dy$

Let O=$(0{,}0), A = (a, 0), B = (a, b)$ and

C=(0,b) are the vertices of the rectangle.

(i)Along the line OA

y=0; dy=0, x ranges from 0 to a.

$\int_{OA} \bar{F}.d\bar{r} = \int_{x=0}^{a} x^2dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$

(ii)Along the line AB

x=a; dx=0, y ranges from 0 to b.

$$\int_{AB} \bar{F}.d\bar{r} = \int_{y=0}^{b}(2xy)\,dy = \left[2a\,\frac{y^3}{2}\right]_0^b = ab^2$$

(iii)Along the line BC

y=b; dy=0, x ranges from a to 0

$$\int_{BC} \bar{F}.d\bar{r} = \int_{x=a}^{0}(x^2-y^2)dx = \left[\frac{x^3}{3}-b^2x\right]_a^0 = 0-\left(\frac{a^3}{3}-b^2a\right)$$

$$=ab^2 - \frac{a^3}{3}$$

(iv) Along the line CO

x=0,dx=0,y changes from b to 0

$$\int_C \bar{F}\,.\,d\bar{r} = \int_{y=b}^{0} 2xydy = 0$$

Adding these four values

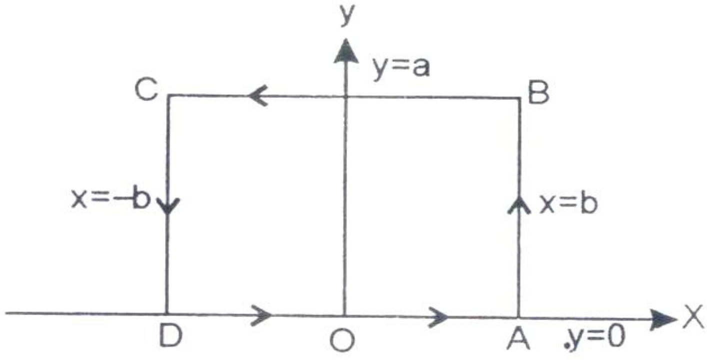
$$\int_{CO} \bar{F}\,.\,d\bar{r} = \frac{a^3}{3} + ab^2 + ab^2 - \frac{a^3}{3} = 2ab^2$$

L.H.S = R.H.S

Hence the verification of the stoke's theorem.

17: Verify Stoke's theorem for $\bar{F}=y^2\,\bar{\imath} - 2xy\bar{\jmath}$ taken round the rectangle bounded by x=$\pm b$, y=0,y=a.

<u>Solution:</u>



$$\text{Curl } \bar{A} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & -2xy & 0 \end{vmatrix} = -4y\bar{k}$$

For the given surface S, $\bar{n} = \bar{k}$

$\therefore (Curl\ \bar{F}).\bar{n} = -4y$

Now $\iint_s (Curl\ \bar{F}).\bar{n} dS = \iint_s -4y dx dy$

$$= \int_{y=0}^{a}\left[\int_{x=-b}^{b} -4y dx\right] dy$$

$$= \int_{0}^{a} [-4xy]_{-b}^{b}\, dy$$

$$= \int_0^a -8by dy = \left[-4by^2\right]_0^a = -4a^2b........(1)$$

$\int_c \bar{F}.d\bar{r} = \int_{DA} + \int_{AB} + \int_{BC} + \int_{CD}$

$\int \bar{F}.d\bar{r} = y^2\, dx - 2xydy$

Along DA , y=0,dy=0 $\Rightarrow \int_{DA} \bar{F}.d\bar{r} = 0$ ($\because \bar{F}.dr = 0$)

Along AB, x=b,dx=0

$\int_{AB} \bar{F}.d\bar{r} = \int_{y=0}^{a} -2by dy = \left[-by^2\right]_0^a = -a^2b$

Along BC,y=a,dy=0

$\int_{BC} \bar{F}.d\bar{r} = \int_b^{-b} a^2\, dx = -2a^2b$

Along CD, x=-b,dx=0

$\int_{CD} \bar{F}.d\bar{r} = \int_a^0 2by dy = \left[-by^2\right]_a^0 = -a^2b$.

$\int_C \bar{F}.d\bar{r} = 0 - a^2b - 2a^2b - a^2b = -4a^2b$ -------(2)

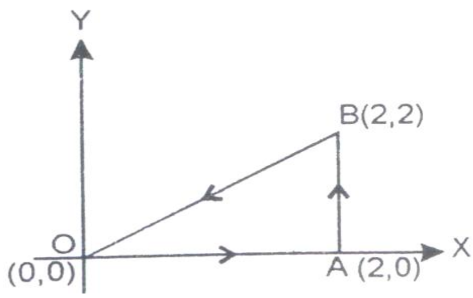
From (1),(2) $\int_C \bar{F}.d\bar{r} = \iint_s (Curl\ \bar{F}).\bar{n} dS$

Hence the theorem is verified.

19: Using Stroke's theorem evaluate the integral $\int_C \bar{F}.d\bar{r}$ where $\bar{F} = 2y^2\,\bar{i} + 3x^2\,\bar{j} - (2x+z)\bar{k}$ and C is the boundary of the triangle whose vertices are (0,0,0),(2,0,0),(2,2,0).

Solution:

$$Curl\ \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y^2 & 3x^2 & -2x - z \end{vmatrix} = 2\bar{j} + (6x-4y)\,\bar{k}$$



Since the z-coordinate of each vertex of the triangle is zero , the triangle lies in the xy-plane .

$\therefore \bar{n}$ =k

$\therefore$ (Curl $\bar{F}$).$\bar{n}$= 6x-4y

Consider the triangle in xy-plane .

Equation of the straight line OB is y=x.

By Stroke's theorem

$$\int_c \overline{F}.d\overline{r} = \iint_s (curl\overline{F}).\overline{n}ds$$

$$=\int_{x=0}^{2} \int_{y=0}^{y=x} (6x-4y)\,dx\,dy \qquad = \int_{x=0}^{2} \left[\int_{y=0}^{x} (6x-4y)\,dy\right] dx$$

$$= \int_{x=0}^{2} \left[6xy - 2y^2\right]_0^x dx = \int_0^2 (6x^2 - 2x^2)\,dx$$

$$= 4\left[\frac{x^3}{3}\right]_0^2 = \frac{32}{3}$$

UNIT-V

1. THE GAMMA FUNCTION

The gamma function may be regarded as a generalization of n! (n-factorial), where n is any positive integer to x!, where x is any real number. (With limited exceptions, the discussion that follows will be restricted to positive real numbers.) Such an extension does not seem reasonable, yet, in certain ways, the gamma function defined by the improper integral

$$\Gamma(x) = \int_0^\infty t^{x-1}\, e^{-t}\, dt \qquad (1)$$

meets the challenge. This integral has proved valuable in applications. However, because it cannot be r presented through elementary functions, establishment of its properties take some effort. Some of the important ones are outlined below.

The gamma function is convergent for x > 0. It follows from eq.(1) that

From (1): $\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt$

Integrating by parts

$$\Gamma(x+1) = \left[t^x\left(\frac{e^{-t}}{-1}\right)\right]_0^\infty + x\int_0^\infty e^{-t}t^{x-1}\,dt$$

$$= \{0-0\} + x\Gamma(x)$$

$$\therefore\ \Gamma(x+1) = x\Gamma(x) \qquad (2)$$

This is a fundamental recurrence relation for gamma functions. It can also be written as

$$\Gamma(x) = (x-1)\Gamma(x-1).$$

A number of other results can be derived from this as follows: If $x = n$, a positive integer, i.e. if $n \geq 1$, then

$$\begin{aligned}
\Gamma(n+1) &= n\Gamma(n).\\
&= n(n-1)\Gamma(n-1) \quad \text{since } \Gamma(n) = (n-1)\Gamma(n-1)\\
&= n(n-1)(n-2)\Gamma(n-2) \quad \text{since } \Gamma(n-1) = (n-2)\Gamma(n-2)\\
&= \ldots\ldots\ldots\ldots\ldots\ldots\\
&= n(n-1)(n-2)(n-3)\ldots 1\Gamma(1)\\
&= n!\Gamma(1)
\end{aligned}$$

But $\Gamma(1) = \int_0^\infty t^0 e^{-t} dt = [-e^{-t}]_0^\infty = 1$

$$\Rightarrow \quad \Gamma(n+1) = n! \qquad (3)$$

Example:

$\Gamma(7) = 6! = 720, \quad \Gamma(8) = 7! = 5040, \quad \Gamma(9) = 40320$

We can also use the recurrence relation in reverse

$\Gamma(x+1) = x\Gamma(x) \quad \Rightarrow \quad \Gamma(x) = \frac{\Gamma(x+1)}{x}$

What happens when $x = \frac{1}{2}$? We will investigate.

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^{\infty} t^{-1/2} e^{-t}\, dt$$

Putting $t = u^2$, $dt = 2u\, du$, then

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^{\infty} u^{-1} e^{-u^2} 2u\, du = 2\int_0^{\infty} e^{-u^2}\, du.$$

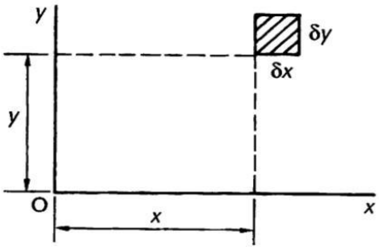
Unfortunately, $\int_0^{\infty} e^{-u^2}\, du$ cannot easily be determined by normal means. It is, however, important, so we have to find a way of getting round the difficulty.

Evaluation of $\int_0^{\infty} e^{-x^2} dx$

Let $I = \int_0^{\infty} e^{-x^2} dx$, then also $I = \int_0^{\infty} e^{-y^2} dy$

$$\therefore\ I^2 = \left(\int_0^{\infty} e^{-x^2} dx\right)\left(\int_0^{\infty} e^{-y^2} dy\right) = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, dx\, dy$$

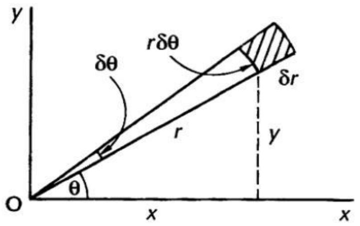
$\delta a = \delta x\, \delta y$ represents an element of area in the x–y plane and the integration with the stated limits covers the whole of the first quadrant.



Converting to polar coordinates, the element of area $\delta a = r\, \delta\theta\, \delta r$. Also, $x^2 + y^2 = r^2$

$$\therefore\ e^{-(x^2+y^2)} = e^{-r^2}$$

For the integration to cover the same region as before,



the limits of r are $r = 0$ to $r = \infty$
the limits of θ are $\theta = 0$ to $\theta = \pi/2$.

$$\therefore\ I^2 = \int_0^{\pi/2}\int_0^{\infty} e^{-r^2} r\, dr\, d\theta = \int_0^{\pi/2}\left[-\frac{e^{-r^2}}{2}\right]_0^{\infty} d\theta$$

$$= \int_0^{\pi/2}\left(\frac{1}{2}\right) d\theta = \left[\frac{\theta}{2}\right]_0^{\pi/2} = \frac{\pi}{4}$$

$$\therefore\ I = \frac{\sqrt{\pi}}{2}$$

$$\therefore\ \int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2} \qquad (5)$$

Before that diversion, we had established that

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^{\infty} e^{-u^2}\,du$$

We now know that $\int_0^{\infty} e^{-u^2}\,du = \dfrac{\sqrt{\pi}}{2}$ $\quad\therefore\ \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$

From this, using the recurrence relation $\Gamma(x+1) = x\Gamma(x)$, we can obtain the following

$$\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\ \Gamma\left(\tfrac{1}{2}\right) = \tfrac{1}{2}(\sqrt{\pi}) \qquad \therefore\ \Gamma\left(\tfrac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{2}\ \Gamma\left(\tfrac{3}{2}\right) = \tfrac{3}{2}\left(\frac{\sqrt{\pi}}{2}\right) \qquad \therefore\ \Gamma\left(\tfrac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}$$

Negative values of x

Since $\Gamma(x) = \dfrac{\Gamma(x+1)}{x}$, then as $x \to 0$, $\Gamma(x) \to \infty$ $\ \therefore\ \Gamma(0) = \infty$.

The same result occurs for all negative integral values of x – which does not follow from the original definition, but which is obtainable from the recurrence relation.

Because at $\quad x = -1, \qquad \Gamma(-1) = \dfrac{\Gamma(0)}{-1} = \infty$

$\qquad\qquad\qquad x = -2, \qquad \Gamma(-2) = \dfrac{\Gamma(-1)}{-2} = \infty$ etc.

Also, at $\quad x = -\tfrac{1}{2}, \qquad \Gamma\left(-\tfrac{1}{2}\right) = \dfrac{\Gamma\left(\tfrac{1}{2}\right)}{-\tfrac{1}{2}} = -2\sqrt{\pi}$

and at $\quad x = -\tfrac{3}{2}, \qquad \Gamma\left(-\tfrac{3}{2}\right) = \dfrac{\Gamma\left(-\tfrac{1}{2}\right)}{-\tfrac{3}{2}} = \dfrac{4}{3}\sqrt{\pi}$

So we have

(a) For n a positive integer

$\Gamma(n+1) = n\Gamma(n) = n!$

$\Gamma(1) = 1; \quad \Gamma(0) = \infty; \quad \Gamma(-n) = \pm\infty$

(b) $\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}; \qquad \Gamma\left(-\tfrac{1}{2}\right) = -2\sqrt{\pi}$

$\Gamma\left(\tfrac{3}{2}\right) = \dfrac{\sqrt{\pi}}{2}; \qquad \Gamma\left(-\tfrac{3}{2}\right) = \dfrac{4}{3}\sqrt{\pi}$

$\Gamma\left(\tfrac{5}{2}\right) = \dfrac{3\sqrt{\pi}}{4}; \qquad \Gamma\left(-\tfrac{5}{2}\right) = -\dfrac{8}{15}\sqrt{\pi}$

$\Gamma\left(\tfrac{7}{2}\right) = \dfrac{15\sqrt{\pi}}{8}; \qquad \Gamma\left(-\tfrac{7}{2}\right) = \dfrac{16}{105}\sqrt{\pi}$

Example:

Evaluate $\int_0^\infty x^7 e^{-x} dx$.

We recognise this as the standard form of the gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \qquad \text{with the variables changed.}$$

It is often convenient to write the gamma function as

$$\Gamma(v) = \int_0^\infty x^{v-1} e^{-x} dx$$

Our example then becomes

$$I = \int_0^\infty x^7 e^{-x} dx = \int_0^\infty x^{v-1} e^{-x}\, dx \qquad \text{where } v = \ldots\ldots\ldots\ldots$$

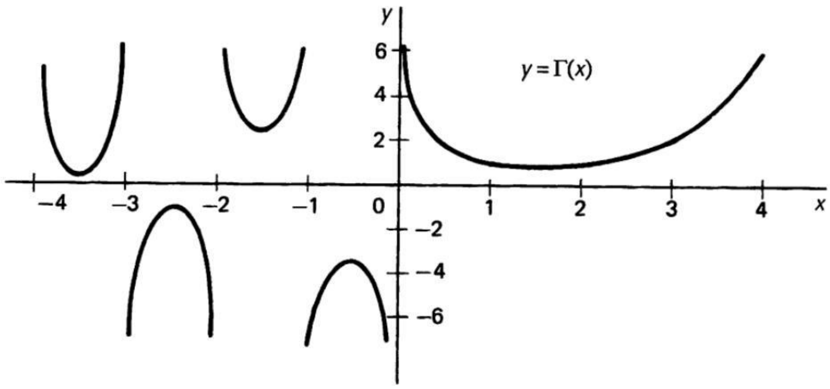
$$\text{i.e. } \int_0^\infty x^7\, e^{-x}\, dx = \Gamma(8) = 7! = 5040$$

Graph of $y = \Gamma(x)$

Values of $\Gamma(x)$ for a range of positive values of x are available in tabulated form in various sets of mathematical tables. These, together with the results established above, enable us to draw the graph of $y = \Gamma(x)$.

x	0	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0
$\Gamma(x)$	∞	1·772	1·000	0·886	1·000	1·329	2·000	3·323	6·000

x	−0·5	−1·5	−2·5	−3·5
$\Gamma(x)$	−3·545	2·363	−0·945	0·270



Example:

Evaluate $\int_0^\infty x^3 e^{-4x}\, dx$.

If we compare this with $\Gamma(v) = \int_0^\infty x^{v-1} e^{-x}\, dx$, we must reduce the power of e to a single variable, i.e. put $y = 4x$, and we use this substitution to convert the whole integral into the required form.

$$y = 4x \qquad \therefore\ dy = 4\,dx \qquad \text{Limits remain unchanged.}$$

The integral now becomes . . . .

$$\boxed{I = \int_0^\infty \left(\frac{y}{4}\right)^3 e^{-y} \frac{dy}{4}}$$

$$\therefore\ I = \frac{1}{4^4}\int_0^\infty y^3 e^{-y}\, dy = \frac{1}{4^4}\Gamma(v) \qquad \text{where } v = \ldots\ldots\ldots\ldots$$

$$\int_0^\infty y^{v-1} e^{-y} dy = \int_0^\infty y^3 e^{-y} dy \qquad \therefore\ v = 4$$

$$\therefore\ I = \frac{1}{4^4}\,\Gamma(4) = \ldots\ldots\ldots\ldots$$

$$I = \frac{1}{256}\Gamma(4) = \frac{1}{256}(3!) = \frac{6}{256} = \frac{3}{128}$$

Review of Properties of Power Series

A power series in (x-a) is an infinite series of the form

$$c_0 + c_1 (x-a) + c_2 (x-a)^2 + \text{- - - -} = \sum_{n=o}^{\infty} c_n (x-a)^n \qquad (6.1)$$

Series of (6.1) is also called a **power series centered** at a. The power series centered at a=0 is often referred as the **power series,** that is, the series $\sum_{n=o}^{\infty} c_n x^n$ A power series centered at a is called **convergent** at a specified value of x if its sequence of partial sums $S_N(x) = \sum_{n=o}^{N} c_n (x-a)^n$, that is, $\{S_N(x)\}$ is convergent. In other words the limit of $\{S_N(x)\}$ exists. If the limit does not exist the power series is called **divergent.** The set of points x at which the power series is convergent is called the **interval of convergence** of the power series**.** For R

>o, a power series $\sum_{n=o}^{\infty} c_n(x-a)^n$ converges if $|x-a|$<R and diverges if $|x-a|$>R. If the series converges only at a then R=0, and if it converges for all x then R=∞. $|x-a|$<R is equivalent to a-R<x<a+R. A power series may or may not converge at the end points a-R and a+R of this interval.

A power series is called absolutely convergent if the series $\sum_{n=o}^{\infty} |c_n(x-a)^n|$ converges. A power series converges absolutely within its interval of convergence. By the Ratio test a power series centered at a, series given in (6.1) is absolutely convergent if L= |x-a| $\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|$ is less than 1, that is, L <1, the series diverges if L>1, and test fails if L=1. A power series defines a function f(x)= $\sum_{n=o}^{\infty} c_n(x-a)^n$ whose domain is the interval of convergence of the series. If the radius of convergence R>o, then f is continuous, differentiable and integrable on the interval (a-R, a+R). Moreover f'(x) and ∫f(x)dx can be found by term by term differentiation and integration. Convergence at an endpoint may be either lost by differentiation or gained through integration.

Let y = $\sum_{n=o}^{\infty} c_n x^n$

$$y' = \sum_{n=o}^{\infty} nc_n x^{n-1}$$

$$y'' = \sum_{n=o}^{\infty} n(n-1)c_n x^{n-2}$$

We observe that the first term in y' and first two terms in y' are zero. Keeping this in mind we can write

$$\left.\begin{aligned} y' &= \sum_{n=1}^{\infty} nc_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \end{aligned}\right\} \quad (6.2)$$

Identity property: If $\sum_{n=o}^{\infty} c_n(x-a)^n$ =0, R>o for all x in the interval of convergence, then c_n=0 for all n.

Analytic at a point:

A function f is analytic at a point a if it can be represented by a power series in x-a with a positive or infinite radius of convergence. A power series where $c_n = \frac{f^{(n)}(a)}{n!}$, that is, the series of the type $\sum_{n=o}^{\infty} c_n \frac{f^{(n)}(a)}{n!}(x-a)^n$ is called the Taylor series. If a=o then Taylor series is called Maclaurin series. In calculus it is shown that e^x, cos x, sin x, ln (x-1) can be written in the form of a power series more precisely in the form of Maclaurin series. For example

$$e^x = 1 + x + \frac{x^2}{2!} + ----$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} ----$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ----$$

for $|x| < \infty$.

Arithmetic of Power Series:

Two power series can be combined through the operation of addition, multiplication, and division. The procedures for power series are similar to those by which two polynomials are added, multiplied, and divided. For example:

$$e^x \sin x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + ----\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + ----\right)$$

$$= (1)x + x^2 + \left(-\frac{1}{6} + \frac{1}{2}\right)x^3 + \left(-\frac{1}{6} + \frac{1}{6}\right)x^4 + ----\left(\frac{1}{120} - \frac{1}{12} + \frac{1}{24}\right)x^5 + ----$$

$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - ----$$

Since the power series for e^x and sin x converge for $|x| < \infty$, the product series converges on the same interval.

Shifting the Summation Index: In order to discuss power series solutions of differential equations it is advisable to learn combining two or more summations as a single summation.

6.2 Solution about Ordinary Point :

We look for power series solution of linear second-order differential equation about a special point:

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad (6.4)$$

where $a_2(x) \neq 0$.

This can be put into the standard form

$$\frac{d^2y}{dx^2}+\frac{a_1(x)}{a_2(x)}\frac{dy}{dx}+\frac{a_0(x)}{a_2(x)}y=0$$

or $$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0 \qquad (6.5)$$

A point x_o is said to be an **ordinary point** of the differential equation (6.4) if P(x) and Q (x) of (6.5) are analytic at x_o, that is, P(x) and are Q(x) represented by a power series. A point that is not an ordinary point is called a singular point.

A solution of the form y = $\sum_{n=0}^{\infty} c_n(x-x_o)^n$ is said to be a **solution about the ordinary point x_0.**

Power series solution about an ordinary point:

Let y= $\sum_{n=0}^{\infty} c_n x^n$ and substitute values of y, $\frac{dy}{dx}=y', \frac{dy^2}{dx^2}=y''$ in (6.5)

Combine series as in Example 6.1, and then equate all coefficients to the right hand side of the equation to determine the coefficients c_n. We illustrate the method by the following examples. We also see through these examples how the single assumption that y= $\sum_{n=0}^{\infty} c_n x^n$ leads to two sets of coefficients, so we have two distinct power series y_1 (x) and y_2(x) both expanded about the ordinary point x=0. The general solution of the differential equation is $y=C_1y_1(x)+C_2y_2(x)$, infact it can been shown that $C_1=c_o$ and $C_2=c_1$.

The differential equation $\frac{d^2y}{dx^2}+xy=0$ is known as Airy's equation and used in the study of diffraction of light, diffraction of radio waves around the surface of the earth, aerodynamics etc. We discuss here power series solution of this equation around its ordinary point x=0.

Example 6.2 Write the general solution of Airy's equation y'+xy=0.

Solution: In view of the remark, two power series solutions centred at 0, convergent for $|x|<\infty$ exist. By substituting y= $\sum_{n=o}^{\infty} c_n x^n$, $y''=\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$ into Airy's differential equation we get

y''+xy= $\sum_{n=2}^{\infty} c_n n\ (n-1)x^{n-2}+x\sum_{n=0}^{\infty} c_n x^n$,

$$= \sum_{n=2}^{\infty} c_n n\ (n-1)x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} \qquad (6.6)$$

As seen in the solution of Example 6.1, (6.6) can be written as $y''+xy=2c_2+\sum_{k=1}^{\infty}$ [(k+1)

$(k+2)c_{k+2}+c_{k-1}]x^k=0$ (6.7)

Since (6.7) is identically zero, it is necessary that coefficient of each power of x be set equal to zero, that is,

$2c_2=0$ (It is the coefficient y x^0) and

$(k+1)(k+2)\ c_{k+2}+c_{k-1}=0$, k=1,2,3 - - - -- - - -.. (6.8)

The above holds in view of the identity property. It is clear that $c_2=0$. The expression in (6.8) is called a recurrence relation and it determines the c_k in such a manner that we can choose a certain subset of the set of coefficients to be non-zero. Since $(k+1)(k+2)\neq 0$ for all values of k, we can solve (6.8) for c_{k+2} in terms of c_{k-1}.

$$c_{k+2} = -\frac{c_{k-1}}{(k+1)(k+2)}, k = 1,2,3, - - - - \qquad (6.9)$$

For k=1, $c_3 = -\frac{c_o}{2.3}$

For k = 2, $c_4 = -\frac{c_o}{3.4}$

For k= 3, $c_5 = -\frac{c_2}{4.5} = 0$ as $c_2=0$

For k= 4, $c_6 = -\frac{c_3}{5.6} = \frac{1}{2.3.5.6.}c_0$

For k= 5, $c_7 = -\frac{-c_4}{6.7} = \frac{1}{3.4.6.7}c_1$

For k= 6. $c_8 = -\frac{-c_5}{7.8} = 0$ as $c_5=0$

For k= 7. $c_9 = -\frac{c_6}{8.9} = -\frac{1}{2.3.5.6.8.9.}c_0$

For k = 8, $c_{10} = -\frac{c_7}{9.10} = -\frac{1}{3.4.6.7.8.10}c_1$

For k = 9, $c_{11} = -\frac{c_8}{10.11} = 0$ as $c_8=0$

and so on,

Substituting the coefficients just obtained into $y=\sum_{n=0}^{\infty} c_n x^n$

$=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5+c_6x^6+c_7x^7+c_8x^8+c_9x^9+c_{10}x^{10}- - - -$

we get

$y=c_0+c_1x+0$

$$-\frac{c_o}{2.3}x^3-\frac{c_1}{3.4}x^4+0+\frac{c_0}{2.3.5.6}x^6+\frac{c_1}{3.4.6.7}x^7+0-\frac{c_0}{2.3.5.6.8.9.}x^9-\frac{c_1}{3.4.6.7.9.10}x^{10}+0+- - - -$$

After grouping the terms containing c_o and the terms containing c_1, we obtain $y=c_0y_1(x)+c_1y_2(x)$, where

$y_1(x)=1-\frac{1}{2.3}x^3+\frac{1}{2.3.5.6}x^6-\frac{1}{2.3.5.6.8.9}x^9+----$

$= 1+\sum_{k=1}^{\infty}\frac{(-1)^k}{2.3----(3k-1)(3k)}x^{3k}$

$y_2(x) = x-\frac{1}{3.4}x^4+\frac{1}{3.4.6.7}x^7-\frac{1}{3.4.6.7.9.10}x^{10}+----$

$= x+\sum_{k=1}^{\infty}\frac{(-1)^k}{3.4----(3k)(3k+1)}x^{3k+1}$

Since the recursive use of (6.9) leaves c_0 and c_1 completely undetermined, they can be chosen arbitrarily.

$y=c_0y_1(x)+c_1y_2(x)$ is the general solution of the Airy's equation.

Example 6.3 : Find two power series solutions of the differential equation y''-xy=0 about the ordinary point x=0.

Solution: Substituting $y=\sum_{n=0}^{\infty}c_n x^n$ into the differential equation we get

$y''-xy=\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}-\sum_{n=0}^{\infty}c_n x^{n+1}$

$=\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k-\sum_{k=1}^{\infty}c_{k-1}x^k$

$= 2c_2+\sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2}-c_{k-1}]x^k$

Thus $c_2 = 0$,

$(k+2)(k+1)c_{k+2}-c_{k-1}= 0$

and

$c_{k+2}=\frac{1}{(k+2)(k+1)}c_{k-1}, k=1,2,3....$

Choosing $c_o= 1$ and $c_1=0$ we find

$c_3=\frac{1}{6}, c_4=c_5=0, c_6=\frac{1}{180}$ and so on.

For $c_0=0$ and $c_1=1$ we obtain

$c_3=0, c_4=\frac{1}{12}, c_5=c_6=0,, c_7=\frac{1}{504}$ and so on. Thus two solutions are

$y_1 = 1+\frac{1}{6}x^3+\frac{1}{180}x^6+----$ and

$y_2=x+\frac{1}{12}x^4+\frac{1}{504}x^7+----$

6.3 Solutions about Regular Singular Points – The Method of Frobenius:

A singular point x_0 of (6.4) is called a **regular singular point** of this equation if the functions $p(x) = (x-x_o) P(x)$ and $q(x)=(x-x_o)^2Q(x)$ are both analytic at x_0. A singular point that is not regular is said to be on **irregular singular point** of the equation. This means that one or both of the functions $p(x)=(x-x_0) P(x)$ and $q(x) = (x-x_0)^2Q(x)$ fail to be analytic at x_0.

In order to solve a differential equation given by (6.4) about a regular singular point we employ the following theorem due to Frobenius.

Theorem 6.1 (Frobenius Theorem)

If $x=x_0$ is a regular singular point of the differential equation (6.4), then there exists at least one solution of the form $y=(x-x_o)^r \sum_{n=o}^{\infty} c_n(x-x_o)^n = \sum_{n=0}^{\infty} c_n (x-x_o)^{n+r}$ where r is constant to be determined. The series will converge at least on some interval $0<x-x_0<R$.

The method of Frobenius:

Finding series solutions about a regular singular point x_0, is similar to the method of previous section in which we substitute $y= \sum_{n=o}^{\infty} c_n(x-x_o)^{n+r}$ into the given differential equation and determine the unknown coefficients c_n by a recurrence relation. However, we have an additional task in this procedure. Before determining coefficients we must find unknown exponent r. Equate to 0 the coefficient of the lowest power of x. This equation is called the **indicial equation** and determines the value(s) of the index r.

If r is found to be number that is not a non negative integer, then the corresponding solution $y= \sum_{n=o}^{\infty} c_n(x-x_o)^{n+r}$ is not a power series. For the sake of simplicity we assume that the regular singular point is x=0.

Example 6.4 Apply the Method of Frobenius to solve the differential equation 2x y''+3y'-y=0 about the regular singular point x=0.

Solution: Let us assume that the solution is of the form

$$y=\sum_{n=o}^{\infty} c_n x^{n+r} \text{ then}$$

$$y' = \sum_{n=o}^{\infty} c_n (n+r)x^{n+r-1}$$

$$y''= \sum_{n=o}^{\infty} c_n (n+r)(n+r-1)x^{n+r-2},$$

Substituting these values of y', y' and y'' into 2x y''+3 y'-y=0, we get

$$\sum_{n=o}^{\infty} c_n (n+r)(n+r-1)x^{n+r-1} + 3\sum_{n=o}^{\infty} c_n (n+r)x^{n+r-1} - \sum_{n=o}^{\infty} c_n x^{n+r} = 0.$$

Shifting the index in the third series and combing the first two yields $\sum_{n=o}^{\infty} c_n (n+r)$

$$(2n+2r+1)x^{n+r-1} - \sum_{n=o}^{\infty} c_{n-1} x^{n+r-1} = 0$$

Writing the term corresponding to n=0 and combining the terms for n≥/ into one series,

$$c_o r(2r+1)x^{r-1} + \sum_{n=1}^{\infty}[c_n (n+r)(2n+2r+1)-c_{n-1}]x^{n+r-1} = 0$$

Equating the coefficients of x^{r-1} to zero yields the indicial equation

$c_0 r(2r+1)=0$

Since $c_0 \neq 0$, either r=0 or = $-\frac{1}{2}$

Hence two linearly independent solutions of the given differential equation have the form

$$y_1 = F_0(x) = \sum_{n=o}^{\infty} c_n x^n \quad \text{and}$$

$$y_2 = F_{-1/2}(x) = x^{-1/2} \sum_{n=o}^{\infty} c^*{}_n x^n$$

Since $c_n(n+r)(2n+2r+1) - c_{n-1}=0$ for all n ≥ 1, we have the following information on the coefficients for the two series:

(i) c_o is arbitrary, and for n≥1, $c_n = \frac{1}{n(2n+1)} c_{n-1}$

(ii) $c^*{}_o$ is arbitrary, and for n≥1, $c_n^* = \frac{1}{n(2n-1)} c_{n-1}{}^*$

Iteration of the formula for c_n yields

n=1, $c_1 = \frac{1}{1.3} c_0 = \frac{2}{1.2.3} c_0 = \frac{2c_0}{3!}$

n= 2, $c_2 = \frac{1}{2.5} c_1 = \frac{1}{2.3.5} c_0 = \frac{2^2 c_0}{5!}$

n= 3, $c_3 = \frac{1}{3.7} c_2 = \frac{1}{3.7}\frac{2^2 c_0}{5!} = \frac{2^3 c_0}{7!}$

Each term of c_n was multiplied by $\frac{2}{2}$ to make the denominator (2n+1)!. The general form of c_n is then

$$c_n = \frac{2^n c_0}{(2n+1)!}$$

Similarly, the general form of c_n^* is found to be $c_n^* = \frac{2^n c_0}{(2n)!}$.

The two solutions are

$$y_1 = c_o \sum_{n=o}^{\infty} \frac{2^n}{(2n+1)!} x^n, y_2 = c_o^* x^{-1/2} \sum_{n=o}^{\infty} \frac{2^n}{(2n)!} x^n$$

y_2 is not a power series.

Example 6.5 Apply the method of Frobenius to obtain two linearly independent series solution of the differential equation $2x\ y'' - y' + 2y = 0$ about a regular singular point x=0 of the differential equation.

Solution: Substituting $y = \sum_{n=o}^{\infty} c_n x^{n+r}$, $y' = \sum_{n=o}^{\infty} c_n (n+r) x^{n+r-1}$ and

$$y'' = \sum_{n=o}^{\infty} c_n (n+r)(n+r-1) x^{n+r-1}$$

into the differential equation and collecting terms, we obtain

$$2x\ y'' - y' + 2y = (2r^2 - 3r) c_0 x^{r-1} + \sum_{k=1}^{\infty} [2(k+r-1)(k+r)c_k - (k+r)c_k + 2c_{k-1}] x^{k+r-1} = 0,$$

which implies that

$$2r^2 - 3r = r(2r-3) = 0$$

and

$$(k+r)(2k+2r-3)c_k + 2c_{k-1} = 0.$$

The indicial roots are r=0 and $r = \frac{3}{2}$. For r=0 the recurrence relation is $c_k = -\frac{2c_{k-1}}{k(2k-3)}$, k= 1,2,3, - - - -

and $c_1 = 2c_0$, $c_2 = -2c_0$, $c_3 = \frac{4}{9}c_0$

For $r = \frac{3}{2}$ the recurrence relation is $c_k = -\frac{2c_{k-1}}{(2k+3)k}$, k=1,2,3,- - - - and

$c_1 = -\frac{2}{5}c_0, c_2 = \frac{2}{35}c_0, c_3 = -\frac{4}{945}c_0.$

The general solution is $y = C_1\left(1+2x-2x^2+\frac{4}{9}x^3+ - - - -\right)+C_2 x^{3/2}\left(1-\frac{2}{5}+\frac{2}{35}x^2-\frac{4}{945}x^3+ - - - -\right)$

SPECIAL FUNCTIONS

Introduction

Many Differential equations arising from physical problems are linear but have variable coefficients and do not permit a general analytical solution in terms of known functions. Such equations can be solved by numerical methods (Unit – I), but in many cases it is easier to find a solution in the form of an infinite convergent series. The series solution of certain differential equations give rise to special functions such as Bessel’s function, Legendre’s polynomial. These special functions have many applications in engineering.

Series solution of the Bessel Differential Equation

Consider the Bessel Differential equation of order n in the form

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0 \qquad \text{(i)}$$

where n is a non negative real constant or parameter.

We assume the series solution of (i) in the form

$$y=\sum_{r=0}^{\infty}a_r x^{k+r} \quad \text{where } a_0 \neq 0 \qquad \text{(ii)}$$

Hence,

$$\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(k+r)x^{k+r-1}$$

$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(k+r)(k+r-1)x^{k+r-2}$$

Substituting these in (i) we get,

$$x^2\sum_{r=0}^{\infty}a_r(k+r)(k+r-1)x^{k+r-2}+x\sum_{r=0}^{\infty}a_r(k+r)x^{k+r-1}+\left(x^2-n^2\right)\sum_{r=0}^{\infty}a_r x^{k+r}=0$$

i.e., $\sum_{r=0}^{\infty}a_r(k+r)(k+r-1)x^{k+r}+\sum_{r=0}^{\infty}a_r(k+r)x^{k+r}+\sum_{r=0}^{\infty}a_r x^{k+r+2}-n^2\sum_{r=0}^{\infty}a_r x^{k+r}=0$

Grouping the like powers, we get

$$\sum_{r=0}^{\infty}a_r\left[(k+r)(k+r-1)+(k+r)-n^2\right]x^{k+r}+\sum_{r=0}^{\infty}a_r x^{k+r+2}=0$$

$$\sum_{r=0}^{\infty}a_r\left[(k+r)^2-n^2\right]x^{k+r}+\sum_{r=0}^{\infty}a_r x^{k+r+2}=0 \qquad \text{(iii)}$$

Now we shall equate the coefficient of various powers of x to zero
Equating the coefficient of x^k from the first term and equating it to zero, we get
$a_0\left[k^2-n^2\right]=0.$ Since $a_0\neq 0$, we get $k^2-n^2=0, \quad \therefore k=\pm n$
Coefficient of x^{k+1} is got by putting r = 1 in the first term and equating it to zero, we get
i.e., $a_1\left[(k+1)^2-n^2\right]=0.$ This gives $a_1=0$, since $(k+1)^2-n^2=0$ gives, $k+1=\pm n$
which is a contradiction to $k=\pm n$.

Let us consider the coefficient of x^{k+r} from (iii) and equate it to zero.
i.e, $a_r\left[(k+r)^2-n^2\right]+a_{r-2}=0.$

$$\therefore a_r=\frac{-a_{r-2}}{\left[(k+r)^2-n^2\right]} \qquad \text{(iv)}$$

If $k=+n$, (iv) becomes

$$a_r=\frac{-a_{r-2}}{\left[(n+r)^2-n^2\right]}=\frac{-a_{r-2}}{\left[r^2+2nr\right]}$$

Now putting r = 1,3,5, ….., (odd vales of n) we obtain,

$$a_3=\frac{-a_1}{6n+9}=0, \quad \because a_1=0$$

Similarly a_5, a_7, ….. are equal to zero.
i.e., $a_1=a_5=a_7=\ldots\ldots=0$
Now, putting r = 2,4,6, ……(even values of n) we get,

$$a_2=\frac{-a_0}{4n+4}=\frac{-a_0}{4(n+1)}; \qquad a_4=\frac{-a_2}{8n+16}=\frac{a_0}{32(n+1)(n+2)};$$

Similarly we can obtain a_6, a_8, …
We shall substitute the values of $a_1,a_2,a_3,a_4,\cdots\cdots$ in the assumed series solution, we get

$$y=\sum_{r=0}^{\infty}a_r x^{k+r}=x^k(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots\cdots)$$

Let y_1 be the solution for $k=+n$

$$\therefore \quad y_1 = x^n\left[a_0 - \frac{a_0}{4(n+1)}x^2 + \frac{a_0}{32(n+1)(n+2)}x^4 - \cdots\cdots\right]$$

$$i.e., \quad y_1 = a_0 x^n\left[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^5(n+1)(n+2)} - \cdots\cdots\right] \qquad \text{(v)}$$

This is a solution of the Bessel's equation.
Let y_2 be the solution corresponding to k = - n. Replacing n be – n in (v) we get

$$y_2 = a_0 x^{-n}\left[1 - \frac{x^2}{2^2(-n+1)} + \frac{x^4}{2^5(-n+1)(-n+2)} - \cdots\cdots\right] \qquad \text{(vi)}$$

The complete or general solution of the Bessel's differential equation is $y = c_1y_1 + c_2y_2$, where c_1, c_2 are arbitrary constants.
Now we will proceed to find the solution in terms of Bessel's function by choosing $a_0 = \frac{1}{2^n\Gamma(n+1)}$ and let us denote it as Y_1.

$$i.e., \quad Y_1 = \frac{x^n}{2^n\Gamma(n+1)}\left[1 - \left(\frac{x}{2}\right)^2\frac{1}{(n+1)} + \left(\frac{x}{2}\right)^4\frac{1}{(n+1)(n+2)\cdot 2} - \cdots\cdots\right]$$

$$= \left(\frac{x}{2}\right)^n\left[\frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2\frac{1}{(n+1)\Gamma(n+1)} + \left(\frac{x}{2}\right)^4\frac{1}{(n+1)(n+2)\Gamma(n+1)\cdot 2} - \cdots\cdots\right]$$

We have the result Γ(n) = (n – 1) Γ(n – 1) from Gamma function
Hence, Γ(n + 2) = (n + 1) Γ(n + 1) and
Γ(n + 3) = (n + 2) Γ(n + 2) = (n + 2) (n + 1) Γ(n + 1)
Using the above results in Y_1, we get

$$Y_1 = \left(\frac{x}{2}\right)^n\left[\frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2\frac{1}{\Gamma(n+2)} + \left(\frac{x}{2}\right)^4\frac{1}{\Gamma(n+3)\cdot 2} - \cdots\cdots\right]$$

which can be further put in the following form

$$Y_1 = \left(\frac{x}{2}\right)^n\left[\frac{(-1)^0}{\Gamma(n+1)\cdot 0!}\left(\frac{x}{2}\right)^0 + \frac{(-1)^1}{\Gamma(n+2)\cdot 1!}\left(\frac{x}{2}\right)^2 + \frac{(-1)^2}{\Gamma(n+3)\cdot 2!}\left(\frac{x}{2}\right)^4 + \cdots\cdots\right]$$

$$= \left(\frac{x}{2}\right)^n\sum_{r=0}^{\infty}\frac{(-1)^r}{\Gamma(n+r+1)\cdot r!}\left(\frac{x}{2}\right)^{2r}$$

$$= \sum_{r=0}^{\infty}(-1)^r\cdot\left(\frac{x}{2}\right)^{n+2r}\cdot\frac{1}{\Gamma(n+r+1)\cdot r!}$$

This function is called the Bessel function of the first kind of order n and is denoted by $J_n(x)$.

Thus $J_n(x) = \sum_{r=0}^{\infty}(-1)^r\cdot\left(\frac{x}{2}\right)^{n+2r}\cdot\frac{1}{\Gamma(n+r+1)\cdot r!}$

Further the particular solution for k = $-n$ (replacing n by –n) be denoted as $J_{-n}(x)$. Hence the general solution of the Bessel's equation is given by $y = AJ_n(x) + BJ_{-n}(x)$, where A and B are arbitrary constants.

Properties of Bessel's function

1. $J_{-n}(x) = (-1)^n J_n(x)$, where n is a positive integer.

Proof: By definition of Bessel's function, we have

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1) \cdot r!} \qquad \text{……….(1)}$$

Hence, $J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{-n+2r} \cdot \frac{1}{\Gamma(-n+r+1) \cdot r!}$ ……….(2)

But gamma function is defined only for a positive real number. Thus we write (2) in the following from

$$J_{-n}(x) = \sum_{r=n}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{-n+2r} \cdot \frac{1}{\Gamma(-n+r+1) \cdot r!} \qquad \text{………..(3)}$$

Let r – n = s or r = s + n. Then (3) becomes

$$J_{-n}(x) = \sum_{s=0}^{\infty} (-1)^{s+n} \cdot \left(\frac{x}{2}\right)^{-n+2s+2n} \cdot \frac{1}{\Gamma(s+1) \cdot (s+n)!}$$

We know that $\Gamma(s+1) = s!$ and $(s + n)! = \Gamma(s+n+1)$

$$= \sum_{s=0}^{\infty} (-1)^{s+n} \cdot \left(\frac{x}{2}\right)^{n+2s} \cdot \frac{1}{\Gamma(s+n+1) \cdot s!}$$

$$= (-1)^n \sum_{s=0}^{\infty} (-1)^{s} \cdot \left(\frac{x}{2}\right)^{n+2s} \cdot \frac{1}{\Gamma(s+n+1) \cdot s!}$$

Comparing the above summation with (1), we note that the RHS is $J_n(x)$.
Thus, $\boldsymbol{J_{-n}(x) = (-1)^n J_n(x)}$

2. $J_n(-x) = (-1)^n J_n(x) = J_{-n}(x)$, *where n is a positive integer*

Proof : By definition, $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1) \cdot r!}$

$$\therefore \qquad J_n(-x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(-\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1) \cdot r!}$$

i.e., $$= \sum_{r=0}^{\infty} (-1)^r \cdot (-1)^{n+2r} \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1) \cdot r!}$$

$$= (-1)^n \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1) \cdot r!}$$

Thus, $\boldsymbol{J_n(-x) = (-1)^n J_n(x)}$

Since, $(-1)^n J_n(x) = J_{-n}(x)$, we have $\boldsymbol{J_n(-x) = (-1)^n J_n(x) = J_{-n}(x)}$

Recurrence Relations:

Recurrence Relations are relations between Bessel's functions of different order.

Recurrence Relations 1: $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

From definition,

$$x^n J_n(x) = x^n \sum_{r=0}^{\infty}(-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1)\cdot r!} = \sum_{r=0}^{\infty}(-1)^r \cdot \left(\frac{x}{2}\right)^{2(n+r)} \cdot \frac{1}{\Gamma(n+r+1)\cdot r!}$$

$$\therefore \quad \frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{r=0}^{\infty}(-1)^r \cdot \frac{2(n+r)x^{2(n+r)-1}}{2^{n+2r}\Gamma(n+r+1)\cdot r!}$$

$$= x^n \sum_{r=0}^{\infty}(-1)^r \cdot \frac{(n+r)x^{n+2r-1}}{2^{n+2r-1}(n+r)\Gamma(n+r)\cdot r!}$$

$$= x^n \sum_{r=0}^{\infty}(-1)^r \cdot \frac{(x/2)^{(n-1)+2r}}{\Gamma(n-1+r+1)\cdot r!} = x^n J_{n-1}(x)$$

Thus, $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$ --------(1)

Recurrence Relations 2: $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

From definition,

$$x^{-n} J_n(x) = x^{-n} \sum_{r=0}^{\infty}(-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1)\cdot r!}$$

$$= \sum_{r=0}^{\infty}(-1)^r \cdot \left(\frac{x}{2}\right)^{2r} \cdot \frac{1}{\Gamma(n+r+1)\cdot r!}$$

$$\therefore \quad \frac{d}{dx}\left[x^{-n} J_n(x)\right] = \sum_{r=0}^{\infty}(-1)^r \cdot \frac{2r\, x^{2r-1}}{2^{n+2r}\Gamma(n+r+1)\cdot r!}$$

$$= -x^{-n} \sum_{r=1}^{\infty}(-1)^{r-1} \cdot \frac{x^{n+1+2(r-1)}}{2^{n+1+2(r-1)}\Gamma(n+r+1)\cdot(r-1)!}$$

Let $k = r - 1$

$$= -x^{-n} \sum_{k=0}^{\infty}(-1)^k \cdot \frac{x^{n+1+2k}}{2^{n+1+2k}\Gamma(n+1+k+1)\cdot k!} = -x^{-n} J_{n+1}(x)$$

Thus, $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$ --------(2)

Recurrence Relations 3: $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

We know that $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

Applying product rule on LHS, we get $x^n J_n'(x) + nx^{n-1} J_n(x) = x^n J_{n-1}(x)$

Dividing by x^n we get $J_n'(x) + (n/x) J_n(x) = J_{n-1}(x)$ --------(3)

Also differentiating LHS of $\frac{d}{dx}\left[x^{-n}J_n(x)\right]=-x^{-n}J_{n+1}(x)$, we get

$$x^{-n}J_n^{/}(x)-nx^{-n-1}J_n(x)=-x^{-n}J_{n+1}(x)$$

Dividing by $-x^{-n}$ we get $-J_n^{/}(x)+(n/x)J_n(x)=J_{n+1}(x)$ --------(4)

Adding (3) and (4), we obtain $2nJ_n(x)=x\left[J_{n-1}(x)+J_{n+1}(x)\right]$

i.e., $J_n(x)=\frac{x}{2n}\left[J_{n-1}(x)+J_{n+1}(x)\right]$

Recurrence Relations 4: $J_n^{/}(x)=\frac{1}{2}\left[J_{n-1}(x)-J_{n+1}(x)\right]$

Subtracting (4) from (3), we obtain $2J_n^{/}(x)=\left[J_{n-1}(x)-J_{n+1}(x)\right]$

i.e., $J_n^{/}(x)=\frac{1}{2}\left[J_{n-1}(x)-J_{n+1}(x)\right]$

Recurrence Relations 5: $J_n^{/}(x)=\frac{n}{x}J_n(x)-J_{n+1}(x)$

This recurrence relation is another way of writing the Recurrence relation 2.

Recurrence Relations 6: $J_n^{/}(x)=J_{n-1}(x)-\frac{n}{x}J_n(x)$

This recurrence relation is another way of writing the Recurrence relation 1.

Recurrence Relations 7: $J_{n+1}(x)=\frac{2n}{x}J_n(x)-J_{n-1}(x)$

This recurrence relation is another way of writing the Recurrence relation 3.

Problems:

Prove that $(a)\quad J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x \quad (b)\quad J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x$

By definition,

$$J_n(x)=\sum_{r=0}^{\infty}(-1)^r\cdot\left(\frac{x}{2}\right)^{n+2r}\cdot\frac{1}{\Gamma(n+r+1)\cdot r!}$$

Putting n = ½, we get

$$J_{1/2}(x)=\sum_{r=0}^{\infty}(-1)^r\cdot\left(\frac{x}{2}\right)^{1/2+2r}\cdot\frac{1}{\Gamma(r+3/2)\cdot r!}$$

$$J_{1/2}(x)=\sqrt{\frac{x}{2}}\left[\frac{1}{\Gamma(3/2)}-\left(\frac{x}{2}\right)^2\frac{1}{\Gamma(5/2)1!}+\left(\frac{x}{2}\right)^4\frac{1}{\Gamma(7/2)2!}-\cdots\cdots\right] \qquad \text{--------(1)}$$

Using the results Γ(1/2) = √π and Γ(n) = (n – 1) Γ(n–1), we get

$\Gamma(3/2)=\frac{\sqrt{\pi}}{2},\Gamma(5/2)=\frac{3\sqrt{\pi}}{4},\Gamma(7/2)=\frac{15\sqrt{\pi}}{8}$ and so on.

Using these values in (1), we get

$$J_{1/2}(x)=\sqrt{\frac{x}{2}}\left[\frac{2}{\sqrt{\pi}}-\frac{x^2}{4}\,\frac{4}{3\sqrt{\pi}}+\frac{x^4}{16}\,\frac{8}{15\sqrt{\pi}.2}-\cdots\cdots\right]$$

$$=\sqrt{\frac{x}{2\pi}}\cdot\frac{2}{x}\left[x-\frac{x^3}{6}+\frac{x^5}{120}-\cdots\cdots\right]=\sqrt{\frac{2}{x\pi}}\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\cdots\right]$$

$$J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x$$

Putting n = - 1/2, we get

$$J_{-1/2}(x)=\sum_{r=0}^{\infty}(-1)^r\cdot\left(\frac{x}{2}\right)^{-1/2+2r}\cdot\frac{1}{\Gamma(r+1/2)\cdot r!}$$

$$J_{-1/2}(x)=\sqrt{\frac{x}{2}}\left[\frac{1}{\Gamma(1/2)}-\left(\frac{x}{2}\right)^2\frac{1}{\Gamma(3/2)1!}+\left(\frac{x}{2}\right)^4\frac{1}{\Gamma(5/2)2!}-\cdots\cdots\right] \quad \text{--------(2)}$$

Using the results Γ(1/2) = √π and Γ(n) = (n – 1) Γ(n–1) in (2), we get

$$J_{-1/2}(x)=\sqrt{\frac{2}{x}}\left[\frac{1}{\sqrt{\pi}}-\frac{x^2}{4}\,\frac{2}{\sqrt{\pi}}+\frac{x^4}{16}\,\frac{4}{3\sqrt{\pi}.2}-\cdots\cdots\right]$$

$$=\sqrt{\frac{2}{x\pi}}\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\cdots\right]$$

$$J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x$$

2. Prove the following results :

$$(a)\quad J_{5/2}(x)=\sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x^2}\sin x-\frac{3}{x}\cos x\right] \text{ and}$$

$$(b)\quad J_{-5/2}(x)=\sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x^2}\cos x+\frac{3}{x}\sin x\right]$$

Solution :

We prove this result using the recurrence relation $J_n(x)=\frac{x}{2n}\left[J_{n-1}(x)+J_{n+1}(x)\right]$ ------ (1).

Putting n = 3/2 in (1), we get $J_{1/2}(x)+J_{5/2}(x)=\frac{3}{x}J_{3/2}(x)$

$$\therefore\quad J_{5/2}(x)=\frac{3}{x}J_{3/2}(x)-J_{1/2}(x)$$

$$i.e.,\quad J_{5/2}(x)=\frac{3}{x}\sqrt{\frac{2}{\pi x}}\left[\frac{\sin x-x\cos x}{x}\right]-\sqrt{\frac{2}{\pi x}}\sin x$$

$$J_{5/2}(x)=\sqrt{\frac{2}{\pi x}}\left[\frac{3\sin x-3x\cos x-x^2\sin x}{x^2}\right]=\sqrt{\frac{2}{\pi x}}\left[\frac{(3-x^2)}{x^2}\sin x-\frac{3}{x}\cos x\right]$$

Also putting n = - 3/2 in (1), we get $J_{-5/2}(x)+J_{-1/2}(x)=-\frac{3}{x}J_{-3/2}(x)$

$$\therefore\quad J_{-5/2}(x)=-\frac{3}{x}J_{-3/2}(x)-J_{-1/2}(x)=\left(\frac{-3}{x}\right)\left(-\sqrt{\frac{2}{\pi x}}\right)\left[\frac{x\sin x+\cos x}{x}\right]-\sqrt{\frac{2}{\pi x}}\cos x$$

$$i.e.,\quad J_{-5/2}(x)=\sqrt{\frac{2}{\pi x}}\left[\frac{3x\sin x+3\cos x-x^2\cos x}{x^2}\right]=\sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\sin x+\frac{3-x^2}{x^2}\cos x\right]$$

3. Show that $\frac{d}{dx}\left[J_n^2(x)+J_{n+1}^2(x)\right]=\frac{2}{x}\left[nJ_n^2(x)-(n+1)J_{n+1}^2(x)\right]$

Solution:

L.H.S = $\frac{d}{dx}\left[J_n^2(x)+J_{n+1}^2(x)\right]=2J_n(x)J_n'(x)+2J_{n+1}(x)J_{n+1}'(x)$ ------- (1)

We know the recurrence relations

$$xJ_n'(x)=nJ_n(x)-xJ_{n+1}(x) \quad \text{------- (2)}$$

$$xJ_{n+1}'(x)=xJ_n(x)-(n+1)J_{n+1}(x) \quad \text{------- (3)}$$

Relation (3) is obtained by replacing ***n*** by ***n+1*** in $xJ_n'(x)=xJ_{n-1}(x)-nJ_n(x)$

Now using (2) and (3) in (1), we get

L.H.S $=\frac{d}{dx}\left[J_n^2(x)+J_{n+1}^2(x)\right]=2J_n(x)\left[\frac{n}{x}J_n(x)-J_{n+1}(x)\right]+2J_{n+1}(x)\left[J_n(x)-\frac{n+1}{x}J_{n+1}(x)\right]$

$$=\frac{2n}{x}J_n^2(x)-2J_n(x)J_{n+1}(x)+2J_{n+1}(x)J_n(x)-2\frac{n+1}{x}J_{n+1}^2(x)$$

Hence, $\frac{d}{dx}\left[J_n^2(x)+J_{n+1}^2(x)\right]=\frac{2}{x}\left[nJ_n^2(x)-(n+1)J_{n+1}^2(x)\right]$

4. Prove that $J_0''(x)=\frac{1}{2}[J_2(x)-J_0(x)]$

Solution :

We have the recurrence relation $J_n'(x)=\frac{1}{2}[J_{n-1}(x)-J_{n+1}(x)]$ -------(1)

Putting n = 0 in (1), we get $J_0'(x)=\frac{1}{2}[J_{-1}(x)-J_1(x)]=\frac{1}{2}[-J_1(x)-J_1(x)]=-J_1(x)$

Thus, $J_0'(x)=-J_1(x)$. Differentiating this w.r.t. x we get, $J_0''(x)=-J_1'(x)$ ----- (2)

Now, from (1), for n = 1, we get $J_1'(x)=\frac{1}{2}[J_0(x)-J_2(x)]$.

Using (2), the above equation becomes

$$-J_0''(x)=\frac{1}{2}[J_0(x)-J_2(x)] \text{ or } J_0''(x)=\frac{1}{2}[J_2(x)-J_0(x)].$$

Thus we have proved that, $J_0''(x)=\frac{1}{2}[J_2(x)-J_0(x)]$

5. Show that (a) $\int J_3(x)dx=c-J_2(x)-\frac{2}{x}J_1(x)$

(b) $\int xJ_0^2(x)dx=\frac{1}{2}x^2\left[J_0^2(x)+J_1^2(x)\right]$

Solution :

(a) We know that $\frac{d}{dx}\left[x^{-n}J_n(x)\right]=-x^{-n}J_{n+1}(x)$ or $\int x^{-n}J_{n+1}(x)dx=-x^{-n}J_n(x)$ ------ (1)

Now, $\int J_3(x)dx=\int x^2\cdot x^{-2}J_3(x)dx+c=x^2\cdot\int x^{-2}J_3(x)dx-\int 2x\left[\int x^{-2}J_3(x)dx\right]dx+c$

$=x^2\cdot\left[-x^{-2}J_2(x)\right]-\int 2x\left[-x^{-2}J_2(x)\right]dx+c$ (from (1) when n = 2)

$=c-J_2(x)-\int\frac{2}{x}J_2(x)dx=c-J_2(x)-\frac{2}{x}J_1(x)$ (from (1) when n = 1)

Hence, $\int J_3(x)dx=c-J_2(x)-\frac{2}{x}J_1(x)$

(b) $\int xJ_0^2(x)dx=J_0^2(x)\cdot\frac{1}{2}x^2-\int 2J_0(x)\cdot J_0'(x).\frac{1}{2}x^2dx$ (Integrate by parts)

$$=\frac{1}{2}x^2J_0^2(x)+\int x^2J_0(x)\cdot J_1(x)dx \qquad \text{(From (1) for n = 0)}$$

$$=\frac{1}{2}x^2J_0^2(x)+\int xJ_1(x)\cdot\frac{d}{dx}[xJ_1(x)]dx \quad \left[\because \quad \frac{d}{dx}[xJ_1(x)]=xJ_0(x) \text{ from recurrence relation (1)}\right]$$

$$=\frac{1}{2}x^2J_0^2(x)+\frac{1}{2}[xJ_1(x)]^2=\frac{1}{2}x^2\left[J_0^2(x)+J_1^2(x)\right]$$

Generating Function for $J_n(x)$

To prove that $e^{\frac{x}{2}(t-1/t)}=\sum_{n=-\infty}^{\infty}t^nJ_n(x)$

or

If n is an integer then $J_n(x)$ is the coefficient of t^n in the expansion of $e^{\frac{x}{2}(t-1/t)}$.

Proof:

We have $e^{\frac{x}{2}(t-1/t)}=e^{xt/2}\times e^{-x/2t}$

$$=\left[1+\frac{(xt/2)}{1!}+\frac{(xt/2)^2}{2!}+\frac{(xt/2)^3}{3!}+\cdots\cdots\right]\bullet\left[1+\frac{(-xt/2)}{1!}+\frac{(-xt/2)^2}{2!}+\frac{(-xt/2)^3}{3!}+\cdots\cdots\right]$$

(using the expansion of exponential function)

$$=\left[1+\frac{xt}{2\cdot 1!}+\frac{x^2t^2}{2^22!}+\cdots+\frac{x^nt^n}{2^nn!}+\frac{x^{n+1}t^{n+1}}{2^{n+1}(n+1)!}+\cdots\cdots\right]\bullet\left[1-\frac{x}{2t\cdot 1!}+\frac{x^2}{2^2t^22!}-\cdots+\frac{(-1)^nx^n}{2^nt^nn!}+\frac{(-1)^{n+1}x^{n+1}}{2^{n+1}t^{n+1}(n+1)!}+\cdots\cdots\right]$$

If we collect the coefficient of $\boldsymbol{t^n}$ in the product, they are

$$=\frac{x^n}{2^nn!}-\frac{x^{n+2}}{2^{n+2}(n+1)!1!}+\frac{x^{n+4}}{2^{n+4}(n+2)!2!}-\cdots\cdots$$

$$=\frac{1}{n!}\left(\frac{x}{2}\right)^n-\frac{1}{(n+1)!1!}\left(\frac{x}{2}\right)^{n+2}+\frac{1}{(n+2)!2!}\left(\frac{x}{2}\right)^{n+4}-\cdots\cdots=\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{\Gamma(n+r+1)r!}=J_n(x)$$

Similarly, if we collect the coefficients of $\boldsymbol{t^{-n}}$ in the product, we get $\boldsymbol{J_{-n}(x)}$.

Thus, $e^{\frac{x}{2}(t-1/t)}=\sum_{n=-\infty}^{\infty}t^nJ_n(x)$

Result: $e^{\frac{x}{2}(t-1/t)}=J_0(x)+\sum_{n=1}^{\infty}\left[t^n+(-1)^nt^{-n}\right]J_n(x)$

Proof :

$$e^{\frac{x}{2}(t-1/t)}=\sum_{n=-\infty}^{\infty}t^nJ_n(x)=\sum_{n=-\infty}^{-1}t^nJ_n(x)+\sum_{n=0}^{\infty}t^nJ_n(x)$$

$$=\sum_{n=1}^{\infty}t^{-n}J_{-n}(x)+J_0(x)+\sum_{n=1}^{\infty}t^nJ_n(x)=J_0(x)+\sum_{n=1}^{\infty}t^{-n}(-1)^nJ_n(x)+\sum_{n=1}^{\infty}t^nJ_n(x) \quad \{\because J_{-n}(x)=(-1)^nJ_n(x)\}$$

Thus, $e^{\frac{x}{2}(t-1/t)}=J_0(x)+\sum_{n=1}^{\infty}\left[t^n+(-1)^nt^{-n}\right]J_n(x)$

Problem 6: Show that

(a) $J_n(x)=\frac{1}{\pi}\int_0^{\pi}\cos(n\theta - x\sin\theta)d\theta$ *, n being an integer*

(b) $J_0(x)=\frac{1}{\pi}\int_0^{\pi}\cos(x\cos\theta)d\theta$

(c) $J_0^2+2J_1^2+2J_2^2+J_3^2+\cdots\cdots=1$

Solution :

We know that $e^{\frac{x}{2}(t-1/t)}=J_0(x)+\sum_{n=1}^{\infty}\left[t^n+(-1)^n t^{-n}\right]J_n(x)$

$=J_0(x)+tJ_1(x)+t^2J_2(x)+t^3J_3(x)+\cdots\cdots+t^{-1}J_{-1}(x)+t^{-2}J_{-2}(x)+t^{-3}J_{-3}(x)+\cdots\cdots$

Since $J_{-n}(x)=(-1)^n J_n(x)$, we have

$$e^{\frac{x}{2}(t-1/t)}=J_0(x)+J_1(x)(t-1/t)+J_2(x)(t^2+1/t^2)+J_3(x)(t^3-1/t^3)+\cdots\cdots \quad \text{----- (1)}$$

Let t = cosθ + i sinθ so that t^p = cospθ + i sinpθ and $1/t^p$ = cospθ - i sinpθ.
From this we get, $t^p + 1/t^p$ = 2cospθ and $t^p - 1/t^p$ = 2i sinpθ
Using these results in (1), we get

$$e^{\frac{x}{2}(2i\sin\theta)}=e^{ix\sin\theta}=J_0(x)+2\left[J_2(x)\cos 2\theta+J_4(x)\cos 4\theta+\cdots\right]+2i\left[J_1(x)\sin\theta+J_3(x)\sin 3\theta+\cdots\right] \quad \text{-----(2)}$$

Since $e^{ix\sin\theta}=\cos(x\sin\theta)+i\sin(x\sin\theta)$, equating real and imaginary parts in (2) we get,

$$\cos(x\sin\theta)=J_0(x)+2\left[J_2(x)\cos 2\theta+J_4(x)\cos 4\theta+\cdots\right] \quad \text{----- (3)}$$

$$\sin(x\sin\theta)=2\left[J_1(x)\sin\theta+J_3(x)\sin 3\theta+\cdots\right] \quad \text{----- (4)}$$

These series are known as ***Jacobi Series***.

Now multiplying both sides of (3) by $\cos n\theta$ and both sides of (4) by $\sin n\theta$ and integrating each of the resulting expression between 0 and π, we obtain

$$\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\cos n\theta d\theta=\begin{cases}J_n(x), & \text{n is even or zero}\\ 0, & \text{n is odd}\end{cases}$$

and
$$\frac{1}{\pi}\int_0^{\pi}\sin(x\sin\theta)\sin n\theta d\theta=\begin{cases}0, & \text{n is even}\\ J_n(x), & \text{n is odd}\end{cases}$$

Here we used the standard result $\int_0^{\pi}\cos p\theta\cos q\theta d\theta=\int_0^{\pi}\sin p\theta\sin q\theta d\theta=\begin{cases}\frac{\pi}{2}, & \text{if } p=q\\ 0, & \text{if } p\neq q\end{cases}$

From the above two expression, in general, if n is a positive integer, we get

$$J_n(x)=\frac{1}{\pi}\int_0^{\pi}\left[\cos(x\sin\theta)\cos n\theta+\sin(x\sin\theta)\sin n\theta\right]d\theta=\frac{1}{\pi}\int_0^{\pi}\cos(n\theta-x\sin\theta)d\theta$$

(b) Changing θ to (π/2) θ in (3), we get

$$\cos(x\cos\theta)=J_0(x)+2\left[J_2(x)\cos(\pi-2\theta)+J_4(x)\cos(\pi-4\theta)+\cdots\right]$$

$$\cos(x\cos\theta)=J_0(x)-2J_2(x)\cos 2\theta+2J_4(x)\cos 4\theta-\cdots$$

Integrating the above equation w.r.t θ from 0 to π, we get

$$\int_0^{\pi}\cos(x\cos\theta)d\theta=\int_0^{\pi}\left[J_0(x)-2J_2(x)\cos 2\theta+2J_4(x)\cos 4\theta-\cdots\right]$$

$$\int_0^{\pi} \cos(x\cos\theta)d\theta = \left| J_0(x)\cdot\theta - 2J_2(x)\frac{\sin 2\theta}{2} + 2J_4(x)\frac{\sin 4\theta}{4} - \cdots \right|_0^{\pi} = J_0(x)\cdot\pi$$

Thus, $J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\theta)d\theta$

(c) Squaring (3) and (4) and integrating w.r.t. θ *from* 0 *to* π and noting that *m* and *n* being integers

$$\int_0^{\pi} \cos^2(x\sin\theta)d\theta = [J_0(x)]^2\cdot\pi + 4[J_2(x)]^2\frac{\pi}{2} + 4[J_4(x)]^2\frac{\pi}{2} + \cdots$$

$$\int_0^{\pi} \sin^2(x\sin\theta)d\theta = 4[J_1(x)]^2\frac{\pi}{2} + 4[J_3(x)]^2\frac{\pi}{2} + \cdots$$

Adding, $\int_0^{\pi} d\theta = \pi = \pi\left[J_0^2(x) + 2J_1^2(x) + 2J_2^2(x) + J_3^2(x) + \cdots\cdots\right]$

Hence, $J_0^2 + 2J_1^2 + 2J_2^2 + J_3^2 + \cdots\cdots = 1$

Orthogonality of Bessel Functions

If α *and* β *are the two distinct roots of* $J_n(x) = 0$*, then*

$$\int_0^{\pi} xJ_n(\alpha x)J_n(\beta x)dx = \begin{cases} 0, & if\ \alpha \neq \beta \\ \frac{1}{2}\left[J_n'(\alpha)\right]^2 = \frac{1}{2}\left[J_{n+1}(\alpha)\right]^2, & if\ \alpha = \beta \end{cases}$$

Proof:

We know that the solution of the equation

$$x^2u'' + xu' + (\alpha^2x^2 - n^2)u = 0 \quad \text{-------- (1)}$$

$$x^2v'' + xv' + (\beta^2x^2 - n^2)v = 0 \quad \text{-------- (2)}$$

are $u = J_n(\alpha x)$ *and* $v = J_n(\beta x)$ respectively.

Multiplying (1) by ***v/x*** and (2) by ***u/x*** and subtracting, we get

$$x(u''v - uv'') + (u'v - uv') + (\beta^2 - \alpha^2)xuv = 0$$

or $\quad \frac{d}{dx}\left\{x\left(u'v - uv'\right)\right\} = \left(\beta^2 - \alpha^2\right)xuv$

Now integrating both sides from 0 to 1, we get

$$\left(\beta^2 - \alpha^2\right)\int_0^1 xuv\,dx = \left[x\left(u'v - uv'\right)\right]_0^1 = \left(u'v - uv'\right)_{x=1} \quad \text{------- (3)}$$

Since $u = J_n(\alpha x)$, $u' = \frac{d}{dx}\left[J_n(\alpha x)\right] = \frac{d}{d(\alpha x)}\left[J_n(\alpha x)\right]\cdot\frac{d(\alpha x)}{dx} = \alpha J_n'(\alpha x)$

Similarly $v = J_n(\beta x)$ *gives* $v' = \frac{d}{dx}\left[J_n(\beta x)\right] = \beta J_n'(\beta x)$. Substituting these values in (3), we get

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \frac{\alpha J_n'(\alpha)J_n(\beta) - \beta J_n(\alpha)J_n'(\beta)}{\beta^2 - \alpha^2} \quad \text{------- (4)}$$

If α and β are the two distinct roots of $\mathbf{J_n(x) = 0}$, then $J_n(\alpha) = 0$ and $J_n(\beta) = 0$, and hence (4) reduces to $\int_0^{\pi} xJ_n(\alpha x)J_n(\beta x)dx = 0$.

This is known as Orthogonality relation of Bessel functions.

When $\beta = \alpha$, the RHS of (4) takes 0/0 form. Its value can be found by considering α as a root of $J_n(x) = 0$ and β as a variable approaching to α. Then (4) gives

$$\underset{\beta\to\alpha}{Lt}\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \underset{\beta\to\alpha}{Lt}\frac{\alpha J_n'(\alpha)J_n(\beta)}{\beta^2-\alpha^2}$$

Applying L'Hospital rule, we get

$$\underset{\beta\to\alpha}{Lt}\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \underset{\beta\to\alpha}{Lt}\frac{\alpha J_n'(\alpha)J_n'(\beta)}{2\beta} = \frac{1}{2}\left\{J_n'(\alpha)\right\}^2 \text{ ---------(5)}$$

We have the recurrence relation $J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$.

$\therefore \quad J_n'(\alpha) = \frac{n}{\alpha}J_n(\alpha) - J_{n+1}(\alpha)$. Since $J_n(\alpha) = 0$, we have $J_n'(\alpha) = -J_{n+1}(\alpha)$

Thus, (5) becomes $\underset{\beta\to\alpha}{Lt}\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \frac{1}{2}\left\{J_n'(\alpha)\right\}^2 = \frac{1}{2}\left\{J_{n+1}(\alpha)\right\}^2$