LECTURE NOTES

ON

COMPLEX ANALYSIS AND PROBABILITY DISTRIBUTION

B. Tech II semester

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SYLLABUS

UNIT-I	COMPLEX FUNCTIONS AND DIFFERENTIATION
Complex functions differentiation and integration: Complex functions and its representation on argand plane, concepts of limit, continuity, differentiability, analyticity, Cauchy-Riemann conditions and harmonic functions; Milne-Thomson method.	
UNIT-II	**COMPLEX INTEGRATION**
Line integral: Evaluation along a path and by indefinite integration; Cauchy's integral theorem; Cauchy's integral formula; Generalized integral formula; Power series expansions of complex functions and contour Integration: Radius of convergence.	
UNIT-III	**POWER SERIES EXPANSION OF COMPLEX FUNCTION**
Expansion in Taylor's series, Maclaurin's series and Laurent series. Singular point; Isolated singular point; Pole of order m; Essential singularity; Residue: Cauchy Residue Theorem. Evaluation of Residue by Laurent Series and Residue Theorem. Evaluation of integrals of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta$ and $\int_{-\infty}^{\infty} f(x)\,dx$ Bilinear Transformation.	
UNIT-IV	**SINGLE RANDOM VARIABLES**
Random variables: Discrete and continuous, probability distributions, mass function-density function of a probability distribution. Mathematical expectation. Moment about origin, central moments, moment generating function of probability distribution.	
UNIT-V	**PROBABILITY DISTRIBUTIONS**
Binomial, Poisson and normal distributions and their properties.	

TEXT BOOKS:

1	Kreyszig, "Advanced Engineering Mathematics", John Wiley & Sons Publishers, 10^{th} Edition, 2010.
2	B. S. Grewal, "Higher Engineering Mathematics", Khanna Publishers, 43^{rd} Edition, 2015.

REFERENCES:

1	T.K.V Iyengar, B.Krishna Gandhi, "Engineering Mathematics - III", S. Chand & Co., 12^{th} Edition, 2015.
2	T.K.V Iyengar, B.Krishna Gandhi, "Probability and Statistics", S. Chand & Co., 7^{th} Edition, 2015.
3	Churchill, R.V. and Brown, J.W, "Complex Variables and Applications", Tata Mc Graw-Hill, 8^{th} Edition, 2012.

UNIT-I
COMPLEX FUNCTIONS AND DIFFERENTIATION

COMPLEX FUNCTIONS

Complex number

For a complex number $z = x + iy$, the number Re $z = x$ is called the real part of z and the number Im $z = y$ is said to be the its imaginary part. If $x = 0$, z is said to be a purely imaginary number.

Definition : Let $z = x + iy \in C$. The complex number $z = x - iy$ is called the complex conjugate of z and $|z| = \sqrt{x^2 + y^2}$ is said to be the absolute value or the modulus of the complex number z.

Functions of a Complex Variable :

Let D be a nonempty set in C. A single-valued complex function or, simply, a complex function $f : D \to C$ is a map that assigns to each complex argument $z = x + iy$ in D a unique complex number $w = u + iv$. We write $w = f(z)$.

The set D is called the domain of the function f and the set f(D) is the range or the image of f. So, a complex-valued function f of a complex variable z is a rule that assigns to each complex number z in a set D one and only one complex number w. We call w the image of z under f.

If $z = x + iy \in D$, we shall write $f(z) = u(x, y) + iv(x, y)$ or $f(z) = u(z) + iv(z)$. The real functions u and v are called the real and, respectively, the imaginary part of the complex function f. Therefore, we can describe a complex function with the aid of two real functions depending on two real variables.

Example 1.The function $f : C \to C$, defined by $f(z) = z^3$, can be written as $f(z) = u(x, y) + iv(x, y)$, with $u, v : R^2 \to R$ given by $u(x, y) = x^3 - 3xy^2$, $v(x, y) = 3x^2 y - y^3$.

Example 2.For the function $f : C \to C$, defined by $f(z) = e^z$, we have $u(x, y) = e^x \cos y$, $v(x, y) = e^x \sin y$, for any $(x, y) \in R^2$.

Limits of Functions : Let $D \subseteq C$, $a \in D'$ and $f : D \to C$. A number $l \in C$ is called a limit of the function f at the point a if for any $V \in V(l)$, there exists $U \in V(a)$ such that, for any $z \in U \cap D \setminus \{a\}$, it follows that $f(z) \in V$. We shall use the notation $l = \lim_{z \to z_0} f(z)$.

Remark : If a complex function $f : D \to C$ possesses a limit l at a given point a, then this limit is unique.

Exercise 1: Prove that $\lim_{z \to z_0} \frac{\bar{z}}{z}$ does not exist.

Solution : To prove that the above limit does not exist, we compute this limit as z → 0 on the real and on the imaginary axis, respectively. In the first situation, i.e. for z = x ∈ R, the value of the limit is 1. In the second situation,

i.e. for z = i y, with y ∈ R, the limit is −1. Thus, the limit depends on the direction from which we approach 0, which implies that the limit does not exist.

Differentiability of complex function :

Let w = f(z) be a given function defined for all z in a neighbourhood of z_0.If $\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ exists,the function f(z) is said to be derivable at z_0 and the limit is denoted by $f'(z_0)$. $f'(z_0)$ if exists is called the derivative of f(z) at z_0.

Exercise : f(z)= $|z^2|$ is a function which is continuous at all z but not derivable at any z $\neq$ 0

Solution: Let f(z)=$|z^2|$ $= z\bar{z}$

Then f(z)= $z_0 \bar{z}_0$

We have to prove that $\underset{z \to z_0}{lt}$ z=z_0 and $\underset{z \to z_0}{lt} \bar{z} = \bar{z}_0$ Thus $\underset{z \to z_0}{lt} z\bar{z} = z_0\bar{z}_0$

$\therefore \underset{z \to z_0}{lt}$ f(z)=f(z_0)

$\therefore$ The function is continuous at all z

$\therefore$ f(z_0+Δz) $= (z_0 + \Delta z)(\bar{z} + \Delta \bar{z}) = z_0\bar{z}_0 + z_0\Delta\bar{z} + \Delta z\bar{z}_0 + \Delta z \Delta \bar{z}$

Now $\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{z_0\Delta\bar{z} + \Delta z\bar{z}_0 + \Delta z\Delta\bar{z}}{\Delta z}$

Consider the limit as $\Delta z \to 0$

Case 1: let $\Delta z \to 0$ along x-axis then $\Delta x = \Delta x, \Delta y = 0 \Rightarrow \Delta z = \Delta x$

$$\therefore \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta x\to 0} \frac{z_0\Delta x+\Delta x\bar{z}_0+\Delta x\Delta x}{\Delta x} = z_0+\bar{z}_0 \qquad \to (1)$$

Case 2: Let $\Delta z \to 0$ along y-axis then $\Delta x = 0, \Delta y = \Delta y \Rightarrow \Delta z = i\Delta y$

$$\lim_{\Delta z\to 0} \frac{f(z_0+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta y\to 0} \frac{z_0(-i\Delta y)+i\Delta y\bar{z}_0+(i\Delta y)(-i\Delta y)}{i\Delta y} = -z_0+\bar{z}_0 \qquad \to (2)$$

Thus , from (1) and (2) for f'(z_0) to exists

i.e., $z_0 = -z_0 \Rightarrow 2z_0 = 0 \Rightarrow z_0 \neq 0$

$\therefore$ f'(z) does not exists though f(z)= $|z^2|$ is continuous at all z.

polar form of Cauchy-Riemann equation:

Theorem:

If $f(z) = f(re^{i\theta}) = u(r,\theta)+iv(r,\theta)$ and f(z) is derivable at $z_0 = r_0 e^{i\theta_0}$ then

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Proof: Let $z = re^{i\theta}$ Then $f(z) = f(re^{i\theta}) = u(r,\theta)+iv(r,\theta)$

Differentiating it with respect to r partially,

$$\frac{\partial}{\partial r} f(z) = f'(z)\frac{\partial z}{\partial r} = f'(z)e^{i\theta}$$

$$\therefore f'(z)= \frac{1}{e^{i\theta}}\frac{\partial f}{\partial r} = \frac{1}{e^{i\theta}}(u_r+iv_r) \qquad \to (1)$$

Similarly differentiating partially with respect to θ

$$\frac{\partial f}{\partial \theta} = f'(z)\frac{\partial z}{\partial \theta} = f'(z).rie^{i\theta}$$

$$\therefore f'(z) = \frac{1}{rie^{i\theta}}(u_\theta+iv_\theta) \qquad \to (2)$$

From (1) and (2) we have

$$\frac{1}{e^{i\theta}}(u_r + iv_r) = \frac{1}{rie^{i\theta}}(u_\theta + iv_\theta)$$

$$\therefore\ u_r + iv_r = \frac{1}{r}\frac{\partial v}{\partial \theta} - i\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Equating real and imaginary parts ,we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}\ and\ \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Analytic function:

A complex function is said to be analytic on a region R if it is complex differentiable at every point in R . The terms holomorphic function, differentiable function, and complex differentiable function are sometimes used interchangeably with "analytic function". Many mathematicians prefer the term "holomorphic function" (or "holomorphic map") to "analytic function" .

If a complex function is analytic on a region R , it is infinitely differentiable in R.

Singularities:

A complex function may fail to be analytic at one or more points through the presence of singularities, or along lines or line segments through the presence of branch cuts.

Eg. f(z)= $\frac{1}{z}$ is analytic every where except at z=0.

At z=0 $f'(z)$ does not exist.

So z=0 is an isolated singular point.

Entire function:

A complex function that is analytic at all finite points of the complex plane is said to be entire. A single-valued function that is analytic in all but possibly a discrete subset of its domain, and at those singularities goes to infinity like a polynomial (i.e., these exceptional points must be poles and not essential singularities), is called a meromorphic function.

Cauchy–Riemann equations:

The Cauchy–Riemann equations on a pair of real-valued functions of two real variables $u(x,y)$ and $v(x,y)$ are the two equations:

1. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

2. $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Typically *u* and *v* are taken to be the real and imaginary parts respectively of a complex-valued function of a single complex variable $z = x + iy$, $f(x + iy) = u(x,y) + iv(x,y)$

Relation with harmonic functions :

Analytic functions are intimately related to harmonic functions. We say that a real-valued function h(x, y) on the plane is harmonic if it obeys Laplace's equation:

$$\frac{\partial^2 h}{\partial^2 x} + \frac{\partial^2 h}{\partial^2 y} = 0 \ .$$

In fact, as we now show, the real and imaginary parts of an analytic function are harmonic. Let f = u + i v be analytic in some open set of the complex plane.

Then, $\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}$

$= \frac{\partial}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial}{\partial y}\frac{\partial u}{\partial x}$ (using Cauchy–Riemann)

$= \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x}$

$=0$

A similar calculation shows that v is also har monic. This result is important in applications because it shows that one can obtain solutions of a second order partial differential equation by solving a system of first order partial differential equations. It is particularly important in this case because we will be able to obtain solutions of the Cauchy–Riemann equations without really solving these equations.

Given a harmonic function u we say that another harmonic function v is its harmonic conjugate if the complex-valued function f = u+i v is analytic.

Conjugate harmonic function:

If two harmonic functions u and v satisfy the Cauchy-Reimann equations in a domain D and they are real and imaginary parts of an analytic function f in D then v is said to be a conjugate harmonic function of u in D.If f(z)=u+iv is an analytic function and if u and v satisfy Laplace's equation ,then u and v are called conjugate harmonic functions.

Polar form of cauchys Riemann equations:

The Cauchy-Riemann equations can be written in other coordinate systems. For instance, it is not difficult to see that in the system of coordinates given by the polar representation z = r $e^{i\theta}$ these equations take the following form:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$
$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Problem: Show that the function f : C → C, defined by f(z) = $\bar{z}$ does not satisfy the Cauchy-Riemann equations.

Solution: Indeed, since u(x, y) = x, v(x, y) = −y, it follows that ∂u /∂x = 1, while ∂v/ ∂y = −1. So, this function, despite the fact that it is continuous everywhere on C, it is R differentiable on C, is nowhere C-derivable.

Problem: Show that the function $f(z) = e^z$ satisfies the Cauchy-Riemann equations.
Solution:
since $e^z = e^x($ cosy+i siny),

Indeed it follows that
u(x, y) =e^xcosy, v(x, y) =e^x siny
and $\frac{\partial u}{\partial x}$ = e ^{x}cosy = $\frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y}$ = e ^{x}siny = $-\frac{\partial v}{\partial x}$;

Moreover, e^z is complex derivable and it follows immediately that its complex derivative is e^z.

Holomorphic functions:

Holomorphic functions are complex functions, defined on an open subset of the complex plane, that are differentiable. In the context of complex analysis, the derivative of f at z_0 is defined to be $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \ z \in c$.

Construction of analytic function whose real or imaginary part is known:

Suppose f(z)=u+iv is an analytic function ,whose real part u is known .We can find v, the imaginary part and also the function f(z).

Problem: Showthat $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log\left|f'(z)\right|=0$ where f(z) is an analytic function.

Solution: Taking $x=\frac{z+\bar{z}}{2}, y=\frac{z-\bar{z}}{2}=\frac{-i}{2}(z-\bar{z})$

We have $\frac{\partial}{\partial z}=\frac{\partial}{\partial x}\left(\frac{\partial x}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial y}{\partial z}\right)=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)$

And $\frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$

$\therefore \frac{\partial^2}{\partial z\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right).\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)=\frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$

Hence $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\log\left|f'(z)\right|=4\frac{\partial^2}{\partial z\partial \bar{z}}\left(\frac{1}{2}\log\left|f'(z)\right|^2\right)$

$=2\frac{\partial^2}{\partial z\partial \bar{z}}[(\log f'(z)f'(\bar{z}))]$ $(\because \left|z\right|^2=z\bar{z})$

$=2\frac{\partial^2}{\partial z\partial \bar{z}}[(\log f'(z)+f'(\bar{z}))]$

$=2\left[\frac{\partial}{\partial \bar{z}}\frac{f''(z)}{f'(z)}+\frac{\partial}{\partial z}\frac{f''(z)}{f'(z\_)}\right]$

= 2(0+0)=0

Since f(z) is analytic , f(z) is analytic, $f'(\bar{z})$ is also analytic and $\frac{\partial f'(z)}{\partial \bar{z}}=0, \frac{\partial f'(\bar{z})}{\partial z}=0$

Problem: Show that f(z)= $\begin{cases}\frac{xy(x+iy)}{x^2+y^4} & , z\neq 0\\ 0 & , z=0\end{cases}$ is not analytic at z=0 although C-R equations satisified at origin.

Solution: $\frac{f(z)-f(0)}{z-0}=\frac{f(z)-0}{z}=\frac{f(z)}{z}$

$$=\frac{xy^2(x+iy)}{(x^2+y^4).z}=\frac{xy^2(z)}{(x^2+y^4).z}=\frac{xy^2}{(x^2+y^4)}$$

Clearly $\lim_{\substack{x\to 0\\ y\to 0}}\frac{xy^2}{(x^2+y^4)}=\lim_{\substack{y\to 0\\ x\to 0}}\frac{xy^2}{(x^2+y^4)}=0$

Along path y=mx

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{x\to 0}\frac{x(m^2.x^2)}{x^2+m^4.x^4}=\lim_{x\to 0}\frac{m^2.x^2}{1+m^4.x^2}=0$$

Along path $x=my^2$

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{y\to 0}\frac{y^2(m.y^2)}{y^4+m^2.y^4}=\lim_{y\to 0}\frac{m}{1+m^2}\neq 0$$

Limit value depends on m i.e on the path of approach and its different for the different paths

Followed and therefore limit does not exists.

Hence f(z) is not differentiable at z=0.Thus f(z) is not analytic at z=0

To prove that C-R conditions are satisified at origin

Let $f(z)=u+iv=\frac{xy^2(x+iy)}{(x^2+y^4)}$

Then u(x,y)= $\frac{x^2y^2}{(x^2+y^4)}$ and v(x,y)= $\frac{xy^3}{(x^2+y^4)}$ for $z\neq 0$

Also u(0,0)=0 and v(0,0)= 0 [$\because$ f(z)=0 at z=0]

Now $\frac{\partial u}{\partial x}=\lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x}=\lim_{x\to 0}\frac{0}{x}=0$

$$\frac{\partial u}{\partial y}=\lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y}=\lim_{x\to 0}\frac{0}{y}=0$$

$$\frac{\partial v}{\partial x}=\lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x}=\lim_{x\to 0}\frac{0}{x}=0$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{x \to 0} \frac{0}{y} = 0$$

Thus C-R equations are satisified are satisified at the origin

Hence f(z) is not analytic at z=0 even C-R equations are satisified at origin.

Milne Thomson method:

Problem : Find the regular function whose imaginary part is $\log(x^2 + y^2) + x - 2y$.

Solution: Given $v = \log(x^2 + y^2) + x - 2y$

$\therefore \frac{\partial v}{\partial x} = \frac{2x}{x^2 + y^2} + 1$ -→(1) and $\frac{\partial v}{\partial y} = \frac{2y}{x^2 + y^2} - 2$ -→(2)

f'(z) = $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$ (Using C-R equation)

$= \frac{2y}{x^2 + y^2} - 2 + \left(\frac{2x}{x^2 + y^2} + 1\right)$ (using (1) ,(2))

By Milne Thomson method ,f'(z) is expressed in terms of z by replacing x z and y by 0.

Hence $f'(z) = -2 + i\left(\frac{2z}{z^2} + 1\right) = -2 + i\left(\frac{2}{z} + 1\right)$

On integrating, $f(z) = \int \left[-2 + i\left(\frac{2}{z} + 1\right)\right] dz + c$

$$= -2z + i(2\log z + z) + c = 2i\log z - (2 - i)z + c \; .$$

Problem: Show that the function $u = 4xy - 3x + 2$ is harmonic .construct the corresponding analytic function f(z)=u+iv in terms of z.

Solution: Given $u = 4xy - 3x + 2$ →(1)

Differentiating (1) partially w.r.t .x, $\frac{\partial u}{\partial x} = 4y - 3$

Again differentiating $\frac{\partial^2 u}{\partial x^2} = 0$

Again differentiating (1) partially w.r.t .y, $\frac{\partial u}{\partial x} = 4x$

Again differentiating $\frac{\partial^2 u}{\partial y^2} = 0$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Hence u is Harmonic.

Now $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \Rightarrow f'(z) = 4y - 3 - i.4x$

Using Milne Thomson method

$f'(z) = -3 - i4z$ (putting x=z and y=0)

Integrating, $f(z) = -3z - i2z^2 + c$

Problem : Find the imaginary part of an analytic function whose real part is $e^x(x\cos y - y\sin y)$
.

Solution: Let $f(z) = u + iv$ where u=$e^x(x\cos y - y\sin y)$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ (using C-R equ)

$= [e^x(x\cos y - y\sin y) + e^x\cos y] - i[e^x(-x\sin y - \sin y - y\cos y)]$

By milne’s method $f'(z) = (ze^z + e^z) - i(0) = ze^z + e^z$

Integrating , We get

f(z)= $\int (ze^z + e^z)dz + c = (z-1)e^z + e^z + c = ze^z + c$

i.e., $u + iv = (x+iy)e^{x+iy} + c$

$= (x + iy)e^x . e^{iy} + c$

$= e^x (x + iy)(\cos y + i \sin y) + c$

$= e^x (x \cos y + ix \sin y + iy \cos y - y \sin y) + c$

$= e^x [(x \cos y - y \sin y) + i(x \sin y + y \cos y)] + c$

EXCERCISE PROBLEMS:

1)Show that the real part of an analytic function f (z) where u = $e^{-2xy} \sin(x^2 - y^2)$ is a harmonic function. Hence find its harmonic conjugate.

2) Prove that the real part of analytic function f (z) where u = $\log|z|^2$ is harmonic function. If so find the analytic function by Milne Thompson method.

3)Obtain the regular function f (z) whose imaginary part of an analytic function is $\frac{x-y}{x^2+y^2}$

4) Find an analytic function f (z) whose real part of an analytic function is u = $\frac{\sin 2x}{\cosh 2y - \cos 2x}$ by Milne-Thompson method.

5) Find an analytic function f (z) = u +iv if the real part of an analytic function is u = a (1+cosθ) using Cauchy-Riemann equations in polar form.

6) Prove that if $u = x^2 - y^2, v = -\frac{y}{x^2+y^2}$ both u and v satisfy Laplace's equation, but u + iv is not a regular (analytic) function of z.

7)Show that the function f (z) = $\sqrt{|xy|}$ is not analytic at the origin although Cauchy – Riemann equations are satisfied at origin.

8) If $w = \emptyset + i\varphi$ represents the complex potential for an electric field where $\varphi = x^2 - y^2 + \frac{x}{x^2+y^2}$ then determine the function φ.

9)State and Prove the necessary condition for f (z) to be an analytic function in Cartesian form.

10)If u and v are conjugate harmonic functions then show that uv is also a harmonic function.

11)Find the orthogonal trajectories of the family of curves $r^2 cos2\theta$ = c

12)Find an analytic function whose real part is u = $\frac{\sin 2x}{\cosh 2y - \cos 2x}$

13)Find an analytic function whose imaginary part is v = $e^x(xsiny + ycosy)$

14)Find an analytic function whose real part is (i) u = $\frac{x}{x^2+y^2}$ (ii) u = $\frac{y}{x^2+y^2}$

15)Find an analytic function whose imaginary part is v =$\frac{2sinxsiny}{cos h2x+cos h2y}$

16)Find an analytic function f(z) = u +iv if u = a(1+cosθ)

17)Find the conjugate harmonic of u = $e^{x^2-y^2}$cos2xyand find f(z) in terms of z.

18)If f(z) is an analytic function of z and if u - v = $e^x(cosy - siny)$ find f(z) in terms of z.

19)If f(z) is an analytic function of z and if u - v = (x-y)$(x^2 + 4xy + y^2)$) find f(z) in terms of z.

20) Find the orthogonal trajectories of the family of curves $x^3y - xy^3$= C = constant

UNIT-II
COMPLEX INTEGRATION

LINE INTEGRAL

Defination: In mathematics, a **line integral** is an integral where the function to be integrated is evaluated along a curve. The terms **path integral**, **curve integral**, and **curvilinear integral** are also used; contour integral as well, although that is typically reserved for line integrals in the complex plane.

The function to be integrated may be a scalar field or a vector field. The value of the line integral is the sum of values of the field at all points on the curve, weighted by some scalar function on the curve (commonly arc length or, for a vector field, the scalar product of the vector field with a differential vector in the curve). This weighting distinguishes the line integral from simpler integrals defined on intervals. Many simple formulae in physics (for example, $W = \mathbf{F} \cdot \mathbf{s}$) have natural continuous analogs in terms of line integrals ($W = \int_C \mathbf{F} \cdot d\mathbf{s}$). The line integral finds the work done on an object moving through an atomic or gravitational field.

In complex analysis, the line integral is defined in terms of multiplication and addition of complex numbers.

Let us consider F(t)= u(t)+i v(t) , $a \le t \le b$. Where u and v are real valued continuous functions of t in [a,b].

we define $\int_a^b F(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$

Thus, $\int_a^b F(t)\,dt$ is a complex number such that real part of $\int_a^b F(t)\,dt$ is $\int_a^b u(t)\,dt$ and imaginary part of $\int_a^b F(t)\,dt$ is $\int_a^b v(t)\,dt$.

Problem: Evaluate $\int_0^{1+i} (x^2 - iy)\,dz$ along the paths 1)y=x 2)$y=x^2$

Solution: 1)along the line y=x, dy= dx so that dz = dx+idx=(1+i) dx

$$\therefore \int_0^{1+i} (x^2 - iy)\,dz = \int_0^1 (x^2 - ix)(1+i)\,dx \,,$$

since y=x

$$=(1+i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1$$

$$=(1+i)\left[\frac{1}{3} - \frac{1}{2}i\right]$$

$$=\frac{5}{6} - \frac{1}{6}i$$

2) alongtheparabola $y=x^2$,dy=2xdx sothatdz=dx+2ixdx

dz=(1+2ix)dx and xvaries from 0to1

$$\therefore \int_0^{1+i} (x^2 - iy)\,dz = \int_0^1 (x^2 - ix^2)(1+2ix)\,dx$$

$$=(1-i)\int_0^1 x^2(1+2ix)\,dx$$

$$=(1-i)\left(\frac{1}{3} + \frac{1}{2}i\right)$$

$$=\frac{(1-i)(2+3i)}{6}$$

$$=\frac{5}{6} + \frac{1}{6}i$$

▯

Problem: Evaluate $\int_{z=0}^{z=1+i} (x^2 + 2xy + i(y^2 - x))\,dz$ along $y=x^2$

Solution: Given $f(z)=x^2 + 2xy + i(y^2 - x))\,dz$

$$Z=x+iy, dz=dx+idy$$

$$\therefore\ the curve\ y = x^2, dy = 2xdx$$

$$\therefore\ dz = dx + 2xidx = (1 + 2ix)dx$$

$f(z)=x^2+2x(x^2)+i(x^4-x)$

$=x^2+2x^3 +i(x^4-x)$

$f(z)\ dz=(x^2 +2x^3)+i(x^4-x)(1+2ix))dx$

$=x^2+2x^3+i(x^4-x)+2ix^3+4ix^4-2x^5+2x^2$

$$\therefore \int_c f(z)dz = \int_{z=0}^{1+i} x^2 + 2xy + i(y^2 - x)dz$$

$$= \int_0^1 (-2x^5 + 3x^2 + 2x^3 + i(5x^4 - x + 2x^3))\,dx\ )$$

$$= \left[-\frac{x^6}{3} + x^3 + \frac{x^4}{2} + i(\frac{5x^5}{5} - \frac{x^2}{2} + \frac{x^4}{2} \right]_0^1$$

$$= \left[\left(\frac{-1}{3} + 1 + \frac{1}{2} \right) + \left(\frac{5}{5} - \frac{1}{2} + \frac{1}{2} \right) \right] - 0$$

$$= \frac{7}{6} + \frac{5}{5} i = \frac{7}{6} + i$$

$$\int_c f(z)dz = \frac{7}{6} + i$$

Cauchy-Goursat Theorem: Let $f(z)$ be analytic in a simply connected domain D. If C is a simple closed contour that lies in D, then

$$\int_C f(z)\,dz = 0$$

Let us recall that e^z, $\cos(z)$, and z^n (where n is a positive integer) are all entire functions and have continuous derivatives. The Cauchy-Goursat theorem implies that, for any simple closed contour,

(a) $\int_C e^z\,dz = 0$,

(b) $\int_C \cos(z)\,dz = 0$, and

(c) $\int_C z^n\,dz = 0$.

Cauchy integral formula

STATEMENT : let F(z)=u(x,y)+iv(x,y) be analytic on and within a simple closed contour (or curve) 'c' and let f '(z) be continuous there,then $\int f(z)dz = 0$

Proof: f (z)=u(x,y)+iv(x,y)

And dz=dx+idy

$\Rightarrow$ f(z).dz = (u(x,y)+iv(x,y))dx+idy

f(z).dz = u(x,y)dx+i u(x,y)dy+iv(x,y)dx+i^2 v(x,y)dy

f(z).dz= u(x,y)dx- v(x,y)dy+i(u(x,y)dy+ v(x,y)dx

Integrate both sides, we get

$$\int f(z)dz = \int (\text{udx} - \text{vdy}) + i(\text{udy} + \text{vdx})$$

By greens theorem ,we have

$$\int Mdx + Ndy = \iint \frac{\partial N}{\partial x} - \frac{\partial M}{\partial Y}\,\text{dxdy}$$

Now $\int f(z)dz = \iint (-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial Y})\text{dxdy} + i(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial Y})\text{dxdy}$

Since $f^{`}(z)$ is continuous &four partial derivatives i.e $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial Y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial Y}$ are also continuous in the region R enclosed by C, Hence we can apply Green's Theorem.

Using Green's Theorem in plane ,assuming that R is the region bounded by C.

It is given that f (z)=u(x,y)+iv(x,y) is analytic on and within c.

Hence $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Using this we have

$$\int_c f(z)dz = \iint_R 0 \quad dxdy + i\iint_R 0 \quad dxdy = 0$$

Hence the theorem.

Cauchy's integral formula:

Cauchy's integral formula states that

$$f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)\,dz}{z - z_0}, \qquad (1)$$

where the integral is a contour integral along the contour γ enclosing the point z_0.

It can be derived by considering the contour integral

$$\oint_\gamma \frac{f(z)\,dz}{z - z_0}, \qquad (2)$$

defining a path γ_r as an infinitesimal counterclockwise circle around the point z_0, and defining the path γ_0 as an arbitrary loop with a cut line (on which the forward and reverse contributions cancel each other out) so as to go around z_0. The total path is then

$$\gamma = \gamma_0 + \gamma_r, \qquad (3)$$

so

$$\oint_\gamma \frac{f(z)\,dz}{z - z_0} = \oint_{\gamma_0} \frac{f(z)\,dz}{z - z_0} + \oint_{\gamma_r} \frac{f(z)\,dz}{z - z_0}. \qquad (4)$$

From the Cauchy integral theorem, the contour integral along any path not enclosing a pole is 0. Therefore, the first term in the above equation is 0 since γ_0 does not enclose the pole, and we are left with

$$\oint_{\gamma} \frac{f(z)\, dz}{z - z_0} = \oint_{\gamma_r} \frac{f(z)\, dz}{z - z_0}. \tag{5}$$

Now, let $z \equiv z_0 + r e^{i\theta}$, so $dz = i r e^{i\theta} d\theta$. Then

$$\oint_{\gamma} \frac{f(z)\, dz}{z - z_0} = \oint_{\gamma_r} \frac{f\left(z_0 + r e^{i\theta}\right)}{r e^{i\theta}} i r e^{i\theta} d\theta \tag{6}$$

$$= \oint_{\gamma_r} f\left(z_0 + r e^{i\theta}\right) i\, d\theta. \tag{7}$$

But we are free to allow the radius r to shrink to 0, so

$$\oint_{\gamma} \frac{f(z)\, dz}{z - z_0} = \lim_{r \to 0} \oint_{\gamma_r} f\left(z_0 + r e^{i\theta}\right) i\, d\theta \tag{8}$$

$$= \oint_{\gamma_r} f(z_0)\, i\, d\theta \tag{9}$$

$$= i f(z_0) \oint_{\gamma_r} d\theta \tag{10}$$

$$= 2\pi i f(z_0), \tag{11}$$

giving (1).

If multiple loops are made around the point z_0, then equation (11) becomes

$$n(\gamma, z_0) f(z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)\, dz}{z - z_0}, \tag{12}$$

where $n(\gamma, z_0)$ is the contour winding number.

A similar formula holds for the derivatives of $f(z)$,

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{13}$$

$$= \lim_{h\to 0} \frac{1}{2\pi i h}\left[\oint_\gamma \frac{f(z)\,dz}{z-z_0-h} - \oint_\gamma \frac{f(z)\,dz}{z-z_0}\right] \tag{14}$$

$$= \lim_{h\to 0} \frac{1}{2\pi i h}\oint_\gamma \frac{f(z)\,[(z-z_0)-(z-z_0-h)]\,dz}{(z-z_0-h)(z-z_0)} \tag{15}$$

$$= \lim_{h\to 0} \frac{1}{2\pi i h}\oint_\gamma \frac{h\,f(z)\,dz}{(z-z_0-h)(z-z_0)} \tag{16}$$

$$= \frac{1}{2\pi i}\oint_\gamma \frac{f(z)\,dz}{(z-z_0)^2}. \tag{17}$$

Iterating again,

$$f''(z_0) = \frac{2}{2\pi i}\oint_\gamma \frac{f(z)\,dz}{(z-z_0)^3}. \tag{18}$$

Continuing the process and adding the contour winding number n,

$$n(\gamma, z_0)\,f^{(r)}(z_0) = \frac{r!}{2\pi i}\oint_\gamma \frac{f(z)\,dz}{(z-z_0)^{r+1}}.$$

Problem: Evaluate using cauchy's integral formula $\int_c \frac{e^{2z}}{(z-1)(z-2)}dz$ where c is the circle $|z| = 3$

Solution: Given $\int_c \frac{e^{2z}}{(z-1)(z-2)}dz$ ……………(1)

Both the points z=1,z=2 line inside $|z| = 3$

Resolving into partial fractions

$$\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

A=-1, B=1

From(1)

$$\int_c \frac{e^{2z}}{(z-1)(z-2)}dz = \int_c \frac{-e^{2z}}{(z-1)}dz + \int_c \frac{e^{2z}}{(z-2)}dz \quad \text{(by cauchy's integral formula)}$$

$=-2\pi i f(1)+2\pi i f(2)$

$=-2\pi i e^{2.1}+2\pi i e^{2.2}$

$=-2\pi i e^{2}+2\pi i e^{4}=2\pi i(e^{4}-e^{2})$

Problem: Using cauchy'sintegralformula to evaluate $\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)z-2)}dz$, where c is the circle $|z| = 3$

Solution: $\int_c \frac{f(z)}{(z-1)z-2)}dz$ $=(\int_c \frac{1}{(z-2)}dz + \int_c \frac{1}{(z-1)}dz$)f(z)dz

$= \int_c \frac{f(z)}{(z-2)}dz + \int_c \frac{f(z)}{(z-1)}dz$

=2Πif(2)- 2∏if(1)

=2Πi(sin4Π+cos4Π)-(sinΠ+cosΠ))

=2Πi(1-(-1))=4Πi

$\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)z-2)}dz$ =4Πi

Problem: Evaluate $\int_c \frac{(z-1)}{(z+1)^2(z-2)}dz$ whrere c is $|Z - i| = 2$

Solution: the singularities of $\frac{(z-1)}{(z+1)^2(z-2)}$ are given by

$(z+1)^2(z-2)=0$

$\Rightarrow$ Z=-1 and z=2

Z=-1 lies inside the circle since $|-1-i| - 2 < 0$

Z=2 lies outside the circle sinceI2-iI-2>0 $|2-i| - 2 > 0$

The given line integral can be written as

$$\int_c \frac{(z-1)}{(z+1)^2(z-2)}dz = \int_c \frac{\frac{(z-1)}{(z-2)}}{(z+1)^2} \text{-----------------------------}(1)$$

The derivative of analytic function is given by

$$\int_c \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i f^n(a)}{n!} \text{-------------------------------------}(2)$$

From (1) and (2) f(z)=$\frac{(z-1)}{(z-2)}$, a=-1,n=1

$$f^1(z) = \frac{1(z-2)-1(z-1)}{(z-2)^2} = \frac{1}{(z-2)^2}$$

$$f^1(-1) = \frac{1}{-9}$$

Substituting in (2),we get

$$\int_c \frac{(z-1)}{(z+1)^2(z-2)}dz = \frac{2\pi i}{1}(-\frac{1}{9})$$

$$= \frac{-2}{9}\Pi i$$

Problem: Evaluate $\int_c \frac{e^{2z}}{(z+1)^4}dz$ where c:$|z-1| = 1$

Solution: the singular points of $\frac{e^{2z}}{(z+1)^4}dz$ are givenby7

$(z+1)^4=0 \Rightarrow z = -1$

The singular point z=-1 lies insidethecirclec:$|z-1| = 3$

Applying cauchy's integral formula for derivatives

$$\int_c \frac{f(z)}{(z-a)^{n+1}}dz = \int_c \frac{2\pi i f^{n}(-1)}{n!}dz \text{ --------------------------(1)}$$

$f(z)=e^{2z}$,n=3,a=-1

$f(z)=2e^{2z}$

$f^{1}(z)=4e^{2z}$

$f^{11}(z)=8e^{2z}$

$f^{111})z)=16e^{2z}$

$f^{111}(-1)=16e^{-2}$

substituting in(1)

$$\int_c \frac{e^{2z}}{(z+1)^4}dz = \int_c \frac{2\pi i f^{111}(-1)}{n!}$$

$$= \frac{2\pi i 16\ e^{-2}}{2!}$$

$=16\Pi ie^{-2}$

Problem: Use cauchy's integral formula to evaluate $\int_c \frac{e^{-2z}}{(z+1)^3}dz$ with c:$|z| = 2$

Solution: Given $\int_c \frac{e^{-2z}}{(z+1)^3}dz$

$f(z)=e^{-2z}$

the singular poin z=-1 lies inside the given circle $|z| = 2$

apply Cauchy's integral formula for derivatives

$$\int_c \frac{e^{-2z}}{(z+1)^3}dz = \frac{2\pi i f^{1}(-1)}{2!} \qquad \left[\because \int_c \frac{f(z)}{(z-a)^3} = \frac{2\pi i f^{1}(a)}{2!}\right]$$

Where $f(z)=e^{-2z}$

$f^{1}(z)=-2\,e^{-2z}$

$f^{11}(z)=4\,e^{-2z}$

$f^{11}(-1)=4\,e^{2}$

$$\therefore \int_c \frac{e^{-2z}}{(z+1)^3}dz = \frac{2\pi i 4 e^2}{2} = 4\pi i e^2$$

Problem: Evaluate $\int_c \frac{dz}{z^8(z+4)}dz$ with c: $|z| = 2$

Solution:

The singularities of $\int_c \frac{dz}{z^8(z+4)}dz$ are given by

$Z^8(z+4) = 0 \Rightarrow z = 0, z = -4$

The point z=0 lie inside and the z=-4 lies outside the circle $|z| = 2$

By the derivative of analytic function.

Problem: Evaluate using integral formula $\oint_c \frac{e^{2z}dz}{(z-1)(z-2)}$ where c is the circle $|z| = 3$

Solution: Let (z)= e^z which is analytic within the circle c: $|z| = 3$ and the two singular points a=1,a=2 lie inside c.

$$\oint_c \frac{e^{2z}dz}{(z-1)(z-2)} = \oint e^{2z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right)dz$$

$$\oint_c \frac{e^{2z}}{z-2}dz - \oint_c \frac{e^{2z}}{z-1}dz$$

Now using cauchy's integral formula ,we obtain

$$\oint_c \frac{e^{2z}dz}{(z-1)(z-2)} = 2\pi i e^4 - 2\pi i e^2$$

$$= 2\pi i(e^4 - e^2)$$

$$\oint_c \frac{e^{2z}dz}{(z-1)(z-2)} = 2\pi i(e^4 - e^2)$$

Problem : Evaluate $\oint_c \frac{3z^2+z}{z^2-1}dz$ where cisthe circle $|z-1| = 1$

Solution: Given f(z)= $3z^2+z$

Z=a=+1or -1

The circle $|z-1| = 1$ has centre at z=1 and radius 1 and includes the point z=1,f(z)=$3z^2+z$ is an analytic function

Also $\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$

$$\oint \frac{3z^2+z}{z^2-1} = \frac{1}{2}\left[\int_c \frac{3z^2+z}{z-1}dz\right] - \frac{1}{2}\left[\int_c \frac{3z^2+z}{z+1}dz\right] \text{----------------(1)}$$

Since z=1 lies inside c,we have by cauchy's integralformula

$$\oint_c \frac{3z^2+z}{z^2-1}dz = 2\pi i f(i)$$

$$= 2\pi i * 4$$

Bycauchy'sintegral theorem ,since z=-1 lies out side c,we have

$$\oint_c \frac{3z^2+z}{z-1}dz = 0$$

From equation(1) we have

$$\oint_c \frac{3z^2+z}{z^2-1}dz = \frac{1}{2}(8\pi i) - 0$$

$$= 4\pi i$$

EXCERCISE PROBLEMS:

1) Evaluate $\int \frac{dz}{z-z_0}$ where c: $\left|z-z_0\right|$ = r

2) Evaluate $\int_{(1,1)}^{(2,2)} (x+y)dx+(y-x)dy$ along the parabola $y^2 = x$

3)Evaluate $\int_c \frac{z^2+4}{z^2-1}dz$ where C: $|z| = 2$ using Cauchy's Integral formula

4)Evaluate $\int_c \frac{e^{2z}}{(z-1)(z-2)}dz$ where C: $|z| = 4$ using Cauchy's integral formula

5) Evaluate $\int_c \frac{z^3-z}{(z-2)^3}$ where $C : |z| = 3$ using Cauchy's integral formula

6) Expand f(z) = $\int_c \frac{e^{2z}}{(z-1)^3}$ at a point z=1

7) Expand f(z)= $\int_c \frac{1}{z^2-4z+3}$ for $1 < |z| < 3$

8)Evaluate $\int (y^2+z^2)dx+(z^2+x^2)dy+(x^2+y^2)dz$ from (0,0,0) to (1,1,1) , where

C is the curve $x = t, x = t^2, x = t^3$

9) Evaluate $\int_{(0,0)}^{(1,1)} (3x^2+4xy+ix^2)dz$ along $y = x^2$

10) Evaluate $\int_0^{1+i} (x-y+ix^2)$ dz

(i) along the straight from z = 0 to z = 1+i .

(ii) along the real axis from z = 0 to z = 1 and then along a line parallel to real axis from z = 1 to z = 1+i

(iii) along the imaginary axis from z = 0 to z = I and then along a line parallel to real axis z = i to z = 1+ i .

11) Evaluate $\int_{1-i}^{2+i} (2x + 1 + iy)$ dz along (1-i) to (2+i)

12) Evaluate $\int_{c} (y^2 + 2xy)dx + (x^2 - 2xy)dy$ where c is boundary of the region y=x^2 and x=y^2

UNIT-III
POWER SERIES EXPANSION OF COMPLEX FUNCTION

Power series:

A series expansion is a representation of a particular function as a sum of powers in one of its variables, or by a sum of powers of another (usually elementary) function f(z).

A power series in a variable z is an infinite sum of the form

$$\sum a_i z^i$$

A series of the form $\sum a_n z^n$ is called as power series.

That is $\sum a_n z^n = a_1 z + a_2 z^2 + \dots\dots\dots + a_n z^n + \dots\dots$

Taylor's series:

Taylor's theorem states that any function satisfying certain conditions may be represented by a Taylor series.

The Taylor series is an infinite series, whereas a Taylor polynomial is a polynomial of degree n and has a finite number of terms. The form of a Taylor polynomial of degree n for a function

f (z) at x = a is

$$f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{(z-a)^2}{2!} + f'''(a)\frac{(z-a)^3}{3!} + \dots\dots\dots + f^n(a)\frac{(z-a)^n}{n!} + \dots\dots ,$$
$$|z-a| < r$$

Maclaurin series:

A Maclaurin series is a Taylor series expansion of a function about x=0,

$$f(z) = f(0) + f'(0)(z) + f''(0)\frac{(z)^2}{2!} + f'''(0)\frac{(z)^3}{3!} + \dots\dots\dots + f^n(0)\frac{(z)^n}{n!} + \dots\dots$$

This series is called as maclurins series expansion of f(z).

Some important result:

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \dots \quad \text{for } -1 < x \le 1$$
$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \dots \quad \text{for } -\infty < x < \infty$$
$$\cos x = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \tfrac{1}{720}x^6 + \dots \quad \text{for } -\infty < x < \infty$$
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots \quad \text{for } -1 < x < 1$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Problems

Problem: Determine the first four terms of the power series for sin 2x using Maclaurin's series.

Solution:

Let

$f(x) = \sin 2x \qquad f(0) = \sin 0 = 0$

$f'(x)= 2 \cos 2x \qquad f'(0) = 2 \cos 0 = 2$

$f''(x)= -4 \sin 2x \qquad f''(0) = -4 \sin 0 = 0$

$f'''(x) = -8 \cos 2x \qquad f'''(0) = -8 \cos 0 = -8$

$f^{iv}(x)= 16 \sin 2x \qquad f^{iv}(0)= 16 \sin 0 = 0$

$f^{v}(x)= 32 \cos 2x(0) \qquad f^{v}(0)= 32 \cos 0 = 32$

$f^{vi}(x)= -64 \sin 2x \qquad f^{vi}(0)= -64 \sin 0 = 0$

$f^{vi\,i}(x)= -128 \cos 2x \qquad f^{vii}(0)= -128 \cos 0 = -128$

$$f(x)= \sin 2x = 0+2x+0x^2+(-8)\frac{x^3}{3!}+0.x^4+32\frac{x^5}{5!}$$

$$= 2x - \frac{4x^3}{3} + \frac{4x^5}{15}$$

Problem : Find the Taylor series about z = -1 for f (x) = 1/z. Express your answer in sigma notation.

Solution:

Let $f(z) = z^{-1} \qquad f(-1) = -1$

$f' = -z^{-2} \qquad f'(-1) = -1$

$f'' = 2z^{-3} \qquad f''(-1) = -2$

$f''' = -6z^{-4} \qquad f'''(-1) = -6$

$f'''' = 24z^{-5} \qquad f''''(-1) = -24$

$$f(z) = -1-1(z+1) - \frac{2}{2!}(z+1)^2 - \frac{6}{3!}(z+1)^3 - \frac{24}{4!}(z+1)^4 - \ldots\ldots\ldots$$

$$= \sum_{n=0}^{\infty} -1(z+1)^n$$

Problem : Find the Maclaurin series for $f(z) = z e^z$ Express your answer in sigma notation.

Solution:

Let $f(z) = z e^z \qquad f(0) = 0$

$f' = e^z + z e^z \qquad f'(0) = 1 + 0 = 1$

$f'' = e^z + e^z + z e^z \qquad f''(0) = 1 + 1 + 0 = 2$

$f''' = e^z + e^z + e^z + z e^z \qquad f'''(0) = 1 + 1 + 1 + 0 = 3$

$f'''' = e^z + e^z + e^z + e^z + z e^z \qquad f''''(0) = 1 + 1 + 1 + 1 + 0 = 4$

$$f(z) = 0 + 1z + \frac{2}{2!}z^2 + \frac{3}{3!}z^3 + \frac{4}{4!}z^4 + \ldots\ldots\ldots\ \ldots$$

$$= z + z^2 + \frac{1}{2}z^3 + \frac{1}{6}z^4 + \ldots\ldots\ldots\ \ldots$$

$$= \sum_{n=1}^{\infty} \frac{z^n}{(n-1)!}$$

Problem: Expand log z by taylor's series about z=1.

Solution:

Let f(z) = log z

Put z-1= w

z= 1+w

log z = log (1+w)

f(z)=log z = log (1+w)

$$= w - \frac{w^2}{2} + \frac{w^3}{3} - \ldots\ldots\ldots + (-1)^n \frac{w^n}{n!} + \ldots\ldots\ldots; \quad |w| < 1$$

$$f(z) = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \ldots\ldots\ldots + (-1)^n \frac{(z-1)^n}{n!} + \ldots\ldots\ldots; \quad |z-1| < 1$$

Laurent series:

In mathematics, the **Laurent series** of a complex function *f*(*z*) is a representation of that function as a power series which includes terms of negative degree. It may be used to express complex functions in cases where a Taylor series expansion cannot be applied.

The Laurent series for a complex function *f*(*z*) about a point *c* is given by:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z-a)^n}$$

where the a_n *and a* are constants.

Laurent polynomials:

A Laurent polynomial is a Laurent series in which only finitely many coefficients are non-zero. Laurent polynomials differ from ordinary polynomials in that they may have terms of negative degree.

Principal part:

The principal part of a Laurent series is the series of terms with negative degree, that is

$$f(z) = \sum_{K=-\infty}^{-1} a_K (z-a)^K$$

If the principal part of *f* is a finite sum, then *f* has a pole at *c* of order equal to (negative) the degree of the highest term; on the other hand, if *f* has an essential singularity at *c*, the principal part is an infinite sum (meaning it has infinitely many non-zero terms).

Two Laurent series with only *finitely* many negative terms can be multiplied: algebraically, the sums are all finite; geometrically, these have poles at *c*, and inner radius of convergence 0, so they both converge on an overlapping annulus.
Thus when defining formal Laurent series, one requires Laurent series with only finitely many negative terms.
Similarly, the sum of two convergent Laurent series need not converge, though it is always defined formally, but the sum of two bounded below Laurent series (or any Laurent series on a punctured disk) has a non-empty annulus of convergence.

Zero's of an analytic function:

A zero of an analytic function f(z) is a value of z such that f(z)=0 .Particularly a point a is called a zero of an analytic function f (z) if f(a) = 0.

Eg: $f(z) = \frac{(z+1)^2}{(z^2+1)^2}$

Now, $(z+1)^2 = 0$

Z = -1, z = -1 are zero's of an analytic function.

Zero's of mth order:

If an analytic function f(z) can be expressed in the form $f(z) = (z-a)^m \Phi(z)$ where $\Phi(z)$ is analytic function and $\Phi(a) \neq 0$ then z=a is called zero of mth order of the function f(z).

- A simple zero is a zero of order 1.

Eg: 1. $f(z) = (z-1)^3$

$\Rightarrow (z-1)^3 = 0$

z=1 is a zero of order 3 of the function f(z).

2. $f(z) = \frac{1}{1-z}$

i.e $z = \infty$ is a simple zero of f(z).

3. $f(z) = \sin z$

i.e $z = n\pi \quad \forall n = 0,1,2,3,......$ are simple zero's of f(z).

Problems

Problem: Find the first four terms of the Taylor's series expansion of the complex function

$f(z) = \frac{z+1}{(z-3)(z-4)}$ about z =2.Find the region of convergence.

Solution:

The singularities of the function $f(z) = \frac{z+1}{(z-3)(z-4)}$ are z = 3 and z = 4

Draw a circle with centre at z=2 and radius 1 .Then the distance of singularities from the centre are 1 and 2.

Hence within the circle $|z-2| = 1$,the given function is analytic .Hence ,it can be extended in Taylor's series within the circle $|z-2| = 1$.

Hence $|z-2| = 1$ is the circle of convergence.

Now $f(z) = \frac{5}{z-4} - \frac{4}{z-3}$ (partial fraction) , f(2)= 3/2

$$f'(z) = -\frac{5}{(z-4)^2} + \frac{4}{(z-3)^2} \quad , \quad f'(2) = \frac{11}{4}$$

$$f''(z) = -\frac{8}{(z-3)^3} + \frac{10}{(z-4)^3} \quad , \quad f''(2) = \frac{27}{4}$$

$$f'''(z) = \frac{24}{(z-3)^4} - \frac{30}{(z-4)^4} \quad , \quad f'''(2) = \frac{177}{8}$$

Taylor's series expansion for f(z) at z=a is

$$f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{(z-a)^2}{2!} + f'''(a)\frac{(z-a)^3}{3!} + \dots\dots\dots + f^n(a)\frac{(z-a)^n}{n!} + \dots\dots.$$

$$\frac{z+1}{(z-3)(z-4)} = \frac{3}{2} + (z-2)\frac{11}{4} + \frac{(z-2)^2}{2!}\left(\frac{27}{4}\right) + \frac{(z-2)^3}{3!}\left(\frac{177}{8}\right)$$

$$f(z) = \frac{3}{2} + (z-2)\frac{11}{4} + (z-2)^2\left(\frac{27}{8}\right) + (z-2)^3\left(\frac{59}{16}\right).$$

Problem: ObtainLaurent series for $f(z) = \frac{e^{2z}}{(z-1)^3}$ about z = 1.

Solution:

Given $f(z) = \frac{e^{2z}}{(z-1)^3}$

Put z-1= w so that z = w+1

$$f(z) = \frac{e^{2(1+w)}}{w^3}$$

$$f(z) = \frac{e^2 e^{2w}}{w^3}$$

$$= \frac{e^2}{w^3}\left[1 + 2w + \frac{(2w)^2}{2!} + \frac{(2w)^3}{3!} + ...\right] \text{ if } w \neq 0$$

$$= e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!} w^{n-3}$$

$$= e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!} (z-1)^{n-3} \quad \text{, if } z - 1 \neq 0$$

$$= e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!} (z-1)^{n-3} \quad \text{, if } |z - 1| \neq 0$$

$$f(z) = e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!} (z-1)^{n-3} \quad \text{, if } |z - 1| > 0$$

Since points $|z-1| \leq 0$ will be singular points.

Singular point of an analytic function: A point at which an analytic function f(z) is not analytic, i.e. at which f '(z) fails to exist, is called a **singular point** or **singularity** of the function.

There are different types of singular points:

Isolated and non-isolated singular points: A singular point z_0 is called an **isolated singular point** of an analytic function f(z) if there exists a deleted ε-spherical neighborhood of z_0 that contains no singularity. If no such neighborhood can be found, z_0 is called a non-isolated singular point. Thus an isolated singular point is a singular point that stands completely by itself, embedded in regular points. In fig 1a where z_1, z_2 and z_3 are isolated singular points. Most singular points are isolated singular points. A non-isolated singular point is a singular point such that every deleted ε-spherical neighborhood of it contains singular points. See Fig. 1b where z_0 is the limit point of a set of singular points. Isolated singular points include poles, removable singularities, essential singularities and branch points.

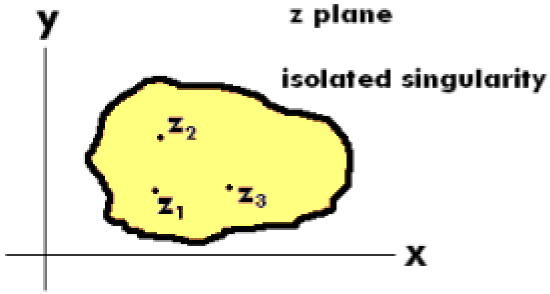

Fig. 1a

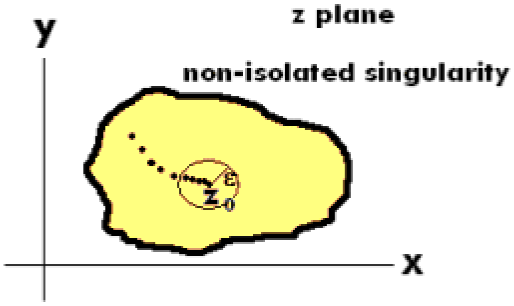

Fig. 1b

Types of isolated singular points:

Pole: An isolated singular point z_0 such that f(z) can be represented by an expression that is of the form

$$f(z) = \frac{\phi(z)}{(z - z_0)^n}$$

Where n is a positive integer, $\phi(z)$ is analytic at z_0, and $\phi(z_0) \neq 0$. The integer n is called the **order** of the pole. If n = 1, z_0 is called a simple pole.

Example: 1. The function

$$f(z) = \frac{5z + 1}{(z-2)^3 (z+3)(z-2)}$$

has a pole of order 3 at $z = 2$ and simple poles at $z = -3$ and $z = 2$.

1. A point z is a pole for f if f blows up at z (f goes to infinity as you approach z). An example of a pole is z=0 for f(z) = 1/z.

Simple pole : A pole of order 1 is called a simple pole whilst a pole of order 2 is called a double pole.

If the principal part of the Laurent series has an infinite number of terms then $z = z_0$ is called an isolated essential singularity of f(z). The function $f(z) = i/ z(z - i) \equiv 1/(z - i) - (1/ z)$ has a simple pole at z = 0 and another simple pole at z = i.

The function $e^{\frac{1}{z-2}}$ has an isolated essential singularity at z = 2. Some complex functions have non-isolated singularities called branch points. An example of such a function is $\sqrt{z}$.

Removable singular point: An isolated singular point z_0 such that f can be defined, or redefined, at z_0 in such a way as to be analytic at z_0. A singular point z_0 is removable if

$\lim_{z \to z_0} f(z)$ exist.

Example: 1.The singular point z = 0 is a removable singularity of f(z) = (sin z)/z since

$$\lim_{z \to 0} \frac{\sin z}{z} = 1$$

A point z is a removable singularity for f if f is defined in a neighborhood of the point z, but not at z, but f can be defined at z so that f is a continuous function which includes z. Here is an example of this: if f(z) = z is defined in the punctured disk, the disk minus 0, then f is not defined at z=0, but it can certainly be extended continuously to 0 by defining f(0) = 0. This means at z=0 is a removable singularity.

Essential singular point: A singular point that is not a pole or removable singularity is called an essential singular point.

Example: 1. $f(z) = e^{1/(z-3)}$ has an essential singularity at z = 3.

2. A point z is an essential singularity if the limit as f approaches z takes on different values as you approach z from different directions. An example of this is exp(1/z) at z=0. As z approaches 0 from the right, exp(1/z) blows up and as z approaches 0 from the left, exp(1/z) goes to 0.

Singular points at infinity: The type of singularity of f(z) at $z = \infty$ is the same as that of f(1/w) at w = 0. Consult the following example.

Example: The function $f(z) = z^2$ has a pole of order 2 at $z = \infty$, since f(1/w) has a pole of order 2 at w = 0.

Using the transformation w = 1/z the point z = 0 (i.e. the origin) is mapped into $w = \infty$, called the point at infinity in the w plane. Similarly, we call $z = \infty$ the point at infinity in the z plane. To consider the behavior of f(z) at $z = \infty$, we let z = 1/w and examine the behavior of f(1/w) at w = 0.

Residues:

The constant a_{-1} in the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{1}$$

of about a point z_0 is called the residue of f(z). If f is analytic at z_0, its residue is zero, but the converse is not always true (for example, $\frac{1}{z^2}$ has residue of 0 at z=0 but is not analytic at z=0 .

The residue of a function f at a point z_0 may be denoted $\operatorname{Res}_{z \to z_0} f(z)$.

Residue: Let $f(z)$ have a nonremovable isolated singularity at the point z_0. Then $f(z)$ has the Laurent series representation for all z in some disk $D_R^*(z_0)$ given by

$$. \quad f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{1}$$

The coefficient a_{-1} of $\frac{1}{z - z_0}$ is called the residue of $f(z)$ at z_0 and we use the notation

$$\text{Res}[f, z_0] = a_{-1}$$

Example : If $f(z) = e^{\frac{2}{z}}$, then the Laurent series of f about the point $z_0 = 0$ has the form

$$f(z) = 1 + 2\frac{1}{z} + \frac{2^2}{2!\,z^2} + \frac{2^3}{3!\,z^3} + \frac{2^4}{4!\,z^4} + \frac{2^5}{5!\,z^5} + \ldots$$, and

Res[f, 0] = a_{-1} =2

The residue of a function f around a point z_0 is also defined by

$$\operatorname{Res} f = \frac{1}{2\pi i}\int_c f(z)\,dz \tag{2}$$

where C is counterclockwise simple closed contour, small enough to avoid any other poles of f. In fact, any counterclockwise path with contour winding number 1 which does not contain any other poles gives the same result by the Cauchy integral formula. The above diagram shows a suitable contour for which to define the residue of function, where the poles are indicated as black dots.

It is more natural to consider the residue of a meromorphic one-form because it is independent of the choice of coordinate. On a Riemann surface, the residue is defined for a meromorphic one-form α at a point p by writing $\alpha = f\,d\,z$ in a coordinate z around p. Then

$$\operatorname*{Res}_{p}\alpha = \operatorname*{Res}_{z=p} f. \tag{3}$$

The sum of the residues of $\int f\,d\,z$ is zero on the Riemann sphere. More generally, the sum of the residues of a meromorphic one-form on a compact Riemann surface must be zero.

The residues of a function $f(z)$ may be found without explicitly expanding into a Laurent series as follows. If $f(z)$ has a pole of order m at z_0, then $a_n = 0$ for $n < -m$ and $a_{-m} \neq 0$. Therefore,

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} a_{-m+n} (z - z_0)^{-m+n} \tag{4}$$

$$(z - z_0)^m f(z) = \sum_{n=0}^{\infty} a_{-m+n} (z - z_0)^n \tag{5}$$

$$\frac{d}{dz}\left[(z - z_0)^m f(z)\right] = \sum_{n=0}^{\infty} n\, a_{-m+n} (z - z_0)^{n-1} \tag{6}$$

$$= \sum_{n=1}^{\infty} n\, a_{-m+n} (z - z_0)^{n-1} \tag{7}$$

$$= \sum_{n=0}^{\infty} (n+1)\, a_{-m+n+1}\, (z - z_0)^n \tag{8}$$

$$\frac{d^2}{d\,z^2} [(z - z_0)^m f(z)] = \sum_{n=0}^{\infty} n\,(n+1)\, a_{-m+n+1}\, (z - z_0)^{n-1} \tag{9}$$

$$= \sum_{n=1}^{\infty} n\,(n+1)\, a_{-m+n+1}\, (z - z_0)^{n-1} \tag{10}$$

$$= \sum_{n=0}^{\infty} (n+1)\,(n+2)\, a_{-m+n+2}\, (z - z_0)^n. \tag{11}$$

Iterating,

$$\frac{d^{m-1}}{d\,z^{m-1}} [(z - z_0)^m f(z)] = \sum_{n=0}^{\infty} (n+1)\,(n+2) \cdots (n+m-1)\, a_{n-1}\, (z - z_0)^n$$

$$= (m-1)!\, a_{-1} + \sum_{n=1}^{\infty} (n+1)\,(n+2) \cdots (n+m-1)\, a_{n-1}\, (z - z_0)^{n-1}. \tag{12}$$

So

$$\lim_{z \to z_0} \frac{d^{m-1}}{d\,z^{m-1}} [(z - z_0)^m f(z)] = \lim_{z \to z_0} (m-1)!\, a_{-1} + 0 \tag{13}$$

$$= (m-1)!\, a_{-1}, \tag{14}$$

and the residue is

$$a_{-1} = \frac{1}{(m-1)!} \frac{d^{m-1}}{d\,z^{m-1}} [(z - z_0)^m f(z)]_{z=z_0}. \tag{15}$$

The residues of a holomorphic function at its poles characterize a great deal of the structure of a function, appearing for example in the amazing residue theorem of contour integration.

If $f(z)$ has a removable singularity at z_0 then a_{-1} =0 for n=1,2,……. Therefore, Res[f, z_0]=0.

Residues at Poles:

(i) If f(z) has a simple pole at z_0 , then $\operatorname{Re} s[f, z_0] = \lim_{z \to z_0} (z - z_0) f(z)$

(ii) If f(z) has a pole of order 2 at z_0 , then $\operatorname{Re} s[f, z_0] = \lim_{z \to z_0} \frac{d}{dz}(z - z_0)^2 f(z)$

(iii) If f(z) has a pole of order 3 at z_0 , then $\operatorname{Re} s[f, z_0] = \frac{1}{2!} \lim_{z \to z_0} \frac{d^2}{dz^2}((z - z_0)^3 f(z))$

(v) If f(z) has a pole of order k at z_0 , then $\operatorname{Re} s[f, z_0] = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}}((z - z_0)^k f(z))$

Cauch's Residue Theorem:

An analytic function f(z) whose Laurent series is given by $f(z) = \lim_{z \to z_0} (z - z_0) f(z)$ (1)

can be integrated term by term using a closed contour C encircling z_0,

$$\int_c f(z)dz = \sum_{n=-\infty}^{\infty} a_n \int_c (z - z_0)^n dz$$
$$= \sum_{n=-\infty}^{-2} a_n \int_c (z - z_0)^n dz + a_{-1} \int_c \frac{dz}{(z - z_0)} + \sum_{n=0}^{\infty} a_n \int_c (z - z_0)^n dz \qquad (2)$$

The Cauchy integral theorem requires that the first and last terms vanish, so we have

$$\int_c f(z)dz = a_{-1} \int_c \frac{dz}{z - z_0} \qquad (3)$$

where a_{-1} is the complex residue. Using the contour $z=c(t)=e^{it}+z_0$ gives

$$\int_c \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{ie^{it}}{e^{it}} dt = 2\pi i \qquad (4)$$

so we have

$$\int_c f(z)dz = a_{-1} 2\pi i \qquad (5)$$

If the contour C encloses multiple poles, then the theorem gives the general result

$$\int_c f(z)dz = 2\pi i \sum_{a\in A} \operatorname{Re}_{z=a_i} s\, f(z) \qquad (6)$$

Where A is the set of poles contained inside the contour. This amazing theorem therefore says that the value of a contour integral for *any* contour in the complex plane depends *only* on the properties of a few very special points *inside* the contour.

Residue at infinity:

The residue at infinity is given by:

$$\operatorname{Re} s[f(z)]_{Z=\infty} = -\frac{1}{2\pi i}\int_C f(z)dz$$

Where f is an analytic function except at finite number of singular points and C is a closed countour so all singular points lie inside it.

Problem: Determine the poles of the function $f(z) = \frac{z+2}{(z+1)^2(z-2)}$ and the residue at each pole.

Solution: The poles of f(z) are given by $(z+1)^2(z-2)=0$

Here z=2 is a simple pole and z= -1 is a pole of order 2 .

Residue at z=2 is

$$\lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}(z-2)\frac{z+2}{(z+1)^2(z-2)} = \frac{4}{9}$$

Residue at z=-1 is

$$\lim_{z\to -1}\frac{d}{dz}(z+1)^2 f(z) = \lim_{z\to -1}\frac{d}{dz}(z+1)^2\frac{z+2}{(z+1)^2(z-2)}$$

$$\lim_{z\to -1}\frac{d}{dz}\frac{(z+2)}{(z-2)} = \lim_{z\to -1}\frac{-4}{(z-2)^2} = \frac{-4}{9}$$

Problem: Find the residue of the function $f(z) = \frac{1-e^{2z}}{z^4}$ at the poles.

Solution: Let $f(z) = \frac{1-e^{2z}}{z^4}$

z =0 is a pole of order 4

Residue of f(z) at z=0 is

$$=\frac{1}{3!}\lim_{z\to 0}\frac{d^3}{dz^3}(z-0)^4\frac{(1-e^{2z})}{z^4}$$

$$=\frac{1}{3!}\lim_{z\to 0}\frac{d^3}{dz^3}(1-e^{2z})$$

$$=\frac{1}{3!}\lim_{z\to 0}\frac{d^2}{dz^2}(-2e^{2z})$$

$$=\frac{1}{3!}\lim_{z\to 0}\frac{d}{dz}(-4e^{2z})$$

$$=\frac{1}{3!}\lim_{z\to 0}(-8e^{2z})$$

$$=\frac{-8}{3!}=\frac{-4}{3}.$$

Problem: Find the residue of the function $f(z)=z^3\cos(\frac{1}{z})$ at $z=\infty$.

Solution: Let $f(z)=z^3\cos(\frac{1}{z})$

$$g(t)=f(\frac{1}{t})=\frac{1}{t^3}\cos t$$

$$=\frac{1}{t^3}\left[1-\frac{t^2}{2!}+\frac{t^4}{4!}-..........\right]$$

$$=\left[\frac{1}{t^3}-\frac{1}{2t}+\frac{t}{24}-..........\right]$$

There fore $\operatorname{Re}\underset{z\to\infty}{sf}(z)$ = - coefficient ot t in the eapansion of g(t) about t=0

= -1/24.

Problem: Evaluate $\int_c \frac{4-3z}{z(z-1)(z-2)}dz$ where c is the circle $|z|=\frac{3}{2}$. Using Residue theorem.

Solution: Let $\int_c \frac{4-3z}{z(z-1)(z-2)}dz$

The poles of f(z) are z (z-1)(z-2)=0

z=0, z= 1, z=2

These poles are simple poles.

The poles z=0 and z=1 lie within the circle c: $|z|=\frac{3}{2}$

Residue of f(z) at z=0 is $R_1=\lim_{z\to 0}(z-0)f(z)=\lim_{z\to 0}z\frac{4-3z}{z(z-1)(z-2)}=\frac{4}{2}=2$

Residue of f(z) at z=1 is $R_2=\lim_{z\to 1}(z-1)f(z)=\lim_{z\to 1}(z-1)\frac{4-3z}{z(z-1)(z-2)}=\frac{4-3}{1-2}=-1$

By Residue theorem, $\int_c \frac{4-3z}{z(z-1)(z-2)}dz=2\pi i(R_1+R_2)=2\pi i(2-1)=2\pi i.$

Problem: Evaluate $\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta}$ Using Residue theorem.

Solution: Let $I = \int_{-\pi}^{\pi} \frac{d\theta}{5 + 4\sin\theta}$

Put $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$ and $\sin\theta = \frac{1}{2i}(z - \frac{1}{z})$

$z = e^{i\theta}$ unit circle c: $|z| = 1$

$$I = \int_c \frac{1}{5 + 4\frac{1}{2i}(z - \frac{1}{z})} \frac{dz}{iz}$$

$$= \int_c \frac{dz}{z^2 + 5iz + 2i^2} = \int_c \frac{dz}{(2z + i)(z + 2i)}$$

$$= \int_c \frac{dz}{(z + \frac{i}{2})(z + 2i)}$$

$$= \frac{1}{2}\int_c f(z)dz$$

Where $f(z) = \left(z + \frac{i}{2}\right)(z + 2i)$

The poles of f(z) are z=-i/2 and z=-2i

The pole z=-i/2 lies inside the unit circle .

Residue of f(z) at z=-i/2 is

$$= \lim_{z \to -i/2} (z + \frac{i}{2}) f(z)$$

$$= \lim_{z \to -i/2} (z + \frac{i}{2}) \frac{1}{(z + \frac{i}{2})(z + 2i)}$$

$$= \lim_{z \to -i/2} \frac{1}{(z + 2i)}$$

$$= \frac{1}{\frac{-i}{2} + 2i} = \frac{2}{3i}$$

By cauchy's residue theorem

$$I = \frac{1}{2}\int_c f(z)dz = \frac{1}{2} 2\pi i \left(\frac{2}{3i}\right) = \frac{2\pi}{3}$$

$$\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta} = \frac{2\pi}{3} .$$

Problem: Prove that $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{a+b} \qquad (a > 0, b > 0, a \neq b)$

Solution: To evaluate the given integral, consider $\int_c \frac{z^2 dz}{(z^2 + a^2)(z^2 + b^2)} = \int_c f(z) dz$

Where c is the contour consisting of the semi circle C_R of radius R together with the real part of the real axis from –R to R.

The poles of $f(z) = \frac{z^2}{(z^2 + a^2)(z^2 + b^2)}$ are $z = \pm ai \quad ; z = \pm bi$

But z=ia and z=ib are the only two poles lie in the upper half of the plane .

$$[\operatorname{Re} sf(z)]_{z=ia} = \underset{z \to ai}{Lt} (z - ia) f(z)$$

$$= \underset{z \to ai}{Lt} \frac{z^2}{(z + ia)(z^2 + b^2)} = \frac{-a^2}{2ia(-a^2 + b^2)} = \frac{a}{2i(a^2 - b^2)}$$

Also $[\operatorname{Re} sf(z)]_{z=ib} = \underset{z \to bi}{Lt} (z - ib) f(z)$

$$= \underset{z \to bi}{Lt} \frac{z^2}{(z + ib)(z^2 + a^2)} = \frac{-b^2}{2ib(a^2 - b^2)} = \frac{-b}{2i(a^2 - b^2)}$$

By Cauchy's Residue theorem ,we have $\int_c f(z) dz = 2\pi i$ (sum of the residues with in C)

$$\int_c f(z) dz = 2\pi i \left[\frac{a}{2i(a^2 - b^2)} - \frac{b}{2i(a^2 - b^2)} \right] = \pi \left[\frac{a - b}{(a^2 - b^2)} \right] = \frac{\pi}{a + b}$$

We have $\int_{-R}^{R} f(x) dx + \int_{C_R} f(z) dz = \frac{\pi}{a + b}$

But $\int_{C_R} f(z) dz \to 0 \quad as \quad z = \operatorname{Re}^{i\theta} \ and \quad R \to \infty$

Hence $\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{a + b}$

EXCERCISE PROBLEMS:

1)Evaluate $\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}$ where C: $|z| = 1$

2)Prove that $\int_{-\infty}^{\infty} \frac{dx}{a^2 + x^2} = \frac{\pi}{a}$

3)Show that $\int_{-\infty}^{\infty} \frac{dx}{(x+1)^3} = \frac{3\pi}{8}$

4)Prove that $\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{ab(a+b)}$

5)Evaluate $\int (y^2 + z^2)dx + (z^2 + x^2)dy + (x^2 + y^2)dz$ from (0,0,0) to (1,1,1) , where C is the curve $x = t, x = t^2, x = t^3$

6) Evaluate $\int_{(0,0)}^{(1,1)} (3x^2 + 4xy + ix^2)dz$ along $y = x^2$

7)Obtain the Taylor series expansion of f(z) = $\frac{1}{z}$ about the point z = 1

8)Obtain the Taylor series expansion of f(z) = e^z about the point z = 1

9)Expand f(z) = $\frac{z-1}{z+1}$ in Taylor's series about the point (i) z = 0 (ii) z = 1

10)Expand f(z) = $\frac{1}{z^2}$ in Taylor's series in powers of z +1

11)Obtain Laurent's series expansion of f(z) = $\frac{z^2 - 4}{z^2 + 5z + 4}$ valid in 1< z < 2

12)Give two Laurent's series expansions in powers of Z for f(z) = $\frac{1}{z^2(1-z)}$

13)Expand f(z)= $\frac{1}{(1-z)(z-2)}$

14)Maclaurin's series expansion of f(z)

15)Laurent's series expansion in the annulus region in

a) $1 < |z| < 2$

16)Find the residue of the function $f(z) = \frac{z^3}{(z^2 - 1)}$ $at\ z = \infty$

17) Find the residue of $\frac{z^2}{z^4+1}$ at these singular points which lie inside the circle $|z|=2$

18) Find the residue of the function $f(z)=\frac{z^2-2z}{(z^2+1)(z+1)^2}$ at each pole

UNIT - IV
SINGLE RANDOM VARIABLE

Probability

Trial and Event: Consider an experiment, which though repeated under essential and identical conditions, does not give a unique result but may result in any one of the several possible outcomes. The experiment is known as **Trial** and the outcome is called **Event**

E.g. (1) Throwing a dice experiment getting the no's 1,2,3,4,5,6 (event)

(2) Tossing a coin experiment and getting head or tail (event)

Exhaustive Events:

The total no. of possible outcomes in any trial is called exhaustive event.

E.g.: (1) In tossing of a coin experiment there are two exhaustive events.

(2) In throwing an n-dice experiment, there are 6^n exhaustive events.

Favorable event:

The no of cases favorable to an event in a trial is the no of outcomes which entities the happening of the event.

E.g. (1) In tossing a coin, there is one and only one favorable case to get either head or tail.

Mutually exclusive Event: If two or more of them cannot happen simultaneously in the same trial then the event are called mutually exclusive event.

E.g. In throwing a dice experiment, the events 1,2,3,------6 are M.E. events

Equally likely Events: Outcomes of events are said to be equally likely if there is no reason for one to be preferred over other. E.g. tossing a coin. Chance of getting 1,2,3,4,5,6 is equally likely.

Independent Event:

Several events are said to be independent if the happening or the non-happening of the event is not affected by the concerning of the occurrence of any one of the remaining events.

An event that always happen is called **Certain event,** it is denoted by 'S'.

An event that never happens is called **Impossible event**, it is denoted by 'ϕ'.

Eg: In tossing a coin and throwing a die, getting head or tail is independent of getting no's 1 or 2 or 3 or 4 or 5 or 6.

Definition: probability (Mathematical Definition)

If a trial results in n-exhaustive mutually exclusive, and equally likely cases and m of them are favorable to the happening of an event E then the probability of an event E is denoted by P(E) and is defined as

$$P(E) = \frac{no\ of\ favourable\ cases\ to\ event}{Total\ no\ of\ exaustive\ cases} = \frac{m}{n}$$

Sample Space:

The set of all possible outcomes of a random experiment is called Sample Space .The elements of this set are called sample points. Sample Space is denoted by S.

Eg. (1) In throwing two dies experiment, Sample S contains 36 Sample points.

S = {(1,1) ,(1,2) ,----------(1,6), --------(6,1),(6,2),--------(6,6)}

Eg. (2) In tossing two coins experiment , S = {HH ,HT,TH,TT}

A sample space is called **discrete** if it contains only finitely or infinitely many points which can be arranged into a simple sequence $w_1, w_2, \ldots\ldots$ .while a sample space containing non denumerable no. of points is called a continuous sample space.

Statistical or Empirical Probability:

If a trial is repeated a no. of times under essential homogenous and identical conditions, then the limiting value of the ratio of the no. of times the event happens to the total no. of trials, as the number of trials become indefinitely large, is called the probability of happening of the event.(It is assumed the limit is finite and unique)

Symbolically, if in 'n' trials and events E happens 'm' times , then the probability 'p' of the happening of E is given by $p = P(E) = \lim_{n \to \infty} \frac{m}{n}$.

An event E is called **elementary event** if it consists only one element.

An event, which is not elementary, is called **compound event.**

Random Variables

- A random variable X on a sample space S is a function $X : S \to R$ from S onto the set of real numbers R, which assigns a real number X (s) to each sample point 's' of S.
- Random variables (r.v.) bare denoted by the capital letters X,Y,Z,etc..
- Random variable is a single valued function.
- Sum, difference, product of two random variables is also a random variable .Finite linear combination of r.v is also a r.v .Scalar multiple of a random variable is also random variable.
- A random variable, which takes at most a countable number of values, it is called a discrete r.v. In other words, a real valued function defined on a discrete sample space is called discrete r.v.
- A random variable X is said to be continuous if it can take all possible values between certain limits .In other words, a r.v is said to be continuous when it's different values cannot be put in 1-1 correspondence with a set of positive integers.
- A continuous r.v is a r.v that can be measured to any desired degree of accuracy. Ex : age , height, weight etc..
- Discrete Probability distribution: Each event in a sample has a certain probability of occurrence . A formula representing all these probabilities which a discrete r.v. assumes is known as the discrete probability distribution.
- The probability function or probability mass function (p.m.f) of a discrete random variable X is the function f(x) satisfying the following conditions.

  i) $f(x) \geq 0$

  ii) $\sum_{x} f(x) = 1$

  iii) $P(X = x) = f(x)$

- Cumulative distribution or simply distribution of a discrete r.v. X is F(x) defined by $F(x) = P(X \leq x) = \sum_{t \leq x} f(t)$ for $-\infty < x < \infty$

- If X takes on only a finite no. of values $x_1, x_2, \ldots x_n$ then the distribution function is given by

$$F(x) = \begin{cases} 0 & -\infty < x < x_1 \\ f(x_1) & x_1 \le x < x_2 \\ f(x_1)+f(x_2) & x_2 \le x < x_3 \\ \ldots & \\ f(x_1)+f(x_2)+\ldots+f(x_n) & x_n \le x < \infty \end{cases}$$

$F(-\infty) = 0$, $F(\infty)=1$, $0 \le F(x) \le 1$, $F(x) \le F(y)$ if $x<y$

$P(x_k) = P(X = x_k) = F(x_k) - F(x_{k-1})$

- For a continuous r.v. X, the function f(x) satisfying the following is known as the probability density function(p.d.f.) or simply density function:

    i) $f(x) \ge 0$, $-\infty < x < \infty$

    ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

    iii) $P(a<X<b) = \int_a^b f(x)\,dx$ = Area under f(x) between ordinates x=a and x=b

- $P(a<X<b) = P(a \le x < b) = P(a < X \le b) = P(a \le X \le b)$

(i.e) In case of continuous it does not matter weather we include the end points of the interval from a to b.This result in general is not true for discrete r.v.

- Probability at a point $P(X=a) = \int_{a-\Delta x}^{a+\Delta x} f(x)\,dx$

- Cumulative distribution for a continuous r.v. X with p.d.f. f(x), the cumulative distribution F(x) is defined as

$$F(x) = P(X \le x) = \int_{-\infty}^{\infty} f(t)\,dt \quad -\infty < x < \infty$$

It follows that $F(-\infty) = 0$, $F(\infty)=1$, $0 \le F(x) \le 1$ for $-\infty < x < \infty$

$f(x) = d/dx(F(x)) = F^1(x) \geq 0$ and $P(a < x < b) = F(b) - F(a)$

- In case of discrete r.v. the probability at a point i.e., P(x=c) is not zero for some fixed c however in case of continuous random variables the probability at appoint is always zero. I.e., P(x=c) = 0 for all possible values of c.
- P(E) = 0 does not imply that the event E is null or impossible event.
- If X and Y are two discrete random variables the joint probability function of X and Y is given by P(X=x,Y=y) = f(x,y) and satisfies

  (i) $f(x,y) \geq 0$ (ii) $\sum_x \sum_y f(x,y) = 1$

  The joint probability function for X and Y can be reperesented by a joint probability table.

Table

X \ Y	y_1	y_2	……	y_n	Totals
x_1	$f(x_1,y_1)$	$f(x_1,y_2)$	……..	$f(x_1,y_n)$	$f_1(x_1) = P(X=x_1)$
x_2	$F(x_2,y_1)$	$f(x_2,y_2)$	……..	$f(x_2,y_n)$	$f_1(x_2) = P(X=x_2)$
……..	…….	………	………	………	……..
x_m	$f(x_m,y_1)$	$f(x_m,y_2)$	…….	$f(x_m,y_n)$	$f_1(x_m) = P(X=x_m)$
Totals	$f_2(y_1) = P(Y=y_1)$	$f_2(y_2) = P(Y=y_2)$	……..	$f_2(y_n) = P(Y=y_n)$	1

The probability of$X = x_j$ is obtained by adding all entries in arrow corresponding to $X = x_j$

Similarly the probability of $Y = y_k$ is obtained by all entries in the column corresponding to Y $=y_k$

$f_1(x)$ and $f_2(y)$ are called marginal probability functions of X and Y respectively.

The joint distribution function of X and Y is defined by $F(x,y)= P(X\le x, Y\le y)= \sum_{u\le x}\sum_{v\le y} f(u,v)$

- If X and Y are two continuous r.v.'s the joint probability function for the r.v.'s X and Y is defined by

  (i) $f(x,y) \ge 0$ (ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = 1$

- $P(a < X < b, c< Y < d) = \int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dxdy$

- The joint distribution function of X and Y is $F(x,y) = P(X \le x, Y \le y)= \int_{u=-\infty}^{\infty}\int_{v=-\infty}^{\infty} f(u,v)\,dudv$

- $\frac{\partial^2 F}{\partial x \partial y} = f(x,y)$

- The Marginal distribution function of X and Y are given by $P(X \le x) = F_1(x)=$
  $\int_{u=-\infty}^{\infty}\int_{v=-\infty}^{\infty} f(u,v)\,dudv$ and $P(Y \le y) = F_2(y) = \int_{u=-\infty}^{\infty}\int_{v=-\infty}^{\infty} f(u,v)\,dudv$

- The marginal density function of X and Y are given by

  $f_1(x) = \int_{v=-\infty}^{\infty} f(x,v)\,dv$ and $f_2(y) = \int_{u=-\infty}^{\infty} f(u,y)\,du$

- Two discrete random variables X and Y are independent iff

  $P(X = x,Y = y) = P(X = x)P(Y = y)\ \forall\ x,y$ (or)

  $f(x,y) = f_1(x)f_2(y) \quad \forall\ x, y$

- Two continuous random variables X and Y are independent iff

$$P(X \le x, Y \le y) = P(X \le x)P(Y \le y) \ \forall\ x,y \quad \text{(or)}$$

$$f(x,y) = f_1(x)f_2(y) \quad \forall\ x, y$$

If X and Y are two discrete r.v. with joint probability function f(x,y) then

$$P(Y = y|X=x) = \frac{f(x,y)}{f_1(x)} = f(y|x)$$

Similarly, $P(X = x|Y=y) = \frac{f(x,y)}{f_2(y)} = f(x|y)$

If X and Y are continuous r.v. with joint density function f(x,y) then $\frac{f(x,y)}{f_1(x)} = f(y|x)$ and $\frac{f(x,y)}{f_2(y)} = f(x|y)$

Expectation or mean or Expected value : The mathematical expectation or expected value of r.v. X is denoted by E(x) or μ and is defined as

$$E(X) = \begin{cases} \sum_i x_i f(x_i) & X \text{ is discrete} \\ \int_{-\infty}^{\infty} xf(x)\,dx & X \text{ is Continuous} \end{cases}$$

- If X is a r.v. then $E[g(X)] = \begin{cases} \sum_x g(x)f(x) & \text{FOR Discrete} \\ \int_{-\infty}^{\infty} g(x)f(x)\,dx & \text{For Continuous} \end{cases}$

- If X, Y are r.v.'s with joint probability function f(x,y) then

$$E[g(X,Y)] = \begin{cases} \sum_x \sum_y g(x,y) f(x,y) & \text{for discrete r.v.'s} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f(x,y)\,dx\,dy & \text{for continuous r.v.'s} \end{cases}$$

If X and Y are two continuous r.v.'s the joint density function f(x,y) the conditional expectation or the conditional mean of Y given X is $E(Y \mid X = x) = \int_{-\infty}^{\infty} y f(y \mid x)\,dy$

Similarly, conditional mean of X given Y is $E(X \mid Y = y) = \int_{-\infty}^{\infty} x f(x \mid y)\,dx$

- Median is the point, which divides the entire distribution into two equal parts. In case of continuous distribution median is the point, which divides the total area into two equal parts. Thus, if M is the median then $\int_{-\infty}^{M} f(x)\,dx = \int_{M}^{\infty} f(x)\,dx = 1/2$. Thus, solving any one of the equations for M we get the value of median. Median is unique
- Mode: Mode is the value for f(x) or $P(x_i)$ at attains its maximum

  For continuous r.v. X mode is the solution of $f^1(x) = 0$ and $f^{11}(x) < 0$

  provided it lies in the given interval. Mode may or may not be unique.
- Variance: Variance characterizes the variability in the distributions with same mean can still have different dispersion of data about their means

  Variance of r.v. X denoted by Var(X) and is defined as

$$Var(X) = E\left[(X - \mu)^2\right] = \begin{cases} \sum_x (x-\mu)^2 f(x) & \text{for discrete} \\ \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx & \text{for continuous} \end{cases}$$

  where $\mu = E(X)$

- If c is any constant then E(cX) = c E(X)

- If X and Y are two r.v.'s then E(X+Y) = E(X)+E(Y)
- IF X,Y are two independent r.v.'s then E(XY) = E(X)E(Y)
- If X_1,X_2,-------,X_n are random variables then $E(c_1X_1 + c_2X_2 + \text{------} + c_nX_n) = c_1E(X_1)+c_2E(X_2)+\text{-----} + c_nE(X_n)$ for any scalars c_1,c_2,------,c_n If all expectations exists
- If X_1,X_2,-------,X_n are independent r.v's then $E\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} E(X_i)$ if all expectations exists.
- $Var(X) = E(X^2) - [E(X)]^2$
- If 'c' is any constant then $var(cX) = c^2 var(X)$
- The quantity $E[(X-a)^2]$ is minimum when $a = \mu = E(X)$
- If X and Y are independent r.v.'s then $Var(X \pm Y) = Var(X) \pm Var(Y)$

UNIT - V
PROBABILITY DISTRIBUTIONS

Binomial Distribution

- A random variable X is said to follow binomial distribution if it assumes only non-negative values and its probability mass function is given by

$$P(X = x) = P(x) = \begin{cases} \binom{n}{x} p^{x} q^{n-x} & \text{where } x = 0,1,2,3,\ldots,n \quad q = 1-p \\ 0 & \text{other wise} \end{cases}$$

where n, p are known as parameters, n- number of independent trials p- probability of success in each trial, q- probability of failure.

- Binomial distribution is a discrete distribution.
- The notation X ~ B(n,p) is the random variable X which follows the binomial distribution with parameters n and p
- If n trials constitute an experiment and the experiment is repeated N times the frequency function of the binomial distribution is given by f(x) = NP(x). The expected frequencies of 0,1,2,….. n successes are the successive terms of the binomial expansion $N(p+q)^n$
- The mean and variance of Binomial distribution are np , npq respectively.
- **Mode of the Binomial distribution**: Mode of B.D. Depending upon the values of (n+1)p

  (i) If (n+1)p is not an integer then there exists a unique modal value for binomial distribution and it is 'm'= integral part of (n+1)p

  (ii) If (n+1)p is an integer say m then the distribution is Bi-Modal and the two modal values are m and m-1
- Moment generating function of Binomial distribution: If X ~ B(n,p)then $M_X(t)=(q+pe^t)^n$
- The sum of two independent binomial variates is not a binomial varaite. In other words, Binomial distribution does not posses the additive or reproductive property.
- For B.D. $\gamma_1= \sqrt{\beta_1} = \frac{1-2p}{\sqrt{npq}}$ $\gamma_2= \beta_2 - 3 = \frac{1-6pq}{npq}$
- If $X_1 \sim B(n_1,p)$ and $X_2 \sim B(n_2,p)$ then $X_1+X_2 \sim B(n_1+n_2,p)$.Thus the B.D. Possesses the additive or reproductive property if $p_1=p_2$

Poisson Distribution

- Poisson Distribution is a limiting case of the Binomial distribution under the following conditions:

  (i) n, the number of trials is infinitely large.

(ii) P, the constant probability of success for each trial is indefinitely small.

(iii) np= λ, is finite where λ is a positive real number.

- A random variable X is said to follow a Poisson distribution if it assumes only non-negative values and its p.m.f. is given by

$$P(x,\lambda)= P(X= x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{x}}{x!} : & x= 0,1,2,3,\ldots\ldots \ \lambda > 0 \\ 0 & \text{Other wise} \end{cases}$$

Here λ is known as the parameter of the distribution.

- We shall use the notation X~ P(λ) to denote that X is a Poisson variate with parameter λ
- Mean and variance of Poisson distribution are equal to λ.
- The coefficient of skewness and kurtosis of the poisson distribution are $\gamma_1 = \sqrt{\beta_1} = 1/\sqrt{\lambda}$ and $\gamma_2 = \beta_2 - 3 = 1/\lambda$. Hence the poisson distribution is always a skewed distribution. Proceeding to limit as λ tends to infinity we get $\beta_1 = 0$ and $\beta_2=3$
- Mode of Poisson Distribution: Mode of P.D. Depending upon the value of λ

(i) when λ is not an integer the distribution is uni- modal and integral part of λ is the unique modal value.

(ii) When λ = k is an integer the distribution is bi-modal and the two modals are k-1 and k.

- Sum of independent poisson variates is also poisson variate.
- The difference of two independent poisson variates is not a poisson variate.
- **Moment generating function of the P.D.**

If X~ P(λ) then $M_X(t) = e^{\lambda(e^{t}-1)}$

- Recurrence formula for the probabilities of P.D. (Fitting of P.D.)

$$P(x+1) = \frac{\lambda}{x+1}\, p(x)$$

- Recurrence relation for the probabilities of B.D. (Fitting of B.D.)

$$P(x+1) = \left\{\frac{n-x}{x+1}.\frac{p}{q}\right\} p(x)$$

Normal Distribution

- A random variable X is said to have a normal distribution with parameters μ called mean and σ^2 called variance if its density function is given by the probability law

$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[\frac{-1}{2}\left\{\frac{x-\mu}{\sigma}\right\}^2\right], \quad -\infty < x < \infty,\ -\infty < \mu < \infty,\ \sigma > 0$$

- A r.v. X with mean μ and variance σ^2 follows the normal distribution is denoted by

  $X \sim N(\mu, \sigma^2)$

- If $X \sim N(\mu, \sigma^2)$ then $Z = \frac{X-\mu}{\sigma}$ is a standard normal variate with E(Z) = 0 and var(Z)=0 and we write $Z \sim N(0,1)$

- The p.d.f. of standard normal variate Z is given by $f(Z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$, $-\infty < Z < \infty$

- The distribution function $F(Z) = P(Z \le z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt$

- $F(-z) = 1 - F(z)$

- $P(a < z \le b) = P(a \le z < b) = P(a < z < b) = P(a \le z \le b) = F(b) - F(a)$

- If $X \sim N(\mu, \sigma^2)$ then $Z = \frac{X-\mu}{\sigma}$ then $P(a \le X \le b) = F\left(\frac{b-\mu}{\sigma}\right) - F\left(\frac{a-\mu}{\sigma}\right)$

- N.D. is another limiting form of the B.D. under the following conditions:

  i) n , the number of trials is infinitely large.

  ii) Neither p nor q is very small

- **Chief Characteristics of the normal distribution and normal probability curve:**

  i) The curve is bell shaped and symmetrical about the line x = μ

  ii) Mean median and mode of the distribution coincide.

  iii) As x increases numerically f(x) decreases rapidly.

  iv) The maximum probability occurring at the point x= μ and is given by

  $[P(x)]_{max} = 1/\sigma\sqrt{2\Pi}$

v) $\beta_1 = 0$ and $\beta_2 = 3$

vi) $\mu_{2r+1} = 0$ (r = 0,1,2……) and $\mu_{2r} = 1.3.5….(2r-1)\sigma^{2r}$

vii) Since f(x) being the probability can never be negative no portion of the curve lies below x- axis.

viii) Linear combination of independent normal variate is also a normal variate.

ix) X- axis is an asymptote to the curve.

x) The points of inflexion of the curve are given by $x = \mu \pm \sigma$, $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2}$

xi) Q.D. **:** M.D.**:** S.D. **::** $\frac{2}{3}\sigma$ **:** $\frac{4}{5}\sigma$ **:** σ **::** $\frac{2}{3}$ **:** $\frac{4}{5}$ **:** 1 Or Q.D. **:** M.D.**:** S.D. **::10:12:15**

xii) Area property: $P(\mu- \sigma < X < \mu + \sigma) = 0.6826 = P(-1 < Z < 1)$

$P(\mu- 2\sigma < X < \mu + 2\sigma) = 0.9544 = P(-2 < Z < 2)$

$P(\mu- 3\sigma < X < \mu +3 \sigma) = 0.9973 = P(-3 < Z < 3)$

$P(|Z| > 3) = 0.0027$

- m.g.f. of N.D. If $X\sim N(\mu, \sigma^2)$ then $M_X(t) = e^{\mu t} + t^2\sigma^2/2$

  If $Z\sim N(0,1)$ then $M_Z(t) = e^{t^2/2}$

Continuity Correction:

- The N.D. applies to continuous random variables. It is often used to approximate distributions of discrete r.v. Provided that we make the continuity correction.

- If we want to approximate its distribution with a N.D. we must spread its values over a continuous scale. We do this by representing each integer k by the interval from k-1/2 to k+1/2 and at least k is represented by the interval to the right of k-1/2 to at most k is represented by the interval to the left of k+1/2.

- **Normal approximation to the B.D:**

$X\sim B(n, p)$ and if $Z = \frac{X - np}{\sqrt{np(1-p)}}$ then $Z \sim N(0,1)$ as n tends to infinity and F(Z) =

$$F(Z)= P(Z \le z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt \quad -\infty < Z < \infty$$

- Use the normal approximation to the B.D. only when (i) np and n(1-p) are both greater than 15 (ii) n is small and p is close to ½

- **Poisson process:** Poisson process is a random process in which the number of events (successes) x occurring in a time interval of length T is counted. It is continuous parameter, discrete stable process. By dividing T into n equal parts of length Δt we have $T = n \cdot \Delta T$. Assuming that (i) $P \propto \Delta T$ or $P = \alpha \Delta t$ (ii) The occurrence of events are independent (iii) The probability of more than one substance during a small time interval Δt is negligible.

  As $n \to \infty$, the probability of x success during a time interval T follows the P.D. with parameter $\lambda = np = \alpha T$ where α is the average(mean) number of successes for unit time.

PROBLEMS:

1:A random variable x has the following probability function:

x	0	1	3	4	5	6	7
P(x)	0	k	2k	2k	3k	k^2	$7k^2$+k

Find (i) k (ii) P(x<6) (iii) P(x>6)

Solution:

(i) since the total probability is unity, we have $\sum_{x=0}^{n} p(x) = 1$

i.e., $0 + k + 2k + 2k + 3k + k^2 + 7k^2 + k = 1$

i.e., $8k^2 + 9k - 1 = 0$

$k = 1, -1/8$

(ii) $P(x<6) = 0 + k + 2k + 2k + 3k$

$= 1+2+2+3 = 8$

iii) P(x>6)= $k^2 + 7k^2$+k
=9

2. Let X denotes the minimum of the two numbers that appear when a pair of fair dice is thrown once. Determine (i) Discrete probability distribution (ii) Expectation (iii) Variance

Solution:

When two dice are thrown, total number of outcomes is 6x6-36

In this case, sample space S=

$$\begin{Bmatrix} (1,1)(1,2)(1,3)(1,4)(1,5)(1,6) \\ (2,1)(2,2)(2,3)(2,4)(2,5)(2,6) \\ (3,1)(3,2)(3,3)(3,4)(3,5)(3,6) \\ (4,1)(4,2)(4,3)(4,4)(4,5)(4,6) \\ (5,1)(5,2)(5,3)(5,4)(5,5)(5,6) \\ (6,1)(6,2)(6,3)(6,4)(6,5)(6,6) \end{Bmatrix}$$

If the random variable X assigns the minimum of its number in S, then the sample space S=

$$\begin{Bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{Bmatrix}$$

The minimum number could be 1,2,3,4,5,6

For minimum 1, the favorable cases are 11

Therefore, P(x=1)=11/36

P(x=2)=9/36, P(x=3)=7/36, P(x=4)=5/36, P(x=5)=3/36, P(x=6)=1/36

The probability distribution is

X	1	2	3	4	5	6
P(x)	11/36	9/36	7/36	5/36	3/36	1/36

(ii)Expectation mean = $\sum p_i x_i$

$$E(x) = 1\frac{11}{36} + 2\frac{9}{36} + 3\frac{7}{36} + 4\frac{5}{36} + 5\frac{3}{36} + 6\frac{1}{36}$$

Or $\mu = \frac{1}{36}[11 + 8 + 21 + 20 + 15 + 6] = \frac{9}{36} = 2.5278$

(ii) variance = $\sum p_i x^2{}_i - \mu^2$

$$E(x) = \frac{11}{36}1 + \frac{9}{36}4 + \frac{7}{36}9 + \frac{5}{36}16 + \frac{3}{36}25 + \frac{1}{36}36 - (2.5278)^2$$

=1.9713

3: A continuous random variable has the probability density function

$$f(x) = \begin{cases} kxe^{-\lambda x}, \ for\ x \geq 0, \lambda > 0 \\ 0, otherwise \end{cases}$$

Determine (i) k (ii) Mean (iii) Variance

Solution:

(i) since the total probability is unity, we have $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{0} 0dx + \int_{0}^{\infty} kxe^{-\lambda x} dx = 1$$

i.e., $\int_{0}^{\infty} kxe^{-\lambda x} dx = 1$

$$k\left[x\left(\frac{e^{-\lambda x}}{-\lambda}\right) - 1\left(\frac{e^{-\lambda x}}{\lambda^2}\right)\right]_0^{\infty} \ or\ k = \lambda^2$$

(ii) mean of the distribution $\mu = \int_{-\infty}^{\infty} xf(x)dx$

$$\int_{-\infty}^{0} 0dx + \int_{0}^{\infty} kx^2 e^{-\lambda x} dx$$

$$\lambda^2\left[x^2\left(\frac{e^{-\lambda x}}{-\lambda}\right) - 2x\left(\frac{e^{-\lambda x}}{\lambda^2}\right) + 2\left(\frac{e^{-\lambda x}}{\lambda^3}\right)\right]_0^{\infty}$$

$= \frac{2}{\lambda}$

Variance of the distribution $\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - \frac{4}{\lambda^2}$$

$$\lambda^2 \left[x^3 \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 3x^2 \left(\frac{e^{-\lambda x}}{\lambda^2} \right) + 6x \left(\frac{e^{-\lambda x}}{\lambda^3} \right) - 6 \left(\frac{e^{-\lambda x}}{\lambda^4} \right) \right]_0^{\infty} - \frac{4}{\lambda^2}$$

$$= \frac{2}{\lambda^2}$$

4:

Out of 800 families with 5 children each, how many would you expect to have (i)3 boys (ii)5girls (iii)either 2 or 3 boys ? Assume equal probabilities for boys and girls

Solution(i)

P(3boys)=P(r=3)=P(3)= $\frac{1}{2^5} {}^5C_3 = \frac{5}{16}$ per family

Thus for 800 families the probability of number of families having 3 boys= $\frac{5}{16}(800) = 250$ families

(iii)

P(5 girls)=P(no boys)=P(r=0)= $\frac{1}{2^5} {}^5C_0 = \frac{1}{32}$ per family

Thus for 800 families the probability of number of families having 5girls= $\frac{1}{32}(800) = 25$ families

(iv) P(either 2 or 3 boys =P(r=2)+P(r=3)=P(2)+P(3)

$\frac{1}{2^5} {}^5C_2 + \frac{1}{2^5} {}^5C_3$ =5/8 per family

Expected number of families with 2 or 3 boys = $\frac{5}{8}(800) = 500$ families.

5: Average number of accidents on any day on a national highway is 1.8. Determine the probability that the number of accidents is (i) at least one (ii) at most one

Solution:

Mean= $\lambda = 1.8$

We have P(X=x)=p(x)$\frac{e^{-\lambda}\lambda^x}{x!}$ =$\frac{e^{-1.8}1.8^x}{x!}$

(i)P (at least one) =P(x≥1)=1-P(x=0)

=1-0.1653

=0.8347

P (at most one) =P (x≤1)

=P(x=0)+P(x=1)

= 0.4628

6: The mean weight of 800 male students at a certain college is 140kg and the standard deviation is 10kg assuming that the weights are normally distributed find how many students weigh I) Between 130 and 148kg ii) more than 152kg

Solution:

Let μ be the mean and σ be the standard deviation. Then μ =140kg and σ =10pounds

(i) When x= 138, $z = \frac{x - \mu}{\sigma} = \frac{138 - 140}{10} = -0.2 = z_1$

When x= 138, $z = \frac{x - \mu}{\sigma} = \frac{148 - 140}{10} = 0.8 = z_2$

$\therefore$ P(138≤x≤148)=P(-0.2≤z≤0.8)

=A(z_2)+A(z_1)

=A(0.8)+A(0.2)=0.2881+0.0793=0.3674

Hence the number of students whose weights are between 138kg and 140kg =0.3674x800=294

(ii) When x=152,$\frac{x-\mu}{\sigma}$=$\frac{152-140}{10} = 1.2$=z_1

Therefore P(x>152)=P(z>z_1)=0.5-A(z_1)

=0.5-0.3849=0.1151

Therefore number of students whose weights are more than 152kg =800x0.1151=92.

Exercise Problems:

1. Two coins are tossed simultaneously. Let X denotes the number of heads then find i) E(X) ii) $E(X^2)$ iii)$E(X^3)$ iv) V(X)
2. If f(x)=k$e^{-|x|}$ is probability density function in the interval, $-\infty < x < \infty$, then find i) k ii) Mean iii) Variance iv) P(0<x<4)
3. Out of 20 tape recorders 5 are defective. Find the standard deviation of defective in the sample of 10 randomly chosen tape recorders. Find (i) P(X=0) (ii) P(X=1) (iii) P(X=2) (iv) P (1<X<4).
4. In 1000 sets of trials per an event of small probability the frequencies f of the number of x of successes are

f	0	1	2	3	4	5	6	7	Total
x	305	365	210	80	28	9	2	1	1000

Fit the expected frequencies.

5.If X is a normal variate with mean 30 and standard deviation 5. Find the probabilities that i) P(26 ≤ X≤40) ii) P(X ≥ 45)

6. The marks obtained in Statistics in a certain examination found to be normally distributed. If 15% of the students greater than or equal to 60 marks, 40% less than 30 marks. Find the mean and standard deviation.

7.If a Poisson distribution is such that $P(X=1)=\frac{3}{2}P(X=3)$ then find (i) $P(X \geq 1)$ (ii) $P(X \leq 3)$ (iii) $P(2 \leq X \leq 5)$.

8. A random variable X has the following probability function:

X	-2	-1	0	1	2	3
P(x)	0.1	K	0.2	2K	0.3	K

Then find (i) k (ii) mean (iii) variance (iv) P(0 < x < 3)